SOME NEW RESULTS ON L^2 COHOMOLOGY OF NEGATIVELY CURVED RIEMANNIAN MANIFOLDS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

Department of Mathematics

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

March 2003

Department of Mathematics
The University of British Columbia
Vancouver, Canada

Date April 4th /2003

Abstract

The present paper is concerned with the study of the L^2 cohomology spaces of negatively curved manifolds. The first half presents a finiteness and vanishing result obtained under some curvature assumptions, while the second half identifies a large class of metrics having the same L^2 cohomology as the Hyperbolic space. For the second part we rely on the Heat-Flow method initiated by M.Gaffney.

Acknowledgement: I wish to thank Professor Jing Yi Chen for his constant encouragement and for the numerous helpful discussions we had regarding the material presented in this paper.

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Introduction

The study of L^2 harmonic forms on a complete Riemannian manifold is a very interesting and important subject; it also has numerous applications in the field of Mathematical Physics (see for example [22], [16]). For topological applications of L^2 harmonic forms on noncompact manifolds see [5], [24]. It is well known that on the Hyperbolic space the harmonic L^2 forms are zero, except in the case of forms of degree equal to the half dimension. A possible generalization of this is the following conjecture of Dodziuk & Singer [9], [23]:

Conjecture: *Let M^n be a complete Riemannian manifold with sectional curvature $K \leq -\delta < 0$. Then M has the same L^2 cohomology as $\mathbb{H}^n$.*

There is another, probably more compelling reason for considering the conjecture above. Namely, it can be used to obtain topological information about compact quotients of M, as we shall now explain following Atiyah [2].

Let Γ be a discrete cocompact group of isometries acting freely on our noncompact manifold M. "Cocompact" means that the quotient M/Γ is compact. As the action of Γ is free, M/Γ is thus a compact Riemannian manifold. Since Γ commutes with the Laplace operator, the Hilbert spaces $\mathcal{H}^k$ are Γ-modules. Atiyah defines real valued L^2-Betti numbers

$$B_\Gamma^k = \dim_\Gamma \mathcal{H}^k,$$

satisfying Poincaré duality, i.e $B_\Gamma^k = B_\Gamma^{\dim M - k}$, and the corresponding L^2 -Euler characteristic

$$\chi(M, \Gamma) = \sum_k (-1)^k B_\Gamma^k.$$

Atiyah shows that $\chi(M,\Gamma)$ equals the ordinary Euler characteristic of M/Γ which is an integer, and this is the basis of the relation between L^2-cohomology of M and the topology of M/Γ alluded to above.

More precisely, Hopf asked whether the sign of the sectional curvature determines the Euler characteristic of a compact Riemannian manifold. For example, if N^{2n} is a compact manifold of dimension $2n$ with negative sectional curvature, one should have

$$(-1)^n \chi(N^{2n}) > 0.$$

Since the sign of the sectional curvature does not determine the sign of the Gauss-Bonnet integrand, this cannot be deduced from algebraic considerations alone.

Hopf conjecture is implied by the above mentioned conjecture as Dodziuck [8] and Singer [22] show by proving

$$B_{\Gamma}^{k} = 0 \text{ if and only if } \mathcal{H}^{k} = 0.$$

However, Anderson [1] constructed simply connected, complete, negatively curved manifolds for which the Dodziuck-Singer conjecture does not hold. This difficulty might be of a purely technical nature as Anderson's examples do not admit compact quotients.

Having in mind the examples produced by Anderson we should obviously modify the conjecture of Dodziuk & Singer to the following:

Conjecture: *Let M^n be the universal cover of a oriented Riemannian manifold with negative sectional curvature. Then M has the same L^2 cohomology as $\mathbb{H}^n$.*

It is now easy to see, although not very interesting from the point of view of Hopf's original conjecture, that in the two dimensional case the conjecture holds. Let us sketch the proof of this conjecture for covers of compact surfaces:

Let $\psi : (\tilde{M}, \tilde{g}) \to (M, g)$ represent the cover map of the negatively curved two dimensional manifold M^2. Since the sectional curvature is negative, it follows the genus of the surface M is strictly larger than one. Consequently M admits a metric h of constant sectional curvature equal to -1. Consider now the pullback of this metric on the universal cover of M. Let us denote this metric by $\tilde{h} = \psi^* h$. According to Cartan-Hadamard theorem [7] the Riemannian manifold $(\tilde{M}, \tilde{h})$ is isometric to the hyperbolic plane, hence they have the same L^2 cohomology. Moreover, since M is compact it is obvious that g and h are quasi-isometric. This also implies that "upstairs" on the universal cover, $\tilde{h}$ and $\tilde{g}$ are also quasi-isometric. Hence the conjecture in dimension two holds.

As we see from the considerations above there are serious reasons to belive the conjecture will probably hold only in the case of universal covers of compact manifolds. Also we think it is very important that such metrics should be compared pointwise with the standard hyperbolic metric. A small step in this direction is done in the last chapter of this paper.

In a positive direction there is the following result of Donnelly and Xavier [11],
Theorem (Donnelly & Xavier): *Let M be a complete, simply connected manifold, with*

pinched sectional curvatures $-1 \leq K \leq -1+\epsilon$ *for* $0 \leq \epsilon < 1$. *If* $p < \frac{n-1}{2}$ *then*

$$\mathcal{H}^p = 0, \ \ for \ \epsilon < 1 - \frac{4p^2}{(n-1)^2}.$$

A more recent refinement of their result is due to Jost and Xin [18],

Theorem (Jost & Xin): *Let M be a Cartan-Hadamard manifold of dimension $m > 2$ whose sectional curvature satisfies $-a^2 \leq K \leq 0$ and whose Ricci curvature is bounded from above by $-b^2$, where a, b are positive constants. If $b \geq 2pa$, then $\mathcal{H}^p = 0$, provided $p \neq m/2$.*

A more complete statement was made by Gromov [14], which settles the conjecture in the case of d-bounded Kähler manifolds,

Theorem (Gromov): *If M^{2n} is a complete d-bounded Kähler manifold (in the sense that $d\eta = \omega$; ω being the Kähler form of M and η is bounded in the supremum norm) then $\mathcal{H}^k = 0$ iff $k \neq n$ and $\mathcal{H}^n \neq 0$.*

Also a very interesting result in this direction is due to J.Dodziuk [10],
Theorem (Dodziuk): *Suppose M is diffeomorphfic with $\mathbb{R}^n$ and has a metric which in terms of geodesic polar coordinates centered at some point of M can be written as*

$$ds^2 = dr^2 + f(r)^2 d\theta^2.$$

Then $\mathcal{H}^k = 0$ if $k \neq n/2$ and $\int_0^\infty f^{n-1} dr = \infty$. If n is even and $\int_1^\infty 1/f dr < \infty$ then $\mathcal{H}^{n/2}$ is infinite dimensional.

One of the first results on the vanishing and finite dimensionality of the space of harmonic L^2 forms were obtained by E.Vesentini in [25]. The main result of the first part of this paper follows his ideas closely. This result can be stated as follows:

Theorem 0.0.1. *Let M be a complete manifold of infinite volume. Then we have the following:*

a) If $\lambda_1 > I_M(R^k)$ then $\mathcal{H}^k = 0$,

b) If $\lambda_1 > I_{M \setminus B_r}(R^k)$ then $\dim \mathcal{H}^k < \infty$.

Here $I_M(R^k)$ is a positive quantity depending on the curvature operator acting on k-forms (see Definition 2.0.17) and λ_1 is the Poincaré constant of the manifold. As a more practical application of Theorem 0.0.1 we also prove vanishing and finiteness of the L^2 spaces if the sectional curvature is appropriately pinched outside some compact set (see Corollary 2.0.20).

Following an observation of Gromov in [14] one can see that the space of L^2 harmonic forms is invariant under bi-Lipschitz homeomorphisms. Using the heat-flow method, initiated by Gaffney in [13], the author proves that if M is a noncompact manifold and g and $\tilde{g}$ are two complete metrics on M such that $L^2\Omega_{\tilde{g}} \subset L^2\Omega_g$ then there exist a map $H_g : \mathcal{H}_{\tilde{g}} \to \mathcal{H}_g$ which is linear and injective. As an application of this statement the author constructs a large class of metrics (which includes the quasi-isometry class of the hyperbolic metric) having the same L^2 cohomology as the Hyperbolic space. The nature of this comes from comparing the metric pointwise with the Hyperbolic one.

Chapter 1
Preliminaries

Let M be a smooth, complete and oriented Riemannian manifold. Let $C^\infty\Omega^k$ denote the space of smooth k-forms. The metric on M induces a natural pointwise scalar product on forms and let us denote this by $\langle\alpha(x), \beta(x)\rangle$ where $x \in M$ and $\alpha, \beta \in C^\infty\Omega^k$. Thus we obtain the square of length at a point $x \in M$ of a form $\alpha \in \Omega^k$ as $\langle\alpha(x), \alpha(x)\rangle \geq 0$. This leads to the definition of a norm of a form (if finite) by

$$||\alpha||^2 = \int_M \langle\alpha(x), \alpha(x)\rangle \, dV(x)$$

where dV is the volume form of the manifold and x represents the variable of integration.

By completing this space with respect to the above mentioned norm we obtain a Hilbert space, henceforth denoted by $L^2\Omega^k$. The inner product in this space will be denoted by

$$(\alpha, \beta) = \int_M \langle\alpha(x), \beta(x)\rangle \, dV(x), \tag{1.1}$$

where $\alpha, \beta \in L^2\Omega^k$. With the help of the metric one could naturally define the Hodge Laplacian of the manifold $\Delta : C^\infty\Omega^k \to C^\infty\Omega^k$, $\Delta = d\delta + \delta d$. In the above formula d is the exterior derivative and δ the formal adjoint. More details of the definition of Δ can be found in [17], [21].

One of the goals of this paper is to study the the space of L^2-harmonic forms (i.e $\Delta\alpha = 0$ and $\alpha \in L^2\Omega^k$). The heat equation will provide an injection from the space of compactly supported de Rham cohomology classes into the space of L^2 harmonic forms. This will be shown in more detail in what follows.

The definition of L^2 cohomology groups is a slight modification of de Rham groups. Next we give the precise definition of these groups as well as the definition of the space of L^2 harmonic forms.

Definition 1.0.2.

$$\mathrm{H}^k(L^2M) = Z^k(L^2M)/\overline{B^k(L^2M)}$$

where

$$Z^k(L^2M) = \{\alpha \,|\, \alpha \in C^\infty\Omega^k \cap L^2\Omega^k,\, d\alpha = 0\},$$

and

$$B^k(L^2M) = \{\beta \,|\, \beta \in C^\infty\Omega^k \cap L^2\Omega^k,\;\; \exists \mu \in C^\infty\Omega^{k-1} \cap L^2\Omega^{k-1},\, \beta = d\mu\}.$$

$\overline{B^k(L^2M)}$ *is the closure with respect to the* L^2 *norm.*

Definition 1.0.3.

$$\mathcal{H}^k = \{\alpha \,|\, \alpha \in C^\infty\Omega^k \cap L^2\Omega^k,\, \Delta\alpha = 0\}$$

If M is compact, B^k is closed in Z^k, i.e. $\overline{B^k} = B^k$, and the L^2 cohomology is the same as the ordinary de Rham cohomology. In the compact case, in turn we have Hodge theory representinig each de Rham cohomology class by a harmonic form. In this sense, L^2-cohomology is the appropriate extension to the noncompact case inasmuch as here every L^2-cohomology class can be represented by an L^2-harmonic form. In the noncompact case, B^k need not be closed in Z^k, essentially because the spectrum of the Laplacian need not have a positive lower bound, or equivalently, the Poincaré inequality need not hold. An example is the euclidean space $\mathbb{R}^n$. For hyperbolic spaces, however, we do have such inequalities, and consequently B^k is closed. In any case, however, in order to have a uniform theory, one considers $\overline{B^k}$ in place of B^k.

Examples:

i) $\mathcal{H}^*(\mathbb{R}^n) = 0$.

ii) $\mathcal{H}^*(\mathbb{H}^n) = 0$ if n is odd, and $\mathcal{H}^k(\mathbb{H}^{2n}) = 0$ for every $k \neq n$. The dimension of $\mathcal{H}^n(\mathbb{H}^{2n})$ is infinite.

The following decomposition theorem due to de Rham is well known ,

Theorem (de Rham): *Let M be a complete Riemannian manifold then we have the following decomposition:*

$$L^2\Omega^k = \overline{dL^2\Omega^{k-1}} \oplus \mathcal{H}^k \oplus \overline{\delta L^2\Omega^{k+1}}.$$

An easy corollary of this is:

Corollary 1.0.4. *If M is a complete Riemannian manifold then $\mathcal{H}^k \simeq H^k(L^2M)$.*

It is well known (cf. [14]) that for the hyperbolic space $\mathbb{H}^{2n}$, $\mathcal{H}^n$ is nonzero. Since $\mathbb{H}^{2n}$ is diffeomorphic to $\mathbb{R}^{2n}$,on which $\mathcal{H}^n = 0$, the $\mathcal{H}^k$'s are not topological invariants. As we have already seen, the $\mathcal{H}^k$'s usually depend on the metric.

What is also known is that in the compact case a form is harmonic ($\Delta\alpha = 0$) if and only if $d\alpha = 0$ and $\delta\alpha = 0$, and this is a consequence of Stoke's Theorem and the very definition of δ operator. The same result remains valid for an L^2-harmonic form (possibly an L^p form) on a complete manifold. The next proposition is due to Andreotti and Vesentini:

Proposition 1.0.5. *Let α be an L^2 harmonic form on a complete Riemannian manifold. Then $d\alpha = 0$ and $\delta\alpha = 0$.*

PROOF. We want to justify the integral identity

$$(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (\delta\alpha, \delta\alpha).$$

We consider a family of cutoff functions a_ϵ satisfying the following conditions:

i) a_ϵ is smooth and takes values in the interval $[0, 1]$; furthermore, a_ϵ has compact support.

ii) The subsets $a_\epsilon^{-1}(1) \subset M$ exhaust M as $\epsilon \to 0$.

iii) $|da_\epsilon|^2 \leq \epsilon a_\epsilon$ everywhere on M

The construction of such a function is always possible on a complete Riemanian manifold and it is a standard technicality. A simple computation will give us

$$0 = \langle \Delta\alpha, a_\epsilon \alpha \rangle = I_1(\epsilon) + I_2(\epsilon),$$

where

$$I_1(\epsilon) = \int_M a_\epsilon(|d\alpha|^2 + |\delta\alpha|^2)$$

and

$$|I_2(\epsilon)| \leq \int_M |da_\epsilon||\alpha|(|d\alpha| + |\delta\alpha|).$$

By Schwarz inequality this yields

$$|I_2(\epsilon)| \leq 2\epsilon||\alpha|| \left(\int_M |a_\epsilon|(|d\alpha|^2 + |\delta\alpha|^2) \right)^{1/2}$$

and hence $I_2(\epsilon) \to 0$ for $\epsilon \to 0$. $\square$

The following proposition will be used in the following sections and it essentially asserts that d and $*$ are adjoint operators on L^2 forms on a complete manifold. The proof is a typical application of a cut-off function argument. We give the proof in detail.

Proposition 1.0.6. *Let M be a complete Riemannian manifold and $\alpha, \beta, d\alpha, \delta\beta$ be square integrable forms. Then*

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

PROOF. Let α and β as above and let $0 \leq \psi_n \leq 1$ be a sequence of cut-off functions with the following two properties: $|d\psi_n| \leq \frac{C}{n}$ for some positive constant $C > 0$ and $\psi_n(x) \longrightarrow 1$ for every $x \in M$.

We have the pointwise identity

$$\langle d(\psi_n \alpha), \beta \rangle = \langle d\psi_n \wedge \alpha + \psi_n d\alpha, \beta \rangle, \tag{1.2}$$

and integrating the left hand side we get

$$\int_M \langle d(\psi_n \alpha), \beta \rangle dV = \int_M \langle \psi_n \alpha, \delta\beta \rangle dV.$$

Since $\psi_n \longrightarrow 1$ pointwise we have, according to the Lebesque Dominated Convergence Theorem

$$\lim_{n \to \infty} \int_M \langle \psi_n \alpha, \delta\beta \rangle dV = (\alpha, \delta\beta). \tag{1.3}$$

Now integrating the right hand side of (1.2) and using the pointwise estimate

$$\langle d\psi_n \wedge \alpha, \beta \rangle \leq |d\psi_n||\alpha||\beta| \leq \frac{C}{n}|\alpha||\beta|$$

toghether with Lebesque' theorem, we get

$$\lim_{n \to \infty} \int_M \langle d\psi_n \wedge \alpha + \psi_n d\alpha, \beta \rangle dV = \int_M \langle d\alpha, \beta \rangle dV, \tag{1.4}$$

Using (1.2), (1.3) and (1.4) we get the desired result. □

As we have already seen, the L^2 cohomology spaces are not topological invariants, yet one recaptures the invariance if one restricts to bi-Lipschitz homeomorphism. More generally, let $f : M \to N$ be a Lipschitz map between Riemannian manifolds, i.e.,

$$dist_N(f(x_1), f(x_2)) \leq C dist_M(x_1, x_2)$$

for all pairs of points x_1 and x_2 in M. (If f is a smooth map, this is equivalent to$||df||_{L^\infty} \leq$ *const.*) Then the induced map on forms, called f^o sends L^2-forms on N to M. The composition of f^o on $\mathcal{H}^*(N)$ with the orthogonal projection $h : L^2\Omega^*(M) \to \mathcal{H}^*(M)$ defines a linear map, between the harmonic spaces

$$f^* : \mathcal{H}^*(N) \to \mathcal{H}^*(M).$$

It is well known that f^* is Lipschitz homotopy invariant. That is, if f_1 andf_2 can be joined by a homotopy $F : M \times [0,1] \to N$, which is a Lipschitz map for the product metric in $M \times [0,1]$, then $f_1^* = f_2^*$.

Remark. If M an N are compact one gets this way the usual homotopy invariance of $\mathcal{H}^*$ as all maps can be approximated by Lipschitz maps. A more interesting case is that where M and N are infinite coverings of compact manifolds, say M_0 and N_0 respectively, and pertinent maps $f : M \to N$ are lifts of continuous maps $f_0 : M_0 \to N_0$. Here again we may assume f_0 and f are Lipschitz, and then we can see that if f_0 is a homotopy equivalence, then the induced map

$$f^* : \mathcal{H}^*(N) \to \mathcal{H}^*(M)$$

is an isomorphism.

Before stating the following useful theorems we need to make a few definitions. In what follows H is a separable Hilbert space and A is a linear operator defined on a dense subset $\mathcal{D} \subset H$.

Definition 1.0.7. *Let A and H be as above. We say the operator A is closed, if and only if its grapf is a closed subset of $H \times H$.*

Let A,D and H as before. We call the operator $\tilde{A}$ having domain $\tilde{D}$ an extension of A if and only if the following hold

i) $D \subset \tilde{D}$

ii) $Ax = \tilde{A}x$ for any $x \in D$

Definition 1.0.8. *We say the operator A is closable if and only if there exist $\tilde{A}$ a closed extension of A.*

The smallest(with respect to the inclusion of the domain) closed extension of an operator A is called the closure of A and is usually denoted by $\overline{A}$.

Let us denote now by $A(D)$ the range of the operator A. Here again we assume that $\overline{D} = H$, i.e. D is dense in H. We shall define the operator A^* by the identity

$$(Au, v) = (u, w), \text{ for every } u \in D, \ A^*v = w \tag{1.5}$$

More precisely, the domain of the operator A^*, here denoted by D^*, is the set of all v for which there exist $w \in H$ such that (1.5) holds. If the vector w exists, for a given v, then it is unique by the classical Riesz Theorem.

Definition 1.0.9. *We call A a self adjoint operator if and only if $A = A^*$.*

Essential in the proof of the existence of the solution of the Heat Equation for forms is the Spectral Theory of Self-Adjoint operators, namely the Spectral Theorem of Von-Neumann and the Friedrichs extension result. An excellent reference for this is [19].

In what follows we state the existence and regularity of the solution to the (abstract) heat equation. This result is mainly due to F. Browder [3]. We consider A as being a second order elliptic operator acting on smooth sections of a fiber bundle endowed with a smooth scalar product.

Suppose the operator fulfills the following conditions:

i) $(A\alpha, \beta) = (\alpha, A\beta)$ for any compactly supported α, β,

ii) $(A\alpha, \alpha) \geq 0$ for any compactly supported α.

We are interested in finding a regular solution to the Cauchy problem:

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = -A\alpha, \\ \alpha(0) = \alpha_0 \in L^2. \end{cases} \tag{1.6}$$

More precisely we are interested in finding a path $\alpha\colon[0,\infty) \to L^2\Gamma$ such that the two conditions within the accolade are fulfilled. Here Γ denotes the space of sections of the fiber bundle. The most important result is contained in the following theorem of Browder, [3]:

Theorem(Browder): *There is always a C^∞ solution to the Cauchy problem if A satisfies conditions (i) and (ii) above. Also* $\lim_{t\to\infty} \alpha(t) \in \ker A$.

Sketch of Proof: Since under this assumptions, A is a semibounded operator, there exist a semibounded self-adjoint extension $\tilde{A}$. Let μ be the spectral measure associated to $\overline{A}$. Then the solution to the Cauchy problem(1.6)is given by

$$\alpha(x,t) = \int_0^\infty e^{-\lambda t} d\mu_\lambda \alpha_0(x).$$

Formally we may write the solution above as

$$\alpha = e^{-t\tilde{A}} \alpha_0$$

and by the spectral theorem we have

$$\tilde{A}^m \alpha = \int_0^\infty \lambda^m e^{-\lambda t} d\mu_\lambda \alpha_0(x)$$

hence

$$\tilde{A}^m \alpha \in L^2.$$

The above inclusion implies

$$\alpha(-,t) \in W^{m,2}$$

for an arbitrarly m. Hence by the Sobolev Imbedding Theorem we get spatial regularity. The temporal regularity is obvious. □

The next theorem is due to M. Gaffney :

Theorem(Gaffney): *Let M^n be a complete Riemannian manifold. Then the closure of the Laplacian on forms is self-adjoint.*

For the proof of this fact see [13]. As a result of these two theorems we have the following:

Proposition 1.0.10. *There is always a unique solution to the Cauchy problem of the heat equation for forms and the solution has the following properties:*

a) $\lim_{t\to\infty} \alpha(t) \in \mathcal{H}^k$,

b) $\alpha, \delta\alpha, d\alpha, \Delta\alpha$ *are all in* L^2 *at any time* $t > 0$,

c) the solution is C^∞ *for all* $t > 0$,

d) if the initial data α_0 *is closed then* α *is closed for all* $t > 0$,

e) the cohomology class is preserved by the flow.

Sketch of Proof: We shall only prove parts (d) and (e), since all the others are consequences of the previous two theorems. Let $\alpha_0 \in L^2$ be a closed initial data for the Cauchy problem (1.6). First we need to prove that $\alpha(t)$ is closed for all $t > 0$. To see this let us consider the following scalar quantity

$$I(t) = \int_M (d\alpha, d\alpha) dV.$$

Differentiating with respect to t we obtain

$$\frac{dI}{dt} = 2\int_M (d\dot{\alpha}, d\alpha) dV.$$

Since α is the solution to the problem (1.6), we get

$$\frac{dI}{dt} = -2\int_M (d\Delta\alpha, d\alpha) dV,$$

and furthermore using the fact that d and δ are self-adjoint

$$\frac{dI}{dt} = -2\int_M |\delta d\alpha|^2 dV.$$

It follows that $I(t)$ is decreasing with respect to t. But obviously $I(t) \geq 0$ and $I(0) = 0$, hence $I(t)$=0 for all time t. This obviously means $d\alpha(t) = 0$, or equivalently $\alpha(t)$ is closed at any instant.

To see that $[\alpha(t)] = [\alpha_0]$, where $[\alpha]$ denotes the DeRham cohomology class of α, we shall consider an arbitrarly closed orientable submanifold of dimension equal to the degrée of the form. Let us denote this submanifold by X. As before let us consider the following scalar quantity

$$Q(t) = \int_X \alpha(t).$$

Differentiating with respect to t, we get

$$\frac{dQ}{dt} = \int_X \frac{d\alpha}{dt} = -\int_X \Delta\alpha,$$

or equivalently

$$\frac{dQ}{dt} = -\int_X (d\delta + \delta d)\alpha.$$

Now using the fact that α is closed at all times we get

$$\frac{dQ}{dt} = -\int_X d\delta\alpha = 0.$$

Here above we have made use of Stokes' Theorem and the fact that X has empty boundary. In conclusion

$$\int_X \alpha(t) = \int_X \alpha_0.$$

□

Chapter 2
The Finite Dimensionality of $\mathcal{H}^k$'s and Some Vanishing Results

This section is concerned with finding sufficient geometric conditions on the manifold M which will guarantee the finite dimensionality of the L^2 cohomology spaces. The techniques are based on the classical Weitzenböck formula and a few standard PDE techniques.

Proposition(Weitzenböck formula): *Let M be a Riemannian manifold (not necessarily complete). Let e_i be a local orthonormal frame and η^i the associated coframe. Then we have the following pointwise identity*

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathcal{R}(\alpha),$$

where ∇ represents the covariant derivative acting on forms, ∇^* represents its formal adjoint and $\mathcal{R}(\alpha) = \eta^i \wedge (i_{e_j} R(e_i, e_j)\alpha)$.

For a proof of this formula see [17].

Definition 2.0.11. *The $\mathcal{R}(\alpha)$ operator defined by the identity in the previous proposition is called the Weitzenböck curvature term. We say $\mathcal{R}$ is positive (negative) if and only if $g(\mathcal{R}(\alpha), \alpha) > 0$ (< 0) for all $\alpha \neq 0$ where g is the Riemannian metric of M.*

We also need the following simple lemma.

Lemma 2.0.12. *Let M be as above and $(\alpha_n)_{n\geq 1}$ be a bounded sequence in $W^{1,2}\Omega$ and ψ be a cut-off function. Then the following sequence $\psi_n = \psi\alpha_n$ is bounded in $W^{1,2}\Omega$ and is compactly supported.*

PROOF. We have $|\psi\alpha_n| = |\psi||\alpha_n|$ which in turn implies $\int_M |\psi_n|^2 dV \leq A \int_M |\alpha_n| dV$ where the constant $A = \sup|\psi|^2$. This takes care of the L^2 part of the norm. For the derivative part we have the following pointwise identity:

$$d(\psi\alpha_n) = d\psi \wedge \alpha_n + \psi d\alpha_n.$$

Therefore

$$|d(\psi\alpha_n)| \leq |d\psi||\alpha_n| + |\psi||d\alpha_n|.$$

Integrating and applying an elementary inequality we get

$$\int_M |d(\psi\alpha_n)|^2 dV \leq C_1||\alpha_n||^2_{L^2} + C_2||d\alpha_n||^2_{L^2}.$$

Where the positive constants C_1, C_2, depend only on ψ. Next we have

$$|\delta(\psi\alpha_n)| = |*d*(\psi\alpha_n)| = |d*(\psi\alpha_n)| = |d(\psi * \alpha_n)|.$$

Hence

$$|\delta(\psi\alpha_n)| = |d\psi \wedge *\alpha_n + \psi d * \alpha_n| \leq |d\psi||\alpha_n| + |\psi||\delta\alpha_n|,$$

now integrating and applying the same inequality again we get:

$$\int_M |\delta(\psi\alpha_n)|^2 dV \leq C_1||\alpha_n||^2_{L^2} + C_2||\delta\alpha_n||^2_{L^2}.$$

This concludes the proof of the lemma. □

The following proposition is essential in proving the main result of this section.

Proposition 2.0.13. *On a complete manifold M, $\dim \mathcal{H}^k < \infty$ if and only if there exist $R > 0$ and $C > 0$ such that*

$$\int_{B_R} |\alpha|^2 dV \geq C \int_M |\alpha|^2 dV$$

for every $\alpha \in \mathcal{H}^k$ (here the positive scalar $C > 0$ may depend on R) .

PROOF. For the "only if" part we observe that both the quantities involved in the above mentioned inequality are norms on a finite dimensional vector space, hence, equivalent.

For the "if" part, let $R > 0$ and $C > 0$ as in the hypothesis. Let ψ be a cut-off function such that $\psi \equiv 1$ on B_R and $\psi \equiv 0$ on B_{2R}. Assume $\dim \mathcal{H}^k = \infty$ and let α_n be a countable L^2-orthonormal sequence of harmonic forms in $\mathcal{H}^k$. Then according to Lemma 2.0.12 the sequence $\psi_n = \psi\alpha_n$ satisfies the conditions of Rellich theorem, hence we can extract a subsequence convergent in L^2. We will use the same notation for the subsequence ψ_n. Let us now estimate the distance between members of this sequence, namely $d(\psi_n, \psi_m)$ (here $d(\, , \,)$denotes the L^2 distance). We have

$$\begin{aligned} d(\psi_n, \psi_m) &= \int_M \langle \psi_n - \psi_m, \psi_n - \psi_m \rangle dV \\ &= \int_M (|\psi_n|^2 + |\psi_m|^2) dV - 2 \int_M \langle \psi_n, \psi_m \rangle dV. \end{aligned}$$

Hence

$$d(\psi_n, \psi_m) \geq \int_{B_R} (|\alpha_n|^2 + |\alpha_m|^2) dV - 2 \int_M \langle \psi_n, \psi_m \rangle dV.$$

Now applying the inequality from hypothesis we can estimate the first term of right-hand side as follows:

$$\int_{B_R} (|\alpha_n|^2 + |\alpha_m|^2) dV \geq 2C.$$

So finally we get

$$d(\psi_n, \psi_m) \geq 2C - 2 \int_M \langle \psi_n, \psi_m \rangle dV.$$

But the sequence ψ_n is obtained by multiplying an orthonormal sequence in L^2 by a cut-off function, so it is weakly convergent to zero in L^2 and using a diagonal argument we can see that the second term on the right-hand side of the inequality above can be made arbitrarily small (as $n, m \to \infty$). By the Rellich theorem so is the left-hand side. It then follows that $2C \leq 0$. Contradiction. □

As this proposition shows, in proving the finite dimensionality of $\mathcal{H}^k$ one could try to get an estimate as above. In fact Vesentini in [25] obtained the first result of this kind. More precisely he proved that if the curvature operator is positive outside some compact subset of the manifold then the desired inequality holds. In a similar fashion he also proved that if the curvature operator is nonnegative then $\mathcal{H}^k = 0$.
A related result was recently obtained by G.Carron in [4]. To state his result we need to make the following definition,

Definition 2.0.14. *The Gauss-Bonnet operator $d+\delta$ of a complete Riemannian manifold (M, g) is called non-parabolic at infinity when there is a compact set K of M such that for any bounded open subset $D \subset M \setminus K$ there is a constant $C(D) > 0$ satisfying the inequality*

$$C(D) \int_D |\alpha|^2 \leq \int_{M \setminus K} |d\alpha|^2 + |\delta\alpha|^2 \textit{ for every } \alpha \in C_0^\infty \Omega^*(M \setminus K).$$

Observation The condition that the Gauss-Bonnet operator is non-parabolic at infinity is similar in nature with the assumption of Proposition 2.0.13.

Carron's result can now be stated as follows

Proposition 2.0.15. *If the Gauss-Bonnet operator of (M, g) is non-parabolic at infinity* *then*

$$\dim \mathcal{H}^* < \infty.$$

Naturally one should try to express the non-parabolicity of the Gauss-Bonnet operator in terms of the curvature operator if possible. Carron proves the following,

Proposition 2.0.16. *If (M, g) is a complete Riemanian manifold whose curvature vanishes outside some compact set, then the Gauss-Bonnet operator is non-parabolic at infinity.*

thus giving another proof of a very well known fact.

In what follows we will give other geometric conditions that imply the required estimate and also will obtain another useful vanishing result. Before going to the main result we need to make a definition:

Definition 2.0.17. *Let M be a complete manifold and let R^k be the Weitzenböck curvature operator acting pointwise on k-forms. Let $D \subseteq M$ be a subset. Then*

$$I_D(R^k) = \inf\{c \mid c \geq 0 \text{ and } \langle R^k\alpha(p), \alpha(p)\rangle \geq -c|\alpha(p)|^2, \ \ \alpha \in \Omega^k, \ \ p \in D\},$$

Remark: Obviously if $D_1 \subset D_2$ then $I_{D_1}(R^k) \leq I_{D_2}(R^k)$

For the proof of the main theorem we will first need to prove one technical lemma.

Lemma 2.0.18. *Let M be a complete manifold whose curvature operator on Ω^k is bounded from below. Then for any $\alpha \in \mathcal{H}$ we have $\int_M |\nabla\alpha|^2 < \infty$. Hence in this case the Weitzenböck formula gives*

$$\int_M |\nabla\alpha|^2 dV + \int_M \langle R\alpha, \alpha\rangle dV = 0.$$

PROOF. Let $\psi \in C_0^\infty\Omega$ be a compactly supported form. According to the Weitzenböck formula we have

$$\Delta\psi = \nabla^*\nabla\psi + R\psi.$$

Taking inner product of both sides with ψ and integrating by parts (we can do this because ψ is compactly supported) we get

$$(\Delta\psi, \psi) = ||\nabla\psi||^2 + (R\psi, \psi).$$

We can rewrite the left-hand side of the above identity in terms of d and δ as follows

$$||d\psi||^2 + ||\delta\psi||^2 = ||\nabla\psi||^2 + (R\psi, \psi).$$

In all of the above formulas $||\psi||$ denotes the L^2 norm of ψ. But the curvature operator is bounded from below i.e. $\langle R\psi, \psi\rangle \geq -c|\psi|^2$ and this is pointwise, or equivalently $-\langle R\psi, \psi\rangle \leq c|\psi|^2$. Hence we get

$$||\nabla\psi||^2 = \int_M |\nabla\psi|^2 dV \leq c\int_M |\psi|^2 dV + ||d\psi||^2 + ||\delta\psi||^2 \tag{2.1}$$

for any ψ compactly supported in M.

Now let $\alpha \in \mathcal{H}$ and ϕ be a cut-off function such that $\phi \equiv 1$ on B_R and zero outside B_{2R} for arbitrary $R > 1$ and also $|\nabla\phi| \leq 1$. Applying (2.1) (taking into account that $0 \leq \phi \leq 1$) to the compactly supported form $\phi\alpha$ we get

$$\int_M |\nabla(\phi\alpha)|^2 dV \leq c\int_M |\alpha|^2 dV + ||d(\phi\alpha)||^2 + ||\delta(\phi\alpha)||^2. \tag{2.2}$$

In order to estimate the last two terms of (2.2) we proceed as follows:

$$|d(\phi\alpha)| = |d\phi \wedge \alpha| \leq |d\phi||\alpha| \leq |\alpha|$$

hence,

$$||d(\phi\alpha)||^2 \leq ||\alpha||^2$$

and also

$$|\delta(\phi\alpha)| = | * d * (\phi\alpha)| = |d(\phi(*\alpha))| = |d\phi \wedge (*\alpha)| \leq | * \alpha| = |\alpha|$$

Integrating the above inequality yields

$$||\delta(\phi\alpha)|| \leq ||\alpha||.$$

To get these two estimates we essentially used the fact that $\alpha \in \mathcal{H}$ and $*$ is a pointwise isometry. Also taking into account that

$$\int_{B_R} |\nabla\alpha|^2 dV = \int_{B_R} |\nabla(\phi\alpha)|^2 dV \leq \int_M |\nabla(\phi\alpha)|^2 dV$$

we get

$$\int_{B_R} |\nabla\alpha|^2 dV \leq c||\alpha||^2 + 2||\alpha||^2. \tag{2.3}$$

Since $R > 1$ was arbitrarily chosen, letting $R \to \infty$ we get the desired result. □

PROOF. **(Theorem 0.0.1)**

As a result of Lemma 2.0.18 we have

$$\int_M |\nabla\alpha|^2 dV + \int_M \langle R\alpha, \alpha \rangle dV = 0. \tag{2.4}$$

By the definition of $I_M(R^k)$ we have

$$\langle R\alpha, \alpha \rangle \geq -I_M(R^k)|\alpha|^2$$

or equivalently

$$-\langle R\alpha, \alpha \rangle \leq I_M(R^k)|\alpha|^2.$$

This together with (2.4) gives

$$\int_M |\nabla\alpha|^2 dV \leq I_M(R^k) \int_M |\alpha|^2 dV. \tag{2.5}$$

Assume there is a nonzero harmonic L^2 form α. Since the volume of M is infinite, it follows that $|\alpha|$ is nonconstant so we may apply the Poincaré inequality to $|\alpha|$. Hence

$$\int_M |\nabla\alpha|^2 dV \geq \lambda_1 \int_M |\alpha|^2 dV,$$

here we made use of the pointwise inequality: $|\nabla\alpha|^2 \geq |\nabla|\alpha||^2$. This together with (2.5) implies

$$\lambda_1 \int_M |\alpha|^2 dV \leq I_M(R^k) \int_M |\alpha|^2 dV.$$

Since α is nonzero we have

$$\lambda_1 \leq I_M(R^k)$$

which contradicts the assumption of part (a) of the theorem.

For part (b) of the theorem let us observe first that there exists constant $C > 0$ such that

$$\langle R\alpha(p), \alpha(p)\rangle \geq -C|\alpha(p)|^2 \tag{2.6}$$

for any $\alpha \in \Omega^k$ and $p \in B_R$. To see this, one should consider the continuous function $f(p,v) = \langle R_p v, v\rangle$ defined on the unit sphere bundle of $\Omega\overline{B_R}$ (since this set is compact, f attains its infimum).

Using Lemma 2.0.18 we have

$$\int_M |\nabla\alpha|^2 dV + \int_{M\backslash B_R} \langle R\alpha, \alpha\rangle dV + \int_{B_R} \langle R\alpha, \alpha\rangle dV = 0.$$

This together with (2.6) gives

$$\int_M |\nabla\alpha|^2 dV + \int_{M\backslash B_R} \langle R\alpha, \alpha\rangle dV \leq C \int_{B_R} |\alpha|^2 dV.$$

Using the definition of $I_{M\backslash B_R}(R^k)$ we have

$$\int_M |\nabla\alpha|^2 dV - I_{M\backslash B_R}(R^k) \int_{M\backslash B_R} |\alpha|^2 dV \leq C \int_{B_R} |\alpha|^2 dV$$

now using the Poincaré inequality as in part (1) we get

$$\lambda_1 \int_M |\alpha|^2 dV - I_{M\setminus B_R}(R^k) \int_{M\setminus B_R} |\alpha|^2 dV \leq C \int_{B_R} |\alpha|^2 dV.$$

And finally

$$\lambda_1 \int_M |\alpha|^2 dV - I_{M\setminus B_R}(R^k) \int_M |\alpha|^2 dV \leq C \int_{B_R} |\alpha|^2 dV.$$

By the hypothesis of part (b)

$$\int_M |\alpha|^2 dV \leq \frac{C}{\lambda_1 - I_{M\setminus B_R}(R^k)} \int_{B_R} |\alpha|^2 dV \tag{2.7}$$

But this is exactly what Proposition 2.0.13 requires. Hence $\dim \mathcal{H}^k < \infty$. □

Next we will show that $I_{\mathbb{H}^n}(R^k) = nk - k^2$. This follows easily from the following lemma:

Lemma 2.0.19. *Let e_i be an orthonormal frame at some point and η_i its associated coframe. Then the following formula holds*

$$\langle R^{(k)}(\eta_1 \wedge \eta_2 \ldots \wedge \eta_k), \eta_1 \wedge \eta_2 \ldots \wedge \eta_k \rangle = \sum_{i=1}^{k} \sum_{j=k+1}^{n} K(e_i, e_j). \tag{2.8}$$

For a proof of this formula, one should consult [21].

By the homogeneity of $\mathbb{H}$ and since $K \equiv -1$ we get the desired formula, namely

$$I_{\mathbb{H}^n}(R^k) = nk - k^2.$$

This together with the well known fact that

$$\lambda_1(\mathbb{H}^n) = \frac{(n-1)^2}{4},$$

imply the vanishing of the $\mathcal{H}^k$ whenever $nk - k^2 < \frac{(n-1)^2}{4}$.

In what follows we shall make use of the estimate obtained by H. McKean in [20],

Theorem (McKean): *Let M^n be a complete manifold of negative sectional curvature $K < -b$, where $b > 0$. Then the Poincaré constant satisfies*

$$\lambda_1 > \frac{(n-1)^2 b}{4}.$$

As a direct application of Theorem 0.0.1 we obtain the following result:

Corollary 2.0.20. *Let M^n be a complete, simply connected, negatively curved manifold with sectional curvature K pinched by $-1 \leq K$ outside some compact set and $K \leq -1+\epsilon$ everywhere . Then the following hold:*

i) If $n \geq 6$ and $\epsilon < 1 - \frac{4}{n-1}$, then $\dim \mathcal{H}^1 < \infty$,

ii) If $n \geq 9$ and $\epsilon < \frac{(n-1)^2 - 8(n-2)}{(n-1)^2 + 8(n-2)^2}$, then $\dim \mathcal{H}^2 < \infty$.

Proof. For the proof of the first part we shall use (2.8) to estimate $I_{M \backslash B_R}(R^1)$. Let α be a unit length 1-form at a point outside B_R where the pinching condition is satisfied. There exist $\alpha_2, \ldots, \alpha_n \in T^*M$ s.t $\alpha, \alpha_2, \ldots, \alpha_n$ form an orthonormal coframe. Then according to (2.8) we have

$$\langle R^1 \alpha, \alpha \rangle = \sum_{i=2}^{n} K(\alpha, \alpha_i) \geq -(n-1).$$

It follows that

$$I_{M \backslash B_R}(R^1) \leq (n-1),$$

and by McKean's estimate of the Poincaré constant of a negatively curved manifold [20] we have

$$\lambda_1 \geq \frac{(n-1)^2(1-\epsilon)}{4}.$$

Hence if $\epsilon < 1 - \frac{4}{n-1}$ we have

$$I_{M \backslash B_R}(R^1) < \lambda_1,$$

which means, according to Theorem 0.0.1, $\dim \mathcal{H}^1 < \infty$. This concludes the proof of part one.

For the proof of part two we have to employ more subtle estimates of the curvature operator in terms of the sectional curvature. We shall use the estimates obtained by Elworthy, Li, and Rosenberg in [12]. Let α be a unit length 2-form. According to Lemma 3.1 in [12] we have

$$\langle R^2(\alpha), \alpha \rangle > B - (A - B)(n - 2),$$

here

$$A = \sup\{\sum_{i=1}^{2} \sum_{3}^{n} K(v_i, v_j) \mid v_1, \ldots, v_n \text{ orthonormal frame}\}$$

and

$$B = \inf\{\sum_{i=1}^{2} \sum_{3}^{n} K(v_i, v_j) \mid v_1, \ldots, v_n \text{ orthonormal frame}\}.$$

Outside of the compact set the pinching condition is satisfied so we have

$$A = 2(n-2)(-1+\epsilon) \text{ and } B = -2(n-2),$$

it follows that, outside B_R

$$\langle R^2(\alpha), \alpha \rangle > -2(n-2) - 2\epsilon(n-2)^2,$$

which is equivalent to

$$I_{M \setminus B_R}(R^2) \leq 2(n-2) + 2\epsilon(n-2)^2.$$

Using the same estimate of McKean [20] and the hypothesis of the second part of the corollary we obtain again

$$I_{M \setminus B_R} R^2 < \lambda_1,$$

which by the conclusion of the Theorem 0.0.1 implies $\dim \mathcal{H}^2 < \infty$. This concludes the proof of the corollary. $\square$

Remarks:

a) If in the hypothesis of the Corollary 2.0.20 one asks for the curvature to be pinched everywhere, then one gets vanishing of the corresponding spaces. However, the ϵ required is much smaller than the one obtained by Donnelly & Xavier.

b) This result relies heavily on being able to estimate the lower bound of the curvature operator in terms of sectional curvature. A better understanding of this relationship, not easy in general, may lead to new results for the vanishing or finite dimensionality of the L^2 cohomology spaces.

Chapter 3
On the Heat-Flow Method of Gaffney

As we have seen before the heat flow takes an L^2-form and transforms it into harmonic L^2 form preserving the cohomology class. A nice differential-topological result is the following corollary:

Corollary 3.0.21. *Let M^n be an n-dimensional noncompact manifold. Then any n-degree compactly supported form is exact.*

The proof relies on the following geometric lemma, which is of independent interest:

Lemma 3.0.22. *Any noncompact manifold admits a complete metric of infinite volume.*

PROOF. Embed the manifold into some large Euclidean space $\mathbb{R}^N$ such that the image of the embeding is closed. This is always possible due to Withney's Embe ding theorem. We denote this metric by g. If the volume of the manifold with respect to this metric is infinite, we are done. If not let us fix a point $p \in M$ and let $r(x) = d(x,p)$ be the geodesic distance to the fixed point. Since the image of the embeding is closed and noncompact, it cannot be bounded, hence $r \to \infty$.
Now choose

$$f(x)^n \geq \frac{1}{V_{m+1} - V_m} + 1 \text{ for } m \leq r \leq m+1, \;\; f \in C^\infty. \tag{3.1}$$

(In (3.1) V_r denotes the volume of the geodesic ball of radius $r = r(x)$). This is always possible.

Let us consider now the metric $\tilde{g} = f^2 g$ and denote the corresponding volume elements by $d\widetilde{V}$ and by dV . Then we have the following identity $d\widetilde{V} = f^n dV$ where n represents the dimension of the manifold.

Now obviously

$$
\begin{aligned}
\int_M 1 d\tilde{V} &\geq \sum_{m=0}^{\infty} \int_{\overline{B(p,m+1)} \setminus B(p,m)} f^n dV \\
&\geq \left(\frac{1}{V_{m+1} - V_m} + 1 \right) (V_{m+1} - V_m) \\
&\geq \sum_{m=0}^{\infty} 1 \geq \infty.
\end{aligned}
\tag{3.2}
$$

In the inequalities above $B(p,r)$ denotes the geodesic ball with respect to the metric g. As a conclusion we see that $(M, \tilde{g})$ has infinite volume. On the other hand $\tilde{g} \geq g$ which implies that any Cauchy sequence w.r.t. $\tilde{g}$ is Cauchy w.r.t. g, hence a convergent sequence. This concludes the proof of the lemma. □

For the proof of Corollary 3.0.21 let us assume the contrary. Endow the manifold M^n with a complete metric of infinite volume. Let $\alpha \in C_0{}^{\infty}\Omega^n$ and $[\alpha] \neq 0$, then let $\alpha(t)$ be the solution to the heat equation with initial data α and let $[\alpha_\infty] = \lim_{t \to \infty} \alpha(t)$. Then we obtain α_∞ a harmonic L^2 n-form which is nontrivial. Contradiction. □

Observations:

a) It is well known that on a noncompact manifold every top-degree form must be exact. For example see [15]. The proof we offered above makes no use of algebraic-topology techniques.

b) The fact that the existence of a compactly supported nontrivial de Rham class induces a nontrivial L^2 harmonic form, was used by Segal in [22] and by Hitchin in [16]. The method used by Segal to prove this is not based on the heat flow method initiated by Gaffney.

Proposition 3.0.23. *Let M be a complete Riemannian manifold. The following two conditions are equivalent:*

i) $\mathcal{H}^k = \{0\}$,

ii) closed L^2 forms are orthogonal to coclosed L^2 forms.

PROOF. Suppose $\mathcal{H}^k \neq 0$ then there exists $\alpha \in \mathcal{H}^k$ and $\alpha \neq 0$. But $\alpha \in L^2, d\alpha = 0, \delta\alpha = 0$ and by assumption $(\alpha, \alpha) \neq 0$. For the converse let us assume there exist $\alpha, \beta \in L^2, d\alpha = 0, \delta\beta = 0$ and $(\alpha, \beta) \neq 0$. Let μ denote the solution to the heat flow having as initial data $\mu(0) = \alpha$. Now consider $Q(t) = (\mu(t), \beta)$. Due to the properties of the solution to the heat equation this is a smooth function in t, for $t > 0$ and continuous for $t \geq 0$.

Differentiating Q we get:

$$\dot{Q}(t) = (\dot{\mu}, \beta) = -(\Delta\mu, \beta).$$

Since μ is closed for all $t > 0$ it follows

$$\dot{Q}(t) = -(d\delta\mu, \beta) = -(\delta\mu, \delta\beta) = 0.$$

This means $Q(t) = Q(0)$ and $(\alpha_\infty, \beta) \neq 0$. Therefore $\mathcal{H}^k \neq 0$. $\square$

Next we will introduce the concept of the heat-flow map.

Proposition 3.0.24. *Let M be a manifold as before and α in $L^2\Omega^*$ a closed form on M. Let $\dot{\mu} = -\Delta\mu$ be the solution to the heat equation having initial data α. Then the following map $H: L^2\Omega^* \to \mathcal{H}^*$, $H(\alpha) = \lim_{n\to\infty} \mu(t) = \mu_\infty$ is well defined and linear.*

PROOF. Obvious from the uniqueness of the heat flow. □

Remark: When considering different metrics on the same manifold we will indicate that the spaces or operators are taken w.r.t. the metric g by an appropriate index. For example the space of harmonic L^2 forms w.r.t. the metric g will be indicated as $\mathcal{H}^k_g$, the Laplacian w.r.t. the metric g will be denoted by Δ_g, the Hodge-star operator by $*_g$, etc.

The next theorem is another immediate application of the principle we used to prove Proposition 3.0.23.

Theorem 3.0.25. *Let M be a noncompact manifold and let g and $\tilde{g}$ be two complete metrics on M such that $L^2\Omega_{\tilde{g}} \subset L^2\Omega_g$. The map $H_g: \mathcal{H}_{\tilde{g}} \to \mathcal{H}_g$ is linear and injective.*

PROOF. All we have to prove is that $ker H_g = 0$. Let $\alpha \in \mathcal{H}_{\tilde{g}}$ such that $H_g(\alpha) = 0$. Since $\alpha \in \mathcal{H}_{\tilde{g}}$ means α is closed and coclosed w.r.t. the $\tilde{g}$ metric. This means $*_{\tilde{g}}\alpha$ is also closed, hence $*_g *_{\tilde{g}} \alpha$ is coclosed w.r.t. the g metric. All the forms here are also L^2 since the $*$-operator preserves length.
Let $\beta = *_g *_{\tilde{g}} \alpha$. We have

$$0 = (H_g(\alpha), \beta) = (\alpha, \beta)_g = \int_M \alpha \wedge *_g\beta.$$

But $*_g\beta = \pm *_{\tilde{g}} \alpha$. In conclusion we have

$$\int_M \alpha \wedge *_{\tilde{g}}\alpha = (\alpha, \alpha)_{\tilde{g}} = 0.$$

Hence $\alpha = 0$. Therefore H_g is injective. □

Remark: Having two conformal metrics $\tilde{g} = f^2 g$ on M, they are quasi-isometric if and only if the conformal factor is bounded (i.e $0 < c \leq f \leq C$). This easily follows from the very definition of quasi-isometric metrics. It also implies that the area of applicability of Theorem 3.0.25 is larger than that of the well known fact that two quasi-isometric metrics have the same L^2 cohomology.

A straight forward application of Theorem 3.0.25 is the following:

Corollary 3.0.26. *Let $\tilde{g} = f^2 g$ be a conformal deformation of the hyperbolic metric g on $\mathbb{H}^{2n}$ such that $f \geq c > 0$, then the corresponding L^2 cohomology spaces are isomorphic.*

PROOF. It suffices to show that for $k < n$ we have $\mathcal{H}^k_{\tilde{g}} = 0$. In mid-dimension degree the fact that the space of harmonic L^2 forms is conformal invariant is well known. We will show that $L^2\Omega_{\tilde{g}} \subset L^2\Omega_g$ and applying Theorem 3.0.25 we conclude $\mathcal{H}^k_{\tilde{g}} = 0$.
Let $e_1, \ldots, e_{2n}$ be a local orthonormal frame w.r.t. the g metric, and $\eta_1, \ldots, \eta_{2n}$ the associated dual frame. It follows that $f^{-1}e_1, \ldots, f^{-1}e_{2n}$ is an orthonormal frame w.r.t. $\tilde{g}$ metric and its associated coframe is $f\eta_1, \ldots, f\eta_{2n}$. Hence if we denote by dV the volume element w.r.t. g and by $\tilde{dV}$ the volume of $\tilde{g}$ we have

$$\tilde{dV} = f^{2n} dV.$$

Now we need to compare the pointwise length of a k-form w.r.t. the two metrics. To do this we notice as usual that a pointwise-orthonormal frame in Ω^n_g is given by

$$\eta_{i_1} \wedge \eta_{i_2} \wedge \ldots \wedge \eta_{i_k}, \ 1 < i_1 < i_2 \cdots < i_k < 2n.$$

Also a pointwise orthonormal frame in $\Omega^n_{\tilde{g}}$ is given by

$$f\eta_{i_1} \wedge f\eta_{i_2} \wedge \ldots \wedge f\eta_{i_k}, \ 1 < i_1 < i_2 \cdots < i_k < 2n.$$

It easily follows that

$$|\alpha|^2_{\tilde{g}} = f^{-2k}|\alpha|^2_g.$$

Integrating and using both identities we get

$$||\alpha||_{\tilde{g}}^2 = \int_M |\alpha|_{\tilde{g}}^2 d\tilde{V} = \int_M f^{-2k}|\alpha|_g^2 f^{2n} dV \geq c^{2(n-k)}||\alpha||_g^2.$$

It follows that $L^2\Omega_{\tilde{g}} \subset L^2\Omega_g$ which in turn implies the conclusion of the corollary. □

In order to give another interesting application of Theorem 3.0.25 we need to make the following definition:

Definition 3.0.27. *Let V be a finite dimensional real vector space and let g and h be two positively defined inner prodoucts on V. Let $G : V \longrightarrow V^*$ and $H : V \longrightarrow V^*$ denote the metric isomorfism induced by g and h respectively. Let $A : V \longrightarrow V$ denote the composition $A = H^{-1}G$. It is well known that A is orthogonally diagonalizable with respect to the metric h and let $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be its eigenvalues. We will call these positive numbers the eigenvalues of g w.r.t h.*

Corollary 3.0.28. *Let (M^n, g) be a complete simply connected Riemannian manifold of negative sectional curvature . Let h denote the complete metric of constant -1 sectional curvature. Let $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalues of g w.r.t. h as being functions on the manifold (i.e $\mu_i = \mu_i(x)$, $x \in M$). The following holds:*

a) if $\sup_{x\in M} \frac{\mu_n}{\mu_1} < \infty$ *then* $\dim \mathcal{H}^{n/2}(M, g) = \infty$*, and*

b) if $\inf_{x\in M} \frac{\mu_n^{n/2}}{\mu_1^k} > 0$ *for* $k \neq n/2$ *then* $\dim \mathcal{H}^k(M, g) = 0$.

PROOF. The idea of the proof is to show that the L^2 norms of a k-form, when considered with the two different metrics are equivalent.

Let us fix a point $x \in M$ and let $e_1, e_2, \ldots, e_n$ be a set of eigenvectors for g which are orthonormal w.r.t. h. Let $e^1, e^2, \ldots, e^n$ be the associated dual coframe. It follows that

$$\eta_i = \frac{1}{\sqrt{\mu_i}} e_i$$

is an orthonormal frame w.r.t. g having associated coframe

$$\eta^i = \sqrt{\mu_i} e^i.$$

Hence, if we denote by ω_h and ω_g the volume forms for the two metrics respectively, we have

$$\omega_g = \eta^1 \wedge \eta^2 \wedge \ldots \wedge \eta^n = \sqrt{\mu_1 \cdots \mu_n}\ \omega_h.$$

Let α be a k-form expressed at $x \in M$ as

$$\alpha = \alpha_{i_1 \cdots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k} = \alpha_{i_1 \cdots i_k} \frac{1}{\sqrt{\mu_{i_1} \cdots \mu_{i_k}}} \eta^{i_1} \wedge \ldots \wedge \eta^{i_k},$$

it follows that

$$|\alpha|_h^2 = \sum_{i_1 < i_2 < \cdots < i_k} \alpha^2_{i_1 \cdots i_k} \text{ and } |\alpha|_g^2 = \sum_{i_1 < i_2 < \cdots < i_k} \alpha^2_{i_1 \cdots i_k} \frac{1}{\mu_{i_1} \cdots \mu_{i_k}}.$$

Next we compare the L^2 norms, we have

$$|\alpha|_g^2\ \omega_g = \sum_{i_1 < i_2 < \cdots < i_k} \alpha^2_{i_1 \cdots i_k} \frac{1}{\mu_{i_1} \cdots \mu_{i_k}} \sqrt{\mu_1 \cdots \mu_n}\ \omega_h,$$

hence

$$\frac{\mu_1^{n/2}}{\mu_n^k} |\alpha|_h^2 \omega_h \leq |\alpha|_g^2\ \omega_g \leq \frac{\mu_n^{n/2}}{\mu_1^k} |\alpha|_h^2 \omega_h \tag{3.3}$$

Using the hypothesis that $\sup_{x \in M} \frac{\mu_n}{\mu_1} < \infty$ and integrating we obtain for $\alpha \in C_0^\infty \Omega^{n/2}$

$$\sup_{x \in M} (\frac{\mu_n}{\mu_1})^{n/2} ||\alpha||_h^2 \geq ||\alpha||_g^2,$$

hence

$$L^2 \Omega_h^{n/2} \subset L^2 \Omega_g^{n/2}$$

and applying the conclusion of Theorem 3.0.25 we get

$$\dim \mathcal{H}_g^{n/2} = \dim \mathcal{H}_h^{n/2} = \infty.$$

To prove that $\mathcal{H}_g^k = 0$ for $k \neq n/2$ we use again (3.3) and the hypothesis that $\inf_{x\in M} \frac{\mu_n^{n/2}}{\mu_1^k} >$ 0 for every $k \neq n/2$ and integrating again we obtain, for any $\alpha \in C_0^\infty\Omega^k$,

$$\inf_{x\in M} \frac{\mu_n^{n/2}}{\mu_1^k} ||\alpha||_h^2 \leq ||\alpha||_g^2,$$

hence

$$L^2\Omega_g^k \subset L^2\Omega_h^k \text{ for } k \neq n/2.$$

By Theorem 3.0.25 we obtain

$$\dim \mathcal{H}_g^k = 0 \text{ for } k \neq n/2.$$

This concludes the proof of the corollary. □

To give a concrete example of an application of the Corollary 3.0.28 we shall construct a metric which has the same L^2 cohomology as the hyperbolic metric but is not conformal nor quasi-isometric to the hyperbolic metric. Let $(x_1, ..., x_n)$ denote the euclidean coordinates on $\mathbb{H}^n$, where n is even. With respect to these coordinates the hyperbolic metric is:

$$h = \frac{1}{x_n^2} \sum_{i=1}^{n} dx_i^2.$$

Consider now the following metric:

$$g = \frac{1}{x_n^2} \sum_{i=1}^{n} (i + x_1^2) dx_i^2.$$

Both g and h are diagonal metrics but they are not conformal to each other. In this particular case it is easy to see that if $\mu_1, ..., \mu_n$ denote the eigenvalues of g w.r.t h then

$$\mu_i = i + x_1^2.$$

It follows, from the fact that μ_i's are not bounded from above, that g is not quasi-isometric to h. Since obviously

$$\frac{\mu_n}{\mu_1} = \frac{n + x_1^2}{1 + x_1^2} \leq n$$

it implies $\dim \mathcal{H}_g^{n/2} = \infty$. On the other hand we have

$$\frac{\mu_n^{n/2}}{\mu_1^k} = \frac{(n + x_1^2)^{n/2}}{(1 + x_1^2)^k},$$

hence if $k < n/2$

$$\inf \frac{\mu_n^{n/2}}{\mu_1^k} = \inf \frac{(n + x_1^2)^{n/2}}{(1 + x_1^2)^k} > 0.$$

In conclusion $\mathcal{H}_g^k = 0$. Now we only need to show that g is a complete metric. We shall do this by comparision with h, namely we shall prove that

$$g_p(X, X) \geq h_p(X, X) \text{ for any } p \in \mathbb{H}^n \text{ and } X \in T_p\mathbb{H}^n.$$

Let $e_1, ..., e_n$ be an orthonormal frame for h at the point $p \in \mathbb{H}^n$ and such that the metric g is diagonal. Obviously

$$g_p(e_i, e_i) = \mu_i.$$

Let $X \in T_p\mathbb{H}^n$ be a vector and $X = X^i e_i$ be its expresion with respect to the chosen frame. Then obviously

$$h_p(X, X) = \sum_{i=1}^{n} (X^i)^2$$

and

$$g_p(X, X) = \sum_{i=1}^{n} \mu_i (X^i)^2.$$

Now taking into account that $\mu_i = i + x_1^2$ we obtain

$$g_p(X, X) \geq h_p(X, X).$$

It is also obvious this metric is not rotationally symmetric in the sense of Dodziuk.

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