The Local Gromov-Witten Invariants of Configurations of Rational Curves

by

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Abstract

We compute the local Gromov-Witten invariants of certain configurations of rational curves in a Calabi-Yau threefold. We first transform this from a problem involving local Gromov-Witten invariants to one involving global or ordinary invariants. We do so by expressing the local invariants of a configuration of curves in terms of ordinary Gromov-Witten invariants of a blowup of $\mathbb{CP}^3$ at points. The Gromov-Witten invariants of a blowup of $\mathbb{CP}^3$ along points have a symmetry, which arises from the geometry of the Cremona transformation, and transforms some difficult to compute invariants into others that are less difficult or already known. This symmetry is then used to compute the global invariants.

Contents

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To my grandparents, Berna and Gabe Allen,
and in loving memory, Sanford and Helen Karp.

And to Juan Carlos Fontanive.

Chapter 1

Introduction

1.1 Motivation

The relationship between contemporary physics and mathematics is exciting and deep. String theory, conformal and topological field theories and mirror symmetry have become tremendously important in representation theory, combinatorics, low-dimensional topology, knot theory and symplectic and algebraic geometry.

Gromov-Witten theory is an example of a particularly rich interaction between these two disciplines. The subject arose from the study of string theory on Calabi-Yau threefolds. It was noticed that the coefficients of certain correlation functions correspond to the number of holomorphic maps from the worldsheet of the string to the Calabi-Yau threefold. Thus, Gromov-Witten theory was seen to predict answers to questions in enumerative geometry. Of particular significance, for instance, was the prediction of the number of rational curves on the quintic threefold [9].

One main reason that one may be motivated to study the local Gromov-Witten invariants of configuration of rational curves is related to the topological vertex. The Gromov-Witten invariants of any toric Calabi-Yau threefold can in principle be computed by the virtual localization of Graber and Pandharipande [16], which reduces the invariants to Hodge integrals, which can then be computed recursively [12, 20, 24]. However this method is impractical for higher genus invariants.

A more efficient algorithm to compute the all genus Gromov-Witten invariants of any toric Calabi-Yau threefold was proposed by Aganagic, Klemm, Mariño and Vafa [1]. The basic ingredients are the topological vertex, which is a generating function for certain open Gromov-Witten invariants, and a gluing algorithm which expresses the Gromov-Witten invariants of any toric Calabi-Yau threefold in terms of the topological vertex. The topological vertex itself is computed using duality be-

tween Chern-Simons theory and Gromov-Witten theory. This work is conjectural, as neither open Gromov-Witten invariants nor Chern-Simons/Gromov-Witten duality are well defined mathematically.

In an attempt to overcome these difficulties, J. Li, K. Liu, C.-C. Liu, and J. Zhou [21] introduce formal relative Gromov-Witten invariants and degeneration as an alternative to open invariants and gluing, and they define an alternate generating function. It is shown that the degeneration formula agrees with the gluing formula. To show that the two algorithms are in complete agreement, it remains to show that the generating functions are equal. In order to prove equality, it is necessary to compute so called one-, two-, and three-partition Hodge integrals. The proof has been completed in case one of the partitions is empty, and the full three-partition case remains open.

Consequently, the local Gromov-Witten invariants of configurations of rational curves, in case the configuration is not contained in a surface in a Calabi-Yau threefold, are conjectured but not proved using the vertex technology. Thus, computations of these local invariants provide a check of the conjecture in that case. For example, the closed topological vertex is defined to be a configuration of three $\mathbb{P}^1$'s meeting in a single triple point in a Calabi-Yau threefold, with some mild assumptions about the formal neighborhood. In general it is not contained in a surface and its local invariants were computed by Bryan and Karp [5], which appears as Theorem 1 here. Theorem 2 provides another infinite family of potential verifications of the vertex conjecture.

In addition to the above motivation, local Gromov-Witten invariants of rational curves are interesting in and of themselves. The local invariants of curves have been studied in [5, 7, 8, 11, 23]. Also, the local invariants of ADE configurations of rational curves were computed by Bryan, Katz and Leung in [6]. Our computations are new and add to this list.

1.2 Background and Notation

1.2.1 Gromov-Witten invariants

We now recall the central objects in the Gromov-Witten theory of (local) Calabi-Yau threefolds and establish notation.

Let X be a smooth complex projective algebraic variety of dimension three. We may identify $H^*(X,\mathbb{Z})$ with $H_*(X,\mathbb{Z})$ as rings via Poincaré duality, where intersection product is dual to cup product.

We use the convention that curve *classes* (and not necessarily curves themselves) will be denoted by lower case letters, and divisor classes will be denoted by

the upper case. For example, we may denote the class of a hyperplane in $\mathbb{P}^3$ by $H \in H_4(\mathbb{P}^3, \mathbb{Z})$ and the class of a line by $h \in H_2(\mathbb{P}^3, \mathbb{Z})$. And we may denote a curve by $C \subset X$, but we will specify its class by $[C] \in H_*(X, \mathbb{Z})$.

We denote the canonical bundle of X by K_X. We say that X is *Calabi-Yau* if it has trivial canonical bundle, $K_X \cong \mathcal{O}_X$, and we say that X is *locally Calabi-Yau* near the curve $C \subset X$, or that C is a *locally Calabi-Yau* curve in X, if $K_X \cdot [C] = 0$.

Let X be locally Calabi-Yau near $C \subset X$ and let $[C] = \beta \in H_2(X, \mathbb{Z})$. Then, we let

$$\overline{M}_g(X, \beta)$$

denote the moduli space of stable genus g maps to X representing β. It is constructed and shown to be a proper algebraic stack by Fulton and Pandharipande in [13]. Li, Tian, Behrend and Fantechi [2, 3, 22] showed that this moduli space comes equipped with a perfect obstruction theory, which defines a virtual fundamental zero cycle. The genus g Gromov-Witten invariant of X in class β is defined to be the degree of this virtual fundamental class; we denote it as follows.

$$\langle\, \rangle^X_{g,\beta} := \deg[\overline{M}_g(X, \beta)]^{\mathrm{vir}} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

For a general reference on Gromov-Witten theory, see the excellent books [10, 18].

1.2.2 Local Gromov-Witten invariants

Let $\iota : Z \hookrightarrow X$ be a closed subvariety of the smooth complex projective threefold X, and suppose that X is a local Calabi-Yau threefold near Z. Let β be the class of a curve class in Z, and let M_Z denote the substack of $\overline{M}_g(X, \iota_*\beta)$ consisting of stable maps with image in Z. If M_Z is a union of path connected components, then it inherits a virtual 0-cycle (by restricting $[\overline{M}_{g(X,\iota_*\beta)}]^{\mathrm{vir}}$ to $H_0(M_Z, \mathbb{Q})$). The *local* Gromov-Witten invariant of $Z \subset X$ is defined to be the degree of this 0-cycle; we denote it by

$$N^g_\beta(Z \subset X)$$

Note that in general $N^g_\beta(Z \subset X)$ depends on a formal neighborhood of $Z \subset X$, and in some cases it only depends on the normal bundle. If the neighborhood is understood, we write $N^g_\beta(Z)$. For a wonderful expository article on the issues surrounding local invariants, see [4].

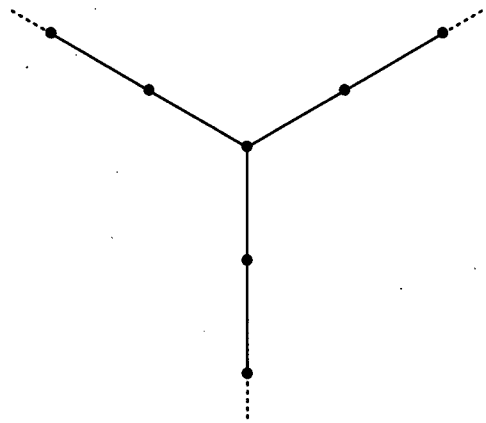

Figure 1.1: The minimal trivalent tree

1.2.3 Trees of rational curves

Let Γ be a connected graph consisting of vertices $\mathcal{V}(\Gamma)$ and edges $\mathcal{E}(\Gamma)$. An edge is specified by an unordered pair of vertices:

$$\mathcal{E}(\Gamma) \subset \mathrm{Sym}\,(\mathcal{V}(\Gamma), \mathcal{V}(\Gamma))$$

Furthermore, we assume throughout that Γ is a tree.

For any vertex v, the *valence* of v is defined to be the number of distinct edges containing v. We denote the valence of v by

$$|v| = \left|\{v' \in \mathcal{V}(\Gamma) \ : \ \{v, v'\} \in \mathcal{E}(\Gamma)\}\right|$$

We say that the tree Γ is *trivalent* if

(i) $|v| \leq 3$ for all $v \in \mathcal{V}(\Gamma)$

(ii) $|v_0| = 3$ for some $v_0 \in \mathcal{V}(\Gamma)$

Thus there is a unique infinite trivalent tree $\Gamma^3_{\min}$ such that

$$|v| = \begin{cases} 3 & \text{if } v = v_0 \\ 2 & \text{otherwise} \end{cases}$$

We call it the *minimal trivalent tree*, and it is shown in Figure 1.1. When there is no chance of confusion, we will also refer to a finite trivalent subtree of the minimal trivalent tree as the minimal trivalent tree.

Also, there is a unique infinite tree $\Gamma^3_{\max}$ such that

$$|v| = 3 \text{ for all } v \in \mathcal{V}(\Gamma^3_{\max})$$

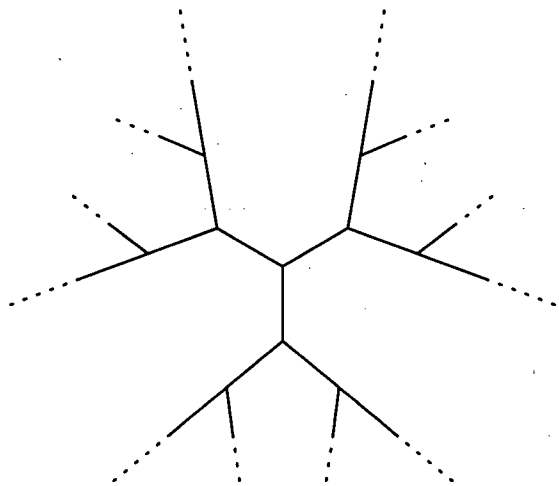

Figure 1.2: The maximal trivalent tree

We call $(\Gamma^3_{\max})$ the *maximal trivalent tree.* It is depicted in Figure 1.2.

A *configuration of rational curves* $\mathcal{C}$ is by definition a union along points of non-singular rational curves.

$$\mathcal{C} = \bigcup_{j \in J} C_j \qquad\qquad C_j \cong \mathbb{P}^1$$

Here J is some indexing set. We specify these points below and we specify the local geometry of the intersection points for the configurations of interest in Assumption 1.

We say that a configuration of rational curves *corresponds to the graph* Γ if there is a one to one correspondence between edges of the graph and irreducible components of $\mathcal{C}$.

$$\{C_j \subset \mathcal{C}\} \longleftrightarrow \{\mathcal{E}_j \in \mathcal{E}(\Gamma)\}$$

Additionally, it is required that there is a one to one correspondence between vertices v with valence greater than one in Γ and intersection points p_v of the corresponding components of $\mathcal{C}$:

$$v \in \mathcal{E}_j \cap \mathcal{E}_k \Longleftrightarrow p_v \in C_j \cap C_k$$

Here the components are necessarily distinct, i.e. $j \neq k$. The above union is then defined to be the union along these intersection points.

$$\mathcal{C} = \bigcup_{\{p_v\}} C_j$$

1.2.4 The blowup of $\mathbb{CP}^3$ at points

We briefly review the properties of the blowup of $\mathbb{P}^3$ at points used here for completeness and to set notation. This material can be found in much greater detail in, for instance, [17].

Let $X \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along M distinct points $\{p_1, \ldots, p_M\}$. We describe the homology of X. All (co)homology is taken with integer coefficients. Note that we may identify homology and cohomology as rings via Poincaré duality, where cup product is dual to intersection product.

Let H be the total transform of a hyperplane in $\mathbb{P}^3$, and let E_i be the exceptional divisor over p_i. Then $H_4(X, \mathbb{Z})$ has a basis

$$H_4(X) = \langle H, E_1, \ldots, E_M \rangle$$

Furthermore, let $h \in H_2(X)$ be the class of a line in H, and let e_i be the class of a line in E_i. The collection of all such classes form a basis of $H_2(X)$.

$$H_2(X) = \langle h, e_1, \ldots, e_M \rangle$$

The intersection ring structure is given as follows. Let $pt \in H_0(X)$ denote the class of a point. Two general hyperplanes meet in a line, so $H \cdot H = h$. A general hyperplane and line intersect in a point, so $H \cdot h = pt$. Also, a general hyperplane is far from the center of a blowup, so all other products involving H or h vanish. The restriction of $\mathcal{O}_X(E_i)$ to $E_i \cong \mathbb{P}^2$ is the dual of the bundle $\mathcal{O}_{\mathbb{P}^2}(1)$, so $E_i \cdot E_i$ is represented by minus a hyperplane in E_i, i.e. $E_i \cdot E_i = -e_i$, and $E_i^3 = (-1)^{3-1} pt = pt$ ([14]). Furthermore, the centers of the blowups are far away from each other, so all other intersections vanish. In summary, the following are the only non-zero intersection products.

$$\boxed{\begin{array}{ll} H \cdot H = h & H \cdot h = pt \\ E_i \cdot E_i = -e_i & E_i \cdot e_i = -pt \end{array}}$$

Also, we point out the important fact that the canonical bundle is easy to describe in this basis. Let K_X denote the canonical bundle of X. Then we have

$$K_X = 4H - 2\sum_{i=1}^{M} E_i$$

Finally, we introduce a notational convenience for the Gromov-Witten invariants of $\mathbb{P}^3$ blown up at points in a Calabi-Yau class. Any curve class is of the form

$$\beta = dh - \sum_{i=1}^{M} a_i e_i$$

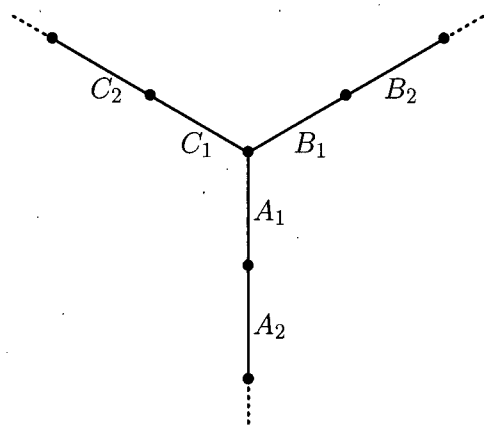

Figure 1.3: The minimal trivalent configuration

for some integers d, a_i where d is non-negative. Thus $K_X \cdot \beta = 0$ if and only if $2d = \sum_{i=1}^{M} a_i$. In that case, $\langle\,\rangle_{g,\beta}^X$ is determined by the discrete data $\{d, a_i, \ldots, a_M\}$. Then, we may use the shorthand notation

$$\langle\,\rangle_{g,\beta}^X = \langle d; a_1, \ldots, a_M\rangle_g^X .$$

For example,

$$\langle\,\rangle_{g,6h-e_1-e_2-2e_3-3e_5-3e_6}^X = \langle 6; 1, 1, 2, 0, 3, 3\rangle_g^X .$$

Furthermore, the Gromov-Witten invariants of X do not depend on ordering of the points p_i, and thus for any permutation σ of M points,

$$\langle d; a_1, \ldots, a_M\rangle_g^X = \langle d; a_{\sigma(1)}, \ldots, a_{\sigma(M)}\rangle_g^X .$$

1.3 Main results

Our main results determine the local Gromov-Witten invariants of certain configurations of rational curves $\mathcal{C} \subset X$ inside a local Calabi-Yau threefold. All of the configurations considered correspond to connected subtrees of the minimal trivalent tree. The infinite minimal trivalent configuration is depicted in Figure 1.3. Fix a non-negative integer N. As indicated, we label the components of $\mathcal{C}$ by

$$\mathcal{C} = \bigcup_{i=1}^{N} A_i \cup B_i \cup C_i,$$

where $A_i \cong B_i \cong C_i \cong \mathbb{P}^1$, reflecting the nature of the configuration.

We denote the genus-g local Gromov-Witten invariants of $\mathcal{C} \subset X$ by $N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C})$ where

$$\mathbf{a} = a_1, \ldots, a_N$$
$$\mathbf{b} = b_1, \ldots, b_N$$
$$\mathbf{c} = c_1, \ldots, c_N$$

and a_i, b_i, c_i is the degree of the map to the component A_i, B_i, C_i respectively.

In order for $N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C})$ to be well defined, we need to specify the formal neighborhood $\mathcal{C} \subset X$ and the local geometry of the intersection points in $\mathcal{C}$.

Assumption 1. *We assume that the local geometry of $\mathcal{C} \subset X$ is as explicitly constructed in section 2.1. In particular, $\mathcal{C}$ is embedded in X such that the normal bundle of each component of $\mathcal{C}$ is given as follows.*

$$N_{A_i/X} = N_{B_i/X} = N_{C_i/X} = \begin{cases} \mathcal{O}(-1) \oplus \mathcal{O}(-1) & \text{if } i = 1 \\ \mathcal{O} \oplus \mathcal{O}(-2) & \text{if } 1 < i \leq N \end{cases}$$

Additionally, for the case of $a_1, b_1, c_1 > 0$, we assume the formal neighborhood of the triple point has the geometry of the coordinate axes in $\mathbb{C}^3$ with respect to the local coordinates defined by the normal bundles. We assume all other intersections are nodal singularities.

We now state the main results.

1.3.1 The closed topological vertex

We now define the closed topological vertex. Let

$$\mathcal{C} = A_1 \cup B_1 \cup C_1 \subset X$$

be a locally Calabi-Yau configuration corresponding to a minimal trivalent configuration satisfying Assumption 1. Then we call $\mathcal{C}$ the *closed topological vertex.*

Theorem 1 (Bryan-Karp). *Assume $a_1, b_1, c_1 > 0$. Then the local invariants of the closed topological vertex are well defined and given as follows.*

$$N^g_{a_1;b_1;c_1}(\mathcal{C}) = 0$$

if $\{a_1, b_1, c_1\}$ contains two distinct non-zero values, otherwise

$$N^g_{a_1;a_1;a_1}(\mathcal{C}) = N^g_{a_1;0;0}(\mathcal{C}).$$

Note that $N^g_{a_1;0;0}(\mathcal{C})$ is the contribution to the genus g Gromov-Witten invariant from a single $\mathbb{P}^1$ smoothly embedded by a degree-a_1 map to a Calabi-Yau threefold with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. These were computed by Faber-Pandharipande [11] to be

$$N^g_{a_1;0;0}(\mathcal{C}) = \frac{|B_{2g}(2g-1)|}{(2g)!} a_1^{2g-3}$$

1.3.2 The minimal trivalent configuration

Theorem 2. *Let*

$$\mathcal{C} = \bigcup_{i=1}^{N} (A_i \cup B_i \cup C_i)$$

be a configuration of rational curves in a local Calabi-Yau threefold X which corresponds to the minimal trivalent tree and satisfies Assumption 1. Let a_i, b_i, c_i denote the degree of the map onto the component A_i, B_i, C_i respectively. Assume

$$a_1 = b_1 = c_1 = 1.$$

The local invariants of $\mathcal{C}$ are well defined and given as follows.

$$N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C}) = 0$$

unless $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfy

$$\begin{aligned} 1 = a_1 \geq \cdots \geq a_N \geq 0 \\ 1 = b_1 \geq \cdots \geq b_N \geq 0 \\ 1 = c_1 \geq \cdots \geq c_N \geq 0 \end{aligned}$$

In that case, for any $1 \leq n, m, l \leq N$, we have

$$\begin{aligned} N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C}) &= N_{1,\ldots,1;1,\ldots,1;1,\ldots,1}(\mathcal{C}) \\ &= N_{\underbrace{1,\ldots,1}_{n},\underbrace{0,\ldots,0}_{N-n};\underbrace{1,\ldots,1}_{m},\underbrace{0,\ldots,0}_{N-m};\underbrace{1,\ldots,1}_{l},\underbrace{0,\ldots,0}_{N-l}}(\mathcal{C}) \\ &= N^g_{1;1;1}(\mathcal{C}). \end{aligned}$$

Note that $N^g_{1;1;1}(\mathcal{C})$ is the genus g, degree (1,1,1) local invariant of the closed topological vertex. This is a special case of Theorem 1.

$$N^g_{1;1;1}(\mathcal{C}) = \frac{|B_{2g}(2g-1)|}{(2g)!}$$

where B_{2g} is the $2g$th Bernoulli number.

1.3.3 A chain of rational curves

Now consider the case that $\mathcal{C} = A_1 \cup \cdots \cup A_N$ is a chain of rational curves; it is shown in Figure 1.4. Note that a chain is of course a subtree of the minimal trivalent tree.

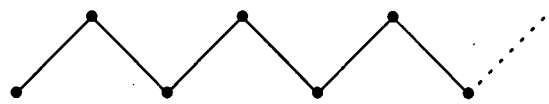

Figure 1.4: A chain of rational curves

Theorem 3. *Let*

$$\mathcal{C} = A_1 \cup \cdots \cup A_N$$

be a chain of rational curves satisfying Assumption 1. Let a_i denote the degree of the map onto the ith component (here $b_i = c_i = 0$). Assume $a_1 > 0$. Then the local invariants of $\mathcal{C}$ are well defined and given as follows.

$$N^g_{\mathbf{a}}(\mathcal{C}) = 0$$

unless

$$a_1 = a_2 = \cdots = a_j = a \qquad\qquad a_{j+1} = a_{j+2} = \cdots = a_N = 0$$

for some $a > 0$ and $1 \leq j \leq N$. Otherwise

$$N^g_{\mathbf{a}}(\mathcal{C}) = N^g_{a,\ldots,a}(\mathcal{C}) = N^g_{a,\ldots,a,0}(\mathcal{C}) = \cdots = N^g_a(\mathcal{C}).$$

Note that $N^g_a(\mathcal{C})$ is again the contribution to the genus g Gromov-Witten invariant from a single $\mathbb{P}^1$ smoothly embedded by a degree-a map to a Calabi-Yau threefold with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Thus

$$N^g_a(\mathcal{C}) = \frac{|B_{2g}(2g-1)|}{(2g)!} a^{2g-3}$$

1.4 Brief overview

We now give a brief description of the organization of this work and the techniques used to obtain the main results.

In chapter 2 we construct the configurations $\mathcal{C}$ of rational curves that are the study of this work. They are constructed as locally Calabi-Yau configurations in

a blowup space which is deformation equivalent to the blowup of $\mathbb{P}^3$ along distinct points.

We next show that the local invariants of the configurations are equal to certain ordinary invariants of the blowup space; this takes place in chapter 3. In order to relate the local invariants to global ones, we first show that inside the blowup space lives a configuration with the correct formal neighborhood. Then, it is left to show that the only contributions to the global invariants in the class of interest come from maps whose image lies in the specified configuration. We do so by using the toric nature of blowup space and using a homological argument.

In chapter 4 we study the properties of the Gromov-Witten invariants of the blowup of $\mathbb{P}^3$ along points. This material provides the tools necessary to complete the proofs of the main theorems. In particular, we prove a lemma showing that a large class of invariants of the blowup space vanish. Finally, we make crucial use of the geometry of the Cremona transformation.

The Cremona transformation admits a resolution on a space which is the blowup of $\mathbb{P}^3$ along points and lines. The resolved map acts on (co)homology and preserves Gromov-Witten invariants, because it is an isomorphism. This results in a symmetry of the invariants on the resolution space. In order to use this symmetry, we show that for a Calabi-Yau class β, the invariants of the resolved space descend to invariants of a blowup of $\mathbb{P}^3$ along only points, and not points and lines. This results in a symmetry of the Gromov-Witten invariants of $\mathbb{P}^3$ blown up at points. This study of the Cremona transformation first appeared in [5], was inspired by the beautiful work of Gathmann [15], and is joint work with Jim Bryan.

In chapter 5 we use the previous results to first relate the local invariants of the configuration to certain ordinary invariants of the blowup of $\mathbb{P}^3$ at points. Then, we use the tools of chapter 4 to compute the ordinary invariants.

As discussed above, the configurations we study are all finite subtrees of the minimal trivalent tree. And what's more, for the most general subtrees we restrict the degree of the invariants. However, in the appendix we construct Calabi-Yau configurations of curves corresponding to any finite subtree of the maximal trivalent tree. We do not compute the invariants of these general configurations or for general degree minimally trivalent configurations because our method fails in those cases. For counter examples, see remark 10. Specifically, it is not the case that the only contributions to the global invariants of the blowup space in the correct class come from maps whose image is the desired configuration.

Chapter 2

Configurations of rational curves

2.1 A geometric construction

We now construct configurations of rational curves, whose local Gromov-Witten invariants are the study of this work. These configurations correspond to finite sub trees of the minimal trivalent tree. We construct these configurations as subvarieties of a locally Calabi-Yau space X, which is obtained via a sequence of toric blowups of $\mathbb{P}^3$:

$$X = X^{N+1} \xrightarrow{\pi_{N+1}} X^N \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_2} X^1 \xrightarrow{\pi_1} X^0 = \mathbb{P}^3$$

In fact, X^{i+1} will be the blowup of X^i along three points. Our rational curves will be labeled by A_i, B_i, C_i, where $1 \leq i \leq N$, reflecting the nature of the configuration. Curves and in intermediary spaces will have super-scripts, and their corresponding proper transforms in X will not.

The standard torus $\mathbb{T} = (\mathbb{C}^\times)^3$ action on $\mathbb{P}^3$ is given by

$$(t_1, t_2, t_3) \cdot (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : t_1 x_1 : t_2 x_2 : t_3 x_3).$$

There are four $\mathbb{T}$-fixed points in $X^0 := \mathbb{P}^3$; we label them $p_0 = (1 : 0 : 0 : 0)$, $q_0 = (0 : 1 : 0 : 0)$, $r_0 = (0 : 0 : 1 : 0)$ and $s_0 = (0 : 0 : 0 : 1)$. Let A^0, B^0 and C^0 denote the (unique, $\mathbb{T}$-invariant) line in X^0 through the two points $\{p_0, s_0\}$, $\{q_0, s_0\}$ and $\{r_0, s_0\}$, respectively.

Define

$$X^1 \xrightarrow{\pi_1} X^0$$

to be the blowup of X^0 at the three points $\{p_0, q_0, r_0\}$, and let $A^1, B^1, C^1 \subset X^1$ be the proper transforms of A^0, B^0 and C^0. The exceptional divisor in X^1 over p_0 intersects A^1 in a unique fixed point; call it $p_1 \in X^1$. Similarly, the exceptional

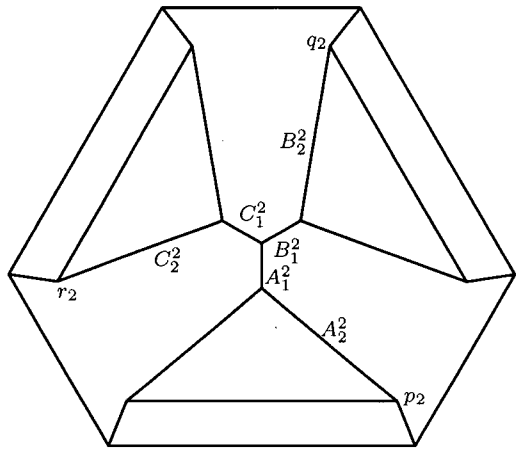

Figure 2.1: The $\mathbb{T}$-invariant curves in X^2

divisor in X^1 also intersects each of B^1 and C^1 in unique fixed points; call them q_1 and r_1.

Now define

$$X^2 \xrightarrow{\pi_2} X^1$$

to be the blowup of X^1 at the three points $\{p_1, q_1, r_1\}$, and let $A_1^2, B_1^2, C_1^2 \subset X^2$ be the proper transforms of A^1, B^1, C^1. The exceptional divisor over p_1 contains two $\mathbb{T}$ fixed points disjoint from A_1^2. Choose one of them, and call it p_2; this choice is arbitrary. Similarly, there are two fixed points in the exceptional divisors above q_1, r_1 disjoint from B_1^2, C_1^2. Choose one in each pair consistent with the choice of p_2 and call them q_2 and r_2. This choice is indicated in Figure 2.1. Let A_2^2 denote the (unique, $\mathbb{T}$ invariant) line intersecting A_1^2 and p_2. Define B_2^2, C_2^2 analogously.

Clearly X^2 is deformation equivalent to a blowup of $\mathbb{P}^3$ at six distinct points. The $\mathbb{T}$-invariant curves in X^1 are depicted in Figure 2.1, where each edge corresponds to a $\mathbb{T}$-invariant curve in X^1, and each vertex corresponds to a fixed point. For simplicity, at this time we only label those curves in the configuration. The remaining $\mathbb{T}$-invariant curves will be discussed in Chapter 3.

We now define a sequence of blowups beginning with X^2. Fix an integer $N \geq 2$. For each $1 < i \leq N$, define

$$X^{i+1} \xrightarrow{\pi_{i+1}} X^i$$

to be the blowup of X^i along the three points p_i, q_i, r_i. Let $A_j^{i+1} \subset X^{i+1}$ denote the proper transform of A_j^i for each $1 \leq j \leq i$. The exceptional divisor in X^{i+1} above p_i contains two $\mathbb{T}$ fixed points, choose one of them and call it p_{i+1}. Similarly choose

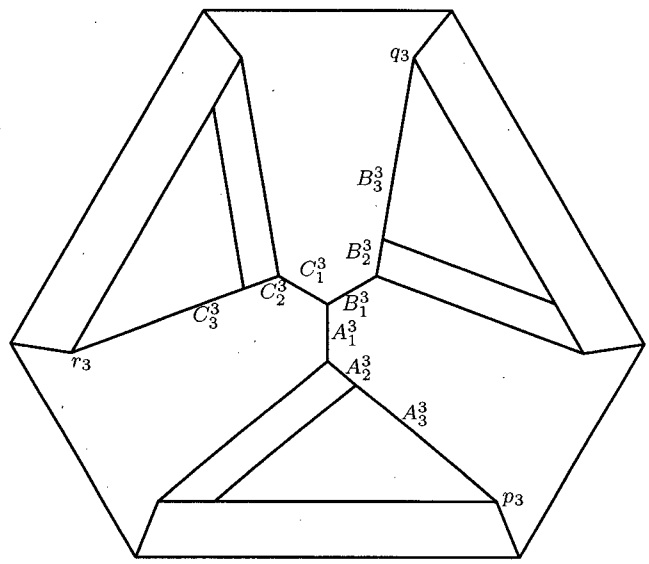

Figure 2.2: The $\mathbb{T}$-invariant curves in X^3

q_{i+1}, r_{i+1}, and define $A_{i+1}^{i+1} \subset X^{i+1}$ to be the line intersecting A_i^{i+1} and p_{i+1}, with $B_{i+1}^{i+1}, C_{i+1}^{i+1}$ defined similarly.

The $\mathbb{T}$ invariant curves in X^2 are shown in Figure 2.2. Terminate this process after obtaining the space X^{N+1}, and define $X = X^{N+1}$.

Finally, define the configuration $\mathcal{C} \subset X$ by

$$\mathcal{C} = \bigcup_{1 \leq j \leq N} A_j \cup B_j \cup C_j,$$

where

$$\begin{aligned} A_j &= A_j^{N+1} \\ B_j &= B_j^{N+1} \\ C_j &= C_j^{N+1}. \end{aligned}$$

The configuration $\mathcal{C}$ is shown in Figure 2.3, along with all other $\mathbb{T}$-invariant curves in X.

2.2 Intersection products and normal bundles

We now compute $H_*(X, \mathbb{Z})$ and identify the class of the configuration $[\mathcal{C}] \in H_2(X, \mathbb{Z})$. All (co)homology will be taken with integer coefficients. We denote divisors by

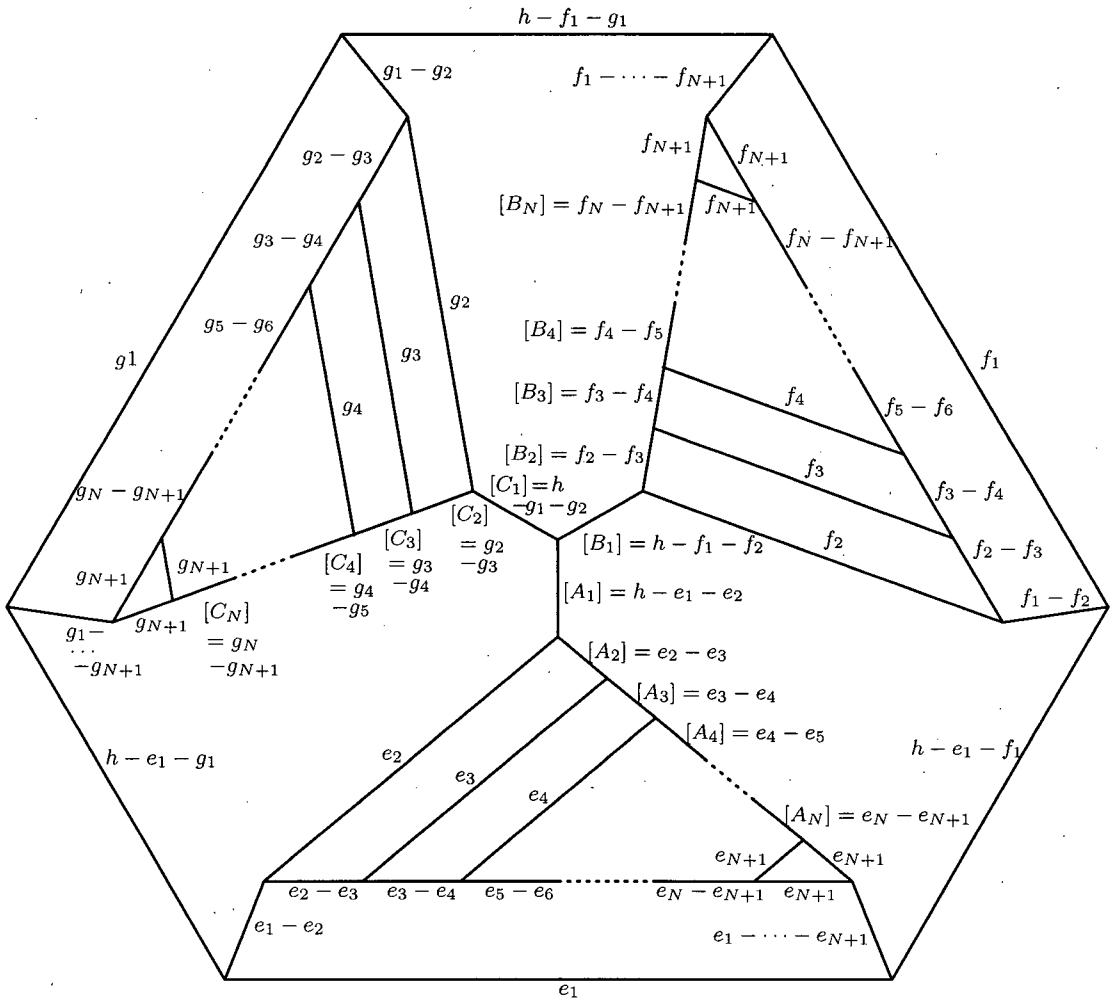

Figure 2.3: The $\mathbb{T}$-invariant curves in X

upper case letters, and curve classes with the lower case. In addition, we decorate homology classes in intermediary spaces with a tilde, and their total transforms in X are undecorated.

Let $\tilde{E}_1, \tilde{F}_1, \tilde{G}_1 \in H_4(X^1)$ denote the exceptional divisors in $X^1 \to X^0$ over the points p_0, q_0 and r_0, and let $E_1, F_1, G_1 \in H_4(X)$ denote their total transforms. Continuing, for each $1 \leq i \leq N+1$, let $\tilde{E}_i, \tilde{F}_i, \tilde{G}_i \in H_4(X^i)$ denote the exceptional divisors over the points $p_{i-1}, q_{i-1}, r_{i-1}$ and let $E_i, F_i, G_i \in H_4(X)$ denote their total transforms. Finally, let H denote the total transform of the hyperplane in $X^0 = \mathbb{P}^3$. The collection of all such classes $\{H, E_i, F_i, G_i\}$, where $1 \leq i \leq N+1$, spans $H_4(X)$.

Similarly, for each $1 \leq i \leq N+1$, let $\tilde{e}_i, \tilde{f}_i, \tilde{g}_i \in H_2(X^{i+1})$ denote the class of a line in $\tilde{E}_i, \tilde{F}_i, \tilde{G}_i$ and let $e_i, f_i, g_i \in H_2(X)$ denote their total transforms. In addition, let $h \in H_2(X)$ denote the class of a line in H. Then $H_2(X)$ has a basis given by $\{h, e_i, f_i, g_i\}$.

The intersection product ring structure is given as follows. Note that X is deformation equivalent to the blowup of $\mathbb{P}^3$ at $3N$ distinct points. Therefore, these

$H \cdot H = h$	$H \cdot h = pt$
$E_i \cdot E_i = -e_i$	$E_i \cdot e_i = -pt$
$F_i \cdot F_i = -f_i$	$F_i \cdot f_i = -pt$
$G_i \cdot g_i = -g_i$	$G_i \cdot g_i = -pt$

are all of the nonzero intersection products in $H_*(X)$. See section 1.2.4 above for details.

Lemma 4. *In the above basis, the classes of the components of C are given as follows.*

$$[A_i] = \begin{cases} h - e_1 - e_2 & \text{if } i = 1 \\ e_i - e_{i+1} & \text{if } 1 < i < N+1 \end{cases}$$

$$[B_i] = \begin{cases} h - f_1 - f_2 & \text{if } i = 1 \\ f_i - f_{i+1} & \text{if } 1 < i < N+1 \end{cases}$$

$$[C_i] = \begin{cases} h - g_1 - g_2 & \text{if } i = 1 \\ g_i - g_{i+1} & \text{if } 1 < i < N+1. \end{cases}$$

PROOF: Recall that A_1 is the proper transform of a line in $\mathbb{P}^3$ through the two points p_1, p_2 which are centers of a blowup. Since A_1 is in particular a curve, and we have the above basis, it must be the case that

$$[A_1] = d_0 h - \sum_{i=1}^{N} d_{1,i} e_i + d_{2,i} f_i + d_{3,i} g_i$$

for some integers $d_{i,j}$ with d_0 non-negative. We study the intersection theory of X in order to determine the coefficients $d_{i,j}$.

By functoriality of blowups, A_1 is isomorphic to a line blown up at the two points p_1, p_2, which of course is also a line. Since a line and a plane generically meet in a point, and using the above ring structure, we calculate

$$\begin{aligned} pt &= [A_1] \cdot H \\ &= d_0(pt) + 0 \end{aligned}$$

Therefore $d_0 = 1$.

Now, to calculate $d_{1,1}, d_{1,2}$, note that the exceptional divisor E_i parameterizes directions in $\mathbb{P}^3$ of intersection with the point p_i. Since A_1 is the proper transform of a line which intersects (contains) p_1 and does so at a unique direction, it must be the case that A_1 and E_1 intersect at a point. We calculate

$$\begin{aligned} pt &= [A_1] \cdot E_1 \\ &= -d_{1,1}(-pt) + 0 \end{aligned}$$

Thus $d_{1,1} = 1$. An identical argument also shows that $d_{1,2} = 1$.

To determine $d_{1,j}$ for $j > 2$, note that A_1 is the proper transform of a line which is far from the other centers of the blowups. So we calculate

$$\begin{aligned} 0 &= [A_1] \cdot E_j \\ &= -d_{1,j}(-pt) \end{aligned}$$

and thus $d_{1,j} = 0$ for $j > 2$. Similarly, we see that $d_{2,j} = d_{3,j} = 0$ for $j > 2$. Thus we have $[A_1] = h - e_1 - e_2$.

Now inspect A_i for $i > 1$. Its homology class is of the form

$$[A_1] = d_0' h - \sum_{i=1}^{N} d_{1,i}' e_i + d_{2,i}' f_i + d_{3,i}' g_i$$

for some integers $d_{i,j}'$ where d_0' is non-negative.

Recall that A_i is the proper transform of a line in $\tilde{E}_i \subset X^i$ containing the point p_i, which is subsequently blown up. Since A_i is contained in the total transform of an exceptional divisor, it pairs zero with the total transform of the hyperplane class:

$$0 = [A_i] \cdot H = d_0'(pt)$$

Therefore $d_0' = 0$.

Now note that by the functoriality of blowups, E_i is isomorphic to the blowup of $\mathbb{P}^2$ at one point, and the class of a line in E_i is e_i and the class of the exceptional line is e_{i+1}. Since A_i meets a generic line in E_i in a point, we calculate

$$\begin{aligned} pt &= [A_i] \cdot e_i \\ &= -d_{1,i}(pt) \end{aligned}$$

where the product is taken in E_i. Thus $d_{1,i} = -1$.

Furthermore, A_i intersects E_{i+1} in a point corresponding to the direction of incident with the point p_i, as E_{i+1} is the exceptional divisor above p_i, and so we calculate

$$pt = [A_i] \cdot E_{i+1} = -d_{1,i+1}(-pt).$$

Therefore $d_{1,i} = 1$.

Finally, note that A_i is far from the centers of all other blowups, and therefore all other coefficients must vanish. Thus $[A_i] = e_i - e_{i+1}$ for each $i > 2$.

Note that an identical argument works for B_i and C_i for each i, and therefore the result holds. □

We now show that $\mathcal{C}$ is a locally Calabi-Yau configuration. Let $[f : \Sigma \to X] \in \overline{M}_g(X, \beta)$ be a stable map with image in $\mathcal{C}$. We show that X is locally Calabi-Yau near $\mathrm{Im}(f)$ counted with multiplicity:

$$K_X \cdot f_*[\Sigma] = 0$$

Here,

$$\begin{aligned} f_*[\Sigma] = \beta = (a_1 + b_1 + c_1)h - a_1 e_1 - b_1 f_1 - c_1 g_1 \\ - \sum_{i=1}^{N} \{(a_i - a_{i+1})e_{i+1} + (b_i - b_{i+1})f_{i+1} + (c_i - c_{i+1})g_{i+1}\}. \end{aligned}$$

where a_i, b_i, c_i is the degree of the map to the component A_i, B_i, C_i and $a_{N+1} = b_{N+1} = c_{N+1} = 0$. So, we compute

$$\begin{aligned} K_X \cdot f_*[\Sigma] = &\left(4H - 2\sum_{i=1}^{N} A_i + B_i + C_i\right) \cdot \\ &\Big((a_1 + b_1 + c_1)h - a_1 e_1 - b_1 f_1 - c_1 g_1 \\ &- \sum_{i=1}^{N} \{(a_i - a_{i+1})e_{i+1} + (b_i - b_{i+1})f_{i+1} + (c_i - c_{i+1})g_{i+1}\}\Big) \\ =&12H \cdot h - 2(2[A_1] \cdot [A_1] + 2[B_1] \cdot [B_1] + 2[C_1] \cdot [C_1]) \\ =&\ 0 \end{aligned}$$

We now describe the normal bundles of the components of $\mathcal{C}$ in X. These are given as follows.

Lemma 5.

$$N_{A_i/X} = N_{B_i/X} = N_{C_i/X} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{if } i = 1 \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \text{if } i > 1. \end{cases}$$

PROOF: The equivalence $N_{A_i/X} \cong N_{B_i/X} \cong N_{C_i/X}$ is easily seen by relabeling points. To calculate $N_{A_1/X}$, let $D^0 \subset X^0$ be a plane containing the line A^0, and let D denote its proper transform in X. Then $A_1 \subset D$, and $N_{A_1/D}$ is a sub bundle of $N_{A_1/X}$ of degree $[A_1] \cdot [A_1]$, where the product is taken in D. Note that D is deformation equivalent to the blowup of a plane at two points, and $[A_1] = h - e_1 - e_2$. Thus, the intersection product in D_1 is given by

$$[A_1] \cdot [A_1] = (h - e_1 - e_2) \cdot (h - e_1 - e_2) = -1.$$

The set of planes containing A_1 span $N_{A_1/X}$, and the above argument holds for any such plane, so we conclude $N_{A_1/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Now consider A_i, where $i > 1$. Note that $A_i \subset E_i$. As above, N_{A_i/E_i} is a sub bundle of $N_{A_i/X}$ of degree $[A_i] \cdot [A_i]$, *where the product is taken in* E_i. Recall that, by the functoriality of blowups, E_i is the blowup of $\mathbb{P}^2$ at a point, and that e_i is the class of a line in E_i, and e_{i+1} is the exceptional divisor. We compute

$$[A_i] \cdot [A_i] = (e_i - e_{i+1}) \cdot (e_i - e_{i+1}) = 1 - 1 = 0.$$

We now show that the total degree of the normal bundle is -2, forcing the result to hold. Inspect the defining exact sequence

$$0 \to T_{A_i} \to T_X \to N_{A_i/X} \to 0.$$

This implies

$$\begin{aligned} c_1\left(N_{A_i/X}\right) &= c_1\left(T_X\right) \cdot [A_i] - c_1\left(T_{A_i}\right) \\ &= \left(4H - 2\sum_{j=1}^{N+1} E_j + F_j + G_j\right) \cdot (e_i - e_{i+1}) - 2pt \\ &= 0 - 2(pt - pt) - 2pt \\ &= -2pt. \end{aligned}$$

Thus the total degree of the normal bundle is -2, and so $N_{A_i/X} = \mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a + b = -2$. Since we have already shown that (without loss of generality) $a = 0$, we conclude

$$N_{A_i/X} \cong \mathcal{O} \oplus \mathcal{O}(-2).$$

□

Chapter 3

From local to global invariants

In this chapter we show that the local Gromov-Witten invariants of the configurations $\mathcal{C}$ of rational curves constructed in Chapter 2 are equal to certain global or ordinary Gromov-Witten invariants of a blowup of $\mathbb{P}^3$ along points.

3.1 The closed topological vertex

Let $\mathcal{C} = A \cup B \cup C$ be the closed topological vertex, and let X be the blowup space constructed in section 2.1. Moreover, in this case let $N = 1$ so that $X = X^2$, and is deformation equivalent to the blowup of $\mathbb{P}^3$ along $2 \cdot 3 = 6$ points. Then the local invariants of the closed topological vertex are equal to certain ordinary invariants of X.

Proposition 6 (Bryan-Karp). *Let $\mathcal{C} = A \cup B \cup C$ be the closed topological vertex, and let $X = X^1$ be the $N = 0$ blowup space constructed in section 2.1. Let*

$$\beta = a(h - e_1 - e_2) + b(h - f_1 - f_2) + c(h - g_1 - g_2)$$

and assume $a, b, c > 0$. Then the local invariants of $\mathcal{C}$ are equal to the ordinary invariants of X in the class β.

$$N^g_{a;b;c} = \langle\, \rangle^X_{g,\beta}$$

PROOF: $\mathcal{C} \subset X$ has the correct local geometry by assumption. So in order to prove the proposition, it suffices to show that the only contributions to $\langle\, \rangle^X_{g,\beta}$ are given by maps with image in $\mathcal{C}$.

Lemma 7. *Let $X, \mathcal{C}$ and β be as above. Assume $a, b, c > 0$. Then*

$$\mathrm{Im}(f) = \mathcal{C}$$

for every stable map $[f] \in \overline{M}_g(X, \beta)$.

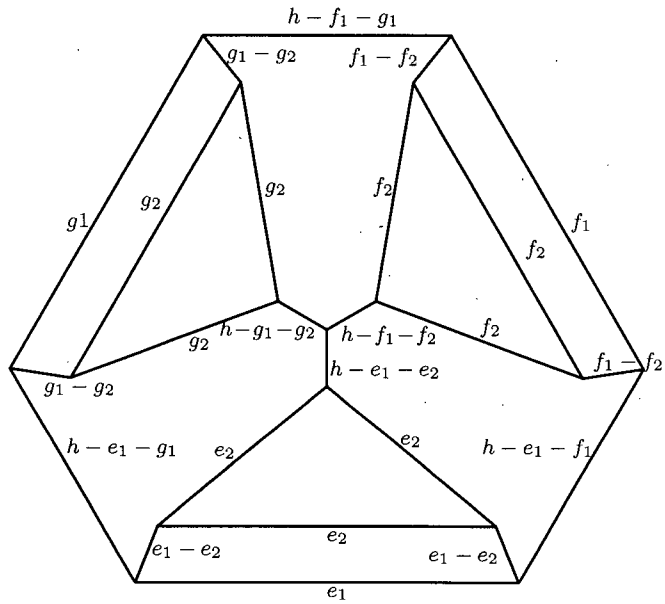

Figure 3.1: The $\mathbb{T}$-invariant curves in X

PROOF: We explicitly use the toric nature of the construction. Note that the torus action on $X^0 = \mathbb{P}^3$ lifts to X since the center of each subsequent blowup is $\mathbb{T}$ fixed. Thus there is a $\mathbb{T}$ action on $\overline{M}_g(X, \beta)$, simply by composition.

Assume that there exists a stable map

$$[f : \Sigma \to X] \in \overline{M}_g(X, \beta)$$

such that $\mathrm{Im}(f) \not\subset \mathcal{C}$. Then there exists a point $x \in \mathrm{Im}(f)$ such that $x \notin \mathcal{C}$. Recall that a one parameter family ψ of $\mathbb{T}$ is defined to be an element $\psi \in \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T}) \cong \mathbb{Z}^3$, where the isomorphism is given by

$$\begin{aligned}(n_1, n_2, n_3) \mapsto \psi_{n_1,n_2,n_3} : \mathbb{C}^\times &\to \mathbb{T} \\ t &\mapsto (t^{n_1}, t^{n_2}, t^{n_3}).\end{aligned}$$

Moreover, recall that $\mathbb{T}$-invariant subvarieties of X are given precisely by orbit-closures of limit points of one parameter subgroups of T.

$$\{\mathbb{T}\text{-invariant subvarieties}\} \longleftrightarrow \left\{\overline{\mathbb{T} \cdot \lim_{t \to 0} \psi(t)}\right\}$$

So, in particular, the limit of the point x under the action of ψ is a fixed point. Moreover, since $x \notin \mathcal{C}$ and every fixed point is the limit of some one parameter subgroup, there exists ψ such that

$$\lim_{t \to 0} \psi(t) \cdot x = q,$$

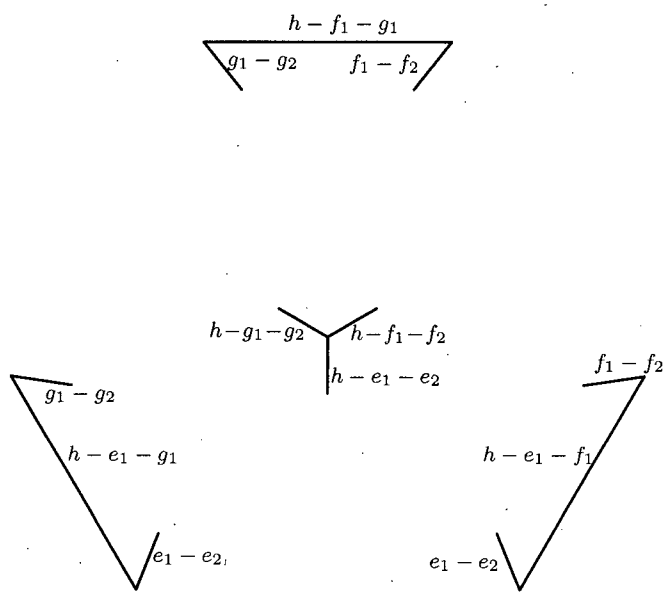

Figure 3.2: The remaining possible curves in $\mathrm{Im}(f'')$

where q is $\mathbb{T}$ fixed and $q \notin \mathcal{C}$. So, the limit of ψ acting on $[f]$ is a stable map f' such that $q \in \mathrm{Im}(f')$. It follows that q is in image of all stable maps in the orbit closure of $[f']$. Thus, there must exist a stable map $[f'' : \Sigma \to X] \in \overline{M}_g(X, \beta)$ such that $\mathrm{Im}(f'')$ is $\mathbb{T}$ invariant and $\mathrm{Im}(f'') \not\subset \mathcal{C}$.

We show that this leads to a contradiction. Let I denote the union of the $\mathbb{T}$ invariant curves in X; it is shown in Figure 3.1.

Inspect the class $f''_*[\Sigma]$. Note that the total multiplicity of the e's is $a+b+c$. Also note that every component of I whose class contains h also contains two $-e$ terms. Therefore the total multiplicity of the terms not containing h must be zero. The same is true for the f and g terms. The curves in I whose class does not contain h either contribute nothing to the multiplicity of the e, f, g's, or they contribute a strictly positive amount. Therefore, the curves whose classes contribute positively to the total multiplicity of the e, f, g's must not be contained in the image of f''. Therefore, the curves in class e_i, f_i, g_i are not in $\mathrm{Im}(f'')$. The remaining possible curves in the image of f'' are shown in Figure 3.2.

Recall that $\mathrm{Im}(f'') \not\subset \mathcal{C}$ by assumption. Therefore the image of f'' is contained in the outer components in Figure 3.2. But the image of f'' is connected and must and $[\mathrm{Im}(f'')]$ contains strictly positive multiples of each of e_1, f_1, g_1, since $a, b, c > 0$. Therefore the image of f'' is a union of the outer components components. Therefore the image of f'' is not connected. This contradiction shows that our original assumption is incorrect. Therefore the result holds.

3.2 The minimal trivalent configuration

We now consider the case when $\mathcal{C}$ is a minimal trivalent configuration.

Proposition 8. *Let*

$$\mathcal{C} = \bigcup_{i=1}^{N} (A_i \cup B_i \cup C_i)$$

be a minimal trivalent configuration of rational curves satisfying Assumptions 1. Let a_i, b_i, c_i denote the degree of the map onto the component A_i, B_i, C_i respectively, and let $\mathbf{a} = (a_1, \ldots, a_N), \mathbf{b} = (b_1, \ldots, b_N), \mathbf{c} = (c_1, \ldots, c_N)$. Also, let X continue to denote the blowup space constructed in Chapter 2. Assume

$$a_1 = b_1 = c_1 = 1.$$

Then the local invariants of $\mathcal{C}$ are equal to the global Gromov-Witten invariants of X in the class β,

$$N^g_{\mathbf{a},\mathbf{b},\mathbf{c}}(\mathcal{C}) = \langle\, \rangle^X_{g,\beta},$$

where

$$\begin{aligned}\beta = {} & (a_1 + b_1 + c_1)h - a_1 e_1 - b_1 f_1 - c_1 g_1 \\ & - \sum_{i=1}^{N} \{(a_i - a_{i+1})e_{i+1} + (b_i - b_{i+1})f_{i+1} + (c_i - c_{i+1})g_{i+1}\}.\end{aligned}$$

and $a_{N+1} = b_{N+1} = c_{N+1} = 0$.

PROOF: By assumption, the formal neighborhood of $\mathcal{C}$ agrees with the construction in X. Thus, in order to prove the Proposition it suffices to show that the only contributions to $\langle\, \rangle^X_{g,\beta}$ are from maps to $\mathcal{C}$.

Lemma 9. *Let $X, \mathcal{C}$ and β be as above, where we assume that*

$$a_1 = b_1 = c_1 = 1.$$

Then every stable map $[f] \in \overline{M}_g(X, \beta)$ has image $\mathcal{C}$.

PROOF: Assume that there exists a stable map $[f : \Sigma \to X] \in \overline{M}_g(X, \beta)$ such that $\mathrm{Im}(f'') \not\subset \mathcal{C}$. Then by the proof of Lemma 7, there exists a stable map $[f''] \in \overline{M}_g(X, \beta)$ such that $\mathrm{Im}(f'')$ is $\mathbb{T}$ invariant but $\mathrm{Im}(f'') \not\subset \mathcal{C}$.

We show that this is a contradiction. Let $F \subset X$ denote the union of the $\mathbb{T}$-invariant curves in X; it's shown in Figure 2.3. We study the possible components of F contained in the image of f''.

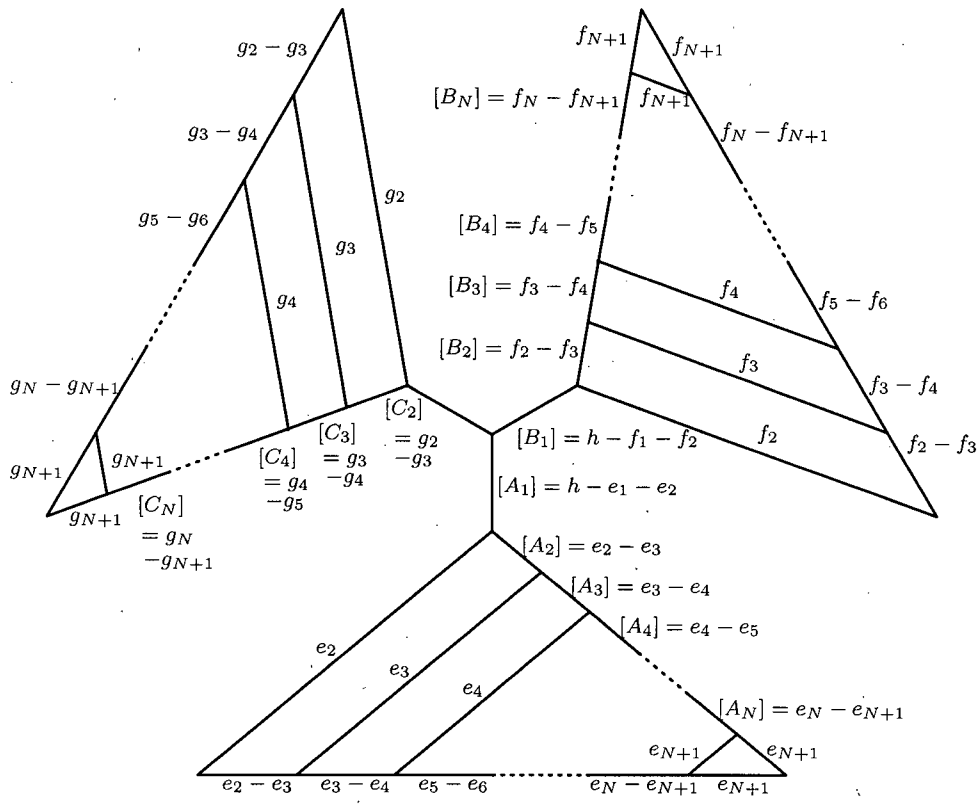

Figure 3.3: The possible curves in $\mathrm{Im}(f'')$

Suppose that $A_1 \cup B_1 \cup C_1 \subset \mathrm{Im}(f'')$. Then $f''_*[\Sigma]$ contains (at least) $3h$. Note that $[F]$ has no $-h$ terms. Therefore $\mathrm{Im}(f'')$ does not contain any of the curves $h - e_1 - f_1, h - e_1 - g_1, h - f_1 - g_1$. And furthermore each of A_1, B_1 and C_1 must have multiplicity one.

There are no remaining terms that contain $-e_1, -f_1$ or $-g_1$. Also, since the image of f'' contains precisely one of A_1, B_1, C_1, we conclude that the multiplicity of terms contain positive e_1, f_1, g_1 must be zero. Thus, $\mathrm{Im}(f'')$ is contained in the configuration shown in Figure 3.3.

Now, note that in β the sum of the multiplicities of the e_i's is -2. This is true of the curve A_1 as well. Therefore the total multiplicity of all other e terms must vanish. But all other e terms are of the form $e_i - e_{i+1}$ or e_j. Since, the former

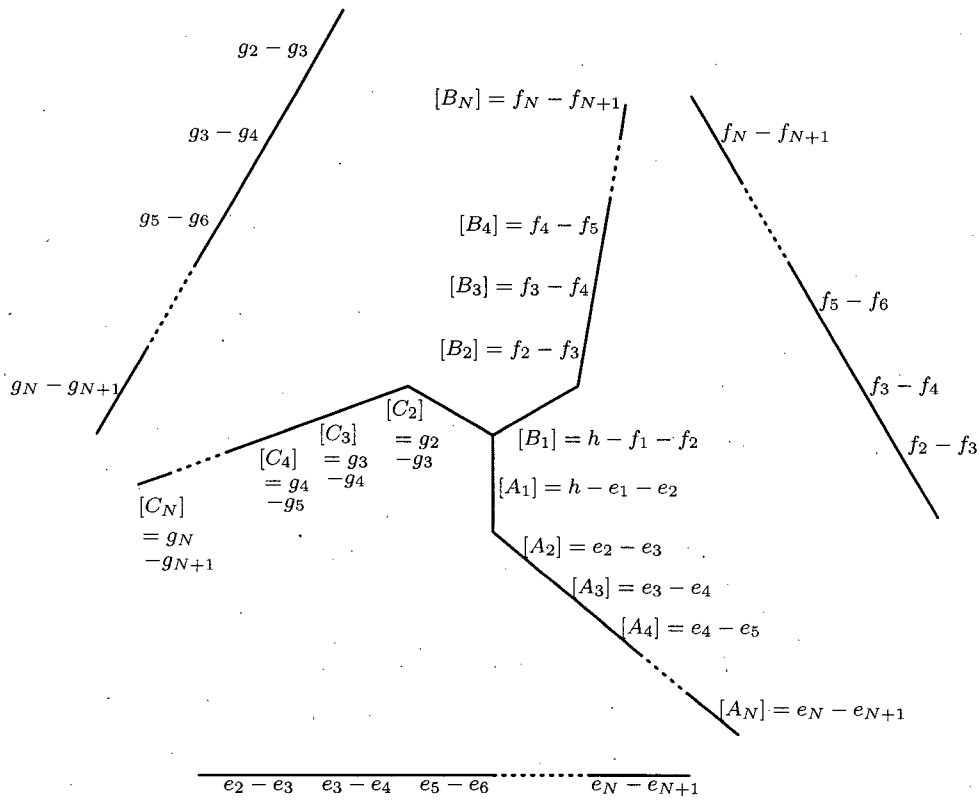

Figure 3.4: The remaining possible curves in $\mathrm{Im}(f'')$

contribute nothing to the total multiplicity, we conclude that there are no e_j terms in the image of f''. Therefore $\mathrm{Im}(f'')$ must be contained in the configuration shown in Figure 3.4.

But $\mathrm{Im}(f'')$ is connected, and contains h terms. Therefore it can not contain nor be contained in any of the three outer parts of Figure 3.4. Therefore $\mathrm{Im}(f'') \subset \mathcal{C}$ and thus $\mathrm{Im}(f'') = \mathcal{C}$. This contradicts our assumption, and therefore at least one of A_1, B_1, C_1 is not in $\mathrm{Im}(f'')$.

Without loss of generality, suppose $A_1 \not\subset \mathrm{Im}(f'')$. Let $d_{e,f}, d_{e,g}, d_{f,g}$ denote the degree of f'' on the components $h - e_1 - f_1, h - e_1 - g_1, h - f_1 - g_1$ respectively.

Since A_1 is not contained in the image of f'', we must have

$$d_{e,f} + d_{e,g} > 0.$$

Furthermore, in order for $\mathrm{Im}(f'')$ to simultaneously be connected and contain $-e_i$ terms for $i > 1$, it must be the case that $\mathrm{Im}(f'')$ contains *two* of

$$\{e_1, e_1 - e_2, e_1 - \cdots - e_{N+1}\}.$$

Thus

$$\begin{aligned} d_{e,f} + d_{e,g} &= 3 \\ d_{f,g} &= 0 \end{aligned}$$

and $B_1, C_1 \not\subset \mathrm{Im}(f'')$. This forces $\mathrm{Im}(f'')$ to be contained in the configuration shown in Figure 3.5.

Again we have that $\mathrm{Im}(f'')$ is connected and contains $-f_i, -g_j$ for some $i, j > 1$. Therefore $\mathrm{Im}(f'')$ contains at least one of $f_1 - f_2, f_1 - \cdots - f_{N+1}$ and also at least one of $g_1 - g_2, g_1 - \cdots - g_{N+1}$. But the multiplicity of f_1 and g_1 in β is -1. Therefore

$$d_{e,f}, d_{e,g} \geq 2.$$

This contradictions shows that our assumption $A_1 \not\subset \mathrm{Im}(f'')$ is incorrect. Therefore $A_1 \subset \mathrm{Im}(f'')$. An identical argument also shows that $B_1, C_1 \subset \mathrm{Im}(f'')$. However we showed above that $A_1, B_1, C_1 \not\subset \mathrm{Im}(f'')$.

This contradiction shows that our original assumption is incorrect. Therefore there does not exist a point $x \in \mathrm{Im}(f'')$ such that $x \notin \mathcal{C}$, and $\mathrm{Im}(f'') \subset \mathcal{C}$. Therefore $\mathrm{Im}(f'') = \mathcal{C}$, and the result holds. □

Remark 10. Note that this argument does not hold for general a_1, b_1, c_1. For instance, it is a fun exercise to show that there is more than one $\mathbb{T}$ invariant configuration of curves in X in the following classes.

$$\begin{aligned} \beta_1 =& 2(h - e_1 - e_2) + (e_2 - e_3) \\ &+ 2(h - f_1 - f_2) + (f_2 - f_3) \\ &+ 2(h - g_1 - g_2) + (g_2 - g_3) \\ \beta_2 =& 4(h - e_1 - e_2) + (e_2 - e_3) + 2(h - f_1 - f_2) + 2(h - g_1 - g_2) + (g_2 - g_3) \\ \beta_3 =& 4(h - e_1 - e_2) + 4(e_2 - e_3) \\ &+ 4(h - f_1 - f_2) + 4(f_2 - f_3) \\ &+ 4(h - g_1 - g_2) + 4(g_2 - g_3). \end{aligned}$$

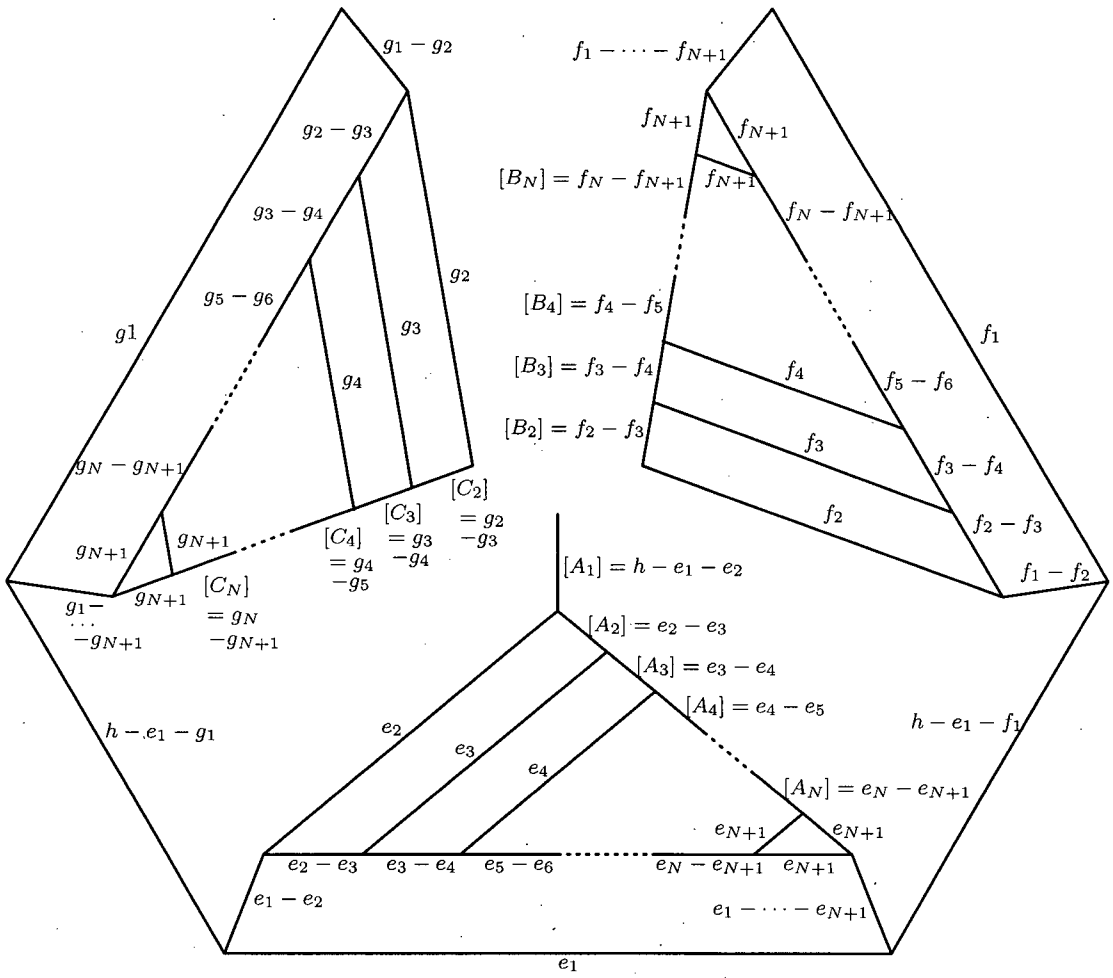

Figure 3.5: The other possibility for curves in $\mathrm{Im}(f'')$

3.3 A chain of rational curves

Next consider the case when $\mathcal{C}$ is a chain of rational curves. More precisely, let

$$\mathcal{C} = A_1 \cup \cdots \cup A_{N+1} \subset X.$$

Since $\mathcal{C}$ does not contain any of the curves B_i, C_i, the blowups with centers p_i and q_i in the construction of X are extraneous. In order to simplify the argument in this case, consider the space

$$Y = Y^{N+1} \xrightarrow{\pi_{N+1}} Y^N \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_1} \mathbb{P}^3,$$

where the construction of Y follows that of X, without the extraneous blowups. So $Y^{i+1} \to Y^i$ is the blowup of Y^i along the point p_i, where p_i is defined in Section 2.1. Thus, Y is deformation equivalent to the blowup of $\mathbb{P}^3$ at $N+1$ points. Since $\mathcal{C}$ does not contain the curves B_i, C_i, clearly the formal neighborhood of $\mathcal{C}$ in Y agrees with the construction in X.

We continue to let E_i be the total transform of the exceptional divisor over p_i, and e_i be the class of a line in E_i. Furthermore, we continue to let H denote the pullback of the class of a hyperplane in $\mathbb{P}^3$, and h be the class of a line in H. Then, $\{H, E_i\}$ is a basis for $H_4(Y)$ and $\{h, e_i\}$ is a basis for $H_2(Y)$. The non-zero intersection pairings are given as follows.

$H \cdot H = h$	$H \cdot h = pt$
$E_i \cdot E_i = -e_i$	$E_i \cdot e_i = -pt$

The $\mathbb{T}$-invariant curves in Y are shown together with their homology classes in Figure 3.6

Proposition 11. *Let the blowup space Y and the chain of rational curves $\mathcal{C} = A_1 \cup \cdots \cup A_N$ be as constructed above. Let $\mathbf{a} = a_1, \ldots, a_N$ where a_i is the degree of the map to A_i. Assume $a_1 > 0$. Then the local invariants of $\mathcal{C}$ are equal to the global invariants of Y in the class β,*

$$N^g_{\mathbf{a}}(\mathcal{C}) = \langle\, \rangle^Y_{g,\beta},$$

where

$$\beta = a_1 h - a_1 e_1 - \sum_{i=2}^{N+1} (a_{i-1} - a_i) e_i$$

and $a_{N+1} = 0$.

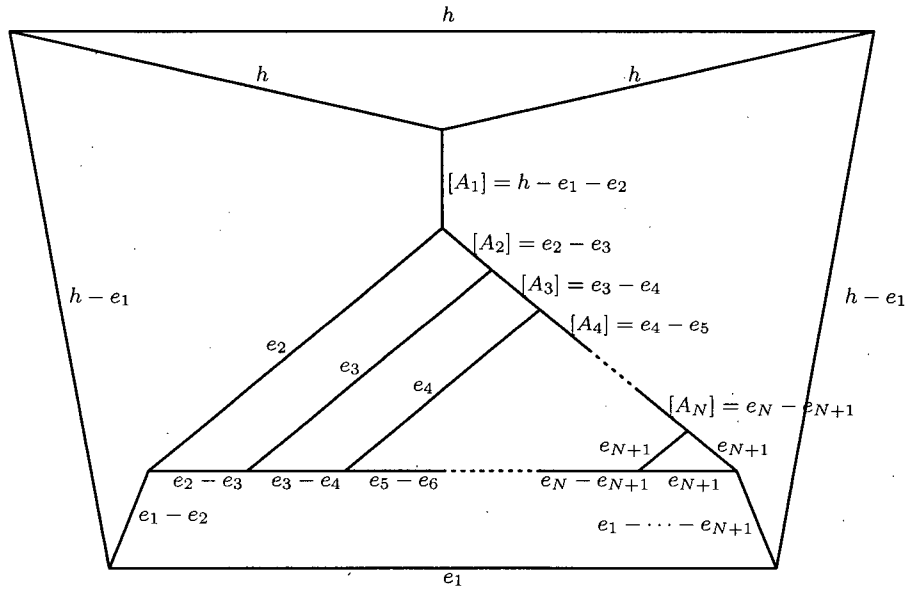

Figure 3.6: The $\mathbb{T}$-invariant curves in Y

PROOF: $\mathcal{C} \subset Y$ has the desired geometry by construction, in order to prove the proposition it suffices to show that the only contributions to the Gromov-Witten invariants of Y in class β are from maps to $\mathcal{C}$.

Lemma 12. *Let Y, $\mathcal{C} = A_1 \cup \cdots \cup A_N$ and β be as above. Then*

$$\mathrm{Im}(f) = \mathcal{C}$$

for any stable map $[f] \in \overline{M}_g(Y, \beta)$.

As shown in Proposition 8, we may use the toric nature of Y to construct a stable map $[f'' : \Sigma \to Y] \in \overline{M}_g(Y, \beta)$ such that $\mathrm{Im}(f'')$ is $\mathbb{T}$ invariant, but $\mathrm{Im}(f'') \not\subset \mathcal{C}$. We show that this leads to a contradiction.

We study the class $f''_*[\Sigma] = \beta$. Note that the multiplicity of the $-e_1$ term is the same as that of h. Furthermore, each $-e_1$ occurs along with h, and there are no $-h$ terms. Therefore $\mathrm{Im}(f'')$ can not contain any terms containing positive e_1, nor can it contain any of the curves in class h. Thus, the image of f'' is contained in the configuration of curves shown in Figure 3.7.

Since $a_1 > 0$, it must be that $f''_*[\Sigma]$ contains at least one e_i term with non-zero multiplicity for $i > 1$. Also, $\mathrm{Im}(f'')$ is connected and so we conclude that the image of f must not contain either of the curves of class $h - e_1$ in Figure 3.7.

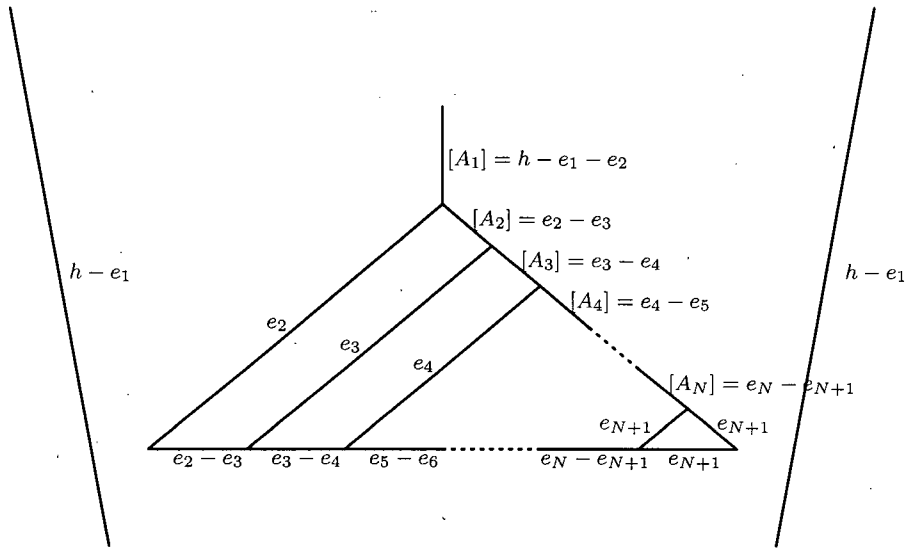

Figure 3.7: The possible curves in $\mathrm{Im}(f'')$

Now, note that the total multiplicity of the e terms is $-2a_1$, and that the curve A_1 must also have this property. Therefore the sum of all other e terms must be zero. Since the other e terms are of the form $e_i - e_{i+1}$ or e_j, we conclude that $\mathrm{Im}(f'')$ does not contain any of the curves e_j. Thus $\mathrm{Im}(f'')$ is contained in the configuration depicted in Figure 3.8.

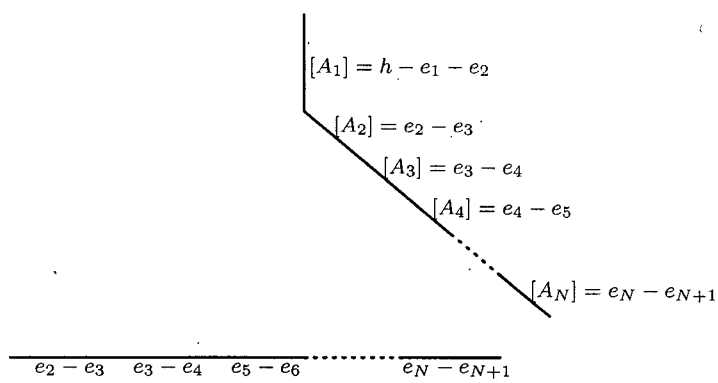

Figure 3.8: The remaining possible curves in $\mathrm{Im}(f'')$

However, since Y is connected and contains h, we conclude that $\mathrm{Im}(f'') \subset \mathcal{C}$, and therefore $\mathrm{Im}(f'') = \mathcal{C}$. This contradiction shows that our original assumption is incorrect, and the result holds. $\square$

Chapter 4

Properties of the invariants of the blowup of $\mathbb{P}^3$ at points

In this chapter we prove results needed for the proofs of the main theorems.

4.1 A vanishing Lemma

We use the notation of subsection 1.2.4.

Lemma 13. *Let X be the blowup of $\mathbb{P}^3$ at n distinct generic points $\{x_1, \ldots, x_n\}$, and $\beta = dh - \sum_{i=1}^n a_i e_i$ with $2d = \sum_{i=1}^n a_i$, and assume that $d > 0$ and $a_i < 0$ for some i. Then*

$$\overline{M}_g(X, \beta) = \emptyset.$$

Corollary 14. *For any n points $\{x_1, \ldots, x_n\}$ and X and β as above the corresponding invariant vanishes;*

$$\langle\, \rangle_{g,\beta}^X = 0.$$

This follows immediately from the deformation invariance of Gromov-Witten invariants and Lemma 13.

Lemma 15. *Let $\widetilde{X}$ be the blowup of $\mathbb{P}^3$ at $\tilde{n}$ distinct generic points $\{\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}\}$ and let $\widetilde{\beta} = \sum_{i=1}^{\tilde{n}} dh - \widetilde{a}_i e_i$, with $2d < \sum_{i=1}^{\tilde{n}} \widetilde{a}_i$, and assume that $\widetilde{a}_i > 0$ for all i. Then*

$$\overline{M}_g(\widetilde{X}, \widetilde{\beta}) = \emptyset.$$

Proposition 16. *For each $d > 0$, Lemma 13 is equivalent to Lemma 15.*

To prove Lemma 13 or Lemma 15, it suffices to prove the Lemmas for some particular choice of $\{x_1, \ldots, x_n\}$ or $\{\tilde{x}_1, \ldots, \tilde{x}_{\tilde{n}}\}$, since if $\overline{M}_g(X, \beta)$ (or $\overline{M}_g(\widetilde{X}, \widetilde{\beta})$) is empty for a specific choice of points, then it is empty for the generic choice.

PROOF: We prove that Lemma 13 is false if and only if Lemma 15 is false. First, without loss of generality, we may reorder the centers of the blowup so that $a_1 \leq \cdots \leq a_m < 0 \leq a_{m+1} \leq \cdots \leq a_n$ for some $1 \leq m < n$.

Choose $\{x_1, \ldots, x_n\}$ so that $\{x_1, \ldots, x_m\}$ are coplanar, as are $\{x_{m+1}, \ldots, x_n\}$. Let D' and D'' be the classes of the proper transforms of those planes, respectively, so that

$$D' = H - E_1 - \cdots - E_m$$
$$D'' = H - E_{m+1} - \cdots - E_n.$$

Assume that Lemma 13 is false. Then there exists $[f] \in \overline{M}_g(X, \beta)$. We show that Lemma 15 fails by studying the image of f. $\mathrm{Im}(f)$ decomposes into several components. Note that $\beta \cdot E_1 = a_1 < 0$. Thus $\mathrm{Im}(f)$ has a component(s) contained in E_1. Denote the union of these components by C_1. If $\mathrm{Im}(f)$ has any components contained in E_j we denote them by C_j as well. As e_j is the class of a line in $E_j \cong \mathbb{P}^2$, it must be that

$$[C_j] = b_j e_j$$

for some $b_j > 0$. Let $J \subset \{1, \ldots, n\}$ be the indexing set of these components,

$$J = \{1 \leq j \leq n | \text{ There exists a component of} \mathrm{Im}(f) \subset E_j\},$$

and let C_E be the union,

$$[C_E] = \sum_{j \in J} C_j.$$

$\mathrm{Im}(f)$ decomposes further. As $2d = \sum_{i=1}^n a_i$, we have $\beta \cdot (D' + D'') = 2d - \sum_{i=1}^n a_i = 0$ and, because $a_1 \leq \cdots \leq a_m < 0 \leq a_{m+1} \leq \cdots \leq a_n$,

$$\beta \cdot D'' = \left(d - \sum_{i=m+1}^{n} a_i\right) < \left(d - \sum_{i=1}^{m} a_i\right) = \beta \cdot D',$$

so

$$d - \sum_{i=m+1}^{n} a_i < 0.$$

Thus

$$(\beta - [C_E]) \cdot D'' = d - \sum_{i=1}^{n} a_i - \sum_{\substack{j>m \\ j \in J}} b_j < 0.$$

Therefore there exists a nonempty closed subscheme of $\mathrm{Im}(f) \cap D''$ which does not have components in the exceptional divisors. Denote it by C''.

Let $[\mathrm{Im}(f)]$ denote the class of $\mathrm{Im}(f)$. Then in summary, we have

$$[\mathrm{Im}(f)] = [C_E] + [C'] + [C''],$$

where

$$\begin{aligned}
[C_E] =& \sum_{j\in J} b_j e_j \\
[C''] =& d''h - \sum_{i=m+1}^{n} a_i'' e_i \\
[C'] =& (d-d'')h - \sum_{i=1}^{n} a_i e_i + \sum_{i=m+1}^{n} a_i'' e_i - \sum_{j\in J} b_j e_j.
\end{aligned}$$

Here $b_j > 0$ as noted above. Also $d'' > 0$, as C'' is not contained in the exceptional divisors E_j, and $d - d'' > 0$, since $\mathrm{Im}(f)$ is connected. For later use, we point out that $b_1 > 0$ and as a result

$$\sum_{j\in J} b_j > 0.$$

We now show this implies Lemma 15 could not be true. Consider the curve C which consists of the components of $\mathrm{Im}(f)$ which are not contained in the exceptional divisors E_j. It is of class

$$\widetilde{\beta} := [\mathrm{Im}(f)] - [C_E] = [C'] + [C''] = dh - \sum_{i=1}^{n} a_i e_i - \sum_{j\in J} b_j e_j.$$

Note that

$$2d = \sum_{i=1}^{n} a_i < \sum_{i=1}^{n} a_i + \sum_{j\in J} b_j.$$

Also $\widetilde{\beta}\cdot E_i \geq 0$ as C has no component in the exceptional divisors E_j. Thus the pair $(X, \widetilde{\beta})$ are as in Lemma 15. Clearly there is a stable map $\widetilde{g}$ such that $[\mathrm{Im}(\widetilde{g})] = [C'] + [C'']$ and consequently

$$[\widetilde{g}] \in \overline{M}_g(X, \widetilde{\beta}),$$

which contradicts Lemma 15.

Conversely, assume that Lemma 15 is false, so that there exists $[\widetilde{f}] \in \overline{M}_g(\widetilde{X}, \widetilde{\beta})$. Let X be the blowup of $\widetilde{X}$ at $\tilde{x}_* \in \mathrm{Im}(\widetilde{f})$ where $\tilde{x}_*$ is not contained in the exceptional divisor. Such a point exists as $\widetilde{\beta}\cdot E_i > 0$ and $\mathrm{Im}(\widetilde{f})$ is connected. Let $\widetilde{a}_* = -\sum_{i=1}^{\tilde{n}} \widetilde{a}_i$. There is a curve $C_* \subset E_*$ in the exceptional divisor over $\tilde{x}_*$ of class $[C_*] = -\widetilde{a}_* e_*$ such that $C_* \cap \mathrm{Im}(f) \neq \emptyset$, where e_* is the class of a line in E_*. Then

Define $\beta := [C_* \cup \mathrm{Im}(f)] = dh - \tilde{a}_* e_* - \tilde{a}_1 e_1 - \cdots - \tilde{a}_{\tilde{n}} e_{\tilde{n}}$. Evidently the pair (X, β) are as in Lemma 13, and as before it is clear that there exists a stable map f with $\mathrm{Im}(f) = C_*$ and

$$[f] \in \overline{M}_g(X, \beta),$$

which contradicts Lemma 15.

□

PROOF (OF LEMMA 13): Suppose there exists $[f] \in \overline{M}_g(X, \beta)$. Then with the above choice of $\{x_1, \ldots, x_n\}$ we have the above decomposition of $\mathrm{Im}(f)$. We use induction on d. Suppose $d = 1$, then we have $1 - d'' > 0$ and $d'' > 0$, which is a contradiction.

We now proceed inductively. Consider the curves C' and C'' defined above. Suppose $2d'' \geq \sum_{i=m+1}^n a_i''$ and $2(d - d'') \geq \sum_{i=1}^n a_i - \sum_{i=m+1}^n a_i'' + \sum_{j \in J} b_j$. Then

$$\sum_{i=m+1}^n a_i'' \leq 2d'' \leq \sum_{i=m+1}^n a_i'' - \sum_{j \in J} b_j.$$

This is impossible. Therefore either $2d'' < \sum_{i=m+1}^n a_i''$ or $2(d - d'') < \sum_{i=1}^n a_i - \sum_{i-m+1}^n a_i'' + \sum_{j \in J} b_j$. Therefore by Lemma 15 and the inductive hypothesis there is no decomposition of $[\mathrm{Im}(f)]$ involving either C' or C'', and therefore there is no $[f] \in \overline{M}_g(X, \beta)$. □

Corollary 17. *Let X be the blowup of $\mathbb{P}^3$ along points and define $\beta = dh - \sum_{i=1}^n a_i e_i$ where $2d = \sum_{i=1}^n a_i$ and $d > 0$. Also define*

$$X' \xrightarrow{\pi} X$$

to be the blowup of X at a generic point p, so that X' is deformation deformation equivalent to the blowup of $\mathbb{P}^3$ at $n+1$ distinct points. Let $\{h', e_1', \ldots, e_{n+1}'\}$ be a basis of $H_2(X')$, and let $\beta' = dh' - \sum_{i=1}^n a_i e_i'$. Then

$$\langle d; a_1, \ldots a_n, 0 \rangle_g^{X'} = \langle d; a_1, \ldots, a_n \rangle_g^X$$

PROOF: This follows from Lemma 15 (or equivalently Lemma 13). The method of proof used here was used in [5] to prove what is included as Lemma 20 here. This result also follows from the more general results of Hu in [19], but the proof is easy in this case, so we include it in order for our results to be more self contained.

We will show that any $[f'] \in \overline{M}_g(X', \beta')$ has an image disjoint from E_{n+1}', the exceptional divisor over p. Note that any $[f] \in \overline{M}_g(X, \beta)$ has an image disjoint from p. It follows that the natural map $\overline{M}_g(X', \beta') \to \overline{M}_g(X, \beta)$ induced by π is an isomorphism of the moduli space and their virtual classes. Indeed, it will follow that both $\overline{M}_g(X', \beta')$ and $\overline{M}_g(X, \beta)$ are canonically identified with $\overline{M}_g(X' \backslash E_{n+1}', \beta')$.

Let $[f' : \Sigma \to X'] \in \overline{M}_g(X', \beta')$. Suppose $\mathrm{Im}(f') \cap E'_{i+1} \neq \emptyset$. Note that $f'_*[\Sigma] \cdot E'_{i+1} = 0$. Therefore $\mathrm{Im}(f')$ has a component C' contained in E'_{i+1}. Since $d > 0$, $\mathrm{Im}(f')$ also has a (union of) component(s) C not contained in E'_{i+1}. The classes of these components are given by

$$[C] = dh - \sum_{i=1}^{n} a_i e'_i - m e'_{i+1} \qquad\qquad [C'] = m e'_{i+1}$$

where $m > 0$. Note that $2d = \sum_{i=1}^{n} a_i$. Therefore

$$2d < \sum_{i=1}^{n} a_i + m$$

Therefore the component C does not exist by Lemma 15. This contradiction shows that our assumption was incorrect. Therefore $\mathrm{Im}(f') \cap E'_{i+1} = \emptyset$, and the corollary is proved. □

4.2 The geometry of the Cremona transformation

This section consists essentially of section 5 from Bryan-Karp [5]. In particular, every result in this section is joint with Jim Bryan. It is included so that this document may be self contained.

Theorem 18. *Let $\beta = dh - \sum_{i=1}^{n} a_i e_i$ with $2d = \sum_{i=1}^{n} a_i$ and assume that $a_i \neq 0$ for some $i > 4$. Then we have the following equality of Gromov-Witten invariants:*

$$\langle\ \rangle^X_{g,\beta} = \langle\ \rangle^X_{g,\beta'}$$

where $\beta' = d'h - \sum_{i=1}^{n} a'_i e_i$ has coefficients given by

$$\begin{aligned}
d' &= 3d - 2(a_1 + a_2 + a_3 + a_4) \\
a'_1 &= d - (a_2 + a_3 + a_4) \\
a'_2 &= d - (a_1 + a_3 + a_4) \\
a'_3 &= d - (a_1 + a_2 + a_4) \\
a'_4 &= d - (a_1 + a_2 + a_3) \\
a'_5 &= a_5 \\
&\vdots \\
a'_n &= a_n.
\end{aligned}$$

In this section, we prove Theorem 18 by studying the geometry of X, the blowup of $\mathbb{P}^3$ at n points, and $\hat{X}$, the blowup of X along a certain configuration of six lines.

Let X be the blowup of $\mathbb{P}^3$ at n distinct points $x_1, \ldots, x_n$ where $n > 4$. We take the first four points to be the fixed points of the standard torus action on $\mathbb{P}^3$ and we take the remaining points to be any fixed points of the Cremona transformation:

$$\begin{aligned} \mathbb{P}^3 &\dashrightarrow \mathbb{P}^3 \\ (z_0 : z_1 : z_2 : z_3) &\mapsto (\frac{1}{z_0} : \frac{1}{z_1} : \frac{1}{z_2} : \frac{1}{z_3}). \end{aligned}$$

Remark 19. The case where $n - 4$ is greater than the number of fixed points is easily handled by including in the blowup locus pairs of points exchanged by the Cremona transformation. However, for notational convenience we will assume that the points $x_1, \ldots, x_n$ are fixed.

Let l_{jk}, $1 \leq j < k \leq 4$ be the proper transform of the line through x_j and x_k. Let

$$\pi : \hat{X} \to X$$

be the blowup of X along the six (disjoint) lines l_{jk}.

$\hat{X}$ admits an involution $\tau : \hat{X} \to \hat{X}$ which resolves the Cremona transformation. The map τ is discussed in more detail by Gathmann in [15], although note that our $\hat{X}$ has the additional blowups at $x_5, \ldots, x_n$ whose corresponding exceptional divisors are simply fixed by τ, or possibly exchanged if the points $x_5 \ldots x_n$ include non-trivial orbits (c.f. Remark 19).

We briefly describe the divisors and the curves on X and $\hat{X}$ and their intersections. Generally, we denote divisor classes with upper case letters and curve classes with lower case letters. Classes on $\hat{X}$ will have a hat, and classes on X will not.

The homology groups $H_4(X;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$ are spanned by the divisor and curve classes respectively:

$$H_4(X;\mathbb{Z}) = \langle H, E_1, \ldots, E_n \rangle, \quad H_2(X;\mathbb{Z}) = \langle h, e_1, \ldots, e_n \rangle.$$

Here H is the pullback of the hyperplane in $\mathbb{P}^3$, h is the class of the line in H, E_i is the exceptional divisor over x_i, and e_i is the class of a line in E_i.

The intersection pairing on X is given by:

$$\begin{aligned} H \cdot H &= h, & E_i \cdot E_i &= -e_i, \\ H \cdot h &= p, & E_i \cdot e_i &= -p \end{aligned}$$

where $p \in H_0(X;\mathbb{Z})$ is the class of the point and all other pairings are zero.

The homology groups $H_4(\hat{X};\mathbb{Z})$ and $H_2(\hat{X};\mathbb{Z})$ are also spanned by divisor and curve classes:

$$H_4(\hat{X};\mathbb{Z}) = \left\langle \hat{H}, \hat{E}_i, \hat{F}_{jk} \right\rangle, \quad H_2(\hat{X};\mathbb{Z}) = \left\langle \hat{h}, \hat{e}_i, \hat{f}_{jk} \right\rangle,$$

where $1 \leq i \leq n$ and $1 \leq j < k \leq 4$. Here $\hat{H}$ is the proper transform of H and $\hat{h}$ is the generic line in $\hat{H}$. $\hat{E}_i$ is the proper transform of E_i and $\hat{e}_i$ is the class of the generic line in $\hat{E}_i$. $\hat{F}_{jk}$ is the component of the exceptional divisor of $\hat{X} \to X$ lying over l_{jk}, and $\hat{f}_{jk}$ is the fiber class of $\pi : \hat{F}_{jk} \to l_{jk}$.

Note that $\hat{F}_{jk} \to l_{jk}$ is the trivial fibration and the class of the section $\hat{s}_{jk}$ is given by

$$\hat{s}_{jk} = \hat{h} - \hat{e}_j - \hat{e}_k + \hat{f}_{jk}.$$

The intersections are given as follows:

$$\begin{aligned}
\hat{H} \cdot \hat{H} &= \hat{h}, & \hat{E}_i \cdot \hat{E}_i &= -\hat{e}_i, & \hat{F}_{jk} \cdot \hat{F}_{jk} &= -\hat{s}_{jk} - \hat{f}_{jk}, \\
\hat{H} \cdot \hat{F}_{jk} &= \hat{f}_{jk}, & \hat{E}_j \cdot \hat{F}_{jk} &= \hat{f}_{jk}, & & \\
\hat{H} \cdot \hat{h} &= \hat{p}, & \hat{E}_i \cdot \hat{e}_i &= -\hat{p}, & \hat{F}_{jk} \cdot \hat{f}_{jk} &= -\hat{p},
\end{aligned}$$

where $\hat{p} \in H_0(\hat{X};\mathbb{Z})$ is the class of the point and all other intersections are zero.

The action of τ on divisors is described by Gathmann [15] in section 6. The action of τ on the curve classes of $\hat{X}$ is then easily obtained using Poincaré duality and is given as follows:

$$\begin{aligned}
\tau_*\hat{h} &= 3\hat{h} - (\hat{e}_1 + \hat{e}_2 + \hat{e}_3 + \hat{e}_4), \\
\tau_*\hat{e}_1 &= 2\hat{h} - (\hat{e}_2 + \hat{e}_3 + \hat{e}_4), \\
\tau_*\hat{e}_2 &= 2\hat{h} - (\hat{e}_1 + \hat{e}_3 + \hat{e}_4), \\
\tau_*\hat{e}_3 &= 2\hat{h} - (\hat{e}_1 + \hat{e}_2 + \hat{e}_4), \\
\tau_*\hat{e}_4 &= 2\hat{h} - (\hat{e}_1 + \hat{e}_2 + \hat{e}_3), \\
\tau_*\hat{e}_5 &= \hat{e}_5, \\
&\vdots \\
\tau_*\hat{e}_n &= \hat{e}_n, \\
\tau_*\hat{f}_{jk} &= \hat{s}_{j'k'},
\end{aligned}$$

where $\{j', k'\}$ is defined by the condition $\{j,k\} \cup \{j',k'\} = \{1,2,3,4\}$.

For a class $\hat{\beta} = d\hat{h} - \sum_{i=1}^n a_i\hat{e}_i$ with $2d = \sum_{i=1}^n a_i$, we have $-K_{\hat{X}} \cdot \hat{\beta} = 0$ and so the degree $\hat{\beta}$ Gromov-Witten invariants have no insertions. Since τ is an isomorphism, it preserves the Gromov-Witten invariants of $\hat{X}$ so in particular,

$$\langle\,\rangle^{\hat{X}}_{g,\hat{\beta}} = \langle\,\rangle^{\hat{X}}_{g,\tau_*\hat{\beta}}$$

where

$$\tau_*\hat{\beta} = d'\hat{h} - \sum_{i=1}^{n} a_i'\hat{e}_i$$

has coefficients $d', a_1', \ldots, a_n'$ given by the equations of Theorem 18.

To prove Theorem 18 then, it suffices to prove the following

Lemma 20. *Let $d, a_1, \ldots, a_n$ be such that $2d = \sum_{i=1}^{n} a_i$ and $a_i \neq 0$ for some $i > 4$. Then*

$$\langle\,\rangle^{X}_{g,\beta} = \langle\,\rangle^{\hat{X}}_{g,\hat{\beta}}$$

where $\beta = dh - \sum_{i=1}^{n} a_i e_i$ and $\hat{\beta} = d\hat{h} - \sum_{i=1}^{n} a_i\hat{e}_i$.

Remark 21. The condition that $a_i \neq 0$ for some $i > 4$ is necessary. For example,

$$1 = \langle\,\rangle^{X}_{0,h-e_1-e_2} \neq \langle\,\rangle^{\hat{X}}_{0,\hat{h}-\hat{e}_1-\hat{e}_2} = 0.$$

PROOF: The lemma follows from the general results of Hu [19]. We warn the reader that the theorems in [19] are incorrect as stated; the above example provides a counterexample. However, the author has informed us that a crucial hypothesis is missing in the main theorems of [19]. Namely, in Hu's notation, he must additionally assume that the class $p_!(A)$ is not exceptional.

The paper [19] uses the machinery of relative Gromov-Witten invariants and gluing. To make our paper self-contained, we provide below an independent proof of Lemma 20 in the case of $n = 6$ which is what is needed for the closed topological vertex in [5].

Assume that $n = 6$. Without loss of generality we may assume that $a_5 \neq 0$. We will show that any $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ has an image which is disjoint from $\hat{F} = \cup_{j<k}\hat{F}_{jk}$, and any $[f] \in \overline{M}_g(X, \beta)$ has an image which is disjoint from $l = \cup_{j<k} l_{jk}$. It follows that the natural map $\overline{M}_g(\hat{X}, \hat{\beta}) \to \overline{M}_g(X, \beta)$ induced by π is an isomorphism of the moduli spaces and their virtual fundamental classes. Indeed, if both $\mathrm{Im}(\hat{f}) \cap \hat{F} = \emptyset$ and $\mathrm{Im}(f) \cap l = \emptyset$ for all stable maps $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ and $[f] \in \overline{M}_g(X, \beta)$, then both $\overline{M}_g(\hat{X}, \hat{\beta})$ and $\overline{M}_g(X, \beta)$ are canonically identified with $\overline{M}_g(\hat{X}\backslash\hat{F}, \hat{\beta})$.

Let $[f : C \to X] \in \overline{M}_g(X, \beta)$ and suppose that $\mathrm{Im}(f) \cap l_{jk} \neq \emptyset$ for some j and k. $\mathrm{Im}(f) \not\subset l_{jk}$ since $a_5 \neq 0$ and so

$$f_*(C) = C' + bl_{jk}$$

where C' meets l_{jk} in a finite set of points (b can be zero here). Let $\hat{C}'$ be the proper transform of C'. Since $C' \cap l_{jk} \neq \emptyset$, we have $\hat{C}' \cdot \hat{F}_{jk} = m > 0$. Therefore we have

$$\hat{C}' = d\hat{h} - \sum_{i=1}^{6} a_i \hat{e}_i - b(\hat{h} - \hat{e}_j - \hat{e}_k) - m\hat{f}_{jk}.$$

Define $\{j', k'\}$ by the condition $\{j', k'\} \cup \{j, k\} = \{1, 2, 3, 4\}$ and let

$$\hat{D}_{jk} = 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - \hat{F}_{jk} - \hat{F}_{j'k'}.$$

Then

$$\hat{D}_{jk} \cdot \hat{C}' = -m < 0.$$

However, this contradicts Lemma 22 which states that $\hat{D}_{jk}$ is nef.

Lemma 22. *Let $1 \leq j < k \leq 4$ and define j', k' by the condition $\{j, k\} \cup \{j', k'\} = \{1, 2, 3, 4\}$. Then the divisor*

$$\hat{D}_{jk} = 2\hat{H} - (\hat{E}_1 + \cdots + \hat{E}_6) - \hat{F}_{jk} - \hat{F}_{j'k'}$$

is nef in $\hat{X}$.

PROOF: Let $\hat{D}'$ and $\hat{D}''$ be the proper transforms of the planes through $\{x_j, x_k, x_5\}$ and $\{x_{j'}, x_{k'}, x_6\}$ respectively. Then

$$\begin{aligned} \hat{D}' &= \hat{H} - \hat{E}_j - \hat{E}_k - \hat{E}_5 - \hat{F}_{jk} \\ \hat{D}'' &= \hat{H} - \hat{E}_{j'} - \hat{E}_{k'} - \hat{E}_6 - \hat{F}_{j'k'} \end{aligned}$$

so $\hat{D}_{jk} = \hat{D}' + \hat{D}''$.

To see that $\hat{D}_{jk}$ is nef, it suffices to check that $\hat{D}_{jk} \cdot C \geq 0$ for any curve $C \subset \hat{D}'$.

$\hat{D}'$ is isomorphic to the blowup of $\mathbb{P}^2$ at three points. Under this identification, the classes of the line and the three exceptional divisors are

$$h' = \hat{h} - \hat{f}_{jk}, \quad e'_j = \hat{e}_j - \hat{f}_{jk}, \quad e'_k = \hat{e}_k - \hat{f}_{jk}, \quad e'_5 = \hat{e}_5.$$

The curve $C \subset \hat{D}'$ has class

$$dh' - a_j e'_j - a_k e'_k - a_5 e'_5$$

and since $h' - e'_5$ is a nef divisor in $\hat{D}'$, we have

$$d \geq a_5.$$

The first Chern class of the normal bundle of $\hat{D}' \subset \hat{X}$ is

$$(\hat{H} - \hat{E}_j - \hat{E}_k - \hat{E}_5 - \hat{F}_{jk})^2 = -\hat{e}_5 = -e'_5$$

and so

$$\hat{D}' \cdot C = -e'_5 \cdot (dh' - a_j e'_j - a_k e'_k - a_5 e'_5) = -a_5$$

where the intersection product on the right hand side is on $\hat{D}'$.

Therefore

$$\begin{aligned} \hat{D}_{jk} \cdot C &= \hat{D}' \cdot C + \hat{D}'' \cdot C \\ &= -a_5 + d \\ &\geq 0. \end{aligned}$$

□

Thus $\mathrm{Im}(f) \cap l = \emptyset$ for all $[f] \in \overline{M}_g(X, \beta)$.

We argue in a similar fashion for $\overline{M}_g(\hat{X}, \hat{\beta})$. Let $[\hat{f} : C \to \hat{X}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ and suppose that $\mathrm{Im}(\hat{f}) \cap \hat{F}_{jk} \neq \emptyset$ for some j and k. Since $\hat{\beta} \cdot \hat{F}_{jk} = 0$, $\hat{f}_*(C)$ must have a component C'' contained in $\hat{F}_{jk}$. We then have

$$\hat{\beta} = \hat{f}_*(C) = C' + C''$$

where C' is non-empty since $\hat{\beta} \cdot \hat{E}_5 = a_5 > 0$.

Since $C'' \subset \hat{F}_{jk}$ is an effective class in $\hat{F}_{jk} \cong \mathbb{P}^1 \times \mathbb{P}^1$, it is of the form $a\hat{s}_{jk} + b\hat{f}_{jk}$ with $a, b \geq 0$ and $a + b > 0$.

Define $\hat{D}_{jk}$ as above. Then $\hat{D}_{jk} \cdot \hat{\beta} = 0$ and $\hat{D}_{jk} \cdot C'' = a + b > 0$ and so $\hat{D}_{jk} \cdot C' < 0$, contradicting the fact that $\hat{D}_{jk}$ is nef.

This proves that $\mathrm{Im}(\hat{f}) \cap \hat{F} = \emptyset$ for all $[\hat{f}] \in \overline{M}_g(\hat{X}, \hat{\beta})$ and Lemma 20 is proved.

This then completes the proof of Theorem 18.

Chapter 5

Proofs of the main theorems

5.1 Proof of Theorem 1

Let $X = X^2$ be as constructed in section 2.1 and let $Y = Y^2$ be as constructed in section 3.3. By Proposition 6, we have

$$N^g_{a;b;c}(\mathcal{C}) = \langle\ \rangle^X_{g,\beta}$$

where

$$\beta = a(h - e_1 - e_2) + b(h - f_1 - f_2) + c(h - g_1 - g_2).$$

So, we inspect the invariant

$$\langle\ \rangle^X_{g,\beta} = \langle a + b + c; a, a, b, b, c, c\rangle^X_g\ .$$

By Theorem 18, we have

$$\langle a + b + c; a, a, b, b, c, c\rangle^X_g = \langle 3c - a - b; c - b, c - b, c - a, c - a, c, c\rangle^X_g$$

Thus, by Corollary 14, $N^g_{a,b,c}(\mathcal{C}) = 0$ unless $a = b = c$. In that case,

$$\begin{aligned}\langle 3c - a - b; c - b, c - b, c - a, c - a, c, c\rangle^X_g &= \langle a; 0, 0, 0, 0, a, a\rangle^X_g \\ &= \langle a; a, a\rangle^Y_g\end{aligned}$$

where the last equality follows from Corollary 17. □

5.2 Proof of Theorem 2

Let $X = X^{N+1}$ be as constructed in section 2.1 By Proposition 8, we have

$$N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C}) = \langle\ \rangle^X_{g,\beta}$$

where

$$\beta = 3h - e_1 - f_1 - g_1 \\ - \sum_{i=1}^{N+1} \{(a_i - a_{i+1})e_{i+1} + (b_i - b_{i+1})f_{i+1} + (c_i - c_{i+1})g_{i+1}\},$$

where $a_1 = b_1 = c_1 = 1$ and $a_{N+1} = b_{N+1} = c_{N+1} = 0$.

Assume that the invariant is non-zero:

$$\begin{aligned} \langle\,\rangle^X_{g,\beta} =& \langle 3; 1, 1 - a_2, \ldots, a_{N-1} - a_N, a_N, \\ & 1, 1 - b_2, \ldots, b_{N-1} - b_N, b_N, \\ & 1, 1 - c_2, \ldots, c_{N-1} - c_N, c_N \rangle^X_g \\ & \neq 0 \end{aligned}$$

Then, by Lemma 13, the coefficient of each e_i, f_i, g_i is non-negative. Thus

$$\begin{aligned} 1 \geq a_2 \geq \cdots \geq a_N \geq 0, \\ 1 \geq b_2 \geq \cdots \geq b_N \geq 0, \\ 1 \geq c_2 \geq \cdots \geq c_N \geq 0. \end{aligned}$$

Therefore we compute

$$\begin{aligned} \langle\,\rangle^X_{g,\beta} =& \langle 3; 1, 0, \ldots, 0, 1, \\ & 1, 0, \ldots, 0, 1, \\ & 1, 0, \ldots, 0, 1 \rangle^X_g \\ =& \langle 3; 1, 1, 1, 1, 1, 1 \rangle^{X^2}_g, \end{aligned}$$

where again the last equality follows from Corollary 17. □

5.3 Proof of Theorem 3

Assume that the invariant is non-zero:

$$\begin{aligned} N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C}) &= \langle\,\rangle^Y_{g,\beta} \\ &= \langle a_1; a_1, a_1 - a_2, \ldots, a_{N-1} - a_N, a_N \rangle^Y_g \\ &\neq 0 \end{aligned}$$

Here Y continues to denote the space constructed in section 3.3.

By Corollary 14 the multiplicities are decreasing.

$$a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$$

Therefore, as $a_1 > 0$, there exists some $1 < j \leq N$ such that

$$a_1 \geq a_2 \geq \cdots \geq a_j > 0 \qquad\qquad a_{j+1} = \cdots = a_N = 0$$

Then by Corollary 17 we compute

$$\begin{aligned} N^g_{\mathbf{a};\mathbf{b};\mathbf{c}}(\mathcal{C}) &= \langle a_1; a_1, a_1 - a_2, \ldots, a_{j-1} - a_j, a_j, 0, \ldots, 0\rangle_g^Y \\ &= \langle a_1; a_1, a_1 - a_2, \ldots, a_{j-1} - a_j, a_j\rangle_g^{Y^{j+1}} \end{aligned}$$

Note that, for any $1 \leq i \leq j+1$, we may reorder

$$\langle a_1; a_1, a_1 - a_2, \ldots, a_{j-1} - a_j, a_j\rangle_g^{Y^{j+1}} =$$
$$\left\langle a_1; a_1, a_i - a_{i+1}, 0, 0, a_1 - a_2, \ldots, a_{i-2} - a_{i-1}, a_{i+1} - a_{i+2}, \ldots, a_{j-1} - a_j, a_j\right\rangle_g^{Y^{j+1}}$$

Applying Cremona invariance (Theorem 18) we compute

$$\begin{aligned} \langle\ \rangle_{g,\beta}^{Y^{j+1}} = \big\langle a_1 - 2(a_i - a_{i+1}); a_1 - (a_i - a_{i+1}), 0, a_{i+1} - a_i, a_{i+1} - a_i, \\ a_1 - a_2, \ldots, a_{j-1} - a_j, a_j\big\rangle_g^{Y^{j+1}} \end{aligned}$$

Then by Lemma 13, $a_{i+1} \geq a_i$. Since this holds for all $1 \leq i \leq j$ we have $a_1 \leq \cdots \leq a_j$. Therefore

$$a_1 = \cdots = a_j$$

Thus the invariant reduces to

$$\begin{aligned} \langle\ \rangle_{g,\beta}^{Y} &= \langle a; a, 0, \cdots, 0, a\rangle_g^{Y^{j+1}} \\ &= \langle a; a, a\rangle_g^{Y^2} \end{aligned}$$

□

Appendix A

Calabi-Yau configurations in blowups of $\mathbb{P}^3$

A.1 Another geometric construction

In this appendix we construct a locally Calabi-Yau configuration of rational curves $\mathcal{C}$ corresponding to any arbitrary finite subtree of the maximal trivalent tree.

We construct these configurations as subvarieties of a space X, which is obtained via a sequence of toric blowups of $\mathbb{P}^3$.

$$X = X^N \xrightarrow{\pi_N} X^{N-1} \xrightarrow{\pi_{N-1}} \cdots \xrightarrow{\pi_2} X^1 \xrightarrow{\pi_1} X^0 \xrightarrow{\pi_0} X' \xrightarrow{\pi'} X'' = \mathbb{P}^3$$

Our rational curves will be labeled by $A_\alpha, B_\alpha, C_\alpha$, where α is a binary number, reflecting the trivalent nature of the configuration. Curves and homology classes in intermediary spaces will have super-scripts, and their corresponding proper transforms in X will not.

As above, the standard torus $\mathbb{T} = (\mathbb{C}^\times)^3$ action on $\mathbb{P}^3$ has four fixed points; we now label the four $\mathbb{T}$-fixed points in $X'' := \mathbb{P}^3$ by $p' = (1:0:0:0)$, $q' = (0:1:0:0)$, $r' = (0:0:1:0)$ and $s' = (0:0:0:1)$. Let A'', B'' and C'' denote the (unique, $\mathbb{T}$-invariant) line in X'' through the two points $\{p', s'\}$, $\{q', s'\}$ and $\{r', s'\}$, respectively.

Define

$$X' \xrightarrow{\pi'} X''$$

to be the blowup of X'' at the three points $\{p', q', r'\}$, and let $A', B', C' \subset X'$ be the proper transforms of A'', B'' and C'. The exceptional divisor in X' over p' intersects A' in a unique fixed point; call it p. Similarly, the exceptional divisor in X' also intersects each of B' and C' in unique fixed points; call them q and r. Now define

$$X^0 \xrightarrow{\pi_0} X'$$

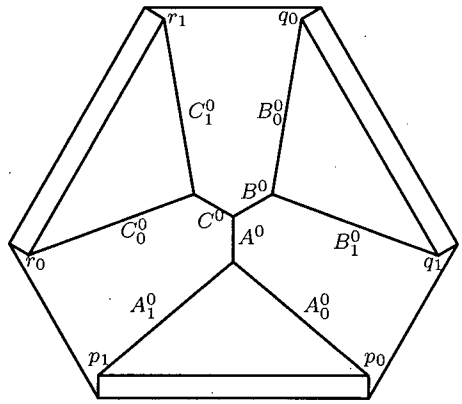

Figure A.1: The $\mathbb{T}$-invariant curves in X^0

to be the blowup of X' at the three points $\{p, q, r\}$, and let $A^0, B^0, C^0 \subset X^0$ be the proper transforms of A', B', C'. Clearly X^0 is deformation equivalent to a blowup of $\mathbb{P}^3$ at six distinct points. The $\mathbb{T}$-invariant curves in X^0 are depicted in Figure A.1, where each edge corresponds to a $\mathbb{T}$-invariant curve in X^0, and each vertex corresponds to a fixed point.

We now construct a sequence of blowups, beginning with X^0. Define p_0, $p_1 \in X^0$ to be the two fixed points of the exceptional divisor over p which are not contained in A^0; let q_0, q_1, r_0 and r_1 be defined similarly, as indicated in Figure A.1. Let $A_0^0, A_1^0 \subset X^0$ denote the $\mathbb{T}$-invariants curve intersecting the pairs $\{A^0, p_0\}$ and $\{A^0, p_1\}$, and define B_0^0, B_1^0, C_0^0 and C_1^0 analogously. Define

$$X^1 \xrightarrow{\pi_1} X^0$$

to be the blowup of X^0 at the six points $\{p_0, p_1, q_0, q_1, r_0, r_1\}$. Let $A^1, B^1, C^1 \subset X^1$ be the proper transforms of A^0, B^0, and C^0.

We iterate this process to construct X, as follows. Fix a non-negative integer N. For any binary number α, let $|\alpha|$ denote its length, i.e. if $\alpha = \alpha_1 \cdots \alpha_l$ where $\alpha_i \in \{0, 1\}$, then $|\alpha| = l$. Define

$$X^j \xrightarrow{\pi_j} X^{j-1}$$

to be the blowup of X^{j-1} along the 3×2^j points $\{p_\alpha, q_\alpha, r_\alpha\}_{|\alpha|=j}$. Let $A_\alpha^j, B_\alpha^j, C_\alpha^j \subset X^j$ denote the proper transforms of $A_\alpha^{j-1}, B_\alpha^{j-1}$ and C_α^{j-1} for all $|\alpha| < j$. Let $p_{\alpha 0}$ and $p_{\alpha 1}$ denote the two fixed points of the exceptional divisor $(\pi_j)^{-1}(p_\alpha)$ not intersecting A_α^j, and define $q_{\beta 0}, q_{\beta 1}, r_{\gamma 0}, r_{\gamma 1}$ similarly. Now, for each $|\alpha| = j - 1$, define $A_{\alpha 0}, A_{\alpha 1}$ to be the unique, $\mathbb{T}$-invariant lines intersecting $\{A_\alpha, p_{\alpha 0}\}, \{A_{\alpha, p_{\alpha 1}}\}$, and define $B_{\alpha 0}, B_{\alpha 1}, C_{\alpha_0, C_{\alpha 1}}$ similarly. The $\mathbb{T}$-invariant curves of X^1 are shown

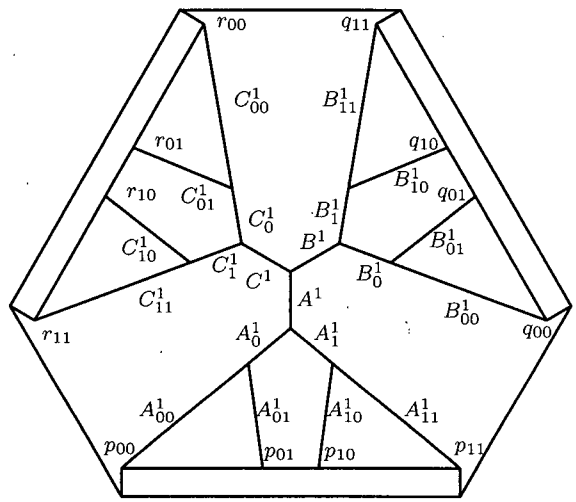

Figure A.2: The $\mathbb{T}$-invariant curves in X^1

in Figure A.2. Terminate this process after obtaining the space X^N, and define $X = X^N$.

Finally, define the configuration $\mathcal{C} \subset X$ by

$$\mathcal{C} = \bigcup_{|\alpha| \leq N} A_\alpha \cup B_\alpha \cup C_\alpha,$$

where

$$\begin{aligned} A_\alpha &= A^N_\alpha \\ B_\beta &= B^N_\beta \\ C_\gamma &= C^N_\gamma \end{aligned}$$

for all $|\alpha|, |\beta|, |\gamma| \leq N+1$. The configuration $\mathcal{C}$ is shown in Figure A.3, along with all other $\mathbb{T}$-invariant curves in X.

A.2 Intersection products and normal bundles

A.2.1 The Calabi-Yau condition

We now describe $H_*(X, \mathbb{Z})$, identify the class of the configuration $[\mathcal{C}] \in H_2(X, \mathbb{Z})$, and show that $\mathcal{C}$ is a locally Calabi-Yau configuration in X. All (co)homology is taken with integer coefficients unless otherwise noted. Let H'' denote the class of a hyperplane in $X'' = \mathbb{P}^3$, and let $H \in H_4(X)$ denote the class of its proper

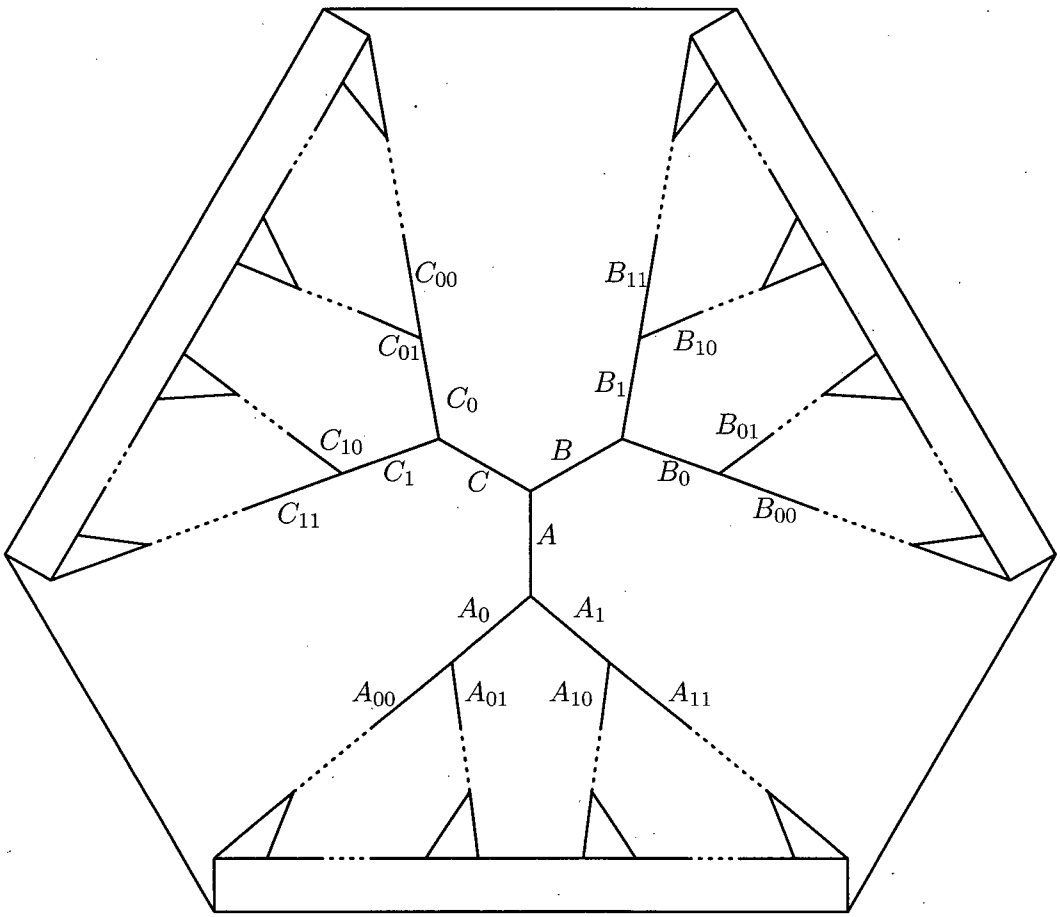

Figure A.3: The $\mathbb{T}$-invariant curves in X

transform. We may slightly abuse notation by not distinguishing between a subscheme and its homology class. Let $\tilde{E}', \tilde{F}', \tilde{G}' \in H_2(X')$ denote the exceptional divisors in $X' \to X''$ over the points p', q' and r', and let $E', F', G' \in H_4(X)$ denote their proper transforms. In addition, let $E_\alpha^{|\alpha|}, F_\alpha^{|\alpha|}, G_\alpha^{|\alpha|} \in H_4(X^{|\alpha|})$ denote the classes of the exceptional divisors $X^{|\alpha|} \to X^{|\alpha|-1}$ over the points $p_\alpha, q_\alpha, r_\alpha$, and let $E_\alpha, F_\alpha, G_\alpha \in H_4(X)$ denote their proper transforms. The collection of all such classes $\{H, E', F', G', E_\alpha, F_\beta, G_\gamma\}$ span $H_4(X)$.

Similarly, let $h'' \in H_2(X'')$ denote the class of a line in H'', and let $h \in H_2(X)$ denote its proper transform. Also, let $\tilde{e}', \tilde{f}', \tilde{g}'$ denote the class of a line in $\tilde{E}', \tilde{F}', \tilde{G}'$, and let e', f', g' denote the class of their proper transforms in X. Similarly, let $e_\alpha^{|\alpha|}, f_\alpha^{|\alpha|}, g_\alpha^{|\alpha|} \in H_4(X^{|\alpha|})$ denote the class of a line in $E_\alpha^{|\alpha|}, F_\alpha^{|\alpha|}, G_\alpha^{|\alpha|}$, and denote their proper transforms by $e_\alpha, f_\alpha, g_\alpha \in H_4(X)$. Then $H_2(X)$ is generated by $\{h, e', f', g', e_\alpha, f_\beta, g_\gamma\}$.

The intersection product ring structure is given as follows. Note that X is deformation equivalent to the blowup of $\mathbb{P}^3$ at $3 + \sum_{j=0}^{N} 3 \times 2^j = 3 \times 2^{N+1}$ distinct points. Therefore, these

$$\boxed{\begin{array}{cc} H \cdot H = h & H \cdot h = pt \\ E' \cdot E' = -e' & E' \cdot e = -pt \\ F' \cdot F' = -f' & F' \cdot f' = -pt \\ G' \cdot G' = -g' & G' \cdot g' = -pt \\ E_\alpha \cdot E_\alpha = -e_\alpha & E_\alpha \cdot e_\alpha = -pt \\ F_\alpha \cdot F_\alpha = -f_\alpha & F_\alpha \cdot f_\alpha = -pt \\ G_\alpha \cdot g_\alpha = -g_\alpha & G_\alpha \cdot g_\alpha = -pt \end{array}}$$

are all of the nonzero intersection products in $H_*(X)$.

In this basis we have, for any $\alpha = \alpha_1 \cdots \alpha_l$,

$$[A_\alpha] = \begin{cases} h - e' - e & \text{if } l = 0 \\ e_{\alpha_1 \cdots \alpha_{l-1}} - e_\alpha & \text{if } 0 < l \leq N \end{cases}$$

$$[B_\alpha] = \begin{cases} h - f' - f & \text{if } l = 0 \\ f_{\alpha_1 \cdots \alpha_{l-1}} - f_\alpha & \text{if } 0 < l \leq N \end{cases}$$

$$[C_\alpha] = \begin{cases} h - g' - g & \text{if } l = 0 \\ g_{\alpha_1 \cdots \alpha_{l-1}} - g_\alpha & \text{if } 0 < l \leq N \end{cases}$$

Thus, we compute

$$\begin{aligned}
K_X \cdot [\mathcal{C}] &= \left(4H - 2(E' + F' + G') - 2\sum_{|\alpha|\leq N} E_\alpha + F_\alpha + G_\alpha\right) \cdot \\
&\quad \Bigg((h - e' - e) + (h - f' - f) + (h - g' - g) + \\
&\quad \sum_{0<l\leq N} (e_{\alpha_1\cdots\alpha_{l-1}} - e_\alpha + f_{\alpha_1\cdots\alpha_{l-1}} - f_\alpha + g_{\alpha_1\cdots\alpha_{l-1}} - g_\alpha)\Bigg) \\
&= (4-2-2) + (4-2-2) + (4-2-2) + \\
&\quad -2\sum_{0<l\leq N} ((1-1) + (1-1) + (1-1)) \\
&= 0
\end{aligned}$$

Therefore $\mathcal{C}$ is a locally Calabi-Yau configuration in X.

A.2.2 Normal bundles

We now describe the normal bundles of the components of $\mathcal{C}$ in X. These are given as follows.

$$N_{A_\alpha/X} = N_{B_\alpha/X} = N_{C_\alpha/X} = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) & \text{if } |\alpha| = 0 \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) & \text{if } 0 < |\alpha| \leq N. \end{cases}$$

The equivalence $N_{A_\alpha/X} = N_{B_\alpha/X} = N_{C_\alpha/X}$ is easily seen by relabeling points. To calculate $N_{A/X}$, let $D_1'' \subset X''$ be a plane containing the line A'', and let D_1 denote its proper transform in X. Then $A \subset D_1$, and N_{A/D_1} is a sub bundle of $N_{A/X}$ of degree $A \cdot A$. Note that D_1 is deformation equivalent to the blowup of a plane at two points, and $[A] = h - e' - e$. Thus, the intersection product in D_1 is given by

$$A \cdot A = (h - e' - e) \cdot (h - e' - e) = -1.$$

The set of planes D_1 containing A span $N_{A/X}$, and the above argument holds for any such plane, so we conclude $N_{A/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Now consider A_α, where $\alpha = \alpha_1 \cdots \alpha_l$ and $l \geq 1$. Note that $A_\alpha \subset E_{\alpha_1\cdots\alpha_{l-1}}$. As above, $N_{A_\alpha/E_{\alpha_1\cdots\alpha_{l-1}}}$ is a sub bundle of $N_{A_\alpha/X}$ of degree $A_\alpha \cdot A_\alpha$, where the product is taken in $E_{\alpha_1\cdots\alpha_{l-1}}$. Recall that, by the functoriality of blowups, E_α is the blowup of $\mathbb{P}^2$ at a point, and that $e_{\alpha_1\cdots\alpha_{l-1}}$ is the class of a line in $E_{\alpha_1\cdots\alpha_{l-1}}$, and e_α is the exceptional divisor. We compute

$$A_\alpha \cdot A_\alpha = (e_{\alpha_1\cdots\alpha_{l-1}} - e_\alpha) \cdot (e_{\alpha_1\cdots\alpha_{l-1}} - e_\alpha) = 1 - 1 = 0.$$

We now show that the total degree of the normal bundle is -2, forcing the result to hold. Inspect the defining exact sequence

$$0 \to T_{A_\alpha} \to T_X \to N_{A_\alpha/X} \to 0.$$

This implies

$$\begin{aligned}
c_1\left(N_{A_\alpha/X}\right) &= c_1\left(T_X\right) \cdot [A_\alpha] - c_1\left(T_{A_\alpha}\right) \\
&= \left(4H - 2(E' + F' + G') - 2 \sum_{|\alpha| \leq N} E_\alpha + F_\alpha + G_\alpha\right) \cdot \\
&\quad \left(e_{\alpha_1 \cdots \alpha_{l-1}} - e_\alpha\right) - 2pt \\
&= 0 - 2(pt - pt) - 2pt \\
&= -2pt.
\end{aligned}$$

Thus the total degree of the normal bundle is -2, and so $N_{A_\alpha/X} = \mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a + b = -2$. Since we have already shown that (without loss of generality) $a = 0$, we conclude

$$N_{A_i/X} \cong \mathcal{O} \oplus \mathcal{O}(-2).$$

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