MULTIPLE CRITICAL POINTS FOR NEAR-SYMMETRIC FUNCTIONALS AND APPLICATION TO A NON-HOMOGENEOUS BOUNDARY VALUE PROBLEM

by

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Abstract

A method of perturbation from symmetry, developed by Bolle [Bol99] in order to prove that a particular non-homogeneous boundary value problem has infinitely many solutions, is presented as an abstract result in critical point theory. The main theorem establishes the existence of multiple critical points for certain "near-symmetric" functionals.

As an application, we consider the problem

\[\begin{array}{ll}
-\Delta u = \lambda |u|^{q-1}u + |u|^{p-1}u + f & \text{on } \Omega \\
u = u_0 & \text{on } \partial \Omega 
\end{array}\]

where \( \Omega \) is a smooth, bounded, open subset of \( \mathbb{R}^n \) (\( n \geq 2 \)), \( \lambda > 0 \), \( 1 \leq q < p \), \( f \in C(\overline{\Omega}, \mathbb{R}) \) and \( u_0 \in C^2(\overline{\Omega}, \mathbb{R}) \). We prove that this equation has an infinite number of solutions for \( p < \frac{n+1}{n-1} \) and that for any sub-critical \( p \) i.e., \( p < \frac{n+2}{n-2} \), there are as many solutions as we like, provided \( \|f\| \) and \( \|u_0\| \) are small enough.
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for my parents,
Karen and Graham Chambers

and

for my best friend,
Michelle M. Munchua
Chapter 1

Introduction

Symmetric functionals hold a privileged position in critical point theory. In general, a functional may have few if any critical points. One may, through various methods be able reveal a handful of critical points for a non-symmetric functional. In contrast, min-max methods for even functionals often yield an infinity of critical points.

Variational methods in partial differential equations has enjoyed the benefits of this phenomenon. Many equations have a symmetry which is reflected in their variational problem. The associated functionals are even and thus, can be shown to have an infinite number of critical points, each corresponding to a distinct solution of the equation.

There are, however, many more equations for which the associated functional is not even. The symmetry is broken, perhaps by a non-homogeneous term in the equation or by a non-homogeneous boundary condition. What happens then? Will a “small” non-symmetric term destroy the multiplicity of solutions? Intuition might suggest the answer should be no. However, the question has yet to be addressed in full generality.

A method of “perturbation from symmetry”, developed by Bahri-Berestycki [BB81] and Struwe [Str90] and formalized by Rabinowitz [Rab82, Rab84], has had some success with this issue for certain variational problems corresponding to partial differential equations where the non-symmetric term is of low order. The approach of this method is to consider the non-even functional $I = I_0 + K$ as a perturbation from its symmetric part $I_0$. One then defines a sequence of min-max values of $I$ (which are critical if the non-symmetric part, $K$, vanishes) and studies the effect of the perturbation, $K$, on the growth of these min-max levels.

This thesis will present an alternative method, introduced by Bolle [Bol99], for dealing with these problems. The idea is, given a non-symmetric functional, $I_1$, to consider a path of functionals $I_\theta$, $\theta \in [0, 1]$, from an even functional, $I_0$, to $I_1$. Bolle’s result says, in essence, that the preservation of critical points along the path depends on the variations $\frac{\partial}{\partial \theta} I_\theta$ at critical points of $I_\theta$. A consequence of Bolle’s theorem, due to Bolle-Ghoussoub-Tehrani [BGT], connects the growth of a sequence of min-max critical levels of $I_0$ with the existence of critical points of $I_\theta$. This result makes it possible to find general conditions under which $I_\theta$ has multiple critical points.
An application of this method to a non-homogeneous boundary value problem will illustrate its advantage over the standard perturbative method - its greater success in dealing with higher order perturbations from symmetry. We will consider the problem

\[
\begin{align*}
-\Delta u &= \lambda |u|^{q-1}u + |u|^{p-1}u + f \quad \text{on } \Omega \\
u &= u_0 \quad \text{on } \partial \Omega
\end{align*}
\]

where $\Omega$ is a smooth, bounded, open subset of $\mathbb{R}^n$ ($n \geq 2$), $\lambda > 0$, $1 \leq q < p$, $f \in C(\overline{\Omega}, \mathbb{R})$ and $u_0 \in C^2(\partial \Omega, \mathbb{R})$. When $u_0 = 0$, the non-symmetric term in the functional associated to this equation is linear. A non-trivial $u_0$, however, produces a higher order perturbation from symmetry. The standard method was used by Candela-Salvatore [CS97, CS98] to show this equation has an infinite number of solutions for $1 \leq q < p < \frac{n+2}{n}$. They also proved that, for any sub-critical $p$, the multiplicity of solutions can be made arbitrarily large, provided $\|f\|$ and $\|u_0\|$ are small enough. By employing Bolle’s method instead, we are able to reproduce these results, improving the range of $p$ values to $1 < p < \frac{n+1}{n-1}$. 
Chapter 2
Preservation of a Min-Max Critical Level Along a Path of Functionals

2.1 Introduction

In this chapter, an arbitrary path of functionals, $I_{\theta}$, on a Hilbert space will be considered. (We will not assume the path begins at an even functional.) We shall construct a min-max critical level for the initial functional, $I_0$. The main result, due to Bolle [Bol99], shows that the variations $\frac{\partial}{\partial \theta} I_\theta$ at critical points of $I_0$ roughly determines whether this critical level persists along the path i.e., we can construct similar critical levels for each functional along the path.

2.2 Bolle's Theorem

Let $E$ be a Hilbert space with inner product denoted by $\langle \cdot , \cdot \rangle$ and the associated norm by $\| \cdot \|$. We consider a $C^2$ functional $I : [0, 1] \times E \to \mathbb{R}$ and the continuum of functionals, $I_\theta := I(\theta, \cdot)$. We make the following hypotheses about $I$.

H1) $I$ satisfies a Palais-Smale condition: Any sequence $(\theta_n, v_n)$ in $[0, 1] \times E$ such that $I(\theta_n, v_n)$ is bounded and $\lim_{n \to \infty} \|I'(\theta_n, v_n)\| = 0$ has a convergent subsequence.

H2) For each $b > 0$, there exist a constant $C(b)$ depending on $b$ such that $|I_\theta(v)| \leq b$ implies that $|\frac{\partial}{\partial \theta} I(\theta, v)| \leq C(b)(\|I_\theta'(v)\| + 1)(\|v\| + 1)$.

H3) There exist two continuous functions $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ with $f_1 \leq f_2$, which are Lipschitz continous with respect to the second variable, such that at any critical point $v$ of $I_\theta$, we have

$$f_1(\theta, I_\theta(v)) \leq \frac{\partial}{\partial \theta} I(\theta, v) \leq f_2(\theta, I_\theta(v)).$$
Chapter 2. Preservation of a Min-Max Critical Level Along a Path of Functionals

We shall define functions \( \psi_1, \psi_2 : [0, 1] \times \mathbb{R} \to \mathbb{R} \) by

\[
\begin{align*}
\psi_1(0, s) &= s \\
\frac{\partial}{\partial \theta} \psi_1(\theta, s) &= f_1(\theta, \psi_1(\theta, s)).
\end{align*}
\]

Then \( \psi_1 \) and \( \psi_2 \) are continuous. Also \( \psi_1 \) and \( \psi_2 \) are increasing in \( s \) and \( \psi_1 \leq \psi_2 \) since \( f_1 \leq f_2 \).

H4) There are two closed subsets of \( E \), \( A \) and \( B \) with \( A \subset B \) such that:

i) \( I_0 \) is bounded on \( B \)

\[ \lim_{|v| \to \infty} \sup_{v \in B} I_\theta(v) = -\infty \]

ii) \( c_{B,A}(0) > c_A(0) \)

where \( c_U(\theta) = \sup_{U} I_\theta, \ c_{V,U}(\theta) = \inf_{g \in S_{V,U}} \sup_{g(V)} I_\theta \)

\[ S_{V,U} = \{ g \in C(V,E) \mid g(v) = v \forall v \in U \text{ and } \exists R > 0 \Rightarrow g(v) = v \forall \|v\| > R \}. \]

Note that if \( I \) satisfies H1) and H4), then it can be shown that \( I_0 \) has a critical point of critical level greater than \( c_A(0) \) as follows. Let \( \varepsilon > 0 \) be such that \( c_A(0) < c_{B,A}(0) - \varepsilon. \)

Since \( c_{B,A}(0) = \inf_{g \in S_{B,A}} \sup_{g(B)} I_0 \), there exists \( h \in S_{B,A} \) with \( \sup_{h(B)} I_0 < c_{B,A}(0) + \varepsilon. \)

Now \( I_0 \) satisfies the Palais-Smale condition by H1). Thus, by the deformation lemma (see [Rab84] or [Str90]), there exist \( 0 < \varepsilon < \varepsilon \) and \( \eta \in C([0,1] \times E, E) \) such that

\[ \begin{align*}
i & \text{ i) } \eta(0,v) = v \text{ for all } v \in E \\
ii & \text{ ii) } \eta(t,v) = v \text{ for all } v \notin I_0^{-1}(c_{B,A}(0) - \varepsilon, c_{B,A}(0) + \varepsilon) \text{ and } c_{B,A}(0) \text{ and } c_{B,A}(0) \text{ and } \\
iii & \text{ iii) } K_{c_{B,A}(0)} = \emptyset \text{ then } \eta(1, A_{c_{B,A}(0)+\varepsilon}) \subset A_{c_{B,A}(0)-\varepsilon} \]
\]

where \( A_d = \{ v \in E \mid I_0(v) \leq d \} \). Let \( \eta_1 = \eta(1, \cdot) \) and consider \( \eta_1 \circ h \in C(B,E) \). If \( v \in A \), then \( I_0(v) \leq c_A(0) < c_{B,A}(0) - \varepsilon \) and \( \eta_1 \circ h(v) = \eta_1(h(v)) = \eta_1(v) = v \). Since \( h \in S_{B,A} \), there is \( R_1 > 0 \) such that \( \|v\| > R_1 \) implies \( h(v) = v \). By H4), \( \lim_{|v| \to \infty} \sup_{v \in B} I_0(v) = -\infty \), so there exists \( R_2 > 0 \) so that \( \|v\| > R_2 \) implies \( I_0(v) < c_{B,A}(0) - \varepsilon \) and thus \( \eta_1(v) = v \).

Let \( R = \max\{ R_1, R_2 \} \). Then if \( \|v\| > R \), we have \( \eta_1 \circ h(v) = \eta_1(h(v)) = \eta(v) = v \). Thus, \( \eta_1 \circ h \in S_{B,A}. \)

Suppose \( c_{B,A}(0) \) is not a critical level of \( I_0 \). Then \( \eta_1 \circ h(B) \subset \eta_1(A_{c_{B,A}(0)+\varepsilon}) \subset A_{c_{B,A}(0)-\varepsilon}. \) This means that \( c_{B,A}(0) = \inf_{g \in S_{B,A}} \sup_{g(B)} I_0 \leq \sup_{\eta_1 \circ h(B)} I_0 \leq c_{B,A}(0) - \varepsilon \).

By this contradiction, \( I_0 \) must have a critical point at critical level \( c_{B,A}(0) > c_A(0) \).

Now if \( I \) also satisfies H2) and H3), which restricts how the functionals \( I_\theta \) vary with \( \theta \) at critical points, and \( \psi_1(\theta, c_{B,A}(0)) > \psi_2(\theta, c_A(0)) \), then this min-max critical level is preserved along the path of functionals. By this we mean that we can find sets \( A(\theta) \) and \( B(\theta) \) in \( E \) such that \( c_B(\theta) > c_A(\theta) \) and thus, by a deformation argument like the one above, \( I_\theta \) has a critical point of critical level greater than \( c_A(\theta) \).

More precisely, we will present a proof of the following theorem.
**Theorem 2.1** (Bolle) Suppose \( I : [0,1] \times E \to \mathbb{R} \) is a \( C^2 \) functional satisfying H1-H4). If
\[
\psi_1(\theta, c_{B,A}(0)) > \psi_2(\theta, c_A(0)) \text{ then for all } \theta \in [0,1], \text{ } I_\theta \text{ has a critical point with critical level in the interval } [\psi_1(\theta, c_{B,A}(0)), \psi_2(\theta, c_{B,A}(0))].
\]

### 2.3 Preliminary Results

Before proving the theorem, we will present a pair of lemmas which are required to show that for the sets \( A(\theta) \) and \( B(\theta) \) which we will define, we have that \( c_{B(\theta),A(\theta)}(\theta) > c_A(\theta) \). In all that follows, we assume the hypotheses of the theorem hold. Namely, we have H1) - H4) and also \( \psi_1(\theta, c_{B,A}(0)) > \psi_2(\theta, c_A(0)) \). Now since \( \psi_1(1, c_{B,A}(0)) > \psi_2(1, c_A(0)) \) and \( \psi_2 \) is continuous, for \( \eta > 0 \) sufficiently small, \( \psi_1(1, c_{B,A}(0)) > \psi_2(1, c_A(0) - \eta) \). Fix such an \( \eta \). Define \( D = \{ v \in E \mid I_0(v) \geq c_{B,A}(0) - \eta \}. \)

**Lemma 2.2** Suppose \( H \in C([0,1] \times E, E) \) satisfies

1. \( H(0, v) = v \) for all \( v \in E \)
2. there exists \( R > 0 \) such that for all \( \phi \in [0, \theta] \), \( H(\phi, v) = v \) for all \( v \in B \) with \( \| v \| > R \)
3. \( H(\phi, A) \cap D = \emptyset \) for all \( \phi \in [0, \theta] \)

for some \( \theta \in [0,1] \). Then \( H(\theta, B) \cap D \neq \emptyset. \)

**Proof.** Since \( H \) is continuous, \( A \) and \( B \) are closed and \( H([0, \theta] \times A) \cap D = \emptyset \), there exists an open neighbourhood, \( U \), of \( A \) so that \( H([0, \theta] \times U) \cap D = \emptyset. \) Let \( V \) be an open set containing \( A \) such that \( \tilde{V} \subset U. \) Then there exists \( l \in C(E, [0, \theta]) \) with \( l(v) = 0 \) for \( v \in V \) and \( l(v) = \theta \) for \( v \notin U. \)

Define \( h : E \to E \) by \( h(v) = H(l(v), v). \) Then if \( v \in A \subset V \), \( h(v) = H(l(v), v) = H(0, v) = v. \) If \( v \in B \) and \( \| v \| > R \) with \( R \) as in ii) above, \( h(v) = H(l(v), v) = v. \) So \( h|_B \in S_{B,A}. \)

Now \( c_{B,A}(0) = \inf_{g \in S_{B,A}} \sup_{g(B)} I_0 \), so \( \sup_{h|_B} I_0 \geq c_{B,A}(0) \) by \( c_{B,A}(0) \). Thus \( \psi_1(0, c_{B,A}(0)) > c_{B,A}(0) \). Now \( v_0 \notin U \) since \( \psi_1(v_0) = \psi_1(1, c_{B,A}(0) - \eta) \). Therefore, \( l(v_0) = \theta \) and \( H(\theta, v_0) = h(v_0) \in D. \) Thus \( H(\theta, B) \cap D \neq \emptyset. \)

**Corollary 2.3** Suppose \( H, G \in C([0,1] \times E, E) \) satisfy i) and ii) in Lemma 2.2 and also satisfy

4. \( G(\phi, A) \cap G(\phi, D) = \emptyset \) for all \( \phi \in [0, \theta] \)

and

5. for all \( \phi \in [0, \theta] \), \( G(\phi, \cdot) \) is a homeomorphism and \( G' : [0, \theta] \times E \to E \) defined by
\[
G'(\phi, v) = [G(\phi, \cdot)]^{-1}(v)
\]

is continuous

for some \( \theta \in [0,1] \). Then \( H(\theta, B) \cap G(\theta, D) \neq \emptyset. \)

**Proof.** Define \( F \in C([0, \theta] \times E, E) \) by \( F(\phi, v) = G'(\phi, H(\phi, v)). \) Now we show that \( F \) satisfies the hypotheses of Lemma 2.2.
First, \( F(0, v) = G'(0, H(0, v)) = G'(0, v) \). Since \( G(0, \cdot) \) is the identity on \( E \), so is its inverse and \( F(0, v) = G'(0, v) = [G(0, \cdot)]^{-1}(v) = v \). So \( F \) satisfies i).

Secondly, since \( H \) and \( G \) satisfy ii), there exist \( R_1, R_2 \) such that \( H(\phi, v) = v \) for all \( v \in B \) with \( \|v\| > R_1 \) for all \( \phi \in [0, \theta] \) and \( G(\phi, v) = v \) for all \( v \in B \) with \( \|v\| > R_2 \) for all \( \phi \in [0, \theta] \). Let \( R = \max\{R_1, R_2\} \). If \( v \in B \) and \( \|v\| > R \) then \( G(\phi, v) = v \) and \( H(\phi, v) = v \), so \( F(\phi, v) = G'(\phi, H(\phi, v)) = G'(\phi, v) = [G(\phi, \cdot)]^{-1}(v) = v \). Thus ii) holds for \( F \).

Finally, suppose \( v \in F(\phi, A) \cap D \) for some \( \phi \in [0, \theta] \). Then there exists \( u \in A \) such that \( v = F(\phi, u) = G'(\phi, H(\phi, u)) = [G(\phi, \cdot)]^{-1}(H(\phi, u)) \). This means \( G(\phi, v) = H(\phi, u) \) i.e., \( G(\phi, D) \cap H(\phi, A) \neq \emptyset \), contradicting iv). Thus \( F(\phi, A) \cap D = \emptyset \) and \( F \) satisfies iii) of the lemma.

Applying Lemma 2.2, we have \( F(\theta, B) \cap D \neq \emptyset \). There exists \( u \in B \) and \( v \in D \) such that \( v = F(\theta, u) \in D \) i.e., \( G(\theta, u) = H(\theta, v) \). So \( H(\theta, B) \cap G(\theta, D) \neq \emptyset \). \( \square \)

### 2.4 Proof of Bolle's Theorem

We begin by defining \( \tilde{\psi}_1, \tilde{\psi}_2 : [0, 1] \times \mathbb{R} \to \mathbb{R} \) by

\[
\begin{aligned}
\tilde{\psi}_1(0, s) &= s \\
\frac{\partial}{\partial \theta} \tilde{\psi}_1(\theta, s) &= f_1(\theta, \tilde{\psi}_1(\theta, s)) - \delta
\end{aligned}
\]

and

\[
\begin{aligned}
\tilde{\psi}_2(0, s) &= s \\
\frac{\partial}{\partial \theta} \tilde{\psi}_2(\theta, s) &= f_2(\theta, \tilde{\psi}_2(\theta, s)) + \delta
\end{aligned}
\]

where \( \delta \) is chosen sufficiently small so that \( \tilde{\psi}_2(1, c_A(0)) < \tilde{\psi}_1(1, c_B, A(0)) \). We may do this since \( \psi_2(1, c_A(0)) < \psi_1(1, c_B, A(0)) \).

Also define \( \varphi_1, \varphi_2, \zeta : [0, 1] \to \mathbb{R} \) by \( \varphi_1(\theta) = \tilde{\psi}_1(\theta, c_B, A(0)) - \eta \), \( \varphi_2(\theta) = \tilde{\psi}_2(\theta, c_B, A(0)) + \eta \) and \( \zeta(\theta) = \tilde{\psi}_2(\theta, c_B, A(0)) + \eta \). Since \( \varphi_2(1) < \varphi_1(1) \) and \( f_1 \leq f_2 \), it follows that \( \varphi_2(\theta) < \varphi_1(\theta) \) for all \( \theta \in [0, 1] \). Since \( \varphi_1(0) < \zeta(0) \), \( \varphi_1(\theta) < \zeta(\theta) \) for all \( \theta \in [0, 1] \). Let \( \alpha = \min_{\theta \in [0, 1]} \varphi_2(\theta) \) and \( \beta = \max_{\theta \in [0, 1]} \zeta(\theta) \). Then \( \alpha < \beta \).

Let \( u \in C^{\infty}(\mathbb{R}, [0, 1]) \) be such that \( u(t) = 0 \) for \( t \in (-\infty, \alpha - 2] \cup [\beta + 2, \infty) \) and \( u(t) = 1 \) for \( t \in [\alpha - 1, \beta + 1] \).

By H1) and H2) there exists \( \gamma > 0 \) such that \( I_\theta(v) \in (\alpha - 2, \beta + 2) \) and \( \|I_\theta'(v)\| < \gamma \) imply

\[ f_1(\theta, I_\theta(v)) - \delta < \frac{\partial}{\partial \theta} I_\theta(v) < f_2(\theta, I_\theta(v)) + \delta. \]

Given this \( \gamma \), let \( w \in C^{\infty}(\mathbb{R}, [0, 1]) \) be such that \( w(t) = 0 \) if \( |t| \leq \frac{\gamma}{2} \) and \( w(t) = 1 \) if \( |t| \geq \gamma \).

Now define \( X_1, X_2 : [0, 1] \times E \to E \) by

\[ X_1(\theta, v) = (J_{I_\theta}^{-1} + f_1^+(\theta, \varphi_1(\theta))u(I_\theta(v)))w(\|I_\theta'(v)\|) \frac{I_\theta'(v)}{\|I_\theta'(v)\|^2} \]

and

\[ X_2(\theta, v) = (J_{I_\theta}^{-1} + f_2^+(\theta, \varphi_2(\theta))u(I_\theta(v)))w(\|I_\theta'(v)\|) \frac{I_\theta'(v)}{\|I_\theta'(v)\|^2} \]
and
\[ X_2(\theta, v) = -\{J_\theta^+ + 1 + f_2^- (\theta, \varphi_2(\theta))\} u(I_\theta(v)) w(\|I_\theta(v)\|) \frac{I_\theta'(v)}{\|I_\theta(v)\|^2} \]

where \( J_\theta(v) = \frac{\partial}{\partial \theta} I(\theta, v) \) and \( g^+ = \max\{g, 0\} \) and \( g^- = \min\{-g, 0\} \). Finally, we define \( \phi_1, \phi_2 : [0, 1] \times E \to E \), as flows of \( X_1 \) and \( X_2 \) respectively, by

\[
\begin{align*}
\phi_1(0, v) &= v \\
\frac{\partial}{\partial \theta}\phi_1(\theta, v) &= X_1(\theta, \phi_1(\theta, v))
\end{align*}
\]

Before continuing, we must check that \( \phi_1 \) and \( \phi_2 \) are well-defined. Clearly, \( X_i \) is continuous in \( \theta \) and recalling that \( I \) is \( C^2 \), it can be shown that \( X_i \) is Lipschitz with respect to the second variable. This implies that for each \( v \in E \), there exists a unique solution \( \phi_i(\cdot, v) : [0, 1] \to E \) of the above. Obviously, \( \phi_i(\cdot, v) \) is continuous for each \( v \in E \). In fact, \( \phi_i \) is continuous on \([0, 1] \times E \).

We are now ready to define the sets \( A(\theta) = \phi_2(\theta, A) \) and \( B(\theta) = \phi_2(\theta, B) \). We will also require the set \( D(\theta) = \phi_1(\theta, D) \). In order to prove that \( c_{B(\theta), A(\theta)}(\theta) > c_{A(\theta)}(\theta) \), we will apply Corollary 2.3 to \( \phi_1 \) and \( \phi_2 \).

First, we must check that \( \phi_1 \) and \( \phi_2 \) satisfy i) and ii). By definition of \( \phi_i \), i) holds. To see that ii) is also satisfied, note that \( X_i(\theta, v) = 0 \) if \( I_\theta(v) \not\in (\alpha - 2, \beta + 2) \). Now, using H4) ii) there exits \( R_0 > 0 \) such that \( I_\theta(v) < \alpha - 2 \) and thus \( X_i(\theta, v) = 0 \) for all \( \theta \in [0, 1] \), for all \( |v| > R_0 \). Choose \( R > R_0 \). Then for \( |v| > R \), we have \( \phi_i(\theta, v) = \phi_i(0, v) = v \) for all \( \theta \in [0, 1] \).

Second, we need to show that \( \phi_1 \) and \( \phi_2 \) satisfy iv) and v) of the corollary. That v) is satisfied follows from the definition of \( \phi_1 \) as a flow of \( X_i \).

To check that iv) holds (ie \( \phi_2(\theta, A) \cap \phi_1(\theta, D) = \emptyset \)) requires a little more work. We begin by proving that if \( I_\theta(v) \leq c_A(0) \) (in particular, if \( v \in A \)) then \( I_\theta(\phi_2(\theta, v)) \leq \varphi_2(\theta) \) for all \( \theta \in [0, 1] \).

Let \( v \in E \) be such that \( I_\theta(v) \leq c_A(0) \) and let \( W(\theta) = I_\theta(\phi_2(\theta, v)) \). We wish to show that \( W(\theta) \leq \varphi_2(\theta) \) for all \( \theta \in [0, 1] \). It is sufficient to prove that \( W(\theta) = \varphi_2(\theta) \) implies \( W'(\theta) < \varphi_2(\theta) \), since \( W(0) = I_0(\phi_2(0, v)) = I_0(v) \leq c_A(0) = \varphi_2(0) \).

Assume \( W(\theta) = \varphi_2(\theta) \). Then
\[
W'(\theta) = J_\theta(\phi_2(\theta, v)) + \{I_\theta'(\phi_2(\theta, v), \frac{\partial}{\partial \theta}\phi_2(\theta, v))
\]
\[
= J_\theta(\phi_2(\theta, v)) + \{I_\theta'(\phi_2(\theta, v), X_2(\theta, \phi_1(\theta, v))
\]
\[
= J_\theta(\phi_2(\theta, v)) - \{J_\theta^+ (2\phi_2(\theta, v)) + 1 + f_2^- (\theta, \varphi_2(\theta))\} \times
\]
\[
\frac{u(I_\theta(\phi_2(\theta, v))) w(\|I_\theta(\phi_2(\theta, v))\|)}{\|I_\theta(\phi_2(\theta, v))\|}
\]
\[
= J_\theta(\phi_2(\theta, v)) - \{J_\theta^+ (2\phi_2(\theta, v)) + 1 + f_2^- (\theta, \varphi_2(\theta))\} \times
\]
\[
\frac{u(W(\theta))) w(\|I_\theta(\phi_2(\theta, v))\|)}{\|I_\theta(\phi_2(\theta, v))\|}.
\]

Since \( \alpha \leq \varphi_2(\theta) < \varphi_1(\theta) < \zeta(\theta) \leq \beta \), we have \( u(W(\theta)) = u(\varphi_2(\theta)) = 1 \) and
\[
W'(\theta) = J_\theta(\phi_2(\theta, v)) - \{J_\theta^+ (2\phi_2(\theta, v)) + 1 + f_2^- (\theta, \varphi_2(\theta))\} w(\|I_\theta(\phi_2(\theta, v))\|).
\]
We consider two cases. First, if $\|I_\theta'(\phi_2(\theta, v))\| < \gamma$, then

$$W(\theta) \leq J_\theta(\phi_2(\theta, v)) < f_2(\theta, I_\theta(\phi_2(\theta, v))) + \delta = f_2(\theta, W(\theta)) + \delta.$$  

Second, if $\|I_\theta'(\phi_2(\theta, v))\| \geq \gamma$, then $w([\|I_\theta'(\phi_2(\theta, v))\|] = 1$ and

$$W'(\theta) = J_\theta(\phi_2(\theta, v)) - \{J_\theta^+ (\phi_2(\theta, v)) - 1 - f_2^+ (\theta, \phi_2(\theta))\}$$

$$= -J_\theta^- (\phi_2(\theta, v)) - 1 + f_2(\theta, W(\theta))$$

$$< f_2(\theta, W(\theta)) + \delta$$

Thus in any case, we have $W'(\theta) < f_2(\theta, W(\theta)) + \delta = \varphi_2'(\theta)$.

In a similar manner, we prove that if $I_\theta(v) \geq c_{B, A}(0) - \eta$ i.e., if $v \in D$, then $I_\theta(\phi_1(\theta, v)) \geq \varphi_1(\theta)$ for all $\theta \in [0, 1]$. Suppose $v \in E$ is such that $I_\theta(v) \geq c_{B, A}(0) - \eta$ and define $U(\theta) = I_\theta(\phi_1(\theta, v))$. As before, it is sufficient to prove that if $U(\theta) = \varphi_1(\theta)$, then $U'(\theta) > \varphi_1'(\theta)$, since $U(0) \geq \varphi_1(0)$.

Assuming $U(\theta) = \varphi_1(\theta)$, we have

$$U'(\theta) = J_\theta(\phi_1(\theta, v)) + \{I_\theta'(\phi_1(\theta, v), \frac{\partial}{\partial \theta}\phi_1(\theta, v))\}$$

$$= J_\theta(\phi_1(\theta, v)) + \{I_\theta'(\phi_1(\theta, v), X_1(\theta, \phi_1(\theta, v)))\}$$

$$= J_\theta(\phi_1(\theta, v)) + \{J_\theta^- (\phi_1(\theta, v)) + 1 + f_1^+ (\theta, \phi_1(\theta))\} \times$$

$$u(I_\theta(\phi_1(\theta, v)))w([\|I_\theta'(\phi_1(\theta, v))\|])$$

$$= J_\theta(\phi_1(\theta, v)) + \{J_\theta^- (\phi_1(\theta, v)) + 1 + f_1^+ (\theta, \phi_1(\theta))\} \times$$

$$u(W(\theta))w([\|I_\theta'(\phi_1(\theta, v))\|])$$

where in the last step we have used the fact that $u(W(\theta)) = u(\varphi_1(\theta)) = 1$.

If $\|I_\theta'(\phi_2(\theta, v))\| < \gamma$, then

$$U'(\theta) \geq J_\theta(\phi_1(\theta, v)) > f_1(\theta, I_\theta(\phi_1(\theta, v))) - \delta = f_1(\theta, U(\theta)) - \delta.$$  

If $\|I_\theta'(\phi_2(\theta, v))\| \geq \gamma$, then $v([\|I_\theta'(\phi_2(\theta, v))\|] = 1$ and

$$U'(\theta) = J_\theta(\phi_1(\theta, v)) + \{J_\theta'(\phi_1(\theta, v)) + 1 + f_1^+ (\theta, \phi_1(\theta))\} \times$$

$$J_\theta(\phi_1(\theta, v)) - \varphi_1(\theta)$$

$$> f_1(\theta, \varphi_1(\theta)) - \delta.$$  

Hence, $U'(\theta) > f_1(\theta, \varphi_1(\theta) - \delta = \varphi_1'(\theta)$.

Finally, we prove that $\phi_2(\theta, A) \cap \phi_1(\theta, D) = \emptyset$ for all $\theta \in [0, 1]$ using these two results. Suppose $v \in A$ and $u \in D$ are such that $\phi_2(\theta, v) = \phi_1(\theta, u)$ for some $\theta \in [0, 1]$. Then $I_\theta(v) \leq c_A(0)$ and $I_\theta(u) \geq c_{B, A}(0) - \eta$ and hence $\varphi_2(\theta) \geq I_\theta(\phi_2(\theta, u)) = I_\theta(\phi_1(\theta, u)) \geq \varphi_1(\theta)$. By contradiction, $\phi_2(\theta, A) \cap \phi_1(\theta, D) = \emptyset$ for all $\theta \in [0, 1]$. 

Chapter 2. Preservation of a Min-Max Critical Level Along a Path of Functionals

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Thus, we may apply the corollary and conclude that \( \phi_2(\theta, B) \cap \phi_1(\theta, D) \neq \emptyset \), i.e., \( B(\theta) \cap D(\theta) \neq \emptyset \). This allows us to prove that \( c_{B(\theta), A(\theta)}(\theta) > c_A(\theta) \) as follows.

We begin by noting that for all \( g \in S_{B(\theta), A(\theta)}, \ g(B(\theta)) \cap D(\theta) \neq \emptyset \) for any \( \theta \in [0, 1] \).

To see this, let

\[
H_\theta(\phi, v) = \begin{cases} 
\phi_2(2\theta \phi, v) & \phi \in [0, \frac{1}{2}] \\
(2 - 2\phi)\phi_2(\theta, v) + (2\phi - 1)g(\phi_2(\theta, v)) & \phi \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

and

\[
G_\theta(\phi, v) = \begin{cases} 
\phi_1(2\theta \phi, v) & \phi \in [0, \frac{1}{2}] \\
\phi_1(\theta, v) & \phi \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

and apply Corollary 2.3 to \( H_\theta \) and \( G_\theta \) at \( \phi = 1 \).

First, we must check the hypotheses of the corollary. Clearly, i) and v) are satisfied.

Since \( \phi_1, \phi_2 \) and \( g \) satisfy ii), so do \( H_\theta \) and \( G_\theta \). Now \( g \in S_{B(\theta), A(\theta)} \) implies that \( g \big|_{A(\theta)} = id_{A(\theta)} \) and thus if \( v \in A(\theta) \), we have

\[
H_\theta(\theta, v) = \begin{cases} 
\phi_2(2\theta \phi, v) & \phi \in [0, \frac{1}{2}] \\
\phi_2(\theta, v) & \phi \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

Recalling, \( \phi_2(\theta, A) \cap \phi_1(\theta, D) = \emptyset \) for all \( \theta \in [0, 1] \) it is easily seen that \( H_\theta \) and \( G_\theta \) satisfy iv).

Applying the corollary, we have \( H_\theta(\phi, B) \cap G_\theta(\phi, D) \neq \emptyset \) for all \( \phi \in [0, 1] \). When \( \phi = 1 \), this yields \( g(\phi_2(\theta, B)) \cap \phi_1(\theta, D) \neq \emptyset \) i.e., \( g(B(\theta)) \cap D(\theta) \neq \emptyset \).

This implies that

\[
c_{B(\theta), A(\theta)}(\theta) = \inf_{g \in S_{B(\theta), A(\theta)}} \sup_{v \in g(B(\theta))} I_\theta \geq \inf_{D(\theta)} I_\theta \geq \varphi_1(\theta) > \varphi_2(\theta) \geq \sup_{A(\theta)} I_\theta = c_A(\theta)(\theta)
\]

and therefore we may apply a deformation argument like the one in section 2.2 to show that \( I_\theta \) has a critical point of critical level \( c_{B(\theta), A(\theta)}(\theta) \).

By above, we have \( c_{B(\theta), A(\theta)}(\theta) \geq \varphi_1(\theta) = \psi_1(\theta, c_{B,A}(0) - \eta) \). Now we will show that \( c_{B(\theta), A(\theta)}(\theta) \leq \psi_2(\theta, c_{B,A}(0) + \eta) \).

We begin by proving that if \( I_0(v) \leq c_{B,A}(0) + \eta \), then \( I_\theta(\phi_2(\theta, v)) \leq \zeta(\theta) \). Consider again \( W(\theta) = I_0(\phi_2(\theta, v)) \). In the same way as before, we can show that if \( W(\theta) = \zeta(\theta) \) then \( W'(\theta) < \zeta'(\theta) \). Then, since \( W(0) = I_0(0) \leq c_{B,A}(0) + \eta = \zeta(0) \), we have \( W(0) \leq \zeta(\theta) \) for all \( \theta \in [0, 1] \).

Now fix \( \theta \in [0, 1] \). Let \( k = \phi_2(\theta, \cdot) \). By the definition of \( c_{B,A}(0) \) there exists \( h \in S_{B,A} \) such that \( \sup_{h(B)} I_0 \leq c_{B,A}(0) + \eta \). Let \( q = k \circ h \circ k^{-1} \) and note that \( q \in S_{B(\theta), A(\theta)} \). We have

\[
c_{B(\theta), A(\theta)}(\theta) = \inf_{g \in S_{B(\theta), A(\theta)}} \sup_{v \in g(B(\theta))} I_\theta(v) \leq \sup_{v \in q(B(\theta))} I_\theta(v) = \sup_{v \in q(h(B))} I_\theta(v) \leq \psi_2(\theta, c_{B,A}(0) + \eta).
\]
We have shown that $I_\theta$ has a critical point with critical level in the interval $[\tilde{\psi}_1(\theta, c_{B,A}(0) - \eta), \tilde{\psi}_2(\theta, c_{B,A}(0) + \eta)]$. Note that this critical level, $c_{B(\theta),A(\theta)}(\theta)$, depends on $\delta$ through $\tilde{\psi}_i$ and on $\eta$ through $\varphi_1$, and that $\delta$ and $\eta$ may be chosen arbitrarily small. Let $u_{\delta,\eta}$ be a critical point at this level, and consider $v_n := u_{\frac{1}{n},\frac{1}{n}}$, $n \in \mathbb{N}$. This is a sequence of critical points of $I_\theta$ with $I_\theta(v_n)$ bounded. By the Palais-Smale condition, $v_n$ has a subsequence converging to some $v \in E$. This $v$ is a critical point of $I_\theta$ and by continuity, the critical level is in the interval $[\varphi_1(\theta, c_{B,A}(0)), \varphi_2(\theta, c_{B,A}(0))]$. 
Chapter 3

Multiplicity of Critical Points Near an Even Functional

3.1 Introduction

A path of functionals satisfying similar conditions to those in chapter two, but also having its initial functional even, will be considered in this chapter. We will define a sequence of min-max levels of this initial functional. Applying the theorem of the previous chapter will yield the main result due to Bolle-Ghoussoub-Tehrani [BGT]. This theorem connects the growth of these min-max values with the existence of critical points of functionals along the path. By this result, we shall derive conditions sufficient for the existence of multiple critical points of functionals near the even one.

3.2 A Consequence of Bolle's Theorem

Let $E$ be a Hilbert space with inner product denoted by $\langle \cdot , \cdot \rangle$ and the associated norm by $\| \cdot \|$. We consider a $C^2$ functional $I : [0, 1] \times E \to \mathbb{R}$ and the continuum of functionals, $I_\theta := I(\theta, \cdot)$. We make the following hypotheses about $I$.

H1) $I$ satisfies a Palais-Smale condition: Any sequence $(\theta_n, v_n)$ in $[0, 1] \times E$ such that $I(\theta_n, v_n)$ is bounded and $\lim_{n \to \infty} \|I'(\theta_n, v_n)\| = 0$ has a convergent subsequence.

H2) For each $b > 0$, there exist a constant $C(b)$ depending on $b$ such that $|I_\theta(v)| \leq b$ implies that $|\frac{\partial}{\partial \theta} I(\theta, v)| \leq C(b)(\|I_\theta'(v)\| + 1)(\|v\| + 1)$.

H3) There exist two continuous functions $f_1, f_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ with $f_1 \leq f_2$, which are Lipschitz continuous with respect to the second variable, such that at any critical point $v$ of $I_\theta$, we have

$$f_1(\theta, I_\theta(v)) \leq \frac{\partial}{\partial \theta} I(\theta, v) \leq f_2(\theta, I_\theta(v)).$$
We shall define functions \( \psi_1, \psi_2 : [0, 1] \times \mathbb{R} \to \mathbb{R} \) by
\[
\begin{cases}
\psi_1(0, s) = s \\
\frac{\partial}{\partial s} \psi_1(\theta, s) = f_i(\theta, \psi_1(\theta, s)) \nendcases}
\]

Then \( \psi_1 \) and \( \psi_2 \) are continuous. Also \( \psi_1 \) and \( \psi_2 \) are increasing in \( s \) and \( \psi_1 \leq \psi_2 \) since \( f_1 \leq f_2 \).

Also define \( \bar{f}_i(\theta, t) = \sup_{\phi \in [0, \theta]} |f_i(\phi, t)| \).

H5) \( I_0 \) is even and for any finite dimensional subspace \( W \) of \( E \), we have \( \lim_{\|w\| \to \infty, w \in W} I_0 = -\infty \).

Now suppose \( E = E_- \oplus E_+ \) with \( E_- \) finite dimensional and \( (E_n)_{n=1}^{\infty} \) is an increasing sequence of subspaces of \( E \) such that \( E_0 = E_- \) and \( E_n = E_{n-1} \oplus \mathbb{R} \cdot e_n \). Let \( H = \{ h \in C(E, E) \mid h \) is odd and \( \exists R > 0 \) \( h(v) = v \forall \|v\| > R \} \) and define \( c_k = \inf_{h \in H} \sup_{h(E_k)} I_0 \).

**Theorem 3.1 (Bolle-Ghoussoub-Tehrani)** Suppose \( I \) satisfies H1-H3) and H5). Then for any \( \theta \in [0, 1] \), for each \( k \) either

i) \( I_0 \) has a critical level \( c_k(\theta) \) with \( \psi_2(\theta, c_k) < \psi_1(\theta, c_{k+1}) \leq c_k(\theta) \)

or

ii) \( c_{k+1} - c_k \leq \theta(\bar{f}_1(\theta, c_{k+1}) + \bar{f}_2(\theta, c_k)) \).

**Proof.** We consider two cases.

First, suppose that \( \psi_2(\theta, c_k) < \psi_1(\theta, c_{k+1}) \). We will apply Theorem 2.1 to show that \( I_0 \) has a critical point of critical level greater than \( \psi_1(\theta, c_{k+1}) \). By our hypothesis, H1-H3) hold. Now choose \( \varepsilon > 0 \) such that \( \psi_2(\theta, c_k + \varepsilon) < \psi_1(\theta, c_{k+1}) \) (note that this implies \( c_k + \varepsilon < c_{k+1} \)) and \( h \in H \) such that \( \sup_{h(E_k)} I_0 < c_k + \varepsilon \). Let \( E_{k+1} = E_k \oplus \mathbb{R}^+ e_{k+1} \) and set \( B_k = h(E_{k+1}) \) and \( A_k = h(E_k) \). Before applying the theorem, we must check that H4) is satisfied for these sets \( A_k \) and \( B_k \), and that \( \psi_1(\theta, c_{B_k, A_k}(0)) > \psi_2(\theta, c_{A_k}(0)) \). It is easy to show, using H5), that \( I_0 \) has an upper bound on \( B_k \) and that \( \lim_{\|u\| \to \infty, u \in B_k} \sup_{g \in H} I_0 = -\infty \).

Now suppose \( g \in S_{B_k, A_k} \). Then \( p = g \circ h \mid_{E_{k+1}^+} \) is odd on \( E_k \) since \( h \) is odd and \( g \) is the identity on \( A_k = h(E_k) \). Since \( E_{k+1} = E_k \oplus \mathbb{R}^+ e_{k+1} \), we may extend \( p \) to an odd function on \( E_{k+1} \). By Tietze extension theorem, we can extend \( p \) to a function on \( E \) \( \bar{p} \) which is odd and \( \bar{p}(v) = v \) for large \( \|v\| \) i.e., \( \bar{p} \in H \). We find \( \sup_{g(A_k)} I_0 = \sup_{p(E_{k+1}^+)} I_0 = \sup_{E_{k+1}} I_0 = \inf_{h \in H} \sup_{h(E_k)} I_0 = c_{k+1} \).

and hence \( c_{B_k, A_k}(0) = \inf_{g \in S_{B_k, A_k}} \sup_{g(A_k)} I_0 \geq c_{k+1} \). Note also that \( c_{A_k}(0) = \sup_{A_k} I_0 = \sup_{h(E_k)} I_0 < c_k + \varepsilon \). These two results imply \( \psi_1(\theta, c_{B_k, A_k}(0)) \geq \psi_1(\theta, c_{k+1}) > \psi_2(\theta, c_k + \varepsilon) \geq \psi_2(\theta, c_{A_k}(0)) \).
Therefore, we can indeed apply Bolle's theorem and conclude that $I_\theta$ has a critical level $c_k(\theta)$ in the interval $[\psi_1(\theta, c_{B_k, A_k}(0)), \psi_2(\theta, c_{B_k, A_k}(0))]$ and $c_k(\theta)$ satisfies

$$c_k(\theta) \geq \psi_1(\theta, c_{B_k, A_k}(0)) \geq \psi_1(\theta, c_{k+1}) > \psi_2(\theta, c_k).$$

In the second case, we have $\psi_2(\theta, c_k) \geq \psi_1(\theta, c_{k+1})$. Note that $-\tilde{f}_i(\theta, t) \leq f_i(\phi, t) \leq \tilde{f}_i(\theta, t)$. This, the fact that $f_i$ is Lipschitz and the definition of $\psi_i$ imply that

$$-\phi \tilde{f}_i(\theta, s) + s \leq \psi_i(\phi, s) \leq \phi \tilde{f}_i(\theta, s) + s$$

for all $\phi \in [0, \theta]$. Thus $|\psi_i(\phi, s) - s| \leq \phi \tilde{f}_i(\theta, s)$ for all $\phi \in [0, \theta]$ and so $|\psi_i(\theta, s) - s| \leq \theta f_i(\theta, s)$. This implies

$$c_{k+1} - c_k \leq \theta \tilde{f}_1(\theta, c_{k+1}) + \psi_1(\theta, c_{k+1}) + \theta \tilde{f}_2(\theta, c_k) - \psi_2(\theta, c_k)$$

$$\leq \theta(f_1(\theta, c_{k+1}) + f_2(\theta, c_k)).$$

\[3.3\] A Result on Multiplicity

The previous result allows one to prove the following corollary on the existence of multiple critical points.

**Corollary 3.2** Suppose $I$ satisfies $H1)$-H3), $H5$).

a) Suppose $\tilde{f}_i(\theta, t) \leq A_1 + A_2 |t|^\alpha$ where $0 \leq \alpha < 1$, $A_1, A_2 \geq 0$ and $c_k \geq B_1 + B_2 k^\beta$ where $\beta > 0, B_1 \in \mathbb{R}, B_2 > 0$. If $\beta > \frac{1}{1-\alpha}$, $I_\theta$ has an infinite number of critical points.

b) If $\lim_{k \to \infty} c_k = \infty$, then for every $N$ there exists $\theta_N \in [0, 1]$ such that for $\theta \in [0, \theta_N]$ has at least $N$ critical points.

**Proof.** By Theorem 3.1, for every $\theta \in [0, 1]$ and each $k$ either

i) $I_\theta$ has a critical level $c_k(\theta)$ with $\psi_2(\theta, c_k) < \psi_1(\theta, c_{k+1}) \leq c_k(\theta)$

or

ii) $c_{k+1} - c_k \leq \theta(\tilde{f}_1(\theta, c_{k+1}) + \tilde{f}_2(\theta, c_k))$.

Now we argue by contradiction. Suppose $I_\theta$ has only finitely many critical points for some $\theta \in [0, 1]$. Then the critical levels of $I_\theta$ are bounded.

Note that as shown in the proof of Theorem 3.1, $|\psi_i(\theta, s) - s| \leq \theta \tilde{f}_i(\theta, s)$ and thus

$$\psi_1(\theta, s) \geq s - \tilde{f}_1(\theta, s)$$

$$\geq s - (A_1 - A_2 |s|^\alpha).$$

(3.1)
Since $c_k$ is unbounded, by (3.1) so is $\psi_\theta(1, c_k + 1)$. Therefore, ii) must hold for large $k$ and we have

$$c_{k+1} - c_k \leq \theta(\bar{f}_1(\theta, c_{k+1})) + \bar{f}_2(\theta, c_k) \leq \theta(A_1 + A_2|c_{k+1}|^\alpha + B_1 + B_2|c_k|^\alpha)$$

for large $k$. This implies $c_k \leq C_1 + C_2k^{\frac{1}{1-\alpha}}$ for some $C_1 \in \mathbb{R}$, $C_2 > 0$. However, by assumption, $c_k \geq B_1 + B_2k^\beta$. We have a contradiction if $\beta > \frac{1}{1-\alpha}$. This establishes part a).

Now we claim that for every $N$ there exists $\theta_N \in [0,1]$ so that for all $\theta \in [0,\theta_N]$, $I_\theta$ has $N$ distinct critical levels $l_1(\theta) < l_2(\theta) < \ldots < l_N(\theta)$. It is enough to show that for fixed $M \in \mathbb{R}$ and $\theta_0 \in [0,1]$, there exists $\bar{\theta} \in [0,\theta_0]$ such that for all $\theta \in [0,\bar{\theta}]$, $I_\theta$ has a critical level $l(\theta)$, with $l(\theta) > M$ and $\sup_{\theta \in [0,\bar{\theta}]} l(\theta) < \infty$. Applying Theorem 3.1 we find this is reduced to proving there is some $\bar{\theta} \in [0,\theta_0]$ so that there exists $k$ such that ii) above is false and $\psi_\theta(\theta, c_{k+1}) > M$ for all $\theta \in [0,\bar{\theta}]$. By the hypothesis, the $c_k$'s are unbounded and thus by (3.1) there exists $L$ such that for all $k > L$, $\psi_\theta(\theta, c_{k+1}) > M$ for all $\theta \in [0,1]$. Therefore, we need only that there exists $k_0 > L$ such that ii) is false.

We argue by contradiction. Fix $M$ and $\theta_0$, and suppose the above is untrue. Then there exists $\theta_n \in [0,\theta_0]$ with $\lim_{n \to \infty} \theta_n = 0$ such that ii) is true for all $k > L$. Thus we have for all $n$ and $k > L$

$$c_{k+1} - c_k \leq \theta_n(\bar{f}_1(\theta_n, c_{k+1}) + \bar{f}_2(\theta_n, c_k)) \leq \theta_n(\bar{f}_1(\theta, c_{k+1}) + \bar{f}_2(\theta, c_k)) \leq \theta_n(A_1 + A_2|c_{k+1}|^\alpha + A_1 + A_2|c_k|^\alpha).$$

Taking the limit as $n \to \infty$, we get $c_{k+1} = c_k$ for all $k$, contradicting the assumption that $\lim_{k \to \infty} c_k = \infty$. Thus part b) holds. □
Chapter 4

Application to a Non-homogeneous Boundary Value Problem

4.1 Introduction

In this chapter, the results of chapter three will be applied to a semi-linear elliptic partial differential equation with non-homogeneous boundary conditions. The variational problem associated to this equation involves a functional which is not even. There is a linear perturbation from symmetry, due a non-homogeneous term in the equation and a higher order perturbation, resulting from the non-homogeneous boundary condition. We shall see that although this problem can be addressed by the method of perturbation from symmetry developed by Bahri-Berestycki [BB81] and Struwe [Str90], Bolle's approach yields stronger results.

4.2 Discussion of Problem and Previous Results

We will consider the problem

$$\begin{cases}
-\Delta u = \lambda |u|^{q-1}u + |u|^{p-1}u + f & \text{on } \Omega \\
u = u_0 & \text{on } \partial \Omega
\end{cases}$$

(4.1)

where $\Omega$ is a smooth, bounded, open subset of $\mathbb{R}^n$ ($n \geq 2$), $\lambda > 0$, $1 \leq q < p$, $f \in C(\overline{\Omega}, \mathbb{R})$ and $u_0 \in C^2(\partial \Omega, \mathbb{R})$.

In the case that $f = 0$ and $u_0 = 0$, the problem is symmetric; the energy functional associated to equation (4.1) is even. Min-max principles for even functionals have been used to show multiplicity of solutions for a certain class of partial differential equations with this type of symmetry. (See [Rab84] or [Str90].) In particular, these results prove that this problem has infinitely many solutions for all subcritical values of $p$ i.e., $1 < p < \frac{n+2}{n-2}$.

If $u_0 = 0$ but $f$ is arbitrary, this symmetry is lost. When $\lambda = 0$, some success in dealing with this problem has been found in methods of perturbation from symmetry. Bahri-Lions [BL88] have the best result in this case, showing that equation (4.1) has an
infinite number of solutions for all \(1 < p < \frac{n}{n-2}\). It is also known that for any subcritical value of \(p\), the multiplicity of solutions can be made arbitrarily large, provided \(\|f\|\) is small enough.

When \(u_0\) is non-trivial, the symmetry is again broken and the perturbation, due to the non-homogeneous boundary condition, is of a higher order. (If \(u_0 = 0\), the non-symmetric term of the functional is at most linear.) The standard perturbative method can be applied but yields the result for a smaller range of \(p\) values. This approach was used by Candela-Salvatore [CS97, CS98] to show that equation (4.1) has an infinity of solutions for \(1 \leq q < p < \frac{n+2}{n}\). They also proved that, for any sub-critical \(p\), equation (4.1) has as many solution as we like, provided \(\|f\|\) and \(\|u_0\|\) are small enough.

Bolle constructed his method to deal with just such a higher order perturbation for a different non-homogeneous boundary value problem. Bolle-Ghoussoub-Tehrani [BGT] have since used his approach on several other problems, including (4.1) when \(\lambda = 0\), proving the existence of infinitely many solutions for a larger range of \(p\), namely, \(1 < p < \frac{n+1}{n-1}\). In the following sections, we will apply Bolle’s method to equation (4.1) in all generality, repeating the results of Candela-Salvatore with this larger range of \(p\) values.

Specifically, we will consider the problem

\[
\begin{cases}
-\Delta u = \lambda |u|^{q-1}u + |u|^{p-1}u + \theta f & \text{on } \Omega \\
u = \theta u_0 & \text{on } \partial\Omega
\end{cases}
\]

where \(\theta \in [0, 1]\) (note that (4.2) reduces to (4.1) when \(\theta = 1\)) and prove the following results.

**Theorem 4.1** If \(\theta = 1\) and \(1 \leq q < p < \frac{n+1}{n-1}\) then the equation (4.2) above has infinitely many solutions.

**Theorem 4.2** Suppose \(1 \leq q < p < \frac{n+2}{n-2}\). Then for any \(N\), there exist \(\theta_N > 0\) such that for all \(\theta \in [0, \theta_N]\), equation (4.2) has at least \(N\) solutions.

### 4.3 Technical Results

Theorems 4.1 and 4.2 follow almost directly from Corollary 3.2. The bulk of the work in proving these theorems is in checking the hypotheses of the corollary. In this section we present technical results necessary to that end.

We begin by setting \(v = u - \theta u_0\) and rewriting equation (4.2) in terms of \(v\). The equation becomes

\[
\begin{cases}
-\Delta v = \lambda |v + \theta u_0|^{q-1}(v + \theta u_0) + |v + \theta u_0|^{p-1}(v + \theta u_0) + \theta f & \text{on } \Omega \\
v = 0 & \text{on } \partial\Omega
\end{cases}
\]
Consider the Hilbert space $E = W^{1,2}(\Omega)$ with inner product $\langle v_1, v_2 \rangle = \int_{\Omega} \nabla v_1 \cdot \nabla v_2 dx$.

For $1 < p \leq \frac{n+2}{n-2}$, we can define $I : [0, 1] \times E \to \mathbb{R}$ by

$$I(\theta, v) = \int_{\Omega} \left( \frac{1}{2} |\nabla v|^2 - \frac{\lambda}{q + 1} |v + \theta u_0|^{q+1} - \frac{1}{p+1} |v + \theta u_0|^{p+1} - \theta f v \right) dx.$$ 

We also define the continuum of functionals, $I_\theta : E \to \mathbb{R}$ by $I_\theta = I(\theta, \cdot)$. Now solutions of equation (4.3) are critical points of $I_\theta$, while $I_0$ is an even functional.

**Lemma 4.3** For all $b > 0$ there exists a constant $C(b)$ such that $|I_\theta(v)| \leq b$ implies $\left| \frac{\partial}{\partial \theta} I(\theta, v) \right| \leq C(b)(\|I_\theta'(v)\| + 1)(\|v\| + 1)$.

**Proof.** Let $b > 0$ and suppose $|I_\theta(v)| \leq b$ i.e.,

$$\int_{\Omega} \left( \frac{1}{2} |\nabla v|^2 - \frac{\lambda}{q + 1} |v + \theta u_0|^{q+1} - \frac{1}{p+1} |v + \theta u_0|^{p+1} - \theta f v \right) dx \leq b. \quad (4.4)$$

First, we will show that there exist constants $A, B > 0$ such that

$$\|v + \theta u_0\|_{p+1}^{p+1} \leq A \int_{\Omega} |\nabla v|^2 + B. \quad (4.5)$$

From the definition of $I$ we have

$$\int_{\Omega} \frac{1}{2} |\nabla v|^2 dx + I_\theta(v) = \int_{\Omega} \left( \frac{\lambda}{q + 1} |v + \theta u_0|^{q+1} + \frac{1}{p+1} |v + \theta u_0|^{p+1} + \theta f v \right) dx$$

$$\geq \int_{\Omega} \frac{1}{p+1} |v + \theta u_0|^{p+1} dx - \int_{\Omega} |\theta f v| dx$$

$$\geq A_1 \|v + \theta u_0\|_{p+1}^{p+1} - A_2 \|v\|_{p+1}$$

$$\geq A_1 \|v + \theta u_0\|_{p+1}^{p+1} - A_2 \|v + \theta u_0\|_{p+1} - A_3$$

$$\geq A_4 \|v + \theta u_0\|_{p+1}^{p+1} - A_5 \quad (4.6)$$

where the $A_i$ are positive constants and (4.5) follows immediately.
Now by (4.4), we have

\[
-\langle I'_\theta(v), v \rangle = \int_\Omega (-|\nabla v|^2 + \lambda |v + \theta u_0|^{q-1}(v + \theta u_0)v + \theta f v) dx
\]

\[
= \int_\Omega (|v + \theta u_0|^{q+1} + |v + \theta u_0|^{p+1}) dx + \int_\Omega (-|\nabla v|^2 - \lambda |v + \theta u_0|^{q-1}(v + \theta u_0)v + \theta f v) dx
\]

\[
\geq \int_\Omega (\lambda |v + \theta u_0|^{q+1} + \frac{q+1}{p+1} |v + \theta u_0|^{p+1}) dx + \int_\Omega (-|\nabla v|^2 - \lambda |v + \theta u_0|^{q-1}(v + \theta u_0)v + \theta f v) dx
\]

\[
\geq -(q+1)(\int_\Omega |\nabla v|^2 dx + b) + \int_\Omega (-|\nabla v|^2 - \lambda |v + \theta u_0|^{q-1}(v + \theta u_0)v + \theta f v) dx
\]

\[
\geq (q+1)(\int_\Omega |\nabla v|^2 dx - \int_\Omega (\lambda |v + \theta u_0|^{q+1} + \frac{q+1}{p+1} |v + \theta u_0|^{p+1})) dx + \int_\Omega q\theta f v dx - (q+1)b
\]

Thus by Hölder inequality, there exist positive constants $B_i$ such that

\[
-\langle I'_\theta(v), v \rangle \geq B_1 |v|_2^2 - B_2 |v + \theta u_0|_{p+1}^q - B_3 |v + \theta u_0|_{p+1}^p - B_4 |v + \theta u_0|_{p+1} - B_5
\]

\[
\geq B_1 |v|_2^2 - B_2 |v + \theta u_0|_{p+1}^q - B_3 |v + \theta u_0|_{p+1}^p - B_4 |v + \theta u_0|_{p+1} - B_6
\]

and using (4.5) we have

\[
-\langle I'_\theta(v), v \rangle \geq B_1 |v|_2^2 - B_7 (|v|_2^q)^{\frac{2q}{p+1}} - B_8 (|v|_2^p)^{\frac{2p}{p+1}} - B_9 (|v|_2^q)^{\frac{2q}{p+1}} - B_{10}
\]

\[
\geq B_{11} |v|_2^2 - B_{12} |v|_2^2 - B_{13}.
\]

Therefore,

\[
|I'_\theta(v)| |v|_2^2 \leq |I'_\theta(v)| |v|_2^2 + B_{12} |v|_2^2 + B_{13}.
\]
Finally, we have positive constants $C_i$ such that

$$\left| \frac{\partial}{\partial \theta} I(\theta, v) \right| = \left| \int_{\Omega} \left( \lambda |v + \theta u_0|^{q-1}(v + \theta u_0)u_0 + |v + \theta u_0|^{p-1}(v + \theta u_0)u_0 + f v \right) dx \right|$$

\begin{align*}
&\leq C_1 \|v + \theta u_0\|_{p+1}^q + C_2 \|v + \theta u_0\|_{p+1} p + C_3 \|v\|_{p+1} \\
&\leq C_1 \|v + \theta u_0\|_{p+1}^q + C_2 \|v + \theta u_0\|_{p+1} p + C_3 \|v\|_{p+1} + C_4 \\
&\leq C_5 \|v + \theta u_0\|_{p+1}^q + C_6 \\
&\leq C_7 \|v\|^2 + C_8 \\
&\leq C_9 \|I_\theta'(v)\| 2 + C_{10} \|v\| + C_{11} \\
&\leq C(\delta)(\|I_\theta'(v)\| + 1)(\|v\| + 1).
\end{align*}

Lemma 4.4 There exists a constant $C > 0$ such that if $v$ is a critical point of $I_\theta$ then

$$\left| \int_{\partial \Omega} \left( \frac{1}{2} |\nabla u|^2 - |\frac{\partial u}{\partial n}|^2 \right) d\sigma \right| \leq C \int_{\Omega} (|\nabla u|^2 + |u|^{p+1} + 1) dx$$

where $u = v + \theta u_0$.

Proof. Suppose $v$ is a critical point of $I_\theta$. This means $v$ satisfies equation (4.3) or $u = v + \theta u_0$ satisfies the equation

$$\begin{cases}
-\Delta u = \lambda |u|^{q-1} u + |u|^{p-1} u + \theta f & \text{on } \Omega \\
\theta u_0 & \text{on } \partial \Omega.
\end{cases} \quad (4.8)$$

We define $n : \overline{\Omega} \to \mathbb{R}^n$ such that $n$ coincides with the inward normal on $\partial \Omega$ as follows. For $x \in \overline{\Omega}$, let $l(x) = d(x, \partial \Omega)$, the distance from the boundary of $\Omega$. Since $\Omega$ is of class $C^2$, for some $\delta > 0$, $l$ is $C^2$ on $N_\delta = \{x \in \overline{\Omega} \mid l(x) < \delta\}$. Now let $n(x) = \nabla l(x)$.

Also let $\varphi : R \to [0, 1]$ be a smooth function such that $\varphi = 1$ on $(-\infty, 0]$ and $\varphi = 0$ on $[\delta, +\infty)$. Let $g(x) = \varphi(l(x))$. Note that $n$ and $g$ are $C^1$.

Now multiply equation (4.8) by $g \nabla u \cdot n$ and integrate over $\Omega$. On the left we have

$$\int_{\Omega} -\Delta g \nabla u \cdot n \, dx = \int_{\partial \Omega} -\frac{\partial u}{\partial n} (g \nabla u \cdot n) \, d\sigma + \int_{\Omega} \nabla u . \nabla (g \nabla u \cdot n) \, dx$$

$$= \int_{\partial \Omega} -g \frac{\partial u}{\partial n} |2 d\sigma + \int_{\Omega} \nabla u . \nabla (g \nabla u \cdot n) \, dx$$

$$= \int_{\partial \Omega} -\frac{\partial u}{\partial n} |^2 \, d\sigma + \int_{\Omega} \nabla u . \nabla (g \nabla u \cdot n) \, dx. \quad (4.9)$$
Now
\[
\int_{\Omega} \nabla u \cdot \nabla (g \nabla u \cdot n) \, dx = \int_{\Omega} \sum_{i,j} u_{x_i} (g u_{x_j} n_j)_{x_i} \, dx
\]
\[
= \int_{\Omega} \sum_{i,j} u_{x_i} u_{x_j} (g n_j)_{x_i} \, dx + u_{x_i} u_{x_j} u_{x_i} (g n_j) \, dx
\]
\[
= \int_{\Omega} \sum_{i,j} u_{x_i} u_{x_j} (g n_j)_{x_i} \, dx + \sum_{i,j} \int_{\Omega} \frac{1}{2} (u_{x_i})^2 x_j g n_j \, dx
\]
\[
= \int_{\Omega} \sum_{i,j} u_{x_i} u_{x_j} (g n_j)_{x_i} \, dx + \sum_{i,j} \int_{\Omega} \frac{1}{2} (u_{x_i})^2 g n_j n_j \, dx
\]
\[
= \int_{\Omega} \sum_{i,j} u_{x_i} u_{x_j} (g n_j)_{x_i} \, dx + \int_{\Omega} \frac{1}{2} |\nabla u|^2 \, d\sigma
\]
\[
= \int_{\partial \Omega} \frac{1}{2} |\nabla u|^2 \, d\sigma + O(\int_{\Omega} |\nabla u|^2 \, dx) . \tag{4.10}
\]

The first term on the right is
\[
\int_{\Omega} \lambda |u|^{q-1} u g \nabla u \cdot n \, dx = \int_{\Omega} \sum_{i} \lambda |u|^{q-1} u g u_{x_i} n_i \, dx
\]
\[
= \sum_{i} \left( \int_{\partial \Omega} \lambda u |u|^{q-1} u g n_i n_i \, d\sigma - \int_{\Omega} \lambda u (|u|^{q-1} u g n_i)_{x_i} \, dx \right)
\]
\[
= \sum_{i} \left( \int_{\partial \Omega} \lambda |u|^{q+1} g n_i n_i \, d\sigma - \int_{\Omega} \lambda |u|^{q+1} (g n_i)_{x_i} \, dx \right)
\]
\[
+ \int_{\Omega} \lambda u g n_i (q |u|^{q-1} u_{x_i}) \, dx
\]
\[
= \int_{\partial \Omega} \lambda |u|^{q+1} - \int_{\Omega} \sum_{i} \lambda |u|^{q+1} (g n_i)_{x_i} + \int_{\Omega} \sum_{i} \lambda |u|^{q+1} g n_i |u|^{q-1} u u_{x_i}
\]
\[
= \int_{\partial \Omega} \lambda |u|^{q+1} u_0 |u|^{q+1} \, d\sigma + O(\int_{\Omega} |u|^{q+1} \, dx) . \tag{4.11}
\]
Similarly, the second term is
\[
\int_{\Omega} |u|^{p-1} g \nabla u \cdot n dx = \int_{\partial \Omega} \theta^{2(p+1)} |u_0|^{p+1} d\sigma + O\left( \int_{\Omega} |u|^{p+1} dx \right) \tag{4.12}
\]
and the last term,
\[
\int_{\Omega} \theta f g \nabla u \cdot n dx = \int_{\partial \Omega} \sum_i f g u_{x_i} n_i dx
\]
\[
= \sum_i \left( \int_{\partial \Omega} f g u_i n_i d\sigma - \int_{\Omega} u(f g n_i)_{x_i} dx \right)
\]
\[
= \int_{\partial \Omega} f \theta u dx - \sum_i \int_{\Omega} u(f g n_i)_{x_i} dx
\]
\[
= \int_{\partial \Omega} \theta f u_0 dx + O\left( \int_{\Omega} |u|^{p+1} dx \right)^{\frac{1}{p+1}}. \tag{4.13}
\]

Combining equations (4.9)-(4.13), we get
\[
\int_{\partial \Omega} \frac{1}{2} |\nabla u|^2 - |\frac{\partial u}{\partial n}|^2 d\sigma + O\left( \int_{\Omega} |\nabla u|^2 dx \right) =
\]
\[
\int_{\partial \Omega} \lambda \theta^{q+1} |u_0|^{q+1} d\sigma + \int_{\partial \Omega} \theta^{p+1} |u_0|^{p+1} d\sigma + \int_{\partial \Omega} \theta f u_0 dx + O\left( \int_{\Omega} |u|^{q+1} dx \right)
\]
\[
+ O\left( \int_{\Omega} |u|^{p+1} dx \right) + O\left( \int_{\Omega} |u|^{p+1} dx \right)^{\frac{1}{p+1}} \tag{4.14}
\]
and so, there exist \( C, C_1 > 0 \) such that
\[
\left| \int_{\partial \Omega} \frac{1}{2} |\nabla u|^2 - |\frac{\partial u}{\partial n}|^2 d\sigma \right| \leq C_0 + C_1 \int_{\Omega} |\nabla u|^2 dx + C_2 \int_{\Omega} |u|^{q+1} dx + C_3 \int_{\Omega} |u|^{p+1} dx + C_4 \left( \int_{\Omega} |u|^{p+1} dx \right)^{\frac{1}{p+1}}
\]
\[
\leq C \left( \int_{\Omega} |\nabla u|^2 + |u|^{p+1} + 1 dx \right).
\]

\[\square\]

**Lemma 4.5** There exists a positive constant \( C > 0 \) such that if \( v \) is a critical point of \( I_\theta \) then
\[
|\frac{\partial}{\partial \theta} I_\theta(\theta, v)| \leq C(I_\theta(v)^2 + 1)^{\frac{1}{2}}.
\]
Proof. First note that by a calculation similar to the one in Lemma 4.3, we have

\[-\langle I_0'(v), v \rangle \geq (\frac{q+1}{2} - 1) \int_\Omega |\nabla v|^2 \, dx - \int_\Omega (\lambda |v + \theta u_0|^{q-1} (v + \theta u_0) \theta u_0 \, dx + |v + \theta u_0|^p - 1 (v + \theta u_0) \theta u_0) \, dx - \int_\Omega q \theta f v \, dx - (q + 1) I_\theta(v)\]

\[
\geq A_1 \int_\Omega |\nabla v|^2 \, dx - A_2 \|v + \theta u_0\|_{p+1}^q - A_3 \|v + \theta u_0\|_{p+1}^p
\]

\[-A_4 \|v\|_{p+1} - A_5 - A_6 I_\theta(v)\]

\[
\geq A_1 \int_\Omega |\nabla v|^2 \, dx - A_2 \|v\|_{p+1}^q - A_3 \|v\|_{p+1}^p - A_4 \|v\|_{p+1}
\]

\[-A_7 - A_6 I_\theta(v)\]

\[
\geq A_1 \int_\Omega |\nabla v|^2 \, dx - A_8 \left( \int_\Omega |\nabla v|^2 \, dx \right)^{\frac{3}{2}} - A_9 \left( \int_\Omega |\nabla v|^2 \, dx \right)^{\frac{p}{2}}
\]

\[-A_{10} \left( \int_\Omega |\nabla v|^2 \, dx \right)^{\frac{1}{2}} - A_7 - A_6 I_\theta(v)\]

\[
\geq A_{11} \int_\Omega |\nabla v|^2 \, dx - A_{12} - A_6 I_\theta(v)\]

for some \( A_i > 0 \). Thus if \( v \) is a critical point of \( I_\theta \) then there exists \( A > 0 \) such that

\[
\int_\Omega |\nabla v|^2 \, dx \leq A (I_\theta(v) + 1). \tag{4.15}
\]

Also, by (4.6), there exist \( D_i > 0 \) such that

\[
\|v + \theta u_0\|_{p+1}^p \leq D_1 \int_\Omega |\nabla v|^2 \, dx + D_2 I_\theta(v) + D_3
\]

\[
\leq D_4 (I_\theta(v) + 1) + D_2 I_\theta(v) + D_3
\]

\[
\leq D_5 (I_\theta(v) + 1). \tag{4.16}
\]

Now suppose \( v \) is a critical point of \( I_\theta \). Then \( v \) satisfies (4.3) i.e.,

\[
\begin{cases}
-\Delta u = \lambda |u|^{q-1} u + |u|^{p-1} u + \theta f & \text{on } \Omega \\
u = \theta u_0 & \text{on } \partial \Omega
\end{cases}
\]
Chapter 4. Application to a Non-homogeneous Boundary Value Problem

and so

\[
\frac{\partial}{\partial \theta} I(\theta, v) = \int_{\Omega} (-\lambda |v + \theta u_0|^{q-1}(v + \theta u_0)u_0 - |v + \theta u_0|^{p-1}(v + \theta u_0)u_0 - f v)dx
\]

\[
= \int_{\Omega} \{ (\Delta v + \theta f)u_0 - f v \}dx
\]

\[
= \left\{ \int_{\partial \Omega} (\Delta v)u_0 dx + \int_{\Omega} (\theta f u_0 - f v)dx \right\}
\]

\[
= \left\{ \int_{\partial \Omega} (-\v + \theta u_0 q - \v + \theta u_0 p - 1 \theta u_0)u_0 - 1 (v + \theta u_0)u_0 - f v)dx \right\}
\]

\[
= \left\{ \int_{\partial \Omega} u_0 \frac{\partial v}{\partial n} ds + \int_{\Omega} (\theta f u_0 - f v)dx \right\}.
\] (4.17)

Considering the second term,

\[
\left| \int_{\partial \Omega} (\theta f u_0 - f v)dx \right| \leq \int_{\Omega} |\theta f u_0|dx + \int_{\Omega} |fv|dx
\]

\[
\leq B_1 + B_2 \|v\|_{p+1}
\]

\[
\leq B_1 + B_3 \|v\|
\]

\[
\leq B_1 + B_4 (I_0(v) + 1)^{\frac{1}{2}}
\]

\[
\leq B_5 + B_6 (I_0(v))^{\frac{1}{2}}
\]

\[
\leq B_7 (I_0(v)^2 + 1)^{\frac{1}{2}}
\] (4.18)

where \( A_i > 0 \). Regarding the first term, recall by Lemma 4.4

\[
\int_{\partial \Omega} (\frac{1}{2} |\nabla u|^2 - |\frac{\partial u}{\partial n}|^2)ds \leq C \int_{\Omega} (|\nabla u|^2 + |u|^{p+1} + 1)dx
\]

where \( u = v + \theta u_0 \). So we have \( C_i > 0 \) such that

\[
\int_{\partial \Omega} (\frac{1}{2} |\nabla u|^2 - |\frac{\partial u}{\partial n}|^2)ds \leq C_0 \int_{\Omega} (|\nabla u|^2 + |u|^{p+1} + 1)dx
\]

\[
\int_{\partial \Omega} (\frac{1}{2} |\nabla u|^2 - |\frac{\partial u}{\partial n}|^2)ds \leq C_0 \int_{\Omega} (|\nabla u|^2 + |u|^{p+1} + 1)dx + \int_{\partial \Omega} \frac{1}{2} |\nabla u|^2 ds
\]

\[
\leq C_1 \int_{\Omega} |\nabla u|^2 dx + C_2 \|u\|_{p+1}^{p+1} + C_3
\]

\[
\leq C_4 (I_0(v) + 1)
\] by (4.15) and (4.16)
and hence

\[ \int_{\partial \Omega} |w_0 \frac{\partial v}{\partial n}| d\sigma \leq C_7 \left( \int_{\partial \Omega} \left| \frac{\partial v}{\partial n} \right|^2 d\sigma \right)^{\frac{1}{2}} \]

\[ \leq C_7 \left( \int_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma \right)^{\frac{1}{2}} + \theta^2 \left( \int_{\partial \Omega} \left| \frac{\partial w_0}{\partial n} \right|^2 d\sigma \right)^{\frac{1}{2}} \]

\[ \leq C_7 \left( \int_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma \right)^{\frac{1}{2}} + C_8 \]

\[ \leq C_9 \left( I_0(v) + 1 \right)^{\frac{1}{2}} + C_8 \]

\[ \leq C_{10} \left( I_0(v)^2 + 1 \right)^{\frac{1}{2}}. \quad (4.19) \]

It follows from (4.18) and (4.19) that there exists a constant \( C > 0 \) such that \( |\frac{\partial}{\partial \theta} I(\theta, v)| \leq C(I_0(v)^2 + 1)^{\frac{1}{2}}. \)

\[ \square \]

### 4.4 Existence of Multiple Solutions

**Proof.** (Theorem 4.1) We must show that if \( \theta = 1 \) and \( 1 \leq q < p < \frac{n+1}{n-1} \), then equation \( (4.3) \) has infinitely many solutions. To do this, we will apply Corollary 3.2 to \( I \).

First we must check that the hypotheses of the corollary hold. By a standard argument, \( I \) satisfies the Palais-Smale condition H1). That H2) is satisfied is proved by Lemma 4.3 and Lemma 4.5 shows that H3) holds with \( f_1(\theta, s) = -C(s^2 + 1)^{\frac{1}{2}} \) and \( f_2(\theta, s) = C(s^2 + 1)\frac{1}{2} \). (So in reference to Corollary 3.2, \( \alpha = \frac{1}{2} \).) We have already noted that \( I_0 \) is even and it is easily shown that in any finite dimensional subspace of \( E \), \( \sup_{\theta \in [0,1]} I(\theta, y) \to -\infty \) as \( \|y\| \to \infty \). Thus H5) is satisfied.

Now let \( E_k \) be the subspace of \( E \) spanned by the first \( k \) eigenfunctions of \( \Delta \). Let \( H = \{ h \in C(E, E) | h \text{ is odd and } \exists R > 0 : h(v) = v \forall \|v\| > R \} \). Set \( c_k = \inf_{h \in H} \sup_{E_k} I_0 \). Note that a result of Tanaka [Tan89] shows that these \( c_k \) are critical levels of \( I_0 \) and \( c_k \geq \gamma k^{p+\frac{1}{2}} n - \mu \) for some \( \gamma, \mu > 0 \). Thus in applying Corollary 3.2 \( \beta = \frac{p+1}{p-1} \frac{2}{n} \). We conclude that \( I_1 \) has an infinite number of solutions when \( \frac{p+1}{p-1} \frac{2}{n} = \beta \geq \frac{1}{1-\alpha} = 2 \) i.e., if \( p < \frac{n+1}{n-1} \).

\[ \square \]

**Proof.** (Theorem 4.2) As we have already checked the hypotheses in the proof above, the result follows directly from Corollary 3.2. \[ \square \]
Chapter 5

Conclusion

The main result of this thesis is Theorem 3.2, an abstraction of Bolle’s method which states conditions sufficient for multiple critical points to be preserved along a path from an even functional to neighbouring, non-symmetric functionals. This theorem in itself is a pleasing result in critical point theory but perhaps, its real merit will be found in applications. If there is symmetry in a variational problem, it can often be exploited to prove multiplicity results. But every problem which can be addressed successfully in this manner raises the issue of similar problems that do not share this symmetry. These are the potential applications of this new method.

In chapter four, Bolle’s method was applied to the non-homogeneous boundary value problem

\[
\begin{align*}
-\Delta u &= \lambda|u|^{q-1}u + |u|^{p-1}u + f & \text{on } \Omega \\
u &= u_0 & \text{on } \partial \Omega
\end{align*}
\]

and was successful in improving results previously attained using the standard perturbative method. This has been the method’s achievement thus far - reproducing and improving multiplicity results for differential equations with non-homogeneous boundary conditions. (See [Bol99] and [BGT].)

The method’s success with these higher order perturbations produced by non-trivial boundary values suggests that it may to useful in dealing more general perturbations from symmetry. For instance, consider the problem

\[
\begin{align*}
-\Delta u &= p(x, u) + f(x, u) & \text{on } \Omega \\
u &= 0 & \text{on } \partial \Omega
\end{align*}
\]

where for all \( x \in \Omega \) \( p(x, \cdot) \) is odd and uniformly superquadratic. The functional associated to this equation is symmetric except for a perturbation resulting from \( f \). As noted by Rabibowitz [Rab82], with conditions on the growth of \( p(x, \cdot) \) (depending on \( f \)), the standard method can be used to reveal an infinity of solutions. It would be interesting to see if these conditions can be relaxed by employing Bolle’s method in this problem instead.
Bibliography


