DOUBLE HOPF BIFURCATIONS IN TWO GEOPHYSICAL FLUID DYNAMICS MODELS

by

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B.Sc. (Physics) McGill University, 1991
M.Sc. (Physics) McGill University, 1993

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics
Institute of Applied Mathematics

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

• April 2000

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Abstract

We analyze the double Hopf bifurcations which occur in two geophysical fluid dynamics models: (1) a two-layer quasigeostrophic potential vorticity model with forcing and (2) a mathematical model of the differentially heated rotating annulus experiment. The bifurcations occur at the transition between axisymmetric steady solutions and non-axisymmetric travelling waves. For both models, the results indicate that, close to the transition, there are regions in parameter space where there are multiple stable waves. Hysteresis of these waves is predicted. For each model, center manifold reduction and normal form theory are used to deduce the local behaviour of the full system of partial differential equations from a low-dimensional system of ordinary differential equations.

In each case, it is not possible to compute the relevant eigenvalues and eigenfunctions analytically. Therefore, the linear part of the equations is discretized and the eigenvalues and eigenfunctions are approximated from the resulting matrix eigenvalue problem. However, the projection onto the center manifold and reduction to normal form can be done analytically. Thus, a combination of analytical and numerical methods are used to obtain numerical approximations of the normal form coefficients, from which the dynamics are deduced.

The first model differs from those previously studied with bifurcation analysis since it supports a steady solution which varies nonlinearly with latitude. The results indicate that the forcing does not qualitatively change the behaviour. However, the form of the bifurcating solution is affected.

The second model uses the Navier-Stokes equations in the Boussinesq approximation, in cylindrical geometry. In addition to the double Hopf bifurcation analysis, a detailed axisymmetric to non-axisymmetric transition curve is produced from the computed eigenvalues. A quantitative comparison with experimental data finds that the computed transition curve, critical wave numbers and drift rates of the bifurcating waves are reasonably accurate. This indicates that the analysis, as well as the approximations which are made, are valid.
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Chapter 1

Introduction

In this thesis, two mathematical models from geophysical fluid dynamics are studied. One possible definition of geophysical fluid dynamics is that it is the study of rotating and stratified fluids, or more precisely, fluid flows in which rotation and stratification play an important role. The goal of studying such fluid systems is the understanding of the large scale dynamics of the atmosphere and ocean. In particular, it is a search for the physical processes or mechanisms which are most important to the character of the large scale dynamics. Thus perhaps a more appropriate definition was put forward by Pedlosky [49]: “Geophysical fluid dynamics is the subject whose concerns are the study of the fundamental dynamical concepts essential to an understanding of the atmosphere and ocean.”

The starting point of this thesis is a system of evolutionary equations which describes the dynamics of a fluid subject to rotation. The premise is that an understanding of geophysical flows can be gained by a careful study of these equations (i.e. that the equations accurately represent the dynamics). This is not an obvious point, since the number of factors which contribute to the dynamics of the atmosphere and ocean is enormous. They encompass all areas of science, from physics to chemistry to biology. If a set of equations which incorporated all the possible factors could be written down, their complexity would be so great that analysis of any kind would be impossible or at least unrevealing. It is therefore necessary to restrict our attention to temporal and spatial scales on which it is possible to argue that many or most of the factors or processes are unimportant. The hope is that the simplified equations that we study capture the character of the real system while being tractable for mathematical analysis.

If we wish to accurately represent the real system, simplifications must be made systematically and with caution. One possible method of simplification is called scaling analysis. Essentially, an estimation is made of the magnitudes of the terms in the dynamical equations. The processes associated with the terms with small magnitude are judged to have little effect and are therefore neglected, thus simplifying the equations. Another method
of simplification is to restrict the study to motions which can be observed in controlled laboratory experiments. In this situation, there are significantly less factors to consider.

We study two geophysical fluid dynamics models: (1) a model which is derived using scaling analysis and (2) a mathematical model of a controlled laboratory experiment. The systems of equations which result from these simplifications are analyzed mathematically. When numerical approximations are made during the analysis, we attempt to justify their accuracy. In this way, it may be argued that the discrepancies between the analysis and the real flow are not an artifact of the numerical approximations. Furthermore, speculation is not used in the analysis. That is, the dynamics are predicted and not presumed. Specifically, a bifurcation analysis is performed. The power of this method is that it finds 'new' steady (time-independent) or time-periodic solutions of the dynamical equations. The analysis also gives the stability of each solution and the location in the parameter space where transitions between solution types occur. If the system 'lives' close to these transitions, this method will be maximally useful. However, even if it does not, the analysis will lead to a better understanding of the dynamical equations which, as mentioned before, we assume represents the real flow.

1.1 General circulation

When rays of solar radiation reach the Earth, they are essentially parallel. Since the surface of the Earth at the equator is less oblique to the sun, more radiation is absorbed per unit area here than at the poles. See Figure 1.1. This causes the differential heating between the equator and the poles which, aside from lunar tides, is ultimately responsible for all motions in the atmosphere and ocean [49].

Consider for a moment the atmosphere on a non-rotating planet where the differential heating is axisymmetric (invariant under rotation about the polar axis). As a caricature, assume that there is a heat source at the equator and a heat sink at the poles. The heating at the equator would cause the surrounding air to become less dense and therefore rise, while the cooling at the poles would cause the air to become more dense and therefore sink. Continuity of mass would close the circuit: the air aloft would move toward the poles and the air along the ground would tend to the equator. See Figure 1.2. Alternatively, one could imagine that the differential heating on the surface would cause a corresponding north–south gradient of the density of the air and therefore a north–south pressure gradient along the surface which induces equator-ward motion. The convergence at the equator causes air to rise, which in turn causes a pressure gradient aloft inducing pole-ward motion. The air sinks at the poles closing the circuit. In either view, the result is an axisymmetric convection
cell in which fluid flow is aligned along lines of constant longitude. If the gradients induced by the differential heating were large enough, presumably more complicated flow patterns could emerge.

Now consider the same scenario, but on a rotating planet where the differential heating is symmetric about the axis of rotation. If observed from a fixed location on the rotating planet, objects in motion which are not subject to external forcing seem to follow curved trajectories (this is a consequence of the Coriolis effect, see below). In a similar manner, the fluid motion induced by the differential heating is deflected perpendicularly to the direction of motion, the direction of the deflection depending on the sense of rotation and the magnitude of the deflection depending on the magnitude of the velocity and rotation rate. This implies that the fluid can no longer stay in a longitudinal plane. However, if the rate of rotation is small, the resulting flow pattern may simply be a tilted (both vertically and horizontally) version of the convection cell of the non-rotating planet, with a mirror symmetry between the hemispheres. That is, the axisymmetry would be maintained. It would not be surprising if, as the parameters (the rotation rate and the differential heating) were increased, the tilting would become severe enough so that the axisymmetric flow pattern would become unstable, leading to the development of a different, perhaps more complicated, flow pattern. In fact, the rotation can lead to instability of the axisymmetric convection pattern at significantly lower values of the differential heating than for a non-rotating system.
An additional interesting feature is that the effects of rotation are felt differently at different locations on the planet. For an observer at the equator, the rotation is felt mostly as a reduction of gravity, while for an observer at the poles, essentially, the rotation is felt only in the horizontal. Due to the consequences this has on the dynamics, it is useful to consider a local 'effective' rotation rate, which has a minimum at the equator and a maximum at the poles. It is also possible to argue that there is a variation of the differential heating with latitude (due to the variation of the rate of change of absorbed solar radiation with respect to latitude; see Figure 1.1). Therefore, since both the local rotation rate and differential heating change with latitude, it leads one to imagine that different bands of latitude might favor flows of different character. Indeed this is the case on the Earth. Near the equator, there is a large scale axisymmetric circulation pattern called the Hadley cell, which, on average, the wind approximately follows. It is characterized by upward motion at the equator, subsidence in the sub-tropics (near 30° latitude) and persistent equator-eastward winds (called the trade winds) at the surface. This deflection of the surface winds is presumably caused by the effect of rotation. In the mid-latitudes, oscillatory non-axisymmetric patterns are often observed which propagate to the east. The motion in this region tends to be predominately horizontal and tends to follow curved trajectories.
In the thesis, we will try to capture, not the coexistence of such patterns, but the changes of such patterns as the rotation rate and differential heating (the parameters) are varied. In particular, we will be concerned with the transitions from axisymmetric to non-axisymmetric flow patterns.

1.2 Experiments

If differential heating and rotation define the character of large scale geophysical flows, then it is useful to study laboratory experiments which attempt to isolate these effects. In particular, if the form of differential heating, the geometry of the system, properties of the fluid, and boundary conditions play a secondary role, the flows observed in experiments may contain the essential character of their geophysical counterparts. Alternatively, if experiments are performed with various configurations, the features which are common to all the experiments may be considered those inherent to differentially heated rotating systems. In this section we discuss some general experimental results and in so doing, present some of the main features of the general circulation of the atmosphere and ocean which our models reproduce.

Many different experiments have been performed in an attempt to develop an understanding of differentially heated rotating fluid systems (see e.g. [27], [44] and [24]). The experiments often take the form of studying fluid flow in a rotating cylindrical annulus, where the differential heating is obtained either by keeping the inner and outer walls of the annulus at different temperatures or by an internal heating of the fluid [27]. See Figure 1.3. Many other configurations have been studied and are mentioned in Section 2.3. The experiments consist of finding the various stable flow patterns which occur at different values of the rotation rate and differential heating. Usually, the results are given in a diagram where the transitions between the different flow types are plotted in parameter space. The most important dimensionless parameters were judged to be the Taylor number $\mathcal{T}$, and the thermal Rossby number $\mathcal{R}$ [26]. The Taylor number

$$\mathcal{T} = \frac{4\Omega^2 R^5}{\nu^2 D}$$

is a measure of the relative importance of rotation to viscosity, where $\Omega$ is the rate of rotation, $R = r_b - r_a$ is the difference between the outer and inner radii of the annulus, $\nu$ is the kinematic viscosity of the fluid and $D$ is the depth of the fluid. The thermal Rossby number

$$\mathcal{R} = \frac{\alpha g D \Delta T}{\Omega^2 R^2}$$
Figure 1.3: The differentially heated rotating annulus experiment, where the annulus is rotated at rate $\Omega$ and the inner wall is held at the fixed temperature $T_a$ and the outer wall at temperature $T_b$, creating a differential heating. $r_a$ and $r_b$ are the radii of the inner and outer cylinders, $R = r_b - r_a$ and $D$ is the height of the annulus.

is a measure of the relative importance of rotation to the differential heating, where $\Delta T = T_b - T_a$ is the imposed horizontal temperature gradient, $\alpha$ is the coefficient of thermal expansion and $g$ is the gravitational acceleration. Note that, if all other parameters are held fixed, there is a one-to-one relationship between the dimensionless parameters (thermal Rossby number and the Taylor number) and the physical parameters (the differential heating and rate of rotation).

If the parameters are held fixed at small values and the flow is allowed to equilibrate (transients are allowed to pass), a stable axisymmetric flow pattern is observed. By stable, it is meant observable: a small perturbation does not cause the pattern to disappear. If the parameters are incremented slowly and at each incrementation the flow is allowed to equilibrate, the region in parameter space where this flow is stable may be mapped out. If the parameters are incremented past certain critical values, the axisymmetric flow becomes unstable and a non-axisymmetric pattern arises. This pattern is usually a travelling wave whose wavelength depends on the parameter values and the experimental configuration.
The set of critical parameter values is called the transition curve and it forms the boundary between axisymmetric and non-axisymmetric flow regimes. If the same procedure is performed for the stable wave motion, transitions to other, more complicated, flow regimes may be found.

Most of the experiments find four main flow regimes in different regions of parameter space (see Figure 1.4):

**Axisymmetric Flow.** This flow is characterized by its azimuthal invariance.

**Steady Waves.** The flow in this region is non-axisymmetric and resembles a rotating wave with constant amplitude and phase. Different wavelengths are seen in different subregions.

**Vacillation.** In this region, the structure or amplitude or wavelength of the observed wave varies periodically in time.

**Irregular Flow.** This region is characterized by its irregular nature in both space and time.

All of the observed flows have their counterparts in the atmosphere. The axisymmetric flow resembles the Hadley cell which is observed in the atmosphere near the equator where the ‘local’ rotation rate and differential heating are relatively small (see Section 2.1). Given a counter-clockwise rotation and a positive temperature gradient between the outer and inner cylinder, the experimental flow in this region rises at the outer cylinder, sinks at the inner, with deflection of the radial motions to the right. That is, a tilted convection cell is observed. In mid-latitude regions of the Earth, the flow sometimes has wave characteristics that resemble the steady waves and vacillations seen in the experiments. In this region in the experiment, the flow trajectories are curved and vertical motion is inhibited.

### 1.3 Analysis

Numerous methods of analysis have been used in the study of geophysical fluid dynamics models. One of the most important was pioneered by Charney [5] and Eady [13] in the late 1940’s. They postulated that mid-latitude cyclones (waves) are generated by the growth of small disturbances on a ‘basic state’, a steady solution of the governing equations. The basic state was taken to be a zonal flow (a flow along isolines of latitude) that depended only on the vertical coordinate. A central theme in the analysis is that the ‘instability’, or growth of disturbances, is induced by the combination of the vertical shear of the basic state and the stratification of the fluid. This important mechanism is called ‘baroclinic instability’.
Figure 1.4: A schematic diagram depicting general experimental results. See e.g. [26]. To the left of all the curves is the axisymmetric regime which is separated into three (dynamically similar) regions: lower symmetric, knee, and upper symmetric. To the right of the curve is the non-axisymmetric regime which is separated into three dynamically distinct regimes: steady waves, vacillation and irregular flow.

The analysis consists of computing the linear stability of the basic state and finding the fastest growing unstable mode, or disturbance. Eady found that, in his model [13], this mode has an azimuthal (zonal) wavelength comparable to those of the finite amplitude waves observed in the atmosphere. Implicit in the analysis are the assumptions that (1) the unstable wave could grow to finite amplitude without being affected by nonlinear processes, (2) the growth stops when some finite amplitude is reached and (3) the growth is restricted to this mode alone (i.e. the growth of the other unstable modes are somehow suppressed). These assumptions were not verified, nor was it shown why the time-dependent flow would approach a region close enough to the unstable basic state where the linear analysis would be approximately valid.

Many studies have applied this idea to models with various features. The following list is far from complete. Barcilon [3] introduced Ekman (boundary) layers. Others [42], [2], [6] used various basic states to include horizontal shear (variations in the meridional direction), while other studies [1] were performed on the 'beta plane', where the variation
of the effective rotation rate with latitude is assumed to be linear and all other curvature
effects are neglected. Most studies used the quasigeostrophic equations (see Section 2.2)
in three spatial dimensions, but some used the two-layer version [49], while others used
more complex models [2]. There were also studies performed on different geometries, in
cylindrical or spherical coordinates. Some of the studies were done analytically and some
numerically. However, they all studied the linear stability of some basic state.

Another method of analysis could be called 'numerical experimentation', since the pro­
cedure of the laboratory experiments is simulated on a numerical model. This method is
often used to attempt a reproduction of the flows observed in the laboratory experiments.
The procedure often begins by finding an axisymmetric steady solution either analytically
or numerically. This solution, to which small amplitude random perturbations are added
to simulate natural fluctuations, is used as the initial condition for a numerical integration,
or time-stepping. If the time-stepping produces solutions which stay near the steady solu­
tion, then the steady solution is labelled as stable in the studies. The parameters are then
varied and the process is repeated. If the time-stepping produces a solution which evolves
away from the steady solution, it is labelled unstable. In this case, the time-stepping is
continued until transient behaviour disappears and the system appears to reach a different
stable steady state, periodic solution, or a more complicated solution.

Again, many different models have been studied using numerical experimentation. Mo
et al. [45] followed the above procedure on a three-dimensional quasigeostrophic model
with Ekman layers at the top and bottom. Hignett et al. [29], James et al. [30], Miller and
Butler [43] and Lu et al. [38] performed numerical experiments with a Navier-Stokes model
in cylindrical geometry for direct comparison to laboratory experiments. Kwak and Hyun
[33] performed the analysis on a model of the Hathaway and Fowlis experiment [24], and
Collins and James [8] studied a simplified global circulation model. Lewis [36] and Mundt
et al. [47] also performed time-integration of basic states on different two-layer models.

1.3.1 A dynamical systems approach

In this thesis, a dynamical systems approach will be taken. The underlying principle is that
a knowledge of the solution structure of the equations will lead to a better understanding
of the system we are trying to model. We do not seek a time-dependent solution given a
single initial condition, but rather we attempt to determine what the dynamics will be, at
least qualitatively, given any initial condition. When the system depends on parameters,
such as the strength of differential heating, the goal is to determine the solution structure
for all relevant parameter values. Although it is rarely possible, if we can find all the
invariant sets of the equations (the steady solutions, periodic orbits, chaotic invariant sets,
etc.) along with their stability, then this gives an idea of what the long-time behaviour will be, regardless of the initial conditions. That is, time-dependent solutions 'move away' from unstable invariant sets and 'approach' stable invariant sets if enough time is allowed to pass. This method, however, often does not provide sufficient information to deduce which stable invariant sets will be approached for a given initial condition and it may not be able to determine how long the approach will take. Moreover, it is possible that the natural fluctuations present in all physical systems could inhibit the convergence to a stable invariant set.

We implicitly assume that the time-dependent solutions of the equations of our model exist and are unique for all the initial conditions of interest. It is possible to prove this for ordinary differential equation models provided certain general conditions are satisfied (see e.g. [56]). However, for the partial differential equation models we use, this has not been explicitly shown. Yet, since the three-dimensional incompressible Navier-Stokes equations and the two-dimensional quasigeostrophic (barotropic) potential vorticity equations [9] exhibit local existence and uniqueness of solutions, it is reasonable to assume that in the parameter range and initial conditions of interest, the models we study do as well.

Bifurcation analysis is a particularly powerful method of finding invariant sets. Given a known steady solution, or other invariant set, for a given set of the parameters of the system, bifurcation methods can not only be used to trace out how this solution changes as the parameters change, but it can also be used to uncover previously unknown invariant sets which 'bifurcate' from the known solution. By bifurcation we mean a qualitative change in the solution structure of the equations as the parameters are varied past some critical values. The location in parameter space where a bifurcation occurs is called a bifurcation point. Under certain generic conditions, it can be shown that close to certain types of bifurcation points, multiple invariant sets exist. For example, when the linearization about a steady solution (see Section 3.1) has a complex conjugate pair of eigenvalues whose real parts cross the imaginary axis as a parameter of the system is varied, a periodic solution appears. This is called a Hopf bifurcation. A bifurcation analysis can not only show the existence of the new invariant set but can also provide the solution form of the invariant set and its stability. For the bifurcation of a steady solution, different forms of the eigenvalues (e.g. real or complex conjugate pairs) and different numbers of the eigenvalues which cross the imaginary axis correspond to different types of bifurcations, in which different kinds of invariant sets may appear. The other kinds of invariant sets have analogous, but more complicated situations which give rise to bifurcations. Theoretically, the newly discovered invariant sets can also be tracked through parameter space until subsequent bifurcations occur.
As previously mentioned, bifurcation analysis attempts to find the solution structure for all relevant parameter values. At the most basic level, the motivation is that this leads to a better understanding of the underlying principles which are being modelled. For example, it is useful to know for what values of the differential heating and rotation rate a certain type of flow pattern is observed. From such an analysis, it can be learned, for example, that axisymmetric flow is expected for small values of the parameters, while wave motion is expected for larger values. Therefore, near the equator, one may look for an axisymmetric flow pattern while in mid-latitudes, wave motion should be seen. In this light, bifurcation analysis and the laboratory experiments have the same goal. However, in a particular naturally occurring fluid flow, it may be argued that the parameter values are constant and thus it is necessary to make a calculation for only one value of each parameter. In fact, the parameters of mathematical models often arise due to simplifications of the real system. For instance, the differential heating of the annulus experiment can be written as the temperature difference between the inner and outer annulus walls, however, in the atmosphere, this forcing is much more complicated, with temporal and spatial dependence. In such a situation, it cannot be determined what value of the parameter will best reproduce the behaviour of the real system. Therefore, a knowledge of the solution structure for a range of parameters will more likely validate a model. Also, in some situations, the parameter may depend on the state of the system and so it may vary in time. If the parameter changes smoothly and slowly in comparison to the dynamics, perhaps it could be assumed that the time dependence of the parameter simply changes the location in parameter space without affecting the solution structure. We therefore think of the atmosphere as existing at different locations in parameter space at different times and, the analysis does not try to address why a particular flow pattern is observed at a given time. This point is most important when relating the results to the mid-latitudes, where flows of different character may be observed at different times.

Ideally, it would be possible to find all the invariant sets of the system for all parameter values of interest. However, this can be a formidable task even for low-dimensional ordinary differential equations, let alone for the infinite-dimensional partial differential equations we are interested in studying.

One method of simplification is to generate a low-dimensional ordinary differential equation (ODE) from the partial differential equation (PDE) model. This can be achieved with a truncated spectral expansion. This method consists of expanding the dependent variables of the PDEs as a series of complete orthogonal spectral functions of the spatial variables, which results in an infinite-dimensional system of ordinary differential equations in the time-dependent coefficients of the spectral functions. The approximation results by consid-
ering only a small number of terms in the expansions. Although this method is very useful for the study of the mathematical mechanisms which possibly exist in the PDEs (see e.g. [37], [53], and also [19]), it is, in general, difficult to know if the behaviour in the ODE corresponds to that of the original equations. Due to the presence of the nonlinear terms in the original equations, inter-scale interactions are fundamental to the dynamics and the low-order truncation, in effect, ignores the interactions between all but finitely many scales. The advantage is tractability. It is often possible to perform a thorough bifurcation analysis, sometimes even analytically [37]. Numerically, a software package such as AUTO [10] can be used for low-dimensional ODEs to find bifurcations and to follow steady and periodic solutions through parameter space.

In order to include more of the inter-scale interactions, it is possible to perform a higher-order spectral truncation or use another method of discretization. The higher-dimensional system, however, becomes difficult to analyze and the inter-scale interactions which are ignored may still not be negligible. Due to the size of the discretized systems, the preferred method of analysis is numerical experimentation. Although such analysis can lead to interesting discoveries, the dynamical behaviour can only be studied for a relatively short integration time and for only relatively few initial conditions. It can be difficult to localize bifurcation points or to determine when the time-dependent solution is close to a stable invariant set. In fact, this method cannot show the existence of invariant sets; the existence must be presumed. For example, to determine whether a periodic orbit has been approached, a judgment must be made that the time-dependent solution has returned closely enough to a previous state and this return must occur many times so that a periodic orbit is not mistakenly presumed in place of a more complicated trajectory. Problems arise also if the period of the orbit is long. This becomes even more difficult, or prohibitive, for more complicated invariant sets. Furthermore, errors are introduced at each time step and long time integrations are necessary to find the behaviour of interest. Quantitative justification that these errors are not important may not be possible. Bifurcation analysis is usually not performed on such systems, although Legras and Ghil [34] performed a detailed investigation of a higher-dimensional spectral truncation model of barotropic flow over topography.

In this thesis, we choose to directly study the partial differential equations at the cost of restricting the analysis to isolated locations in parameter space where a particular type of bifurcation occurs. In particular, we study double Hopf bifurcation points which occur when the linearization about a steady axisymmetric solution has two pairs of complex conjugate eigenvalues that simultaneously cross the imaginary axis as the parameters are varied. Close to such points, interesting behaviour may be discovered that could otherwise have been
Center manifold reduction is used to find the dynamics of the partial differential equations close to the bifurcation point. This is a method of simplifying the equations in a way which takes into account all the nonlinear inter-scale interactions. The reduction method is described in detail in Section 3.2. The results are valid for parameter values close to the bifurcation point and when the bifurcating solution (invariant set) is close to the axisymmetric solution. This method is sometimes referred to as weakly nonlinear analysis, since the nonlinear terms in the equations are assumed to be small but not negligible. Theoretically, it is possible to write the bifurcating solution as a series in a small parameter and thus expand the range of validity. In actuality, this is rarely practical and the solution is only estimated to first order in the small parameter. Essentially, the technique is able to show the existence and stability of the bifurcating solution and give a first-order estimate of the solution itself, but it is not able to determine if the solution persists for values of the parameters far from the bifurcation point.

This type of bifurcation analysis has been successful in various applications. Perhaps the most well-known is the onset of convection in a fluid heated from below, the Rayleigh-Bénard problem (see [51]). Another application of note is the Taylor-Couette problem (see [7] and the references contained therein), which is a fluid annulus experiment without differential heating where the inner and outer cylinders rotate at different rates generating a shear flow in the interior of the annulus. A rich variety of behaviour was uncovered using bifurcation analysis, much of which was confirmed by experimental results. In the geophysical fluid dynamics literature, an asymptotic method, formally equivalent to center manifold reduction, was used to analyze 'weakly nonlinear' wave-wave interactions (double Hopf bifurcations) in the two-layer quasigeostrophic potential vorticity equations with the beta effect (see Section 2.2) and a linear basic state in [46], [39], and [54]. Also, Hart [22] studied similar equations with no beta effect and Drazin [12] studied Eady's model [13], both using similar methods. The results indicated multiple stable wave solutions and hysteresis of these solutions. Moroz and Holmes [46] found stable quasi-periodic dynamics close to one of the bifurcation points. However, our reproduction of the analysis found that due to a mistake in the formula for one of the normal form coefficients, this result was in error. For all of these models, it is possible to find the results analytically.

In the field of geophysical fluid dynamics, few models exist which can be studied purely analytically. Since the models studied in this thesis do not fall into this category, we use a combination of weakly nonlinear analysis with numerical methods. The result is an analytical–numerical hybrid analysis technique. Analytically, it is possible to reduce the time-dependent nonlinear PDEs to a series of steady linear PDE problems. These linear systems are then solved numerically. Not only are the linear problems less difficult
to numerically approximate, but also the validity of the approximations is more easily verified. Essentially equivalent methods are used in the Couette-Taylor problem [7] and in [20], where a double Hopf bifurcation was analyzed in a barotropic quasigeostrophic model.

It should be restated that a priori it is not known if this analysis will lead to the discovery of physically realizable invariant sets. For example, in Poiseuille flow, experimental observations indicate that instability sets in for significantly lower parameter values than the linear stability analysis predicts [40] and the wave solution predicted by the bifurcation analysis is not observed. The cause of this has been attributed to the non-orthogonality of the eigenfunctions [14], due to the non-self-adjoint linearization about the steady solution (see Section 3.1). In this case, it is possible that even when the steady solution is linearly stable, small perturbations can grow to appreciable size before they ultimately decay. Thus, small perturbations may take the flow into the nonlinear regime where the stability calculation is not valid. Alternatively, considering that small perturbations will always be present, there will always be growth and thus the solution will never converge to the stable flow pattern, i.e. the 'transients' will never disappear. It should be noted that not all non-self-adjoint systems will have this property (e.g. the Couette-Taylor problem [7], where linear stability was used to reproduce the experimentally observed transitions between different flow regimes). However, since the equations we wish to study have non-self-adjoint linear parts, this problem may arise for our models. It is therefore useful to analyze a model where the results can be quantitatively verified by the results of laboratory experiments. If the results of the analysis can be confirmed, then not only can something be learned of the structure of the solution space, but it also validates the method for these and similar models.

1.4 Outline of thesis

In this thesis, we study the transitions from axisymmetric steady solutions to non-axisymmetric travelling waves in two geophysical fluid dynamics models: (1) a two-layer quasigeostrophic potential vorticity model with a nonlinear basic state and (2) a model of the differentially heated rotating annulus experiment. Using the analytical–numerical center manifold reduction at the double Hopf bifurcation points which occur at this transition, it is shown that there are regions, in the respective parameter spaces, which support multiple stable wave solutions and hysteresis of these solutions.

The first model differs from those previously studied with bifurcation analysis in that it contains a latitudinally varying forcing term which leads to a basic state which is nonlinear. The model has been proposed to study the internally heated rotating annulus experiment,
as well as the general circulation of the great planets [36]. As a consequence of the form of the basic state, the linearization about the axisymmetric solution is non-self-adjoint and results cannot be computed analytically. The results indicate that the forcing term in the equation does not qualitatively affect the behaviour. However, there are some minor differences which are of interest, in particular, the form of the bifurcating solution.

The second model uses the Navier-Stokes equations in the Boussinesq approximation, in cylindrical geometry. The dimensions of the domain and the properties of the fluid are chosen to imitate the differentially heated rotating annulus experiment of Fein [15]. The transition curves are approximated by calculating the eigenvalues of the discretization of the linearization about the axisymmetric solutions. To my knowledge, this is the first time a detailed investigation of this kind has been performed for the differentially heated rotating system. Double Hopf points are located and analyzed, and an approximation of the region of hysteresis is also found. Again, the linearizations are non-self-adjoint and numerical methods must be implemented. The transition curve, critical wave number and drift rate of the wave are all found to reproduce well the experimental observations.

The type of analysis that has been employed in this thesis has been predominately overlooked in the geophysical fluid dynamics community. Perhaps this is due to the difficulty of the analysis. However, the work in this thesis indicates that a bifurcation analysis of this kind can lead to valid results and thus supports the further application to other, similar, problems. The ubiquity of the region containing multiple stable waves leads to the speculation that the existence of the region may be a fundamental property of all differentially heated rotating systems in the parameter ranges applicable to geophysical fluids.

In Chapter 2, the models which will be analyzed are presented. An overview of the basic concepts and approximations leading to the dynamical equations is included in this chapter. In Chapter 3, the analytical–numerical center manifold reduction is outlined using the simpler example of a Hopf bifurcation in the two-layer model. In the second part of Chapter 3 and Chapter 4, the results of the double Hopf bifurcations applied to the two models are presented. A conclusion follows.
Chapter 2

The Models

The atmosphere is a very complex system. Its dynamics are influenced by the moisture carried by the air (e.g. vapour and clouds) and the fact that the oceans, which are themselves complex dynamical systems, are intricately coupled to the atmosphere. There are variations of thermal forcing over many space and time scales which are themselves the result of many factors, and the boundaries introduce additional complexity. Furthermore, there are nonlinear processes which cannot be ignored. The nonlinearity couples the small and large scales and since the atmosphere and oceans exhibit important dynamics at all scales (they are turbulent), all scales have to be well resolved if we wish to reproduce all the flow features. At present, the most powerful computers cannot compute the range of scales necessary for such a reproduction and it will be a very long time, if ever, before one is built.

Because of the complexity, it is useful, or necessary, to study simplified models of the system that isolate particular flow features and that are also tractable to analysis. If the simplifications are arrived upon systematically, then much can be learned from these models. As a simple example, if a model neglects all but one physical process and is still able to reproduce an observed flow pattern, then it is plausible to postulate that it is this physical process which is most important in the generation of this flow. The fact that such methods have been very successful in the field of geophysical fluid dynamics gives evidence that there are a relatively small number of dominant processes that shape the main structure of the flows of interest.

Here we are interested in studying two particular simple models for the large scale dynamics of the atmosphere (or ocean). First, we will study the two-layer quasigeostrophic potential vorticity equations. The central assumption of this model is that the characteristic length and velocity of the flow are such that the Earth’s rotation becomes a dominant effect. The motivation for studying this model is that it is the most simple model available for studying baroclinic effects (see below). The model we analyze differs from the standard
model in that it contains a basic state which not only has vertical shear but also nonlinear horizontal shear.

We also study a controlled laboratory experiment in which the changes in the flow patterns in a differentially heated rotating annulus are observed as the imposed temperature gradient and rate of rotation are varied. We attempt to reproduce quantitatively some of the observed results using an accurate mathematical model. The study of this system will not only validate the accuracy of the analysis, but will help develop an understanding of the mechanisms at work in flows with differential heating and rotation. These mechanisms will invariably be present in all large scale geophysical flows.

2.1 The general equations of motion

Before the simplified models are presented, it is useful to discuss the general equations of motion for a rotating fluid of variable density subject to gravity (referred to below as a stratified fluid). It is from these equations that many geophysical fluid models are derived, including those of interest here. However, the general equations already contain important simplifications.

In this section, we write down the general equations and discuss some of the related fundamental concepts, in particular, the aspects that characterize large scale flow. We start with a definition of 'large scale', which leads to the consideration of studying the flow in a rotating frame of reference. This will lead to additional terms in the equations of motion of the fluid. Since these terms are essential for characterizing large scale flow, a discussion of their influence on the flow will be given. Afterwards, the important concept of baroclinicity will be introduced. This feature of the flow has been attributed with causing the instability which leads to the transition from axisymmetric to non-axisymmetric flow.

2.1.1 The Rossby number: large scale flow

We are interested in large scale flows on the Earth, which have the feature that rotation is a dominant factor in its characterization. A useful measure of the importance of rotation is the Rossby number

$$\epsilon = \frac{V}{2\Omega L}$$  \hspace{1cm} (2.1)

where $V$ and $L$ are the characteristic velocity and length scales of the flow and $\Omega$ is the rotation rate of the Earth. It should be noted that $V$ is measured in a frame of reference rotating at a rate of $\Omega$; it is the characteristic velocity relative to the surface of the Earth. When the Rossby number $\epsilon$ is small, rotation is important. One way to see this is to look
at $\epsilon$ as a ratio of time scales. If a fluid element is moving with speed $V$, then to move a distance $L$ would take a time interval $L/V$. Thus, a characteristic time scale of the fluid motion is $\tau_M = L/V$. The period of rotation is proportional to $\tau_R = 1/(2\Omega)$, so the Rossby number is the ratio of the period of rotation to the characteristic time scale of the motion: $\epsilon = \tau_R/\tau_M$ (the factor of $1/2$ in $\tau_R$ is there by convention, see below). If the Rossby number $\epsilon$ is very small, then the time scale of the fluid motion is much longer than the period of rotation. This implies that rotation could play an important role, since the rotation has time to influence changes in the motion. It is possible to define flows as 'large scale' if they have $\epsilon < 1$, although we will be considering flows with $\epsilon << 1$.

Another way to look at this is to notice that, if $L$ is the same order of magnitude as the radius of the Earth, then $\Omega L$ is the same order of magnitude as the velocity of the surface of the Earth. Thus if the Rossby number is small, the relative velocity (measured in a frame of reference rotating with the Earth) is small compared to the velocity of the surface of the Earth. For this reason, and because it is the deviations from solid-body rotation that are interesting, it is useful to study large scale flows in a rotating reference frame.

2.1.2 The equations of motion

The starting point will be Newton's second law for a fluid continuum, in a reference frame rotating uniformly at a rate $\Omega$:

$$\rho \left( \frac{du}{dt} + 2\Omega \times u + \Omega \times (\Omega \times r) \right) = -\nabla p + \rho \nabla \Phi + \mathcal{F}$$

(2.2)

i.e. the change in momentum equals the force per unit volume, where $u = u(r,t)$ is the fluid velocity, $r$ is the position vector, $t$ is time, $p = p(r,t)$ is the pressure, $\rho = \rho(r,t)$ is the density of the fluid, $\nabla$ is the usual three-dimensional gradient operator, $\rho \nabla \Phi$ is the body force (e.g. gravity), $\mathcal{F}$ are the non-conservative forces (e.g. frictional forces) and $\times$ is the vector cross product. The vector quantity $\Omega$ represents the rate of rotation $\Omega$ multiplied by the unit vector that is pointing along the axis of rotation such that the rotation is counter-clockwise with respect to the unit vector. The equation presented is in vector form (the bold type represents three-dimensional vectors). The form that the equations take when expanded will depend on the coordinate system which is used. Since the equations describe the change in momentum of fluid elements, they are sometimes referred to as the momentum equations.

The material derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla,$$

also called the substantive or total derivative, represents the rate of change of the given quantity (e.g. $u$) that a fluid element experiences as it moves with the flow. That is, it
is the rate of change following the flow. The partial derivative $\partial / \partial t$ is the rate of change of the quantity at a fixed location in space. The changes at a fixed location are due to changes effected by 'external' variables and the amount of the quantity which is advected to (carried to) the location by the flow. The term $\mathbf{u} \cdot \nabla$ represents the advection, which is the source of the nonlinearity of the equations.

The term $2\Omega \times \mathbf{u}$ in the momentum equations is the Coriolis acceleration and $\Omega \times (\Omega \times \mathbf{r})$ is the centrifugal acceleration. These are the extra terms that arise due to the application of Newton’s law in a non-inertial frame of reference. Multiplying these terms by the density $\rho$ gives the associated forces, which are often referred to as imaginary forces. These terms are responsible for the unique character of geophysical flows and will be discussed in more detail below.

For a Newtonian fluid, the frictional force can be written as

$$\mathcal{F} = \mu \nabla^2 \mathbf{u} + \frac{\mu}{3} \nabla (\mathbf{\nabla} \cdot \mathbf{u}) \quad (2.3)$$

where $\mu$ is the molecular viscosity. This is exact only if $\mu$ does not depend on the velocity of the fluid (more precisely, when $\mu$ does not depend on the rate-of-strain tensor; see [51]). In general, the molecular viscosity $\mu$ may be a function of temperature, density and pressure.

One of the difficulties with considering large scale flows is the nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ which is contained in the material derivative $d\mathbf{u}/dt$. If this term was not present, i.e. if the equations were linear, then different scales would not influence each other and therefore could be solved for independently. In the nonlinear case, this is not so, and the effects that the small scales have on the large scales cannot be ignored. Attempting to account for the effects of the small scale flow on the large scale flow is the so-called turbulent closure problem and is one of the great unsolved problems in geophysical fluid dynamics and in turbulence modelling. The idea is to separate the large scales by writing the small scale effects in terms of large scale variables. See [49] for a discussion. It is common to include these effects with the non-conservative forces $\mathcal{F}$, which, therefore, will not necessarily have the above form (2.3).

Also needed is the equation for mass conservation, called the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{d\rho}{dt} + \rho (\nabla \cdot \mathbf{u}) = 0. \quad (2.4)$$

This equation states that the rate of change of density of a fluid element is balanced by the divergence of the velocity of the fluid at the location of the element.

If the density is not constant, the first law of thermodynamics gives another equation, which after the assumption that $\rho = \rho(p,T)$ and some manipulation (see [49]), becomes
\[ C_p \frac{dT}{dt} - \frac{T}{\rho} \alpha \frac{dp}{dt} = \frac{k}{\rho} \nabla^2 T + Q \]  \hspace{1cm} (2.5)

where \( T = T(r, t) \) is the temperature, \( C_p \) is the specific heat at constant pressure, \( \alpha = -\left( \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)_p \) is the coefficient of thermal expansion (the subscript \( p \) indicates that the derivative is taken at constant pressure), \( k \) is the thermal conductivity (assumed to be constant) and \( Q \) is due to the internal heat sources. The assumption that \( \rho = \rho(p, T) \) is valid for fluids consisting of a single substance. With this assumption, it is not possible to take into account salinity effects in the ocean or water vapour in the atmosphere.

At present there are six unknowns (\( u, T, \rho \) and \( p \)), but only five equations. Therefore, to form a well-posed problem, the equation \( \rho = \rho(p, T) \), describing the thermodynamic properties of the fluid, must be specified.

In summary, the evolutionary equations for a stratified fluid in a rotating frame of reference are

\[ \rho \left( \frac{du}{dt} + 2\Omega \times u + \Omega \times (\Omega \times r) \right) = -\nabla p + \rho \nabla \Phi + \mathcal{F} \]  \hspace{1cm} (2.6)

\[ \frac{d\rho}{dt} + \rho \nabla \cdot u = 0 \]  \hspace{1cm} (2.7)

\[ C_p \frac{dT}{dt} - \frac{T}{\rho} \alpha \frac{dp}{dt} = \frac{k}{\rho} \nabla^2 T + Q \]  \hspace{1cm} (2.8)

\[ \rho = \rho(p, T) \]  \hspace{1cm} (2.9)

where the differential heating may be introduced either via the internal heat source term \( Q \) or the boundary conditions.

As yet, the domain or the boundary conditions have not been discussed. Considering realistic domains and boundary conditions will greatly increase the complexity of the problem and so, simplifications of these are often central features of a model. At this time, we defer further discussion to the sections below, where we present the particular simplifications made in the models of interest. Here, it will simply be mentioned that, usually, topography is ignored (or only a very simple undulating surface is considered) and the coupling of the atmosphere and ocean are ignored (unless this itself is being studied).

The choice of coordinate system also might add an element of complexity. To consider the fluid flow on the surface of a rotating planet, spherical coordinates are often used (see below). However, when the equations are written in expanded form, the use of spherical coordinates leads to additional (curvature) terms in the equations. Therefore, the equations
are more easily dealt with when considered in a Cartesian coordinate system, as long as
there are accompanying approximations.

2.1.3 The Coriolis and centrifugal forces

We are considering the motion of a fluid in a rotating system. As anyone who has attempted
to walk on a merry-go-round would attest, the same laws of motion do not apply to a
rotating system as to a system which is at rest. It is the imaginary forces (the Coriolis
and centrifugal forces) that arise from writing the equations of motion in a non-inertial
reference frame, which lead to the unintuitive behaviour of a rotating system. For this
reason, it is useful to examine them further.

If the Coriolis term is brought to the right-hand side of the equation (2.6), it becomes
\(-\rho (2\Omega \times \mathbf{u})\), and can be interpreted as a force affecting the acceleration of the fluid. Note
that this force is only felt when the velocity is non-zero, and that it acts at right angles to
the velocity. This particular effect leads to surprising behaviour. For the moment, consider
a Cartesian coordinate system which has its vertical \(z\) axis coinciding with the rotation
vector. In this case, for objects in motion, the force will be exclusively in the horizontal
and will tend to push them to the right. See Figure 2.1. Thus, objects in motion, which are
not subjected to external forces, will appear to follow curved trajectories when observed in
a rotating reference frame. See Figure 2.1.

![Diagram of Coriolis force](image)

Figure 2.1: The Coriolis force. A particle moving with velocity \(\mathbf{u}\), in a frame of reference rotating
counter-clockwise at rate \(\Omega\), will feel a Coriolis force \(\mathbf{C}\) pulling it to its right. A possible path the particle
may follow is indicated with a dashed line.

It is possible to reinterpret the Rossby number \(\epsilon\) using the Coriolis acceleration. Given
\(V\) and \(L\) as above, a characteristic scale for the acceleration of the fluid (see equation (2.6))
is \(V^2/L\) (velocity over time) and for the Coriolis acceleration is \(2\Omega V\). Thus, the Rossby

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number is the ratio of the relative acceleration to the Coriolis acceleration (note the factor of 1/2 in the Rossby number is now present). And so, if the Rossby number is small, then it is the Coriolis force which approximately balances the forces on the right-hand side of equation (2.6). If all forces except the pressure term are assumed to be of order \( \epsilon \) or smaller, then the Coriolis force approximately balances the pressure gradient. This is called the geostrophic approximation, and is the fundamental assumption of the quasigeostrophic potential vorticity equations discussed in the next section.

The centrifugal force \(-\rho[\Omega \times (\Omega \times \mathbf{r})]\) acts to pull an object away from the axis of rotation. Consider an object travelling with uniform speed on a circular path about the origin, such that its angular velocity is \( \Omega \). If we observe the object from a stationary (inertial) reference frame, it is clear that a force, acting towards the origin, is needed to keep the object on this path (if there is no force, there is no change in momentum and the object would travel in a straight line). See Figure 2.2. Such a force is called a centripetal force. Now, if we look at the object in a frame of reference rotating at rate \( \Omega \), then the object will appear to be stationary. However, the force acting on the object in the inertial frame will still be acting in the rotating frame. In order for the object to be stationary, by Newton’s second law, the sum of the forces must be zero. Newton’s laws apply in inertial frames of reference, so if we would like to use these laws in the non-inertial (rotating) frame, it is necessary to add an imaginary force which balances the centripetal force. This additional force is the centrifugal force. Thus an object in a rotating reference frame always feels the centrifugal force trying to pull it away from the axis of rotation (i.e. an applied force is needed to keep an object stationary). The centrifugal force is greater the further the object is away from the axis of rotation and the higher the rotation rate and it does not depend on the velocity of the object.

It is instructive to write out the Coriolis and centrifugal terms explicitly in spherical coordinates, which is the most applicable geometry for motion on the Earth. In this way, some of the important effects of the curvature can be seen, perhaps the most important being the variations of the imaginary forces as a function of latitude. A common approximation assumes that this variation of the Coriolis force is the most important effect associated with the curvature of the planet.

The spherical coordinates will be denoted \((\varphi, \theta, r)\), see Figure 2.3, where \( \varphi \in [0, 2\pi) \) is the longitudinal coordinate (azimuthal angle), \( \theta \in [-\pi/2, \pi/2] \) is the latitudinal coordinate (equals \( \pi/2 \) minus the polar angle) and \( r \in [0, \infty) \) is the radial coordinate (distance from the origin, the center of the sphere). Write \( \mathbf{u} = (u, v, w) = u\hat{\varphi} + v\hat{\theta} + w\hat{r} \), where the hats represent unit vectors in the coordinate directions. Then the Coriolis acceleration is

\[
2\Omega \times \mathbf{u} = 2\Omega \left[ (w \cos \theta - v \sin \theta) \hat{\varphi} + (u \sin \theta) \hat{\theta} - (u \cos \theta) \hat{r} \right]. \tag{2.10}
\]
Generally, the vertical velocity $w$ is assumed to be an order of magnitude smaller than the horizontal velocities ($u$ and $v$), so the Coriolis acceleration in the horizontal is then approximately

$$2\Omega \sin \theta \left(-v\dot{\phi} + u\dot{\theta}\right).$$

At the equator, where $\theta = 0$, the horizontal Coriolis acceleration is zero, while at the poles ($\theta = \pm\pi/2$) it reaches a maximum. For this reason, it is useful to define the local Rossby number,

$$\epsilon = \frac{V}{fL}$$

(2.11)

where $f = 2\Omega \sin \theta$ is the Coriolis parameter, and $V$ and $L$ are the characteristic velocity and length scales of the flow. This local parameter is a more accurate measure, for instance, of the validity of the geostrophic approximation, which, it can been seen, cannot hold near the equator.

In spherical coordinates, the centrifugal term

$$\bm{\Omega} \times (\bm{\Omega} \times \mathbf{r}) = r\Omega^2 \left[(\sin \theta \cos \theta) \hat{\theta} - \left(\cos^2 \theta\right) \hat{r}\right]$$

(2.12)

is also dependent on latitude. At the equator, it acts purely in the vertical while at the poles, it is zero. Usually, the latitudinal component is neglected and the vertical component is included as a reduction in gravity, since it can be written as a body force.
2.1.4 Baroclinic instability

Stratification combined with the vertical shear of the axisymmetric solution has been stipulated to be the origin of the instability which causes the transition from axisymmetric to non-axisymmetric flow in stratified rotating systems. The vertical shear implies the existence of a horizontal temperature gradient within the fluid \((via\) the geostrophic approximation\), therefore, the mechanism has been called baroclinic instability. Alternatively, it is a mechanism of interest since its existence depends on the presence of rotation and a horizontal temperature gradient (which may imply differential heating).

Before the presentation of the particular models, we will make a short digression to discuss this important topic. A fluid is called baroclinic if surfaces of constant density are not aligned with surfaces of constant pressure. It should be noted that for a barotropic fluid, these surfaces coincide, in which case the effects of stratification may be neglected. Below, we give a simplified explanation of the basic mechanism of baroclinic instability which was originally argued by Eady [13]. See also [49] or [19] for more detail.

Consider a stratified fluid in equilibrium such that there is lighter fluid over heavier fluid. If a fluid element (a small volume of fluid) is adiabatically displaced vertically upward, then the fluid element will find itself surrounded by lighter fluid. Due to gravity, the heavier fluid will sink back in the direction it came, tending to decrease the displacement. This property suggests that small perturbations from the original fluid structure will not significantly change the structure. Thus, the equilibrium can be called stable. This argument is relevant,
for example, for Boussinesq fluids, where the density does not depend on the ambient pressure. A modification is necessary for atmospheric application, however, the idea is identical.

Figure 2.4: The mechanism of baroclinic instability. The isolines of constant density $\rho$ of a fluid are tilted by an angle $\psi$ from the horizontal. If a fluid element at position A is moved to position B, the particle will move further away from A. If the angle of displacement $\phi$ is smaller than $\psi$, the same instability will be observed.

Now consider a fluid in an equilibrium, where the lighter fluid lies above the heavier fluid, but the surfaces of constant density are tilted by an angle $\psi$ in the meridional plane. See Figure 2.4. Such a situation can arise, for instance, in a geostrophic zonal flow with vertical shear where the horizontal density gradient is needed to satisfy the thermal wind equations (see [49]). If a fluid element is displaced from position A in Figure 2.4, to position B, it will be surrounded by fluid which is heavier and will feel a force which will accelerate it upward. This tends to push the fluid element further from its original position, i.e. there is an instability. If the angle of displacement, $\phi$, is smaller than $\psi$ and above the horizontal, the instability will be observed.

This argument does not discuss under what conditions there will be perturbations of this type which lead to the instability of the equilibrium. In fact, if this idea is correct, these perturbations cannot be present (or at least effective) in all flows with tilting density curves, since there exist stable flows with this property.

Finally, the mechanism depends crucially on the existence of tilting surfaces of constant density, which can be realized in a rotating fluid if the equilibrium flow has vertical shear.


2.1.5 The general equations of motion: summary

The general equations of motion have been presented for a stratified fluid subject to rotation and there has been a discussion of some of the main features of such flows, namely, the possibility of baroclinic instability and the Coriolis and centrifugal terms. The equations and the concepts they incorporate are the core of geophysical fluid dynamics and are present in most models. Ideally, it would be possible to work directly with these equations, but realistically, it is necessary to work with simplified models. Below, different types of simplifications are used to formulate two different models. One uses scaling arguments and the other model studies a more simple physical system without simplifying the equations significantly. It is hoped, however, that with the simple models something can be learned of the general system.

2.2 The two-layer quasigeostrophic potential vorticity equations

The fundamental assumption of the quasigeostrophic potential vorticity equations is that the flows being modeled have a small local Rossby number $\epsilon$. That is, the effects of rotation are important. With the local Rossby number as the small parameter, the equations can be derived from the general governing equations (2.6) – (2.9) using a formal asymptotic expansion. The expansion results in evolutionary equations for the zeroth-order stream functions (from which the zeroth-order velocities can be calculated). The zeroth-order dependent variables are accurate to $O(\epsilon)$, where the notation $O(\epsilon)$ indicates equality to a multiplicative constant in the limit that $\epsilon$ goes to zero.

The two-layer version of the model assumes that the fluid consists of two layers, where each layer has constant density and there is no mixing of the layers. The rationale behind this assumption is that it leads to the most simple system in which to study baroclinic instability (which requires vertical shear and stratification). The simplification results in the elimination of the vertical dependence of the dependent variables. The two-layer quasigeostrophic potential vorticity equations have been shown to be a very useful tool, and have been used by many researchers including Pedlosky [48], Lorenz [37], Lewis [36], Moroz and Holmes [46], Anderson [1] and Whitaker and Barcilon [55].

An outline of the derivation of the two-layer model is presented below. For more detail see Appendix B or [49]. Here, we will also discuss the approximations involved and attempt to describe their validity and usefulness.

In this model, it is assumed that the fluid consists of two immiscible layers and that
the fluid in each layer is incompressible. The fluid is assumed to be contained in a periodic channel of width $L$ and height $2D$ (flat upper and lower boundaries). See Figure 2.5. The dependent variables are functions of $x$, $y$ and $t$. The $x$-coordinate gives the zonal direction and is analogous to the longitude on the Earth. The $y$-coordinate defines the meridional variable and is analogous to the latitude. $z$ is the vertical coordinate and $t$ is time.

![Figure 2.5: A two-layer fluid in a periodic channel of width $L$ and height $2D$. $h = h(x,y,t)$ gives the height of the interface between the layers and $\rho_n$ is the constant density in the $n$th layer.](image)

It is assumed that the Coriolis parameter is $f = f_0 + \beta_0 y$, where $f_0 = 2\Omega \sin \theta_0$, $\beta_0 = (2\Omega/r_0) \cos \theta_0$, $\Omega$ is the rotation rate of the Earth, $r_0$ is the radius of the Earth, and $\theta_0$ is a reference latitude (which determines $y = 0$). That is, the Coriolis force is assumed to vary linearly with latitude. It can be argued that this channel flow, called the $\beta$-plane approximation, models the flow of a fluid on a sphere contained between two latitudes as long as $L/r_0$ is small (range of latitude is small). That is, it is assumed that in this case all curvature effects are negligible except the variation of the Coriolis force with latitude.

The boundary conditions at the upper and lower surfaces are determined through a matching with an Ekman layer, which is a boundary layer which assumes that the Rossby number $\epsilon$ is small, frictional forces are important and the fluid velocity is zero (no-slip) at the boundary. It is assumed that there are no edge effects at the lateral boundaries ($y = \pm L/2$), and that there is no flow through these boundaries. Periodicity is required in the $x$-direction.

The derivation starts with the general equations (2.6) – (2.9) where the dependent
variables are functions of $x$, $y$, $z$ and $t$ and the non-conservative forces are given by

$$\mathcal{F} = \left[ A_H \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + A_V \frac{\partial^2}{\partial z^2} \right] \mathbf{u} \quad (2.13)$$

That is, the usual coefficient of viscosity is replaced with two larger ‘coefficients of viscosity’: $A_H$ for the horizontal variations and $A_V$ for the vertical variations [49], which are considered as parameters that depend on the characteristics of the small scale flow. This assumption is made to allow the dependent variables to be interpreted as large scale, where the effects of the turbulent smaller scales are assumed to be parameterized by the increased viscosity. Although this is an ad hoc assumption, it is common to make a simplification of this kind to obtain a tractable problem. It is hoped that this assumption brings the solutions into a laminar as opposed to a turbulent regime. Since there is little small scale variation in a laminar flow, it can be hoped that estimation techniques will produce accurate results.

After a rescaling of the equations, it is assumed that the dependent variables can be expanded in a series of powers of $\epsilon$. At this point several assumptions are made concerning the magnitude of some of the dimensionless parameters. In particular, the effective horizontal Reynolds number, $Re = VL/A_H$, and the variations of the height of the interface are both assumed to be of order $\epsilon$. The horizontal length scale $L$ is assumed to be much larger than the vertical length scale $D$ and the horizontal velocities are assumed to be much larger than the vertical velocities. The horizontal pressure gradient is assumed to be of the same order as the Coriolis parameter and the vertical pressure gradient is assumed to be given by the hydrostatic approximation. Finally, the vertical coefficient of viscosity $A_V$ is assumed to be $O(\epsilon^2)$, see [49].

Considering all these assumptions, the terms in the equations which are $O(1)$ (zeroth-order) are collected to give the geostrophic approximation, where the Coriolis force balances the pressure gradient. This, however, does not give evolutionary equations for the zeroth-order dependent variables and so it is necessary to collect the $O(\epsilon)$ terms. Here, it is possible to eliminate the first-order dependent variables. Upon integration in the vertical, and application of the conditions at the boundaries, the layered quasigeostrophic potential vorticity equations are obtained. The equations describe the time evolution of the zeroth-order stream function in each layer (from which the flow velocity can be calculated), which are independent of the vertical coordinate $z$. Note that it is possible to introduce the two-layer approximation only at this last step, with the vertical integration being performed on the three-dimensional quasigeostrophic potential vorticity equations.

Consider the two-layer, non-mixing assumption. In the annulus experiment, most of the vertical motion occurs close to the annulus boundaries, and the flow in the interior is predominantly horizontal. Observations show that large scale mid-latitude flows in the
atmosphere are also predominantly horizontal. Thus, it may be argued that the vertical motion is most important for the creation of the vertical structure of the fluid and otherwise has secondary effects. Furthermore, in Pedlosky [49], it is shown that there is a 'one-to-one' relationship between the layered model and a finite difference approximation in the vertical. These arguments, however, would seem more plausible in the case when there are multiple layers. We will simply restate that the purpose of the two-layer approximation is to develop the most simple model possible in which baroclinic effects can be studied. Upon comparing the results of, for instance, Barcilon [3] and Pedlosky [48], it can be seen that the two-layer model contains some of the same characteristics of the transition from axisymmetric to non-axisymmetric flow as the three-dimensional model. Thus, it is reasonable to assume that results from a two-layer model will be qualitatively useful.

Although many assumptions regarding the magnitude of the parameters are made, these are chosen with the aid of observations [49]. Even so, it can be argued that the model reproduces flows which are in the region of parameter space where the assumptions are valid. If we wish to observe flows outside this region, the derivation has to be repeated while considering new assumptions. For this reason, it is necessary to remember which approximations are made so that we do not inadvertently move 'too far' from the range where the approximations are reasonable. It should also be noted that the two-layer equations are only for the zeroth-order dependent variables and therefore the best we can hope to have is an $O(\epsilon)$ approximation. For typical large scale atmospheric flows of interest, the observed Rossby number is approximately 0.1 [49]. Therefore, the errors regarding the assumptions of the size of the parameters seem acceptable in relation to the potential errors due to the assumption that the flow can be described by two layers of a non-mixing incompressible fluid, and the unprovable assumption that the small scale turbulent motion can be accounted for with the coefficients $A_H$ and $A_V$.

Considering the assumptions that have been made, the model is not expected to give quantitatively accurate results. However, the purpose of the model is to provide a simple tool with which to qualitatively study certain characteristics of geophysical fluid dynamics. In fact, this model exhibits solutions which can be associated with observed atmospheric flows [48], [36].

The particular model studied in this thesis contains a sinusoidal forcing term which models, in a simple way, the latitudinal variations of radiative heating. The forcing results in a basic state (steady solution) which has sinusoidal horizontal shear as well as vertical shear. The horizontal shear is an additional potential source of instability. It should be noted that an alternative view is often taken; the sinusoidal form of the basic state is assumed and the forcing is prescribed so that the basic state satisfies the equations. Lewis
[36] studied this model using numerical experimentation, and argued that it could be used to study the atmospheres of the giant planets and the internally heated rotating annulus experiment.

The two-layer quasigeostrophic potential vorticity equations are given by

$$\frac{\partial}{\partial t} \nabla^2 \sigma + \beta \frac{\partial \sigma}{\partial x} + J \left( \sigma, \nabla^2 \sigma \right) + J \left( \delta, \nabla^2 \delta \right) = -r^{-1} \nabla^2 \sigma + Re^{-1} \nabla^4 \sigma$$  \hspace{1cm} (2.14)

$$\frac{\partial}{\partial t} \left( \nabla^2 - 2F \right) \delta + \beta \frac{\partial \delta}{\partial x} + J \left( \sigma, \nabla^2 \delta \right) + J \left( \delta, \nabla^2 \sigma \right) + 2FJ \left( \delta, \sigma \right) = -r^{-1} \nabla^2 \delta + Re^{-1} \nabla^2 \left( \nabla^2 - 2F \right) \delta - Q$$  \hspace{1cm} (2.15)

where

$$Q = -4\pi^2 \left( r^{-1} + Re^{-1} \left( 4\pi^2 + 2F \right) \right) \cos 2\pi y$$

and $J(u,v) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$, $\sigma = \sigma(x,y,t) = (\chi_1 + \chi_2)/2$, $\delta = \delta(x,y,t) = (\chi_1 - \chi_2)/2$ (where $\chi_n$ is the stream function in the $n$th layer, $n = 1, 2$), $\nabla^2$ is the Laplacian in two dimensions and $r, Re, \beta$ and $F$ are real parameters. The parameter $r = \epsilon D \left( 2f_0/A_V \right)^{\frac{1}{2}}$ is the inverse of a rescaled Ekman number, $Re$ is the effective Reynolds number, $\beta = \beta_0 L/ (\epsilon f_0)$, is the scaled $\beta_0$, and $F = f_0^2 L^2 / (g (\Delta \rho/\rho_0) D)$ is the internal rotational Froude number, where $\Delta \rho = \rho_1 - \rho_2$, $\rho_n$ being the density of the fluid in the $n$th layer, $\rho_0$ is the average density and $g$ is the gravitational acceleration. See Appendix B. The domain is given by: $-\infty < x < \infty$, $0 < y < 1$, where $x$ and $y$ have been scaled and $y$ has been shifted for convenience. The rescaled Ekman number $r$ can be interpreted as representing the importance of the 'viscous' effects that are imposed via the Ekman layer. As $r$ decreases, the associated dissipation increases. In fact, this parameter is more important than the effective Reynolds number, which represents the viscous effects within the layer. The Froude number $F$ is a non-dimensional measure of the relative importance of rotation and stratification and can be written in terms of a Burger number [49]. If the density difference between the layers is assumed to be generated by differential heating, it can be interpreted as a measure of the relative importance of rotation to differential heating.

The boundary conditions are:

(i) $\sigma(x,y,t) = \sigma(x + \gamma, y, t)$ and $\delta(x,y,t) = \delta(x + \gamma, y, t)$, i.e. $\sigma$ and $\delta$ are periodic in $x$ with period $\gamma$, which is a parameter.

(ii) $\frac{\partial \sigma}{\partial x} = \frac{\partial \delta}{\partial y} = 0$ at $y = 0$ and $y = 1$.

This ensures no tangential velocity at the latitudinal boundaries.
(iii) \[ \lim_{x \to \infty} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \sigma}{\partial y \partial t} \, dx = \lim_{y \to 0} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \delta}{\partial y \partial t} \, dx = 0 \] at \( y = 0 \) and \( y = 1 \).

This means there is no change in circulation about the latitudinal boundaries.

With \( Q \) chosen as above, a steady solution is:

\[
\delta = -\cos(2\pi y) \quad (2.16)
\]

\[
\sigma = 0. \quad (2.17)
\]

This solution corresponds to a stream function in the first layer \( \chi_1 = -\cos(2\pi y) \) and the opposite sign in the second layer. The zonal fluid velocity \( u_n \) and the meridional (latitudinal) velocity \( v_n \), in each layer \((n = 1, 2)\), can be calculated from the stream functions \( \chi_n \) using \( u_n = -\partial \chi_n / \partial y \) and \( v_n = \partial \chi / \partial x \).

The steady solution is therefore given by

\[
\begin{align*}
  u_1 &= -2\pi \sin(2\pi y) \\
  u_2 &= 2\pi \sin(2\pi y)
\end{align*} \quad (2.18)
\]

with no flow in the latitudinal direction in both layers \((v_n = 0, n = 1, 2)\), where \( u_n \) is the fluid velocity along the channel in the \( n \)th layer. This is a purely zonal flow which is independent of \( x \) and has nonlinear latitudinal shear. There are two oppositely directed ‘jets’ in each layer and also oppositely directed in the vertical. See Figure 2.6.

### 2.3 The differentially heated rotating annulus experiment

A simple experimental system which has been used to study rotating stratified flow is the differentially heated rotating cylindrical annulus. In this system, the annulus is placed on a turn-table with fluid filling the space between the cylinder walls. The differential heating is imposed by maintaining the side-walls of the annulus at different temperatures. See Figure 2.7. The details of the flow are observed as the temperature difference and rotation rate of the turn-table are varied. Here, we are interested in studying a mathematical model with the intent of quantitatively reproducing the observations of the laboratory experiment. For this reason, we try to keep the approximations to a minimum. Below, we describe the equations of the model which are very similar to the general equations \((2.6) - (2.9)\). However, to begin, we will discuss the laboratory experiments in more detail and in so doing, describe some of the results we hope to reproduce.
2.3.1 The experiments

Many experimental configurations have been used to study stratified fluids subject to rotation. See [26] or [52] for a review. Many of the first experiments were not in cylindrical annuli, but in a rotating dish pan, where the edge of the pan was heated (Exner, 1923 vis. [18], [17]). Although the geometry seems to be more applicable to the atmosphere and ocean (smaller height to width ratio as well as no inner boundary), some of the flow features were not reproducible due to experimental difficulties [26]. There has also been an experiment which has simulated the differential heating by studying a two-layer fluid where the flow was generated by rotating the top and bottom surfaces at different rates [22].

Here, we concentrate on the differentially heated rotating annulus experiments. Even within this class of experiments, many variations are possible. Hide and Mason [27] used internal heating via an electrical current to differentially heat the fluid with the idea that it models solar heating more realistically. Also, for similar reasons, experiments have been done with differential heating from below [24]. However, most often, the differential heating
is produced by maintaining the inner and outer cylinder walls at different temperatures. Many experiments of this kind have been performed with different experimental variables (e.g. different fluid properties and annuli of different dimensions) which, we will see, has an effect on the results. Usually, the outer cylinder is warmer than the inner cylinder. However, the opposite was used to study the effects of the centrifugal buoyancy [31] (see below).

An important distinction between some of the experiments is in the nature of the upper boundary. Some experiments allowed the upper surface to be free (no lid), in which case tracers were introduced into the liquid and the observations were made visually by the experimenter [16], [11]. Other experiments had a rigid lid [29], [30], [32], [35], where the flow was observed via temperature fluctuations measured by thermocouples placed at various locations inside the annulus. Using this technique, the vertical structure of the flow is more easily observed by placing thermocouples at various depths. However, the thermocouples are invasive. A discussion of the possible effects of the thermocouples is given by Fein [15]. With the use of multi-level tracer streak photography [11], it is also possible to obtain the vertical structure without thermocouples. However, it seems that this technique has not been widely used. Fein [15] gives an experimental comparison of the rigid versus free upper boundary. In the free upper boundary experiments, the upper surface of the fluid tends to have a parabolic shape with concavity depending on the rate of rotation. It has been argued that the parabolic upper surface simulates a radial variation of the rotation rate similar to the $\beta$ effect [41], [15]. The variation of the upper surface in the free upper boundary case makes the mathematical analysis more difficult than for the rigid lid case. However, if the variations in height are negligible, the assumption of a flat upper surface with stress free boundary conditions at the top of the annulus may be reasonable. See, for instance, Miller and Butler [43] who performed numerical experiments using this assumption.

There are also variations in the experimental procedures, e.g. the initialization of the flow. Some experiments perform a 'spin-up' from rest to the desired rotation rate [29]; transients are allowed to pass before the flow property is determined. Others increment the parameters slowly, allowing equilibration to take place before changing the parameters. In this manner, the transition curves are more easily traced out and the bifurcating flow may be observed to grow slowly as the parameters are increased. This is the procedure which is most like the bifurcation analysis which we will perform. It should be noted however that theoretically, if only the incremental approach is used, then all the possible long time behaviour may not be observed. The idea of the 'spin-up' is that the noise in the initial conditions may cause the flow to approach different equilibria. Another method which has
been used in an attempt to reproduce all possible solutions is the mixing method [30]. Here, the fluid is allowed to equilibrate at a given rotation rate and temperature differential. It is then thoroughly mixed and the resulting flow is observed. The hope is that the mixing creates a variety of initial conditions which are 'different' enough that all the possible equilibria (stable attractors) might be found.

As discussed in Chapter 1, the results are usually presented in a log-log plot of the Taylor number $T = 4\Omega^2 R^5/(\nu^2 D)$ versus the thermal Rossby number $\mathcal{R} = \alpha g D \Delta T / (\Omega^2 R^2)$. $\Omega$ is the rate of rotation, $R = r_b - r_a$ is the difference between the outer and inner radii of the annulus, $\nu$ is the kinematic viscosity of the fluid and $D$ is the depth of the fluid, $\Delta T = T_b - T_a$ is the imposed horizontal temperature gradient, $\alpha$ is the coefficient of thermal expansion and $g$ is the gravitational acceleration.

Although there have been many configurations and dimensions of the rotating annulus experiments, many of the observations are strikingly similar. In particular, generally, the experiments find four major flow regimes: axisymmetric, steady wave, vacillation and irregular. See Figure 1.4 for a schematic plot of a typical experiment. The shape of the transition curve between the axisymmetric and non-axisymmetric regimes is generally of the same shape as well. There is a critical value which the Taylor number must exceed for the axisymmetric solution to be unstable. For values of the Taylor number less than this value, the axisymmetric solution is stable and for values greater, there is a range of the thermal Rossby number for which it is unstable. This leads to the knee shape of the transition curve. Two other important common features are (1) the variation of the critical wave number (the wave number of the bifurcating wave) along the transition curve between the axisymmetric and non-axisymmetric regimes and (2) the hysteresis of steady waves of different wave numbers in the steady wave regime. The wave number gives the number of spatial undulations in a unit distance (e.g. the number of crests per unit distance).

Although the different dimensions of the experimental annulus lead to similar shaped regime diagrams, there are significant differences in the critical wave numbers and the locations in parameter space where the transitions occur. This can be seen by comparing the results of James et al. [30] and those of Fein [15].

Different experiments have revealed hysteresis in various regions. The most marked, however, seems to be in the steady wave regime; different wavelengths are seen in the same subregions. See [28], [32] and [29]. The hysteresis is usually studied by allowing equilibration of a stable wave solution of a certain wave number for a given value of the parameters. One of the parameters is incremented until the wave becomes unstable and another wave of a different wave number is allowed to equilibrate. The process is then reversed until the second wave becomes unstable and the original wave is again observed.
Hysteresis occurs when the forward and backward transitions between the two waves do not occur at the same location in parameter space. In the annulus experiments, the hysteresis of the waves occurs in all regions of the steady wave regime, and seems to be associated with the critical wave number transitions along the axisymmetric to non-axisymmetric transition curve [32]. We hope the analysis will shed light on the mechanism of this hysteresis.

2.3.2 The mathematical model

In modeling the experiment directly, we choose the same annulus dimensions and fluid properties that were used in the experiments of Fein [15]. In order to keep the approximations to a minimum, we start with the general equations (2.6) - (2.9) where we assume the non-conservative forces \( F \) are given by equation (2.3). We also adopt the Boussinesq approximation. In this approximation, variations of all fluid properties are assumed to be negligible and the effects of the variation of the density of the fluid are assumed to be evident only in the buoyancy forces. In particular, the equation of state of the fluid is assumed to be

\[
\rho = \rho_0 \left(1 - \alpha (T - T_0)\right)
\]  

(2.19)

where \( \rho \) is the density of the fluid, \( T \) is the temperature, \( \alpha \) is the coefficient of thermal expansion, \( \rho_0 \) is the density at a reference temperature \( T_0 \) and \( \alpha (T - T_0) \) is assumed to be small compared to the quantity one. With this approximation, the fluid can be considered incompressible as long as the temperature gradients are not too large.

The boundary conditions which are chosen require the velocity of the fluid to be zero on all boundaries (no-slip conditions), and we consider the temperature at the inner and outer cylinder held fixed at \( T_a \) and \( T_b \), respectively, while the bottom and top are insulating (this requires the derivative of the temperature with respect to the vertical coordinate to vanish at the boundaries). We consider the fluid in a reference frame rotating with the cylinder, which gives rise to terms in the equation which represent the Coriolis and centrifugal forces. Circular cylindrical coordinates are used, where the radial, azimuthal and vertical (or axial) coordinates will be denoted \( r \), \( \phi \) and \( z \), respectively, with unit vectors \( \hat{i} \), \( \hat{\varphi} \) and \( \hat{k} \) (see Figure 2.7).

In the Boussinesq approximation, the nonconservative forces are given by \( F = \mu \nabla^2 u \), and the continuity equation is reduced to the incompressibility condition \( \nabla \cdot u = 0 \), where \( u = u\hat{i} + v\hat{\varphi} + w\hat{k} \) is the velocity of the fluid. Upon dividing by \( \rho \), the vector form of the general equations (2.6) - (2.9) in cylindrical coordinates becomes:

\[
\frac{\partial u}{\partial t} = \nu \nabla^2 u - 2\Omega \hat{k} \times u + (g\hat{k} - \Omega^2 r\hat{i}) \alpha (T - T_0) - \frac{1}{\rho_0} \nabla p - (u \cdot \nabla) u
\]  

(2.20)
Figure 2.7: The differentially heated rotating annulus. Circular cylindrical coordinates are used: the radial coordinate $r$ is the distance from the cylinders' axes, $\varphi$ is the azimuthal coordinate and the vertical coordinate $z$ is the distance from the floor of the cylinders. The annulus is rotated at rate $\Omega$ and the inner wall is held at the fixed temperature $T_a$ and the outer wall at temperature $T_b$, creating a differential heating. The inner and outer radii of the annulus are at $r = r_a$ and $r = r_b$. The width of the annulus is $R = r_b - r_a$ and $D$ is the height of the annulus.

\[
\frac{\partial T}{\partial t} = \kappa \nabla^2 T - (\mathbf{u} \cdot \nabla) T
\]

(2.21)

\[
\nabla \cdot \mathbf{u} = 0
\]

(2.22)

where $p$ is the pressure deviation from $p_0 = \rho_0 g (D - z) + \frac{1}{2} \rho_0 \Omega^2 r^2$, $\nu = \mu / \rho_0$ is the kinematic viscosity, $\kappa = k / (\rho_0 C_p)$ is the coefficient of thermal diffusivity, $g$ is the gravitational acceleration and the domain is $r_a < r < r_b$, $0 \leq \varphi < 2\pi$, $0 < z < D$. The kinematic viscosity $\nu$ and the coefficient of thermal diffusivity $\kappa$ are assumed to be constants. With the rotation vector aligned with the vertical axis, the Coriolis acceleration $2\Omega \hat{k} \times \mathbf{u}$ and centrifugal acceleration $-\Omega^2 r \alpha (T - T_0) \hat{i}$ act only in the horizontal. Since the constant density part of the centrifugal force has been eliminated by assuming the form of $p_0$, the remaining part depends only on the variation of the fluid density and is therefore often called the centrifugal buoyancy term. The other body force is given by the gravitational buoyancy term $g \alpha (T - T_0) \hat{j}$.

The boundary conditions are

\[
\mathbf{u} = 0 \quad \text{on} \quad r = r_a, r_b \text{ and } z = 0, D
\]

\[
T = T_a \quad \text{on} \quad r = r_a
\]
\[
T = T_b \quad \text{on} \quad r = r_b \\
\frac{\partial T}{\partial z} = 0 \quad \text{on} \quad z = 0, D
\]  
(2.23)

with \(2\pi\) periodicity in \(\varphi\).

In order to generalize the problem and simplify the boundary conditions, we make a change of variables:

\[
\begin{align*}
  r &= Rr' \\
  z &= Dz'
\end{align*}
\]  
(2.24)

and write

\[
T = T' + \Delta T r' - \frac{\Delta T r_a}{R} + T_a
\]  
(2.25)

where \(R = r_b - r_a\) and \(\Delta T = T_b - T_a\). Upon dropping the primes, the equations in expanded form become

\[
\begin{align*}
  \frac{\partial u}{\partial t} &= \nu_s \left( \nabla^2 u - \frac{u}{r^2} - 2 \frac{\partial v}{r^2 \partial \varphi} \right) + 2\Omega v - \frac{1}{\rho_0 R} \frac{\partial p}{\partial r} \\
  & \quad - \Omega^2 \rho_0 R T - \frac{\Omega^2 \rho_0 R}{r^2} \left( \frac{\Delta T r - \Delta T r_a}{R} + T_a - T_b \right) - N_r \\
\frac{\partial v}{\partial t} &= \nu_s \left( \nabla^2 v - \frac{v}{r^2} + 2 \frac{\partial u}{r^2 \partial \varphi} \right) - 2\Omega u - \frac{1}{\rho_0 R} \frac{\partial p}{\partial \varphi} - N_\varphi \\
\frac{\partial w}{\partial t} &= \nu_s \nabla^2 w + g\alpha T + g\alpha \left( \Delta T r - \Delta T r_a \frac{r_a}{R} + T_a - T_b \right) - \frac{1}{\rho_0 D} \frac{\partial p}{\partial z} - N_z \\
\frac{\partial T}{\partial t} &= \kappa_s \nabla^2 T + \frac{\Delta T}{R} - \frac{\Delta T}{R} u - N_T \\
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial \varphi} + \frac{1}{\delta} \frac{\partial w}{\partial z} &= 0
\end{align*}
\]  
(2.26) - (2.30)

where \(u = \hat{u} + v\hat{\varphi} + w\hat{k}, \delta = \frac{D}{R}, \nu_s = \frac{\nu}{R^2}, \kappa_s = \frac{\kappa}{R^2}\) and

\[
\nabla^2 s = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\delta^2} \frac{\partial^2}{\partial z^2}
\]  
(2.31)

\[
\begin{align*}
N_r &= \frac{1}{R} \left( \hat{u} \cdot \nabla s u - \frac{v^2}{r} \right) = \frac{1}{R} \left( \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \varphi} + \frac{1}{\delta} \frac{\partial u}{\partial z} - \frac{v^2}{r} \right) \\
N_\varphi &= \frac{1}{R} \left( \hat{u} \cdot \nabla s v - uv \right) = \frac{1}{R} \left( \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \varphi} + \frac{1}{\delta} \frac{\partial v}{\partial z} \right) \\
N_z &= \frac{1}{R} \left( \hat{u} \cdot \nabla s w \right) = \frac{1}{R} \left( \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \varphi} + \frac{1}{\delta} \frac{\partial w}{\partial z} \right) \\
N_T &= \frac{1}{R} \left( \hat{u} \cdot \nabla s T \right) = \frac{1}{R} \left( \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \varphi} + \frac{1}{\delta} \frac{\partial T}{\partial z} \right).
\]  
(2.32)
The boundary conditions are now

\[ u = 0 \quad \text{on} \quad r = \frac{r_a}{R}, \frac{r_b}{R} \quad \text{and} \quad z = 0, 1 \]

\[ T = 0 \quad \text{on} \quad r = \frac{r_a}{R}, \frac{r_b}{R} \]

\[ \frac{\partial T}{\partial z} = 0 \quad \text{on} \quad z = 0, 1 \quad (2.33) \]

with \(2\pi\) periodicity in \(\varphi\).

It is now instructive to non-dimensionalize the system. We rescale variables as

\[ t \rightarrow \tau t, \quad \tau = \frac{R^2}{\nu} \]

\[ u \rightarrow Vu, \quad V = \frac{R^3\Omega^2}{\nu} \]

\[ T \rightarrow (\Delta T) T \]

\[ p \rightarrow Pp, \quad P = \frac{\rho_0\nu V}{R} \quad (2.34) \]

where \(R, \tau, V\) and \(P\) are the characteristic length scale, time scale, velocity and pressure of the flow. The characteristic length has been chosen as the gap width \(R\) and the viscous time scale has been chosen. The characteristic velocity may be interpreted as a mixture of a diffusive velocity \((\nu/R)\) and the annulus velocity \((R\Omega)\). Note that the spatial coordinates have already been scaled using \(R\) and \(D\) (see above). With this scaling the equations become

\[
\frac{\partial u}{\partial t} = \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} + \sqrt{\tau} \frac{v}{r} \frac{\partial p}{\partial r} - (\alpha \Delta T r) \tilde{T} + -\frac{T}{4} N_r \quad (2.35)
\]

\[
\frac{\partial v}{\partial t} = \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} - \sqrt{\tau} \frac{u}{r} \frac{\partial p}{\partial r} - \frac{1}{r} \frac{\partial p}{\partial \varphi} - \frac{T}{4} N_\varphi \quad (2.36)
\]

\[
\frac{\partial w}{\partial t} = \nabla^2 w + R \tilde{T} - \frac{1}{\delta} \frac{\partial p}{\partial z} - \frac{T}{4} N_z \quad (2.37)
\]

\[
\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{1}{r} \frac{T}{4} u - \frac{T}{4} N_T \quad (2.38)
\]

where \(\tilde{T} = T + r - r_a/R + (T_a - T_0) / \Delta T\), the definitions of \(\nabla^2 T\) and \(N_\varphi\) and the incompressibility equation are unchanged, and

\[
\mathcal{T} = \frac{4\Omega^2 R^4}{\nu^2}
\]

\[
\mathcal{R} = \frac{g\alpha \Delta T}{\Omega^2 R}
\]

\[
\delta = \frac{R}{D}
\]

are the Taylor number, thermal Rossby number and aspect ratio, respectively. These parameters, along with \(\kappa = \kappa / \nu\) and \(\alpha \Delta T\), are the dimensionless parameters of the problem.
(the parameters contained in the definition of $T$ can be shown to be insignificant for our purposes). It is possible to write $\alpha \Delta T$ as $\mathcal{R} T \left\{ v^2 / \left( 4 g R^3 \right) \right\}$, where the expression inside the curly bracket is another dimensionless parameter. Most experimental results are presented as regime diagrams with Taylor number on the horizontal axis and thermal Rossby number on the vertical axis. For a given experiment (fluid properties and annulus geometry fixed), these two parameters have a one-to-one relation to $\Omega$ and $\Delta T$, the physical parameters which are varied. Thus, as long as the other dimensionless parameters are equivalent, the results should be identical from experiment to experiment. However, changing the aspect ratio, for instance, will change the regime diagram. This is immediately obvious when comparing the results of Fein [15] to those of James et al. [29]. Although the shape of the transition curves are similar, the locations of the transitions as well as the critical wave numbers are different. The definitions of the Taylor and thermal Rossby numbers sometimes contain the aspect ratio (see Section 1.2), however, it is not possible to derive a similarity principle from the equations with these definitions. Also, the adjusted definitions do not reconcile the differences between the experimental results mentioned above.

Since the dimensionless parameters are not all independent and because an important parameter appears in the nonlinear part, the analysis will be easier on the dimensional equations.
Chapter 3

Bifurcations in the two-layer model

In this chapter, we show how center manifold reduction is used to find the behaviour close to the bifurcation points. The method produces, from the partial differential equations, a finite-dimensional system which retains all the important local dynamics. See Appendix A, which contains a brief discussion of some general theory. For details see Hassard et al. [23]. Once the finite-dimensional system is obtained, it can be converted to normal form, from which the dynamics may be more readily deduced.

The application of immediate interest is the two-layer model. First, an overview of the center manifold reduction will be given. Then the example of a Hopf bifurcation will be given in some detail. Following is the application of the reduction at a double Hopf point. In this latter description, less detail will be given since the fundamentals are the same and the higher dimension of the equations on the center manifold makes the details cumbersome.

3.1 The two-layer model and linear stability

The analysis will be performed on the two-layer quasigeostrophic potential vorticity equations:

\[
\frac{\partial}{\partial t} \nabla^2 \sigma + \beta \frac{\partial \sigma}{\partial x} + J(\sigma, \nabla^2 \sigma) + J(\delta, \nabla^2 \delta) = -r^{-1} \nabla^2 \sigma + Re^{-1} \nabla^4 \sigma \quad (3.1)
\]

\[
\frac{\partial}{\partial t} (\nabla^2 - 2F) \delta + \beta \frac{\partial \delta}{\partial x} + J(\sigma, \nabla^2 \delta) + J(\delta, \nabla^2 \sigma) + 2F J(\delta, \sigma) = -r^{-1} \nabla^2 \delta + Re^{-1} \nabla^2 (\nabla^2 - 2F) \delta - Q \quad (3.2)
\]

where

\[
Q = -4\pi^2 \left( r^{-1} + Re^{-1} (4\pi^2 + 2F) \right) \cos 2\pi y,
\]
\[ J(u,v) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}, \quad \sigma = \sigma(x,y,t) = (\chi_1 + \chi_2)/2, \quad \delta = \delta(x,y,t) = (\chi_1 - \chi_2)/2 \] (\chi_n is the stream function in the nth layer, n = 1, 2), \( \nabla^2 \) is the Laplacian in two dimensions and \( r, Re, \beta \) and \( F \) are positive real parameters. See Section 2.2.

The boundary conditions are:

(i) \( \sigma(x,y,t) = \sigma(x+\gamma,y,t) \) and \( \delta(x,y,t) = \delta(x+\gamma,y,t) \)

i.e., \( \sigma \) and \( \delta \) are periodic in \( x \) with period \( \gamma \), which is a parameter.

(ii) \( \frac{\partial \sigma}{\partial x} = \frac{\partial \delta}{\partial x} = 0 \) at \( y = 0 \) and \( y = 1 \).

This ensures no tangential velocity at the latitudinal boundaries.

(iii) \( \lim_{x \to \infty} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \sigma}{\partial y \partial t} dx = \lim_{x \to \infty} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \delta}{\partial y \partial t} dx = 0 \) at \( y = 0 \) and \( y = 1 \).

This means there is no change in circulation about the inner and outer boundary.

With \( Q \) chosen as above, a steady solution is:

\[ \sigma = -\cos 2\pi y \quad (3.3) \]
\[ \delta = 0. \quad (3.4) \]

See Section 2.2 for the fluid velocity to which this solution corresponds.

Setting \( \delta = -\cos 2\pi y + \tilde{\delta} \) and \( \sigma = \tilde{\sigma} \) and substituting these into the above equations, the 'perturbation equations' for \( \tilde{\sigma} \) and \( \tilde{\delta} \) are obtained. Dropping the tildes the equations can be written as:

\[ \mathbf{A}\mathbf{U} = \mathbf{LU} + \mathbf{N(U,U)} \quad (3.5) \]

where

\[ U = \begin{pmatrix} \sigma \\ \delta \end{pmatrix} \quad (3.6) \]

\[ \mathbf{A} = \begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 - 2F \end{pmatrix} \quad (3.7) \]

\[ \mathbf{L} = \begin{pmatrix} -r^{-1} \nabla^2 + Re^{-1} \nabla^4 - \beta \frac{\partial}{\partial x} & 2\pi \sin 2\pi y \frac{\partial}{\partial x} \left( 4\pi^2 + \nabla^2 \right) \\ 2\pi \sin 2\pi y \frac{\partial}{\partial x} \left( 4\pi^2 + 2F + \nabla^2 \right) & -r^{-1} \nabla^2 + Re^{-1} \nabla^2 \left( \nabla^2 - 2F \right) - \beta \frac{\partial}{\partial x} \end{pmatrix} \quad (3.8) \]
The same boundary conditions apply and $U = 0$ is now a steady solution. The nonlinear part, $N(U,U)$, is written as such to emphasize the fact that it is a quadratic nonlinearity. Note also that $U$ is a vector of functions and that $L$ and $A$ are matrices of linear differential operators.

Neutral stability curves (or surfaces, depending on the dimension of parameter space) are the boundaries between the regions in parameter space where the steady solution of interest is linearly stable and those regions where it is linearly unstable. In our case, we have a neutral stability curve for each zonal (or azimuthal) wave number (every integer). Figure 3.1 shows the neutral stability curves for the present model when the parameter space is considered two-dimensional ($r$ and $F$ are considered the parameters while all other parameters are considered constants). In the linearized equations, to the left of each curve all small perturbations of the given wave number decay to zero, whereas to the right there is a perturbation which grows exponentially. In the region of parameter space which is to the left of all the neutral stability curves, the solution is linearly stable. In the region to the right of any of the curves, the solution is unstable. If the parameters are varied such that there is a crossing from the stable region to the unstable region, we expect a bifurcation to take place.

The linear stability of a steady solution is defined in terms of the eigenvalues of the system linearized about that solution. If the real parts of all the eigenvalues are negative, then the solution is said to be linearly stable. If any of the eigenvalues has positive real part, then the solution is linearly unstable. To see this, write

$$U(x, y, t) = e^{\lambda t} \bar{U}(x, y)$$

where we will assume that $\bar{U}$ is some small perturbation from the trivial solution. If we substitute this into the linear part of equation (3.5), we get

$$\lambda A \bar{U} = L \bar{U}$$

i.e. a generalized eigenvalue problem, where $\lambda$ are the eigenvalues. Presently, we are concerned with how all small perturbations about the given steady solution behave. If we perturb in the direction of an eigenfunction (multiplication by a small constant), then the corresponding eigenvalue will give us the linear time dependence via equation (3.10). If the operators $A$ and $L$ have certain properties, then the eigenfunctions form a complete
Figure 3.1: The neutral stability curves for the two-layer model for the azimuthal wave numbers $m = 2, 3, 4$. To the left of all the curves, the solution is linearly stable; to the right of any of the curves, the solution is linearly unstable. See Section 3.2.4 and Section 4.2.3 for details of the computation of the neutral stability curves. The points where the curves cross, $p_{23}$ and $p_{34}$, are double Hopf bifurcation points and are discussed in Section 3.3.

basis, and any function can be written as a linear combination of the eigenfunctions. If the perturbations $\tilde{U}$ are expanded in such a way, then it can be seen how the set of eigenvalues give the time behaviour of all perturbations. That is, if all eigenvalues have negative real part, then in equation (3.10), the norm of each term in the expansion of any perturbation decays exponentially since each term is an eigenfunction multiplied by a constant. Thus, (the norms of) all perturbations eventually decay exponentially as long as the perturbations are small enough that the nonlinear terms are negligible. Note that the imaginary part causes oscillation, not growth or decay. If only one eigenvalue has positive real part, the part of a perturbation which is in the direction of the corresponding eigenfunction will grow exponentially (until nonlinearity becomes important). Thus, if all eigenvalues have negative real part, the solution is called linearly stable. If there is any eigenvalue with positive real
part, then the solution is linearly unstable. If there are only eigenvalues with zero real part and negative real part it is called neutrally stable, since there is neither growth nor decay of the part of the perturbation which is in the direction of the eigenfunction corresponding to the eigenvalue with zero real part. In the present case, for the parameter values to the left of the neutral stability curves, the eigenvalues corresponding to the given wave number all have negative real part, and to the right at least one has positive real part. This definition of stability says nothing about transient behaviour. For this reason linear stability is referred to as the long-time behaviour.

Linear stability analysis is justified by the Hartman-Grobman theorem (see e.g. [56] for a precise statement of the theorem). A hyperbolic steady solution is one whose linearization has no eigenvalues with zero real part. The Hartman-Grobman theorem states that if the steady solution is hyperbolic, then the linearization about the solution has qualitatively the same behaviour as the full nonlinear system. This implies that if the solution is linearly stable, then it is indeed a stable solution. More precisely, if a solution is linearly stable, then there is a neighbourhood about the solution where all initial conditions for the nonlinear system will tend to the solution as $t \to \infty$. However, the theorem does not state the size of the neighbourhood about the solution, and so it could be that the neighbourhood is very small. This, however, depends on the specific equations and solution.

If the parameters are varied (e.g., the parameter $r$ in Figure 3.1) such that there is a transition from the stable region to the unstable region, the steady solution is expected to undergo a bifurcation. The form of the bifurcation depends on the type and number of eigenvalues which cross the imaginary axis, on the eigenfunctions, and on the nonlinear terms in the dynamical equations. In this problem, each of the stability curves in Figure 3.1 corresponds to two eigenvalues, which form a complex conjugate pair (one each for the positive and negative integer wave number). Since the original equations are real, the eigenvalues will be real, or complex conjugate pairs. If the transition between the regions occurs at a location where two neutral stability curves cross (see points $p_{23}$ and $p_{34}$ on Figure 3.1), then at this location there are a total of four eigenvalues with zero real part (two complex conjugate pairs). At this point a double Hopf bifurcation occurs. Two parameters are required to describe all the possible behaviour near this bifurcation (here we use $r$ and $F$) and so it is a codimension two bifurcation. It is about these locations that we are able to show that hysteresis of the bifurcating solutions can occur. If the transition between regions occurs at any other location, a single complex conjugate pair of eigenvalues crosses the imaginary axis. Here a Hopf bifurcation occurs and only a single parameter is necessary to describe all local behaviour.

Before we proceed, a few definitions are necessary. As mentioned above, the linear
stability problem for the trivial solution amounts to solving a linear eigenvalue problem. That is, the eigenvalues, \( \lambda_j \), and eigenfunctions, \( \Psi_j \), satisfy

\[
L \Psi_j = \lambda_j A \Psi_j \tag{3.12}
\]

with the same boundary conditions as above.

Also, define the adjoint eigenfunction \( \Psi_j^* \) as a vector function which satisfies

\[
L^* \Psi_j^* = \bar{\lambda}_j A^* \Psi_j^* \tag{3.13}
\]

where \( L^* \) and \( A^* \) are the adjoints of \( L \) and \( A \) respectively and the over-line represents the operation of complex conjugation. The adjoints satisfy

\[
(LV,W) = (V,L^*W) \tag{3.14}
\]

and

\[
(AV,W) = (V,A^*W) \tag{3.15}
\]

for all functions \( V \) and \( W \). The inner product, denoted by the angled braces, is defined as

\[
(V,W) = \int_0^1 \int_{-\frac{T}{2}}^{\frac{T}{2}} (V \cdot W) \, dx \, dy \tag{3.16}
\]

where the dot represents the usual vector dot product (in particular, there is complex conjugation of the components of \( W \)). The eigenvalues of the adjoint problem are the complex conjugates of those of the original problem (equation (3.12)), so the adjoint eigenvalues and eigenfunctions have been labeled in equation (3.13) to match their corresponding solutions of equation (3.12).

The adjoint operators \( L^* \) and \( A^* \) can be calculated from equations (3.14) and (3.15), using Green's identity and integration by parts. It turns out that \( A^* = A \) and

\[
L^* = \begin{pmatrix}
-r^{-1} \nabla^2 + Re^{-1} \nabla^4 + \beta \frac{\partial}{\partial x} & -2\pi \sin 2\pi y \frac{\partial}{\partial x} (2F + \nabla^2) - 8\pi^2 \cos 2\pi y \frac{\partial^2}{\partial x \partial y} \\
-2\pi \sin 2\pi y \frac{\partial}{\partial x} \nabla^2 - 8\pi^2 \cos 2\pi y \frac{\partial^2}{\partial x \partial y} & -r^{-1} \nabla^2 + Re^{-1} \nabla^2 (\nabla^2 - 2F) + \beta \frac{\partial}{\partial x}
\end{pmatrix}
\]

Since \( L^* \neq L \), \( L \) is not self-adjoint. This is a consequence of the nonlinear steady solution. Because \( L \) is not self-adjoint, the adjoint eigenfunctions are needed to project an arbitrary vector function in the direction of an eigenfunction. In particular, it can be shown that if \( i \neq j \), then

\[
\langle \Psi_i, A^* \Psi_j^* \rangle = 0. \tag{3.17}
\]

This means that an adjoint eigenfunction is orthogonal to the span of all eigenfunctions except its corresponding eigenfunction. This important property will be used below to define a projection which will lead to a simplification of the dynamical equations.
3.2 Hopf bifurcation

Suppose we fix all parameters except $r$, which we consider as the bifurcation parameter. Define $r_0$ as the value of $r$ for which there is a single pair of complex conjugate eigenvalues with zero real part ($\lambda_1 = i\omega_0$ and $\lambda_2 = \bar{\lambda}_1 = -i\omega_0$), while all the other eigenvalues have negative real part. Therefore, as $r$ is varied past $r_0$, the trivial solution loses its stability as a single complex conjugate pair of eigenvalues crosses the imaginary axis, and a Hopf bifurcation is expected to occur. However, in order to see the details of the bifurcation, the normal form coefficients must be computed.

3.2.1 The center eigenspace

Define $\Phi_0$ as a vector function which satisfies

$$L\Phi_0 = i\omega_0 A\Phi_0$$

(3.18)

when $r = r_0$. Its complex conjugate $\bar{\Phi}_0$ is a vector function which satisfies

$$L\bar{\Phi}_0 = -i\omega_0 A\bar{\Phi}_0$$

(3.19)

when $r = r_0$.

An adjoint eigenfunction $\Phi_0^*$ satisfies

$$L^*\Phi_0^* = -i\omega_0 A^*\Phi_0^*$$

(3.20)

when $r = r_0$, where $L^*$ and $A^*$ are the adjoints of $L$ and $A$ respectively (see equations (3.14) and (3.15)).

We normalize $\Phi_0^*$ and $\Phi_0^*$ so that

$$\langle \Phi_0, A^*\Phi_0^* \rangle = 1$$

(3.21)

and

$$\langle \Phi_0^*, A^*\Phi_0 \rangle = 1.$$  

(3.22)

There is also a normalization constant associated with $\Phi_0$, but it is arbitrary.

Define the center eigenspace, $E^c = \text{span}\{\Phi_0, \bar{\Phi}_0\}$, and the stable eigenspace, $E^s$, as the span of all other eigenfunctions (those corresponding to eigenvalues with negative real part). We may decompose any vector function $U$ into a part which is in $E^c$ and a part which is in $E^s$. That is, we can write

$$U = z\Phi_0 + \bar{z}\bar{\Phi}_0 + \zeta$$

(3.23)

where $z$ is some complex number, and $\zeta = U - z\Phi_0 - \bar{z}\bar{\Phi}_0$. This may be done by taking $z = \langle U, A^*\Phi_0^* \rangle$, then by the properties of the adjoint, $z\Phi_0 + \bar{z}\bar{\Phi}_0 \in E^c$ and $\zeta \in E^s$. 

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3.2.2 The center manifold

The analysis of the system would be greatly simplified if the stable and center parts could be decoupled. In fact, using center manifold reduction, this can essentially be done close to the bifurcation point.

Define the projection \( P \) of a vector function \( U \) as

\[
PU = \langle U, \Phi_0^* \rangle \Phi_0 + \langle U, \Phi_0^* \rangle \Phi_0.
\]  

This projection is chosen since it decouples the linear part of the dynamical equations into a part which is in the center eigenspace and a part which is in the stable eigenspace.

Thus, taking the projection \( P \) of equation (3.5), we get

\[
\dot{z}\Phi_0 + \dot{z}\Phi_0 = i\omega_0 z\Phi_0 - i\omega_0 z\Phi_0 + \langle N, \Phi_0^* \rangle \Phi_0 + \langle N, \Phi_0^* \rangle \Phi_0
\]

and so

\[
\dot{z} = i\omega_0 z + \langle N, \Phi_0^* \rangle.
\]

However, \( \langle N, \Phi_0^* \rangle \) still depends on the stable space variables and so the system has not yet been decoupled. This is done using the center manifold theorem.

The center manifold theorem states that, given certain technical conditions on \( L \) and \( A \), there exists a differentiable center manifold for equation (3.5):

\[
W_{loc}^c = \left\{ U = z\Phi_0 + \dot{z}\Phi_0 + H (z\Phi_0, \dot{z}\Phi_0) ; \|z\Phi_0 + \dot{z}\Phi_0\| \text{ small}, \ H : E^c \to E^s \right\}
\]  

which is locally invariant, tangent to \( E^c \) at \( 0 (z = 0 \text{ and } r = r_0) \) and locally exponentially attracting (see Appendix A or [23]). In (3.27), \( \| \cdot \| \) is the norm that corresponds to the inner product (3.16). We will assume, without proof, that the conditions of the theorem are satisfied. It has been shown that the incompressible Navier-Stokes equations and the Navier-Stokes equations in the Boussinesq approximation do satisfy the conditions (see [25] and [7]), therefore it may not be unreasonable to assume that they are also satisfied by the quasigeostrophic equations (see also [9]).

Because the center manifold is locally exponentially attracting, any trajectory whose initial condition is close to the center manifold will approach it exponentially quickly as \( t \to \infty \). This implies that the long-time dynamics of the full system of equations are deducible from the dynamics on the center manifold. If there are any fixed points or periodic orbits that exist close to the bifurcation point, they must exist on the center manifold. By close, it is meant that both \( \|z\Phi_0 + \dot{z}\Phi_0\| \) and \( |r - r_0| \) are small. Therefore, if we find an equation which describes the dynamics on the center manifold, we have found a lower-dimensional system from which we can deduce the dynamics. To find this equation,
Note that the center manifold theory states that on the center manifold, $\zeta = H(z, \bar{z})$ for some smooth function $H$. That is, there is a one-to-one correspondence between the coordinate $z$ and the center manifold. See Figure 3.2. So if $H(z, \bar{z})$ is used in place of $\zeta$ in the $z$ equation (3.26), then the equation will describe the dynamics on the center manifold and the only dependent variable will be $z$. This is a consequence of the invariance property (which means that if a trajectory originates on the center manifold it will stay on the center manifold). The invariance can also be used to calculate a Taylor series approximation to the function $H$. Note that the tangency condition already implies that $H = O(|z, \bar{z}|^2)$.

![Figure 3.2: A schematic diagram of a center manifold. The center eigenspace is the horizontal plane and $\zeta$ is written as a function of $z$, the coordinate on the center eigenspace.](image)

Thus, $\zeta$ may be written as

$$\zeta = H(z, \bar{z}) = O\left(|z, \bar{z}|^2\right) = H_{20}z^2 + H_{11}z\bar{z} + H_{02}\bar{z}^2 + \ldots \quad (3.28)$$

where

$$H_{ij} = \frac{1}{i!j!} \frac{d^{(i+j)}}{dz^i d\bar{z}^j} H(0, 0) \quad (3.29)$$

are the Taylor series coefficients of $H(z, \bar{z})$, and close to the bifurcation point, the interesting dynamics of equation (3.5) are contained in

$$\dot{z} = i\omega_0 z + G(z, \bar{z}) \quad (3.30)$$

where

$$G(z, \bar{z}) = \langle N \left( z\Phi_0 + \bar{z}\Phi_0 + H \right), \Phi_0^* \rangle = G_{20}z^2 + G_{11}z\bar{z} + G_{02}\bar{z}^2 + G_{21}z^2\bar{z} + \ldots \quad (3.31)$$
and $G_{ij}$ are the Taylor series coefficients of $G(z, \bar{z})$ as in (3.29). That is,

$$G_{ij} = \langle N_{ij}, \Phi_0^* \rangle \quad (3.32)$$

where

$$N(z, \bar{z}) = N_{20}z^2 + N_{11}z\bar{z} + N_{02}\bar{z}^2 + N_{21}z^2\bar{z} + \ldots \quad (3.33)$$

and $N(z, \bar{z})$ is the nonlinear part (3.9) written in terms of $z$ using the decomposition of $U$ in equation (3.23) with equation (3.28) substituted.

### 3.2.3 The normal form

We want to reduce equation (3.30) to normal form. The reduction to normal form is a series of near-identity transformations which transform the equations to a coordinate system where they take on the most simple form possible. More specifically, the nonlinear part is expanded in a Taylor series (as $G$ above) and the transformations eliminate as many of the terms as possible. The truncation occurs when it can be shown that (for small $z$) the addition of extra terms will not qualitatively effect the solutions of the equations; by this we mean that the addition of extra terms does not alter the existence and stability of the solutions (see e.g. [56]). Writing $z = \rho e^{i\theta}$, where $\rho$ is the magnitude of $z$ in the complex plane and $\theta$ is the phase of $z$, the normal form of equations (3.30) in the polar coordinates $(\rho, \theta)$ are

$$\dot{\rho} = B (r - r_0) \rho + R\rho^3 + O(\rho^4)$$
$$\dot{\theta} = \omega_0 + E (r - r_0) + K\rho^2 + O(\rho^3) \quad (3.34)$$

where $B, R, E, K \in \mathbb{R}$ are the normal form coefficients. If $B \neq 0$ and $R \neq 0$, then for all $\rho$ and $|r - r_0|$ sufficiently small, we have $\dot{\theta} \neq 0$, and there exists a periodic solution of equation (3.34). Thus, there is a Hopf bifurcation at $r = r_0$. If $\dot{\theta} \neq 0$ (for all time), $\dot{\rho} = 0$, and $\rho = \rho_p \neq 0$, then $\rho = \rho_p$ describes the periodic orbit. If $B \neq 0$, $R \neq 0$ and $\rho \neq 0$, then it can be shown that the $O(\rho^4)$ terms in the $\dot{\rho}$ equation and the $O(\rho^2)$ and $O(|r - r_0|)$ terms in the $\dot{\theta}$ equation may be neglected when considering the existence and stability of solutions close to the bifurcation point.

If the reduction to normal form is performed for a general equation, the following formulae can be obtained:

$$B = \left. \frac{d}{dr} \text{Real} \left( \lambda_1 \right) \right|_{r=r_0} \quad (3.35)$$

and

$$R = \text{Real} \left\{ G_{21} + \frac{i}{\omega_0} \left( G_{20}G_{11} - |G_{11}|^2 - \frac{2}{3}|G_{02}|^2 \right) \right\} \quad (3.36)$$

where 'Real' means 'the real part of' and $\lambda_1$ is the eigenvalue which has real part zero when $r = r_0$. 

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In the present application, it can be easily shown that \( B > 0 \) (which implies that the steady solution is stable for \( r < r_0 \)), so the sign of \( R \) will indicate the direction of the bifurcation. If \( R \) is positive, there will be a sub-critical Hopf bifurcation with an unstable periodic orbit \((i.e. \) there exists an unstable orbit for \( r < r_0 \)). If \( R \) is negative, the bifurcation will be supercritical with a stable periodic orbit \((i.e. \) for \( r > r_0 \)).

In order to compute \( R \), we need compute only \( G_{20}, G_{11}, G_{02} \) and \( G_2 \). Thus, the \( N_{ij} \) of interest in 3.33 are:

\[
\begin{align*}
N_{20} &= N(\Phi_0, \Phi_0) \\
N_{11} &= N(\Phi_0, \Phi_0) + N(\Phi_0, H_0) + N(H_0, \Phi_0) + N(H_0, H_0) \\
N_{21} &= N(\Phi_0, H_11) + N(H_11, \Phi_0) + N(\Phi_0, H_20) + N(H_20, \Phi_0)
\end{align*}
\]  

(3.37)

where \( N(\cdot, \cdot) \) on the right hand side is \( N(U, U) \) of (3.9). See Appendix C. It can be seen that \( G_{20}, G_{11}, \) and \( G_{02} \) depend only on the eigenfunctions \( \Phi_0, \Phi_0 \) and \( \Phi_0^* \), while \( G_{21} \) depends only on \( H_{11}, H_{20} \) and the eigenfunctions. This is a consequence of the fact that \( N \) is quadratic.

Now we use the invariance property of the center manifold and apply \((I - AP)\) to equation (3.5) to find \( H_{11} \) and \( H_{20} \). Taking \((I - AP)\) of equation (3.5), we get

\[
(I - AP)\dot{A} = (I - AP) L\zeta + (I - AP) N
\]

(3.38)

where equations (3.23), (3.17) and (3.24) have been used to eliminate the \( z \)-dependence in the linear part.

The invariance of the center manifold gives

\[
\dot{\zeta} = D_z H(z, \bar{z}) \dot{z} + D_{\bar{z}} H(z, \bar{z}) \dot{\bar{z}}
\]

(3.39)

where \( D_z \) indicates derivative with respect to \( z \). So,

\[
A\dot{\zeta} = AD_z H(z, \bar{z}) \dot{z} + AD_{\bar{z}} H(z, \bar{z}) \dot{\bar{z}}
\]

(3.40)

and using equations (3.26) and (3.28) we get

\[
A\dot{\zeta} = A \left( 2zH_{20} + \bar{z}H_{11} + \ldots \right) (i\omega_0 \dot{z} + \ldots) \\
+ A \left( zH_{11} + 2\bar{z}H_{02} + \ldots \right) (-i\omega_0 \dot{z} + \ldots)
\]

\[
= 2i\omega_0 A H_{20}z^2 + i\omega_0 A H_{11}z\bar{z} - i\omega_0 A H_{11}z\bar{z} + \ldots
\]

thus,

\[
A\dot{\zeta} = 2i\omega_0 A H_{20}z^2 + O\left( |z|^2, |z|^3 \right).
\]

(3.41)
Substituting (3.41) into equation (3.38) and collecting terms of order $z^2$, it can be seen that $H_{20}$ satisfies

$$(I - AP)(2i\omega_0 A - L) H_{20} = (I - AP) N_{20}. \quad (3.42)$$

Collecting terms of order $z^2$ in a similar way gives the equation

$$(I - AP) LH_{11} = -(I - AP) N_{11}, \quad (3.43)$$

from which $H_{11}$ can be calculated.

To recapitulate, the following equations must be solved for $r_0, \lambda_1 = i\omega_0, \Phi_0, \Phi_0^*, H_{20}$ and $H_{11}$:

$$L\Phi_0 = i\omega_0 A\Phi_0 \quad (3.44)$$

$$L^*\Phi_0^* = -i\omega_0 A^*\Phi_0^* \quad (3.45)$$

$$(I - AP)(2i\omega_0 A - L) H_{20} = (I - AP) N_{20} \quad (3.46)$$

$$(I - AP) LH_{11} = -(I - AP) N_{11} \quad (3.47)$$

and their solutions substituted into the formulae for $G_{20}, G_{11}, G_{02}$ and $G_{21}$. The equations (3.44) – (3.47) are linear partial differential equations. By using this method, the dynamics near the bifurcation point for the full time-dependent nonlinear partial differential equations can be found by solving steady linear problems (but still partial differential equations).

These equations are general and apply to any Hopf bifurcation analysis. In the present application, we can simplify things a little more by using some information specific to the problem. In particular, the equations can be reduced to a series of ordinary differential equations by assuming a form of the $x$-dependence for the eigenfunctions. However, it is not possible to analytically solve for the $y$-dependence, and so this is done numerically. Using a numerical approximation of the relevant inner products leads to numerical approximations of the normal form coefficient, $R$.

### 3.2.4 Computation and Results

We look for eigenfunctions of the form

$$\Psi = \tilde{\Psi}(y)e^{ikmx} \quad (3.48)$$

where $k = 2\pi/\gamma$ and $m$ is an integer which gives the number of crests of the wave along the channel. Technically, the wave number is given by $km$. However, since $m$ is a conceptually more appealing means of distinguishing between the different waves, we use $m$ to unambiguously label the wave number with the understanding that the wave number is actually $km$. 

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Upon substitution of this form (3.48) in the eigenvalue problem (3.12), the problem is reduced to a set of eigenvalue problems in one spatial variable:

\[ \hat{L}^{(m)} \hat{\psi}^{(m)} = \lambda \hat{A}^{(m)} \hat{\psi}^{(m)} \]  

(one equation must be solved for every integer \( m \)), where \( \hat{L}^{(m)} = e^{-ikmx} \hat{L} e^{ikmx} \) (likewise for \( \hat{A}^{(m)} \)). In practice, for each \( m \), \( r = r_0^{(m)} \) is found such that all eigenvalues of (3.49) have real part negative except a finite number (here a complex conjugate pair). Then \( r_0 \) is taken as \( \min_m (r_0^{(m)}) \). This separation of the \( x \)-dependence is the reason that when two parameters are considered, there are different neutral stability curves for each wave number.

It is necessary to look at the boundary conditions carefully. Given solutions as in equation (3.48), boundary condition (i) will always be satisfied. However, the other conditions will be different for different values of \( m \).

For \( m \neq 0 \), (iii) will always be satisfied. To satisfy (ii), we must have:

\[ \hat{\psi}^{(m)} = 0, \quad \text{at} \quad y = 0, 1. \]  

(3.50)

For \( m = 0 \), (ii) will always be satisfied and we must have

\[ \frac{d\hat{\psi}^{(m)}}{dy} = 0, \quad \text{at} \quad y = 0, 1 \]  

(3.51)  

so that (iii) is satisfied. This is the case for the exact solution.

It is easy to show analytically that for \( m = 0 \), all eigenvalues will be negative as long as \( r > 0, \Re > 0 \) and \( F > 0 \). For \( m \neq 0 \), the eigenvalues and eigenfunctions must be found numerically. The problem, however, is underdetermined (the only boundary conditions are given in equation (3.50)). To complete the problem, the condition

\[ \frac{d^2\hat{\psi}^{(m)}}{dy^2} = 0, \quad \text{at} \quad y = 0, 1 \]  

(3.52)  

will be chosen. This corresponds to the condition on the meridional fluid velocity that \( \partial^2 v/\partial y^2 = 0 \) at \( y = 0, 1 \). This choice is difficult to argue on physical grounds, and so it will be considered as an additional approximation of the model.

The critical parameter value \( r_0 \) can be found by solving the nonlinear equation

\[ \Re (\lambda_1 (r)) = 0 \]  

(3.53)  

where \( \lambda_1 \) is the eigenvalue with the largest real part, regardless of \( m \). Denote \( m_c \) as the \( m \) to which \( \lambda_1 \) corresponds. The discretization of the operators in equation (3.49) and the eigenvalue and eigenfunction approximation are discussed later.
Write
\[ \Phi_0 = \hat{\Phi}(y)e^{ikmcx} = \left( \hat{\phi}_1 \hat{\phi}_2 \right) e^{ikmcx} \]
and
\[ \Phi_0^* = \hat{\Phi}^*(y)e^{-ikmcx}. \] (3.54)

It is $\hat{\Phi}$ and $\hat{\Phi}^*$ which must be calculated numerically.

For the present application, it can be shown that
\[ G_{20}, \ G_{11}, \ G_{02} = 0 \]
because none of the second-order terms of the expansion of the nonlinear part ($N_{ij}$ with $i + j = 2$) will have integer $m_c$ in their (exponential) $x$-dependence, which means that they will be orthogonal to the adjoint eigenfunction. Thus, the transformation to normal form will affect only the fourth-order terms.

In addition to the eigenfunctions, the coefficient $G_{21}$ depends only on the following two single-variate (in $y$) functions: $\hat{H}^{(0)}_{11}$ and $\hat{H}^{(2m_c)}_{20}$, where $\hat{H}^{(m)}_{ij}$ is defined by
\[ H_{ij}(x, y) = \sum_m \hat{H}^{(m)}_{ij}(y)e^{imkx} \] (3.55)
and $H_{ij}$ is defined by equation (3.28).

To see this, expand $H_{20}(x, y) = \sum_m \hat{H}^{(m)}_{20}(y)e^{imkx}$ and write
\[ LH_{20} = \sum_m e^{imkx} \hat{L}^{(m)} \hat{H}^{(m)}_{20} \] (3.56)
and similarly for $A H_{20}$, $L H_{11}$ and $A H_{11}$. Then, $P \hat{L}^{(m)} \hat{H}^{(m)}_{20} e^{imkx} = 0$ and $P \hat{A}^{(m)} \hat{H}^{(m)}_{20} e^{imkx} = 0$ for all $m \neq m_c$. Also note that $PN_{20} = 0$ and $PN_{11} = 0$. So, collecting terms of $e^{2ikmcx}$ in equation (3.42), we get:
\[ (\hat{L}^{(2m_c)} - 2i\omega_0 \hat{A}^{(2m_c)}) \hat{H}^{(2m_c)}_{20} = \hat{N}^{(2m_c)}_{20} \] (3.57)
and collecting terms of $e^0$ in equation (3.43) we get:
\[ \hat{L}^{(0)} \hat{H}^{(0)}_{11} = -\hat{N}^{(0)}_{11}. \] (3.58)

Solving the last two equations give $\hat{H}^{(0)}_{11}$ and $\hat{H}^{(2m_c)}_{20}$.

The boundary conditions for equation (3.57) are the same as those for equation (3.49) with $m \neq 0$, while the boundary conditions for equation (3.58) are also the same as those for equation (3.49) but with $m = 0$. For the latter case, the problem is underdetermined.
Therefore, as we did for the case \( m \neq 0 \), we make the same argument and choose an additional boundary condition. The condition is:

\[
\frac{d^3 \hat{H}^{(0)}_{11}}{dy^3} = 0 \quad \text{at} \quad y = 0, 1. \tag{3.59}
\]

The coefficient \( G_{21} = \langle N_{21}, \Phi^* \rangle \) can be written as an integral in \( y \) which depends only on \( \hat{H}^{(0)}_{20}, \hat{H}^{(2mc)}_{20}, \hat{\Phi}, \hat{\Phi}, \hat{\Phi}^* \) and \( \hat{\Phi}^* \) and some of their derivatives. All of these functions and the integral must be calculated numerically.

The trapezoidal rule is used to approximate the integral:

\[
\int_0^1 \xi(y) \, dy = h \left( \frac{\xi_0}{2} + \xi_1 + \ldots + \xi_N + \frac{\xi_{N+1}}{2} \right) \tag{3.60}
\]

where \( \xi_i = \xi(y_i) \), \( y_i \) is the position of the \( i \)th grid point, \( y_i - y_{i-1} = h = 1/(N + 1) \), for all \( i \), \( N \) is the number of uniformly spaced interior grid points and the boundaries are at \( y_0 = 0 \) and \( y_{N+1} = 1 \).

Centered finite-differencing is used in the discretization of the eigenvalue and eigenfunction problem, in the solution of \( \hat{H}^{(0)}_{11} \) and \( \hat{H}^{(2mc)}_{20} \), and also in calculating the derivatives involved in \( G_{21} \). The fourth derivatives of functions \( \xi(y) \) are approximated using

\[
\left. \frac{d^4 \xi}{dy^4} \right|_{y=y_i} = \frac{\xi_{i+2} - 4\xi_{i+1} + 6\xi_i - 4\xi_{i-1} + \xi_{i-2}}{h^4} + O(h^2) \tag{3.61}
\]

and the third derivatives with

\[
\left. \frac{d^3 \xi}{dy^3} \right|_{y=y_i} = \frac{\xi_{i+2} - 2\xi_{i+1} + 2\xi_{i-1} - \xi_{i-2}}{2h^3} + O(h^2) \tag{3.62}
\]

and second derivative with

\[
\left. \frac{d^2 \xi}{dy^2} \right|_{y=y_i} = \frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{h^2} + O(h^2) \tag{3.63}
\]

and the first derivative with

\[
\left. \frac{d \xi}{dy} \right|_{y=y_i} = \frac{\xi_{i+1} - \xi_{i-1}}{2h} + O(h^2). \tag{3.64}
\]

For \( m \neq 0 \), in order to satisfy the boundary conditions, we must have that \( \xi_0 = 0 \) and \( \xi_{N+1} = 0 \) and

\[
\left. \frac{d^2 \xi}{dy^2} \right|_{y=y_0} = \frac{\xi_{-1} - 2\xi_0 + \xi_1}{h^2} = 0 \tag{3.65}
\]

\[
\left. \frac{d^2 \xi}{dy^2} \right|_{y=y_N} = \frac{\xi_N - 2\xi_{N+1} + \xi_{N+2}}{h^2} = 0. \tag{3.66}
\]

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Therefore we take \( \xi_{-1} = -\xi_1 \) and \( \xi_{N+2} = -\xi_N \), which are needed for the fourth-order derivatives at \( y_1 \) and \( y_{N-1} \). Similar conditions apply for the \( m = 0 \) case.

Upon discretization, the continuous eigenvalue problem

\[
\hat{L}^{(m)} \hat{\Psi}^{(m)} = \lambda \hat{A}^{(m)} \hat{\Psi}^{(m)}
\]

(3.67)

with

\[
\hat{\Psi}^{(m)} = 0, \quad \frac{d^2 \hat{\Psi}^{(m)}}{dy^2} = 0, \quad \text{at } y = 0, 1
\]

(3.68)

\( m = \pm 1, \pm 2, \ldots \), is reduced to the generalized matrix eigenvalue problem

\[
L^{(m)}_h X = \lambda A^{(m)}_h X
\]

(3.69)

(one problem for each \( m \)), where \( L^{(m)}_h \) and \( A^{(m)}_h \) are \( 2N \times 2N \) real-valued matrices and \( X \), which is of size \( 2N \), is the discrete solution vector of the eigenfunctions. The first \( N \) components of \( X \) are the approximate values of the first component of the unknown vector functions (e.g. \( \tilde{\phi}_1 \)) at the \( N \) grid points and the last \( N \) components of \( X \) are the approximations for the second component (e.g. \( \tilde{\phi}_2 \)). It is not known if this method for calculating eigenvalues converges in general. However, it is reasonable to assume that it does (see e.g. [7]). See Chapter 4 for further discussion.

Similarly, discretization of the problems (3.57) and (3.58) for \( \tilde{H}^{(0)}_{11} \) and \( \tilde{H}^{(2mc)}_{20} \) lead to systems of linear algebraic equations which can be readily solved. The only exceptional note is that in the computation of \( \tilde{H}^{(0)}_{11} \), the solution is not known at \( y_0 \) and \( y_{N+1} \) (since this function satisfies the boundary conditions for \( m = 0 \)) and therefore, an approximate solution must be found at these points. The boundary conditions are used to find \( \xi_{-2}, \xi_{-1}, \xi_{N+2} \) and \( \xi_{N+3} \) in terms of interior points (as in equations (3.65) and (3.66)), since these are needed for derivatives at \( y_0, y_1, y_{N-1} \) and \( y_N \).

Finally, in the computation of the \( N_{ij} \), derivatives of the approximated functions must be calculated. It can be shown that the approximations of these derivatives are still \( O(h^2) \). See Chapter 4 for further discussion. All the computations were performed in Matlab programming environment.

The parameters are set at \( F = 5.5, \beta = 0.5, \) \( Re = 240 \) and \( \gamma = \sqrt{12} \), while \( r \) is considered the primary bifurcation parameter. The results for an example of a Hopf bifurcation at different number of grid points, \( N \), are summarized in Table 3.1.

A few remarks should be made. First, the normal form coefficient \( R \) is negative and therefore a supercritical Hopf bifurcation occurs at \( r_0 \). That is, there is a stable periodic orbit for values of \( r > r_0 \), for \( r \) close enough to \( r_0 \). The convergence of the method will not be proven, mostly because the accuracy of the eigenvalues is not known. It will be
Table 3.1: Numerical results. \( N \) is number of grid points, \( R \) is the normal form coefficient, \( r_0 \) is the value of \( r \) where the bifurcation occurs and \( m_c \) gives the value of \( m \) corresponding to the zero eigenvalue.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( R )</th>
<th>( r_0 )</th>
<th>( m_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-50.95</td>
<td>0.1219</td>
<td>2</td>
</tr>
<tr>
<td>40</td>
<td>-55.15</td>
<td>0.1215</td>
<td>2</td>
</tr>
<tr>
<td>80</td>
<td>-56.26</td>
<td>0.1212</td>
<td>2</td>
</tr>
<tr>
<td>160</td>
<td>-56.53</td>
<td>0.1212</td>
<td>2</td>
</tr>
</tbody>
</table>

noted that the discretizations that are used and the numerical integration are \( O(h^2) \), and so it is reasonable to expect that the method is convergent. This seems to be what is observed, since the differences between the values of \( R \) obtained at the various \( N \) appear to decrease like \( h^2 \). The normal form coefficient \( R \) was also calculated at several other points in parameter space, where only a single pair of eigenvalues crossed the imaginary axes as \( r \) was increased. It was found that \( R \) was always negative.

From equation (3.34), the periodic orbit, to lowest order in \( r - r_0 \), is given by

\[
\begin{align*}
\rho & = \sqrt{\frac{B(r - r_0)}{-R}} + O(r - r_0) \\
\theta & = \omega_0 t + O(r - r_0)
\end{align*}
\] (3.70)

where \( z = \rho e^{i\theta} \), or in terms of \( z \):

\[
z = \sqrt{\frac{B(r - r_0)}{-R}} e^{i\omega t} + O(r - r_0)
\] (3.71)

which describes a near-circular periodic orbit. Finally, in the original variables, where to first-order, \( U = z\Phi_0 + \overline{z}\Phi_0 = \text{Real}(z\Phi_0) \), the periodic solution is given by:

\[
U = \text{Real} \left[ \sqrt{\frac{B(r - r_0)}{-R}} e^{i\omega t} \Phi e^{ikm_c x} \right] + O(r - r_0)
\]

\[
= \sqrt{\frac{B(r - r_0)}{-R}} \left[ \hat{\Phi}_r \cos(km_c x + \omega t) - \hat{\Phi}_i \sin(km_c x + \omega t) \right] + O(r - r_0)
\] (3.72)

where \( \hat{\Phi}_r \) and \( \hat{\Phi}_i \) are the real and imaginary parts of \( \hat{\Phi} \), respectively. This is a travelling wave of azimuthal (zonal) wave number \( m_c \), moving in the \( x \) direction with the structure of the latitudinal dependence given by \( \hat{\Phi} \) and with magnitude growing as \( \sqrt{r - r_0} \). See Figure 3.3 for an example of the wave form with \( m_c = 2 \).
3.3 Double Hopf bifurcation in the two-layer model

In this section, we discuss an extension of the analysis of the previous section. In particular, the results will be shown for the double Hopf bifurcations that occur in the two-layer model. We show that there is a region in parameter space where there are two simultaneously stable periodic orbits which correspond to two wave solutions. Hysteresis of these solutions is predicted.

Again we start with the equations of the two-layer model (3.2) written as

\[ A\dot{U} = LU + N(U, U) \]  

(3.73)

where

\[ U = \begin{pmatrix} \sigma \\ \delta \end{pmatrix} \]  

(3.74)

and \( A, L \) and \( N(U, U) \) are given in equations (3.7), (3.8) and (3.9). Recall that these are the perturbation equations about the axisymmetric solution

\[ \delta = -\cos 2\pi y \]  

(3.75)

\[ \sigma = 0. \]  

(3.76)

A double Hopf bifurcation (sometimes referred to as a Hopf-Hopf bifurcation) is characterized by two complex conjugate pairs of eigenvalues simultaneously crossing the imaginary axis as the parameters are varied. It is a codimension two bifurcation, which means that two parameters are needed to describe all the possible local behaviour. The two parameters that we use are \( r \) and \( F \). See Section 2.2. This choice follows work by Pedlosky [48], Mansbridge [39], Moroz and Holmes [46], and Lewis [36].

Most of the analysis is similar to the Hopf bifurcation case given in detail in the previous section. For this reason, only the main points and the differences in notation are given. In particular, the center manifold reduction is the same in principle, except that the dimension of the center manifold is four instead of two. This leads to a normal form equation which is a four-dimensional ordinary differential equation. Thus, much more complicated dynamics are possible.

The first step is to plot the neutral stability curves in the two-dimensional parameter space. As described in Section 3.1, there will be one stability curve for each azimuthal wave number. The results have already been presented in Figure 3.1. Since each curve corresponds to a complex conjugate pair of eigenvalues with zero real part, double Hopf bifurcation points occur at intersections of the curves. The points of interest are on the right-most boundary of the stable regime (all other curves are to the right). These points
are indicated in the figure with the corresponding wave numbers of the neutrally stable waves.

It must be mentioned that it is impossible to calculate the neutral stability curves for all wave numbers. It is possible to show [3], [28] that in certain quasigeostrophic dynamics with dissipation, there is a critical wave number where perturbations of higher wave number cannot be unstable in the relevant parameter range. Although it cannot be shown that this will follow in the present application, we quote this result and show that as the wave number increases, the stability curves monotonically shift to higher parameter values (see Figure 3.1).

Label the double Hopf bifurcation point as \((r_0, F_0)\). That is, at \(r = r_0\) and \(F = F_0\) the linear eigenvalue problem

\[
\lambda A U = LU
\]  

(3.77)

has two complex conjugate pairs of eigenvalues with zero real part (for a total of four), which will be denoted by:

\[
\lambda_1 = \mu_1 + i\omega_1, \quad \lambda_2 = \mu_2 + i\omega_2, \quad \lambda_2^* = \lambda_1^* + i\omega_2,
\]  

(3.78)

where at \(r_0\) and \(F_0\), \(\mu_1 = \mu_2 = 0\). Assume also that all other eigenvalues have negative real part.

The eigenfunctions corresponding to the above eigenvalues are written as

\[
\Phi, \quad \Phi^*, \quad \Psi, \quad \Psi^*,
\]

so that

\[
\lambda_1 A \Phi = L\Phi
\]  

(3.79)

and

\[
\lambda_2 A \Psi = L\Psi.
\]  

(3.80)

Note the change in notation from the previous section: \(\Psi\) now refers to a single eigenfunction corresponding to an eigenvalue with zero real part.

Define the projection

\[
PU = \langle U, \Phi^* \rangle \Phi + \langle U, \Phi^* \rangle \Phi^* + \langle U, \Psi^* \rangle \Psi + \langle U, \Psi^* \rangle \Psi^*
\]  

(3.81)

where \(\Phi^*\) and \(\Psi^*\) are the adjoint eigenfunctions which satisfy the adjoint eigenvalue problem associated with \(\lambda_1\) and \(\lambda_2\) respectively (see equation (3.13)) and the angled braces are the inner product defined in equation (3.16).

Writing

\[
U = z\Phi + \bar{z}\Phi + w\Psi + \bar{w}\Psi + \zeta
\]  

(3.82)
where \( z = z(t) = \langle U, A^* \Phi^* \rangle \) and \( w = w(t) = \langle U, A^* \Psi^* \rangle \) are complex numbers and \( z\Phi + \overline{z}\Phi + w\Psi + \overline{w}\Psi \in E^c \), and \( \zeta \in E^s \) \((E^c \text{ and } E^s \text{ are the center and stable eigenspaces, respectively, discussed in the previous section)}\). Taking the projection of equation (3.73), we get,

\[
\dot{z} = \lambda_1 z + \langle N, \Phi^* \rangle \\
\dot{w} = \lambda_2 w + \langle N, \Psi^* \rangle.
\] (3.83)

The center space is now four-dimensional where previously it was two-dimensional.

Now assume that the center manifold theorem applies (see Section 3.2). That is, there is a locally invariant, locally exponentially attracting differentiable center manifold which is tangent to the center space at \( 0 \) \((r = r_0, F = F_0, z = 0 \text{ and } w = 0)\). Then on the center manifold, we can write

\[
\zeta = H(z, \overline{z}, w, \overline{w}) = O(|z, \overline{z}, w, \overline{w}|^2).
\] (3.84)

Expand \( H \) in a Taylor series as

\[
H(z, \overline{z}, w, \overline{w}) = H_{2000}z^2 + H_{1100}z\overline{z} + H_{0200}w^2 + H_{0011}w\overline{w} \\
+ H_{1010}zw + H_{1001}z\overline{w} + \text{c.c.} + O(3)
\] (3.85)

where

\[
H_{ijkl} = \frac{1}{i!j!k!l!} \frac{\partial^{i+j+k+l}}{\partial z^i \partial \overline{z}^j \partial w^k \partial \overline{w}^l} H(0,0,0,0)
\]

are the Taylor series coefficients, c.c. are the complex conjugates of the terms which are written explicitly, and \( O(n) = O(|z, \overline{z}, w, \overline{w}|^n) \).

Also write \( N(z, \overline{z}, w, \overline{w}) \) in the same form, where \( N(z, \overline{z}, w, \overline{w}) \) is the nonlinear term (3.9) written in terms of \( z, \overline{z}, w \) and \( \overline{w} \), using the decomposition of \( U \) given in equation (3.82), with \( \zeta \) written using equations (3.84) and (3.85). See Appendix C for the formulae for the coefficients of \( N \).

The normal form equations for the non-resonant case (see below) are

\[
\dot{z} = \lambda_1 z + G_{11}z^2\overline{z} + G_{12}zw\overline{w} + O(4) \\
\dot{w} = \lambda_2 w + G_{21}z\overline{z}w + G_{22}w^2\overline{w} + O(4)
\] (3.86)

where \( G_{kl} \) are the normal form coefficients, \( \lambda_j = \lambda_j(r, F) \) are the eigenvalues and it can be shown that truncating the \( O(4) \) terms does not qualitatively change the dynamics close to the origin. This normal form requires the condition that there is no resonance. This simply means that the imaginary parts of the eigenvalues \((\omega_1, \omega_2)\), must satisfy \( m\omega_1 + n\omega_2 \neq 0, |m| + |n| \leq 4 \). If there is resonance, then there are extra terms in the normal form. See e.g. [56] or [21].
The dynamics are more readily deduced when the normal form equations are written in polar coordinates (i.e. the magnitudes \( \rho_j \) and phases \( \theta_j \) of the complex dependent variables). Write \( z = \rho_1 e^{i\theta_1}/\sqrt{|G_{11}|} \) and \( w = \rho_2 e^{i\theta_2}/\sqrt{|G_{22}|} \), where \( G_{ij}^r \) is the real part of \( G_{ij} \), and substitute these in equation (3.86). Then in polar coordinates, the truncated normal form equations are

\[
\begin{align*}
\dot{\rho}_1 &= \rho_1 \left( \mu_1 + a\rho_1^2 + b\rho_2^2 \right) \\
\dot{\rho}_2 &= \rho_2 \left( \mu_2 + c\rho_1^2 + d\rho_2^2 \right) \\
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2
\end{align*}
\] (3.87)

where

\[
\begin{align*}
a &= \frac{G_{11}^r}{|G_{11}|} = \pm 1, \\
b &= \frac{G_{12}^r}{|G_{22}|}, \\
c &= \frac{G_{21}^r}{|G_{11}|}, \\
d &= \frac{G_{22}^r}{|G_{22}|} = \pm 1,
\end{align*}
\] (3.88)

\( \lambda_j = \mu_j + i\omega_j \), the \( O(|\rho_1, \rho_2|^4) \) terms are ignored in the \( \dot{\rho}_j \) equations, and the \( O(|\rho_1, \rho_2|^2) \) terms are ignored in the \( \dot{\theta}_j \) equations. This choice of coordinates eliminates the discrepancies in the numerical values of the coefficients that can arise due to different choices of the arbitrary normalization condition (see Section 3.2.1). Below, the parameters will be considered to be \( \mu_1 \) and \( \mu_2 \) with the understanding that they are dependent upon \( r \) and \( F \).

To see the possible behaviour that these equations could describe, look for fixed points of the reduced system

\[
\begin{align*}
\dot{\rho}_1 &= \rho_1 \left( \mu_1 + a\rho_1^2 + b\rho_2^2 \right) \\
\dot{\rho}_2 &= \rho_2 \left( \mu_2 + c\rho_1^2 + d\rho_2^2 \right).
\end{align*}
\] (3.89)

That is, find the \( \rho_1 \) and \( \rho_2 \) such that \( \dot{\rho}_1 = \dot{\rho}_2 = 0 \). The truncated equations for the \( \dot{\theta}_j \) imply that \( \theta_j \) increases or decreases linearly (and so monotonically) in time. This adds a continuous rotation to the dynamics of the reduced system. Therefore, fixed points of the reduced system (3.89) correspond to one of three different types of solutions for the equations (3.87), depending on the values of \( \rho_1 \) and \( \rho_2 \). If \( \dot{\rho}_1 = \dot{\rho}_2 = 0 \) and

1. \( \rho_1 = \rho_2 = 0 \), then this is also a fixed point for equations (3.87).
2. \( \rho_1 = 0 \) and \( \rho_2 \neq 0 \), or \( \rho_1 \neq 0 \) and \( \rho_2 = 0 \), then there is a periodic orbit for equations (3.87).

3. \( \rho_1 \neq 0 \) and \( \rho_2 \neq 0 \), then there is an invariant torus for equations (3.87).

The periodic orbit occurs because the radius is fixed, and so the rotation will necessarily bring an initial point back to itself after one period of rotation. The torus results since there are two fixed radii and about each, a continuous rotation. Although, this cannot be pictured in four dimensions, an analogy in three dimensions is the surface of a doughnut. Consider motion on a circle with (constant) radius \( r_1 \), where the position along the circle is given by the angle \( \varphi_1 \). At \( r_1 \), \( \varphi_1 \), add another circle, of radius \( r_2 \) (with origin at \( r_1 \), \( \varphi_1 \)), in a perpendicular plane. See Figure 3.4. The position along this second circle is given by the angle \( \varphi_2 \). Then if \( r_1 \) and \( r_2 \) are fixed and \( \varphi_1 \) and \( \varphi_2 \) are varied, the trajectory will fall on the surface of a doughnut. This is the same situation as above (fixed point 3), with a different interpretation for the variables. A note should be made that the torus is invariant and that if \( \omega_1/\omega_2 \) is irrational, then no trajectory will ever pass through its initial position. However, it will come arbitrarily close an infinite number of times as \( t \to \infty \). That is, there will be quasi-periodic dynamics. However, if \( \omega_1/\omega_2 \) is rational, then there will be an infinite number of periodic solutions on the torus and every point on the torus will be on a periodic orbit. It can be shown that the inclusion of the higher-order terms complicates the behaviour, but, close to the bifurcation point, does not affect the existence and stability of the solutions described above.

See Guckenheimer and Holmes [21] for a summary of the different behaviours found for all possible combinations of the values of the normal form coefficients. This includes the regions in parameter space where the different types of solutions are observed, as well as their linear stabilities. Below we briefly discuss the specific behaviour that is observed in the two-layer model.

In the case of the two-layer model the form of the \( x \)-dependence of the eigenfunctions is known. We use this knowledge to write them as

\[
\Phi = \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} e^{ikm_1x} \\
(3.90)
\]

\[
\Psi = \begin{pmatrix} \hat{\psi}_1(y) \\ \hat{\psi}_2(y) \end{pmatrix} e^{ikm_2x} \\
(3.91)
\]

where the \( \hat{\phi}_j, \hat{\psi}_j \) for \( j = 1, 2 \) are functions only of the \( y \), \( k = 2\pi/\gamma \), and \( m_j \) is an integer which we use to label the wave number \( km_j \).
The normal form coefficients are given by

\[
\begin{align*}
G_{11} & = \langle N_{2100}, \Phi^* \rangle, \\
G_{12} & = \langle N_{1011}, \Phi^* \rangle, \\
G_{21} & = \langle N_{1110}, \Psi^* \rangle, \\
G_{22} & = \langle N_{0021}, \Psi^* \rangle,
\end{align*}
\]  

(3.92)

where \( N_{ijkl} \) are the Taylor coefficients of \( N(z, \bar{z}, w, \bar{w}) \) (see Section 3.2.2). In general, the formulae also depend on the terms that are quadratic in \( z \) and \( w \) (e.g. \( N_{2000} \)). However, here these terms will vanish in the projection (3.83) since they are orthogonal to the adjoint eigenfunctions (due to their \( z \)-dependence).

Also, in the same manner (see the previous section), it can be shown that to find \( \langle N_{2100}, \Phi^* \rangle \), only \( N_{(m_1)}^{(m)} \) is needed, where \( N_{2100} = \sum_m N_{2100}^{(m)}(y)e^{ikmx} \), since all terms with a factor \( e^{ikmx} \), \( m \neq m_1 \) will be orthogonal to \( \Phi^* \) and therefore vanish in the inner product. The only possible pairs of \( z\Phi, w\Psi \) and \( z^2\bar{z}w^2\bar{w}H_{ijkl} \) whose product results in a \( z^2\bar{z} \) term with a factor \( e^{ikmx} \) are \( z\Phi, z\bar{z}\hat{H}_{1100}^{(0)} \) and \( \bar{z}\Phi, z^2\hat{H}_{2000}^{(2m_1)} \). Therefore, only \( \Phi, \bar{\Phi}, \hat{H}_{1100}^{(0)} \) and \( \hat{H}_{2000}^{(2m_1)} \) appear in the formula for \( \hat{N}_{2100}^{(m_1)} \). This is due to the specific form of nonlinearity and the fact that derivatives leave the form of \( e^{ikmx} \) invariant.

Thus, in addition to the eigenfunctions,

- to compute \( \langle N_{2100}, \Phi^* \rangle \), we only need \( \hat{H}_{1100}^{(0)} \) and \( \hat{H}_{2000}^{(2m_1)} \).

Similarly,

- to compute \( \langle N_{1011}, \Phi^* \rangle \), we only need \( \hat{H}_{0011}^{(0)}, \hat{H}_{1001}^{(m_1-m_2)} \) and \( \hat{H}_{1010}^{(m_1+m_2)} \),
- to compute \( \langle N_{1110}, \Psi^* \rangle \), we only need \( \hat{H}_{1100}^{(0)}, \hat{H}_{0110}^{(m_2-m_1)} \) and \( \hat{H}_{1010}^{(m_1+m_2)} \),
- to compute \( \langle N_{0021}, \Psi^* \rangle \), we only need \( \hat{H}_{0011}^{(0)} \) and \( \hat{H}_{0020}^{(2m_2)} \).

The equations satisfied by the \( H_{ijkl} \) are derived using the invariance of the center manifold at various orders in \( z \) and \( w \). The derivations are exactly analogous to those of equations (3.42) and (3.43) in the previous section. The relevant \( H_{ijkl} \) satisfy

\[
\begin{align*}
(I - AP)[2\lambda_1 A - L] H_{2000} & = (I - AP) N_{2000} \\
(I - AP)L H_{1100} & = - (I - AP) N_{1100} \\
(I - AP)[2\lambda_2 A - L] H_{0020} & = (I - AP) N_{0020} \\
(I - AP)L H_{0011} & = - (I - AP) N_{0011} \\
(I - AP)[(\lambda_1 + \lambda_2) A - L] H_{0100} & = (I - AP) N_{0100} \\
(I - AP)[(\lambda_1 + \lambda_2) A - L] H_{1001} & = (I - AP) N_{1001}.
\end{align*}
\]  

(3.93)
The \((I - AP)\) on the right hand side are reminders that the solutions \(H_{ijkl}\) must be in the correct function space. Subsequently, the hatted \(H_{ijkl}\) satisfy

\[
\begin{align*}
2\lambda_1 \hat{A}^{(2m_1)} - \mathbf{L}^{(2m_1)} \hat{H}^{(2m_1)}_{2000} &= \hat{N}_{2000}^{(2m_1)} \\
\hat{L}^{(0)} \hat{H}^{(0)}_{1100} &= -\hat{N}_{1100}^{(0)} \\
2\lambda_2 \hat{A}^{(2m_2)} - \mathbf{L}^{(2m_2)} \hat{H}^{(2m_2)}_{0020} &= \hat{N}_{0020}^{(2m_2)} \\
\hat{L}^{(0)} \hat{H}^{(0)}_{0011} &= -\hat{N}_{0011}^{(0)} \\
[\lambda_1 + \lambda_2] \hat{A}^{(m_1+m_2)} - \mathbf{L}^{(m_1+m_2)} \hat{H}^{(m_1+m_2)}_{1010} &= \hat{N}_{1010}^{(m_1+m_2)} \\
[\lambda_1 + \lambda_2] \hat{A}^{(m_1-m_2)} - \mathbf{L}^{(m_1-m_2)} \hat{H}^{(m_1-m_2)}_{1001} &= \hat{N}_{1001}^{(m_1-m_2)}.
\end{align*}
\]

Therefore, apart from the eigenfunctions \((\Phi, \Phi^*, \Psi, \Psi^*)\), an additional six functions must be calculated: \(\hat{H}^{(0)}_{1100}, \hat{H}^{(2m_1)}_{2000}, \hat{H}^{(m_1-m_2)}_{1010}\) and \(\hat{H}^{(m_1+m_2)}_{1001}, \hat{H}^{(0)}_{0011}\) and \(\hat{H}^{(2m_2)}_{0020}\). Note that \(\hat{H}^{(m_2-m_1)}_{0110} = \hat{H}^{(m_1-m_2)}_{1001}\). All these are functions of \(y\) only and are the solutions of ordinary differential boundary value problems. These equations cannot be solved analytically, therefore numerical methods are employed. The details are given in the previous section. Here the results will be quoted with a discussion to follow (see Tables 3.2 – 3.5).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(m_1)</th>
<th>(m_2)</th>
<th>(r_0)</th>
<th>(F_0)</th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2</td>
<td>3</td>
<td>0.09807</td>
<td>11.66</td>
<td>0.04963</td>
<td>0.05163</td>
<td>-1</td>
<td>-1.234</td>
<td>-2.494</td>
<td>-1</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
<td>3</td>
<td>0.09715</td>
<td>11.82</td>
<td>0.04949</td>
<td>0.05135</td>
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<td>-1.160</td>
<td>-2.563</td>
<td>-1</td>
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<tr>
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<td>3</td>
<td>0.09692</td>
<td>11.86</td>
<td>0.04945</td>
<td>0.05150</td>
<td>-1</td>
<td>-1.141</td>
<td>-2.591</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 3.2: Numerical results for the \(m_1 = 2, m_2 = 3\) double Hopf bifurcation point denoted by \(p_{32}\) on Figure 3.1. \(N\) is number of grid points, \((r_0, F_0)\) is the location in parameter space where the bifurcation occurs, \(\omega_1, \omega_2\) are the imaginary parts of the eigenvalues at \((r_0, F_0)\) and \(a, b, c\) and \(d\) are the normal form coefficients in (3.87).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(m_1)</th>
<th>(m_2)</th>
<th>(r_0)</th>
<th>(F_0)</th>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
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<td>44.75</td>
<td>0.03403</td>
<td>0.03444</td>
<td>-1</td>
<td>-1.640</td>
<td>-1.808</td>
<td>-1</td>
</tr>
<tr>
<td>40</td>
<td>3</td>
<td>4</td>
<td>0.04260</td>
<td>45.04</td>
<td>0.03397</td>
<td>0.03439</td>
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<td>-1.632</td>
<td>-1.813</td>
<td>-1</td>
</tr>
<tr>
<td>80</td>
<td>3</td>
<td>4</td>
<td>0.04255</td>
<td>45.12</td>
<td>0.03395</td>
<td>0.03438</td>
<td>-1</td>
<td>-1.630</td>
<td>-1.815</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 3.3: Numerical results for the \(m_1 = 3, m_2 = 4\) double Hopf bifurcation point denoted by \(p_{34}\) on Figure 3.1. \(N\) is number of grid points, \((r_0, F_0)\) is the location in parameter space where the bifurcation occurs, \(\omega_1, \omega_2\) are the imaginary parts of the eigenvalues at \((r_0, F_0)\) and \(a, b, c\) and \(d\) are the normal form coefficients in (3.87).
For all double Hopf points, the results are that the coefficients \( a, b, c, \) and \( d \) are all negative and their relative magnitudes imply that there is a region in parameter space in which the two bifurcating periodic orbits are both stable. The qualitative behaviour of the solutions in the different regions in parameter space is presented in Figure 3.5.

As discussed above, the results follow from an investigation of the fixed points of the reduced normal form equations (3.89), which occur when both \( \dot{\rho}_1 = 0 \) and \( \dot{\rho}_2 = 0 \). Here, there are fixed points when

1. \( \rho_1 = \rho_2 = 0 \)

2. \( \rho_2 = 0 \) and \( \rho_1 = \rho_2 = \sqrt{\frac{\mu_1}{-a}} \)

3. \( \rho_1 = 0 \) and \( \rho_2 = \rho_2 = \sqrt{\frac{\mu_2}{-d}} \)

4. \( \rho_1 = \rho_1^{(T)} = \sqrt{\frac{-d\mu_1 + b\mu_2}{A}} \) and \( \rho_2 = \rho_2^{(T)} = \sqrt{\frac{c\mu_1 - a\mu_2}{A}} \)

where \( A = ad - bc \) and with the condition that the quantities inside the square root signs must be positive.

Fixed point (1.) is a fixed point of the full normal form equations for all values of the parameters. In this case, it is fairly easy to see that for small \( \rho_1 \) and \( \rho_2 \), \( \dot{\rho}_1 \) and \( \dot{\rho}_2 \) will
have the same sign as $\mu_1$ and $\mu_2$, respectively. This means that solution (1.) will be stable if both $\mu_1$ and $\mu_2$ are negative and unstable if either one is greater than zero.

Fixed points (2.) and (3.) correspond to periodic solutions of the full normal form equations and exist when $\mu_1 > 0$ and $\mu_2 > 0$, respectively (when the quantity inside the radical is positive).

The torus (4.) exists when $-(d\mu_1 + b\mu_2)/A > 0$ and $(c\mu_1 - a\mu_2)/A > 0$. It can be shown that if $A < 0$, $a < 0$, $b < 0$, $c < 0$ and $d < 0$ as is the case for all results, then the torus exists in the wedge, $d\mu_1/b < \mu_2 < c\mu_1/a$, $\mu_1 > 0$, $\mu_2 > 0$. See Figure 3.5.

In order to decipher the stability regions for the periodic orbits, first, for the periodic orbit (2.), consider,

$$\dot{\rho}_1 = \rho_1 \left( \mu_1 + a\rho_1^2 + b\rho_2^2 \right)$$

(3.95)

If $\rho_2$ is small, then if $\rho_1$ is larger (smaller) than $\rho_p$, then the quantity in the bracket will be negative (positive), and so $\dot{\rho}_1$ will be negative (positive). This implies that $\rho_2 = 0$ is part of the stable manifold of fixed point (2.) for all $\mu_1 > 0$. Now consider

$$\dot{\rho}_2 = \rho_2 \left( \mu_2 + c\rho_1^2 + d\rho_2^2 \right)$$

(3.96)

and write $\mu_2 = c\mu_1/a + \varepsilon_\mu$, $\rho_1 = \rho_p + \varepsilon_1$ and $\rho_2 = \varepsilon$. Then

$$\dot{\rho}_2 = \varepsilon \left( \varepsilon_\mu + 2c\rho_p\varepsilon_1 + O(\varepsilon^2) \right)$$

(3.97)

since the $\mu_1$ term cancels $\rho_2^2$. Then for any $\varepsilon_\mu$ we can choose $|\varepsilon_1|$ small enough such that the sign of $\dot{\rho}_2$ is given by the sign of $\varepsilon_\mu$. This means that for $\mu_1 > 0$, when $\mu_2 < (>) c\mu_1/a$, then $\dot{\rho}_2 < (>) 0$ and so fixed point (2.) is stable (unstable).

Similarly, it can be shown that, for $\mu_1 > 0$, fixed point (3.) is stable (unstable) when $\mu_2 > (<) d\mu_1/b$. Therefore, since $ad < bc$ ($A < 0$), there is a region given by $d\mu_1/b < \mu_2 < c\mu_1/a$, $\mu_1 > 0$, $\mu_2 > 0$ where there are two stable periodic orbits. Recall that a periodic solution of the normal form equations corresponds to a travelling wave solution for the PDE. See the previous section for an example.

The linear stability for the torus can be found by finding the eigenvalues of the linearization of the reduced equations about the fixed point (4.). For all cases, the torus is unstable.

The phase portraits in the various regimes are given in Figure 3.5. Note that since only one solution loses stability on each of the edges of the wedge, there is hysteresis, and the lines $\mu_2 = d\mu_1/b$ and $\mu_2 = c\mu_1/a$ give the boundaries of the hysteresis region. See Figure 3.6 for a bifurcation diagram of a possible one parameter transect of the parameter space showing the hysteresis. The parameter $s$ could be either $\tau$ or $F$ or a function of both, depending on the particular circumstances. An example of a possible path is indicated on
Figure 3.5. The bifurcation point $s_0$ is a Hopf bifurcation from a stable fixed point to a stable periodic orbit. As the parameter $s$ is increased, there is a bifurcation at $s_1$ from the unstable fixed point to a different, unstable periodic orbit. At $s_2$, the second periodic orbit becomes stable and an unstable invariant torus bifurcates. Finally, the invariant torus is annihilated at $s_3$, where the first periodic orbit loses stability. The hysteresis occurs in the region between $s_1$ and $s_2$. See Figure 3.7. Imagine an experiment (laboratory or numerical) which starts at a stable fixed point. If the parameter is slowly incremented, the solution will stay near the fixed point until $s_0$ is reached. As $s_0$ is passed, the solution will move away from the unstable fixed point and come to rest on the stable periodic orbit. Now if the parameter is increased, the experimenter will not observe the bifurcations at $s_1$ and $s_2$, and the solution will reside near the periodic orbit until it becomes unstable at $s_3$. At this point, the solution will move away from the unstable periodic orbit and (after transients die down) come to reside near the second (now stable) periodic orbit. If the parameter is now decreased, the solution will reside near the second periodic orbit until it becomes unstable at $s_2$, where it will move back to the first periodic orbit. The transition from the first periodic orbit to the second does not occur at the same value of the parameter as the reverse transition, and so this is hysteresis.

In a comparison of the results presented above to the previously published results on the two-layer quasigeostrophic equations without the forcing [48], [46], [39], some interesting points arise. The qualitative dynamical behaviour with and without forcing is identical. That is, the analysis of both models indicates that there are several regions in parameter space where there are two stable waves. The shape of the axisymmetric to non-axisymmetric transition curves and the wave numbers at transition are also similar. These observations support the idea that it is the baroclinic as opposed to the barotropic effects that are most important in the realization of the dynamics. An interesting difference, however, is in the form of the bifurcating wave. When the forcing is omitted, the basic state is linear and the $y$-dependence of the bifurcating wave is simply a sine function of wave number one. The form of the bifurcating wave with forcing is shown in Figure 3.3. In this wave, the oscillatory part of the wave is more dominant for higher latitudes than for the lower latitude where the wave is closer to being axisymmetric. In the lower (second) layer the opposite occurs. For the linear basic state, there is no such latitudinal variation. It is difficult to say exactly which feature of the forcing led to the observed wave. However, investigating the form of this wave for different forms of forcing would be an interesting future study.

In Table 3.6, the numerical differences between the results at the different levels of discretization (from Tables 3.2 – 3.5 ) are listed as evidence of the convergence of the method. If the sequence of differences of the results from adjacent discretization levels
(e.g. results for \( N = 20 \) and \( N = 40 \)), converges to zero, this indicates convergence of the numerical approximation. In the limit as the grid spacing \( h \) goes to zero, the rate at which this convergence takes place depends on the order of the method. By estimating the rate of convergence from two of the numerical differences, an approximation of the order can be calculated.

\[
\begin{array}{|c|c|c|c|c|}
\hline
m_1 & m_2 & N_1, N_2 & b_{\text{diff}} & c_{\text{diff}} \\
\hline
2 & 3 & 20,40 & 0.074 & -0.069 \\
2 & 3 & 40,80 & 0.019 & -0.028 \\
2 & 3 & \text{order} & 2 & 1.3 \\
3 & 4 & 20,40 & 0.008 & -0.005 \\
3 & 4 & 40,80 & 0.002 & -0.002 \\
3 & 4 & \text{order} & 2 & 1.3 \\
4 & 5 & 20,40 & 0.0025 & -0.0014 \\
4 & 5 & 40,80 & 0.0006 & -0.0005 \\
4 & 5 & \text{order} & 2 & 1.5 \\
5 & 6 & 20,40 & -0.0029 & -0.0019 \\
5 & 6 & 40,80 & -0.0008 & -0.0006 \\
5 & 6 & \text{order} & 1.9 & 1.7 \\
\hline
\end{array}
\]

Table 3.6: Estimation of the order of the numerical approximation. \( N_1 \) and \( N_2 \) are the number of grid points at the two levels which are being compared, \( m_1 \) and \( m_2 \) indicate the wave numbers of the bifurcating waves at the double Hopf point and \( a, b, c, d \) are the normal form coefficients.

The results indicate convergence since the numerical differences are significantly smaller for the \( N_1 = 40, N_2 = 80 \) difference. However, it is hoped that, since the discretization formulae were second-order, the results could be accurate to second-order. In fact, the results indicate that the approximation of the normal form coefficient \( b \) is of order 2, while for \( c \), it seems to be approximately 1.5. It is not known what causes this. It is possible that \( N_1 \) and \( N_2 \) are not large enough to bring us into the asymptotic range, and so we cannot accurately estimate the order of convergence. However, the results indicate convergence of at least order 1.5. A minimum requirement for the accuracy of the results is that the maximum error should be too small to effect the associated qualitative dynamics, which are deduced from the signs of \( a, b, c, d \) and \( ad - bc \). Here, it is quite reasonable to assume that this requirement is satisfied.
Figure 3.3: The form of $\chi_1$, the bifurcating stream function in the upper layer. The actual solution is a travelling wave of this form whose amplitude grows as $\sqrt{r - r_0}$. a) Surface plot, b) contour plot (lines of constant value of the stream function); the fluid tends to follow these lines.
Figure 3.4: A torus. If $r_1$ and $r_2$ are fixed, then $\phi_1$ and $\phi_2$ give a position on the torus. An example of a dynamical path is also plotted.
Figure 3.5: The two-dimensional bifurcation diagram. The dynamical behaviour corresponding to the calculated normal form coefficients is shown. The regions of different character are separated by solid lines. In each region, the corresponding phase portrait is drawn showing the behaviour in the region. The dotted line indicates a possible one-parameter path which will lead to hysteresis of the wave solutions. The bifurcation points along this path are indicated by $s_0$, $s_1$, $s_2$ and $s_3$ (see Figure 3.6).
Figure 3.6: The one-dimensional bifurcation diagram depicting the bifurcation observed along the path indicated with the dotted line in Figure 3.5. The bifurcation points are labelled as $s_0$, $s_1$, $s_2$ and $s_3$. See text for details. $\|U\|$ is a measure of the size of the solution and $s$ is the bifurcation parameter.

Figure 3.7: The hysteresis loop for the path indicated with the dotted line in Figure 3.5. $\|U\|$ is a measure of the size of the solution and $s$ is the bifurcation parameter. The arrows show how the solution will change as the parameter is increased or decreased.
Chapter 4

Double Hopf bifurcation in the differentially heated rotating annulus

Below, we present the analysis for double Hopf bifurcation points which occur in a realistic model of the differentially heated rotating annulus experiment. Although previously written, for completeness and easy referral, the model is written down again here.

The dynamical equations of the model are essentially the general momentum equations of fluid motion in a rotating reference frame (2.6) – (2.9) simplified using the Boussinesq approximation. In particular, we consider the variations of all fluid properties to be negligible and the equation of state of the fluid is assumed to be

$$\rho = \rho_0 (1 - \alpha (T - T_0)) \quad (4.1)$$

where $\rho$ is the density of the fluid, $T$ is the temperature, $\alpha$ is the coefficient of thermal expansion, $\rho_0$ is the density at a reference temperature $T_0$ and $\alpha (T - T_0)$ is assumed to be small. A significant simplification due to the Boussinesq approximation is that the fluid can be considered approximately incompressible. The boundaries are the inner and outer walls of the cylindrical annulus as well as a rigid flat top and bottom. At the boundaries, the no-slip condition is imposed on the fluid, and the temperature is $T_a$ and $T_b$ at the inner and outer walls, respectively. The bottom and top are thermally insulating. The equations are written in circular cylindrical coordinates in a frame of reference rotating with the annulus (at rate $\Omega$). The radial, azimuthal and vertical (or axial) coordinates will be denoted $r$, $\varphi$ and $z$, respectively with unit vectors $\hat{i}$, $\hat{\varphi}$ and $\hat{k}$.

The equations describing the evolution of the vector fluid velocity, $\mathbf{u} = \mathbf{u}(r, \varphi, z, t) = u \hat{i} + v \hat{\varphi} + w \hat{k}$ and the temperature of the fluid, $T = T(r, \varphi, z, t)$ are as follows:

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} - 2\Omega \hat{k} \times \mathbf{u} + \left( g \hat{k} - \Omega^2 r \hat{i} \right) \alpha (T - T_0) - \frac{1}{\rho_0} \nabla p - \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} \quad (4.2)$$

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T - \left( \mathbf{u} \cdot \nabla \right) T \quad (4.3)$$
\[ \nabla \cdot \mathbf{u} = 0 \quad (4.4) \]

where \( p \) is the pressure deviation from \( p_0 = \rho_0 g (D - z) + \rho_0 \Omega^2 r^2 / 2 \), \( \nu \) is the kinematic viscosity, \( \kappa \) is the coefficient of thermal diffusivity, \( g \) is the gravitational acceleration and the domain is \( r \in (r_a, r_b), \varphi \in [0, 2\pi), z \in (0, D) \). See Figure 2.7. The boundary conditions are

\[
\begin{align*}
\mathbf{u} &= 0 \quad \text{on} \quad r = r_a, r_b \quad \text{and} \quad z = 0, D \\
T &= T_a \quad \text{on} \quad r = r_a \\
T &= T_b \quad \text{on} \quad r = r_b \\
\frac{\partial T}{\partial z} &= 0 \quad \text{on} \quad z = 0, D 
\end{align*}
\quad (4.5)
\]

with 2\( \pi \)-periodicity in \( \varphi \).

Making a change of variables:

\[
\begin{align*}
\tau &\to Rr' \\
\varphi &\to \varphi \\
z &\to Dz'
\end{align*}
\quad (4.6)
\]

and writing

\[
T \to T' + \Delta T r' - \Delta T \frac{r_a}{R} + T_a
\quad (4.7)
\]

where \( R = r_b - r_a \) and \( \Delta T = T_b - T_a \), then upon dropping the primes, the equations in scalar form become

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nu_s \left( \nabla^2 u - \frac{u}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} \right) + 2\Omega v - \frac{1}{\rho_0 R} \frac{\partial p}{\partial r} - \Omega^2 R \alpha r T - \Omega^2 R \alpha r \left( \Delta T r - \Delta T \frac{r_a}{R} + T_a - T_0 \right) - N_r \\
\frac{\partial v}{\partial t} &= \nu_s \left( \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} \right) - 2\Omega u - \frac{1}{\rho_0 R} \frac{\partial p}{\partial \varphi} - N_\varphi \\
\frac{\partial w}{\partial t} &= \nu_s \nabla^2 w + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} - \frac{1}{\rho_0 D} \frac{\partial p}{\partial z} - N_z \\
\frac{\partial T}{\partial t} &= \kappa_s \nabla^2 T + \Delta T \frac{u}{r} - \frac{\Delta T}{R} u - N_T
\end{align*}
\quad (4.8)
\]

where \( \mathbf{u} = u\hat{i} + v\hat{\varphi} + w\hat{k}, \delta = D/R, \nu_s = \nu/R^2, \kappa_s = \kappa/R^2 \) and

\[
\nabla^2_s = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\delta^2} \frac{\partial^2}{\partial z^2}
\]

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\[ N_r = \frac{1}{R} \left( u \cdot \nabla_z u - \frac{v^2}{r} \right) = \frac{1}{R} \left( \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \varphi} + \frac{1}{\delta} \frac{\partial u}{\partial z} - \frac{v^2}{r} \right) \]

\[ N_{\varphi} = \frac{1}{R} \left( u \cdot \nabla_z v - \frac{u v}{r} \right) = \frac{1}{R} \left( \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \varphi} + \frac{1}{\delta} \frac{\partial v}{\partial z} - \frac{u v}{r} \right) \]

\[ N_z = \frac{1}{R} \left( u \cdot \nabla_z w \right) = \frac{1}{R} \left( \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \varphi} + \frac{1}{\delta} \frac{\partial w}{\partial z} \right) \]

\[ N_T = \frac{1}{R} \left( u \cdot \nabla_z T \right) = \frac{1}{R} \left( \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \varphi} + \frac{1}{\delta} \frac{\partial T}{\partial z} \right) \tag{4.13} \]

The boundary conditions are now:

\[ u = 0 \text{ on } r = \frac{r_a}{R}, \frac{r_b}{R} \text{ and } z = 0, 1 \]
\[ T = 0 \text{ on } r = \frac{r_a}{R}, \frac{r_b}{R} \]
\[ \frac{\partial T}{\partial z} = 0 \text{ on } z = 0, 1 \tag{4.14} \]

with 2π-periodicity in \( \varphi \).

### 4.1 The analysis

The rotation rate \( \Omega \) and the temperature difference between the inner and outer annulus walls \( \Delta T \) are the parameters of interest. These are the physical quantities (external variables) which are easily varied in an experiment. The other parameters describe the geometry of the annulus or properties of the fluid. Another possible choice is to use the dimensionless parameters, the Taylor number \( \mathcal{T} \) and the thermal Rossby number \( \mathcal{R} \) (see Section 1.2), which have a one-to-one correspondence with \( \Omega \) and \( \Delta T \). The results are quoted in terms of these parameters since experimental results are usually presented on a log-log plot of \( \mathcal{T} \) versus \( \mathcal{R} \). However, the analysis was carried out using the dimensional parameters \( \Omega \) and \( \Delta T \), since non-dimensionalization did not simplify the equations (see Section 2.3). In principle, the choice of parameters will not change the procedure or the results.

A short summary of the main steps of the analysis are as follows:

1. Plot the neutral stability curves, by

   (a) calculating the steady axisymmetric solution at a particular location in parameter space,

   (b) solving the eigenvalue problem for this solution to find its linear stability,
(c) repeating steps (a) and (b) at various locations in parameter space to find the parameter values where the solution is neutrally stable.

2. Localize the point in parameter space where the double Hopf bifurcation occurs (the intersections of the neutral stability curves).

3. Calculate the eigenvalues and eigenfunctions at the bifurcation point.

4. Compute the appropriate normal form coefficients, which involves
   (a) calculating the adjoint eigenfunctions,
   (b) calculating the center manifold coefficients.

The procedure is almost the same as that for the two-layer model, which was presented in the previous chapter. One main difference is that an analytical form for the steady axisymmetric solution is not known and so this too has to be approximated numerically. In the analysis, this is dealt with by leaving the axisymmetric solution unresolved when the perturbation equations are written down. Then, for the numerical approximation of the eigenfunctions and center manifold coefficients, the values of the axisymmetric solution are only needed at specific locations (the grid points) and numerical approximations are used. Other differences occur only in the details of the computational procedure. For this reason, it is the differences that are emphasized. For a more detailed explanation of the center manifold reduction and normal form equations, see the previous chapter.

4.1.1 Steady axisymmetric solution

The analysis begins with the computation of a steady axisymmetric solution. That is, we look for solutions of equations (4.8) — (4.12), with the boundary conditions (4.14), in the form

\[ u = u(r, z), \quad v = v(r, z), \quad w = w(r, z), \quad T = T(r, z), \]

\[ i.e. \quad \text{the dependent variables are independent of } \varphi \text{ (axisymmetric) and } t \text{ (steady).} \]

Note that the solutions also depend on the parameters.

Stream functions are used to solve for the axisymmetric solutions. With the form of the solutions as above, the incompressibility equation (4.12) becomes

\[ \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{\delta} \frac{\partial w}{\partial z} = 0. \]

If \( u \) and \( w \) are written in terms of a stream function \( \xi \),

\[ u = \frac{1}{r} \frac{\partial \xi}{\partial z} \]
\[ w = -\delta \frac{1}{r} \frac{\partial \xi}{\partial r} \]  

then the incompressibility condition (4.16) is automatically satisfied. After using (4.17) to replace \( u \) and \( w \) in the axisymmetric equations, the pressure terms can be eliminated by taking

\[
\frac{1}{R} \frac{\partial}{\partial r} \left[ \text{equation (4.10)} \right] - \frac{1}{D} \frac{\partial}{\partial z} \left[ \text{equation (4.8)} \right].
\]  

This results in three equations in the three unknown functions \( v, \xi \) and \( T \). The equations were computed symbolically using Maple, and are sufficiently complicated that no insight is gained by explicitly writing them here.

The boundary conditions are the same for \( v \) and \( T \) and the no-slip conditions on \( u \) and \( w \) become

\[
\frac{\partial \xi}{\partial r} = \frac{\partial \xi}{\partial z} = 0 \quad \text{on} \quad r = \frac{r_a}{R}, \frac{r_b}{R} \quad \text{and} \quad z = 0, 1.
\]

This condition implies that \( \xi \) is constant on the boundaries and since there is a freedom to choose \( \xi \) up to an additive constant, the additional boundary condition is chosen to be

\[
\xi = 0 \quad \text{on} \quad r = \frac{r_a}{R}, \frac{r_b}{R} \quad \text{and} \quad z = 0, 1.
\]

### 4.1.2 The perturbation equations

The next set of equations that are needed are the perturbation equations. It is this system on which the center manifold reduction is performed. The perturbation equations are obtained by writing the solutions in the form

\[
\begin{align*}
    u &= f_u + u', \\
    v &= f_v + v', \\
    w &= f_w + w', \\
    T &= f_T + T', \\
    p &= f_p + p',
\end{align*}
\]  

where \((f_u, f_v, f_w, f_T, f_p)\) is a steady solution of the axisymmetric equations.

Upon substitution of (4.19) into equations (4.8) – (4.12), and dropping primes, the perturbation equations are obtained:

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= \nu_s \left( \nabla_s^2 u - \frac{u}{r^2} - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} \right) + 2\Omega v - \frac{1}{\rho_0 R} \frac{\partial p}{\partial r} - \Omega^2 R \alpha r T \\
    & \quad - \frac{1}{R} \left( F_u \cdot \nabla_s u + u \cdot \nabla_s f_u - 2 \frac{f_u v}{r} \right) - N_r
\end{align*}
\]  

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\[
\frac{\partial v}{\partial t} = \nu_s \left( \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} \frac{\partial u}{\partial \varphi} \right) - 2\Omega u - \frac{1}{\rho_0 R} \frac{\partial p}{\partial \varphi} - \frac{1}{R} \left( F_u \cdot \nabla_s v + u \cdot \nabla_s f_v + \frac{f_v u}{r} + \frac{f_u v}{r} \right) - N \varphi
\]  
(4.21)

\[
\frac{\partial w}{\partial t} = \nu_s \nabla^2 w + g\alpha T - \frac{1}{\rho_0 D} \frac{\partial p}{\partial z} - \frac{1}{R} \left( F_u \cdot \nabla_s w + u \cdot \nabla_s f_w \right) - N_z
\]  
(4.22)

\[
\frac{\partial T}{\partial t} = \kappa_s \nabla^2 T - \frac{\Delta T}{R} u - \frac{1}{R} \left( F_u \cdot \nabla_s T + u \cdot \nabla_s f_T \right) - N_T
\]  
(4.23)

\[
\frac{\partial U}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{1}{\delta} \frac{\partial w}{\partial z} = 0
\]  
(4.24)

where \( F_u = f_u \hat{\varphi} + f_v \hat{r} + f_w \hat{z} \), \( \nabla_s = \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \varphi} + \frac{k}{\delta} \frac{\partial}{\partial z} \) and the nonlinear terms \( N \) again are given by the formulae in equation (4.13).

4.1.3 The eigenvalue problem

If we assume that the all the unknowns may be written as \( u = u(r, \varphi, z, t) = e^{i\lambda t} \hat{u}_m(r, z) e^{im\varphi} \), with \( m \) an integer, and the perturbation equations (4.24) are linearized, then a linear eigenvalue problem is obtained for each azimuthal wave number \( m \). From these equations the linear stability of the axisymmetric solution is found (see Section 3.1). Note that the form of the azimuthal dependence is a result of the \( 2\pi \) periodicity in \( \varphi \). The eigenvalue problem is:

\[
\lambda \hat{u}_m = \nu_s \left( \nabla^2 \hat{u}_m - \frac{\hat{u}_m}{r^2} - \frac{2im}{r^2} \hat{v}_m \right) + 2\Omega \hat{u}_m - \frac{1}{\rho_0 R} \frac{\partial \hat{p}_m}{\partial \varphi} - \frac{1}{R} \left( F_u \cdot \nabla_s \hat{u}_m + \hat{u}_m \cdot \nabla_s f_u - 2 \frac{f_v \hat{v}_m}{r} \right)
\]  
(4.25)

\[
\lambda \hat{v}_m = \nu_s \left( \nabla^2 \hat{v}_m - \frac{\hat{v}_m}{r^2} + \frac{2im}{r^2} \hat{u}_m \right) - 2\Omega \hat{u}_m - \frac{im}{\rho_0 R} \frac{\partial \hat{p}_m}{\partial \varphi} - \frac{1}{R} \left( F_u \cdot \nabla_s \hat{v}_m + \hat{u}_m \cdot \nabla_s f_v + \frac{f_v \hat{u}_m}{r} + \frac{f_u \hat{v}_m}{r} \right)
\]  
(4.26)

\[
\lambda \hat{w}_m = \nu_s \nabla^2 \hat{w}_m + g\alpha \hat{T}_m - \frac{1}{\rho_0 D} \frac{\partial \hat{p}_m}{\partial z} - \frac{1}{R} \left( F_u \cdot \nabla_s \hat{w}_m + \hat{u}_m \cdot \nabla_s f_w \right)
\]  
(4.27)

\[
\lambda \hat{T}_m = \kappa_s \nabla^2 \hat{T}_m - \frac{\Delta T}{R} \hat{u}_m - \frac{1}{R} \left( F_u \cdot \nabla_s \hat{T}_m + \hat{u}_m \cdot \nabla_s f_T \right)
\]  
(4.28)

\[
\frac{\partial \hat{u}_m}{\partial r} + \frac{\hat{u}_m}{r} + \frac{im}{r} \hat{v}_m + \frac{1}{\delta} \frac{\partial \hat{w}_m}{\partial z} = 0
\]  
(4.29)

where

\[
\nabla^2 = -\frac{m^2}{r^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{\delta^2} \frac{\partial^2}{\partial z^2}
\]
\[ \nabla_m = \left( \frac{\partial}{\partial r}, \frac{im}{r}, \frac{1}{\delta \partial z} \right) \]

and

\[ \hat{\mathbf{u}}_m = (\hat{u}_m, \hat{v}_m, \hat{w}_m). \]

If \( m \neq 0 \), it is possible to eliminate \( \hat{p}_m \) and \( \hat{v}_m \). From the incompressibility equation (4.29), we obtain

\[ \hat{v}_m = \frac{im}{m} \left( \frac{\partial \hat{u}_m}{\partial r} + \hat{u}_m + \frac{1}{\delta} \frac{\partial \hat{w}_m}{\partial z} \right) \tag{4.30} \]

and from equation (4.26) we have

\[ \hat{p}_m = -i \rho_0 R r \left[ -\lambda \hat{v}_m + \nu_s \left( \nabla_m^2 \hat{v}_m - \frac{\hat{v}_m}{r^2} + \frac{2im}{r^2} \hat{u}_m \right) - 2\Omega \hat{u}_m \right. \\
- \frac{1}{R} \left( F_{u} \cdot \nabla_m \hat{v}_m + \hat{u}_m \cdot \nabla_s f_v + \frac{f_v \hat{u}_m}{r} + \frac{f_u \hat{v}_m}{r} \right). \tag{4.31} \]

The resulting three equations in the three remaining unknowns \( \hat{u}_m, \hat{w}_m, \hat{T}_m \) may be written in the form

\[ \lambda \hat{A}_m \hat{U}_m = \hat{L}_m \hat{U}_m \tag{4.32} \]

where

\[ \hat{U}_m = \begin{pmatrix} \hat{u}_m \\ \hat{w}_m \\ \hat{T}_m \end{pmatrix} \]

and \( \hat{A}_m \) and \( \hat{L}_m \) are \( 3 \times 3 \) matrices of linear operators. If \( m = 0 \), a stream function method can be used in exactly the same manner as in the calculation of the axisymmetric solution. Again the equations were calculated using the Maple symbolic computation package and are too complicated to write here.

The inner product of two vector functions \( U = (u, v, w, T) \) and \( U' = (u', v', w', T') \) is taken to be

\[ \langle U, U' \rangle = \int_0^1 \int_0^{2\pi} \int_{R_0}^{R} (uu' + vv' + ww' + TT') r \, dr \, d\varphi \, dz \tag{4.33} \]

Equations (4.32) are solved numerically and equation (4.30) is used to restore the missing component.

### 4.1.4 The adjoint eigenvalue problem

Finally, the adjoint equations are necessary to calculate the adjoint eigenfunctions. The equations are obtained in the same manner as in Section 3.3. Note that in the derivation of the adjoint operators, it is possible to eliminate the pressure term using the incompressibility condition. The adjoint eigenvalue problem is
\[
\bar{\lambda}u^* = \nu_s \left( \nabla^2 \bar{u}^* - \frac{u^*}{r^2} \frac{\partial v^*}{r^2} \frac{\partial \varphi}{\partial \varphi} \right) + 2\Omega v^* - \frac{1}{\rho_0 R} \frac{\partial p^*}{\partial r} - \frac{\Delta T}{R} T^* \\
+ \frac{1}{R} \left( F_u \cdot \nabla \bar{s} u^* - V_a^* \cdot \frac{\partial F_a}{\partial r} - \frac{f_v v^*}{r} \right)
\]
(4.34)

\[
\bar{\lambda}v^* = \nu_s \left( \nabla^2 \bar{v}^* - \frac{v^*}{r^2} \frac{\partial u^*}{r^2} \frac{\partial \varphi}{\partial \varphi} \right) - 2\Omega u^* - \frac{1}{\rho_0 R} \frac{\partial p^*}{\partial \varphi} \\
+ \frac{1}{R} \left( F_u \cdot \nabla \bar{s} v^* - \frac{1}{r} V_a^* \cdot \frac{\partial F_a}{\partial \varphi} + \frac{2}{r} f_v u^* - \frac{f_u v^*}{r} \right)
\]
(4.35)

\[
\bar{\lambda}w^* = \nu_s \nabla^2 \bar{w}^* - \frac{1}{\rho_0 D} \frac{\partial p^*}{\partial z} + \frac{1}{R} \left( F_u \cdot \nabla \bar{s} w^* - \frac{1}{\delta} V_a^* \cdot \frac{\partial F_a}{\partial z} \right)
\]
(4.36)

\[
\bar{\lambda}T^* = \kappa_s \nabla^2 \bar{T}^* - \Omega^2 R a r u^* + g a w^* + \frac{1}{R} F_u \cdot \nabla \bar{s} T^*
\]
(4.37)

\[
\frac{\partial u^*}{\partial r} + \frac{u^*}{r} + \frac{1}{r} \frac{\partial v^*}{\partial \varphi} + \frac{1}{\delta} \frac{\partial w^*}{\partial z} = 0
\]
(4.38)

where \( V_a^* = (u^*, v^*, w^*, T^*) \), \( F_a = (f_u, f_v, f_w, f_T) \), and where, for example,

\[
V_a^* \cdot \frac{\partial F_a}{\partial r} = u^* \frac{\partial f_u}{\partial r} + v^* \frac{\partial f_v}{\partial r} + w^* \frac{\partial f_w}{\partial r} + T^* \frac{\partial f_T}{\partial r}.
\]

Again solutions are sought in the form \( u^*(r, \varphi, z) = \hat{u}_m(r, z)e^{im\varphi} \), and again it is possible to eliminate \( v_m^* \) and \( p_m^* \).

### 4.1.5 Normal form coefficients

In this section the definitions needed for the normal form coefficient formulae will be given. This section is essentially an overview of Section 3.3, except that the material is presented in the context of the rotating annulus model. Details are kept to a minimum and the reader is referred to Section 3.3 for more detail.

Assume that the bifurcation of interest occurs at \( \Omega = \Omega_0 \) and \( \Delta T = \Delta T_0 \). That is, at \( \Omega_0 \) and \( \Delta T_0 \), the linear eigenvalue problem given in equations (4.25) — (4.29) has two complex conjugate pairs of eigenvalues with zero real part (for a total of four), which will be denoted by

\[
\lambda_1 = \mu_1 + i\omega_1, \quad \bar{\lambda}_1, \quad \lambda_2 = \mu_2 + i\omega_2, \quad \bar{\lambda}_2,
\]
(4.39)

where at \( \Omega_0 \) and \( \Delta T_0 \), \( \mu_1 = \mu_2 = 0 \). Assume also that all other eigenvalues have negative real part.

The eigenfunctions corresponding to the above eigenvalues will be written as

\[
\Phi, \quad \bar{\Phi}, \quad \Psi, \quad \bar{\Psi},
\]

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where they have the form
\[ \Phi = \hat{\Phi}(r, z)e^{im_1\varphi} = (\phi_u, \phi_v, \phi_w, \phi_T)e^{im_1\varphi}, \]
with \( m_1 \) being the azimuthal wave number corresponding to \( \Phi \). \( \Psi \) has the same form with \( m_2 \) labeling its wave number.

The adjoint eigenfunctions corresponding to \( \Phi \) and \( \Psi \) are denoted by \( \Phi^* \) and \( \Psi^* \), respectively, where \( \Phi^* \) and \( \Psi^* \) are found from equations (4.34) – (4.38), with their respective wave numbers. The normalization will be chosen such that
\[ \langle \Phi, \Phi^* \rangle = 1. \quad (4.40) \]
This is slightly different than previously (equation (3.21)), since in the present case \( A = I \) (where \( I \) is the identity matrix). See equations (4.25) — (4.29).

We write the vector of unknowns as
\[ U = (u, v, w, T) = z\Phi + \bar{z}\Phi + w\Psi + \bar{w}\Psi + \zeta \quad (4.41) \]
where \( z = z(t) = \langle U, \Phi^* \rangle \) and \( w = w(t) = \langle U, \Psi^* \rangle \) are complex numbers and \( z\Phi + \bar{z}\Phi + w\Psi + \bar{w}\Psi \in E^c \), and \( \zeta \in E^s \) (where \( E^c \) and \( E^s \) are the center and stable eigenspaces, respectively). Note that the the definitions of \( z \) and \( w \) are slightly different again since \( A = I \).

On the center manifold, we can write
\[ \zeta = H(z, \bar{z}, w, \bar{w}) = O(|z, \bar{z}, w, \bar{w}|^2) \quad (4.42) \]
and expand \( H \) in a Taylor series as
\[ H(z, \bar{z}, w, \bar{w}) = H_{2000}z^2 + H_{1100}z\bar{z} + H_{0020}w^2 + H_{0011}w\bar{w} + H_{1010}zw + H_{1001}z\bar{w} + c.c. + O(3) \quad (4.43) \]
where \( H_{ijkl} \) are the Taylor coefficients of \( H(z, \bar{z}, w, \bar{w}) \), \( O(n) = O(|z, \bar{z}, w, \bar{w}|^n) \) and c.c. are the complex conjugates of the terms that are written explicitly. We also write \( N(z, \bar{z}, w, \bar{w}) \) in the same form, where \( N(z, \bar{z}, w, \bar{w}) \) is the nonlinear term (4.13) written in terms of \( z, \bar{z}, w \) and \( \bar{w} \), using the decomposition of \( U \) given in equation (4.41), with \( \zeta \) written using (4.42) and (4.43). See Appendix C for formulae for the coefficients of \( N \).

The normal form equations for the non-resonant case are
\begin{align*}
\dot{z} &= \lambda_1 z + G_{11}z^2\bar{z} + G_{12}zw\bar{w} + O(4) \quad (4.44) \\
\dot{w} &= \lambda_2 w + G_{21}z\bar{z}w + G_{22}w^2\bar{w} + O(4) \quad (4.45)
\end{align*}
where $\lambda_j = \lambda_j(\Omega, \Delta T)$, and the normal form coefficients $G_{kl}$ are given by

\[
\begin{align*}
G_{11} &= \langle N_{2100}, \Phi^* \rangle, \\
G_{12} &= \langle N_{1011}, \Phi^* \rangle, \\
G_{21} &= \langle N_{1110}, \Psi^* \rangle, \\
G_{22} &= \langle N_{0021}, \Psi^* \rangle.
\end{align*}
\] (4.46)

We write $z = \rho_1 e^{i\theta_1} / \sqrt{|G_{11}|}$ and $w = \rho_2 e^{i\theta_2} / \sqrt{|G_{22}|}$, where $G_{ij}^r$ is the real part of $G_{ij}$ and substitute these expressions into (4.45). In these polar coordinates, the truncated normal form equations are

\[
\begin{align*}
\dot{\rho}_1 &= \rho_1 \left( \mu_1 + a\rho_1^2 + b\rho_2^2 \right) \\
\dot{\rho}_2 &= \rho_2 \left( \mu_1 + c\rho_1^2 + d\rho_2^2 \right) \\
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2
\end{align*}
\] (4.47)

where

\[
\begin{align*}
a &= \frac{G_{11}^r}{|G_{11}|} = \pm 1, \\
b &= \frac{G_{12}^r}{|G_{22}|}, \\
c &= \frac{G_{21}^r}{|G_{11}|}, \\
d &= \frac{G_{22}^r}{|G_{22}|} = \pm 1,
\end{align*}
\] (4.48)

and $\lambda_j = \mu_j + i\omega_j$. The $O(|\rho_1, \rho_2|^4)$ terms are ignored in the $\dot{\rho}_j$ equations and the $O(|\rho_1, \rho_2|^2)$ terms are ignored in the $\dot{\theta}_j$ equations.

Given $m_1$ and $m_2$, it can be shown that the normal form coefficients $a, b, c, d$ can be written in terms of the following functions, which are all functions of $r$ and $z$ only:

- the eigenfunctions
  \[ \hat{\Phi}, \hat{\Phi}^*, \hat{\Psi}, \hat{\Psi}^* \],

- the coefficients of the center manifold:
  \[ \hat{H}_{1100}^{(0)}, \hat{H}_{2000}^{(2m_1)}, \hat{H}_{1011}^{(m_1-m_2)}, \hat{H}_{1010}^{(m_1+m_2)}, \hat{H}_{0011}^{(0)} \text{ and } \hat{H}_{0020}^{(2m_2)} \].
where $H = \sum m \hat{H}_m e^{im\phi}$. The center manifold coefficients are found from equations (3.94), with $\hat{A}(m) = I$ and where $\hat{L}(m)$ is the linear part in equations (4.25) – (4.29). For $m \neq 0$, the same solution method that was used for the eigenvalue problem can be used here (i.e. elimination of the pressure term and one velocity component). For $m = 0$, the stream function method (as for the axisymmetric solution) can be used.

4.2 Numerics

Analytic solutions for the unknown functions (the axisymmetric solution, eigenfunctions and center manifold coefficients) are not possible, therefore they are approximated numerically. As in the analysis of the two-layer model, centered finite differencing is used to discretize the spatial derivatives.

Upon discretization, the axisymmetric solution is approximated from a system of nonlinear algebraic equations, the partial differential eigenvalue problems become matrix eigenvalue problems and the partial differential boundary value problems for finding the center manifold coefficients become systems of linear equations.

4.2.1 The ordering of the unknowns

We approximate the value of the unknown functions at the locations of the $N \times N$ grid points in the interior of the domain defined by: $r = r_k, 1 \leq k \leq N$ and $z = z_l, 1 \leq l \leq N$, where $N, k, l$ are positive integers, where $r_0 = \frac{r_a}{R}, r_{N+1} = \frac{r_b}{R}, z_0 = 0$ and $z_{N+1} = 1$. Define $\xi(r_k, z_l) = \xi_{kl}$, and similarly for the other unknown functions. In the numerical procedures, it is necessary to arrange the unknowns, $\xi_{kl}$, in vector form, as opposed to the gridded notation. We order the unknowns in the conceptually most simple way and write the vector of unknowns as

$$X_{\xi} = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \vdots \\ \xi_{1N} \\ \xi_{21} \\ \vdots \\ \xi_{N1} \\ \vdots \\ \xi_{NN} \end{pmatrix}$$ (4.49)
where \( X_\xi \) is of size \( N^2 \). That is, we stack the rows (constant \( k \)) of the unknowns (transposed) on top of each other. This could also be done such that the columns are stacked, however, since the order of the derivatives with respect to \( r \) and \( z \) are the same, it is unlikely that one ordering is better than the other.

In fact, since the original problems are systems of partial differential equations, there are three unknown functions which must be written in vector form. Again the most simple method of doing this is to stack the three vectors of the form (4.49). For example, the vector of unknowns for equation (4.32) is

\[
\hat{U}_m = \begin{pmatrix} X_{um} \\ X_{wm} \\ X_{Tm} \end{pmatrix} \tag{4.50}
\]

where each \( X_{um} \) and \( X_{wm} \) are of size \( N^2 \), while \( X_{Tm} \) is size \( N(N+2) \) (this is due to the insulating boundary conditions at \( z = 0,1 \); the values of \( T \) at these grid points must also be considered as unknowns). Thus, \( \hat{U}_m \) is a vector of size \( 3N^2 + 2N \).

It is easy to see how the discretization reduces the problems to those of linear algebra. When the derivatives at a particular grid location are approximated with the differencing formulae, an algebraic equation in terms of \( \xi_{kl} \) is formed, since the \( \xi_{kl} \) are simply real variables. If this is done for every grid point, a system of algebraic equations in the \( \xi_{kl} \) is produced. It is useful to write the linear part of this equation in terms of the coefficient matrix, where the entries of the matrix are the coefficients of the \( \xi_{kl} \) in the equations (each row corresponds to an equation and each column to the \( \xi_{kl} \), i.e. when the matrix is multiplied to the vector of unknowns, the linear part of the system of equations results). Given that \( \hat{U}_m \) is a vector of size \( 3N^2 + 2N \), this implies that the matrices will be of size \( (3N^2 + 2N) \times (3N^2 + 2N) \).

The matrices associated with the chosen ordering of the unknowns is sparse, however, their bandwidth are relatively large. This comes from the fact that the centered differencing formulae for the derivatives with respect to \( r \) at \((r_{k}, z_i)\) depend on neighbours along a column of the square grid (the unknowns at \((r_{k-1}, z_i), (r_{k+1}, z_i), \text{etc.}\)). These unknowns are separated by \( N \) entries in the vector of unknowns (see equation (4.49)), and therefore, so are the associated entries in the coefficient matrix. Therefore, with third- and fourth-order derivatives in \( r \), the bandwidth of the matrix will be about \( 4N \). However, the linear solver that was used only takes advantage of the sparseness property and not the structure. Since this is adequate for our purposes, the ordering is not considered further. The eigenvalue solver, which also does not take advantage of the structure, is discussed below.
4.2.2 The mesh: non-uniform spacing

With the combination of the no-slip boundary conditions and the small parameter \((\nu)\) multiplying the second derivative term, boundary layers in the solution of the axisymmetric problem are expected. The naïve approach would ignore this fact and attempt to approximate the solution at locations on a uniform grid. For small values of the adjustable parameters, this may be sufficient. However, as the parameters increase, non-physical high frequency oscillations are observed near the boundary if the mesh size is not sufficient to resolve the steep gradients in the boundary layer. The options are to increase the number of grid points, to use non-centered differencing or to use a non-uniform grid. Unfortunately, given the available resources, the first is not sufficient to solve the problem. The second option was also passed over since non-centered differencing results in a lower order accurate solution. It was therefore necessary to look for a method of choosing a non-uniform grid.

Since the large changes in the solution occur at the boundaries, a higher density of grid points should be located here. The solution has slower changes away from the boundary, so it would be inefficient to have too many points in the interior. The options are to manually choose the grid points or to use a scaling method. An example of the first option would be to modify a uniform grid by halving the mesh size for points within a certain distance of the boundary. This was not done since the third and fourth-order derivatives would make the coding for the coefficient matrix very laborious.

A scaling method was therefore chosen. This method consists of making a change of coordinates and calculating the solutions on a uniform grid in the new coordinates. That is, the inverse of the transformation takes a uniform grid to a grid with many points near the boundary.

The transformation which takes \((r, z)\) to the new coordinates \((x, y)\), is

\[
\begin{align*}
    r &= \frac{\tan^{-1}(\eta x)}{2 \tan^{-1} \left( \frac{y}{2} \right)} + \frac{1}{2} + \frac{r_{a}}{R} \\
    z &= \frac{\tan^{-1}(\eta y)}{2 \tan^{-1} \left( \frac{y}{2} \right)} + \frac{1}{2}
\end{align*}
\]

where \(\eta\) is a scaling factor which determines the magnitude of compression near the boundary. See Figure 4.1. The domain \(r \in [r_{a}/R, r_{b}/R], z \in [0, 1]\) goes to \(x \in [-1/2, 1/2], y \in [-1/2, 1/2]\).

The equations are transformed simply by writing \(u(r, z) = u'(x, y)\) (likewise for other functions) and using the chain rule to write the equations in terms of derivatives with respect to \(x\) and \(y\). For example,

\[
\frac{\partial u}{\partial r} = \frac{\partial u'}{\partial x} \frac{\partial x}{\partial r}
\]

(4.52)
Figure 4.1: The transformation of the grid points. (a) A uniform grid (equally spaced grid points) of \( N = 20 \) (b) the grid obtained by applying the change of coordinates (4.51) with \( \eta = 6 \); it is on this non-uniform grid which the solutions are approximated.

where, here, 

\[
\frac{\partial x}{\partial r} = \frac{2}{\eta} \tan^{-1} \left( \frac{\eta}{2} \right) \left( 1 + \eta^2 x^2 \right). \tag{4.53}
\]

A note should be made that the boundary layers observed in the solutions of the eigenvalue problem are not as severe as those in the axisymmetric solutions. In fact, significant errors are introduced in the eigenvalues and eigenfunctions if the points in the interior are too sparse. This occurs even if the axisymmetric solutions appear to be well represented. This problem suggests that different scaling factors should be used for the axisymmetric and eigenvalue problems. However, the errors introduced in the interpolation seemed to negate the benefit of using multiple scaling factors. In the calculations presented, the scaling factor \( \eta = 6 \) is the smallest value possible which gives qualitatively good results for the axisymmetric problem when \( N = 20 \). It seems that smaller \( \eta \) (for this \( N \)) do not resolve the boundary layer well enough and larger \( \eta \) do not contain enough interior points to describe the eigenfunctions well enough. Note however, that for larger values of \( N \) the results are consistent.

4.2.3 Solution techniques

For the computation of the axisymmetric solution, Newton’s method is used [4]. This method is very effective since a continuation technique can be used to generate a very good guess of the solution for any parameter value. It is known that if \( \Omega = \Delta T = 0 \) then the
trivial solution satisfies the axisymmetric equations. Thus for $\Omega$ and $\Delta T$ small, the zero vector is a good guess of the solution. For small increments from these parameter values, the previous solution is a good guess. As long as the incrementation of the parameters is small enough, a solution can be obtained for any parameter value. In addition, between increments, a secant line approximation can be made so that the guess for the next step is even better. For example, given a solution at $s - 1, X_{s-1}$ and $s, X_s$, a guess for the solution at $s + 1$ could be $X_s + (X_s - X_{s-1})/\Delta s$, where $\Delta s$ is the change in the parameter value. This guess is expected to be better than $X_s$.

The neutral stability curves are found using an iterative secant method. Varying one parameter while the other is held fixed, the largest real part of the eigenvalues is found at two different locations. Considering the largest real part as a function of the parameter, the line passing through both points is calculated. A hopefully improved guess of the point on the neutral stability curve is the location where the line passes through zero. The largest real part is calculated at the new guess and a line is drawn between this point and the better guess of the two previous points. This is repeated until the desired accuracy is obtained.

The double Hopf bifurcation points occur at intersections of two neutral stability curves. Therefore, the following procedure is used to locate them. Two points are found on each neutral stability curve and a line in parameter space is drawn through each pair. The intersection of these lines is an initial guess of the location of the bifurcation point. From this guess, one parameter is varied while the other is fixed so that a new point on each of the neutral stability curves can be found. For each curve, a new pair of points is formed with the new point and one of the previous points. The intersection of the lines drawn through the new pairs is (hopefully) a better guess. This is repeated until the magnitudes of the real parts of the two relevant eigenvalues are less than a specified tolerance.

After discretization, the eigenvalue problem (4.32) is solved using the Matlab command ‘eigs’. This method computes a given number of eigenvalues (and corresponding eigenvectors) using Arnoldi iteration [50]. In particular, it is able to calculate the eigenvalues with smallest magnitude. Although we are interested in the eigenvalues with largest real part (when this is close to zero), it was judged adequate since experimental results have shown the phase speed of the observed waves to be very slow (see below). The phase speed is given approximately by the imaginary part of the critical eigenvalue, which implies that the magnitude of the critical eigenvalue is small. This hypothesis was tested on the $N = 20$ case by computing all the eigenvalues. In all cases, it was found that the eigenvalue with smallest magnitude was indeed the eigenvalue with largest real part. This does not guarantee that for smaller grid spacings this is still true, however, the correspondence between the different grid spacings suggests that the assumption is valid. Also, to reduce the risk
of error, not only the smallest magnitude eigenvalues were computed, but the $p$ smallest, where in most cases $p = 12$.

The 'eigs' function is able to use the sparseness properties of the matrices. However, for the general matrix eigenvalue problem, $(\lambda Bv = Av$, where $A$ and $B$ are matrices, $v$ is the eigenvector, $\lambda$ is the eigenvalue), Arnoldi iteration needs $B$ to be symmetric and positive definite. Unfortunately, this is not the case with the present problem, so it is necessary to invert $B$ to obtain the usual matrix eigenvalue problem (i.e. $\lambda v = B^{-1}Av$). This destroys most of the sparseness of the matrices. Using the sparseness feature, however, still saves a factor of two in memory. It should also be noted that the non-symmetric property of $B$ is not due to the computation on the non-uniform grid, since there are variable coefficients in the pre-transformed equations.

The discretized (transformed) equations and the entries of the coefficient matrices were computed symbolically using Maple. The results were then transferred to Matlab, to find the numerical approximations.

4.2.4 Discussion of convergence

A couple of notes should be made concerning the approximation by centered finite differencing. The formulae that were used are derived using truncated Taylor series expansions. The error in the value of the derivative at any grid point (called the local truncation error) is $O(h^2)$ (i.e. approximately a constant times $h^2$, as $h \to 0$), where $h$ is the (local) mesh size. This means that the error would be expected to decrease by a factor of approximately four when $h$ is halved. The constant multiplying the $h^2$ depends on higher-order derivatives than the derivative that is being approximated. For instance, for the approximation of the fourth-order derivative, the constant depends on the sixth-order derivative. More specifically, consider the first-order derivative of the function $\xi(r)$ at $r = r_k$. The truncation error for the approximation of $\frac{d\xi}{dr}_{r = r_k}$, is $\frac{h^2}{12} \frac{d^3\xi}{dr^3}_{r = \zeta}$, where $r_{k-1} \leq \zeta \leq r_{k+1}$. That is, the constant depends on the third-order derivative evaluated at some point in the given interval. An upper bound on the error depends on the maximum value of the third-order derivative in the interval. It should be noted that the constant is only 'constant' locally. That is, it varies as the grid point varies. Considering all this, implicitly we are assuming that the solutions we are approximating are smooth enough that these derivatives exist everywhere and are small enough that the errors are not too large.

It can be shown that derivatives of the approximated solutions are also $O(h^2)$, if the approximate solution and the differencing scheme for the derivative are $O(h^2)$. This is because the 'constant' in the $O(h^2)$ of the approximate solution is a function of location.
Thus, the derivative will be $O(h^2)$, but with a different 'constant'.

Considering the above discussion, it seems reasonable to assume that given an $O(h^2)$ local truncation error in the derivatives, then the approximation would be convergent (the error goes to zero as $h$ goes to zero) and that the accuracy of the approximation will also be $O(h^2)$. In fact, this can be shown to be true for the boundary value problems, but not for the eigenvalue problem. It is obvious that the matrix approximation to the eigenvalue problem will not be able to contain all the solutions of the continuous problem. (There are at most $n$ eigenvectors of an $n \times n$ matrix where there are an infinite number of eigenfunctions of the continuous problem.) Since we are expecting the eigenfunctions to be of relatively low wave number, we hope that it is these functions which are approximated by the matrix problem. This seems reasonable since the discretization is not able to resolve the high wave numbers (which are highly oscillatory) and if the eigenfunctions are of low wave number, then the errors in the differencing are relatively small. Below, the results indicate that the method seems to be convergent. However, it seems that $h$ could not be taken small enough to obtain an accurate estimate of the order.

4.3 Results

The results of the analysis are presented in this section. The geometry of the annulus and fluid properties are listed in Table 4.1. These values correspond to the experiments performed by Fein [15]. Our results are compared with those obtained in that study.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_a$</td>
<td>3.48 cm</td>
</tr>
<tr>
<td>$r_b$</td>
<td>6.02 cm</td>
</tr>
<tr>
<td>$R$</td>
<td>2.54 cm</td>
</tr>
<tr>
<td>$D$</td>
<td>5 cm</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$1.01 \times 10^{-2}$ cm$^2$/sec</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$1.41 \times 10^{-3}$ cm$^2$/sec</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$2.06 \times 10^{-4}$ $1/\circ$ C</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>0.998 gm cm$^{-3}$</td>
</tr>
<tr>
<td>$T_0$</td>
<td>20.0 $\circ$ C</td>
</tr>
<tr>
<td>$g$</td>
<td>980 gm/cm$^3$</td>
</tr>
</tbody>
</table>

Table 4.1: The annulus geometry and fluid properties used in the analysis, after [15]. See text for definitions of symbols.
4.3.1 The axisymmetric solution

An example of the axisymmetric solution is plotted in Figure 4.2. Qualitatively, the form of the solution is the same for all values of the parameters.

![Figure 4.2: The axisymmetric solution](image)

The figure shows that the fluid velocity in the interior of the fluid is predominantly in the azimuthal direction. The radial velocity is almost zero everywhere except at the upper and lower boundaries, where it is negative and positive, respectively. The vertical velocity is largest at the inner and outer walls, where there is rising at the outer wall and sinking at the inner wall. The azimuthal velocity exhibits an almost linear shear in the vertical in
the interior with a positive velocity in the upper half of the annulus and negative velocity in the lower half. The resulting circulation is a convection cell which is tilted from the radial plane such that, at the upper and lower boundaries, the inward and outward motion is deflected to the right.

4.3.2 Neutral stability and transition curves

The neutral stability curves are presented in Figure 4.3. There is a separate curve for each azimuthal wave number. The curves are the points in the parameter space where, for the given wave number, there are eigenvalues with zero real part while all other eigenvalues associated to that wave number have negative real part. Figure 4.3 shows the neutral stability curves for the wave numbers \( m=3 \) to 8 and for clarity, Figure 4.5 shows the ‘knee’ area and the upper transition (high thermal Rossby number, see Section 1.2). \( N = 25 \) for all curves shown. The other wave numbers which were calculated, \( m = 2, 9 \) and 10, were found to be to the right of at least one other curve. That is, the critical wave number is never \( m = 2, 9 \) or 10, where the critical wave number is the wave number of the bifurcating wave at the axisymmetric to non-axisymmetric transition. It is not possible to calculate the neutral stability curves of all wave numbers, and so we repeat the argument of Chapter 3 with additional justification coming from comparison with the experimental results. The stability curves in the lower transition region are so closely grouped that a plot of this region does not reveal any more detail.

In Figure 4.4, the curve which delineates the axisymmetric from the non-axisymmetric regimes is plotted. Along this curve it can be seen that there are transitions of the critical wave number. These transitions occur at intersections of the neutral stability curves and correspond to the double Hopf bifurcation points. Also plotted on Figure 4.4 is the experimentally observed transition curve taken from Fein [15], with critical wave number transitions. This is the curve where a transition from the axisymmetric to steady wave flow was observed. All curves are plotted on a log-log graph of Taylor number \( T \) versus thermal Rossby number \( \mathcal{R} \). This is so that the results could be easily compared to the experiments.

A number of observations can be made.

- There is a good correspondence between the numerical and experimental results.

- The upper transition seems to be better represented than the lower transition. However, the experimental errors are much higher for the lower part of the transition curve than for other regions. Because of the log scale, small errors in \( \Omega \) and \( \Delta T \), either numerical or experimental, will stand out more in the lower transition. See Fein [15]. This alone may account for the discrepancies.
There is cusping along the upper transition curve, associated with changes in the critical wave number, in both the experimental and numerical results.

- There is a local maximum of critical wave number \( m = 8 \) along both of the lower transition curves.

- It seems that the discrepancies in the wave number transitions along the stability curve are relatively large. This could be due to the difficulty in locating these transitions, both numerically and experimentally.

- The theoretical lower transition curve is not linear on the graph. Fein [15] believed that his experimental data showed evidence (albeit inconclusive) of this claim.

In fact, all the main features of the transition curves observed in the experiments are replicated with the numerical results. Neither the cusping along the upper transition nor the critical wave number maximum of \( m = 8 \) along the lower transition has been predicted before in such a realistic model. Miller and Butler [43], using numerical experimentation (see Section 1.3), did not locate enough points along the transition curve to reproduce the curve, and so could not make these predictions. In fact, their results showed a critical wave number maximum of \( m = 7 \).

Also, the 'drift rates' \( \omega_d = \omega_j/r_{ij} \) at the axisymmetric to non-axisymmetric transition are plotted in Figure 4.6, where the drift rate is the frequency that full wavelengths drift past a fixed point on the annulus. The solid line in the figure is a line which is consistent with experimental data [15]. Again there is good correspondence. The experimental results do not cover the whole transition curve since some of the wave speeds were judged to be too slow to measure accurately [15].

### 4.3.3 Double Hopf normal form coefficients: hysteresis

The results of the location of the double Hopf bifurcation points and normal form coefficients, as well as the imaginary parts of the eigenvalues, are presented in Tables 4.2 – 4.7. The double Hopf points are labelled in terms of the associated critical wave numbers \( m_1 \) and \( m_2 \). For all double Hopf points, all the normal form coefficients \( a, b, c \) and \( d \) are negative and also satisfy \( ad - be < 0 \). This implies that near the bifurcation point there is a region where the two bifurcating waves are stable, and hysteresis of these waves is predicted. The results also indicate the existence of an unstable invariant torus. See Section 3.3 for a detailed description and bifurcation diagram (Figure 3.5) showing the regions in parameter space where the different behaviours are observed. In Figure 4.7, the approximate boundaries of the region of hysteresis are drawn, for the double Hopf points for which they were
calculated. These boundaries were calculated from the condition $d\mu_1/b < \mu_2 < c\mu_1/a$, \(\mu_1 > 0, \mu_2 > 0\) (see Section 3.3).

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\(N\) & \(m_1\) & \(m_2\) & \(\Omega_0\) & \(\Delta T_0\) & \(\omega_1\) & \(\omega_2\) \\
\hline
20 & 5 & 6 & 0.6354 & 1.543 & $-7.946 \cdot 10^{-3}$ & $-1.039 \cdot 10^{-2}$ \\
25 & 5 & 6 & 0.6171 & 1.489 & $-8.467 \cdot 10^{-3}$ & $-1.101 \cdot 10^{-2}$ \\
30 & 5 & 6 & 0.6102 & 1.490 & $-8.462 \cdot 10^{-3}$ & $-1.091 \cdot 10^{-2}$ \\
\hline
\end{tabular}
\caption{Numerical results for the \(m_1 = 5, m_2 = 6\) double Hopf bifurcation point. \(a = -1\) and \(d = -1\) for all \(N\). \(N\) is number of grid points on one side, \(\Omega_0, \Delta T_0\) is the location in parameter space where the bifurcation occurs, \(\omega_1\) and \(\omega_2\) are the imaginary parts of the eigenvalues at \(\Omega_0, \Delta T_0\) and \(a,b,c\) and \(d\) are the normal form coefficients.}
\end{table}

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\(N\) & \(m_1\) & \(m_2\) & \(\Omega_0\) & \(\Delta T_0\) & \(\omega_1\) & \(\omega_2\) \\
\hline
20 & 6 & 7 & 0.6117 & 0.6972 & $-3.711 \cdot 10^{-3}$ & $-4.960 \cdot 10^{-3}$ \\
25 & 6 & 7 & 0.5927 & 0.6950 & $-3.953 \cdot 10^{-3}$ & $-5.289 \cdot 10^{-3}$ \\
30 & 6 & 7 & 0.5838 & 0.6944 & $-4.046 \cdot 10^{-3}$ & $-5.398 \cdot 10^{-3}$ \\
\hline
\end{tabular}
\caption{Numerical results for the \(m_1 = 6, m_2 = 7\) double Hopf bifurcation point. \(a = -1\) and \(d = -1\) for all \(N\). \(N\) is number of grid points on one side, \(\Omega_0, \Delta T_0\) is the location in parameter space where the bifurcation occurs, \(\omega_1\) and \(\omega_2\) are the imaginary parts of the eigenvalues at \(\Omega_0, \Delta T_0\) and \(a,b,c\) and \(d\) are the normal form coefficients.}
\end{table}

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\(N\) & \(m_1\) & \(m_2\) & \(\Omega_0\) & \(\Delta T_0\) & \(\omega_1\) & \(\omega_2\) \\
\hline
20 & 7 & 8 & 0.8699 & 0.3959 & $-8.582 \cdot 10^{-4}$ & $-1.161 \cdot 10^{-3}$ \\
25 & 7 & 8 & 0.8169 & 0.3820 & $-9.017 \cdot 10^{-4}$ & $-1.237 \cdot 10^{-3}$ \\
30 & 7 & 8 & 0.7925 & 0.3758 & $-9.294 \cdot 10^{-4}$ & $-1.283 \cdot 10^{-3}$ \\
\hline
\end{tabular}
\caption{Numerical results for the \(m_1 = 7, m_2 = 8\) double Hopf bifurcation point. \(a = -1\) and \(d = -1\) for all \(N\). \(N\) is number of grid points on one side, \(\Omega_0, \Delta T_0\) is the location in parameter space where the bifurcation occurs, \(\omega_1\) and \(\omega_2\) are the imaginary parts of the eigenvalues at \(\Omega_0, \Delta T_0\) and \(a,b,c\) and \(d\) are the normal form coefficients.}
\end{table}

The numerical differences between the normal form coefficients at the different levels of discretization, decrease with increasing discretization level. This is an indication of convergence. However, it appears that \(N\) is not large enough to make an estimation of the order of convergence. Yet, the differences in the normal form coefficients at different \(N\) are quite small. In particular, for the cases on the lower part of the transition curve they are approximately 1\%, which indicates that these results are probably at least qualitatively...
Table 4.5: Numerical results for the $m_1 = 8$, $m_2 = 7$ double Hopf bifurcation point. $a = -1$ and $d = -1$ for all $N$. $N$ is number of grid points on one side, $\Omega_0, \Delta T_0$ is the location in parameter space where the bifurcation occurs, $\omega_1$ and $\omega_2$ are the imaginary parts of the eigenvalues at $\Omega_0, \Delta T_0$ and $a,b,c$ and $d$ are the normal form coefficients.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$\Omega_0$</th>
<th>$\Delta T_0$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>8</td>
<td>7</td>
<td>1.603</td>
<td>0.4692</td>
<td>$-4.010 \cdot 10^{-4}$</td>
<td>$-3.493 \cdot 10^{-4}$</td>
<td>-2.309</td>
<td>-1.748</td>
</tr>
<tr>
<td>25</td>
<td>8</td>
<td>7</td>
<td>1.622</td>
<td>0.4613</td>
<td>$-3.849 \cdot 10^{-4}$</td>
<td>$-3.368 \cdot 10^{-4}$</td>
<td>-2.281</td>
<td>-1.728</td>
</tr>
<tr>
<td>30</td>
<td>8</td>
<td>7</td>
<td>1.635</td>
<td>0.4581</td>
<td>$-3.748 \cdot 10^{-4}$</td>
<td>$-3.284 \cdot 10^{-4}$</td>
<td>-2.274</td>
<td>-1.723</td>
</tr>
</tbody>
</table>

Table 4.6: Numerical results for the $m_1 = 7$, $m_2 = 6$ double Hopf bifurcation point. $a = -1$ and $d = -1$ for all $N$. $N$ is number of grid points on one side, $\Omega_0, \Delta T_0$ is the location in parameter space where the bifurcation occurs, $\omega_1$ and $\omega_2$ are the imaginary parts of the eigenvalues at $\Omega_0, \Delta T_0$ and $a,b,c$ and $d$ are the normal form coefficients.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$\Omega_0$</th>
<th>$\Delta T_0$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$b$</th>
<th>$c$</th>
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<tbody>
<tr>
<td>20</td>
<td>7</td>
<td>6</td>
<td>2.226</td>
<td>0.4625</td>
<td>$-1.5529 \cdot 10^{-4}$</td>
<td>$-1.3613 \cdot 10^{-4}$</td>
<td>-2.311</td>
<td>-1.734</td>
</tr>
<tr>
<td>25</td>
<td>7</td>
<td>6</td>
<td>2.224</td>
<td>0.4522</td>
<td>$-1.5098 \cdot 10^{-4}$</td>
<td>$-1.3029 \cdot 10^{-4}$</td>
<td>-2.308</td>
<td>-1.723</td>
</tr>
<tr>
<td>30</td>
<td>7</td>
<td>6</td>
<td>2.231</td>
<td>0.4457</td>
<td>$-1.4406 \cdot 10^{-4}$</td>
<td>$-1.2289 \cdot 10^{-4}$</td>
<td>-2.309</td>
<td>-1.719</td>
</tr>
</tbody>
</table>

accurate. To say this with certainty, the analysis must be performed on a higher level of discretization. This was not possible with the available resources. It should be noted that in numerical experiments on similar systems (e.g. [43], [29]) usually $N = 25$ was used. Finally, since the results accurately reproduce the experimental results, we conclude that the approximations are satisfactory.

We also include the incomplete results for the $m_1 = 3$, $m_2 = 4$ and $m_1 = 4$, $m_2 = 5$ double Hopf points in Tables 4.8 and 4.9. For these wave number pairs, the eigenfunctions for the $N = 20$ case were not well resolved and so the eigenvalues were inaccurate. For the $m_1 = 3$, $m_2 = 4$, $N = 30$ case, the iterative process to locate the bifurcation point did not

Table 4.7: Numerical results for the $m_1 = 6$, $m_2 = 5$ double Hopf bifurcation point. $a = -1$ and $d = -1$ for all $N$. $N$ is number of grid points on one side, $\Omega_0, \Delta T_0$ is the location in parameter space where the bifurcation occurs, $\omega_1$ and $\omega_2$ are the imaginary parts of the eigenvalues at $\Omega_0, \Delta T_0$ and $a,b,c$ and $d$ are the normal form coefficients.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$\Omega_0$</th>
<th>$\Delta T_0$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>6</td>
<td>5</td>
<td>3.843</td>
<td>0.2559</td>
<td>$-2.064 \cdot 10^{-5}$</td>
<td>$-2.083 \cdot 10^{-5}$</td>
<td>-2.376</td>
<td>-1.746</td>
</tr>
<tr>
<td>25</td>
<td>6</td>
<td>5</td>
<td>4.238</td>
<td>0.2087</td>
<td>$-2.262 \cdot 10^{-5}$</td>
<td>$-2.211 \cdot 10^{-5}$</td>
<td>-2.361</td>
<td>-1.738</td>
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<tr>
<td>30</td>
<td>6</td>
<td>5</td>
<td>4.696</td>
<td>0.1718</td>
<td>$-2.278 \cdot 10^{-5}$</td>
<td>$-2.198 \cdot 10^{-5}$</td>
<td>-2.350</td>
<td>-1.733</td>
</tr>
</tbody>
</table>
converge. As an initial guess for this iteration, we used the location of the bifurcation point for \( N = 25 \), and for this case, it was not a good enough guess. The increase in numerical difficulty seen as the differential heating is increased is not caused by the inability to resolve the boundary layer in the axisymmetric solution, but rather by the inability to resolve the eigenfunctions in the interior of the domain. The results, however, are still consistent with the experimental results and so are quoted here.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( \Omega_0 )</th>
<th>( \Delta T_0 )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>3</td>
<td>4</td>
<td>1.062</td>
<td>14.25</td>
<td>-0.03019</td>
<td>-0.04416</td>
<td>-0.9754</td>
<td>-2.139</td>
</tr>
</tbody>
</table>

Table 4.8: Numerical results for the \( m_1 = 3, m_2 = 4 \) double Hopf bifurcation point. \( a = -1 \) and \( d = -1 \) for all \( N \). \( N \) is number of grid points on one side, \( \Omega_0, \Delta T_0 \) is the location in parameter space where the bifurcation occurs, \( \omega_1 \) and \( \omega_2 \) are the imaginary parts of the eigenvalues at \( \Omega_0, \Delta T_0 \) and \( a,b,c \) and \( d \) are the normal form coefficients.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( \Omega_0 )</th>
<th>( \Delta T_0 )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>4</td>
<td>5</td>
<td>0.7383</td>
<td>3.785</td>
<td>-0.01525</td>
<td>-0.02060</td>
<td>-1.280</td>
<td>-2.236</td>
</tr>
<tr>
<td>30</td>
<td>4</td>
<td>5</td>
<td>0.7313</td>
<td>3.772</td>
<td>-0.01450</td>
<td>-0.01936</td>
<td>-1.332</td>
<td>-2.273</td>
</tr>
</tbody>
</table>

Table 4.9: Numerical results for the \( m_1 = 4, m_2 = 5 \) double Hopf bifurcation point. \( a = -1 \) and \( d = -1 \) for all \( N \). \( N \) is number of grid points on one side, \( \Omega_0, \Delta T_0 \) is the location in parameter space where the bifurcation occurs, \( \omega_1 \) and \( \omega_2 \) are the imaginary parts of the eigenvalues at \( \Omega_0, \Delta T_0 \) and \( a,b,c \) and \( d \) are the normal form coefficients.

### 4.3.4 The eigenfunctions: bifurcating wave form

An example of an eigenfunction is plotted in Figure 4.8. This is the eigenfunction with \( m = 6 \) which is observed at the \( m_1 = 6, m_2 = 7 \) double Hopf point (see Table 4.3).

From equation (4.47), the periodic orbit, to lowest order in \( \mu_j \), is given by

\[
\rho_j = \sqrt{\frac{-\mu_j}{a}} + O(\mu_j)
\]

\[
\theta_j = \omega_j t + O(\mu_j)
\]

or in terms of \( z \):

\[
z = \sqrt{\frac{-\mu_j}{a}} e^{i\omega_j t} + O(\mu_j)
\]
which describes a near-circular periodic orbit. Finally, for the $m_1$ wave in the original variables, where to first order, $U = z\Phi + \overline{z\Phi} = \text{Real}(z\Phi)$, the periodic solution is given by:

\[ U = \text{Real} \left[ \sqrt{-\mu_1} e^{i\omega_1 t} \hat{\Phi} e^{i m_1 \varphi} \right] + O(\mu_1) \]

\[ = \sqrt{-\mu_1} \left[ \hat{\Phi}^r \cos (m_1 \varphi + \omega_1 t) - \hat{\Phi}^i \sin (m_1 \varphi + \omega_1 t) \right] + O(\mu_1) \quad (4.56) \]

where $\Phi = \hat{\Phi} e^{i m_1 \varphi}$, $\hat{\Phi}^r$ is the real part of $\hat{\Phi}$ and $\hat{\Phi}^i$ is the imaginary part of $\hat{\Phi}$. Note that if $t$ is fixed, then at different $\varphi$, $U$ is a different linear combination of $\hat{\Phi}^r$ and $\hat{\Phi}^i$, and so the form of the eigenfunction gives the form of the bifurcating wave.

The temperature profile of the bifurcating wave is consistent with experiment. Measurements indicated that the temperature has a maximum at mid-radius mid-depth [15]. Other experiments [11] (with no upper lid) showed via streak photography a decreased horizontal fluid speed at mid-depth, which is also consistent. The experiments do not resolve enough levels for the fine detail of the wave forms to be validated.
Figure 4.3: Neutral stability curves are plotted for the wave numbers $m = 3$ to $m = 8$. The curves are calculated by finding the parameter values where for each $m$, the eigenvalues of (4.32) all have negative real part except one with zero real part. The curves are plotted on a log-log graph of thermal Rossby number versus Taylor number.
Figure 4.4: Transition curves for theory and experiment delineating the axisymmetric from the non-axisymmetric regimes. The critical wave number transitions (double Hopf bifurcation points), labelled as $(m_1, m_2)$, are also plotted along the curve.
Figure 4.5: Neutral stability curves: upper transition. Same as Figure 4.3, but only top part is shown.

Figure 4.6: Theoretical and experimental drift rates of bifurcating waves close to bifurcation point, where the drift rate is the frequency that full wavelengths drift past a fixed point on the annulus. The solid line is consistent with the experimental results (within experimental error) of [15]. Theoretical results are calculated from the imaginary parts of the critical eigenvalues. $\alpha \Delta T/\Omega$ is the intensity of the 'thermal wind', see [15].
Figure 4.7: Theoretical transition curve between the axisymmetric and the non-axisymmetric regimes with the double Hopf bifurcation points (critical wave number transitions) plotted along the curve. The area between the solid lines attached to the double Hopf points is the region where there are multiple stable wave solutions.
Figure 4.8: An example of the radial and vertical dependence of an eigenfunction with $m = 6$ at $\Omega = 0.5927$ and $\Delta T = 0.6950$: (a) real part and (b) imaginary part of the radial component of the eigenfunction, (c) real part and (d) imaginary part of the azimuthal component of the eigenfunction, (e) real part and (f) imaginary part of the vertical component of the eigenfunction, (g) real part and (h) imaginary part of the temperature component of the eigenfunction. That is, the actual components of the eigenfunctions are the plotted functions multiplied by $e^{im\varphi}$. 
Chapter 5

Conclusion

In this thesis, we study the transitions from axisymmetric steady solutions to non-axisymmetric travelling waves in two geophysical fluid dynamics models: (1) a two-layer quasi-geostrophic potential vorticity model with a nonlinear basic state, and (2) a model of the differentially heated rotating annulus experiment. Using the analytical–numerical center manifold reduction at the double Hopf bifurcation points which occur at this transition, it is shown that there are regions in the respective parameter spaces which support multiple stable wave solutions. Hysteresis of these solutions is observed.

The results on the first model indicate that the forcing term in the equation does not qualitatively effect the behaviour. However, there are some minor differences which are of interest, in particular in the form of the bifurcating solution. This supports the idea that the baroclinic effects as opposed to the barotropic effects are most important in the realization of the dynamics.

The second model consists of the Navier-Stokes equations in the Boussinesq approximation in cylindrical geometry. The dimensions of the domain and the properties of the fluid are chosen to imitate the differentially heated rotating annulus experiment of Fein [15]. The axisymmetric to non-axisymmetric transition curve, critical wave numbers and drift rates of the waves are all found to well reproduce the experimental observations. This supports not only the validity of the approximations which are made, but also the validity of the analysis itself.

This is the first theoretical study on a Navier-Stokes model of the differentially heated rotating annulus which uses eigenvalues to trace out the transition curve between the axisymmetric and non-axisymmetric regimes. A number of features of the experimental data have been reproduced here for the first time: (1) the cusping of the upper transition curve associated with critical wave number transitions, (2) the reproduction of \( m = 8 \) as the local maximum of the critical wave number along the lower transition curve, (3) a detailed reproduction of the drift rate of the bifurcating wave, and (4) the indication that
the lower transition curve is not linear, a speculation that Fein felt his data supported but could not confirm.

The bifurcation analysis presented in this thesis is the first of its kind which has been performed on such a model. The results show that there are stable waves which bifurcate from the axisymmetric solution and that hysteresis of the bifurcating waves occurs near critical wave number transitions. Associated with the hysteresis is the existence of an unstable torus. The results, which are obtained by numerically approximating the normal form coefficients, are consistent with both laboratory and numerical experiments. Although there is evidence of numerical convergence, due to the inability to prove the convergence, it is not possible to prove completely rigorously that the predicted behaviour occurs in the full partial differential equation model. The method does prove the existence and stability of the bifurcating solutions in the ordinary differential equations which are formed by discretizing the original model.

The type of analysis that has been employed in this thesis has not been exploited by the geophysical fluid dynamics community. However, the work in this thesis indicates that bifurcation analysis can lead to valid results and thus supports the further application to similar problems. The method was able to highlight the dynamical similarity between two geophysical fluid dynamics models of vastly different scales. This not only supports the use of the method but also is an argument for the usefulness of studying these models, i.e. they incorporate the fundamental properties of differentially heated rotating systems. Alternatively, it can be argued that the dynamics the models share are the fundamental characteristics of the flows in differential heated rotating systems.

The study presented here is a beginning, and there are many possible directions future work could take. In the analysis of the annulus experiment, the effect of the upper boundary could be studied using the assumption of stress-free boundary conditions (as in [43]). In this case, perhaps the interesting behaviour at the upper transition could be explained. There is also the possibility of resonant behaviour close to what is called the triple-point, which is the point where three regimes meet (axisymmetric, wave and irregular). Very close to the experimentally located triple-point, the imaginary parts of the two eigenvalues with largest real part are equal. Although no double Hopf point was observed in the vicinity, the dynamics found close to a resonant double Hopf bifurcation may still be found here. This may explain the existence of the triple-point.

Another interesting direction, would be to attempt to follow the bifurcating solutions as the parameters move away from the bifurcation point. An interesting flow that is observed in the annulus (both experimentally and numerically) is vacillation. It has been hypothesized [46] that the mechanism responsible for vacillation is the interaction of two
waves via a stable torus. Vacillation-type behaviour is sometimes observed during equili-
bration of steady waves. This is consistent with the existence of an unstable torus, which
we have shown can occur in the steady wave regime. Thus, it is possible that if the unstable
torus could be followed further into the steady wave regime, a bifurcation to a stable torus
(and vacillation) may be discovered. At the moment, such a study is computationally pro-
hibitive. However, if the curvature of the annulus is neglected, a symmetry of the resulting
system leads to a bifurcation to a steady solution as opposed to a periodic orbit. In this
case, the computation may be possible.

The comparison of theoretical and experimental results which took place in the inves-
tigation of the Taylor-Couette flow led to many more discoveries about the system than
otherwise would have been possible. Work presented in this thesis starts such a comparison
for the differentially heated rotating annulus flow. The hope for future work is that the
use of such techniques will lead to further discovery of new dynamics, both theoretical and
experimental, which in turn will lead to a better understanding of all differentially heated
rotating systems.
Bibliography


Appendix A

Center manifolds for partial differential equations

Consider the nonlinear equation

\[ \dot{u} = Lu + N(u), \quad u \in \mathcal{D}, \quad u(0) = u_0 \in \mathcal{H} \]  

(A.1)

where \( \mathcal{H} \) is a Banach space, \( \mathcal{D} \subset \mathcal{H} \), \( L : \mathcal{D} \to \mathcal{H} \), generates a linear semigroup\(^1\), \( N : \mathcal{H} \to \mathcal{H} \) is \( C^2 \) with \( N(0) = 0, N'(0) = 0 \) and the dot represents differentiation with respect to time. See Hazard et al. [23]. Note that since \( N(0) = 0 \), \( u = 0 \) is a fixed point of equation A.1. Suppose \( L \) has some eigenvalues, \( \lambda \), with \( \text{Real}(\lambda) = 0 \) and all the rest satisfying \( \text{Real}(\lambda) < \epsilon < 0 \). Write \( \mathcal{H} = E^c \oplus E^s \) where \( E^c \) and \( E^s \) are invariant under \( L \), i.e. \( u \in E^c \Rightarrow Lu \in E^c, \sigma = c, s \) and \( E^c \) is finite dimensional. Denote \( B = L|_{E^c} \) (\( B : E^c \to E^c, Bu = Lu \) if \( u \in E^c \)) and \( C = L|_{E^s} \), then all the eigenvalues of \( B \) have zero real part and all eigenvalues of \( C \) have negative real part. There is a projection \( P \) which takes \( u \) to \( E^c \) along \( E^s \) (i.e. \( Pu \in E^c \) for all \( u \) and \( P\phi = \phi \) for all \( \phi \in E^c \), \( P\psi = 0 \) for all \( \psi \in E^s \)). Then, any \( u \in \mathcal{H} \) may be written uniquely as \( u = \phi + \psi \) where \( \phi = Pu \in E^c \), \( \psi = (I - P)u \in E^s \). If \( u = u(t) \) is differentiable, then equation A.1 may be written as,

\[ \dot{\phi} + \dot{\psi} = L\phi + L\psi + N(\phi + \psi). \]  

(A.2)

Taking \( P \) and \( (I - P) \) of equation A.2, and noting that \( P L \phi = L \phi = B \phi \) and \( (I - P)L \psi = L \psi = C \psi \), we get,

\[
\begin{align*}
\dot{\phi} &= B \phi + P N(\phi + \psi) \\
\dot{\psi} &= C \psi + (I - P) N(\phi + \psi).
\end{align*}
\]

(A.3)

\(^1\)A linear semigroup is a family, \( (T_t)_{t \geq 0} \), of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{H} \), satisfying (i) \( T_0 = I \), (ii) \( T_{s+t} = T_s T_t \) for all \( s, t \geq 0 \), (iii) \( \lim_{t \to 0} ||T_t u_0 - u_0|| = 0 \) for all \( u_0 \in \mathcal{H} \). A linear operator, \( L \), generates the linear semigroup if \( L u = \lim_{t \to 0} \frac{L u_t - u_0}{t} \) [23].
Now, given our assumptions, the center manifold theorem states that there exists a differentiable center manifold for equation A.1

\[ W_{loc}^c = \{ u = \phi + h(\phi) : \phi \in E^c, \| \phi \| \text{ small}, \ h : E^c \to E^s \} \]  

(A.4)

which is locally invariant under the (nonlinear) local semiflow\(^2\) of equation A.1, tangent to \( E^c \) at 0 (\( h(0) = 0, h'(0) = 0 \)), and locally exponentially attracting. The local semiflow is topologically equivalent to

\[
\begin{align*}
\dot{\phi} &= B\phi + PN(\phi + h(\phi)) \\
\dot{\psi} &= C\psi
\end{align*}
\]  

(A.5)

(in particular, an asymptotic stable fixed point in \( E^c \), for the first equation, corresponds to an asymptotic stable fixed point in \( \mathcal{H} \)). Note that this is now a decoupled system. The equation in \( \phi \) is finite dimensional and can be written as an ordinary differential equation. The equation in \( \psi \) is linear with all eigenvalues of \( C \) negative. This is valid for \( \| u \| \) sufficiently small (\( i.e. \ u \) close to \( u = 0 \)) and \( h(\phi) \) can be found using the invariance of \( W_{loc}^c \).

\(^2\)(\( \Phi_t \))\(_{t \geq 0}\) is called a (nonlinear) local semiflow if the unique solution to equation A.1 is \( u(t) = \Phi_t(u_0) \), where \( \Phi_0 = I \), \( \Phi_{s+t} = \Phi_s \circ \Phi_t \) for all \( s, t \geq 0 \) (whenever both sides are defined) and \( \lim_{t \to 0} \| \Phi_t(u_0) - u_0 \| = 0 \) for all \( u_0 \in \mathcal{H} \). It can be proven that there is a semiflow for \( 0 \leq t < T \) for some \( T > 0 \) for all \( \| u_0 \| \) sufficiently small (\( i.e. \ u_0 \) 'near' the fixed point \( u = 0 \)) [23].
Appendix B

Outline of derivation of two-layer quasigeostrophic potential vorticity equations

Contained here is an outline of the derivation of the two layer quasigeostrophic potential vorticity model. See Pedlosky [49].

Assume a fluid consists of two layers and that the density is constant within each layer. See Figure B.1. The fluid occupies the region $0 < z^* < 2D$, $0 < y^* < L$ and there is periodicity in $x^*$, with period $\gamma$. There are rigid boundaries at $z^* = 0$, $2D$, $y^* = 0$, $L$. The * indicates dimensional variables. Edge effects are neglected at $y^* = 0$, $L$ and for now (see below) at the interface between the two layers, but an Ekman layer will be included at $z^* = 0$, $2D$. It is assumed that there is no mixing of the two layers. The height of the interface will be given by $z^* = h^*(x^*, y^*, t^*)$. If the fluid is at rest, both layers have height $D$. That is, the interface between the two layers will be at $z^* = D$.

With the given assumptions, the boundary conditions are as follows: no flow through the boundaries at $z^* = h^*$, $y^* = 0, L$; periodicity in $x^*$; at $z^* = 0, 2D$ there will be a vertical velocity determined by the Ekman layer.

The velocity of the fluid in each layer, in the $x^*$, $y^*$ and $z^*$ directions, respectively, is given by $u_n^*$, $v_n^*$ and $w_n^*$, $n = 1, 2$. The velocities are all functions of $t^*$, $x^*$, $y^*$ and $z^*$. The equations of motion in each layer are:

\[
\frac{d^*}{dt^*}u_n^* - fv_n^* = -\frac{1}{\rho_n^*} \frac{\partial p_n^*}{\partial x^*} + \frac{\mathcal{F}_n^*}{\rho_n^*}
\]  \hspace{1cm} (B.1)

\[
\frac{d^*}{dt^*}v_n^* + fu_n^* = -\frac{1}{\rho_n^*} \frac{\partial p_n^*}{\partial y^*} + \frac{\mathcal{F}_n^*}{\rho_n^*}
\]  \hspace{1cm} (B.2)

\[
\frac{d^*}{dt^*}w_n^* = -\frac{1}{\rho_n^*} \frac{\partial p_n^*}{\partial z^*} - g + \frac{\mathcal{F}_n^*}{\rho_n^*}
\]  \hspace{1cm} (B.3)
Figure B.1: A two-layer fluid in a periodic channel of width $L$ and height $2D$.

\[ \frac{\partial u_n^*}{\partial x^*} + \frac{\partial v_n^*}{\partial y^*} + \frac{\partial w_n^*}{\partial z^*} = 0 \]  

(B.4)

where

\[ \frac{d_n^*}{dt^*} = \frac{\partial}{\partial t^*} + u_n^* \frac{\partial}{\partial x^*} + v_n^* \frac{\partial}{\partial y^*} + w_n^* \frac{\partial}{\partial z^*} \]  

(B.5)

$F_i^*$ are the frictional forces and $f$ is the Coriolis parameter. These equations are meant to give the evolution of a (( time-average of the dependent variables, )) where the effects of the turbulent smaller scales are assumed to be seen only in the frictional forces (see below). It is hoped that these equations will describe the large-scale behaviour of atmospheric flow.

It will be assumed that $f = f_0 + \beta_0 y^*$ where $f_0 = 2\Omega \sin \theta_0$ and $\beta_0 = (2\Omega \cos \theta_0) / r_0$ and $\Omega$ is the rotation rate of the earth, $r_0$ is the radius of the earth, and $\theta_0$ is a reference latitude (this determines the $y = 0$). This simplification is valid for a range of latitude about $\theta_0$ where it can be assumed that the Coriolis varies linearly with latitude ($y$ small enough).

Now the variables will be non-dimensionalized by rescaling them as follows: $x^* = Lx$, $y^* = Ly$, $z^* = Dz$, $u_n^* = Vu_n$, $v_n^* = Vv_n$, $w_n^* = (D/L)Vw_n$ and $t^* = (L/V)t$, where $V$ is a characteristic velocity. The scaling of $w_n^*$ reflects the difference in the horizontal and vertical scales. The scaling of $t^*$ is chosen to ensure that the advective terms are of the same order of magnitude as the time evolution term.

The pressure in the two layers will be scaled as $p_1^* = \rho_1 g (2D - z^*) + \rho_1 f_0 V L p_1$, $p_2^* = \rho_1 g D + \rho_2 g (D - z^*) + \rho_2 f_0 V L p_2$, i.e. $p_n$ are the non-dimensional deviations in each layer.
from the hydrostatic approximation:

\[
\frac{\partial p_n^*}{\partial z^*} = -\rho_n g
\]  

(B.6)

which holds in the absence of motion. The scaling of the deviations, \( p_n \), was chosen in anticipation that the pressure gradients are of the same order as the Coriolis acceleration \((f u_n^*, f v_n^*)\).

The frictional terms will be rescaled as follows:

\[
\begin{align*}
\frac{\mathcal{F}_x^*}{\rho_n^*} &= \frac{V A_H}{L^2} \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) + \frac{V A_V}{D^2} \left( \frac{\partial^2 u_n}{\partial z^2} \right) \\
\frac{\mathcal{F}_y^*}{\rho_n^*} &= \frac{V A_H}{L^2} \left( \frac{\partial^2 v_n}{\partial x^2} + \frac{\partial^2 v_n}{\partial y^2} \right) + \frac{V A_V}{D^2} \left( \frac{\partial^2 v_n}{\partial z^2} \right) \\
\frac{\mathcal{F}_z^*}{\rho_n^*} &= \frac{D A_H}{L^3} \left( \frac{\partial^2 w_n}{\partial x^2} + \frac{\partial^2 w_n}{\partial y^2} \right) + \frac{V A_V}{L D} \left( \frac{\partial^2 w_n}{\partial z^2} \right)
\end{align*}
\]  

(B.7)

(B.8)

(B.9)

i.e. the magnitude of the frictional forces is estimated in terms of 'the characteristic turbulent mixing coefficients', where the molecular viscosity is relatively negligible. This comes from an ad hoc theory which assumes that the effect of the small scale motion on the large scale motion can be parameterized by an increased frictional coefficient. Note, that \( A_H \) is a parameter as opposed to a fixed property of the fluid.

Finally, we will rescale the height of the interface as

\[
h^* = D h = D (1 + R\eta)
\]  

(B.10)

where \( \eta \) is called the surface deflection which is the non-dimensional deviation of the interface from its height when the fluid is at rest \((h^* = D \text{ implies } h = 1)\) and \( R \) is a scaling factor chosen such that \( \eta \) is order 1.

To determine \( R \), we will use the requirement that the pressure must be continuous at the interface. Choosing \( R \) as

\[
R = \frac{\rho_0 f_0 V L}{g D \Delta \rho}
\]  

(B.11)

(where \( \rho_0 = (\rho_1 + \rho_2)/2 \)) will ensure that \( \eta \) is of order 1. With this choice of \( R \) we get that

\[
\eta = \frac{(\rho_2 p_2 - \rho_1 p_1)}{\rho_0}.
\]  

(B.12)

Define the Rossby number to be \( \epsilon = V / (f_0 L) \) and the other dimensionless parameters to be \( \delta = D / L, \text{Re} = (V L) / A_H, \) which is the Reynolds number, \( \beta = (L^2 \beta_0) / V, \) and \( E_V = (2 A_V) / (f_0 D^2) \). It will be assumed that the characteristic length and velocity of the flows of interest are such that \( \epsilon = O(L/r_0) \ll 1, \) which implies that \( \beta = O(1) \). This
assumption essentially means that rotation in the form of Coriolis is important, but that the characteristic length is small enough that there is only small variation of the Coriolis over the domain.

If it is assumed that \( Re = O(1) \), \( \delta < O(\epsilon) \), \( R = O(\epsilon) \) and \( E_V = O(\epsilon^2) \) and that the dependent variables can be expanded in powers of \( \epsilon \), for example:

\[
  u_n = u_n^{(0)} + \epsilon u_n^{(1)} + \ldots \tag{B.13}
\]

then the \( O(1) \) (zeroth-order) terms of the rescaled dimensionless equations form the geostrophic approximation:

\[
  v_n^{(0)} = \frac{\partial p_n^{(0)}}{\partial x} \tag{B.14}
\]

\[
  u_n^{(0)} = -\frac{\partial p_n^{(0)}}{\partial y} \tag{B.15}
\]

\[
  \frac{\partial p_n^{(0)}}{\partial z} = 0 \tag{B.16}
\]

\[
  \frac{\partial u_n^{(0)}}{\partial x} + \frac{\partial v_n^{(0)}}{\partial y} + \frac{\partial w_n^{(0)}}{\partial z} = 0. \tag{B.17}
\]

Equations (B.14) and (B.15) indicate that to first order, the Coriolis acceleration balances the pressure gradient which is the geostrophic approximation. Note that \( p_n^{(0)} \) is independent of \( z \) and therefore, by equations (B.14) and (B.15), so are \( u_n^{(0)} \) and \( v_n^{(0)} \). Taking the derivative of equation (B.14) with respect to \( y \) and the derivative of equation (B.15) with respect to \( x \), it can be seen that

\[
  \frac{\partial u_n^{(0)}}{\partial x} + \frac{\partial v_n^{(0)}}{\partial y} = 0 \tag{B.18}
\]

and so, from equation (B.17)

\[
  \frac{\partial w_n^{(0)}}{\partial z} = 0. \tag{B.19}
\]

The vertical velocity at \( z = 0 \) is \( O(\epsilon) \), [49], which implies that

\[
  w_n^{(0)} = 0. \tag{B.20}
\]

If the \( O(\epsilon) \) terms are collected and the first-order dependent variables (e.g. \( u_n^{(1)} \)) are eliminated, the following equation can be obtained:

\[
  \frac{\partial \zeta_n^{(0)}}{\partial t} + u_n^{(0)} \frac{\partial \zeta_n^{(0)}}{\partial x} + v_n^{(0)} \frac{\partial \zeta_n^{(0)}}{\partial y} + \beta v_n^{(0)} = \frac{\partial w_n^{(0)}}{\partial z} + \frac{1}{Re} \left( \frac{\partial^2 \zeta_n^{(0)}}{\partial x^2} + \frac{\partial^2 \zeta_n^{(0)}}{\partial y^2} \right) \tag{B.21}
\]

where

\[
  \zeta_n^{(0)} = \frac{\partial u_n^{(0)}}{\partial x} - \frac{\partial v_n^{(0)}}{\partial y} \tag{B.22}
\]
Since \( u_n^{(0)} \) and \( v_n^{(0)} \) are independent of \( z \), equation (B.21) can be easily integrated with respect to \( z \). Integrating equation (B.21) with \( n = 1 \) from \( z = h \) to \( z = 2 \), we obtain:

\[
(2 - 1 - R\eta) \left\{ \frac{\partial \zeta_1^{(0)}}{\partial t} + u_1^{(0)} \frac{\partial \zeta_1^{(0)}}{\partial x} + v_1^{(0)} \frac{\partial \zeta_1^{(0)}}{\partial y} + \beta v_1^{(0)} - \frac{1}{Re} \left( \frac{\partial^2 \zeta_1^{(0)}}{\partial x^2} + \frac{\partial^2 \zeta_1^{(0)}}{\partial y^2} \right) \right\} = w_1^{(1)}\big|_{z=2} - w_1^{(1)}\big|_{z=h}
\]

Likewise, integrating equation (B.21) with \( n = 2 \) from \( z = 0 \) to \( z = h = 1 + R\eta \), we obtain:

\[
(1 + R\eta) \left\{ \frac{\partial \zeta_2^{(0)}}{\partial t} + u_2^{(0)} \frac{\partial \zeta_2^{(0)}}{\partial x} + v_2^{(0)} \frac{\partial \zeta_2^{(0)}}{\partial y} + \beta v_2^{(0)} - \frac{1}{Re} \left( \frac{\partial^2 \zeta_2^{(0)}}{\partial x^2} + \frac{\partial^2 \zeta_2^{(0)}}{\partial y^2} \right) \right\} = w_2^{(1)}\big|_{z=h} - w_2^{(1)}\big|_{z=0}
\]

Now the boundary conditions at \( z = 0 \), \( z = 2 \) and at the interface will be used to eliminate the vertical velocities in equation (B.23) and (B.24). The condition that there is no mixing of the layers implies that there is no flow through the interface. This in turn implies that a fluid element initially adjacent to the interface will never move from the interface (it will remain on the interface for all time). It can be shown that the conditions on the first-order vertical velocity of the fluid at the interface are given by

\[
w_1^{(1)}\big|_{z=h} = F \left( \frac{\partial \eta^{(0)}}{\partial t} + u_1^{(0)} \frac{\partial \eta^{(0)}}{\partial x} + v_1^{(0)} \frac{\partial \eta^{(0)}}{\partial y} \right)
\]

\[
w_2^{(1)}\big|_{z=h} = F \left( \frac{\partial \eta^{(0)}}{\partial t} + u_2^{(0)} \frac{\partial \eta^{(0)}}{\partial x} + v_2^{(0)} \frac{\partial \eta^{(0)}}{\partial y} \right)
\]

where

\[
F = \frac{R}{\epsilon} = \frac{f_0^2 L^2 \rho_0}{gD\Delta \rho}
\]

Because it has been assumed that \( R = O(\epsilon) \), this implies that \( F = O(1) \).

The vertical velocities at the upper and lower boundaries will be determined through a matching with the Ekman layer (a boundary layer which assumes \( \epsilon \) is small, friction is important and no-slip at the boundary is required). The \( O(\epsilon) \) conditions are

\[
w_1^{(1)}\big|_{z=2} = -r^{-1} \zeta_2^{(0)}
\]

\[
w_2^{(1)}\big|_{z=0} = r^{-1} \zeta_2^{(0)}
\]

where

\[
r^{-1} = \frac{E_\psi^{1/2}}{2\epsilon} = O(1)
\]

If we write,

\[
\psi_1 = p_1^{(0)}
\]

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and
$$\psi_2 = p_2^{(0)}$$  \hspace{1cm} (B.32)

then
$$\frac{\partial \psi_n}{\partial x} = v_n^{(0)}$$  \hspace{1cm} (B.33)

and
$$\frac{\partial \psi_n}{\partial y} = -u_n^{(0)}$$  \hspace{1cm} (B.34)

and
$$\zeta_n^{(0)} = \frac{\partial^2 \psi_n}{\partial x^2} + \frac{\partial^2 \psi_n}{\partial y^2} = \nabla^2 \psi_n.$$  \hspace{1cm} (B.35)

Also, if we assume that
$$\rho_n = \rho_0 + \epsilon \rho_n^{(1)} + \ldots$$  \hspace{1cm} (B.36)

then by equation (B.12)
$$\eta^{(0)} = p_2^{(0)} - p_1^{(0)} = \psi_2 - \psi_1.$$  \hspace{1cm} (B.37)

Then upon substituting the conditions of equations (B.25), (B.26), (B.28) and (B.29) into equations (B.23) and (B.24) and neglecting terms $O(\epsilon)$ (remembering $R = O(\epsilon)$), the equations for $\psi_1$ and $\psi_2$ are obtained:

$$\left\{ \frac{\partial}{\partial t} + \frac{\partial \psi_1}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \psi_1}{\partial y} \frac{\partial}{\partial x} \right\} \left\{ \nabla^2 \psi_1 + F (\psi_2 - \psi_1) + \beta y \right\} = -r^{-1} \nabla^2 \psi_1 + \frac{1}{Re} \nabla^4 \psi_1$$  \hspace{1cm} (B.38)

$$\left\{ \frac{\partial}{\partial t} + \frac{\partial \psi_2}{\partial x} \frac{\partial}{\partial y} - \frac{\partial \psi_2}{\partial y} \frac{\partial}{\partial x} \right\} \left\{ \nabla^2 \psi_2 - F (\psi_2 - \psi_1) + \beta y \right\} = -r^{-1} \nabla^2 \psi_2 + \frac{1}{Re} \nabla^4 \psi_2$$  \hspace{1cm} (B.39)

Equivalently,

$$\frac{\partial \nabla^2 \psi_1}{\partial t} - F \frac{\partial}{\partial t} (\psi_2 - \psi_1) + J \left( \psi_1, \nabla^2 \psi_1 \right) + FJ (\psi_1, \psi_2) + \beta \frac{\partial \psi_1}{\partial x} = -r^{-1} \nabla^2 \psi_1 + \frac{1}{Re} \nabla^4 \psi_1$$ \hspace{1cm} (B.40)

$$\frac{\partial \nabla^2 \psi_2}{\partial t} + F \frac{\partial}{\partial t} (\psi_2 - \psi_1) + J \left( \psi_2, \nabla^2 \psi_2 \right) - FJ (\psi_1, \psi_2) + \beta \frac{\partial \psi_2}{\partial x} = -r^{-1} \nabla^2 \psi_2 + \frac{1}{Re} \nabla^4 \psi_2$$ \hspace{1cm} (B.41)

where $J(A, B) = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}$.

Defining
$$\sigma = \frac{\psi_1 + \psi_2}{2}, \quad \delta = \frac{\psi_1 - \psi_2}{2}$$  \hspace{1cm} (B.42)

then
$$\psi_1 = \sigma + \delta, \quad \psi_2 = \sigma - \delta.$$  \hspace{1cm} (B.43)

$\sigma$ can be called the barotropic mode, since it is an average of the stream functions of the two layers and $\delta$ can be called the first baroclinic mode, since it contains the first approximation.
of the stratification effects. Substituting this into equations (B.40) and (B.41) the equations for \( \sigma \) and \( \delta \) can be obtained:

\[
\frac{\partial \nabla^2 \sigma}{\partial t} + J \left( \sigma, \nabla^2 \sigma \right) + J \left( \delta, \nabla^2 \delta \right) + \beta \frac{\partial \sigma}{\partial x} = -r^{-1} \nabla^2 \sigma + \frac{1}{Re} \nabla^4 \sigma \quad (B.44)
\]

\[
\frac{\partial \nabla^2 \delta}{\partial t} - 2F \frac{\partial \delta}{\partial t} + J \left( \sigma, \nabla^2 \delta \right) + J \left( \delta, \nabla^2 \sigma \right) - 2FJ \left( \sigma, \delta \right) + \beta \frac{\partial \delta}{\partial x} = -r^{-1} \nabla^2 \delta + \frac{1}{Re} \nabla^4 \delta \quad (B.45)
\]

where the boundary conditions are:

(i) \( \sigma(x) = \sigma(x + \gamma) \) and \( \delta(x) = \delta(x + \gamma) \)

i.e. \( \sigma \) and \( \delta \) are periodic in \( x \) with period \( \gamma \), which will be a parameter.

(ii) \( \frac{\partial \sigma}{\partial x} = \frac{\partial \delta}{\partial x} = 0 \) at \( y = 0 \) and \( y = 1 \)

This ensures no tangential velocity at the boundary.

(iii) \( \lim_{x \to \infty} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \sigma}{\partial y \partial t} \, dx = \lim_{x \to \infty} \frac{1}{2X} \int_{-X}^{X} \frac{\partial^2 \delta}{\partial y \partial t} \, dx = 0 \) at \( y = 0 \) and \( y = 1 \).

This condition ensures that there is no change in circulation about the inner and outer boundary, which comes from the fact that the velocities are geostrophic [49].

If it is assumed that the velocities along the interface in the two layers need not be the same, yet there is a drag between the layers which is proportional to the difference in the velocities at the interface (with a constant of proportionality, \( F/Re \)), then it can be shown that this results in the addition of the term, \( (F/Re) \nabla^2 \delta \) to equation (B.45).

A forcing term was also added to the equations used for the analysis in this thesis.
Appendix C

Normal form formulae

In this appendix, the general formulae for the center manifold reduction are given. The specific formulae for the applications discussed in this thesis are easily computed from the general formulae. The full details are not presented since they are very lengthy and do not lead to further insight. However, one of the simplest formula is written in detail to clarify the notation.

The double Hopf bifurcation analyzed in this thesis, the normal form coefficients are given by

\[
\begin{align*}
G_{11} &= \langle N_{2100}, \Phi^* \rangle, \\
G_{12} &= \langle N_{1011}, \Phi^* \rangle, \\
G_{21} &= \langle N_{1110}, \Psi^* \rangle, \\
G_{22} &= \langle N_{0021}, \Psi^* \rangle,
\end{align*}
\]

where \( N_{ijkl} \) are the Taylor coefficients of \( N(z, \bar{z}, w, \bar{w}) \). That is,

\[
N_{ijkl} = \frac{1}{i!j!k!l!} \frac{\partial^{i+j+k+l}}{\partial z^i \partial \bar{z}^j \partial w^k \partial \bar{w}^l} N(0, 0, 0, 0)
\]

where \( N(z, \bar{z}, w, \bar{w}) \) is the nonlinear part \( N(U, U) \) with \( U \) written as

\[
U = z \Phi + \bar{z} \bar{\Phi} + w \Psi + \bar{w} \bar{\Psi} + z^2 H_{2000} + z \bar{z} H_{1100} + w^2 H_{002} + w \bar{w} H_{0011} + z w H_{1010} + z \bar{w} H_{1001} + c.c. + O(3)
\]

where \( \Phi \) and \( \Psi \) are the eigenfunctions, \( H_{ijkl} \) are the center manifold coefficients and \( c.c. \) represents the complex conjugates of the \( H_{ijkl} \) which are written explicitly. Thus, the detail of the normal form coefficients are contained in the \( N_{ijkl} \).
First, we will write the formulae for the relevant $N_{ijkl}$ which are quadratic in $z, \overline{z}, w, \overline{w}$.

\begin{align*}
N_{2000} & = N(\Phi, \Phi) \\
N_{1100} & = N(\Phi, \overline{\Phi}) + N(\overline{\Phi}, \Phi) \\
N_{0020} & = N(\Psi, \Psi) \\
N_{1010} & = N(\Phi, \Psi) + N(\Psi, \Phi) \\
N_{1001} & = N(\Phi, \overline{\Psi}) + N(\overline{\Psi}, \Phi)
\end{align*}

The remaining formulae for the quadratic terms can be found by taking the complex conjugates of these. These will be necessary for the computation of the center manifold coefficients.

The formulae for the relevant coefficients which are cubic in $z, \overline{z}, w, \overline{w}$ are as follows:

\begin{align*}
N_{2100} & = N(\Phi, H_{1100}) + N(H_{1100}, \Phi) + N(\overline{\Phi}, H_{2000}) + N(H_{2000}, \overline{\Phi}) \\
N_{1011} & = N(\Phi, H_{0011}) + N(H_{0011}, \Phi) + N(\overline{\Phi}, H_{1010}) + N(H_{1010}, \overline{\Phi}) \\
& \quad + N(\Psi, H_{1001}) + N(H_{1001}, \Psi) \\
N_{1110} & = N(\Phi, H_{0110}) + N(H_{0110}, \Phi) + N(\overline{\Phi}, H_{1010}) + N(H_{1010}, \overline{\Phi}) \\
& \quad + N(\Psi, H_{1100}) + N(H_{1100}, \Psi) \\
N_{0021} & = N(\Psi, H_{0011}) + N(H_{0011}, \Psi) + N(\overline{\Psi}, H_{0020}) + N(H_{0020}, \overline{\Psi})
\end{align*}

These formulae are the same for all non-resonant double Hopf bifurcations. In each application, the formulae must be expanded using the appropriate definitions of the non-linear part $N(U, U)$. Note that for the two-layer model, $U$ is a two-dimensional vector of functions and therefore each $\Phi, \Psi, H_{ijkl}$ and $N(U, U)$ are two-dimensional. For the annulus experiment application, $U$ is a four-dimensional vector of functions.

Here we will not explicitly write out all the formulae, however, the formula for $N_{1010}$ for the two-layer case will be given as an example in order to clarify the notation. For the two-layer case, $N(U, U)$ is given by:

\begin{equation}
N(U_1, U_2) = \begin{pmatrix}
-J(\sigma_1, \nabla^2 \sigma_2) - J(\delta_1, \nabla^2 \delta_2) \\
-J(\sigma_1, \nabla^2 \delta_2) - J(\delta_1, \nabla^2 \sigma_2) - 2FJ(\delta_1, \sigma_2)
\end{pmatrix}
\end{equation}

where

\begin{equation}
U_j = \begin{pmatrix}
\sigma_j \\
\delta_j
\end{pmatrix},
\end{equation}

with $J(u, v) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$ and the subscripts have been added to the $U$ for clarification.
Writing

\[ \Phi = \begin{pmatrix} \Phi^{(1)} \\ \Phi^{(2)} \end{pmatrix} \quad \text{(C.6)} \]

\[ \Psi = \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \end{pmatrix} \quad \text{(C.7)} \]

we get that

\begin{align*}
N_{1010}^{(1)} &= -J \left( \Phi^{(1)}, \nabla^2 \Psi^{(1)} \right) - J \left( \Phi^{(2)}, \nabla^2 \Psi^{(2)} \right) \\
&\quad - J \left( \Psi^{(1)}, \nabla^2 \Phi^{(1)} \right) - J \left( \Psi^{(2)}, \nabla^2 \Phi^{(2)} \right) \\
N_{1010}^{(2)} &= -J \left( \Phi^{(1)}, \nabla^2 \Psi^{(2)} \right) - J \left( \Phi^{(2)}, \nabla^2 \Psi^{(1)} \right) - 2FJ \left( \Phi^{(2)}, \Psi^{(1)} \right) \\
&\quad - J \left( \Psi^{(1)}, \nabla^2 \Phi^{(2)} \right) - J \left( \Psi^{(2)}, \nabla^2 \Phi^{(1)} \right) - 2FJ \left( \Psi^{(2)}, \Phi^{(1)} \right) \quad \text{(C.8)}
\end{align*}

That is, this is found from \( N_{1010} = N(\Phi, \Psi) + N(\Psi, \Phi) \), where, for \( N(\Phi, \Psi) \), the substitutions \( \sigma_1 \rightarrow \Phi^{(1)} \), \( \delta_1 \rightarrow \Phi^{(2)} \), \( \sigma_2 \rightarrow \Psi^{(1)} \) and \( \delta_2 \rightarrow \Psi^{(2)} \) have been made in \( N(U_1, U_2) \) and for \( N(\Psi, \Phi) \), the substitutions \( \sigma_1 \rightarrow \Psi^{(1)} \), \( \delta_1 \rightarrow \Psi^{(2)} \), \( \sigma_2 \rightarrow \Phi^{(1)} \) and \( \delta_2 \rightarrow \Phi^{(2)} \) have been made in \( N(U_1, U_2) \).