# HEEGAARD DIAGRAMS AND APPLICATIONS

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## Abstract

The main objective of this thesis is to study Heegaard diagrams and their applications. First, we investigate Heegaard diagrams of closed 3-manifolds and introduce the circle and chord presentation for a connected, closed 3-manifold. The equivalence problem for Heegaard diagrams after connected sum moves and Dehn twists will be investigated. Presentations will be used to detect reducible Heegaard diagrams and homeomorphic 3-manifolds.

We also investigate Heegaard diagrams of the 3-sphere. The main result of this part is that if two Heegaard diagrams of the 3-sphere have the same genus, then there is a sequence of connected sum moves and Dehn twists to pass from one to the other. If we use connected sum moves only, Heegaard curves can be changed to primitive curves and if we use Dehn twists only Heegaard curves can be brought into a simple position.

Finally, we construct an immersion of a compact, orientable, connected 3-manifold with non-empty boundary into $\mathbb{R}^3$ with at most double and triple points as singularities. Further, we prove that if the boundary of the 3-manifold consists of 2-spheres and the 3-manifold can immerse into $\mathbb{R}^3$ with only double points as singularities, then the 3-manifold must be a punctured 3-sphere or a punctured $(\mathbb{S}^1 \times \mathbb{S}^2)\sharp \cdots \sharp(\mathbb{S}^1 \times \mathbb{S}^2)$.

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## Introduction

In the early 1960s, W.Haken [13], [12] introduced the theory of normal surfaces and he exhibited an algorithm to detect embedded, 2-sided, closed, incompressible surfaces in closed, irreducible 3-manifolds. His algorithm will eventually stop if such surfaces exist. But the algorithm had no bound for termination. W.Jaco and U.Oertel [19] obtained an efficient algorithm with such a bound. Jaco and Oertel's algorithm and a result on the conjugacy problem in the mapping class group of surfaces (Hemion [16] or Hatcher and Thurston [14]) gives an algorithm to decide if two closed, irreducible 3-manifolds are homeomorphic assuming one of the 3-manifolds contains an embedded 2-sided incompressible surface. J.H.Rubinstein [33], [34] generalized normal surfaces to almost normal surfaces and used them to solve the recognition problem for the 3-sphere, i.e., to decide if a given closed 3-manifold is the 3-sphere. A.Thompson [38] also gave an algorithm to recognize the 3-sphere by using thin position.

In the first part of the thesis, we investigate Heegaard diagrams of closed 3-manifolds and introduce a new representation method for connected, closed 3-manifolds. Recall that a Heegaard splitting of genus $g$ of a closed 3-manifold $M$ decomposes $M$ into two handlebodies $V, W$ of genus $g$. The boundary curves $b_1, b_2, \cdots, b_g$ of the disks in a complete meridian disk system in $W$ (i.e., cutting $W$ along the disks is a 3-cell) lie in the surface $\partial V$ as a complete system (i.e., cutting $\partial V$ along the curves is a disk). The (one-sided) Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ completely determines $M$. After choosing a suitable complete meridian disk system $D_1, D_2, \cdots, D_g$ of $V$, we can find sub-arcs $c_1, c_2, \cdots, c_{2g-1}$ of the curves $b_1, b_2, \cdots, b_g$ whose end-points lie in the curves $\partial D_1, \partial D_2, \cdots, \partial D_g$ such that cutting $\partial V$ along the sub-arcs $c_1, c_2, \cdots, c_{2g-1}$ and the curves $\partial D_1, \partial D_2, \cdots, \partial D_g$ leaves a disk. The remaining parts of the Heegaard curves $b_1, b_2, \cdots, b_g$ are chords in the disk. We can use the disk, the chords and the labels which denote the intersection points between the curve set $\{b_1, b_2, \cdots, b_g\}$ and the curve set $\{\partial D_1, \partial D_2, \cdots, \partial D_g\}$ to represent the 3-manifold $M$. We give a necessary and sufficient condition that a circle with some chords and labeled endpoints represents a 3-manifold (Theorem 2.6).

We also investigate the equivalence problem for Heegaard splittings of closed 3-manifolds. We prove that a Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ can be changed to another Heegaard diagram $(V; b_1', b_2', \cdots, b_g')$ by using a sequence of connected sum moves if and only if $b_1', b_2', \cdots, b_g'$ have trivial reduced words corresponding to the complete system $b_1, b_2, \cdots, b_g$ of the surface $\partial V$ (Theorem 2.2). Properties of connected sum moves and Dehn twists are investigated. Combining these results with results about stabilization of Heegaard splittings ([11], [36], [23], [20], [35]), we obtain an algorithm to detect if two closed, connected 3-manifolds are homeomorphic (Theorem 2.12). It is an open problem to find a bound for this algorithm.

In the second part of the thesis, Heegaard diagrams of the 3-sphere are investigated. The main result of this part is that if two Heegaard diagrams of the 3-sphere have the same genus, then there is a sequence of connected sum moves and Dehn twists to pass from one to the other (Theorem 3.3). For every Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ of the 3-sphere, we prove the following two properties. First, if the handlebody $V$ lies in the 3-dimensional Euclidean space $\mathbb{R}^3$ standardly (i.e., $\mathbb{R}^3 - Int\ V$ is homeomorphic to a handlebody of genus $g$ with a 3-cell removed), then we can use Dehn twists on $V$ to change the Heegaard curves $b_1, b_2, \cdots, b_g$ to new Heegaard curves $b_1', b_2', \cdots, b_g'$ such that $b_1', b_2', \cdots, b_g'$ bound pairwise disjoint disks in $\mathbb{R}^3 - Int\ V$ (Theorem 3.5). Second, we can use connected sum moves on the curves $b_1, b_2, \cdots, b_g$ to obtain new Heegaard curves $b_1', b_2', \cdots, b_g'$ such that $b_1', b_2', \cdots, b_g'$ are primitive curves of $V$; i.e., they form a free base of the fundamental group of the handlebody $V$ (Theorem 3.4).

In the third part of the thesis, we construct an immersion of any given compact, orientable, connected 3-manifold with non-empty boundary into $\mathbb{R}^3$ with at most double and triple points as singularities by using a Heegaard diagram of the 3-manifold (Theorem 4.1). This is stronger version of a special case of a result of J.H.C.Whitehead [41] which does not have a bound on the multiplicity of the singularities. Further, we prove that if a compact, orientable, connected punctured 3-manifold can immerse into $\mathbb{R}^3$ with only double points as singularities, then the 3-manifold must be a punctured 3-sphere or a punctured $(S^1 \times S^2)\sharp(S^1 \times S^2)\sharp \cdots \sharp(S^1 \times S^2)$ (Theorem 4.2).

# Chapter 1

## Notation and Preliminaries

We will work throughout in the piecewise linear category. All manifolds and all maps are piecewise linear. All intersection points are in general position. The term homeomorphism always means a piecewise linear homeomorphism. Our reference is [17] and [31].

A 3-manifold $V$ is called a *handlebody of genus $g$* if there exist $g$ disjoint proper 2-cells $D_1, \cdots, D_g$ in $V$ such that if $D_i \times [-\epsilon, \epsilon], i = 1, \cdots, g$ are disjoint regular neighborhoods of $D_i = D_i \times 0, i = 1, \cdots, g$ in $V$ then $C = V - (D_1 \times (-\epsilon, \epsilon) \cup \cdots \cup D_g \times (-\epsilon, \epsilon))$ is a 3-cell. The disks $D_1, \cdots, D_g$ are called a *complete meridian system* of $V$. The 3-cell $C$ is called a *cut* of $V$ along $D_1, \cdots, D_g$.

A *Heegaard splitting* $(V, W)$ *of genus $g$* of the closed connected 3-manifold $M$ consists of two handlebodies $V, W$ of genus $g$ in $M$ such that $M = V \cup W$ and $V \cap W = \partial V = \partial W$. If $\{B_1, \cdots, B_g\}$ is a complete meridian disk system of $W$ and $\partial B_i = b_i, i = 1, \cdots, g$, then $(V; b_1, \cdots, b_g)$ is called a *(one-sided) Heegaard diagram of genus $g$* of $M$. Note that a Heegaard diagram determines the 3-manifold $M$ (i.e., $M$ can be obtained by attaching 2-handles along disjoint regular neighborhoods $b_i \times [-\epsilon, \epsilon], i = 1, \cdots, g$ in $\partial V$, plus one 3-handle.

Two Heegaard splittings $(V, W), (V', W')$ of $M$ are called *strongly equivalent* if there exists an ambient isotopy $h_t : M \longrightarrow M, 0 \leq t \leq 1$, such that $h_0 = \mathit{identity}$ and $h_1(V, W) = (V', W')$. Following Birman [2] , we call two Heegaard splittings $(V, W), (V', W')$ *equivalent* if there exist a homeomorphism $h : M \longrightarrow M$ with $h(V) = V', h(W) = W'$

or $h(V) = W', h(W) = V'$.

Let $(V, W)$ be a Heegaard splitting of genus $g$ of the closed connected 3-manifold $M$. Let $\gamma \subset V$ be an arc and $D \subset V$ be a disk such that $\partial D = \gamma \cup (D \cap \partial V), \partial\gamma = \gamma \cap \partial V$. If $N(\gamma)$ is a closed regular neighborhood of $\gamma$, then $V' = V \cup N(\gamma)$ and $W' = Cl(W - N(\gamma))$ are handlebodies of genus $g+1$. Therefore, $(V', W')$ is a Heegaard splitting of genus $g+1$ of $M$. This construction from the Heegaard splitting $(V, W)$ to $(V', W')$ is called *handle addition.* The inverse procedure is called a *reduction.* A Heegaard splitting is called *minimal* or *irreducible* if it can not be reduced.

**Reidemeister-Singer Theorem.** *Let $M$ be a connected, closed 3-manifold and let $(V_1, W_1)$, $(V_2, W_2)$ be two Heegaard splittings of $M$. Then there exist Heegaard splitting $(\tilde{V}_1, \tilde{W}_1), (\tilde{V}_2, \tilde{W}_2)$ of $M$ such that $(\tilde{V}_1, \tilde{W}_1)$ reduces to $(V_1, W_1)$ and $(\tilde{V}_2, \tilde{W}_2)$ reduces to $(V_2, W_2)$ and $(\tilde{V}_1, \tilde{W}_1)$, $(\tilde{V}_2, \tilde{W}_2)$ are strongly equivalent.*

Now, we state the definitions of three well-known fundamental moves on some curves in the boundary surface of a handlebody here.

**Definition 1.1** *Suppose that $V$ is a handlebody of genus $g$ and $c_1, c_2, \cdots, c_m$ are pairwise disjoint, simple closed curves in $\partial V$. Then we can use the following three kinds of moves to change the positions of the curves in $\partial V$.*

*I. Connected sum or handle sliding. We replace the curve $c_i$ by $c_i \sharp_\alpha c_j$ in the curve set $c_1, c_2, \cdots, c_m$ as in the following figure (where $i, j = 1, 2 \cdots, m; i \neq j$, and $\alpha$ is an arc in $\partial V$ whose endpoints $P$, $Q$ lie in the curves $c_i$ and $c_j$ respectively such that $\alpha \cap (\cup_{k=1}^m c_k) = \{P, Q\}$. (See Fig. 1.1.)*

**Note.** *Connected sum moves are defined in an orientable surface, i.e., we can ignore the handlebody itself and only consider the moves in its boundary.*

**Note.** *If $(V, W)$ is a Heegaard splitting of $M$ and $(V; b_1, b_2, \cdots, b_g)$ a corresponding Heegaard diagram defined by a complete meridian system $B_1, B_2, \cdots, B_g$ of $W$ with $\partial B_i =$*

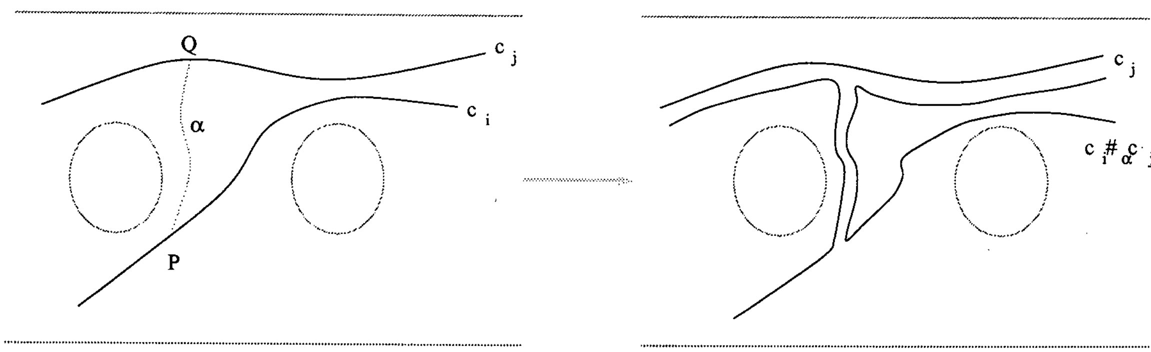



Figure 1.1: Connected sum move

*$b_i, i = 1, 2, \cdots, g$, and if $b_i \sharp_c b_j$ is a connected sum move (to replace $b_i$) on $\partial V = \partial W$, then there is a complete meridian system $B_1, \cdots, B_{i-1}, B, B_{i+1}, \cdots, B_g$ of $W$ with $\partial B = b_i \sharp_c b_j$.*

*II. Dehn twist. Suppose $D$ is a meridian disk in $V$, i.e., cutting $V$ along $D$ is connected. Then we use the following Dehn twist on the curve set. (See Fig. 1.2.)*

**Note.** *If $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of $M$ and $b'_1, \cdots, b'_g$ are the curves obtained by applying a Dehn twist $T_D$ (where $D$ is a properly embeded, non-separating disk in $V$) to $b_1, b_2, \cdots, b_g$. Then $(V; b'_1, b'_2, \cdots, b'_g)$ is a Heegaard diagram of $M'$ with Heegaard splitting $(V, W')$ such that there is a homeomorphism $h : M \longrightarrow M'$ with $h(V, W) = (V, W')$ and $h \mid_V = T_D$.*

**Note.** *Sometimes we need to use one kind of move similar to a Dehn twist. Suppose $D$ is a proper disk in $V$ that separates $V$ into two connected components. We rotate one of the connected components with angle $\pi$ along the disk $D$. (See Fig. 1.3)*

*III. Handle addition (or stabilization) and reduction. Suppose $D_1, D_2$ are two small disjoint disks in $\partial V$ and $(D_1 \cup D_2) \cap (\cup_{i=1}^{m} c_i) = \emptyset$. Then we add a 1-handle to $V$ along*

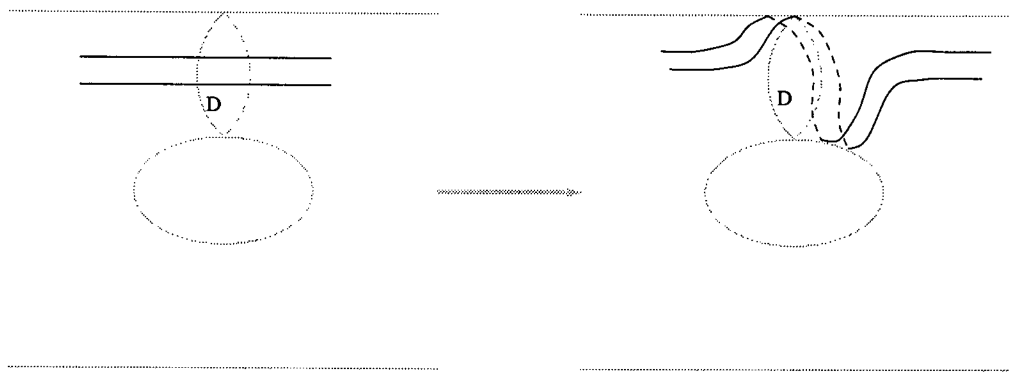



Figure 1.2: Dehn twist

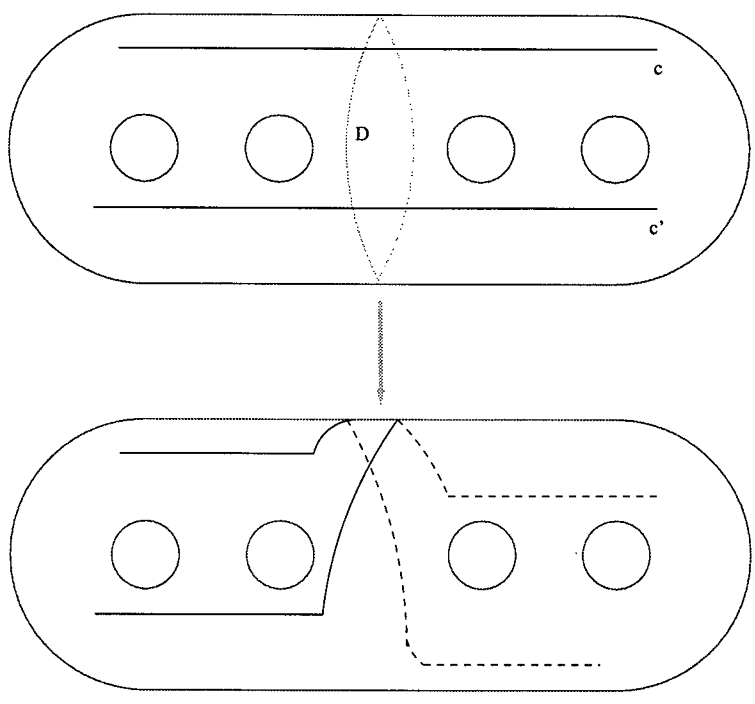



Figure 1.3: Twist handle

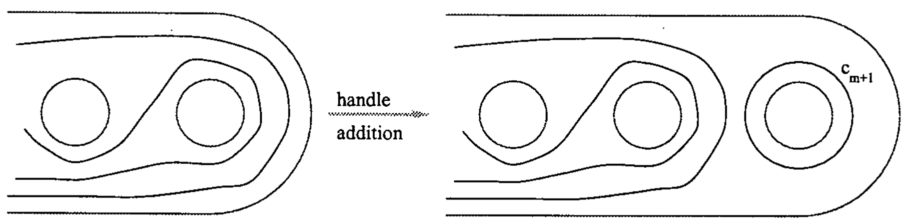



Figure 1.4: Handle addition

*$D_1, D_2$ and let $c_{m+1}$ be a simple closed curve in the boundary of the 1-handle as in Fig. 1.4. That is, $c_{m+1}$ is the union of an arc in $\partial V - (Int(D_1) \cup Int(D_2) \cup (\cup_{i=1}^{m} c_i))$ and an arc in the handle. The inverse move of a stabilization is called a reduction. A Heegaard splitting is called minimal or irreducible if it cannot be reduced, that is, it can not be obtained by a stabilization of a Heegaard splitting of smaller genus.*

**Remark.** It is clear that the inverse move of a connected sum or a Dehn twist is still a connected sum or a Dehn twist, respectively. And the inverse move of a handle addition is a handle reduction, i.e. removing a 1-handle from $H$ and removing the curve $c_m$ if there exists a proper embedded disk $D$ in the 1-handle and a curve, say $c_m$, in the curve set such that $\partial D \cap (\cup_{i=1}^{m} c_i) = \partial D \cap c_m$ and $\partial D$ and $c_m$ intersect transversely at one point.

Sometimes, we will consider other special curves in the boundary of a handlebody $V$. A simple closed curve in $\partial V$ is called *primitive* if there is a meridian curve in $\partial V$ such that the two curves intersect transversely at one point. *A complete primitive system* of $V$

is a complete system $\{b_1, b_2, \cdots, b_g\}$ of $\partial V$ which in addition satisfies the property that there exists a complete meridian system $\{d_1, d_2, \cdots, d_g\}$ of $V$ such that $b_i$ intersects $d_i$ transversely at one point and $b_i$ does not intersect $d_j$, where $i, j = 1, 2, \cdots, g; i \neq j$. The concept *primitive* refers to primitive elements of free groups. (See Definition 1.2.)

**Definition 1.2** *A set of primitive elements of a free group $F$ is a set of elements which can be completed to a set of free generators for $F$.*

Although primitive curves are curves which have a simple property, they may have infinite possible positions in the surface. By a result of Zieschang [43], also [10], if we choose a complete meridian system for the handlebody, then the word of a primitive curve corresponding to the complete system is conjugate to a word which can be extended to a basis of the fundamental group of the handlebody. It is clear that the above condition is a necessary and sufficient condition. By Whitehead's results about automorphisms of free groups [42], there is an algorithm to determine if an element $w$ of a free group $F$ of finite rank is or is not primitive. Therefore, this gives an algorithm to determine whether a curve in the boundary of a handlebody is a primitive curve or not. Gordon [10] proved another necessary and sufficient condition: A curve in the boundary of a handlebody of genus $g$ is primitive if and only if adding a 2-handle to the handlebody along the curve yields a handlebody with genus $g - 1$.

The above results of Zieschang and Gordon imply that to check whether a simple closed curve in the boundary of a handlebody is primitive or not it is only necessary to check whether or not the word of the curve with respect to a complete meridian system of the handlebody is conjugate to a primitive element in the free group generated by the elements which are corresponding to the curves in the complete meridian system.

There is a well-known simple method to check whether a simple closed curve bounds a disk in a handlebody or not.

**Lemma 1.1** *Supposse $V$ is a handlebody of genus $g$ and $d_1, d_2, \cdots, d_g$ is a complete meridian system of $V$. If $D$ is a properly embedded disk in $V$ then the reduced word of $\partial D$ corresponding to the complete meridian system is 1, i.e., the empty word. On the other hand, suppose $c$ is a simple closed curve on $\partial V$. If the cyclically reduced word of $c$ corresponding to a complete meridian system is trivial, then $c$ bounds a disk in the handlebody.*

*Proof.* Since the fundamental group of $V$ is a free group generated by the generators corresponding to the complete meridian system and $\partial D$ can shrink to one point in $V$, then $[\partial D]$ representes the trivial element 1 in the free group. Therefore, the reduced word of $\partial D$ corresponding to the complete system is 1.

On the other hand, if the reduced word of $c$ corresponding to a complete meridian system is 1, then we consider a regular neighborhood $N$ of the curve $c$ in the surface $\partial V$. Suppose $i : N \to V$ is the inclusion map. Since $ker(i_* : \pi_1(N) \to \pi_1(V)) \neq \emptyset$, then by the loop theorem [29], there exists a properly embedded disk $D$ in $V$ such that $\partial D \subset N$ and $\partial D$ can not be moved to a point in $N$ continuously. Note that $\partial D$ is a simple closed curve. Thus we may assume that the curve $\partial D$ is just the curve $c$. Therefore, $c$ bounds a disk $D$ in the handlebody.

□

There are some fundamental results on free groups which are useful for determining primitive elements of a free group.

**Definition 1.3** *(Nielsen transformation). Suppose that $w_1, w_2, \cdots, w_n$ are words in the generator set $X = \{x_1, x_2, \cdots, x_n\}$ of the free group $F(X)$. An elementary Nielsen transformation of the word set $W = \{w_1, w_2, \cdots, w_n\}$ is of one of the following three types: for some $i, 1 \leq i \leq n$,*

*(1) replace $x_1$ by $x_1^{-1}$,*

*(2) interchange $x_1$ and $x_i$, leaving others the same,*

*(3) replace $x_1$ with $x_1x_2$, all others fixed.*

**Nielsen Theorem.** Suppose $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ are two sets of words in the generator set $X = \{x_1, x_2, \cdots, x_n\}$ of a free group $F(X)$. If $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ are bases of $F(X)$, then a finite sequence of elementary Nielsen transformations will change $u_1, u_2, \cdots, u_n$ to $v_1, v_2, \cdots, v_n$.

On the other hand, Birman established a simple criterion to show whether or not a word set is a basis of a free goup $F$ in terms of the free differential calculus which was introduced by Fox in [Fox].

**Birman Theorem.** A set of words $\{w_1, w_2, \cdots, w_n\}$ in the generator set $X = \{x_1, x_2, \cdots, x_n\}$ of the free group $F(X)$ is a basis for $F(X)$ if and only if Fox matrix

$$(\frac{\partial w_i}{\partial x_j})_{n\times n}$$

is invertible in the group ring $\mathbb{Z}[F(X)]$, where the Fox derivative

$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F(X)] \to \mathbb{Z}[F(X)]$$

is given by

$$\frac{\partial}{\partial x_j}(x_{\mu_1}^{\epsilon_1} \cdot x_{\mu_2}^{\epsilon_2} \cdot \cdots \cdot x_{\mu_r}^{\epsilon_r}) = \sum_{i=1}^{r} \epsilon_i \delta_{\mu_i, j} x_{\mu_1}^{\epsilon_1} \cdot \cdots \cdot x_{\mu_{i-1}}^{\epsilon_{i-1}} \cdot x_{\mu_i}^{\frac{1}{2}(\epsilon_i - 1)};$$

(where $\epsilon_i = 1$ or $-1$; and $\delta_{\mu_i,j}$ is the Kronecker $\delta$).

Birman's theorem determines an algorithm to see whether a curve on the boundary surface of a handlebody is primitive or not.

# Chapter 2

# Geometry and Algebra of Heegaard splittings

In this chapter, $M$ will always denote a closed, connected, orientable 3-manifold.

In this chapter, we consider two operations on Heegaard diagrams of a 3-manifold: *Connected sum move* ( or called *handle sliding* by some authors ) and *Dehn twist.*

We also introduce a new tool, the circle and chord presentation of a closed, orientable 3-manifolds. Using this presentation, we will obtain the following results.

1. An algorithm to list all possible circle and chord presentations of closed, orientable 3-manifolds. Furthermore, we obtain a method to list all closed orientable 3-manifolds.

2. An algorithm to detect whether the corresponding Heegaard diagram of a circle and chord presentation is reducible or not. This will also give a method to detect whether or not a closed, orientable 3-manifold is the 3-dimensional sphere.

3. Relations between strong equivalence and equivalence.

## 2.1 Connected sum move

**Definition 2.1** *Let $F$ be an orientable closed surface of genus $g$. A set of $k$ pairwise disjoint, oriented, simple closed curves in $F$ is called non-separating if the result of cutting $F$ along these curves is connected. If furthermore, $k = g$, then we call the curve set a complete system of $F$. If $V$ is a handlebody of genus $g$, then an oriented curve $c$ is called a meridian curve of $V$ if $c$ is a non-separating curve on $\partial V$ and $c$ bounds a proper disk $D$ in $V$. $D$ is called a meridian disk of $V$. A complete meridian system of $V$ is a complete system of $\partial V$ which in addition satisfies the property that every curve in the system is a*

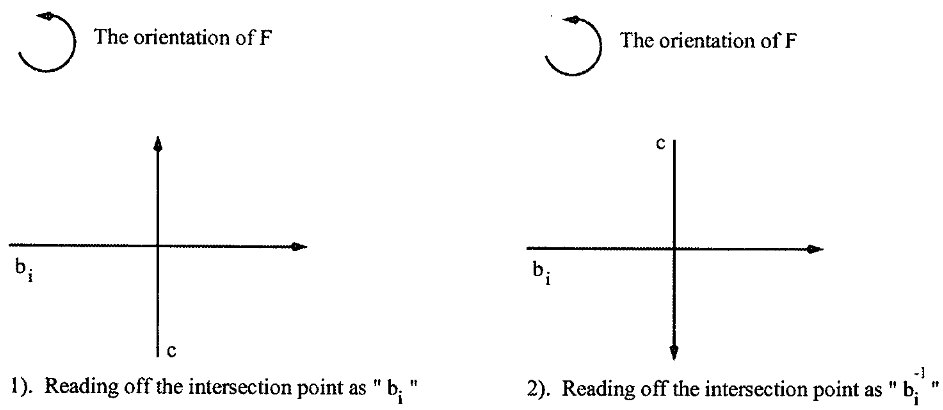



Figure 2.5: Reading word of the intersection points

*meridian curve of $V$.*

**Definition 2.2** *Suppose $\{b_1, b_2, \cdots, b_g\}$ is a complete system of an oriented closed surface $F$ of genus $g$. Suppose $c$ is an oriented simple closed curve in $F$. If we travel along $c$ in direction of its orientation starting at a point on $c$, and reading off the intersection points between $c$ and the curve set $\{b_1, b_2, \cdots, b_g\}$ according to the orientations of the respective two subarcs near each intersection point ( see Fig. 2.5 ), then we obtain a word $w(b_1, b_2, \cdots, b_g)$. We call the word $w(b_1, b_2, \cdots, b_g)$ the word of $c$ corresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$.*

*Note.* In Definition 2.2, since $\{b_1, b_2, \cdots, b_g\}$ is a complete system of $F$, there is another complete system $\{a_1, a_2, \cdots, a_g\}$ such that $b_i \cap a_j = \emptyset$ for $i, j = 1, 2, \cdots, g; i \neq j$ and $b_i$ intersects $a_i$ at one point. After isotopically moving the curves $b_1, \cdots, b_g, a_1, \cdots, a_g$ in $F$ to let them be attached to one point $P \in F$, cutting $F$ along the new curves which are still denoted $b_1, \cdots, b_g, a_1, \cdots, a_g$ leaves an open disk. Therefore, the fundamental group $\pi_1(F)$ has $2g$ generators $b_1, b_2, \cdots, b_g, a_1, a_2, \cdots, a_g$ and a relator $R =$

$a_1b_1a_1{}^{-1}b_1{}^{-1}a_2b_2a_2{}^{-1}b_2{}^{-1}\cdots a_gb_ga_g{}^{-1}b_1{}^{-1}$. By the Independence Theorem ( see Theorem 4.10 in [26] ), the subgroup of $\pi_1(F)$ generated by $b_1, b_2, \cdots, b_g$ is a free group and $w(b_1, \cdots, b_g)$ is an element of this free group.

**Note.** When we isotopically move the curve $c$ to a new curve $c'$ in $F$, the word $w'$ of $c'$ corresponding to the complete system is given by inserting or removing some cancelling pairs of generators.

Among the equivalent moves of Heegaard diagrams, *connected sum moves* ( or *handle sliding* ) is the most important of the moves because of the following two reasons. One is that only *connected sum moves* can decrease lengths of cyclically reduced words of Heegaard curves. Another reason is that only *connected sum moves* can be used to obtain new primitive curves which can be used to reduce Heegaard diagram.

We generalize a result in [37] to obtain an algebraic property of the connected sum move.

**Theorem 2.1** *Suppose $F$ is an oriented surface of genus $g$ and curve set $\{b_1, b_2, \cdots, b_g\}$ is a complete system in $F$. Suppose $a_1$ is a simple closed curve in $F$ and $a_1$ does not separate $F$. If the reduced word $w$ of $a_1$ corresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$ is 1, i.e., empty word, then we can use a sequence of connected sum moves on the curve set $\{b_1, b_2, \cdots, b_g\}$ to obtain a new complete system $\{a_1, a_2, \cdots, a_g\}$. In particular, if the length of $w$ is $s$, then at most $(s+1)(g-1)$ moves are required..*

In the proof of Theorem 2.1, we will make use of the following two lemmas.

**Lemma 2.1** *Suppose $F$, $\{b_1, b_2, \cdots, b_g\}$ and $a_1$ are the same as in Theorem 2.1. Suppose that we use a connected sum move on the curves $b_1, b_2$ along an arc $\alpha$ to replace $b_1$ by $b_1' = b_1 \sharp_\alpha b_2$ to obtain a new complete system $\{b_1', b_2, \cdots, b_g\}$. Suppose that the words of $a_1$ corresponding to the old complete system $\{b_1, b_2, \cdots, b_g\}$ and the new complete system*

*$\{b'_1, b_2, \cdots, b_g\}$ are $w(b_1, b_2, \cdots, b_g)$ and , $w'(b'_1, b_2, \cdots, b_g)$ respectively. If the reduced word of $w(b_1, b_2, \cdots, b_g)$ is 1, then the reduced word of $w'(b'_1, b_2, \cdots, b_g)$ is also 1.*

*Proof.* Without loss of generality, we may suppose that the orientation of $b'_1$ matches the orientations of both $b_1$ and $b_2$. In an intersection point between $\alpha$ and $a_1$, we read nothing for the old complete system and read a cancelling pair $b'_1 \cdot (b'_1)^{-1}$ or $(b'_1)^{-1} \cdot b'_1$ for the new complete system. In an intersection point between $b_1$ and $a_1$, if we read $b_1$ or $(b_1)^{-1}$ for the old complete system, then we read $b'_1$ or $(b'_1)^{-1}$ respectively for the new complete system. In an intersection point between $b_2$ and $a_1$, if we read $b_2$ or $(b_2)^{-1}$ for the old complete system, then we read $b'_1 \cdot b_2$ or $(b'_1 \cdot b_2)^{-1}$ respectively for the new complete system. In all other intersection points, we read the same results for both the complete systems. Therefore, $w'(b'_1, b_2, \cdots, b_g)$ has the same reduced word with a word $w(b'_1, b'_1 \cdot b_2, \cdots, b_g)$. The reduced word of $w(b'_1, b'_1 \cdot b_2, \cdots, b_g)$ is 1 since the reduced word of $w(b_1, b_2, \cdots, b_g)$ is 1. Thus, the reduced word of $w'(b'_1, b_2, \cdots, b_g)$ is also 1.

□

**Lemma 2.2** *Suppose $F$, $\{b_1, b_2, \cdots, b_g\}$ and $a_1$ are the same as in Theorem 2.1. If $a_1 \cap (\cup_{j=1}^g b_j) = \emptyset$, then we can use at most $g-1$ connected sum moves to change the complete system $\{b_1, b_2, \cdots, b_g\}$ to a new complete system $\{a_1, a_2, \cdots, a_g\}$.*

*Proof.* The result of cutting $F$ along the curves $b_1, b_2, \cdots, b_g$ is a 2-sphere with $2g$ holes. Denote this surface as $\Sigma$ and the holes as $b_1^1, b_1^2, b_2^1, b_2^2, \cdots, b_g^1, b_g^2$, where $b_j^1, b_j^2$ are two copies of $b_j$ for $j = 1, 2, \cdots, g$. If we cut $\Sigma$ along $a_1$, we obtain two surfaces $\Sigma_1$, $\Sigma_2$ which are 2-spheres with holes. One of the surface $\Sigma_1$, $\Sigma_2$ has at most $g+1$ boundary connected components. Without loss of generality, we assume the surface is $\Sigma_1$. Since $a_1$ does not separate $F$, there is some $i \in \{1, 2, \cdots, g\}$ such that $b_i^1 \subset \Sigma_1$, $b_i^2 \subset \Sigma_2$ or $b_i^1 \subset \Sigma_2$, $b_i^2 \subset \Sigma_1$. Without loss of generality, we may assume that $b_i^1 \subset \Sigma_1$, $b_i^2 \subset \Sigma_2$. Suppose that the number of the holes of $\Sigma_1$ is $k$. We draw $k-2$ properly embedded, pairwise disjoint arcs in $\Sigma_1$ such that each hole of $\Sigma_1$ except $b_i^1$ and $a_1$ is connected to the hole $b_i^1$ by exactly one arc. Now, these arcs define $k-2$ connected sum moves. The result curve of the moves is isotopic to $a_1$ in the surface $F$. Therefore, we obtain a new complete system $\{a_1, a_2, \cdots, a_g\}$. In fact, the curves $a_2, \cdots, a_g$ are $b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_g$. The number of moves is $k-2 \leq g-1$ since $k \leq g+1$.

□

*Proof of Theorem 2.1.* If $a_1 \cap (\cup_{j=1}^{g} b_j) \neq \emptyset$, we read the word $w$ of $a_1$ cooresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$. Since the cyclically reduced word of $w$ is 1, the empty word, we can reduce $w$ to 1 by cancelling reduced pairs of $w$ step by step. Suppose the first such reduced pair in $w$ is $b_i$ and $b_i^{-1}$ for some $i \in \{1, 2, \cdots, g\}$. Then $b_i$ and $b_i^{-1}$ are adjacent in the presentation of $w$. This fact indicates that there exists an arc $c \subset a_1$ so that $c$ intersects $(\cup_{j=1}^{g} b_j)$ by two points $A$ and $B$ which are the endpoints of $c$. The points $A, B$ separate the curve $b_i$ into two arcs $a$ and $b$. The result of cutting $F$ along the curves $b_1, b_2, \cdots, b_g$ is a 2-sphere with $2g$ holes and we denote this surface as $\Sigma$. If we cut the surface $\Sigma$ along $c$, we obtain two surfaces $\Sigma_1$ and $\Sigma_2$. One of the surfaces $\Sigma_1$ and $\Sigma_2$ does not include the hole which is the copy of $b_i$. We assume, without loss of generality, this surface is $\Sigma_1$ and assume that $\Sigma_1$ includes the hole $c \cup a$. Now, we view $\Sigma_1$ as a disk bounded by $c \cup a$ and with $k$ holes inside. We draw $k$ properly embedded, pairwise disjoint arcs in $\Sigma_1$ such that each of the holes is connected to the curve $a$ by exactly one arc. Now, we go back to consider the surface $F$. Along each of the arcs, we use the respective connected sum move. These moves do not remove the curves $b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_g$ from the new complete system, i.e., we replace the curve $b_i$ in the complete system $\{b_1, b_2, \cdots, b_g\}$ with the curve which is the connected sums of $b_i$ with the $k$ curves in the curve set $\{b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_g\}$ along the $k$ arcs. Note that this curve (up to isotopy in $\Sigma$) is just $c \cup b$. Then, we obtain a new complete system

$\{b_1, \cdots, b_{i-1}, c \cup b, b_{i+1}, \cdots, b_g\}$ which has at least two less intersection points with the curve $a_1$ and the reduced word of $a_1$ corresponding to the new complete system is still 1 by Lemma 2.1. Note that the number of our moves is $k$ which is at most $2g-2$. Further, we reduced one cancelling pair $b_i$ and $b_i^{-1}$ in the representation of $w$.

We continue the above step to the adjacent cancelling pair of the word $w$ and finally we will get a complete system $\{b'_1, b'_2, \cdots, b'_g\}$ of the surface $F$ so that every curve in this complete system does not intersect the curve $a_1$. The number of the moves is at most $s/2 \times (2g-2)$.

Now, $a_1 \cap (\cup_{i=1}^{g} b'_i) = \emptyset$. Therefore, by Lemma 2.2, we can use at most $g-1$ connected sum moves on the complete system $\{b'_1, b'_2, \cdots, b'_g\}$ to obtain a new complete system $\{a_1, a_2, \cdots, a_g\}$.

Since we only need $s/2 \times (2g-2)$ times of connected sum moves to cancel the $s$ intersection points between $a_1$ and the complete system $b_1, b_2, \cdots, b_g$ and need at most $g-1$ more moves to get the curve $a_1$, the number of total connected sum moves is at most $s/2 \times (2g-2) + (g-1) = (s+1)(g-1)$. This completes the proof of the theorem.

□

Now, we apply Theorem 2.1 to get several results.

**Theorem 2.2** *Suppose $F$ is an orientable surface of genus $g$ and $\{b_1, b_2, \cdots, b_g\}$ is a complete system in $F$. Suppose $\{a_1, a_2, \cdots, a_n\} (n \leq g)$ is a non-separating curve set in $F$, i.e., cutting $F$ along $\{a_1, a_2, \cdots, a_n\}$ is connected. If the respective reduced words $w_1, w_2, \cdots, w_n$ of $a_1, a_2, \cdots, a_n$ corresponding to the complete system are all equal to 1, then we can use a sequence of connected sum moves on the complete system to obtain a new complete system $\{a_1, a_2, \cdots, a_n, b'_{n+1}, \cdots, b'_g\}$. If the sum of the word lengths of $w_1, w_2, \cdots, w_n$ is $s$, then at most $(s+n)(g-1)$ moves are required. In particular, if $n = g$, then the requirement that all $w_i$ reduce to 1 gives a necessary and sufficient* *condition passing one Heegaard diagram to another Heegaard diagram by using connected sum moves.*

*Proof.* The proof is similar to Theorem 2.1. We cancel all intersection points between the curve set $\{b_1, b_2, \cdots, b_g\}$ and $\{a_1, a_2, \cdots, a_n\}$ first, that is, we can use at most $s(g-1)$ moves on the complete system $\{b_1, b_2, \cdots, b_g\}$ to obtain a new complete system $\{b_1'', b_2'', \cdots, b_g''\}$ such that $b_1, b_2, \cdots, b_g, b_1'', b_2'', \cdots, b_g''$ are pairwise disjoint curves in $F$. We cut $F$ along $\{b_1'', b_2'', \cdots, b_g''\}$ to obtain a 2-sphere with $2g$ holes. Denote this surface as $\Sigma$. There is a curve, say $a_1$ without loss of generality, in the curve set $\{a_1, a_2, \cdots, a_n\}$ such that $a_1$ bounds a disk with $m$ holes in $\Sigma$ with $m \leq g$ and no curves in the curve set $\{a_1, a_2, \cdots, a_n\}$ lie inside this punctured disk. Denote this punctured disk as $\Sigma_1$. By the proof of Theorem 2.1, we can use at most $g-1$ connected sum moves to obtain $a_1$. We continue the similar steps for other curves in the curve set $\{a_1, a_2, \cdots, a_n\}$ and finally get a complete system $\{a_1, a_2, \cdots, a_n, b_{n+1}', \cdots, b_g'\}$. The number of total moves is at most $s(g-1) + n(g-1) = (s+n)(g-1)$.

□

**Theorem 2.3** *Suppose that $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of genus $g$ of $M$. If one of the following conditions is true, then the Heegaard diagram is reducible ( i.e., we can use a reduction move on the Heegaard diagram to obtain a Heegaard diagram of genus $g-1$ ).*

*1). One of the Heegaard curves $b_1, b_2, \cdots, b_g$ is primitive in the handlebody $V$.*

*2). There is a simple closed curve $c$ in $\partial V$ such that $c$ is primitive in $V$ and $c$ can be obtained by using a sequence of connected sum moves on the curve set $b_1, b_2, \cdots, b_g$.*

*3). There is a simple closed curve $c$ in $\partial V$ such that $c$ is primitive in $V$ and the reduced word of the curve $c$ corresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$ of $\partial V$ is 1.*

*Proof.* 1). If one, say $b_1$, without loss of generality, of the Heegaard curves $b_1, b_2, \cdots, b_g$ is primitive in the handlebody $V$, then adding 2-handle to $V$ along the Heegaard curve is a handlebody $V'$ of genus $g-1$ by Gordon's Theorem[Gordon]. Now, $(V'; b_2, \cdots, b_g)$ is a Heegaard diagram of genus $g-1$ of $M$.

2). If $c$ is a simple closed curve in $\partial V$ such that $c$ is primitive in $V$ and $c$ can be obtained by using a sequence of connected sum moves on the curve set $b_1, b_2, \cdots, b_g$, then the connected sum moves give us a Heegaard diagram $(V; c, b'_2, \cdots, b'_g)$. Since $c$ is primitive, by 1), the Heegaard diagram $(V; c, b'_2, \cdots, b'_g)$ is reducible. Therefore, the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ is also reducible.

3). If $c$ is a simple closed curve in $\partial V$ such that $c$ is primitive in $V$ and the reduced word of the curve $c$ corresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$ of $V$ is 1, then by Theorem 2.1, we can use connected sum moves on the complete system to obtain $c$. Therefore, by 2), the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ is reducible.

□

Now we consider the converse of Theorem 2.3 and obtain the following result.

**Theorem 2.4** *Suppose that $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of genus $g$ of $M$. If the Heegaard diagram is reducible, then the following conditions are true.*

*1). There is a simple closed curve $c$ in $\partial V$ such that $c$ is primitive in $V$ and $c$ can be obtained by using a sequence of connected sum moves on the complete system $\{b_1, b_2, \cdots, b_g\}$ of $\partial V$.*

*2). There is a simple closed curve $c$ in $\partial V$ such that $c$ is primitive in $V$ and the reduced word of the curve $c$ corresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$ of $V$ is 1.*

*Proof.* The condition 1) is equivalent to the condition 2) by Theorem 2.1. Therefore, we only need to prove the theorem in the case of condition 1). Suppose the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ is reducible and $(V, W)$ is the respective Heegaard splitting of the Heegaard diagram. Then there are properly embedded disks $D \in V$ and $D' \in W$ respectively such that $D \cap D' = \partial D \cap \partial D'$ is one point, that is, the curve $\partial D'$ is a primitive curve of $V$. Note the word of $\partial D'$ corresponding to the complete meridian system $\{b_1, b_2, \cdots, b_g\}$ of the handlebody $W$ is 1 since $\partial D'$ bounds a disk $D'$ in $W$. Therefore, we can use a sequence of connected sum moves on the complete system $b_1, b_2, \cdots, b_g$ of $\partial V$ to obtain the curve $\partial D'$.

□

It is a consequence of Theorem 2.4 that if we can find an algorithm to obtain all primitive curves in the boundary surface of a handlebody, then we have an algorithm to detect if a Heegaard diagram $(V; a_1, a_2, \cdots, a_g)$ of a 3-manifold $M$ is reducible. The steps are as follows.

Step 1. List all primitive curves according to their word lengths as $c_1, c_2, \cdots, c_m, \cdots$.

Step 2. For $i = 1, 2, \cdots$, read the word $w_i$ of $c_i$ corresponding to the complete system $\{b_1, b_2, \cdots, b_g\}$. If $w_i$ is 1, then the Heegaard diagram is reducible and we can stop. Otherwise we continue to consider the next primitive curve.

If for a sufficiently large $N$, all of the words $w_1, w_2, \cdots, w_m$ are not 1, then the Heegaard diagram is irreducible.

**Note.** It is an open problem to find a bound $N$.

**Example.** The following figure shows a Heegaard diagram $(V; b_1, b_2)$ of the 3-sphere [17]. The corresponding fundamental group presentation of the Heegaard diagram is

$$\pi_1(S^3) = < x_1, x_2 : x_1^2 \cdot x_2^3, x_1^3 \cdot x_2^4 > .$$

It is clear that $c$ is a simple closed curve which satisfies condition 3) in the above theorem. Therefore the Heegaard diagram is reducible.

Now we discuss the algebraic properties of connected sum moves.

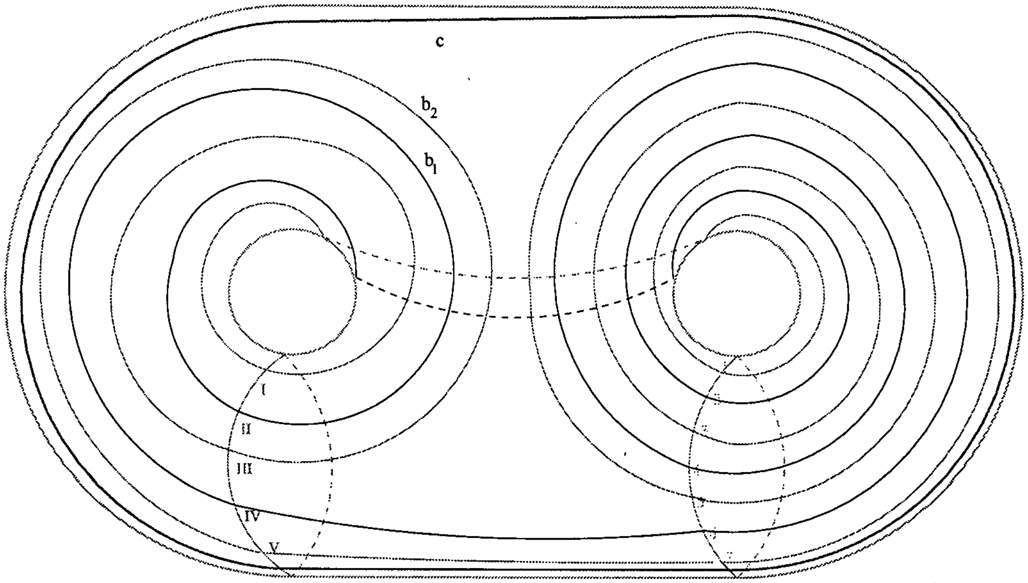



Figure 2.6: A Heegaard diagram of the 3-sphere

The words of a simple closed curve in the boundary surface of a handlebody corresponding to different complete meridian systems are in general completely different. We need to know how to choose suitable complete meridian systems to make such words simpler. We know that one complete meridian system can be changed to another complete meridian system by using connected sum moves ( [37] ). Now, we investigate transformation properties of word presentations under the connected sum moves on a complete meridian systems.

Suppose $V$ is a handlebody of genus $g$ with a complete meridian system $\{d_1, \cdots, d_g\}$ and $\alpha$ is a simple curve in $\partial V$ which connects two points in different curves of the complete meridian system, and $\alpha$ has no other intersection points with the curves in the complete meridian system. Without loss of generality, we may assume that the endpoints of $\alpha$ are $P_1, P_2$ and $P_1 \in d_1$, $P_2 \in d_2$. Suppose $c$ is a simple, oriented closed curve in $\partial V$ and $c$ intersects $\alpha$ at points $Q_1, Q_2, \cdots, Q_n$, where we list the intersection points according to the order of passing through them when we read the word of $c$. Suppose the word of $c$

corresponding to the complete meridian system is

$$\begin{aligned} w_c(d_1, d_2, \cdots, d_g) &= d_1^{\epsilon_{11}} d_2^{\epsilon_{12}} \cdots d_g^{\epsilon_{1g}} \\ & d_1^{\epsilon_{21}} d_2^{\epsilon_{22}} \cdots d_g^{\epsilon_{2g}} \cdots \\ & d_1^{\epsilon_{k_1 1}} d_2^{\epsilon_{k_1 2}} \cdots d_{t_1}^{\epsilon_{k_1, t_1}} d_{t_1+1}^{\epsilon_{k_1, t_1+1}} \cdots d_g^{\epsilon_{k_1 g}} \cdots \\ & d_1^{\epsilon_{k_2 1}} d_2^{\epsilon_{k_2 2}} \cdots d_{t_2}^{\epsilon_{k_2, t_2}} d_{t_2+1}^{\epsilon_{k_2, t_2+1}} \cdots d_g^{\epsilon_{k_2 g}} \cdots \\ & d_1^{\epsilon_{k_n 1}} d_2^{\epsilon_{k_n 2}} \cdots d_{t_n}^{\epsilon_{k_n, t_n}} d_{t_n+1}^{\epsilon_{k_1, t_n+1}} \cdots d_g^{\epsilon_{k_n g}} \cdots \\ & d_1^{\epsilon_{m1}} d_2^{\epsilon_{m2}} \cdots d_g^{\epsilon_{mg}} \end{aligned}$$

where $1 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq m$; $\epsilon_{ij} \in \{-1, 0, +1\}$, $t_i \in \{1, 2, \cdots, g\}$ for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, g$ and $Q_i$ lies in the subarc of $c$ which corresponds to the subword $d_{t_s}^{\epsilon_{k_s, t_s}} d_{t_s+1}^{\epsilon_{k_s, t_s+1}}$ ( or $d_g^{\epsilon_{k_s, g}} d_1^{\epsilon_{k_s+1, 1}}$ if the former $t_s = g$ ) of $w_c$ for $s = 1, 2, \cdots, n$. Then we apply the proof of the lemma 2 in the section 2.3 and obtain the following proposition.

**Proposition 2.1** *Suppose we use the sum $d_1' = d_1 \sharp_\alpha d_2$ of $d_1$ and $d_2$ along $\alpha$ ( we assume that their orientations are the same as in Lemma 2.1. ) to replace $d_1$ to obtain a new complete meridian system $\{d_1', d_2, \cdots, d_g\}$. Then the word of $c$ corresponding to the new system is*

$$\begin{aligned} w_c'(d_1', d_2, \cdots, d_g) &= {d_1'}^{\epsilon_{11}} (d_1' d_2)^{\epsilon_{12}} \cdots d_g^{\epsilon_{1g}} \\ & {d_1'}^{\epsilon_{21}} (d_1' d_2)^{\epsilon_{22}} \cdots d_g^{\epsilon_{2g}} \cdots \\ & {d_1'}^{\epsilon_{k_1 1}} (d_1' d_2)^{\epsilon_{k_1 2}} \cdots {d_{t_1}}^{\epsilon_{k_1, t_1}} (d_1' {d_1'}^{-1})^{\epsilon_1} {d_{t_1+1}}^{\epsilon_{k_1, t_1+1}} \cdots d_g^{\epsilon_{k_1 g}} \cdots \\ & {d_1'}^{\epsilon_{k_2 1}} (d_1' d_2)^{\epsilon_{k_2 2}} \cdots {d_{t_2}}^{\epsilon_{k_2, t_2}} (d_1' {d_1'}^{-1})^{\epsilon_2} {d_{t_2+1}}^{\epsilon_{k_2, t_2+1}} \cdots d_g^{\epsilon_{k_2 g}} \cdots \\ & {d_1'}^{\epsilon_{k_n 1}} (d_1' d_2)^{\epsilon_{k_n 2}} \cdots {d_{t_n}}^{\epsilon_{k_n, t_n}} (d_1' {d_1'}^{-1})^{\epsilon_n} {d_{t_n+1}}^{\epsilon_{k_1, t_n+1}} \cdots d_g^{\epsilon_{k_n g}} \cdots \\ & {d_1'}^{\epsilon_{m1}} (d_1' d_2)^{\epsilon_{m2}} \cdots d_g^{\epsilon_{mg}}; \end{aligned}$$

*where $\epsilon_1, \epsilon_2, \cdots, \epsilon_n \in \{-1, 1\}$.*

*Proof.* We can directly write the word $w'_c(d'_1, d_2, \cdots, d_g)$ according to the discussion in the proof of the lemma 2. Note that $\epsilon_i$ is equal to $-1$ or $1$ is decided by the orientations of $c$ and $d'_1$ near $Q_i$ for $i = 1, 2, \cdots, n$.

□

**Note.** There is a similar result for Proposition 2.1 for any complete system on an oriented surface. But the result is important to a complete meridian system of a handlebody since the above group transformation can be used to present the respective transformation of presentations of fundamental group of a 3-manifold corresponding to different bases. ( We know that every base corresponds to a complete meridian system of the respective Heegaard handlebody. )

It is clear that connected sum moves on a complete meridian system do not change the primitive property of a curve since the primitive property is independent of complete meridian systems.

## 2.2 Dehn twists

Sometimes we can not use connected sum moves to reduce cancelling pairs of words of Heegaard curves. Then we consider the use of the Dehn twists.

The word presentations of Heegaard curves corresponding to a complete meridian system have the following transformations under Dehn twists.

**Proposition 2.2** *Suppose $V$ is a handlebody of genus $g$ with a complete meridian system $\{d_1, \cdots, d_g\}$ and $D$ is a properly embedded disk in $V$. Suppose $d = \partial D$ and the word $w_d$ of $d$ corresponding to the complete meridian system is*

$$w_d = d_{i_1}^{\epsilon_1} \cdot d_{i_2}^{\epsilon_2} \cdot \cdots \cdot d_{i_k}^{\epsilon_k},$$

*where $i_1, i_2, \cdots, i_k \in \{1, 2, \cdots, g\}$ and $\epsilon_1, \epsilon_2, \cdots, \epsilon_k \in \{1, -1\}$. Suppose $c$ is a simple* *closed curve in $\partial V$. Suppose the word $w_c$ of the curve $c$ corresonding to the system is*

$$w_c = d_{j_1}^{\alpha_1} \cdot d_{j_2}^{\alpha_2} \cdot \cdots \cdot d_{j_n}^{\alpha_n},$$

*where $j_1, j_2, \cdots, j_n \in \{1, 2, \cdots, g\}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \{1, -1\}$. Suppose that $c$ intersects the curve $\partial D$ at $m$ points $P_1, P_2, \cdots, P_m$ in order and $P_s$ lies in the subarc of $c$ which corresponds to the subword $d_{j_{\mu_s}}^{\alpha_{\mu_s}} \cdot d_{j_{\mu_s \oplus_n 1}}^{\alpha_{\mu_s \oplus_n 1}}$ of $w_c$ and $P_s$ lies in the subarc of $\partial D$ which corresponds to the subword $d_{i_{\nu_s}}^{\epsilon_{\nu_s}} \cdot d_{i_{\nu_s \oplus_k 1}}^{\epsilon_{\nu_s \oplus_k 1}}$ of $w_{\partial D}$ for $s = 1, 2, \cdots, m$, where $\mu_s \in \{1, 2, \cdots, n\}$,$\nu_s \in \{1, 2, \cdots, k\}$ for $s = 1, 2, \cdots, m$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$, and for positive integers $p, q, r$, we define $q \oplus_p r$ as the least positive number with $q \oplus_p r \equiv q + r (mod\, p)$. If we use Dehn twist one time on $c$ which corresponds to a small neighborhood of the disk $D$ in $H$ and is along the direction of $\partial D$, then $c$ is changed to a new curve $c'$ whose word $w_{c'}$ corresonding to the system $\{d_1, d_2, \cdots, d_g\}$ is*

$$\begin{aligned} w_{c'} = \quad & d_{j_1}^{\alpha_1} \cdot d_{j_2}^{\alpha_2} \cdot \cdots \cdot d_{j_{\mu_1 - 1}}^{\alpha_{\mu_1 - 1}} \cdot d_{j_{\mu_1}}^{\alpha_{\mu_1}} \cdot d_{i_{\nu_1 \oplus_k 1}}^{\epsilon_{\nu_1 \oplus_k 1}} \cdot d_{i_{\nu_1 \oplus_k 2}}^{\epsilon_{\nu_1 \oplus_k 2}} \cdot \cdots \\ & \cdot d_{i_{\nu_1 \oplus_k k}}^{\epsilon_{\nu_1 \oplus_k k}} \cdot d_{j_{\mu_1 + 1}}^{\alpha_{\mu_1 + 1}} \cdot \cdots \cdot d_{j_{\mu_2 - 1}}^{\alpha_{\mu_2 - 1}} \cdot d_{j_{\mu_2}}^{\alpha_{\mu_2}} \cdot d_{i_{\nu_2 \oplus_k 1}}^{\epsilon_{\nu_2 \oplus_k 1}} \cdot d_{i_{\nu_2 \oplus_k 2}}^{\epsilon_{\nu_2 \oplus_k 2}} \cdot \cdots \\ & \cdot d_{i_{\nu_2 \oplus_k k}}^{\epsilon_{\nu_2 \oplus_k k}} \cdot d_{j_{\mu_2 + 1}}^{\alpha_{\mu_2 + 1}} \cdot \cdots \cdot d_{j_n}^{\alpha_n}. \end{aligned}$$

*Proof.* We can directly check the move result for each small neighborhood of $P_s$ in the curve $c$ for $s = 1, 2, \cdots, m$. The s-th small subarc of $c$ has been changed to a simple curve $c_s$ which is just like a copy of $\partial D$ and which has word

$$w_{c_s} = d_{i_{\nu_s \oplus_k 1}}^{\epsilon_{\nu_s \oplus_k 1}} \cdot d_{i_{\nu_s \oplus_k 2}}^{\epsilon_{\nu_s \oplus_k 2}} \cdot \cdots \cdot d_{i_{\nu_s \oplus_k k}}^{\epsilon_{\nu_s \oplus_k k}}$$

for $s = 1, 2, \cdots, m$. Adding all the $m$ words $w_{c_1}, w_{c_2}, \cdots, w_{c_m}$ to the word $w_c$ in suitable places according to the positions of respective subarcs, we obtain the word $w_{c'}$ whose form is as in the proposition.

□

If we only consider word presentations of Heegaard curves, connected sum moves on complete meridian systems produce the similar results with Dehn twist.

Now, we note that Dehn twists do not change the primitive property of a curve in the boundary of a handlebody.

**Proposition 2.3** *Suppose $V$ is a handlebody of genus $g$ and $c$ is a simple closed curve in $\partial V$. Suppose $D$ is a proper embedded disk in $V$ and $c'$ is the curve obtained by using Dehn twist along $D$ on the curve $c$. Then $c$ is a primitive curve in $\partial V$ if and only if $c'$ is.*

□

## 2.3 Stabilization and reduction

Although stabilization and reduction are not powerful methods to simplify Heegaard diagram of a 3-manifold, they change the genus of a Heegaard diagram. Therefore sometimes we need it to obtain minimal Heegaard diagrams. ( A *minimal or irreducible Heegaard diagram* of a 3-manifold $M$ is a Heegaard diagram of $M$ which can not be reduced, that is, does not result from a stabilization of a Heegaard diagram of smaller genus.)

According to Boileau and Zieschang [3], there exists a 3-manifold with two irreducible Heegaard splittings, one of genus 2, the other of genus 3. Thus minimal does not mean smallest genus.

The transformations of word presentations for stabilization and reduction moves are as follows:

**Proposition 2.4** *Suppose $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of genus $g$ of $M$. Suppose we use a stabilization move on this Heegaard diagram to obtain a new Heegaard* *diagram* $(V'; b'_1, b'_2, \cdots, b'_{g+1})$*. If the associated fundamental group presentation of the Heegaard diagram* $(V; b_1, b_2, \cdots, b_g)$ *is* $\pi(M) =< x_1, x_2, \cdots, x_g : R_1, R_2, \cdots, R_g >$*, then the associated foundamental group presentation of the Heegaard diagram* $(V'; b'_1, b'_2, \cdots, b'_{g+1})$ *is* $\pi(M) =< x_1, x_2, \cdots, x_g, x_{g+1} : R_1, R_2, \cdots, R_g, x_{g+1} >$*. And the word of a curve in* $\partial V$ *is same with the word of the respective curve in* $\partial V'$*.*

*Proof.* By the definition of stabilization.

□

We need to prove that stabilizations have the same property as connected sum moves and Dehn twists, that is, stabilizations do not change the primitive property of a curve in the boundary of a handlebody.

**Proposition 2.5** *Suppose* $(V; b_1, b_2, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of a closed orientable 3-manifold* $M$ *and* $c$ *is a simple closed curve in* $\partial V$*. Suppose we use a stabilization move on this Heegaard diagram to obtain a new Heegaard diagram* $(V'; b'_1, b'_2, \cdots, b'_{g+1})$*. If the curve* $c$ *is a primitive curve for the Heegaard diagram* $(V; b_1, b_2, \cdots, b_g)$*, then the corresponding curve* $c'$ *of* $c$ *in* $\partial V'$ *is a primitive curve for the Heegaard diagram* $(V; b'_1, b'_2, \cdots, b'_{g+1})$*.*

*Proof.* Since $c$ is a primitive curve for the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$, then there is a proper embedded disk $D$ in the handlbody $V$ such that $D \cap c = \partial D \cap c$ is one point and we can use connected sum moves in $\partial V$ on the curves $b_1, b_2, \cdots, b_g$ to obtain $c$.

Suppose $D'$ is the respective proper embedding disk in $V'$ associated to $D$. Then clearly, $D' \cap c' = \partial D' \cap c'$ is one point in $\partial V'$. Note if we try to use connected sum moves on the curves $b'_1, b'_2, \cdots, b'_g$ to obtain the curve $c'$, there are obstructions when we try to isotopically move a curve passing through the disk $B$ which corresponds to the 1-handle that we add to $V$ to obtain $V'$. We solve this problem by using the connected sum moves in the following figure.

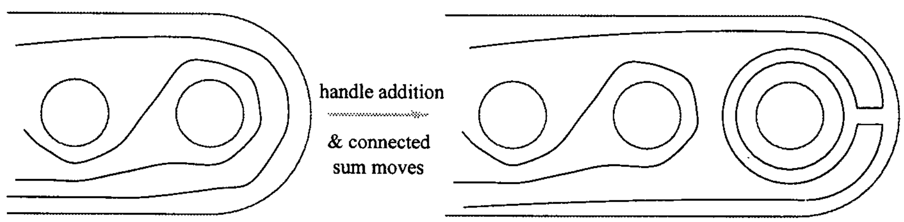



Figure 2.7: Handle addition move and connected sum move

Thus, we can use connected sum moves on the curves $b'_1, b'_2, \cdots, b'_{g+1}$ in the surface $\partial V'$ to obtain the curve $c'$. Then, we have proven that $c'$ satisfies the two conditions of primitive curve in $\partial V'$. Therefore, $c'$ is a primitive curve for the Heegaard diagram $(V'; b'_1, b'_2, \cdots, b'_{g+1})$.

□

The similar result for reduction moves is also true.

**Theorem 2.5** *Suppose $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of genus $g(g > 1)$ of $M$ and $c$ is a simple closed curve in $\partial V$. Suppose $D$ is a properly embedded disk in $V$ such that $D \cap (\cup_{i=1}^{g} b_i) = \emptyset$. Suppose cutting $V$ along $D$ has two connected components: One of them is a handlebody $V'$ of genus $g-1$ and the other connected component is a solid torus $T$ so that $b_g$ is a standard meridian curve in $\partial T$ and $T \cap (\cup_{i=1}^{g-1} b_i) = \emptyset$. We use a reduction move on the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ along the disk $D$ to obtain a new Heegaard diagram $(V'; b_1, b_2, \cdots, b_{g-1})$ (for the sake of simplication, we do not change symbols for the Heegaard curves this time). If $c$ is a primitive curve for the*

*Heegaard diagram* $(V; b_1, b_2, \cdots, b_g)$ *and* $c \subset \partial V'$*, then* $c$ *is also a primitive curve for the Heegaard diagram* $(V'; b_1, b_2, \cdots, b_{g-1})$.

*Proof.* Suppose $B$ is a properly embedded disk in $V$ such that $c \cap B = c \cap \partial B$ is one point. If $B \cap D \neq \emptyset$, then each connected component of $(B \cap D) \cup (\partial B \cap T)$ bounds a disk in $T$. Then, we can isotopically move $B$ in the handlebody $V$ such that $B \cap D = \emptyset$. This indicates that there is a properly embedded disk $B'$ in $V'$ such that $c \cap B' = c\partial B'$ is one point.

On the other hand, since the curve $c$ is a primitive curve for the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$, the cyclically reduced word of $c$ corresponding to the curves $b_1, b_2, \cdots, b_g$ is trivial. But $c \cap b_g = \emptyset$. Then, the reduced word of $c$ corresponding to the curves $b_1, b_2, \cdots, b_{g-1}$ is also trivial.

The above two conditions indicate that $c$ is a primitive curve for the Heegaard diagram $(V'; b_1, b_2, \cdots, b_{g-1})$.

□

## 2.4 A property of moves

The following theorem implies that we can reverse the orders between a connected sum move and a Dehn twist.

**Proposition 2.6** *Suppose* $(V; b_1, b_2, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of* $M$*. Suppose that* $D$ *is a properly embedded disk in the handlbody* $V$ *and* $T_D$ *is a Dehn twist along* $D$ *in* $V$*. Then* $T_D(b_1, \cdots, b_i \sharp_c b_j, \cdots, b_g) = (T_D(b_1), \cdots, T_D(b_i) \sharp_{T_D(c)} T_D(b_j), \cdots, T_D(b_g))$. *That is applying first a connected sum move along an arc* $c$ *and then a Dehn twist along* $D$ *is equivalent to applying first the Dehn twist and then a connected sum move along an arc* $T_D(c)$.

*Proof.* If $c \cap \partial D = \emptyset$, then the proposition is true. If $c \cap \partial D \neq \emptyset$, we may assume that $\partial c \cap \partial D = \emptyset$. Note that in this case the disk in $\partial V$ given by applying the Dehn twist on a neighborhood of $c$ in $\partial V$ is the same with the disk which is a neighborhood of $T_D(c)$ in $\partial V$.

□

## 2.5 Circle and chord presentations of closed 3-manifolds

In this section, we introduce a new method to represent 3-manifolds. We call this method the *circle and chord presentations of closed 3-manifolds* .

**Definition 2.3** *Suppose that $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of $M$. Suppose that $\{d_1, d_2, \cdots, d_g\}$ is a complete meridian system of the handlebody $V$. Further suppose there are connected curves $c_1, c_2, \cdots, c_{2g-1}$ such that the following conditions hold.*

*1). For each $i \in \{1, 2, \cdots, 2g-1\}$, $c_i \subset b_j$ for some $j \in \{1, 2, \cdots, g\}$ and the intersection points between $c_i$ and the curve set $\{d_1, d_2, \cdots, d_g\}$ are the endpoints of $c_i$.*

*2). Cutting $\partial V$ along the curves $d_1, d_2, \cdots, d_g, c_1, c_2, \cdots, c_{2g-1}$ is a disk.*

*Then we can use the boundary circle of the disk and some disjoint chords which are the connected components of the set $\cup_{i=1}^{g} b_i - \cup_{i=1}^{2g-1} c_i$ to represent the 3-manifold $M$.*

*We call such a presentation a circle and chord presentation of $M$.*

*Note.* We can obtain a circle and chord presentation from each Heegaard diagram after we isotopically move Heegaard curves in $\partial V$.

**Example.** We can obtain a circle and chord presentation of the Poincaré homology 3-sphere in Fig. 2.8 by cutting the surface along $d_1, d_2, c_1, c_2, c_3$.

From the preceding example, we know that a circle and chord presentation of a closed 3-manifold consists of a circle and some chords. The circle consists of the d-curves $d_1, \cdots, d_g$ and the c-curves $c_1, \cdots, c_{2g-1}$ in Definition 2.3. The chords consist of

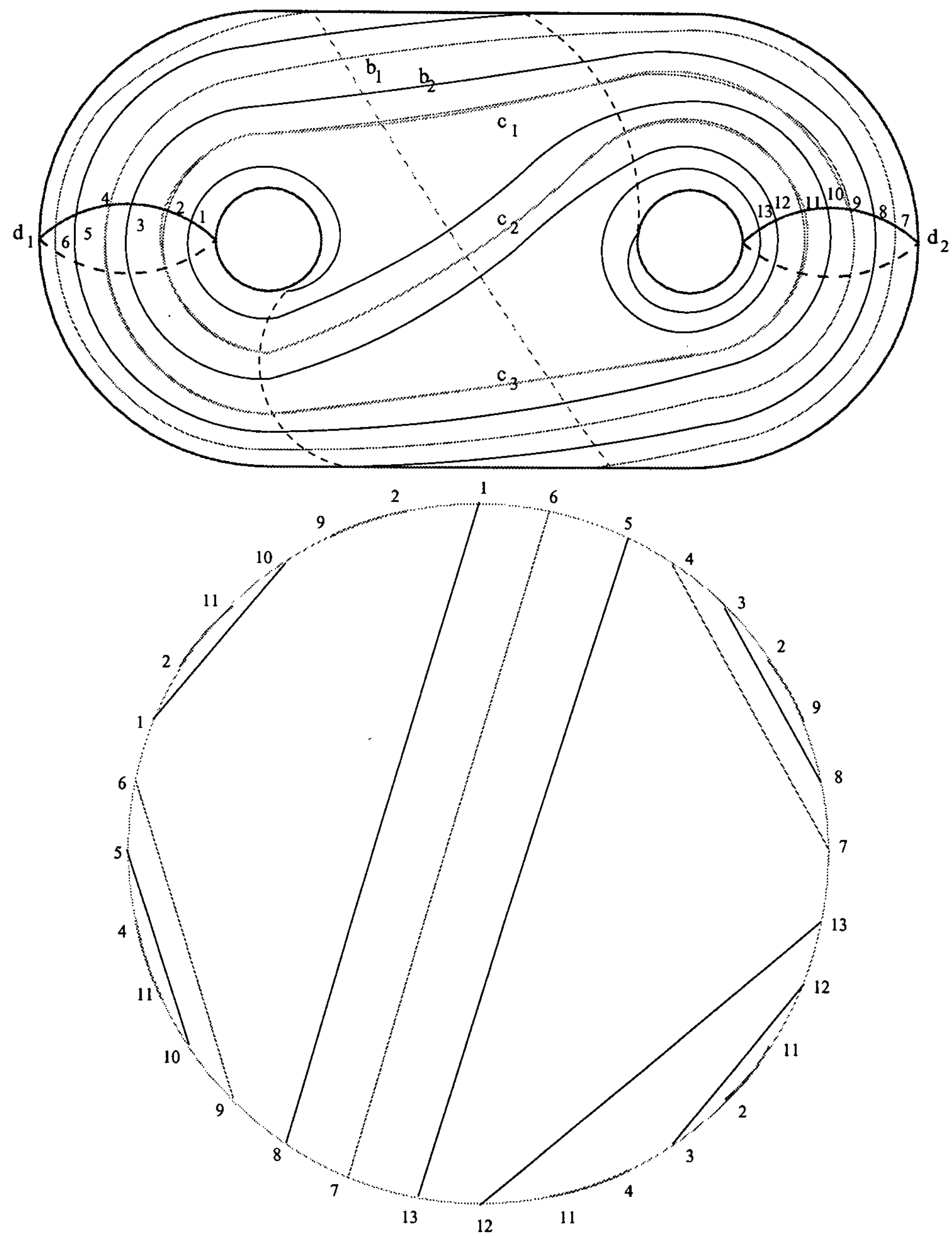



Figure 2.8: A circle and chord presentation for the Poincaré homology 3-sphere

the connected components of $\cup_{i=1}^{g} b_i - (\cup_{i=1}^{2g-1} c_i)$. And we use some marking numbers to denote the respective intersection points of the Heegaard curves and the curves in the complete meridian system.

On the other hand, a graph consisting of a circle with some marking signs, such as numbers, on it and some chords may correspond to a Heegaard diagram of genus $g$ of a closed 3-manifold. In the following we give conditions for such a labelled graph to represent a 3-manifold.

**Definition 2.4** *We classify the marking numbers assigned to vertices in a circle and chord presentation by the following three types.*

*Type I. The marking number which appears four times and each appearance is not an endpoint of any chord.*

*Type II. The marking number which appears three times and among three appearances of such a marking number exactly one is an endpoint of a chord.*

*Type III. The marking number which appears two times and each appearance is an endpoint of a chord.*

**Definition 2.5** *An h-arc of a circle and chord presentation is the arc of the circle bounded by two adjacent marking numbers each of which is not endpoint of any chord. m-arcs are the connected components of the complement of all h-arcs in the circle.*

**Example.** In Fig 2.8, we have two marking numbers of Type I ( 2, 11 ), two marking numbers of Type II ( 4, 9 ), nine marking numbers of Type III ( 1, 3, 5, 6, 7, 8, 10, 12, 13 ), six h-arcs ( endpoints marked by ( 2, 9 ), ( 11, 2 ), ( 4, 11 ), ( 11, 4 ), ( 2, 11 ), ( 9, 2 ) respectively ) and six m-arcs ( passing through ( 9, 10, 11 ), ( 2, 1, 6, 5, 4 ), ( 11, 10, 9, 8, 7, 13, 12, 11 ), ( 4, 3, 2 ), ( 11, 12, 13, 7, 8, 9 ), ( 2, 3, 4, 5, 6, 1, 2 ) respectively ).

Now, we obtain a necessary and sufficent condition for a graph to correspond a circle and chord presentation of a 3-manifold.

Suppose that $G$ is a planar graph consisting of a circle and disjoint chords and all vertices of $G$ lie on the circle. Suppose that the vertices listed according to the anti-clockwise direction of the circle are $P_1, P_2, \cdots, P_m$ and $P_i$ has been assigned a marking number $f(P_i)$ for $i = 1, 2, \cdots, m$. Then $G$ is a circle and chord presentation of a 3-manifold if and only if it satisfies the following conditions.

Condition 1. All vertices are classified by four types:

Type 1. A vertex of degree 2 whose marking number is of Type I ( i.e., there are exactly three other vertices with the same marking number ).

Type 2. A vertex of degree 2 whose marking number is of Type II ( i.e., there are exactly two other vertices with the same marking number ).

Type 3. A vertex of degree 3 whose marking number is of Type II ( i.e., there are exactly two other vertices with the same marking number ).

Type 4. A vertex of degree 2 whose marking number is of Type III ( i.e., there is exactly one other vertex with the same marking number ).

Condition 2. The circle is a union of two Types of arcs. An h-arc only includes two adjacent vertices of degree 2. An m-arc is a connected component of the closure of the complement of all h-arcs in the circle.

Two h-arcs have no common endpoints and for each h-arc bounded by $P_i, P_{i \oplus_m 1}$, there is exactly one h-arc bounded by $P_j, P_{j \oplus_m 1}$ such that $f(P_i) = f(P_{j \oplus_m 1}), f(P_{i \oplus_m 1}) = f(P_j)$.

From now on, we suppose that the number of h-arcs is $\alpha$.

Two m-arcs have no common endpoints and each m-arc does not include two vertices which have the same marking number. We define an equivalence relation between m-arcs as follows: Two m-arcs are *m-equivalent* if and only if a vertex in an m-arc and a vertex in the other m-arc have the same marking number, then we obtain $\alpha/2 - 1$ equivalence classes. And if we list all the marking numbers of the vertices in an equivalence class according to anti-clockwise direction, we obtain $i_1, i_2, \cdots, i_k, i_k, i_{k-1}, \cdots, i_1$, where $2k$ is the number of the vertices in the equivalence class, $i_s \in \{f(P_1), \cdots, f(P_m)\}$ for $s = 1, 2, \cdots, k$.

From now on, we suppose that the number of equivalence classes is $g$.

Condition 3. Consider the set consisting of all chords and h-arcs. Define an equivalence relation on this set as follows: Call two elements in the set *h-equivalent* if they have vertices whose marking numbers are equal. Then we have $g$ equivalence classes for this equivalence relation.

Condition 4. All chords separate the disk $D$ bounded by the circle as disjoint regions. Call two such regions $D', D''$ *adjacent* if there exist adjacent vertices $P_i, P_{i\oplus_m 1} \in \partial D'$ and $P_j, P_{j\oplus_m 1} \in \partial D''$ such that $f(P_i) = f(P_j), f(P_{i\oplus_m 1}) = f(P_{j\oplus_m 1})$ or $f(P_i) = f(P_{j\oplus_m 1}), f(P_{i\oplus_m 1}) = f(P_j)$. "Adjacent" is an equivalence relation. Condition 4 requires that all regions are in the same equivalence class.

Therefore, we have the following theorem.

**Theorem 2.6** *The preceding 4 conditions are necessary and sufficient that a circle and chords with marking endpoints can represent a closed, orientable 3-manifold.*

*Proof.* According to Condition 2, if we pairwise attach the respective vertices in an m-equivalence class which have the same marking number and the respective arc bounded by these vertices and pairwise attach the two h-arcs which have the same marking number, we obtain a connected, orientable, closed surface $F$ of genus $g$. Condition 3 defines $g$ simple closed curves in $F$. Condition 4 ensures the surface and curves can be used to present a Heegaard diagram.

□

The following theorem is a criterion to check whether a circle and chord presentation correspods to a Heegaard diagram and a complete meridian system exists or not.

**Theorem 2.7** *Suppose $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of the 3-manifold $M$ and $\{d_1, d_2, \cdots, d_g\}$ is a complete meridian system of the handlebody $V$. Let $w_1, w_2, \cdots, w_g$ be the cyclical words of the respective Heegaard curves associated with the complete meridian system. Then the Heegaard diagram and complete meridian system can give a circle and chord presentation of $M$ if and only if the words $w_1, w_2, \cdots, w_g$ include the nontrivial subwords $x_{s_1}^{\epsilon_1} \cdot x_{t_1}^{\delta_1}, x_{s_2}^{\epsilon_2} \cdot x_{t_2}^{\delta_2}, \cdots, x_{s_{2g-1}}^{\epsilon_{2g-1}} \cdot x_{t_{2g-1}}^{\delta_{2g-1}}$; where $\epsilon_1, \delta_1, \epsilon_2, \delta_2, \cdots, \epsilon_{2g-1}, \delta_{2g-1} \in \{1, -1\}$ and $x_{s_i}^{\epsilon_i} \cdot x_{t_i}^{\delta_i} \neq (x_{s_k}^{\epsilon_k} \cdot x_{t_k}^{\delta_k})^{\delta}$ for $i \neq k, i, k \in \{1, 2, \cdots, 2g-1\}, \delta \in \{1, -1\}$.*

**Note.** If $w_i = x_k$ for some $i, k \in \{1, 2, \cdots, g\}$, then we regard $w_i$ as including the subword $x_k \cdot x_k$.

*Proof.* Cutting $V$ along the curves $d_1, d_2, \cdots, d_g$ is a 2-sphere with $2g$ holes. Subword $x_{s_i}^{\epsilon_i} \cdot x_{t_i}^{\delta_i}$ indicates that there exists a subarc of the Heegaard curves to connect two holes. The condition $x_{s_i}^{\epsilon_i} \cdot x_{t_i}^{\delta_i} \neq (x_{s_k}^{\epsilon_k} \cdot x_{t_k}^{\delta_k})^{\delta}$ indicates that there are no two holes which are connected by two such subarcs. Therefore the union of the boundary curves of the holes and the subarcs is a connected graph.

From a suitable circle and chord presentation, we can form some Heegaard diagrams. It is clear that all these Heegaard diagrams determine a homeomorphic 3-manifold. In fact, all these Heegaard diagrams are equivalent and we only need move II to change one to another. This is because if we let the respective simple closed curves in the two complete meridian system match and let the respective subarcs in the circles of the presentations match, all other part will match.

□

**Definition 2.6** *The complexity of a circle and chord presentation of a 3-manifold $M$ corresponding to a Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ and complete meridian system $\{d_1, d_2, \cdots, d_g\}$ is defined to be the sum of the lengths of words of the curves $b_1, b_2, \cdots, b_g$ corresponding to the complete meridian system. The genus of such a presentation is the* *genus of $V$.*

It is clear that for fixed positive integer $m$, there are only finitely many circle and chord presentations with complexity $< m$ for all possible 3-manifolds and these presentations and the 3-manifolds can be found. Therefore, we have the following theorem.

**Theorem 2.8** *There is an algorithm to list all circle and chord presentations and then to obtain all closed 3-manifolds. There is also an algorithm to list all possible primitive curves for a circle and chord presentation.*

*Proof.* By Birman's theorem in section 1, one can list all primitive curves.

□

**Example.** The 6 circle and chord presentations with complexity 7 and genus 1 include all lens spaces with fundamental group $\mathbb{Z}_7$.

**Theorem 2.9** *There is an algorithm to detect whether a Heegaard diagram is reducible or not.*

*Proof.* Suppose the Heegaard diagram is $(V; b_1, b_2, \cdots, b_g)$. It is easy to find a complete meridian system $\{d_1, d_2, \cdots, d_g\}$ such that a circle and chord presentation can be obtained. Now, we can draw primitive curves and check whether we can use connected sum moves on the Heegaard curves $b_1, b_2, \cdots, b_g$ to obtain a primitive curve. If such a primitive curve exists, then the Heegaard diagram is reducible.

□

**Example.** The following example is a Heegaard diagram of the 3-sphere ( see Fig. 2.10 ). We find it is reducible by drawing respective circle and chord presentation.

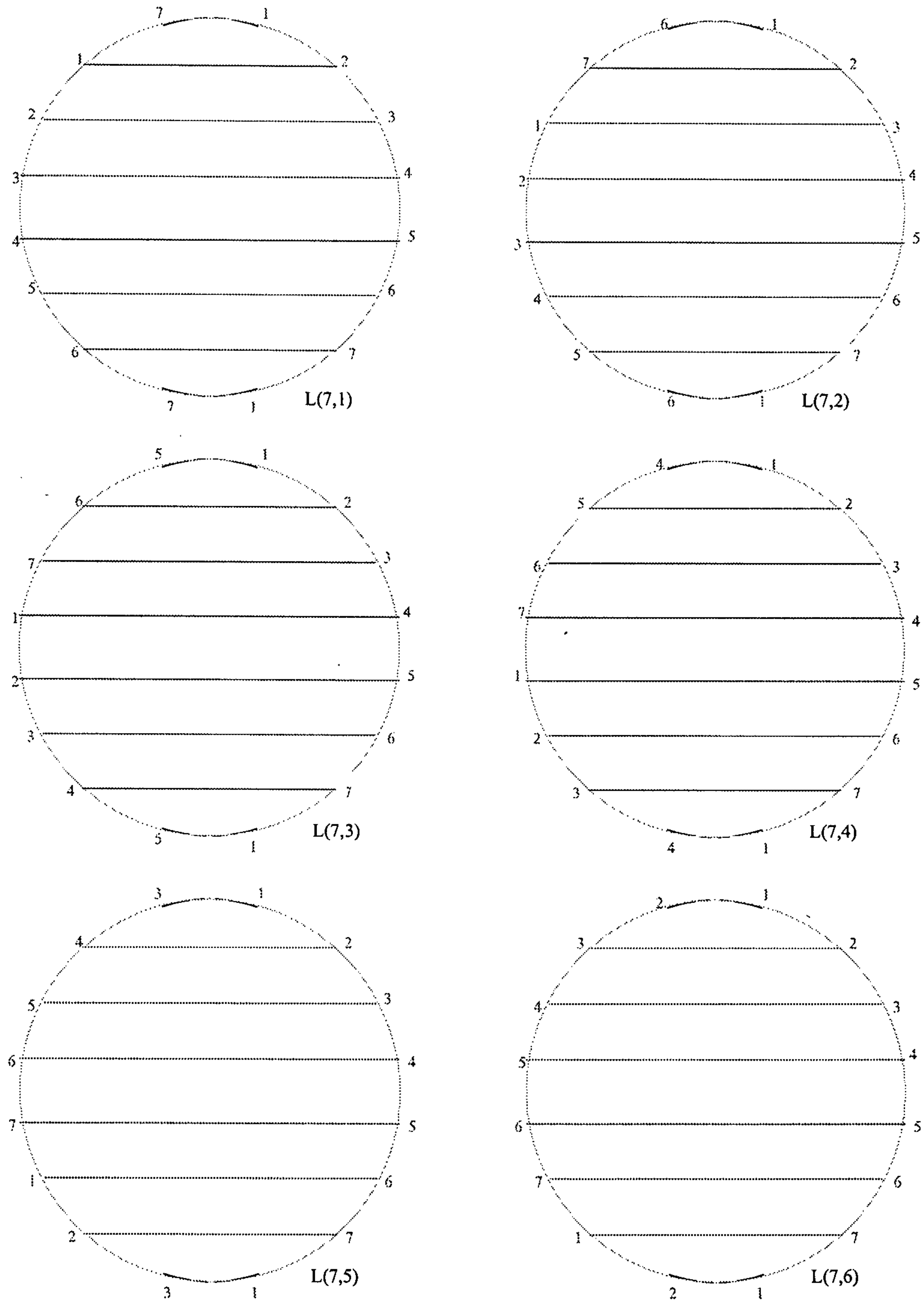


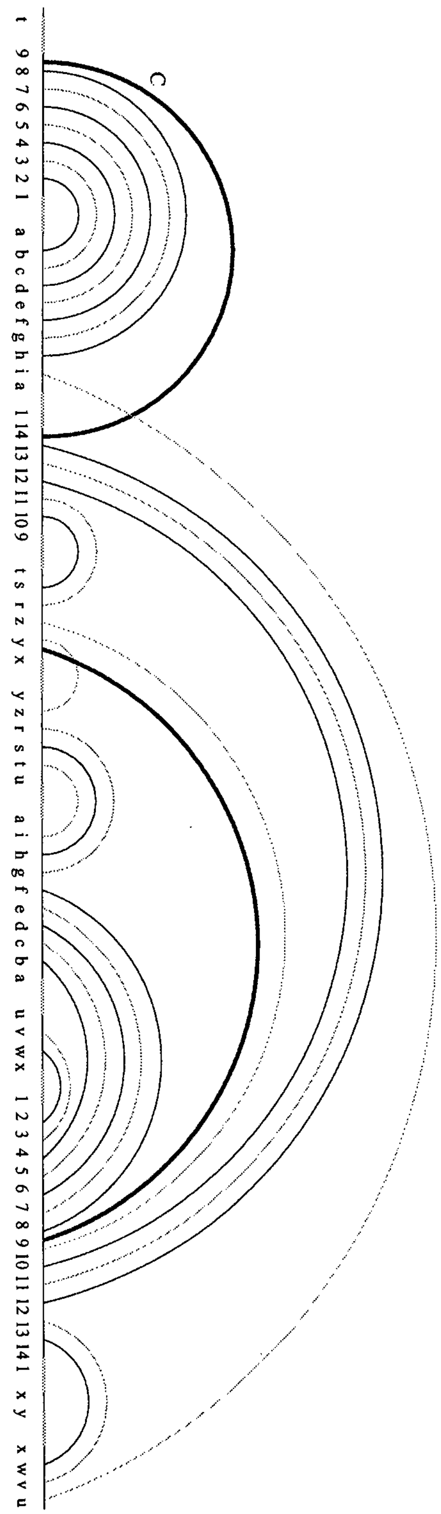

Figure 2.9: Circle and chord presentations for the lens spaces $L(7, q)$, $q = 1, 2, 3, 4, 5, 6$

We know that a Heegaard splitting $(V, W)$ of genus $g$ of $M$ produces a Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ where $b_1, b_2, \cdots, b_g$ are the respective boundary curves of the disks in a compeltely meridian disk system $\{B_1, B_2, \cdots, B_g\}$ of the handlebody $W$. Suppose that $\{d_1, d_2, \cdots, d_g\}$ is a complete meridian system of the handlebody $V$ which corresponds to a complete meridian disk system $\{D_1, D_2, \cdots, D_g\}$ of $V$. We need to know how to draw the curves $d_1, d_2, \cdots, d_g$ on the surface $\partial W$. The following theorem solves this problem.

**Theorem 2.10** *There exists an algorithm to obtain a circle and chord presentation corresponding to the Heegaard diagram $(W; d_1, d_2, \cdots, d_g)$, where $d_i = \partial D_i$ for $i = 1, 2, \cdots, g$ and the complete meridian system $\{b_1, b_2, \cdots, b_g\}$ of $W$ directly from a circle and chord presentation corresponding to the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ and the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ of $V$.*

*Proof.* The necessary and sufficient condition that there exists a circle and chord presentation corresponding to the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ and the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ of $V$ is that there does not exist a simple closed curve $c \subset \partial V$ such that $c \cap (\cup_{i=1}^{g} (b_i \cup d_i)) = \emptyset$ and $c$ does not bound a disk in $\partial V$. Thus, there exists a circle and chord presentation corresponding to the Heegaard diagram $(W; d_1, d_2, \cdots, d_g)$ and the complete meridian system $\{b_1, b_2, \cdots, b_g\}$ of $W$ if and only if there exists a circle and chord presentation corresponding to the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ and the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ of $V$.

Now, suppose the latter exists. To obtain the former, we determine subarcs of the curves $d_1, d_2, \cdots, d_g$ and use them and the curves $b_1, b_2, \cdots, b_g$ to obtain the circle in a circle and chord presentation. This procedure means that we find subarcs $c_1, c_2, \cdots, c_{2g-1}$ of $d_1, d_2, \cdots, d_g$ which corresponds to the above subarcs of the curves $d_1, d_2, \cdots, d_g$ such that cutting $V$ along the simple closed curves $b_1, b_2, \cdots, b_g$ and the subarcs $c_1, c_2, \cdots, c_{2g-1}$




A Heegaard splitting of the 3-sphere. The two wider semicircles consist of a primitive curve C since this primitive curve has trivial reduced word corresponding to the Heegaard curves, the Heegaard splitting is reducible.

Figure 2.10: A circle and chord presentation of $\mathbb{S}^3$

is a disk. Then the other parts of the curves $d_1, d_2, \cdots, d_g$ are the chords in our circle and chord presentation.

□

**Definition 2.7** *We call the two circle and chord presentations in Theorem 2.10 dual circle and chord presentations.*

**Example.** The dual circle and chord presentation of the standard Heegaard diagram of the lens space $L(p, q)$ is just the circle and chord presentation of the standard Heegaard diagram of the lens space $L(p, q')$, where $q'q \equiv 1 mod\ p$.

**Definition 2.8** *Two circle and chord presentations are equivalent if the respective Heegaard diagrams corresponding to them are strongly equivalent.*

**Theorem 2.11** *Suppose $(V, W)$ and $(V', W')$ are two Heegaard splittings of $M$. Suppose their respective Heegaard diagrams are $(V; b_1, b_2, \cdots, b_g)$ and $(V'; b'_1, b'_2, \cdots, b'_g)$. Then the two Heegaard diagrams are equivalent but not strongly equivalent if and only if the circle and chord presentation given by the first Heegaard diagram is equivalent to the dual presentation of the circle and chord presentation given by the second Heegaard diagram.*

*Proof.* A dual circle and chord presentation of the Heegaard diagram $(V'; b'_1, b'_2, \cdots, b'_g)$ corresponds to the Heegaard diagram $(W'; d'_1, d'_2, \cdots, d'_g)$, where the curves $d'_1, d'_2, \cdots, d'_g$ consist of a complete meridian system of the handlebody $V'$. Therefore, according to Birman's definition about strong equivalences, the theorem is true.

□

**Example.** The circle and chord representations of $L(7, 2)$ and $L(7, 3)$ drawing in this section are dual. Therefore, $L(7, 2) \cong L(7, 3)$. In fact, we can extend this fact to give a new proof of the classification theorem of the lens spaces since for $q \equiv \pm q'$

(mod $p$), the circle and chord representatins of $L(p,q)$ and $L(p,q')$ are same or are same after changing the direction of a Heegaard curve; for $q \cdot q' \equiv \pm 1 \pmod p$, the circle and chord representatins of $L(p,q)$ and $L(p,q')$ are dual or are dual after changing the direction of a Heegaard curve.

**Example.** An application of circle and chord presentations is to decide whether a word of a free group generated by generators $d_1, d_2, \cdots, d_g$ is the word of a curve in the boundary surface of a handlebody of genus $g$ corresponding to a complete meridian system $d_1, d_2, \cdots, d_g$. For example, Gillman [8] gave a balanced presentation of the trivial group $< a, b : aba^{-1}b^{-2}, bab^{-1}a^{-2} >$. In this presentation, the words $aba^{-1}b^{-2}, bab^{-1}a^{-2}$ can not be the words of the Heegaard curves $b_1, b_2$ corresponding to a complete meridian system $\{a, b\}$ of the handlebody $V$ for any Heegaard diagram $(V; b_1, b_2)$ of genus 2 of a closed 3-manifold. In fact, we can not draw a curve in $\partial V$ with the word $aba^{-1}b^{-2}$ since the sub-curves $ab, ba^{-1}, a^{-1}b^{-1}$ gives us a unique circle in the circle and chord presentation and there exists at least one intersection point when we try to draw the sub-curves $b^{-1}b^{-1}, b^{-1}a$. This method can be extended to list all the words or word sets which correspond to a Heegaard diagram of a 3-manifold.

## 2.6 Detection of closed homeomorphic 3-manifolds

In this section, we use the circle and chord presentation to detect two homeomorphic 3-manifolds.

We will analyse the stable equivalence relation of two Heegaard diagrams step by step. Since we can determine equivalence of Heegaard diagrams through strong equivalence of Heegaard diagrams and their dual Heegaard diagrams by the last section, we will only consider strong equivalence.

Through out this section, we use the following definitions.

(1). $M$ is a connected, closed, orientable 3-manifold.

(2). $(V, W)$, $(V', W')$ are two stably equivalent Heegaard splittings of $M$ of genus $g$.

(3). The strong equivalent map of the two Heegaard splitting is a homeomorphic map $h : M \longrightarrow M$ with $h(V) = V'$, $h(W) = W'$.

(4). $W, V, W', V'$ have complete meridian disk systems $B_1, B_2, \cdots, B_g$; $D_1, D_2, \cdots, D_g$; $B'_1, B'_2, \cdots, B'_g$; $D'_1, D'_2, \cdots, D'_g$; respectively.

(5). $b_i = \partial B_i$; $d_i = \partial D_i$; $b'_i = \partial B'_i$ and $d'_i = \partial D'_i$ for $i = 1, 2, \cdots, g$.

(6). $(V; b_1, b_2, \cdots, b_g)$ and $(V'; b'_1, b'_2, \cdots, b'_g)$ are the respective Heegaard diagrams of the above two Heegaard splittings, i.e., $b_i$ ( or $b'_i$ ) is the image of $b_i$ in the surface $\partial V$ ( or the image of $b'_i$ in the surface $\partial V'$; respectively ) for $i = 1, 2, \cdots, g$.

(7). Words of the curves $b_1, b_2, \cdots, b_g$ corresponding to the complete meridian system $\{d'_1, d'_2, \cdots, d'_g\}$ are $w_1, w_2, \cdots, w_g$ respectively.

*Case 1.* $h(B_i) = B'_i, h(D_i) = D'_i$ *for* $i = 1, 2, \cdots, g$.

Even in this simple case, the two Heegaard diagrams $(V; b_1, \cdots, b_g)$ and $(V'; b'_1, \cdots, b'_g)$ may be different. For example, we may use a Dehn twist on $V'$ along a disk which does not intersect the complete meridian disk system of $V'$. This does not change the equivalence relation but will change the presentation of $b', \cdots, b'_g$. However, if we use the circle and chord presentations and choose the same sub-curves for the two Heegaard curve systems, then the respective two circle and chord presentations will be exactly the same since the circles in the two presentations are equal and the chords have unique positions in the presentations. Therefore, this case will be easily decided by using circle and chord presentations. We only need to find all such presentations for the two Heegaard diagrams and compare them.

*Case 2.* $h(D_i) = D'_i$ *for* $i = 1, 2, \cdots, g$.

In this case, the curves $b_1, b_2, \cdots, b_g$ are not the respective curves in the curve set $b'_1, b'_2, \cdots, b'_g$. Since $h(B_1), h(B_2), \cdots, h(B_g)$ are disks in the handlebody $W'$, the curves

$b_1, b_2, \cdots, b_g$ are connected sums of the curves $b'_1, b'_2, \cdots, b'_g$. Then, $b_1, b_2, \cdots, b_g$ are connected sums of the curves $b'_1, b'_2, \cdots, b'_g$.

For each circle and chord presentation of the Heegaard diagram $(V'; b'_1, b'_2, \cdots, b'_g)$, we draw the chords which correspond to the words $w_1, w_2, \cdots, w_g$. By the last section, we have only finitely many possible positions for these chords. For each possibility, we read the words of the chord set corresponding to the complete meridian system $\{b'_1, b'_2, \cdots, b'_g\}$, where the chord set represent the words $w_1, w_2, \cdots, w_g$ respectively. Then in this case, at least for one possibility, all the words that we just read have trivial reduced form. Thus, there is an algorithm to determine whether two Heegaard diagrams represent the same manifold so that the equivalent relation belongs to this case or not.

*Case 3. The general case.*

In this case, even the curves $b_1, b_2, \cdots, b_g$ will not match with the curves $b'_1, b'_2, \cdots, b'_g$ respectively. By the same reason as in Case 2, we know that $b_1, b_2, \cdots, b_g$ are connected sums of the curves $b'_1, b'_2, \cdots, b'_g$ and $\{b_1, b_2, \cdots, b_g\}$ are connected sums of the curves $\{b'_1, b'_2, \cdots, b'_g\}$.

Since we can not directly know the connected sum moves on the curves $\{b'_1, b'_2, \cdots, b'_g\}$ to obtain the curves $\{b_1, b_2, \cdots, b_g\}$, we need to consider many complete meridian disk systems of the handlebody $V$. Suppose that we list all such systems ( according to the numbers of the intersection points with the original complete meridian system $\{d_1, d_2, \cdots, d_g\}$ ) as $\{D_{11}, \cdots, D_{1g}\}; \{D_{21}, \cdots, D_{2g}\}; \cdots, \{D_{n1}, \cdots, D_{ng}\}; \cdots$. Then since the two Heegaard diagrams are stably equivalent, there exists a number $n$ such that $h(D_{ni}) = D'_i$ for $i = 1, 2, \cdots, g$. The remaining problem is to apply the method in Case 2 to find such an $n$.

Combining the Reidemeister-Singer Theorem and the above algorithm, we have the following result.

**Theorem 2.12** *There is an algorithm to detect if two closed, orientable 3-manifolds are homeomorphic.*

**Note.** The method in Case 3 can only detect the stable equivalence property of two Heegaard diagrams if they are really stably equivalent. For two Heegaard diagrams which are not stably equivalent, the above algorithm will not stop since the $n$ does not exist. Therefore, we need to find a bound for $n$, i.e., we need to find a number $M(g)$ for any two Heegaard diagrams of genus $g$ such that we do not need to consider the cases $n > M(g)$ when we use our algorithm to determine the two Heegaard diagrams are stably equivalent or not.

Problem. Find a bound $M(g)$ for the above algorithm.

# Chapter 3

## Heegaard diagrams of the 3-sphere $\mathbb{S}^3$

In this chapter, we will prove that for any two Heegaard diagrams of the same genus of $\mathbb{S}^3$, there is a sequence of connected sum moves and Dehn twists to pass from one to the other. In particular, there is a sequence of connected sum moves to change the Heegaard curves of a Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ of $\mathbb{S}^3$ to become primitive curves of the handlebody $V$ and there is also a sequence of Dehn twists on the Heegaard curves to change their positions such that the new Heegaard curves bound disjoint disks in $\mathbb{R}^3 - Int\ V$.

**Definition 3.1** *Let $V$ be a handlebody of genus $g$ in $\mathbb{R}^3$ or $\mathbb{S}^3$. We say that $V$ lies in $\mathbb{R}^3$ standardly if $V$ lies in $\mathbb{R}^3$ as in Fig. 3.11.*

Let the handlebody lie in the $\mathbb{R}^3$ standardly. We know that there exists a simple Heegaard diagram of $\mathbb{S}^3$ for each genus. We call this Heegaard diagram a standard Heegaard diagram of $\mathbb{S}^3$ in $\mathbb{R}^3$ .

**Definition 3.2** *The following Heegaard diagram $(V; e_1, e_2, \cdots, e_g)$ of genus $g$ of $\mathbb{S}^3$ in $\mathbb{R}^3$ is called the standard Heegaard diagram of genus $g$ of $\mathbb{S}^3$ in $\mathbb{R}^3$ ; where $V$ is the corresponding Heegaard handlebody and $e_1, e_2, \cdots, e_g$ are Heegaard curves. We also call the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ in the figure the standard complete meridian system of the handlebody.*

One important fact for Heegaard diagrams of $\mathbb{S}^3$ is that we can use a sequence of connected sum moves and Dehn twists to change them to the standard Heegaard diagram

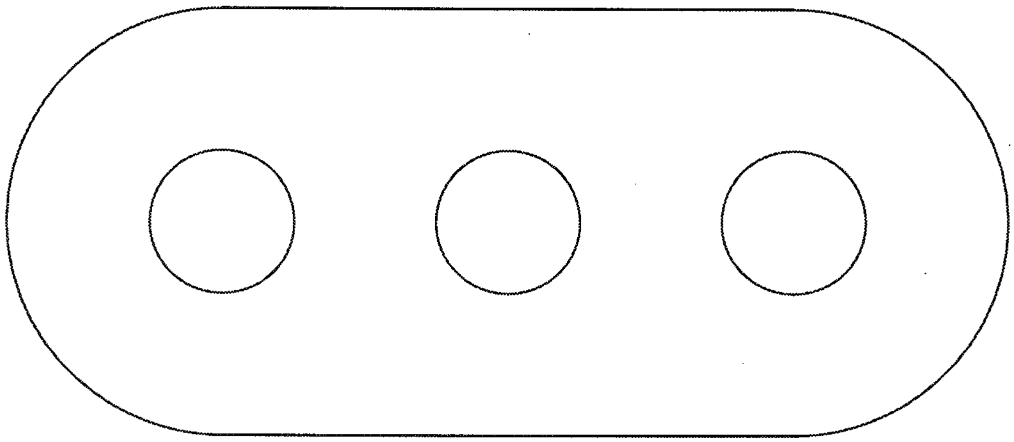


Figure 3.11: Standard handlebody in $\mathbb{R}^3$

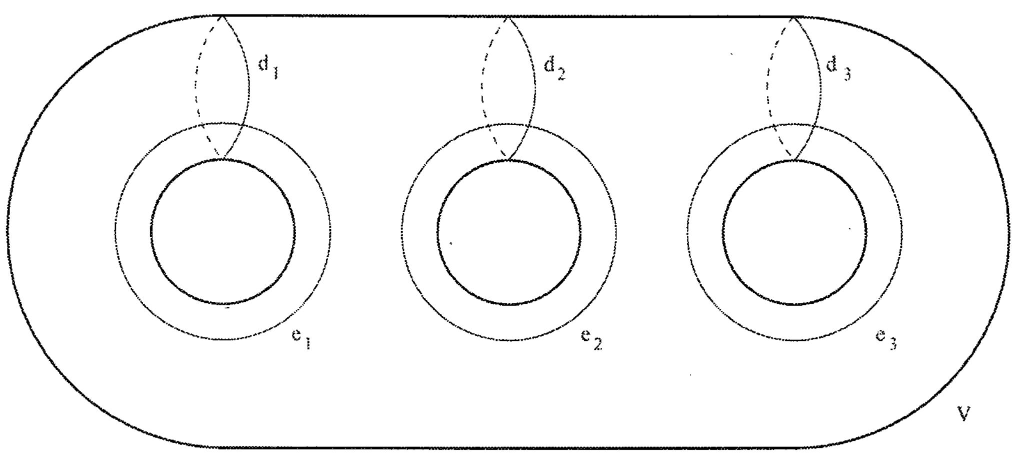



Figure 3.12: Standard Heegaard diagram of $\mathbb{S}^3$

of the same genus if the corresponding Heegaard handlebodies lie in $\mathbb{R}^3$ standardly.

**Definition 3.3** *Let $A$ be an annulus in 3-space. If there exists a 3-cell $B$ in 3-space such that $A \subset \partial B$, then we say $N$ is attached to the 3-cell $B$ in $\mathbb{R}^3$ .*

We also discuss an algorithm to determine Heegaard diagrams of $\mathbb{S}^3$.

## 3.1 Simplifying Heegaard diagrams of $\mathbb{S}^3$

We use Dehn twists and connected sum moves to change Heegaard curves of any Heegaard diagram of $\mathbb{S}^3$ to new positions such that the new Heegaard curves have the simplest forms in $\mathbb{R}^3$ .

**Theorem 3.1** *Suppose $(V; b_1, b_2, \cdots, b_g)$ is a Heegaard diagram of genus $g$ of $\mathbb{S}^3$. Let the handlebody $V$ lie in a 3-space $\mathbb{R}^3$ or $\mathbb{S}^3$ standardly. If the complete system $\{b_1, b_2, \cdots, b_g\}$ consists of primitive curves, then we can use a sequence of Dehn twists and connected sum moves on the complete system $\{b_1, b_2, \cdots, b_g\}$ up to ambient isotopy to obtain the standard Heegaard diagram of genus $g$ of $\mathbb{S}^3$.*

In the proof of the theorem, we will use the following lemmas.

**Lemma 3.1** *Suppose $V$ is a handlebody of genus $g$ and $\{D_1, D_2, \cdots, D_g\}$ is a complete meridian disk system of $V$ and $\{e_1, e_2, \cdots, e_g\}$ is a dual complete system of the system $\{D_1, D_2, \cdots, D_g\}$ (see Fig. 3.13). Suppose $c$ is a simple curve on $\partial V$ such that its end-points $P, Q$ lie in the curves $\partial D_s, \partial D_t$ respectively, where $s, t \in \{1, 2, \cdots, g\}, s \neq t$. If $c \cap (\cup_{i=1}^{g} \partial D_i) = \{P, Q\}$, then we can use Dehn twists along some disks which do not intersect the disks in the complete meridian disk system and ambient isotopy on $\partial V$ to change $c$ to a curve $c'$ such that $c \cap (\cup_{i=1}^{g} e_i) = \emptyset$.*

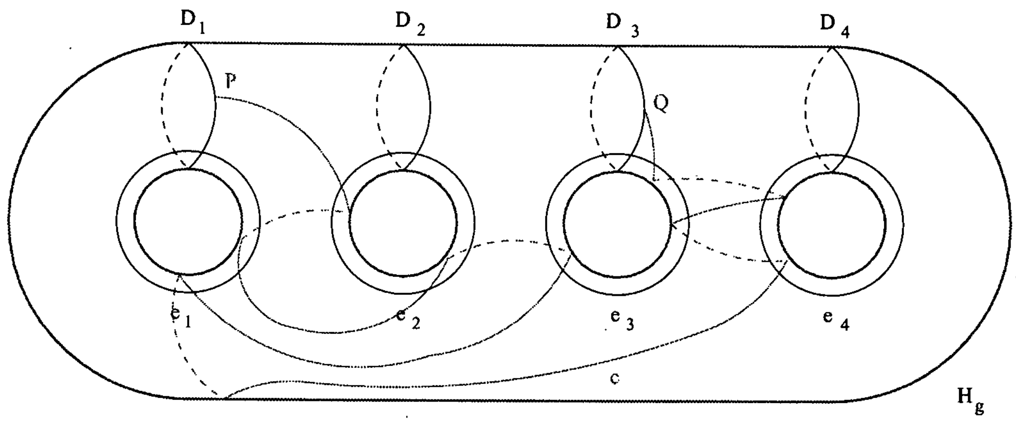



Figure 3.13: A handlebody with two standard complete systems in $\mathbb{R}^3$

*Note.* We use Dehn twists to change the position of $c$ only, i.e., we keep the curves $\partial D_1, \partial D_2, \cdots, \partial D_g$ fixed in $\partial V$.

*Proof.* Denote the boundary curves of the disks $D_1, D_2, \cdots, D_g$ as $d_1, d_2, \cdots, d_g$ respectively. Suppose the word of the curve $c$ – beginning from the endpoint $P$ – corresponding to the curve system $\{d_1, d_2, \cdots, d_g, e_1, e_2, \cdots, e_g\}$ is $w_c = d_s^\mu e_{i_1}^{\alpha_1} e_{i_2}^{\alpha_2} \cdots e_{i_n}^{\alpha_n} d_t^\nu$; where $\mu, \nu \in \{1, -1\}$; $\alpha_1, \alpha_2, \cdots, \alpha_n \in \{-1, 1\}$; and $i_1, i_2, \cdots, i_n \in \{1, 2, \cdots, g\}$. We will prove that we can use a sequence of Dehn twists on the curve $c$ to change it to a curve $c'$ whose word corresponding to the same curve system is $d_s^\mu d_t^\nu$.

We cut the surface $\partial V$ along the curves in the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ and obtain a 2-sphere with $2g$ holes $d_1^1, d_1^2, d_2^1, d_2^2, \cdots, d_g^1, d_g^2$, where $d_i^1, d_i^2$ are the two respective copies of $d_i$ for $i = 1, 2, \cdots, g$. We denote this surface as $\Omega$ and use $A_k^1, A_k^2$ to denote the two endpoints of $e_k$, $k = 1, \cdots, g$. (See Fig. 3.14)

For any $k \in \{1, 2, \cdots, g\}$ and $k \neq s, k \neq t$, if $c \cap e_k \neq \emptyset$, there exists a point $R \in c \cap e_k$ such that the portion $e'$ of $e_k$ bounded by $A_k^1, R$ satisfies $e' \cap c = R$, i.e., $c$ does not intersect

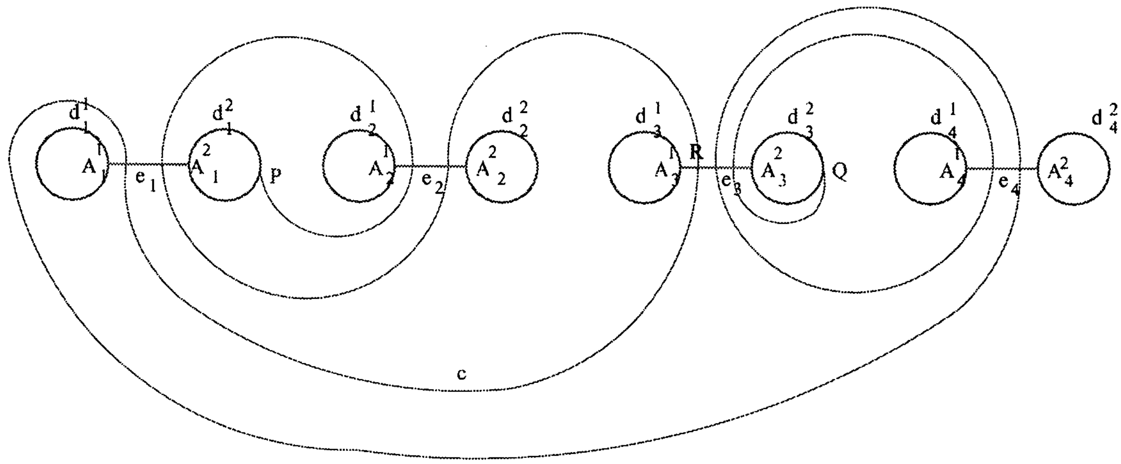



Figure 3.14: Cutting $\partial V$ along the complete meridian system

the interior of $e'$. Suppose $R$ separates $c$ into two simple curves $c', c''$ whose endpoint sets are $\{P, R\}$, $\{R, Q\}$ respectively. Then the respective words of $c', c''$ corresponding to the the curve system $\{d_1, d_2, \cdots, d_g, e_1, e_2, \cdots, e_g\}$ are $w_{c'} = d_s^{\mu} e_{i_1}^{\alpha_1} e_{i_2}^{\alpha_2} \cdots e_{i_m}^{\alpha_m}$, $w_{c''} = e_{i_m}^{\alpha_m} \cdots e_{i_n}^{\alpha_n} d_t^{\nu}$, where $i_m = k$. Then, $c' \cup e'$ is a simple curve whose endpoints $P, R$ lie on the curves $d_s^{\epsilon}, d_k^1$ respectively for some $\epsilon \in \{1, 2\}$. Now, consider a small regular neighborhood $N$ of graph $d_s^{\epsilon} \cup c' \cup e' \cup d_k^1$ in the surface $\Omega$. Let $d$ be the boundary connected component of $N$ not parallel to either $d_k^1$ or $d_s^{\epsilon}$. Then $d$ bounds a disk $D$ in the handlebody $V$. We use a Dehn twist along $D$ in a suitable direction to change $c$ to a curve $c_1$ whose word is $w_{c_1} = d_s^{\mu} d_s^{\alpha} e_{i_1}^{\alpha_1} e_{i_2}^{\alpha_2} \cdots e_{i_{m-1}}^{\alpha_{m-1}} e_{i_{m+1}}^{\alpha_{m+1}} \cdots e_{i_n}^{\alpha_n} d_t^{\nu}$, where $\alpha \in \{1, -1\}$. We use one more Dehn twist along $D_s$ to cancel the new intersection point between $c_1, e_s$. Thus, the new curve $c_2$ has at least one less intersection points with the curves in the dual complete system. In fact, the word of $c_2$ is $w_{c_2} = d_s^{\mu} e_{i_1}^{\alpha_1} e_{i_2}^{\alpha_2} \cdots e_{i_{m-1}}^{\alpha_{m-1}} e_{i_{m+1}}^{\alpha_{m+1}} \cdots e_{i_n}^{\alpha_n} d_t^{\nu}$ with reducing some adjacent cancel pairs. (See Fig. 3.15)

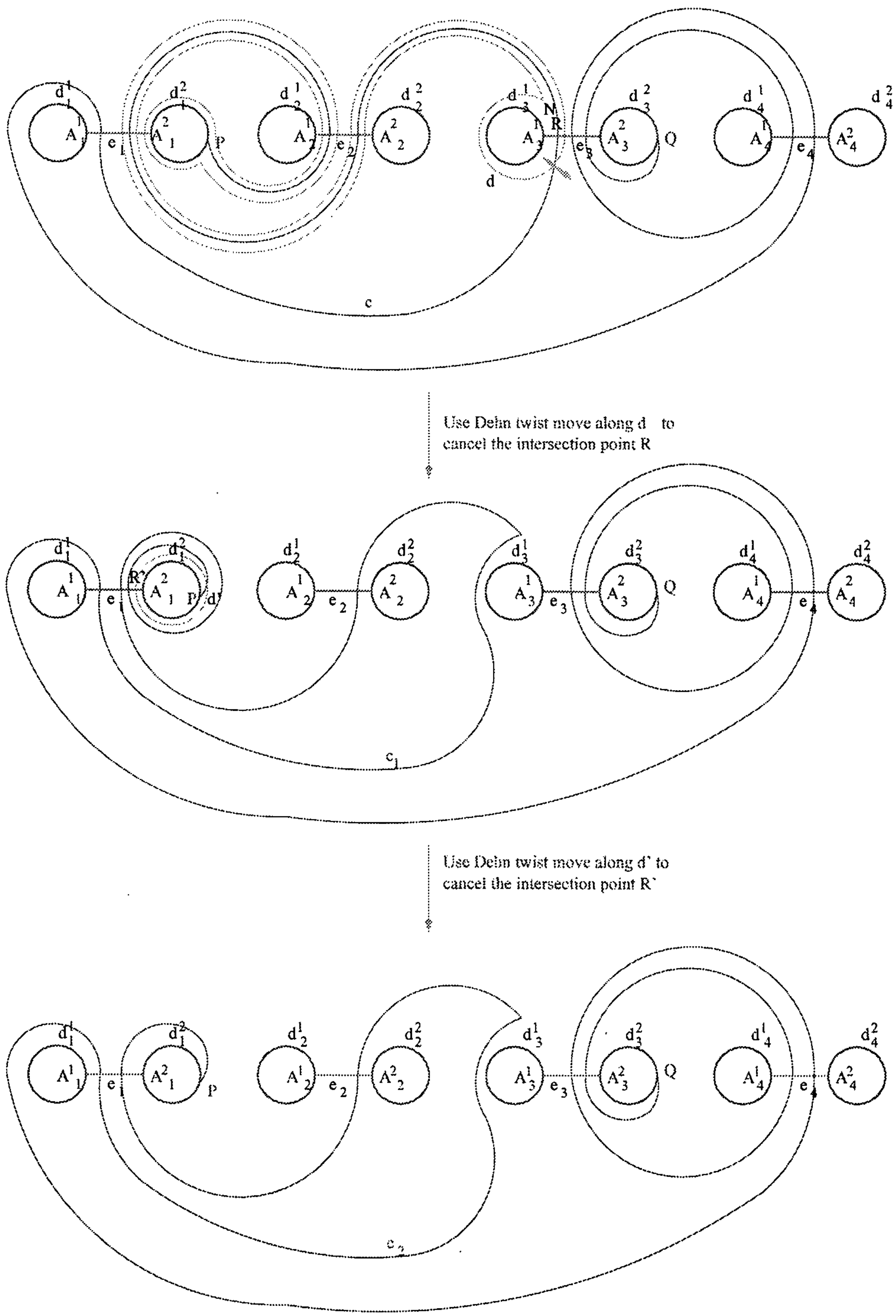



Figure 3.15: Using two Dehn twists to remove intersection points

Applying the preceding construction repeatedly, we may assume that the curve $c$ has been changed to a new curve also denoted by $c$ which does not intersect the curves in the dual complete system $\{e_1, e_2, \cdots, e_g\}$ except possibly the curves $e_s, e_t$. Finally, we will apply a similar construction as before to cancel the remainning intersection points between $c$ and $e_s \cup e_t$.

If $c \cap e_t \neq \emptyset$ and $Q \in d_t^2$ in $\Omega$, suppose that $Z$ is one of the intersection points between $c$ and $e_t$ such that $c$ does not intersect with the interior of the simple sub-curve $a'$ of $e_t$ whose endpoints are $A_t^1$ and $Z$. Suppose $Z$ separates $c$ into two simple curves also denoted by $c', c''$ whose endpoint sets are $\{P, Z\}$, $\{Z, Q\}$ respectively. Consider a small regular neighborhood $U$ of graph $d_s^\epsilon \cup c' \cup a' \cup d_t^1$ in the surface $\Omega$. The boundary connected components of $U$ consist of three simple closed curves: $d_t^1$, $d_s^\epsilon$ and another curve $d'$. $d'$ bounds a disk in $V$. After we suitably use a Dehn twist on $c$ along $d'$ we change it to a new curve which has one less intersection point with $e_t$. This surgery lets the new curve have one more intersection point $Y$ with $e_s$. But $Y$ can be removed by using one Dehn twist move along $d_s^\epsilon$ as before.

There are three other cases we need to consider: $c \cap e_t \neq \emptyset$ and $Q \in d_t^1$ in $\Omega$; or $c \cap e_s \neq \emptyset$ and $P \in d_s^2$ in $\Omega$; or $c \cap e_s \neq \emptyset$ and $P \in d_s^1$ in $\Omega$. It is clear that they can be solved by using the same method as the case we just solved.

Therefore, we can use Dehn twists to change $c$ to a new curve which does not intersect with the curves $e_1, e_2, \cdots, e_g$.

□

Lemma 3.1 shows how to change the position of $c$ in the surface $\partial V$. Now, we consider changes to the curves $e_1, e_2, \cdots, e_g$ that remove the intersection points between $c$ and these curves.

**Lemma 3.2** *Suppose the hypothesis of Lemma 3.1. Then we can use Dehn twists along* *some disks which do not intersect the curves in the complete meridian system on the curves* $e_1, e_2, \cdots, e_g$ *to obtain a new dual complete system* $\{e'_1, e'_2, \cdots, e'_g\}$ *such that* $c$ *does not intersect the curves of the new dual complete system.*

*Proof.* It follows from Lemma 3.1 that there is a sequence of Dehn twists $T_1, T_2, \cdots, T_n$ along disks $B_1, B_2, \cdots, B_n$ respectively and ambient isotopies which applied to $c$ change $c$ to $c'$ such that $c' \cap (\cup_{i=1}^{g} e_i) = \emptyset$. We also know that the disks $B_1, B_2, \cdots, B_n$ do not intersect the curves $d_1, d_2, \cdots, d_g$. Now we use a sequence of Dehn twists $T_n^{-1}, T_{n-1}^{-1}, \cdots, T_1^{-1}$ and ambient isotopies on both $c'$ and all the curves in the complete system $\{e_1, \cdots, e_g\}$, where $T_i^{-1}$ is the inverse Dehn twist of $T_i$ for $i = 1, 2, \cdots, n$. These moves change $c'$ back to $c$ and change the curves $e_1, e_2, \cdots, e_g$ to curves $e'_1, e'_2, \cdots, e'_g$. $c' \cap (\cup_{i=1}^{g} e_i) = \emptyset$ implies $c \cap (\cup_{i=1}^{g} e'_i) = \emptyset$ . Since $B_1, B_2, \cdots, B_n$ do not intersect with the curves $d_1, d_2, \cdots, d_g$, the intersections between the curves $e'_1, e'_2, \cdots, e'_g$ and the curves $d_1, d_2, \cdots, d_g$ are the same as the corresponding intersections between the curves $e_1, e_2, \cdots, e_g$ and the curves $d_1, d_2, \cdots, d_g$. Therefore, $\{e'_1, e'_2, \cdots, e'_g\}$ is a dual complete system of the complete meridian system $\{d_1, d_2, \cdots, d_g\}$

□

Next, we consider the use of connected sum moves to change a complete meridian system and its dual complete system.

**Lemma 3.3** *Suppose that* $\{d_1, d_2, \cdots, d_g\}$ *is a complete meridian system of a handlebody* $V$ *of genus* $g$ *and* $\{b_1, b_2, \cdots, b_g\}$ *is a dual complete system, i.e.,* $d_i \cap b_i$ *is one point* $P_i$ *and* $d_i \cap b_k = \emptyset$ *for* $i, k = 1, 2, \cdots, g; i \neq k$. *Suppose* $c$ *is a simple curve in* $\partial V$ *whose endpoints* $P, Q$ *respectively lie in the curves* $d_s, d_t$ *for some* $s, t \in \{1, 2, \cdots, g\}, s < t$ *and the interior of* $c$ *does not intersect the curves of the two complete systems. Then* $\{d_1, d_2, \ldots, d_s, \cdots, d_{t-1}, d_s \sharp_c d_t, d_{t+1}, \cdots, d_g\}$ *is a complete meridian system of* $V$. *It has a dual complete system* $\{b_1, b_2, \cdots, b_{s-1}, b_s \sharp_{c'} b_t, b_{s+1}, \cdots, b_t, \cdots, b_g\}$, *where we isotopically* *move $P, Q$ together with their neighbor part sub-curves of $c$ in small neighborhoods of $d_s \cup b_s, d_t \cup b_t$ in the surface $V$ to obtain the simple curve $c'$ whose endpoints lie in $b_s, b_t$ respectively.*

*Proof.* $\{d_1, d_2, \ldots, d_s, \cdots, d_{t-1}, d_s\sharp_c d_t, d_{t+1}, \cdots, d_g\}$ is a complete meridian system of $V$. Note that the simple closed curve $d_s\sharp_c d_t$ intersects the curves $b_s, b_t$ at $P', Q'$ respectively. $P', Q'$ separate $d_s\sharp_c d_t$ into two simple curves. Denote one of them as $c'$. Then the simple closed curve $b_s\sharp_{c'} b_t$ intersects $d_s, d_t$ at $P, Q$ respectively and it does not intersect with the curve $d_s\sharp_c d_t$. Since the interior of $c$ does not intersect the curves $d_1, \cdots, d_g, b_1, \cdots, b_g$, $b_s\sharp_{c'} b_t \cap ((\cup_{i=1}^g d_i) \cup (\cup_{i=1}^g b_i)) = (b_s\sharp_{c'} b_t \cap d_s) \cup (b_s\sharp_{c'} b_t \cap d_t) = \{P, Q\}$ and $(d_s\sharp_c d_t) \cap (\cup_{i=1}^g b_i) = (d_s\sharp_c d_t) \cap (b_s \cup b_t) = \{P', Q'\}$.

Therefore, $\{d_1, d_2, \ldots, d_s, \cdots, d_{t-1}, d_s\sharp_c d_t, d_{t+1}, \cdots, d_g\}$ has a dual complete system $\{b_1, b_2, \cdots, b_{s-1}, b_s\sharp_{c'} b_t, b_{s+1}, \cdots, b_t, \cdots, b_g\}$ and $\{d_1, \ldots, d_{s-1}, d_s\sharp_c d_t, d_{s+1}, \cdots, d_t, \cdots, d_g\}$ has a dual complete system $\{b_1, \cdots, b_s, \cdots, b_{t-1}, b_s\sharp_{c'} b_t, b_{t+1}, \cdots, b_g\}$.

□

Now, we are ready to prove Theorem 3.1. Our main idea is to change a complete meridian system of $\mathbb{S}^3$ to the standard complete meridian system and at the same time to obtain new dual complete systems by using connected sum moves.

*Proof of Theorem 3.1* Since $b_1, b_2, \cdots, b_g$ are primitive curves, there exists a complete meridian system $\{d_1, d_2, \cdots, d_g\}$ of the handlebody $V$ such that $b_s \cap d_t = \emptyset$ and $b_s \cap b_s$ is one point $P_s$ for $s, t = 1, 2, \cdots, g, s \neq t$ , i.e., $\{b_1, b_2, \cdots, b_g\}$ is a dual complete system of the complete meridian system.

It is easy to see that the reduced words of the curves in a complete meridian system corresponding to another complete meridian system are all trivial. Thus, by the chapter 1, we know that we can use a sequence of connected sum moves on the curves in the first complete meridian system to change them to the curves in the second complete meridian system. Therefore, we can use a sequence of connected sum moves $T_1, T_2, \cdots, T_m$ to transform the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ to the standard complete meridian system of $V$. Suppose that $d_1^0 = d_1, d_2^0 = d_2, \cdots, d_0^g = d_g$. Then for $i = 1, 2, \cdots, m$, $T_i$ is a connected sum move on the curves $d_1^{i-1}, d_2^{i-1}, \cdots, d_g^{i-1}$ which transforms the above curves to the respective curves in a new complete meridian system $d_1^i, d_2^i, \cdots, d_g^i$, i.e., there are $s_i, t_i \in \{1, 2, \cdots, g\}, s_i \neq t_i$ and a simple curve $c_i$ in $\partial V$ whose endpoints lie in $d_{s_i}^{i-1}, d_{t_i}^{i-1}$ respectively such that the interior of $c_i$ does not intersect the curves in the complete meridian system $\{d_1^{i-1}, d_2^{i-1}, \cdots, d_g^{i-1}\}$ and $d_{t_i}^i = d_{s_i}^{i-1} \sharp_{c_i} d_{s_i}^{i-1}$, $d_k^i = d_k^{i-1}$ for $k \in \{1, 2, \cdots, g\}, k \neq t_i$; and $\{d_1^m, d_2^m, \cdots, d_g^m\}$ is the standard complete meridian system of $V$.

Now, we use connected sum moves and Dehn twists according to the following steps.

Step 1. Use Dehn twists stated in Lemma 3.1 on the curves in the complete system $\{b_1, b_2, \cdots, b_g\}$ to transform them to the curves in a new complete system $\{b_1^1, b_2^1, \cdots, b_g^1\}$ such that $c_1 \cap (\cup_{i=1}^g b_i) = \emptyset$. Note that $\{b_1^1, b_2^1, \cdots, b_g^1\}$ still is a dual complete system of the complete meridian system $\{d_1, d_2, \cdots, d_g\}$

Step 2. Use one connected sum move $T_1$ introduced in the above on the curves $d_1^0, d_2^0, \cdots, d_g^0$ to transform them to the curves $d_1^1, d_2^1, \cdots, d_g^1$.

Step 3. Use one connected sum move stated in Lemma 3.3 on the curves in the complete system $\{b_1^1, b_2^1, \cdots, b_g^1\}$ to transform them to curves which form a new complete system $\{b_1^2, b_2^2, \cdots, b_g^2\}$. Note that this new complete system is a dual complete system of the complete meridian system $\{d_1^1, d_2^1, \cdots, d_g^1\}$.

Repeat the above steps on the complete meridian system $\{d_1^1, d_2^1, \cdots, d_g^1\}$ and its dual complete system $\{b_1^2, b_2^2, \cdots, b_g^2\}$. $\cdots, \cdots$. Finally, after using the above three steps $m$ times, we change the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ to the standard complete meridian system of $V$ and change the complete system $\{b_1, b_2, \cdots, b_g\}$ to a dual complete system $\{b_1^{2m}, b_2^{2m}, \cdots, b_g^{2m}\}$ of the standard complete meridian system $\{x_1, x_2, \cdots, x_g\}$.

That is $j_1^{2m} \cap x_j$ is one point if $i = j$ or is an empty set if $i \neq j$ for $i, j = 1, 2, \cdots, g$.

If $\{b_1^{2m}, b_2^{2m}, \cdots, b_g^{2m}\}$ is not the standard dual complete system of the standard complete meridian system (see the following figure), then we can easily find Dehn twists on the curves $b_1^{2m}, b_2^{2m}, \cdots, b_g^{2m}$ along disks which do not intersect the disks in the standard complete meridian disk system to change the curves $b_1^{2m}, b_2^{2m}, \cdots, b_g^{2m}$ to their standard positions one by one. (To move one such a curve, for example $b_1^{2m}$, to its respective standard position $e_1$, we can use a similar method as in the proof of the lemma 6 to change $b_1^{2m}$ to a curve $b_1''$ which does not intersect with the curves $e_2, e_3, \cdots, e_g$. Then we use Dehn twists suitable many times along $x_1$ to change $b_1''$ to a curve $\ell_1$ such that $\ell_1 \cap (\cup_{i=1}^{g} e_i) = \emptyset$, i.e., $\ell$ does not intersect with the curves $e_1, e_2, \cdots, e_g, x_2, \cdots, x_g$ and $\ell$ intersects with $x_1$ at one point. Perhaps $\ell$ still is not $e_1$. In this case, we can use Dehn twists along some disks $\Delta_1, \Delta_2, \cdots, \Delta_m$ to change the curve $\ell_1$ to curve $\ell$ such that $\ell$ lies in a neighborhood of $e_1 \cup x_1$ in $\partial V$, where the disk $\Delta_j$ intersects $e_1$ at one point and $\Delta_j$ does not intersect with the curves $e_2, \cdots, e_g, x_1, x_2, \cdots, x_g$ for $j = 1, 2, \cdots, m$. Now, we can use Dehn twists along $x_1$ to change $\ell$ to the standard curve $e_1$. After we move $a_1^{2m}$ to $e_1$, we use the similar method to move $a_2^{2m}$ to $e_2$. It is clear that when we move $y_2^{2m}$, we do not need to care about the curve $e_1$ since the curves which we use Dehn twists along does not intersect with $e_1$. We continue to move the remaining curves to the positions of the respective curves in the complete system $\{e_1, e_2, \cdots, e_g\}$ and at the same time keep the curves which are already in their standard position fixed. Finally, we change all the curves $b_1^{2m}, b_2^{2m}, \cdots, b_g^{2m}$ to their standard positions.)

□

**Theorem 3.2** *Suppose that* $(V; b_1, b_2, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of* $\mathbb{S}^3$. *Then we can use a sequence of connected sum moves and Dehn twists on the Heegaard curves* $b_1, b_2, \cdots, b_g$ *to transform these curves to the positions of the Heegaard curves of*

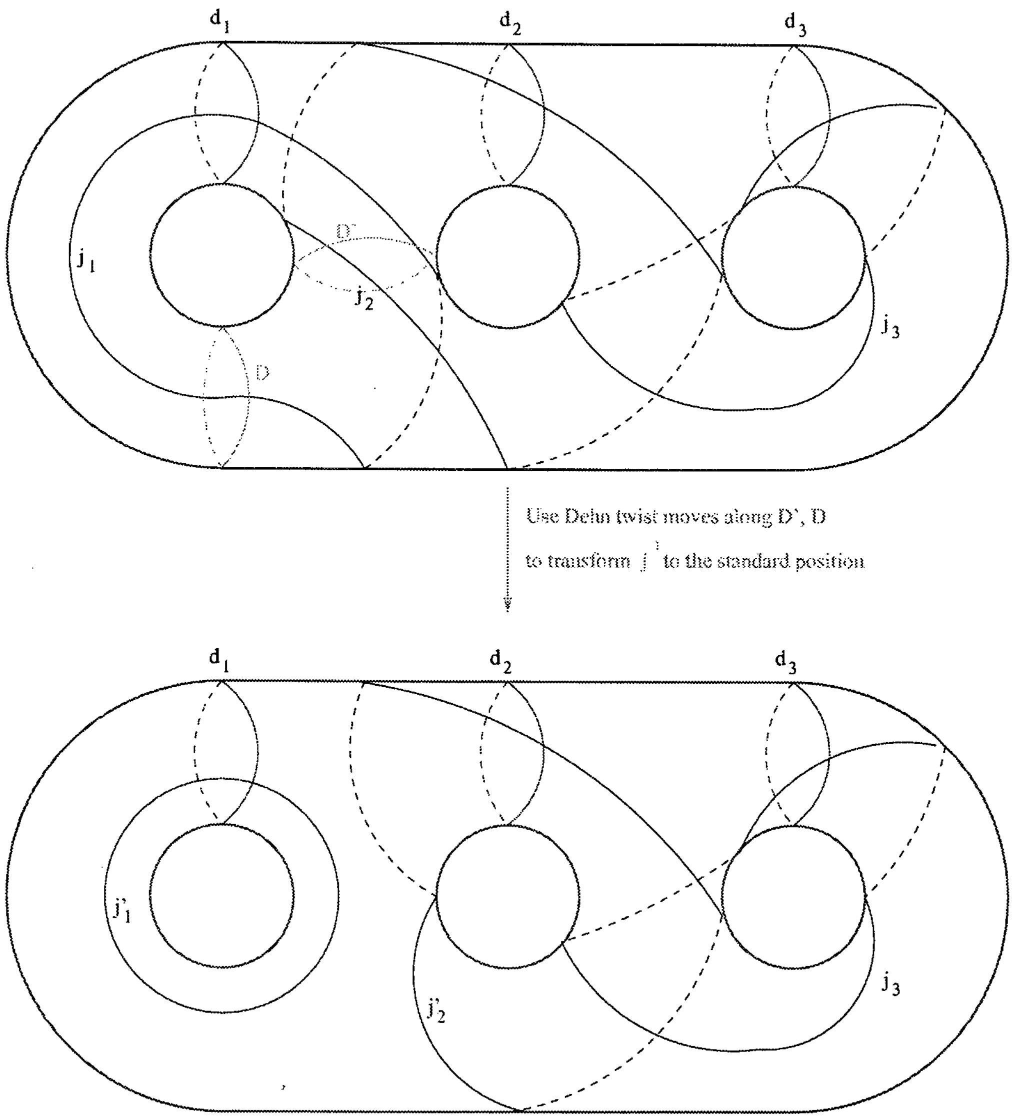



Figure 3.16: Using Dehn twists to change curves to standard positions

*the standard Heegaard diagram of genus $g$ of* $\mathbb{S}^3$.

*Proof.* By a result of Waldhausen [40], every Heegaard diagram of genus $g$ of $S^3$ is strongly equivalent to the standard Heegaard diagram $(V'; e_1, e_2, \cdots, e_g)$ of genus $g$ of $S^3$. Note $\{b_1, b_2, \cdots, b_g\}$ consists of a complete meridian system of the handlebody $Cl(S^3 - V)$, and the Heegaard curves $e_1, e_2, \cdots, e_g$ of the standard Heegaard diagram also consist of a complete meridian system of the handlebody $Cl(S^3 - V')$. Since these two Heegaard diagrams are equivalent, there exists a homeomorphism $\phi : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ such that $\phi(V) = V'$ and $\phi(Cl(S^3 - V)) = Cl(S^3 - V')$. Therefore, $\phi(e_1), \phi(e_2), \cdots, \phi(e_g)$ bound $g$ pairwise disjoint disks in the handlebody $S^3 - V$ and cutting $S^3 - V$ along these disks is a 3-cell. Therefore, $\phi(e_1), \phi(e_2), \cdots, \phi(e_g)$ consist of a complete meridian system of $S^3 - V$. Now, let $b_1, b_2, \cdots, b_g$ be the standard complete meridian system of $V'$. Then by the same reason, $\{\phi(b_1), \phi(b_2), \cdots, \phi(b_g)\}$ is a complete meridian system of $V$. Note $\{e_1, e_2, \cdots, e_g\}$ is the standard dual complete system of the complete meridian system $\{b_1, b_2, \cdots, b_g\}$ of the handlebody $V'$. Then, $\phi(e_1), \phi(e_2), \cdots, \phi(e_g)$ consist of a dual complete system of the complete meridian system $\{\phi(b_1), \phi(b_2), \cdots, \phi(b_g)\}$ of the handlebody $V$. Therefore $\phi(e_1), \phi(e_2), \cdots, \phi(e_g)$ consist of primitive curves in $V$. Since $\{\phi(e_1), \phi(e_2), \cdots, \phi(e_g)\}$ and $\{b_1, b_2, \cdots, b_g\}$ are two complete meridian systems of the handlebody $Cl(S^3 - V)$, the reduced words of the curves $b_1, b_2, \cdots, b_g$ corresponding to the complete system $\{\phi(e_1), \phi(e_2), \cdots, \phi(e_g)\}$ are trivial. Therefore, we can use a sequence of connected sum moves to change the complete system $\{b_1, b_2, \cdots, b_g\}$ to the complete system $\{\phi(e_1), \phi(e_2), \cdots, \phi(e_g)\}$ (Theorem 2.2). Since $\phi(e_1), \phi(e_2), \cdots, \phi(e_g)$ consist of primitive curves in $V$, we can use connected sum moves and Dehn twists to change them to the positions of the standard Heegaard curves in the handlebody $V$ by Theorem3.1. This completes the proof of the theorem.

□

The proof of the above theorem implies the following properties of Heegaard diagrams of $\mathbb{S}^3$.

**Corollary 3.1** *Suppose that* $(V; b_1, b_2, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of* $\mathbb{S}^3$*. Then there is a sequence of connected sum moves on the curves* $b_1, b_2, \cdots, b_g$ *to obtain a new Heegaard diagram* $(V; b_1', b_2', \cdots, b_g')$ *of* $\mathbb{S}^3$ *such that* $b_1', b_2', \cdots, b_g'$ *are primitive curves of* $V$*.*

*Proof.* In the proof of Theorem 3.2, we have proven that we can use connected sum moves on the curves $b_1, b_2, \cdots, b_g$ to obtain primitive curves $\phi(e_1), \phi(e_2), \cdots, \phi(e_g)$.

□

**Theorem 3.3** *Suppose that* $(V; b_1, b_2, \cdots, b_g)$ *and* $(V; b_1', b_2', \cdots, b_g')$ *are two Heegaard diagrams of genus* $g$ *of* $\mathbb{S}^3$*. Then we can use a sequence of connected sum moves and Dehn twists to pass from one to the other.*

*Proof.* By Theorem 3.2, we can use a sequence of connected sum moves and Dehn twists $T_1, T_2, \cdots, T_m$ (or $T_1', T_2', \cdots, T_n'$) to change the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ (or the the Heegaard diagram $(V; b_1', b_2', \cdots, b_g')$ respectively) to the standard Heegaard diagram $(V; e_1, e_2, \cdots, e_g)$. Therefore, the moves $T_1, T_2, \cdots, T_m, T_n'^{-1}, \cdots, T_2'^{-1}, T_1'^{-1}$ will change the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ to the Heegaard diagram $(V; b_1', \cdots, b_g')$; where we use $T^{-1}$ to denote the inverse move of a move $T$. Note the inverse move of a connected sum move (or a Dehn twist) still is a connected sum move (or a Dehn twist respectively). Therefore the corollary is true.

□

**Theorem 3.4** *Suppose that* $(V; b_1, b_2, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of* $\mathbb{S}^3$ *and* $\{d_1, d_2, \cdots, d_g\}$ *is a complete meridian system of* $V$*. Then there is a sequence of connected* *sum moves on the curves* $b_1, b_2, \cdots, b_g$ *to obtain a new Heegaard diagram* $(V; b'_1, b'_2, \cdots, b'_g)$ *of* $\mathbb{S}^3$ *such that the new Heegaard curves have cyclically reduced words* $d_1, d_2, \cdots, d_g$ *corresponding to the complete meridian system* $\{d_1, d_2, \cdots, d_g\}$ *respectively.*

*Proof.* By Theorem 3.2, we can use connected sum moves and Dehn twists to change the Heegaard curves $b_1, b_2, \cdots, b_g$ to the dual Heegaard curves $e_1, e_2, \cdots, e_g$ of the complete meridian system $\{d_1, d_2, \cdots, d_g\}$. Note the words of $e_1, e_2, \cdots, e_g$ corresponding to the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ are $d_1, d_2, \cdots, d_g$ respectively. By Lemma 11 of Section 1.7, we can use the corresponding connected sum moves first to obtain a Heegaard diagram $(V; b'_1, b'_2, \cdots, b'_g)$. Then we use the corresponding Dehn twists to change the Heegaard diagram $(V; b'_1, b'_2, \cdots, b'_g)$ to the standard Heegaard diagram. The cyclically reduced words of the Heegaard curves $b'_1, b'_2, \cdots, b'_g$ corresponding to the complete meridian system $\{d_1, d_2, \cdots, d_g\}$ are $\{d_1, d_2, \cdots, d_g\}$ respectively since Dehn twists do not change such cyclically reduced words.

**Remark.** We can not generalize the above corollary to two equivalent Heegaard diagrams of a 3-manifold. In fact, we can not even generalize it to two strongly equivalent Heegaard diagrams. We proved Theorem 3.1 by using a very special property of a Heegaard diagram of $\mathbb{S}^3$: there exist free primitive curves which consist of a complete system of the respective handlebody. For other 3-manifolds, this kind of free primitive curves do not exist. Therefore, the above method does not work.

For example, the standard Heegaard diagrams corresponding to Lens space $L(7,2)$ and $L(7,3)$ are stably equivalent. But we can not pass from one to the other by using connected sum moves and Dehn twists. The reason is that the Heegaard curve set consists of one curve we can not use connected sum moves and if we use Dehn twists on the standard Heegaard diagram of $L(7,2)$ we will obtain $L(7, 2+7k)$ for some integer $k$.

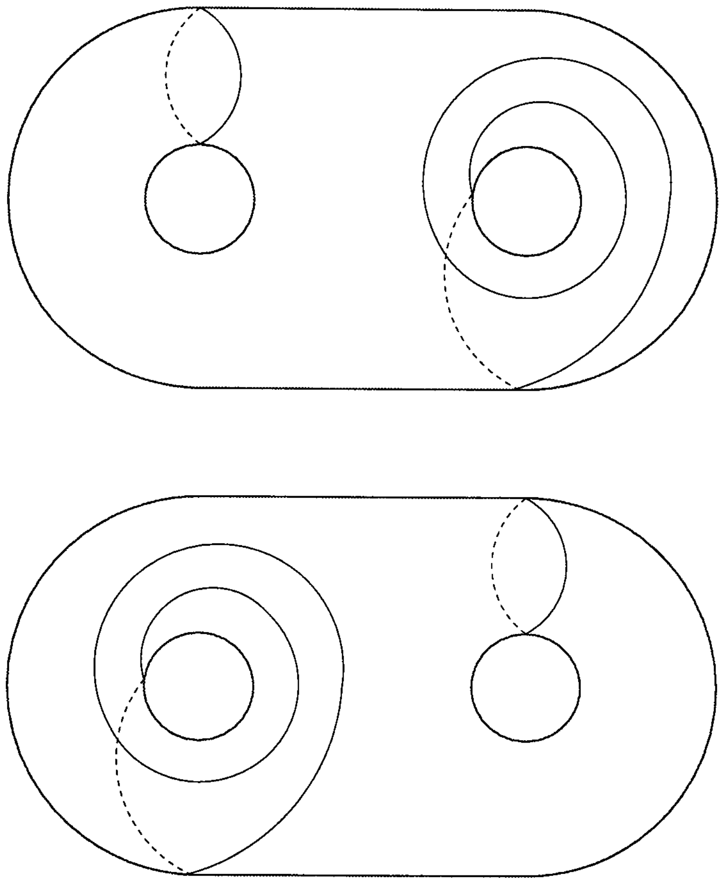


Figure 3.17: Two strongly equivalent Heegaard diagrams of $(\mathbb{S}^1 \times \mathbb{S}^2)\sharp\mathbb{R}P^3$

Figure 3.17 is another example. Both the Heegaard diagrams in the figure are of $(\mathbb{S}^1 \times \mathbb{S}^2)\sharp\mathbb{R}P^3$ and they are strongly equivalent. But we can not use connected sum moves and Dehn twists to pass from one to the other, i.e., in the upper Heegaard diagram, we can not exchange the positions of the two Heegaard curves.

## 3.2 An example

**Example.** We consider the example given in Section 1.7. The Heegaard diagram $(V; j_1, j_2, j_3)$ of $\mathbb{S}^3$ is drawn in $\mathbb{R}^3$ as Fig. 3.18.

The corresponding fundamental group representation of $\mathbb{S}^3$ for this Heegaard diagram

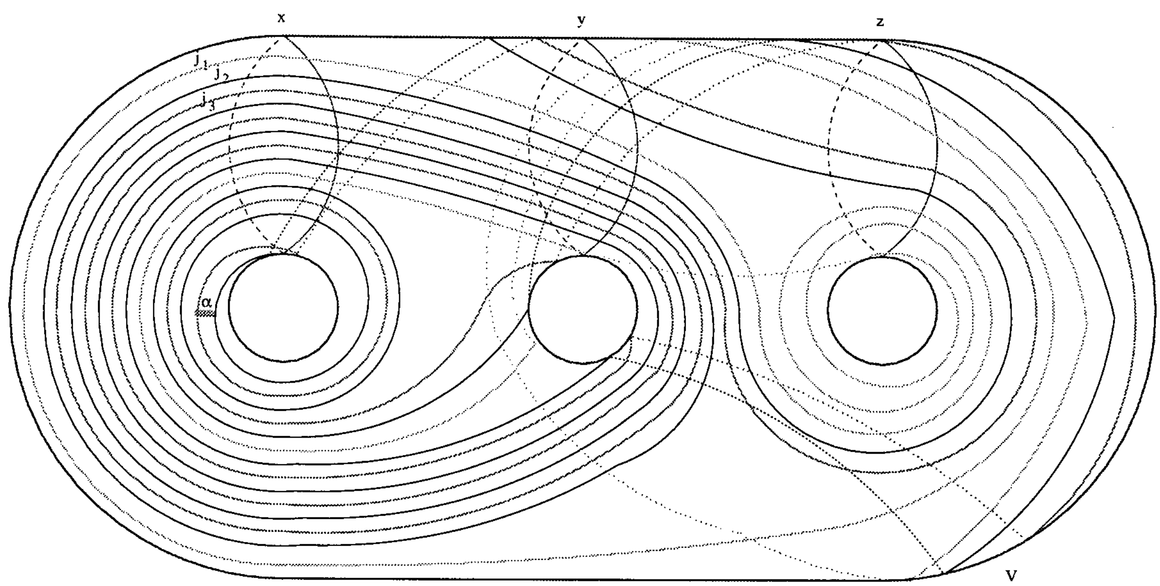



Figure 3.18: A Heegaard diagram of $\mathbb{S}^3$

is:

$$\pi(\mathbb{S}^3) =< x, y, z : xyz^{-3}y^{-1}x^{-1}(yz)^2, xyz^{-1}y^{-1}x^{-3}yz(xy)^3, xyz^{-1}y^{-1}x^{-2}yz(xy)^2 > .$$

We use a connected sum move on the curves $j_2, j_3$ along $\alpha$ (see the above figure), i.e., we use $j_2' = j_2 \sharp_\alpha j_3$ to replace $j_2$. Then we obtain a new Heegaard diagram $(V; j_1, j_2', j_3)$ for $\mathbb{S}^3$. It is easy to see that the curve $j_2'$ is a primitive curve for the handlebody. (See Fig. 3.19)

Fig. 3.20 draws the Heegaard diagram after several more Dehn twists.

## 3.3 Using Dehn twists on Heegaard diagrams of $\mathbb{S}^3$

In Proposition 2.6 of Chapter 2, we proved that we can change the move order of a connected sum move and a Dehn twist without changing the move result.

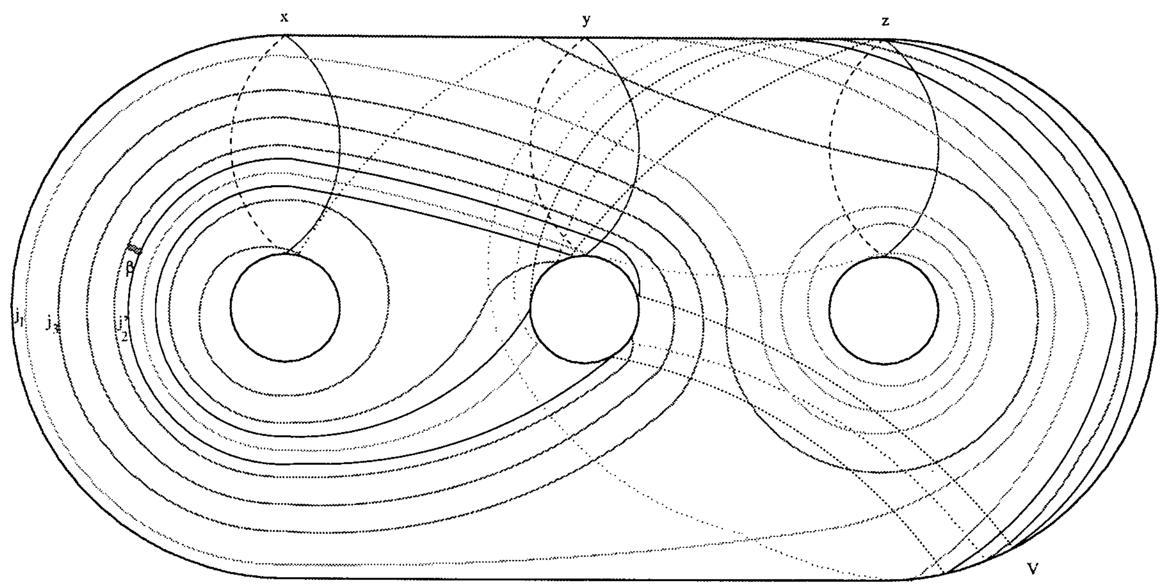
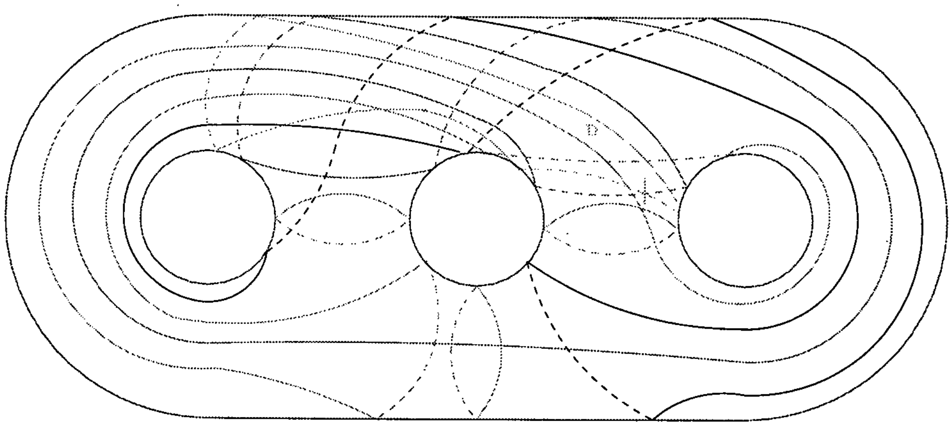



Figure 3.19: Using a connected sum move to simplify the Heegaard diagram

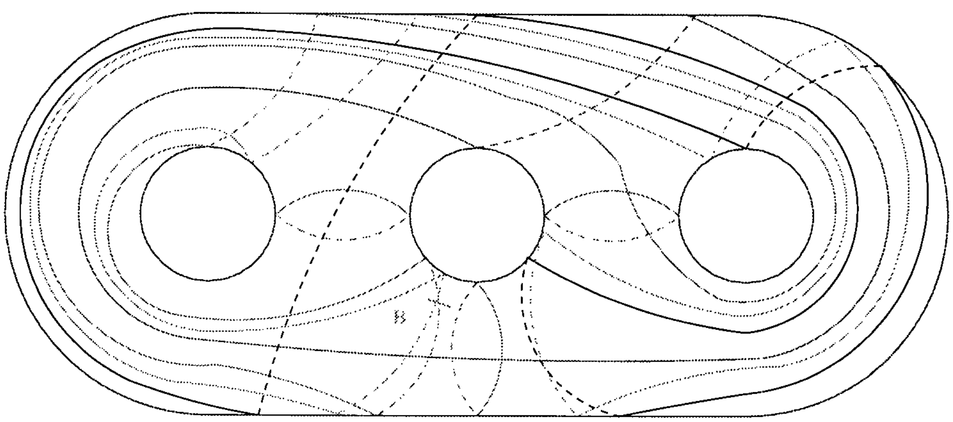
**Theorem 3.5** *Suppose that* $(V; b_1, b_2, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of* $\mathbb{S}^3$*. Let the handlebody* $V$ *lie in* $\mathbb{R}^3$ *standardly. Then there is a sequence of Dehn twists on the Heegaard curves to obtain a new Heegaard diagram* $(V; b_1', b_2', \cdots, b_g')$ *such that the curves* $b_1', b_2', \cdots, b_g'$ *bound pairwise disjoint 2-cells in the closure of the complement of* $V$ *in* $\mathbb{R}^3$ .

First, we prove the following lemma.

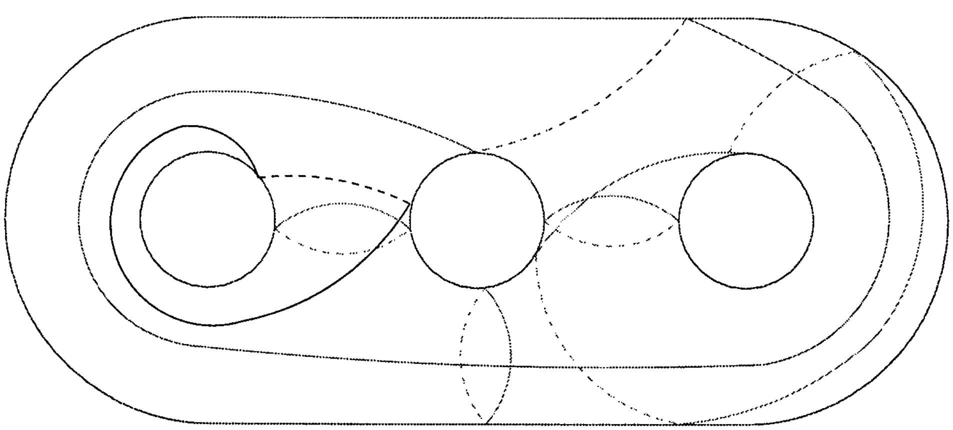
**Lemma 3.4** *Suppose that* $(V; b_1, \cdots, b_g)$ *is a Heegaard diagram of genus* $g$ *of* $\mathbb{S}^3$*. Let the handlebody* $V$ *lie in 3-space standardly. Suppose that* $T$ *is a connected sum move which moves the above Heegaard diagram to a new Heegaard diagram* $(V; b_1', \cdots, b_g')$*. Let* $N_1, \cdots N_g, N_1', \cdots, N_g'$ *be small regular neighborhoods of the curves* $b_1, \cdots, b_g, b_1', \cdots, b_g'$ *in the surface* $\partial V$ *respectively. Then if the* $2g$ *curves* $\partial N_1, \cdots, \partial N_g$ *form a trivial link in* $\mathbb{R}^3$ *, then the* $2g$ *curves* $\partial N_1', \cdots, \partial N_g'$ *also form a trivial link in* $\mathbb{R}^3$ *, and vice versa.*



1. After using Dehn twist on $j''_2$ along $d'_2$



2. After using Dehn twist on the Heegaard curves along D ( D is the disk defined in the picture 1)



3. After use Dehn twist move along B in the B ( B is defined in the picture 2 )

Figure 3.20: Using Dehn twists to simplify the Heegaard diagram

*Proof.* Since the inverse move of a connected sum is also a connected sum move, then we can use the inverse move $T^{-1}$ of $T$ to change the Heegaard diagram $(V; b'_1, b'_2, \cdots, b'_g)$ to the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$. If $\partial N'_1, \partial N'_2, \cdots, \partial N'_g$ is a trivial link in 3-space, then $\partial N_1, \partial N_2, \cdots, \partial N_g$ is a trivial link in 3-space too. Therefore, both the curve sets form a trivial link in 3-space or both of them do not form a trivial link in $\mathbb{R}^3$ .

□

*Proof of Theorem 3.5.* By Theorem 3.2, we can use connected sum moves and Dehn twists to change the Heegaard curves $b_1, b_2, \cdots, b_g$ to the Heegaard curves $e_1, e_2, \cdots, e_g$ of the standard Heegaard. Since we can exchange the move order of one connected sum move and one Dehn twist by Proposition 2.6, then we can use the Dehn twists first and then use the connected sum moves. That is, if we suppose these Dehn twists change the Heegaard diagram $(V; b_1, b_2, \cdots, b_g)$ in order to Heegaard diagram $(V; b''_1, b''_2, \cdots, b''_g)$, then after we use the connected sum moves on this new Heegaard splitting in order we obtain the standard Heegaard diagram $(V; e_1, e_2, \cdots, e_g)$. Denote the respective small neighborhoods of $b''_1, b''_2, \cdots, b''_g$ in $\partial V$ as $N_1, N_2, \cdots, N_g$. If $2g$ curves $\partial N_1, \partial N_2, \cdots, \partial N_g$ do not consist of a trivial link in 3-space, then after each connected sum move, the $2g$ boundary curves of the small neighborhoods of new Heegaard diagram in $\partial V$ do not consist of a trivial link in 3-space by Lemma 3.4. This implies that $e_1, e_2, \cdots, e_g$ do not bound pairwise disjoint disks in the closure of the complement of $V$ in 3-space. This clearly contradicts the definition of the standard Heegaard diagram of $\mathbb{S}^3$. Therefore, $\partial N_1, \partial N_2, \cdots, \partial N_g$ is a trivial link in 3-space.

□

We know that every Dehn twist $T$ twists a small neighborhood of a disk $D$ in the handlebody $V$. That is, the disk $D$ and twist direction completely determine the move. Note the complexity of the disk $D$ can be determined by the length of the word of $\partial D$ corresponding to the standard complete meridian system of $V$. To move Heegaard curves of a Heegaard diagram to a trivial link in $\mathbb{R}^3$ we only need to use some Dehn twists along not too complicated disks (i.e., the disks for our Dehn twists can be chosen such that the length of the word of each of these disks corresponding to the standard complete meridian system of $V$ is less than a constant $k$, where $k$ is completely determined by the Heegaard diagram). Furthermore, given a Heegaard diagram, we think that we only need a constant number $m$ times Dehn twists to change the Heegaard curves of the Heegaard diagram to trivial link in $\mathbb{R}^3$ . For exmple, perhaps $m$ is less than the cross number of all the Heegaard curves.

If the above statement is correct, then we would have a good algorithm to determine Heegaard diagrams of $\mathbb{S}^3$. Given a Heegaard diagram in $\mathbb{R}^3$ , we compute the numbers $m, k$ first. Then we check all the possible combinations of $n$ Dehn twists to see whether we get trivial link such that each move is along a disk which satisfies that the length of the word of the disk corresponding to the standard complete meridian system of $V$ is less than $k$, where $n \leq m$. Since we only have finite possible combinations, this algorithm can finally determine whether the Heegaard diagram is a Heegaard diagram of $\mathbb{S}^3$ or not.

# Chapter 4

## Immersing orientable 3-manifolds into $\mathbb{R}^3$

It is well known that every compact, connected, orientable 2-manifold with non-empty boundary immerses into $\mathbb{R}^2$. The immersion can actually be used to describe the 2-manifold ( see [27] ). J.H.C.Whitehead [41] proved that a connected, orientable polyhedral 3-manifold that is not closed immerses piecewise linearly into $\mathbb{R}^3$.

In this chapter, we will give a stronger version of Whitehead's theorem in the special case of compact, connected, orientable 3-manifolds with non-empty boundary by constructing an immersion with singularities that are at most double and triple points. Our proof uses a Heegaard diagram. The application of algebraic linking theory and $\mathbb{Z}_2$-homology in [41] is replaced by a direct and more transparent geometric construction. This geometric method of proof can be adapted to give a proof of the general case of the Whitehead theorem with the additional property that the singularities of the immersion are double and triple points only.

**Theorem 4.1** *Suppose $M$ is a compact, connected, orientable 3-manifold and $\partial M \neq \emptyset$. Then $M$ immerses into $\mathbb{R}^3$ such that the singularities are only double and triple points.*

If $M$ is a compact, connected, orientable 3-manifold with $\partial M \neq \emptyset$, we embed $M$ into a closed, connected, orientable 3-manifold $\hat{M}$ by attaching a handlebody or a 3-cell to each component of $\partial M$. Note, there is some choice involved in the attachment of a handlebody, but any such $\hat{M}$ will do. If $(V, W)$ is a Heegaard splitting of $\hat{M}$ and $(V; b_1, \cdots, b_g)$ a Heegaard diagram, let $B_1, \cdots, B_g \subset W$ be disjoint proper 2-cells such that $\partial B_i = b_i, i = 1, \cdots, g$. Let $B_i \times [-\epsilon, \epsilon], i = 1, \cdots, g$, be disjoint regular neighborhoods of $B_i = B_i \times 0$ in $W$ with $(B_i \times [-\epsilon, \epsilon]) \cap \partial W = \partial B_i \times [-\epsilon, \epsilon]$, $i = 1, \cdots, g$. Consider the punctured manifold $\hat{M}_0 = V \cup (\cup_{i=1}^{g} B_i \times [-\epsilon, \epsilon])$ with $\partial \hat{M}_0$ the 2-sphere $[\partial W - \cup_{i=1}^{g} \partial B_i \times (-\epsilon, \epsilon)] \cup (\cup_{i=1}^{g} B_i \times (-\epsilon)) \cup (\cup_{i=1}^{g} B_i \times \epsilon)$. By the homogeneity of closed 3-manifolds, we may assume that $M \subset \hat{M}_0$ ( there is a 3-cell $B^3 \subset \hat{M} - M$ and there is an ambient isotopy on $\hat{M}$ that moves $B^3$ to the 3-cell $W - (\cup_{i=1}^{g} B_i \times (-\epsilon, \epsilon))$ ).

An immersion of $\hat{M}_0$ into $\mathbb{R}^3$ will restrict to an immersion of $M$ into $\mathbb{R}^3$. Therefore it will suffice to prove the theorem for compact, connected, orientable 3-manifold $M$ with $\partial M$ a 2-sphere with the presentation $M = V \cup (\cup_{i=1}^{g} B_i \times [-\epsilon, \epsilon])$. as above. We use $\hat{M}$ to denote the corresponding closed 3-manifold obtained by attaching a 3-cell to $\partial M$.

Let $\iota : V \longrightarrow \mathbb{R}^3$ be an embedding such that $\iota(V)$ is in the upper half space $\mathbb{R}^3_+$ of $\mathbb{R}^3$ in the standard position Fig. 4.22. We assume from now on that $V = \iota(V)$. Let $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 = \mathbb{R}^2 \times 0$ be the projection $\pi(x, y, z) = (x, y, 0)$.

**Definition 4.1** *Let $(V; b_1, \ldots, b_g)$ be a Heegaard diagram of genus $g$ of $\hat{M}$. Let $A_1, \ldots, A_g$ be small disjoint regular neighborhoods of $b_1, \ldots, b_g$ in $\partial V$. We call $A_1, \ldots, A_g$ Heegaard annuli of the respective Heegaard curves $b_1, \ldots, b_g$.*

**Definition 4.2** *Let $(V; b_1, \ldots, b_g)$ be a Heegaard diagram of genus $g$ of $\hat{M}$ with Heegaard annuli $A_1, \cdots, A_g$. We call the Heegaard curve $b_i$ an even ( odd ) Heegaard curve if the following conditions are satisfied:*

*1) $\pi \mid : A_i \longrightarrow \mathbb{R}^2$ is an immersion.*

*2) The number of self intersection points of $\pi(b_i)$ is even ( odd respectively ).*

*If all Heegaard curves are even, then we call the Heegaard diagram an even Heegaard diagram.*

A Heegaard diagram being even or not even only depends on the position of the Heegaard curves in $\partial V$.

We will prove that an even Heegaard diagram of $M$ defines an immersion of $M$ into $\mathbb{R}^3$. ( Section 4.2. )

In [24], Kirby posed the following problem.

Problem 3.19 of Kirby's problem list: Which immersed 2-spheres in $\mathbb{R}^3$ bound immersed 3-cells?

We note that in particular each even Heegaard diagram of the 3-cell defines an immersion of the 3-cell.

We will prove that we can use Dehn twists and connected sum moves on the Heegaard curves of a Heegaard diagram of $M$ to obtain an *even* Heegaard diagram of $M$ ( Theorem 4.3 ).

We also classify all compact, connected, orientable 3-manifolds with 2-sphere boundaries which can be immersed into $\mathbb{R}^3$ with singularities that are at most double points. We will prove the following theorem.

**Theorem 4.2** *Suppose $M$ is a compact, connected, orientable 3-manifold with $\partial M$ consisting of 2-spheres. If there exists an immersion $\tau : M \longrightarrow \mathbb{R}^3$ whose singularities are at most double points, then $M$ is either a punctured 3-sphere or a punctured* $(\mathbb{S}^1 \times \mathbb{S}^2)\sharp(\mathbb{S}^1 \times \mathbb{S}^2)\sharp \cdots \sharp(\mathbb{S}^1 \times \mathbb{S}^2)$.

**Corollary 4.1** *Suppose that $(V; b_1, \cdots, b_g)$ is a Heegaard diagram of genus $g$ of a compact, connected, orientable 3-manifold $M$. If $M$ is not homeomorphic to $\mathbb{S}^3$ or $(\mathbb{S}^1 \times \mathbb{S}^2)\sharp(\mathbb{S}^1 \times \mathbb{S}^2)\sharp \cdots \sharp(\mathbb{S}^1 \times \mathbb{S}^2)$, then there does not exist an embedding of $V$ into $\mathbb{R}^3$ such that the boundary of the Heegaard annuli $A_1, \cdots, A_g$ of the respective Heegaard curves form a trivial link in $\mathbb{R}^3$.*

At the end of the chapter, we give an explicit immersion of the punctured real projective space $\mathbb{R}P^3_0$ into $\mathbb{R}^3$.

## 4.1 Even Heegaard diagrams

We have already defined an even Heegaard diagram. Not all Heegaard diagrams are even. For example, the Heegaard diagram of $\mathbb{S}^3 = L(1,1)$ is not even. Note that the boundary components of its Heegaard annulus are trivial but linked. ( See Fig. 4.21. )

In this section, we will prove that by applying Dehn twists and connected sum moves to a Heegaard diagram we can obtain an even Heegaard diagram.

At first, we isotopically move all Heegaard curves in the boundary of the handlebody into a neighborhood of $3g-1$ standard simple closed curves.

**Lemma 4.1** *Suppose that $M$ is a compact, connected, orientable 3-manifold with boundary and $(V; b_1, \cdots, b_g)$ is a Heegaard diagram of genus $g$ of $\hat{M}$. Assume that the simple closed curves $e_1, \cdots, e_g, d_1, \cdots, d_{2g-1}$ on the surface $\partial V$ are as in Fig. 4.22.*

*Let $N$ be a small regular neighborhood of the set $\cup_{i=1}^{g} e_i \cup \cup_{i=1}^{2g-1} d_i$ in $\partial V$. We can isotopically move the Heegaard curves $b_1, \cdots, b_g$ in the surface $\partial V$ to obtain new Heegaard curves $b'_1, \cdots, b'_g$ such that the new Heegaard curves lie in $N$.*

*Proof.* Note that $\partial V - IntN$ is a union of $g$ disks. The Heegaard curves $b_1, \cdots, b_g$ can isotopically be pushed out from these disks into $N$.

□

From now on, we assume that all Heegaard diagrams in this section satisfy the conclusion of Lemma 4.1. Let $(V; b_1, \cdots, b_g)$ be such a Heegaard diagram of genus $g$ of $M$. Note that $\pi|_N$ is an immersion with at most double points as singularities.

It follows Fig. 4.23 that whether the Heegaard diagram $(V; b_1, \cdots, b_g)$ is even or is not even is completely determined by the image of the Heegaard curves $b_1, \cdots, b_g$ under the

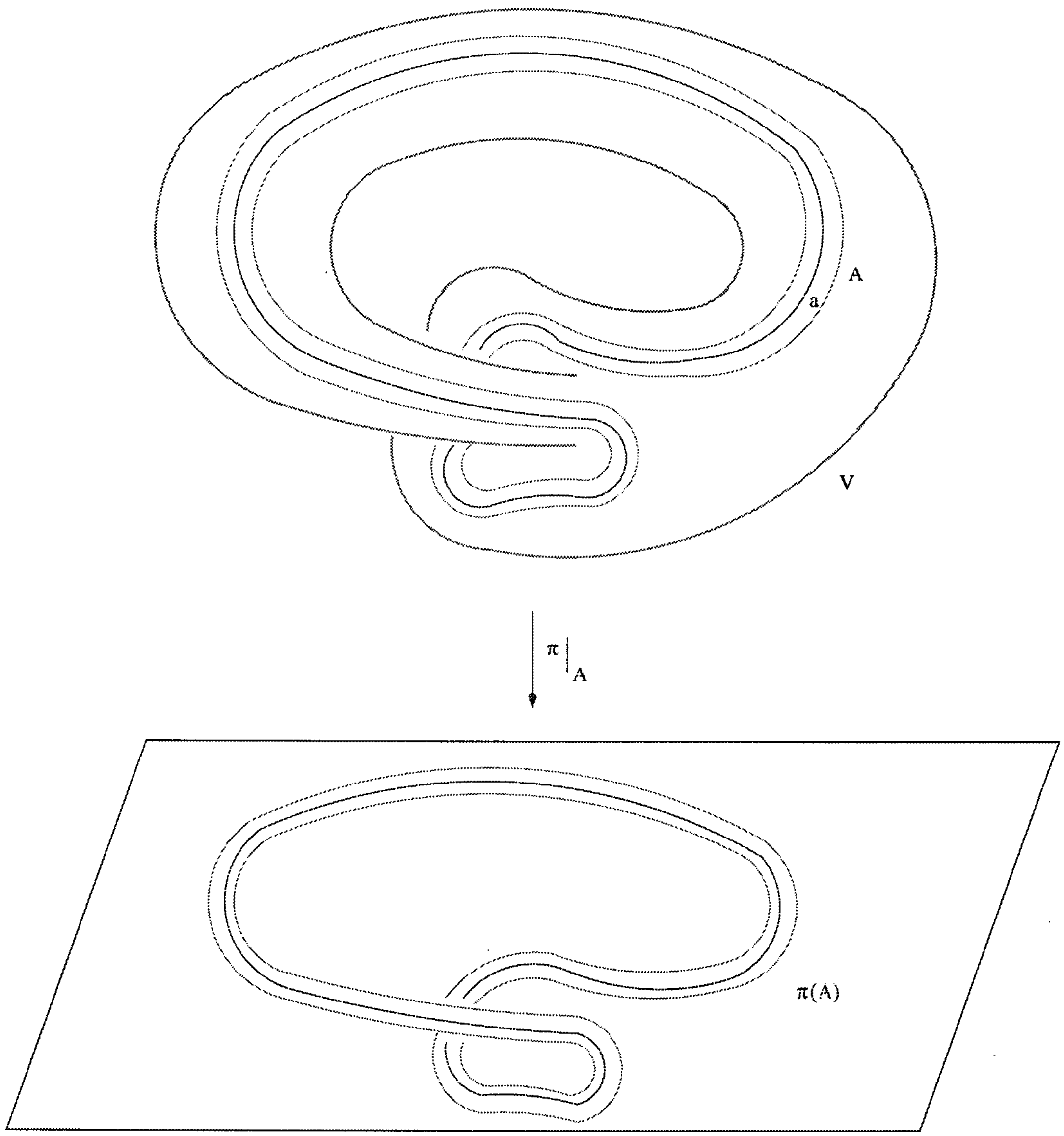



Figure 4.21: A Heegaard diagram of $\mathbb{S}^3$ that is not even

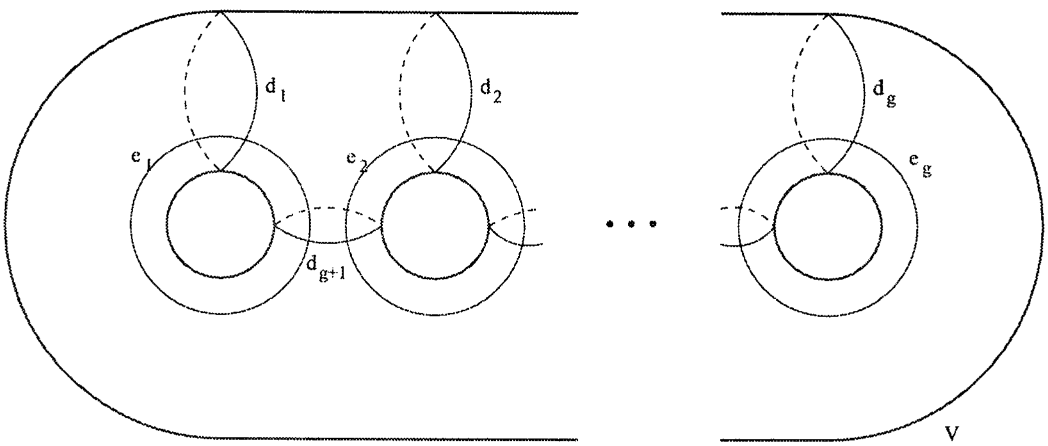



Figure 4.22: Curves in $\partial V$ that separate $\partial V$ as disks

projection $\pi$. If for $k \in \{1, 2, \cdots, g\}$, simple closed curve $\pi(b_k)$ has an even number of self-intersection points, then the Heegaard diagram is even. Suppose that for $k = 1, 2, \cdots, g$, the word $w_k$ of the curve $a_k$ corresponding to the curve set $\{e_1, e_2, \cdots, e_g; d_1, d_2, \cdots, d_g\}$ is the following.

$$w_k = e_{\alpha_{1k}}^{\mu_{1k}} \cdot d_{\beta_{1k}}^{\nu_{1k}} \cdot e_{\alpha_{2k}}^{\mu_{2k}} \cdot d_{\beta_{2k}}^{\nu_{2k}} \cdot \cdots \cdot e_{\alpha_{m_k k}}^{\mu_{m_k k}} \cdot d_{\beta_{m_k k}}^{\nu_{m_k k}};$$

where $\mu_{ik}, \nu_{ik} \in \{-1, 0, 1\}$ for $k = 1, \cdots, g; i = 1, 2, \cdots, m_k$ and $\alpha_{jk}, \beta_{jk} \in \{1, 2, \cdots, g\}$ for $k = 1, 2, \cdots, g; j = 1, 2, \cdots, m_k$. We assume that $\mu_{ik} + \nu_{ik} > 0$.

**Proposition 4.1** *For $s = 1, 2, \cdots, g$, let $n_{s,k} = \nu_{1k} + \nu_{2k} + \cdots + \nu_{m_k k}$. If $n_{1,k}, n_{2,k}, \cdots, n_{n,k}$ are all even numbers, then the Heegaard curve $a_k$ is even.*

*Proof.* The fact that $n_{s,k}$ is an even number indicates that the curve $c_k = \pi(a_k)$ has an even number of intersection points with the simple closed curve $\pi(d_s)$ in the $xy$-plane. Note that if $c_k$ has an even number of intersection points with $\pi(d_s)$ and $\pi(d_{s+1})$,

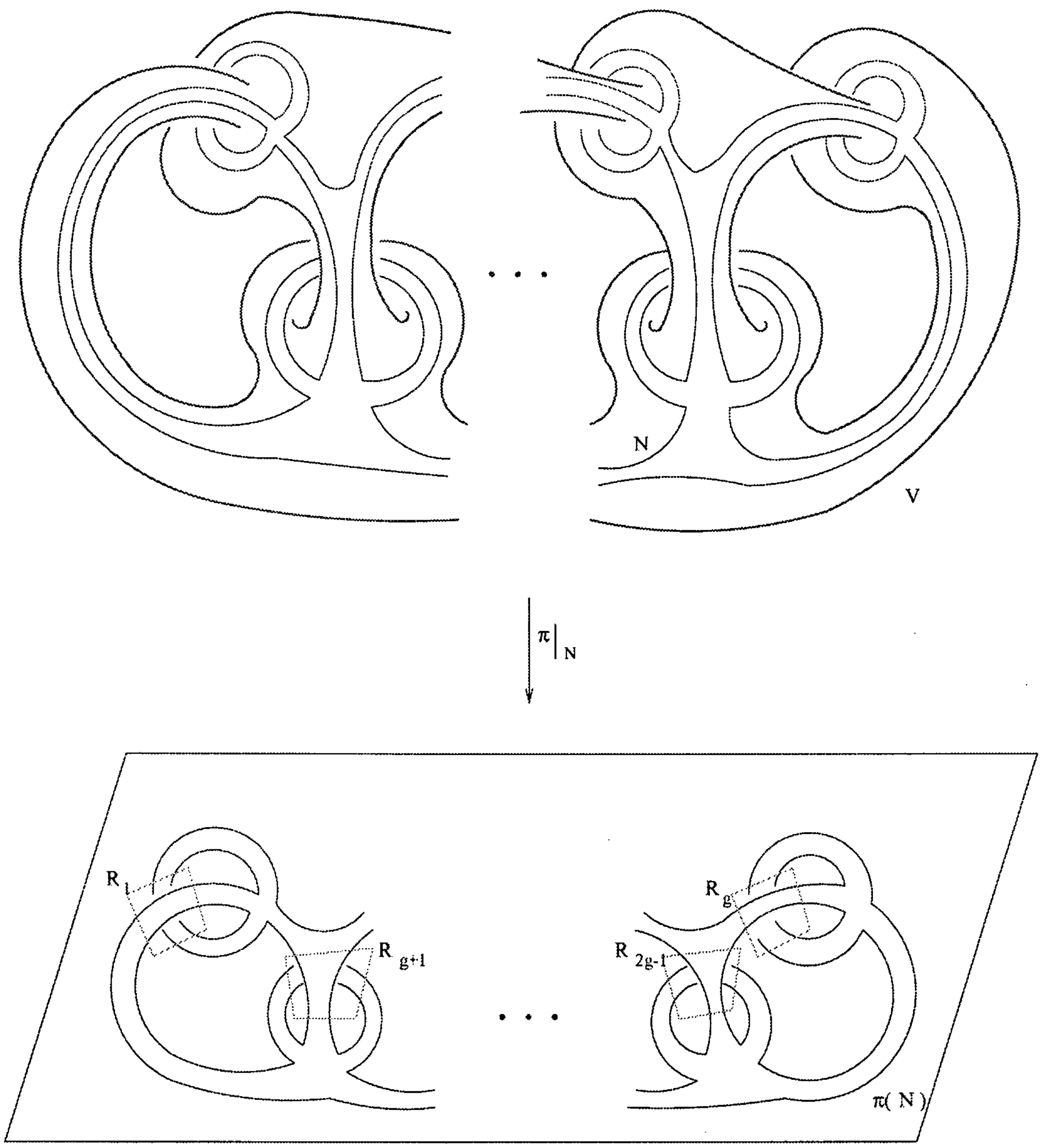



Figure 4.23: Getting a diagram for Heegaard annuli

then it has also an even number of self-intersection points with $\pi(d_{s+g})$ since cutting $V$ along the disks bounded by $d_s, d_{s+1}, d_{s+g}$ is a 3-cell for $s = 1, 2, \cdots, g-1$. Therefore, the hypothesis of the lemma implies that $c_k$ has an even number of self-intersection points in the $xy$-plane. Thus, the Heegaard curve $a_k$ is even.

□

Proposition 4.1 shows that some Heegaard diagrams are even. For example, the standard Heegaard diagram of genus 1 of $\mathbb{R}P^3$ is even since the Heegaard curve intersects the meridian curve in exactly two points ( see Fig. 4.29 ). But for the lens space $L(3,1)$, the standard Heegaard diagram is not even. However, we can apply a Dehn twist along the meridian circle to change this Heegaard diagram to an even Heegaard diagram. We note that in this case, every time we use a Dehn twist we alternatively exchange the odd/even number of the intersection points. Thus in the case of genus 1, we can always apply Dehn twists to obtain even Heegaard diagrams.

For higher genus cases, using Dehn twists alone will in general not suffice to obtain an even Heegaard diagram since a Dehn twist may change an odd Heegaard curve to become even but at the same time it may change another even Heegaard curve to become odd.

**Lemma 4.2** *Fix $i \in \{1, 2, \cdots, 2g-1\}$. Suppose that $d_i = \partial D_i$ is the meridian curve in Fig. 4.22 and $d_i$ intersects a Heegaard curve $a$ at $r$ points. Suppose that along $D_i$ we apply a Dehn twist on $a$ to obtain a new Heegaard curve $a'$. If $r$ is even, then both of $a$ and $a'$ have even numbers of crossings in the box $R_i$. If $r$ is odd, then $a$ has an even number of crossings in $R_i$ if and only if $a'$ has an odd number of crossings in $R_i$ and hence $a$ has an odd number of crossings in $R_i$ if and only if $a'$ has an even number of crossings in $R_i$.*

*Proof.* In the $xy$-plane, suppose that $m$ sub-arcs of the curve $\pi(a)$ pass through the box $R_i$ directly, $n$ sub-arcs of $\pi(a)$ move along $d_i$ once and then pass through $R_i$ and $k$

sub-arcs of $\pi(a)$ move around $d_i$ once and then do not pass through the box ( see the figure a) in Fig. 4.24 ). The number of the crossings of $\pi(a)$ in $R_i$ is $(k+n)(n+m)$. Then, after a Dehn twist along $d_i$ in the direction of Fig. 4.24, $a$ changes to $a'$ such that $\pi(a')$ has $m$ sub-arcs move around $d_i$ once ( in the other direction ) and then pass through the box , $n$ sub-arcs pass through the box directly and $k$ sub-arcs of $\pi(a)$ move around $d_i$ once and then do not pass through the box ( see the figure b) in Fig. 4.24 ). Therefore, the number of the crossings of $\pi(a')$ in $R_i$ is $(k+m)(n+m)$.

Note that $r = m + n$. Therefore, if $r$ is even, then both $a$ and $a'$ have even numbers of crossings in the box $R_i$.

If $r$ is odd, then $m$ is odd if and only if $n$ is even. Thus $k + m$ is odd if and only if $k + n$ is even. Therefore, $a$ has an even number of crossings in $R_i$ if and only if $a'$ has an odd number of crossings in $R_i$.

If we use Dehn twist along $d_i$ in the other direction, then the number of crossings of $a'$ in $R_i$ is $(k + m + 2n)(n + m)$. Since $(k + m + 2n)(n + m)$ is even if and only if $(k + m)(n + m)$ is even, the theorem is true by the preceding proof.

□

Next we consider connected sum moves.

**Lemma 4.3** *Fix $i \in \{1, 2, \cdots, 2g-1\}$. Suppose that $c$ is a simple curve whose endpoints lie in two Heegaard curves $a', a''$ respectively and $Int\ c \cap (a' \cup a'') = \emptyset$. Let $a = a' \sharp_c a''$ be a connected sum move on $a', a''$ along $c$. Then $a$ has an even number of crossings in $R_i$ if and only if both $a', a''$ have even numbers of crossings in $R_i$ or both of them have odd numbers of crossings in $R_i$.*

*Proof.* The number of crossings of $a$ in $R_i$ is the sum of the number of crossings of $a'$ in $R_i$, the number of crossings of $a''$ in $R_i$ and the number of crossings in $R_i$ produced by the two copies of $c$ used as connected sum move. The two copies of $c$ always produces

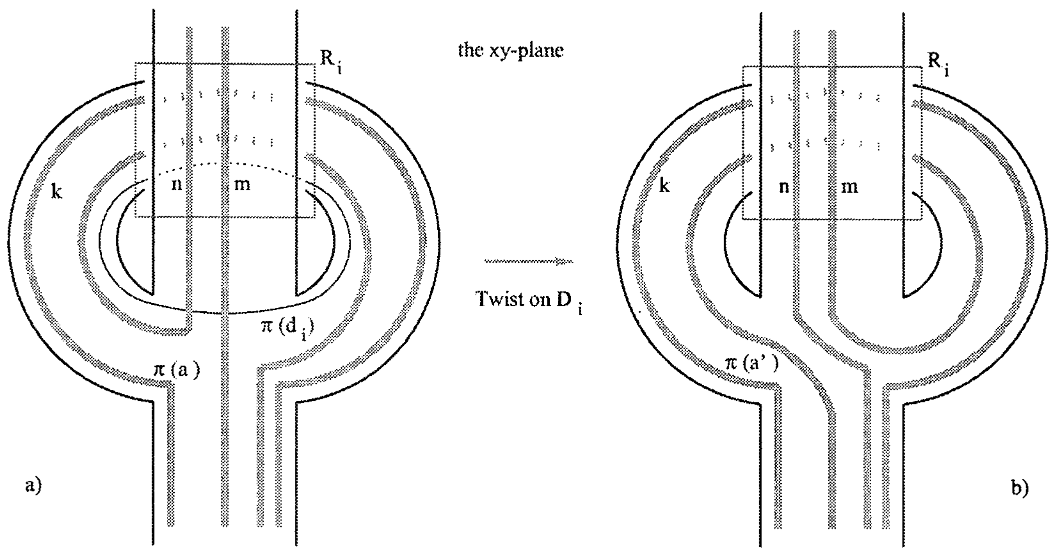



Figure 4.24: Dehn twist along D

an even number of crossings. Therefore, the number of crossings of $a$ in $R_i$ is even if and only if the sum of the number of crossings of $a'$ in $R_i$ plus the number of crossings of $a''$ in $R_i$ is even.

□

**Theorem 4.3** *Suppose that* $(V; b_1, \cdots, b_g)$ *is a Heegaard diagram of a connected, orientable, closed 3-manifold* $M$*. Then we can use a sequence of connected sum moves and Dehn twists on the Heegaard diagram to obtain an even Heegaard diagram.*

*Proof.* If each Heegaard curve has an even number of crossings in each box, then each Heegaard curve is even and then the Heegaard diagram is even.

Now, we assume that there exists a box $R_j$ such that the Heegaard curves $b_{i_1}, \cdots, b_{i_k}$ have odd numbers of crossings in $R_i$ and all other Heegaard curves $b_{i_{k+1}}, \cdots, b_{i_g}$ have even numbers of crossings in $R_i$. Then we replace each $b_{i_s}$ ( $s = k+1, \cdots, g$ ) in the

Heegaard diagram $(V; b_1, \cdots, b_g)$ by a connected sum $b'_s = b_{i_1} \sharp_{c_s} b_{i_s}$ to obtain a new Heegaard diagram $(V; b_{i_1}, \cdots, b_{i_k}, b'_{k+1}, b'_{k+2}, \cdots, b'_g)$. Note that for $s = k+1, k+2, \cdots, g$, the new Heegaard curve $b'_s$ has an even number of crossings in box $R_{j'}, j' \in \{1, \cdots, j-1, j+1, \cdots, 2g-1\}$ if and only if the old Heegaard curve $b_{i_s}$ has an even number of crossings in the box by Lemma 4.3. Now, each Heegaard curve in the new Heegaard diagram has an odd number of crossings in $R_j$. We apply a Dehn twist along $d_j$ to the Heegaard curves $b_{i_1}, \cdots, b_{i_k}, b'_{k+1}, b'_{k+2}, \cdots, b'_g$ to obtain a Heegaard diagram $(V; b'_{i_1}, \cdots, b'_{i_k}, b''_{k+1}, b''_{k+2}, \cdots, b''_g)$. Then each Heegaard curve of this Heegaard diagram has an even number of crossings in $R_j$ by Lemma 4.2.

Since our connected moves and Dehn twists do not change the odd/even property of the numbers of crossings in other boxes, we can use the above method to let each Heegaard curve have an even number of crossings in each box.

□

## 4.2 Proof of Theorem 4.1

**Definition 4.3** *Let $K$ be a simple closed curve in $\mathbb{R}^3$. An annulus $A = K \times [-1, 1]$ in $\mathbb{R}^3$ with $K = K \times 0$ is trivial in $\mathbb{R}^3$ if there exists a 2-handle $B \times [-1, 1]$ in $\mathbb{R}^3$ with $\partial B \times [-1, 1] = A$.*

We need some concepts from knot and link theory here.

The projection $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 = R^2 \times 0$ in 4.23 defines a regular projection on a link $L$ in $R^3$ consisting of all the Heegaard curves $b_1, \cdots, b_g$. For each crossing $c$ of $L$, $\pi^{-1}(c) \cap L$ consists of two points $c_+ = (x_c, y_c, z_{c_+}), c_- = (x_c, y_c, z_{c_-})$ with $z_{c_+} > z_{c_-}$. We say that $c_+$ is an *overcrossing* and $c_-$ is an *undercrossing*. The segment of $L$ that contains the overcrossing or undercrossing of $c$ is called the *overpass* or *underpass* of $c$, respectively.

We will apply the following well-known lemma.

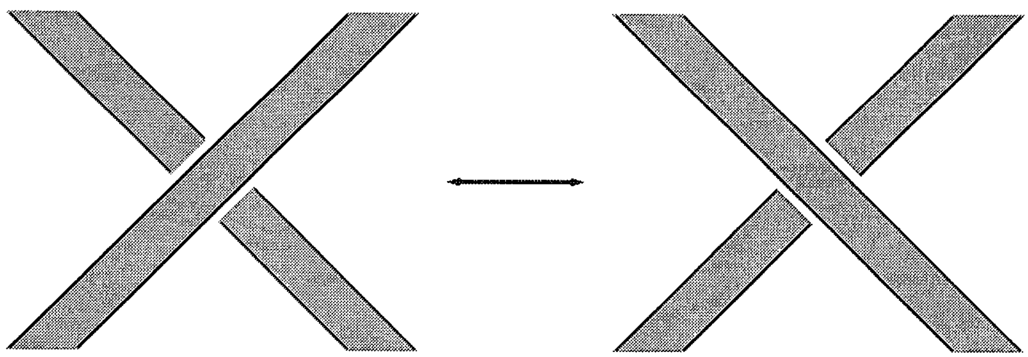


Figure 4.25: Changing crossings in a diagram of annuli

**Lemma 4.4** *Let $K$ be an oriented simple closed curve in $R^3$ such that $\pi(K)$ in $\mathbb{R}^2$ has only transversal intersection points. We move along $K$ once beginning from a point on $K$ according to its orientation. If for each crossing, we alway pass through the respective overcrossing first, then $K$ bounds a 2-cell in $\mathbb{R}^3$.*

□

**Lemma 4.5** *Let $K \subset \mathbb{R}^3$ be a knot such that $\pi(K)$ has only transversal intersection points and their number is even. Let $A = K \times [-\epsilon, \epsilon], K = K \times 0$ be an annulus with $\pi \mid: A \longrightarrow \mathbb{R}^2 = \mathbb{R}^2 \times 0$ an immersion. Then we can change some crossings of $A$ according to Fig. 4.25 to obtain an annulus $A' = K' \times [-\epsilon, \epsilon]$ such that $\pi(K) = \pi(K')$ and $A'$ is trivial.*

*Proof.* If $K$ is a standard trivial knot, i.e., $\pi(K)$ is a simple closed curve in the $xy$-plane, then let $K' = K, A' = A$.

From now on suppose that $K$ is not a standard trivial knot.

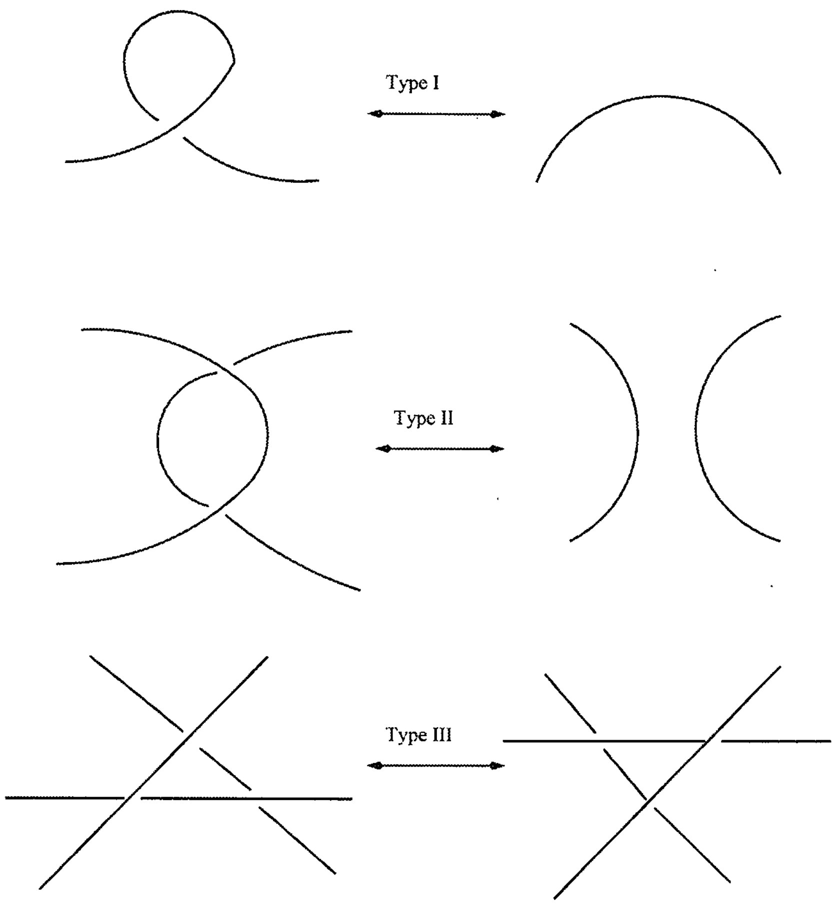



Figure 4.26: Reidemeister moves

Let $P = (x_0, y_0, 0) \in \pi(K)$ satisfy that $y_0 = max\{y : (x, y, 0) \in \pi(K)\}$ and $P$ is not a crossing of $K$.

We move along $K$ once beginning from $\pi^{-1}(P)$ in $\mathbb{R}^3$ according to the orientation of $K$. That is, we define a map $t : [0,1] \longrightarrow K$ such that $t(0) = t(1) = \pi^{-1}(P)$ and $t \mid: (0,1) \longrightarrow (K - \pi^{-1}(P))$ is an orientation preserving homeomorphism ( where the orientation of $(0,1)$ is the induced orientation of $x$-axis ).

Since $K$ is not a standard trivial knot, then there exists $\alpha_+^1, \alpha_-^1 \in (0,1), \alpha_+^1 < \alpha_-^1$ such that $\pi(t(\alpha_+^1)) = \pi(t(\alpha_-^1))$ and for any $\beta_+, \beta_- \in (0,1)$ with $\pi(t(\beta_+)) = \pi(t(\beta_-))$ we have $\beta_- > \alpha_-^1$. That is $c^1 = \pi(t(\alpha_+^1)) = \pi(t(\alpha_-^1))$ is the first crossing we pass through both of its overcrossing and undercrossing.

Now, let $S_t = \{\alpha_+^1, \alpha_-^1, \alpha_+^2, \alpha_-^2, \cdots, \alpha_+^m, \alpha_-^m\}$ be the maximum set such that the following conditions are true:

1). For $i = 1, 2, \cdots, m$, $\alpha_+^i, \alpha_-^i \in (0,1), \alpha_+^i < \alpha_-^i$ and $\pi(t(\alpha_+^i)) = \pi(t(\alpha_-^i))$. That is $c^i = \pi(t(\alpha_+^i)) = \pi(t(\alpha_-^i))$ is a crossing of $K$.

2). For $i, j = 1, 2, \cdots, m; i < j$, $\alpha_-^i < \alpha_-^j$ and $\alpha_+^j, \alpha_-^j \notin (\alpha_+^i, \alpha_-^i)$. That is, after we pass through $c^i$, we do not need to consider the crossings which lie in the closed, passed curve $\ell^i = \pi(t(\alpha_+^i, \alpha_-^i))$ any more.

Then, $\alpha_+^1, \alpha_-^1 \in S_t$, i.e., $S_t \neq \emptyset$.

Now, we change the crossings of $K$ to obtain a knot $K'$ according to the following rules:

1). If a crossing $c \notin \{c^1, c^2, \cdots, c^m\}$ and $c_+, c_- = t^{-1}(\pi^{-1}(c))$ with $c_+ < c_-$, then $\pi(c_+)$ should be the overcrossing of $c$ and $\pi(c_-)$ should be the undercrossing of $c$.

2). For $i \leq m$, if $i$ is an odd number, then the overcrossing of $c_i$ should be $\pi(\alpha_+^i)$ and the undercrossing of $c_i$ should be $\pi(\alpha_-^i)$.

3). For $i \leq m$, if $i$ is an even number, then the overcrossing of $c_i$ should be $\pi(\alpha_-^i)$ and the undercrossing of $c_i$ should be $\pi(\alpha_+^i)$.

Note that when we reverse an overpass segment as an underpass segment we always move the underpass segment upward and we assume that the pre-image of the points of the underpass segments for the map $t$ remain fixed ( that is if we move $(x, y, z) \in K$ upward to $(x, y, z + \epsilon)$, $t^{-1}((x, y, z + \epsilon)) := t^{-1}((x, y, z))$ ).

Suppose the above moves change $A$ to an annulus $A'$ in $R^3$. Then $A' = K' \times [-\epsilon, \epsilon]$ and $\pi \mid: A' \longrightarrow \mathbb{R}^2 = \mathbb{R}^2 \times 0$ is an immersion. Further, $\pi(A') = \pi(A)$.

Note the projection $\pi$ gives a diagram $\kappa'$ of the knot $K'$.

*Claim 1. We can use a sequence of Reidemeister moves of Type II and Type III on the diagram $\kappa'$ to obtain a knot diagram $\kappa''$ which has only $m$ crossings $c^1, c^2, \cdots, c^m$.*

*Proof of Claim 1.* Using Reidemeister moves of Type II and Type III on the respective part of $\pi(\ell^1)$ in $\kappa'$, we can move $\pi(\ell^1)$ into a small neighborhood of $c^1$ since $\pi(\ell^1)$ is an open arc in the $xy$-plane and it only corresponds to overpass segments of $K'$. Now, $\pi(\ell^2)$ is an open arc in $\kappa'$ and it only corresponds to overpass segments of $K'$, thus we can use a sequence of Reidemeister moves of Type II and Type III to cancel all crossings in $\pi(\ell^2)$ perhaps except $c^1$. We use this method step by step and finally obtain a knot diagram $\kappa''$ which has only $m$ crossings $c^1, c^2, \cdots, c^m$.

*Claim 2. m is an even number.*

*Proof of Claim 2.* By the hypothesis of the theorem, $K$ has an even number of crossings. $K'$ has the same number of crossings with $K$. Since each Reidemeister move of Type II does not change the number of crossings and each Reidemeister move of Type III changes the number by 2, by Claim 1 implies that $m$ is even.

*Claim 3. $A'$ is trivial.*

*Proof of Claim 3.* Consider the diagram $\kappa''$ in Claim 1. There are two cases for the portion of $\kappa''$ including $c^1, c^2$. ( See a) and b) in Fig. 4.27. ) But Case b) can be moved to Case a) by using two Reidemeister moves of Type II and a Reidemeister move of Type III. The c) and d) show that the portion of $A'$ will be changed to standard case after attaching a 3-cell to it. Therefore by induction, $A'$ is trivial.

□

**Lemma 4.6** *Let $A = K \times [-1,1]$ be a trivial annulus in $\mathbb{R}^3$. Then there are two 2-handles $B_+ \times [-1,1], B_- \times [-1,1]$ with $A = (B_+ \times [-1,1]) \cap (B_- \times [-1,1]) = \partial(B_+ \times [-1,1]) \cap \partial(B_- \times [-1,1]) = \partial B_+ \times [-1,1] = \partial B_+ \times [-1,1]$.*

*Proof.* Let $B_+ \times [-1,1]$ be a 2-handle with $A = \partial B_+ \times [-1,1]$. Let $S^2$ be a 2-sphere in $\mathbb{R}^3$ and let $S^2 = D_+ \cup D_-, D_+, D_-$ 2-cells with $\partial D_+ = \partial D_-$. There is an ambient isotopy $h_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, 0 \leq t \leq 1$ with $h_1(B_+ \times 0) = D_+$. Let $S^2 \times [-1,1]$ be a regular

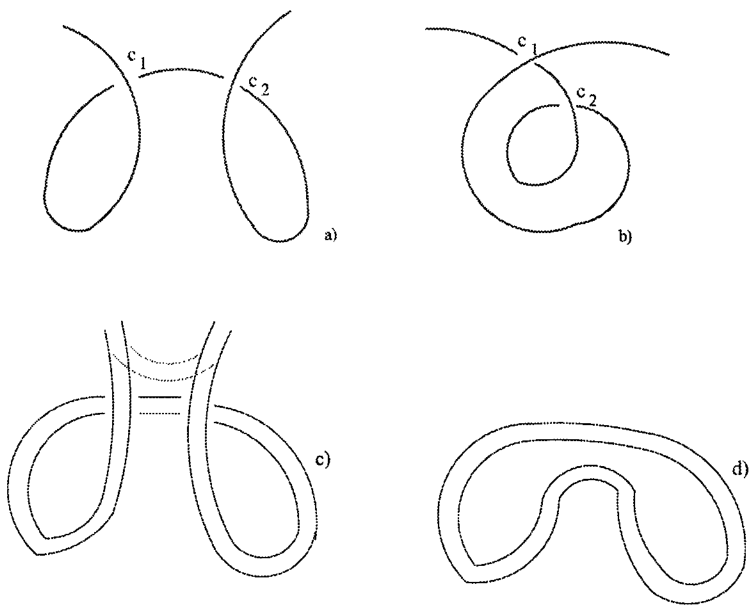



Figure 4.27: Changing crossings in a diagram of annuli

neighborhood of $S^2 = S^2 \times 0$. We may assume that $h_1(B_+ \times [-1,1]) = D_+ \times [-1,1]$. Then $B_- \times [-1,1] = h^{-1}(D_- \times [-1,1]$ has the required property.

□

**Theorem 4.4** *Let $(V; b_1, b_2, \cdots, b_g)$ be an even Heegaard diagram and $A_1, A_2, \cdots, A_g$ its Heegaard annuli. Then there is an immersion $\tau : V \longrightarrow \mathbb{R}^3$ with at most double points as singularities such that $\tau(A_1), \tau(A_2), \cdots, \tau(A_g)$ bound disjoint 2-handles in $\mathbb{R}^3$.*

*Proof.* Let $V$ be embedded in $\mathbb{R}^3$ as in Fig. 4.23. Then $\pi \mid_{\cup_{i=1}^g A_i}$ is an immersion from the Heegaard annuli into the $xy$-plane. This immersion ensures that the positive side of the Heegaard annuli is upward where the positive side means the outside of the handlebody $V$. This projection also gives us a diagram for the annuli $A_1, A_2, \cdots, A_g$ in $\mathbb{R}^3$.

By Lemma 4.5, we can suitably change the crossings in the annulus diagram to obtain a trivial annulus. That is, we can change the crossings of the Heegaard annuli

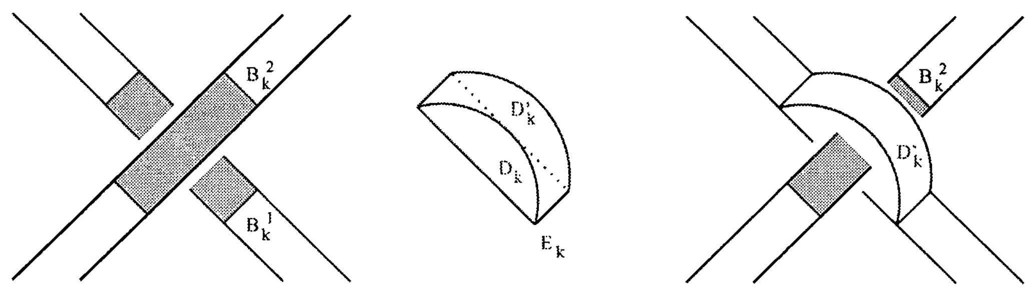



Figure 4.28: Adding a 3-cell to change crossing

$A_1, A_2, \ldots, A_g$ in $\mathbb{R}^3$ to obtain new annuli $A'_1, A'_2, \ldots, A'_g$ such that $\partial A'_1, \partial A'_2, \cdots, \partial A'_g$ is trivial in $\mathbb{R}^3$.

Similar to the proof of Lemma 4.5, we can suitably change the crossings produced by different annuli to let $A_i$ lie above $A_{i+1}$ for $i = 1, \cdots, g-1$.

Now, suppose that we need to change $n$ crossing relationships of the Heegaard annuli which are near $B_1^1, B_1^2; B_2^1, B_2^2; \cdots; B_n^1, B_n^2$ respectively; where $B_i^1, B_i^2$ is a disk pair such that two disks $\pi(B_i^1)$, $\pi(B_i^2)$ form a crossing in the $xy$-plane and $B_i^1$ lies below $B_i^2$ in the 3-space for $i = 1, 2, \cdots, n$. That is, if we move the parts of the Heegaard annuli near $B_1^1, B_2^1, \cdots, B_n^1$ upward to let $B_i^2$ lie below $B_i^1$ in the 3-space for $i = 1, 2, \cdots, n$, then the new annuli are trivial in $\mathbb{R}^3$.

The above moves can be completed by attaching 3-cells $E_1, \cdots, E_n$ to $B_1^1, \cdots, B_n^1$ respectively according to Fig 4.28. That is, we attach $E_k$ to $B_k^1$ along a disk $D_k \in \partial E_k$ and let the disk $D'_k$ lie over the disk $B_k^1$, where $D_k \cup D'_k$ is an annulus in $\partial E_k$ for $k = 1, 2, \cdots, n$. ( See Fig. 4.28. )

Now, we replace $B^1_k, B^1_2, \cdots, B^1_n$ by $D'_1, D'_2, \cdots, D'_n$ respectively in the corresponding annuli to obtain new annuli $A'_1, A'_2, \cdots, A'_g$ such that each annulus is trivial in $\mathbb{R}^3$ and $A'_i$ lie above $A'_{i+1}$ for $i = 1, \cdots, g-1$. .

By Lemma 4.5, $A'_1, A'_2, \cdots, A'_g$ bound disjoint 2-handles.

□

*Proof of Theorem 4.1* By Theorem 4.4, we can immerse $V$ into $\mathbb{R}^3$ with at most double points as singularities such that the Heegaard annuli can be attached to disjoint 2-handles. That is, we can immerse $\hat{M}_0$ into $\mathbb{R}^3$ with at most double and triple points as singularities.

□

## 4.3 Proof of Theorem 4.2

We will apply the following lemma.

**Lemma 4.7** *Let $S$ be a set of disjoint 2-spheres and let $\tau : S \longrightarrow \mathbb{R}^3$ be an immersion such that $\tau(S)$ is in general position in $\mathbb{R}^3$. If $\tau$ has at most double points as singularities, then the image of the singular points consists of disjoint simple closed curves which form a trivial link $L$ in $\mathbb{R}^3$. The components of $\tau(S) - L$ are open disks with holes.*

*Proof.* Each connected component of the image of singular points is a simple closed curve since the singular points are double points and the map $\tau$ is an map in general position. There exists a simple closed curve $c$ in $\tau(S)$ such that $c$ is a connected component of the image of singular points and $c$ bound a disk $D \in \tau(S)$ so that there are no connected components of the image of singular points in the interior $IntD$ of $D$, i.e., $\tau \mid: \tau^{-1}(IntD) \longrightarrow IntD$ is a homeomorphism. Then we may use the following surgery on $\tau(S)$. Suppose that $D$ belongs to the image surface $S_1$ of $\tau$ and surface $S_2$ is the other image surface which includes the curve $c$. Consider a small regular neighborhood

$N$ of $D$ in $\mathbb{R}^3$ bounded by the surface $S_2$. We view $N$ as $D \times [-\epsilon, \epsilon]$ with $D \times 0 = D$ and $\partial D \times [-\epsilon, \epsilon] \subset S_2$. We remove the open annulus $\partial D \times (-\epsilon, \epsilon)$ from $S_2$ and add the disks $D \times (-\epsilon)$ and $D \times \epsilon$ to $S_2 - \partial D \times (-\epsilon, \epsilon)$ by attaching $\partial D \times (-\epsilon)$ and $\partial D \times \epsilon$ to the respective boundary connected components of $S_2 - \partial D \times (-\epsilon, \epsilon)$. This surgery can cancel the singular connected components of $\tau(S)$ step by step. Note that every simple closed curve in a 2-sphere separates the sphere into two disks. Then, the result of the above surgeries is a disjoint union of 2-spheres. Therefore the image of the singular points consists of disjoint simple closed curves which form a trivial link in $R^3$. It is clear that the result of cutting the image along the curves is an open disk with some closed disk removed since each connected component of the resulting surface is a homeomorphic image of a portion of a 2-sphere in the set $S$.

$\square$

Now, suppose that $M$ is a 3-manifold with $\partial M$ consisting of 2-spheres. Suppose that $\tau : M \longrightarrow \mathbb{R}^3$ is an immersion with at most double points as singularities. Then $\tau$ restricted to $\partial M$ is an immersion of $\partial M$ into $\mathbb{R}^3$. We may assume that $\tau(\partial M)$ is in general position.

In the proof of Lemma 4.7, there exists a disk $D$ so that $D \subset \tau(\partial M)$ and $\partial D$ consists of a connected component of the image set of the singular points of the immersion map $\tau$ restricted to $\partial M$ and there are no other such connected components in the interior of the disk $D$. Note that $\tau^{-1}(\partial D)$ consists of two simple closed curves $c_1, c_2$ in $\partial M$ such that $c_1$ bounds a disk $D_1$ in $\partial M$ with $\tau(D_1) = D$. Let the annulus $N$ be a small regular neighborhood of $c_2$ in $\partial M$.

We consider the two cases.

Case 1. $D$ is part of the boundary surface of a connected component of the image set of the singular points of the immersion map $\tau$, i.e., the curve $c_2$ bounds an open disk $D_2$ in the interior of $M$ so that $\tau(D_2) = D$ and $N$ is in the boundary of a 3-cell $B$ which is a small regular neighborhood of $D_2$ in $M$. Thus $B = D_2 \times [-\epsilon, \epsilon]$ with $D_2 \times 0 = D_2$.

Case 2. *Int* $D$ does not intersect the image set of the singular points of the immersion map $\tau$.

In Case 1, we remove $D_2 \times (-\epsilon, \epsilon)$ from $M$ and obtain the 3-manifold $M'$ whose boundary continues to consist of 2-spheres, and we restrict the immersion $\iota : M \longrightarrow \mathbb{R}^3$ to $M' \longrightarrow \mathbb{R}^3$. In Case 2, we attach a 2-handle $B$ to $M$ along the annulus $N$ ( i.e., $B = D_3 \times [-\epsilon, \epsilon]$ with $\partial D_3 \times [-\epsilon, \epsilon] = N$ ). The resulting 3-manifold we again denote by $M'$. We extend the immersion $\iota : M \longrightarrow \mathbb{R}^3$ by homeomorphically mapping the 2-handle $B$ to the corresponding 3-cell $B'$, that is a small regular neighborhood of $D$ with $\tau(D) \subset \partial B'$.

Both of the above surgeries cancel one connected component of the image set of the singular points of the immersion $\tau$ restricted to $\partial M$. Thus, after finitely many steps we will obtain a 3-manifold $M''$ with $\partial M''$ consisting of 2-spheres and an immersion $\tau'' : M'' \longrightarrow \mathbb{R}^3$ such that the map $\tau''$ restricted to $\partial M''$ is a homeomorphism. Therefore, $\tau''$ restricted to each connected component of $M''$ is a homeomorphism. Suppose $M_0$ is a connected component of $M''$, then $\tau''(M_0)$ is a compact, connected 3-manifold embedded in $\mathbb{R}^3$ and $\partial(\tau''(M_1))$ consists of 2-spheres. Therefore, $\tau''(M_0)$ is a 3-sphere with holes. Thus, $M''$ is the disjoint union of several 3-spheres with holes.

Note that in Case 1, we remove a 2-handle from $M$ to obtain the 3-manifold $M'$, and in Case 2, we add a 2-handle to $M$ to obtain the 3-manifold $M'$. Thus we can add 1-handles to $M''$ to obtain a 3-manifold $M^*$ such that $M \subset M^*$. Note that $\partial M^*$ consists again of 2-spheres. Therefore, $M^*$ is a punctured 3-sphere or a punctured $(S^1 \times S^2)\sharp(S^1 \times S^2)\sharp \cdots (S^1 \times S^2)$. Since $M \subset M^*$ and $\partial M$ consists of 2-spheres, then $M$ embeds in $Triad \times I \times I$. By Rolfsen and Li [32], $M$ is a punctured 3-sphere or a punctured $(S^1 \times S^2)\sharp(S^1 \times S^2)\sharp \cdots (S^1 \times S^2)$.

$\square$

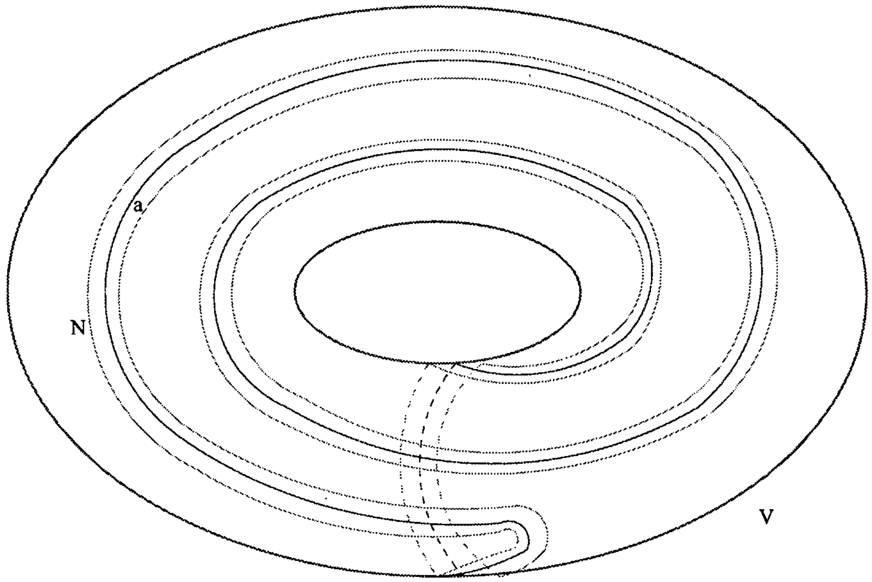



Figure 4.29: A Heegaard diagram of $\mathbb{R}P^3$

## 4.4 An immersion of $\mathbb{R}P^3_0$, the punctured 3-dimensional projective space, into $\mathbb{R}^3$

The real projective plane $\mathbb{R}P^2$ has a complicated immersion into $\mathbb{R}^3$ ( [4], [18], [7], [1] ). Note that $\mathbb{R}P^2$ embeds in $\mathbb{R}P^3_0$. Thus the following lemma gives an immersion of $\mathbb{R}P^2$ into $\mathbb{R}^3$.

**Lemma 4.8** *$\mathbb{R}P^3_0$ immerses into $\mathbb{R}^3$.*

*Proof.* $\mathbb{R}P^3_0$ has a Heegaard diagram $(V; a)$ as showing in Fig. 4.29. After adding a 3-cell $D^2 \times [-\epsilon, \epsilon]$ to the solid torus $V = D^2 \times \mathbb{S}^1$ by identifying $\partial D^2 \times [-\epsilon, \epsilon]$ with the annulus $N$ a regular neighborhood of the Heegaard curve $a$, we obtain the 3-manifold $\mathbb{R}P^3_0$. That is, let $h : \partial D^2 \times [-\epsilon, \epsilon] \longrightarrow N$ be the identifying homeomorphism, then $\mathbb{R}P^3_0 = V \cup_h (D^2 \times [-\epsilon, \epsilon])$.

Now, we isotopically move $V$ in $\mathbb{R}^3_+$ such that one side of the annulus $N$ always points upward ( see Fig. 4.30 ). Let $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 = \mathbb{R}^2 \times 0$, $\pi(x, y, z) = (x, y, 0)$, be the projection onto the $xy$-plane in $\mathbb{R}^3$. Then $\pi\,|_N$ is an immersion. Fig. 4.30 shows that the Heegaard diagram is *even* .

After we change the crossing according to Fig. 4.31, the new annulus is trivial in $\mathbb{R}^3$. Therefore, we can attach a 2-handle to its positive side.

□

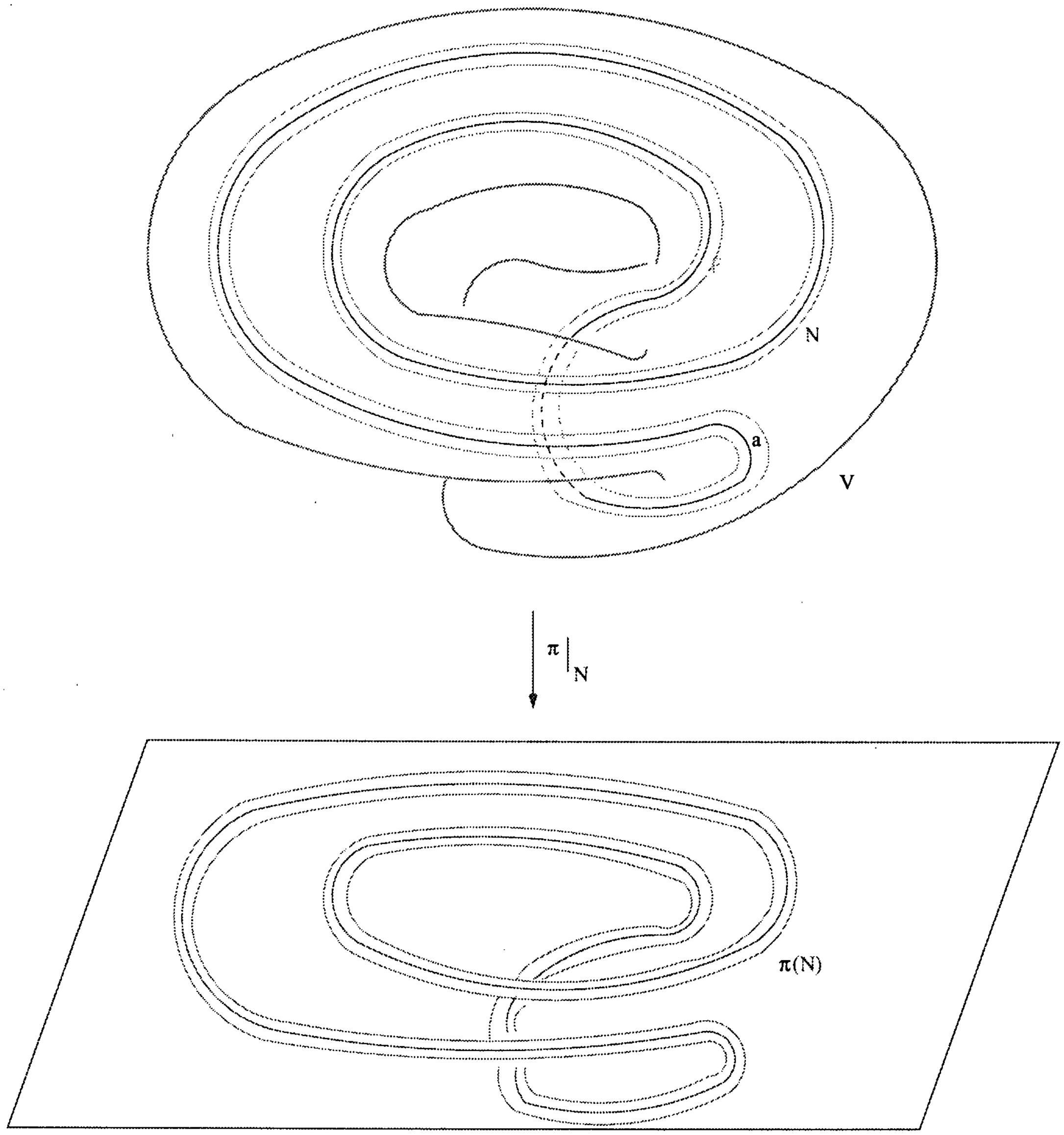



Figure 4.30: A projection from $\mathbb{R}^3_+$ to the $xy$-plane

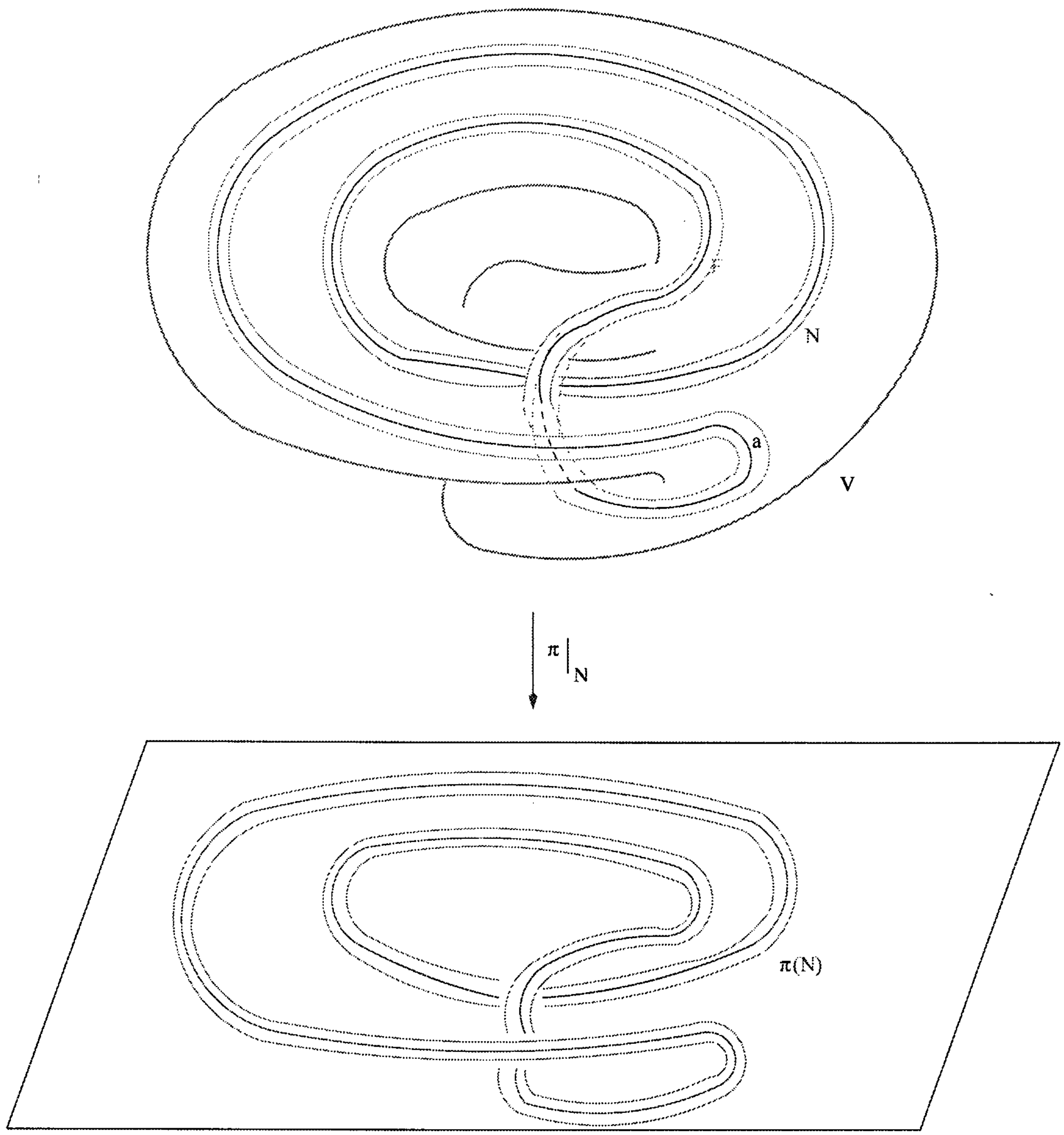



Figure 4.31: An immersion of the solid torus into $\mathbb{R}^3$

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