Asymmetric Fermi Surfaces for Periodic Schrödinger Operators

by

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Abstract

We consider Schrödinger Operators with periodic electric and magnetic field with zero flux through a fundamental cell of the periodic lattice with dimension d. We show that, for a generic small electric/magnetic field and a generic small Fermi energy, the corresponding Fermi surface is at most dimension d-2, convex and not invariant under inversion at any point.

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Part I

Thesis

Chapter 1

Preliminaries

Definition : Let $f(x)$ be a function in $\mathbb{R}^d$. Then the vector γ is called a period of f if $f(x+\gamma) = f(x)$ $\forall x \in \mathbb{R}^d$. If $\gamma_1, \cdots, \gamma_d \in \mathbb{R}^d$ are independent vectors then $\Gamma = \{n_1\gamma_1 + n_2\gamma_2 + \cdots n_d\gamma_d \mid n_j \in \mathbb{Z}\}$ is called a non-degenerate lattice. Let Γ be such a lattice. Suppose there is a crystal lattice with ions at Γ that generate electric and magnetic potentials V(x) and A(x) periodic with respect to Γ. Then the Hamiltonian for a single electron moving in the crystal is

$$H = \frac{1}{2m}(i\nabla + A(x))^2 + V(x)$$

This Hamiltonian commutes with all of the translation operators

$$(T_\gamma\phi)(x) = \phi(x+\gamma) \qquad \gamma \in \Gamma$$

Suppose for now H and T_γ were matrices, then we could find an orthonormal basis of simultaneous eigenvectors for both H and T and these eigenvectors obey

$$\begin{aligned} H\phi_\alpha &= e_\alpha\phi_\alpha \\ T_\gamma\phi_\alpha &= \lambda_{\alpha,\gamma}\phi_\gamma \qquad \forall\gamma \in \Gamma \end{aligned}$$

As T_γ is unitary, all its eigenvalues must be complex numbers of modulus one. So there must exist real numbers $\beta_{\alpha,\gamma}$ such that $\lambda_{\alpha,\gamma} = e^{i\beta_{\alpha,\gamma}}$. Now because $T_\gamma T_{\gamma'} = T_{\gamma+\gamma'}$, we have

$$e^{i\beta_{\alpha,\gamma+\gamma'}}\phi_\alpha = T_{\gamma+\gamma'}\phi_\alpha = T_\gamma T_{\gamma'}\phi_\alpha = T_\gamma e^{i\beta_{\alpha,\gamma'}}\phi_\alpha = e^{i\beta_{\alpha,\gamma}}e^{i\beta_{\alpha,\gamma'}}\phi_\alpha = e^{i(\beta_{\alpha,\gamma}+\beta_{\alpha,\gamma'})}\phi_\alpha$$

which gives

$$\beta_{\alpha,\gamma} + \beta_{\alpha,\gamma'} = \beta_{\alpha,\gamma+\gamma'} \mod 2\pi \qquad \forall\gamma,\gamma' \in \Gamma$$

Then given any d numbers $\beta_1, \cdots, \beta_d$ the system of linear equations (with unknowns $k_1, \cdots k_d$)

$$\begin{aligned} \gamma_i \cdot k &= \beta_i \qquad 1 \le i \le d \\ \text{that is } \sum_{j=1}^{d} \gamma_{i,j}k_j &= \beta_i \qquad 1 \le i \le d \end{aligned}$$

(where $\gamma_{i,j}$ is the j^{th} component of γ_i) has a unique solution because the linear independence of $\gamma_i, \cdots \gamma_d$ implies that the matrix $[\gamma_{i,j}]_{1\le i,j\le d}$ is invertible. So for each α, there exists a $k_\alpha \in \mathbb{R}^d$ with

$$\beta_{\alpha,\gamma} = k_\alpha \cdot \gamma \mod 2\pi \qquad \forall\gamma \in \Gamma$$

Notice that, for each α, k_α is not uniquely determined. Because

$$\begin{aligned} &\beta_{\alpha,\gamma} = k_\alpha \cdot \gamma \bmod 2\pi \text{ and } \beta_{\alpha,\gamma} = k'_\alpha \cdot \gamma \bmod 2\pi && \forall\gamma \in \Gamma \\ \Longleftrightarrow & (k_\alpha - k'_\alpha) \cdot \gamma \in 2\pi\mathbb{Z} && \forall\gamma \in \Gamma \\ \Longleftrightarrow & k_\alpha - k'_\alpha \in \Gamma^{\#} \end{aligned}$$

where $\Gamma^{\#} = \{b \in \mathbb{R}^d \mid b \cdot \gamma \in 2\pi\mathbb{Z}\ \forall\gamma \in \Gamma\}$, the dual lattice. For example, if $\Gamma = \mathbb{Z}^d$ then $\Gamma^{\#} = 2\pi\mathbb{Z}^d$. Now relabel the eigenvalues and eigenvectors, replacing the index α by the corresponding value of $k \in \mathbb{R}^d/\Gamma^{\#}$ and another index n. The index n is needed because many k_α's with different values of α can be equal. With the new labelling, the equations are now

$$\begin{aligned} H\phi_{n,k} &= e_n(k)\phi_{n,k} \\ T_\gamma\phi_{n,k} &= e^{ik\cdot\gamma}\phi_{n,k} \qquad \forall\gamma \in \Gamma \end{aligned} \tag{P.1}$$

Now fix any k and observe that "$T_\gamma\phi_{n,k} = e^{ik\cdot\gamma}\phi_{n,k}$ for all $\gamma \in \Gamma$" means that

$$\phi_{n,k}(x+\gamma) = e^{ik\cdot\gamma}\phi_{n,k}(x)$$

for all x$\in \mathbb{R}^d$ and $\gamma \in \Gamma$. If the $e^{ik\cdot\gamma}$ were not there, this would mean that $\phi_{n,k}$ is periodic with respect to Γ. We can make a simple change of variables to eliminate the $e^{ik\cdot\gamma}$. Define

$$\psi_{n,k}(x) = e^{-ik\cdot\gamma}\phi_{n,k}(x)$$

Then subbing it into (P.1) gives

$$\begin{aligned} \frac{1}{2m}(i\nabla + A - k)^2\psi_{n,k} + V\psi_{n,k} &= \frac{1}{2m}(i\nabla + A - k)^2 e^{-ik\cdot x}\phi_{n,k} + Ve^{-ik\cdot x}\phi_{n,k} \\ &= \frac{1}{2m}(i\nabla + A - k)(e^{-ik\cdot x}(i\nabla + A)\phi_{n,k}) + e^{-ik\cdot x}V\phi_{n,k} \\ &= e^{-ik\cdot x}\frac{1}{2m}(i\nabla + A)^2\phi_{n,k} + e^{-ik\cdot x}V\phi_{n,k} = e_n(k, A, V)\psi_{n,k} \end{aligned}$$

Denote by $\mathbb{N}_k$ the set of values of n that appear in pairs $\alpha = (k, n)$ and define

$$\mathcal{H}_k = \operatorname{span}\{\phi_{n,k} \mid n \in \mathbb{N}_k\}$$

Then, formally, ignoring that k runs over an uncountable set,

$$L^2(\mathbb{R}^d) = \operatorname{span}\{\phi_{n,k} \mid k \in \mathbb{R}^d/\Gamma^{\#}, n \in \mathbb{N}_k\} = \oplus_{k\in\mathbb{R}^d/\Gamma^{\#}}\mathcal{H}_k$$

Set

$$\tilde{\mathcal{H}}_k = \operatorname{span}\{\psi_{n,k} \mid n \in \mathbb{N}_k\}$$

As multiplication by $e^{ik\cdot x}$ is a unitary operator, $\mathcal{H}_k$ is unitarily equivalent to $\tilde{\mathcal{H}}_k$ and $L^2(\mathbb{R}^d)$ is unitarily equivalent to $\oplus_{k\in\mathbb{R}^d/\Gamma^{\#}}\tilde{\mathcal{H}}_k$. The restriction of the Schrödinger operator H to $\tilde{\mathcal{H}}_k$ is $\frac{1}{2m}(i\nabla + A - k)^2 + V$ applied to functions that

are periodic with respect to Γ. Therefore, at least formally, we know that in order to find the spectrum of $H = \frac{1}{2m}(i\nabla + A)^2 + V$ acting on $L^2(\mathbb{R}^d)$, it suffices to find, for each $k \in \mathbb{R}^d/\Gamma^{\#}$ the spectrum of $H_k = \frac{1}{2m}(i\nabla + A - k)^2 + V$.
To make this rigorous, we shall make $L^2(\mathbb{R}^d)$ unitarily equivalent to $\oplus_{k\in\mathbb{R}^d/\Gamma^{\#}}\tilde{\mathcal{H}}_k$ by constructing a unitary operator U from the space of L^2 functions $f(x), x \in \mathbb{R}^d$ to the space of L^2 functions $\psi(k,x), k \in \mathbb{R}^d/\Gamma^{\#}, x \in \mathbb{R}^d/\Gamma$ with the property that

$$(UHU^*\psi)(k,x) = H_k\psi(k,x)$$

Now define

$$\mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma) = \{\psi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \mid \begin{array}{ll} \psi(k, x+\gamma) = \psi(k,x) & \forall \gamma \in \Gamma \\ e^{ib\cdot x}\psi(k+b, x) = \psi(k,x) & \forall b \in \Gamma^{\#}\} \end{array}$$

and

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \sup_x \left|(1+x^{2n})(\prod_{j=1}^{d} \frac{\partial^{i_j}}{\partial x_j^{i_j}} f(x))\right| < \infty \qquad \forall n, i_1, \cdots, i_d \in \mathbb{N}\}$$

With the inner product on $\mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$ given by

$$\langle \psi, \phi \rangle_\Gamma = \frac{1}{|\Gamma\#|}\int_{\mathbb{R}^d/\Gamma^{\#}} dk \int_{\mathbb{R}^d/\Gamma} dx\ \overline{\psi(k,x)}\phi(k,x)$$

$\mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$ is almost a Hilbert space. The only missing axiom is completeness. Call the completion $L^2(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$ Now set

$$(u\psi)(x) = \frac{1}{|\Gamma\#|}\int_{\mathbb{R}^d/\Gamma^{\#}} d^dk e^{ik\cdot x}\psi(k,x)$$
$$(\tilde{u}f)(k,x) = \sum_{\gamma\in\Gamma} e^{-ik\cdot(x+\gamma)}f(x+\gamma)$$

Then notice $u : \mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma) \to \mathcal{S}(\mathbb{R}^d)$
Proof: Let $\alpha \in \mathbb{N}^d$ and denote $x^\alpha = \sum_{j=1}^d x_j^{\alpha_j}$, $|\alpha| = \sum_{j=1}^d \alpha_j$. Then

$$\begin{aligned} & e^{ib\cdot x}\psi(k,x) \text{ is periodic in k} \\ \Longleftrightarrow & e^{ib\cdot x}\psi(k+b,x) = \psi(k,x)\ \forall\ b \in \Gamma^{\#} \\ \Longleftrightarrow & e^{ib\cdot x}\frac{\partial^\alpha}{\partial k^\alpha}\psi(k+b,x) = \frac{\partial^\alpha}{\partial k^\alpha}\psi(k,x)\ \forall\ b \in \Gamma^{\#} \\ \Longleftrightarrow & e^{ik\cdot x}\frac{\partial^\alpha}{\partial k^\alpha}\psi(k,x) \text{ is periodic in k} \end{aligned}$$

Therefore using integration by parts with periodic boundary conditions, we get

$$x^\alpha \int e^{ik\cdot x}\psi(k,x)d^dk = \int (-i)^{|\alpha|}\frac{\partial^\alpha}{\partial k^\alpha}(e^{ik\cdot x})\psi(k,x)d^dk = \int e^{ik\cdot x} i^{|\alpha|}(\frac{\partial}{\partial k})^\alpha\psi(k,x)d^dk$$

And also because $\psi(k,x)$ and all its derivatives are bounded,

$$|\partial_x^\beta\Big(x^\alpha\int_{\mathbb{R}^d/\Gamma^\#}e^{ik\cdot x}\psi(k,x)d^dk\Big)| = \Big|\int_{\mathbb{R}^d/\Gamma^\#}\partial_x^\beta\big(e^{ik\cdot x}\, i^{|\alpha|}\frac{\partial^\alpha}{\partial k^\alpha}\psi(k,x)\big)d^dk\Big| < \infty$$

Now the operator

$$\partial_x^\delta x^\mu f = x^\mu\partial_x^\delta f + \sum_{|\alpha|<|\delta|,\alpha_i\le\delta_i,|\beta|<|\mu|,\beta_i\le\mu_i\ ;\ \alpha,\beta\in\mathbb{N}^d} C_{\alpha,\beta}x^\beta\partial_x^\alpha f$$

So by induction we have the desired result.

$\tilde{u}:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d/\Gamma^\#\times\mathbb{R}^d/\Gamma)$,
Proof: Fix $f\in\mathcal{S}(\mathbb{R}^d)$ and set

$$\psi(k,x)=\sum_{\gamma\in\Gamma}e^{-ik\cdot(x+\gamma)}f(x+\gamma)$$

As $f(x)$ and all of its derivatives are bounded by $\frac{const}{1+|x|^{d+1}}$ the series

$$\sum_{\gamma\in\Gamma}\prod_{\ell=1}^{d}\frac{\partial^{i_\ell}}{\partial x_\ell^{i_\ell}}\frac{\partial^{j_\ell}}{\partial k_\ell^{j_\ell}}e^{-ik\cdot(x+\gamma)}f(x+\gamma)$$

converges absolutely and uniformly in k and x (on any compact set) for all $i_1,\cdots,i_d,j_1,\cdots j_d$. Consequently $\psi(k,x)$ exists and is C^∞. As for the periodicity conditions, if $\gamma\in\Gamma$,

$$\begin{aligned}\psi(k,x+\gamma)&=\sum_{\gamma'\in\Gamma}e^{-ik\cdot(x+\gamma+\gamma')}f(x+\gamma+\gamma')\\&=\sum_{\gamma''\in\Gamma}e^{-ik\cdot(x+\gamma'')}f(x+\gamma'')\qquad\text{where }\gamma''=\gamma+\gamma'\\&=\psi(k,x)\end{aligned}$$

and if $b\in\Gamma^\#$

$$\begin{aligned}e^{i(k+b)\cdot x}\psi(k+b,x)&=\sum_{\gamma\in\Gamma}e^{i(k+b)\cdot x}e^{-i(k+b)\cdot(x+\gamma)}f(x+\gamma)=\sum_{\gamma\in\Gamma}e^{-i(k+b)\cdot\gamma}f(x+\gamma)\\&=\sum_{\gamma\in\Gamma}e^{-ik\cdot\gamma}f(x+\gamma)=e^{ik\cdot x}\sum_{\gamma\in\Gamma}e^{-ik\cdot(x+\gamma)}f(x+\gamma)\\&=e^{ik\cdot x}\psi(k,x)\end{aligned}$$

So by proofs similar to the lecture notes [FN] we have propositions that state:
Proposition P.1 *Let A and V be C^∞* functions that are periodic with respect to Γ and set

$$\begin{aligned}H&=(i\nabla+A(x))^2+V(x)\\H_k&=(i\nabla+A(x)-k)^2+V(x)\end{aligned}$$

with domains $\mathcal{S}$ and $\mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$ respectively. Then,

$$(\tilde{u}Hu\psi)(k,x) = (H_k\psi)(k,x)$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$

Proposition P.2

i) *The operators u and $\tilde{u}$ have unique bounded extensions $U : L^2(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma) \to L^2(\mathbb{R}^d)$* and $\tilde{U} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$ and

$$\tilde{U}U = \mathbb{1}_{L^2(\mathbb{R}^d/\Gamma^{\#}\times\mathbb{R}^d/\Gamma)} \qquad U\tilde{U} = \mathbb{1}_{L^2(\mathbb{R}^d)} \qquad \tilde{U} = U^* \qquad U = \tilde{U}^*$$

Proof: with similar treatment in the study of Fourier series, with the periodicity condition, we get

$$\tilde{u}u\psi = \psi \text{ for all } \psi \in \mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma) \tag{P.2}$$
$$u\tilde{u}f = f \text{ for all } f \in \mathcal{S}(\mathbb{R}^d) \tag{P.3}$$
$$\langle \tilde{u}f, \tilde{u}g \rangle = \langle f, g \rangle \text{ for all } f,g \in \mathcal{S}(\mathbb{R}^d) \tag{P.4}$$

Set $f = u\psi$ and $g = u\phi$. Then by (P.2), $\tilde{u}f = \psi$ and $\tilde{u}g = \phi$, so that by (P.4)

$$\langle u\psi, u\phi \rangle = \langle f, g \rangle = \langle \tilde{u}f, \tilde{u}g \rangle_\Gamma = \langle \psi, \phi \rangle_\Gamma \tag{P.5}$$

Next set g $= u\phi$. Then by (P.2), $\tilde{u}g = \phi$ so that by (P.4)

$$\langle f, u\phi \rangle = \langle f, g \rangle = \langle \tilde{u}f, \tilde{u}g \rangle_\Gamma = \langle \tilde{u}f, \phi \rangle_\Gamma \tag{P.6}$$

So now $\tilde{u}$ and u are bounded by (P.4) and (P.5) respectively. As $\mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$ and $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, $\tilde{u}$ and u have unique bounded extensions $\tilde{U}$ and U. The remaining claims follow from (P.2), (P.3), (P.6) and (P.6) respectively by continuity.

ii) *The operators H(defined on $\mathcal{S}(\mathbb{R}^d)$) and H_k (defined on $\mathcal{S}(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$) have some unique self-adjoint extensions to $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d/\Gamma^{\#} \times \mathbb{R}^d/\Gamma)$. Call these extension H and H_k, then they obey $U^*HU = H_k$.* [FN]

This gives the unitary equivalence we needed.

Chapter 2

The Main Result

Let Γ be a lattice in $\mathbb{R}^d$, $d \geq 2$ and let $r > d$. Define

$$\mathcal{A} = \{A = (A_1, \cdots, A_d) \in (L^r_{\mathbb{R}}(\mathbb{R}/\Gamma))^d | \int_{\mathbb{R}^d/\Gamma} A(x)dx = 0\}$$

$$\mathcal{V} = \{V \in L^{r/2}_{\mathbb{R}}(\mathbb{R}/\Gamma)| \int_{\mathbb{R}^d/\Gamma} V(x)dx = 0\}$$

For $(A, V) \in \mathcal{A} \times \mathcal{V}$ set

$$H_k(A, V) = (i\nabla + A(x) - k)^2 + V(x)$$

When d = 2,3, this operator $H_k(A, V)$ describes an electron in $\mathbb{R}^d$ with quasimomentum k moving under the influence of the magnetic field with periodic vector potential $A(x) = (A_1(x), \cdots, A_d(x))$ and electric field with periodic potential V(x). Later we shall show that

$$e_1(k, A, V) \leq e_2(k, A, V) \leq \cdots$$

are the eigenvalues of the operator $H_k(A, V)$ on $L^r(\mathbb{R}^d/\Gamma)$. The restriction of $e_n(k, A, V)$ to the first Brillouin zone $\mathcal{B}$ of Γ is called the n-th band function of A. Observe that $H_k(0, 0) = (i\nabla - k)^2$ and so the eigenvalues are $(b - k)^2$, with the corresponding eigenvectors $e^{-ib\cdot x}, b \in \Gamma^{\#}$. In particular,

$$e_1(k, 0, 0) = |k|^2$$

The Fermi surface of (A,V) with energy λ is defined as

$$F_\lambda(A, V) = \{k \in \mathcal{B} \mid e_n(k, A, V) = \lambda \text{ for some n}\}$$

Because H has real eigenvalues, let $H_k(A, V)\phi_n = e_n(k, A, V)\phi_n$

$$\begin{aligned} e_n(k, A, V)\overline{\phi_n} &= \overline{e_n(k, A, V)\phi_n} = \overline{H_k(A, V)\phi_n} = \overline{((-i\nabla + A(x) - k)^2 + V(x))\phi_n} \\ &= ((i\nabla - A(x) + k)^2 + V(x))\overline{\phi_n} = H_{-k}(-A, V)\overline{\phi_n} \end{aligned}$$

Therefore

$$e_n(-k, -A, V) = e_n(k, A, V)$$

for all $n \geq 1$. In particular, when A = 0,

$$e_n(-k, 0, V) = e_n(k, 0, V)$$

for all $n \geq 1$, so that $F_\lambda(0,V) = -F_\lambda(0,V)$ for all λ and V. For all $(A,V) \in \mathcal{A} \times \mathcal{V}$, $\lambda \in \mathbb{R}$ and $p \in \mathbb{R}^d$, define

$$p - F_\lambda(A,V) = \{p - k \mid k \in F_\lambda(A,V)\}$$

The main result of the paper is:
Theorem *There is a neighbourhood $\mathcal{A}_0 \times \mathcal{V}_0$ of the origin in $\mathcal{A} \times \mathcal{V}$ and $\lambda_0 > 0$ such that*
(i) for all $(\lambda, A, V) \in (-\infty, \lambda_0) \times \mathcal{A}_0 \times \mathcal{V}_0$, $F_\lambda(A,V)$ is either a strictly convex (d-1)-dimensional real anaylytic submanifold of $\mathcal{B}$, or consists of one point, or is empty
(ii)there is an open dense subset $\mathcal{S}$ of $\mathcal{A}_0 \times \mathcal{V}_0$ of the origin in $\mathcal{A} \times \mathcal{V}$ such that for all $(\lambda, A, V) \in \mathcal{S}$ and all $p \in \mathbb{R}^d$

$$F_\lambda(A,V) \cap (p - F_\lambda(A,V))$$

has dimension at most d-2. Furthermore $\mathcal{S} \cap ((-\infty, \lambda_0) \times \mathcal{A}_0 \times \{0\})$ is open and dense in $(-\infty, \lambda_0) \times \mathcal{A}_0 \times \{0\}$
The theorem shows that for generic small periodic magnetic fields of mean zero and generic small Fermi energies, the Fermi surface is strictly convex and does not have inversion symmetry about any point. In particular, when d=2, the intersection of the Fermi surface and its inversion in any point is generically a finite set of points. Similar statements will hold if electric fields are also added into the formula.

Chapter 3

Analyticity of the Fermi surfaces

Let

$$\mathcal{A}_{\mathbb{C}} = \{A = (A_1, \cdots, A_d) \in (L^r_{\mathbb{R}}(\mathbb{R}/\Gamma))^d | \int_{\mathbb{R}^d/\Gamma} A(x)dx = 0\}$$

$$\mathcal{V}_{\mathbb{C}} = \{V \in L^{r/2}_{\mathbb{R}}(\mathbb{R}/\Gamma) | \int_{\mathbb{R}^d/\Gamma} V(x)dx = 0\}$$

be the complexifications for $\mathcal{A}$ and $\mathcal{V}$

Theorem : *There exists an analytic function F on $\mathbb{C}^d \times \mathbb{C} \times \mathcal{A}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$ such that for k,A, V real,*

$$\lambda \in \mathrm{Spec}(H_k(A, V)) \Longleftrightarrow F(k, \lambda, A, V) = 0$$

Corollary *Fix an open ball D in the first Brillouin zone $\mathcal{B}$ that contains 0 such that $\bar{D} \subset \mathcal{B}^0$.*
There is a neighbourhood U of the origin in $\mathcal{A} \times \mathcal{V}$ and there is $\lambda_0 > 0$ such that
i) The map

$$D \times U \longrightarrow \mathbb{R}, \quad (k, A, V) \longmapsto e_1(k, A, V)$$

is real analytic

ii) For all $(A, V) \in U$ and all $k \in D$

$$e_1(k, A, V) < e_2(k, A, V)$$

iii) For each fixed $(A, V) \in U$ the Hessian of the map $k \longmapsto e_1(k, A, V)$ is positive definite. Furthermore $inf_{k \in D} e_1(k, A, V) < \lambda_0$

iv) For each $(A, V) \in U$ and each $\lambda < \lambda_0$ the Fermi surface $F_\lambda(A, V)$ is either empty, or consists of one point only, or is a real analytic smooth strictly convex (d-1)-dimensional real analytic manifold that is completely contained in D.

Proof of the Corollary : First we state the Implicit Function Theorem below.[L]

The Implicit Function Theorem *let U, V be open sets in complex Banach spaces E, F respectively and let*

$$f : U \times V \to G$$

be analytic, and let $(a,c) \in U \times V$. If the partial derivative of f at (a,c) with respect to c,

$$\partial_c f : F \to G$$

is a linear isomorphism, then there exists an open neighbourhood A of a and a unique analytic map $u : A \to V$ such that

$$f(x, u(x)) = f(a,c), \qquad u(a) = c$$

on A.

Proof of the Corollary : i) This is the direct application of the Implicit Function Theorem with a = (k,A,V) = (k,0,0) and c = $\lambda = k^2$. Now to check linear isomorphism, with the $F(k,\lambda,A,V)$ and $u(k,\lambda)$ we construct later, we have

$$\begin{aligned}
\frac{d}{d\lambda}F|_{\lambda=k^2,A=V=0} &= \frac{d}{d\lambda}[exp(\sum_{i=1}^{d}\frac{(-1)^i}{i}\mathrm{tr}\,(\frac{1}{\sqrt{\mathbb{1}-\Delta}}u(k,\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}})^j) \\
&\qquad \det(\mathbb{1}+\frac{1}{\sqrt{\mathbb{1}-\Delta}}u(k,\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}})]|_{\lambda=k^2} \\
&= \frac{d}{d\lambda}(exp(\sum_{i=1}^{d}\frac{(-1)^i}{i}(\sum_{b\in\Gamma\#}(\frac{2k\cdot b+k^2-\lambda-1}{1+b^2})^i)) \\
&\qquad \prod_{b\in\Gamma\#}(1+\frac{2k\cdot b+k^2-\lambda-1}{1+b^2}))|_{\lambda=k^2} \\
&= \frac{d}{d\lambda}(\prod_{b\in\Gamma\#}(1+\frac{2k\cdot b+k^2-\lambda-1}{1+b^2})\cdot e^{f(d,k,\lambda,b)})|_{\lambda=k^2} \\
&= \frac{d}{d\lambda}(1+\frac{2k\cdot b+k^2-\lambda-1}{1+b^2})|_{\lambda=k^2,b=0}\prod_{b\in\Gamma\#,b\neq 0}(1+\frac{2k\cdot b-1}{1+b^2})\prod_{b\in\Gamma\#}e^{f(d,k,k^2,b)} \\
&\quad +(1+\frac{2k\cdot b+k^2-\lambda-1}{1+b^2})|_{\lambda=k^2,b=0}\frac{d}{d\lambda}(\prod_{b\in\Gamma\#,b\neq 0}g(d,k,\lambda,b))|_{\lambda=k^2,b=0} \\
&= -\prod_{b\in\Gamma\#,b\neq 0}(1+\frac{2k\cdot b-1}{1+b^2})\prod_{b\in\Gamma\#}e^{f(d,k,k^2,b)}\neq 0
\end{aligned}$$

Because we are restricting ourselves within the first Brillouin zone.
ii) The spectrum of $H_k(0,0)$ is $\{|k+b|^2 \mid b \in \Gamma\#\}$. Hence for $k \in D$, $e_1(k,0,0) = |k|^2$ and $e_2(k,0,0) > e_1(k,0,0)$. Since $(k,A,V) \longrightarrow e_1(k,A,V)$ is real analytic, we just have to choose U sufficiently small and by continuity $e_1(k,A,V) < e_2(k,A,V)\ \forall\ (A,V) \in U$

iii) Here we define the Hessian to be the matrix of the 2nd derivatives of the the mapping $k \longrightarrow e_1(k, A, V)$. For $(A, V) = (0,0)$, the Hessian at any point is given by $2 \cdot \mathbb{1}$ and therefore is positive definite. Here by continuity, for U small enough, the Hessian is positive definite for all $(A, V) \in U$.
iv) For U sufficiently small the Hessian is positive definite and therefore by [M], we may write $e_1(k, A, V)$ as $c + \sum_{i=1}^{d} p^2$;so if $\lambda < c$ then $F_\lambda(A, V)$ is empty, if $\lambda = c$ then $F_\lambda(A, V)$ is one point and if $\lambda > c$ then $F_\lambda(A, V)$ is real analytic smooth strictly convex (d-1)-dimensional real analytic manifold that is completely contained in D. But because the mapping is diffeomorphic, we have the desired results in the original k co-ordinates.

Before we proceed to prove the theorem, we present some lemmas first.

Lemma1. *Let* $\|B\|_r = [\mathrm{tr}\,(B^*B)^{r/2}]^{1/r}$ *where* tr *denotes the trace defined on trace class operators on* $L^2(\mathbb{R}^d/\Gamma)$. *If B is a compact operator on* $L^2(\mathbb{R}^d/\Gamma)$ *and* $r \geq 2$, *then*
a) $\|B\| \leq \|B\|_r$
b) $\|B\|_r = \|B^*\|_r$

Proof of the Lemma 1: a) Since B is compact, we can write

$$B = \sum_{n=1}^{\infty} \lambda_n (\psi_n, \cdot)\phi_n$$

,with $\{\lambda_n\}_{n=1}^{\infty} \in \mathbb{R}^+, \lambda_n \to 0, \{\psi_n\}_{n=1}^{\infty}$ and $\{\phi_n\}_{n=1}^{\infty}$ orthonormal sets being the singular value decomposition. Then $B^*B = \sum_{n=1}^{\infty} \lambda_n^2 (\phi_n, \cdot)\phi_n$, $(B^*B)^{r/2} = \sum_{n=1}^{\infty} \lambda_n^r (\phi_n, \cdot)\phi_n$, so

$$\|B\| \leq max\{\lambda_n \mid n \in \mathbb{N}\} \leq \big(\sum_{n=1}^{\infty} \lambda_n^r\big)^{1/r} = \|B\|_r$$

b) From the singular value decomposition we see that $BB^* = \sum_{n=1}^{\infty} \lambda_n^2 (\psi, \cdot)\psi$ and so $\|B^*\|_r = \big(\sum_{n=1}^{\infty} \lambda_n^r\big)^{1/r} = \|B\|_r$

Lemma2. *Write*

$$(i\nabla + A(x) - k)^2 + V(x) - \lambda = \mathbb{1} - \Delta + u(k, \lambda) + w(k, A, V)$$

with

$$u(k, \lambda) = -2ik \cdot \nabla + k^2 - \lambda - \mathbb{1}$$
$$w(k, A, V) = i\nabla \cdot A + iA \cdot \nabla - 2k \cdot A + A^2 + V$$

Then there is a constant $const_{\Gamma,r,d}$ *such that*

$$a) \left\| \frac{1}{\sqrt{\mathbb{1} - \Delta}} w(k, A, V) \frac{1}{\sqrt{\mathbb{1} - \Delta}} \right\|_r \leq const_{\Gamma,r,d}\big((1 + |k|) \|A\|_{L^r} + \|A\|_{L^r}^2 + \|V\|_{L^{r/2}}\big)$$

$$b) \left\| \frac{1}{\sqrt{\mathbb{1} - \Delta}} u(k, \lambda) \frac{1}{\sqrt{\mathbb{1} - \Delta}} \right\|_r \leq const_{\Gamma,r,d}(1 + |k|^2 + |\lambda|)$$

c) *Let* $0 \le \epsilon \le \frac{r-d}{2r}$. *There is a constant* $const_{\Gamma,r,d,k,\lambda,A,V}$ *such that*

$$|\langle (u(k,\lambda)+w(k,A,V))\phi,\psi\rangle| \le const_{\Gamma,r,d,k,\lambda,A,V}\Big(\left\|(\mathbb{1}-\Delta)^{(1-\epsilon)/2}\phi\right\| \left\|(\mathbb{1}-\Delta)^{1/2}\psi\right\| + \left\|(\mathbb{1}-\Delta)^{1/2}\phi\right\| \left\|(\mathbb{1}-\Delta)^{(1-\epsilon)/2}\psi\right\| \Big)$$

for all $\psi,\phi \in L^2(\mathbb{R}^d/\Gamma)$

Proof of the Lemma 2: a) we repeatedly apply the result that, for any $r \ge 2$ and any$f \in \ell^r(\Gamma^\#)$ and g $\in L^r(\mathbb{R}^d/\Gamma)$

$$\|f(i\nabla)g(x)\| \le vol(\mathbb{R}^d/\Gamma)^{-1/r}\,\|f\|_{\ell^r(\Gamma^\#)}\,\|g\|_{L^r(\mathbb{R}^d/\Gamma)} \qquad (*)$$

This is proven just as in [S] Theorem 4.1, except with the periodic domain. One first proves that the Hilbert-Schmidt norm of $f(i\nabla)g(x)$ is bounded by $vol(\mathbb{R}^d/\Gamma)^{1/2}\,\|f\|_{\ell^2(\Gamma^\#)}\,\|g\|_{L^2(\mathbb{R}^d/\Gamma)}$ and that the operator norm of $f(i\nabla)g(x)$ is bounded by $\|f\|_{\ell^\infty(\Gamma^\#)}\,\|g\|_{L^\infty(\mathbb{R}^d/\Gamma)}$. One then interpolate using [S], Theorem 2.9.

As the operator norms $\left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}\nabla\right\| \le 1, \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}k\right\| \le |k|, \|BB'\|_r \le \|B\|_r\,\|B'\|,$ and with lemma 1b, we have

$$\begin{aligned}
\left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}w\frac{1}{\sqrt{\mathbb{1}-\Delta}}\right\|_r &\le (2+|k|)\left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A\right\|_r + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A\cdot\nabla\frac{1}{\sqrt{\mathbb{1}-\Delta}}\right\|_r \\
&\quad + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A^2\frac{1}{\sqrt{\mathbb{1}-\Delta}}\right\|_r^2 + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}V\frac{1}{\sqrt{\mathbb{1}-\Delta}}\right\|_r \\
&\le (2+|k|)\left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A\right\|_r + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}(-\nabla\cdot\overline{A})^*\frac{1}{\sqrt{\mathbb{1}-\Delta}}\right\|_r \\
&\quad + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A\right\|_r^2 + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}\sqrt{V}\right\|_r^2 \\
&\le (2+2|k|)\left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A\right\|_r + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}A\right\|_r^2 + \left\|\frac{1}{\sqrt{\mathbb{1}-\Delta}}\sqrt{V}\right\|_r^2
\end{aligned}$$

Write

$$\frac{1}{\sqrt{\mathbb{1}-\Delta}} = f(i\nabla) \qquad \textit{with} \qquad f(b) = \begin{cases} 1 & \text{if b} = 0 \\ \frac{1}{\sqrt{1+b^2}} & \text{if } b \ne 0 \end{cases}$$

This $f \in \ell^r(\Gamma^\#)$ for all $r > d$ because by the integral test

$$\sum_{b\in\Gamma^\#}\left(\frac{1}{1+b^2}\right)^{r/2} \le const_{\Gamma^\#}\int \frac{1}{(1+x^2)^{r/2}}dx^d = C_{\Gamma^\#}\int_0^\infty \frac{s^d}{(1+s^2)^{r/2}}ds\int_{\text{solid angle}} d\Omega < \infty$$

, so the desired result follows from (*) with $g = A_1,\cdots,A_d,\sqrt{V}$

b) With the eigenfunctions $e^{ib\cdot x}, b\in\Gamma^\#$, the spectrum of $\frac{1}{\sqrt{\mathbb{1}-\Delta}}u(k,\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}}$ is

$$\left\{\frac{2k\cdot b+k^2-\lambda-1}{1+b^2}\;\middle|\; b\in\Gamma^\#\right\}$$

For any $r > d$, the $\ell^r(\Gamma^\#)$-norm of $\frac{2k\cdot b+k^2-\lambda-1}{1+b^2}$, which is also the $\|\cdot\|_r$ norm of $\frac{1}{\sqrt{\mathbb{1}-\Delta}}u(k,\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}}$, is bounded by

$$\begin{aligned}\left\|\frac{2k\cdot b+k^2-\lambda-1}{1+b^2}\right\|_{\ell^r} &\le 2|k|\left\|\frac{b}{1+b^2}\right\|_{\ell^r} + (|k|^2+|\lambda|+1)\left\|\frac{1}{1+b^2}\right\|_{\ell^r}\\ &\le (1+|k|^2)C_{\Gamma^\#,r}\int_0^\infty \left(\frac{s}{1+s^2}\right)^r s^d ds + C'_{\Gamma^\#,r}(|k|^2+|\lambda|+1)\int_0^\infty \left(\frac{1}{1+s^2}\right)^r s^d ds\\ &\le const_{\Gamma,r,d}(1+|k|^2+|\lambda|)\end{aligned}$$

c) Denote $D=\sqrt{\mathbb{1}-\Delta}$. The condition on ϵ implies that $r(1-\epsilon)\ge d+\frac{r-d}{2}>d$ so that $\frac{1}{(1+b^2)^{r(1-\epsilon)/2}}$ is still summable. So as in part a),

$$\begin{aligned}\left\|\frac{1}{D^{1-\epsilon}}A\right\| &\le \left\|\frac{1}{D^{1-\epsilon}}A\right\|_r \le const_{\Gamma,r}\|A\|_{L^r}\\ \left\|\frac{1}{D^{1-\epsilon}}V\frac{1}{D}\right\| &\le \left\|\frac{1}{D^{1-\epsilon}}\sqrt{V}\right\|_r\left\|\sqrt{V}\frac{1}{D}\right\|_r \le const_{\Gamma,r}\|V\|_{L^{r/2}}\\ \left\|u\frac{1}{D}\right\| &\le const_{\Gamma,r}(1+|k|^2+|\lambda|)\end{aligned}$$

Consequently,

$$\begin{aligned}|\langle(i\nabla A)\phi,\psi\rangle| &= |\langle(i\nabla A)\phi,D^{-1}D\psi\rangle| = |\langle D^{-1}(i\nabla A)\phi,D\psi\rangle|\\ &\le \|D^{-1}(i\nabla A)\phi\|\,\|D\psi\| = \|D^{-1}(i\nabla A)D^{-1+\epsilon}D^{1-\epsilon}\phi\|\,\|D\psi\|\\ &\le \|D^{-1}(i\nabla A)D^{-1+\epsilon}\|\,\|D^{1-\epsilon}\phi\|\,\|D\psi\|\\ &\le \|D^{-1}i\nabla\|\,\|AD^{-1+\epsilon}\|\,\|D^{1-\epsilon}\phi\|\,\|D\psi\|\\ &\le 1\cdot const_{\Gamma,r}\|A\|_{L^r}\|D^{1-\epsilon}\phi\|\,\|D\psi\|\\ |\langle(Ai\nabla)\phi,\psi\rangle| &= |\langle(Ai\nabla)D^{-1}D\phi,D^{-1+\epsilon}D^{1-\epsilon}\psi\rangle|\\ &\le \|D^{-1+\epsilon}(Ai\nabla)D^{-1}\|\,\|D\phi\|\,\|D^{1-\epsilon}\psi\|\\ &\le const_{\Gamma,r}\|A\|_{L^r}\|D\phi\|\,\|D^{1-\epsilon}\psi\|\\ |\langle(2k\cdot A)\phi,\psi\rangle| &\le const_{\Gamma,r}|k|\,\|A\|_{L^r}\|D^{1-\epsilon}\phi\|\,\|\psi\|\\ |\langle(A\cdot A)\phi,\psi\rangle| &\le const_{\Gamma,r}\|A\|_{L^r}^2\|D^{1-\epsilon}\phi\|\,\|D^{1-\epsilon}\psi\|\\ |\langle(V\phi,\psi\rangle| &\le const_{\Gamma,r}\|V\|_{L^{r/2}}\|D^{1-\epsilon}\phi\|\,\|D\psi\|\\ |\langle(u\phi,\psi\rangle| &\le const_{\Gamma,r}(1+|k|^2+|\lambda|)\,\|\phi\|\,\|D\psi\|\end{aligned}$$

Putting all these together we get the desired expression for part c).

Proof of the Theorem: Because $L^s(\mathbb{R}^d/\Gamma)\supset L^{s'}(\mathbb{R}/\Gamma)$ for all $1\le s\le s'$, we may assume without loss of generality that $r\le d+1$. Then the lemma implies that

$$F(k,\lambda,A,V)=det_{d+1}\big(\mathbb{1}+\frac{1}{\sqrt{\mathbb{1}-\Delta}}u(k,\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}}+\frac{1}{\sqrt{\mathbb{1}-\Delta}}w(k,A,V)\frac{1}{\sqrt{\mathbb{1}-\Delta}}\big)$$

is a well-defined analytic function on $\mathbb{C}^d \times \mathbb{C} \times \mathcal{A}_{\mathbb{C}} \times \mathcal{V}_{\mathbb{C}}$. Here, $\det_{d+1}(\mathbb{1} + B)$ is the regularised determinant which, for matrices , is defined by

$$\det_{d+1}(\mathbb{1}+B) = exp(\sum_{i=1}^{d} \frac{(-1)^i}{i} trB^i) \det(\mathbb{1}+B)$$

This regularised determinant is defined for B with $\|B\|_{d+1}$ finite. See [S], Theorem 9.2. It is analytic since one can take limits of finite rank approximations of B.
Let $\mathcal{D}$ be the domain of $\sqrt{\mathbb{1}-\Delta}$. And let $q : \mathcal{D} \times \mathcal{D} \to \mathbb{C}, q(\phi, \phi) = (\phi, (H-\lambda)\phi)$ be the form, then by the lemma,

$$(i\Delta + A(x) - l)^2 + V(x) - \lambda = \mathbb{1} - \Delta + u(k, \lambda) + w(k, A, V)$$

gives a well-defined quadratic form on $\mathcal{D} \times \mathcal{D}$. Furthermore, for $\phi \in \mathcal{D}$, by lemma c)

$$\begin{aligned}\langle(\mathbb{1} - \Delta + u(k, \lambda) + w(k, A, V))\phi, \phi)\rangle &= \langle(\mathbb{1} - \Delta)\phi, \phi\rangle - \langle(u(k, \lambda) + w(k, A, V))\phi, \phi)\rangle \\ &\geq \left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2 - const\left\|(\mathbb{1}-\Delta)^{(1-\epsilon)/2}\phi\right\| \left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|\end{aligned}$$

For any $\delta > 0$ there is a constant c_δ such that

$$\left\|(\mathbb{1}-\Delta)^{(1-\epsilon)/2}\phi\right\| \leq \delta\left\|\sqrt{\mathbb{1}-\Delta}\phi\right\| + c_\delta \|\phi\|$$

This is so because

$$\begin{aligned}\left\|(\mathbb{1}-\Delta)^{(1-\epsilon)/2}\phi\right\|^2 &= \sum_{b\in\Gamma^\#} (1+b^2)^{(1-\epsilon)/2} |\hat{\phi}(b)|^2 \\ &= \sum_{\{b \mid \frac{1}{(1+b^2)^\epsilon} \leq \delta^2\}} (1+b^2)|\hat{\phi}(b)|^2 \frac{1}{(1+b^2)^\epsilon} + \sum_{\{b \mid \frac{1}{(1+b^2)^\epsilon} > \delta^2\}} (1+b^2)|\hat{\phi}(b)|^2 \frac{1}{(1+b^2)^\epsilon} \\ &\leq \delta^2 \sum_{b\in\Gamma^\#} (1+b^2)|\hat{\phi}(b)|^2 + c_\delta^2 \sum_{b\in\Gamma^\#} |\hat{\phi}(b)|^2 \\ &= \delta^2 \left\|\sqrt{1-\nabla}\phi\right\|^2 + c_\delta^2 \|\phi\|^2 \leq (\delta\left\|\sqrt{1-\nabla}\phi\right\| + c_\delta\|\phi\|)^2\end{aligned}$$

Choosing $\delta = \frac{1}{2\,\text{const}}$ then

$$\langle(\mathbb{1}-\Delta+u(k,\lambda)+w(k,A,V))\phi,\phi)\rangle \geq \frac{1}{2}\left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2 - \text{const } c_\delta \|\phi\| \left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|$$

However notice $(\alpha \pm \beta)^2 \geq 0$, so $|\alpha\beta| \leq \frac{1}{2}(\alpha^2 + \beta^2)$ and so

$$\left|\text{const}c_\delta \|\phi\| \left\|\sqrt{1-\Delta}\phi\right\|\right| \leq \frac{1}{2}\text{const}^2 c_\delta^2 \|\phi\|^2 + \frac{1}{2}\left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2$$

Therefore

$$\langle(\mathbb{1} - \Delta + u(k, \lambda) + w(k, A, V))\phi, \phi)\rangle \geq -\frac{1}{2}(\text{const } c_\delta \|\phi\|)^2$$

And the form is semibounded. Again since

$$\sqrt{2}\ \text{const}\ c_\delta \|\phi\| \frac{1}{\sqrt{2}} \left\|\sqrt{\mathbb{1}-\Delta}\phi\right\| \leq 2\ \text{const}^2 c_\delta^2 \|\phi\|^2 + \frac{1}{4}\left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2$$

We have .

$$\frac{1}{4}\left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2 - \text{const}\ \|\phi\|^2 \leq |\langle(\mathbb{1}-\Delta+u(k,\lambda)+w(k,A,V))\phi,\phi\rangle| \leq \text{const}\ \left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2$$

In order to show that $\mathbb{1}-\Delta+u(k,\lambda)+w(k,A,V)$ is closed, we must show it is complete under the norm $\|\phi\|_{+1}^2 = \langle(\mathbb{1}-\Delta+u(k,\lambda)+w(k,A,V))\phi,\phi\rangle + const_1 \|\phi\|^2$. From above we then have

$$\frac{1}{4}\left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2 \leq \|\phi\|_{+1}^2 \leq const_2 \left\|\sqrt{\mathbb{1}-\Delta}\phi\right\|^2$$

Now if ϕ_n are Cauchy with respect to $\|\phi\|_{+1}$ then

$$\|\phi_n-\phi_m\|_{+1} \overset{n,m\to\infty}{\longrightarrow} 0 \Longrightarrow \left\|\sqrt{\mathbb{1}-\Delta}(\phi_n-\phi_m)\right\| \longrightarrow 0$$
$$\Longrightarrow \left\|\sqrt{\mathbb{1}-\Delta}(\phi_n-\phi)\right\| \longrightarrow 0 \Longrightarrow \|\phi_n-\phi\|_{+1} \longrightarrow 0$$

Here we used the fact that $\sqrt{\mathbb{1}-\Delta}$ is complete. Hence there is a unique associated self-adjoint semibounded operator $H_k(A,V)$
The resolvent of $(\mathbb{1}-\Delta)^{-1}$ at i is given by $\frac{1}{\mathbb{1}-\Delta+i}$, which is compact because the eigenvalues $\{\frac{1}{1+b^2+i} \mid b \in \Gamma^{\#}\}$ converges to zero. Then by the resolvent identity

$$\frac{1}{H_k(A,V)-\lambda+i} = \frac{1}{\mathbb{1}-\Delta+i+u+k} = \frac{1}{\mathbb{1}-\Delta+i} - \frac{1}{\mathbb{1}-\Delta+i+u+w}(u+w)\frac{1}{\mathbb{1}-\Delta+i}$$

and lemma 2 a), b), the resolvent of $H_k(A,V)$ at i is also compact. Hence the spectrum of $H_k(A,V)$ is discrete. Then $\lambda \in \text{Spec}(H_k(A,V))$ if and only if there exists $\psi \in \mathcal{D}_{H_k(A,V)} \subset \mathcal{D}$ such that

$$(H_k(A,V)-\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}}\sqrt{\mathbb{1}-\Delta}\psi = 0$$

This is the case if and only if $\frac{1}{\sqrt{\mathbb{1}-\Delta}}(H_k(A,V)-\lambda)\frac{1}{\sqrt{\mathbb{1}-\Delta}}$ has a non-trivial kernel. By [S] Theorem 9.2e), this is the case if and only if $F(k,\lambda,A,V)=0$

Chapter 4

Proof of the Main Theorem

For simplicity we write e(k,A,V) = $e_1(k, A, V)$. By part (iii) of the corollary in secion II, for each (A,V) $\in$ U the function e(-,A,V) has a unique extremum k_{min}(A,V). This extremum is a non-degenerate minimum. From the implicit function we know that k_{min}(A,V) depends analytically on (A,V). The same is true for the corresponding value λ_{min}(A,V) = e(k_{min}(A,V), A,V). From part (iii) of the corollary in section II we have λ_{min}(A,V) $< \lambda_0$. Now let P = $\{(\lambda$, A, V) $\in \mathbb{R}\times$ U $\mid \lambda_{min}$(A,V) $< \lambda < \lambda_0\}$. Then for each (λ, A, V) $\in$ P the Fermi surface F_λ(A,V) is a smooth, real analytic, strictly convex (d-1)-dimensional manifold which is not empty. For k $\in F_\lambda$(A,V) denote the outward unit normal vector to F_λ(A,V) at k by n(k). If (λ,A,V) $\in$ P then for each ξ on the unit sphere S^{d-1} there is a unique point $k_\lambda(\xi, A, V) \in F_\lambda$ (A, V)such that n(k_λ(ξ,A,V)) = ξ.
Again it follows from the implicit function theorem that

$$S^{d-1} \times P \longrightarrow D, (\xi, \lambda, A, V) \longmapsto k_\lambda(\xi, A, V)$$

is a real analytic map.
To prove the theorem stated in the Introduction we have to show that for (λ, A, V) in an open dense subset of P and all $p \in \mathbb{R}^d$ the intersection $F_\lambda(A, V) \cap (p - F_\lambda(A, V)$ has dimension at most d-2. Since for all $(\lambda, A, V) \in P$ the manifold is real analytic smooth and strictly convex, one has either

$$dim\big(F_\lambda(A, V) \cap (p - F_\lambda(A, V)\big) \le d - 2 \; or \; F_\lambda(A, V) = (p - F_\lambda(A, V)$$

If the first case is true then we're done. If $F_\lambda(A, V) = (p - F_\lambda(A, V))$ then the inversion in the point p/2 maps the point of $F_\lambda(A, V)$ with norml vector ξ to the point with $-\xi$. i.e.

$$F_\lambda(A, V) = (p - F_\lambda(A, V)) \Rightarrow k_\lambda(\xi, A, V) + k_\lambda(\xi, A, V) + k_\lambda(-\xi, A, V) = p \,\forall\, \xi \in S^{d-1}$$

Therefore the set of all $(\lambda, A,) \in P$ for which there is a point $p \in \mathbb{R}^d$ such that $\dim\big(F_\lambda(A, V) \cap (p - F_\lambda(A, V))\big) > d - 2$ is contained in the set

$$S' = \{(\lambda, A, \acute{V}) \in P \mid \nabla_\xi(k_\lambda(\xi, A, V) + k_\lambda(-\xi, A, V)) = 0 \,\forall\, \xi \in S^{d-1}\}$$

Observe that S' is the interection of the analytic hypersurfaces

$$S' = \bigcap_{\xi \in S^{d-1}} \{(\lambda, A, V) \in P \mid \nabla_\xi(k_\lambda(\xi, A, V) + k_\lambda(-\xi, A, V)) = 0\}$$

Hence to show that the complement of S' is open and dense it suffices to find one $(\lambda, A, V) \in P$ that does not lie in S'. We can do this by choosing V = 0 and a particular vector potential A which is only 2 dimensional, and showing that for small t and some λ the triple $(\lambda, t \cdot A, 0)$ does not in S'. This then also shows that the complement of $S' \cap \{(\lambda, A, V) \in P \mid V = 0\}$ is open and dense in $\{(\lambda, A, V) \in P \mid V = 0\}$
Hence in the following calculation we will only consider the points $(\lambda, t \cdot A, 0)$ of P with a suitably choosen two dimensional vector potential A. Therefore we restrict ourselves to the case d = 2 and delete the V-variable in the notation. We begin by computing the first 3 derivatives of $e(k, t \cdot A)$ at the origin for arbitrary A. We use notation $\dot{f}(k) = \frac{d}{dt} f(k,t)|_{t=0}$

Lemma. Let $A \in \mathcal{A}$. Denote $\epsilon(k,t) = e(k, t \cdot A)$. Then there exists a constant C such that for all $k \in D$

$$\begin{aligned}
\dot{\epsilon}(k) =& 0 \\
\ddot{\epsilon}(k) =& C - 2 \sum_{b \in \Gamma^{\#} \setminus \{0\}} \frac{1}{b^2 + 2k \cdot b} |(2k + b) \cdot \hat{A}(b)|^2 \\
\dddot{\epsilon}(k) =& 12 Re \sum_{b,c \in \Gamma^{\#} \setminus \{0\}} \frac{1}{b^2 + 2k \cdot b} [\hat{A}(-c) \cdot \hat{A}(c - b)[(2k + b) \cdot \hat{A}(b)] \\
& - 6 \sum_{b,c \in \Gamma^{\#} \setminus \{0\}} \frac{[(2k + c) \cdot \hat{A}(-c)]}{c^2 + 2k \cdot c} [(2k + b + c) \cdot \hat{a}(c - b) \frac{[(2k + b) \cdot \hat{A}(b)]}{b^2 + 2k \cdot b}
\end{aligned}$$

Here $A(x) = \sum_{b \in \Gamma^{\#}} \hat{A}(b) e^{ib \cdot x}$ with $\hat{A}(b) = (\hat{A}_1(b), \hat{A}_2(b))$ being its Fourier coefficients. Also, for each $\lambda \in (0, \lambda_0)$ and every $\xi \in S^1$

$$\begin{aligned}
\frac{d}{dt} k_\lambda(\xi, t \cdot A)|_{t=0} &= 0 \\
2\sqrt{\lambda}\xi \cdot \frac{d^2}{dt^2} k_\lambda(\xi, t \cdot A)|_{t=0} &= -\ddot{\epsilon}(\sqrt{\lambda}\xi) \\
2\sqrt{\lambda}\xi \cdot \frac{d^3}{dt^3} k_\lambda(\xi, t \cdot A)|_{t=0} &= -\dddot{\epsilon}(\sqrt{\lambda}\xi)
\end{aligned}$$

Proof : Let $\psi_k(t)$ be the eigenfunction with eigenvalue $\epsilon(k,t)$ for the operator $H_k(t \cdot A)$ normalised by

$$\psi_k(0) = \frac{1}{\sqrt{vol}} \text{ and } < \psi_k(0), \psi_k(t) >= 1$$

, where vol is the volume of $\mathbb{R}^2/\Gamma$. The constant function is an acceptable eigenfunction since at t = 0, $H_k(0 \cdot A) = (i\nabla - k)^2$. Then for small t and k $\in$ D, $\psi_k(t)$ is an analytic function of t and k, so we may differentiate

$$H\psi = \epsilon$$

and get

$$\dot{H}\psi + H\dot{\psi} = \dot{\epsilon}\psi + \epsilon\dot{\psi} \tag{4.1}$$

$$\ddot{H}\psi + 2\dot{H}\dot{\psi} + H\ddot{\psi} = \ddot{\epsilon}\psi + 2\dot{\epsilon}\dot{\psi} + \epsilon\ddot{\psi} \tag{4.2}$$

$$\dddot{H}\psi + 3\ddot{H}\dot{\psi} + 3\dot{H}\ddot{\psi} + H\dddot{\psi} = \dddot{\epsilon}\psi + 3\ddot{\epsilon}\dot{\psi} + 3\dot{\epsilon}\ddot{\psi} + \epsilon\dddot{\psi} \tag{4.3}$$

Next we can take the inner product of these with $\psi(0)$ to get

$$\dot{\epsilon} =< \dot{H}\psi, \psi > \tag{4.4}$$

$$\ddot{\epsilon} =< \ddot{H}\psi, \psi > +2 < \dot{H}\dot{\psi}, \psi > \tag{4.5}$$

$$\dddot{\epsilon} =< \dddot{H}\psi, \psi > +3 < \ddot{H}\dot{\psi}, \psi > +3 < \dot{H}\ddot{\psi}, \psi > \tag{4.6}$$

Since we have $< \psi_k(0), \psi_k(t) >= 1$, differentiate and we will get $\frac{d}{dt} < \psi_k(t), \psi_k(0) >$ $=< \dot{\psi}_k(t), \psi_k(0) >= 0$. So similarly $< \ddot{\psi}_k(t), \psi_k(0) > =< \dddot{\psi}_k(t), \psi_k(0) >= 0$ and $< H\dot{\psi}_k(t), \psi_k(0) > =< H\ddot{\psi}_k(t), \psi_k(0) > =< H\dddot{\psi}_k(t), \psi_k(0) >= 0$.
But now

$$\begin{aligned} \frac{d}{dt}H_k(t\cdot A) &= A\cdot(i\nabla + t\cdot A - k) + (i\nabla + t\cdot A - k)\cdot A \\ &= 2A\cdot(i\nabla + t\cdot A - k) + i(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2}) \end{aligned} \tag{4.7}$$

So at t = 0, we have

$$\dot{H}_k\psi = \frac{1}{\sqrt{vol}}\dot{H}_k 1 = \frac{1}{\sqrt{vol}}(-2k\cdot A + i(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2})) \tag{4.8}$$

Now form the inner product with $\psi(0) = \frac{1}{\sqrt{vol}} \cdot 1$ we will get

$$\dot{\epsilon} =< \dot{H}\psi, \psi >= \frac{1}{vol} < (-2k\cdot A + i(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2})), 1 >$$

But recall in the beginning of the paper we specified $\mathbb{A} = \{A = (A_1, \cdots, A_d) \in (L^r_{\mathbb{R}}(\mathbb{R}^d/\Gamma))^d$
$\mid \int_{\mathbb{R}^d/\Gamma} A(x)dx = 0\}$, so

$$< k\cdot A, 1 >=< k_1A_1(x), 1 > + < k_2A_2(x), 1 >= \int_{\mathbb{R}^d/\Gamma} k_1A_1(x)dx + \int_{\mathbb{R}^d/\Gamma} k_2A_2(x)dx = 0$$

Now $< \frac{\partial A_1}{\partial x_1}, 1 >=< \frac{\partial A_2}{\partial x_2}, 1 >= \int_{\mathbb{R}^d/\Gamma} \frac{\partial A_2}{\partial x_2}dx = A_2(x)|_{evaluated\,at\,boundaries} = 0$ because $A(x)$ is periodic. Hence we have

$$\dot{\epsilon} = 0$$

From (4) we have

$$\dot{\psi} = -(H - \epsilon)^{-1}\dot{H}\psi \tag{4.9}$$

Here we define $(H-\epsilon)^{-1}$ to be 0 on $\psi(0)$ and the inverse of $(H-\epsilon)$ on the orthogonal complement of $\psi(0)$. Plugging into (8)

$$\ddot{\epsilon} = <\ddot{H}\psi, \psi> -2 < \dot{H}(H-\epsilon)^{-1}\dot{H}\psi, \psi>$$

From (7)

$$\ddot{H} = 2(A_1^2 + A_2^2)$$

so that $<\ddot{H}\psi, \psi>$ is a constant $C = \frac{2}{vol} < (A_1^2 + A_2^2), 1 >$ independent of k. Using (8) we get

$$\ddot{\epsilon} = C - \frac{2}{vol} < (H_k - \epsilon)^{-1}(-2k\cdot A + i(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2})), -2k\cdot A + i(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2})) >$$

Since we can write A(x) as $\sum_{b\in\Gamma^\#} \hat{A}(b)e^{ib\cdot x}$, we have

$$(-2k\cdot A + i(\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2})) = -\sum_{b\in\Gamma^\#}(2k+b)\cdot\hat{A}(b)e^{ib\cdot x}$$

Also because $H_k(0,0) = (i\nabla - k)^2$, $H_k(0,0)\,1 = \epsilon(0)\,1$, $\epsilon(0) = k^2$. So

$$(H_k(0,0) - \epsilon(0))e^{ib\cdot x} = (b^2 + 2k\cdot b)e^{ib\cdot x}$$

We finally get

$$\begin{aligned}\ddot{\epsilon} &= C - \frac{2}{vol}\langle \sum_{b\in\Gamma^\#\backslash\{0\}} \frac{1}{b^2+2k\cdot b}((2k+b)\cdot\hat{A}(b)e^{ib\cdot x}, \sum_{b\in\Gamma^\#}((2k+b)\cdot\hat{A}(b))e^{ib\cdot x}\rangle \\ &= C - 2\sum_{b\in\Gamma^\#\backslash\{0\}} \frac{1}{b^2+2k\cdot b}|(2k+b)\cdot\hat{A}(b)|^2\end{aligned}$$

Now, from (2) we get

$$\begin{aligned}\ddot{\psi} &= -(H-\epsilon)^{-1}(\ddot{H}\psi - \ddot{\epsilon}\psi - 2\dot{\epsilon}\dot{\psi} + 2\dot{H}\dot{\psi}) \\ &= -(H-\epsilon)^{-1}\ddot{H}\psi + \ddot{\epsilon}(H-\epsilon)^{-1}\psi(0) - 2\cdot 0\cdot\dot{\psi} - 2(H-\epsilon)^{-1}\dot{H}(H-\epsilon)^{-1}\dot{H}\psi \\ &= -(H-\epsilon)^{-1}\ddot{H}\psi + 2(H-\epsilon)^{-1}\dot{H}(H-\epsilon)^{-1}\dot{H}\psi \qquad (4.10)\end{aligned}$$

Now combine (6),(9),(10) and the fact that $\dddot{H} = 0$, we get

$$\begin{aligned}\dddot{\epsilon} &= -3< \ddot{H}(H-\epsilon)^{-1}\dot{H}\psi, \psi > - 3 < \dot{H}(H-\epsilon)^{-1}\ddot{H}\psi, \psi > \\ &\quad + 6 < \dot{H}(H-\epsilon)^{-1}\dot{H}(H-\epsilon)^{-1}\dot{H}\psi, \psi > \\ &= -6Re < (H-\epsilon)^{-1}\dot{H}\psi, \ddot{H}\psi > +6 < \dot{H}(H-\epsilon)^{-1}\dot{H}(H-\epsilon)^{-1}\dot{H}\psi, \psi >\end{aligned}$$

Since

$$(H-\epsilon)^{-1}\dot{H}\psi = -\frac{1}{\sqrt{vol}}(H-\epsilon)^{-1}\sum_{b\in\Gamma^\#}(2k+b)\cdot\hat{A}(b)e^{ib\cdot x}$$
$$= -\frac{1}{\sqrt{vol}}\sum_{b\in\Gamma^\#\setminus\{0\}}\frac{1}{b^2+2k\cdot b}((2k+b)\cdot\hat{A}(b))e^{ib\cdot x}$$
$$\ddot{H}\psi = \frac{2}{\sqrt{vol}}(A_1^2+A_2^2) = \sum_{b\in\Gamma^\#}\sum_{c\in\Gamma^\#}\hat{A}(b)\cdot\hat{A}(c)e^{i(b+c)\cdot x}$$
$$= \frac{2}{\sqrt{vol}}\sum_{b,c\in\Gamma^\#}\hat{A}(b-c)\cdot\hat{A}(c)e^{ib\cdot x}$$

and

$$\frac{1}{vol} < \dot{H}e^{ib\cdot x}, e^{ic\cdot x} > = \frac{1}{vol} < [-2A\cdot(b+k)+i\nabla\cdot A]e^{ib\cdot x}, e^{ic\cdot x} >$$
$$= \frac{1}{vol}\int_{\mathbb{R}^d/\Gamma}[-2A\cdot(b+k)+i\nabla\cdot A]e^{i(b-c)\cdot x}dx$$
$$= \frac{1}{vol}\int_{\mathbb{R}^d/\Gamma} -2(b+k)\cdot\hat{A}(c-b)+i\,i(c-b)\hat{A}(c-b)dx$$
$$= -(2k+b+c)\cdot\hat{A}(c-b)$$

we get

$$\ddot{\epsilon} = 12Re\sum_{b,c\in\Gamma^\#\setminus\{0\}}\frac{1}{b^2+2k\cdot b}[(2k+b)\cdot\hat{A}(b)][\hat{A}(b-c)\cdot\hat{A}(c)]$$
$$-6\sum_{b,c\in\Gamma^\#\setminus\{0\}}\frac{[(2k+b)\cdot\hat{A}(b)]}{b^2+2k\cdot b}[(2k+b+c)\cdot\hat{A}(c-b)]\frac{[(2k+c)\cdot\hat{A}(c)]}{c^2+2k\cdot c}$$

The above completes the statement about the derivatives of ϵ.

We now prove the statement about the derivatives of $t\to k_\lambda(\xi, t\cdot A)$ for fixed $\lambda\in(0,\lambda_0)$ and $\xi\in S^1$. To simplify notation put

$$\kappa(\xi,t) = k_\lambda(\xi, t\cdot A)$$

Differentiating the identity

$$\epsilon(\kappa(\xi,t),t) = \lambda$$

we get

$$\nabla_k\epsilon(\kappa(\xi,t),t)\cdot\frac{\partial}{\partial t}\kappa(\xi,t)+\frac{\partial}{\partial t}\epsilon(\kappa(\xi,t),t) = 0 \tag{4.11}$$

Since $\dot{\epsilon}=0$ and $\nabla_k\epsilon(\kappa(\xi,0),0) = 2k = 2\sqrt{\lambda}\xi$, setting t = 0 gives

$$\xi\cdot\dot{\kappa}(\xi) = 0 \tag{12.a}$$

Let $\xi^{\perp}$ denote the vector $(-\xi_2, \xi_1)$ perpendicular to $\xi = (\xi_1, \xi_2)$, then by the definition of k_λ we have $\xi^{\perp} \cdot \nabla_k \epsilon(\kappa(\xi, t), t) = 0$. Differentiating this identity we get

$$\xi^{\perp} \cdot (\text{Hessian}(\epsilon) \cdot \dot{\kappa}(\xi) + \nabla_k \dot{\epsilon}) = 0$$

Since for t = 0, we have Hessian(ϵ) = Hessian$(2k) = 2 \times \mathbb{1}$ and $\dot{\epsilon} = 0$, we get

$$\xi^{\perp} \cdot \dot{\kappa}(\xi) = 0 \tag{12.b}$$

Combining (12a) and (12b) gives

$$\dot{\kappa} = 0 \tag{4.12}$$

Differentiating (11) again and letting t = 0 gives

$$\dot{\kappa}(\xi) \cdot Hessian_k(\epsilon) \cdot \dot{\kappa}(\xi) + 2\nabla_k \dot{\epsilon}(\kappa(\xi, 0)) \cdot \dot{\kappa}(\xi) + \nabla_k \epsilon(\kappa(\xi, 0)) \cdot \ddot{\kappa}(\xi) + \ddot{\epsilon}(\kappa(\xi, 0)) = 0$$

Using (12) we have

$$2\sqrt{\lambda}\xi \cdot \ddot{\kappa}(\xi) = -\ddot{\epsilon}(\kappa(\xi, 0)) = -\ddot{\epsilon}(\sqrt{\lambda}\xi)$$

Differentiating (11) twice, we will get

$$\begin{aligned}(\frac{d}{dt}\dot{\kappa}(\xi)) \cdot Hessian_k(\epsilon) \cdot \dot{\kappa}(\xi) + \dot{\kappa}(\xi) \cdot (\frac{d}{dt}Hessian_k(\epsilon) \cdot \dot{\kappa}(\xi)) + \\ 2\big(\dot{\kappa}(\xi) \cdot Hessian_k(\dot{\epsilon}) \cdot \dot{\kappa}(\xi) + \nabla_k \ddot{\epsilon}(\kappa(\xi, t)) \cdot \dot{\kappa}(\xi) + \nabla_k \dot{\epsilon}(\kappa(\xi, t)) \cdot \ddot{\kappa}(\xi)\big) + \\ \dot{\kappa}(\xi) \cdot Hessian_k(\epsilon) \cdot \ddot{\kappa}(\xi) + \nabla_k \dot{\epsilon}(\kappa(\xi, t)) \cdot \ddot{\kappa}(\xi) + \nabla_k \epsilon(\kappa(\xi, t)) \cdot \dddot{\kappa}(\xi) + \\ \nabla_k \ddot{\epsilon}(\kappa(\xi, t)) \cdot \dot{\kappa}(\xi) + \dddot{\epsilon}(\kappa(\xi, t)) = 0\end{aligned}$$

Setting t= 0 and using $\dot{\kappa} = 0$ we get

$$3\nabla_k \dot{\epsilon}(\kappa(\xi, 0)) \cdot \ddot{\kappa}(\xi) + 2\sqrt{\lambda}\xi \cdot \dddot{\kappa}(\xi) + \dddot{\epsilon}(\kappa(\xi, 0)) = 0$$

Since $\dot{\epsilon} = 0$

$$2\sqrt{\lambda}\xi \cdot \dddot{\kappa}(\xi) = -\dddot{\epsilon}(\kappa(\xi, 0)) = -\dddot{\epsilon}(\sqrt{\lambda}\xi)$$

We now proceed to prove the main theorem and consider a fixed vector potential a and a fixed $\lambda \in (0, \lambda_0)$. If

$$F_\lambda(t \cdot A) = (p(\lambda, t) - F_\lambda(t \cdot A)) \text{ for all small t}$$

then

$$k_\lambda(\xi, t \cdot A) + k_\lambda(-\xi, t \cdot A) = p(\lambda, t)$$

for all $\xi \in S^1$and all small t. Multiplying this equality with $2\sqrt{\lambda}\xi$ and take the third derivative with respect to t and evaluating at t = 0 gives

$$2\sqrt{\lambda}\xi \cdot (\frac{d^3}{dt^3}k_\lambda(\xi, t \cdot A)|_{t=0} + \frac{d^3}{dt^3}k_\lambda(-\xi, t \cdot A)|_{t=0}) = 2\sqrt{\lambda}\xi \cdot \dddot{p}(\lambda)$$

By the previous lemma, this becomes

$$\ddot{\epsilon}(-k) - \ddot{\epsilon}(k) = 2k \cdot \ddot{p}(\lambda) \text{ for all k with } |k|^2 = \lambda$$

or, since $e_n(-k, -A, V) = e_n(k, A, V)$ as mentioned earlier, $\ddot{\epsilon}(-k) = -\ddot{\epsilon}(k)$, we have

$$\ddot{\epsilon}(k) = -k \cdot \ddot{p}(\lambda) \text{ for all k with } |k|^2 = \lambda \qquad (4.13)$$

As mentioned above, it suffices to find one vector potential A and one $\lambda \in (0, \lambda_0)$ such that (13) fails. Since we have the freedom to limit the size of λ_0, we may also choose it to be less than 3/4 and proceed to present one specific A such that (13) fails for all $\lambda \in (0, \lambda_0)$ Fix a non-zero vector $d \in \Gamma^{\#}$. Without loss of generality up to scaling we may set d = (1,0). Let

$$\begin{aligned}
\ddot{\epsilon} =& 12 Re \sum_{b,c \in \Gamma^{\#} \setminus \{0\}} \frac{1}{b^2 + 2k \cdot b}[(2k+b) \cdot \hat{A}(b)[\hat{A}(b-c) \cdot \hat{A}(c)] \\
&- 6 \sum_{b,c \in \Gamma^{\#} \setminus \{0\}} \frac{[(2k+b) \cdot \hat{A}(b)]}{b^2 + 2k \cdot b}[(2k+b+c) \cdot \hat{A}(c-b)]\frac{[(2k+c) \cdot \hat{A}(c)]}{c^2 + 2k \cdot c} \\
=& 24 k \cdot d^{\perp} \sum_{b,c \in \{\pm d, \pm 2d\}, b-c \in \{\pm d, \pm 2d\}} \frac{1}{b^2 + 2k \cdot b} \\
&- 48 (k \cdot d^{\perp})^3 \sum_{b,c \in \{\pm d, \pm 2d\}, b-c \in \{\pm d, \pm 2d\}} \frac{1}{c^2 + 2k \cdot c} \frac{1}{b^2 + 2k \cdot b} \\
=& 24 k_2 \left(\frac{2}{1+2k_1} + \frac{2}{1-2k_1} + \frac{1}{4(1+k_1)} + \frac{1}{4(1-k_1)}\right) \\
&- 48 k_2^3 \left(\frac{1}{2(1+2k_1)(1+k_1)} + \frac{2}{(1+2k_1)(1-2k_1)} + \frac{1}{2(1-2k_1)(1-k_1)}\right)
\end{aligned}$$

If (13) were true, i.e. if the above quantity were of the form $-k \cdot \ddot{p}(\lambda)$ for all k with $|k|^2 = \lambda$ then one must have $\ddot{p}(\lambda) = \mu(\lambda) d^{\perp}$ because the right hand side vanishes for $k_2 = 0$. Hence

$$\begin{aligned}
\mu(\lambda) =& -24\left(\frac{2}{1+2k_1} + \frac{2}{1-2k_1} + \frac{1}{4(1+k_1)} + \frac{1}{4(1-k_1)}\right. \\
&\left. - \frac{k_2^2}{(1+2k_1)(1+k_1)} - \frac{4k_2^2}{(1+2k_1)(1-2k_1)} - \frac{k_2^2}{(1-2k_1)(1-k_1)}\right) \qquad (4.14)
\end{aligned}$$

If (13) were true, then the right hand side of (14) would have to be constant on the circle $\{(k_1, k_2) \in \mathbb{R}^2 | k_1^2 + k_2^2 = \lambda\}$. Since the right hand side is a meromorphic function of $f(k_1, k_2)$, it should also be constant on the complex quadric

$$Q_\lambda = \{(k_1, k_2) \in \mathbb{C} \mid k_1^2 + k_2^2 = \lambda\}$$

On the other hand, $f(k_1, k_2)$ has a pole with residue $24(1 - 2k_2^2)$ along the complex line $L = \{(k_1, k_2) \in \mathbb{C}^2 \mid k_1 = \frac{1}{2}\}$. Consequently $f|_{Q_\lambda}$ is infinite on the points of $Q_\lambda \cap L$ different from $(\frac{1}{2}, \pm\frac{1}{\sqrt{2}})$. This shows that $f|_{Q_\lambda}$ cannot be constant unless $Q_\lambda \cap L \subset \{(\frac{1}{2}, \pm\frac{1}{\sqrt{2}})\}$. But since earlier we chose $\lambda < \frac{3}{4}$, this is not the case and (13) is not true.

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