NON-STANDARD ANALYSIS

by

GLEN RUSSELL COOPER
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Department of Mathematics

The University of British Columbia
Vancouver 8, Canada

Date April, 1975
In this thesis some classical theorems of analysis are provided with non-standard proofs.

In Chapter I some compactness theorems are examined. In 1.1 the monad $\mu(p)$ of any point $p$ contained in a set $X$ (and relative to a family $H$ of subsets of $X$) is defined. Using monads, a non-standard characterization of compact families of subsets of $X$ is given. In 1.2 it is shown that the monad $\mu(p)$ of any point $p \in X$ (relative to $H$) remains unchanged if $H$ is extended to the smallest topology $\tau(H)$ on $X$ containing $H$. Then, as an immediate consequence, the Alexander Subbase theorem is proved. In 1.3 monads are examined in topological products of topological spaces. Then, in 1.4 and 1.5 respectively, both Tychonoff's theorem and Alaoglu's theorem are easily proved.

In Chapter 2 various extension results of Tarski and Nikodým (in the theory of Boolean algebras) are presented with rather short proofs. Also, a result about Boolean covers is proved.

The techniques of non-standard analysis contained in Abraham Robinson's book [Robinson 1974] are used throughout.

The remarks at the end of each chapter set forth pertinent references.
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This thesis is dedicated to my wife, Adriana (née Buonanno) Cooper.
INTRODUCTION


Recall that if \( \{X_t\} \) is the full, normal higher order structure on a set \( X \) then there exists a normal higher order structure \( \{X^*_t\} \) on a set \( ^*_X \) such that \( \{X^*_t\} \) extends \( \{X_t\} \) (in particular \( ^*_X \) contains \( X \)) and such that the following properties are satisfied. If \( S \) is a higher order sentence in \( \{X_t\} \) and \( ^*_S \) is its interpretation in \( \{X^*_t\} \) then \( S \) is true in \( \{X_t\} \) iff \( ^*_S \) is true in \( \{X^*_t\} \). This property is sometimes called the transfer principle.

For the next property we recall some definitions. Suppose \( Q(x,y) \) is a binary relation in \( \{X_t\} \). The domain of \( Q \), domain \( (Q) \), is defined as the set of all \( x \) in \( \{X_t\} \) such that \( Q(x,y) \) for some \( y \) in \( \{X_t\} \). The range of \( Q \), range \( (Q) \), is defined similarly. A binary relation \( Q(x,y) \) in \( \{X_t\} \) is called concurrent if whenever \( x_1, \ldots, x_n \in \text{domain}(Q) \) there exists some \( y \in \text{range}(Q) \) such that \( Q(x_k,y) \) for \( k = 1, \ldots, n \). If \( Q(x,y) \) is a concurrent, binary relation in \( \{X_t\} \) then there exists some \( y_Q \) in \( \{X^*_t\} \) (called an ideal element for \( Q \)) such that \( Q(x,y_Q) \) is true in \( \{X^*_t\} \) for \( x \in \text{domain}(Q) \).

An enlargement of the full, normal higher order structure \( \{X^*_t\} \) on a set \( X \) is any normal higher order structure \( \{X^*_t\} \) on a set \( ^*_X \) satisfying the above properties. Thus, by the above, any full, normal higher order structure possesses and enlargement (which therefore satisfies the transfer principle and assigns ideal elements to concurrent binary relations).
Suppose a normal higher order structure \( \{ *X^*_t \} \) on a set \( *X \) is an enlargement of the full, normal higher order structure \( \{ X^*_t \} \) on a set \( X \). Recall that, in general, \( \{ *X^*_t \} \) need not be full. Thus if \( \{ **X^*_t \} \) is the full, normal higher order structure on \( *X \) then, for each \( \tau \), \( **X^*_t - *X^*_t \) need not be empty. Any relation contained in some \( **X^*_t - *X^*_t \) is called external while the relations contained in \( *X^*_t \) are called internal. Any relation contained in some \( *X^*_t \) is called standard.

Some general properties of enlargements are recalled here. Suppose the normal higher order structure \( \{ *X^*_t \} \) on the set \( *X \) is an enlargement for the full, normal higher order structure \( \{ X^*_t \} \) on the set \( X \). Let \( A \) and \( B \) be sets (that is, unary relations) in \( \{ X^*_t \} \). Then \( A \cup B \) is contained in \( \{ X^*_t \} \) since \( \{ X^*_t \} \) is full. Let \( *A \), \( *B \) and \( *(A \cup B) \) be the extensions in \( \{ X^*_t \} \) of \( A \), \( B \) and \( A \cup B \) respectively. Then \( *A \cup *B = *(A \cup B) \). Similar results hold for all finite Boolean combinations of sets.

Let \( \{ *R^*_t \} \) be a normal higher order structure on a set \( *R \) which is an enlargement of the full, normal higher order structure \( \{ R^*_t \} \) on the set \( R \) of real numbers. Recall that, by transfer, the sentence in \( \{ R^*_t \} \) which states that \( R \) is an ordered field under the usual relations also states (when interpreted in \( \{ R^*_t \} \)) that \( *R \) is an ordered field under the same relations (when extended in \( \{ R^*_t \} \)). Thus \( *R \) (strictly speaking \( \{ *R^*_t \} \)) may be called a field of non-standard real numbers. Let \( || \) be the absolute value function in \( \{ R^*_t \} \). Recall that a non-standard real number \( x \in *R \) is called finite if
\[ |x| \leq y \text{ for some } y \in \mathbb{R} \text{ (otherwise it is called infinite). Also,} \]
a non-standard real number \( x \in \mathbb{R}^* \) is called infinitesimal if
\[ |x| < y \text{ for all positive } y \in \mathbb{R} . \text{ Clearly } 0 \in \mathbb{R} \text{ is infinitesimal.} \]

But the binary relation \( Q(x,y) \) in \( \mathbb{R}^* \) meaning \( o < y < x \) is
concurrent. Thus there exists some \( y_Q \) in \( \mathbb{R}^* \) such that \( Q(x,y_Q) \)
for all positive \( x \in \mathbb{R} \). Hence \( y_Q \) is a non-zero infinitesimal
non-standard real number. Since \( \mathbb{R}^* \) contains non-zero infinitesimals
it contains infinite numbers. From this it is easy to show that \( \mathbb{R}^* \)
is non-archimedean. Recall that if \( x \in \mathbb{R}^* \) is a finite non-standard
real number then there exists a unique standard real number \( \mathbb{O}x \in \mathbb{R} \)
called the standard part of \( x \) such that \( x - \mathbb{O}x \) is infinitesimal
(write \( x = \mathbb{O}x \) and call \( x \) infinitely close to \( \mathbb{O}x \) ). It is easy
to show that if \( x, y \in \mathbb{R}^* \) are finite then \( \mathbb{O}(x+y) = \mathbb{O}x + \mathbb{O}y \) and
\( \mathbb{O}(xy) = \mathbb{O}x \cdot \mathbb{O}y \). Obviously \( x = \mathbb{O}x \) if \( x \in \mathbb{R} \). If \( x \in \mathbb{R} \) let the
monad \( \mu(x) \) of \( x \) be defined as the set of all finite \( y \in \mathbb{R}^* \)
satisfying \( y \leq x \). Recall that a set \( A \subseteq \mathbb{R} \) of standard real numbers
is open iff whenever \( x \in A \) then \( \mu(x) \subseteq A^* \). Hence a set \( B \subseteq \mathbb{R} \)
of standard real numbers is closed iff whenever \( x \in \mathbb{R} \) and
\( \mu(x) \cap B \neq \emptyset \) then \( x \in B \). Note that a set \( A \subseteq \mathbb{R} \) of standard real
numbers is finite iff there exists a bijection from \{1,\ldots,n\} onto
\( A \) (for some \( n \in \mathbb{N} \) ). An internal set \( A \subseteq \mathbb{R}^* \) of non-standard real
numbers is called \( *\)-finite iff there exists an internal bijection
from \{1,\ldots,\omega\} onto \( \mathbb{A}^* \) (for some \( \omega \in \mathbb{N} \) ). A similar definition
exists for \( *\)-finite internal sequences.
Finally, recall that it may be assumed that the full, higher order structure \( \{X_\tau\} \) on a set \( X \) contains the set \( \mathbb{R} \) of standard real numbers (extend \( X \) to \( X \cup \mathbb{R} \) if necessary) and thus any enlargement of \( \{X_\tau\} \) contains a set \( \mathbb{R}^* \) of non-standard real numbers. From now on such enlargements are assumed fixed in any discussion.

CHAPTER 1

1.1 Compact Families of Sets

definition

Let \( X \) be a set and let \( S(X) \) be the power set of \( X \) and let \( H \subseteq S(X) \) be a set of subsets of \( X \). If \( x \in X \) then \( \mu(x) = \bigcap \{ A : x \in A \text{ and } A \in H \} \) is called the monad of \( x \) (relative to \( H \)). A point \( p \in \mathbb{R}^* \) is called near-standard (relative to \( H \)) if \( p \in \mu(x) \) for some \( x \in X \) (write \( p \approx x \) (relative to \( H \))). A set \( X \) is called \( H \)-compact if every cover of \( X \) by members of \( H \) has a finite sub-cover. Note that if \( H \) is a topology on \( X \) then \( H \)-compactness is just compactness in the sense of topology.

Lemma

Let \( X \) be a set and let \( H \) be a set of subsets of \( X \). Then \( X \) is \( H \)-compact iff each \( p \in \mathbb{R}^* \) is near-standard (relative to \( H \)).

Proof

Suppose \( X \) is \( H \)-compact but assume some \( p \in \mathbb{R}^* \) is not
near-standard (relative to $H$). Then for each $x \in X$ there exists some $A_x \in H$ such that $x \in A_x$ but $p \notin A_x$. However

$\{A_x : x \in X\}$ covers $X$ so there exists some finite set

$\{x_1, \ldots, x_n\} \subset X$ such that $X = A_{x_1} \cup \cdots \cup A_{x_n}$. Thus

$X = (A_{x_1} \cup \cdots \cup A_{x_n}) = A_{x_1}^* \cup \cdots \cup A_{x_n}^*$ so $p \in A_{x_K}$ for some $K = 1, \ldots, n$. This contradiction implies that $p$ is near-standard (relative to $H$). Thus each $p \in X$ is near-standard (relative to $H$).

Suppose, conversely, each $p \in X$ is near-standard (relative to $H$) but assume $X$ is not $H$-compact. Then there exists some

$\{A_\alpha\} \subset H$ covering $X$ but possessing no finite sub-cover. Thus the binary relation $Q(A_\alpha, x)$ meaning $x \notin A_\alpha$ is concurrent so there exists some $x_Q \in X$ such that $x_Q \notin A_\alpha$ for all $\alpha$. However

$x_Q = x$ (relative to $H$) for some $x \in X$. But $x \in A_\alpha$ for some $\alpha$. Thus $x_Q \notin A_\alpha$. This contradiction implies that $X$ is $H$-compact.

1.2 Alexander Subbase Theorem

definition

Let $X$ be a set and let $H \subset S(X)$ be a set of subsets of $X$. Let $\tau(H)$ denote the smallest topology on $X$ containing $H$. Recall that $\tau(H) = H^- \cup \{X\}$ where $H^-$ is the set of all unions of finite intersections of members of $H$. Note that if $x \in A$ and $A \in H^-$ then $x \in A_1 \cap \cdots \cap A_n \subset A$ for some finite set $\{A_1, \ldots, A_n\} \subset H$. 


Theorem

Let \( X \) be a set and let \( H \) be a set of subsets of \( X \) and let \( \tau(H) \) be the smallest topology on \( X \) containing \( H \). Furthermore, let \( p \in X \) and \( x \in X \). Then \( p \preceq x \) (relative to \( H \)) iff \( p \preceq x \) (relative to \( \tau(H) \)).

Proof

Suppose \( p \preceq x \) (relative to \( H \)). It suffices to show that \( p \preceq x \) (relative to \( \tau(H) \)). Suppose \( x \in A \) and \( A \in \tau(H) \). It suffices to show that \( p \in ^*A \). If \( A = X \) then \( p \in ^*A \). Thus, assume \( A \neq X \). Then \( A \in H' \) where \( H' \) is the set of all unions of finite intersections of members of \( H \). Thus \( x \in A \cap \cdots \cap A_n \subseteq A \) for some finite set \( \{A_1, \ldots, A_n\} \subseteq H \). In particular, \( x \in A_K \) for \( K = 1, \ldots, n \). Hence \( p \in ^*A_K \) for \( K = 1, \ldots, n \) since \( p \preceq x \) (relative to \( H \)). But then \( p \in ^*A_1 \cap \cdots \cap ^*A_n = ^*(A_1 \cap \cdots \cap A_n) \subseteq ^*A \) so \( p \in ^*A \) as desired.

The reverse implication is clear.

Corollary (Alexander Subbase)

Let \( X \) be a set and let \( H \) be a set of subsets of \( X \) and let \( \tau(H) \) be the smallest topology on \( X \) containing \( H \). If \( X \) is \( H \)-compact then \( X \) is \( \tau(H) \)-compact.

Proof

Suppose \( X \) is \( H \)-compact. Let \( p \in ^*X \). By 1.1 there
exists some \( x \in X \) such that \( p \sim x \) (relative to \( H \)). By 1.1 it suffices to show that \( p \sim x \) (relative to \( \tau(H) \)). But this follows from the above theorem.

1.3 Products of Topological Spaces

**definition**

If \( \{(X_i, \tau_i) : i \in I\} \) is a family of topological spaces (where \( \tau_i \) is the topology on \( X_i \)) let \( (\Pi X_i, \Pi \tau_i) \) be its topological product (where \( \Pi \tau_i \) is the product topology on \( \Pi X_i \)). Recall that the product topology on \( \Pi X_i \) is the smallest topology on \( \Pi X_i \) containing all sets of the form

\[
A = \{ f \in \Pi X_i : f(i_1) \in A_{i_1} \}
\]

where \( i_1 \in I \) and \( A_{i_1} \in \tau_{i_1} \). Hence the product topology on \( \Pi X_i \) is the set of all unions of sets of the form

\[
A = \bigcap_{K=1}^{n} \{ f \in \Pi X_i : f(i_K) \in A_{i_K} \}
\]

where \( i_1, \ldots, i_n \in I \) and \( A_{i_K} \in \tau_{i_K} \) for \( K = 1, \ldots, n \).

**Lemma**

Let \( \{(X_i, \tau_i) : i \in I\} \) be a family of topological spaces and let \( (\Pi X_i, \Pi \tau_i) \) be its topological product. Suppose \( g \in \star(\Pi X_i) \) and \( h \in \Pi X_i \). Then \( g \sim h \) (relative to \( \Pi \tau_i \)) iff \( g(i) \sim h(i) \) (relative
Proof

Suppose \( g \sim h \) (relative to \( \mathbb{P}\tau_1 \)). Let \( i_1 \in I \). It suffices to show that \( g(i_1) \sim h(i_1) \) (relative to \( \tau_{i_1} \)). Let \( h(i_1) \in \mathbb{A}_{i_1}^1 \) and \( \mathbb{A}_{i_1}^1 \in \tau_{i_1} \). It suffices to show that \( g(i_1) \in \mathbb{A}_{i_1}^1 \).

But \( h \in \{ f \in \prod X_i \mid f(i) \in \mathbb{A}_i \} \) and \( \{ f \in \prod X_i \mid f(i) \in \mathbb{A}_i \} \in \mathbb{P}\tau_i \).

Hence \( g \in \{ f \in \prod X_i \mid f(i) \in \mathbb{A}_i \} \) since \( g \sim h \) (relative to \( \mathbb{P}\tau_i \)).

Thus \( g(i_1) \in \mathbb{A}_{i_1}^1 \) as desired.

Suppose, conversely, \( g(i) \sim h(i) \) (relative to \( \tau_i \)) for \( i \in I \). It suffices to show that \( g \sim h \) (relative to \( \mathbb{P}\tau_i \)).

Let \( h \in \mathbb{A} \) and \( \mathbb{A} \in \mathbb{P}\tau_i \). Then it suffices to show that \( g \in \mathbb{A} \).

But \( h \in \bigcap_{K=1}^{n} \{ f \in \prod X_i \mid f(i_K) \in \mathbb{A}_{i_K} \} \subseteq \mathbb{A} \) for some \( i_1, \ldots, i_n \in I \) and some \( \mathbb{A}_{i_K} \in \tau_{i_K} \) for \( K = 1, \ldots, n \). In particular \( h(i_K) \in \mathbb{A}_{i_K} \) and \( \mathbb{A}_{i_K} \in \tau_{i_K} \) for \( K = 1, \ldots, n \). But \( g(i_K) \sim h(i_K) \) (relative to \( \tau_{i_K} \)) for \( K = 1, \ldots, n \) so \( g(i_K) \in \mathbb{A}_{i_K}^1 \) for \( K = 1, \ldots, n \). Hence \( g \in \bigcap_{K=1}^{n} \{ f \in \prod X_i \mid f(i_K) \in \mathbb{A}_{i_K} \} \) for \( K = 1, \ldots, n \).

Therefore, \( g \in \bigcap_{K=1}^{n} \{ f \in \prod X_i \mid f(i_K) \in \mathbb{A}_{i_K} \} = \bigcap_{K=1}^{n} \{ f \in \prod X_i \mid f(i_K) \in \mathbb{A}_{i_K}^1 \} \subseteq \mathbb{A} \). Thus \( g \in \mathbb{A} \) as desired.

1.4 Tychonoff's Theorem

Theorem (Tychonoff)
Let \( \{(X^i, \tau_i) : i \in I\} \) be a family of topological spaces and let 
\((\Pi X^i, \Pi \tau_i)\) be its topological product. Suppose \( X^i \) is \( \tau_i \)-compact 
for \( i \in I \). Then \( \Pi X^i \) is \( \Pi \tau_i \)-compact.

**Proof**

Suppose \( X^i \) is \( \tau_i \)-compact for \( i \in I \). Let \( g \in (\Pi X^i) \).
By 1.1 it suffices to show that \( g \) is near-standard (relative to 
\( \Pi \tau_i \)). But \( g(i) \in X^i \) for \( i \in I \). By 1.1, \( g(i) \) is near-standard 
(relative to \( \tau_i \)) since \( X^i \) is \( \tau_i \)-compact.

Thus \( g(i) \approx x^i \) (relative to \( \tau_i \)) for some \( x^i \in X^i \).
Let \( h(i) = x^i \) for \( i \in I \). Then \( h \in \Pi X^i \) and \( g(i) \approx h(i) \)
(relative to \( \tau_i \)) for \( i \in I \). By 1.3, \( g \approx h \) (relative to \( \Pi \tau_i \)).
Thus \( g \) is near-standard (relative to \( \Pi \tau_i \)) as desired.

1.5 *Alaoglu's Theorem*

**Definition**

Let \( X \) be a real normed linear space and let \( L(X) \) be the 
set of bounded, linear functionals on \( X \). Recall that \( x \in X \) and 
\( f \in L(X) \) implies \( ||f(x)|| \leq ||f|| ||x|| \) (where \( || \mid \mid \mid \) denotes the 
norms on \( X \) and \( L(X) \)). Note that \( L(X) \) is contained in the 
Cartesian power \( \mathbb{R}^X \). Let \( \tau_p \) be the topology of pointwise convergence 
on \( \mathbb{R}^X \). By 1.3, if \( g \in (K^X) \) and \( h \in \mathbb{R}^X \) then \( g \approx h \) (relative 
to \( \tau_p \)) iff \( g(x) \approx h(x) \) for \( x \in X \). Let \( B = \{ f \in L(X) : ||f|| \leq 1 \} \).
Theorem (Alaoglu)

B is $\tau_p$-compact

Proof

Let $g \in \ast B$. By 1.1 it suffices to show that $g \simeq h$ (relative to $\tau_p$) for some $h \in B$. If $x \in \ast X$ then

$\ast \|g(x)\| \leq \ast \|g\| \ast |x| \leq \ast |x|$ (since $g \in \ast (L(X))$ and $\ast \|g\| \leq 1$). In particular, if $x \in X$ then $\ast \|x\| = |x|$ so

$\ast \|g(x)\| \leq |x|$. Hence $g(x)$ is finite and $^0g(x)$ is defined if $x \in X$. Let $h(x) = ^0g(x)$ for $x \in X$. Then $h \in B$. But $g(x) = h(x)$ for $x \in X$. Thus, by 1.3, $g \simeq h$ (relative to $\tau_p$).

1.7 Remarks

In 1.1 a generalization is provided for Robinson's non-standard characterization of compactness in topological spaces [Robinson 1974]. This is done by observing that the definition of compactness, as well as Robinson's characterization of it, does not require the axioms for a topological space. Then, in 1.2, this generalization yields the Alexander subbase theorem. In 1.3 the near-standard relation in topological products is reduced to one involving the coordinate spaces. Then, in 1.4, Tychonoff's theorem is easily proved. Both 1.3 and 1.4 appear in a book of Robinson [Robinson 1974]. The usefulness of 1.3 may be seen elsewhere, however. For instance, Alaoglu's theorem is proved in 1.5. This is possible since Alaoglu's theorem deals with
compactness relative to a topology of pointwise convergence, and thus relative to a product topology.

One might observe that classical proofs of the above results follow similar lines. Hence the Alexander Subbase theorem provides an immediate proof of Tychonoff's theorem [Kelley 1955]. But Tychonoff's theorem yields Alaoglu's theorem by providing a compact topological product in which the unit ball of the dual of a normed linear space exists as a closed (hence compact) subspace [Bachman-Narici 1966]. Thus in the standard proof of Alaoglu's theorem some analysis is required to show that the above subspace is closed (hence compact). The lemma in 1.3 eliminates this requirement by providing compactness directly.

CHAPTER 2

2.1 Tarski's Result

Let \( B \) be a Boolean algebra. By the Stone Representation theorem \( B \) is isomorphic to a field of subsets \( F \) of some set \( X \). A measure on \( B \) is any function \( m : B \to \mathbb{R} \) such that \( m(\emptyset) = 0 \) and 
\( m(x) \geq 0 \) for \( x \in B \) and \( m(x \lor y) = m(x) + m(y) \) for \( x, y \in B \) such that \( x \land y = \emptyset \). A measure \( m : B \to \mathbb{R} \) is strictly positive if 
\( m(x) > 0 \) for \( x \neq \emptyset \).
Theorem (Tarski)

Any measure $m_0$ defined on a subalgebra $B_0$ of a Boolean algebra $B$ may be extended to a measure $m$ on $B$ such that the range of $m$ lies within the closure of the range of $m_0$.

Proof

By the Stone Representation theorem it may be assumed that $B_0 \subseteq B$ are fields of subsets of some set $X$. Let any finite, disjoint subset $\Pi \subseteq B_0$ of nonvoid sets such that $X = \bigcup \Pi$ be called a $B_0$-measurable partition of $X$. For such partitions let the binary relation $Q(\Sigma, \Pi)$ mean that $\Pi$ refines $\Sigma$. Thus if

$$\Sigma = \{\Sigma_1, \ldots, \Sigma_m\} \quad \text{and} \quad \Pi = \{\Pi_1, \ldots, \Pi_n\} \quad \text{and} \quad Q(\Sigma, \Pi)$$

then

$$\Sigma_j = \bigcup_{\Pi_K \subseteq \Sigma_j} \Pi_K \quad \text{for} \quad j = 1, \ldots, m.$$  

Evidently $Q$ is concurrent (since $B_0$ is a field of subsets). Thus there exists some $\Pi_Q$ such that $Q(\Sigma, \Pi_Q)$ for all $B_0$-measurable partitions $\Sigma$ of $X$. In particular, $\Pi_Q$ is an internal, *-finite, $B_0$-measurable partition of $X$.

Write $\Pi_Q = \{\Pi_1, \ldots, \Pi_0\}$. Consequently

$$1) \quad ^*A = \bigcup_{\Pi \in \Pi_Q} \Pi_K \quad \Pi_K \subseteq ^*A \quad \Pi_K \in \Pi_Q$$

for $A \in B_0$.  

Let \( \{x_1, \ldots, x_\omega\} \) be an internal, \( * \)-finite sequence in \( \mathbb{X} \) such that \( x_K \in \Pi_K \) for \( K = 1, \ldots, \omega \). Thus (1) may be written as

\[
\text{(2)} \quad *A = \bigcup_{x_K \in A} \Pi_K
\]

for \( A \in B_\circ \). Therefore

\[
\text{(3)} \quad m_0(A) = *m_0(*A) = \sum_{x_K \in A} *m_0(\Pi_K)
\]

for \( A \in B_\circ \). Extend \( m_0 \) on \( B_\circ \) to a *measure \( m_1 \) on \( B \) by letting

\[
\text{(4)} \quad m_1(A) = \sum_{x_K \in A} *m_0(\Pi_K)
\]

for \( A \in B \). Note that \( m_1(A) \) is finite for \( A \in B \). Now, extend \( m_0 \) on \( B_\circ \) to a measure \( m \) on \( B \) by letting

\[
\text{(5)} \quad m(A) = m_1(A)
\]

for \( A \in B \). Finally, let \( A \in B \). Then \( m_1(A) = m(A) \) but \( m_1(A) \in \text{range } (m) \). Hence \( m(A) \) lies within the closure of \( \text{range } (m) \).

2.2 Nikodým's Result

Theorem (Nikodým)

Let \( B \) be a Boolean algebra. Then there exists an ordered field \( F \) and a measure \( m : B \to F \) which is strictly positive.
Proof

By the Stone Representation theorem it may be assumed that $B$ is a field of subsets of some set $X$. Let $\mathcal{P} = \{\Pi_1, \ldots, \Pi_\omega\}$ be an internal, *-finite, *-$B$-measurable partition of $X$ such that if $A \in B$ then $A = \bigcup_{\Pi_K \subseteq A} \Pi_K$. Let $m(A) = \sum_{\Pi_K \subseteq A} m(\Pi_K)$ where $m(\Pi_K) = \frac{1}{\omega}$ for $K = 1, \ldots, \omega$. Evidently $m : B \to \mathbb{R}$ is a strictly positive measure. But $\mathbb{R}$ is an ordered field.

2.3 Boolean Covers

definition

Let $X$ be a set and let $F$ be a field of subsets of $X$ and let $\tau$ be a function from $F$ into $S(X)$. Then $\tau$ is called pre-Boolean if

1. $\tau(X) = X$
2. $\tau(\emptyset) = \emptyset$
3. $\tau(A \cap B = \tau(A) \cap \tau(B)$

Furthermore, if

4. $\tau(A \cup B) = \tau(A) \cup \tau(B)$

then $\tau$ is called Boolean. A cover for $\tau$ is any function $\sigma$ from $F$ into $S(X)$ such that $\tau(A) \subseteq \sigma(A)$ for $A \in F$. 
Lemma

Let \( \tau : F \to S(X) \) be pre-Boolean. Then for each \( x \in X \) there exists some \( A_x \in F \) such that \( \tau(A) = \{ x \in X : A_x \subseteq A \} \) for \( A \in F \).

Proof

For each \( x \in X \) let \( F_x = \{ A \in F : x \in \tau(A) \} \). Then \( F_x \) has the finite intersection property so the binary relation \( Q(A_1, A_2) \) on \( F_x \) meaning \( A_2 \subseteq A_1 \) is concurrent. Thus there exists some \( A_Q \in F_x \) such that \( Q(A,Q) \) for \( A \in F_x \). In particular, \( A_Q \subseteq A \) for \( A \in F_x \). Let \( A_x = A_Q \). Now, let \( A \in F \). It suffices to show that \( \tau(A) = \{ x \in X : A_x \subseteq A \} \). To show that \( \tau(A) \subseteq \{ x \in X : A_x \subseteq A \} \) let \( x \in \tau(A) \). Then \( A \in F_x \) so \( A_x \subseteq A \). To show that \( \tau(A) \supseteq \{ x \in X : A_x \subseteq A \} \) let \( x \in X \) and \( A_x \subseteq A \). Then the sentence

\[
\exists B (B \subseteq A \text{ and } x \in \tau(B))
\]

is true in \( ^*X \) (let \( B = A_x \)) so it is true in \( X \). But \( x \in \tau(B) \) and \( \tau(B) = \tau(B \cap A) = \tau(B) \cap \tau(A) \) so \( x \in \tau(A) \).

Theorem

Let \( \tau : F \to S(X) \) be pre-Boolean. Then \( \tau \) has a Boolean cover.

Proof

For each \( x \in X \) select some \( A_x \in F_x \) such that \( A_x \subseteq A \) for \( A \in F_x \). Also, for each \( x \in X \) select some \( y_x \in A_x \). Let
\[ \sigma(A) = \{ x \in X : y_x \in A \} \]. Then \( \sigma \) covers \( \tau \). Evidently \( \sigma \) is Boolean. Thus \( \sigma \) is a Boolean cover for \( \tau \).

2.4 Remarks

In 1962, W. A. J. Luxemburg [Luxemburg 1962a, 1962b] employed ultrapowers to prove both Tarski's result [Tarski 1930] and Nikodym's result [Nikodym 1956, 1960] concerning extensions of measures on Boolean algebras. In both proofs, however, the existence of a measure about a single point in a Boolean algebra was required before the methods of ultrapowers could be applied.

For example, Tarski's result depends upon the following. Any measure \( m_0 \) defined on a subalgebra \( B_0 \) of a Boolean algebra \( B \) may be extended to a measure \( m_\perp \) on \( B_\perp \) such that the range of \( m_\perp \) lies within the closure of the range of \( m_0 \) (where \( B_\perp \) is the subalgebra generated by \( B_0 \) and some point \( x_0 \)). The result of Nikodym, moreover, requires the following. If \( B \) is a Boolean algebra and \( o \neq x \in B \) then there exists a measure \( m \) on \( B \) such that \( m(x) \neq 0 \).

In this thesis we have used Robinson's enlargements [Robinson 1974] to remove such requirements by providing the desired extensions immediately. The proof of the result concerning Boolean covers is an improvement of a note in the theory of lifting due to Eifrig [Eifrig 1972]. Though Eifrig employs enlargements his method is unnecessarily complicated. Some remarks about Boolean covers and lifting serve to demonstrate this. Let \( \langle \Omega, F, P \rangle \) be a complete probability space
(where $P$ is the countably additive measure on the Borel field $F$ of subsets of $\Omega$) and let $N = \{A \in F : P(A) = 0\}$. It is easy to show that the relation $A \sim B$ meaning $A \Delta B \in N$ is an equivalence relation. A density on $<\Omega, F, P>$ is any function $\theta : F \to F$ satisfying

1. $\theta(A) \sim A$
2. $A \sim B$ implies $\theta(A) = \theta(B)$
3. $\theta(\phi) = \phi$ and $\theta(\Omega) = \Omega$
4. $\theta(A \cap B) = \theta(A) \cap \theta(B)$.

Furthermore, if $\theta$ satisfies

5. $\theta(A \cup B) = \theta(A) \cup \theta(B)$

then it is called a lifting on $<\Omega, F, P>$. Eifrig provides a non-standard proof of the well-known result that any density on $<\Omega, F, P>$ may be extended to a lifting on $<\Omega, F, P>$ by employing the fact that for sets of measure 1, inclusion is a concurrent binary relation. But such considerations may be avoided by noting that any density is pre-Boolean and hence possesses a Boolean cover. Furthermore, it is easy to show that any Boolean cover of a density (in a complete probability space) is already a lifting.
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