EXOTIC GROUP ACTIONS ON HOMOLOGY 3-SPHERES

by

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Abstract

We present two contrasting methods for constructing group actions on homology 3-spheres. These actions are characterized as being exotic - not conjugate to a standard linear action.

The first method is to modify a standard linear action on the 3-sphere by performing surgery on an invariant link. A family of exotic actions is produced for a class of metacyclic groups. This method is certainly applicable to any group admitting a linear representation.

The second method is a computer aided study using the program Twiggy, software written by the author. Starting from a knot in the 3-sphere, a representation is found from the knot group to a finite group $G$. The branched cover associated to the representation has a natural $G$-action. The program looks for representations, determines if the cover is a homology sphere and tests if the action constructed is exotic. The primary test of whether or not an action is exotic is by examining the linking numbers of the fixed point curves.
# Table of Contents

Abstract  

Table of Contents  

List of Figures  

Chapter 1. Introduction  

Chapter 2. Covering Spaces  

2.1 Unbranched Covers  
2.2 Application: Reidemeister-Schreier Rewriting  
2.2.1 Free Groups and Group Presentations  
2.2.2 A Modicum of Geometry  
2.2.3 Rewriting a Subgroup  
2.3 Branched Covers  
2.3.1 The Fundamental Group of a Branched Cover  
2.3.2 Branched Covers of Knots  

Chapter 3. Linking Numbers  

3.1 Some Basic Algebraic Topology  
3.1.1 Manifolds and Simplicial Complexes  
3.1.2 Poincaré Duality  
3.2 Linking Numbers  
3.2.1 Intersection Forms  
3.2.2 Linking  
3.3 Branched Covers and Linking Numbers  

Chapter 4. Surgery  

4.1 The Basic Construction  
4.2 Homology Calculations  
4.3 Linking Number Calculations  

Chapter 5. Results  

5.1 Method 1 — Modifying the Linear Action  
5.1.1 The Standard Linear Action  

iii
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1.2 Modifying the Action</td>
<td>39</td>
</tr>
<tr>
<td>5.1.3 Finding Non-homogeneous Linking</td>
<td>41</td>
</tr>
<tr>
<td>5.2 Method 2 — Branched Covers of $S^3$</td>
<td>43</td>
</tr>
<tr>
<td>Chapter 6. Conclusion</td>
<td>46</td>
</tr>
<tr>
<td>Bibliography</td>
<td>48</td>
</tr>
<tr>
<td>Appendix A. Table of Results</td>
<td>50</td>
</tr>
</tbody>
</table>
List of Figures

2.1 A monodromy example. ......................................................... 8
2.2 The branched cover of Example 2.4. ....................................... 13
2.3 A typical branched cover of a knot. ....................................... 15

3.1 A 2-simplex in $\mathbb{R}^3$. ................................................. 17
3.2 A triangulated disc and the dual cellulation. ............................. 22
3.3 The square knot (left) and granny knot (right). ......................... 27

4.1 Examples of surgery on $S^3$. ............................................... 31
4.2 Surgery link for Example 4.2. ............................................... 33

5.1 Fixed point curves for $D(2,3)$ in toroidal coordinates. ............... 40
5.2 Invariant surgery link for $D(2,3)$. ....................................... 41
5.3 Knot 10_132 from the standard knot tables. ............................. 44
Chapter 1
Introduction

This thesis is primarily concerned with the spherical space form problem, that is examining group actions on spheres and objects related to spheres. In [11] Milnor produced a complete list of groups which could potentially act fixed point freely on some homology sphere of odd dimension. Notably missing from his list were the dihedral groups: given a dihedral group $D_{2n}$ the best one could come up with is an action where the $n$-fold cyclic action is fixed point free while the involutions have fixed points.

A simple construction of such an action on the $(2k - 1)$-sphere is the standard linear embedding into $U(k)$. As will be seen later, the fixed point set of such an action is a link of $n$ unknotted $(2k - 3)$-spheres where the linking numbers between any two are $\pm 1$. Any action on the sphere which is not conjugate to a linear action is termed exotic. A simple test of whether or not a group action is exotic is obtained by examining the linking numbers of the fixed curves. Conjugating the action by any homeomorphism will leave the linking numbers unchanged, hence they are an invariant of the action.

In dimension 3, the primary focus of this paper, all dihedral actions on the 3-sphere are conjugate to a linear one (following from a result of Thurston [15].) However, one can construct exotic actions on homology 3-spheres. In this case, exotic refers to the fact that by the double suspension theorem any action on a homology 3-sphere can be lifted to an action on the 5-sphere, and from there to all odd-dimensional spheres. The lift preserves linking numbers, hence the term exotic is valid when applied to homology 3-spheres, even if strictly speaking it is out of context.

Some other related results stem from work done by Davis and tom Dieck.
In [1] they looked at the Brieskorn variety $B^2_d$ which is the intersection of the algebraic curve
\[ z_d^2 + z_1^2 + \cdots + z_n^2 = 0 \]
with the sphere
\[ z_0^2 + \cdots + z_n^2 = 1 \]
Any element of $O(n)$ acts on $B$ by acting on the last $n$ coordinates, hence there is a natural dihedral action on $B$. However, the fixed point set in this case is a link with exotic linking numbers. To complete the construction, they performed surgery away from the branch set in order to kill the homotopy. There are certain cases when the obstruction vanishes and exotic actions on homotopy spheres are possible. Their constructions resulted in the following theorem:

**Theorem 1.1.** Let $d$ be relatively prime to $2m$ and suppose $\pm d$ is a square mod $8m$. Then there exists a smooth $D_{2m}$-action on a homotopy sphere $\Sigma^{8k+3}$ ($k \geq 1$) so that the fixed sets of the involutions have pairwise linking number $d$.

However, in dimension 3 their methods fail since one cannot construct an exotic dihedral action on the 3-sphere. In [9], Livingston reinterpreted the methods of Davis and tom Dieck and applied them to construct actions on homology 3-spheres with exotic linking numbers. His method was a bottom-up approach: starting from a knot in $S^3$, he examined the manifold resulting from the 2-fold branched cover of the knot followed by a regular $n$-fold covering. The composition of the two is a dihedral covering. The statement of his result is:

**Theorem 1.2.** For every integer $x$, there is a 2n-fold regular dihedral covering of $S^3$ branched over the pretzel knot $K(p, q, r)$ with $p = 1 + 2xn$, $q = -2 - 2xn$ and $r = -n - (4x^2n^2 + 6xn + 2)$. That cover is a homology sphere, and the linking number of any two components of the branch set is $-4x(xn + 1) - 1$.

This paper presents two contrasting methods for finding exotic group actions. The first method is similar to the one used by Livingston in [9], except the approach is more algebraic and experimental in nature. Starting from a knot in $S^3$, a computer program, Twiggy, was used to find representations of the knot group onto a dihedral group. Corresponding to this representation is a branched cover of the 3-sphere, with branch set the knot in question. Using a
Chapter 1. Introduction

Reidemeister-Schreier rewriting procedure, one can determine the homology of the space above, and furthermore calculate the linking numbers of the upstairs branch link. It turns out that exotic dihedral actions are not exceptional, in fact they turn up with an amazing regularity. However, attempts to extend these methods to other groups, notably metacyclic groups, were met with complete failure.

The other method, which is much more topological in flavour, can be extended very easily to any group with a linear representation. The basic procedure is to start from the standard linear action on \( S^3 \), then perform surgery on a link invariant under the action in such a way as to be able to extend the action to the new manifold. If the surgery curves link up with the branch set, then performing surgery modifies the linking numbers between the branch curves, thus obtaining exotic actions.

Two comments are in order about the previously obtained results by Davis and tom Dieck, as well as by Livingston. First, there are rather strong constraints on the allowed linking numbers. In both cases it is a comparatively small subset of the integers, depending rather strongly on the order of the dihedral group involved. Secondly, the linking between any two of the branch components is constant. Although Livingston's method can be applied to produce examples with non-homogeneous linking, there appears to be no systematic method for constructing such cases. This paper attempts to ameliorate both of the above concerns. The surgery method produces actions where the constraints on the linking number are not so severe as either the Davis or the Livingston examples. As well, a family of examples involving non-homogeneous linking was produced. One of the interesting conclusions to be drawn from the computer aided study is that homogeneous actions are exceptionally rare. In fact, not a single case was found, other than the trivial one, by a brute force search.
Chapter 2
Covering Spaces

2.1 Unbranched Covers

Let $X$ be a topological space.

Definition 2.1. A space $\tilde{X}$ is an unbranched cover of $X$ with covering map $p : \tilde{X} \to X$ if the following conditions are satisfied. There is an open cover $\{U_\alpha\}$ of $X$ (i.e. each $U_\alpha \subset X$ is open and $\cup U_\alpha = X$) such that for each $\alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically by $p$ onto $U_\alpha$.

Examples. Consider the unit circle in the complex plane, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ Then the map $p : \mathbb{R} \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ makes $\mathbb{R}$ a covering space of $S^1$. In fact, take the open cover of $S^1$ to be any two open arcs whose union is the circle.

Similarly, the map $p_n : S^1 \to S^1$ described by $p_n(z) = z^n$ is also a covering space map, as can be easily verified.

There are many useful properties of covering spaces which can be derived directly from the lifting propositions described below. Let $p : \tilde{X} \to X$ be a covering map. If $f : Y \to X$ is any map then a lift of $f$ is a map $\tilde{f} : Y \to \tilde{X}$.
Chapter 2. Covering Spaces

satisfying $p \circ \tilde{f} = f$. In other words, the following diagram commutes.

\[ \begin{array}{ccc}
    & \tilde{X} & \\
\hat{f} & \downarrow p & \\
 Y & \xrightarrow{f} & X
\end{array} \]

The proof of the following can be found in any introductory text on algebraic topology, such as [7].

Proposition 2.1. (*Homotopy lifting property*) If $F : Y \times I \to X$ is a homotopy, denoted by $F(y,t) = f_t(y)$, and $f_0$ lifts to a map $\tilde{f}_0$ then there exists a unique homotopy

\[ \tilde{F} : Y \times I \to \tilde{X} \]

which lifts $F$ and is a homotopy of $\tilde{f}_0$ (i.e. $\tilde{F}|Y \times 0 = \tilde{f}_0$.)

Taking $Y$ to be a point in the above proposition leads to:

Proposition 2.2. (*Path lifting property*) For every path

\[ \gamma : I \to X \]

starting at a point $x_0 \in X$ and for every $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique path

\[ \tilde{\gamma} : I \to \tilde{X} \]

which lifts $\gamma$ and starts at the point $\tilde{x}_0$.

In particular, uniqueness of lifts implies that any lift of a constant path is itself constant.

Finally, for the case of a general map $h : Y \to X$, with $Y$ path connected, there is the following:

Proposition 2.3. (*Lifting criterion*) $h : Y \to X$ lifts to a map $\tilde{h} : Y \to \tilde{X}$ exactly when $h_* \left( \pi_1(Y) \right) \subset p_* \left( \pi_1(\tilde{X}) \right)$.

There is an intimate relationship between covering spaces of a topological space $X$, transitive permutation representations of its fundamental group and conjugacy classes of subgroups.
Given an $n$-sheeted cover $p : \tilde{X} \to X$ one may construct a permutation representation of the fundamental group of $X$, $\pi_1(X, b)$. Let $\tilde{B} = \{\tilde{b}_1, \ldots, \tilde{b}_n\}$ be the the preimage of $b$. Then one can find a representation

$$\phi : \pi_1(X, b) \to S(\tilde{B})$$

as follows. (Note that $S(\tilde{B})$ is the symmetry group of the finite set $\tilde{B}$.)

By the path lifting property of covering spaces, any closed loop $x$ in $\pi_1(X, b)$ will lift to $n$ distinct paths in $\tilde{X}$, each starting from a different point in $\tilde{B}$. Label the path based at $\tilde{b}_i$ as $\tilde{x}_i$. The permutation $\phi(x) : \tilde{B} \to \tilde{B}$ is the map which sends $\tilde{b}_i$ to $\tilde{x}_i(1)$, i.e. the end point of the path which lifts to a path based at $\tilde{b}_i$.

Note. Implicit in this discussion is the assumption that permutations are multiplied left to right,

$$\sigma \tau = \tau \circ \sigma$$

In other words if $\sigma(i) = j$ and $\tau(j) = k$ then $\sigma \tau$ is the permutation which sends $i$ to $k$.

If one uses the opposite multiplication convention, right to left, then an element $x \in \pi_1(X, b)$ would be represented by the inverse of the permutation determined above.

If the covering space $\tilde{X}$ is path connected, then its associated permutation representation will be transitive. A transitive representation is one where given $\tilde{b}_i, \tilde{b}_j \in \tilde{B}$ then there is an element $x$ where $\phi(x)(\tilde{b}_i) = \tilde{b}_j$. In the present case, choose a path $\tilde{y}$ which starts at $\tilde{b}_i$ and ends at $\tilde{b}_j$. Then notice that $p(\tilde{y})$ is a closed loop with the desired property.

As a converse to the above discussion, there is an elegant classification of covering spaces in terms of permutation representations of the fundamental group. Assume $X$ is a connected, locally simply connected topological space. In particular, $X$ has a universal cover. Let $\phi : \pi_1(X, b) \to S_n$ be a transitive permutation representation.

**Theorem 2.4.** There is a unique (up to isomorphism) path connected covering space $p : \tilde{X}_\phi \to X$ whose induced permutation representation as described above is precisely $\phi$.

**Proof.** Let $Y$ be the universal cover of $X$, which always exists under the hypotheses of the theorem. One may view $Y$ as the set of homotopy classes
of paths based at \( b \), that is \( Y = \{ [\gamma] \mid \gamma(0) = b \} \). Fix some \( k \) from the set \( \{1, \ldots, n\} \). Then define \( \tilde{X}_\phi \) by \( Y / \sim \) where \([\gamma] \sim [\gamma']\) exactly when \( \gamma(1) = \gamma'(1) \) and \( \phi([\gamma\gamma'^{-1}]) = k \). It is straightforward to verify that \( \tilde{X}_\phi \) has all the required properties. Uniqueness follows from Proposition 2.6 below.

The connection to subgroups of the fundamental group is contained in the next proposition.

Proposition 2.5. For a transitive representation \( \phi : \pi_1(X, b) \to S_n \), the subgroup \( \text{Stab}_\phi(k) = \{ x \in \pi_1(X, b) \mid \phi_x(k) = k \} \) is of index \( n \) in \( \pi_1(X, b) \).

Conversely, given a subgroup \( H \subset \pi_1(X, b) \) of index \( n \), and an integer \( k \), \( 1 \leq k \leq n \), there is a transitive representation

\[
\phi : \pi_1(X, b) \to S_n
\]

such that \( \text{Stab}_\phi(k) = H \). Furthermore, if \( \tilde{X} \) is the cover associated with \( \phi \), then \( \pi_1(\tilde{X}, b_k) = H \).

In particular, finite-sheeted connected covering spaces of \( X \) are in one-to-one correspondence with subgroups of its fundamental group.

Proof. Let \( H_i = \{ x \in \pi_1(X, b) \mid \phi_x(k) = i \} \). It is clear that \( \bigcup H_i = \pi_1(X, b) \) and \( H_i \cap H_j = \emptyset \) if \( i \neq j \). This confirms that \( \pi_1(X, b) : \text{Stab}_\phi(k) = n \).

Now consider a subgroup \( H \) of index \( n \). Let \( x_1H, \ldots, x_nH \) be the cosets with \( H = x_kH \). Define a map \( \phi : \pi_1(X, b) \to S_n \) by the rule \( \phi_x(i) = j \) precisely when \( xx_iH = x_jH \). Certainly this is a well defined homomorphism with the property that \( \text{Stab}_\phi(k) = H \). Now let \( \tilde{X}_\phi \) be the associated cover. From the preceding theorem, with \( Y \) the universal cover of \( X \), we have \( \tilde{X}_\phi = Y / \sim \) with \([\gamma] \sim [\gamma']\) if and only if \( \gamma(1) = \gamma'(1) \) and \( \phi([\gamma\gamma'^{-1}]) = k \). In fact, a loop \( \gamma \) from \( X \) lifts to a loop \( \tilde{\gamma} \) in \( \tilde{X}_\phi \) exactly when \([\gamma]\) is an element of \( H \).

However, \( \pi_* : \pi_1(\tilde{X}) \to \pi_1(X) \) is injective (this follows from the homotopy lifting property). Hence \( \pi_1(\tilde{X}) \) is isomorphic to \( H \).

Proposition 2.6. If \( X \) is path connected and locally path connected then two covering spaces \( p_i : \tilde{X}_i \to X \) \( (i = 1, 2) \) are isomorphic if and only if the image of \( p_{1*} \) is the same as the image of \( p_{2*} \).

Proof. Let \( h : \tilde{X}_1 \to \tilde{X}_2 \) be an isomorphism. Then since \( p_1 = p_2 \circ h \) and \( p_2 = p_1 \circ h^{-1} \) one arrives at the desired conclusion.

Conversely, we may apply the lifting criterion to construct maps \( \tilde{p}_1 : \tilde{X}_1 \to \tilde{X}_2 \) and \( \tilde{p}_2 : \tilde{X}_2 \to \tilde{X}_1 \) with \( p_1\tilde{p}_2 = p_2 \) and \( p_2\tilde{p}_1 = p_1 \). So by uniqueness of lifts, \( \tilde{p}_1\tilde{p}_2 = \text{id} = \tilde{p}_2\tilde{p}_1 \), making them inverse isomorphisms.
The above discussion leads to the following definition:

**Definition 2.2.** The *monodromy*, \( \Sigma_p \) for the covering space map \( p \) is the representation \( \Sigma_p : \pi_1(X) \to S_k \) associated to it.

The preceding theorem guarantees that the monodromy is well defined, and furthermore, that any finite-sheeted covering space map is completely determined by its monodromy.

**Example 2.1.** Figure 2.1 shows a three fold cover of the wedge of two circles. The monodromy for this cover is \( \alpha \mapsto (12), \beta \mapsto (23) \).

### 2.2 Application: Reidemeister-Schreier Rewriting

#### 2.2.1 Free Groups and Group Presentations

Let \( S \) be any set. The free group generated by \( S \) is the unique group \( F_S \) (denoted \( F_n \) if \( |S| = n \) is finite) satisfying the following universal mapping property:

If \( G \) is any group and \( \phi : S \to G \) a map then there is a unique homomorphism \( \eta : F_S \to G \) where

\[
\begin{array}{ccc}
S & \xrightarrow{\iota} & F_S \\
\downarrow{\phi} & & \downarrow{\eta} \\
G & & 
\end{array}
\]

commutes. The map \( \iota : S \to F_S \) is the natural inclusion map.
Let $S$ be the set $\{x_a\}_{a \in \Gamma}$. Then the free group $F_S$ may be given a presentation $F_S = \langle x_a \mid - \rangle_{a \in \Gamma}$. In fact $F_S$ can be regarded as the group of equivalence classes of words, with letters in $S \cup S^{-1}$ ($S^{-1} = \{x_a^{-1}\}$, the set of formal inverses of $S$). Multiplication is defined by concatenation and two words are equivalent if and only if they have the same reduced form (a word is said to be reduced if no letter lies next to its inverse). Refer to [10] for a thorough treatment from this point of view.

Now consider an arbitrary group $G$. $G$ is said to be presented with presentation $\langle x_a \mid r_b \rangle_{a \in \Gamma, b \in \Lambda}$ if there is a surjective homomorphism $\eta : F_{\{x_a\}} \to G$ and the kernel of $\eta$ is the normal closure of the words $\{r_b\} \subset F_{\{x_a\}}$. The set $\{x_a\}$ are the generators of $G$ while the words $\{r_b\}$ are called relators.

Often the map $\eta$ will be suppressed from the notation. As an example, for the group $G$ as presented above, referring to the element $x_1x_2^{-1}$ in $G$ means the element $\eta(x_1x_2^{-1})$.

### 2.2.2 A Modicum of Geometry

There is a nice result which gives group presentations a useful geometric structure:

**Theorem 2.7.** Given a group $G$ with presentation $\langle x_a \mid r_b \rangle$ there is a 2-complex $C$ with $\pi_1(C) = G$. Further, $G$ is a free group if and only if $C$ is homotopic to a wedge of circles.

For a complete proof, refer to Sillwell [13]. However, the actual construction is decidedly simple. The 1-skeleton for the complex is a wedge of circles, with each loop labelled uniquely by a generator of $G$. Finally, one completes the 2-complex by gluing on discs, one disc for each relator. The boundary of each disc is determined by following the corresponding path traced out on the wedge of circles.

**Example 2.2.** For the group $G = \langle x \mid x^2 \rangle$ (isomorphic to $\mathbb{Z}_2$), the associated 2-complex is obtained by gluing the boundary of a disc together by its antipodal points, i.e. one obtains the projective plane.

### 2.2.3 Rewriting a Subgroup

Now that there is some geometric structure associated with a group, we may apply the full power of covering space theory to extract some useful information
about the group. One notable case is finding a presentation for a subgroup.

Let \( G = \langle x_a \mid r_b \rangle \) be a presented group and \( H \) a subgroup. Let \( X \) be the 2-complex with fundamental group \( G \). Since \( H \) is a subgroup of the fundamental group of \( X \), there must be a covering space \( p : \tilde{X} \to X \) corresponding to this subgroup. In fact, \( \pi_1(\tilde{X}) \) is isomorphic to \( H \). Furthermore, let \( \Sigma \) denote the monodromy for the cover.

Since covering spaces preserve dimension, \( \tilde{X} \) will be a 2-complex. The idea is to use the monodromy to reconstruct the structure of \( \tilde{X} \) and hence determine its fundamental group, \( H \). The first step is to reconstruct the 1-skeleton.

For simplicity of presentation, assume that \( H \) is of finite index \( n \) in \( G \). The same method works equally well for infinite sheeted covers, except the notation becomes a mess.

Note that each loop (labelled with an \( x_a \)) will lift to \( n \) distinct paths in \( \tilde{X} \), which will be labelled \( Y_{a,1}, \ldots, Y_{a,n} \). In fact, the 1-skeleton structure for \( \tilde{X} \) will be a graph. A standard result from graph theory states that any graph has the homotopy type of a wedge of circles, which is obtained by contracting along all the edges in a maximal or Schreier tree. This is any simply connected subset of edges which meets every vertex of the graph. Let \( \{s_c\} \) denote the edges in a particular Schreier tree.

Next note that each disc in the 2-skeleton of \( X \) will lift to \( n \) distinct discs upstairs. The boundaries of these discs are obtained by lifting the boundary path of the downstairs disc to the \( n \) different paths upstairs which cover it. Let \( \mathcal{R}_1, \ldots, \mathcal{R}_n \) be \( n \) functions

\[
\mathcal{R}_i : \pi_1(X) \to \{ \gamma : I \to \tilde{X} \mid \gamma(0), \gamma(1) \in \tilde{b}_1, \ldots, \tilde{b}_n \}
\]

with the property that if \( \gamma \) is a path in \( X \) then \( \mathcal{R}_i(\gamma) \) is the path in \( \tilde{X} \) which is the lift of \( \gamma \) to the path based at \( \tilde{b}_i \).

Combining these results gives a presentation for \( H \):

\[
H = \langle Y_{a,i} \mid \mathcal{R}_i(r_b), s_c \rangle
\]

One can easily verify that the following is the correct formula for \( \mathcal{R}_i \), where \( u \) and \( w \) are arbitrary loops in \( X \).

\[
\mathcal{R}_i(x_a) = Y_{a,i} \\
\mathcal{R}_i(uv) = \mathcal{R}_i(u)\mathcal{R}_{u(i)}(w)
\]
Note that these rewriting functions are uniquely defined inductively since \( R_t(1) = 1 \) and \( R_t(x^{-1}) = Y^{-1} \). An algebraic proof of the above discussion is contained in [6] while a more geometric approach is covered in [13]. Finally, for a treatment of Reidemeister-Schreier rewriting which does not utilize covering space theory, refer to [10].

**Example 2.3.** As an example of the above procedure, consider the group

\[
G = \langle x, y, z \mid xyx = yxy, xzx = zxz \rangle
\]

with monodromy

\[
\Sigma_x = (12), \\
\Sigma_y = (23), \\
\Sigma_z = (23)
\]

It so happens that \( G \) is the knot group for both the square and granny knots. In fact, later in the paper it will be shown that these two knots are not the same using a related technique.

A Schreier tree is obtained by taking the edges labelled \( Y_xY_yY_x \) and \( Y_yY_xY_y \), while applying the three rewriting functions to the relators yield

\[
\begin{align*}
Y_xY_yY_x &= Y_yY_xY_y \\
1 & 2 3 \quad & 1 & 1 2 \\
2 & 1 1 \quad & 2 & 3 3 \\
3 & 3 2
\end{align*}
\]

and

\[
\begin{align*}
Y_xY_yY_x &= Y_yY_xY_y \\
1 & 2 3 \quad & 1 & 1 2 \\
2 & 1 1 \quad & 2 & 3 3 \\
3 & 3 2 \quad & 3 & 2 1
\end{align*}
\]

Where the equations above are shorthand notation. For example, the expansion of the first line would be \( Y_{x,1}Y_{y,2}Y_{x,3} = Y_{y,1}Y_{x,1}Y_{y,2} \). Performing a routine simplification and making the substitutions \( a = Y_{x,3}, b = Y_{y,3} \) and \( c = Y_{z,2} \) results in a presentation for the subgroup as

\[
H = \langle a, b, c \mid \begin{array}{l}
abab = baba \\
ca\bar{c}aba\bar{c} = ab\bar{a}c\bar{a}b\bar{a} \end{array} \rangle
\]

### 2.3 Branched Covers

Let \( M \) and \( N \) be \( n \)-dimensional manifolds.
Definition 2.3. A map \( p : M \to N \) is called a \textit{branched cover} if it is finite-to-one and open. A point \( x \) in \( M \) is called a \textit{regular point} if \( p \) is a local homeomorphism at \( x \). In other words, there is a neighbourhood \( U \subset M \) of \( x \) for which \( p|U \) is a homeomorphism. The set of non-regular points is called the \textit{branch cover}, while the image of the branch cover is the \textit{branch set}. Any point of \( N \) which is not in the branch set is called a \textit{regular value}. Equivalently, a point \( y \) in \( N \) is a regular value if and only if the pre-image of \( y \), \( p^{-1}(y) \), is disjoint from the branch cover.

It is easy to see that restricting \( p \) to the set of regular points gives an unbranched cover of the set of regular values (called the associated unbranched cover of \( p \)), hence all results related to covering spaces apply. In particular, \( |p^{-1}(y)| \) is locally constant for all regular values \( y \in N \).

The \textit{branching index} at a point \( \tilde{x} \) in the branch cover is the degree of the covering map for regular points when restricted to a neighbourhood of \( \tilde{x} \). A simple calculation shows this to be locally constant on connected components of the branch cover.

Let \( B^{n-2} \subset N^n \) be a locally flat codimension 2 submanifold. By a more general result of Fox [4], any unbranched cover \( p : M \to N \setminus B \) can be completed in a unique way to a branched cover \( \tilde{p} : \tilde{M} \to N \) where the branch set is \( B \). As well, if the branch cover is \( A \subset \tilde{M} \) then there is a covering isomorphism from \( M \) to \( \tilde{M} \setminus A \).

Example 2.4. Consider a 3-fold branched cover of the disc with branch set equal to two points. Note that the set of regular values deformation retracts onto the wedge of two circles, so let the associated unbranched cover be the one from Example 2.1. Completing the cover to the branch points gives Figure 2.2, a branched cover of a disc by a disc, with branch cover a set of 4 points, two of branch index 1 and two of branch index 2. Note that this is the same branched cover from [12, page 317]. The lines indicate where to cut the base space open, so as to fit the three pieces together according to the monodromy.

2.3.1 The Fundamental Group of a Branched Cover

There is a simple and effective method, due to Fox [4], for calculating the fundamental group of a branched covering space.

Let \( X \) be an \( n \)-dimensional manifold, with \( S \) an open subset. An element of \( \pi_1(X) \) is \textit{represented} in \( S \) if it is represented by a loop \( \alpha \gamma \alpha^{-1} \) where \( \gamma \) is a
Figure 2.2: The branched cover of Example 2.4.

loop in $S$ and $\alpha$ is a path in $X$ from the basepoint of the fundamental group to the basepoint of $\gamma$.

Let $B$ be a subcomplex of the triangulated manifold $X$ with triangulation $T$, denoted by $(X, T)$. If $v$ is any vertex of $B$ let $S(v)$ denote the set $(X - B) \cap \text{star}(v, T)$, and let $S = \bigcup_{v \in B} S(v)$.

**Lemma 2.8.** Let $(X, T)$ be a triangulated manifold and $B$ a codimension 2 submanifold which is a locally flat subcomplex. Then the inclusion homomorphism

$$\iota: \pi_1(X - B) \to \pi_1(X)$$

is onto and its kernel is the normal closure of those elements of $\pi_1(X - B)$ which are represented in $\bigcup_{v \in B} S(v)$.

Note that if $X$ is a closed 3-manifold, making $B$ a link, then for any vertex $v$ of $B$, $S(v)$ deformation retracts onto a meridian of the link.

**Proof.** Under the hypotheses, $S(v)$ is non-empty and connected for each vertex. Construct $R$, a one-dimensional subcomplex of $T$ which is a tree meeting each $S(v)$ at exactly one point. Hence $R \cup S$ is connected and the image of the inclusion map $\pi_1(R \cup S) \to \pi_1(X - B)$ is generated by elements which are represented in $S$. Furthermore, the inclusion $\pi_1(R \cup S) \to \pi_1(R \cup \bigcup_{v \in B} \text{star}(v))$
is certainly trivial as each star(v) is a ball. Finally, decompose X as \((X - B) \cup (R \cup \bigcup_v \text{star}(v))\) and apply van Kampen’s theorem.

Theorem 2.9. Under the hypotheses of the previous lemma, let \(p : \tilde{X} \to X\) be a branched cover with downstairs branch set \(B\). If \(H\) is the subgroup of \(\pi_1(X - B)\) corresponding to the associated unbranched cover, then \(\pi_1(\tilde{X})\) is isomorphic to \(H/N\) where \(N\) is the normal closure of those elements of \(H\) which are represented in \(\bigcup_{v \in B} S(v)\).

Proof. Let \(A \subset \tilde{X}\) be the preimage of \(B\). Note that \(A\) and \(\tilde{X}\) satisfy the hypotheses of the preceding lemma. Then there is a loop in \(\pi_1(\tilde{X} - A)\) covering an element \([\gamma]\) of \(\pi_1(X - B)\) if and only if \([\gamma]\) is in \(H\). Finally, an element of \(\pi_1(\tilde{X} - A)\) is represented in \(\bigcup_{u \in A} S(u)\) if and only if it covers an element of \(H\) which is represented in \(\bigcup_{v \in B} S(v)\).

2.3.2 Branched Covers of Knots

Consider \(K\), an oriented knot contained in an oriented 3-manifold \(X\). Let a regular neighbourhood of \(K\) be denoted \(N(K)\). The set of paths of \(\pi_1(X - K)\) represented in the boundary of the knot neighbourhood forms a subgroup, generated by a meridian–longitude pair of the knot.

If \(\Sigma : \pi_1(X - K) \to S_n\) is a transitive representation then consider the branched covering space \(\tilde{X}\) associated to it with downstairs branch set \(K\). If \(b \in X\) is the basepoint for the fundamental group then label the \(n\) points lying above \(b\) by \(\tilde{b}_1, \ldots, \tilde{b}_n\) where a loop \(\gamma\) based at \(b\) lifts to a path starting from \(\tilde{b}_i\) and ending at \(\tilde{b}_j\) if and only if \(\Sigma_\gamma(i) = j\). For simplicity’s sake, assume that \(b\) lies on \(\partial N(K)\).

Let \(\mu\) and \(\lambda\) be a meridian–longitude pair for \(K\).

Theorem 2.10. The number of components of the covering link \(L = p^{-1}(K)\) is the number of orbits of the points \(\{b_1, \ldots, b_n\}\) under the action of \(\Sigma_\mu\) and \(\Sigma_\lambda\).

Denote the orbit space to which the point \(\tilde{b}_i\) belongs by \(\text{Orbit}(\tilde{b}_i)\).

Proof. First let \(\tilde{b}_i, \tilde{b}_j\) belong to the same component, \(\partial N(\tilde{K}_k)\) of \(\partial N(\tilde{L})\). Hence there is a path \(\tilde{x}\) connecting \(\tilde{b}_i\) to \(\tilde{b}_j\) with \(\tilde{x} \subset \partial N(\tilde{K}_k)\). Then \(\tilde{x}\) projects to a loop on \(\partial N(K)\) hence is represented by some path \(\mu^s\lambda^t\) of \(\pi_1(\partial N(K))\). This proves that \(\tilde{b}_j\) is in \(\text{Orbit}(\tilde{b}_i)\).
For the opposite implication, let $\tilde{b}_j$ belong to Orbit($\tilde{b}_i$). Then there are integers $s, t$ for which $\mu^s \lambda^t$ lifts to a path connecting $\tilde{b}_i$ to $\tilde{b}_j$. Since any such path must lie on a single component of $\partial N(\tilde{L})$, this completes the proof. □

Proposition 2.11. The branching index of $\tilde{K}_j$, branch($\tilde{K}_j$), is the length of any cycle of $\Sigma_{\mu}$ contained in Orbit($\tilde{b}_j$). The covering index of $\tilde{K}_j$, deg($\tilde{K}_j \rightarrow K$) (defined to be the degree of the covering of $K$ by $\tilde{K}_j$) is given by the formula

$$\frac{|\text{Orbit}(\tilde{b}_j)|}{\text{branch}(\tilde{K}_j)}.$$

Proof. Refer to Figure 2.3 for an illustrative picture. Take a meridional disc $D$ with boundary $\mu$ for the downstairs knot $K$. The preimage of $D$ is a disjoint collection of discs $\tilde{D}_1, \ldots, \tilde{D}_r$. Assume the disc $\tilde{D}_j$ is pierced by the branch component $\tilde{K}_i$. Note that the restriction of the covering map $p : \tilde{D}_j \rightarrow D$ is a branched cover, and must have the same branch index as the associated component $\tilde{K}_i$. In particular, this implies that each cycle of $\Sigma_{\mu}$ must be the same length. The monodromy of $p : \tilde{D}_j \rightarrow D$ corresponds to the cycle $(i_1 i_2 i_3 \cdots)$ of $\Sigma_{\mu}$. In fact, the restricted map gives the branched cover of a disc, with a single point as the branch set and cyclic monodromy. Hence branch($\tilde{K}_j$) is precisely the length of the cycle $(i_1 i_2 i_3 \cdots)$ of $\Sigma_{\mu}$.
For the second half of the theorem, it is clear that the number of discs connected to a particular branch component is the \( \deg(\tilde{K}_i \to K) \). However, any point \( \tilde{b}_j \) lying on the boundary of \( N(\tilde{K}_i) \) will be on the boundary of such a disc. Since each disc has \( \text{branch}(\tilde{K}_i) \) distinct points on it, this completes the proof. 

We will use Theorem 2.9 to compute the fundamental group of the covering space. For a triangulation of \( X \) for which \( K \) is a subcomplex, any element represented in the set \( \bigcup_{v \in K} S(v) \) can be expressed as a product of meridians of \( K \). Furthermore, all meridians are conjugate to each other. Hence choose some meridian \( \mu \) of \( K \).

To compute the fundamental group of \( \tilde{X} \), we need a generating set for the meridians of \( \tilde{L} \). If \( \tilde{\mu}_i \) is a meridian of \( \tilde{K}_i \) and the point \( \tilde{b}_j \) is the lift of the basepoint to a point on \( \partial(N(\tilde{K}_i)) \) then \( \tilde{\mu}_i \) covers the element \( \mu^{m_j} \) down below, where \( m_j \) is the smallest integer for which \( \mu^{m_j} \) is in \( \text{Stab}_Z(j) \). In fact, \( \tilde{\mu}_i \) is conjugate to \( R_j(\mu^{m_j}) \). Hence one may set \( N = \{ R_j(\mu^{m_j}) \mid \tilde{b}_j \in W \} \) where \( W \) is a set with one representative from each orbit. Applying Theorem 2.9 results in:

\[
\pi_1(\tilde{X}) = \pi_1(X - K)/N
\]
Chapter 3
Linking Numbers

3.1 Some Basic Algebraic Topology

3.1.1 Manifolds and Simplicial Complexes

The following somewhat non-standard notation will be used.

Definition 3.1. The simplicial span of a set of linearly independent vectors \( \{v_i\} \) (all contained in some finite-dimensional real vector space) is the collection of points \( \sum t_i v_i \) with \( t_i \geq 0 \) and \( \sum t_i = 1 \). This is sometimes referred to as the convex hull of the vectors \( \{v_i\} \).

An \( n \)-simplex \( \Delta = [v_0 \cdots v_n] \) is the simplicial span of \( (n + 1) \) independent vectors \( v_0, \ldots, v_n \). A face of \( \Delta \) is the simplicial span of any subset of the vectors \( \{v_i\} \).

For example, Figure 3.1 shows the 2-simplex \( \Delta_2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 \mid t_i \geq 0, \sum t_i = 1\} \). It has seven faces: three of dimension 1 (corresponding to the edges of \( \Delta_2 \), three of dimension 0 (the vertices of \( \Delta_2 \)) and the empty face.

![Figure 3.1: A 2-simplex in \( \mathbb{R}^3 \).](image-url)
Note that permuting the vertices of $\Delta = [v_0 \cdots v_n]$ results in the same (set-wise equivalent) $n$-simplex; however, it reverses the orientation of the simplex if the permutation is odd. That is, an orientation reversing transformation is required to map one simplex onto the other. The simplex $-\Delta$ will refer to the simplex $\Delta$ with the opposite orientation, i.e. $- [v_0 \cdots v_n] = [v_1 v_0 \cdots v_n]$.

From an algebraic point of view, choosing an orientation for $\Delta$ is equivalent to choosing a generator for $H_n(\Delta, \partial \Delta) = H_n(D^n, \partial D^n) = \mathbb{Z}$.

For any $n$-simplex $\Delta = [v_0 \cdots v_n]$, define the boundary map by

$$\partial([v_0 \cdots v_n]) = \sum_{i=0}^{n}(-1)^i [v_0 \cdots \hat{v}_i \cdots v_n]$$

Definition 3.2. A simplicial complex $T$ is a collection of simplices in $\mathbb{R}^m$ (finite) satisfying the following properties.

(i) If $\Delta$ is a simplex in $T$, then all faces of $\Delta$ are in $T$.
(ii) If $\Delta_1, \Delta_2$ are both simplices in $T$ than $\Delta_1 \cap \Delta_2$ is a face of both simplices (possibly empty).
(iii) If $x$ is a point of $|T|$ then $x$ is contained in only finitely many simplices of $T$ ($|T| \subset \mathbb{R}^m$ is the set of all points belonging to at least one simplex of $T$).

A subcomplex of a simplicial complex $T$ is a subset of $T$ which is itself a complex. A subdivision $T'$ of $T$ is a complex with the properties that $|T'| = |T|$ and every simplex of $T'$ is contained in a simplex of $T$. One of the most useful subdivisions is the barycentric subdivision described below.

For an $n$-simplex $\Delta = [v_0 \cdots v_n]$ the barycentre of $\Delta$, denoted $c_\Delta$, is the point $\frac{1}{n+1} \sum v_i$. Starting from a complex $T$, call $T^{(1)}$ the barycentric subdivision of $T$. Inductively define the subdivision of an $n$-cell, $\Delta_n^{(1)}$ as follows.

The subdivision of a vertex is itself. Furthermore, the subdivision of an $n$-simplex $\Delta$ is the set of all simplices of the form

$$[w_0 \cdots w_{n-1} c_\Delta]$$

where $[w_0 \cdots w_{n-1}]$ is a simplex in the subdivision of a face of $\Delta$.

\footnote{The $\wedge$ symbol is a magic helmet which makes the wearer invisible.}
A somewhat cleaner description, as discussed in [3] is the following. First let the vertices of \( T^{(1)} \) be the set of barycentres \( c_\Delta \) where \( \Delta \) is a simplex of \( T \). Note that there is a partial ordering on the vertices of \( T^{(1)} \) by the rule \( c_\Delta < c_{\Delta'} \) when \( \Delta \) is a proper face of \( \Delta' \) (neither empty nor the entire simplex). Hence let vertices \( v_0, \ldots, v_p \) of \( T^{(1)} \) span a simplex exactly when \( v_0 < \cdots < v_p \).

Let \( \Delta = [v_0 \cdots v_n] \) be an \( n \)-simplex. A map

\[ f : \Delta \to \mathbb{R}^m \]

is linear provided \( f(\sum t_i v_i) = \sum t_i f(v_i) \) for all points in \( \Delta \). If \( S \) and \( T \) are complexes a map \( f : S \to T \) is simplicial provided it carries each simplex of \( S \) linearly into a simplex of \( T \), while the map \( f : S \to T \) is piecewise-linear if there are subdivisions \( S' \) of \( S \) and \( T' \) of \( T \) for which \( f : S' \to T' \) is simplicial.

A triangulation of a space \( X \) is a pair \( (T, h) \) where \( T \) is a simplicial complex and \( h : |T| \to X \) is a homeomorphism. Two triangulations \( (T, h), (S, g) \) are compatible provided \( g^{-1} \circ h : |T| \to |S| \) is piecewise linear.

If \( x \) is a point in \( |T| \) then the star of \( x \) in \( T \), \( \text{star}(x, T) \) is the union of the interiors of all simplices which contain \( x \). The link of \( x \) in \( T \), \( \text{link}(x, T) \) is the union of all simplices which are disjoint from \( x \) but which are a face of a simplex containing \( x \).

Definition 3.3. An \( n \)-dimensional manifold \( M \) is a Hausdorff space with every point in \( M \) having a neighbourhood homeomorphic to an open set in \( \mathbb{R}^{n,+} = \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n \geq 0\} \). In other words, if \( x \) is a point of \( M \), then there is an open neighbourhood \( U \subset M \) of \( x \) and a homeomorphism \( h : U \to V \subset \mathbb{R}^{n,+} \). The set of points which are mapped by a homeomorphism to the plane \( y_n = 0 \) is the boundary of \( M \), denoted \( \partial M \) and by invariance of domain does not depend on the homeomorphism chosen. Furthermore \( \partial M \) is an \((n-1)\)-dimensional manifold without boundary.

A triangulated \( n \)-manifold \( (M, T) \) with boundary \( \partial M \) is said to be oriented if an orientation is chosen for each of the \( n \)-cells \( \{\Delta_n^i\} \in T \) so that \( \sum \partial \Delta_n^i \subset \partial M \). In other words, an orientation can be chosen for each \( n \)-simplex so that if two simplices meet at an \((n-1)\) dimensional face then the two orientations inherited from each simplex are opposite on the face.

Further, the triangulation is combinatorial provided the link of every vertex is a PL \( n-1 \) sphere for points in the interior and a PL \( n-1 \) disc for points on
the boundary. For the remainder of the paper assume that any triangulation of a manifold is combinatorial.

Given a ring $R$, any triangulation $T$ of $M$ exhibits a chain complex

$$\delta : C_p((M,T); R) \rightarrow C_{p-1}((M,T); R) \rightarrow \cdots$$

where $C_p((M,T); R)$ is the free $R$-module with generators the set of $n$-simplices of $T$.

Furthermore, one has a co-chain complex obtained by dualizing the chain complex:

$$\delta : C^p((M,T); R) \rightarrow C^{p-1}((M,T); R) \rightarrow \cdots$$

where as usual $C^p((M,T); R) = \text{Hom}_R(C_p((M,T); R), R)$ is the set of $R$-linear maps into $R$ and $\delta$ is defined by the identity

$$\delta \mu^p(v_{p+1}) = \mu^p(\partial v_{p+1})$$

for all $p$-cochains $\mu^p$ and $(p+1)$-chains $v_{p+1}$.

Call $Z_n((M,T); R) = \ker \partial$ the set of $n$-cycles, $B_n((M,T); R) = \text{Im} \partial$ the $n$-boundaries, $Z^n((M,T); R) = \ker \delta$ cocycles and $B^n((M,T); R) = \text{Im} \delta$ coboundaries.

Since $\partial \partial = 0 = \delta \delta$ one can define the homology groups of $M$ by

$$H_n(M; R) = Z_n/B_n$$

and the cohomology groups by

$$H^n(M; R) = Z^n/B^n.$$  

Note that the triangulation $T$ is suppressed from the notation since by a basic result of homology theory (c.f. [7]) the homology and cohomology groups of a manifold are independent of the triangulation chosen.

For the remainder of the paper, we will restrict our attention to homology and cohomology with rational coefficients (i.e $R = \mathbb{Q}$) and the abbreviation $C_p(M,T)$ (resp. $C^p(M,T)$) will mean $C_p((M,T); \mathbb{Q})$ (resp. $C^p((M,T); \mathbb{Q})$).

Note that as dual $\mathbb{Q}$-modules, $C_p(M,T)$ and $C^p(M,T)$ are isomorphic via the map

$$\xi : C_p(M,T) \rightarrow C^p(M,T)$$
defined on the basis elements by \( \xi(\Delta^i) = \sigma_i \) where

\[
\sigma_i(\Delta^j) = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]

In fact, \( \{\sigma_i\} \) can be regarded as a basis for \( C^p(M,T) \). This allows for the definition of a geometric coboundary map \( \delta(g) : C_n(M,T) \to C_{n+1}(M,T) \)
(denoted \( \delta \) when there is no ambiguity).

\[
\delta(g) = \delta \circ \xi
\]

As well, one can use \( \xi \) to define a symmetric bilinear inner product which is easily seen to be orthonormal on the basis of \( C_n(M,T) \). Given two \( n \)-chains \( u \) and \( v \) define \( \langle u, v \rangle \) by:

\[
\langle u, v \rangle = (\xi u)(v)
\]

### 3.1.2 Poincaré Duality

Now consider an oriented, closed manifold \( M^n \) with a finite triangulation \( T \). One may construct the dual triangulation \( T^* \) of \( M \) as follows.

Let \( T^{(1)} \) be the barycentric subdivision of \( T \). For every \( p \)-cell \( \Delta_p \in T \), there is a corresponding \( (n - p) \)-cell in \( T^* \) obtained by taking the union of all cells in \( T^{(1)} \) of the form \([c_{\Delta} v_1 \cdot \cdot \cdot v_r]\). Recall that \( c_{\Delta} \) is the barycentre of \( \Delta \) and if \( v_i \) is the barycentre of the simplex \( \Delta_i \in T \) then \( \Delta \) is a proper face of \( \Delta_1 \) and each \( \Delta_i \) is a proper face of \( \Delta_{i+1} \). In fact, the dual cell is the join of the barycentre with its link, \( D(\Delta) = [c_{\Delta} \text{link}(\Delta, T^{(1)})] \), and hence is an \( n - p \) cell.

If one proceeds with a little more care, there is a natural way to orient \( (M,T^*) \) so it is compatible with the orientation of \( (M,T) \). Compatibility in this case means that the following diagram commutes.

\[
\begin{array}{ccc}
C_p(M,T) \xrightarrow{\delta(g)} & C_{p-1}(M,T) \xrightarrow{\delta(g)} & C_{p-2}(M,T) \\
\downarrow \delta & \quad & \downarrow \delta \\
C_{n-p}(M,T^*) \xleftarrow{\partial} & C_{n-p-1}(M,T^*) \xleftarrow{\partial}
\end{array}
\]

And one concludes that \( D_* : H^p(M) \to H_{n-p}(M) \) is an isomorphism. Consult [3] for a proof. Furthermore, by considering the above diagram, as well as
the dual diagram (i.e. reverse all arrows and change boundary to co-boundary and vice-versa) one obtains the following relations:
\[ \mathcal{D} \partial u = \delta^{(g)} \mathcal{D} u \]
\[ \mathcal{D} \delta^{(g)} u = \partial \mathcal{D} u \]
as well, it is straightforward to show that
\[ \langle u, v \rangle = \langle \mathcal{D} u, \mathcal{D} v \rangle \]

3.2 Linking Numbers

The following discussion is borrowed mainly from Murasugi and Hartley [6]. Let \( M \) be a closed 3-manifold and \( T \) a finite triangulation. Consider \( L \subset M \) a link of \( r \) components, \( L = K_1 \cup \cdots \cup K_r \), which is a subcomplex of \( T \). If necessary, replace \( T \) with a subdivision so that a simplicial neighbourhood of \( L \) is also a regular neighbourhood (in particular it contains an open neighbourhood containing \( L \)). With this condition, the dual complex of \( L, N(L) \), is a regular open neighbourhood of \( L \).

3.2.1 Intersection Forms

Definition 3.4. Let \( u \) be any 2-chain in \( C_2(M,T) \) and \( v^* \) a 1-chain in \( C_1(M,T^*) \). Define the intersection number, \( \text{int}(u,v^*) \) by
\[ \text{int}(u,v^*) = \langle \mathcal{D} u, v^* \rangle \]
Note that if \( u_p \) is in \( C_p(L, T) \) and \( v^*_{3-p} \) is contained in \( C_{3-p}(M - N(L), T^*) \) then
\[
\langle D u_p, v^*_{3-p} \rangle = 0
\]  
(3.4)

**Proposition 3.1.** The intersection form \( \text{int} \) induces a map

\[
\text{int}: H_2(M, L) \times H_1(M - N(L)) \rightarrow \mathbb{Q}
\]

**Proof.** First choose two representatives \( z_2, z_2' \in Z_2((M, L), T) \) of a class in \( H_2(M, L) \). This occurs when \( z_2 - z_2' = \partial u_3 + u_2 \) where \( u_3 \) is in \( C_3(M, T) \) and \( u_2 \) lies in \( C_2(L, T) \). For any \( z_1^* \) in \( Z_1(M - N(K), T^*) \) we have

\[
\text{int}(u_2, z_1^*) = \langle D u_2, z_1^* \rangle = 0 \quad \text{by (3.4)}
\]

and

\[
\begin{align*}
\text{int}(\partial u_3, z_1^*) &= \langle D \partial u_3, z_1^* \rangle \\
&= \langle \delta D u_3, z_1^* \rangle \quad \text{by (3.1)} \\
&= \langle D u_3, \partial z_1^* \rangle \\
&= 0 \quad \text{since } z_1^* \in \ker \partial
\end{align*}
\]

Now consider \( z_1^*, z_1^* \in Z_1(M - N(L), T^*) \) two representatives of a homology class. Hence \( z_1^* - z_1^* = \partial v_2^* \) with \( v_2^* \) a chain in \( C_2(M - N(L), T^*) \). Then

\[
\begin{align*}
\text{int}(z_2, \partial v_2^*) &= \langle D z_2, \partial v_2^* \rangle \\
&= \langle \delta D z_2, v_2^* \rangle \\
&= \langle D \partial z_2, v_2^* \rangle = 0 \quad \text{by (3.4) since } \partial z_2 \in C_1(L)
\end{align*}
\]

This completes the proof. \( \square \)

**Proposition 3.2.** Let \( \theta : H_1(M - N(L)) \rightarrow \mathbb{Q} \) be a homomorphism. Then there exists \( \alpha \in H_2(M, L) \) such that for any \( \beta \in H_1(M - N(L)) \), \( \theta(\beta) = \text{int}(\alpha, \beta) \).

**Proof.** Given \( \theta \), there is a homomorphism \( \theta' : Z_1(M - N(L), T^*) \rightarrow \mathbb{Q} \) such that \( \theta'(z_1^*) = \theta([z_1^*]) \). Since \( C_1(M - N(K), T^*) \) is a free \( \mathbb{Q} \)-module, \( Z_1(M - N(L), T^*) \) is a direct summand and so \( \theta' \) can be extended to a homomorphism \( \theta'' \) on \( C_1(M - N(L), T^*) \).
Chapter 3. Linking Numbers

Let \( b_1^* = \sum \theta''(\Delta^1) \Delta^1 \) where the sum is taken over all 1-simplices \( \{\Delta^1\} \) in \( T^* \). If \( a_2 = D^{-1}b_1^* \) then it is easy to see that

\[
\int(a_2, z_1^*) = \theta''(z_1^*)
\]

for all \( z_1^* \) in \( Z_1(M - N(K), T^*) \).

There remains to prove that \( a_2 \) is an element of \( Z_2((M, L), T) \). In other words, that \( \partial a_2 \in C_1(L, T) \). For any basis simplex \( \Delta_1 \) in \( C_1(M, T) \) we have:

\[
\partial a_2 = \sum_{\Delta_1 \in C_1(M)} \theta((\partial D \Delta_1)) \Delta_1
\]

And so \( \partial a_2 = \sum_{\Delta_1 \in C_1(L)} \theta((\partial D \Delta_1)) \Delta_1 \). However, for any \( \Delta_1 \) not in \( L, D \partial \Delta_1 \) belongs to \( C_2(M - N(L), T^*) \) hence \( \partial D \Delta_1 \), being a boundary in \( M - N(L) \), is homologous to zero. In fact,

\[
\partial a_2 = \sum_{\Delta_1 \in C_1(L)} \theta((\partial D \Delta_1)) \Delta_1
\]

which means \( a_2 \) is in \( Z_2((M, L), T) \). Setting \( \alpha = [a_2] \in H_2(M, L) \) completes the proof.

Note. For \( K_i \) a component of the link \( L, \) if \( \Delta_1 \) is a 1-cell of \( K_i \) then \( \partial D \Delta_1 \) is homologous to zero in \( N(K_i) \) but not on the boundary of the knot neighbourhood, \( \partial N(K_i) \). The first statement following from the fact that \( D \Delta_1 \) is a 2-cell contained in \( N(K_i) \), while the second is a result of \( D \Delta_1 \) not separating \( N(K_i) \) into two pieces. These are the two requirements for a meridian curve of \( K_i \), any two of which are homologous. Applying this result to (3.5), the formula for \( \alpha \) in the above proposition, we arrive at the result:

\[
\partial a_2 = \sum_{K_i \in L} \theta(\mu_i)|K_i|
\]

where \( \mu_i \) is any meridian of \( K_i \) and |\( K_i | \) is the sum of the 1-simplices which make up \( K_i \).
3.2.2 Linking

Let \((M, P)\) be a pair (i.e. \(P \subset M\)). Then define the set of linkable cycles of \(P\), \(\Lambda(P)\), to be the set of homology classes in \(H_1(P)\) which are trivial in \(H_1(M)\). In other words, a 1-cycle \(\alpha \in H_1(P)\) is linkable if and only if there is a 2-chain in \(M\) with boundary \(\alpha\). Note that \(\Lambda(P)\) is the kernel of the inclusion map \(i_*\) arising from the exact sequence of the pair \((M, P)\):

\[
\cdots \rightarrow H_2(M, P) \xrightarrow{\partial} H_1(P) \xrightarrow{i_*} H_1(M) \rightarrow 0
\]

Definition 3.5. The linking number \(\text{link} : \Lambda(L) \times \Lambda(M - L) \rightarrow \mathbb{Q}\) is defined by

\[
\text{link}(\alpha, \beta) = \int(u_2, v_1^*)
\]

where \(u_2\) is a 2-cycle in \(Z_2(M, L)\) with \([\partial u_2] = \alpha\) and \(v_1^*\) is a representative of \(\beta\) in \(Z_1(M - N(L), T^*)\).

One must confirm that the definition does not depend on the particular choice of \(u_2\). Since \(\beta\) is linkable in \(M - K\), \(v_1^*\) can be written as \(\partial v_2^*\) for some \(v_2 \in C_2(M, T^*)\). Then

\[
\int(u_2, v_1^*) = \langle D u_2, \partial v_2^* \rangle = \langle D \partial u_2, v_2^* \rangle
\]

which only depends on \(\partial u_2\) as required.

The following theorem restates the above results in a concise way:

**Theorem 3.3.** Let \(L\) be an oriented link of \(r\) components \(K_1, \ldots, K_r\) in \(M\), a closed and oriented 3-manifold, and let \(\mu_i\) be a meridian of \(K_i\). Let \(\theta : H_1(M - L) \rightarrow \mathbb{Q}\) be a homomorphism and \(\alpha = \sum \theta(\mu_i) K_i\). Then \(\alpha\) is linkable (i.e. belongs to \(\Lambda(K)\)) and for any \(\beta\) in the knot complement which is linkable,

\[
\text{link}(\alpha, \beta) = \theta(\beta)
\]

3.3 Branched Covers and Linking Numbers

Let \(p : M \rightarrow S^3\) be an \(n\)-fold branched cover with branch set a knot \(K \subset S^3\). In general, the branch cover upstairs will be a link \(\tilde{L}\) of \(r\) components \(\tilde{K}_1, \ldots, \tilde{K}_r\). Using the results of the previous chapter, as well as Theorem 3.3 from the
current chapter, there is a straightforward method for calculating the linking numbers between knots of the branch cover, \( \text{link}(\tilde{K}_i, \tilde{K}_j) \).

Let \( \mu \) be a meridian of \( K \) and \( \lambda \) a longitude. Denote \( l_k \) to be the smallest positive integer for which \( \lambda^{l_k} \in \text{Stab}_C(k) \) and \( m_k \) the smallest positive integer for which \( \mu^{m_k} \in \text{Stab}_C(k) \). Hence they lift to closed curves based at \( \tilde{b}_k \). In fact, they are the lengths of the cycles of \( \Sigma_\lambda \) and \( \Sigma_\mu \) respectively which contain \( k \).

Recall that \( \mathcal{R}_f(\gamma) \) is the lift of \( \gamma \) to the path based at \( \tilde{b}_j \).

**Proposition 3.4.** If \( \text{Orbit}(\tilde{b}_k) \subset \partial N(\tilde{K}_j) \) then the lift of \( \lambda^{l_k} \) to the basepoint \( \tilde{b}_k \) is a loop homologous to \( cK_j \) where \( c = \frac{l_k}{\text{deg}(\tilde{K}_j \to K)} \).

**Proof.** \( \tilde{K}_j \) covers \( K \) with a multiplicity of \( \text{deg}(\tilde{K}_j \to K) \), and the lift of \( \lambda^{l_k} \) covers \( \lambda \) with a multiplicity of \( l_k \). Since \( \lambda \) and \( K \) are homologous it folows that \( \text{deg}(\tilde{K}_j \to K) \cdot \mathcal{R}_k(\lambda^{l_k}) \) is homologous to \( l_k \cdot \tilde{K}_j \). \( \square \)

In this case, we can declare

\[
\text{link}(\tilde{K}_i, \tilde{K}_j) = \frac{\text{deg}(\tilde{K}_j \to K)}{l_k} \text{link}(\tilde{K}_i, \mathcal{R}_k(\lambda^{l_k}))
\]

where \( k \) is such that \( \tilde{b}_k \) lies on \( \partial N(\tilde{K}_j) \).

Further note that as per the discussion from Section 2.3.2, a meridian \( \tilde{\mu}_i \) of the branch link component is the lift \( \mathcal{R}_j(\mu^{m_j}) \) where \( \tilde{b}_j \) is a point on \( \partial N(\tilde{K}_i) \). The goal is to construct a homomorphism \( \theta_i : H_1(M - \tilde{L}) \to \mathbb{Q} \) satisfying the relations

\[
\theta_i(\tilde{\mu}_i) = 1 \\
\theta_i(\tilde{\mu}_j) = 0 \quad \text{if } i \neq j
\]

In other words, a linking function for the loop \( \tilde{K}_i \).

However, from the preceding discussion, as well as the techniques of Reide-meister-Schreier rewriting, finding a relation matrix for \( H_1(M - \tilde{L}) \) is a straightforward matter of finding a presentation for a stabilizer subgroup of the monodromy and then abelianizing. Moreover, the meridians of \( \tilde{L} \) are easy to find: \( \tilde{\mu}_i = \mathcal{R}_j(\mu^{m_j}) \). The result being that calculating a linking homomorphism is equivalent to solving a system of linear equations.

If one is working with a representation where the longitude of the downstairs knot lies in the kernel of the monodromy, which covers a significant
Chapter 3. Linking Numbers

Figure 3.3: The square knot (left) and granny knot (right).

portion of the cases studied in the literature, then the linking number formula becomes much simpler:

$$\text{link}(K_i, K_j) = \text{link}(\tilde{K}_i, \mathcal{R}_j(\lambda))$$

Example 3.1. As an illustration of the techniques covered in this chapter, it will be shown that the square and granny knots, as pictured in Figure 3.3, are distinct.

The knots have identical groups

$$G = \langle x, y, z \mid xy = yx, xzx = zzx \rangle$$

for which there are four distinct representations onto $S_3$:

$$\begin{align*}
\phi_1(x) &= (23) & \phi_2(x) &= (23) & \phi_3(x) &= (23) & \phi_4(x) &= (23) \\
\phi_1(y) &= (13) & \phi_2(y) &= (23) & \phi_3(y) &= (12) & \phi_4(y) &= (13) \\
\phi_1(z) &= (13) & \phi_2(z) &= (13) & \phi_3(z) &= (13) & \phi_4(z) &= (23)
\end{align*}$$

The longitudes of each knot are:

$$\begin{align*}
L_{sq} &= z^2x^2yxyxy \\
L_{gr} &= z^2x^2yxyxyx^6
\end{align*}$$

and it is easily verified that they are represented by a trivial permutation for every monodromy listed above. Note the use of the notation $\bar{x} = x^{-1}$.

Restricting our attention to $\phi_1$, consider the meridian $x$ for each knot. Since $\phi_1(x)$ has two orbits, there will be two knots in the branch link upstairs, the first of index one, the other of index two. The rewriting process for $\text{Stab}_{\phi_1}(1)$ is the same for both knots:
A Schreier tree is obtained by making the generators $Y_{x,3}$ and $Y_{y,1}$ trivial. Following the procedure and notation of Example 2.3, the other relators are given by the equations

$$Y_y Y_x = Y_x Y_y Y_x$$
$$Y_z Y_x Y_z = Y_x Y_z Y_x$$

$$R_{11} : 1 \ 3 \ 2 \quad 1 \ 1 \ 3 \quad (**) \quad and \quad R_{21} : 1 \ 3 \ 2 \quad 1 \ 1 \ 3 \quad (**)$$
$$R_{12} : 2 \ 2 \ 3 \quad 2 \ 3 \ 1$$
$$R_{13} : 3 \ 1 \ 1 \quad 3 \ 2 \ 2$$

Abelianizing the starred relators implies $Y_{x,1} = Y_{y,2} = Y_{z,2}$. Hence a relation matrix for the upstairs link complement is given by

$$\begin{array}{cccccc}
Y_{x,1} & Y_{x,2} & Y_{z,1} & Y_{z,3} & Y_{y,3} \\
R_{12} & 0 & 0 & 0 & 0 & 0 \\
R_{13} & 0 & -1 & 0 & 0 & 1 \\
R_{22} & 0 & 0 & 0 & 0 & 0 \\
R_{23} & 0 & -1 & 1 & 1 & 0 \\
\end{array}$$

Finally, to compute a linking function $\theta$ for the component of index 1, set $\theta(Y_{x,1}) = 1$ and $\theta(Y_{x,2}) = 0$. Solving the resulting system of linear equations gives

$$\begin{align*}
\theta(Y_{x,1}) &= 1 \\
\theta(Y_{y,1}) &= 0 \\
\theta(Y_{x,2}) &= 0 \\
\theta(Y_{y,2}) &= 1 \\
\theta(Y_{x,3}) &= 0 \\
\theta(Y_{y,3}) &= 0 \\
\theta(Y_{z,2}) &= 1 \\
\theta(Y_{z,3}) &= -1
\end{align*}$$

The linking between the two components if given by the formula $\theta(R_2(L))$ which is:

$$\begin{align*}
\hat{Y}_2 \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \\
R_2(L_{sq}) : 2 \ 3 \ 1 \ 3 \ 3 \ 1 \ 3 \ 3 \ 2 \\
\theta(R_2(L_{sq})) : -1 \ 0 \ -1 \ -1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \\
\end{align*}$$

and

$$\begin{align*}
Y_x Y_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \hat{Y}_y \hat{Y}_x \\
R_2(L_{gr}) : 2 \ 2 \ 1 \ 1 \ 1 \ 3 \ 1 \ 3 \ 3 \ 2 \ 3 \ 2 \ 3 \ 2 \ 3 \ 2 \\
\theta(R_2(L_{gr})) : 1 \ 0 \ -1 \ -1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0
\end{align*}$$
Chapter 3. Linking Numbers

The result is a linking number of 0 for the square knot and 4 for the granny. Performing the same operation for the other three representations gives a linking signature of \((0, -2, 0, 2)\) for the square knot and \((4, 2, 4, 2)\) for the granny. In other words, they cannot be the same knot. Note further that an orientation reversing ambient isotopy will reverse the linking numbers when lifted to the cover. Hence the granny knot cannot be amphicheiral as no negative linking numbers appear in the signature. Furthermore, no such obstruction is present in the linking numbers of the square knot, which is most certainly amphicheiral by a reflection across the two trefoils.
4.1 The Basic Construction

Let $L = K_1 \cup \ldots \cup K_r$ be a link of $r$ components contained in a closed 3-manifold $M$. There is a technique for constructing a new manifold $\tilde{M}$ by performing Dehn surgery along the link $L$. The construction proceeds as follows. Choose disjoint tubular neighbourhoods $N(K_i)$ for each component of the link. Let $N \subset M$ be the union of these neighbourhoods. The manifold $M \setminus N$ has boundary homeomorphic to $r$ disjoint tori. Consider $W = V_1 \cup \ldots \cup V_r$, $r$ copies of the solid torus $D^2 \times S^1$. Hence for any homeomorphism

$$h : \partial W \to \partial M \setminus N$$

one can form a new manifold $\tilde{M} = (M \setminus N) \cup_h W$ by gluing the two manifolds together along their boundaries via the homeomorphism $h$.

Let $K \subset M^3$ be a knot with $V \subset M$ a tubular neighbourhood of $K$ and let

$$\tilde{M}_1 = (M \setminus N(K)) \bigcup_{h_1} V$$

and

$$\tilde{M}_2 = (M \setminus N(K)) \bigcup_{h_2} V$$

be manifolds resulting from Dehn surgery along $K$, with attaching maps $h_1$ and $h_2$ respectively. There is a simple classification of when $\tilde{M}_1$ will be homeomorphic to $\tilde{M}_2$. Let $[m] \in H_1(\partial V)$ be the homology class of a meridional curve
and \([\mu], [\lambda] \in H_1(\partial N(K))\) the meridian-longitude basis for the knot neighbourhood. If \(h_i^*\) is the homology homomorphism induced by the map \(h_i\) then

\[
\begin{align*}
\lambda_1^*([m]) &= a_1[\mu] + b_1[\lambda] \\
\lambda_2^*([m]) &= a_2[\mu] + b_2[\lambda]
\end{align*}
\]

for some integers \(a_1, b_1, a_2, b_2\). The rational number \(\frac{a_1}{b_1}\) associated with the attaching map \(h_1\) is called the surgery coefficient.

**Theorem 4.1.** If \(\frac{a_1}{b_1} = \frac{a_2}{b_2}\) then \(\hat{M}_1\) is homeomorphic to \(\hat{M}_2\). In other words, it suffices to specify the surgery coefficient to determine the homeomorphism type of a manifold obtained as Dehn surgery along a knot.

**Proof.** Consider \(\hat{M}_i\) as its two pieces, \(M \setminus N(K)\) and \(V_i\). A homeomorphism \(\hat{M}_1 \rightarrow \hat{M}_2\) is constructed by taking the identity map on \(M \setminus N(K)\) and \(h_2^{-1} \circ h_1\) on \(\partial(V_i)\). The homeomorphism can be extended throughout the interior of the solid torus if and only if \(h_2^{-1} \circ h_1\) takes a meridian to a meridian. Since this is certainly true when the surgery coefficients match, the homeomorphism can be completed. This finishes the proof. \(\square\)

The above theorem can be easily extended to links of more than one component with only a slight increase in notational complexity. In practice, one normally starts from the 3-sphere and describes a manifold by drawing a link diagram with each component of the link labelled with a rational number (or \(\infty = \frac{1}{0}\) for trivial surgery).

**Example 4.1.** Performing surgery on the knot on the left in Figure 4.1 yields the lens space \(L(a, b)\) while the right hand diagram gives the Poincaré homology sphere, \(P^3\). Since the two knots can be separated by a ball, performing the two surgeries simultaneously would result in a connected sum \(L(a, b) \# P^3\).
For more details about Dehn surgery, including allowable transformations of links and surgery coefficients, refer to [12].

### 4.2 Homology Calculations

There is a straightforward method for calculating the first homology group of a manifold obtained by doing surgery on a link in the 3-sphere. Let \( L = K_1 \cup \cdots \cup K_r \) be the link in question with \( a_i \) the surgery coefficient for the component \( K_i \). As well, specify an orientation for each of the knots. In which case we can define the linking matrix, \([l_{ij}]\) for this link by

\[
l_{ij} = \begin{cases} 
0 & \text{if } i = j \\
l(\langle K_i, K_j \rangle) & \text{if } i \neq j
\end{cases}
\]

By Alexander duality, the manifold \( X = S^3 \setminus N(L) \) has first Betti number equal to \( r \), the number of components in the link. In fact \( H_1(S^3 \setminus N(L)) \) is a free abelian group of rank \( r \) and is generated by the meridians of the link, \( \{\mu_i\} \) (\( \mu_i \) is a meridian of the knot \( K_i \)). To form the surgered manifold \( M \), the solid tori are sewn to \( \partial X \) in a two stage process. First a disc whose boundary is a meridian is sewn to the image of a meridian by the attaching map. That is, to a simple closed curve which wraps \( a_i \) times around the meridian and \( b_i \) times around the longitude. The second stage is to sew on the remainder of the solid torus, which is homeomorphic to a 3-ball. The second step does not affect 1-dimensional homology, while the first step makes the homology class \( a_i[\mu_i] + b_i[\lambda_i] \) homologous to zero.

By examining a link projection, it is apparent that

\[
[\lambda_i] = \sum_{i=1}^{r} l_{ij} [\mu_j]
\]

(Recall that \( \text{link}(\lambda_i, K_j) = \text{link}(K_i, K_j) = l_{ij} \) if \( i \neq j \) and longitudes are chosen so that \( \text{link}(\lambda_i, K_i) = 0 = l_{ii} \).)

Hence one arrives at a relation matrix \( A = [\alpha_{ij}] \) for \( H_1(M) \) with

\[
\alpha_{ii} = a_i \\
\alpha_{ij} = b_i l_{ij} \quad (i \neq j)
\]

Note that \( M \) is a homology 3-sphere if and only if \( \det(A) = \pm 1 \).
Example 4.2. Consider the collection of knots shown in Figure 4.2 with surgery coefficients as indicated. Performing the specified Dehn surgery results in a manifold $M$ with relation matrix for $H_1(M)$:

$$A = \begin{bmatrix} a & b & b & -b \\ b & a & -b & b \\ d & -d & c & d \\ -d & d & d & c \end{bmatrix}$$

The first homology group of $M$ has order $|\det(A)| = |(d + c)(b + a)(ad - ac + 3bd + bc)|$ and $M$ is a homology sphere precisely when $\det(A) = \pm 1$. This has many nontrivial solutions. For example, Maple computes one solution set to be

$$a = d$$
$$b = 1 - d$$
$$c = 1 - d$$

with $d$ an arbitrary integer.

4.3 Linking Number Calculations

Let $\hat{L} \subset M^3$ be a link of $q$ components, $\hat{L} = \hat{K}_1 \cup \cdots \cup \hat{K}_q$, with $\hat{\mu}_i$ a meridian and $\hat{\lambda}_i$ a longitude of $\hat{K}_i$. From Chapter 3, recall that one method of calculating
link($\tilde{K}_i, \tilde{K}_j$) is to find a homomorphism

$$\theta_i : H_1(M \setminus \tilde{L}) \rightarrow \mathbb{Q}$$

with the property that

$$\theta_i(\mu_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

For such a homomorphism (termed a linking function for $\tilde{K}_i$), one has the simple formula $\text{link}(\tilde{K}_i, \tilde{K}_j) = \theta_i(\lambda_j)$.

Now consider the situation where the 3-manifold $M$ is constructed from the 3-sphere by performing Dehn surgery along a link $L$ of $r$ components. Following the notation from last section, the $i$-th link component is labelled $K_i$ with surgery coefficient $a_i/b_i$ and the meridian-longitude basis of the knot neighbourhood is $(\mu_i, \lambda_i)$. Hence the relation matrix for $H_1(M)$ is $A = [\alpha_{ij}]$ as per Equation 4.1. If one forms the matrix $B = [\beta_{ij}]$ by the rule

$$\beta_{ij} = \text{link}(K_i, K_j)a_i b_i$$

then the same argument as before makes the matrix $[A \mid B]$ the relation matrix for the group $H_1(M \setminus \tilde{L})$. Finding a linking function $\theta_i$ for the knot $\tilde{K}_i$ is then equivalent to solving the system of $r$ linear equations:

$$\sum_{j=1}^{r} \alpha_{kj} \theta_i(\mu_j) = -\beta_{ki} \quad (k = 1, \ldots, r)$$

In other words, if $A^{-1} = [a_{ij}]$ then

$$\theta_i(\mu_j) = \sum_{k=1}^{r} a_{ik} \beta_{kj}$$

alternatively stated, if

$$\Theta_i = \begin{bmatrix} \theta_i(\mu_1) \\ \vdots \\ \theta_i(\mu_r) \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} \beta_{i1} \\ \vdots \\ \beta_{ri} \end{bmatrix}$$

then

$$\Theta_i = -A^{-1}B_i$$
Finally one can conclude that

$$\text{link}_M(\bar{K}_i, \bar{K}_j) = \theta_i(\lambda_j) = \left( \sum_{k=1}^{r} \text{link}_{S^3}(K_k, \bar{K}_j)\theta_i(\mu_k) \right) + \text{link}_{S^3}(\bar{K}_i, \bar{K}_j)$$

Note that in this formula $\text{link}_{S^3}(K_k, \bar{K}_j)$ and $\text{link}_{S^3}(\bar{K}_i, \bar{K}_j)$ are the linking numbers of the curves before performing surgery, while $\text{link}_M(\bar{K}_i, \bar{K}_j)$ is the linking number after the operation.

**Example 4.3.** Consider the manifold $M$ obtained by performing the surgery as indicated in Figure 4.2. It was shown in Example 4.2 that $M$ is a homology sphere exactly when

$$(a + b)(c + d)(ad - ac + bc + 3bd) = \pm 1$$  \hspace{1cm} (4.2)

If one considers a link $\bar{L} = \bar{K}_1 \cup \cdots \cup \bar{K}_n$ with

$$\text{link}(K_1, \bar{K}_i) = \text{link}(K_2, \bar{K}_i) = 1$$
$$\text{link}(K_3, \bar{K}_i) = \text{link}(K_4, \bar{K}_i) = -1$$
$$\text{link}(\bar{K}_i, \bar{K}_j) = 1$$

then the relation matrix for $M - \bar{L}$ is

$$\begin{bmatrix}
a & b & b & -b & b & \cdots & b \\
b & a & -b & b & b & \cdots & b \\
d & d & c & d & -d & \cdots & -d \\
-d & d & c & -d & \cdots & -d
\end{bmatrix}$$

And one calculates

$$\Theta_i = \begin{bmatrix}
-\frac{b}{a+b} \\
-\frac{a+b}{a+b} \\
\frac{c+d}{c+d} \\
\frac{c+d}{c+d}
\end{bmatrix}$$

for $1 \leq i \leq n$

so that

$$\text{link}_M(\bar{K}_i, \bar{K}_j) = -\frac{ad - ac + bc + 3bd}{(c + d)(a + b)}$$  \hspace{1cm} (4.3)

If one requires $M$ to be a homology sphere then by comparing Equation 4.2 with Equation 4.3 it is clear that the linking numbers will always be $\pm 1$!

This example is interesting because it comes up when examining the linear dihedral action on $S^3$. $\bar{L}$ is the fixed point set of the action while the surgery link is left invariant by the action.
Chapter 5

Results

Using the techniques of the preceding chapters, two distinct methods are presented for finding examples of exotic group actions on homology 3-spheres. Recall that an action is exotic if the linking numbers of the fixed points differ from those of a standard linear action, that is, are different from $\pm 1$.

The first method starts from a standard linear action on $S^3$. Surgery is performed on a link which is invariant under the action, yet disjoint from the fixed point set. The surgery is performed in such a way that the group action can be extended to the new manifold. The surgery has the effect of modifying the linking numbers.

**Theorem 5.1.** Given a metacyclic group $D(2n, m)$ with $n$ and $m$ pairwise relatively prime odd integers, and any integer $x \equiv 1 \pmod{4(n - 1)}$, there exists a group action of $D(2n, m)$ on some homology 3-sphere $H$. The action has fixed point set a link of $n$ components where the linking number between any two of the branch curves is $x$.

As well, it is possible to construct non-homogenous actions by this technique, as demonstrated by the next theorem.

**Theorem 5.2.** Under the hypotheses of the above theorem, with $n \equiv \pm 1 \pmod{6}$ and $n \geq 7$, and the choice of any positive even integer $2k$, there exists a group action of $D(2n, m)$ on some homology 3-sphere $H$. The action has fixed point set a link of $n$ components with the linking number between any two adjacent curves given by $x$ while between any two non-adjacent curves is $x - 2k$.

The other method, which relies heavily on computer calculations, was inspired by the work of Livingston in [9] and relies on techniques developed in [6].

36
Starting with a knot $K$ in the 3-sphere, one finds a representation of the knot group onto a finite group $G$. That is, one wants an epimorphism

$$\Sigma : \pi_1(S^3 - K) \to G$$

Associated to such a representation is a branched covering space $\tilde{X}$ where the 3-sphere is the quotient of $\tilde{X}$ by the action of $G$ on $\tilde{X}$ and the branch set below is $K$.

Using algorithms based on Reidemeister-Shreier rewriting, a computer program searches for those knots $K$ and groups $G$ for which the cover $\tilde{X}$ is a homology 3-sphere. Afterwards, another algorithm is used to calculate the linking numbers of the branch link upstairs.

After an exhaustive search of knots of up to 11 crossings, using the knot tables provided in SnapPea [2], as well as a large family of pretzel knots, some interesting conclusions can be drawn. Firstly, many examples of dihedral actions arise in this manner. In fact, every knot with non-trivial Alexander polynomial admits at least one dihedral action (see [5] for a proof) and for the knots studied 47 percent of the time the space upstairs was a homology sphere. Of those cases 48 percent exhibited exotic linking numbers while 38 were confirmed to be simply connected. The remaining examples had non-exotic linking numbers but the simplification algorithms of the program could not reduce the fundamental group to the trivial one. Interestingly, every single exotic action found was non-homogeneous in that the linking numbers between different pairs of branch curves were not the same.

The situation for more general metacyclic groups, however, was rather bizarre. In fact, not a single example was found where the cover was a homology sphere. The homology groups obtained were exceptionally large for the order of groups considered and were predictable in only a few exceptional cases.

### 5.1 Method 1 — Modifying the Linear Action

#### 5.1.1 The Standard Linear Action

Let $m, n$ be relatively prime odd integers. Let the group denoted by $D(2m, n)$ have the presentation

$$D(2m, n) = \langle \alpha, \beta \mid \alpha^n, \beta^{2m}, \beta \alpha \beta^{-1} \alpha \rangle$$
Chapter 5. Results

It has a natural representation into the unitary group $U(2)$. Let

$$
A = \begin{bmatrix}
\zeta & 0 \\
0 & \zeta^{-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 1 \\
\rho & 0
\end{bmatrix}
$$

where $\zeta = e^{2\pi i/n}$, $\rho = e^{2\pi i/m}$.

It is straightforward to verify that $A$ and $B$ respect the relators of $D(2n, m)$. Typical group elements are

$$
A^s B^{2t} = \begin{bmatrix}
\zeta^s \rho^t & 0 \\
0 & \zeta^{-s} \rho^t
\end{bmatrix}, \quad A^s B^{2t+1} = \begin{bmatrix}
0 & \zeta^s \rho^t \\
\zeta^{-s} \rho^{t+1} & 0
\end{bmatrix}
$$

Finding the fixed circles is similarly simple. The nontrivial elements of the form $A^s B^{2t}$ have no fixed points whatsoever, while

$$
A^s B^{2t+1} \begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
x \\
y
\end{bmatrix} \iff \zeta^s \rho^t y = x \quad \text{and} \quad \zeta^{-s} \rho^{t+1} x = y
$$

$$
\iff \zeta^s \rho^t y = x \quad \text{and} \quad \rho^{2t+1} x = x
$$

So to have nontrivial solutions, we require $\rho^{2t+1} = 1$, in other words $2t+1 = m$. Therefore, the group elements with fixed circles are those of the form

$$
A^s B^m
$$

with corresponding fixed point set

$$
\begin{bmatrix}
x \\
\zeta^s \rho^k x
\end{bmatrix}
$$

Where $k = \frac{m-1}{2}$ and $||x|| = 1/\sqrt{2}$. Denote this fixed point circle by $F_s$. 

5.1.2 Modifying the Action

Now consider a link $L \subset S^3$ of $n$ components $K_1, \ldots, K_n$, with the following properties. Note that $D(2m, n)$ acts on the 3-sphere by matrix multiplication.

(i) $B(K_1) = K_1$

(ii) $K_{i+1} = A(K_i)$

(iii) link($K_i, K_j$) = 0

(iv) link($F_i, K_j$) = \begin{cases} 
1 & \text{if } i = j \\
2 & \text{if } i \neq j 
\end{cases}

Assume for now that such a link can be found. Let $I_n$ be the $n \times n$ identity matrix while $J_n$ is the $n \times n$ matrix with every entry 1.

If one performs surgery along the link with surgery coefficient $1/y$ applied to each component, the first homology of the resultant manifold $M$ has relation matrix $I_n$. In other words, $M$ is a homology sphere. Furthermore, the relation matrix for $M - (\cup F_i)$, the surgered manifold with the branch link removed, is

$$[I_n, 2yJ_n - yI_n]$$

This leads to the conclusion $\text{link}_M(F_i, F_j) = 1 - 4(n - 1)y$.

To complete the proof of Theorem 5.1, we must show that the link $L$ exists and that the group action on $S^3$ can be extended to a group action on $M$.

First note that the branch curves all lie on the torus $\{(z_1, z_2) \in \mathbb{C}^2 : \|z_1\| = \|z_2\| = 1/\sqrt{2}\}$. Introducing a new coordinate system for the 3-sphere will simplify the discussion. Consider the set of points $(\xi, \eta, r)$ with $0 \leq \xi, \eta < 2\pi$ and $0 \leq r < 1$. There is a mapping onto the 3-sphere given by

$$(\xi, \eta, r) \mapsto (re^{i\xi}, \hat{r}e^{i\eta})$$

with $\hat{r} = \sqrt{1 - r^2}$. Note that the surfaces of constant $r$ correspond to tori in the 3-sphere (except the degenerate circles at $r = 0$ and $r = 1$). Hence in this toroidal coordinate system, the fixed point circles lie on the torus $r = 1/\sqrt{2}$ and are lines, each parallel to the line $\xi = \eta$ and shifted by an amount $2\pi k/n$. The case $n = 3$ is illustrated in Figure 5.1.
Chapter 5. Results

Figure 5.1: Fixed point curves for $D(2,3)$ in toroidal coordinates.

Connect the points $(\frac{\pi}{6}, 2\pi - \frac{\pi}{6}, \frac{1}{\sqrt{2}}), (\pi, \pi, \frac{1}{\sqrt{2}} + \epsilon)$ and $(2\pi, \frac{11\pi}{6}, \frac{1}{\sqrt{2}})$ by line segments to form the arc $\gamma$. By joining $\gamma$ with $B\gamma$, we arrive at a loop $K_1$ invariant under $B$ and which links each of the branch curves twice, except for the curve that has been reflected across, which links only once.

Note that $A'(K_1)$ intersects $K_1$ at four points. However, by weaving the upper right arc alternately over and under the other curves (as depicted for $n = 3$ in Figure 5.2), and then propagating those changes through the link using the group elements, the result is that each component is unlinked from all others. The reason this works is because of the way the reflection flips the levels of the two arcs in question and because there are an odd number of surgery curves.

To confirm that $1/y$ surgery on each component respects the group action, let $N(K_i) = D^2 \times S^1$ be a neighbourhood chosen so that $B(N(K_i)) = N(K_i)$ and $A(N(K_i)) = N(K_{i+1})$. As well, realize each solid torus as a quotient space:

$N(K_i) = (I \times S^1) \times S^1 / (0, \theta, \phi) \sim (0, \theta', \phi)$

To simplify notation, denote $N$ to be the union of all these solid tori.

Extend the surgery homeomorphism

$h : \partial N \to \partial (S^3 \setminus N)$
2b. Translate by $A^{-1}$

$A^{-1}(K_1)$

1. Shift arc $A(K_1)$

$K_1$

2a. Reflect by $B$

3. Translate by $A$

Figure 5.2: Invariant surgery link for $D(2,3)$.

to the homeomorphism

$$\tilde{h} : I \times \partial N \to \partial \left( S^3 - \overline{N} \right)$$

by $\tilde{h} = \text{id} \times h$.

Let $\tau$ be an element of $D(2n, m)$, that is a homeomorphism $\tau : \partial S^3 - \overline{N} \to \partial S^3 - \overline{N}$. Then for any point $u \in N$, define $\tilde{\tau}(u)$ by

$$\tilde{\tau}(u) = \tilde{h} \circ \tau \circ \tilde{h}^{-1}(u).$$

It remains to prove that this map respects the quotient space. However, this is a simple, though tedious, verification, so it will be omitted.

5.1.3 Finding Non-homogeneous Linking

We now turn to the proof of Theorem 5.2, namely constructing an example of nonhomogeneous linking. This is really an addendum to Theorem 5.1 as the proof follows along a similar vein. The starting point is the standard linear action of $D(2n, m)$, with $n \equiv \pm 1 \pmod{6}$ and $n \geq 7$.

To construct the surgery link, begin with a loop invariant under a reflection. However, this loop only links with three of the $n$ fixed circles. By translating this loop by $A$, one arrives at an invariant set, where each loop intersects with the preceding and following loops exactly twice, and is disjoint from the
other loops. By shifting the arcs away from the intersection point, one has a link where each component links once with each of its neighbours - a necklace of sorts. By the discussion of [12, page 304], performing $-1$ surgery on each component gives the cyclic $n$-fold cover of the trefoil. It is a well known result that this manifold is a homology sphere for $n \equiv \pm 1 \pmod{6}$ (cf. [12, page 150]).

Further analysis shows the linking between the fixed circles which are grouped within each surgery loop to be unchanged. However, for two knots which are not adjacent, the linking number is reduced by 2. Note that this still does not result in exotic linking, as the $+1$ linking has been changed to $-1$.

In order to complete the construction, one must start with $k$ disjoint loops, each invariant under reflection and linking the same three fixed circles. This collection of loops is extended to a large link of $k \times n$ components following the same procedure as for a single loop. It is easy to see that the change in linking is additive amongst disjoint and unlinked surgery links. Hence repeating the process $k$ times results in nonadjacent fixed circles having a linking number $2k$ less than the linking amongst adjacent curves.

Some observations are in order. First, the main impediment to constructing such actions is arriving at a homology sphere - many promising surgery links which certainly affect the linking numbers fail for that reason. Example 4.3 exhibits this property. However, given a particular group and a desired combination of linking numbers it seems likely that given the patience and computing power a surgery curve resulting in that outcome is possible. The complication is finding examples which work for general groups.

Secondly, the above example does work with $D(10, m)$. In fact, performing $-1$ surgery along a necklace of 5 components results in the Poincaré homology sphere, as demonstrated in [12, page 268]. Hence, a metacyclic action can be constructed for this manifold. However, it is not exotic - the linking numbers remain unchanged by the surgery. This is not surprising as the Poincaré sphere has spherical geometry so it seems likely that the standard action would display the same linking characteristics as the 3-sphere.
5.2 Method 2 — Branched Covers of $S^3$

The second method of finding exotic actions is more experimental than constructive; however, some interesting patterns emerged which are not apparent by the surgery method. The main method of attack is to look for representations of a knot complement onto a particular finite group. This forms the monodromy for a branched cover, with branch image the original knot. With this information, a computer program written by the author is used to calculate the homology of the covering space. If it turns out to be a homology sphere, the linking numbers between the branch curves are calculated and compared to linear linking.

Finding a dihedral covering of a knot $K$ is relatively straightforward. In fact, a representation

$$\Sigma : \pi_1(S^3 - K) \rightarrow D(2, n)$$

is easily constructed whenever a knot projection of $K$ can be $n$-coloured in a compatible fashion. This means that each arc can be labelled with an integer such that at each crossing

$$x - 2y + z \equiv 0 \pmod{n}$$

Where $x$ and $z$ are the labels on the undercrossings and $y$ is the label on the overcrossing. With a colouring as described above, a representation is determined by sending the meridian of an arc labelled with an integer $x$ to the element $\alpha^{2x}\beta$. Such an assignment respects the relations of the dihedral group and is surjective whenever there are arcs labelled with relatively prime integers $x$ and $y$. A regular branched cover results by considering the $2n$-fold permutation representation of $D(2, n)$ where

$$\alpha \mapsto (1 \; 2 \cdots n)(n + 1 \cdots 2n)$$
$$\beta \mapsto (1 \; 2) \cdots (2n - 1 \; 2n)$$

Consider the 2-fold branched cover of $K$ and denote by $r$ the order of the largest cyclic factor of the homology of this space. Then a result of Fox [5] shows that $K$ can be $n$-coloured if and only if $n$ divides $r$. He further showed that the longitude of the knot is represented by an element in the second commutator subgroup. Since this subgroup is trivial for metacyclic groups,
the examples studied in this paper have the simplifying assumption that the longitude is always represented by a trivial permutation.

Furthermore, given any $n$-colouring of a knot, one may extend it to an $nm$-colouring which allows for a representation onto the group $D(2m, n)$. One must solve the system of congruences

\[
\begin{align*}
  x - 2y + z &\equiv 0 \pmod{n} \\
  x - z &\equiv 0 \pmod{m}
\end{align*}
\]

which is always possible when $n$ and $m$ are relatively prime.

The computer program Twiggy is a realization of the algorithms described in the preceding chapters. An example of an exotic action found by it is discussed below.

**Example 5.1.** If one considers the knot $10_{132}$, as pictured in Figure 5.3 then it has a representation onto $D(2,5)$ determined by

\[
\begin{align*}
  x_1 &\mapsto (01)(23)(45)(67)(89) \\
  x_5 &\mapsto (01)(23)(45)(67)(89) \\
  x_8 &\mapsto (09)(12)(34)(56)(78)
\end{align*}
\]

Constructing the branched cover of this monodromy results in a homology sphere with branch set a link of five components. The linking between the
components is given by

\[
\begin{pmatrix}
7 & -3 & -3 & 7 \\
7 & -3 & -3 \\
7 & -3 \\
7
\end{pmatrix}
\]
Chapter 6
Conclusion

Certainly there are many unanswered questions regarding the spherical space
form problem, especially in dimension 3. This thesis has provided a method for
finding group actions on homology 3-spheres which is generally applicable to
any group admitting a linear representation. However, there are many limita­
tions to this approach. The most interesting and least understood groups from
Milnor’s list [11] are the Milnor Q-groups - cyclic extensions of the quaternions.
These have been shown to have no faithful fixed point free linear representa­
tion (cf [11], [14], [16]); however, there are homology spheres on which some of
the Q-groups can act fixed point freely.

One possible approach to these groups is performing a computer study
- much like what was done with the dihedral groups in the present paper.
However, even this line of attack is quite delicate as evidenced by the failure
to find any meaningful results with metacyclic groups. There are, as well,
many avenues for improvement in this regard. The program did not take into
account the many distinct representations of a knot onto a particular group. It
seems possible that a different representation would give a different covering
manifold. However, the success in reproducing Livingston’s results, using an
altogether separate procedure from his, suggests there is a certain invariance
when working with regular covers as was done here.

A more interesting yet overlooked question is the idea of a standard action
on a given homology sphere. If the sphere has a geometric structure, then
a standard action would be a subgroup of the isometries of the sphere. It
is conjectured that all actions on a geometric 3-manifold are conjugate to a
standard action in the sense described above (cf [8]). However, it would be
gratifying to check the examples found in this paper with the standard ones.

Another topic is the totality of examples constructed using the methods of this thesis. That is, given a homology sphere and an action on it, is it possible to perform surgery on the manifold while preserving the group action and arrive at a standard linear action on the 3-sphere? Similarly, one can ask if all such actions are conjugate to a branched cover over the 3-sphere. To the author's knowledge, this question has not even been answered for the linear action - where the quotient manifold will be a circle bundle over an orbifold. The experience with the metacyclic groups suggests that the answer to the second question may be false; however, the first question seems much more likely to be true. In any case, they both reflect deep and not well understood properties of 3-manifolds.
Bibliography


Appendix A
Table of Results

The following table lists a sample of the data obtained using Twiggy. The knots listed are the prime knots of up to ten crossings, with labelling determined by the minimal number of crossings in the knot and subscript by the order in which it appears in the list provided by [2]. A description of the knot can be obtained by accessing it through Twiggy.

For each knot, a representation is found onto the dihedral group of maximal allowable order (corresponding to the largest cyclic factor of the 2-fold branched cover). The knots listed are those for which some dihedral cover is a homology sphere. For these cases, the linking numbers of the branched curves are calculated, which indicates whether or not the action is exotic. The program also applies a simplifying algorithm to the fundamental group of the cover, and the table indicates if the group was simplified to be trivial. Finally, the number of distinct linking numbers in the cover is listed, which could be compared to the maximal possible number of \((n - 1)/2\) for the group \(D_{2n}\).

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<th>Group</th>
<th>Exotic</th>
<th>(\pi_1 = 1) Distinct Link No.s</th>
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