AYMPTOTIC ANALYSIS OF INTERACTION OF A SURFACE WAVE WITH TWO INTERNAL WAVES

by

MIRMOSADEGH JAMALI

B.A.Sc., Sharif University of Technology, 1987
M.A.Sc., Sharif University of Technology, 1990
Ph.D., The University of British Columbia, 1998

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Abstract

The motion of a surface wave in a two-layer fluid can lead to generation of two internal waves through a resonance mechanism under certain circumstances. Two subjects related to this interaction are studied theoretically here. These are the behavior of the waves in three-dimensional interaction when the density difference between the two layers is small, and the effect of a diffuse interface on the interaction.

In the first study, the three-dimensional interaction of a surface wave with two oblique internal waves is analyzed asymptotically in an attempt to obtain simple approximate expressions for the growth rate as well as the kinematic properties of the internal waves. The non-dimensional density difference δ is taken as a perturbation parameter, and the first few terms in the expansions of the desired quantities are derived. The results indicate that the internal-wave numbers are $O(\delta^{-1})$, one order larger than the surface-wave number. Also, at leading order the internal wave frequencies are equal to $\omega_0 / 2$, and the directions of the two internal waves differ by 180°. An important finding is that an immediate consequence of taking δ as a small parameter is that the internal waves become deep-water waves in both layers. According to the asymptotic analysis, the interaction coefficients α_1 and α_2 are $O(1)$ and are equal at leading order.

The second study concerns the generation of two internal waves by a surface wave on a thin diffuse interface. As in the first analysis, the non-dimensional density difference δ is taken as a small perturbation parameter. In addition, it is assumed that the diffuse interface is small compared to the internal wavelengths by taking it to be order δ^2. A three-layer system admits two modes of internal wave motion, and similarly two modes of interaction are found possible through the analysis. These are interaction between a surface wave and two first-

mode internal waves, and interaction between a surface wave, a first-mode and a second-mode internal wave. It is shown that, contrary to the first mode, in the second mode of interaction the waves are not sub-harmonic to the surface wave. An important finding is that the growth rate in the first mode is higher than in the second. This implies that in a real situation the interaction between a surface wave and two first-mode internal waves has more chance to occur.

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List of Major Symbols

a_i	half of the amplitude of wave i at the free surface
α	mean forcing coefficient of the internal waves, equal to $\sqrt{\overline{\alpha}_1 \alpha_2}$
α_i	forcing coefficient of internal wave i
b_i	half of the amplitude of wave i at the interface
d	depth of the lower layer
δ	$(\rho_l - \rho_u)/\rho_m$
ε	non-dimensional wave amplitude
η	interface displacement
ϕ	potential in the lower layer
ϕ'	potential in the upper layer
ϕ''	potential in the middle layer
g	gravity constant
H	total depth of the layers
h	depth of the upper layer
h_ρ	half of the thickness of the interface
$Im[\ \ ]$	imaginary part
k_i	$\lvert\vec{k}_i\rvert$
$\vec{k}_i$	vector wave number of wave i
ξ	free-surface displacement
n	number of peaks and troughs of the standing internal wave across the flume
ϑ_i	$k_i\, x - \omega_i\, t$
ω_i	frequency of wave i
θ_i	direction angle of internal wave i with respect to the surface wave

$Re[\ ]$	real part
r	density ratio ρ_l / ρ_u
ρ_u	density of the upper layer
ρ_m	density of the middle layer
ρ_l	density of the lower layer
t	time
T_0	surface wave period
$\vec{x}$	(x, y)
x	horizontal coordinate
y	horizontal coordinate normal to x
z	vertical coordinate

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CHAPTER 1

INTRODUCTION

1.1 MOTIVATION

The layers of fluid mud found at the bottom of many lakes, estuaries, and coastal waters, and the unconsolidated sludge at the bottom of mine-tailings ponds can often be treated as viscous fluids. Re-suspension of material from these layers can be of significant practical importance. In coastal waters it can lead to the need for substantial dredging or sediment replenishment (U.S. Army Coastal Engineering Research Center, 1984; Mehta et al., 1994). In mine-tailings ponds it can cause blockages in processing plants when the pond water is recycled, and adverse environmental impacts if the pond water flows into natural water courses (Luettich et al., 1990; Lawrence et al., 1991). Re-suspension can be the result of surface wave action triggering instabilities at the interface between the fluid mud (or mine-tailings) and the overlying water.

To investigate the interfacial instabilities, the model of surface wave motion in a two-layer fluid can be adopted. This simplified problem was first studied by Wen (1995) in the context of surface wave motion over a highly viscous sub-layer. Wen's (1995) study was motivated by her qualitative observations of interfacial wave generation and growth by a surface wave over a fine-sediment bed in a laboratory flume. She found that a resonant interaction between the surface wave and two opposite-travelling internal waves leads to the instability of the interface.

The interaction was subsequently investigated by Hill and Foda (1996), and Hill (1997). The first work (Hill and Foda, 1996) was limited to a two-dimensional analysis of the interaction. In the subsequent work, Hill (1997) investigated the interaction both experimentally and theoretically and showed that the interaction has in fact a three-dimensional nature. More recently, Jamali (1998) studied the interaction both theoretically and experimentally,in an attempt to investigate the phenomenon in more detail and to address some issues surrounding Hill's (1997) results.

Jamali (1998) also made some recommendations for further study of the interaction and pointed out a few subjects in this regard. Two of them were the derivation of a simple relation for the growth rate of the internal waves and study of effect of a diffuse interface on the interaction. In the theoretical analysis of Jamali (1998), the equations for the growth rates of the internal waves were found quite long and complicated. Since the interaction is of great importance to the internal mixing in a two-layer fluid subject to surface wave motion, a simple approximate equation to predict the onset of the interfacial instability is needed. The second subject was set forth by the laboratory observation of Jamali (1998) that the diffuse interface of the two layers had an affect on the growth rate of the internal waves. In this study, these two subjects are studied on a theoretical basis. Before proceeding further, it is instructive to first review the past studies on resonant wave interaction and the interaction of a surface wave with two internal waves in particular.

1.2 WAVE INTERACTION

Nonlinear wave interactions are considered to be an important aspect of the dynamics of the oceans (Philips, 1977; Komen et al., 1994) and the atmosphere (Yi and Xiao, 1996). Of particular interest are resonant interactions, which are important in the redistribution of energy among wave modes with different spatial and temporal scales. To study the characteristics of this energy transfer, the theory of resonant wave interaction has been used extensively (Philips, 1981; McComas and Muller, 1981; Hammack and Henderson, 1993; Komen et al., 1994). In principle, the theory addresses the problem of wave generation by weakly non-linear interaction of a group of waves. Each wave in the group can be treated as linear, but when the waves satisfy certain resonance conditions, energy is interchanged preferentially between them (Turner, 1973).

Resonant wave interaction can be described as a non-linear process in which energy is transferred between different natural modes of an oscillatory system by resonance. Consider a non-linear system that is oscillating by one or more of its natural modes. As the system is non-linear, the motion is not simply a summation of the linear modes, but consists of the linear harmonics plus their non-linear coupling. Under resonance conditions the non-linear coupling between some modes may lead to excitation of a natural mode or modes. The behavior of this excited mode(s) depends on the properties of the original modes and the system. An interesting situation occurs when the created mode(s) grows rapidly in time, being of primary importance in studies of hydrodynamic stability.

It is well known that the development of water waves is non-linear in character, and resonant interactions are of particular importance in this regard (Komen et al., 1994). Two examples of such interactions are the generation of an internal wave by two surface waves (Ball, 1964) and the interaction of an internal wave with two higher-mode internal waves on a thin density interface (Davis and Acrivos, 1967). In the study of oceanic internal gravity waves a considerable amount of work has been based on the concept of resonant wave interaction (Hasselmann, 1966; McComas and Bretherton, 1977; McComas and Muller, 1981, Muller et al., 1986). Resonant interactions are a source of internal wave energy and a mechanism for surface wave modification. Furthermore, they contribute to the redistribution of energy among different modes in the spectrum of internal waves in a continuously stratified fluid (Philips 1981). For similar applications of the theory to atmospheric gravity waves, reference can be made to the works of Yeh and Liu (1981), Fritts et al. (1993), and Yi and Xiao (1996).

The phenomenon of resonant wave interaction was first studied by Philips (1960) and subsequently by Longuett-Higgins (1962). Textbooks by Drazin and Reid (1981), Craik (1985), and Komen et al. (1994) as well as articles by Philips (1981) and Hammack and Henderson (1993) give excellent reviews of the subject. Philips (1960) showed that energy can be exchanged among three deep-water surface waves 1, 2, and 3 provided their frequencies and wave numbers meet the following kinematic conditions.

$$\begin{aligned} 2\vec{k}_1 - \vec{k}_2 &= \vec{k}_3 \\ 2\omega_1 - \omega_2 &= \omega_3 \end{aligned} \tag{1.1}$$

where $\vec{k}_i$ and ω_i are the vector wave number and frequency of the i-th wave respectively. Equation 1.1 is called the kinematic conditions of resonance. In wave interaction problems, the resonance conditions are expressed in terms of certain relations between the wave numbers and the frequencies of the waves involved. Note that the form of equation 1.1 is specific to deep-water waves, and the resonance conditions may differ from one class of waves to another.

Philips (1960) showed that under the resonance conditions (1.1), the amplitude of the third wave, if initially infinitesimal, grows in time due to the transfer of energy from finite-amplitude waves 1 and 2. It should be noted that resonant wave interaction, in general, does not necessarily lead to instability (Hasselmann 1967).

Although the analysis by Philips (1960) showed the possibility of energy transfer between deep-water surface waves, it did not address their long-term behavior. Benny (1962) extended Philips's (1960) analysis by adopting the technique used by Bogoliubov and Mitropolski (1959) in non-linear oscillations. Benny (1962) was able to derive a complete set of equations for the time evolution of the wave amplitudes.

After Philips's (1960) work on deep-water waves, the idea of resonant wave interaction was soon extended to other classes of water waves: McGoldrick (1965, 1970, and 1972) studied interaction between capillary-gravity waves in a series of papers. McGoldrick studied the problem when the following conditions of resonance hold between three capillary-gravity waves 1, 2, and 3.

$$\begin{aligned} \vec{k}_1 &= \vec{k}_2 + \vec{k}_3 \\ \omega_1 &= \omega_2 + \omega_3 \end{aligned} \tag{1.2}$$

where $\omega = (gk + \gamma_{capillary} k^3)^{1/2}$, and $\gamma_{capillary}$ is the capillary constant for the interface of air and water. McGoldrick derived the following evolution equations for the wave amplitudes.

$$\begin{aligned} \frac{da_1}{dt} &= i\alpha\omega_1 a_2 a_3 \\ \frac{da_2}{dt} &= i\alpha\omega_2 a_1 \bar{a}_3 \\ \frac{da_3}{dt} &= i\alpha\omega_3 a_1 \bar{a}_2 \end{aligned} \tag{1.3}$$

where α is a constant, and a_i is the amplitude of the i-th wave. The over-bar denotes the complex conjugate.

Simmons (1969) used a variational method and obtained equations 1.3 more quickly. His work was inspired by Whitham's (1965, 1967) averaged Lagrangian method. Simmons (1969) formulated his method quite generally and showed how to find the evolution equations as well as conservation relations for a general wave interaction problem. A variational formulation systematizes and shortens the detailed calculations. This lessens the likelihood of making elementary errors in the long calculations of the interaction coefficients of the evolution equations. Variational formulation also leads more readily to conservation laws such as energy and momentum relations.

Ball (1964) was the first to investigate the resonant interaction in stratified fluids. He studied the resonant interaction between two surface waves and one internal wave in a two-layer fluid and showed that the two surface waves can excite the internal wave to a large amplitude. Denoting the two surface waves as waves 1 and 2 and the internal wave as wave 3, the resonance conditions are the same as given by equation 1.2. As a result, the internal wave has much larger period and wavelength than the surface ones. Ball's (1964) analysis was limited to shallow-water waves. Brekhovskikh et al. (1972) removed this limitation and considered the problem for the whole range of shallow-water to deep-water waves.

Experimental works on the interaction of two surface waves and one internal wave were conducted by Lewis et al. (1974) and Koop and Redekopp (1981). In the former, a layer of fresh water overlay a denser freon-kerosene mixture. Since the density difference between the two layers was small, the surface waves in the triad had close frequencies and wavelength, and the internal wave was a long wave. Hence, according to the resonance conditions the internal wave phase velocity is expected to be close to the group velocity of the surface waves. In the experiments, one surface wave and one internal wave train were generated mechanically as primary waves with the same direction of propagation. The observations confirmed that the strongest modulations of the primary surface wave occurred when the group velocity of the surface waves was close to the phase velocity of the internal wave, in agreement with the theory. The study of Koop and Redekopp (1981) concerned similar interaction of long and short waves on the two interfaces of a three-layer configuration.

Using a different approach from the conventional wave interaction theory, Gargett and Hughes (1972) studied the same interaction theoretically. They modeled the process as one in which the short surface waves interact with a slowly-varying, propagating current supposed and produced by the long internal wave. It then became possible to remove the restriction on the internal wave amplitude and to use the conservation laws for wave trains in slowly-varying media. They found that the variations in the direction and magnitude of the current induced by the internal wave cause local concentrations and reductions in the surface wave amplitudes. Their theoretical analysis was complemented by the field observations of the phenomenon made in the Strait of Georgia, British Columbia.

Resonant wave interaction among internal waves was first studied by Davis and Acrivos (1967). They showed that the lowest-mode internal wave in an infinite two-layer fluid with a diffuse interface is unstable. The wave forms a resonant triad with two second-mode internal waves that modulate the original wave by extracting energy from it.

In summary, since the original work by Philips (1960), resonant wave interaction has been studied in different classes of wave motion in fluids. In the present study, the focus is on the interaction mechanism responsible for instability of the interface in a two-layer fluid subject to surface wave action.

1.3 INTERNAL WAVE GENERATION BY A SURFACE WAVE

1.3.1 Two-dimensional Interaction

The resonant interaction of two internal waves with a surface wave in a two-layer fluid has been studied in two dimensions by Wen (1995), Hill and Foda (1996), and Jamali (1997a and b). Consistent with the experimental observations, the theoretical analyses of all the above authors indicated that two internal waves of nearly the same wavelength moving in opposite directions combine to form a short standing internal wave whose frequency is approximately half of that of the surface wave. The results of the studies mainly differ as to the role of viscosity of the lower layer in the interaction. The following summarizes these studies.

The configuration of the problem is shown in figure 1.1. Wen (1995) analyzed the problem for both cases of an inviscid and a viscous lower layer. She found that viscosity has a destabilizing effect on the interface and is essential to the growth of the internal waves. Wen's (1995) theoretical work was prompted by her observations of the interaction in a laboratory wave flume where a surface wave was allowed to travel over a fluidized silt bed. In the experiments, fine silt with mean grain size $d_{50} = 50\ \mu m$ constituted the sediment bed. Wen (1995) reported two opposite-traveling internal waves formed at the interface of the clear water and the fluidized silt bed. The internal waves had nearly the same frequencies and wavelengths, and hence formed almost a standing internal wave. The internal waves were also short compared to the surface wave and had a frequency close to the half of the surface wave frequency. Although in the experiments by Hill (1997) and Jamali (1998) the interaction was observed to be three-dimensional, Wen (1995) did not report such a situation in her experiments. However, Jamali's (1998) theoretical analysis indicates that when there is a considerable density difference between the layers such as that in Wen's (1995) experiments where the density difference between the fresh water and the fluidized sediment was large, the two-dimensional interaction is possible.

Wen's (1995) work was followed by Hill and Foda (1996), who by taking a similar approach arrived at nearly the same theoretical results regarding the significance of viscosity in the interaction. In both analyses of Wen (1995) and Hill and Foda (1996) the predicted kinematic properties of the waves were in qualitative agreement with the experimental observations. As part of their study, Hill and Foda (1996) also made qualitative observations on the generation of the internal waves at the interface of fresh water and fluidized sediment by a surface wave travelling over a bed of fine silt with the same grain size as in Wen's (1995) experiment.

According to the theoretical studies of Wen (1995) and Hill and Foda (1996), viscosity has a destabilizing effect on the interface and is essential to the growth of the internal waves. Jamali (1997 a and b) performed a two-dimensional, inviscid analysis of the interaction and found that in contrast with the above authors' result, the internal waves may grow easily in an inviscid two-layer fluid. Jamali verified his result later in a series experiments (Jamali, 1998) in a wave flume with fresh and salt water as the upper and lower layer respectively. In addition, Jamali (1998) reported that a series of experiments with fresh water as the upper layer and corn syrup as the lower layer did not result in appearance of the internal waves even though different experimental conditions were examined. Note that compared to the corn syrup, salt water can be considered as an inviscid fluid. These observations led to the conclusion that viscosity does not facilitate generation of the internal waves (Jamali, 1998).

1.3.2 Three-dimensional Interaction

The three-dimensional interaction was observed in both experiments of Hill (1997) and Jamali (1998). This interaction was investigated independently by these authors theoretically and experimentally. Hill (1997) carried out a series of experiments in a small wave flume containing a light mineral oil overlying fresh water. In Jamali's (1998) experiments fresh water was used as the upper layer and salt water as the lower layer. A three-dimensional standing wave pattern was observed in both authors' experiments. However, in Hill's (1997) experiments the wavelength of the 3D standing wave across the flume was always twice the flume width, corresponding to the first mode of a standing wave across the flume, while throughout Jamali's (1998) experiments a variety of modes were observed. Some of the modes observed by Jamali (1998) are shown in figure 1.2, where $n =$ the total number of peaks and troughs of the three-dimensional standing wave across the flume. In the experiments of both authors the internal wave frequencies were measured to be almost half of the surface wave frequency, and the internal waves were short compared to the surface wave.

Hill (1997) and Jamali (1998) both made a theoretical analysis of the three-dimensional interaction. The calculated frequencies in both studies were in good agreement with the experimental measurements. However, in Hill's (1997) study there was poor agreement between the calculated and measured growth rates of the internal waves. This can be attributed to the fact that although mineral oil has an appreciable viscosity, Hill (1997) used the results of an inviscid analysis for comparison with the measurements.

Hill's (1998) theoretical analysis also suggested that there are narrow bands of frequency, density ratio, and direction angle of the internal waves only within which growth of the internal waves is possible. However, Jamali (1998) showed that these are in contrast with both his experimental observations and theoretical results, and the interaction is not limited to narrow bands of the parameters.

1.4 PRESENT STUDY

The present study is aimed at a theoretical study of two subjects related to the interaction of a surface wave with two internal waves in a two-layer fluid: the asymptotic behavior of the waves at small density difference and the effect of a diffuse interface on the interaction. In the theoretical analysis of Jamali (1998) the equations for the growth rates of the internal waves were found to be quite long and complicated. Since this interaction is of great importance to the engineering study of internal mixing in a two-layer fluid subject to surface wave motion, a simple relation to predict the onset of the instability of the interface is a need. In the first study, by assuming that the density difference between the layers is small, the three-dimensional interaction is analyzed asymptotically in an attempt to find useful expressions for the interaction quantities. In the second study, the effect of a diffuse interface on the two-dimensional interaction is analyzed asymptotically. According to the laboratory observation of Jamali (1998), the diffuse interface of the two layers had an effect on the growth rate of the internal waves.

In chapter 2, the theoretical analysis of Jamali (1998) on the three-dimensional interaction is reexamined, and an attempt is made to obtain an approximate relation for the growth rate of the internal. The density difference of the layers is taken as a small perturbation parameter, and asymptotic equations are obtained for the growth rate and the kinematic properties of the internal waves. The effects of different parameters on the interaction are explored in light the asymptotic analysis as well.

In chapter 3, the effect of a diffuse interface on the two-dimensional interaction is investigated theoretically. The interface thickness as well as the density difference of the two layers is assumed to be small, and an asymptotic analysis of the interaction is performed to obtain the growth rate and the kinematic properties of the interacting waves asymptotically.

Finally, a summary of the earlier chapters along with the conclusions and recommendations for future studies is presented in chapter 4.

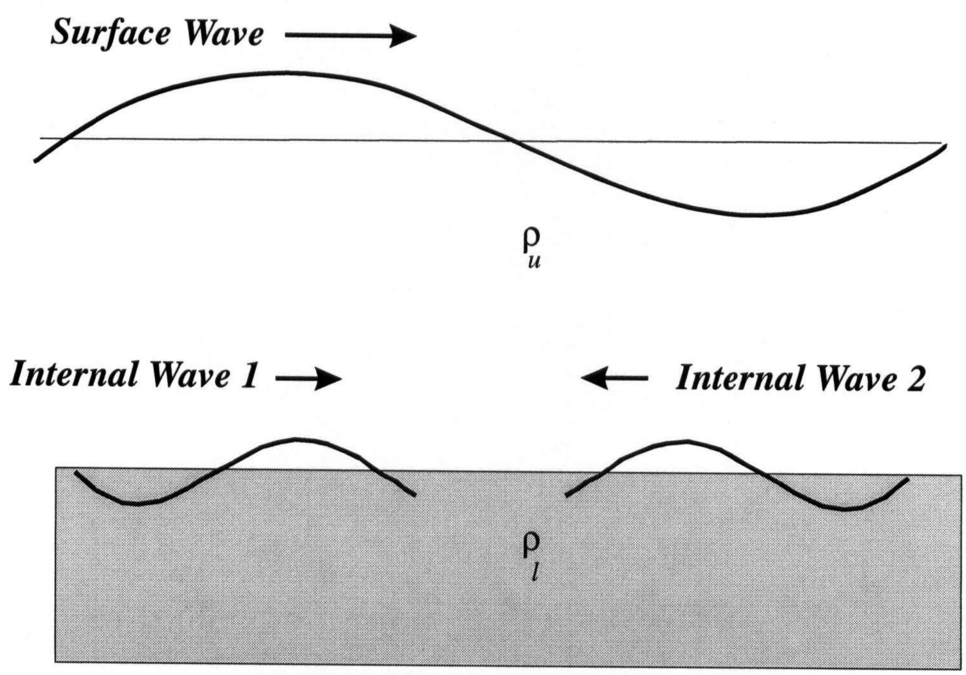
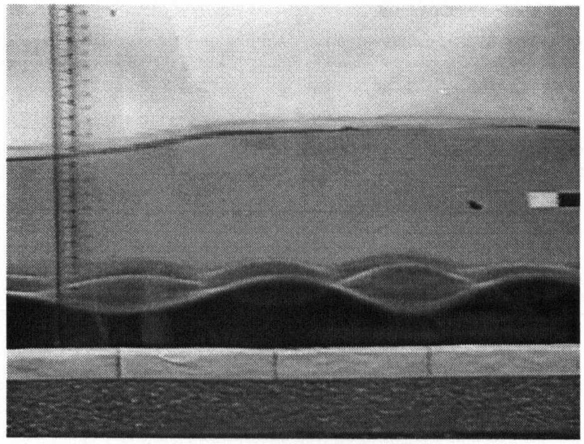


Figure 1.1 Configuration of the problem



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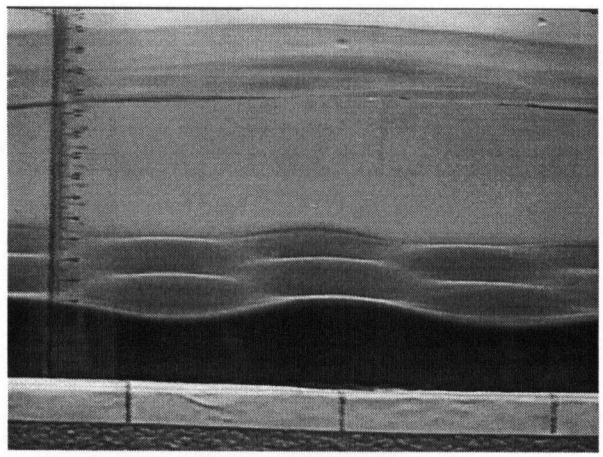

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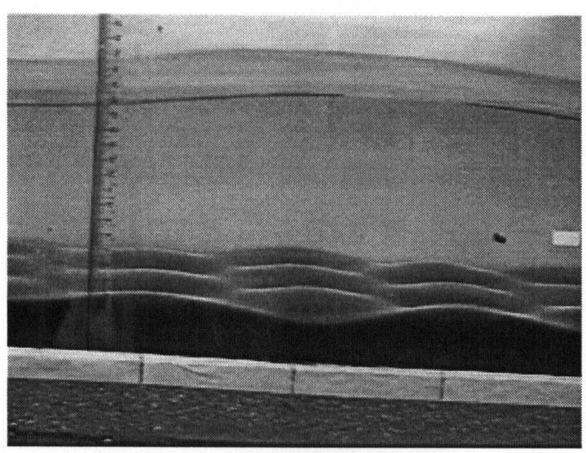


n=6

Figure 1.2 Top-side views of 3D standing internal waves with different *n* 's.

CHAPTER 2

THREE-DIMENSIONAL INTERACTION IN A TWO-LAYER FLUID

2.1 INTRODUCTION

The interaction of a surface wave with two oblique internal waves was studied by Jamali (1998). In the theoretical part of the study the problem was formulated for a three-dimensional, inviscid, two-layer fluid system as shown in figure 2.1. Taking the typical wave amplitude as a small parameter, a perturbation technique was used to find the solution to the interaction problem. At first order, the linear wave theory was obtained, and at second order, the evolution equations of the waves were derived. Taking a_0, b_1, and b_2 to be the complex amplitudes of the surface wave 0 and the internal waves 1 and 2 respectively, the evolution equations of the internal waves were given by

$$\frac{db_1}{dt} = \alpha_1 a_0 \overline{b}_2 \qquad \text{and} \qquad \frac{db_2}{dt} = \alpha_2 a_0 \overline{b}_1 \tag{2.1}$$

where the overbar denotes complex conjugate, and α_1 and α_2 are the interaction coefficients. These equations were obtained from appropriate solvability conditions at second order. As the energy content of the surface wave is much more than those of the internal waves, the amplitude a_0 can be assumed constant in (2.1). Subsequently, the evolution equations can be solved to yield the growth rate of the internal waves in terms of α_1, α_2, and a_0.

Jamali (1998) reported that the expressions for the interaction coefficients were too long and complicated to yield a useful equation for the growth rate of the internal waves. In the present study, which is a follow-up of Jamali (1998), an attempt is made to obtain simple approximate expressions for the interaction coefficients α_1, α_2 as well as the growth rate. As in many situations the density difference between the layers is small, the non-dimensionalized density difference $\delta = 1 - r$, where $r = \rho_u / \rho_l$, is taken as the perturbation parameter, and the leading order terms in the perturbation expansions of the interaction coefficients are derived. The leading order terms are then used to obtain a simple expression for the growth rate.

2.2 REVIEW OF FORMULATION OF THE INTERACTION PROBLEM

Consider the two-layer fluid system shown in figure 2.1. The system is assumed to be infinite horizontally and three-dimensional. The coordinate system xyz is located on the interface. The depth of the upper layer is denoted by h, the depth of the lower layer by d, and the total depth by H. The densities of the upper and lower layers are ρ_u and ρ_l respectively. The surface wave is denoted as wave 0 and the two opposite-traveling internal waves as waves 1 and 2. Without loss of generality, wave 0 is assumed to travel in the positive x direction and the two internal waves in the $x-y$ plane. The internal wave 1 has an arbitrary directional angle θ_1 with respect to the surface wave. As the three waves are in resonance, certain kinematic conditions hold between their frequencies and wavelengths. The kinematic conditions of resonance are given by

$$\begin{aligned} \vec{k}_0 &= \vec{k}_1 + \vec{k}_2 \\ \omega_0 &= \omega_1 + \omega_2 \end{aligned} \tag{2.2}$$

where for each wave i the wave number $\vec{k}_i$ and the frequency ω_i are related by a dispersion relation. For a two-layer inviscid fluid the dispersion relation is given by

$$\frac{\dfrac{\rho_u}{\rho_l}(\omega^4 - g^2k^2)tanh(kh)}{(gk\,tanh(kh) - \omega^2)} + gk - \omega^2 coth(kd) = 0 \tag{2.3}$$

(see Appendix A), where the parameters are defined in figure 2.1. These equations ensure a continuous and effective energy transfer between the waves. Note that from the resonance conditions the direction angle of internal wave 2 is obtained as a function of θ_1.

With the assumptions of incompressible fluid layers and irrotational flows in the layers, the fluid motion can be described by velocity potentials $\phi'(x,y,z,t)$ and $\phi(x,y,z,t)$ in the upper and lower layers respectively. The potentials satisfy Laplace's equation in the two domains.

$$\nabla^2\phi' = 0, \qquad 0 < z < h \tag{2.4}$$

$$\nabla^2\phi = 0, \qquad -d < z < 0 \tag{2.5}$$

The above equations are subject to the boundary conditions at the free surface, the interface of the layers, and the solid bed. On the free surface, the boundary conditions are:

$$\xi_t + \phi'_x \xi_x + \phi'_y \xi_y = \phi'_z, \qquad z = h + \xi(x, y, t) \tag{2.6}$$

$$\rho_u [\phi'_t + \frac{1}{2}(\phi'^2_x + \phi'^2_y + \phi'^2_z) + gz] = C'(t), \quad z = h + \xi(x, y, t) \tag{2.7}$$

where $\xi(x, y, t) =$ the displacement of the free surface. The first equation represents a kinematic boundary condition while the second equation corresponds to a dynamic one. On the two-layer interface, the kinematic boundary conditions are:

$$\eta_t + \phi'_x \eta_x + \phi'_y \eta_y = \phi'_z, \qquad z = \eta(x, y, t) \tag{2.8}$$

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y = \phi_z, \qquad z = \eta(x, y, t) \tag{2.9}$$

where $\eta(x, y, t) =$ the displacement of the interface, and the dynamic boundary condition is:

$$\rho_u [\phi'_t + \frac{1}{2}(\phi'^2_x + \phi'^2_y + \phi'^2_z) + gz - C'(t)] = \rho_l [\phi_t + \frac{1}{2}(\phi^2_x + \phi^2_y + \phi^2_z) + gz - C(t)],$$

$$z = \eta(x, y, t) \tag{2.10}$$

On the bed, the problem is subject to a kinematic boundary condition requiring the normal velocity be zero, i.e.,

$$\phi_z = 0, \qquad z = -d \tag{2.11}$$

For the purpose of the interaction analysis, it is assumed that the amplitudes of the waves are sufficiently small that a weakly nonlinear interaction analysis can be performed. This implies that terms of order ε^3 and higher, where ε is non-dimensional wave amplitude, may be neglected. Accordingly, the following expansions in ε are considered for ξ, η, ϕ' and ϕ.

$$\xi(x, y, t) = a_0\ Exp[i(\vec{k}_0.\vec{x} - \omega_0 t)] + a_1\ Exp[i(\vec{k}_1.\vec{x} - \omega_1 t)] + a_2\ Exp[i(\vec{k}_2.\vec{x} - \omega_2 t)]$$

$$+ \sum_{i=0}^{2} \sum_{j=i}^{2} \xi_{ij}(x, y, t) + complex\ conjugate$$

$$\eta(x,y,t) = b_0\ Exp[i(\vec{k}_0.\vec{x} - \omega_0 t)] + b_1\ Exp[i(\vec{k}_1.\vec{x} - \omega_1 t)] + b_2\ Exp[i(\vec{k}_2.\vec{x} - \omega_2 t)] + \sum_{i=0}^{2}\sum_{j=i}^{2}\eta_{ij}(x,y,t) + complex\ conjugate$$

$$\phi'(x,y,z,t) = \phi_0'(x,y,z,t) + \phi_1'(x,y,z,t) + \phi_2'(x,y,z,t) + \sum_{i=0}^{2}\sum_{j=i}^{2}\phi_{ij}'(x,y,z,t) + complex\ conjugate$$

$$\phi(x,y,z,t) = \phi_0(x,y,z,t) + \phi_1(x,y,z,t) + \phi_2(x,y,z,t) + \sum_{i=0}^{2}\sum_{j=i}^{2}\phi_{ij}(x,y,z,t) + complex\ conjugate$$

(2.12)

where $\vec{x} = (x,y)$; a_i and b_i are half of the amplitudes of wave i at the free surface and the interface respectively. The amplitudes are assumed to be complex numbers in general. The three interacting waves constitute the wave field at first order. The single-indexed terms, such as ϕ_i, are $O(\varepsilon)$ while the double-indexed terms such as ϕ_{ij} are $O(\varepsilon^2)$. Expansions (2.12) follow the standard procedure for three-wave interaction (e.g., see Craik, 1985).

In a weakly non-linear interaction, as far as the short-time behavior of the waves is concerned, the component waves can be regarded as independent and be treated by the linear theory. However, energy is exchanged between the waves as a result of resonance, and the amplitudes of the interacting waves undergo changes with time, but the rate of energy exchange is small, and so are the time variations of the amplitudes. The amplitudes have, in fact, a time scale much greater than the individual wave periods. The time variations of the amplitudes are functions of the amplitudes of the waves; the higher the amplitudes, the faster the variations. To solve the above perturbation problem, a commonly used technique proposed by Benny (1962) for the solution of weakly non-linear interaction problems is employed. The technique makes use of the large-time behavior of the waves, and is quite efficient in predicting their dynamics. Benny (1962) assumed that the wave amplitudes are slowly varying functions of time, and that the time-derivative of each amplitude is a function of the product of the amplitudes of the other waves, and hence is a second-order quantity. These imply that in the present problem the amplitudes of the interacting waves can be taken as:

$$\frac{da_0}{dt} = O(b_1 b_2), \qquad \frac{db_1}{dt} = O(a_0 \bar{b}_2), \qquad \frac{db_2}{dt} = O(a_0 \bar{b}_1), \tag{2.13}$$

where symbol overbar denotes complex conjugate. The quantity a_0 is the amplitude of the surface wave at the free surface, and b_1 and b_2 are the amplitudes of the internal waves 1 and 2 at $z=0$ respectively. The above assumption makes the time derivatives of a_0, b_1, and b_2 appear in the perturbed equations at second order. For a lucid discussion of this technique, the interested reader is referred to Drazin and Reid (1981) and Craik (1985).

Substituting (2.12) in the governing equations and collecting first order terms results in the linear wave theory for waves 0, 1 and 2. The solution to the linear problem is given in Appendix A. At second order, the nonlinear interaction terms appear in the forcing functions of the resulting inhomogeneous systems of partial differential equations. The forcing terms contain the time derivatives of the wave amplitudes. Due to the resonance conditions, the forcing functions are of the form that produces secular solutions at $O(a_0\bar{b}_1)$, $O(a_0\bar{b}_2)$, $O(b_1b_2)$, and at their complex-conjugate counterparts. A secular solution grows in time and hence becomes unbounded as time becomes arbitrarily large. From a physical point of view secular solutions are not acceptable as the energy is bounded. Mathematically, secular solutions destroy the uniformity of the asymptotic expansions. To avoid secular solutions it is necessary to impose a solvability condition on the forcing functions. The desired solvability condition is the requirement that the forcing functions and the homogeneous solution of the adjoint system be orthogonal (Drazin and Reid, 1981, p 385). Applying the solvability condition to the forcing functions result in three equations from which da_0 / dt, db_1 / dt, and db_2 / dt can be explicitly found in the following forms.

$$\frac{da_0}{dt} = \alpha_0 b_1 b_2 , \qquad \frac{db_1}{dt} = \alpha_1 a_0 \bar{b}_2 , \qquad \frac{db_2}{dt} = \alpha_2 a_0 \bar{b}_1 \tag{2.14}$$

where α_0, α_1 and α_2 are constant. The direct derivation of the interaction coefficients α_1 and α_2 is presented in Appendix B of Jamali (1998). The analysis indicates that α_1 and α_2 are purely imaginary.

Since the energy content of the surface wave is much more than those of the internal waves, variation of the surface wave amplitude a_0 with time is negligible. Hence, the first equation in (2.14) can be eliminated to yield (2.1) with the amplitude a_0 taken as constant. The evolution equations can then be solved to yield the growth rate of the internal waves in terms of α_1, α_2, and a_0. In the following, an attempt is made to obtain asymptotic expressions for the interaction coefficients and hence the growth rate.

2.3 PERTURBATION EXPANSIONS OF THE INTERACTION PARAMETERS

In this section, first an asymptotic expansion for α_2 is obtained, and then the analysis is extended to include α_1. The equation yielding the evolution equation for internal wave 2 is the solvability equation at $O(a_0 \bar{b}_1)$. It is given by (Jamali, 1998)

$$\left\{\frac{F(z)}{g}(m_1 + 2i\omega_2 \hat{f}_2(z)\frac{db_2}{dt}\right\}\Bigg|_{z=h} + \left\{\left(G(z) - \frac{g}{\omega_2^{\,2}}\frac{dG(z)}{dz}\right)m_3\right\}\Bigg|_{z=0} +$$

$$\left\{\frac{1}{\omega_2^{\,2}}\frac{dG(z)}{dz}\left(m_2 + 2i\omega_2\left(\hat{g}_2(z) - r\hat{f}_2(z)\right)\frac{db_2}{dt}\right)\right\}\Bigg|_{z=0} = 0$$

(2.15)

where m_1, m_2, and m_3 are given by

$$m_1 e^{i\vartheta_2} = \left\{-(\phi_0')_x(\overline{\phi_1'})_x - (\phi_0')_y(\overline{\phi_1'})_y - (\phi_0')_z(\overline{\phi_1'})_z + \frac{1}{g}(\phi_0')_t(\overline{\phi_1'})_{zt} + \frac{1}{g}(\phi_1')_t(\overline{\phi_0'})_{zt}\right\}_t -$$
$$\left\{(\phi_0')_x(\overline{\phi_1'})_t + (\phi_0')_t(\overline{\phi_1'})_x\right\}_x - \left\{(\phi_0')_y(\overline{\phi_1'})_t + (\phi_0')_t(\overline{\phi_1'})_y\right\}_y$$

(2.16)

$$\begin{aligned} m_2 e^{i\vartheta_2} &= g\left\{\left((\phi_0)_x - r(\phi_0')_x\right)\overline{\eta}_1 + \left((\phi_1)_x - r(\phi_1')_x\right)\overline{\eta}_0\right\}_x \\ &+ g\left\{\left((\phi_0)_y - r(\phi_0')_y\right)\overline{\eta}_1 + \left((\phi_1)_y - r(\phi_1')_y\right)\overline{\eta}_0\right\}_y \\ &+ \left\{r\left((\phi_0')_x(\overline{\phi_1'})_x + (\phi_0')_y(\overline{\phi_1'})_y + (\phi_0')_z(\overline{\phi_1'})_z\right) - \left((\phi_0)_x(\overline{\phi}_1)_x + (\phi_0)_y(\overline{\phi}_1)_y + (\phi_0)_z(\overline{\phi}_1)_z\right)\right\}_t \\ &- \left\{\left((\phi_0)_{zt} - r(\phi_0')_{zt}\right)\overline{\eta}_1 + \left((\phi_1)_{zt} - r(\phi_1')_{zt}\right)\overline{\eta}_0\right\}_t \end{aligned}$$

(2.17)

and

$$m_3 e^{i\vartheta_2} = \left\{\left((\phi_0)_x - (\phi_0')_x\right)\overline{\eta}_1 + \left((\phi_1)_x - (\phi_1')_x\right)\overline{\eta}_0\right\}_x + \left\{\left((\phi_0)_y - (\phi_0')_y\right)\overline{\eta}_1 + \left((\phi_1)_y - (\phi_1')_y\right)\overline{\eta}_0\right\}_y$$

(2.18)

and the functions $\hat{f}_2(z)$ and $\hat{g}_2(z)$ are defined by

$$\phi_2'(x,y,z,t) = \hat{f}_2(z)e^{i\vartheta_2}$$

$$\phi_2(x,y,z,t) = \hat{g}_2(z)e^{i\vartheta_2}$$

where $\vartheta_2 = \vec{k}_2 \cdot \vec{x} - \omega_2 t$. The functions $F(z)$ and $G(z)$ are the solution to the adjoint system of equations corresponding to internal wave 2. It can be shown that if $F(z)$ and $G(z)$ in the adjoint system are replaced with $f(z)$ and $g(z)/r$, the linear system of equations corresponding to the motion of internal wave 1 is obtained. Hence, $F(z)$ and $G(z)$ turn out to be

$$F(z) = \hat{f}_2(z) \tag{2.19}$$

$$G(z) = \frac{\hat{g}_2(z)}{r} \tag{2.20}$$

As seen all the terms in the solvability conditions are in terms of the linear solutions of waves 0, 1, and 2. Hence, first the asymptotic expansions of the linear solutions are found.

2.3.1 Asymptotic Expansion of Linear Solution

In general the independent variables of the problem can be chosen to be δ, k_0, θ_1, d, h, a_0. The dependent variables are ω_0, ω_1, ω_2, k_1, k_2, b_1, and b_2. The resonance conditions can be used along with the dispersion relations of the waves to yield the relations for ω_0, ω_1, ω_2, k_1, k_2. To guess the correct forms of the asymptotic series, the order of each variable has to be determined first. Given k_0 is $O(1)$, ω_0 turns out to be $O(1)$ from the surface-wave dispersion relation. From the previous studies (Wen, 1995; Hill, 1997; Jamali, 1998) it is known that $\omega_0 \approx \omega_2 \approx \omega_0 / 2$. This implies that both ω_1 and ω_2 are order 1. It can be shown that for an internal wave, the wave number is proportional to δ^s, where $s = -1$ for deep waters (Turner, 1973). Later it will be shown that by assumption of small density difference the internal waves turn out to be deep-water waves. Hence, it follows that k_1 and k_2 are $O(\delta^{-1})$.

After ordering the kinematic variables, the resonance conditions and the dispersion relations are solved to find the first few terms in the asymptotic expansions of these quantities. In the scalar form the resonance conditions (2.2) can be written as

$$\begin{aligned} \omega_0 &= \omega_1 + \omega_2 \\ k_0 &= k_1 \cos\theta_1 + k_2 \cos\theta_2 \\ 0 &= k_1 \sin\theta_1 + k_2 \sin\theta_2 \end{aligned} \tag{2.21}$$

Note that as the surface wave is moving in the x direction, $\vec{k}_0$ doesn't have any component in the y direction. Next, the dispersion relation (2.3) is simplified for the internal waves knowing that k_1 and k_2 are $O(\delta^{-1})$. For each internal wave the terms $tanh(kd)$ and $coth(kd)$ are asymptotically equal to 1 correct to $O(\delta^n)$ for any $n \geq 0$. Consequently, the dispersion relation for the internal waves reduces to

$$\omega^2 = gk\frac{\delta}{2-\delta} \tag{2.22}$$

From now on, when a quantity is accurate to $O(\delta^n)$ for all $n \geq 0$, it is called to be accurate to $O(\delta^\infty)$. Simultaneous solution of the resonance conditions (2.21) and the individual internal wave dispersion relations gives the following expressions for ω_1, ω_2, k_1, k_2 in terms of k_0, ω_0, and θ_2.

$$k_1 = \frac{-k_0 \sin\theta_2}{\sin(\theta_1 - \theta_2)} \tag{2.23}$$

$$k_2 = \frac{k_0 \sin\theta_1}{\sin(\theta_1 - \theta_2)} \tag{2.24}$$

$$\omega_1^{\,2} = \frac{-gk_0 \sin\theta_2}{\sin(\theta_1 - \theta_2)}\frac{\delta}{2-\delta} \tag{2.25}$$

$$\omega_2^{\ 2} = \frac{gk_0 \sin\theta_1}{\sin(\theta_1 - \theta_2)} \frac{\delta}{2-\delta} \tag{2.26}$$

Also, the following relation between θ_1 and θ_2 is obtained.

$$\frac{\left(\sqrt{\sin\theta_1} + \sqrt{-\sin\theta_2}\right)^2}{\sin(\theta_1 - \theta_2)} = \frac{\omega_0^{\ 2}}{gk_0} \frac{2-\delta}{\delta} \tag{2.27}$$

The above equations are correct to $O(\delta^{\infty})$. Note that in the above equations ω_0 and θ_2 themselves are dependent variables.

For later use, the first few terms in the asymptotic expansions of θ_2 and $\sin\theta_2$, and $\cos\theta_2$ are obtained. The following expansion is assumed for θ_2.

$$\theta_2 = \theta_2^0 + \theta_2^1\delta + O(\delta^2) \tag{2.28}$$

Substitution of the above in (2.27) and collection of the terms at $O(1)$ yields

$$Sin(\theta_1 - \theta_2^0) = 0$$

Knowing that the two internal waves have nearly opposite directions (e.g., see Hill, 1997, and Jamali, 1998), θ_2^0 is obtained as

$$\theta_2^0 = \pi + \theta_1 \tag{2.29}$$

At $O(\delta)$, the following expression for θ_2^1 is obtained.

$$\theta_2^1 = \frac{2\sin\theta_1 gk_0}{\omega_0^{\ 2}} \tag{2.30}$$

The dispersion relation for the surface wave can be written as

$$\omega_0^{\ 2} = gk_0 \tanh(k_0 H) + O(\delta) \tag{2.31}$$

Substituting for $\omega_0^{\ 2}$ from (2.31) in (2.30), the asymptotic expansion of θ_2 is obtained as

$$\theta_2 = \pi + \theta_1 + \frac{2\sin\theta_1}{\tanh(k_0 H)}\delta + O(\delta^2) \tag{2.32}$$

Using (2.32), the asymptotic expansions of $\sin\theta_2$ and $\cos\theta_2$ are obtained as follows.

$$\sin\theta_2 = -\sin\theta_1 - \frac{\sin 2\theta_1}{\tanh(k_0 H)}\delta + O(\delta^2) \tag{2.33}$$

$$\cos\theta_2 = -\cos\theta_1 + \frac{2\sin^2\theta_1}{\tanh(k_0 H)}\delta + O(\delta^2) \tag{2.34}$$

The linear solution of the internal wave motion is given in Appendix A. Considering the fact that the internal waves turn out to be deep-water waves due to the assumption of small density difference, the expressions for ϕ' and ϕ can be greatly simplified. Knowing $\tanh(kh)$ and $\tanh(kd)$ are asymptotically equal to one, correct to $O(\delta^\infty)$, due to the fact $k \sim O(\delta^{-1})$, ϕ' and ϕ reduce to

$$\phi'(x,y,z) = \frac{ib\omega}{k} e^{-kz} e^{i\vartheta} \tag{2.35}$$

$$\phi(x,y,z) = \frac{-ib\omega}{k} e^{kz} e^{i\vartheta} \tag{2.36}$$

The above equations are correct to $O(\delta^\infty)$. Using (2.35) and (2.36), the asymptotic expressions for $\hat{f}_2(z)$, $\hat{g}_2(z)$, $F(z)$, and $G(z)$ can be obtained as follows.

$$\hat{f}_2(z) = \frac{i\omega}{k} e^{-kz} \tag{2.37}$$

$$\hat{g}_2(z) = \frac{-i\omega}{k} e^{kz} \tag{2.38}$$

$$F(z) = \frac{i\omega}{k} e^{-kz} \tag{2.39}$$

$$G(z) = \frac{-i\omega}{k}\frac{e^{kz}}{r} \tag{2.40}$$

Again, the above equations are correct to $O(\delta^{\infty})$.

2.3.2 Asymptotic Form of Solvability Condition

Next, the asymptotic form of the solvability condition (2.15) is obtained. Substituting for $F(z)$, $G(z)$, $\hat{f}_2(z)$, and $\hat{g}_2(z)$ from (2.37) to (2.40) in (2.15) gives

$$\frac{i\omega_2}{gk_2}e^{-k_2h}\left(m_1\big|_{z=h}-2\frac{\omega_2^{\,2}}{k_2}e^{-k_2h}\frac{db_2}{dt}\right)+\frac{-i\omega_2}{rk_2}\left(1-\frac{gk_2}{\omega_2^{\,2}}\right)m_3\big|_{z=0}+ \\ \frac{-i}{r\omega_2}\left(m_2\big|_{z=0}+\frac{2\omega_2^{\,2}}{k_2}(1+r)\frac{db_2}{dt}\right)=0 \tag{2.41}$$

Using the dispersion relation for the internal wave 2, and noting that $e^{-k_2h}\sim 0$ correct to $O(\delta^{\infty})$, the above equation simplifies to

$$2m_3\big|_{z=0}-\frac{2-\delta}{1-\delta}\left(m_2\big|_{z=0}+\frac{2\omega_2^{\,2}}{k_2}(1+r)\frac{db_2}{dt}\right)=0 \tag{2.42}$$

So, the contribution from the free surface is eliminated from the solvability condition. Next, using (2.35) and (2.36) asymptotic solutions of the potentials are substituted in (2.17) and (2.18) to obtain the corresponding expressions for $m_2\big|_{z=0}$ and $m_3\big|_{z=0}$ in (2.42). As a result, (2.42) becomes

$$-\frac{\delta}{1-\delta}k_2\left(\omega_0\cos\theta_2\left(2\coth(k_0d)-\frac{g\delta k_0}{\omega_0^{\,2}}\right)+\omega_1\cos(\theta_1-\theta_2)\delta\right)ib_0\bar{b}_1- \\ \frac{2-\delta}{g(1-\delta)}\left(-\omega_0\omega_1\omega_2\left(-\delta+\frac{g\delta k_0}{\omega_0^{\,2}}\cos\theta_1-2\coth(k_0d)\cos\theta_1\right)-\omega_2\left(\omega_0^{\,2}+\omega_1^{\,2}\right)\delta\right)ib_0\bar{b}_1- \\ \frac{2(2-\delta)\delta}{(1-\delta)}\frac{db_2}{dt}=0 \tag{2.43}$$

where b_0 is the complex amplitude of the surface wave at the interface given by

$$b_0 = a_0 \cosh(k_0 h)\left(1 - \frac{g k_0 \tanh(k_0 h)}{\omega_0^{\,2}}\right) \qquad (2.44)$$

Equation 2.43 is the asymptotic form of the solvability condition and is correct to $O(\delta^\infty)$. Solving this equation for db_2/dt gives

$$\frac{db_2}{dt} = -\frac{1}{2(2-\delta)}k_2\left(\omega_0 Cos\theta_2\left(2Coth(k_0 d) - \frac{g\delta k_0}{\omega_0^{\,2}}\right) + \omega_1 Cos(\theta_1 - \theta_2)\delta\right)ib_0\bar{b}_1 -$$
$$\frac{1}{2g\delta}\left(-\omega_0\omega_1\omega_2\left(-\delta + \frac{g\delta k_0}{\omega_0^{\,2}}Cos\theta_1 - 2Coth(k_0 d)Cos\theta_1\right) - \omega_2\left(\omega_0^{\,2} + \omega_1^{\,2}\right)\delta\right)ib_0\bar{b}_1$$

$$(2.45)$$

Since from the early discussions

$$\omega_0, \omega_1, \omega_2, k_0, \theta_1, \theta_2 \sim O(1) \qquad \text{and} \qquad k_1, k_2 \sim O(\delta^{-1}) \qquad (2.46)$$

db_2/dt is expected to be $O(\delta^{-1})$ according to (2.45). However, rearrangement and simplification of the terms on the right-hand side of (2.45) shows that db_2/dt is indeed $O(1)$. The new expression for db_2/dt is

$$\frac{db_2}{dt} = \left(\frac{-coth(k_0 d)k_0}{2-\delta}\left(\omega_0 + \omega_1 \cos\theta_1 \frac{sin((\theta_1+\theta_2)/2)}{sin((\theta_1-\theta_2)/2)}\right)\right)ib_0\bar{b}_1 +$$
$$\left(\frac{\omega_2^{\,2}}{2g}\left(\frac{cos\theta_2 g k_0}{\omega_0} - cos(\theta_1 - \theta_2)\omega_1\right) + \frac{1}{2g}(\omega_0\omega_1\omega_2)\left(\frac{gk_0}{\omega_0^{\,2}}cos\theta_1 - 1\right) + \frac{\omega_2}{2g}\left(\omega_0^{\,2} + \omega_1^{\,2}\right)\right)ib_0\bar{b}_1$$

$$(2.47)$$

The above equation is correct to $O(\delta^\infty)$. Now it is clear from the above equation that db_2/dt is $O(1)$.

2.3.3 Leading Order Approximation of α_2

To obtain the leading order approximation of α_2, the leading order approximations for ω_0, ω_1, ω_2 and θ_2 are needed. These are

$$\omega_1 \sim \omega_2 \sim \frac{\omega_0}{2} \tag{2.48}$$

$$\omega_0 \sim \sqrt{g k_0 \tanh(k_0 H)} \tag{2.49}$$

$$\theta_2 \sim \theta_1 + \pi \tag{2.50}$$

By substituting the above in (2.47) and noting that

$$b_0 \sim a_0 \cosh(k_0 h)\left(1 - \frac{\tanh(k_0 h)}{\tanh(k_0 H)}\right) \tag{2.51}$$

the leading order approximation for db_2 / dt is obtained as follows.

$$\frac{db_2}{dt} = k_0 \omega_0 \left(\frac{\coth(k_0 d)(-3 + \cos(2\theta_1)) + 2\tanh(k_0 H)}{8}\right) \cosh(k_0 h)\left(1 - \frac{\tanh(k_0 h)}{\tanh(k_0 H)}\right) i a_0 \bar{b}_1 \tag{2.52}$$

The above equation can be simplified to yield the following leading-order approximation for α_2.

$$\alpha_2 = \frac{-i}{4} \frac{\omega_0^3}{g \sinh^2(k_0 H)} \left(\cosh(k_0 H)\cosh(k_0 d)\sin^2\theta_1 + \cosh(k_0 h)\right) \tag{2.53}$$

Note that by substituting for ω_0 from (2.49) in (2.53) α_2 is obtained merely in terms of the independent variables of the problem.

2.3.4 Leading-order Approximation of α_1

To obtain the leading-order approximation of α_1, the following relation between the interaction coefficients (Jamali, 1998) is used.

$$\frac{\alpha_1}{\omega_1} = \frac{\alpha_2}{\omega_2} \tag{2.54}$$

Given that to the leading order $\omega_1 \sim \omega_2$, the above equation implies that

$$\alpha_1 \sim \alpha_2 \tag{2.55}$$

Hence, the leading order expression for α_1 is the same as that for α_2, given by (2.53).

2.4 NUMERICAL EXAMPLE

Consider a test case where $d = 4.0\ cm$, $H = 16.0\ cm$, $\omega_0 = 2\pi / 0.8\ rad / sec$, $\rho_u = 1.00\ gr / cm^3$, $\rho_l = 1.04\ gr / cm^3$, and $\theta_1 = 75^\circ$. From the surface-wave dispersion relation $k_o = 7.54\ rad/m$. To compare the exact and the asymptotic values of the interaction coefficients, variations of the non-dimensional parameter $\alpha H / \omega_0$ with the non-dimensional parameters ρ_l / ρ_u, $k_0 H$, d / H, and θ_1 from the two solutions are compared in the following. It can be shown (Jamali, 1998) that at large times the solution to (2.1) is given by

$$b_1(t), b_2(t) \approx e^{\alpha|a_0|t} \tag{2.56}$$

where the forcing parameter α is equal to $\sqrt{\alpha_1 \overline{\alpha}_2}$. From the asymptotic solution, it can be shown

$$\alpha \sim |\alpha_1| \sim |\alpha_2| \tag{2.57}$$

correct to the leading order.

Variations of $\alpha H / \omega_0$ with the density ratio ρ_l / ρ_u from the two solutions are shown in figure 2.2. As expected, the two curves are asymptotic when the density ratio approaches unity. The asymptotic solution deviates from the exact solution for density ratios greater than 1.06. It is expected that by finding the next terms in the asymptotic expansions of α_1 and α_2 the range of validity of the asymptotic solution increases.

Variations of $\alpha H / \omega_0$ with $k_0 H$ from the two solutions are demonstrated in figure 2.3. It is recalled that a surface wave with $k_0 H < \pi / 10$ is a shallow-water wave, and one with $k_0 H > \pi$ is a deep-water wave. From the graphs, it is seen that the asymptotic equation closely reproduces the exact value of the forcing parameter when the surface wave is basically a deep-water wave. However, the asymptotic solution deviates from the exact solution when the surface wave becomes a shallow-water wave. This can be explained by the fact that as $k_0 H$, and hence the surface wave frequency, decreases, the frequencies of the internal waves decrease too, see (2.48). This implies that for sufficiently low ω_0 the internal waves are not deep-water internal waves any more in the two layers. However, it was shown before that by taking the density difference as the small parameter the internal waves were found to be deep-water waves in the two layers. Therefore, the asymptotic theory is unable to predict the interaction when $k_0 H$ is small. According to this discussion, even the complete asymptotic series of α will not be asymptotic to the exact solution when $k_0 H$ is small.

Variations of $\alpha H / \omega_0$ with d / H from the two solutions are demonstrated in figure 2.4. It is seen that the asymptotic solution closely reproduces the exact solution except for low values of d / H. This can be explained by the fact that for a constant total depth H when the depth of the lower layer approaches zero, the internal waves become shallow-water waves in the lower layer. However, this violates the assumption of the asymptotic theory that the internal waves are deep-water waves. From this discussion it can be also concluded that even the complete series of the forcing parameter α can not capture the exact solution when d / H is close to zero.

Variations of $\alpha H / \omega_0$ with θ_1 from the two solutions are demonstrated in figure 2.5. It is seen that the approximate leading-order solution has closely predicted the exact values for different directional angles of the internal wave 1.

From the above discussion, it is interesting to note that the leading-order asymptotic solution closely predicts the forcing parameter of the interaction when the medium of interest is a deep ocean. In deep oceans, the density ratio is always less than 1.05, and the surface waves are deep-water waves.

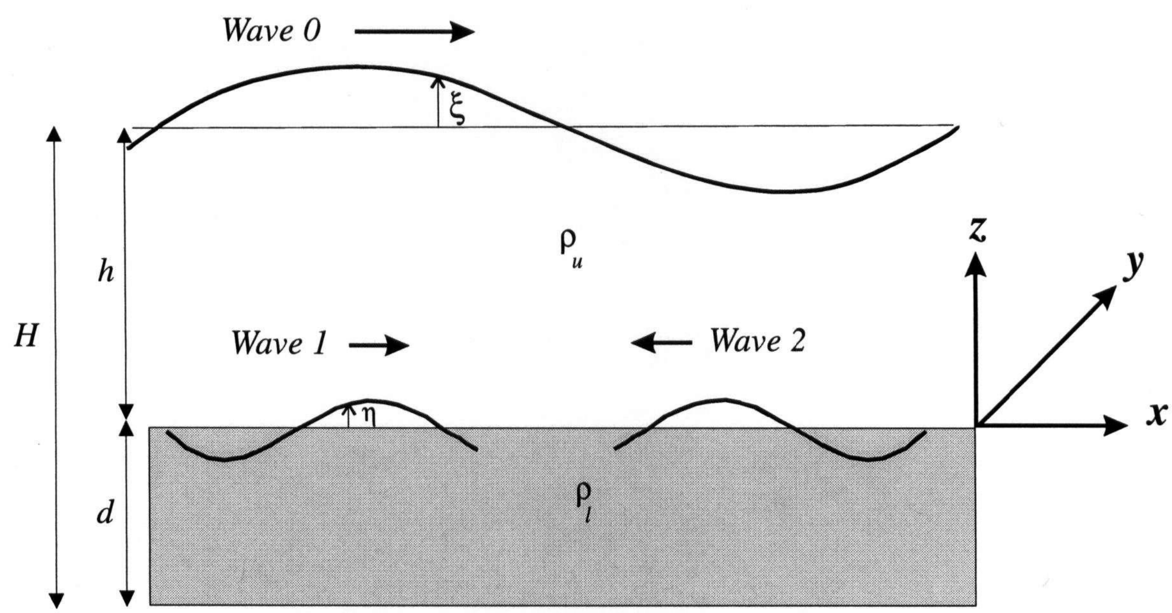


Figure 2.1 Configuration of the problem in the three-wave resonant interaction.

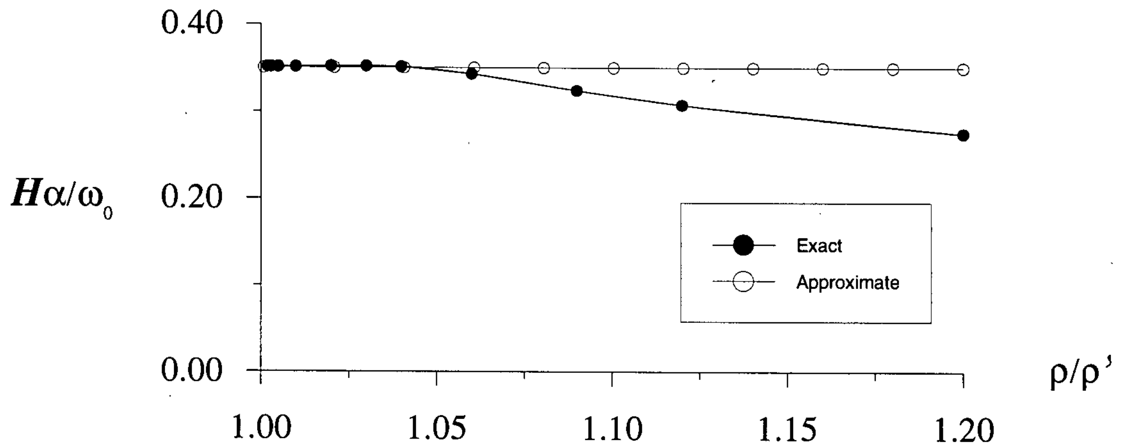


Figure 2.2 Comparison of variations of $H\alpha / \omega_0$ from the exact and the leading-order asymptotic solutions with ρ_l / ρ_u.

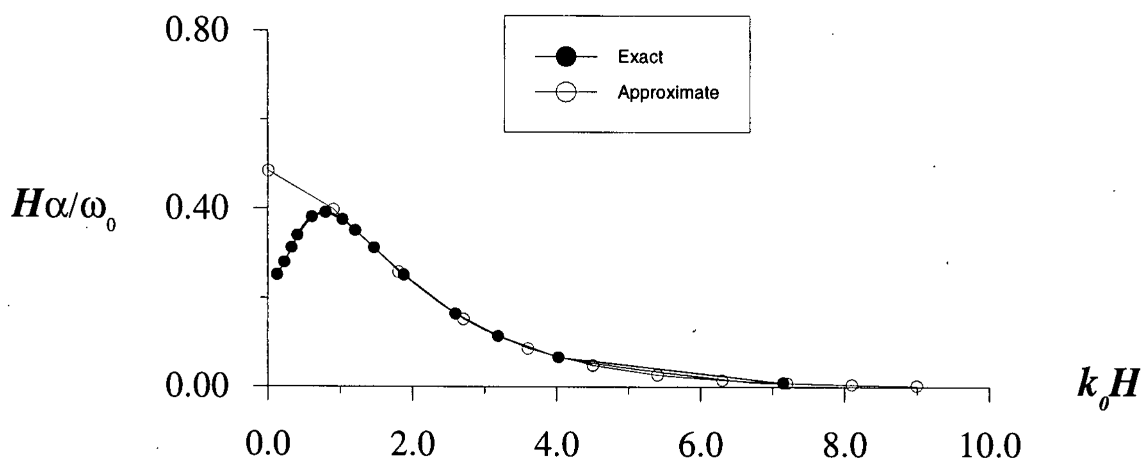


Figure 2.3 Comparison of variations of $H\alpha / \omega_0$ from the exact and the leading-order asymptotic solutions with $k_0 H$.

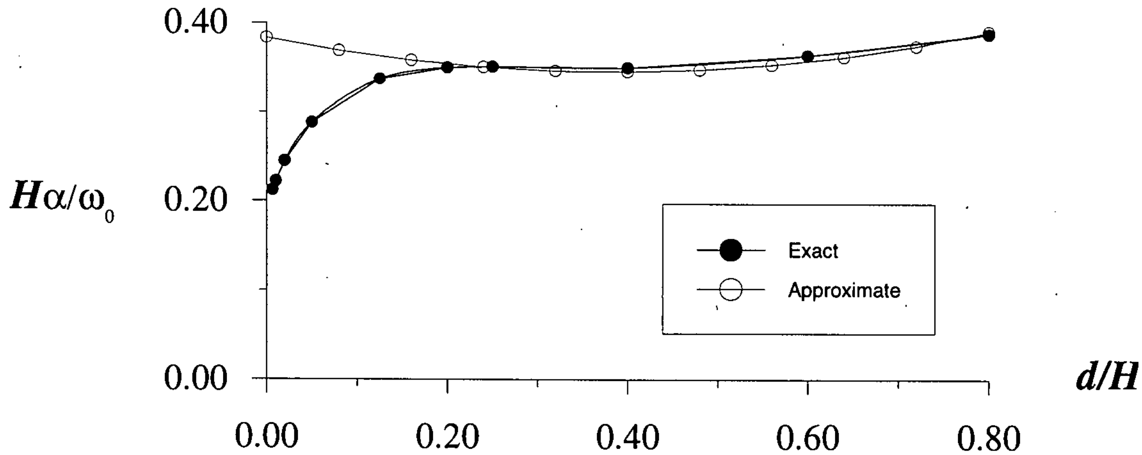


Figure 2.4 Comparison of variations of $H\alpha / \omega_0$ from the exact and the leading-order asymptotic solutions with d / H.

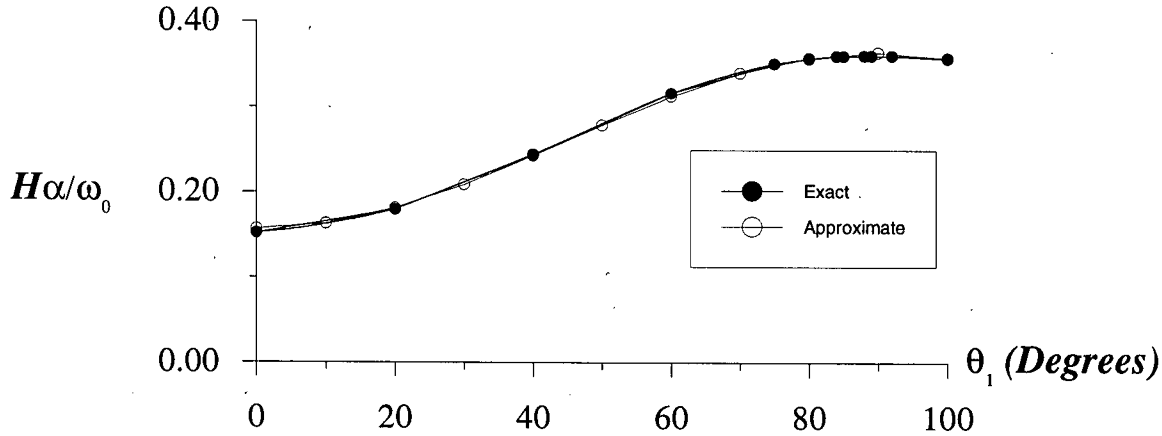


Figure 2.5 Comparison of variations of $H\alpha / \omega_0$ from the exact and the leading-order asymptotic solutions with θ_1.

CHAPTER 3

INTERACTION ON A DIFFUSE INTERFACE

3.1 INTRODUCTION

In this chapter, a two-dimensional analysis of the generation of internal waves by a progressive surface wave on a thin diffuse interface is presented. The fluid system is modeled as a combination of upper and lower layers, divided by a thin third layer. A standard weakly nonlinear wave interaction analysis is performed. By taking the non-dimensional density difference δ as the small perturbation parameter, the evolution equations of the internal waves are derived asymptotically. In this analysis, consistent with some experimental observations (Jamali, 1998), the thickness of the intermediate (diffuse) layer is assumed to be of order δ^2.

In general, a three-layer system admits two modes of internal wave motion. Accordingly, two possibilities for the interaction are obtained during the analysis: 1- interaction between a surface wave and two first-mode internal waves, 2- interaction between a surface wave, a first-mode and a second-mode internal wave. The case of interaction between a surface wave and two second-mode internal waves is shown to be impossible. The asymptotic analysis indicates that the internal waves have a higher growth rate in the first case, so in a real situation only the interaction between a surface wave and two first-mode internal waves is perceptible.

3.1.1 Resonant Triad

The present study considers a triad consisting of a surface wave (denoted as wave 0) and two internal waves (denoted as waves 1 and 2) as shown in figure 3.1. Without loss of generality, wave 0 is assumed to travel in the positive x direction. The three waves satisfy the resonance conditions

$$\begin{aligned} k_0 &= k_1 + k_2 \\ \omega_0 &= \omega_1 + \omega_2 \end{aligned} \tag{3.1}$$

where for each wave i the wave number k_i and the frequency ω_i are related by a dispersion relation. The resonance conditions (3.1) and the individual dispersion relation of each wave form a system of algebraic equations from which the wave numbers and the frequencies of the interacting waves can be determined. A three-layer system with a free surface admits a surface-wave mode and two internal-wave modes. A plot of the dispersion relations for surface and internal waves in a typical three-layer system is given in figure 3.2. It can be seen that there are two possible resonant triads: one composed of a surface wave and two first-mode internal waves, and the other composed of a surface wave and a pair of internal waves of different modes. Later, this point will be confirmed analytically.

3.2 FORMULATION

In this section, a standard three-wave interaction analysis (e.g., refer to Davis and Acrivos, 1967 and Craik, 1985) is performed. Consider a three-layer inviscid fluid system with a thin middle layer as shown in figure 3.1. The system is assumed to be two-dimensional, and horizontally infinite. The coordinate system *xyz* is located on the mid-height of the interface. The surface of the upper layer, and the bottom of the lower layer are located at $z = h$ and $z = -d$ respectively. The middle (diffuse) layer has a thickness of $2h_\rho$ as indicated. The total depth is denoted by H. The densities of the upper, the middle, and the lower layer are ρ_u, ρ_m, and ρ_l respectively, where $\rho_m = (\rho_u + \rho_l)/2$. Each layer has a constant density.

With the assumption of incompressible fluid layers and irrotational flows in the layers, the fluid motion can be described by velocity potentials $\phi'(x,z)$, $\phi''(x,z)$, and $\phi(x,z)$ in the upper, middle, and lower layer respectively. The potentials satisfy Laplace's equation in their respective layer.

$$\nabla^2\phi' = 0, \qquad h_\rho < z < h \tag{3.2}$$

$$\nabla^2\phi'' = 0, \qquad -h_\rho < z < h_\rho \tag{3.3}$$

$$\nabla^2\phi = 0, \qquad -d < z < h_\rho \tag{3.4}$$

The above equations are subject to the boundary conditions at the free surface, the interfaces of the layers, and the solid bed. On the free surface, the boundary conditions are:

$$\xi_t + \phi'_x\xi_x = \phi'_z, \qquad z = h + \xi(x,t) \tag{3.5}$$

$$\rho_u[\phi'_t + \frac{1}{2}(\phi_x'^2 + \phi_z'^2) + gz] = C'(t), \qquad z = h + \xi(x,t) \tag{3.6}$$

where $\xi(x,t) =$ the displacement of the free surface. The first equation represents a kinematic boundary condition while the second equation corresponds to a dynamic condition. At the interfaces, $z = h_\rho + \eta_u(x,t)$ and $z = -h_\rho + \eta_l(x,t)$ where η_u and η_l are the displacements of the upper and the lower interface respectively, there are four boundary conditions. Transferring these boundary conditions to the corresponding undisturbed interfaces yields the following, correct to $O(\varepsilon^2)$ where ε is a typical non-dimensional wave amplitude:

$$\phi'_z - \phi''_z = \{(\phi' - \phi'')_z \eta_u\}_x, \qquad z = h_\rho \tag{3.7}$$

$$\begin{aligned}(1-\delta/2)(\phi'_{tt} + g\phi'_z) - (\phi''_{tt} + g\phi''_z) = &\{((1-\delta/2)\phi' - \phi'')_x g\eta_u\}_x \\ &+ \tfrac{1}{2}\{(\phi''^2_x + \phi''^2_z) - (1-\delta/2)(\phi'^2_x + \phi'^2_z)\}_t \\ &- \{((1-\delta/2)\phi'_{zt} - \phi''_{zt})\eta_u\}_t \\ & \qquad z = h_\rho \end{aligned} \tag{3.8}$$

$$\phi_z - \phi''_z = \{(\phi - \phi'')_x \eta_l\}_x, \qquad z = -h_\rho \tag{3.9}$$

$$\begin{aligned}(1+\delta/2)(\phi_{tt} + g\phi_z) - (\phi''_{tt} + g\phi''_z) = &\{((1+\delta/2)\phi - \phi'')_x g\eta_l\}_x \\ &+ \tfrac{1}{2}\{(\phi''^2_x + \phi''^2_z) - (1+\delta/2)(\phi_x^2 + \phi_z^2)\}_t \\ &- \{((1+\delta/2)\phi_{zt} - \phi''_{zt})\eta_l\}_t \\ & \qquad z = -h_\rho \end{aligned} \tag{3.10}$$

where $\delta = (\rho_2 - \rho_1)/\rho_3$. At the bed, the problem is subject to a kinematic boundary condition requiring the normal velocity be zero, i.e.,

$$\phi''_z = 0, \qquad z = -d \tag{3.11}$$

3.3 PERTURBATION SOLUTION

For the purpose of the interaction analysis, it is assumed that the amplitudes of the waves are sufficiently small that a weakly nonlinear interaction analysis similar to that in chapter 2 can be performed. This implies that terms of order ε^3 and higher may be neglected. Accordingly, the following expansions in ε are considered for $\xi, \eta_u, \eta_l, \phi', \phi''$ and ϕ.

$$\xi(x,t)=\xi_0(x,t)+\xi_1(x,t)+\xi_2(x,t)+\sum_{i=0}^{2}\sum_{j=i}^{2}\xi_{ij}(x,y,t)+\textit{complex conjugate}$$

$$\eta_u(x,t)=\eta_{u,0}(x,t)+\eta_{u,1}(x,t)+\eta_{u,2}(x,t)+\sum_{i=0}^{2}\sum_{j=i}^{2}\eta_{u,ij}(x,t)+\textit{complex conjugate}$$

$$\eta_l(x,t)=\eta_{l,0}(x,t)+\eta_{l,1}(x,t)+\eta_{l,2}(x,t)+\sum_{i=0}^{2}\sum_{j=i}^{2}\eta_{l,ij}(x,t)+\textit{complex conjugate}$$

$$\phi'(x,z,t)=\phi_0'(x,z,t)+\phi_1'(x,z,t)+\phi_2'(x,z,t)+\sum_{i=0}^{2}\sum_{j=i}^{2}\phi_{ij}'(x,z,t)+\textit{complex conjugate}$$

$$\phi''(x,z,t)=\phi_0''(x,z,t)+\phi_1''(x,z,t)+\phi_2''(x,z,t)+\sum_{i=0}^{2}\sum_{j=i}^{2}\phi_{ij}''(x,z,t)+\textit{complex conjugate}$$

$$\phi(x,z,t)=\phi_0(x,z,t)+\phi_1(x,z,t)+\phi_2(x,z,t)+\sum_{i=0}^{2}\sum_{j=i}^{2}\phi_{ij}(x,z,t)+\textit{complex conjugate}$$

(3.12)

where k_i and ω_i are wave number and frequency of wave i respectively. The three interacting waves constitute the wave field at first order. The single-indexed terms, such as $\phi_{u,i}$, are first-order terms, while the double-indexed terms, such as $\phi_{u,ij}$, are second-order terms. Expansions (3.12) follow the standard procedure for three-wave interaction (e.g., see Craik, 1985).

As the three waves are in resonance, certain kinematic conditions hold between their frequencies and wavelengths. These conditions, known as kinematic conditions of resonance, are given by (3.1). They ensure a continuous and effective energy transfer between the waves.

Next, as in the previous analysis it is assumed that the time-derivative of each amplitude is a function of the product of the amplitudes of the other waves as follows.

$$\frac{da_0}{dt}=O(b_1b_2), \qquad \frac{db_1}{dt}=O(a_0\bar{b}_2), \qquad \frac{db_2}{dt}=O(a_0\bar{b}_1), \qquad (3.13)$$

where symbol overbar denotes complex conjugate. The quantity a_0 is the amplitude of the surface wave at the free surface, and b_1 and b_2 are the amplitudes of the internal waves 1 and 2 at $z=0$ respectively. The above assumption makes the time derivatives of a_0, b_1, and b_2 appear in the perturbed equations at second order. Substituting (3.12) in the governing equations and collecting first order terms results in the linear wave theory for waves 0, 1 and 2. At second order, the nonlinear interaction terms appear in the forcing functions of the resulting inhomogeneous systems of partial differential equations. The forcing terms also contain the time derivatives of the wave amplitudes. Due to the resonance conditions, the forcing functions are of the form that produces secular solutions at $O(a_0\bar{b}_1)$, $O(a_0\bar{b}_2)$, $O(b_1b_2)$, and at their complex-conjugate counterparts. To avoid secular solutions it is necessary to impose a solvability condition on the forcing functions. As in the previous analysis, applying appropriate solvability condition to the forcing functions result in three equations from which da_0/dt, db_1/dt, and db_2/dt can be explicitly found in the following forms.

$$\frac{da_0}{dt}=\alpha_0 b_1 b_2\,, \qquad \frac{db_1}{dt}=\alpha_1 a_0 \bar{b}_2\,, \qquad \frac{db_2}{dt}=\alpha_2 a_0 \bar{b}_1 \tag{3.14}$$

where α_0, α_1 and α_2 are constant.

As in the previous analysis, a_0 is assumed constant in the last two equations of (3.14), and hence the need for computing α_0 is eliminated. Simultaneous solution of the last two equations yields variations of the internal wave amplitudes with time.

In the following the linear and the second-order solutions are derived asymptotically.

3.3.1 Linear Solutions

As discussed, at first order the linear solutions of the surface and internal waves are obtained. Assuming $h_p=O(\delta^2)$, for the surface wave the leading order solution corresponds to the motion in a homogenous medium. The effects of density variation in the surface wave solution appear at higher orders.

For the internal waves, knowing that the internal wavelengths are $O(1/\delta)$ (Jamali, 1998), the free surface and the bed boundary conditions turn into infinite boundary conditions (the same is true for the system of equations at second order). As a result, for an internal wave 1 ($k>0$) the linear solution is obtained as follows.

$$\phi'(x,z,t)=C_1 e^{-kz} e^{i(kx-\omega t)}, \qquad h_\rho<z<h$$

$$\phi''(x,z,t)=\left(-\frac{ib\omega}{k} sinh(kz)+C_3 cosh(kz)\right) e^{i(kx-\omega t)}, \quad -h_\rho<z<h_\rho$$

$$\phi(x,z,t)=C_2 e^{kz} e^{i(kx-\omega t)}, \qquad -d<z<-h_\rho$$

(3.15)

where

$$C_1=\frac{-2ibe^{kh_\rho}\omega^3}{k\left(\omega^2\left(-2cosh\left(kh_\rho\right)+(\delta-2)sinh\left(kh_\rho\right)\right)+gk\delta\, sinh\left(kh_\rho\right)\right)}$$

$$C_2=\frac{-ibe^{-kh_\rho}\omega\left(\omega^2\left(-4e^{4kh_\rho}+\left(e^{4kh_\rho}-1\right)\delta\right)+\left(e^{4kh_\rho}-1\right)k\delta g\right)}{2k\left(\omega^2\left(-2cosh\left(kh_\rho\right)+(\delta-2)sinh\left(kh_\rho\right)\right)+gk\delta\, sinh\left(kh_\rho\right)\right)}$$

$$C_3=\frac{ib\omega\left(\omega^2\left((-2+\delta)cosh\left(kh_\rho\right)-2\, sinh\left(kh_\rho\right)\right)+gk\delta\, cosh\left(kh_\rho\right)\right)}{k\left(\omega^2\left(-2cosh\left(kh_\rho\right)+(\delta-2)sinh\left(kh_\rho\right)\right)+gk\delta\, sinh\left(kh_\rho\right)\right)}$$

(3.16)

and b is the amplitude of the wave at $z=0$. The corresponding dispersion relation is

$$\omega^4\left(-16e^{4kh_\rho}+\left(e^{4kh_\rho}-1\right)\delta^2\right)+8\omega^2 e^{4kh_\rho} gk\delta-\left(e^{4kh_\rho}-1\right)g^2k^2\delta^2=0, \qquad (3.17)$$

Solving the above equation for ω^2 yields two roots; the smaller corresponds to the first mode and the bigger to the second mode. Assuming $h_\rho=O(\delta^2)$, a three-term expansion for ω^2 of the first mode in terms of δ can be obtained from (3.17) as follows.

$$\omega^2 = \frac{gk\delta}{2} - \frac{gk^2 h_\rho}{2}\delta + \frac{gk^3 h_\rho^{\ 2}}{2\delta}\delta^2 + O(\delta^3), \tag{3.18}$$

For the second-mode the corresponding expansion is

$$\omega^2 = \frac{gk^2 h_\rho}{2}\delta - \frac{gk^3 h_\rho^{\ 2}}{2\delta}\delta^2 + \frac{gk^4 h_\rho^{\ 3}}{3\delta^2}\delta^3 + O(\delta^4), \tag{3.19}$$

From the above equations it can be seen that the frequency of the first mode is $O(1)$ while that of the second mode is $O(\delta^{1/2})$.

By replacing k by $-k$ in (3.16) through (3.19), the corresponding equations for internal wave 2 ($k < 0$) is obtained.

3.3.2 Solution of Resonance Conditions

Having dispersion relations for the interacting waves, the solutions to (3.1) are obtained asymptotically in terms of δ with the assumption $h_\rho = O(\delta^2)$. When both internal waves are both primary, simultaneous solution of (3.1) with the expanded dispersion relation (3.18) for waves 1 and 2 yields the following solution for ω_1, ω_2, k_1, and k_2.

$$\omega_1 = \frac{\omega_0}{2} + \frac{gk_0}{4\omega_0}\delta - \frac{h_\rho k_0 \omega_0}{4\delta^2}\delta^2 + O(\delta^3) \tag{3.20}$$

$$\omega_2 = \frac{\omega_0}{2} - \frac{gk_0}{4\omega_0}\delta + \frac{h_\rho k_0 \omega_0}{4\delta^2}\delta^2 + O(\delta^3) \tag{3.21}$$

$$k_1 = \left(\frac{\omega_0^{\ 2}}{2g}\right)\frac{1}{\delta} + \left(\frac{k_0}{2} + \frac{\omega_0^{\ 4} h_\rho}{4g^2\delta^2}\right) + \left(\frac{\omega_0^{\ 8} h_\rho^{\ 2} + k_0^{\ 2} g^4 \delta^4}{8g^3\omega_0^{\ 2}\delta^4}\right)\delta + O(\delta^2) \tag{3.22}$$

$$k_2 = -\left(\frac{\omega_0^{\ 2}}{2g}\right)\frac{1}{\delta} + \left(\frac{k_0}{2} - \frac{\omega_0^{\ 4} h_\rho}{4g^2\delta^2}\right) - \left(\frac{\omega_0^{\ 8} h_\rho^{\ 2} + k_0^{\ 2} g^4 \delta^4}{8g^3\omega_0^{\ 2}\delta^4}\right)\delta + O(\delta^2) \tag{3.23}$$

It is seen that in this mode of interaction $\omega_1, \omega_2 = O(1)$ and $k_1, k_2 = O(1/\delta)$.

When the internal waves are of different modes, e.g., when internal wave 1 is primary, and wave 2 is secondary, simultaneous solution of (3.1) with the expanded dispersion relation (3.18) for wave 1 and (3.19) for wave 2 yields the following solution for ω_1, ω_2, k_1, and k_2.

$$\omega_1 = \omega_0 - \frac{\sqrt{2}\omega_0^2\sqrt{h_\rho}}{\sqrt{g\delta}}\delta^{1/2} + \frac{4\omega_0^3 h_\rho}{g\delta^2}\delta - \frac{\sqrt{h_\rho}\left(26\omega_0^4 h_\rho - g^2 k_0\delta^2\right)}{\sqrt{2}g^{3/2}\delta^3}\delta^{3/2} + O\left(\delta^2\right) \quad (3.24)$$

$$\omega_2 = 0 + \frac{\sqrt{2}\omega_0^2\sqrt{h_\rho}}{\sqrt{g\delta}}\delta^{1/2} - \frac{4\omega_0^3 h_\rho}{g\delta^2}\delta + \frac{\sqrt{h_\rho}\left(26\omega_0^4 h_\rho - g^2 k_0\delta^2\right)}{\sqrt{2}g^{3/2}\delta^3}\delta^{3/2} + O\left(\delta^2\right) \quad (3.25)$$

$$k_1 = \left(\frac{2\omega_0^2}{g}\right)\frac{1}{\delta} - \left(\frac{4\sqrt{2}\omega_0^3\sqrt{h_\rho}}{g^{3/2}\delta}\right)\frac{1}{\delta^{1/2}} + O(1) \quad (3.26)$$

$$k_2 = -\left(\frac{2\omega_0^2}{g}\right)\frac{1}{\delta} + \left(\frac{4\sqrt{2}\omega_0^3\sqrt{h_\rho}}{g^{3/2}\delta}\right)\frac{1}{\delta^{1/2}} + O(1) \quad (3.27)$$

From the above it can be seen that in this mode of interaction $\omega_1 = O(1)$, $\omega_2 = O\left(\delta^{1/2}\right)$, and $k_1, k_2 = O(1/\delta)$.

Since second-mode internal waves have frequencies of order $\delta^{1/2}$ whereas a surface wave has a frequency of order 1, the second equation of (3.1) can never be satisfied by two second-mode internal waves. This proves that interaction between a surface wave and two second-mode internal waves is impossible.

3.3.3 Second-order Solution

At second order the evolution equations of the waves are obtained. For instance, consider the equations at $O\left(a_0\bar{b}_1\right)$. The unknowns are the terms $\phi'_{01(x,z,t)}$, $\phi''_{01(x,z,t)}$, $\phi_{01(x,z,t)}$ of the expansions for $\phi'_{(x,z,t)}$, $\phi''_{(x,z,t)}$, $\phi_{(x,z,t)}$ respectively. Since the forcing functions turn out to be in phase with internal wave 2, the following forms are considered for $\phi'_{01(x,z,t)}$, $\phi''_{01(x,z,t)}$, $\phi_{01(x,z,t)}$.

$$\phi'_{01}(x,z,t) = f_{01}(z)e^{i(k_2x-\omega_2t)}$$

$$\phi''_{01}(x,z,t) = s_{01}(z)e^{i(k_2x-\omega_2t)} \qquad (3.28)$$

$$\phi_{01}(x,z,t) = g_{01}(z)e^{i(k_2x-\omega_2t)}$$

As a result, the governing equations in terms of f_{01}, s_{01}, and g_{01} are

$$\frac{d^2 f_{01}}{dz^2} - k_2^{\,2} f_{01} = 0, \qquad h_\rho < z < h \qquad (3.29)$$

$$\frac{d^2 s_{01}}{dz^2} - k_2^{\,2} s_{01} = 0, \qquad -h_\rho < z < h_\rho \qquad (3.30)$$

$$\frac{d^2 g_{01}}{dz^2} - k_2^{\,2} g_{01} = 0, \qquad -d < z < -h_\rho \qquad (3.31)$$

$$f_{01} = 0, \qquad z = \infty \qquad (3.32)$$

$$\frac{df_{01}}{dz} - \frac{ds_{01}}{dz} = Int^*_{01}[(\phi' - \phi'')_z \eta_u](ik_2), \qquad z = h_\rho \qquad (3.33)$$

$$\begin{aligned}(1-\delta/2)\left(-\omega_2^{\,2} f_{01} + g\frac{df_{01}}{dz}\right) - \left(-\omega_2^{\,2} s_{01} + g\frac{ds_{01}}{dz}\right) &= Int^*_{01}[((1-\delta/2)\phi' - \phi'')_x g\eta_u](ik_2) \\ &+ \tfrac{1}{2} Int^*_{01}\left[\left(\phi_x''^2 + \phi_z''^2\right) - (1-\delta/2)\left(\phi_x'^2 + \phi_z'^2\right)\right](-i\omega_2) \\ &- Int^*_{01}[((1-\delta/2)\phi'_{zt} - \phi''_{zt})\eta_u](-i\omega_2) \\ &- (1-\delta/2)\left(-2i\omega_2 f_2(z)\frac{db_2}{dt}\right) - 2i\omega_2 s_2(z)\frac{db_2}{dt}\end{aligned}$$

$$z = h_\rho \qquad (3.34)$$

$$\frac{dg_{01}}{dz} - \frac{ds_{01}}{dz} = Int^*_{01}[(\phi - \phi'')_z \eta_l](ik_2), \qquad z = h_\rho \qquad (3.35)$$

$$(1+\delta/2)\left(-\omega_2^{\,2}g_{01}+g\frac{dg_{01}}{dz}\right)-\left(-\omega_2^{\,2}s_{01}+g\frac{ds_{01}}{dz}\right)=Int_{01}^{*}\left[\left((1+\delta/2)\phi-\phi''\right)_x g\eta_l\right](ik_2)$$
$$+\frac{1}{2}Int_{01}^{*}\left[\left(\phi_x''^2+\phi_z''^2\right)-(1+\delta/2)\left(\phi_x^{\,2}+\phi_z^{\,2}\right)\right](-i\omega_2)$$
$$-Int_{01}^{*}\left[\left((1+\delta/2)\phi_{zt}-\phi_{zt}''\right)\eta_l\right](-i\omega_2)$$
$$-(1+\delta/2)\left(-2i\omega_2 g_2(z)\frac{db_2}{dt}\right)-2i\omega_2 s_2(z)\frac{db_2}{dt}$$
$$z=-h_\rho \qquad (3.36)$$

$$g_{01}=0, \qquad z=-\infty \qquad (3.37)$$

where $f_2(z)$, $s_2(z)$, $g_2(z)$ are determined from

$$\begin{aligned}\phi_2'(x,z,t)&=b_2 f_2(z)e^{i(k_2x-\omega_2t)}\\ \phi_2''(x,z,t)&=b_2 s_2(z)e^{i(k_2x-\omega_2t)}\\ \phi_2(x,z,t)&=b_2 g_2(z)e^{i(k_2x-\omega_2t)}\end{aligned} \qquad (3.38)$$

and the functional Int_{01}^{*} of two functions $\phi(x,z,t)$ and $\varphi(x,z,t)$ is defined below.

$$Int_{01}^{*}\left[\phi(x,z,t)\varphi(x,z,t)\right]=\left(\phi_0(x,z,t)\overline{\varphi}_1(x,z,t)+\overline{\phi}_1(x,z,t)\varphi_0(x,z,t)\right)e^{-i(k_2x-\omega_2t)} \qquad (3.39)$$

where the indices refer to the waves, and overbar means complex conjugate. Due to the resonant conditions (3.1), all Int_{01}^{*} terms in (3.35) become purely a function of z.

In general, for the above system to have a solution, a certain solvability condition should be satisfied by the forcing terms on the right hand sides. It can be shown that the above system has the following adjoint system.

$$\frac{d^2F(z)}{dz^2}-k_2^{\,2}F(z)=0, \qquad h_\rho<z<h \qquad (3.40)$$

$$\frac{d^2S(z)}{dz^2}-k_2^{\,2}S(z)=0, \qquad -h_\rho<z<h_\rho \qquad (3.41)$$

$$\frac{d^2G(z)}{dz^2} - k_2^{\,2}G(z) = 0, \qquad -d < z < -h_\rho \tag{3.42}$$

$$F = 0, \qquad z = \infty \tag{3.43}$$

$$\frac{dF}{dz} - \frac{dS}{dz}(1 - \delta/2) = 0, \qquad z = h_\rho \tag{3.44}$$

$$S - F - \frac{\delta g}{2\omega_2^{\,2}}\frac{dS}{dz} = 0, \qquad z = h_\rho \tag{3.45}$$

$$\frac{dG}{dz} - \frac{dS}{dz}(1 + \delta/2) = 0, \qquad z = -h_\rho \tag{3.46}$$

$$S - G + \frac{\delta g}{2\omega_2^{\,2}}\frac{dS}{dz} = 0, \qquad z = -h_\rho \tag{3.47}$$

$$G = 0, \qquad z = -\infty \tag{3.48}$$

It can be easily shown that if the following substitutions are made, the above system changes to that of linear internal wave motion.

$$\begin{aligned} F(z) &= (1 - \delta/2)f(z) \\ S(z) &= s(z) \\ G(z) &= (1 + \delta/2)g(z) \end{aligned} \tag{3.49}$$

Having $f(z)$, $s(z)$, and $g(z)$ from the linear internal wave solution, the solution to the adjoint system is obtained from (3.49).

Having the adjoint system, it can be shown that the solvability condition can be expressed as follows.

$$\left\{\left(\frac{g}{\omega_2^{\;2}}\frac{dS(z)}{dz}-S(z)\right)L_1-\frac{1}{\omega_2^{\;2}}\frac{dS(z)}{dz}L_2\right\}_{z=h_p}-\left\{\left(\frac{g}{\omega_2^{\;2}}\frac{dS(z)}{dz}-S(z)\right)L_3-\frac{1}{\omega_2^{\;2}}\frac{dS(z)}{dz}L_4\right\}_{z=-h_p}=0 \tag{3.50}$$

where L_1, L_2, L_3, and L_4 are right-hand sides (3.33), (3.34), (3.35), and (3.35) respectively.

Substituting for db_2/dt from (3.14), (3.50) can be solved for the interaction coefficient α_2. Since the resulting expression is long, an attempt is made to find asymptotic expansions for α_2 and the other growth parameters. This is done below for the two modes of interaction discussed before. In the following, the interaction between a surface wave and two first-mode internal waves is referred to as mode 1, and the interaction between a surface wave and two internal waves of different modes is refereed to as mode 2.

a) First Mode of Interaction

Substituting the asymptotic forms of the linear solutions of the surface wave and the first modes of the internal waves 1 and 2 in (3.50), and solving for α_2 yields the following two-term expansion.

$$\alpha_2 \sim \alpha_2^0 + \alpha_2^1 \delta \tag{3.51}$$

where

$$\alpha_2^0 = \frac{i\omega_0\left(\omega_0^{\;2}\, sinh(k_0 d) - g k_0\, cosh(k_0 d)\right)}{4g\, sinh(k_0 H)}. \tag{3.52}$$

A Mathematica® output of α_2^1 is given in figure 3.3. It is interesting to note that the effect of the diffuse interface appears at second order. It can be easily shown that α_2^0 is also the leading order solution of the interaction in a purely two-layer fluid system. This can be explained by noting that in the present problem, the density gradient in the middle layer is of order δ^{-1} and hence goes to infinity when $\delta \to 0$. Therefore, the present problem becomes asymptotic to the two-layer problem at small δ.

To obtain α_1, use is made of the following relation between α_1 and α_2.

$$\frac{\alpha_1}{\omega_1} = \frac{\alpha_2}{\omega_2} = \frac{\alpha_0}{\omega_0 \delta} \tag{3.53}$$

The above is correct to the leading order in δ. A discussion of the above relation is given in Jamali (1998). From $\omega_1 \sim \omega_2 = O(1)$, correct to the leading order, it follows that the leading term of α_1 is given also by (3.52).

From (3.53) and the facts that both α_1 and α_2 are of order 1, it is found that $\alpha_0 = O(\delta)$. This verifies the assumption made earlier that a_0 can be taken constant in the last two equations of (3.14).

Having α_1 and α_2, the last two equation of (3.14) can be combined to yield the following equation for b_1.

$$b_1''(t) - (\alpha_1 \overline{\alpha}_2 a_0 \overline{a}_0) b_1(t) = 0 . \tag{3.54}$$

The equation for b_2 has a similar form. At large times, the solution to (3.54) is given by

$$b_1(t) \approx C_1 e^{\alpha |a_0| t} , \tag{3.55}$$

where

$$\alpha = \sqrt{\alpha_1 \overline{\alpha}_2} . \tag{3.56}$$

It can be shown that (3.55) is also the long-term solution for b_2. It can also be seen from (3.55) that the higher $\alpha|a_0|$, the faster the growth of the internal waves. The parameter $\alpha|a_0|$ is referred to as the growth parameter of the internal waves. Since α_1 and α_2 are equal at the leading order, it follows that the growth parameter $\alpha|a_0|$ is of order 1 in this mode of the interaction.

A plot of $H|\alpha_2|/\omega_0$ against $k_0 h_\rho/\delta^2$ is shown in figure 3.4. The plot corresponds to the test case $d = 0.1\,m$, $h = 0.1\,m$, $\delta = 0.05$, and $k_0 = 5.02\, rad/m$. It can be seen that the effect of diffusion appears at the second order, and from the two-term solution $|\alpha_2|$ is an increasing function of h_ρ.

The plot of $H|\alpha_2|/\omega_0$ against δ for the same case is given in figure 3.5. It can be seen that the leading-order and the two-term solutions are asymptotic when $\delta \to 0$, and $|\alpha_2|$ is an increasing function δ as well.

b) Second Mode of Interaction

In this mode of interaction, internal wave 1 is primary, and wave 2 secondary. It should be noted that since the interaction is symmetric with respect to waves 1 and 2 (Jamali, 1998), the same results are obtained if the two waves exchange their mode numbers. Substituting the asymptotic forms of the linear solutions of the respective waves in the solvability equation and solving for db_2/dt reveals that the leading order of α_2, given below, is of order δ in this mode of interaction.

$$\alpha_2 \sim \alpha_2^1 \delta \tag{3.57}$$

where

$$\alpha_2^1 = \frac{i(cosh(k_0 d) + sinh(k_0 d))\left(\dfrac{26 h_\rho \omega_0{}^4}{\delta^2} - g^2 k_0\right)^2}{\omega_0 g^3\, sinh(k_0 H)} \tag{3.58}$$

Knowing that in this mode of interaction $\omega_1 = O(1)$ and $\omega_2 = O(\delta^{1/2})$, from (3.53) it follows that $\alpha_1 = O(\delta^{1/2})$. Consequently, α turns out to be $O(\delta^{3/4})$, and hence it goes to zero when $\delta -> 0$. This result implies that in this mode of interaction the internal waves have a lower growth rate than that in the previous mode, and hence in a real situation the interaction is most anticipated between a surface wave and two first-mode internal waves.

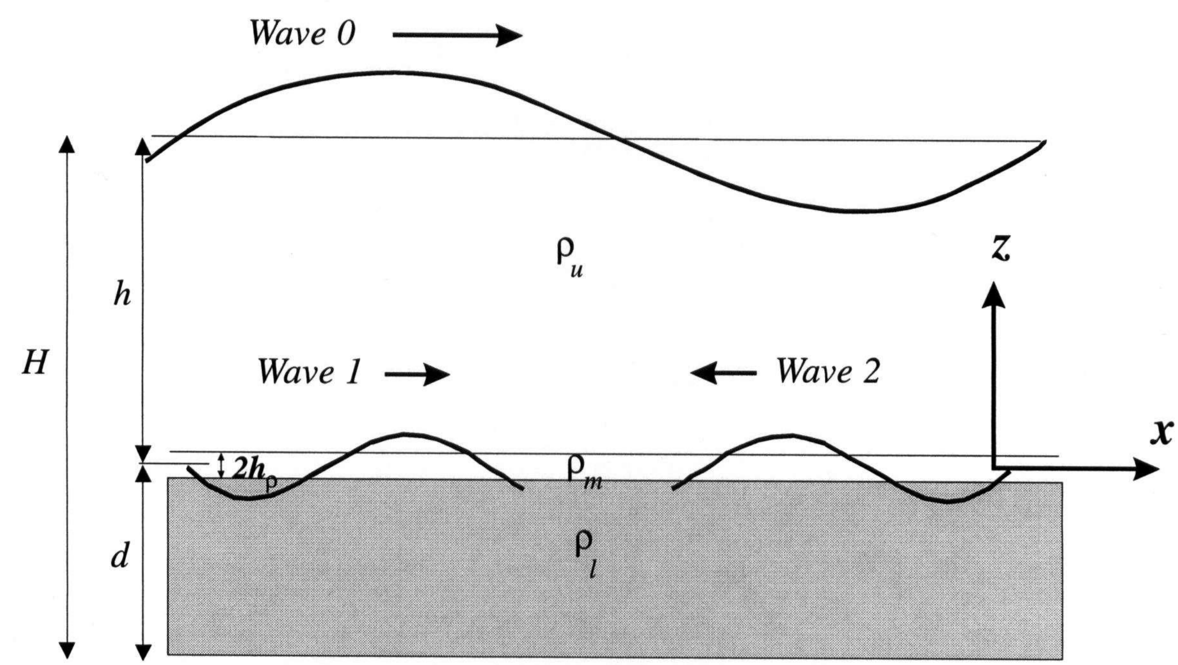
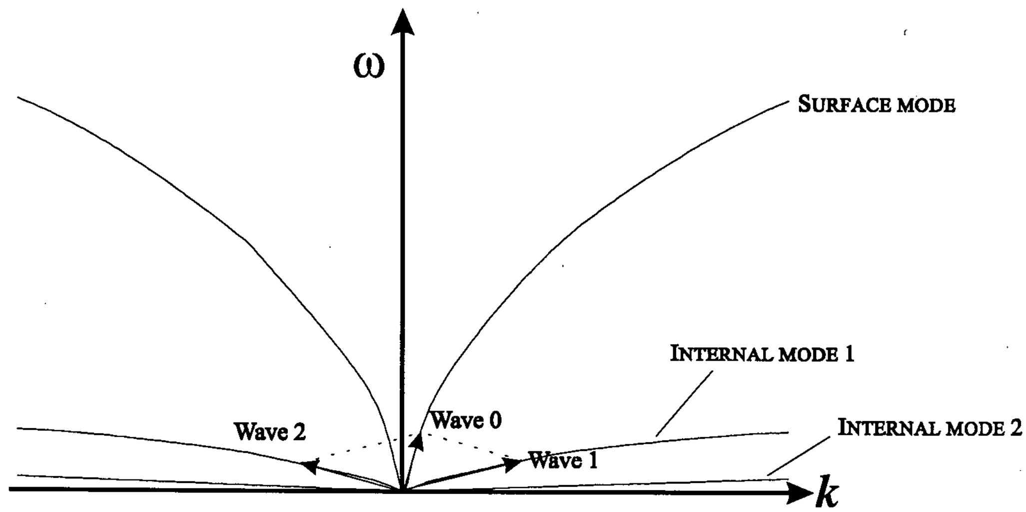
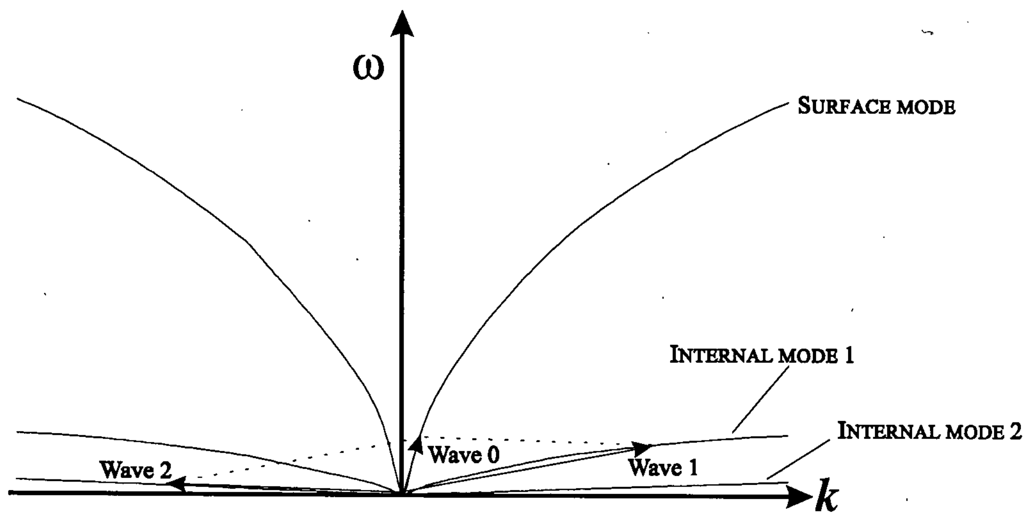


Figure 3.1 Configuration of the problem.



(a)



(b)

Figure 3.2 Graphical demonstration of resonant triads: **a)** Interaction of a surface wave with first-mode internal waves; **b)** Interaction a surface wave with internal waves of different modes.

$$
\begin{aligned}
&\frac{1}{64\,\delta^2\,\omega_0}\Big(I\,g\,\mathrm{Csch}[(d+h)\,k_0]^2\,\mathrm{Sech}[(d+h)\,k_0]\,k_0^2\,\big(2\,\delta^2\,(-2\,\mathrm{Cosh}[d\,k_0]+\\
&\quad 2\,\mathrm{Cosh}[(d-2\,h)\,k_0]+\mathrm{Sinh}[(d+2\,h)\,k_0]+\mathrm{Sinh}[(3\,d+2\,h)\,k_0])+\\
&\delta^2\,(\mathrm{Cosh}[(d+2\,h)\,k_0]-\mathrm{Cosh}[(3\,d+2\,h)\,k_0]+\\
&\quad 4\,\mathrm{Sinh}[d\,k_0]-2\,\mathrm{Sinh}[(d-2\,h)\,k_0]-2\,\mathrm{Sinh}[(d+2\,h)\,k_0])\\
&\big(\delta\,(\mathrm{Cosh}[(-d+h)\,k_0]-\mathrm{Cosh}[(d+h)\,k_0])\\
&\quad\quad \mathrm{Csch}[(d+h)\,k_0]\,\mathrm{Sech}[(d+h)\,k_0]^2+2\,\mathrm{Tanh}[(d+h)\,k_0]\big)-\\
&(\mathrm{Sinh}[(d+2\,h)\,k_0]+\mathrm{Sinh}[(3\,d+2\,h)\,k_0])\\
&h_\rho\,k_0\,\big(\delta\,(\mathrm{Cosh}[(-d+h)\,k_0]-\mathrm{Cosh}[(d+h)\,k_0])\\
&\quad\quad \mathrm{Csch}[(d+h)\,k_0]\,\mathrm{Sech}[(d+h)\,k_0]^2+2\,\mathrm{Tanh}[(d+h)\,k_0]\big)\hat{}\\
&\quad 2\big)\Big)
\end{aligned}
$$

Figure 3.3 Mathematica® output of α_2^1 in the first mode of the interaction.

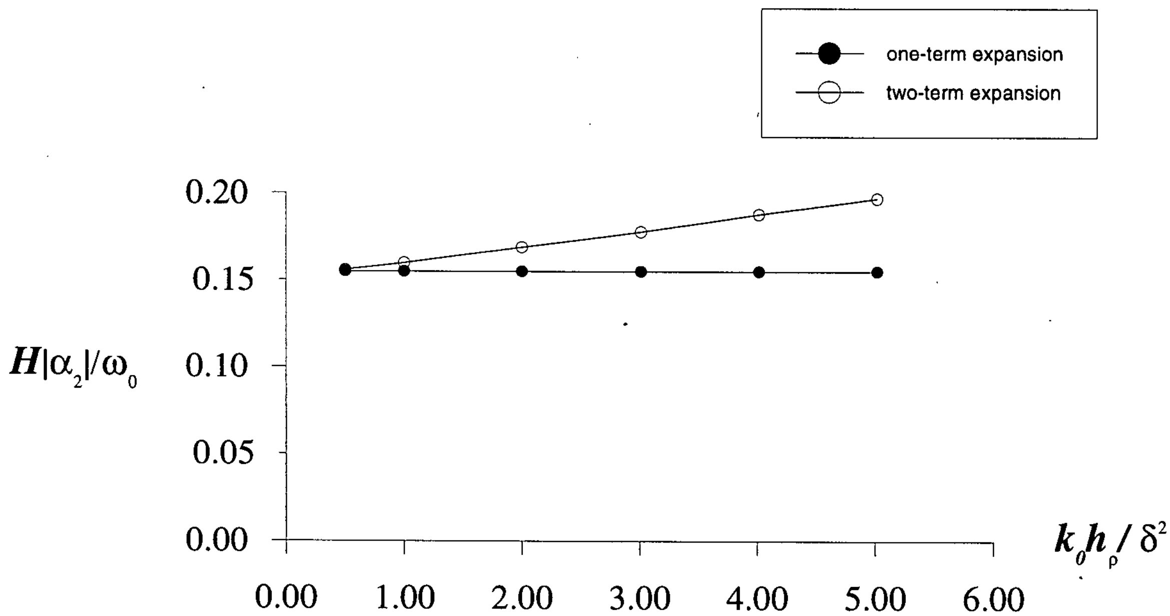


Figure 3.4 Variation of $H|\alpha_2|/\omega_0$ with k_0h_ρ/δ^2.

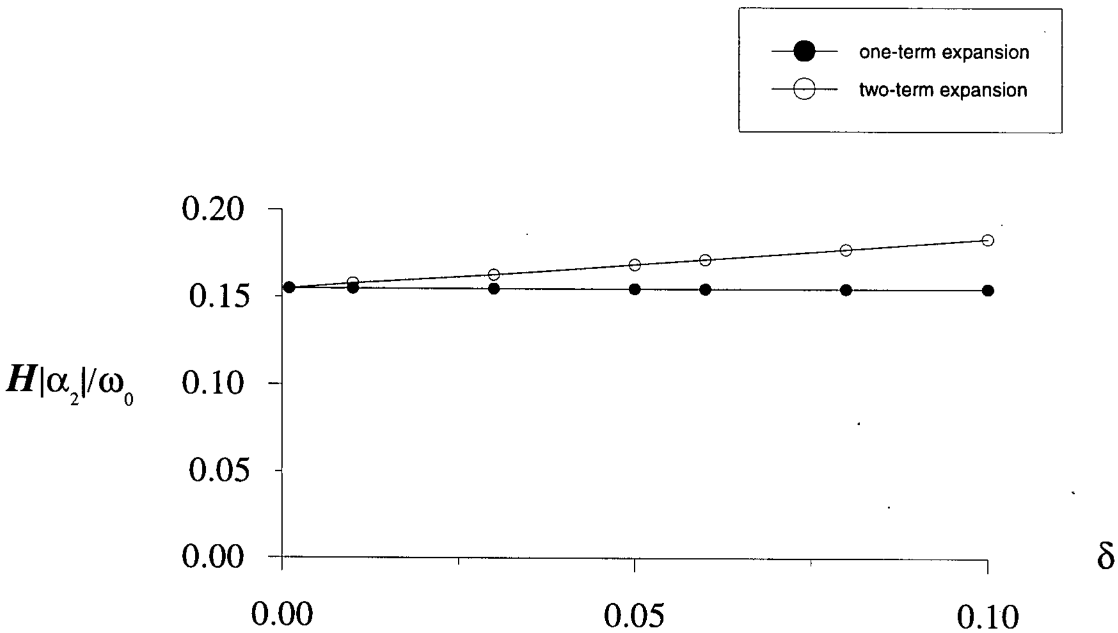


Figure 3.5 Variation of $H|\alpha_2|/\omega_0$ with δ.

CHAPTER 4

CONCLUSIONS AND RECOMMENDATIONS

4.1 ASYMPTOTIC ANALYSIS OF INTERACTION

Two subjects related to the interaction of a surface wave with two internal waves in a two-layer fluid were studied theoretically in chapters 2 and 3. These were the asymptotic behavior of the waves in three-dimensional interaction at small density difference and the effect of a diffuse interface on the interaction. The following is a summary of the two studies.

4.1.1 Three-dimensional Interaction in a Two-layer fluid

The interaction of a surface wave with two oblique internal waves were studied asymptotically in chapter 2 in an attempt to obtain simple approximate expressions for the growth rate as well as the kinematic properties of the internal waves. The non-dimensional density difference δ was taken as the perturbation parameter, and the first few terms in the expansions of the desired quantities were derived. The results indicated that the internal-wave numbers are $O(\delta^{-1})$, one order larger than the surface-wave number. It was also found that at leading order the frequency of both internal waves is equal to $\omega_0/2$, and the directions of the two internal waves differ by 180°.

According to the asymptotic analysis, the interaction coefficients α_1 and α_2 are $O(1)$ and are equal at leading order. As expected, the leading-order term of the growth parameter asymptotes the exact value when δ goes to zero. It was found that an immediate consequence of taking δ as a small parameter is that the internal waves are deep-water waves in both layers. For this reason, the asymptotic solution was found to be valid only for that range of frequency in which the surface wave motion results in excitation of deep-water internal waves.

4.1.2 Interaction on a Diffuse Interface

In chapter 3 a two-dimensional analysis of the generation of two internal waves by a surface wave on a thin diffuse interface was presented. As in the previous analysis, the non-dimensional density difference δ was taken as the small perturbation parameter. The diffuse interface was assumed to be small compared to the internal wavelengths. It was taken to be order δ^2.

A three-layer system admits two modes of internal wave motion, and similarly two modes of interaction were found to be possible. These were interaction between a surface wave and two first-mode internal waves, and the interaction between a surface wave, a first-mode and a second-mode internal wave. The case of interaction between a surface wave and two second-mode internal waves was shown to be impossible. The asymptotic analysis indicated that the growth rate in the first mode is higher than in the second. This implies that in a real situation the interaction emerges as between a surface wave and two first-mode internal waves.

4.2 RECOMMENDATIONS

In many real situations, a water body is stratified into two layers and is subject to the action of surface waves. For instance, in many lakes, tailings ponds, and muddy coastal regions a layer of fluid mud is present beneath the clear water, and the surface waves continuously disturb the interface of the two layers. Similarly, in many stratified estuaries and oceans the water body is almost two-layered, and the interface oscillates under the influence of the surface waves. The interaction of a surface wave with two sub-harmonic internal waves was found to be a strong mechanism for the instability of an interface subject to surface wave motion (Jamali, 1998). Considering that in real situations there is always a diffuse interface between the layers, the study of interaction in presence of a diffuse interface has considerable applications in mixing studies of two-layer fluids. The interaction on a diffuse interface was investigated theoretically in the present study. However, the study had two limitations: the analysis was confined to two dimensions, and the stratification considered was discontinuous, and hence unrealistic. In a real situation, the interaction is more likely to be three-dimensional as the internal waves have a greater growth rate when they are not in the same plane as the surface wave (Jamali, 1998). Also, in stratified aquatic systems, the density varies continuously across the interfacial layer. To have a better understanding of the interaction in real situations, it is suggested the mentioned limitations be removed from the interaction analysis.

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APPENDIX A

LINEAR SOLUTION

Here the solution to the linear equations of motion of a wave in a two-layer inviscid medium is presented. The equations of motion are given by (2.4) to (2.11). The wave is assumed to move in the $x-y$ plane with wave number $\vec{k}=(k_x,k_y)$ and frequency ω. The solution to the linearized equations of motion for a surface wave can be obtained as (e.g., see Lamb 1934)

$$\phi'(x,y,z)=\{C_1 Sinh(kz)+C_2 Cosh(kz)\}e^{i(k_x x+k_y y-\omega t)} \tag{A.1}$$

$$\phi(x,y,z)=\{D_1 Sinh(k(z+d))+D_2 Cosh\{k(z+d)\}\}e^{i(k_x x+k_y y-\omega t)} \tag{A.2}$$

$$\xi(x,y,t)=ae^{i(k_x x+k_y y-\omega t)} \tag{A.3}$$

$$\eta(x,y)=be^{i(k_x x+k_y y-\omega t)} \tag{A.4}$$

where

$$C_1=\frac{ia(gkSinh(kh)-\omega^2 Cosh(kh))}{k\omega} \tag{A.5}$$

$$C_2=\frac{-ia(gkCosh(kh)-\omega^2 Sinh(kh))}{k\omega} \tag{A.6}$$

$$D_1 = 0 \tag{A.7}$$

$$D_2 = \frac{ia(\, gkSinh(\, kh\,) - \omega^2 Cosh(\, kh\,))}{k\omega Sinh(\, kd\,)} \tag{A.8}$$

$$b = a\left(Cosh(\, kh\,) - \frac{gkSinh(\, kh\,)}{\omega^2} \right) \tag{A.9}$$

In the above equations, a is the amplitude of the surface wave, and the system parameters are defined in figure 2.1. The dispersion relation can be written as

$$\frac{\frac{\rho_u}{\rho_l}(\, \omega^4 - g^2 k^2\,) tanh(\, kh\,)}{(\, gk\, tanh(\, kh\,) - \omega^2\,)} + gk - \omega^2\, coth(\, kd\,) = 0 \tag{A.10}$$

For the motion of an internal wave, the above equations are still valid. However, It is convenient to write the coefficients in equations A.1 to A.4 in terms of the internal wave amplitude b:

$$C_1 = \frac{-ib\omega}{k} \tag{A.11}$$

$$C_2 = \frac{ib\omega\left(gkCosh(kh) - \omega^2 Sinh(kh)\right)}{-k\left(\omega^2 Cosh(kh) - gkSinh(kh)\right)} \tag{A.12}$$

$$D_1 = 0 \tag{A.13}$$

$$D_2 = \frac{-ib\omega Csch(\, kd\,)}{k} \tag{A.14}$$

$$a = \frac{b\omega^2}{\left(\omega^2 Cosh(\, kh\,) - gkSinh(\, kh\,)\right)} \tag{A.15}$$

The dispersion relation remains the same.