ESTIMATING THE INTENSITY FUNCTION OF THE NONSTATIONARY POISSON PROCESS

by

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ABSTRACT

Let $\{N(t), -T<t<T\}$ be a nonstationary Poisson process with intensity function, $\lambda(t)>0$, assumed integrable on $[-T,T]$. The optimal linear estimator, $\hat{\lambda}_L$, of the intensity function is considered in this thesis.

Chapter 1 discusses $\hat{\lambda}_L$ as a function of $h(t;s)$, which is the unique solution of the Fredholm integral equation of the second kind,

$$m(s)h_t(s) + \int_a^b K(s;u)h_t(u)du = K(t;s), \quad a<s<b.$$

Chapters 2 and 3 are respectively devoted to a discussion of some of the exact and approximate methods for solving the above integral equation.

To illustrate the use of the techniques devised, three numerical examples are treated. Chapter 4 deals with data on oilwell discoveries in Alberta, Canada. Finally, in Chapter 5, the model is applied to data on traffic counts on the Lions Gate Bridge, Vancouver, and to data on coal-mining disasters in Great Britain. Computer programs and numerous diagrams are also presented.

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CHAPTER 1

ON THE ESTIMATION OF THE INTENSITY FUNCTION OF THE NONSTATIONARY POISSON PROCESS: INTRODUCTION, PRELIMINARIES

INTRODUCTION

In this chapter we consider linear estimators of the intensity function of a nonstationary Poisson process. Using the formulation given by Clevenson and Zidek (7), we consider a point process

$$\{N(t), -T \leq t \leq T\}, \quad 0 < T \leq \infty$$

with independent increments and with

$$P[N(b)-N(a)=n] = \frac{1}{n!}[\Lambda(a,b)]^n \exp[-\Lambda(a,b)]$$

where

$$\Lambda(a,b) = \int_a^b \lambda(t)dt, \qquad -T \leq a < b \leq T,$$

and the intensity function $\lambda(t) > 0$, is assumed to be Riemann integrable on $[-T,T]$. The unknown $\lambda(t)$ does not have parametric form. For details we refer to Grandell (10). In section 1.1, we consider linear estimators of the intensity function, $\lambda(t)$, in

general, and two natural estimators, in particular, the histogram and the moving-average. In section 1.2, Grandell's criterion is imposed and $\hat{\lambda}(t)$, the best estimator of $\lambda(t)$, is determined in a particular form. Following the work of Clevenson and Zidek (7), a Fredholm integral equation of the second kind is obtained; the solution of which is required in order to obtain the best linear estimator $\hat{\lambda}_L(t)$ in the given form.

1.1 OPTIMUM LINEAR ESTIMATORS OF THE INTENSITY FUNCTION

Let $\{N(t), -T \leq t \leq T\}$ be a nonstationary Poisson process with intensity function, $\lambda(t) > 0$, assumed integrable on $[-T,T]$. We seek an estimate of $\{\lambda(t); -T \leq t \leq T\}$. Following the approach of Grandell (10), as extended by Clevenson and Zidek (7), $\lambda(t)$ is assumed to be square integrable function on $[-T,T]$. Denoting by E the expectation with respect to the joint distribution of N and λ, the performance of an estimator, $\hat{\lambda}$, is measured by

$$(1.1.1) \qquad E\int_{-T}^{T}\left[\hat{\lambda}(t)-\lambda(t)\right]^2 dt$$

In the present work as well as in the references cited above, the estimate is constrained to be a linear function of the counting record . The motivation for this constraint is that it leads to estimators that are relatively easy to compute in practice with numerical algorithms. Also, two well known estimators, the histogram and the moving average, are linear. A disadvantage, however, is that for many applications of interest, linear

estimates may not be sufficiently accurate. One may wish to consider the estimation problem without the linearity constraint. Here, the major difficulty is analytic intractability.

In general, by the linearity constraint, the estimate $\hat{\lambda}(t)$ of $\lambda(t)$ must have the form

$$(1.1.2) \qquad \hat{\lambda}(t)=a(t)+\int_{-T}^{T} h(t,s)d(N(s))$$

for some deterministic function $a(t)$.

As an example, let

$$(1.1.3) \qquad a(t)=0 \text{ and } h(t,s)=\begin{cases}\frac{1}{\Gamma}, & t-\Gamma\leq s<t\\ 0, & \text{otherwise.}\end{cases}$$

For this choice,

$$(1.1.3) \qquad \hat{\lambda}(t)=\frac{1}{\Gamma}\int_{t-\Gamma}^{t} dN(s)=(N(t)-N(t-\Gamma))/\Gamma.$$

This is a moving-average, histogram estimate of the intensity process. Such an estimate is widely used because it has the advantage of requiring almost no knowledge about the intensity process beyond that needed to select the averaging time Γ. It is common practice to evaluate the moving-average estimate at discrete times; for instance, at integral multiples of Γ, and then to least-squares curve fit an assumed time function to the sampled values.

Clevenson and Zidek (7) assumed prior knowledge about $\lambda(t)$ is representable by the condition that $\{\lambda(t);-T<t<T\}$ is a wide-sense stationary stochastic process with constant mean μ and covariance function K.

DEFINITION: A stochastic process $\{x(t);-\infty<t<\infty\}$ is said to be wide-sense stationary if its mean-value function $m(t)=E\{x(t)\}$ is a constant independent of t and its covariance function $K(t,s)=E\{x(t)x(s)\}-E\{x(t)\}E\{x(s)\}$ is a function only of $(t-s)$ and not t and s separately.

In (7) Clevenson and Zidek define the histogram estimator

$$(1.1.4)\qquad \hat{\lambda}_H(t)=[N(t_i)-N(t_{i-1})][2\Delta_i]^{-1}, t_{i-1}<t<t_i,$$

where $-T=t_0<t_1<\ldots<t_p=T$ is any partition of $[-T,T]$ and $2\Delta_i=t_i-t_{i-1}$. The moving-average estimator, $\hat{\lambda}_M$, is defined by

$$(1.1.5)\qquad \hat{\lambda}_M(t)=[N(t+\Delta)-N(t-\Delta)][2\Delta]^{-1},$$

provided $-T+\Delta<t<T-\Delta$.

The optimal class width for the histogram estimator and the optimal window width for the moving-average estimator were also obtained. We do not, however, intend to discuss the details here.

1.2 THE BEST LINEAR ESTIMATOR

It is at least intuitively evident that estimators having performance superior to the moving-average and histogram estimators can be designed if more knowledge about the statistics of the intensity process is available.

Let $\hat{\lambda}_L$ denote the linear estimate of $\lambda(t)$ that results in (1.1.2) when both $a(t)$ and $h(s,t)$ are selected to minimize the mean-square error $E[(\lambda(t)-\hat{\lambda}_L(t))^2]$. This linear minimum mean-square error estimate is given by Grandell (10). The method we consider here follows the Clevenson-Zidek (7) generalization of Grandell's technique. Thus we seek an estimator of the form

$$\hat{\lambda}_L(t)=a(t)+\int_{-T}^{T}h(t,s)d(N(s)-M(s)). \tag{1.2.1}$$

Here, we drop the assumptions in section 1.1 that $\lambda(t)$ is wide-sense stationary and that $m(t)=E\lambda(t)\equiv\mu$. We take $M(s)=\int_0^s m(t)dt$.

The problem now is to determine the functions $a(t)$ and $h(t;.)$ to minimize the functional $E[\hat{\lambda}_1(t)-\lambda(t)]^2$. If the resulting optimal choices are, say, $a^\circ(t)$ and $h^\circ(t;.)$, then these will in turn minimize (1.1.1). Grandell, and subsequently Clevenson and Zidek, show, after some manipulation, that

$$E[\lambda_L(t)-\lambda(t)]^2=(a(t)-m(t))^2+(h(t;.),h(t;.))-2L_t(h(t;.)). \tag{1.2.2}$$

where (.,.) is defined by

$$(1.2.3) \quad (x(.),y(.))=\int_{-T}^{T}x(s)y(s)m(s)ds+\int_{-T}^{T}\int_{-T}^{T}x(s)y(s)K(s,u)dsdu$$

for all functions x,y for which $(x,x)<\infty$; and because K(.,.) is nonnegative definite, $(x,x)<\infty$ is an inner product. The linear functional L_t is defined by $L_t x(.)=\int_{-T}^{T}x(s)K(t;s)ds$,

for all x such that $(x,x)<\infty$. It is clear from (1.2.2) that the optimal choice for a(t) is

$$(1.2.4) \quad a^{o}(t)=m(t) \qquad \text{for all } t.$$

Assume that $0<\inf_{s} m(s)$ and that m and K are bounded, it follows that L_t is a continuous linear functional on the Hilbert space of functions

$$H=\{x:(x,x)<\infty\}.$$

It in turn follows by the Riesz representation theorem that

$$L_t x(.)=(x(.),g_t(.))$$

for all $x\varepsilon H$ where $g_t(.)\varepsilon H$. Thus the optimal choice for h,

$$(1.2.5) \quad h^{o}(t;s)=g_t(s),$$

is the unique solution in H of the integral equation,

$$(1.2.6) \quad x(s)m(s)+\int_{-T}^{T}x(u)K(s,u)du=K(t;s).$$

This is the well known Fredholm integral equation of the second kind which occurs frequently in many areas of applied mathematics such as in Communication and Information theory.

The integral equation (1.2.6) has been studied extensively in connection with the linear filtering problem for observations that contain additive noise (see, for example, H. Van Trees (19), chapters 4 and 6).

The solution to equation (1.2.6) is our main concern in this work and exact methods as well as approximation techniques for this problem are given in subsequent chapters.

REMARK: It is perhaps worth emphasizing that theoretically only the mean and covariance functions are needed to solve the integral equation (1.2.6) and thus to design the estimator. However, for numerical and approximate solutions the data in hand is very useful.

In the sequel (e.g., in chapter 4), we shall consider the special case where

$$(1.2.7) \quad m(t)\equiv\mu, \text{ and } K(t;s)\equiv K(t-s).$$

Thus (1.2.6) becomes

$$(1.2.8)\qquad \mu x(s)+\int_{T}^{T}x(u)K(s-u)\,du=K(t-s),\ -T<s<T.$$

It follows that our linear estimator now takes the special form

$$(1.2.9)\qquad \hat{\lambda}_{L}(t)=\mu+\int_{-T}^{T}H(t;s)\,d(N(s)-\mu s),$$

obtained from (1.2.1) by setting $a^{\circ}(t)=m(t)=\mu$. This is the form in which our model will be used in the applications in chapters 4 and 5.

CHAPTER 2

SURVEY OF EXACT METHODS FOR SOLVING FREDHOLM INTEGRAL EQUATIONS (OF TYPE II)

SUMMARY

This chapter is devoted to a discussion of some of the exact methods for solving Fredholm Integral Equations (of the second kind), together with some theoretical background, and a number of applications. The basic methods are treated in some detail, and recent developments are also discussed and compared. By means of examples, some motivation will be given for the differences in theory and methodology underlying these methods and their investigation. The necessary background in linear algebra will be sketched and some aspects of Hilbert space theory will be presented. Subsequently, all integral operators will be viewed as acting on suitable Hilbert spaces.

1. INTRODUCTION

An integral equation is an equation in which the unknown function appears under the integral sign.

Integral equations have been encountered in mathematics for a number of years, originally in the theory of Fourier Integrals. The actual development of the theory of integral equations began, however, only at the end of the nineteenth century due to the

works of the Italian mathematician V. Volterra, and principally in the year 1900, when the Swedish mathematician Ivar Fredholm published his famous work on a new method of solving the Dirichlet problem*. From then on, up to the present, integral equations have been the subject of research for numerous mathematicians.

The theory of integral equations has close contacts with many different areas of mathematics. Indeed, many problems of applied mathematics can be stated in the form of integral equations. To make a list of such applications would be almost impossible. Suffice it to say that there is almost no area of applied mathematics and mathematical physics where integral equations do not play a role.

It is worth mentioning, at this stage, that in dealing with linear integral equations the fundamental concepts of linear vector spaces, eigenvalues and eigenfunctions play a significant role.

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*"Sur une nouvelle méthode pour la résolution du problème de Dirichlet". Öfvers. af Kungl. Vetensk. Akad. Förh., Stockholm, 57, nr. 1 (10 Jan. 1900), 39-46.

*"Sur une classe d'équations fonctionnelles". Acta Mathematica, Stockholm, 27 (1903), 365-390.

The most frequently studied integral equations are the following:

$$\int_a^b K(t,s)f(s)ds = g(t) \tag{2.1.1}$$

$$f(t) + \lambda \int_a^b K(t,s)f(s)ds = g(t) \tag{2.1.2}$$

$$m(t)f(t) + \lambda \int_a^b K(t,s)f(s)ds = g(t) \tag{2.1.3}$$

The above equations are generally known as Fredholm equations of the first, second, and third kind, respectively. The interval (a,b) may in general be a finite interval or $(-\infty,b]$, $[a,\infty)$, or $(-\infty,\infty)$, where a and b are finite. We note that we may divide (2.1.3) by m(t) to reduce it to (2.1.2).

The function K(.,.), which is generally known as the kernel, and g(.) are assumed known and f(.) is sought. All the above equations are linear; that is, the function f(.) enters the equations in a linear manner so that

$$\int_a^b K(t,s)[c_1f_1(s)+C_2f_2(s)]ds=C_1\int_a^b K(t,s)f_1(s)ds+C_2\int_a^b K(t,s)f_2(s)ds.$$

As stated earlier, the equations (2.1.1)-(2.1.3) arise in many situations; in statistical problems the kernels are usually symmetric and often also nonnegative definite.

In the next section we shall briefly state some mathematical results, including a discussion of the necessary background of Hilbert space theory, as useful tools in the sequel. The classical Fredholm expansion techniques will be considered in Section 3.

In Section 4 we shall consider Hilbert-Schmidt, Karhunen-Loève, and Grandell (7) type solutions and the resulting expansions. We shall discuss some other methods in the literature including the detection theoretic approach by Vantrees (19), Slepian (18), and others.

2. SOME USEFUL MATHEMATICAL TOOLS

Although we are dealing with exact methods, we find it appropriate to briefly consider finite difference approximations at this stage because finite difference approximations are not only of great practical utility, but also provide certain insight into the nature of integral equations. We shall also state Hadamard's inequality as well as some aspects of Hilbert space theory.

2.1 FINITE DIFFERENCE APPROXIMATIONS

If, in the equation

$$f(t)-\lambda\int_0^1 K(t,s)f(s)ds = g(t) \tag{2.2.1}$$

we replace the integral by a suitable sum:

$$(2.2.2) \qquad f(t)-\lambda\sum_{i=1}^{n}\frac{1}{n}K(t,\frac{i}{n})f(\frac{i}{n}) = g(t)$$

for large n and a continuous kernel K(t,s) and continuous f(t), the sum in (2.2.2) represents a close approximation to the integral in (2.2.1). If, furthermore, we evaluate (2.2.2) only at n discrete points

$$(2.2.3) \qquad f(\frac{j}{n})-\lambda\sum_{i=1}^{n}\frac{1}{n}K(\frac{j}{n},\frac{i}{n})f(\frac{i}{n}) = g(\frac{j}{n}),\ j=1,2,\ldots,n,$$

we have replaced the integral equation (2.2.1) by the algebraic system (2.2.3) which may be rewritten in matrix form

$$(2.2.4) \qquad (I-\lambda H)F = G$$

where the i-jth element in the matrix H is $(\frac{1}{n})K(\frac{j}{n},\frac{i}{n})$, and F is a vector with components $f(\frac{i}{n})$ and G has components $g(\frac{i}{n})$.

To solve (2.2.4) we invert the matrix and find

$$(2.2.5) \qquad F=(I-\lambda H)^{-1}G.$$

The above inverse will exist for all λ, with the exception of at most n values. These are the roots of the characteristic determinantal equation,

$$(2.2.6) \qquad |I-\lambda H| = 0.$$

In (2.2.5) we see that there may be special values of λ for which no solution exists. Such values are commonly known as eigenvalues. We also note that every finite algebraic system (2.2.3) necessarily has eigenvalues, even though the integral equation (2.2.1) need not have eigenvalues.

2.2.1 Hadamard's Inequality

We state this as a theorem.

THEOREM 2.2.1 Let H be a matrix with the general element h_{ij}. An upper estimate for its determinant is given by

$$(2.2.7) \qquad |H|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}|^2.$$

2.2.2 Hilbert Spaces

So far we have not really addressed ourselves to the question of what we mean by a solution of an integral equation. Basically, of course, a solution of any equation must reduce the equation to an identity. But often, however, we may impose additional restrictions on the solution, such as demanding that it should belong to a particular class of functions. For these purposes it will prove to be convenient to work in the so-called Hilbert spaces.

DEFINITION 2.3.1 A linear space X is said to be an inner product space if an inner product is defined on it. Such an inner product assigns to every pair f and g in X a complex number denoted by (f,g) with the following properties:

1. $(f,g) = \overline{(g,f)}$

2. $(\alpha f+\beta g,h) = \alpha(f,h)+\beta(g,h)$

3. $(f,f)\geq 0$ and $(f,f) = 0$ if and only if $f = 0$.

We note that (f,f) is real, since by property (1) $(f,f) = \overline{(f,f)}$. We let $(f,f)^{\frac{1}{2}} = ||f||$ and call it the Norm of f.

DEFINITION 2.3.2 Let H be an inner product space and $\{f_n\}$ a Cauchy sequence in H. Then H is said to be a Hilbert space if every Cauchy sequence converges to an element in H.

In the study of integral equations, we find that the notion of an integral operator is fundamental. Such an operator assigns to an element f in H a new element, say Kf in H.

If the operator satisfies the condition

(2.2.1) $K(\alpha f+\beta g) = \alpha Kf+\beta Kg$

we say that K is a linear operator. An example of a linear operator is

$$Kf = \int_0^1 K(t,s)f(s)ds,$$

where $f \in L_2[0,1]$ and $K(t,s)$ is continuous. Such an operator may or may not be defined on the whole space H.

THEOREM 2.3.1 Consider $L_2[a,b]$, where the interval $[a,b]$ may be infinite. If

$$\int_a^b \int_a^b |K(t,s)|^2 dtds = M^2 < \infty$$

then the operator $Kf = \int_a^b K(t,s)f(s)ds$ is bounded.

The stage is now set for our main concern in the sections ahead.

2.3 THE CLASSICAL FREDHOLM TECHNIQUES

To begin with, in order to fix our ideas, we shall consider the Fredholm equation

$$f(t) = g(t) + \lambda \int_a^b K(t,s)f(s)ds \tag{2.3.1}$$

with a Riemann integral in a given interval (a,b).

Fredholm was the first person to give the solution of equation (2.3.1) in the general form for all values of the parameter λ. The results of Fredholm's investigations are contained in three theorems which are among the most important and beautiful

mathematical discoveries. The method used by Fredholm consisted in replacing the integral in (2.3.1) by a sum, the reduction of this equation to a system of linear equations and letting the number of terms of the sum tend to infinity.

In accordance with Fredholm's method, we partition the interval (a,b) into n equal parts by the points

$$a=t_1,\ t_2,\ \ldots,\ t_n=b,\ h=t_{i+1}-t_i=\frac{b-a}{n}$$

and replace equation (2.3.1) by

$$(2.3.2)\qquad f(t)=g(t)+\lambda h\sum_{i=1}^{n}K(t,t_i)f(t_i).$$

Let $f(t_i)=f_i$, $K(t_j,t_i)=K_{ji}$; then we may rewrite the system of equations as

$$(2.3.3)\qquad \begin{aligned}(1-\lambda hK_{11})f_1-\lambda hK_{12}f_2-\ldots-\lambda hK_{in}f_n&=g(t_1)\\ &\vdots\\ -\lambda hK_{n1}f_1-\lambda hK_{n2}-\ldots+(1-\lambda hK_{nn})f_n&=g(t_n).\end{aligned}$$

The solutions $f_1,f_2,\ldots,f_n$ of (2.3.3) can be expressed in the form of ratios of certain determinants by the common characteristic determinant

$$(2.3.4)\qquad D_n(\lambda) = \begin{vmatrix} 1-\lambda hK_{11} & -\lambda hK_{12} & \dots & -\lambda hK_{1n} \\ -\lambda hK_{21} & 1-\lambda hk_{22} & \dots & -\lambda hK_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda hK_{n1} & -\lambda hK_{n2} & \dots & 1-\lambda hK_{nn} \end{vmatrix}$$

provided that this determinant is not equal to zero.

An expansion of (2.3.4) may be expressed in the form

$$(2.3.5)\qquad \begin{aligned} D_n(\lambda) = 1-\lambda h\sum_{i=1}^{n} K_{ii} + \frac{(-\lambda h)^2}{2!} \sum_{i,j=1}^{n} \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} \\ + \dots + \frac{(-\lambda h)^n}{n!} \sum_{i_1,i_2,\dots i_n=1} \begin{vmatrix} K_{i_1 i_1} & K_{i_1 i_2} & \dots & K_{i_1 i_n} \\ K_{i_2 i_1} & K_{i_2 i_2} & \dots & K_{i_2 i_n} \\ \cdot & \cdot & \cdot & \cdot \\ K_{i_n i_1} & K_{i_n i_2} & \dots & K_{i_n i_n} \end{vmatrix} \end{aligned}$$

For the sake of simplicity, let

$$(2.3.6)\qquad \begin{vmatrix} K(t_1,S_1) & K(t_1,S_2) & \dots & K(t_1,S_n) \\ \cdot & \cdot & \cdot & \cdot \\ K(t_n,S_1) & K(t_n,S_2) & \dots & K(t_n,S_n) \end{vmatrix} = K\begin{pmatrix} t_1,t_2\dots t_n \\ S_1,S_2\dots S_n \end{pmatrix}$$

The determinant (2.3.6) is called Fredholm's determinant, and the above symbol is taken to be defined for every kernel $K(t,s)$ also in a multi-dimensional domain.

The fundamental property of Fredholm's determinant is that, if any pair of arguments in the upper or the lower sequence is transposed, the value of the determinant changes the sign.

Using the symbol (2.3.6), we may write the expansion in the form

$$(2.3.7)\quad D_n(\lambda)=1-\lambda h\sum_{i=1}^{n} K(t_i,t_i)+\frac{(-\lambda h)^2}{2!}\sum_{i,j=1}^{n} K\begin{pmatrix}t_i,t_j\\t_i,t_j\end{pmatrix}$$

$$+\frac{(-\lambda h)^3}{3!}\sum_{i,j,r=1}^{n} K\begin{pmatrix}t_i,t_j,t_r\\t_i,t_j,t_r\end{pmatrix}+\ldots$$

Now suppose that $h\to 0$ and $n\to\infty$, then each of the terms of sum (2.3.7) tends to some single, double, triple integral, etc. Thus arises the series

$$(2.3.8)\quad D(\lambda)=1-\lambda\int_a^b K(s,s)\,ds+\frac{\lambda^2}{2!}\int_a^b\int_a^b K\begin{pmatrix}s_1,s_2\\s_1,s_2\end{pmatrix}ds_1ds_2$$

$$-\frac{\lambda^3}{3!}\int_a^b\int_a^b\int_a^b K\begin{pmatrix}s_1,s_2,s_3\\s_1,s_2,s_3\end{pmatrix}ds_1ds_2ds_3+\ldots,$$

which was shown by Fredholm, on the basis of Hadamard's theorem, to converge for every value of λ.

In this vein the Fredholm's function $D(\lambda)$ may be expanded in a convergent power series (Fredholm's first series) of the form,

$$(2.3.9)\qquad D(\lambda)=1+\sum_{i=1}^{\infty}\frac{(-\lambda)^i}{i!}\int_{\Omega}\cdots\int_{\Omega}K\begin{pmatrix}s_1,s_2,\ldots,s_i\\ s_1,s_2,\ldots,s_i\end{pmatrix}ds_1ds_2\ldots ds_i$$

in an arbitrary domain Ω.

We now seek a solution of the form

$$(2.3.10)\qquad f(t)=g(t)+\lambda\int_{\Omega}N(t,s,\lambda)g(s)ds$$

where the resolvent kernel $N(t,s,\lambda)$ is the product

$$(2.3.11)\qquad N(t,s,\lambda)=\frac{D(t,s,\lambda)}{D(\lambda)},$$

$D(t,s,\lambda)$ is the sum of a certain sequence,

$$(2.3.12)\qquad D(t,s,\lambda)=C_0(t,s)+\sum_{i=1}^{\infty}\frac{(-\lambda)^i}{i!}C_i(t,s),$$

and

$$(2.3.13)\qquad C_i(t,s)=\cdots\int_{\Omega}\cdots\int_{\Omega}K\begin{pmatrix}t,s_1,s_2,\ldots,s_i\\ s,s_1,s_2,\ldots,s_i\end{pmatrix}ds_1ds_2\ldots ds_i.$$

This leads to the Fredholm series:

$$(2.3.14)\qquad D(t,s,\lambda)=K(t,s)+\sum_{i=1}^{\infty}\frac{(-\lambda)^i}{i!}\int_{\Omega}\cdots\int_{\Omega}K\begin{pmatrix}t,s_1,\ldots,s_i\\ s,s_1,\ldots,s_i\end{pmatrix}ds_1\ldots ds_i$$

which has the same convergence properties as Fredholm's first series.

We are now in a position to state the three Fredholm theorems.

THEOREM 2.3.1 Fredholm's equation of the second kind, under the assumption that the functions $g(t)$ and $K(t,s)$ are integrable, has in the case $D(\lambda)\neq 0$ a unique solution, which is of the form (2.3.10).

THEOREM 2.3.2 If λ_0 is a zero of multiplicity q of $D(\lambda)$, then the homogeneous equation

$$f(t)=\lambda_0\int_{\Omega}K(t,s)f(s)ds \tag{2.3.15}$$

possesses at least one, and at most q, linearly independent solutions

$$f_j(t)=D_i\begin{pmatrix} t_1,t_2,\ldots,t_{j-1},\ldots,t_i \\ s_1,\ldots\ldots\ldots s_i \end{pmatrix}\lambda_0$$

for $j=1,2,\ldots,i;\ 1\leq i\leq q$,

not identically zero, and any other solution is a linear combination of these solutions. D_i denotes a Fredholm minor of order i relative to the kernel $K(t,s)$.

THEOREM 2.3.3 For the nonhomogeneous equation to possess a solution in the case $D(\lambda_0)=0$, it is necessary and sufficient that the given function $g(t)$ be orthogonal to all the characteristic solutions $\psi_j(t)$ $(j=1,2,\ldots,i)$ of the associated homogeneous equation corresponding to the eigenvalue λ_0, and forming the fundamental system. The general solution then has the form

$$(2.3.16)\quad f(t)=g(t)+\lambda_0\int_{\Omega}\frac{D_{p+1}\begin{pmatrix}t,t_1,\ldots,t_p\\ s,s_1,\ldots,s_p\end{vmatrix}\lambda_0\end{pmatrix}}{D_p\begin{pmatrix}t_1,\ldots,t_p\\ s_1,\ldots,s_p\end{vmatrix}\lambda_0\end{pmatrix}}g(s)\,ds$$

$$+\sum_{j=1}^{p}C_j\phi_j(t),\ \text{where}\ \phi_j(t)=\frac{D_q\begin{pmatrix}t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_p\\ s_1,\ldots,s_p\end{vmatrix}\lambda_0\end{pmatrix}}{D_q\begin{pmatrix}t_1,\ldots,t_{j-1},t_j,t_{j+1},\ldots t_p\\ s_1,\ldots,s_p\end{vmatrix}\lambda_0\end{pmatrix}}$$

2.3.2 Fredholm's Equation with Degenerate Kernel

We call the kernel $K(t,s)$ degenerate if it is the sum of products of functions of one variable

$$(2.3.17)\quad K(t,s)=\sum_{i+1}^{n}N_i(t)L_i(S)$$

Substituting (2.3.17) into

$$f(t)=g(t)+\lambda\int_{\Omega}K(t,s)f(s)\,ds$$

we notice immediately that a solution is the sum of the function $g(t)$ and of a certain linear combination of the functions $N_i(t)$:

$$(2.3.18)\quad f(t)=g(t)+\sum_{i=1}^{n}A_iN_i(t),\ A_i\ \text{being constant}$$

In order to determine the constants A_i, we substitute expression (2.3.18) in the integral equation. We illustrate this by an example:

EXAMPLE 2.3.1 Suppose we are given the integral equation

$$f(t)=g(t)+\lambda\int_0^1(t+s)f(s)\,ds$$

The solution should have the form

$$f(t)=g(t)+A_1t+A_2,$$

whence we have the identity

$$g(t)+A_1t+A_2\equiv g(t)+\lambda\int_0^1(t+s)\,[g(s)+A_1s+A_2]ds$$

and the system of equations

$$A_1(1-\tfrac{1}{2}\lambda)-A_2\lambda=\lambda\int_0^1 g(s)ds,$$

$$-\tfrac{1}{3}A_1\lambda+A_2(1-\tfrac{1}{2}\lambda)=\lambda\int_0^1 sg(s)ds.$$

Hence we obtain A_1 and A_2 and the solution of the given integral equation in the form

$$f(t)=g(t)+2\lambda\int_0^1\frac{6\lambda tS+3(2-\lambda)(t+1)+2\lambda}{12-12\lambda-\lambda^2}\,g(S)ds$$

provided λ is not one of the eigenvalues. The eigenvalues are roots of the equation

$$12-12\lambda-\lambda^2=0$$

and hence

$$\lambda_1=-6+4\sqrt{3},\quad \lambda_2=-6-4\sqrt{3}.$$

The corresponding characteristic solutions are

$$f_1(t)=C(\lambda_1 t+1-\frac{1}{2}\lambda_1),\ f_2(t)=C(\lambda_2 t+1-\frac{1}{2}\lambda_2)$$

where C is an arbitrary constant.

REMARK The Fredholm equation with degenerate kernel plays an important role in the theory and applications of integral equations, since it can easily be solved by a finite number of integrations.

2.3.3 Existence of Solutions. The Fredholm Alternative

Equations of the second kind have an existence theory: Suppose that the range of integration is finite, then we have the Fredholm Alternative as follows:

Either: If λ is a regular value, then the equation $f(t)=g(t)+\lambda\int_a^b K(t;s)f(s)ds$ has a unique solution for any arbitrary $g(t)$;

Or: If λ is a characteristic value, then the homogeneous equation

(2.3.19) $$f(t)=\lambda\int_a^b K(t,s)f(s)ds$$

has a finite number (p, say) of linearly independent solutions $\phi_1(t),\ldots,\phi_p(t)$. In this case the transposed homogeneous equation

(2.3.20) $\Psi(t)=\lambda\int_a^b K(t,s)\Psi(s)ds$

also has p solutions $\Psi_1(t),\ldots,\Psi_p(t)$; and the nonhomogeneous equation has a solution if and only if g(t) is orthogonal to all theΨ_i,; that is, if and only if

$$\int_a^b g(t)\Psi_i(t)dt=0, \quad i=1,\ldots,p.$$

This solution is clearly not unique, since we can add to it any linear combinations of the ϕ_i's, and obtain another solution.

Stated in another form, the Fredholm Alternative says: If the homogeneous equation (2.3.19) has the unique solution $f\equiv 0$, then the nonhomogeneous equation is solvable for arbitrary g (and the solution must evidently be unique). In other words, uniqueness implies existence.

Before we turn to the next section we make the following observation concerning the above.

REMARK The Fredholm classical techniques yield rather precise information regarding solvability of equations and existence and distribution of eigenvalues, but often require rather tedious and wearisome analysis. Consequently Fredholm's formulae have so far found only a few applications, either analytical or numerical, apart from providing a foundation for the theory of integral equations.

2.4 HILBERT-SCHMIDT, KARHUNEN-LOEVE, AND GRANDELL TYPE SOLUTIONS

2.4.1 Hilbert-Schmidt Theory

In the first part of this section we define the so-called selfadjoint compact operators and obtain some information regarding their eigenvalues, eigenfunctions and the associated expansion theorems. These results are then applied to compact integral operators.

As before, we shall be concerned with integral equations of the type

$$f(t)=g(t)+\lambda\int_a^b K(t,s)f(s)ds \tag{2.4.1}$$

in the Hilbert space $L_2[a,b]$. The function $g(t)$ will be assumed to belong to $L_2[a,b]$ and the kernel will be assumed to be square-integrable, so that

$$\int_a^b\int_a^b |K(t,s)|^2 dtds<\infty. \tag{2.4.2}$$

DEFINITION 2.4.1 Let K be a bounded, linear operator on a Hilbert space H. Then K will be said to be a compact operator if from the sequence $\{Kf_n\}$ we can extract a subsequence $\{Kf_{n_K}\}$ that is a Cauchy sequence, for any uniformly bounded sequence, $\{f_n\}$, in H.

Now let $\{\phi_i\}$ be an orthonormal set in H, and let

$$K_{nf}=K_f-\sum_{i=1}^{n}\mu_i(f,\phi_i)\phi_i \qquad n=1,2,\ldots$$

where $\{\mu_i\}$ is the sequence of eigenvalues of K ordered such that

$$|\mu_1|>|\mu_2|>\ldots,$$

Then if K has an infinity of nonzero eigenvalues, these accumulate at the origin, and $\{K_{nf}\}$ is a convergent sequence in H. We thus have the following results:

THEOREM 2.4.1 Let f be an element in H. Then f can be represented in the form

$$f=\sum_{i}(f,\phi_i)\phi_i+f_0 \tag{2.4.3}$$

where f_0 is a suitable element in the nullspace of K (i.e., $Kf_0=0$).

THEOREM 2.4.2 Let $\{\phi_i\}$ be the corresponding eigenvectors associated with K, and suppose H is $L_2[a,b]$, then

$$\lim_{n\to\infty}\int_a^b\int_a^b\left|K(t,s)-\sum_{i=1}^{n}\mu_i\phi_i(t)\overline{\phi_i(s)}\right|^2 dtds=0 \tag{2.4.4}$$

so that

$$K(t,s)=\sum_{i=1}^{\infty}\mu_i\phi_i(t)\overline{\phi_i(s)}$$

converges in the mean (in the sense of (2.4.4)).

It then follows that

$$(2.4.5) \qquad \int_a^b \int_a^b |K(t,s)|^2 dt ds = \sum_{i=1}^{\infty} \mu_i^2$$

We note that in the above results the fact that $K(t,s)$ is square-integrable is vital, since for arbitrary compact operators the sum $\sum_{i=1}^{\infty} \mu_i^2$ need not be finite.

We also recall that operators for which $K(t,s)$ is an L_2 kernel are referred to as Hilbert-Schmidt operators. We now state the important

HILBERT-SCHMIDT THEOREM 2.4.3 Every function $f(t)$ of the form

$$(2.4.6) \qquad f(t) = \int_{\Omega} K(t,s)h(s)ds$$

is almost everywhere the sum of its Fourier series with respect to the orthonormal system $\phi_i(t)$ of eigenfunctions of the symmetric kernel K.

The above theorem implies that

$$(2.4.7) \qquad f(t) = \sum_{i=1}^{\infty} f_i \phi_i(t) \qquad \text{converges,}$$

where the coefficients f_i are the Fourier coefficients of the function $f(t)$ with respect to the system $\{\phi_i(t)\}$, that is

$$(2.4.8)\qquad f_i=\int_\Omega f(t)\overline{\phi_i(t)}\,dt=\frac{h_i}{\lambda_i}$$

and the h_i are the Fourier coefficients of the given function h with respect to the system $\{\phi_i(t)\}$:

$$(2.4.9)\qquad h_i=\int_\Omega h(t)\overline{\phi_i(t)}\,dt$$

Consequently, the function f(t) is almost everywhere equal to the sum of its absolutely and uniformly convergent Fourier series:

$$(2.4.10)\qquad f(t)=\sum_{i=1}^{\infty}\frac{h_i}{\lambda_i}\phi_i(t).$$

2.4.2 Application of the Hilbert-Schmidt Theorem

Using the Hilbert-Schmidt Theorem, it is possible to obtain a series expansion of the solution f(t) of the integral equation

$$(2.4.11)\qquad f(t)=g(t)+\lambda\int_\Omega K(t,s)f(s)ds$$

with respect to the system of eigenfunctions $\{\phi_i(t)\}$ of a symmetric kernel K(t,s). Assuming that λ is not equal to any of the eigenvalues, $\lambda\neq\lambda_i$, there exists a unique solution f(t) in $L_2(\Omega)$.

Specifically equation (2.4.11) may be expanded in the form

$$(2.4.12)\qquad \frac{f(t)-g(t)}{\lambda}=\sum_{i=1}^{\infty}C_i\phi_i(t)$$

which may be substituted in (2.4.11) and integrated term by term to obtain

$$(2.4.13)\quad \sum_{i=1}^{\infty} C_i\left(1-\frac{\lambda}{\lambda_i}\right)\phi_i(t)=\int_{\Omega} K(t,s)g(s)ds$$

But, according to the Hilbert-Schmidt theorem, we have again

$$(2.4.14)\quad \int_{\Omega} K(t,s)g(s)ds=\sum_{i=1}^{\infty}\frac{g_i}{\lambda_i}\phi_i(t),$$

where the f_i are the Fourier coefficients of the function $g(t)$. From (2.4.13) and (2.4.14) it follows that

$$(2.4.15)\quad \sum_{i=1}^{\infty}\left[C_i\left(1-\frac{\lambda}{\lambda_i}\right)-\frac{g_i}{\lambda_i}\right]\phi_i(t)=0.$$

Multiplying both sides in turn by $\phi_1,\phi_2,\ldots$ and integrating, because of the orthogonality we obtain

$$(2.4.16)\quad C_i\left(1-\frac{\lambda}{\lambda_i}\right)-\frac{g_i}{\lambda_i}=0 \qquad \text{for every } i.$$

Consequently $$C_i=\frac{g_i}{\lambda_i-\lambda}$$

Substituting these values in the series (2.4.12), we obtain the required expansion of the solution of equation (2.4.11) as an absolutely and uniformly convergent series in terms of eigenfunctions of the kernel:

$$(2.4.17)\quad f(t)=g(t)-\lambda\sum_{i=1}^{\infty}\frac{g_i}{\lambda-\lambda_i}\phi_i(t) \quad (\lambda\neq\lambda_i).$$

REMARK If λ were equal to one of the eigenvalues $\lambda=\lambda_p=\lambda_{p+1}=\lambda_{p+2}=\ldots=\lambda_{p+q-1}$ with rank q, then equation (2.4.16) would be satisfied if and only if

$$(2.4.18)\quad g_{p+i}=\int_\Omega g(t)\overline{\phi_{p+i}(t)}\,dt=0 \qquad (i=0,1,2,\ldots,q-1)$$

Condition (2.4.18) is in accordance with the Fredholm Third Theorem and it has to be extended to all the eigenfunctions corresponding to the value λ_p which is repeated in the above series as many times as the number of its rank. In that case the solution takes the form

$$(2.4.19)\quad f(t)=g(t)-\lambda\sum_{i=1}^{\infty}{}^{*}\frac{g_i}{\lambda-\lambda_i}\phi_i(t)+C_1\phi_p(t)+$$

$$+C_2\phi_{p+1}(t)+\ldots+C_q\phi_{p+q-1}(t)$$

where $\sum^{*}$ denotes that in the summation we have excluded all the values of i equal to p, p+1, ..., p+q-1, for which

$$\lambda_p=\lambda_{p+1}=\lambda_{p+2}=\ldots=\lambda_{p+q-1},$$

where q is the rank of that eigenvalue.

By a similar method we may find the expansion of the resolvent kernel N using the integral equation satisfied by N:

$$(2.4.20)\quad N(t,s,\lambda)=K(t,s)+\lambda\int_\Omega K(t,y)N(y,s,\lambda)\,dy.$$

We obtain in this case

$$(2.4.21)\quad \sum_{i=1}^{\infty}\left[b_i(s)\left(1-\frac{\lambda}{\lambda_i}\right)-\frac{\overline{\phi_i(s)}}{\lambda^2{}_i}\right]\phi_i(t)=0,$$

whence, as before, on account of the orthogonality of $\{\phi_i(t)\}$, it follows that

$$(2.4.22)\quad b_i(s)=\frac{\phi_i(s)}{\lambda_i(\lambda_i-\lambda)}$$

The required expansion of the resolvent kernel is therefore obtained in the form

$$(2.4.23)\quad N(t,s,\lambda)=K(t,s)-\lambda\sum_{i=1}^{\infty}\frac{\phi_i(t)\phi_i(s)}{\lambda_i(\lambda-\lambda_i)}\qquad(\lambda=\lambda_i),$$

the convergence of which is evident in view of the similar absolute and uniform convergence of the series $\sum_{i=1}^{\infty}\frac{\phi_i(t)\phi_i(s)}{\lambda_i^2}$.

REMARK The expansions (2.4.17) and (2.4.23) are clearer and seem more convenient to apply than Fredholm's formulae, but the functional analytic techniques often may not lead to the quantitative results provided by the classical techniques.

2.4′ THE KARHUNEN-LOEVE EXPANSION

The approach will be to expand the function f(t) in a particular kind of series. This is analogous to a Fourier series

expansion in terms of sine and cosine functions with appropriate weighting coefficients; a good reference is Helstrom (126 pp.124-133).

To be more precise, we desire an expansion in terms of a set of orthonormal functions $\phi_i(t)$ with uncorrelated weighting coefficients r_i.

Recall that a set of functions $\{\phi_i(t):i=1,2,\ldots\}$ defined over the interval (O,T) is orthonormal on this interval if

$$\int_0^T \phi_i(t)\,\phi^*_j(t)\,dt=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

If the orthonormal set is complete, then a square-integrable function f(t), defined on (O,T) may be represented as

$$(2.4.1)' \qquad f(t)=\sum_i r_i\phi_i(t).$$

The coefficients r_i may be determined by multiplying each side of (2.4.1)' by $\phi^*_j(t)$ and integrating over the interval (O,T),

$$\int_0^T f(t)\,\phi^*_j(t)\,dt=\sum_i \int_0^T r_i\phi_i(t)\,\phi^*_j(t)\,dt.$$

Since the functions are orthonormal, the right-hand side is zero except for j=i, and so

$$(2.4.2)' \qquad r_i=\int_0^T f(t)\,\phi^*_k(t)\,dt.$$

For illustrative purposes we consider the homogeneous integral equation

$$(2.4.3)' \quad \int^T K(t-s)\phi_i(s)ds=\lambda_i\phi_i(t)$$

where $K(t-s)$ is the kernel and where we assume that $\phi_i(t)$ is an eigenfunction, and λ_i and eigenvalue.

We know that for a positive definite kernel, the eigenvalues are strictly positive. Furthermore, if $K(t,s)$ is positive definite, the eigenfunctions form a complete set. By definition, the set of functions $\phi_i(t)$ is said to be complete if the only function $g(t)$ satisfying

$$\int_0^T g(t)\phi_i(t)dt=0$$

for all i is the function $g(t)\equiv 0$. In essence this means that if a function $g(t)$ not identically zero, is orthogonal to the set of functions $\phi_i(t)$, then $g(t)$ is also an eigenfunction.

The significance of the completeness property of the functions $\phi_i(t)$ is that the series expansion (2.4.1) converges in the mean to $f(t)$. Convergence in the mean means

$$\lim_{n\to\infty} E\{[\sum_{i=1}^{n} r_i\phi_i(t)-f(t)]^2\}=0$$

The series expansion of equation (2.4.1)' with the functions $\phi_i(t)$ chosen as the eigenfunctions of the integral equation (2.4.3)' is the Karhunen-Loève expansion.

Some other facts will be of eventual interest. One is Mercer's theorem which states if K(t,s) is positive semidefinite it can be expanded in terms of eigenvalues and eigenfunctions as

$$(2.4.4)' \qquad K(t,s)=\sum_{i=1}^{\infty} \lambda_i \phi_i(t) \phi^*_j(s).$$

In some circumstances it is convenient to use the inverse kernel $K^{-1}(u,v)$ defined by

$$(2.4.5)' \qquad \int_0^T K^{-1}(t,u)K(u,v)du=\delta(t-v), \quad 0\leq t, \ v\leq T.$$

The inverse kernel has an expansion in terms of $\phi_i(t)$ and λ_i given by

$$(2.4.6)' \qquad K^{-1}(t,u)=\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \phi_i(t) \phi^*_i(u)$$

The usefulness of the inverse kernel is, however, mainly analytical. In practice, it may be difficult to determine.

In summary, we may represent a square-integrable function of a random process over a finite observation interval in a series of orthonormal functions. The coefficients may be made

to have the useful property of being uncorrelated.

REMARK Homogeneous Fredholm integral equations play an important role in communication theory. As a theoretical tool, they are used in determining the Karhunen-Loève expansion theory of a random process. A difficult aspect of this theory, however, is that it is often difficult to find solutions to the equations involved.

2.5 GRANDELL TYPE SOLUTION

In (10), Grandell used a method similar to that used in chapter 1, to obtain the best linear estimator. In this section we briefly review Grandell's method and make some comments thereon.

With the assumption that only the covariance is known, Grandell adopts as a criterion for the choice of the estimate, the quadratic mean. Thus he seeks estimates of the type

$$(2.5.1) \qquad \lambda^*(t)=\alpha(t)+\int_0^T \beta_t(s)\,d(N(s)-s).$$

where the functions $\alpha(t)$ and $\beta_t(s)$ (as in chapter 1) are determined so that

$$E\{\lambda^*(t) - \lambda(t)\}^2$$

is minimized.

By virtue of the Karhunen-Loève expansion and Mercer's theorem of the preceding section, the covariance kernel K(s,t) can be represented by

$$(2.5.2) \qquad K(s,t)=\sum_{i=1}^{\infty}\frac{\phi_i(t)\phi_i(s)}{\mu_i}$$

the μ_i and ϕ_i being eigenvalues and eigenfunctions, respectively.

By assumption $\alpha(t)$ and $\beta_t(s)$ are finite and $E\{\lambda^*(t)-\lambda(t)\}^2$ exists, so $\beta_t(s)$ has the form

$$(2.5.3) \qquad \beta_t(s)=\sum_{i=1}^{\infty}\beta_i\phi_i(s)+b_t(s)$$

where $b_t(s)$ is orthogonal to $\phi_1,\phi_2,\ldots$.

We now proceed to minimize the expression

$$(2.5.4) \qquad E\{\lambda^*(t)-\lambda(t)\}^2=E\{(\alpha(t)-1)+\int_0^T \beta_t(s)\,dN(s)-s)-(Nt)-1)\}^2$$

$$=d(\alpha(t)-1)^2+\sum_{i=1}^{\infty}\beta_i^2+\int_0^T b^2{}_t(s)\,ds+\sum_{i=1}^{\infty}\frac{\beta_i^2}{\mu_i}+$$

$$+\sum_{i=1}^{\infty}\frac{\phi_i^2(t)}{\mu_i}-2\sum_{i=1}^{\infty}\frac{\beta_i\phi_i(t)}{\mu_i}$$

Since (2.5.4) is being minimized, $\alpha(t)=1$ and $b_t(s)=0$. Thus

$$E\{\lambda^*(t)-\lambda(t)\}^2=\sum_{i=1}^{\infty}\{\beta_i^2+\frac{(\beta_i-\phi_i(t))^2}{\mu_i}\}.$$

so that

$$\frac{\partial E\{\lambda^*(t)-\lambda(t)\}^2}{\partial \beta_i} = 2\beta_i + \frac{2(\beta_i-\phi_i(t))}{\mu_i}$$

setting this equal to zero we obtain

$$\beta_i = \frac{\phi_i(t)}{1+\mu_i}$$

which corresponds to a minimum since

$$\frac{\partial^2 E\{\lambda^*(t)-\lambda(t)\}^2}{\partial \beta_i^2} = 2+\frac{2}{\mu_i} > 0.$$

It follows that (substitution in (2.5.3)),

$$\beta_t(s) = \sum \frac{\phi_i(t)\phi_i(s)}{1+\mu_i}, \tag{2.5.5}$$

and (2.5.1) in turn becomes

$$\lambda^*(t) = 1 + \int_0^T \sum_{i=1}^{\infty} \frac{\phi_i(t)\phi_i(s)}{1+\mu_i}\, d(N(s)-s); \tag{2.5.6}$$

and

$$E\{\lambda^*(t)-\lambda(t)\}^2 = \sum_{i=1}^{\infty} \frac{\phi_i^2(t)}{1+\mu_i} \tag{2.5.7}$$

If we now multiply expressions (2.5.2) and (2.5.5), and integrate over the interval (O,T), we obtain

$$\int_0^T \beta_t(u)K(u,s)\,du = \int_0^T \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\phi_i(t)\phi_i(u)}{1+\mu_i}\,\frac{\phi_j(u)\phi_j(s)}{\mu_j}\,du$$

$$= \sum_{i=1}^{\infty} \frac{\phi_i(t)\phi_i(s)}{(1+\mu_i)\mu_i}$$

$$= \sum_{i=1}^{\infty} \frac{\phi_i(t)\phi_i(s)}{\mu_i} - \sum_{i=1}^{\infty} \frac{\phi_i(t)\phi_i(s)}{1+\mu_i}$$

$$= K(s,t) - \beta_t(s)$$

Thus $\beta_t(s)$ is a solution of the integral equation

$$(2.5.8) \qquad \beta_t(s) + \int_0^T K(u,s)\,\beta_t(u)\,du = K(t,s).$$

REMARK Although equation (2.5.8) is a Fredholm integral equation of the second kind, Grandell fails to note this fact in (10).

Grandell shows that $\beta_t(t)$ is a unique solution. Further, in the case of a degenerate covariance kernel, there exists an unbiased estimate of $\lambda(t)$. On the other hand, if the number of eigenfunctions is infinite, it is impossible to find an unbiased estimate of $\lambda(t)$ which is useful for every t in (O,T). The interested reader may refer to Grandell's paper.

To illustrate this theory, we consider an example.

EXAMPLE (2.5). Suppose the kernel is given as

$$K(s,t) = \frac{1}{p}$$

As was noted in the previous section, all eigenvalues, μ, of

$$\phi(t) = \mu \int_0^T K(t,s)\,\phi(s)\,ds$$

are positive. Thus in this example we get the equation

$$\phi(t) = \mu \frac{1}{p} \int_0^T \phi(s)\,ds, \tag{2.5.9}$$

the only solution being

$$\phi(t) = C, \text{ a constant.}$$

The orthogonality requirement of the $\phi_i(t)$ implies

$$\int_0^T \phi^2(t) = Tp^2 = 1,$$

so that

$$\phi(t) = \frac{1}{\sqrt{T}}.$$

Substitution in equation (2.5.9) gives

$$\frac{1}{\sqrt{T}} = \mu \frac{1}{P} \sqrt{T}$$

so that

$$\mu = \frac{P}{T} .$$

Now to get the best linear estimator, λ^*, we use equation (2.5.6), thus

$$\lambda^*(t) = 1 + \int_0^T \frac{1/T}{1+(P/T)} \, d\{N(s)-s\}$$

from which we obtain the best linear estimate as:

$$\lambda^* = \frac{P+N(T)}{P+T} \qquad (2.5.10)$$

REMARK We note that in the above example the best linear estimate is dependent only on the covariance function and so the estimate, (2.5.10), is the best linear estimate for every process with the given covariance function.

2.6 SOME OTHER METHODS

We now turn to other solution techniques which appear mainly in the engineering literature as described by Van Trees (19), Helstrom (12b) and others.

Integral equations are frequently encountered in the theory of signal detection and estimation. Two such equations encountered in connection with the detection of signals in nonwhite noise are:

$$(2.6.1) \qquad \int_0^T F(t-s)f(s)ds = \lambda f(t) \qquad 0 \leq t \leq T$$

where $f(t)$ and λ are to be determined, and

$$(2.6.2) \qquad \int_0^T F(t-s)f(s)ds = g(t)$$

where $f(t)$ is to be determined. By assumption the covariance function is the sum of parts corresponding to white and nonwhite noise, so

$$R(t-s) = \frac{N_0}{2}\delta(t-s)+K(t-s)$$

and substituting this into equation (2.6.2) produces the integral equation

$$(2.6.3) \qquad \frac{N_0}{2}f(t)+\int_0^T K(t-s)f(s)ds = g(t) \qquad 0 \leq t \leq T$$

which we immediately recognize as a Fredholm integral equation of the second kind.

As mentioned in sections (2.3) and (2.4), a solution to (2.6.3) will generally exist unless $(-N_0/2)$ is an eigenvalue

of the homogeneous integral equation (2.6.1). Since K(t-s) is a positive-definite kernel, the integral equation cannot have a negative eigenvalue, and there is no trouble about the existence of a solution.

2.6.1 Applications of Fourier Transforms

In numerous applications, integral equations of the type

$$(2.6.4) \quad \int_{-\infty}^{\infty} K(t-s)f(s)ds = g(t)$$

are encountered. The integral on the left is a convolution. If $K(\tau)$, $g(\tau) \varepsilon L_2[-\infty,\infty]$, we can use the Fourier transform of both sides to obtain

$$(2.6.5) \quad \sqrt{2\pi}G(K)G(f) = G(g), \qquad \text{and so}$$

$$(2.6.6) \quad G(f) = \frac{G(g)}{\sqrt{2\pi}G(K)},$$

provided G(K) does not vanish.

If the right side, as a function of u, is in $L_2[-\infty,\infty]$ we finally obtain

$$(2.6.7) \quad f = \frac{1}{\sqrt{2}} G^*\left(\frac{G(g)}{G(K)}\right) .$$

EXAMPLE 2.6.2. Consider

$$(2.6.8) \quad f(t)-\lambda\int_{-\infty}^{\infty} e^{-|t-s|}f(s)ds = g(t).$$

By a direct integration

$$G(e^{-|t|}) = \frac{\sqrt{2/\Pi}}{1+u^2}$$

so that

$$G(f)-\lambda\sqrt{2\Pi}\ \frac{\sqrt{2/\Pi}}{1+u^2}G(f) = G(g)$$

and so

$$(2.6.9) \quad G(f) = \frac{1+u^2}{1+u^2-2\lambda}\ G(g)$$

where we require $\lambda<\frac{1}{2}$. Then

$$f = G^*\left(\frac{\sqrt{2/\Pi}}{1+u^2-2\lambda}\right) = \frac{1}{\Pi}\int_{-\infty}^{\infty} \frac{e^{-iut}}{1+u^2-2\lambda}\ du$$

and for $\lambda<\frac{1}{2}$, we obtain the solution

$$(2.6.10) \quad f = \frac{e^{-\gamma|t|}}{\gamma} \qquad \text{where } \gamma = \sqrt{1-2\lambda}$$

REMARK Techniques and properties of Laplace, Hankel, and Mellin transforms (Helstrom (12b)) can be derived by relating them to Fourier transforms; we do not, however, intend to discuss these transforms in the present work.

2.6.3 Equations with Separable Kernels

In this case we may expand the kernel in the form

$$(2.6.11) \quad K(t,s) = \sum_{j=1}^{n} \lambda_j \phi_i(t) \phi_i(s),$$

where λ_j and $\phi_j(t)$ are the eigenvalues and eigenfunctions of $K(t,s)$. Thus we can use the techniques of sections (2.4) and (2.5) to solve the given integral equation.

REMARK Lack of space prevents us from discussing all the numerous methods which are found in the literature. A noteworthy omission for finite observation and nonstationary processes, is the state-variable formulation of Kalman and Bucy (6b) which leads to a complete solution.

CHAPTER 3

A SURVEY OF APPROXIMATE METHODS FOR THE SOLUTION OF FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

3.1 INTRODUCTION

Since exact solutions of the Fredholm integral equation are usually difficult to obtain, it is natural to consider, as we do here, the possibility of finding approximate solutions. In a private communication, Professor James Varah suggested the use of quadrature methods of the form

$$(3.11) \qquad \int_a^b f(t)\,dt \doteq \sum_{i=1}^{n} w_i f(t_i),$$

i.e., the integral is represented by a weighted sum of values of the integrand at a finite number of points t_i. In this chapter we explore this and other approximate methods; in general, the numerical methods are explained without proof of their validity, for which we can refer to the cited literature (6), (12), (14).

In section (3.2) we shall deal with simple quadrature rules; in particular the Trapezium and Simpson's rules. We shall briefly consider generalizations to these rules in section (3.3), and then consider the method of collocation in section (3.4). In section (3.5) we tackle the question of errors and the subsequent suggested improvements on the methods discussed thus far. Finally,

in section (3.6) we shall mention other methods which are not discussed in detail here. Then we shall end the chapter with a summary and some concluding remarks.

3.2 QUADRATURE RULES

We summarize the principal formulae of type (3.1.1) with their error terms. Let us write I(f) for the integral $\int_a^b f(t)dt$. For the repeated forms, the interval $[a,b]$ is divided into n equal steps of length h, so that h = (b-a)/n, and the formula is applied over sub-intervals. The error depends on some high order derivative of f(t) at a point ξ in $[a,b]$, where we assume that this derivative is continuous. The following three formulae use equally spaced points, so we let $t_i = a+ih$. $i=1,2,\ldots,n$.

(3.2.1) The Mid-Point Rule

$$I(f) = h\sum_{i=1}^{n} f(t_{i-\frac{1}{2}}) + R_1$$

where

$R_1 = \frac{1}{24}(b-a)h^2 f''(\xi)$, is the remainder term .

(3.2.2) The Trapezium Rule

$$I(f) = \tfrac{1}{2}h\{f(a)+2\sum_{i=1}^{n-1} f(t_i)+f(b)\}+R_2$$

where

$$R_2 = -\frac{1}{12}(b-a)h^2 f''(\xi).$$

(3.2.3) Simpson's Rule

$$I(f) = \frac{1}{3}h\{f(a)+4\sum_{i=1}^{\frac{1}{2}n} f(t_{2i-1})+2\sum_{i=1}^{\frac{1}{2}n-1} f(t_{2i})+f(b)\}+R_3$$

where

$$R_3 = -\frac{1}{180}(b-a)h^4 f^{iv}(\xi).$$

REMARK In formula (3.2.3), n must be even. The above formulae all use equally spaced points, and higher-order formulae of the same type may be derived. However, the above three are the most important in practice.

(3.2.4) Formulation of Discrete Equations

Fredholm equations of the second kind can be approximated in a straightforward way by means of quadrature formulae. The general case is now clear. We choose any quadrative formula

$$\int_a^b f(t)dt = \sum_{i=1}^{n} w_i f(t_i)$$

involving the n points t_i and the corresponding weights w_i. The general Fredholm integral equation of the second kind

$$(3.3.1) \qquad \int_a^b K(t,s)f(s)ds+g(t) = f(t)$$

is then replaced by a system of n linear algebraic equations for the unknowns $\hat{f}(t_i)$ (to indicate that this is only an approximation to the integral equation f(t) has been replaced by $\hat{f}(t)$). Written in matrix form, this system of equations becomes

$$(3.3.2) \qquad (I-KD)\underline{\hat{f}} = \underline{g},$$

where the matrix K has the elements $K_{ij}=K(t_i,t_j)$ and D has the diagonal elements w_i. The solution of these equations, $\hat{f}_i$, represents the approximate values of f(t) at the points $t=t_i$.

We illustrate the above methods by two simple examples:

EXAMPLE 3.2.1 Consider

$$f(t) = t+\int_0^1 K(t,s)f(s)ds$$

where the kernel is of the form

$$K(t,s) = \begin{cases} t(1-s) & \text{if } t<s \\ s(1-t) & \text{if } t>s \end{cases}$$

(Hilderbrand (13) notes that this kernel is weakly singular.)
Take $t_1 = 0$, $t_2 = \frac{1}{4}$, ..., $t_5 = 1$ and $h = \frac{1}{4}$ (trapezium rule)
Then $K_{11} = t_1(1-t_1)$, $K_{12} = t_1(1-t_2)$, ..., $K_{35} = t_3(1-t_5)$ ...
i.e., $K_{ij} = t_i(1-t_j)$, $i<j$, $i,j = 1,\ldots,5$.

so that

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3/16 & 1/8 & 1/16 & 0 \\ 0 & 1/8 & 1/4 & 1/8 & 0 \\ 0 & 1/16 & 1/8 & 3/16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$D = \text{diag } h(1/2,\ 1,\ 1,\ 1,\ 1/2)$$

Hence from $(I-KD)\hat{f}=g$ with $g^1=[0 \quad 1/4 \quad 1/2 \quad 3/4 \quad 1]$, we obtain the system of equations

$$\begin{aligned} \hat{f}_1 &= 0 \\ 61/64\hat{f}_2-1/32\hat{f}_3-1/64\hat{f}_4 &= 1/4 \\ -1/32\hat{f}_2+15/16\hat{f}_3-1/32\hat{f}_4 &= 1/2 \\ -1/64\hat{f}_2-1/32\hat{f}_3+61/64\hat{f}_4 &= 3/4 \\ \hat{f}_5 &= 1 \end{aligned}$$

The solution to this set of equation is

$$\hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \hat{f}_4 \\ \hat{f}_5 \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 0.2943 \\ 0.5702 \\ 0.8104 \\ 1.0000 \end{bmatrix}$$

EXAMPLE 3.2.2 Suppose we are required to find an approximate solution of the integral equation

$$\int_0^1 K(t,s)f(s)ds+g(t) = f(t).$$

We use Simpson's rule to approximate the integral in the form

$$\int_0^1 u(t)dt = 1/6\{u(o)+4u(1/2)+u(1)\}+E_2$$

where E_2 represents the error or remainder term. Neglecting this remainder for the moment, we obtain the relation

$$1/6\{K(t,o)\hat{f}(o)+4K(t,\tfrac{1}{2})\hat{f}(\tfrac{1}{2})+K(t,1)\hat{f}(1)\}+g(t) = \hat{f}(t).$$

In this equation we write

$t = 0, \frac{1}{2}$ and 1 successfully, and obtain

$$1/6\{K(o,o)\hat{f}(o)+4K(o,\tfrac{1}{2})\hat{f}(\tfrac{1}{2})+K(o,1)\hat{f}(1)\}+g(o) = \hat{f}(o)$$
$$1/6\{K(\tfrac{1}{2},o)\hat{f}(o)+4K(\tfrac{1}{2},\tfrac{1}{2})\hat{f}(\tfrac{1}{2})+K(\tfrac{1}{2},1)\hat{f}(1)\}+g(1/2) = \hat{f}(1/2)$$
$$1/6\{K(1,o)\hat{f}(o)+4K(1,\tfrac{1}{2})\hat{f}(1/2)+K(1,1)\hat{f}(1)\}+g(1) = \hat{f}(1)$$

which may be written as

$$(I-KD)\hat{f} = g$$

with D = diag(1/6, 4/6, 1/6).

We can, therefore, solve for the unknown values $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3$.

3.3 GENERALIZED QUADRATURE

Assume $f \in C[a,b]$ and $\phi(t)$ is Lebesque integrable on $[a,b]$. Consider the problem of numerically integrating $f(t)\phi(t)$ over $[a,b]$. The approach here is to use piecewise polynomial interpolation to develop the generalizations of the trapezoidal rule and Simpson's rule.

3.3.1 The Generalized Trapezoidal Rule

As in the preceding section, let

$h = (b-a)/n$ and define $t_i = a+ih$, $i = 0, 1, \ldots, n$. Let $f_n(t)$ be the piecewise linear interpolation function of $f(t)$ at the nodal points $t_0, t_1, \ldots, t_n$; i.e.,

$$(3.3.1) \quad f_n(t) = \frac{1}{h}\{(t_i-t)f(t_{i-1})+(t-t_{i-1})f(t_i)\}, \quad t_{i-1} \le t \le t_i \quad i=1,\ldots,n.$$

Substituting (3.3.1) into $\int_a^b f_n(t)\phi(t)dt$, we obtain

$$(3.3.2) \quad \int_a^b f_n(t)\phi(t)dt = \sum_{i=1}^{n} \{\alpha_i f(t_{i-1})+\beta_i f(t_i)\}$$

where α_i and β_i are given by:

$$(3.3.3) \quad \alpha_i = \frac{1}{h}\int_{t_{i-1}}^{t_i}(t_i-t)\phi(t)dt, \quad \beta_i = \frac{1}{h}\int_{t_{i-1}}^{t_i}(t-t_{i-1})\phi(t)dt$$

REMARKS For this form of quadrature, it is necessary to evaluate the integrals of $\phi(t)$ and $t\phi(t)$ over arbitrary intervals. Since usually the singularity of an integrand can be isolated as a simple function, these integrations may, in general, not be difficult.

We also note that when $\phi(t)\equiv 1$, we obtain the ordinary trapezoidal rule since in this case

$$\alpha_i = \beta_i = \frac{h}{2}$$

3.3.2 The Generalized Simpson's Rule

Here we let $h=(b-a)/2n$, $n>1$, and define f_n as the piecewise quadratic interpolation function to f on $t_0, t_1, \ldots, t_{2n}$; f_n being quadratic on each subinterval $[t_{2i-2}, t_{2i}]$, $i=1, \ldots, n$. The quadrature formula beomces:

$$(3.3.4) \qquad \int_a^b f_n(t)\phi(t)dt = \sum_{i=1}^{n} \{\alpha_i f(t_{i-2}) + \beta_i f(t_{2i-1}) + \gamma_i f(t_{2i})\}$$

where

$$\alpha_i = \frac{1}{2h^2}\int_{t_{2i-2}}^{t_{2i}} (t-t_{2i})(t-t_{2i-1})\phi(t)dt,$$

$$(3.3.5) \qquad \beta_i = \frac{-1}{h^2}\int_{t_{2i-2}}^{t_{2i}} (t-t_{2i-2})(t-t_{2i})\phi(t)dt$$

$$\gamma_i = \frac{1}{2h^2}\int_{t_{2i-2}}^{t_{2i}} (t-t_{2i-1})(t-t_{2i-2})\phi(t)dt$$

We shall see the usefulness of these generalizations when we are dealing with error analysis in section (3.5).

3.4 COLLOCATION METHOD

The above direct methods often have the disadvantage of concealing anomalous errors. In a sense, they are too general, and need to be modified in order to take advantage of any special features which the particular equation may posses.

Following Green (12), let $\phi_1, \phi_2, \ldots, \phi_n$ form a set of linearly independent functions on $[a,b]$. We assume that the ϕ_i's form part of an orthonormal basis for $L_2[a,b]$. Now consider the integral equation

$$f(t) = g(t)+\lambda\int_a^b K(t,s)f(s)ds, \tag{3.4.1}$$

and assume an approximate solution of the form

$$f_{(n)} = \sum_{i=1}^{n} C_i \phi_i \tag{3.4.2}$$

where the C_i's are undetermined constant coefficients. Substituting (3.4.2) into (3.4.1) yields

$$\sum_{i=1}^{n} C_i \phi_i(t) = g(t)+\lambda\sum_{i=1}^{n} C_i \int_a^b K(t,s)\phi_i(s)ds+\varepsilon_n(t:C_1,\ldots,C_n) \tag{3.4.3}$$

where ε_n denotes the error involved as a result of assuming the

solution $f_{(n)}$. We aim to choose the coefficients in such a way as to minimize the error ε_n.

In this vein we choose a set of points $t_1, t_2, \ldots, t_n$, and determine the coefficients by the requirement that $\varepsilon_{(n)}(t)$ should vanish at each of these points.

Let

$$b_j = g(t_j)$$

and

$$a_{ij} = \phi_i(t_j) - \lambda \int_a^b K(t_j, s)\phi_i(s)ds,$$

substituting these into the expression (3.4.3), we obtain a system of n linear equations:

$$(3.4.4) \qquad \sum_{i=1}^{n} a_{ij}C_i = b_j \qquad j = 1, 2, \ldots, n$$

from which the C_i's are to be determined.

Now to solve (3.4.4), we choose the points t_j from the numerical data at hand, and the numbers a_{ij} are obtained by use of a quadrature formula.

Conditions for solvability:

(i) The determinant of the system (3.4.4) must be non-zero.

(ii) At least one of the b_j's must be non-zero.

If these conditions are satisfied, the solution of (3.4.4) gives $f_{(n)}$ in the form of a polynomial approximation. We now give an example.

EXAMPLE 3.3.1 Consider the equation

$$f(t) = t+\int_0^1 K(t,s)f(s)ds \tag{3.4.5}$$

with

$$K(t,s) = \begin{cases} s & s\leq t \\ t & s>t \end{cases}$$

Suppose we have the points

$$t_i,\ s_i = 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1.$$

Let us assume a solution of the form

$$f_{(3)}(t) = C_1+C_2t+C_3t^2$$

with the condition that the error should vanish at the points $t_1=0$, $t_2=\frac{1}{2}$, $t_3=1$. We use the analytic form of the kernel to simplify the computation.

The coefficients a_{ij} are given by

$$a_{1j} = 1-\int_0^{t_j} sds-\int_{t_j}^1 t_jds$$

$$a_{2j} = t_j-\int_0^{t_j} s^2ds-\int_{t_j}^1 t_jsds$$

$$a_{3j} = t^2_j-\int_0^{t_j} s^3ds-\int_{t_j}^1 t_js^2ds$$

and

$$b_j = t_j.$$

Thus $a_{11} = 1-\int_0^0 sds-0\int_0^1 ds = 1$

$a_{12} = 1-\int_0^{1/2} sds-\frac{1}{2}\int_{1/2}^1 ds = 5/8$

.
.
.

$b_1 = u_1$, etc.

Hence the system (3.4.4) assumes the form

$$\begin{aligned} C_1 &= 0 \\ 120\,C_1+52\,C_2+17\,C_3 &= 96 \\ 6\,C_1+8\,C_2+9\,C_3 &= 12 \end{aligned}$$

which leads to the approximation:

$$f_{(3)}(t) = 1.988t-0.434t^2$$

which is automatically best for the chosen points $t=0, \frac{1}{2}, 1$, where $\hat{f}_1=0$, $\hat{f}_3=0.836$, $\hat{f}_5=1.554$, and gives the solutions

$$\hat{f}_2 = 0.470;\ \hat{f}_4 = 1.247$$

at the points $t_2=\frac{1}{4}$ and $t_4=3/4$, respectively. Thus the approximate solution to equation (3.4.5) is:

$$\underline{\hat{f}} = \begin{bmatrix} 0.000 \\ 0.470 \\ 0.836 \\ 1.247 \\ 1.554 \end{bmatrix} \tag{3.4.6}$$

REMARKS The collocation method suffers from the disadvantage that an exact matching of the solution at certain points does not control the size of the deviation between the exact and the approximating solutions at other points (unless we want to choose all the given points which may not be computationally feasible).

At any rate, the collocation method is worth considering because at least it is an improvement on the direct (and widely used) methods of successive approximations and the quadrature methods.

3.5 ERROR ANALYSIS

3.5.1

We now consider the problem of estimating the accuracy of the result of the calculation for any of the quadrature methods discussed. In the sequel we shall denote the norm of any function $\phi(t)$ by

$$||\phi|| = \int_a^b |\phi(t)|\, dt$$

Returning to the problem of estimation we know that the function $f(t)$ is required to satisfy the integral equation

$$(3.5.1) \qquad \int_a^b K(t,s)f(s)ds+g(t) = f(t)$$

The computed function $\hat{f}(t)$ actually satisfies

$$(3.5.2) \qquad \int_a^b K(t,s)\hat{f}(s)ds+g(t) = \hat{f}(t)-E(t)$$

where $E(t)$ denotes the error term.

If we write

$$e(t) = f(t)-\hat{f}(t)$$

for the error in our result, we find by subtraction that

$$(3.5.3) \qquad \int_a^b K(t,s)e(s)ds = e(t)+E(t).$$

In the usual way, writing K for the integral operator:

$$K(f) = \int_a^b K(t,s)f(s)ds,$$

the error is given by

$$(3.5.4) \qquad e(t) = -(1-K)^{-1}E(t), \quad \text{and so}$$

$$(3.5.5) \qquad ||e|| < ||(1-K)^{-1}||\,||E||.$$

Now $||E||$ can be estimated. E(t) will usually be expressed in terms of a derivative, either of the complete integrand in the nonsingular case, or perhaps of $\hat{f}(t)$ only in a singular case. Under suitable assumptions about the smoothness of the various terms we can estimate the magnitudes of the derivatives by examining finite differences.

It remains to estimate $||(1-K)^{-1}||$. A well known result [6], [9], states that

$$(3.5.6) \qquad ||(1-K)^{-1}|| \leq 1/(1-||K||)$$

where K is any bounded linear operator in a Banach space, provided that $||K||<1$. If we choose the norm $||f||=\max|f|$, over the interval $[a,b]$, then the norm of K is

(3.5.7) $||K|| = \max_{a\leq t\leq b} \int_a^b |K(t,s)|ds$, and (3.5.6) holds provided

$$||K||<1.$$

We therefore have a rigorous bound for the error in the result.

3.5.2 Generalized Error

Turning to the corresponding error term of the generalized quadrature rules, and using the notation of section (3.3), we have

(3.5.8) $E_n(f) = \int_a^b \{f(t)-f_n(t)\}\phi(t)dt$, and

(3.5.9) $|E_n(f)|\leq||\phi||\,||f-f_n||$.

Assuming that f^{11} is continuous, and using the error formula for Lagrange interpolation on each subinterval $[t_{j-1},t_j]$, Atkinson (3) has shown that the generalized trapezoidal rule has the error bound

(3.5.10) $|E_n(f)|\leq\frac{1}{8}h^2||f^{11}||\,||\phi||$.

REMARK We note that the order of convergence is the same as that of the ordinary trapezoidal rule, but, as we shall find below, this will not be true for the generalization of all quadrature rules.

Considering Simpson's rule, the corresponding bound for the error is:

$$(3.5.11) \quad |E_n(f)| < \frac{\sqrt{3}}{27} h^3 ||f^{111}|| \; ||\phi||.$$

As hinted earlier, we find that, unlike the case of the trapezoidal rule, the generalized Simpson's rule only has an h^3 order of convergence whereas the regular Simpson's rule is of a higher order h^4.

Thus, at least, the generalized Simpson's rule provides an improvement on the ordinary method.

3.6 SUMMARY AND CONCLUSION

We have considered in some detail the trapezium and Simpson's rules plus their generalizations as convenient quadrature methods for the numerical approximation to the solution of the Fredholm integral equation of the second kind. We found that the method of collocation may yield a significant improvement on the results of the former methods.

We do not claim to have mentioned all, or even most, of the available methods. Indeed, for quadrature rules, some other methods go by various names as: Simple Gauss rule, "Three-Eighths" rule, Newton-Cotes rule, Radau quadrature, Lobatto rule, etc.

Although we do not have an opportunity to discuss all methods here, yet it is worth exploring some further possibilities. For example, there are expansion methods such as the Galerkin and the Rayleigh-Ritz techniques, all of which may be found in the cited literature.

In conclusion, we wish to point out that finding explicit solutions of integral equations is in general difficult. Only in exceptional cases can such solutions be found. Generally, various approximate and numerical methods have to be used.

We do not feel, however, that an approximate method is in any way too inferior to one giving an exact solution. A solution in closed form may certainly be convenient, but is rarely absolutely necessary, bearing in mind that the integral equation, considered as a model for some real system, is almost certainly only an approximate representation of the system in the first place.

CHAPTER 4

COMPARISON: AN EXACT VERSUS AN APPROXIMATE SOLUTION IN A SPECIAL CASE

INTRODUCTION

Whittle (22) does obtain Bayes linear estimators and equation (1.2.6) for certain special cases. Whittle's work suggests a method of obtaining the optimal linear estimator using linear smoothing techniques. We shall use a slightly modified form of Whittle's method to obtain an exact solution to the Fredholm integral equation (1.2.6) and consequently obtain the linear estimator (1.2.1).

To illustrate the use of the techniques devised, a numerical example is treated. The data consists of successive 30-day totals of oil wells discovered by wildcat exploration in Alberta for the period 1953 to 1971. Clevenson and Zidek (7) consider this practical problem and compute the approximate optimal linear estimator $\hat{\lambda}_{\infty}$. We compare our exact and Clevenson-Zidek's approximate results.

4.1 A BRIEF OUTLINE OF WHITTLE'S DERIVATION OF (1.2.6)

Estimate $\lambda(t)$ by a linear smoothing formulae so that $\lambda_L(t)$ will be estimated by statistics of the type

$$(4.1.1) \qquad \hat{\lambda}_L(t) = \int_a^b w_t(s)\,dN(s),$$

for which, conditionally, given λ

$$(4.1.2) \qquad E\hat{\lambda}_L(t) = \int_a^b w_t(s)\lambda(s)\,ds,$$

and

$$(4.1.3) \qquad \mathrm{var}\hat{\lambda}_L(t) = \int_a^b w_t^2(s)\lambda(s)\,ds.$$

Assume that the particular function, $\lambda(t)$, is a member of a population of such functions, so that the $\lambda(t)$'s have an a priori distribution of values. The optimum linear estimator will then be obtained by minimizing

$$(4.1.4) \qquad C^*[t;w] = E\left[\int w^2\lambda\,ds+\left(\int w\lambda\,ds\right)^2-2\lambda(t)\int w\lambda\,ds+\lambda^2(t)\right],$$

where E denotes expectation with respect to prior distribution of the $\lambda(t)$'s.

If $E\lambda(t)=\mu(t)$ and $E[\lambda(t)\lambda(s)]=\mu(t,s)$ then minimizing (4.1.4) with respect to $w_t(s)$ yields the integral equation

$$(4.1.5) \qquad \mu(s)w_t(s)+\int \mu(s,u)w_t(u)\,du = \mu(t,s).$$

By a normalization of the form

$$\xi_t(s) = w_t(s)\sqrt{\{\mu(s)/\mu(t)}$$

$$r(t,s) = \mu(t,s)/\sqrt{\mu(t)\mu(s)}$$

we get

$$\mu(t,s) = r(t,s)\sqrt{\mu(t)\mu(s)}, \tag{4.1.6}$$

$$w_t(s) = \xi_t(s)\sqrt{\mu(t)/\mu(s)} \tag{4.1.7}$$

Substituting these into (4.1.5) we obtain, after some manipulation,

$$\xi_t(s)+\int_a^b r(s;u)\,\xi_t(u)\,du = r(t;s), \tag{4.1.8}$$

which is obviously of the required form.

4.2 THE EXACT SOLUTION

We seek the optimal linear estimator of the form

$$\hat{\lambda}_L(t) = \mu+\int_{-T}^{T} h(t;s)\,d(N(s)-\mu s) \tag{4.2.1}$$

where $h(t;s)$ is the solution of the integral equation

$$\mu h_t(s)+\int_{-T}^{T} K(s;u)\,h_t(u)\,du = K(t;s), \quad -T<s<T. \tag{4.2.2}$$

Consider the special case where the kernel $K(t;s)$ depends only upon $(t-s)$ so that (4.2.2) may be written as

$$(4.2.3) \qquad \mu h(s)+\int_a^b K(s-u)h(u)du = K(t-s),$$

where, for convenience, we have dropped the subscript, t, of h and $[a,b] \equiv [-T,T]$.

We further specialize our results by assuming

$$\begin{aligned} \mathrm{Cov}(\lambda(t),\lambda(s)) &= E\lambda(t)\lambda(s)-E\lambda(t)E\lambda(s) \\ &= \mu(t,s)-\mu^2 \\ &= \sigma^2 p(t-s) = \sigma^2 e^{-\alpha|t-s|}, \end{aligned}$$

so that

$$\mu(t,s) = \sigma^2 e^{-\sigma|t-s|}+\mu^2.$$

Since

$$K(t,s) = \frac{\mu(t,s)}{\sqrt{\mu(t)\mu(s)}},$$

we obtain

$$(4.2.4) \qquad K(t,s) = K(t-s) = \frac{\sigma^2}{\mu} e^{-\alpha|t-s|}+\mu.$$

Whittle treats the special case where the kernel is of the form

$$K(t) = a(\gamma+\beta e^{-\alpha|t|})$$

which in our example means that

$$a\gamma = \mu \qquad \text{and} \quad a\beta = \sigma^2/\mu$$

Following Whittle we find that the integral equation (4.2.3) can be converted by repeated differentiation to

$$(4.2.5) \qquad h''(s) - \Theta^2 h(s) = -2\frac{\sigma^2}{\mu}\,\delta(t-s) + \mu\alpha^2\left[\int_a^b h(u)\,du - 1\right],$$

with $$\Theta^2 = 2\frac{\sigma^2}{\mu}\alpha + \alpha^2.$$

The δ-function arises here because of the discontinuity of the derivative of $e^{-\alpha|t-s|}$ at s=t. Whittle asserts that (4.2.5) must hold without the δ-function for $s \neq t$, however for s=t the derivatives of h(s) and $\frac{\sigma^2}{\mu} e^{-\alpha|t-s|}$ must have step discontinuities of the same magnitude.

Equation (4.2.5) has a general solution of the form

$$(4.2.6) \qquad h(s) = \frac{\sigma^2}{\mu}\frac{\alpha}{\Theta} e^{-\Theta|t-s|} + Pe^{-\Theta(s-a)} + Qe^{-\Theta(b-s)} + R.$$

We determine the quantities P, Q and R by substituting (4.2.6) back into (4.2.3) to get

$$\begin{aligned}(4.2.7) \qquad & \lambda e^{-\Theta|s-t|} + Pe^{-\Theta(s-a)} + Qe^{-(b-s)} + R \\ & + \int_a^b \Big[\Big(\frac{\sigma^2}{\mu} e^{-\alpha|s-u|} + \mu\Big)\Big(\lambda e^{-\Theta|s-u|} + Pe^{-\Theta(u-a)} \\ & + Qe^{-\Theta(b-u)} + R\Big)\Big]\,du \\ & = \frac{\sigma^2}{\mu} e^{-\alpha|s-t|} + \mu,\end{aligned}$$

where $\lambda \equiv \frac{\alpha\sigma^2}{\mu\theta}$

Recalling that $[a,b] \equiv [-T,T]$, and performing the integration involved, (4.2.7) yields the solution,

$$(4.2.8) \quad h_t(s) = \lambda\left[e^{-\theta|s-t|}+e^{-\theta(t-a)-\theta(s-a)}+e^{-\theta(b-t)-\theta(b-s)}\right].$$

If (4.2.8) is substituted into (4.2.1) we obtain

$$(4.2.9) \quad \hat{\lambda}_L(t) = \mu+I_1+I_2+I_3,$$

where

$$I_1 = \lambda\int_a^b e^{-\theta|s-t|}\,d(N(s)-\mu s)$$

$$I_2 = \lambda\int_a^b e^{-\theta(t+s-2a)}\,d(N(s)-\mu s)$$

$$I_3 = \lambda\int_a^b e^{-\theta(-t-s+2b)}\,d(N(s)-\mu s)$$

with

$$\lambda = \frac{\alpha\sigma^2}{\mu\theta}; \quad \theta^2 = \frac{2\alpha\sigma^2}{\mu}+\alpha^2$$

The expression (4.2.9) is the exact optimal linear estimator of the intensity function, $\lambda(t)$, and it is the form which is used to obtain the exact solution to the oilwell discovery problem.

Figure (b) displays the exact and the approximate results graphically for comparison. A Computer Program to compute (4.2.9) is given in Appendix (1).

4.3 AN APPROXIMATE SOLUTION OF 1.2.6

This presentation follows Clevenson and Zidek (7). Denote by $h_T(t;s)$ the solution of

$$(4.3.1) \qquad \mu x(s)+\int_{-T}^{T} x(u)K(s-u)du = K(t-s), \qquad -T<s<T,$$

and let $h_\infty(t;s)$ be our approximation to $h_T(t;s)$; that is, $h_\infty(t;s)$ is the solution of

$$(4.3.2) \qquad \mu x(s)+\int_{-\infty}^{\infty} x(u)K(s-u)du = K(t-s), \quad -\infty<s<\infty,$$

with $\int_{-\infty}^{\infty} x^2(s)ds<\infty$.

Using Fourier transform techniques, Clevenson and Zidek (7) show that

$$\varepsilon_T(t;s) = h_T(t;s)-h_\infty(t;s)\to 0$$

as $T\to\infty$ for each fixed (t,s) under suitable regularity conditions. Their argument also gives a bound for ε_T in the form,

$$(4.3.3) \qquad |\varepsilon_T(t;s)| \le A\mu^{-1}\{\int_{|u|>T} h_\infty^2(t;u)du\}^{\frac{1}{2}}\{T^{-\frac{1}{2}}+B\mu^{-1}\},$$

with

$$A = \{\int_{-\infty}^{\infty} |s|k^2(s)ds\}^{\frac{1}{2}}$$

and

$$B = \{\int_{-\infty}^{\infty} K^2(s)ds\}^{\frac{1}{2}}$$

Corresponding to equations (4.3.1) and (4.3.2), let

$$\hat{\lambda}_T(t) = \mu + \int_{-T}^{T} h_T(t;s)d(N(s)-\mu s) \tag{4.3.4}$$

and

$$\hat{\lambda}_\infty(t) = \mu + \int_{-\infty}^{\infty} h_\infty(t;s)d(N(s)-\mu s) \tag{4.3.5}$$

respectively be the optimal linear estimator and the approximation to $\hat{\lambda}_T$. Then by applying the bound in inequality (4.3.3), a bound for $E|\Delta| = E|\hat{\lambda}_T(t) - \hat{\lambda}_\infty(t)|$ is obtained in the form:

$$E|\Delta| \leq 4TA\mu^{-1}\{\int_{|u|>T} h_\infty^2(t;u)du\}^{\frac{1}{2}}\{T^{-\frac{1}{2}} + B\mu^{-1}\}. \tag{4.3.6}$$

Clevenson and Zidek (7) observe that this bound is small only when t is not too near the boundaries of the observation period, $[-T,T]$. Thus $\hat{\lambda}_\infty$ will not be a good approximation to $\hat{\lambda}_T$ near the boundaries of that period. Note that this is equivalent to our remark after (4.2.8).

Further, Clevenson and Zidek give the large time, T, approximation of the optimal linear estimator $\hat{\lambda}_L(t)$ as

$$(4.3.7)\qquad \hat{\lambda}_{\infty}(t) = \mu+\beta\int_{-T}^{T} e^{-\gamma|t-s|}\,d(N(s)-\mu s), \quad -T<t<T,$$

where

$$\beta = \sigma^2\mu^{-1}(1+2\sigma^2\mu^{-1}\alpha^{-1})^{-\frac{1}{2}}$$

and

$$\gamma = \alpha(1+2\sigma^2\mu^{-1}\alpha^{-1})^{\frac{1}{2}}$$

To determine the accuracy of the approximation (4.3.7) the bound given in inequality (4.3.6) was evaluated, and found to be

$$(4.3.8)\qquad P\cosh(s\gamma t),$$

where

$$P = 4AT\{T^{-\frac{1}{2}}+B\mu^{-1}\}\beta^2\gamma^{-1}e^{-2\gamma T}.$$

4.4 COMPUTATION: OIL WELLS DISCOVERY DATA

From the data the mean, μ, was estimated as $\hat{\mu}=0.70$. We used $\mathrm{Cov}(\lambda(t),\lambda(s)) = \sigma^2 p(|t-s|)$, where $p(u) = e^{-\alpha|u|}$, with $\alpha=0.05$, was chosen without explicitly using the data. The form of p is that of a one-step autoregressive process; we believe that p is symmetric and decreasing on $[0,\infty)$. Also to express our uncertainty about the choice $\mu=0.70$, the value $\hat{\sigma}^2=0.25$ was chosen, with support from the data, as an estimate of σ^2.

Time was measured in 30-day intervals and there were 108 observations in the period $[-T,T]$. For this particular

case T=110.5, and with reference to the constants in section (4.3), the following values were obtained:

$$A = 5/\sqrt{2}, \qquad B = \sqrt{5}/2,$$

$$\beta = 0.0913, \qquad \gamma = 0.195.$$

The bound given in inequality (4.3.6) was found to be less than 0.01 provided $|t| \leq 87$.

As mentioned earlier, we display in Figure (B) for comparison the exact linear estimator and the large time approximation.

REMARK The graphs in Figure (b) support the above calculations; that is, the bound is small only when t is not too near the boundaries of the observation period, $[-T,T]$. This indicates that $\hat{\lambda}_{\infty}$ will not be a good approximation to the optimal linear estimator, $\hat{\lambda}_{L}$, near the boundaries of that period.

CHAPTER 5

APPLICATIONS

5.1 INTRODUCTION

The main purpose of this chapter is to apply the techniques developed thus far to practical situations which may be slightly different from the special case considered in chapter 4. Recall that we have been concerned with the optimal linear estimator, $\hat{\lambda}_L$, of the intensity function, $\lambda(t)$, of a nonstationary poisson process. It has been shown that $\hat{\lambda}_L(t)$ is a function of $h(t;s)$ which is the solution of the integral equation

$$(5.1.1) \quad m(s)h_t(s)+\int_a^b K(s;u)h_t(u)du = K(t;s) \qquad a<s<b,$$

where $m(s) = \mu$ is a constant (in chapter 4).

In many cases the assumption that $m(t)$ is a constant over the entire period of observation $[a,b]$ is unrealistic. It is therefore of interest to study other special situations where we drop that assumption. To facilitate the application of our general model to such practical situations, consider the integral equation (5.1.1) where $m(t)$ is not a constant but is a prescribed function of t. An integral equation with this property is sometimes called a Fredholm integral equation of the third type. However, by suitably redefining the unknown function, $h_t(s)$,

and/or the kernel, K(t;s), it is always possible to rewrite such an equation in the form of the second type.

In particular, when m(t) is positive throughout the interval [a,b], Hilderbrand (13), shows that the equation (5.1.1) can be rewritten in the form

$$(5.1.2) \qquad \sqrt{m(s)}h_t(s)+\int_a^b \frac{K(s;u)}{\sqrt{m(s)m(u)}} \sqrt{m(u)}h_t(u)du = \frac{K(t;s)}{\sqrt{m(s)}} ,$$

or

$$(5.1.3) \qquad x_t(s)+\int_a^b r(s;u)x_t(u)du = g_t(s).$$

Thus in this form one must recompute

$$h_t(s) = \frac{x_t(s)}{\sqrt{m(s)}}$$

after $x_t(s)$ has been found.

Having shown that by appropriately redefining the functions involved, we can rewrite (5.1.1) in a suitable form as a second type equation, we give two practical examples of such situations.

5.2 ESTIMATION OF TRAFFIC DENSITIES AT THE LIONS GATE BRIDGE

Volumes of data have been collected for the distribution of cars on the Lions Gate Bridge in Vancouver. Figure (C) gives five-minute counts of traffic for the total traffic (southbound) crossing a detector on a "typical" day in 1974. An estimator of the intensity function $\lambda(t)$ was sought. To say the least, this intensity function reflects the effects of weather, time of day, and other exogenous variables and knowledge about these effects are useful in decision-making. For example, should another lane be added to the existing ones?

A look at the counts at an individual detector convinces one that, on the Lions Gate Bridge, counts are highly reproducible from day to day. In particular, on working days one would expect traffic to be mainly composed of cars which pass the particular location of the detector at nearly the same time every day; thus there appears to be an underlying schedule. If one were to make traffic counts over a time which is large compared with the uncertainty in an individual arrival time, the variance in counts from one day to the next should be quite small. In order to treat the apparent underlying schedule mathematically, a realization of a nonstationary Poisson process is assumed to be observed. This enables us to apply our model and the techniques developed in the previous chapters to study the underlying intensity process. Specifically, we seek a linear

estimator, $\hat{\lambda}_L$, of the intensity function, $\lambda(t)$, in the form

$$(5.2.1) \quad \hat{\lambda}_L(t) = \mu_t + \int_{-T}^{T} h(t;s)d(N(s)-M(s));$$

where $h(t;s)$ is a solution of (5.1.1). Note that here we write subscripted μ_t to indicate that in this example we drop the assumption that $m(t)=\mu$ is constant. With the assumptions in chapter 4 about the kernel, we require that

$$(5.2.2) \quad \mathrm{Cov}(\lambda(t),\lambda(s)) = \sigma^2 e^{-\alpha|t-s|},$$

and

$$(5.2.3) \quad K(t;s) = K(t-s) = \frac{\sigma^2\rho(|t-s|)+\mu_t\mu_s}{\mu_t}$$

We are now faced with the problem of finding estimates for the constants σ^2 and α, and also a method of obtaining the μ_t's. This is where, following the Bayesian recipe, introspection comes in. The motivation for Bayesian methods is essentially the desire to base calculations and decisions on any available information, whether it is sample information or information of some other nature, such as that based on past experience.

We shall use empirical Bayes' methods as suggested by Barnett (4, pp. 189-200), the 1974 data at hand, and also some values published by Lea, N.D. and Associates (16b) where data collected from the same location was used.

CALCULATIONS:

Consider the data in Table 5.1.

TABLE 5.1

HOURLY VARIATION: LIONS GATE BRIDGE, 1966

Time of Day	Volume of Cars
9 - 10	1,550
10 - 11	1,240
11 - 12	1,140
12 - 13	950
13 - 14	1,400
14 - 15	1,150
15 - 16	1,175
16 - 17	1,300
17 - 18	1,300

Source: Traffic Unit, City of Vancouver

In order to use this information to obtain the prior mean μ_t for the 1974 data, it is convenient to let μ_t change hourly, thus in our calculations μ_t varies for every 12th t since we are dealing with five-minute vehicle counts. One intuitively appealing possibility would be to divide the 1966 figures by 12 and use the resulting values as the μ_t's. But this cannot be entirely satisfactory; one would expect that the volume of traffic might have increased since 1966. Calculations show the

hourly volume for 1974 was, on the average, 1,650, while that of 1966 was 1,050; thus in the ratio of 11:7. Hence, in order to step up the 1966 values, the correction factor 11/7 was used in the calculation of the $\hat{\mu}_t$. However, to make room for sampling fluctuations and other considerations, the factor 3/2 was used in the computer program presented in Appendix (B).

Recall that our choice of $p(u)=e^{-\alpha|u|}$ is in the form of a one-step autoregressive process. This knowledge may be exploited in arriving at a value for α, and also in choosing an estimate for σ^2.

Computer programs were run for the following values of α and σ:

α	$\hat{\sigma}$
0.01	1.0
0.01	2.0
0.05	2.0
0.2	2.0

The resulting estimates of the intensity function, $\lambda(t)$, are shown by the graph of Figures (D, E).

REMARK The graphs indicate that, as in the oilwell example for fixed α, as σ increases, the estimator $\hat{\lambda}_L$ becomes increasingly data-sensitive, and hence irregular.

We now consider another example where the mean $E\lambda(t)=m(t)$ is a prescribed nonconstant function of t.

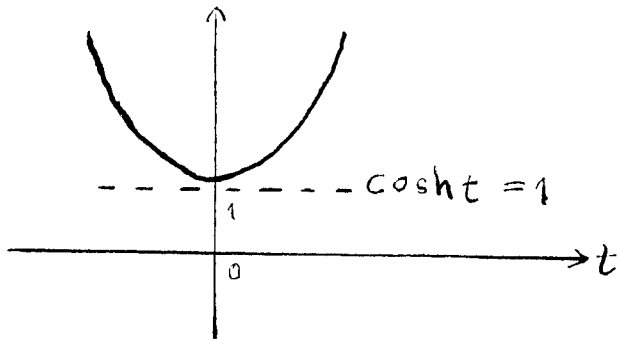
5.3 COAL-MINING DISASTERS

Figure (F) gives the numbers, in successive 400-day periods, of coal-mining disasters in Great Britain for the period 1875 to 1951. The data are taken from Cox and Lewis (8 , pp. 2-6). A disaster is defined as a mining accident involving the death of 10 or more men. Cox and Lewis discuss more formal statistical methods for analyzing this set of data; here as an alternative to Cox and Lewis analysis, we apply our model.

It is hoped that this example will illustrate in a little more detail the sort of problem discussed in this work. In particular, this section may be regarded as an extension to the preceding section where empirical Bayes ideas were cited as justification for our choice of σ^2 and α. Suppose the assumptions of section (5.2) still hold with the exception that we now require m(t) to be a continuous function of t.

Fig. (F) suggests consideration of the function

$$\text{(5.3.1)} \qquad \text{Cosht} = \frac{1}{2}(e^{t}+e^{-t}), \qquad -\infty<t<\infty.$$

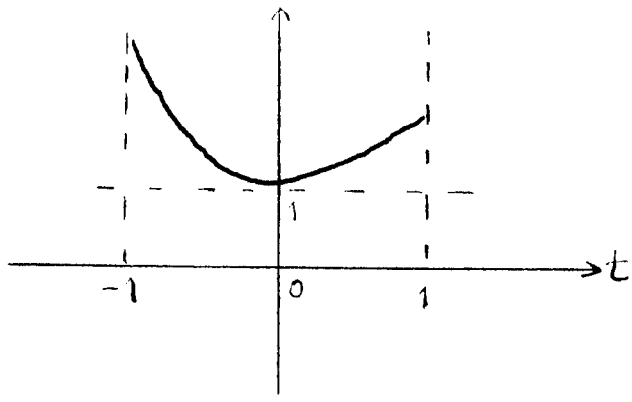


Since, as Fig. (F) indicates, the average rate of occurrence of disasters is decreasing with time, the positive side of the function (5.3.1) is unsatisfactory. Let us choose, instead, m(t) to have the form

$$(5.3.2) \qquad m(t) = \frac{1}{2}(e^{at}+e^{-bt}), \qquad -1 \leq t \leq 1$$

where $0<a,b\leq 1$ and $b>a$.

The restriction on t is to prevent m(t) from becoming too large. The restrictions on a and b are imposed for similar reasons. Note that with these restrictions m(t) takes the following form:



The sample mean of the given data is about 2; with this knowledge, and also to simplify the calculations, we choose a=0.5 and b=1.0. Using the techniques of section (5.2), $\hat{\sigma}^2$=0.25 is chosen as an estimate of σ^2. Also, the value α=0.05 is found to be a good choice.

COMPUTATION: As has already been said, the form of the kernel and the method of calculation are essentially those of the preceding section. It is interesting to note that in conformity with the restriction on t in (5.3.2), in the subprogram to compute the function m(t), each time value t is divided by 30. Note that in this example $t \in [-30,30]$, while for m(t) we require $t \in [-1,1]$.

Another interesting feature of the present example is that, curiously, the values a=1.75 and b=1.0 were used with the same values of $\hat{\sigma}^2$ and α as above. It turned out that both programs produced values of the estimator of the intensity function, $\lambda(t)$, which are pretty much the same. Nevertheless, the cost and time of computation as given by the computer are as follows:

Values	Time (sec.)	Cost ($)
a=0.5; b=1.0	2.784	1.34
a=0.75; b=1.0	2.791	1.39

Thus one is persuaded to stick to the values a=0.5 and b=1.0, and this is what was done here.

Figures (G) and (H) display $\hat{\lambda}_L$ for various values of σ and α so that one can see the effects of varying σ and α. Here again it is seen that for fixed α, an increase in σ causes the estimator to become more data-sensitive and vice versa.

5.4 CONCLUDING REMARKS

Chapter 2 presents various exact methods of solving the Fredholm integral equation of type II encountered in our quest for the optimal linear estimator of the intensity function, λ, of a nonstationary Poisson process. Several approximation techniques are also presented in chapter 3. These chapters are intended as a survey, necessarily incomplete, of some of the available methods of solving the Fredholm type equations which frequently occur in many areas of applied mathematics.

We agree with Grandell (10), Whittle (22), and Clevenson and Zidek (7) that only the optimal linear estimator, $\hat{\lambda}_L$, which is Bayes with respect to a restricted class of linear estimation, is generally applicable when λ is any second order process. Thus, in chapters 1, 4 and 5 the main emphasis was placed on obtaining the optimal linear estimator, $\hat{\lambda}_L$, which involves the numerical solution of equation (1.2.6).

In the numerical examples, a special form of the kernel was chosen to reflect our belief about the covariance structure. The calculations required to obtain $\hat{\lambda}_L$ in the numerical example where

the assumption of constant mean, $m(t)=\mu$, is dropped are much more extensive than those required for the simple case where the assumption is upheld.

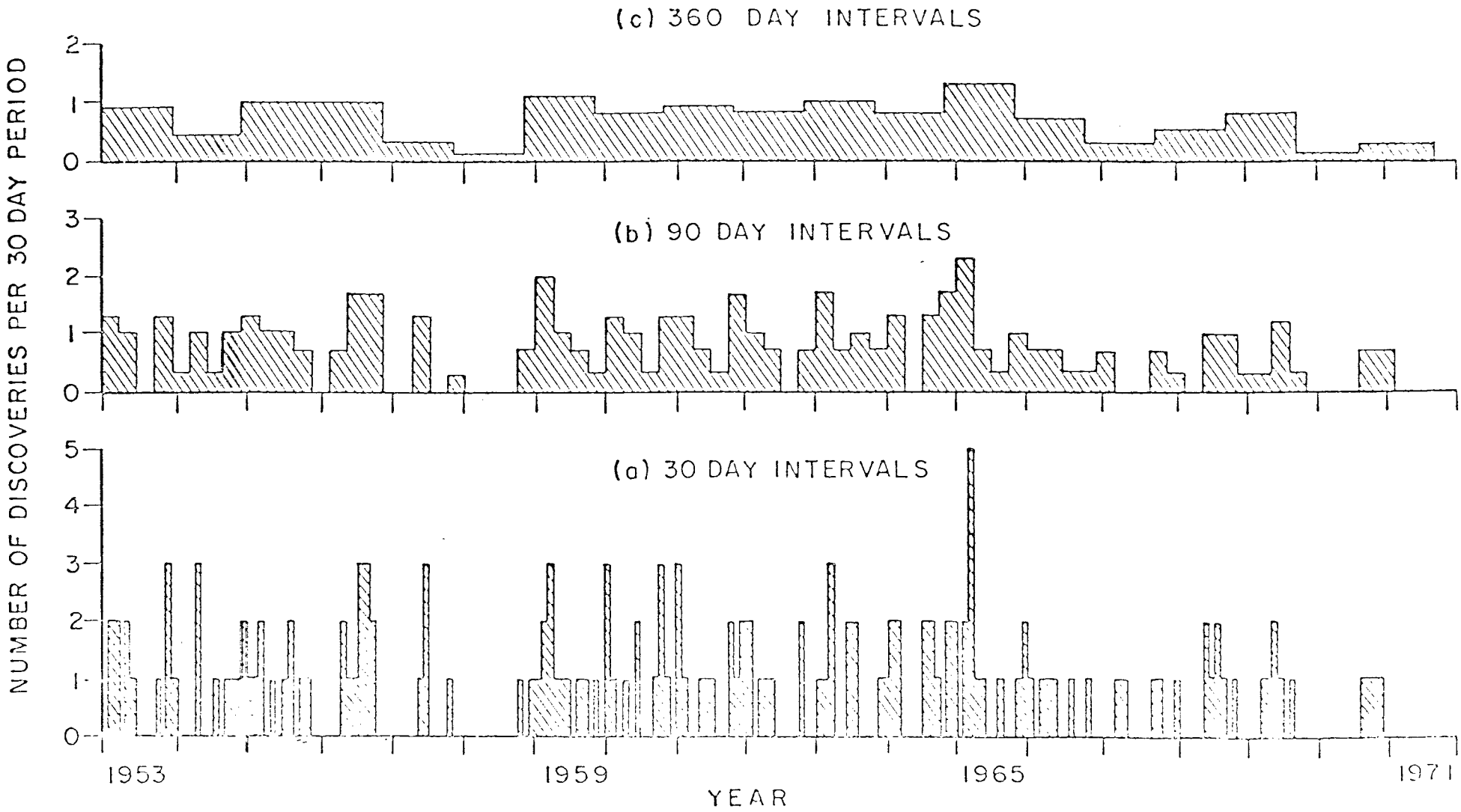
To the author's knowledge, the exact techniques considered here had not been applied to real-life situations before this work was started, even though the methods had frequently been suggested in the literature. It is gratifying to note that in chapter 4, a comparison of the approximate estimator, $\hat{\lambda}_{\infty}$, and the exact optimal linear estimator, $\hat{\lambda}_{L}$, shows that the bound, given by Clevenson and Zidek (7, p. 21), is satisfied in the case of the oilwell discovery example.

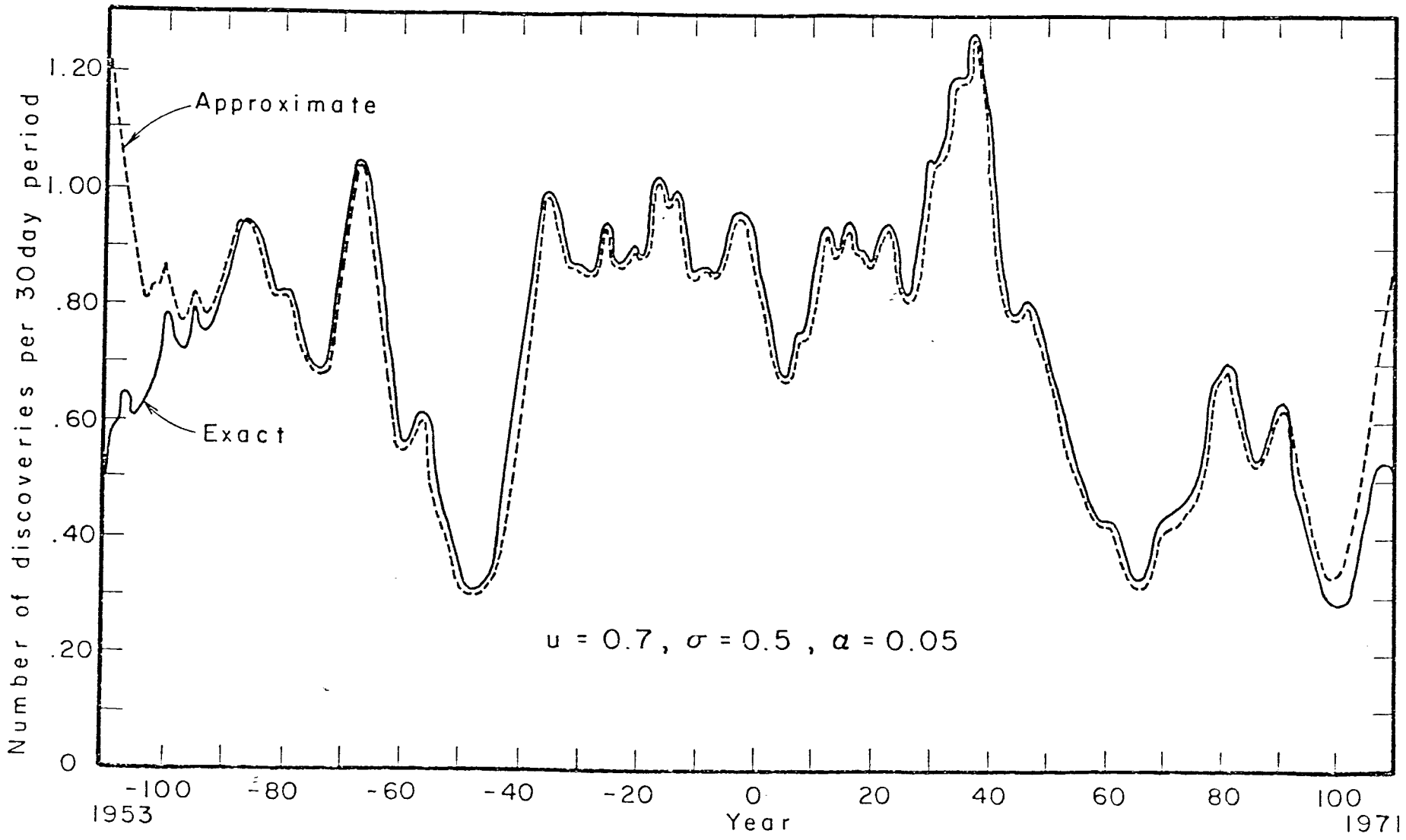
Another interesting observation is that, in choosing $\hat{\mu}$, $\hat{\sigma}^2$, α and the other constants which occur in our model, the Bayesian approach is a very useful tool. At least it is if one may use empirical Bayes methods as was done here.

It is worth noting that in some situations it is much more satisfactory not to assume that the mean is constant since, from our results, it appears that such an assumption may cause the estimators of λ to become more data-sensitive. A word of caution is in order here about the choice of the mean function, $m(t)$; an "off-the-mark" choice is very likely to yield intangible results. A careful examination of the data, at least in graphical form, is usually helpful. In many special cases, however, one may safely maintain the assumption that $m(t)=\mu$, a constant. Also one

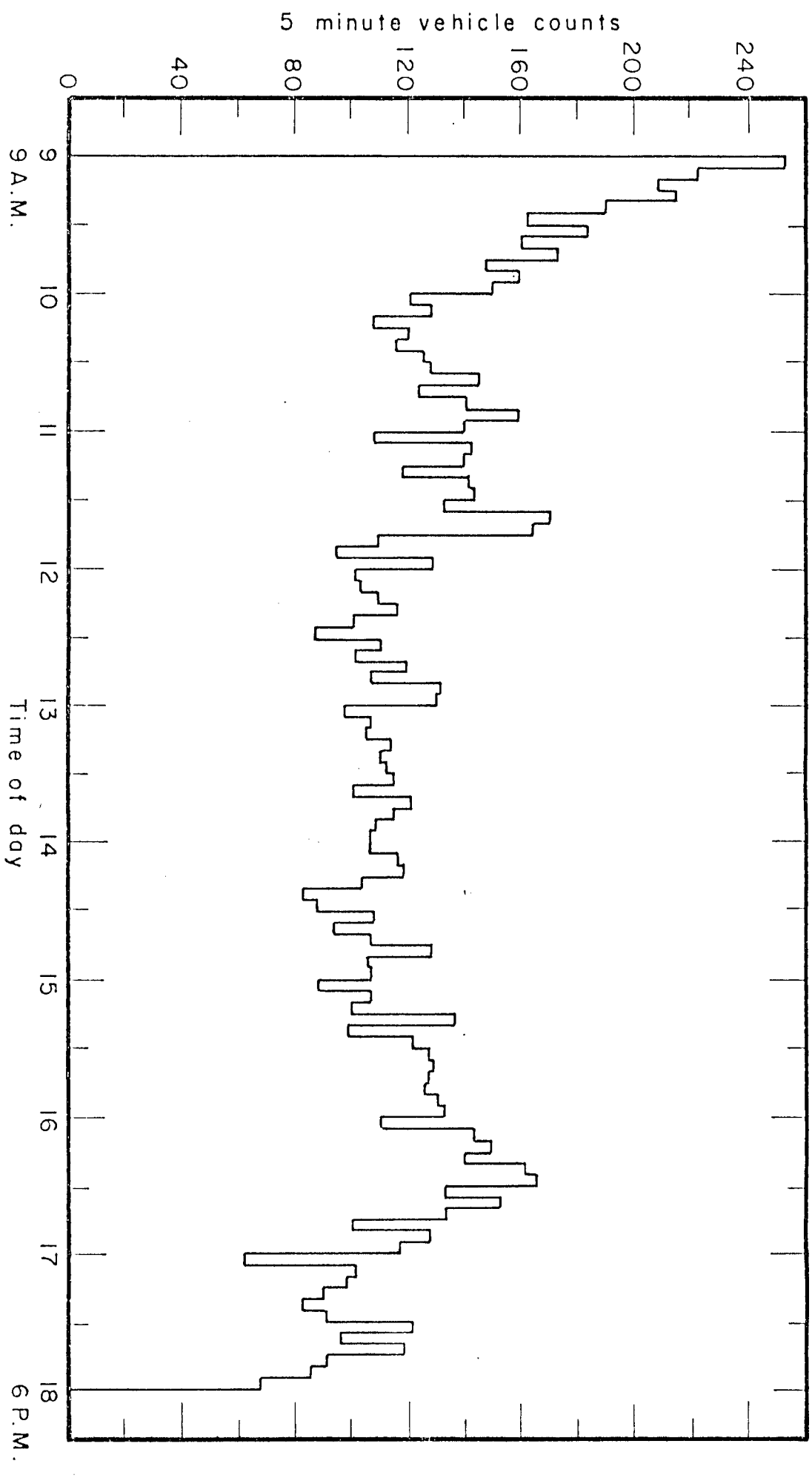
may use the large time approximate estimator, $\hat{\lambda}_{\infty}$, instead of the optimal linear estimator, $\hat{\lambda}_{L}$, since these methods often yield fairly good results for the estimation of the intensity function, $\lambda(t)$.

A. Histogram estimators of the intensity of wildcat oil well discoveries using class widths of (a) 30 days, (b) 90 days and (c) 360 days.

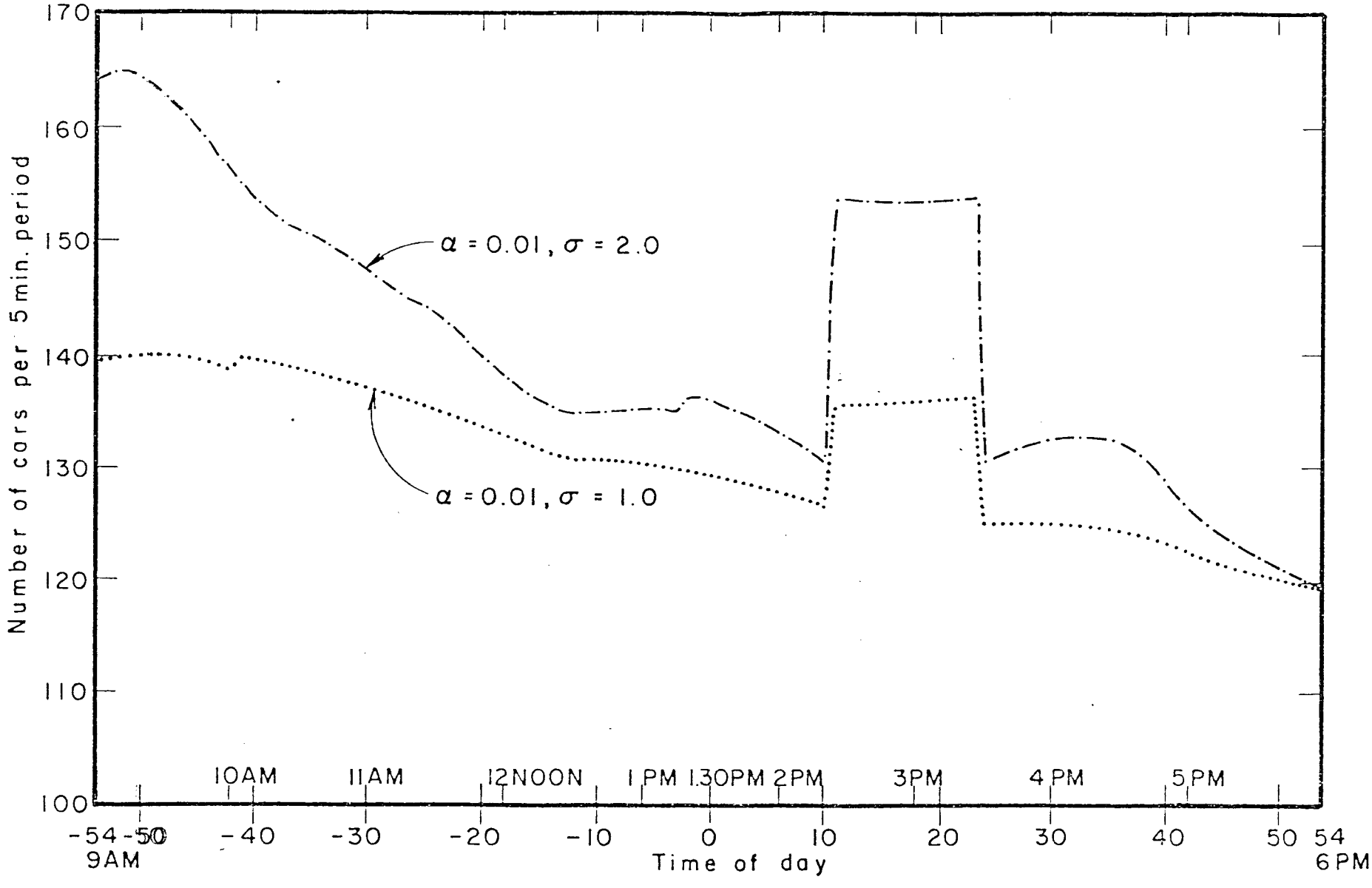




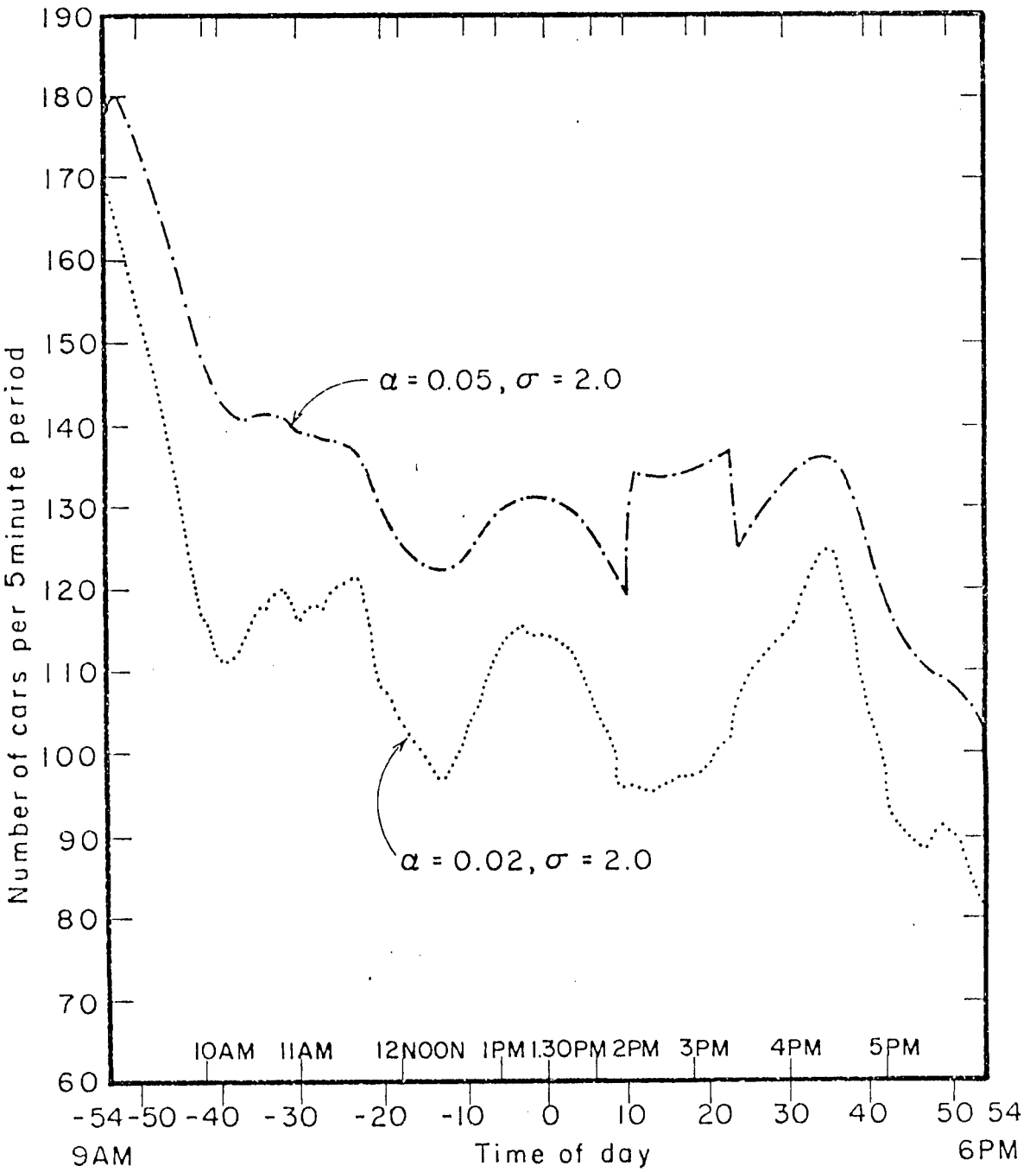
B. A comparison of the exact optimal linear estimator and the approximate solution of the intensity function of the oilwell discovery process.



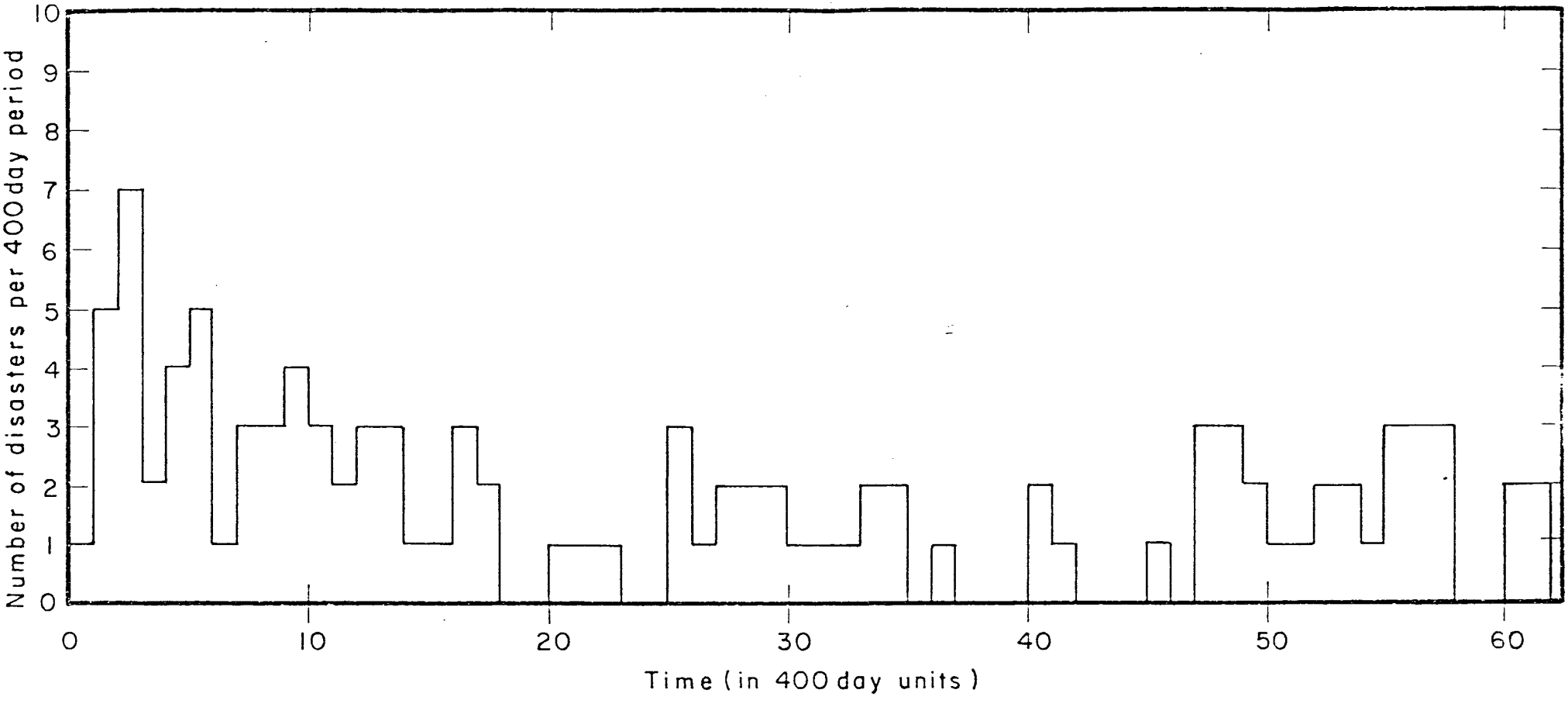
C. 5 minute vehicle counts on the Lions Gate bridge.



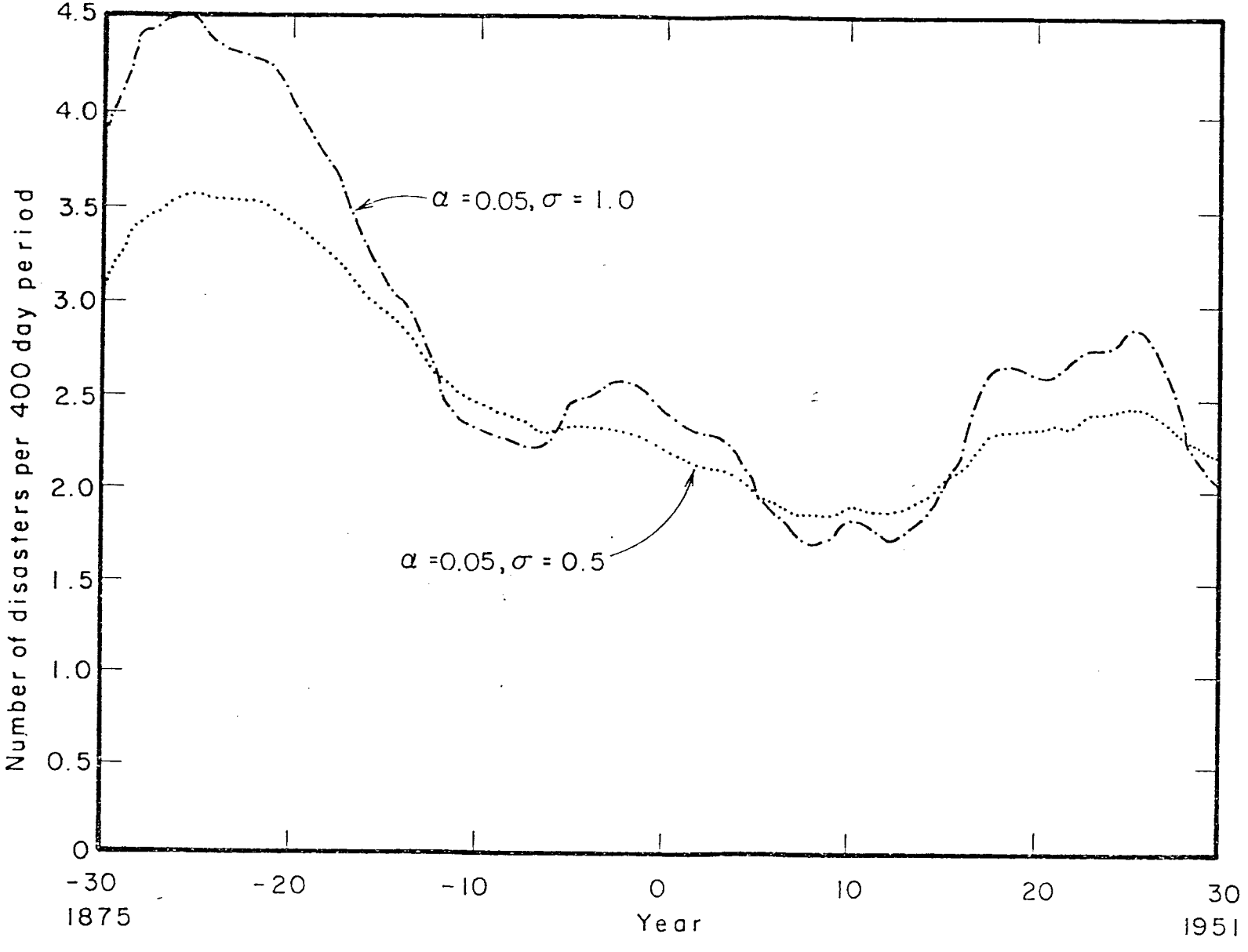
D. The optimal linear estimator of the intensity function of the Lions Gate bridge process using various prior parameters.



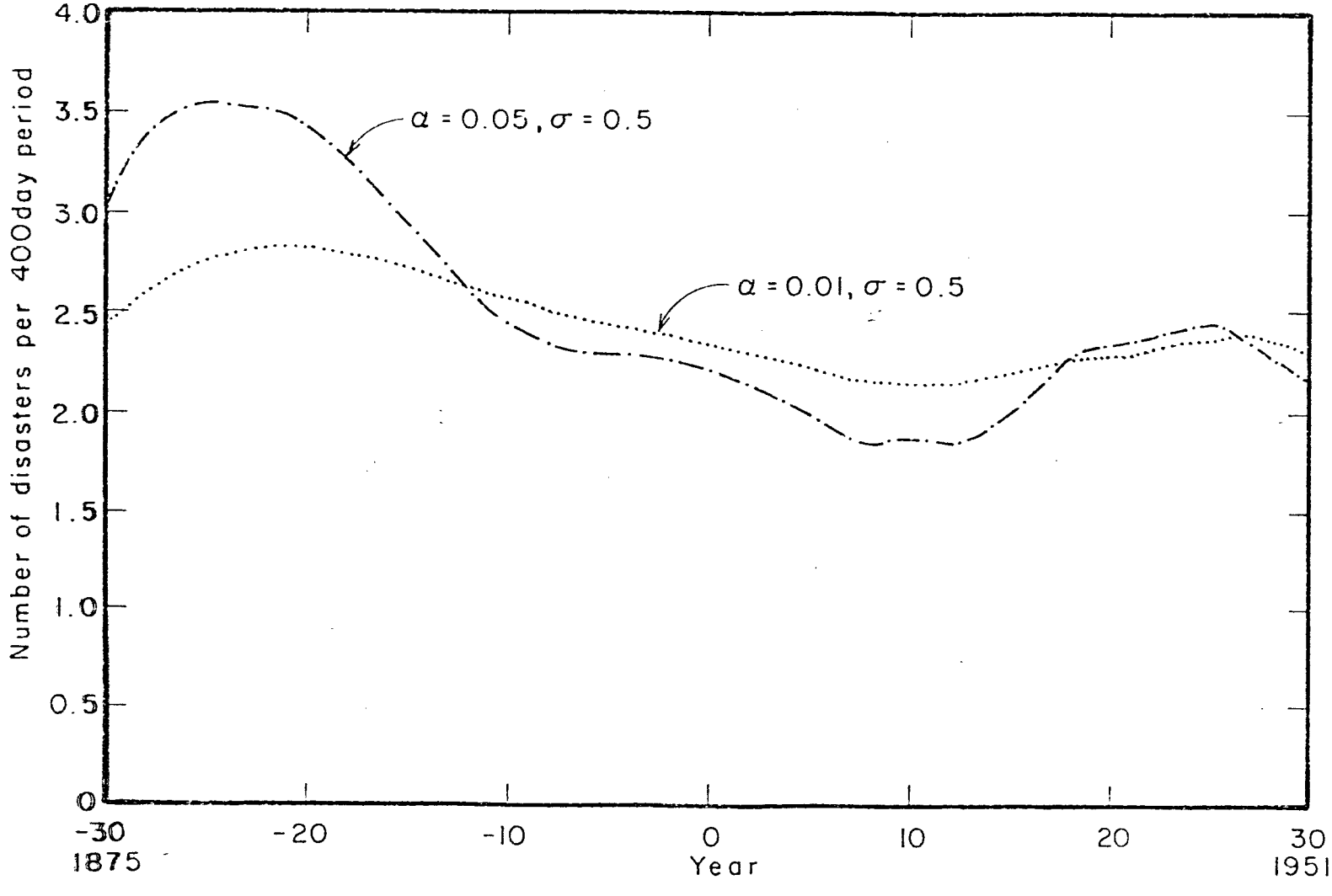
E. The optimal linear estimator of the intensity function of the Lions Gate bridge process using various prior parameters.



F. Coal-mining disasters in Great Britain for the period 1875-1951.
Numbers in successive 400-day periods.



G. The optimal linear estimator of the intensity function of the mining disaster process using various prior parameters.



H. The optimal linear estimator of the intensity function of the mining disaster process using various prior parameters.

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APPENDIX 1

COMPUTER PROGRAM: THE OPTIMAL LINEAR ESTIMATOR OF THE INTENSITY FUNCTION OF THE OILWELL DISCOVERY PROCESS

```
C     ***        OILWELL DISCOVERY    ***
C     ***        EXACT SOLUTION USING WHITTLE'S SUGGESTION     ***
      REAL MU,SIGMA,ALFA,EXACT
      DIMENSION P(120), DS(120), EXACT(225)
      DIMENSION TM(225), CEX(225), CAT(225)
      READ2, MU, SIGMA, ALFA
    2 FORMAT(3F4.2)
      N=108
      READ4, (P(I), DS(I), I=1,N)
    4 FORMAT(F5.0, F3.0)
      WK = ALFA*SIGMA*SIGMA/MU
      TTA = SQRT(2.*WK + ALFA*ALFA)
      VK = WK/TTA
      PRINT6, WK,TTA,VK
    6 FORMAT(' ',3F10.4)
      UT1 = 110.5
      UT =  221.
      T= -110.
      K=0
    8 IF(T.GT.110.) GOTO 20
      SUM= 0.
      J = 0
   10 J = J+1
```

```
      IF(J.GT.N) GOTO 12
      R = ABS(T-P(J))
      S=T+P(J)
      PA1=-TTA*R
      IF(PA1.LT.-100.) A=0.
      A=EXP(PA1)
      PA2=-TTA*(UT+S)
      IF(PA2.LT.-100.) B=0.
      B=EXP(PA2)
      PA3=-TTA*(UT-S)
      IF(PA3.LT.-100.) C=0
      C= EXP(PA3)
      ABC = A+ B + C
      IF(ABC.EQ.O.) GOTO 10
      SUM=SUM+DS(J)*ABC
      GOTO 10
12    CVA = VALU(TTA,UT1,T)
      ADVA = MU*VK*(1. - CVA)
      K = K + 1
      EXACT(K) = VK*SUM + ADVA
      PRINT14, T, EXACT(K)
14    FORMAT(' ', F6.0,F10.2)
      T = T + 1
      GOTO 8
20    NT=221
      M=NT - 1
      N=NT
```

```
      TM(1)=1.
      DO 22 I=1,M
      TM(I+1)=TM(I)+1.
22    CONTINUE
      DO 24 I=1,NT
      CAT(I)=TM(I)
      CEX(I)=EXACT(I)
24    CONTINUE
C     *** NOW CALL SUBROUTINE AJOA TO DO  THE PLOTTING ***
      CALL AJOA(CAT,CEX,NT)
C     *  TERMINATE PLOTTING AND STOP **
      CALL PLOTND
      STOP
      END
C
C
      SUBROUTINE AJOA(X, Y, N)
      DIMENSION X(N), Y(N)
      CALL SCALE(X,N,10.0,XMIN,DX,1)
      CALL SCALE(Y,N,10.0,YMIN,DY,1)
      CALL AXIS(0.,0., 'TIME', -4, 10., 0., XMIN, DX)
      CALL AXIS(0., 0., 'EXACT', 5, 10., 90., YMIN, DY)
      CALL LINE(X,Y,N,1)
      CALL PLOT(12.0, 0., -3)
      RETURN
      END
C
```

```
C
      FUNCTION VALU(X,Y,Z)
      REAL X,Y,Z
      A1= -X*ABS(-Y-Z)
      B1= -X*ABS(Y-Z)
      IF(A1.LT.-100..AND.B1.LT.-100.) PF = 0.
      IF(A1.GE.-100..AND.B1.LT.-100.) PF=EXP(A1)
      IF(A1.LT.-100..AND.B1.GE.-100.) PF=-EXP(B1)
      IF(A1.GE.-100..AND.B1.GE.-100.) PF=EXP(A1)-EXP(B1)
      A2= -X*(Z+Y)
      B2 = -X*(3*Y +Z)
      IF(A2.LT.-100..AND.B2.LT.-100.) QF=0.
      IF(A2.GE.-100..AND.B2.LT.-100.) QF=EXP(A2)
      IF(A2.LT.-100..AND.B2.GE.-100.) QF= -EXP(B2)
      IF(A2.GE.-100..AND.B2.GE.-100.) QF=EXP(A2) - EXP(B2)
      A3 = -X*(Y-Z)
      B3 = -X*(3*Y-Z)
      IF(A3.LT.-100..AND.B3.LT.-100.) RF = 0.
      IF(A3.GE.-100..AND.B3.LT.-100.) RF= EXP(A3)
      IF(A3.LT.-100..AND.B3.GE.-100.) RF= -EXP(B3)
      IF(A3.GE.-100..AND.B3.GE.-100.) RF=EXP(A3) - EXP(B3)
      VALU = (PF + QF + RF)/X + X
      RETURN
      END
```

COMPUTER OUTPUT: OILWELL DISCOVERIES;

THE APPROXIMATE AND THE EXACT LINEAR ESTIMATORS, $\hat{\lambda}_{\infty}$ AND $\hat{\lambda}_{L}$.

RESULTS FOR $\sigma=0.5$, $\alpha=0.05$.

T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$	T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$
-110.	1.21	0.50	-89.	0.88	0.88
-109.	1.16	0.58	-88.	0.91	0.91
-108.	1.08	0.60	-87.	0.94	0.94
-107.	1.03	0.64	-86.	0.93	0.93
-106.	0.96	0.63	-85.	0.92	0.92
-105.	0.88	0.61	-84.	0.91	0.91
-104.	0.83	0.62	-83.	0.86	0.86
-103.	0.81	0.64	-82.	0.84	0.84
-102.	0.83	0.68	-81.	0.81	0.82
-101.	0.83	0.72	-80.	0.82	0.82
-100.	0.87	0.78	-79.	0.81	0.82
-99.	0.83	0.76	-78.	0.77	0.77
-98.	0.79	0.73	-77.	0.75	0.76
-97.	0.77	0.72	-76.	0.73	0.73
-96.	0.78	0.74	-75.	0.69	0.69
-95.	0.82	0.79	-74.	0.68	0.68
-94.	0.79	0.76	-73.	0.69	0.70
-93.	0.78	0.76	-72.	0.73	0.73
-92.	0.80	0.78	-71.	0.79	0.80
-91.	0.81	0.80	-70.	0.89	0.89
-90.	0.85	0.84	-69.	0.94	0.94

T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$	T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$
-65.	0.93	0.94	-40.	0.60	0.61
-64.	0.81	0.81	-39.	0.67	0.68
-63.	0.71	0.71	-38.	0.77	0.77
-62.	0.64	0.64	-37.	0.86	0.86
-61.	0.59	0.59	-36.	0.94	0.94
-60.	0.56	0.56	-35.	0.98	0.99
-59.	0.55	0.55	-34.	0.96	0.96
-58.	0.56	0.56	-33.	0.93	0.93
-57.	0.59	0.59	-32.	0.90	0.90
-56.	0.60	0.60	-31.	0.86	0.87
-55.	0.53	0.53	-30.	0.86	0.87
-54.	0.47	0.48	-29.	0.86	0.86
-53.	0.43	0.44	-28.	0.85	0.85
-52.	0.41	0.42	-27.	0.86	0.87
-51.	0.37	0.37	-26.	0.88	0.88
-50.	0.33	0.34	-25.	0.93	0.93
-49.	0.31	0.32	-24.	0.90	0.90
-48.	0.30	0.31	-23.	0.87	0.87
-47.	0.30	0.30	-22.	0.87	0.87
-46.	0.31	0.31	-21.	0.87	0.87
-45.	0.32	0.33	-20.	0.89	0.90
-44.	0.35	0.36	-19.	0.88	0.89
-43.	0.39	0.40	-18.	0.90	0.91
-42.	0.44	0.45	-17.	0.96	0.96
-41.	0.51	0.52	-16.	1.01	1.01

T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$	T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$
-15.	0.99	0.99	10.	0.77	0.78
-14.	0.97	0.98	11.	0.83	0.83
-13.	0.99	0.99	12.	0.88	0.88
-12.	0.93	0.94	13.	0.92	0.93
-11.	0.88	0.88	14.	0.89	0.90
-10.	0.85	0.86	15.	0.90	0.90
-9.	0.86	0.86	16.	0.93	0.94
-8.	0.86	0.86	17.	0.93	0.94
-7.	0.85	0.86	18.	0.89	0.90
-6.	0.85	0.85	19.	0.89	0.89
-5.	0.87	0.87	20.	0.87	0.88
-4.	0.92	0.93	21.	0.90	0.90
-3.	0.94	0.94	22.	0.91	0.92
-2.	0.95	0.96	23.	0.93	0.93
-1.	0.93	0.93	24.	0.91	0.91
0.	0.86	0.87	25.	0.84	0.85
1.	0.83	0.84	26.	0.81	0.82
2.	0.80	0.80	27.	0.81	0.82
3.	0.76	0.76	28.	0.84	0.84
4.	0.70	0.71	29.	0.89	0.90
5.	0.68	0.68	30.	0.98	0.99
6.	0.67	0.68	31.	1.04	1.04
7.	0.70	0.70	32.	1.06	1.07
8.	0.74	0.75	33.	1.08	1.09
9.	0.74	0.75	34.	1.15	1.15

T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$	T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$
35.	1.18	1.19	60.	0.42	0.42
36.	1.18	1.19	61.	0.42	0.43
37.	1.23	1.24	62.	0.40	0.41
38.	1.26	1.26	63.	0.37	0.37
39.	1.15	1.15	64.	0.34	0.34
40.	1.04	1.05	65.	0.32	0.33
41.	0.94	0.94	66.	0.32	0.32
42.	0.87	0.88	67.	0.32	0.33
43.	0.83	0.84	68.	0.34	0.34
44.	0.79	0.80	69.	0.37	0.37
45.	0.78	0.78	70.	0.41	0.41
46.	0.79	0.80	71.	0.42	0.43
47.	0.80	0.80	72.	0.42	0.42
48.	0.76	0.77	73.	0.43	0.44
49.	0.71	0.72	74.	0.45	0.46
50.	0.70	0.70	75.	0.46	0.46
51.	0.67	0.67	76.	0.48	0.49
52.	0.62	0.63	77.	0.52	0.52
53.	0.57	0.57	78.	0.57	0.58
54.	0.53	0.53	79.	0.65	0.65
55.	0.51	0.52	80.	0.67	0.68
56.	0.48	0.48	81.	0.69	0.69
57.	0.46	0.46	82.	0.66	0.66
58.	0.45	0.46	83.	0.61	0.62
59.	0.43	0.43	84.	0.59	0.59

T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$	T	$\hat{\lambda}_{\infty}(T)$	$\hat{\lambda}_{L}(T)$
85.	0.55	0.55	98.	0.34	0.31
86.	0.53	0.54	99.	0.33	0.29
87.	0.53	0.53	100.	0.34	0.29
88.	0.55	0.55	101.	0.35	0.29
89.	0.59	0.59	102.	0.38	0.30
90.	0.61	0.61	103.	0.42	0.32
91.	0.62	0.62	104.	0.47	0.36
92.	0.58	0.57	105.	0.54	0.40
93.	0.52	0.52	106.	0.63	0.46
94.	0.49	0.47	107.	0.71	0.50
95.	0.43	0.41	108.	0.78	0.52
96.	0.39	0.37	109.	0.83	0.52
97.	0.36	0.33	110.	0.89	0.50

APPENDIX 2

COMPUTER PROGRAM: THE OPTIMAL LINEAR ESTIMATOR OF THE INTENSITY FUNCTION OF THE LIONS GATE BRIDGE PROCESS

```
C          ** LIONS GATE BRIDGE EXACT SOLUTION **
C     **   DATA FROM 9A.M. TO 6 P.M.
           REAL MU,SIGMA,ALFA,FACT
C     **   FACT IS A MULTIPLYING FACTOR TO SCALE UP M(T) **
           DIMENSION P(120),DS(120),SK(10),EXACT(120)
           DIMENSION TM(120), CEX(109), CAT(109)
           READ 2, SIGMA,ALFA,FACT
      2    FORMAT(3F6.3)
           N=109
           M=9
           T=-54.
           UT=109.
           UT1=54.5
           READ4, (P(I), DS(I), I=1,N)
      4    FORMAT(F6.0,F6.0)
           READ5, (SK(I), I=1,M)
      5    FORMAT(F6.01)
           L=0
           K=0
      7    KT=0
           L=L+1
           Y=SK(L)
```

```
C     CALL FUNCTION UTVAR TO COMPUTE NEW M(T)
            MU=UTVAR(FACT,Y)
            RMT=SQRT(MU)
            WK=ALFA*SIGMA*SIGMA/MU
            TTA=SQRT(2.*WK + ALFA*ALFA)
            VK=WK/TTA
      8     IF(T.GT.54) GOTO 20
C       ** CHANGE M(T) AFTER EVERY 12TH INTERVAL **
            IF(KT.GT.12) GOTO 7
            SUM = 0.
            J=0
     10     J=J+1
            IF(J.GT.N) GOTO 12
            R=ABS(T-P(J))
            S=T+P(J)
            PA1=-TTA*R
            IF(PA1.LT.-100.) A=0
            A=EXP(PA1)
            PA2=-TTA*(UT+S)
            IF(PA2.LT.-100.) B=0.
            B=EXP(PA2)
            PA3=-TTA*(UT-S)
            IF(PA3.LT.-100.) C=0.
            C=EXP(PA3)
            ABC= A+B+C
            IF(ABC.EQ.0.) GOTO 10
            SUM = SUM + DS(J)*ABC
```

```
      GOTO 10
C     **   TO COMPUTE XT(S)
   12 CVA = VALU(TTA, UT1, T)
C     **   TO COMPUTE HT(S) AFTER CALCULATING XT(S).  **
      HAVE = CVA/RMT
      ADVA = MU*VK*(1. - HAVE)
      K=K+1
      EXACT(K)=VK*SUM + ADVA + MU
      PRINT14, T, EXACT(K)
   14 FORMAT(' ', F6.0,F10.2)
      KT=KT+1
      T=T+1
      GOTO 8
   20 NT=109
      TM(1)=1.
      DO 21 I=1,N
      TM(I+1)=TM(I)+1.
   21 CONTINUE
      DO 22 I=1,NT
      CAT(I)=TM(I)
      CEX(I)=EXACT(I)
   22 CONTINUE
C     **   NOW CALL SUBROUTINE AJOA TO DO THE PLOTTING **
      CALL AJOA(CAT,CEX,NT)
C     *    TERMINATE PLOTTING AND STOP **
      CALL PLOTND
      STOP
```

```
      END
C
      SUBROUTINE AJOA(X, Y, N)
      DIMENSION X(N), Y(N)
      CALL SCALE(X,N,10.0,XMIN,DX,1)
      CALL SCALE(Y,N,10.0,YMIN,DY,1)
      CALL AXIS(0.,0., 'TIME', -4, 10., 0., XMIN, DX)
      CALL AXIS(0., 0., 'EXACT', 5, 10., 90., YMIN, DY)
      CALL LINE(X,Y,N,1)
      CALL PLOT(12.0, 0., -3)
      RETURN
      END
C
C
      FUNCTION UTVAR(X,Y)
      REAL X,Y
      PRO=X*Y
      UTVAR=SQRT(PRO)
      RETURN
      END
C
      FUNCTION VALU(X,Y,Z)
      REAL X,Y,Z
      A1= -X*ABS(-Y-Z)
      B1= -X*ABS(Y-Z)
      IF(A1.LT.-100..AND.B1.LT.-100.) PF = 0.
      IF(A1.GE.-100..AND.B1.LT.-100.) PF=EXP(A1)
```

```
IF(A1.LT.-100..AND.B1.GE.-100.) PF=-EXP(B1)
IF(A1.GE.-100..AND.B1.GE.-100.) PF=EXP(A1)-EXP(B1)
A2= -X*(Z+Y)
B2 = -X*(3*Y +Z)
IF(A2.LT.-100..AND.B2.LT.-100.) QF=0.
IF(A2.GE.-100..AND.B2.LT.-100.) QF=EXP(A2)
IF(A2.LT.-100..AND.B2.GE.-100.) QF= -EXP(B2)
IF(A2.GE.-100..AND.B2.GE.-100.) QF=EXP(A2) - EXP(B2)
A3 = -X*(Y-Z)
B3 = -X*(3*Y-Z)
IF(A3.LT.-100..AND.B3.LT.-100.) RF = 0.
IF(A3.GE.-100..AND.B3.LT.-100.) RF= EXP(A3)
IF(A3.LT.-100..AND.B3.GE.-100.) RF= -EXP(B3)
IF(A3.GE.-100..AND.B3.GE.-100.) RF=EXP(A3) - EXP(B3)
VALU = (PF + QF + RF)/X + X
RETURN
END
```

COMPUTER OUTPUT: LIONS GATE BRIDGE PROCESS

THE OPTIMAL LINEAR ESTIMATOR, $\hat{\lambda}_L$.

RESULTS FOR $\sigma=2.0$, $\alpha=0.2$.

T	$\hat{\lambda}_L(T)$	T	$\hat{\lambda}_L(T)$
-54.	167.99	-33.	118.86
-53.	167.06	-32.	119.81
-52.	164.14	-31.	118.14
-51.	160.44	-30.	116.04
-50.	154.74	-29.	117.08
-49.	148.83	-28.	117.80
-48.	145.01	-27.	117.38
-47.	140.18	-26.	119.11
-46.	136.26	-25.	120.07
-45.	131.25	-24.	120.30
-44.	127.12	-23.	121.23
-43.	121.96	-22.	118.29
-42.	116.97	-21.	111.63
-41.	114.64	-20.	107.17
-40.	111.57	-19.	106.10
-39.	111.05	-18.	103.75
-38.	111.39	-17.	102.03
-37.	113.03	-16.	100.53
-36.	114.99	-15.	99.50
-35.	117.48	-14.	97.59
-34.	117.46	-13.	96.70

T	$\hat{\lambda}_L(T)$	T	$\hat{\lambda}_L(T)$
-12.	98.48	13.	95.62
-11.	100.00	14.	95.71
-10.	102.80	15.	96.28
-9.	104.89	16.	96.85
-8.	108.44	17.	96.80
-7.	110.80	18.	96.85
-6.	112.50	19.	96.94
-5.	113.69	20.	97.86
-4.	114.57	21.	98.90
-3.	115.16	22.	100.45
-2.	113.98	23.	101.10
-1.	114.13	24.	105.68
0.	114.05	25.	108.08
1.	113.75	26.	109.82
2.	113.05	27.	111.08
3.	111.85	28.	112.17
4.	110.20	29.	113.41
5.	108.18	30.	114.50
6.	106.06	31.	115.23
7.	103.85	32.	118.75
8.	101.46	33.	121.37
9.	98.84	34.	122.75
10.	95.49	35.	124.38
11.	95.90	36.	123.76
12.	95.41	37.	119.45

T	$\hat{\lambda}_L(T)$	T	$\hat{\lambda}_L(T)$
38.	116.63	47.	88.13
39.	111.52	48.	89.70
40.	105.69	49.	91.04
41.	102.30	50.	89.87
42.	97.44	51.	89.08
43.	91.68	52.	85.83
44.	90.67	53.	82.93
45.	89.47	54.	80.64
46.	88.36		

APPENDIX 3

COMPUTER PROGRAM: THE OPTIMAL LINEAR ESTIMATOR OF THE INTENSITY FUNCTION OF THE COAL-MINING DISASTER PROCESS

```
C       ***  COAL MINING DISASTERS * EXACT SOLUTION ***
C
        REAL MU, SIGMA, ALFA, DIV, FACT
        DIMENSION P(100), DS(100), SK(100), EXACT(100)
        DIMENSION TM(100), CEX(60), CAT(60)
        READ2, SIGMA, ALFA, DIV, FACT, AA, BB
    2   FORMAT(6F6.2)
        N = 60
        M = 60
        T = -30.
        UT = 61.
        UT1 = 30.5
        READ4, (P(I), DS(I), I = 1,N)
    4   FORMAT(F6.0, F6.0)
        DO 5 I = 1,N
        SK(I) = P(I)/DIV
    5   CONTINUE
        L = 0
        K = 0
    8   IF(T.GE.30.) GOTO 20
        L = L + 1
        Y=AA*SK(L)
```

```
      Z=BB*SK(L)
C     ***  CALL FUNCTION UTVAR TO COMPUTE NEW M(T)  ***
      MU=UTVAR(FACT,Y,Z)
      QMT=SQRT(MU)
      WK = ALFA*SIGMA*SIGMA/MU
      TTA = SQRT(2.*WK + ALFA*ALFA)
      VK = WK/TTA
      SUM= 0.
      J = 0
   10 J = J+1
      IF(J.GT.N) GOTO 12
      R = ABS(T-P(J))
      S=T+P(J)
      PA1=-TTA*R
      IF(PA1.LT.-100.) A=0.
      A=EXP(PA1)
      PA2=-TTA*(UT+S)
      IF(PA2.LT.-100.) B=0.
      B=EXP(PA2)
      PA3=-TTA*(UT-S)
      IF(PA3.LT.-100.) C=0.
      C= EXP(PA3)
      ABC = A+ B + C
      IF(ABC.EQ.0.) GOTO 10
      SUM=SUM+DS(J)*ABC
      GOTO 10
C     TO COMPUTE XT(S)
```

```
   12    CVA = VALU(TTA,UT1,T)
C        TO COMPUTE HT(S)
         QCVA = CVA/QMT
         ADVA = MU*VK*(1. - QCVA)
         K = K + 1
         EXACT(K) = VK*SUM + ADVA + MU
         PRINT14, T, EXACT(K)
   14    FORMAT(' ', F6.0,F10.2)
         T = T + 1
         GOTO 8
   20    NT = 60
         TM(1) = 1.
         DO 21 I = 1,N
         TM(I+1) = TM(I) + 1.
   21    CONTINUE
         DO 22 I = 1,NT
         CAT(I)=TM(I)
         CEX(I)=EXACT(I)
   22    CONTINUE
C        *** NOW CALL SUBROUTINE AJOA TO DO THE PLOTTING ***
         CALL AJOA(CAT,CEX,NT)
C    *   TERMINATE PLOTTING AND STOP **
         CALL PLOTND
         STOP
         END
C
```

```
      SUBROUTINE AJOA(X, Y, N)
      DIMENSION X(N), Y(N)
      CALL SCALE(X,N,10.0,XMIN,DX,1)
      CALL SCALE(Y,N,10.0,YMIN,DY,1)
      CALL AXIS(0.,0., 'TIME', -4, 10., 0., XMIN, DX)
      CALL AXIS(0., 0., 'EXACT', 5, 10., 90., YMIN, DY)
      CALL LINE(X,Y,N,1)
      CALL PLOT(12.0, 0., -3)
      RETURN
      END
C
C
      FUNCTION UTVAR(X, Y, Z)
      REAL X,Y,Z
      A = EXP(Y)
      B= EXP(-Z)
      C = A + B
      CASH = X*C
      UTVAR = CASH
      RETURN
      END
C
```

T	$\hat{\lambda}_L(T)$	T	$\hat{\lambda}_L(T)$
-30.	3.05	-5.	2.32
-29.	3.25	-4.	2.31
-28.	3.40	-3.	2.31
-27.	3.47	-2.	2.29
-26.	3.53	-1.	2.26
-25.	3.56	0.	2.21
-24.	3.54	1.	2.17
-23.	3.53	2.	2.13
-22.	3.52	3.	2.10
-21.	3.48	4.	2.05
-20.	3.42	5.	1.97
-19.	3.33	6.	1.92
-18.	3.25	7.	1.87
-17.	3.15	8.	1.85
-16.	3.03	9.	1.85
-15.	2.93	10.	1.88
-14.	2.84	11.	1.87
-13.	1.71	12.	1.87
-12.	2.59	13.	1.89
-11.	2.50	14.	1.94
-10.	2.44	15.	2.02
-9.	2.38	16.	2.10
-8.	2.34	17.	2.22
-7.	2.30	18.	2.28
-6.	1.20	19.	2.31

T	$\hat{\lambda}_L(T)$	T	$\hat{\lambda}_L(T)$
20.	2.32	26.	2.42
21.	2.34	27.	2.36
22.	2.38	28.	2.27
23.	2.40	29.	2.22
24.	2.41	30.	2.15
25.	2.43		