Adjunctions and Monads in Categories and 2-Categories

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A THESIS SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

(Department of Mathematics)

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

October, 2003

Abstract

The study of 2-categories extends many of the constructions within category theory itself. In particular, this thesis investigates the important categorical constructions of an adjunction and a monad within the context of a 2-category.

Chapter 1 introduces the fundamental notions of category theory, with an emphasis on presenting a wide variety of examples. It is a goal of this chapter that a reader unfamiliar with category theory is provided with sufficient background to follow subsequent chapters, as well as gain an appreciation for the power of category theory throughout mathematics.

"The slogan is: 'Adjoints arise everywhere'," writes Saunders MacLane, one of the founders of Category Theory, in the preface to his book *Categories for the Working Mathematician*. Chapter 2 begins by defining the concept of adjoint functors; the second focus of this chapter is that of a monad. The relationship between these two is then discussed in detail.

The notion of 2-categories is defined in Chapter 3. We go on to extrapolate many of the categorical structures discussed in Chapters 1 and 2 to 2-categorical structures.

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1 Categories

1.1 Introduction

The notions of a category, functor, and natural transformation were first introduced in 1942 in a joint paper by Samuel Eilenberg and Saunders MacLane on Cěch cohomology [5]. They proceeded to abstract the ideas and treat them directly in a paper entitled *General Theory of Natural Equivalences* published in 1945 [6].

Since then, Category Theory has grown rapidly. Initially studied as a language for other branches of mathematics (like topology), it has emerged and developed as an autonomous field of study.

Unlike a set which is completely defined by the elements which belong to it, category theory puts the emphasis on the "morphisms" between the elements.

1.2 Axiomatic Definition of a Category

A *category* $\mathcal{C}$ is comprised of

- a collection of *objects* $A, B, \ldots$ (sometimes called $\mathcal{C}$-objects)
- for every pair of objects A,B, a collection of *arrows* or *morphisms* $f, g, \ldots$, ($\mathcal{C}$-arrows), written $\mathcal{C}(\mathcal{A}, \mathcal{B})$ (or $Hom(A, B)$ if $\mathcal{C}$ is clear from the context)
- for every triple A,B,C of objects, a composition law:

$$\begin{array}{ccccc} Hom(A,B) & \times & Hom(B,C) & \longrightarrow & Hom(A,C) \\ f & \times & g & \longrightarrow & g \circ f \end{array}$$

which obeys the following associativity axiom:

If $f \in Hom(A,B)$, $g \in Hom(B,C)$, $h \in Hom(C,D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

- for every object A, an arrow $1_A \in Hom(A,A)$, called the *identity* on A, which obeys the following identity axiom:

  Given arrows $f \in Hom(A,B)$ and $g \in Hom(B,C)$, then $1_B \circ f = f$ and $g \circ 1_B = g$.

An arrow $f \in Hom(A,B)$ will often be represented by the notation $f : A \to B$. A is called the *domain* of f and B is called the *codomain.*

We can express the axioms above by simply saying the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, "h \circ g"] & \\ & C \arrow[r, "h"] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \arrow[d, "1_B"] \arrow[dr, "g"] & \\ & B \arrow[r, "g"] & C \end{tikzcd}$$

(associativity) (identity)

1.2.1 Examples of categories:

1. The most fundamental of all categories is *Set*, whose objects are all sets and whose arrows are all set functions.

2. There are a host of examples in which the objects are "sets" with an underlying structure and the arrows are the functions between such sets. For

example, *Grp* is the category of groups and group homomorphisms, *Top* is the category of topological spaces and continuous functions. The following chart lists the most common of such categories:

Grp	Groups and group homomorphisms
Ab	Abelian groups and group homomorphisms
Top	Topological spaces and continuous functions
Rng	Rings and ring homomorphisms (preserving units)
CRng	Commutative rings and ring homomorphisms (preserving units)
Vct_k	Vector spaces over a field k and linear maps

3. (a) **0** is the empty category with no objects and no arrows.

   (b) **1** is the unit category with one object and one arrow, namely the identity arrow of the object.

$$\begin{tikzcd} A \arrow[loop above, "1_A"] \end{tikzcd}$$

   (c) **2** is the category with two objects and three arrows: one identity arrow for each object and one non-identity arrow from one object to the other. Composition can be defined in only one way and clearly both the identity and associativity axioms are satisfied.

$$\begin{tikzcd} A \arrow[loop above, "1_A"] \arrow[r] & B \arrow[loop above, "1_B"] \end{tikzcd}$$

   (d) **3** is the category with three objects (A, B, C), an identity arrow for each object and three other arrows $f : A \to B$, $g : B \to C$ and $h : A \to C$. Again, composition can be defined in only one way ($g \circ f = h$) and both axioms are satisfied.

$$
\begin{tikzcd}
& C \arrow[loop above, "1_C"] & \\
A \arrow[ur, "h"] \arrow[rr, "f"'] \arrow[loop left, "1_A"] & & B \arrow[ul, "g"'] \arrow[loop right, "1_B"]
\end{tikzcd}
$$

4. Take all the natural numbers to be the objects of a category and define an arrow from n to m to be all the real-valued $n \times m$ matrices. The composition will be matrix multiplication; the identity arrow for an object n is the $n \times n$ identity matrix. The associativity and identity axioms follow from properties of matrix multiplication.

5. **Monoids:**

   Let $\mathbb{N}$ be the set of natural numbers. Form a category with one object $\mathbb{N}$ and arrows the natural numbers $n, m, \ldots$. Let the composition of the arrows be addition. The identity arrow will be $1_{\mathbb{N}}$ the number 0. We label this category $< \mathbb{N}, +, 0 >$.

   This is an example of a *monoid*. A monoid is a triple: $< M, *, 1_M >$ in which M is a set, $*$ is a binary operation on M (i.e., $* : M \times M \longrightarrow M$) that is associative, and 1_M is an identity element of M ($1_M * x = x = x * 1_M$ for all $x \in M$). In the manner described above, every monoid can be seen as a category with one object.

6. **Posets:**

   A partial ordering ($\leq$) on a set P is a reflexive ($p \leq p$), transitive ($p \leq q, q \leq r \Rightarrow p \leq r$), and antisymmetric ($p \leq q, q \leq p \Rightarrow p = q$) relation on the elements of P. We can form the category *Poset* whose objects are all partially ordered sets and whose arrows are continuous functions between them.

7. A partially ordered set itself $(P, \leq)$ can be considered a category. The objects are the elements of the set P. The arrows will be determined by the partial ordering. Define there to be a single arrow from p to q if $p \leq q$; otherwise, there exists no arrow between elements. The identity law is satisfied by the reflexivity of $\leq$, composition of arrows follows from the transitivity of $\leq$ and since there is at most one arrow between any two elements, the composition must be associative.

8. A set itself can be viewed as a category whose objects are the elements of the set and whose only arrows are the identities on each element. A category whose only morphisms are identities is called a *discrete* category.

9. **Comma categories:**

   Given any category $\mathcal{C}$ and an object A of $\mathcal{C}$, we can form the category $\mathcal{C} \downarrow A$ of *objects over* A. The objects of $\mathcal{C} \downarrow A$ are the arrows of $\mathcal{C}$ with codomain A and the arrows of $\mathcal{C} \downarrow A$ from $f : B \to A$ to $g : C \to A$ are the $\mathcal{C}$-arrows $k : B \to C$ such that $\begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & A & \end{tikzcd}$ commutes. (i.e., $g \circ k = f$).

   Similarly, we can define the category $\mathcal{C} \uparrow A$ of *objects under* A where the objects are arrows with domain A and the arrows from $f : A \to B$ to $g : A \to C$ are the arrows $k : B \to C$ such that $\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[rr, "k"'] & & C \end{tikzcd}$ commutes. (i.e., $k \circ f = g$). Categories of these type are called *comma categories.*

1.3 Subcategories

A category $\mathcal{B}$ is a *subcategory* of a category $\mathcal{C}$ if:

(i) each object of $\mathcal{B}$ is an object of $\mathcal{C}$;

(ii) for all $\mathcal{B}$-objects B and B', $\mathcal{B}(B,B') \subseteq \mathcal{C}(B,B')$; and

(iii) composites and identity arrows are the same in $\mathcal{B}$ as in $\mathcal{C}$

The subcategory is called *full* if for any objects B, $B' \in \mathcal{B}$, if $f : B \to B'$ is an arrow in $\mathcal{C}$, then f is an arrow in $\mathcal{B}$. In other words, $\mathcal{B}(B,B') = \mathcal{C}(B,B')$.

The category *Ab* of abelian groups is an example of a full subcategory of the category of groups *Grp*, since the arrows of *Ab* are all the group homomorphisms between abelian groups.

Special Objects and Arrows

1.4 Monomorphisms

An arrow $f : A \rightarrow B$ in a category $\mathcal{C}$ is a *monomorphism* (or is *monic*) if for any pair of $\mathcal{C}$-arrows $g, h : C \rightrightarrows A$ such that $f \circ g = f \circ h$ then $g = h$. (i.e, f is left-cancellable).

In Set, the monic arrows correspond exactly with injective functions. Given $f \in Set(A, B)$ such that f is injective and arrows $g, h \in Set(C, A)$ such that $f \circ g = f \circ h$, then we have:

$$\begin{aligned} f(g(c)) &= f(h(c)) \text{ for all elements } c \in C \\ \Rightarrow g(c) &= h(c) \text{ for all } c \in C \text{ since } f \text{ is injective} \\ \Rightarrow g &= h \end{aligned}$$

So f is monic. Conversely, let $f \in Set(A, B)$ be a monomorphism and assume $f(a) = f(\tilde{a})$ for some a, $\tilde{a} \in A$. Construct the maps:

$$\begin{aligned} g : \ \{0\} &\rightarrow A \quad \text{and} \quad h : \ \{0\} \rightarrow A \\ g(0) &= a \qquad\qquad\quad h(0) = \tilde{a} \end{aligned}$$

then

$$\begin{aligned} f(g(0)) &= f(h(0)) \\ \Rightarrow f \circ g &= f \circ h \\ \Rightarrow g &= h \quad \text{since } f \text{ is monic} \\ g(0) &= h(0) \\ \Rightarrow a &= \tilde{a} \end{aligned}$$

Therefore, f is injective. $\square$

In the monoid $< \mathbb{N}, +, 0 >$, every arrow is monic since $m + n = m + p$ certainly implies $n = p$.

1.5 Epimorphisms

An arrow $f : A \to B$ is an *epimorphism* (or is *epic*) in a category $\mathcal{C}$ if for any pair of $\mathcal{C}$-arrows $g, h : B \rightrightarrows C$ such that $g \circ f = h \circ f$ then $g = h$. (i.e., f is right-cancellable).

In *Set*, the epic arrows are the surjective functions. (The justification for this is comparable to the justification above that the injective functions are monic.) In $< \mathbb{N}, +, 0 >$, all arrows are epic since $m + n = p + n$ clearly implies $m = p$.

1.6 Isomorphisms

A $\mathcal{C}$-arrow $f : A \to B$ is an *isomorphism* or is *invertible* (or is *iso*) if there is a $\mathcal{C}$-arrow $g : B \to A$ such that

$$
\begin{aligned}
g \circ f &= 1_A \text{ and} \\
f \circ g &= 1_B
\end{aligned}
$$

In *Set*, the isomorphisms are exactly the bijective functions, which are the functions that are both injective and surjective, and hence both monic and epic. For a general category $\mathcal{C}$, it is easy to show that all isomorphisms are both monomorphisms and epimorphisms. However, the converse is not true in general. For example, in the monoid $< \mathbb{N}, +, 0 >$ all arrows are both monic and epic but the only arrow that is iso is $0 : \mathbb{N} \to \mathbb{N}$. Indeed, if the arrow m has an inverse n, then $m + n = 0$ but m and n are natural numbers so we must have $m = n = 0$.

As a special case of a monoid, any group G can be characterized as a category with one object G such that every morphism has an inverse.

1.7 Initial and Terminal Objects

An object 0 is *initial* in a category $\mathcal{C}$ if for every $\mathcal{C}$-object A there exists *exactly* one arrow from 0 to A in $\mathcal{C}$.

A category may have many initial objects but they will all be *isomorphic*. (Note: two objects A, B are isomorphic in $\mathcal{C}$ if there exists an invertible $\mathcal{C}$-arrow $f : A \to B$.) This also holds for terminal objects.

An object 1 is *terminal* in a category $\mathcal{C}$ if for every $\mathcal{C}$-object A, there exists exactly one arrow from A to 1 in $\mathcal{C}$.

In *Set*, there is only one initial object, namely the empty set, whereas every singleton is a terminal object. Specifically, given a set A, the empty function is the unique function from the empty set to A. Further, for each set A, there is exactly one function from A to a singleton set $\{x\}$, namely the map sending all elements of A to x. In *Grp*, the group with one element is both initial and terminal. The same argument as above can be used to show the group with one element is terminal. It is initial since the single element must be an identity element, and hence must be sent to the identity element of any other group by properties of group homomorphisms.

1.8 Duality

Given a category $\mathcal{C}$ we can construct another category, called $\mathcal{C}^{op}$, whose objects are the objects of $\mathcal{C}$ and whose arrows are the arrows of $\mathcal{C}$ "reversed". That is, for each $\mathcal{C}$-arrow $f : A \to B$, we form an arrow $f^{op} : B \to A$ in $\mathcal{C}^{op}$. Clearly $\mathcal{C}^{op}$ satisfies the axioms for a category. $f^{op} \circ g^{op}$ will exist precisely when $g \circ f$ exists and $f^{op} \circ g^{op} = (g \circ f)^{op}$. The identity morphisms 1_A will be the same as the identity morphisms in $\mathcal{C}$. We call $\mathcal{C}^{op}$ the *dual* or *opposite* category of $\mathcal{C}$. Its existence has far-reaching implications. Monic arrows in $\mathcal{C}$ are epic in $\mathcal{C}^{op}$; initial objects in $\mathcal{C}$ are terminal objects in $\mathcal{C}^{op}$.

More generally, given any categorical statement Σ, define its dual, Σ^{op}, as the statement obtained by replacing "domain" with "codomain" and vice-versa. If Σ is true for a category $\mathcal{C}$ then Σ^{op} will be true for $\mathcal{C}^{op}$. As well, if Σ is true for all categories (i.e., derived from the axioms of a category), then since it can be shown that any category $\mathcal{D}$ can be expressed as $\mathcal{C}^{op}$ for some category $\mathcal{C}$, Σ^{op} must also be true for all categories. This is the *Duality Principle.*

1.9 Equalizers

Given a pair of $\mathcal{C}$-arrows, $f, g : A \rightrightarrows B$, an arrow $i : E \to A$ is an *equalizer* of f and g if:

(i) $f \circ i = g \circ i$ and

(ii) Whenever $h : C \to A$ is such that $f \circ h = g \circ h$, then there exists a unique arrow $k : C \to E$ such that $i \circ k = h$ (i.e., h factors uniquely through the equalizer). That is, given:

$$\begin{tikzcd} C \arrow[d, dashed, "k"'] \arrow[dr, "h"] & & \\ E \arrow[r, "i"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

there is exactly one way to fill in the dotted arrow to make the diagram commute. An arrow will be called an equalizer in $\mathcal{C}$ if it is an equalizer for any pair of $\mathcal{C}$-arrows.

For example, in *Set*, given $f, g : A \rightrightarrows B$, let E be the subset of A on which f and g agree. That is $E = \{x \mid x \in A \text{ and } f(x) = g(x)\}$. Then the inclusion function $i : E \hookrightarrow A$ is an equalizer of f and g. In *Grp*, the equalizer of $f, g : A \rightrightarrows B$ is the kernel $\{a \in A \mid f(a) = g(a)\}$.

1.10 Co-Equalizers

The dual notion to an equalizer is that of a co-equalizer. Given a pair of arrows $f, g : B \rightrightarrows A$, a *co-equalizer* is an arrow $e : A \to E$ such that:

(i) $e \circ f = e \circ g$ and

(ii) Whenever $h : A \to C$ is such that $h \circ f = h \circ g$ then there exists a unique arrow k such that $k \circ e = h$. That is, within the diagram:

$$\begin{tikzcd} E \arrow[d, dashed, "k"'] & A \arrow[l, "e"'] \arrow[dl, "h"] & B \arrow[l, shift right, "g"'] \arrow[l, shift left, "f"] \\ C & & \end{tikzcd}$$

there is only one way to fill in the dotted arrow so that the diagram commutes.

In *Set*, the co-equalizer of $f, g : A \rightrightarrows B$ is the quotient of B by the equivalence relation generated by the pairs $(f(a), g(a))$ for all $a \in A$.

1.11 Pullbacks

Given two morphisms with common co-domain, $f : A \to C$ and $g : B \to C$, in a category $\mathcal{C}$, then a *pullback* of (f, g) is a triple (D, f', g') such that:

(i) D is an object of $\mathcal{C}$

(ii) $f' : D \to B$, $g' : D \to A$ are morphisms of $\mathcal{C}$ such that $f \circ g' = g \circ f'$ and

(iii) whenever there exists an object X and morphisms $h : X \to A$ and $j : X \to B$ such that $f \circ h = g \circ j$ then there exists a unique arrow $k : X \to D$ such that $h = g' \circ k$ and $j = f' \circ k$

$$\begin{tikzcd} X \arrow[dr, dashed, "k"] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "h"'] & & \\ & D \arrow[r, "f'"'] \arrow[d, "g'"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

i.e., given the above diagram, when h and j are such that the outer square commutes, there is only one way to fill in the broken arrow to make the whole diagram commute.

The inner square of this diagram is called a *pullback square.*

In *Set*, the pullback object of $f : A \to C$ and $g : B \to C$ is given by the set $\{< a, b > \mid a \in A, b \in B, f(a) = g(b)\}$.

1.12 Pushouts

The dual to a pullback is called a pushout. Given the morphisms $f : A \to B$ and $g : A \to D$, a *pushout* of (f, g) is comprised of the triple (C, f', g') such that

(i) C is an object of $\mathcal{C}$

(ii) $g' : B \to C$ and $f' : D \to C$ are arrows in $\mathcal{C}$ such that $g' \circ f = f' \circ g$ and

(iii) whenever there exists an object X and arrows $h : B \to X$, $j : D \to X$ such that $h \circ f = j \circ g$, then there exists a unique arrow $k : C \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \arrow[ddr, bend left, "h"] & \\
D \arrow[r, "f'"] \arrow[drr, bend right, "j"'] & C \arrow[dr, dashed, "k"] & \\
& & X
\end{tikzcd}$$

The inner square is called a *pushout square.*

In *Set*, the pushout of arrows f and g always exists and is the disjoint union of their co-domains, $B \cup D$ with the element $f(a)$ and $g(a)$ identified for each $a \in A$.

1.13 Products

Generalizing the notion of the Cartesian product of two sets, we obtain the product of two objects in a category.

A *product* in a category $\mathcal{C}$ of two objects A and B is the triple $(A \times B, p_A, p_B)$ such that

(i) $A \times B$ is a $\mathcal{C}$-object

(ii) $p_A : A \times B \to A$ and $p_B : A \times B \to B$ are $\mathcal{C}$-morphisms and

(iii) for any pair of arrows of the form $f : C \to A$ and $g : C \to B$ there is exactly one arrow $k : C \to A \times B$ for which $k \circ p_A = f$ and $k \circ p_B = g$. That is, there is only one morphism k that makes the following diagram commute:

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, dashed, "k"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}$$

Note: k is often written $< f, g >$.

1.13.1 Examples:

1. In *Set*, the product of two sets A and B is simply the Cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$ and the arrows p_A and p_B are the

projections of $A \times B$ onto A and B respectively. The unique arrow $k : C \to A \times B$ is $k(c) = (f(c), g(c))$.

2. In a poset, the product of two elements a and b is their infimum (greatest lower bound), denoted $inf(a, b)$, if it exists. Notice that $inf(a, b) \leq a$ and $inf(a, b) \leq b$ and if there exists a c such that $c \leq a$ and $c \leq b$, then by definition of infimum, $c \leq inf(a, b)$. Since there is at most one arrow between two elements in a poset (defined by $\leq$), this arrow is unique.

1.14 Co-Products

Dually, the *co-product* of two objects A, B in a category $\mathcal{C}$ is a triple $(A+B, i_A, i_B)$ such that:

(i) $A + B$ is a $\mathcal{C}$-object

(ii) $i_A : A \to A + B$ and $i_B : B \to A + B$ are morphisms and

(iii) whenever there exists a pair of morphisms $f : A \to C$, $g : B \to C$, then there is a unique arrow $k : A + B \to C$ such that

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "i_A"'] & A + B \arrow[u, dashed, "k"] & B \arrow[ul, "g"'] \arrow[l, "i_B"] \end{tikzcd}$$

is commutative.

1.14.1 Examples:

1. The co-product of two sets A, B in *Set* is the disjoint union $A \cup B$ and i_A, i_B are the inclusion maps.

2. In a poset viewed as a category, the supremum (least upper bound), $sup(a, b)$, of two elements a and b, if it exists, has the property that $a \leq sup(a, b)$, $b \leq sup(a, b)$ and given any c such that $a \leq c$ and $a \leq b$ then $sup(a, b) \leq c$. There is at most one arrow between any two element in a poset, so this arrow is unique. Hence $sup(a, b)$, when it exists, is a co-product in a poset.

Beyond Categories:
Functors and Natural Transformations

1.15 Functors

After gaining an understanding of categories, it is natural to look for a way to relate different categories. A *functor* is essentially a morphism of categories that preserves categorical structure.

Precisely, a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of:

(i) a mapping that assigns to each $\mathcal{C}$-object A, a $\mathcal{D}$-object $F(A)$.

(ii) a mapping that assigns to each $\mathcal{C}$-arrow $f : A \to B$, a $\mathcal{D}$-arrow $F(f) : F(A) \to F(B)$ such that

- $F(1_A) = 1_{F(A)}$ for all $\mathcal{C}$-objects A. (The identity arrow on A is assigned to the identity arrow on $F(A)$.)
- $F(g \circ f) = F(g) \circ F(f)$ whenever $g \circ f$ is defined in $\mathcal{C}$.

  That is, whenever $\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$ commutes in $\mathcal{C}$ then $\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)"'] & F(B) \arrow[d, "F(g)"] \\ & F(C) \end{tikzcd}$

  commutes in $\mathcal{D}$.

1.15.1 Examples of Functors:

1. The identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ has $1_{\mathcal{C}}(A) = A$ and $1_{\mathcal{C}}(f) = f$. If $\mathcal{C}$ is a subcategory of $\mathcal{D}$, then $1_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{D}$ is an inclusion functor.

2. There is a functor $U : Grp \to Set$ which takes a group A to its underlying set A and a group homomorphism f to the corresponding set function f.

U is an example of a *forgetful functor* which forgets the structure of *Grp* objects. There is such a functor from $\mathcal{C} \to Set$ for any category $\mathcal{C}$ whose objects are sets with additional structure (eg, *Top*, *Rng*, *Ab*, etc.).

3. A forgetful functor need not discard all the structure of the objects in its domain. $U : Rng \to Ab$ is a forgetful functor assigning each ring R to its additive abelian group and each ring homomorphism $f : R \to S$ to the same function regarded simply as an additive homomorphism.

4. Let $F : Set \to Grp$ be the map which brings a set A to the free group generated by A (the group of finite sequences of elements of A and A^{-1} formal inverses of elements of A where composition is concatenation and cancellation). If $f : A \to B$ is a set function, there is a unique group homomorphism $F(f) : F(A) \to F(B)$ obtained by applying f to each element of a finite sequence in $A \cup A^{-1}$ such that $F(f) \circ i = j \circ f$ where $i : A \to F(A)$ and $j : B \to F(B)$ and $f(a^{-1}) = (f(a))^{-1}$ for $a \in A$ are the inclusion maps. F is a functor called the *free group functor*.

5. The power set functor $P : Set \to Set$ maps each set A to its power set $P(A)$ and each arrow $f : A \to B$ to the arrow $P(f) : P(A) \to P(B)$ that assigns each $A_0 \subseteq A$ to its image under f, $f(A_0) \subseteq B$.

6. (a) Any functor $F : \mathbf{1} \to \mathcal{C}$ picks out an object of $\mathcal{C}$, as F sends the object of $\mathbf{1}$ to some $C \in \mathcal{C}$, and its identity arrow to 1_C, the identity arrow of the object C.

   (b) A functor $G : \mathbf{2} \to \mathcal{C}$ picks out an arrow of $\mathcal{C}$, as $\mathbf{2}$ contains a single non-identity arrow that is sent to a non-identity arrow in $\mathcal{C}$.

7. There is a functor $H_n : Top \to Ab$ for each $n \in \mathbb{Z}^+$ which maps a topological space X to its n^{th} homology group, $H_n(X)$. Notice that continuous maps (arrows in Top) induce homomorphisms (arrows in Ab) between homology groups.

8. The functor $F : Rng \to Grp$ maps a ring to its group of units. (A ring homomorphism maps units to units and induces a group homomorphism on the group of units.)

9. $GL_n : Rng \to Grp$ maps a ring with identity to the group $GL_n(R)$ of invertible $n \times n$ matrices over R.

10. If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets, then a functor between them is a non-decreasing set function.

11. Given a category $\mathcal{C}$ and fixed object $A \in \mathcal{C}$, define a functor:

$$\mathcal{C}(A, -) : \mathcal{C} \to Set$$

from $\mathcal{C}$ to the category of sets by setting

$$\mathcal{C}(A, -)(B) = \mathcal{C}(A, B) \quad \text{for all objects } B \in \mathcal{C}$$

Given the morphism $f : B \to C$ in $\mathcal{C}$, define the mapping $\mathcal{C}(A, -)(f)$, written $\mathcal{C}(A, f)$ from $\mathcal{C}(A, B) \to \mathcal{C}(A, C)$ by

$$\mathcal{C}(A, f)(g) = f \circ g$$

for an arrow $g \in \mathcal{C}(A, B)$. This functor is called a *hom-functor* or a *representable functor.*

1.16 Contravariant Functors

There is a natural mapping $\tilde{I} : \mathcal{C} \to \mathcal{C}^{op}$ for which $\tilde{I}(A) = A$ and $\tilde{I}(f) = f^{op}$. However this mapping contradicts the composition axiom for functors since $\tilde{I}(g \circ f) = (g \circ f)^{op} = f^{op} \circ g^{op} = \tilde{I}(f) \circ \tilde{I}(g)$. Motivated by this, it is useful to define *contravariant functors* which "reverse" the direction of arrows (i.e., the domain of an arrow is assigned to the codomain of the image arrow and vice-versa). Formally, a contravariant functor consists of:

(i) a mapping that assigns to each $\mathcal{C}$-object A, a $\mathcal{D}$-object $F(A)$.

(ii) a mapping that assigns to each $\mathcal{C}$-arrow $f : A \to B$, a $\mathcal{D}$-arrow $F(f) : F(B) \to F(A)$ such that

- $F(1_A) = 1_{F(A)}$ for all $\mathcal{C}$-objects A. (The identity arrow on A is assigned to the identity arrow on $F(A)$.)
- $F(g \circ f) = F(f) \circ F(g)$ whenever $g \circ f$ is defined in $\mathcal{C}$.

1.16.1 Examples:

1. The mapping $\tilde{I} : \mathcal{C} \to \mathcal{C}^{op}$ defined above is a contravariant functor.
2. The contravariant power set functor $\bar{P} : Set \to Set$ takes each set A to its power set $P(A)$ and each set function $f : A \to B$ to $\bar{P}(f) : P(B) \to P(A)$ that takes each $B_0 \subseteq B$ to its inverse image $f^{-1}(B_0) \subseteq A$.
3. Given the posets $(P, \leq_P)$ and $(Q, \leq_Q)$, a contravariant functor between them is an order reversing set function.

1.17 Limits

Initial objects, equalizers, pullbacks, and products are all examples of a more general construction called a *limit.*

Before defining a limit, it is useful to have the following definition: given a functor $F : \mathcal{D} \to \mathcal{C}$, a *cone* on F consists of:

- an object $C \in \mathcal{C}$
- for every object $D \in \mathcal{D}$, a morphism $f_D : C \to F(D)$ in $\mathcal{C}$ in such a way that for every morphism $d : D \to D'$ in $\mathcal{D}$, $f_{D'} = F(d) \circ f_D$.

Such a cone is denoted $(C, (f_D)_{D \in \mathcal{D}})$.

A *limit* of a functor $F : \mathcal{D} \to \mathcal{C}$ is a cone $(L, (f_D)_{D \in \mathcal{D}})$ on F such that for every cone $(M, (g_D)_{D \in \mathcal{D}})$ on F there is a unique morphism $m : M \to L$ for which every object $D \in \mathcal{D}$, $g_D = f_D \circ m$.

The cones for any diagram formed from a category (collection of objects as vertices and arrows as edges) form a category themselves. A limit in this category is a terminal object. It follows that since terminal objects are unique up to isomorphism, then so are limits.

1.18 Co-Limits

Examples of co-limits include terminal objects, co-equalizers, pushouts and co-products. To define a co-limit, first define a *co-cone* on a functor $F : \mathcal{D} \to \mathcal{C}$

as:

- an object $C \in \mathcal{C}$
- for every object $D \in \mathcal{D}$, a morphism $f_D : F(D) \to C$ in $\mathcal{C}$ such that for every morphism $d : D' \to D$ in $\mathcal{D}$, $f_{D'} = f_D \circ F(d)$.

Denote this co-cone by $(C, (f_D)_{D \in \mathcal{D}})$. These are distinguished from cones by the context.

A *co-limit* of a functor $F : \mathcal{D} \to \mathcal{C}$ is a co-cone $(L, (f_D)_{D \in \mathcal{D}})$ on F such that for every co-cone $(M, (g_D)_{D \in \mathcal{D}})$ on F, there exists a unique morphism $m : L \to M$ such that for every object $D \in \mathcal{D}$, $g_D = m \circ f_D$.

A category $\mathcal{C}$ is called *complete* when all functors of the form $F : \mathcal{D} \to \mathcal{C}$ have a limit. The category $\mathcal{C}$ is called *finitely complete* when all such functors with finite domain $\mathcal{D}$ have a limit.

For example, if $F : \mathcal{D} \to \mathit{Set}$ is a functor then the set

$$L = \{(x_D)_{D \in \mathcal{D}} \mid x_D \in F(D); \forall f : D \to D' \text{ in } \mathcal{D}, (F(f))x_D = x_{D'}\}$$

provided with the projections $p_D : L \to F(D)$ is the limit of F. Therefore, *Set* is a complete category.

1.19 Natural Transformations

In keeping with category theory's emphasis on mappings, we will now examine structure-preserving morphisms between functors called natural transformations. In fact, Eilenberg and MacLane's initial motivation to define categories and functors was in order to study natural transformations.

Given two functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$, a *natural transformation* $\eta : F \dot{\to} G$ from F to G is a collection of morphisms $\eta_A : F(A) \to G(A)$ of $\mathcal{D}$, indexed by the objects of $\mathcal{C}$, such that for each $f : A \to B$ in $\mathcal{C}$, $\eta_B \circ F(f) = G(f) \circ \eta_A$. (The arrows η_A are called the components of η.) i.e., given the following arrow in $\mathcal{C}$ the given square in $\mathcal{D}$ commutes:

$$
\begin{tikzcd}
A \arrow[d, "f"] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}
$$

If each component η_A of a natural transformation $\eta : F \dot{\to} G$ is invertible in $\mathcal{D}$, then η is called a *natural isomorphism* or a *natural equivalence*, written $\eta : F \cong G$. The inverses η_A^{-1} in $\mathcal{D}$ also form a natural isomorphism $\eta^{-1} : T \dot{\to} S$.

1.19.1 Examples:

1. There is an identity natural transformation from a functor F to itself.

2. Recall the categories $CRng$ and Grp of commutative rings with identity and groups, respectively. Recall also the functors $F : CRng \to Grp$ which maps a ring to its group of units, and $GL_n : CRng \to Grp$ which maps a ring R to the group of $n \times n$ invertible matrices over R. We claim that the

determinant $\det : GL_n \dot{\rightarrow} F$ is a natural transformation.

$$\begin{tikzcd}
GL_n(R) \arrow[r, "\det_R"] \arrow[d, "GL_n(f)"'] & F(R) \arrow[d, "F(f)"] \\
GL_n(S) \arrow[r, "\det_S"] & F(S)
\end{tikzcd}$$

For any ring R, det is a group homomorphism from $GL_n(R)$ to $F(R)$ since $\det(AB) = (\det A)(\det B)$. Further, let $f : R \rightarrow S$ be an arrow of $CRng$. Since f is a ring homomorphism, $F(f)(\det_R(A)) = \det_S(GL_n(f)(A))$. Hence det is indeed natural.

3. Recall the power set functor $P : Set \rightarrow Set$ and the identity functor $1_{Set} : Set \rightarrow Set$ defined previously. If $A \in Set$, there is a map

$$\begin{aligned} \sigma_A : A &\rightarrow P(A) \\ x &\mapsto \{x\} \end{aligned}$$

which takes x, an element of A to the singleton set $\{x\}$. σ is a natural transformation $1_{Set} \dot{\rightarrow} P$.

4. Given a category $\mathcal{C}$ and a morphism $f : A \rightarrow B$ of $\mathcal{C}$, there is a natural transformation

$$\mathcal{C}(f, -) : \mathcal{C}(B, -) \dot{\rightarrow} \mathcal{C}(A, -)$$

where $\mathcal{C}(B, -)$ and $\mathcal{C}(A, -)$ are hom-functors (representable functors). This natural transformation is defined by putting, for every object $C \in \mathcal{C}$, and every morphism $g : B \rightarrow C$,

$$\mathcal{C}(f, -)(g) = g \circ f$$

An Application of Category Theory to Topology

1.20 Classifying Spaces

Previously, we described how any monoid can be viewed as a category with one object. Conversely, any category $\mathcal{C}$ with a single object A can be realized as a monoid $(M, *, 1_A)$ where M is the collection of $\mathcal{C}$-arrows, $*$ is the composition of arrows, and 1_A is the identity arrow on the object A.

More generally, every category $\mathcal{C}$ can be realized as a *semi-simplicial* set. Further, any semi-simplicial set has a realization as a topological space. A formal definition of a semi-simplicial set can be found in Godement [8]. Informally, it is a sequence of sets $A_0, A_1, A_2, \ldots$, together with certain boundary and degeneracy maps.

Associated to any category $\mathcal{C}$, it is possible to consider the *nerve* of $\mathcal{C}$, denoted $N_{\mathcal{C}}$. For any poset P, $N_{\mathcal{C}}(P)$ is defined to be the set of functors from the category P to $\mathcal{C}$. Specifying P to be the ordered sets $\{0, 1, 2, \ldots, n\}$ now gives a semi-simplicial set A [8]. This semi-simplicial set can be realized, via well-known procedures, as a topological space $\Delta(A)$, called the *classifying space* of $\mathcal{C}$ [16], otherwise denoted by $B\mathcal{C}$.

In this way, an abstract notion, i.e., a category, leads back to a structure that is much more concrete, namely a topological space.

2 Adjunctions and Monads

2.1 Adjoint Functors

The idea of adjointness was developed by Daniel Kan in 1958 [11]. It is considered one of the most important contributions of category theory to other branches of mathematics.

2.1.1 Definition:

Given two categories $\mathcal{C}$ and $\mathcal{D}$, and a pair of functors between them, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, adjointness occurs when there is a bijective correspondence between $\mathcal{D}$-arrows from $F(C)$ to D and $\mathcal{C}$-arrows from C to $G(D)$.

We can define a function η which assigns to each pair of objects $C \in \mathcal{C}$, $D \in \mathcal{D}$, a bijection of hom-sets:

$$\eta = \eta_{C,D} : \mathcal{C}(C, G(D)) \cong \mathcal{D}(F(C), D)$$

The functions $\eta_{C,D}$ form the components of a natural transformation $\eta : F \dot{\to} G$. In this case, F is called *left adjoint* to G, written $F \dashv G$, while G is said to be *right adjoint* to F, which is written $G \vdash F$.

2.1.2 An Alternative Definition:

An adjunction consists of a pair of categories $\mathcal{C}$, $\mathcal{D}$, and a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation: $\eta : I_{\mathcal{C}} \dot{\to} (G \circ F)$ such that for each $\mathcal{C}$-object A and $\mathcal{C}$-arrow $f : A \to G(B)$, where B is a $\mathcal{D}$-object, there is a unique $\mathcal{D}$-arrow $f^{\#} : F(A) \to B$ so that the following triangle in $\mathcal{C}$

commutes: $\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & G(F(A)) \arrow[d, dashed, "G(f^{\#})"] \\ & G(B) \end{tikzcd}$

η is called the *unit* of adjunction.

Equivalently, associated with every adjunction there is another natural transformation $\varepsilon : (F \circ G) \dot{\to} I_{\mathcal{D}}$ called the *co-unit* of adjunction. ε has the property that for each $\mathcal{D}$-arrow $g : F(A) \to B$ there is a unique $\mathcal{C}$-arrow $g^* : A \to G(B)$ for which the following diagram in $\mathcal{D}$ commutes: $\begin{tikzcd} F(G(B)) \arrow[r, "\varepsilon_B"] & B \\ F(A) \arrow[u, dashed, "F(g^*)"] \arrow[ur, "g"'] & \end{tikzcd}$

It can be shown that the existence of a unit implies the existence of a co-unit and vice-versa (see Barr and Wells [2]). Furthermore, either one implies the existence of an adjunction.

2.1.3 Examples:

1. Consider the categories *Set* and *Grp* and the functors $F : Set \to Grp$ and $U : Grp \to Set$ where F is the free group functor and U is the forgetful functor introduced earlier. Then F and U are an adjoint pair $(F \dashv U)$. i.e.,

$$Set(A, U(G)) \cong Grp(F(A), G)$$

where $A \in Set$ and $G \in Grp$. This is equivalent to saying for any set map from A to $U(G)$ (the underlying set of the group G), there is a unique group homomorphism from $F(A)$, the free group with basis A, to G which extends the given set map. Namely, given $f : A \to U(G)$, define $g : F(A) \to G$ to be $g(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) = f(a_1)^{\varepsilon_1} \cdot \ldots \cdot f(a_n)^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$ and $\cdot$ denotes the group operation of G. The empty sequence is mapped to the identity element of G.

2. Let A be a set. Let $P(A)$ be the power set of A. Observe that $P(A)$ can be viewed as a category when inclusion is used to determine an ordering of the subsets (in this way, $P(A)$ can be considered a poset). If $f : A \to B$ is a function between sets, then there is a direct image functor we will also call $f : P(A) \to P(B)$ assigning each $A_0 \subseteq A$ to its image under f, $f(A_0) \subseteq B$. There is also a functor $f^{-1} : P(B) \to P(A)$ sending each B_0 to its inverse image $f^{-1}(B_0) \subseteq A$. Clearly f is left adjoint to the inverse image functor f^{-1}.

3. An initial object 0 in a category $\mathcal{C}$ arises as the image of the unique object of the category $\mathbf{1}$ under left adjoint to the constant functor $\mathcal{C} \to \mathbf{1}$. The unit of adjunction must be the identity natural transformation $\eta : I_1 \dot{\to} I_1$, whereas ε picks out the unique $\mathcal{C}$-arrow from the initial object 0 to each $\mathcal{C}$-object. $\mathcal{C} \to \mathbf{1}$ also has a right adjoint, namely a terminal object, call it t of $\mathcal{C}$. The unit of adjunction is the unique arrow from each $\mathcal{C}$-object to the terminal object where the co-unit ε is the the identity.

4. Let $Int = (\mathbb{Z}, \leq)$ and $Real = (\mathbb{R}, \leq)$ be the integers and the reals with the usual ordering which can both be considered categories as they are examples of posets. Define $I : Int \hookrightarrow Real$ to be the inclusion map which is clearly a functor. Define $F : Real \to Int$ to be the ceiling function, denoted $F(r) = \lceil r \rceil$, bringing each object $r \in \mathbb{R}$ to the smallest integer greater than or equal to r. F is also a functor since if $r \leq r'$ then $\lceil r \rceil \leq \lceil r' \rceil$ (recall $\leq$ represents an arrow). Further, F is left adjoint to I. Observe that

$r \leq I(\lceil r \rceil)$ for each r. (Adjunctions between partial orders are also known as *Galois connections.*)

One of the useful properties of adjoints is the fact that left adjoints preserve co-limits (i.e., left adjoints map colimiting cones in the source category to colimiting cones in the target category)and dually, right adjoints preserve limits. See MacLane [14] for a proof.

2.2 Monads

Closely related to the theory of adjoint functors is the categorical structure called a *monad* or a *triple*. The first appearance of monads was in 1958 by Godement [8] who used them for computing sheaf cohomology. Throughout the 1960's work by various mathematicians began to slowly reveal the significance and power of monads within category theory.

2.2.1 Definition:

A *monad* or a *triple* in a category $\mathcal{C}$, denoted by $T =< T, \eta, \mu >$, consists of

- a functor $T : \mathcal{C} \to \mathcal{C}$
- two natural transformations $\eta : I_{\mathcal{C}} \dot{\to} T$ and $\mu : T \circ T \dot{\to} T$ which make the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "="'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "="] \\ & T & \end{tikzcd}$$

(associativity law) (left and right identity law)

In the diagram above, T^n is used to mean T iterated n times (i.e., $T \circ T \circ \ldots \circ T$). Also, we have dropped symbols for operations when the operation is obvious. η is called the *unit* (*identity*) element of the monad, and μ is called the *multiplication*.

2.2.2 Examples:

1. Let M be a monoid. Define the functor $T : Set \to Set$ by $T(A) = M \times A$. Let $\eta_A : A \to M \times A$ take $a \in A$ to $(1_M, a)$ and let $\mu_A : M \times (M \times A) \to M \times A$ take (m, n, a) where $m, n \in M$ to (mn, a). The associative and identity laws follow from those of the monoid. Then (T, η, μ) is a monad in Set.

2. Let $T : Set \to Set$ take a set X to the underlying set of the free group generated by X. Thus $T(X)$ is the set of words made up of symbols x and x^{-1} for all $x \in X$ plus the empty word []. To simplify, we will actually take $T(X)$ to be the equivalence classes of words where any word containing xx^{-1} or $x^{-1}x$ is equivalent to the word obtained by deleting those elements. Denote the equivalence class of a word w by $[w]$. We get that η_X takes each $x \in X$ to $[x]$ and $\mu_X : T(T(X)) \to T(X)$ gives the concatenation of any two words generated by X. It follows that (T, η, μ) is a monad in Set.

3. Let $(P, \leq)$ be a poset and hence a category. A functor $T : P \to P$ is a non-decreasing function. If there is a monad in P, then there are natural transformations $\eta : I_P \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ such that

$$x \leq T(x) \quad \text{and} \quad T(T(x)) \leq T(x)$$

for all $x \in P$. The diagrams for a monad necessarily commute since there is at most one arrow between objects. Notice that $x \leq T(x)$ implies that

$T(x) \leq T(T(x))$, so together the conditions above imply $T(T(x)) = T(x)$. Thus a monad T in a partial order P is a *closure* operation t in P. That is, $t : P \to P$ is a monotonic function with $x \leq t(x)$ and $t(t(x)) = t(x)$ for all $x \in P$.

4. Recall the power set functor $P : Set \to Set$ (see Section 1.17, Example 5) and the natural transformation $\sigma : 1_{Set} \dot{\to} P$ taking an element to its singleton set (See Section 1.19, Example 3). Let $\mu : P^2 \dot{\to} P$ bring a set of subsets to its union. Then $< P, \eta, \mu >$ with $\eta = \sigma$ is a monad on Set.

2.3 Co-Monads

Dually, a *co-monad* L in a category $\mathcal{C}$ is a monad in the category $\mathcal{C}^{op}$. $\mathrm{L} =< L, \varepsilon, \delta >$ where $L : \mathcal{C} \to \mathcal{C}$ is a functor, $\varepsilon : L \dot{\to} I_{\mathcal{C}}$ and $\delta : L \dot{\to} L \circ L$ are natural transformations satisfying the commutative diagrams:

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"'] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[d, "\delta"] & \\ I \circ L \arrow[ur, "="] & L^2 \arrow[l, "\varepsilon L"] \arrow[r, "L\varepsilon"'] & L \circ I \arrow[ul, "="'] \end{tikzcd}$$

2.3.1 Example:

Let M be a monoid and let L be the representable functor $L = Hom(M, -) : Set \to Set$. If X is a set, let $f : M \to X$ be an arrow in Set, and define $\varepsilon : L \dot{\to} I_{Set}$ by $\varepsilon(X(f)) = f(1)$, and $\delta : L \dot{\to} L^2$ by $[\delta(X(f))][(m)(n)] = f(mn)$ for $m, n \in M$. Then (L, ε, δ) is a co-monad in Set^{op}.

2.4 Adjunction and Monads

Monads are closely related to adjunctions. In 1961, P. Huber [10] proposed and proved the following theorem:

2.4.1 Theorem:

Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors $F \dashv G$, where $\eta : I_{\mathcal{C}} \dot{\to} G \circ F$ and $\varepsilon : F \circ G \to I_{\mathcal{D}}$ are the unit and co-unit of adjunction, then $T =< GF, \eta, G\varepsilon F >$ is a monad on $\mathcal{C}$.

Proof:

$G \circ F = T$ is indeed a functor $\mathcal{C} \to \mathcal{C}$. The unit diagram of the monad is:

$$\begin{tikzcd} I_{\mathcal{C}}GF \arrow[r, "\eta GF"] \arrow[dr, "="'] & GFGF \arrow[d, "\varepsilon"] & GFI_{\mathcal{C}} \arrow[l, "GF\eta"'] \arrow[dl, "="] \\ & GF & \end{tikzcd}$$

which follow from the identity natural transformations $I_G : G\varepsilon \cdot \eta G : G \dot{\to} G$ and $I_F : \varepsilon F \cdot F\eta : F \dot{\to} F$ of the adjunction.

The associative law of the monad reduces to the diagram:

$$\begin{tikzcd} GFGFGF \arrow[r, "GFG\varepsilon F"] \arrow[d, "G\varepsilon FGF"'] & GFGF \arrow[d, "G\varepsilon F"] \\ GFGF \arrow[r, "G\varepsilon F"'] & GF \end{tikzcd}$$

This diagram is simply G applied to the following diagram, evaluated at F:

$$\begin{tikzcd} FGFG \arrow[r, "FG\varepsilon"] \arrow[d, "\varepsilon FG"'] & FG \arrow[d, "\varepsilon"] \\ FG \arrow[r, "\varepsilon"'] & I_{\mathcal{D}} \end{tikzcd}$$

The commutativity of this diagram follows from the naturality of $\varepsilon : FG \dot{\to} I_{\mathcal{D}}$ which implies the horizontal composite $\varepsilon(FG\varepsilon) = \varepsilon(\varepsilon FG)$.

Thus $< GF, \eta, G\varepsilon F >$ is a monad, generated by the adjunction $F \dashv G$, with unit η and co-unit ε. $\square$

For example, the free group monad defined in Section 1.2.2, Example 2, arises from the adjunction of the forgetful functor $U : Grp \to Set$ and the free group functor $F : Set \to Grp$, described in Section 1.1.3, Example 1.

After Huber showed that every adjunction gives rise to a monad, it was a natural question to ask the converse: can every monad be defined by an adjunction? In 1965, both Kleisli and the pair Eilenberg and Moore showed independently that this is indeed the case.

2.4.2 Theorem:

Let $T = < T, \eta, \mu >$ be a monad on $\mathcal{C}$. Then there is a category $\mathcal{D}$ and an adjoint pair $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that $T = G \circ F$, $\eta : I_{\mathcal{C}} \dot{\to} G \circ F$ is the unit of the adjunction and $\mu = G\varepsilon F$ where ε is the co-unit of the adjunction.

We will now state the two very distinct constructions of Kleisli and of Eilenberg-Moore that can be used to prove this theorem.

Proof 1: (Kleisli [13])

First observe that if a category $\mathcal{D}'$ and an adjoint pair $F : \mathcal{C} \to \mathcal{D}'$, $G : \mathcal{D}' \to \mathcal{C}$ exists and $T = G \circ F$, then the full subcategory $\mathcal{D}$ of $\mathcal{D}'$ of objects of the form

$F(A)$ for $A \in \mathcal{C}$ must, by definition, have the property that

$$\mathcal{D}(F(A), F(B)) \cong \mathcal{C}(A, G(F(B))) = \mathcal{C}(A, T(B))$$

This observation will allow us to define $\mathcal{D}$ in terms of the information given from $\mathcal{C}$ and T.

Define the category $\mathcal{D}$ as follows:

- the objects of $\mathcal{D}$ will be the objects of $\mathcal{C}$
- the hom-sets of $\mathcal{D}$ will be $\mathcal{D}(A, B) = \mathcal{C}(A, T(B))$
- if $f : A \to T(B) \in \mathcal{D}(A, B)$ and $g : B \to T(C) \in \mathcal{D}(B, C)$, then let $g \circ f \in \mathcal{D}(A, C)$ be the composite:

$$A \xrightarrow{f} T(B) \xrightarrow{Tg} T^2(C) \xrightarrow{\mu_C} T(C)$$

- the identity arrow on an object A is η_A.

The associativity and identity laws of monads and the naturality of η make this construction $\mathcal{D}$ a category.

Define the functor $G_K : \mathcal{D} \to \mathcal{C}$ by $G_K(A) = T(A)$. If $f : A \to B \in \mathcal{C}(A, B)$ then $G_K(f)$ is defined by the composite:

$$T(A) \xrightarrow{Tf} T^2(B) \xrightarrow{\mu_B} T(B)$$

The functor F_K is defined by $F_K(A) = A$. If $f : A \to B$ is an arrow in $\mathcal{C}$ then $F_K(f)$ is:

$$A \xrightarrow{\eta_A} T(A) \xrightarrow{Tf} T(B) \quad \text{which is equal to} \quad A \xrightarrow{f} B \xrightarrow{\eta_B} T(B)$$

Finally the required equivalence $\mathcal{C}(A, G_K(B)) \cong \mathcal{D}(F_K(A), B)$ is the same as

$$\mathcal{C}(A, T(B)) \cong \mathcal{D}(A, B)$$

which is true by definition. □

This is called the *Kleisli category* and is denoted $\mathcal{K}(T)$.

Proof 2: (Eilenberg-Moore [7])

In order to build Eilenberg-Moore's category, we need to define a *T-algebra* in a category $\mathcal{C}$.

Definition: A *T-algebra* is a pair (A, a) where A is an object of $\mathcal{C}$ and $a : T(A) \to A$ is an arrow of $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & T(A) \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2(A) \arrow[r, "T(a)"] \arrow[d, "\mu_A"'] & T(A) \arrow[d, "a"] \\ T(A) \arrow[r, "a"'] & A \end{tikzcd}$$

The arrow a is called the *structure map* of the algebra. A map between T-algebras $f : (A, a) \to (B, b)$ is a map $f : A \to B$ of $\mathcal{C}$ for which
$$\begin{tikzcd} T(A) \arrow[r, "T(f)"] \arrow[d, "a"'] & T(B) \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$
commutes. Together, these form a category of T-algebras and T-algebra maps, denoted $\mathcal{C}^T$.

Define the forgetful functor $G^T : \mathcal{C}^T \to \mathcal{C}$ on the T-algebra $\mathcal{C}^T$ by $G^T(A, a) = A$

and $G^T(f) = f$. Define the functor $F^T : \mathcal{C} \to \mathcal{C}^T$ by $F^T(A) = (T(A), \mu_A : T(T(A)) \to T(A))$ and $F^T(f) = T(f)$. Notice that for each $A \in \mathcal{C}$, $F^T(A)$ is also a T-algebra (the commutativity requirements follow from those of a monad). This is called the *free T-algebra.* We have $G^T F^T(A) = G^T(T(A), \mu_A) = T(A)$, and the unit η of the monad is a natural transformation $\eta = \eta^T : I_{\mathcal{C}} \dot{\to} G^T F^T$. Conversely, $F^T G^T(A, a) = (T(A), \mu_A)$. Applying the structure map $a : T(A) \to A$ to this, we get $(T(A), \mu_A) \to (A, a)$ by the associativity of a T-algebra (described in the second commutative diagram in the definition above). The family of arrows $\varepsilon^T_{(A,a)} = a : F^T G^T(A, a) \to (A, a)$ is natural by the definition of a morphism of T-algebras. Thus $\eta^T : I_{\mathcal{C}} \dot{\to} G^T F^T$ and $\varepsilon^T : F^T G^T \dot{\to} I_{\mathcal{C}^T}$ define an adjunction. □

For example, recall that a closure operation T on a poset P is a monad in P. A T-algebra in P is an element x with $T(x) \leq x$ (the structure map). Since $x \leq T(x)$ for all x, we get $x \leq T(x) \leq x$ implying $x = T(x)$. So a T-algebra in P is simply an element in the poset which is *closed.*

2.4.3 Relating $K(T)$ and $\mathcal{C}^T$

Proposition: *The Kleisli category $K(T)$ is embedded in the Eilenberg-Moore category (the category of T-algebras), $\mathcal{C}^T$) as the full subcategory generated by the image of F.*

Proof:

The embedding is $\Phi : K(T) \to \mathcal{C}^T$ where $\Phi(A) = (T(A), \mu_A)$ and $\Phi(f)$, where $f : A \to T(B)$, is the composite:

$$T(A) \xrightarrow{T(f)} T^2(B) \xrightarrow{\mu_B} T(B)$$ □

In fact, the relationship runs deeper than this. There is a category $\mathcal{B}$ in which an object is a category $\mathcal{D}$, together with an adjoint pair $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ ($F \dashv G$) which induces the monad $T =< GF, \eta, G\varepsilon F >$ where η and ε are, as usual, the unit and co-unit of adjunction. An arrow in $\mathcal{B}$ from an object $(\mathcal{D}, F, G)$ to an object $(\mathcal{D}', F', G')$ is a functor $H : \mathcal{D} \to \mathcal{D}'$ for which $G' \circ H = G$ and $H \circ F = F'$.

Theorem: *$K(T)$ is an initial object and $\mathcal{C}^T$ is a terminal object in the category $\mathcal{B}$ defined above.*

Proof: $\mathcal{C}^T$ is terminal in $\mathcal{B}$ (We will omit the proof that $K(T)$ is initial.) $(\mathcal{C}^T, F^T, G^T)$ is an object in $\mathcal{B}$. There is a functor $\Phi : \mathcal{D} \to \mathcal{C}^T$ which takes an object $D \in \mathcal{D}$ to $(G(D), G\varepsilon_D)$ and an arrow $f \in \mathcal{D}$ to $G(f)$. Φ is called the *Eilenberg-Moore comparison functor.* Observe that

$$G^T\Phi(D) = G^T(G(D), G\varepsilon_D) = G(D) \text{ and}$$

$$G^T\Phi(f) = G^T(G(f)) = G(f)$$

verifying that $G^T\Phi = G$. Further, since

$$\Phi(F(C)) = (GF(C), G\varepsilon_{F(C)}) = (T(C), \mu_C) = F^T(C) \text{ and}$$

$$\Phi(F(f)) = GF(f) = T(f) = F^T(f)$$

we have that $\Phi F = F^T$, as required.

It is also necessary to show that Φ is the only such function. Assume it is not, and let $H : \mathcal{D} \to \mathcal{C}^T$ be another functor satisfying the commutativity conditions. Notice that $H(D)$ is a T-algebra. By the requirement $G^T H = G$ and the fact that G^T is simply a forgetful functor on a T-algebra, we must have $H(f) = G(f)$ and the object of the T-algebra $H(D)$ must be $G(D)$. Thus $H(D) = (G(D), h)$ where h is the structure map for the T-algebra. It remains to determine h. We will use the fact that $H\varepsilon = \varepsilon^T H$. Apply both the right and left side of this equation to the object D. On the left, we get $H\varepsilon_D = G\varepsilon_D$ since H is simply G on arrows. On the right, $\varepsilon^T H(D) = \varepsilon^T(G(D), h) = h$ by definition of ε^T. Thus $h = G\varepsilon_D$ and so in fact the functor H is the functor Φ. Thus Φ is unique and $\mathcal{C}^T$ is terminal in the category of categories and adjoint pairs that induce the monad T. $\square$

3 2-Categories

Before defining a 2-category, let's introduce several new examples of 1-dimensional categories. Taking all categories as objects and functors as arrows, form an important category called Cat. Going further, take functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ as objects and natural transformations between them as arrows and this will also satisfy the axioms for a category, denoted $Funct(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$. At times, it may be convenient to combine these categories to form a single structure.

Categories consist of a collection of objects and a collection of morphisms connecting the objects. In some cases, the morphisms themselves can be connected by morphisms. This adds another dimension to the idea of a category, leading to the general notion of a 2-category (or a 2-dimensional category).

3.1 Definition

A *2-category* K consists of:

- a collection of objects A,B,C,... called "0-cells"
- for each pair of objects A,B, a collection of morphisms $f : A \to B$, called "1-cells", such that taken together with the objects, they form a category K_0 (sometimes called the underlying category of K)
- for each pair of morphisms $f, g : A \rightrightarrows B$, a collection of morphisms $\alpha : f \Rightarrow g$, called "2-cells" and represented pictorially as: $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$

- two composition laws on the 2-cells:

  Vertical composition:

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & B \end{tikzcd}$$

  Horizontal composition:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "u\circ f"] \arrow[r, bend right, "v\circ g"'] \arrow[r, phantom, "\Downarrow\gamma*\alpha"] & C \end{tikzcd}$$

  such that the 1-cells $f, g, \ldots$ together with the 2-cells $\alpha, \beta, \ldots$ form a category under both the operation of vertical composition and horizontal composition with identities: $\begin{tikzcd} A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "1_A"'] \arrow[r, phantom, "\Downarrow 1_{1_A}"] & A \end{tikzcd}$

- Finally, we require that horizontal and vertical composition commute: i.e,

  given $\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \arrow[r, bend left=60, "u"] \arrow[r, "v" description] \arrow[r, bend right=60, "w"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & C \end{tikzcd}$ we have $(\delta \cdot \gamma) * (\beta \cdot \alpha) = (\delta * \beta) \cdot (\gamma * \alpha)$

  called the *middle four exchange* or simply the *interchange law*; and given the identities: $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1_f"] & B \arrow[r, bend left, "u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow 1_u"] & C \end{tikzcd}$ we have $1_u * 1_f = 1_{(u \circ f)}$

Some notes about notation:

(i) $K(A, B)$ denotes the category formed by taking objects as the arrows from A to B and the arrows as the corresponding 2-cells under vertical composition.

(ii) composition of 1-cells will be denoted by $\circ$ (i.e., $f \circ g$)

(iii) vertical composition of 2-cells will be denoted by $\cdot$ (i.e., $\alpha \cdot \beta$)

(iv) horizontal composition of 2-cells will be denoted by $*$ (i.e., $\gamma * \alpha$)

3.1.1 Examples:

1. The quintessential example of a 2-category is Cat. Cat can be viewed as a 2-category with categories as objects, functors as arrows and natural transformations as 2-cells.

2. Form a 2-category with 0-cells topological spaces, 1-cells continuous functions between them and 2-cells homotopy classes of homotopies (i.e., homotopies which are themselves deformable into each other).

3. The category K of ordered objects can be viewed as a 2-category, observing there is a natural order to the set of morphisms $(Hom(A, B))$.

4. Just as every ordinary set can be viewed as a discrete category, every ordinary category $\mathcal{C}$ can be viewed as a 2-category when we take the 2-cells to be the identity morphisms of the arrows (i.e., for each pair of objects A,B, the category $\mathcal{C}(A, B)$ is discrete).

5. Let $\mathcal{C}$ be a 1-dimensional category. Form a 2-category called an *arrow category*, denoted $Arr(\mathcal{C})$ or $\mathcal{C}^{\rightarrow}$ by taking 0-cells as the objects of C, 1-cells as the arrows of C, and 2-cells as the pair of arrows $< k, l >$ in $\mathcal{C}$ such that given the arrows $f : A \rightarrow B$, $g : C \rightarrow D$, then $k : A \rightarrow C$, $l : B \rightarrow D$ such that
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[d, "l"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$
commutes (i.e., $l \circ f = k \circ g$). The identity 2-cell is $< 1_A, 1_B >$. (Note: often the term *arrow category* refers to the 1-dimensional category formed by taking the 1-cells and 2-cells described above as the objects and arrows

of the category, respectively.)

6. Take 0-cells to be groups $G, H, \ldots$, 1-cells to be group homomorphisms $f, g, \ldots$ and define 2-cells to be the maps $\alpha : f \Rightarrow g$ where $f, g : G \rightrightarrows H$, and $\alpha \in H$ such that for all $x \in G$, we have $f(x) \cdot \alpha = \alpha \cdot g(x)$ (where $\cdot$ represents the group operation in H). i.e., $\alpha \in Aut(H)$, $y \mapsto \alpha \cdot y \cdot \alpha^{-1}$.

3.2 Dual of a 2-category

The dual K^{op} of a 2-category K is defined as the 2-category with the same objects as K, the same 2-cells as K but $K^{op}(A, B) = K(B, A)$. That is, the 1-cells are reversed, but not the 2-cells.

Clearly, there is a second dual where the 2-cells are reversed, but not the 1-cells, labelled K^{co}, so $K^{co}(A, B) = K(A, B)^{op}$. It is not hard to show that $K^{coop} = K^{opco}$.

3.3 2-Functors

A 2-Functor $D : K \to L$ between 2-categories K and L sends objects of K to objects of L, arrows of K to arrows of L, and 2-cells of K to 2-cells of L, preserving domains, co-domains, and all types of composition and identity.

Formally, a *2-Functor* $F : K \to L$ between two 2-categories K, L consists of:

(i) for each object $A \in K$, an object $F(A) \in L$.

(ii) for each pair of objects $A, A' \in K$, a functor $F_{A,A'} : K(A, A') \to L(F(A), F(A'))$ which satisfies the commutative diagrams:

$$\begin{tikzcd}
K(A,A') \times K(A',A'') \arrow[rr] \arrow[d, "F_{AA'} \times F_{A'A''}"'] & & K(A,A'') \arrow[d, "F_{AA''}"] \\
L(F(A),F(A')) \times L(F(A'),F(A'')) \arrow[rr] & & L(F(A),F(A''))
\end{tikzcd}$$

associativity axiom

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "1_A"] \arrow[dr, "1_{F(A)}"'] & K(A,A) \arrow[d, "F_{AA}"] \\
& L(F(A),F(A))
\end{tikzcd}$$

identity axiom

Observe that a 2-functor induces an ordinary functor on the 0-cells and 1-cells (i.e., the underlying category) of a 2-category.

3.3.1 Example:

Given a 2-category K and the 2-category Cat, fix an object A of K. Then there exists a functor:

$$K(A, -) : K \to Cat$$

This functor maps an object B of K to the category $K(A, B)$, an element of Cat. A morphism $f : B \to C$ in K gets mapped to the functor:

$$K(A, f) : K(A, B) \to K(A, C)$$

where $K(A, f)(g) = f \circ g$ and $K(A, f)(\alpha) = 1_f * \alpha$. A 2-cell $\beta : f \Rightarrow f'$ is mapped to the 2-natural transformation:

$$K(A, \beta) : K(A, f) \Rightarrow K(A, f')$$

where $K(A, \beta)_g = \beta * 1_g$. $K(A, -)$ is a *representable* 2-functor.

3.4 2-natural transformations

A *2-natural transformation* $\eta : D \dot{\rightarrow} E : K \rightarrow L$ between 2-functors D and E, assigns to each object A of K, an arrow $\eta_A : D(A) \rightarrow E(A)$ in L which is natural in the usual sense (i.e., for an arrow $f : A \rightarrow B$ in K, $\eta_B \circ D(f) = E(f) \circ \eta_A$) but as well is "2-natural" in the sense that for each 2-cell $\alpha : f \Rightarrow g$ in K, we have:

$$\begin{tikzcd} D(A) \arrow[r, bend left, "D(f)"] \arrow[r, bend right, "D(g)"'] \arrow[r, phantom, "\Downarrow D(\alpha)"] & D(B) \arrow[r, "\eta_B"] & E(B) = D(A) \arrow[r, "\eta_A"] & E(A) \arrow[r, bend left, "E(f)"] \arrow[r, bend right, "E(g)"'] \arrow[r, phantom, "\Downarrow E(\alpha)"] & E(B) \end{tikzcd}$$

In light of these new definitions, we can define a new 2-category called *2-Cat* whose 0-cells are 2-categories, whose 1-cells are 2-functors, and whose 2-cells are 2-natural transformations.

3.4.1 Example:

Given the representable 2-functors $K(A, -)$ and $K(A', -)$. Let $f : A \rightarrow A'$ be an arrow in K. There is a 2-natural transformation:

$$K(f, -) : K(A', -) \Rightarrow K(A, -)$$

defined by, for each object $B \in K$,

$$K(f, -)_B : K(A', B) \Rightarrow K(A, B)$$

where $K(f,-)_B(g) = g \circ f$ and $K(f,-)_B(\alpha) = \alpha * 1_f$.

3.5 Modifications and 3-Categories

In Chapter 1, we defined categories consisting of objects and morphisms of objects; functors which are morphisms of categories; and natural transformations which are morphisms of functors.

In this chapter, we extended the definition of category to a 2-category which includes, along with objects and morphisms of objects, 2-cells, which are morphisms of the morphisms of the objects. Further, we've defined 2-functors and 2-natural transformations. Following in the spirit of category theory, it seems natural to continue extending this idea of morphisms of morphisms, leading us to the definition of a *modification* (as named by Benabou), a morphism of a 2-natural transformation.

Definition: Given 2-categories K, L, 2-functors $D, E : K \rightrightarrows L$ and 2-natural transformations $\alpha, \beta : D \Rightarrow E$, assign to each object A of K, a 2-cell $\rho_A : \alpha_A \Rightarrow \beta_A$ in L in such a way that for every pair of arrows $f, g : A \rightrightarrows B$ of K and every 2-cell $\gamma : f \Rightarrow g$ of K then

$$\rho_B * D\gamma = E\gamma * \rho_A$$

holds in L. This equality can be represented by the diagram:

$$\begin{tikzcd} D(A) \arrow[r, bend left, "D(f)"] \arrow[r, bend right, "D(g)"'] \arrow[r, phantom, "\Downarrow D\gamma"] & D(B) \arrow[r, bend left, "\alpha_B"] \arrow[r, bend right, "\beta_B"'] \arrow[r, phantom, "\Downarrow \rho_B"] & E(B) \end{tikzcd} = \begin{tikzcd} D(A) \arrow[r, bend left, "\alpha_A"] \arrow[r, bend right, "\beta_A"'] \arrow[r, phantom, "\Downarrow \rho_A"] & E(A) \arrow[r, bend left, "E(f)"] \arrow[r, bend right, "E(g)"'] \arrow[r, phantom, "\Downarrow E\gamma"] & E(B) \end{tikzcd}$$

A *modification* $\rho : K \rightsquigarrow L$ is defined as the collection of 2-cells $(\rho_A)_{A \in K}$.

Given 2-categories K, L, 2-functors $D, E : K \rightrightarrows L$ and 2-natural transformations α, β, $\gamma : D \Rightarrow E$ and modifications $\rho : \alpha \rightsquigarrow \beta$ and $\sigma : \beta \rightsquigarrow \gamma$, then $\sigma \diamond \rho : \alpha \rightsquigarrow \gamma$ is defined by putting

$$(\sigma \diamond \rho)_A : \sigma_A \cdot \rho_A$$

Given 2-functors D, E, $F : K \rightarrow L$ and 2-natural transformations $\alpha, \beta, \gamma, \delta$, $\alpha, \beta : D \Rightarrow E$, $\gamma, \delta : E \Rightarrow F$, and modifications $\rho : \alpha \rightsquigarrow \beta$, $\sigma : \gamma \rightsquigarrow \delta$, then the composite modification $\sigma \star \rho : \gamma \cdot \alpha \rightsquigarrow \delta \cdot \beta$ is defined by

$$(\sigma \star \rho)_A = \sigma_A * \rho_A$$

3.5.1 3-Categories

Just as *Set* is the paradigmatic category, *Cat* is the paradigmatic 2-Category, *2-Cat* taken together with modifications is the paradigmatic example of a *3-Category.*

Although we will not formalize the definition here, a 3-Category contains objects (0-cells), arrows (1-cells), 2-cells, as well as morphisms between 2-cells called 3-cells. These are required to satisfy associativity and identity laws for each kind of composition within the 3-Category. As stated above, *2-Cat* can be considered a 3-Category when we take modifications as its 3-cells.

3.6 n-Categories

It is clear that we can continue with the process of defining morphisms between morphisms, ad infinitum. This leads to the theory of *n-Categories*. The classical example of any n-Category is always *(n-1)-Cat.*

It is actually quite rare to find examples of n-Categories where the associativity and identity laws hold precisely. Rather, it is more interesting to study the cases when they hold up to isomorphism or even by a mere morphism. The isomorphisms or morphisms in these cases must then be required to satisfy some "coherence axioms". Although we will not investigate this further here, along with any mention of n-Categories, it is important to note the language used to distinguish between the different situations. The original definition (where associativity and identity hold with equality) is labelled a "strict" n-Category; the case when equality is replaced with isomorphism is a "pseudo" n-Category (or sometimes "strong" n-Category, depending on the context); finally the case when isomorphism is replaced with a simple morphism is a "lax" n-Category (or sometimes "weak" n-Category). These labels can extend to functors, natural transformations, monads, and so on. A pseudo-2-Category is called a *bicategory.* Most study of n-Categories centres around weak (or lax) n-Categories.

3.7 Adjunctions in a 2-category

An adjunction in a 2-category K consists of a pair of arrows (1-cells) $f : A \to B$ and $g : B \to A$, together with 2-cells $\eta : 1_A \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow 1_B$ satisfying:

$$\begin{aligned} (1_g * \varepsilon) \cdot (\eta * 1_g) &= 1_g \text{ and} \\ (\varepsilon * 1_f) \cdot (1_f * \eta) &= 1_f \end{aligned}$$

We say that f is left adjoint to g (written $f \dashv g$) and g is right adjoint to f (written $g \vdash f$).

Observe that when $K = Cat$, f and g satisfy the usual definition of adjoint functors where η and ε are the unit and co-unit of adjunction, respectively.

If $D : K \to L$ is a 2-functor, an adjunction $\eta, \varepsilon : f \dashv u : A \to B$ in K clearly gives a second adjunction $D(\eta)$, $D(\varepsilon) : D(f) \dashv D(u) : D(A) \to D(B)$ in L.

3.7.1 2-Adjunction

We say 2-functors $D : K \to L$ and $E : K \to L$ are *2-adjoint* when there is an isomorphism of 2-categories $K(E(B), A) \cong L(B, D(A))$ which is 2-natural in A and B. It has been shown by Kelly that this is equivalent to having 2-natural transformations $\eta : 1 \Rightarrow D \circ E$ and $\varepsilon : E \circ D \Rightarrow 1$ satisfying the usual conditions. So in fact, 2-adjunction is just adjunction in the 2-category *2-Cat*.

3.8 Monads in a 2-Category

Just as the notion of an adjunction makes sense in a 2-category, so does the notion of a monad.

A monad in a 2-category K on the object B of K consists of:

- an arrow $t : B \to B$
- 2-cells $\eta : 1_B \Rightarrow t$ and $\mu : t^2 \Rightarrow t$ called the *unit* and *multiplication* of the monad respectively. These are required to satisfy the usual commutativity restraints. As equations these are:

$$\mu \cdot t\eta = 1_B, \quad \mu \cdot \eta t = 1_B, \quad \mu \cdot t\mu = \mu \cdot \mu t$$

The classical case is, of course, when $K = Cat$. This definition immediately produces the usual definition of a monad in a category stated in Chapter 2.

3.9 Adjunctions and Monads

If $f : A \to B$, $g : B \to A$ is an adjunction in a 2-category K with unit η and co-unit ε, then $< t, \eta, \mu >$ is a monad on B where $t = gf$ and $\mu = g\varepsilon f$. This is the monad generated by the adjunction of f and g.

Notice that this situation is identical to the 1-dimensional case described in Section 2.3.1 and the proof can be essentially duplicated here.

3.10 t-Algebras in a 2-Category

Unlike adjunctions and monads, it is a little more difficult to find an analogous construction for a T-algebra within a 2-category. Instead of defining a T-algebra with an object and a structure map, we define it with an arrow (1-cell) and a 2-cell.

Definition: A *t-algebra* on a monad $< t, \eta, \mu >$ in a 2-category K is an arrow (with domain A), $s : A \to B$ together with a 2-cell $\nu : ts \Rightarrow s$ satisfying:

$$\nu \cdot \eta s = 1, \quad \nu \cdot t\nu = \nu\mu s$$

This t-algebra is denoted (s, ν). ν is called an *action* of the monad. A *morphism of t-algebras with domain A* is a 2-cell $\sigma : s \Rightarrow s'$ such that

$$\nu' \cdot t\sigma = \sigma \cdot \nu$$

The t-algebras with common domain A, together with their morphisms form a category denoted $Alg(A, t)$.

When $K = Cat$, the 2-category definition of a t-algebra is commonly restricted to those with domain the unit category $\mathbf{1}$. Then the arrow $s : \mathbf{1} \to B$ can be identified with the corresponding object s of B. Further $\nu : ts \Rightarrow s$ and $\sigma : s \Rightarrow s'$ will be morphisms in B. Denote the category $Alg(\mathbf{1}, t)$ of t-algebras by B^t. Under this restriction, we get exactly the classical definition of a T-algebra.

Recall that in Chapter 2, T-algebras were introduced as a tool to construct an adjunction induced by a monad T. Such an adjunction exists here as well. There is a forgetful functor $G_A : Alg(A, t) \to K(A, B)$ such that $G_A(s, \nu) = s$ and

$G_A(\sigma) = \sigma$. Given any $r : A \to B$, a morphism in K, then $(tr, \mu r : t^2 r \Rightarrow tr)$ is a t-algebra. For any $\rho : r \Rightarrow r'$, $t\rho : tr \Rightarrow tr'$ is a morphism of t-algebras. Thus $F_A : K(A, B) \longrightarrow Alg(A, t)$ where $F_A(r) = (tr, \mu r)$ is a functor. The t-algebras $(tr, \mu r)$ are called the *free t-algebras.* F_A is left adjoint to G_A.

Return to the restriction $K = Cat$ and the domain $A = \mathbf{1}$, the unit category. Then $F_A = F_{\mathbf{1}}$, $G_A = G_{\mathbf{1}}$, and $K(A, B) = K(\mathbf{1}, B)$ can be identified with the category B. Thus, $F_{\mathbf{1}} = F^T : B \to B^t$ and $G_{\mathbf{1}} = G^T : B^t \to B$ are exactly the functors defined in Section 2.4.2, verifying that under the given restrictions, we produce the classical Eilenberg-Moore category and adjunction.

These constructions, $Alg(A, t)$, G_A, F_A are extrapolations of the Eilenberg-Moore category and the corresponding adjoint functors defined in the 1-dimensional case.

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