ON THE FIXED POINT PROPERTIES OF GRASSMAN MANIFOLDS

by

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Abstract:

In this thesis we show that for n even, the Grassman manifolds $\mathbb{C}G_{3,n}$ have the fixed point property and that $\mathbb{C}G_{3,4}$ has Lusternik - Schnirelmann Category 13.

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INTRODUCTION

In the first chapter we give a proof of the famous Lefschetz - Hopf Fixed Point Theorem. In the second chapter we use this theorem to show that various examples of familiar manifolds have the fixed point property. We also give examples of manifolds which do not have the fixed point property and mention how manifolds with this property may be constructed. In the third chapter we find the cohomology of the Grassman manifolds. Due to the nature of these cohomology rings, it does not appear likely at first glance that one can prove using only the Lefschetz - Hopf Fixed Point Theorem, as is done with the projective spaces, that certain of these manifolds have the fixed point property. Bachmann, Glover and O'Neill, however, have shown that for some Grassman manifolds (See Theorem 4.1), the endomorphism of the cohomology ring induced by any self map of such a manifold must take the first characteristic class to a times the first characteristic class, the second characteristic class to a^2 the second characteristic class and so on. Such a self map is known as an Adams type mapping. If a self map of an appropriate Grassman manifold is an Adams type mapping, it is easy to show that it's Lefschetz - Hopf number is nom - zero through the use of well known theorems concerning the Euler number of these manifolds. We know from Theorem 4.1 that if n is even and greater than or equal to ten that $\mathbb{C}G_{3,n}$ has the fixed point property. O'Neill, in his doctoral thesis, proved that $\mathbb{C}G_{3,2}$ has the fixed point property, and so to prove that $\mathbb{C}G_{3,n}$ has the fixed point property for all even n we only have to show that the remaining three manifolds have the fixed point property. In proving that $\mathbb{C}G_{3,4}$ has the fixed point property, we also prove that it has maximal cuplength and thus we are able to determine it's Lusternik - Schnirelmann Category. (See Page 39). It should be noted that since $H^*(\mathbb{C}G_{n,k};\mathbb{Z})$ has no torsion and there is the natural injection of the integers into the rationals, the Lefschetz - Hopf number will be the same whether calculated with integral or rational coefficients and thus in all the calculations of Chapter 5 we may assume that the variables involved take their values only in the integers.

1. The Lefschetz - Hopf Fixed Point Theorem

In the following chapter, let X be a connected n - dimensional manifold and R a commutative ring. For more details see (1).

Locally Constant Lemma: Let j_x^U: $H_n(X,X-U;R) \longrightarrow H_n(X,X-x;R)$ be the homomorphism induced by the inclusion $X-U \longrightarrow X-x$ for an open neighbourhood U of a point x in X. Every neighbourhood W of x in X then contains a neighbourhood V of x such that for every y in V, j_y^V is an isomorphism. Hence a generator ξ_x of $H_n(X,X-x;R) \cong R$ has a unique continuation in U.

Definitions:

Given a subspace $U \subset X$, an element $\xi \in H_n(X,X-x;R)$ such that $j_y^U(\xi)$ generates $H_n(X,X-y;R)$ for each $y \in U$ is called a local R - orientation of X along U. An R - orientation system is a family of open subspaces U_i which cover X such that for each i there is a local R - orientation $\xi_i \in H_n(X,X-U_i;R)$ of X along U_i and if $x \in U_i \cap U_{i'}$, then $j_x^{U_i}(\xi_i) = j_x^{U_{i'}}(\xi_{i'})$. X is said to be R - orientable (orientable) if an R - orientation ($\mathbb{Z}$ - orientation) system for X exists.

Let $X^0 = \{(x, \xi_x) | x \in X, \xi_x \in H_n(X,X-x;R)\}$ and $p: X^0 \longrightarrow X$ be defined by $p(x, \xi_x) = x$. The sets $\langle U, \xi_U \rangle = \{(x, \xi_x) | x \in U, \xi_x = j_x^U(\xi_U)\}$ form a base for the topology on X^0. X^0 is known as the R - orientation sheaf of X. For any subspace $A \subset X$, a map $s: A \longrightarrow X$ such that ps = inclusion is called a section over A. Let $s(x) = (x, s'(x))$ define a section over A. Then $x \longmapsto (x, \lambda s'(x))$ for some λ in R also defines a section over A. Also, if s_1 and s_2 are sections over A, then $x \longmapsto (x, s_1'(x) + s_2'(x))$ is a section over A.

Therefore the set of all sections over A is an R - module ΓA. There is a canonical homomorphism $j_A: H_n(X,X-A;R) \longrightarrow \Gamma A$ defined by $j_A(\xi)(x) = (x, j_x(\xi))$ for x in A.

Theorem 1.1: Suppose $A \subset X$ is closed. Then

i) $H_q(X,X-A;R) = 0$ for $q > n$

ii) j_A is a monomorphism and its image is the submodule $\Gamma_C A$ of sections over A with compact support. A section has compact support if it agrees with the zero section outside some compact subset of A. In particular, if A is compact, $\Gamma_C A = \Gamma A$, and if X is compacttthen $j_A: H_n(X;R) \xrightarrow{\cong} X$ is an isomorphism.

Corollary 1.2: Let X be a compact connected manifold and R an integral domain. Then $H_n(X;R)$ is isomorphic to R if X is R - orientable and 0 if not.

An R - orientation of a compact connected manifold X is therefore determined by a generator of ΓX or a generator β of $H_n(X;R)$ which is called the fundamental class of the R- orientation of X. The local R - orientation at each point x in X is then $j_x^X(\beta)$.

Given an R - orientation of X, that is, given a global section $s: X \longrightarrow X^0$ such that for each x, $s'(x)$ generates $H_n(X,X-x;R)$, there exists the dual sheaf $X^{0^*} \longrightarrow X$ whose fibre over a point x in X is the local cohomology module $H^n(X,X-x;R)$ and a global section $s^*: X \longrightarrow X^{0^*}$ which is characterized by $\langle s^*(x), s(x)\rangle = 1$ for x in X. Suppose U is open in X. Denote by $\Gamma^* U$ the module of all sections over U of the dual sheaf. If Δ is the diagonal of

$X \times X$, define $U^{i_X}: (X, X-x) \longrightarrow (X\times U, X\times U - \Delta)$ by $U^{i_X}(x') = (x', x)$, x' in X, x in U.

<u>Theorem 1.3</u>: Let X be an R - oriented n - dimensional manifold, U an open subspace. Then $H^q(X\times U, X\times U - \Delta) = 0$ for all $q < n$, and there is a unique isomorphism $\phi: H^n(X\times U, X\times U - \Delta) \longrightarrow \Gamma^* U$ such that $\phi(\alpha)(x) = H^n(U^{i_X})(\alpha)$ for all $\alpha \in H^n(X\times U, X\times U - \Delta)$, x in U.

<u>Corollary 1.4</u>: There is a unique cohomology class $\mu = \mu_X$ in $H^n(X\times X, X\times X - \Delta)$ such that for all x in X, $s^*(x) = H^n(X^{i_X})(\mu)$. This class μ is called the <u>Thom class</u> of the given orientation.

<u>Corollary 1.5</u>: Suppose X is compact. Let $\beta \in H_n(X;R)$ be the fundamental class of the R - orientation. Let $H^n(j): H^n(X\times X, X\times X - \Delta) \longrightarrow H^n(X\times X)$ be the homomorphism induced by inclusion and let $\mu' = H^n(j)(\mu)$. Then $\mu'/\beta = 1$, where / is the homology slant product.

<u>Proof</u>: For any x in X consider the commutative diagram:

$$\begin{tikzcd}
(X, X-x) \arrow[r, "X^{i_X}"] & (X\times X, X\times X - \Delta) \\
X \arrow[u, "j_X"] \arrow[r, "i_X"] & X\times X \arrow[u, "j"']
\end{tikzcd}$$

Then $1 = \langle H^n(X^{i_X})\mu, s(x)\rangle = \langle H^n(X^{i_X})\mu, H(j_x)\beta\rangle = \langle H^n(X^{i_X} j_x)\mu, \beta\rangle =$ $\langle H^n(j i_x)\mu, \beta\rangle = \langle \mu', H_n(i_x)\beta\rangle$. However, $H_n(i_x)\beta = \beta \times x$ is a property of the exterior homology product and $\langle \mu', \beta \times x\rangle = \langle \mu'/\beta, x\rangle$, a property of the homology slant product. (See appendix) Q.E.D.

Before proceeding to the proof of the Lefschetz - Hopf Fixed Point Theorem, we need some additional facts about compact manifolds.

i) A compact manifold can be embedded in a Euclidean space.

ii) A space is called an <u>Absolute Neighbourhood Retract</u>, (ANR), if for any normal space Y and a map $f: B \longrightarrow X$ of a closed subspace B of Y into the space X, f extends to a map of an open neighbourhood of B into X. It is known (see for instance (2)) that every compact manifold is an ANR and consequently if a compact manifold is embedded in some Euclidean space, it is the retract of some open neighbourhood (just apply the universal property to B = the manifold and f = the identity).

<u>Lemma 1.6</u>: If X is a compact manifold and Δ is the diagonal in $X \times X$, then there is an open neighbourhood V of Δ such that the identity map of V is homotopic in $X \times X$ to a retraction of V onto Δ.

<u>Proof</u>: Embed X in R^N, and let U be an open neighbourhood having a retraction $r: U \longrightarrow X$. Let ε = the distance from X to $R^N - U$, and let V be the ε - neighbourhood of Δ in $X \times X$. Define $F: X \times X \times I \longrightarrow R^N$ by $F(x,x',t) = (1-t)x + tx'$. Then F maps $V \times I$ into U. Let $G = r(F|V \times I): V \times I \longrightarrow X$ so that $G(x,x',0) = r(x) = x$, $G(x,x',1) = x'$. Define $H: V \times I \longrightarrow X \times X$ by $H(x,x',t) = (x, G(x,x',t))$ which is the required homotopy. Q.E.D.

<u>Lemma 1.7</u>: If $\gamma \in H^p(X \times X, X \times X - \Delta)$, $\eta \in H^q(X)$, then $H^p(j)(\gamma) \cup (\eta \times 1) = H^p(j)(\gamma) \cup (1 \times \eta)$ where $j: X \times X \longrightarrow (X \times X, X \times X - \Delta)$ is the inclusion.

<u>Proof</u>: By Lemma 1.6, there is an open neighbourhood V of Δ in $X \times X$ and a

retraction $r: V \longrightarrow \triangle$ such that $ir \simeq k$, where $i: \triangle \longrightarrow X \times X$ and $k: V \longrightarrow X \times X$ are inclusion maps. Denote the inclusion $(V, V-\triangle) \longrightarrow (X\times X, X\times X - \triangle)$ by k', and note that k' is an excision. The following diagram is commutative:

$$
\begin{tikzcd}
 & H^q(X\ \ X) \arrow[r] \arrow[d, "H^q(k)"] & H^{p+q}(X\ \ X, X\ \ X - \triangle) \arrow[d, "H^{p+q}(k')"] \\
H^q(\triangle) \arrow[ur] \arrow[r] & H^q(V) \arrow[r] & H^{p+q}(V, V-\triangle)
\end{tikzcd}
$$

Let $p_i: X\times X \longrightarrow X$, $i = 1,2$; be the projections. Then $1\times\eta = H^0(p_1)(1) \cup H^q(p_2)(\eta) = H^q(p_2)(\eta)$, $\eta\times 1 = H^p(p_1)(\eta)$. Let $p: \triangle \longrightarrow X$ be the common restriction of p_1 and p_2 to the diagonal. From the diagram we have
$\gamma \cup H^q(p_i)(\eta) = H^{p+q}(k')^{-1}H^p(k')(\gamma) \cup H^q(p_i ir)(\eta) = H^{p+q}(k')^{-1}H^p(k')(\gamma) \cup H^q(pr)(\eta)$ for both $i = 1,2$. By the properties of the cup product we have $H^p(j)(\gamma) \cup H^q(p_i)(\eta) = H^{p+q}(\gamma \cup H^q(p_i)(\eta))$ which proves the lemma. Q.E.D.

<u>Lemma 1.8</u>: The basic formula relating all the products is:
$$\{(\xi\times\eta)\cup\gamma\}/\alpha = (-1)^{q(p+q+r-s)}\eta\cup\{\gamma/\alpha\cap\xi\}$$
for $\gamma \in H^p(X\times Y)$, $\xi \in H^q(X)$, $\eta \in H^r(Y)$, $\alpha \in H_s(X)$.

<u>Proof</u>: Suppose $\psi \in H_{p+q+r-s}(Y)$. Then

$[\eta\cup\{\xi\cap\gamma/\alpha\},\psi] = [\xi\cap\gamma/\alpha, \eta\cap\psi]$ - a property of $\cap$

$= [\gamma, (\xi\cap\alpha)\times(\eta\cap\psi)]$ - a property of/

$= (-1)^{q(p+q+r-s)}\,[\gamma, (\xi\times\eta)\cap(\alpha\times\psi)]$ - a property of $\cap$

$= (-1)^{q(p+q+r-s)}\,[(\xi\times\eta)\cup\gamma, \alpha\times\psi]$

$= (-1)^{q(p+q+r-s)}\,[\{(\xi\times\eta)\cup\gamma\}/\alpha, \psi]$ Q.E.D.

<u>Theorem 1.9</u>: Let X be a compact R - oriented n - dimensional manifold with fundamental class $\beta \in H_n(X)$. Then for any $p \leq n$, the inverse to the Poincare Duality isomorphism $H^p(X) \longrightarrow H_{n-p}(X)$ is given by $\alpha \longrightarrow (-1)^p \mu'/\alpha$ for $\alpha \in H_{n-p}(X)$.

<u>Proof</u>: If $\eta \in H^p(X)$, then $\beta \frown \eta$ is its image in $H_{n-p}(X)$ and $\mu'/\beta \frown \eta$
$= 1 \cup \{\mu'/\beta \times \eta\} = (-1)^{p(n+p+0-n)} \{(\eta \times 1) \cup \mu'/\beta\}$ by Lemma 1.8
$= (-1)^{p^2}\{(1 \times \eta) \cup \mu'/\beta\}$ by Lemma 1.7
$= (-1)^p(-1)^0 \eta \cup \{\mu'/1 \frown \beta\}$ by Lemma 1.8
$= (-1)^p \eta \cup (\mu'/\beta)$
$= (-1)^p \eta \cup 1$ by Corollary 1.5
$= (-1)^p \eta$ Q.E.D.

Suppose we have a map $f: X \longrightarrow Y$, where Y is another compact R - oriented manifold of dimension m. We define the cohomology class μ_f of the graph of f by $\mu_f = H^m(f \times id)(\mu') \in H^m(X \times Y)$ where $\mu' \in H^m(Y\ \ Y)$ is the image of the Thom class of Y. The class μ_f completely determines the homomorphism induced by f on the cohomology.

<u>Lemma 1.10</u>: For any $\eta \in H^p(Y)$, $H^p(f)(\eta) = (-1)^p \mu_f/\beta_Y \frown \eta$ where $\beta_Y \in H_m(Y)$ is the fundamental class of Y.

<u>Proof</u>: $\mu_f/\eta \frown \beta_Y = H^{p+m-p}(f \times id)\mu'_Y/\eta \frown \beta_Y$
$= H^p(f)(\mu'_Y)/H_{n-p}(id)(\eta \frown \beta_Y)$ by naturality of /
$= H^p(f)(\mu'_Y)/\eta \frown \beta_Y$
$= (-1)^p H^p(f)\eta$ by Theorem 1.9 Q.E.D.

<u>Theorem 1.11</u>: Let $f\colon X \longrightarrow X$ where X is a connected compact R - oriented manifold. If $\mu_f \neq 0$ then f has a fixed point.

<u>Proof</u>: If f has no fixed point, then we have the factorization

$$\begin{tikzcd} & X \times X - \Delta \arrow[d, "i"] \\ X \times X \arrow[ur] \arrow[r, "f \times id"'] & X \times X \end{tikzcd}$$

where i is the inclusion. Since $H^n(i)H^n(j) = 0$ and $\mu'_X = H^n(j)(\mu_X)$, $\mu_f = H^n(f \times id)(\mu'_X) = 0$. Q.E.D.

Next, define the <u>Lefschetz - Hopf class</u> $L_f = H^n(f,id)(\mu')$ and the <u>Lefschetz - Hopf number</u> $\Lambda_f = \langle L_f, \beta \rangle$.

<u>Theorem 1.12</u>: <u>The Lefschetz - Hopf Fixed Point Theorem</u>

Let X be a compact connected R - oriented manifold, where R is a field. If $f\colon X \longrightarrow X$ is any map, then the Lefschetz - Hopf number of f is given by $\Lambda_f = \sum_q (-1)^q \text{Trace } H^q(f)$. If $\Lambda_f \neq 0$, then f has a fixed point.

<u>Proof</u>: Let $\{\alpha_i\}$ be a basis for $H^*(X)$, where i runs through a finite set; let q_i be the integer such that $\alpha_i \in H^{q_i}(X)$. By the Kunneth formula (since R is a field), the $\alpha_i \times \alpha_j$ form a basis for $H^*(X \times X)$, so that

$\mu' = \sum_{i,j} c_{ij} \alpha_i \times \alpha_j$ (where $c_{ij} = 0$ if $q_i + q_j \neq n$)

Let $H^*(f)(\alpha_i) = \sum_k a_{ki} \alpha_k$ (where $a_{ki} = 0$ if $q_k \neq q_i$)

Let $y_{kj} = \langle \alpha_j, \alpha_k \frown \beta \rangle = \langle \alpha_k \smile \alpha_j, \beta \rangle$ so that $y_{jk} = (-1)^{q_k(n-q_k)} y_{kj}$ when $q_k + q_j = n$, and $y_{kj} = 0$ when $q_k + q_j \neq n$.

$\Lambda_f = \sum_{i,j} c_{ij} \langle H^*(f,id)(\alpha_i \times \alpha_j), \beta \rangle = \sum_{i,j} c_{ij} \langle H^*(f)(\alpha_i) \cup \alpha_j, \beta \rangle$
$= \sum_{i,j} c_{ij} a_{ki} \langle \alpha_k \cup \alpha_j, \beta \rangle = \sum_{i,j} c_{ij} a_{ki} y_{kj} = \sum_{i,k} a_{ki} (\sum_j c_{ij} y_{kj}).$

However:

$(-1)^{q_k} \alpha_k = \mu' / \alpha_k \cap \beta$ by Lemma 1.10
$= (\sum_{j,i} c_{ji} \alpha_j \times \alpha_i) / \alpha_k \cap \beta$
$= \sum_{j,i} c_{ji} (-1)^{q_k(n-q_k)} \langle \alpha_j, \alpha_k \cap \beta \rangle \alpha_i$ by Lemma 1.8
$= \sum_i (\sum_{j,k} c_{ji} y_{jk}) \alpha_i$ so that $\sum_j c_{ji} y_{jk} = (-1)^{q_k} \delta_{ik}$ because the α_i's form a basis. Since the right inverse of a matrix is also its left inverse,

$\sum_j c_{ij} y_{kj} = (-1)^{q_k} \delta_{ik}.$

Thus $\Lambda_f = \sum_k (-1)^{q_k} a_{kk}.$ Q.E.D.

2. Manifolds with the Fixed Point Property

A topological space, X, is said to have the fixed point property if every self map of X has a fixed point. That is, if $f: X \longrightarrow X$ is a map then $f(x) = x$ for some x in X. There are many examples of topological spaces that have the fixed point property, (see for example (3)), however relatively few of these are manifolds. The following is a list of some of these manifolds.

1. A compact manifold X is said to be Q - acyclic if $H^p(X;Q) = 0$ for all $p \neq 0$, where Q denotes the field of rational numbers. Such a manifold has the fixed point property since $\Lambda_f = 1$ for all maps $f: X \longrightarrow X$. In particular, the n - discs $D^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ have the fixed point property. This result is known as the Brouwer Fixed Point Theorem.

2. Real projective 2n - space, $\mathbb{R}P^{2n}$, $n \geq 1$ has the fixed point property since it is Q - acyclic. The manifolds, $\mathbb{R}P^{2n+1}$, however, do not have the fixed point property. Since $\mathbb{R}P^{2n+1}$ = the set of equivalence classes of vectors $(r_0, \ldots, r_{2n+1}) \in \mathbb{R}^{2n+2} - (0,0,\ldots,0)$ where $(r_0,\ldots,r_{2n+1})$ is equivalent to $(r'_0,\ldots,r'_{2n+1})$ if and only if $(r_0,\ldots,r_{2n+1}) = c(r'_0,\ldots,r'_{2n+1})$ for some non - zero real number c, the function $f: \mathbb{R}^{2n+2} - (0,0,\ldots,0) \longrightarrow \mathbb{R}^{2n+2} - (0,0,\ldots,0)$ defined by $f(r_0,\ldots,r_{2n+1}) = (-r_1, r_0, \ldots, -r_{2n+1}, r_{2n})$ induces a self map of $\mathbb{R}P^{2n+1}$ which fixes no point.

3. Complex projective 2n - space, $\mathbb{C}P^{2n}$, $n \geq 1$, has the fixed point property.

The cohomology ring of $\mathbb{C}P^{2n}$ with rational coefficients is the truncated polynomial algebra generated by a generator c of $H^2(\mathbb{C}P^{2n};Q)$ where $c^{2n+1} = 0$. Therefore if f is a self map of $\mathbb{C}P^{2n}$ and $f^*c = ac$ then $\Lambda_f = 1 + a + a^2 + \dots + a^{2n}$. If $a \neq 1$ then $\Lambda_f = (1 - a^{2n+1})/(1-a) \neq 0$. If $a = 1$ then $\Lambda_f = 2n + 1$ and hence $\mathbb{C}P^{2n}$ has the fixed point property. However, if $f\colon \mathbb{C}P^{2n+1} \longrightarrow \mathbb{C}P^{2n+1}$ is defined by $f[(c_0,\dots,c_{2n+1})] = [(-c_1,c_0,\dots,-c_{2n+1},c_{2n})]$, we see that the $\mathbb{C}P^{2n+1}$'s do not have the fixed point property.

4. <u>Quaternionic projective space</u>, $\mathbb{H}P^n$, has the fixed point property for all n greater than 1. ($\mathbb{H}P^1 \approx S^4$ does not)

<u>Proof</u>: $H^*(\mathbb{H}P^n;Q) = Q[h_1]/h_1^{n+1}$, the truncated polynomial algebra generated by h_1, a generator of $H^4(\mathbb{H}P^n;Q)$. Therefore the Lefschetz - Hopf number, Λ_f for a self map f of $\mathbb{H}P$ is $\sum_0^n (-1)^i a^i$ where $f^*h_1 = ah_1$. $\Lambda_f = 0$ only if $a = -1$. We shall show however that a cannot assume the value -1

Let $P^1\colon H^q(X;\mathbb{Z}_3) \longrightarrow H^{q+4}(X;\mathbb{Z}_3)$ be the reduced power operation. (See (4)). There exists a map g which makes the following diagram of Hopf bundles commute and is such that $g^*h_1 = c_1^2$ where c_1 is the generator of $H^2(\mathbb{H}P^{2n};\mathbb{Z})$

$$\begin{tikzcd}
S^1 \arrow[d] & S^3 \arrow[d] \\
S^{4n-1} \arrow[d] \arrow[r, phantom, "="] & S^{4n-1} \arrow[d] \\
\mathbb{C}P^{2n} \arrow[r, "g"'] & \mathbb{H}P^n
\end{tikzcd}$$

It is sufficient to show this for the case $n = 1$. Here the fibre of g is just $\mathbb{C}P^1 \approx S^2$ and we have the result from the Gysin cohomology sequence. Since $\dim c_1 = 2$, $P^1 c_1 = c_1^3$ and the Cartan formula gives $P^1(c_1^2) = 2c_1^4$. Since P^1 is natural $P^1(h\ ) = 2h_1^2$ and by the Cartan formula $P^1(h_1^2) = 4h_1^3$. Therefore

$$\begin{tikzcd}
h_1^2 \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & a^2h_1^2 \arrow[ddd, mapsto] \\
& H^8(\mathbb{H}P^n;\mathbb{Z}_3) \arrow[r, "f^*"] \arrow[d, "P^1"'] & H^8(\mathbb{H}P^n;\mathbb{Z}_3) \arrow[d, "P^1"] & \\
& H^{12}(\mathbb{H}P^n;\mathbb{Z}_3) \arrow[r, "f^*"] & H^{12}(\mathbb{H}P^n;\mathbb{Z}_3) & \\
4h_1^3 \arrow[rrr, mapsto] & & & 4a^3h_1^3 = 4a^2h_1^3
\end{tikzcd}$$

implies that $4a^3 \equiv 4a^2 \pmod 3$ which implies that $a \neq -1$.

5. The Cayley Plane

The Cayley numbers are ordered pairs of quaternions. They are added by adding coordinates and multiplication is defined by $(q_1,q_2)(q_1',q_2') = (q_1q_1' - \bar{q}_2'q_2, q_2'q_1 + q_2\bar{q}_1')$. $(1,0)$ is a two sided unit. Also if the conjugate of $c = (q_1,q_2)$ is defined to be $\bar{c} = (\bar{q}_1,-q_2)$ then $c\bar{c} = |c|^2$ is real and non-negative and equals zero if and only if $c = (0,0) = 0$. It can also be shown that multiplication is distributive with respect to addition and $cd = 0$ implies that either $c = 0$ or $d = 0$. Thus the set of all Cayley numbers forms a division algebra. The associative law does not hold in general and so we cannot use the equivalence relation used for the real, complex and quaternionic spaces to define projective spaces based on the Cayley numbers

Using the fact that any two Cayley numbers generate an associative algebra isomorphic to a subalgebra of the quaternions, we can construct a fibering of S^{15} over S^8 with S^7 for a fibre to define the notion of a Cayley plane. (See (5)). It is, in fact, the homogeneous space F /Spin(9) where F is the Lie group which is the quotient group SO(8)/U(4). Its rational cohomology ring is the truncated polynomial ring generated by a generator c in dimension 8 such that $c^3 = 0$; and hence by the Lefschetz - Hopf Fixed Point Theorem it has the fixed point property.

Since all that is used to show that these manifolds have the fixed point property is the Lefschetz - Hopf Fixed Point Theorem, any manifold with the same rational cohomology algebras as these will have the fixed point property. Such manifolds may be obtained for instance by mixing homotopy types. (See (6)).

Using the Kunneth formula we see that cartesian products of the above manifolds also have the fixed point property, however, in (7) Husseini constructs manifolds with the fixed point property such that certain cartesian products of them do not.

In (2), Brown shows that if X is a compact manifold and M is an n - manifold with the fixed point property and $n \geq 3$ then the mapping cylinder of any map $f: X \longrightarrow M$ has the fixed point property. In particular if I is the unit interval then $M \times I$ has the fixed point property.

3. Cohomology of the Grassman Manifolds

Let K be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the real, complex, and quaternionic fields respectively, and let $G_{n,k}$ denote the Grassman manifold of n - dimensional subspaces of K^{n+k}. Let γ^n_k denote the bundle (and total space of the bundle) of $G_{n,k}$ which is the set whose points are the pairs: (n - plane, point in the n - plane) and $\overline{\gamma}^n_k$ the set of pairs: (n - plane, point in the complementary k - plane). For a vector bundle $\xi = (E,B,\pi)$ let the projectivization, $P(\xi)$, be the space of one - dimensional subspaces of the fibres of E, and let $l(\xi)$ be the bundle over $P(\xi)$ consisting of the pairs (σ,e) where σ is a one - dimensional subspace of a fibre and e is a point in that subspace.

Proposition 3.1: (See (8) for further details of Chapter 3)

Let ξ be a locally trivial vector bundle over a compact Hausdorff space B. There exists a finite dimensional vector space V over K and a surjective bundle map $e: V\times B \longrightarrow B$ and a commutative diagram:

$$\begin{tikzcd} P(E) \arrow[r] \arrow[d] \arrow[rrr, bend left, "j"] & P(V\times B) \arrow[d] \arrow[r, phantom, "\cong"] & P(V)\times B \arrow[r] & P(V) \\ B & B & & \end{tikzcd}$$

For example, if $\xi = \gamma^n_k$ then $V = K^n$ and P(V) is K- projective (n-1) - space, KP^{n-1}.

Now let $A = \mathbb{Z}$ if $K = \mathbb{C}$ or $\mathbb{H}$ and $\mathbb{Z}_2$ if $K = \mathbb{R}$. Let α_V be the class in $H^k(P(V);A)$ so that $H^*(P(V);A)$ is the free A - module on $1, \alpha_V, \alpha_V^2, \ldots, \alpha_V^{\dim V - 1}$.

<u>Theorem 3.2</u>: Let $c \in H^k(P(E);A)$ be the class $j^*(\alpha_V)$. Then $H^*(P(E);A)$ is the free $H^*(B;A)$ module (via π^*) on the classes $1, c, \ldots, c^{n-1}$ and there exist unique classes $\sigma_i(\xi) \in H^{ki}(B;A)$, (where k = the real dimension of K) and $\sigma_0(\xi) = 1$ such that $c^n - c^{n-1}\pi^*(\sigma_1(\xi)) + \ldots + (-1)^{n-1} c\pi^*(\sigma_{n-1}(\xi)) + (-1)^n \pi^*(\sigma_n(\xi)) = 0$.

The class $\sigma(\xi) = 1 + \sigma_1(\xi) + \ldots + \sigma_n(\xi)$ is called the total characteristic class of ξ. These classes are also called:

i) $K = \mathbb{R}$: Stiefel - Whitney class $w(\xi)$

ii) $K = \mathbb{C}$: Chern class $c(\xi)$

iii) $K = \mathbb{H}$: (Symplectic) Pontrjagin class $p(\xi)$

<u>Theorem 3.3</u>: The total characteristic class $\sigma(\xi)$ has the following properties:

1. $\sigma_i(\xi) = 0$ if $i > \dim \xi$.
2. $\sigma(\xi)$ is natural: if $f: B' \longrightarrow B$ then $\sigma(f^*\xi) = f^*\sigma(\xi)$.
3. If ξ and η are two vector bundles over B then $\sigma(\xi \oplus \eta) = \sigma(\xi) \cup \sigma(\eta)$
4. If l is the canonical line bundle over $P(V)$, then $\sigma_1(l) = \alpha_V$.

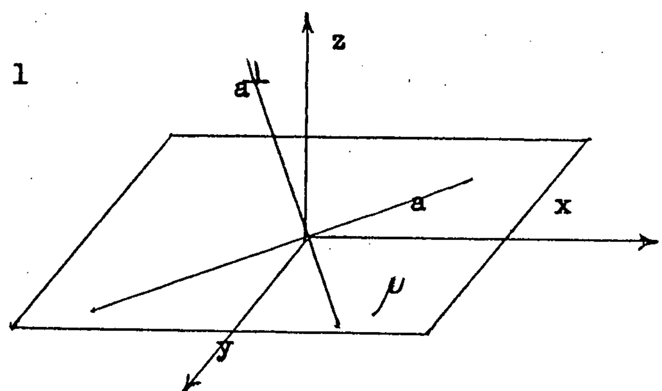
Now, let $\sigma_i = \sigma_i(\gamma_k^n)$, $\overline{\sigma}_i = \sigma(\overline{\gamma}_k^n)$. Property 3. of the previous theorem implies that $\sigma \cup \overline{\sigma} = 1$.

<u>Proposition 3.4</u>: $H^*(G_{n,k};A)$ is the quotient of the polynomial algebra over A on the σ_i, $i \leq n$, by the relations imposed by $\overline{\sigma}_j = 0$ for $j > k$. (Note: $\overline{\sigma}_j$ is the polynomial of degree j in σ_i given by the formal inversion of σ.)

<u>Proof</u>: This polynomial algebra is certainly mapped into $H^*(G_{n,k};A)$ and to prove that this is indeed an isomorphism we induct on n. For $n = 1$, $G_{1,k} = KP^{k+1}$ so that $H(G_{1,k};A)$ is generated by $\alpha = \sigma_1$ with the one relation $\alpha^{k+1} = 0$ but since $\overline{\sigma} = 1 - \alpha + \alpha^2 - \dots + (-1)^k \alpha^k + \dots$, the condition $\overline{\sigma}_j = 0$ for $j > k$ gives this and only this relation.

If the result holds for all $G_{n,k}$ with $n < s$, then consider $G_{s,t}$. A point in $P(\gamma^s_t)$ is a line a in an s - plane μ. The orthogonal complement of a in μ is an s-1 - plane $a^\perp$ and hence a point of $G_{s-1,t+1}$. Therefore $P(\gamma^s_t) = P(\overline{\gamma}^{s-1}_{t+1})$.

e.g. $s = 2,\ t = 1$



$a \in P(\gamma^2_1)$ and $a \in P(\overline{\gamma}^1_2)$

We have the diagram:

$$\begin{array}{ccc} P(\gamma^s_t) = P(\gamma^{s-1}_{t+1}) & \xrightarrow{\overline{\pi}} & G_{s-1,t+1} \\ \big\downarrow \pi & & \\ G_{s,t} & & \end{array}$$

Letting $l = l(\gamma^s_t) = l(\bar{\gamma}^{s-1}_{t+1})$ one has that $\pi^*(\gamma^s_t) = \xi \oplus l$, $\pi^*(\bar{\gamma}^{s-1}_{t+1}) = \eta \oplus l$ with $\xi \oplus \eta \oplus l$ being the trivial bundle. Since $c = \sigma_1(l)$ we have $0 = \sum_0 (-1)^i c^{s-i} \pi^*(\sigma_i(\gamma^s_t)) = \sum (-1)^i \sigma_1(l)^{s-i}(\sigma_i(\xi \oplus l))$ by naturality $= \sum (-1)^i \sigma_1(l)^{s-i}(\sigma_i(\xi) + \sigma_{i-1}(\xi)\sigma_1(l)) = \sigma_s(\xi)$ (the other terms canceling in pairs).

Therefore regarding $P(\gamma^s_t)$ as a bundle over $G_{s-1,t+1}$, the above property and the induction hypothesis say that $H^*(P(\gamma^s_t);A)$ is generated by the characteristic classes of ξ, l, and η subject only to relations imposed by the conditions:

1. $\dim \sigma(\eta \oplus l) = t$
2. $\sigma(\eta \oplus l \oplus \xi) = 1$

Since $\sigma \cup \bar{\sigma} = 1$ we already have that $H^*(G_{s,t};A)$ is generated by σ_i, $i \leq s$ with one of the relations being that $\dim(\sigma(\gamma^s_t)^{-1}) = t$. Now looking at $P(\gamma^s_t)$ as a bundle over $G_{s,t}$ we see that this must be the only relation.

Q.E.D.

4. Fixed Point Properties of the Grassman Manifolds

In (9), F. Bachmann, H.H. Glover and L.S. O'Neill prove the following:

Theorem 4.1:

i) For $q \geq p^2$, $\mathbb{H}G_{p,q}$ has the fixed point property.

ii) For $q \geq p^2$ and pq even, $\mathbb{C}G_{p,q}$ has the fixed point property.

iii) For $q \geq p^2$; $\binom{p+q}{q}$ odd and $p \neq 2^r - 1$, $\mathbb{R}G_{p,q}$ has the fixed point property.

This theorem partially satisfies their conjecture:

Conjecture 4.2: $\mathbb{H}G_{p,q}$ has the fixed point property if and only if $p \neq q$. For $F = \mathbb{R}$ or $\mathbb{C}$, $FG_{p,q}$ has the fixed point property if and only if $p \neq q$ and pq even.

If this conjecture is true then the question of which Grassman manifolds have the fixed point property is completely settled. This is because the self map of $KG_{p,p}$, where $K = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, induced by orthogonal complementation is fixed point free and also because of the following theorem:

Theorem 4.3: For p + q even let $f: F^{p+q} \longrightarrow F^{p+q}$ be defined by $f(u_1,\ldots,u_{p+q}) = (-\bar{u}_2,\bar{u}_1,\ldots,-\bar{u}_{p+q},\bar{u}_{p+q-1})$ and f_V be the induced map of Stiefel manifolds $f_V: FV_{p,q} \longrightarrow FV_{p,q}$ which is seen to be well defined if $F = \mathbb{R}$ or $\mathbb{C}$. If p and q are both odd and $F = \mathbb{R}$ or $\mathbb{C}$, then the induced map $f_G: FG_{p,q} \longrightarrow FG_{p,q}$ is fixed point free.

Proof: In the case that $F = \mathbb{R}$, f is given by the direct sum of matrices of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It follows then that the characteristic equation of f is $(\lambda^2 + 1)^{(p+q)/2} = 0$, which is seen to have no real roots and thus f has no eigenvectors. Now if W is an odd dimensional invariant subspace of f then $f|W$ must have at least one eigenvector, a contradiction.

For the case $F = \mathbb{C}$, we refer the reader to (10), page 304 for a proof using differential geometry. Q.E.D.

Theorem 4.1 follows from an application of the Lefscheta - Hopf Fixed Point Theorem and the following theorems also proved by Bachmann, Glover and O'Neill in (9).

Theorem 4.4: Let R be $\mathbb{Z}$ if $F = \mathbb{C}$ or $\mathbb{H}$, $\mathbb{Z}_2$ if $F = \mathbb{R}$. Let f be a self map of $FG_{p,q}$. Then clearly $f^*\sigma_1 = k\sigma_1$ for some k in R, σ_1 being the first characteristic class. Suppose $q \geq p^2$ (and $p \neq 2^r - 1$ if $F = \mathbb{R}$). Then $f^*c_i = k^i c_i$ for $i = 1, \ldots, p$.

A self map f with the above property is called an Adams type mapping.

Corollary 4.5: For $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $q \geq p^2$ (and $p \neq 2^r - 1$ if $F = \mathbb{R}$) every self map f of $FG_{p,q}$ has Lefschetz - Hopf number $\Lambda_f = \sum_0^{pq} \dim H^{di}(FG_{p,q}; R)\, k^i$ where $d = 1,2,4$ if $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ respectively.

Corollary 4.6:

i) For $F = \mathbb{C}$ or $\mathbb{H}$ and $q \geq p^2$ and f a self map of $FG_{p,q}$, $\Lambda_f \equiv 1 \pmod k$

ii) For $q \geq p^2$, $\binom{p+q}{p}$ odd and $p \neq 2^r - 1$ and f is a self map of $\mathbb{R}G_{p,q}$; $\Lambda_f \equiv 1 \pmod 2$.

Theorem 4.1 now follows from the Lefschetz - Hopf Fixed Point Theorem. If $F = \mathbb{C}$ or $\mathbb{H}$, $\Lambda_f = 0$ only if $k = -1$ perhaps. But then $\Lambda_f =$

$\sum(-1)^i \dim H^{di}(FG_{p,q};\mathbb{Z}) = \sum(-1)^i \beta_i(\mathbb{R}G_{p,q}) = \chi\mathbb{R}G_{p,q}$, the Euler number,

since $\dim H^{di}(FG_{p,q};\mathbb{Z})$ is the number of cells in dimension di in the canonical CW structure for $FG_{p,q}$ where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ ($d = 1, 2$, or 4) (See (11)) It is well known, however, that $\chi\mathbb{R}G_{p,q}$ for $p \neq q$, pq even, is greater than zero, (See (10) page 303). In the case where $F = \mathbb{H}$ and pq is odd we may use P^1 in the Steenrod Algebra mod 3 to eliminate the case $k = -1$ in the same way it was eliminated for $\mathbb{H}G_{1,2n+1} = \mathbb{H}P^{2n+1}$. In the case $F = \mathbb{R}$, $\binom{p+q}{p}$ odd, $\Lambda_f \equiv \dim H^{pq/2}(\mathbb{R}G_{p,q}; \mathbb{Z}_2) \pmod 2 \equiv \binom{p+q}{p} \pmod 2 \equiv 1 \pmod 2$ and thus Theorem 4.1 is proved.

5. Properties of some of the $\mathbb{C}G_{3,k}$'s

Theorem 5.1: Any self map of $\mathbb{C}G_{3,2}$ is an Adams type mapping and hence $\mathbb{C}G_{3,2}$ has the fixed point property.

Proof: Since $\mathbb{C}G_{3,2}$ is homeomorphic to $\mathbb{C}G_{2,3}$, its cohomology ring is generated by a generator x in dimension 2 and a generator y in dimension 4 subject to the relations imposed by the sums of elements of the same dimension in the formal inverse of the total class, $1 + x + y$. $(1 + x + y)^{-1}$ $= 1 - x - y + x^2 + 2xy + y^2 - x^3 - 3x^2y - 3xy^2 - y^3 + x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 - x^5 - 5x^4y - 10x^3y^2 - 10x^2y^3 - 5xy^4 - y^5 + x^6 + \dots$

Thus dimension 8 is completely determined by three generators, x^4, x^2y, and y^2 with the one relation : $x^4 - 3x^2y + y^2 = 0$. We can therefore choose x^2y and y^2 as a basis for dimension 8 with

1. $x^4 = 3x^2y - y^2$

Dimension 10 is generated by x^5, x^3y, and xy^2 subject to the two relations:

2. $x^5 = 3x^3y - xy^2$ (equation 1. multiplied by x)

3. $-x^5 + 4x^3y - 3xy^2 = 0$

Solving this system of equations gives us the single element xy^2 as a basis with

4. $x^3y = 2xy^2$ and

5. $x^5 = 5xy^2$

Let $f\colon \mathbb{C}G_{3,2} \longrightarrow \mathbb{C}G_{3,2}$ with $f^*(x) = ax$ and $f^*(y) = bx^2 + cy$. Using equation 1. and the fact that f^* is a ring homomorphism we have $a^4x^4 = f^*(x^4) = f^*(3x^2y - y^2) = 3a^2x^2(bx^2 + cy) - (b^2x^4 + 2bcx^2y + c^2y^2) = 3a^2bx^4 + 3a^2cx^2y - b^2x^4 - 2bc\,x^2y - c^2y^2 = 3a^2b(3x^2y - y^2) + 3a^2cx^2y$

$- b^2(3x^2y - y^2) - 2bcx^2y - c^2y^2 = (9a^2b + 3a^2c - 3b^2 - 2bc)x^2y +$
$(-3a^2b + b^2 - c^2)y^2.$

Since $a^4x^4 = 3a^4x^2y - a^4y^2$ and x^2y, y^2 form a basis we have:

6. $3a^4 = 9a^2b + 3a^2c - 3b^2 - 2bc$

7. $-a^4 = -3a^2b + b^2 - c^2$

Adding three times 7. to 6. gives us:

8. $3a^2c - 2bc - 3c^2 = 0$

Using equations 4. and 5. in a similar manner gives us:

$5a^5xy^2 = a^5x^5 = f^*(x^5) = f^*(5xy^2) = 5ax(b^2x^4 + 2bcx^2y + c^2y^2) = 5ab^2x^5 +$
$10abcx^2y + 5ac^2xy^2 = (25ab^2 + 20abc + 5ac^2)xy^2.$ Therefore

9. $a^5 = 5ab^2 + 4abc + ac^2.$

Case 1: $a = 0.$

If $b = 0$ as well then equation 8. becomes $-3c^2 = 2bc = 0$ which means that $c = 0$ as well as a and b.

If $b \neq 0$ then after dividing b into equation 6. we have $3b = -2c$ or $b = (-2/3)c$. Substituting this into equation 8. gives us $3c^2 = (4/3)c^2$ which implies that $c = 0$. But if $c = 0$ then according to equation 6. b must also $= 0$, contradicting the assumption. Therefore if $a = 0$ then $b = c = 0$.

Case 2: $a \neq 0$

If $a \neq 0$ then dividing equation 9. by a gives us the equation:

10. $a^4 = 5b^2 + 4bc + c^2$

Now if $b \neq 0$ then $c \neq 0$ for if $c = 0$ then equation 10. becomes $a^4 = 5b^2$ to which there are no solutions consisting of non - zero integers.

Adding equation 7. to equation 10. produces

11. $6b^2 + 4bc - 3a^2b = 0$ and dividing this by b gives

12. $3a^2 = 6b + 4c$

Substituting for $3a$ in equation 8. and then dividing equation 8. by c:

13. $c = -4b$

Substituting for c in equation 12. gives $3a^2 = -10b$ and therefore $9a^4 = 100b^2$

Substituting for c in equation 10. gives $a^4 = 5b^2$ and thus $9a^4 = 45b^2 = 100b^2$

This implies that $b = 0$ contradicting the assumption. Thus if $a \neq 0$ then b $= 0$ and by equation 9. we see that $c \neq 0$ and then by equation 8. we see that $c = a^2$ and thus f is an Adams type mapping. Q.E.D.

<u>Theorem 5.2</u>: Any self map of $\mathbb{C}G_{3,4}$ is an Adam's type mapping and hence $\mathbb{C}G_{3,4}$ has the fixed point property.

<u>Proof</u>: The ideal of zeros of $H^*(\mathbb{C}G_{3,4};\mathbb{Z})$ are generated by the following relations in dimensions 10, 12, 14, 16, 18, 20, 22, 24 respectively:

1. $-x^5 + 4x^3y - 3xy^2 - 3x^2z + 2yz = 0$ (z being the third Chern class)
2. $x^6 - 5x^4y + 6x^2y^2 - y^3 + 4x^3z - 6xyz + z^2 = 0$
3. $-x^7 + 6x^5y - 10x^3y^2 + 4xy^3 - 5x^4z + 12x^2yz - 3y^2z - 3xz^2 = 0$
4. $x^8 - 7x^6y + 15x^4y^2 - 10x^2y^3 + y^4 + 6x^5z - 20x^3yz + 12xy^2z + 6x^2z^2 - 3yz^2 = 0$
5. $-x^9 + 8x^7y - 21x^5y^2 + 20x^3y^3 - 5xy^4 - 7x^6z + 30x^4yz - 30x^2y^2z + 4y^3z - 10x^3z^2 + 12xyz^2 - z^3 = 0$
6. $x^{10} - 9x^8y + 28x^6y^2 - 35x^4y^3 + 15x^2y^4 - y^5 + 8x^7z - 42x^5yz + 60x^3y^2z - 20xy^3z + 15x^4z^2 - 30x^2yz^2 + 6y^2z^2 + 4xz^3 = 0$
7. $-x^{11} + 10x^9y - 36x^7y^2 + 56x^5y^3 - 35x^3y^4 + 6xy^5 - 9x^8z + 56x^6yz - 105x^4y^2z + 60x^2y^3z - 5y^4z - 21x^5z^2 + 60x^3yz^2 - 30xy^2z^2 - 10x^2z^3 + 4yz^3 = 0$

8. $x^{12} - 11x^{10}y + 45x^8y^2 - 84x^6y^3 + 70x^4y^4 - 21x^2y^5 + y^6 + 10x^9z - 72x^7yz$
$+ 168x^5y^2z - 140x^3y^3z + 30xy^4z + 28x^6z^2 - 105x^4yz^2 + 90x^2y^2z^2 - 10y^3z^2$
$+ 20x^3z^3 - 20xyz^3 + z^4 = 0$

Dimension 10 has a basis of four elements: x^3y, xy^2, x^2z and yz with

9. $x^5 = 4x^3y - 3xy^2 - 3x^2z + 2yz$

Dimension 12 has two relations: equation 2 and

10. $-x^6 + 4x^4y - 3x^2y^2 - 3x^3z + 2xyz = 0$ (equation 1. multiplied by x)

By row reducing the associated 2 by 7 matrix we get a basis of 5 elements: x^2y^2, x^3z, y^3, xyz, and z^2 with

11. $x^4y = 3x^2y^2 + x^3z - y^3 - 4xyz + z^2$
12. $x^6 = 9x^2y^2 + x^3z - 4y^3 - 14xyz + 4z^2$

Dimension 14 has four relations: equation 3 and

13. $-x^7 + 4x^5y - 3x^3y^2 - 3x^4z + 2x^2yz = 0$ (equation 1. multiplied by x)
14. $-x^5y + 4x^3y^2 - 3xy^3 - 3x^2yz + 2y^2z = 0$ (equation 1. multiplied by y)
15. $x^7 - 5x^5y + 6x^3y^2 - xy^3 + 4x^4z - 6x^2yz + xz^2 = 0$ (equation 2. mult. by x)

By linear algebra we have a basis of 4 elements: xy^3, x^2yz, y^2z, and xz^2 with

16. $x^7 = 14xy^3 + 7x^2yz - 28y^2z - 7xz^2$ (-14(13) - (14) - 23(15) - 10(3))
17. $x^5y = 5xy^3 + 5x^2yz - 10y^2z - 4xz^2$ (-4(13) - (14) - 8(15) - 4(3))
18. $x^3y^2 = 2xy^3 + 2x^2yz - 3y^2z - xz^2$ ((13) + 2(15) + (3))
19. $x^4z = 3x^2yz - y^2z - 2xz^2$ ((14) + (15) + (3))

By Poincare Duality we know that Dimension 16 is spanned by 4 elements. To find 4 such elements we multiply equations 16, 17, 18, and 19 by x to get:

20. $x^8 = 14x^2y^3 + 7x^3yz - 28xy^2z - 7x^2z^2$

21. $x^6y = 5x^2y^3 + 5x^3yz - 10xy^2z - 4x^2z^2$

22. $x^4y^2 = 2x^2y^3 + 2x^3yz - 3xy^2z - x^2z^2$

23. $x^5z = 3x^3yz - xy^2z - 2x^2z^2$

Multiplying equation 1. by z and substituting for x^5z with equation 23. gives:

24. $x^3yz = -2yz^2 + 2xy^2z + x^2z^2$

Substituting x^3yz with equation 24. in equations 20,21,22, and 23 and collecting terms gives:

25. $x^8 = 14x^2y^3 - 14xy^2z - 14yz^2$

26. $x^6y = 5x^2y^3 + x^2z^2 - 10yz^2$

27. $x^4y^2 = 2x^2y^3 + xy^2z + x^2z^2 - 4yz^2$

28. $x^5z = 5xy^2z + x^2z^2 - 6yz^2$

Multiplying equation 11. by y and substituting gives:

29. $y^4 = x^2y^3 - 3xy^2z + 3yz^2$

Since $\mathbb{C}G_{3,4}$ is an orientable 24 - dimensional manifold, we know that $H^{24}(\mathbb{C}G_{3,4};\mathbb{Z})$ can be spanned by a single element. To find such an element we solve the following system of 22 equations (equation 8. plus equations 30 to 50 inclusive) in 19 variables. Equation 30. is equation 24. multiplied by xz, 31 is 25. multiplied by xz, 32. is 26. by xz, 33. is 27. by xz, 34. is 28. by xz, 35. is 29. by xz, 36. is 11. by z^2, 37. is 12 by z^2, 38. is 16. by yz, 39. is 17. by yz, 40. is 18. by yz, 41. is 19. by yz, 42. is 20. by x^4, 43. is 21. by x^4, 44. is 22. by x^4, 45. is 4. by x^4, 46. is 5. by xy 47. is 5. by x^3, 48. is 6. by y, 49. is 6. by x^2, and 50. is 7. by x.

We find that z^4 generates dimension 24 with $xyz^3 = z^4$, $x^3z^3 = z^4$, $y^3z^2 = z^4$,

$x^2y^2z^2 = 2z^4$, $x^4yz^2 = 3z^4$, $x^6z^2 = 5z^4$, $xy^4z = 3z^4$, $x^3y^3z = 6z^4$, $x^5y^2z = 11z^4$, $x^7yz = 21z^4$, $x^9z = 42z^4$, $y^6 = 5z^4$, $x^2y^5 = 11z^4$, $x^4y^4 = 23\ z^4$, $x^6y^3 = 47z^4$, $x^8y^2 = 98z^4$, $x^{10}y = 210z^4$, $x^{12} = 462z^4$, which of course can be checked by substituting these values into equations 8. and 30 to 50.

Let $h: \mathbb{C}G_{3,4} \longrightarrow \mathbb{C}G_{3,4}$ where a,b,c,e,f, g are integers such that

$h^*(x) = ax$

$h^*(y) = bx^2 + cy$

$h^*(z) = ex^3 + fxy + gz$

Using equation 1. and the fact that h^* is a ring homomorphism we have:

$h^*(x^5) = h^*(4x^3y - 3xy^2 - 3x^2z + 2yz) = 4a^3x^3(bx^2 + cy) - 3ax(b^2x^4 + 2bcx^2y + c^2y^2) - 3a^2x^2(ex^3 + fxy + gz) + 2(bx^2 + cy)(ex^3 + fxy + gz) = 4a^3bx^5 + 4a^3cx^3y - 3ab^2x^5 - 6abcx^3y - 3ac^2xy^2 - 3a^2ex^5 - 3a^2fx^3y - 3a^2gx^2z + 2bex^5 + 2bfx^3y + 2bgx^2z + 2cex^3y + 2cfxy^2 + 2cgyz = (16a^3b - 12ab^2 - 12a^2e + 8be + 4a^3c - 6abc - 3a^2f + 2bf + 2ce)x^3y + (-12a^3b + 9ab^2 + 9a^2e - 6be - 3ac^2 + 2cf)xy^2 + (-12a^3b + 9ab^2 + 9a^2e - 6be - 3a^2g + 2bg)x^2z + (8a^3b - 6ab^2 - 6a^2e + 4be + 2cg)yz$

Since $h^*(x^5)$ also equals $a^5x^5 = 4a^5x^3y - 3a^5xy^2 - 3a^5x^2z + 2a^5yz$ and x^3y, xy^2, x^2z and yz form a basis, we have the following equations:

51. $4a^5 = 16a^3b - 12ab^2 - 12a^2e + 8be + 4a^3c - 6abc - 3a^2f + 2bf + 2ce$

52. $-3a^5 = -12a^3b + 9ab^2 + 9a^2e - 6be - 3ac^2 + 2cf$

53. $-3a^5 = -12a^3b + 9ab^2 + 9a^2e - 6be - 3a^2g + 2bg$

54. $2a^5 = 8a^3b - 6ab^2 - 6a^2e + 4be + 2cg$

Subtracting 53. from 52. gives:

55. $-3ac^2 + 2cf + 3a^2g - 2bg = 0$

Adding three times 54. to twice 53. we have:

56. $6cg - 6a^2g + 4bg = 0$

Adding four times 52. to three times 51. we have:

57. $-12ac^2 + 8cf + 12a^3c - 18abc - 9a^2f + 6bf + 6ce = 0$

Using equations 11. and 12. and the given basis of dimension 12 we have:

$h^*(x^6) = h^*(9x^2y^2 + x^3z - 4y^3 - 14xyz + 4z^2)$ giving us the equations:

58. $9a^6 = 81a^2b^2 + 9a^3e - 36b^3 - 126abe + 36e^2 + 54a^2bc + 3a^3f - 36b^2c$
$-42abf - 42ace + 24ef + 9a^2c^2 - 12bc^2 - 14acf + 4f^2$

59. $a^6 = 9a^2b^2 + a^3e - 4b^3 - 14abe + 4e^2 + 18a^2bc + a^3f - 12b^2c - 14abf$
$-14ace + 8ef + a^3g - 14abg + 8eg$

60. $-4a^6 = -36a^2b^2 - 4a^3e + 16b^3 + 56abe - 16e^2 - 18a^2bc - a^3f + 12b^2c$
$+ 14abf + 14ace - 8ef - 4c^3$

61. $-14a^6 = -126a^2b^2 - 14a^3e + 56b^3 + 196abe - 56e^2 - 72a^2bc - 4a^3f + 48b^2c$
$+ 56abf + 56ace - 32ef - 14acg + 8fg$

62. $4a^6 = 36a^2b^2 + 4a^3e - 16b^3 - 56abe + 16e^2 + 18a^2bc + a^3f - 12b^2c - 14abf$
$- 14ace + 8ef + 4g^2$

Subtracting three times equation 59. from 58. gives:

63. $6a^6 = 54\,a^2b^2 + 6a^3e - 24b^3 - 84abe + 24e^2 + 9a^2c^2 - 12bc^2 - 14acf$
$+ 4f^2 - 3a^3g + 42abg - 24eg$

Adding equation 59. to 60. gives:

64. $-27a^2b^2 - 3a^3e + 12b^3 + 42abe - 12e^2 + a^3g - 14abg + 8eg - 4c^3 = -3a^6$

Subtracting four times equation 60. from 61. yields:

65. $18a^2b^2 + 2a^3e - 8b^3 - 28abe + 8e^2 + 16c^3 - 14acg + 8fg = 2a^6$

Adding equation 60. to equation 62. provides:

66. $g^2 = c^3$

Since $x^{10}y = 210z^4$ and $x^{12} = 462z^4$, $11x^{10}y = 5x^{12}$. $h^*(11x^{10}y) =$
$11a^{10}x^{10}(bx^2 + cy) = 11a^{10}bx^{12} + 11a^{10}cx^{10}y = (5082a^{10}b + 2310a^{10}c)z^4 =$
$h^*(5x^{12}) = 5a^{12}x^{12} = 2310a^{12}z^4$. Therefore:

67. $5a^{12} = 11a^{10}b + 5a^{10}c$

Since $x^8y^2 = 98z^4$ and $x^{12} = 462z^4$, $33x^8y^2 = 7x^{12}$. $h^*(33x^8y^2) =$
$33a^8x^8(b^2x^4 + 2bcx^2y + c^2y^2) = 33a^8b^2x^{12} + 66a^8bcx^{10}y + 33a^8c^2x^8y^2 =$
$(15{,}246a^8b^2 + 13{,}860a^8bc + 3234a^8c^2)z^4 = h^*(7x^{12}) = 7a^{12}x^{12} = 3234a^{12}z^4$. So:

68. $7a^{12} = 33a^8b^2 + 30a^8bc + 7a^8c^2$

Since $x^9z = 42z^4$ and $x^{12} = 462z^4$, $11x^9z = x^{12}$. $h^*(11x^9z) =$
$11a^9x^9(ex^3 + fxy + gz) = 11a^9ex^{12} + 11a^9fx^{10}y + 11a^9gx^9z = (5082a^9e +$
$2310a^9f + 462a^9g)z^4 = h^*(x^{12}) = a^{12}x^{12} = 462a^{12}z^4$. Therefore:

69. $a^{12} = 11a^9e + 5a^9f + a^9g$

Since $y^6 = 5z^4$ and $x^2y^5 = 11z^4$, $11y^6 = 5x^2y^5$. $h^*(11y^6) = 11(bx^2 + cy)^6 =$
$11(b^6x^{12} + 6b^5cx^{10}y + 15b^4c^2x^8y^2 + 20b^3c^3x^6y^3 + 15b^2c^4x^4y^4 + 6bc^5x^2y^5 + c^6y^6)$
$= h^*(5x^2y^5) = 5a^2x^2(b^5x^{10} + 5b^4cx^8y + 10b^3c^2x^6y^2 + 10b^2c^3x^4y^3 + 5bc^4x^2y^4 +$
$c^5y^5)$ Using the basis in dimension 24 gives us:

70. $11(462b^6 + 1260b^5c + 1470b^4c^2 + 940b^3c^3 + 345b^2c^4 + 66bc^5 + 5c^6) =$
$\quad 5(2310a^2b^5 + 5250a^2b^4c + 4900a^2b^3c^2 + 2350a^2b^2c^3 + 575a^2bc^4 + 55a^2c^5)$

<u>Case 1</u>: $a = 0$, $g = 0$. If $g = 0$ then according to equation 66, $c = 0$

If $c = 0$ and $a = 0$ then according to equation 70, $b = 0$

If $a = b = c = g = 0$ then according to equation 64, $e = 0$ and then it follows from equation 63. that $f = 0$ as well.

<u>Case 2</u>: $a = 0,\ g \neq 0$: If $g \neq 0$ then according to equation 66, $c \neq 0$. If $a = 0$ then equation 56. becomes $6cg + 4bg = 0$ and if we are assuming that $g \neq 0$ then we have that $c = (-2/3)b$. Substituting $a = 0$ and $c = (-2/3)b$ into equation 70. gives us $(41{,}342/729)b^6 = 0$ which means that $b = 0$ and c and g are both zero contradicting the assumption. Therefore if $a = 0$ then $b = c = e = f = g = 0$.

<u>Case 3</u>: $a \neq 0$: If $a \neq 0$ then dividing equations 67. and 68. by a^8 gives:

71. $5a^2 = 11b + 5c$

72. $7a^4 = 33b^2 + 30bc + 7c^2$

Squaring 71. and multiplying by 7 produces:

73. $175a^4 = 847b^2 + 175c^2 + 770bc$

Multiplying equation 72. by 25 produces:

74. $175a^4 = 825b^2 + 175c^2 + 750bc$

Subtracting 74. from 73. yields:

75. $0 = 22b^2 + 20bc = 2b(11b + 10c)$

Therefore if $a \neq 0$ then either $b = 0$ and $c = a^2$ (by equation 71.) or $b = (-10/11)c$ and $c = -a^2$. Since by equation 66. $g^2 = c^3$, if $c = -a^2$, then $g^2 = -a^6$ which means that both a and g are equal to zero, contradicting the assumption. Therefore if $a \neq 0$ then $b = 0$ and $c = a^2$ and $g = a^3$. Equation 55. becomes $2a^2f = 0$ which means that $f = 0$ and after dividing equation 69. by a^9 and substituting we have that $e = 0$ as well, hence h is an Adams type mapping. Q.E.D.

<u>Theorem 5.3</u>: Any self map of $\mathbb{C}G_{3,6}$ is an Adams type mapping and hence $\mathbb{C}G_{3,6}$ has the fixed point property.

<u>Proof</u>: Let x,y, and z be the first, second, and third universal Chern classes as usual. Dimension 14 is spanned by seven elements with

1. $x^7 = 6x^5y - 10x^3y^2 + 4xy^3 - 5x^4z + 12x^2yz - 3y^2z - 3xz^2$

If h is a self map of $G_{3,6}$ such that $h^*(x) = ax$, $h^*(y) = bx^2 + cy$, and $h^*(z) = ex^3 + fxy + gz$, then taking h^* of both sides of equation 1. and comparing the coefficients of the x^3y^2, x^4z, xy^3, x^2yz, y^2z, and xz^2 terms gives us the equations:

2. $-10M - 10a^3c^2 + 12abc^2 + 12a^2cf - 6bcf - 3c^2e - 3af^2 = -10a^7$

3. $-5M - 5a^4g + 12a^2bg - 3b^2g - 6aeg = -5a^7$

4. $4M + 4ac^3 - 3c^2f = 4a^7$

5. $12M + 12a^2cg - 6bcg - 6afg = 12a^7$

6. $-3M - 3c^2g = -3a^7$

7. $-3M - 3ag^2 = -3a^7$

where $M = 6a^5b - 10a^3b^2 + 4ab^3 - 5a^4e + 12a^2be - 3b^2e - 3ae^2$

From these equations we get:

8. $c^2g = ag^2$ $\quad$ ((6) - (7))

9. $12a^2cg - 6bcg - 6afg = 12c^2g$ $\quad$ ((5) + 4(6))

10. $15a^4g - 36a^2bg + 9b^2g + 18aeg = 15c^2g$ $\quad$ (5(6) - 3(3))

11. $-12ac^3 + 9c^2f = -12c^2g$ $\quad$ (3(4) + 4(6))

12. $30a^3c^2 - 36abc^2 - 36a^2cf + 18bcf + 9c^2e + 9af^2 = 30c^2g$ $\quad$ (-3(2) + 10(6))

Dimension 16 is generated by 10 elements and 2 relations:

13. $x^8 - 7x^6y + 15x^4y^2 - 10x^2y^3 + y^4 + 6x^5z - 20x^3yz + 12xy^2z + 6x^2z^2 - 3yz^2 = 0$ as well as equation 1. multiplied by x.

By row reduction, we find that dimension 16 is spanned by 8 elements with:

14. $x^6y = 5x^4y^2 - 6x^2y^3 + y^4 + x^5z - 8x^3yz + 9xy^2z + 3x^2z^2 - 3yz^2$

15. $x^8 = 20x^4y^2 - 32x^2y^3 + 6y^4 + x^5z - 36x^3yz + 51xy^2z + 15x^2z^2 - 18yz^2$

Taking h* of equation 14. and comparing the coefficients of y^4, yz^2, and x^5z of both sides gives us the equations:

16. $6M + N + c^4 = 6a^6b + a^6c$

17. $-18M - 3N - 3cg^2 = -18a^6b - 3a^6c$

18. $M + N + a^5g - 8a^3bg + 9ab^2g + 6a^2eg - 6beg = a^6b + a^6c$

where $M = 5a^4b^2 - 6a^2b^3 + b^4 + a^5e - 8a^3be + 9ab^2e + 3a^2e^2 - 3be^2$ and $N = 10a^4bc - 18a^2b^2c + 4b^3c + a^5f - 8a^3bf - 8a^3ce + 9ab^2f + 18abce + 6a^2ef - 6bef - 3ce^2$. From these equations we get:

19. $c^4 = cg^2$ (3(16) + (17))

Dimension 18 is generated by 12 elements and 4 relations:

20. $-x^9 + 8x^7y - 21x^5y^2 + 20x^3y^3 - 5xy^4 - 7x^6z + 30x^4yz - 30x^2y^2z + 4y^3z - 10x^3z^2 + 12xyz^2 - z^3 = 0$.

21. equation 13. multiplied by x

22. equation 1. multiplied by x

23. equation 1. multiplied by y

Solving the associated matrix, we find that dimension 18 is spanned by 8 elements with:

24. $x^9 = 48x^3y^3 - 54xy^4 + 9x^4yz - 135x^2y^2z + 81y^3z - 9x^3z^2 + 108xyz^2 - 21z^3$ (-47(21) - 21(20) - 27(22) - (23))

25. $x^7y = 14x^3y^3 - 14xy^4 + 7x^4yz - 42x^2y^2z + 21y^3z - 6x^3z^2 + 33xyz^2 - 6z^3$ (-6(20) - 12(21) - 6(22) - (23))

26. $x^5y^2 = 4x^3y^3 - 3xy^4 + 2x^4yz - 9x^2y^2z + 4y^3z - x^3z^2 + 6xyz^2 - z^3$

((20) + 2(21) + (22))

27. $x^6z = 5x^4yz - 6x^2y^2z + y^3z - 4x^3z^2 + 6xyz^2 - z^3$ ((20) + (21) + (23))

Taking h* of equation 27. and comparing the coefficients of z^3, x^3y^3, and xy^4 we have:

28. $-21M - 6N - P - Q - g^3 = -21a^6e - 6a^6f - a^6g$

29. $48M + 14N + 4P - 6a^2c^2f + 3bc^2f + c^3e + 6acf^2 - f^3 = 48a^6e + 14a^6f$

30. $-54M - 14N - 3P + c^3f = -54a^6e - 14a^6f$

where $M = 5a^4be - 6a^2b^2e + b^3e - 4a^3e^2 + 6abe^2 - e^3$ and $N = 5a^4bf + 5a^4ce - 6a^2b^2f - 12a^2bce + b^3f + 3b^2ce - 8a^3ef + 12abef + 6ace^2 - 3e^2f$ and $P = 5a^4cf - 12a^2bcf - 6a^2c^2e + 3b^2cf + 3bc^2e - 4a^3f^2 + 6abf^2 + 12acef - 3ef^2$ and $Q = 5a^4bg - 6a^2b^2g + b^3g - 8a^3eg + 12abeg - 3e^2g$

Case 1a: $g \neq 0$, $c = 0$

If $g \neq 0$ and $c = 0$ then according to equation 8, $a = 0$ and consequently by equation 10. we have that $b = 0$ as well. However, if $a = b = c = 0$ and $g \neq 0$ then equations 28., 29., and 30. become:

31. $21e^3 + 18e^2f + 3ef^2 + 3e^2g - g^3 = 0$

32. $-48e^3 - 42e^2f - 12ef^2 - f^3 = 0$

33. $54e^3 + 42e^2f + 9ef^2 = 0$

Equation 31. implies that $e \neq 0$ and therefore equation 32. implies that $f \neq 0$. Adding 9 times equation 32. to 8 times equation 33. and then dividing this sum by f gives:

34. $-42e^2 - 36ef - 9f^2 = 0$

Dividing equation 33. by e and then adding to equation 34. produces:

35. $12e^2 + 6ef = 0$ or $f = -2e$

But if $f = -2e$ then according to equation 33. $6e^3 = 0$ which implies that $e = 0$. Therefore if $g \neq 0$ we only have to consider the case:

Case 1b: $g \neq 0$, $c \neq 0$

Equation 19 when divided by c becomes:

36. $c^3 = g^2$

Equations 36. and 8. then imply that $c = a^2$ and $g = a^3$. Equation 11. then implies that $f = 0$. Equation 9. then implies that $b = 0$ and then equation 10 implies that $e = 0$. Thus if $g \neq 0$ then h is an Adams type mapping.

Case 2a: $g = 0$, $a = 0$

If $g = 0$ then according to equation 18. $c = 0$ and if $a = 0$ as well then equations 16. and 18. become:

37. $6b^4 - 18be^2 - 6bef = 0$

38. $b^4 - 3be^2 - 6bef = 0$

Subtracting 6 times equation 38. from equation 37. yields

39. $30bef = 0$

Subtracting equation 38. from equation 37. makes

40. $5b(\ b^3 - 3e^2\ ) = 0$ so that either $b = 0$ or $0 \neq b^3 = 3e^2$

Now if $a = b = c = g = 0$ then equation 28. becomes:

41. $21e^3 + 18e^2f + 3ef^2 = 0$

and we also have in this case equations 32. and 33. Equations 41. and 32. imply that $e = 0$ iff $f = 0$, however, as we have shown before with equations 34. and 35. e and f must equal zeros in this case. On the other hand, if $0 \neq b^3 = 3e^2$ then by equation 39. we have that $f = 0$. But if $a = c = f = g = 0$

then equation 28. becomes:

42. $-21b^3e + 21e^3 = 0$

which contradicts $0 \neq b^3 = 3e^2$.

By Proposition 3.4, the universal Chern classes x, y, and z are subject to the following equations:

43. $x^{10} - 9x^8y + 28x^6y^2 - 35x^4y^3 + 15x^2y^4 - y^5 + 8x^7z - 42x^5yz + 60x^3y^2z$
$- 20xy^3z + 15x^4z^2 - 30x^2yz^2 + 6y^2z^2 + 4xz^3 = 0$

44. $-x^{11} + 10x^9y - 36x^7y^2 + 56x^5y^3 - 35x^3y^4 + 6xy^5 - 9x^8z + 56x^6yz - 105x^4y^2z$
$+ 60x^2y^3z - 5y^4z - 21x^5z^2 + 60x^3yz^2 - 30xy^2z^2 - 10x^2z^3 + 4yz^3 = 0$

45. $x^{12} - 11x^{10}y + 45x^8y^2 - 84x^6y^3 + 70x^4y^4 - 21x^2y^5 + y^6 + 10x^9z - 72x^7yz$
$+ 168x^5y^2z - 140x^3y^3z + 30xy^4z + 28x^6z^2 - 105x^4yz^2 + 90x^2y^2z^2 - 10y^3z^2$
$+ 20x^3z^3 - 20xyz^3 + z^4 = 0$

We now continue with the final case:

<u>Case 2b</u>: $g = 0$, $a \neq 0$

If $c = g = 0$ and $a \neq 0$ then we have by equation 12. that $f = 0$. We further subdivide this case into three subcases: i) $b = 0$, ii) $e = 0$, and iii) $b \neq 0$, $e \neq 0$.

<u>Case 2bi</u>: If $b = c = f = g = 0$ then h^* of equations 20, 43, and 44. become:

46. $-a^9 - 7a^6e - 10a^3e^2 - e^3 = 0$

47. $a^{10} + 8a^7e + 15a^4e^2 + 4ae^3 = 0$

48. $-a^{11} - 9a^8e - 21a^5e^2 - 10a^2e^3 = 0$

Equation 46. implies that $e \neq 0$. Dividing the sum of 4a times equation 46 plus equation 47. by a gives us the quadratic equation:

49. $25e^2 + 20a^3e + 3a^6 = 0$

Equation 49. has two solutions: $e = (-3/5)a^3$ or $e = (-1/5)a^3$ both of which when substituted into equation 48. imply that $a = 0$.

Case 2bii: If $c = e = f = g = 0$ then h^* of equations 20, 43, and 44. become:

50. $-a^9 + 8a^7b - 21a^5b^2 + 20a^3b^3 - 5ab^4 = 0$

51. $a^{10} - 9a^8b + 28a^6b^2 - 35a^4b^3 + 15a^2b^4 - b^5 = 0$

52. $-a^{11} + 10a^9b - 36a^7b^2 + 56a^5b^3 - 35a^3b^4 + 6ab^5 = 0$

Equation 50. implies that $b \neq 0$. The sum of a^2 times equation 50. plus 2a times equation 51. plus equation 52. when divided by ab^2 is:

53. $-a^6 + 6a^4b - 10a^2b^2 + 4b^3 = 0$

Letting $c = e = f = g = 0$ in equation 16. and then dividing by 6b results in:

54. $-a^6 + 5a^4b - 6a^2b^2 + b^3 = 0$

Subtracting 4a times equation 54. from equation 53. gives us the quadratic

55. $3a^6 - 14a^4b + 14a^2b^2 = 0$

which is readily seen to have no non-zero integer solutions.

Case 2biii: $c = f = g = 0,\ a \neq 0,\ b \neq 0,\ e \neq 0.$

According to Proposition 3.4 we have

56. $1 = (1 + x + y + z)(\overline{\sigma}_1 + \overline{\sigma}_2 + \dots)$

where the $\overline{\sigma}_i$ are the polynomials of degree i in the formal inversion of $1 + x + y + z$ and all but a finite number of the $\overline{\sigma}_i$ are equal to zero.

If we have $c = f = g = 0$ then h^* of equation 56. gives us the equation:

57. $1 = (1 + a + b + e)(\overline{\sigma}_1 + \overline{\sigma}_2 + \dots)$

where the $\overline{\sigma}_i$ are now the polynomials of degree i in the formal inversion of 1 + a + b + e and since all but a finite number of these are equal to zero, we have that any integral solution of this equation must satisfy:

58. $a + b = -e$

Also if $c = f = g = 0$ then equations 20, 43, and 45. become:

59. $-a^9 + 8a^7b - 21a^5b^2 + 20a^3b^3 - 5ab^4 - 7a^6e + 30a^4be - 30a^2b^2e + 4b^3e$
$- 10a^3e^2 + 12abe^2 - e^3 = 0$

60. $a^{10} - 9a^8b + 28a^6b^2 - 35a^4b^3 + 15a^2b^4 - b^5 + 8a^7e - 42a^5be + 60a^3b^2e$
$- 20ab^3e + 15a^4e^2 - 30a^2be^2 + 6b^2e^2 + 4ae^3 = 0$

61. $a^{12} - 11a^{10}b + 45a^8b^2 - 84a^6b^3 + 70a^4b^4 - 21a^2b^5 + b^6 + 10a^9e - 72a^7be$
$+ 168a^5b^2e - 140a^3b^3e + 30ab^4e + 28a^6e^2 - 105a^4be^2 + 90a^2b^2e^2 - 10b^3e^2$
$+ 20a^3b^3 - 20abe^3 + e^4 = 0$

Equations 58. and 59. imply that a, b, and e must all be even. If $2^k|a$, $2^k|b$, and $2^k|e$ for some $k \geq 1$ ($m|n$ means m divides n), then equation 59. implies that $2^{4k}|e^3$ and therefore $2^{k+1}|e$.

If $2^{k+1}\nmid b$ then according to equation 60. we have $2^{5k}|6b^2e^2$(because $2^{4k}|e^3$, $2^{5k}|4ae^3$and clearly 2^{5k} is a factor of the other terms) and thus we also have $2^d|e$ where $d = 3k/2$ if k is even and $d = (3k - 1)/2$ if k is odd. If k is even then according to equation 60. $2^{5k+1}|-b^5$ (since 2^{5k+1} now divides all the other terms of equation 60. in particular $2^{5k+1}|6b^2e^2$) and therefore $2^{k+1}|b$ which contradicts our assumption. If k is odd then according to equation 61. $2^{6k}|e^4$ (in particular $2^{6k}|-10b^3e^2$ since $2^{3k-1}|e^2$) and thus $2^d|e$ where $d = (3k + 1)/2$ (since k is odd) and therefore again by equation 60. we have that $2^{k+1}|b$, a contradiction.

Therefore, if $2^k|a$ and $2^{k+1}|e$ then $2^{k+1}|b$. However, by equation 58. if

$2^{k+1}|b$ and $2^{k+1}|e$ then $2^{k+1}|a$ and we have thus shown that $2^k|a$, $2^k|b$, and $2^k|e$ for all positive integers k which is impossible.

Therefore if $g = 0$ then $a = b = c = e = f = 0$ and if $g \neq 0$ then $c = a^2$, $g = a^3$ and $b = e = f = 0$. Therefore any self map of $\mathbb{C}G_{3,6}$ is an Adams type mapping and hence $\mathbb{C}G_{3,6}$ has the fixed point property. Q.E.D.

Theorem 5.4: Any self map of $\mathbb{C}G_{3,8}$ is an Adams type mapping and hemce $\mathbb{C}G_{3,8}$ has the fixed point property.

Proof: Let x, y, and z be the first, second and third Chern classes. As for $H^*(\mathbb{C}G_{3,6};\mathbb{Z})$ we also have for $H^*(\mathbb{C}G_{3,8};\mathbb{Z})$ the following:

1. $x^9 = 8x^7y - 21x^5y^2 + 20x^3y^3 - 5xy^4 - 7x^6z + 30x^4yz - 30x^2y^2z + 4y^3z$
$- 10x^3z^2 + 12xyz^2 - z^3$
2. $x^{10} - 9x^8y + 28x^6y^2 - 35x^4y^3 + 15x^2y^4 - y^5 + 8x^7z - 42x^5yz + 60x^3y^2z$
$- 20xy^3z + 15x^4z^2 - 30x^2yz^2 + 6y^2z^2 + 4xz^3 = 0$
3. $-x^{11} + 10x^9y - 36x^7y^2 + 56x^5y^3 - 35x^3y^4 + 6xy^5 - 9x^8z + 56x^6yz - 105x^4y^2z$
$+ 60x^2y^3z - 5y^4z - 21x^5z^2 + 60x^3yz^2 - 30xy^2z^2 - 10x^2z^3 + 4yz^3 = 0$

Dimension 18 is spanned by 11 elements with one dependent element, x , as shown by equation 1.

Dimension 20 has two relations: equation 2. and equationl. multiplied by x. It is therefore spanned by 12 elements with:

4. $x^{10} = 35x^6y^2 - 100x^4y^3 + 75x^2y^4 - 8y^5 + x^7z - 66x^5yz + 210x^3y^2z - 124xy^3z$
$+ 30x^4z^2 - 132x^2yz^2 + 48y^2z^2 + 23xz^3$
5. $x^8y = 7x^6y^2 - 15x^4y^3 + 10x^2y^4 - y^5 + x^7z - 12x^5yz + 30x^3y^2z - 16xy^3z$
$+ 5x^4z^2 - 18x^2yz^2 + 6y^2z^2 + 3xz^3$

Let $h: \mathbb{C}G_{3,8} \longrightarrow \mathbb{C}G_{3,8}$ be a map with $h^*(x) = ax$, $h^*(y) = bx^2 + cy$, and

$h^*(z) = ex^3 + fxy + gz$. Taking h^* of equation 1. and comparing the coefficients of the terms x^3y^3, xy^4, x^4yz, y^3z, x^3z^2, xyz^2, and z^3, we get the following equations:

6. $20M + 20a^3c^3 - 20abc^3 - 30a^2c^2f + 12bc^2f + 4c^3e + 12acf^2 - f^3 = 20a^9$

7. $-5M - 5ac^4 + 4c^3f = -5a^9$

8. $30M + 30a^4cg - 60a^2bcg + 12b^2cg - 20a^3fg + 24abfg + 24aceg - 6efg = 30a^9$

9. $4M + 4c^3g = 4a^9$

10. $-10M - 10a^3g^2 + 12abg^2 - 3eg^2 = -10a^9$

11. $12M + 12acg^2 - 3fg^2 = 12a^9$

12. $-M - g^3 = -a^9$

where $M = 8a^7b - 21a^5b^2 + 20a^3b^3 - 5ab^4 - 7a^6e + 30a^4be - 30a^2b^2e + 4b^3e - 10a^3e^2 + 12abe^2 - e^3$.

Taking h^* of equation 5. and comparing the coefficients of the y^5, xy^3z, y^2z^2, and xz^3 terms gives us the following equations:

13. $-8N - P - c^5 = -8a^8b - a^8c$

14. $-124N - 16P - 16ac^3g + 12c^2fg = -124a^8b - 16a^8c$

15. $48N + 6P + 6c^2g^2 = 48a^8b + 6a^8c$

16. $23N + 3P + 3ag^3 = 23a^8b + 3a^8c$

where $N = 7a^6b^2 - 15a^4b^3 + 10a^2b^4 - b^5 + a^7e - 12a^5be + 30a^3b^2e - 16ab^3e + 5a^4e^2 - 18a^2be^2 + 6b^2e^2 + 3ae^3$ and $P = 14a^6bc - 45a^4b^2c + 40a^2b^3c - 5b^4c + a^7f - 12a^5bf - 12a^5ce + 30a^3b^2f + 60a^3bce - 16ab^3f - 48ab^2ce + 10a^4ef - 36a^2bef - 18a^2ce^2 + 12b^2ef + 12bce^2 + 9ae^2f$. From these equations we get the following:

17. $c^3g = g^3 \quad (4(12) + (9))$

18. $-5ac^4 + 4c^3f = -5g^3 \quad ((7) - 5(12))$

19. $12acg^2 - 3fg^2 = 12g^3$ ((11) + 12(12))

20. $-10a^3g^2 + 12abg^2 - 3eg^2 = -10g^3$ ((10) - 10(12))

21. $30a^4cg - 60a^2bcg + 12b^2cg - 20a^3fg + 24abfg + 24aceg - 6efg = 30g^3$

22. $20a^3c^3 - 20abc^3 - 30a^2c^2f + 12bc^2f + 4c^3e + 12acf^2 - f^3 = 20g^3$

23. $c^5 = c^2g^2$

24. $16c^5 - 16ac^3g + 12c^2fg - 12c^2g^2 + 12ag^3 = 0$

(The linear combinations for equations 21. to 24. being ((8) + 30(12)), ((6) + 20(12)), (6(13) + (15)), and (-16(13) + (14) - 2(15) + 4(16)))

Case 1: $g \neq 0$

If $g \neq 0$ then equation 17. tells us that $c^3 = g^2$ and thus $c \neq 0$. Dividing equation 18. by c^3 gives $4f = 5(ac - g)$ and dividing equation 19. by g^2 gives $3f = 12(ac - g)$ and therefore $15(ac - g) = 48(ac - g)$ which implies that $ac = g$ and $f = 0$. From this we have by equation 24. that $c^5 = ag^3$. Thus $c^{10} = a^2g^6 = a^2c^9$ and therefore $c = a^2$. Also $g^{10} = c^{15} = a^3g^9$ and therefore $g = a^3$ and $a \neq 0$. We then have from equation 20. that $e = 4ab$ and thus $b = 0$ iff $e = 0$. Equation 21. reduces to $-60a^2b + 12b^2 + 96a^2b = 0$. If $b \neq 0$ then $b = -3a^2$ and $e = -12a^3$ but these values along with equation 22. imply that $12a^9 = 0$ and thus if $g \neq 0$ then $c = a^2$, $g = a^3$, and $b = e = f = 0$.

Case 2: $g = 0$

If $g = 0$ then according to equation 23. $c = 0$ and then by equation 22. $f = 0$. We have 4 subcases to consider:

Case 2a: $b = c = f = g = 0$

Case 2b: $c = e = f = g = 0$

Case 2c: $a \neq 0,\ b \neq 0,\ e \neq 0$

Case 2d: $a = 0$

Cases 2a, 2b, and 2c have all ready been covered in the proof of the previous theorem because the only equations used there were equations 1. 2. and 3. We need only consider the remaining case:

Case 2d: If $a = c = f = g = 0$ then equations 1. and 2. become:

26. $4b^3e - e^3 = 0$

27. $-b^5 + 6b^2e^2 = 0$

Equations 26. and 27. imply that $b = 0$ iff $e = 0$. The sum of $6b^2$ times equation 26. plus e times equation 27. is

28. $23b^5e = 0$

which implies that $b = e = 0$. Therefore we have shown that if $g = 0$ then $a = b = c = f = e = 0$ and thus any self map of $\mathbb{C}G_{3,8}$ is an Adams type mapping. Q.E.D.

Combining Theorems 5.1, 5.2, 5.3, 5.4, and 4.1 gives us the following:

Theorem 5.5: For n even, $\mathbb{C}G_{3,n}$ has the fixed point property.

The Lusternik - Schnirelmann category of a topological space X is the smallest integer k 1 such that X may be covered by k open subsets which are contractible in X. The category of the sphere, S^2, is then 2 and the category of the torus, $S^1 \times S^1$, is 3. It is well known that the Lusternik - Schnirelmann category is greater or equal to the cuplength of the space plus one. The cuplength of a space X over a ring R is the largest number m such that there exist elements $x_1, \ldots, x_m \in H^*(X;R)$ with the cup product $x_1 x_2 \ldots x_m \neq 0$. It is also well known that the category of a manifold is less than or equal to it's dimension plus one. Thus we have from these upper and lower bounds, for instance, that the category of the real projective space, $\mathbb{R}P^k = \mathbb{R}G_{1,k}$ is $k + 1$. In the complex case, we observe that since there exists a cell decomposition of $\mathbb{C}G_{1,k}$ which contains only cells of even dimension, (See (11)), cat $\mathbb{C}G_{1,k} = k + 1$ as well. Heinz and Singhof (See (12)) have shown that the cuplength (with $\mathbb{Z}$ coefficients) of $\mathbb{C}G_{2,k}$ is $2k$ and thus cat $\mathbb{C}G_{2,k} = 2k + 1$ for any k 1. In the proof of Theorem 5.2 we found that the cuplength of $\mathbb{C}G_{3,4}$ is 12 (since $x^{12} = 462z^3$) and so we have:

<u>Theorem 5.6</u>: The Lusternik - Schnirelmann category of $\mathbb{C}G_{3,4}$ is 13.

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Appendix - Products

Eilenberg - Zilber Theorem: The functors $S(X)\otimes S(Y)$ from Top $\times$ Top to the category of chain complexes are homotopy equivalent. More precisely, there are unique (up to homotopy) natural chain maps $\Phi : S(X)\otimes S(Y) \longrightarrow S(X\times Y)$ and $\Psi : S(X\times Y) \longrightarrow S(X)\otimes S(Y)$ such that $\Phi_0(\sigma\otimes\tau) = (\sigma,\tau)$ and $\Psi_0(\sigma,\tau) = \sigma\otimes\tau$ for zero simplices $\sigma : \Delta_0 \longrightarrow X$, $\tau : \Delta_0 \longrightarrow Y$. Any such chain map is a homotopy equivalence; in fact, there are natural homotopies $\Psi\Phi \simeq id$, $\Phi\Psi' \simeq id$. Any such chain map will be called an Eilenberg - Zilber map and will be denoted EZ. For more details and proofs see (13).

Corollary: For arbitrary Eilenberg - Zilber maps the following diagrams are homotopy commutative:

$$\begin{tikzcd} SX\otimes SY \arrow[r, leftrightarrow, "EZ"] \arrow[d, "\tau"'] & S(X\times Y) \arrow[d, "S(t)"] \\ SY\otimes SX \arrow[r, leftrightarrow, "EZ"] & S(Y\times X) \end{tikzcd} \qquad \begin{tikzcd} SX\otimes SP \arrow[r, leftrightarrow, "EZ"] \arrow[d, "id\otimes\eta"] & S(X\times P) \arrow[d, "proj."] \\ SX\otimes(\mathbb{Z},0) \arrow[r, leftrightarrow, "id"] & SX \end{tikzcd}$$

$$\begin{tikzcd} SX\otimes SY\otimes SZ \arrow[r, leftrightarrow, "EZ\otimes id"] \arrow[d, leftrightarrow, "id\otimes EZ"] & S(X\times Y)\otimes SZ \arrow[d, leftrightarrow, "EZ"] \\ SX\otimes S(Y\times Z) \arrow[r, leftrightarrow, "EZ"] & S(X\times Y\times Z) \end{tikzcd}$$

where $t(x,y) = (y,x)$, $\tau(u\otimes v) = (-1)^{|u||v|}v\otimes u$ ($|\ |$ denotes gradation), P is a point and η is augmentation

Proof: In each case the two ways of going from corner to corner diagonally induce the identity in dimension 0 (or on H_0), hence are (naturally) homotopic. Q.E.D.

<u>Corollary</u>: For arbitrary EZ maps Φ, Ψ and arbitrary pairs of spaces (X,A) and (Y,B) we have commutative diagrams with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & SA \otimes SY + SX \otimes SB & \longrightarrow & SX \otimes SY & \longrightarrow & SX/SA \otimes SY/SB & \longrightarrow & 0 \\
 & & \Phi' \big\downarrow \big\uparrow \Psi' & & \Phi \big\downarrow \big\uparrow \Psi & & \Phi'' \big\downarrow \big\uparrow \Psi'' & & \\
0 & \longrightarrow & S\{A \times Y, X \times B\} & \longrightarrow & S(X \times Y) & \longrightarrow & \dfrac{S(X \times Y)}{S\{A \times Y, X \times B\}} & \longrightarrow & 0
\end{array}$$

The vertical maps are induced by Φ, Ψ and $S\{A \times Y, X \times B\} = \mathrm{im}\ S(A \times Y) \oplus S(X \times B) \longrightarrow S(X \times Y)$. Moreover, there are natural homotopies $\Phi'\Psi' \simeq \mathrm{id}$, $\Psi'\Phi' \simeq \mathrm{id}$, $\Psi''\Phi'' \simeq \mathrm{id}$.

<u>Proof</u>: Naturality of Φ applied to $j: A \xrightarrow{\subset} X$ and id shows $\Phi(SA \otimes SY) \subset S(A \times Y)$ similarly $\Phi(SX \otimes SB) \subset S(A \times Y)$ and analagously for Ψ. This gives the desired maps. Since the homotopy $\Phi\Psi \simeq \mathrm{id}$ is natural it maps $S\{A \times Y, X \times B\}$ into itself and hence induces $\Phi'\Psi' \simeq \mathrm{id}$, $\Phi''\Psi'' \simeq \mathrm{id}$. The other homotopies are similar. Q.E.D.

<u>The Exterior Homology Product</u>

Consider the composite chain map

$$\left(\frac{SX}{SA} \otimes L\right) \otimes_R \left(\frac{SY}{SB} \otimes M\right) \xrightarrow{EZ} \frac{S(X \times Y)}{S\{A \times Y, X \times B\}} \otimes (L \otimes_R M) \xrightarrow{j} \frac{S(X \times Y)}{S(A \times Y \cup X \times B)} \otimes (L \otimes_R M)$$

where $(X,A),(Y,B)$ are arbirary pairs of spaceá and L,M are R - modules. Passage to homology and composition with the unique map

$$\alpha : H\left(\frac{SX}{SA}\otimes L\right) \otimes_R H\left(\frac{SY}{SB}\otimes M\right) \longrightarrow H\left(\frac{SX}{SA}\otimes L \otimes_R \frac{SY}{SB}\otimes M\right)$$

such that $\alpha([x]\otimes[y]) = [x\otimes y]$ for $x \in Z\left(\frac{SX}{SA}\right)$ and $y \in Z\left(\frac{SY}{SB}\right)$ gives

$$j_*(EZ)_*\alpha : H_i(X,A;L) \otimes_R H_k(Y,B;M) \longrightarrow H_{i+k}(X\times Y, A\times Y\cup X\times B; L\otimes_R M)$$

This map is called the exterior homology product and we write $\xi\times\eta = j_*(EZ)_*\alpha(\xi\otimes\eta)$. In terms of representative cycles this reads as $[a]\times[b] = [EZ(a\otimes_R b)]$ where $a\in SX\otimes L$, $\partial a \in SA\otimes L$, $b\in SY\otimes M$, $\partial b\in SB\otimes M$.

Properties:

Naturality: If $f: (X,A)\longrightarrow(X',A')$ and $g: (Y,B)\longrightarrow(Y',B')$ are maps then the naturality of EZ implies $(f\times g)_*(\xi\times\eta) = (f_*\xi)\times(g_*\eta)$

Commutativity: $t_*(\xi\times\eta) = (-1)^{|\xi||\eta|}\eta\times\xi$

Associativity: $(\xi\times\eta)\times\zeta = \xi\times(\eta\times\zeta)$

Unit element: If $Y = P$ is a point, $B = \emptyset$, and $1^P = 1^P \in R = H_0(Y;R)$ then $(X\times Y, A\times Y\cup X\times B) = (X,A)$ and $1^P\times\xi = \xi\times 1^P = \xi$.

These properties follow from the corresponding ones for EZ maps. We also have that the following diagram is commutative: (coefficients ommitted)

$$\begin{tikzcd}
H(X,A)\otimes H(Y,B) \arrow[r, "\times"] \arrow[dd, "(\partial_*\otimes\, id,\,(-1)^{dim} id\otimes\partial_*)"'] & H(X\times Y, A\times Y\cup X\times B) \arrow[d, "\partial_*"] \\
& H(A\times Y\cup X\times B, A\times B) \\
(HA\otimes H(Y,B)) \oplus (H(X,A)\otimes HB) \arrow[r, "\times\oplus\times"] & H(A\times Y, A\times B)\oplus H(X\times B, A\times B) \arrow[u, "(i_{1*},\, i_{2*})"']
\end{tikzcd}$$

where i_1, i_2 are inclusions. That is $\partial_*(\xi \times \eta) = i_{1*}((\partial_*\xi)\times\eta) + i_{2*}((-1)^{|\xi|}\xi \times \partial_*\eta)$. When $B = \emptyset$ we have $i_1 = id$, $i_{2*} = 0$ and $\partial_*(\xi\times\eta) = (\partial_*\xi)\times\eta$ where $\xi \in H(X,A)$, $\eta \in HY$.

Proof: Let $a \in SX$, $b \in SY$ be representatives of ξ, η; in particular $\partial a \in SA$, $\partial b \in SB$. $EZ(\partial a \otimes b) \in S(A\times Y \cup X\times B)$ represents $i_{1*}((\partial_*\xi)\times\eta)$, $EZ(a\otimes\partial b) \in S(X\times B) \subset S(A\times Y\cup X\times B)$ represents $i_{2*}(\xi\times\partial_*\eta)$ and $\partial(EZ)(a\otimes b) = (EZ)\partial(a\otimes b) = EZ(\partial a\otimes b) + (-1)^{|\xi|}\, EZ(a\otimes\partial b)$ represents $\partial_*(\xi\times\eta)$. Q.E.D.

The Exterior Cohomology Product:

Let $(X,A),(Y,B)$ be pairs of spaces such that $(X\times Y; A\times Y, X\times B)$ is an excisive triad and let L, M be R - modules. Consider the composite chain map:

$$\mathrm{Hom}\left(\frac{SX}{SA}, L\right)\otimes_R \mathrm{Hom}\left(\frac{SY}{SB}, M\right) \xrightarrow{\ \ \gamma\ \ } \mathrm{Hom}\left(\frac{SX}{SA}\otimes\frac{SY}{SB}, L\otimes_R M\right) \xrightarrow{\ \ EZ\ \ }$$

$$\mathrm{Hom}\left(\frac{S(X\times Y)}{S\{A\times Y, X\times B\}}, L\otimes_R M\right) \xrightarrow{\ \ j\ \ } \mathrm{Hom}\left(\frac{S(X\times Y)}{S(A\times Y\cup X\times B)}, L\otimes_R M\right)$$

where the chain map γ is defined by $(\gamma(\varphi\otimes\psi))(c\otimes d) = (-1)^{|c||d|}\varphi(c)\otimes\psi(d)$ and j is induced by the inclusion $S\{A\times Y, X\times B\} \subset S(A\times Y\cup X\times B)$, which like EZ, is a homotopy equivalence since $(X\times Y; A\times Y, X\times B)$ is an excisive triad. Passage to homology and composition with

$$\alpha: H^*(X,A;L)\otimes_R \mathrm{Hom}^*(Y,B;M) \longrightarrow H(S^*(X,A;L)\otimes_R S^*(Y,B;M)) \quad \text{gives}$$

$$(j^*)^{-1}(EZ)^*\gamma_*\alpha: H^i(X,A;L)\otimes_R H^k(Y,B;M) \longrightarrow H^{i+k}(X\times Y, A\times Y\cup X\times B; L\otimes_R M)$$

This map is called the exterior cohomology product and it is written: $x\times y = (j^*)^{-1}(EZ)^*\gamma_*\alpha(x\otimes y)$ or in terms of cocycles $(\varphi)\times(\psi) =$

$(\gamma(\varphi\otimes\psi)\circ EZ)$ where $\varphi \in S^*(X;L)$, $\varphi|SA = 0$, $\varphi\circ\partial = 0$, $\psi \in S^*(Y;M)$, $\psi|SB = 0$, $\psi\circ\partial = 0$.

Properties:

Naturality: If $f\colon (X,A) \longrightarrow (X',A')$, $g\colon (Y,B) \longrightarrow (Y',B')$ are maps of pairs such that $(X'\times Y';A'\times Y',X'\times B')$ is also an excisive triad then $(f\times g)^*(x'\otimes y') = (f^*x')\times(g^*y')$.

Commutativity: $t^*(x\times y) = (-1)^{|x||y|}y\times x$ where $t\colon X\times Y \longrightarrow Y\times X$ commutes factors.

Associativity: $(x\times y)\times z = x\times(y\times z)$

Units: If $Y = P$ is a point, $B = \emptyset$, and $1_P \in H^0(P;R)$ is the cohomology class of the augmentation $\eta\colon S_0P \longrightarrow R$, $P \longmapsto 1$, then $1_P\times x = x = x\times 1_P$ (where $P\times(X,A) = (X,A) = (X,A)\times P$). If Y is an arbitrary space again, and $\pi\colon Y \longrightarrow P$ then $1_Y = \pi^*(1_P) \in H^0(Y;R)$ is the class of the augmentation $S_0Y \longrightarrow R$ and naturality gives $x\times 1_Y = (id\times\pi)^*(x\times 1_P) = p^*(x)$ where $p\colon (X,A)\times Y \longrightarrow (X,A)\times P$ is the projection.

The following diagram (coefficients ommitted) is comutative:

$$\begin{array}{ccc}
H^*A \otimes H^*(Y,B) & \xrightarrow{\quad\times\quad} & H^*(A\times Y,A\times B) \cong H^*(A\times Y\cup X\times B,X\times B) \\
\big\downarrow{\scriptstyle \delta^*\otimes id} & & \big\downarrow{\scriptstyle \delta^*} \\
H^*(X,A)\otimes H^*(Y,B) & \xrightarrow{\quad\times\quad} & H^*(X\times Y,A\times Y\cup X\times B)
\end{array}$$

or $\delta^*(i^*)^{-1}(a\times y) = (\delta^*a)\times y$ for $a \in H^*A$, $y \in H^*(Y,B)$. In the case where $B = \emptyset$, $i = id$ and $\delta^*(a\times y) = (\delta^*a)\times y$.

<u>Duality</u>: If $\xi \in H(X,A;R)$, $\eta \in H(Y,B;R)$, $x \in H^*(X,A;L)$, $y \in H^*(Y,B;M)$ then $\langle x \times y, \xi \times \eta \rangle = (-1)^{|y||\xi|} \langle x, \xi \rangle \otimes \langle y, \eta \rangle$.

<u>The Interior Cohomology Product</u> (<u>The Cup Product</u>):

This product is equivalent to the exterior cohomology product because Eilenberg - Zilber maps EZ: $S(X \times Y) \longrightarrow SX \otimes SY$ and natural diagonals D: $SX \longrightarrow SX \otimes SX$ are formally equivalent notions and we get the definition of the interior cohomology product by replacing the EZ map which occurs in the definition of the exterior cohomology product with a diagonal map. Consider the composite chain map:

$$\mathrm{Hom}\left(\frac{SX}{SA_1}, M_1\right) \otimes_R \mathrm{Hom}\left(\frac{SX}{SA_2}, M_2\right) \xrightarrow{\ \ \gamma\ \ } \mathrm{Hom}\left(\frac{SX}{SA_1} \otimes \frac{SX}{SA_2}, M_1 \otimes_R M_2\right) \xrightarrow{\ D\ }$$

$$\mathrm{Hom}\left(\frac{SX}{S\{A_1,A_2\}}, M_1 \otimes_R M_2\right) \xleftarrow{\ \ \ j\ \ \ } \mathrm{Hom}\left(\frac{SX}{S(A_1 \cup A_2)}, M_1 \otimes_R M_2\right)$$

where, as before, $(\gamma(\varphi_1 \otimes \varphi_2))(a_1 \otimes a_2) = (-1)^{|\varphi_2||a_1|} (\varphi_1 a_1) \otimes (\varphi_2 a_2)$

and j is the homotopy equivalence induced by inclusion. Passage to homology and composition with α as before gives

$$(j^*)^{-1} D^* \gamma_* \alpha : H^i(X,A_1;M_1) \otimes H^k(X,A_2;M_2) \qquad H^{i+k}(X,A_1 \quad A_2; M_1 \otimes_R M_2)$$

We write the cup - product as $x_1 \cup x_2 = (j^*)^{-1} D^* \gamma_* \alpha (x_1 \otimes x_2)$ and in terms of representative cocycles φ_1, φ_2, $(\varphi_1) \cup (\varphi_2) = (\gamma(\varphi_1 \otimes \varphi_2) \circ D)$ where $\varphi_i \in S^*(X;M_i)$, $\varphi_i | SA_i = 0$, $\varphi_i \circ \delta = 0$.

<u>Properties</u>:

<u>Naturality</u>: If f: $(X;A_1,A_2) \longrightarrow (X';A_1',A_2')$ is a map of excisive triads

then $f^*(y_1 \cup y_2) = (f^*y_1) \cup (f^*y_2)$ for $y_i \in H^*(Y,B;M_i)$.

Commutativity: $x_1 \cup x_2 = (-1)^{|x_1|\,|x_2|} x_2 \cup x_1$.

Associativity: $x_1 \cup (x_2 \cup x_3) = (x_1 \cup x_2) \cup x_3$

Units: $1_X \cup x = x = x \cup 1_X$, where $1_X \in H^0(X;R)$ is the augmentation class.

The following diagram is communist:

$$\begin{tikzcd}
H^*A_1 \otimes H^*(X,A_2) \arrow[r, "id \otimes i^*"] \arrow[d, "\delta^* \otimes id"] & H^*A_1 \otimes H^*(A_1, A_1 \cap A_2) \arrow[r, "\cup"] & H^*(A_1, A_1 \cap A_2) \\
& & H^*(A_1 \cup A_2, A_2) \arrow[u, "j^*"] \arrow[d, "\delta^*"] \\
H^*(X,A_1) \otimes H^*(X,A_2) \arrow[rr, "\cup"] & & H^*(X, A_1 \cup A_2)
\end{tikzcd}$$

That is: $\delta^*(j^*)^{-1}(a \cup i^*x) = (\delta^*a) \cup x$ for $a \in H^*A_1$, $x \in H^*(X,A_2)$ and if $A_2 = \emptyset$, $\delta^*(a \cup i^*x) = (\delta^*a) \cup x$ for $a \in H^*A_1$, $x \in H^*X$.

$x_1 \cup x_2 = \Delta^*(x_1 \times x_2)$, $x_i \in H^*(X,A_i)$, where $\Delta: (X, A_1 \cup A_2) \longrightarrow (X \times X, A_1 \times X \cup X \times A_2)$ is the diagonal map.

$x \times y = (p^*x) \cup (q^*y)$, if $x \in H^*(X,A)$, $y \in H^*(Y,B)$ and $p: (X \times Y, A \times Y) \longrightarrow (X,A)$, $q: (X \times Y, X \times B) \longrightarrow (Y,B)$ are projections and $(X \times Y; A \times Y, X \times B)$ is excisive. Therefore we have:

Multiplicativity: $(x_1 \times y_1) \cup (x_2 \times y_2) = (-1)^{|y_1|\,|x_2|}(x_1 \cup x_2) \times (y_1 \cup y_2)$ if $x_i \in H^*(X,A_i)$, $y_i \in H^*(Y,B_i)$ and $(X;A_1,A_2)$, $(Y;B_1,B_2)$ are triads such that the products above are defined.

The Cohomology Slant Product:

Consider the chain map:

$$E: \mathrm{Hom}(D,M)\otimes(C\otimes D\otimes L) \overset{\omega}{\cong} (C\otimes L)\otimes(\mathrm{Hom}(D,M)\otimes D) \xrightarrow{id\otimes e} C\otimes L\otimes M$$

where C, D are R - complexes, and L, M are R - modules, ω permutes factors and e is the evaluation map. $E(\psi\otimes c\otimes d\otimes 1) = (-1)^{|\psi||c|}\, c\otimes 1\otimes\psi(d)$. Passage to homology and composition with α gives

$$E_*\alpha: H^i(D,M)\otimes H_n(C\otimes D\otimes L) \longrightarrow H_{n-i}(C\otimes L\otimes M)$$

This map is called the cohomology slant product (for complexes) and is written $y\backslash\xi = E_*\alpha(y\otimes\xi)\in H_{n-i}(C\otimes L\otimes M)$ for $y\in H^i(D,M)$, $\xi\in H_n(C\otimes D\otimes L)$. The cohomology slant product for spaces $(X,A),(Y,B)$ is obtained by taking $C = S(X,A;R)$, $D = S(Y,B;R)$ and replacing $S(X,A;R)\otimes S(Y,B;R)$ by the homotopy equivalent complex

$$S(X\times Y, A\times Y\cup X\times B) \simeq \frac{S(X\times Y;R)}{S\{A\times Y, X\times B;R\}} \overset{EZ}{\simeq} S(X,A;R)\otimes S(Y,B;R)$$

assuming that $(X\times Y;A\times Y,X\times B)$ is an excisive triad. In terms of representative cocycles $[\psi]\backslash[z] = (-1)^{|\psi|(|z|-|\psi|)}\left[\sum_i a_i\otimes\psi(b_i)\right]$ where $EZ(z) = \sum a_i\otimes b_i$, $a_i\in S(X;L)$, $b_i\in S(Y;R)$ for $\psi\in S^i(Y;M)$, $z\in S(X\times Y;L)$. The representative z must be such that $\partial z\in S\{A\times Y, X\times B;L\}$ as well as being in $S(A\times Y\cup X\times B;L)$.

Properties: (ommitting coefficients)

Naturality: If $f: (X,A)\longrightarrow(X',A')$, $g: (Y,B)\longrightarrow(Y',B')$ are maps of pairs then $f_*(g^*y'\backslash\xi) = y'\backslash(f\times g)_*\xi$, for $y'\in H^*(Y',B')$, $\xi\in H(X\times Y, A\times Y\cup X\times B)$.

<u>Associativity</u>: $(x\times y)\backslash\gamma = x\backslash(y\backslash\gamma)$, for $x\in H^*(X,A)$, $y\in H^*(Y,B)$, $\gamma\in H((W,U)\times(X,A)\times(Y,B))$. In particular if W is a point and $U=\emptyset$ we have

<u>Duality</u>: $\langle x\times y,\varphi\rangle = \langle x, y\backslash\varphi\rangle$, for $x\in H^*(X,A)$, $y\in H^*(Y,B)$, $\varphi\in H(X\times Y, A\times Y\cup X\times B)$.

The following diagrams are commutative:

$$\begin{tikzcd}
H^*(Y,B)\otimes H(X\times Y, A\times Y\cup X\times B) \arrow[rr, "\backslash"] \arrow[d, "(-1)^{\dim}\mathrm{id}\otimes\partial_*"] & & H(X,A) \arrow[d, "\partial"] \\
H^*(Y,B)\otimes H(A\times Y\cup X\times B, X\times B) \arrow[r, "\mathrm{id}\otimes j_*", "\cong"'] & H^*(Y,B)\otimes H(A\times Y, A\times B) \arrow[r, "\backslash"] & HA
\end{tikzcd}$$

That is: $\partial_*(y\backslash\varphi) = (-1)^{|y|}y\backslash j_*^{-1}\partial_*\varphi$, if $y\in H^*(Y,B)$, $\varphi\in H(X\times Y, A\times Y\cup X\times B)$.

$$\begin{tikzcd}
H^*B\otimes H(X\times Y, A\times Y\cup X\times B) \arrow[rr, "\delta^*\otimes\mathrm{id}"] \arrow[d, "-(-1)^{\dim}\mathrm{id}\otimes\partial_*"] & & H^*(Y,B)\otimes H(X\times Y, A\times Y\cup X\times B) \arrow[d, "\backslash"] \\
H^*B\otimes H(A\times Y\cup X\times B, A\times Y) \arrow[r, "\mathrm{id}\otimes j_*", "\cong"'] & H^*B\otimes H(X\times B, A\times B) \arrow[r, "\backslash"] & H(X,A)
\end{tikzcd}$$

That is: $(\delta^*b)\backslash\varphi + (-1)^{|b|}b\backslash j_*^{-1}\partial_*\varphi = 0$, if $b\in H^*B$, $\varphi\in H(X\times Y, A\times Y\cup X\times B)$.

<u>Multiplicativity</u>: $y\backslash\omega\times\varphi = (-1)^{|y||\omega|}\omega\times(y\backslash\varphi)$, if $y\in H^*(Y,B)$, $\omega\in H(W,U)$, $\varphi\in H(X\times Y, A\times Y\cup X\times B)$, and $(W,U),(X,A),(Y,B)$ are pairs of spaces such that the products above are defined.

<u>Units</u>: $1_Y\backslash\varphi = p_*\varphi$, where $1_Y\in H^0(Y;R)$, $\varphi\in H(X\times Y, A\times Y)$, and $p\colon (X\times Y, A\times Y)\qquad (X,A)$ is the projection.

<u>The Cap Product</u>:

Let $(X;A_1,A_2)$ be an excisive triad, and let M_1, M_2 be R - modules. Consider

$$\mathrm{Hom}\left(\frac{SX}{SA_2},M_2\right)\otimes_R\left(\frac{SX}{S\{A_1,A_2\}}\right)\otimes M_1\xrightarrow{\ id\otimes D\ }\mathrm{Hom}\left(\frac{SX}{SA_2},M_2\right)\otimes_R\left(\frac{SX}{SA_1}\otimes\frac{SX}{SA_2}\otimes M_1\right)$$

$$\xrightarrow{\qquad E\qquad}\frac{SX}{SA_1}\otimes M_1\otimes_R M_2$$

where D is a natural diagonal and E is the same as in the slant product. Passing to homology (using $H(\frac{SX}{S\{A_1,A_2\}})\cong H(X,A_1\cup A_2)$) and composing with α we obtain $E_*(id\otimes D)_*\alpha$: $H^k(X,A_2;M_2)\otimes H_n(X,A_1\ \ A_2;M_1)\longrightarrow H_{n-k}(X,A_1;M_1\otimes_R M_2)$. This map is called the cap product. We write $x\frown\xi = E_*(id\otimes D)_*\alpha(x\otimes\xi)$, if $x\in H^*(X,A_2;M_2)$, $\xi\in H(X,A_1\cup A_2;M_1)$. In terms of representatives this reads $[\varphi]\frown[c] = (-1)^{|\varphi|(|c|-|\varphi|)}[\sum c_i^1\otimes\varphi(c_i^2)]$ where $Dc=\sum c_i^1\otimes c_i^2$; assuming $\varphi\in S^*X$, $\varphi|SA_2=0$, $\delta\varphi=0$, $c\in SX$, $\partial c\in S\{A_1,A_2\}$.

<u>Properties</u>:

<u>Naturality</u>: $f_*((f^*x')\frown\xi) = x'\frown(f_*\xi)$, if f is a map of excisive triads and $x'\in H^*(X',A_2')$, $\xi\in H(X,A_1\cup A_2)$.

<u>Associativity</u>: $(x_1\cup x_2)\frown\xi = x_1\frown(x_2\frown\xi)$ if $x_i\in H^*(X,A_{i+1})$, $\xi\in H(X,A_1\cup A_2\cup A_3)$.

<u>Duality</u>: $\langle x_1\cup x_2,\xi\rangle = \langle x_1,x_2\frown\xi\rangle$, if $x_i\in H^*(X,A\ )$, $\xi\in H(X,A_1\cup A_2)$. In particular, $\langle 1,x\frown\xi\rangle = \langle x,\xi\rangle$, for $x\in H^j(X,A)$, $\xi\in H_j(X,A)$.

<u>Units</u>: $1\frown\xi=\xi$, if $\xi\in H(X,A)$, and $1\in H^0(X;R)$ is the augmentation class.

The following diagrams are commutative:

$$\begin{tikzcd}
H^*(X,A_2)\otimes H(X,A_1\cup A_2) \arrow[rr, "\cap"] \arrow[d, "(-1)^{\dim} i^*\otimes\partial_*"] & & H(X,A_1) \arrow[d, "\partial_*"] \\
H^*(A_1,A_1\cap A_2)\otimes H(A_1\cup A_2,A_2) \arrow[r, "\underset{\cong}{id\otimes j_*}"] & H^*(A_1,A_1\cap A_2)\otimes H(A_1,A_1\cap A_2) \arrow[r, "\cap"] & HA_1
\end{tikzcd}$$

That is: $\partial_*(x\cap\xi) = (-1)^{|x|}(i^*x)\cap(j_*^{-1}\partial_*\xi)$ if $x\in H^*(X,A_2)$, $\xi\in H(X,A_1\cup A_2)$.

$$\begin{tikzcd}
H^*A_2\otimes H(X,A_1\cup A_2) \arrow[r, "\partial^*\otimes id"] \arrow[d, "-(-1)^{\dim} id\otimes\partial_*"] & H^*(X,A_2)\otimes H(X,A_1\cap A_2) \arrow[r] & H(X,A_1) \\
H^*A_2\otimes H(A_1\cup A_2,A_1) \arrow[r, phantom, "\cong"] & H^*A_2\otimes H(A_2,A_1\cap A_2) \arrow[r] & H(A_2,A_1\cup A_2) \arrow[u, "i^*"]
\end{tikzcd}$$

That is: $(\delta^*a)\cap\xi + (-1)^{|a|}i_*(a\cap j_*^{-1}\partial_*\xi) = 0$ if $a\in H^*A_2$, $\xi\in H(X,A_1\cup A_2)$

$x\cap\xi = x\backslash\Delta_*\xi$, if $x\in H^*(X,A_2)$, $\xi\in H(X,A_1\cup A_2)$ and $\Delta: (X,A_1\cup A_2)\longrightarrow(X\times X,A_1\times X\cup X\times A_2)$ is the diagonal map and the triads are excisive.

$y\backslash\xi = p_*(q^*y\cap\xi)$, if $y\in H^*(Y,B)$, $\xi\in H(X\times Y,A\times Y\cup X\times B)$ and $p: (X\times Y,A\times Y)\longrightarrow(X,A)$, $q: (X\times Y,X\times B)\longrightarrow(Y,B)$ are projections and $(X\times Y;A\times Y,X\times B)$ is excisive.

<u>Multiplicativity</u>: $(x\times y)\cap(\xi\times\eta) = (-1)^{|y||\xi|}(x\cap\xi)\times(y\cap\eta)$ if $x\in H^*(X,A_2)$, $y\in H^*(Y,B_2)$, $\xi\in H(X,A_1\cup A_2)$, $\eta\in H(Y,B_1\cup B_2)$ and the triads are such that the products are defined.