ORDERABLE GROUPS AND TOPOLOGY

by

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Abstract

This thesis examines some connections between topology and group theory, in particular the theory of orderable groups. It investigates in close detail some landmark results on this mathematical interface, beginning with Hölder's Theorem, and touches upon some recent results in this expanding field of research.

Simply stated, Hölder's Theorem asserts that Archimedean orderable groups are none other than subgroups of the group of real numbers under addition. Since Hölder proved this in 1902, only one significant refinement, due to Paul Conrad, has been made, so these powerful theorems provide the foundation for our understanding of orderable groups. In particular this understanding has served topologists well. This thesis is mostly a distillation of work done in connection with topological applications of the theory, which are surprisingly varied and diverse. Burns and Hale's work on local indicability and right orderability is considered, as well as Bergman's study of the universal covering group of $\mathbf{SL}(2, \mathbb{R})$. In addition N. Smythe's extension of a classical result of Alexander's via the left orderability of the fundamental groups of certain surfaces is investigated.

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Chapter 1
Introduction

In 1994, P. Dehornoy [11] proved that the braid group B_n is right orderable: the proof, considered difficult, conveyed no geometric intuition, and a second proof approaching the result from this latter perspective has since been added to the literature. This startling result, which opened the door to further insight into braid theory (for instance, it has recently been shown that P_n, the pure braid group, is bi-orderable) was the inspiration and genesis of this master's thesis, which undertakes a retrospective of some important results in topology that have involved, specifically, the theory of orderable groups. The diversity of these topological applications suggest some of the power and flexibility that order considerations bring to bear on topology, and contains the promise that this conjunction of mathematics will continue to yield interesting research in the future.

We begin with the following definition.

Definition 1 *A group G is said to be* **right orderable** *if there exists a total order relation $\leq$ on the set G such that $a \leq b$ implies $ay \leq by$ for all $a, b, y \in G$; in this case we say that $\leq$ is invariant under right multiplication. A group G is* **left orderable** *if it is invariant under left multiplication. A group G is* **bi-orderable** *or* **orderable** *if $a \leq b$ implies $xay \leq xby$ for all $a, b, x, y \in G$.*

Now it is easily shown that if G is right orderable, then it can also be left-ordered (not necessarily by the same ordering!), and vice versa; but groups abound which possess one-sided orderings but which are not bi-orderable. A well-known example is the fundamental group G of the Klein bottle, which has presentation $G = \langle a, b \mid aba^{-1} = b^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}$. It is known (for details, see [9]) that if $N \lhd G$ is right orderable and G/N is right orderable, then G is right orderable. We have, in the example of the Klein bottle, $\mathbb{Z} \cong \langle b \rangle$ normal in G by virtue of the semi-direct product construction, and $G/\langle b \rangle \cong \mathbb{Z}$, as $\langle a, b \mid aba^{-1} = b^{-1}, b = 1 \rangle = \langle a; - \rangle$; so G is right orderable. But if G were orderable we would have $b > e \Rightarrow ab > a \Rightarrow aba^{-1} > e \Rightarrow b^{-1} > e$, a contradiction as $b > e \Rightarrow b^{-1} < e$.

It is natural to inquire, in the wake of such definitions and distinctions, why orderable groups might be worth studying. To give an idea of the value of orderability as a mathematical tool, we note that if G is right orderable, then it is torsion free. For take $g \neq e$, say $g > e$. Then $g^2 > g \Rightarrow g^3 > g^2 \Rightarrow \ldots \Rightarrow g^n > e$ for every $n \in \mathbb{N}$, implying that $g^n \neq e \ \forall n$. It is known that if G is a right-orderable group and R is a domain, then the group ring RG has no zero divisors and only trivial units. It is also known that if G is an orderable group, then $\mathbb{Z}G$ embeds in a division algebra. So knowing that a group is right-orderable or orderable gives important information, not just about the group itself, but about some related mathematical structures as well.

The purpose of this thesis is to investigate the theory of orderable groups in relation to topology, and is organized as follows. The second chapter is devoted to Hölder's Theorem, which states that a group G is and Archimedean if and only if it is isomorphic to a subgroup of $(\mathbb{R}, +)$. Paul Conrad's refinement of this result, which allows for a weakening of the hypotheses of the theorem to right orderability and Archimedean, is noted. The next chapter positions right orderability in the larger group theoretic context with respect to the class of locally indicable groups. In this chapter major contributions by Burns and Hale and Bergman are investigated. In the final chapter, orderable groups are featured in their interactions with knot theory, and Neville Smythe's generalization of a result in classical knot theory is studied.

Chapter 2
Hölder's Theorem

As the study of orderable groups properly begins with Hölder's characterization of Archimedean ordered groups, and as the proof of the result is obtainable in English only in the form of a brief sketch [22], it seems appropriate to begin this study of orderable groups and topology by working out the details of Hölder' s argument. But first we will need some preliminary definitions.

Definition 2 *Define the* **positive cone** $P \subset G$ *of an orderable group* G *to be the set* $\{x \in G : x > e\}$.

Definition 3 *Let* $\leq$ *be an order on an orderable group* G. *For* $x \in G$ *define the* **absolute value** *of* x *as*

$$|x| = \begin{cases} x & \text{if } x \geq e, \\ x^{-1}, & \text{if } x < e. \end{cases} \tag{2.1}$$

For $x, y \in G$ write $x \ll y$ (in words, x is infinitely smaller than y) to mean that $x^n < |y|$ for all $n \in \mathbb{Z}$. If neither $x \ll y$ nor $y \ll x$, write $x \sim y$.

Observe that for every $x \in G, x \sim x$ as $x^1 \not< x$. Moreover, the relation $\sim$ is symmetric by definition. Suppose finally that $x \sim y$ and $y \sim z$. Then there exist $s, t \in \mathbb{Z}$ satisfying

$x^s \geq |y|$ and $y^t \geq |x|$, so $x^{st} = (x^s)^t \geq |y|^t \geq y^t \geq |z|$; it follows easily that $x \sim z$. So $\sim$ is an equivalence relation; the equivalence classes so defined are called Archimedean classes.

Definition 4 *If all the non-identity elements of G are equivalent, then $(G, \leq)$ is an* **Archimedean group**, *that is, an Archimedean order is one in which G has only two Archimedean classes.*

Note: in an Archimedean group, for all x, y such that $x \leq y$, $x \neq e$ there exists $n \in \mathbb{Z}$ with $|x^n| > y$.

Now it is not the case that every ordered group is Archimedean. For instance, consider $(\mathbb{Z} \times \mathbb{Z}, +)$ with the dictionary order $(a_1, b_1) < (a_2, b_2)$ if $a_1 < a_2$, or if $a_1 = a_2$ and $b_1 < b_2$. Observe that $(0, x) << (y, z)$ if $y > 0$.

Theorem 1 (Hölder) *Let $\leq$ be an ordering on an orderable group G. Then $\leq$ is an Archimedean order if and only if G is order-isomorphic to a subgroup of the additive group of the real numbers under the natural order.*

Proof.

($\Leftarrow$) $(\mathbb{R}, \leq)$ is Archimedean, and so any subgroup of it is.

($\Rightarrow$) The strategy is to show first that G is an abelian group, and then to use this fact to define an isomorphism. Assume that $\leq$ is an archimedean order on G. The claim is that G is abelian. For consider any $t \in$ G such that $t > e$.

Case 1. Suppose that $e < t \leq x$ for every $x \in P \subseteq G$. Then G Archimedean implies the existence of an integer n_x for each x such that

$$\begin{aligned} t^{n_x} \leq x < t^{n_x+1} &\Rightarrow e \leq xt^{-n_x} < t \\ &\Rightarrow e = xt^{-n_x}, \end{aligned}$$

since t is smaller than any element in the positive cone,

$$\begin{aligned} &\Rightarrow x = t^{n_x} \\ &\Rightarrow G = \langle t \rangle \end{aligned}$$

So G is cyclic and hence abelian.

Case 2. Assume now that there exists $u \in G$ such that $e < u < t$. If $t < u^2$, that is, if $e < u < t < u^2$, then

$$\begin{aligned} t < u^2 &\Rightarrow u^{-1}t < u \\ &\Rightarrow u^{-1}tu^{-1} < e \\ &\Rightarrow (u^{-1}t)^2 < t. \end{aligned}$$

So relabelling we have found u (choose $u^{-1}t$ above) such that $e < u < u^2 \leq t$; thus in all cases we may assume that there exists u such that $e < u < u^2 \leq t$. Now if G is not abelian, then there exist $x, y \in G$ such that commutator $[x, y] \neq e$. Without loss of generality take $[x, y] > e$. Then with $[x, y]$ playing the role of t in the inequalities above we obtain, as G Archimedean implies that there exist $m, n \in \mathbb{Z}$ such that $u^m \leq x < u^{m+1}$ and $u^n \leq y < u^{n+1}$, that $[x, y] < u^{-m}u^{-n}u^{m+1}u^{n+1} = u^2$, which is a contradiction as $e < u < u^2 \leq [x, y]$. Therefore we conclude that G is abelian.

The next step is to show that G is order-isomorphic to a subgroup of $(\mathbb{R}, +)$ by defining explicitly an isomorphism $G \to (\mathbb{R}, +)$ taking $x \in G$ to the limit of a certain sequence to be defined for each x. First fix some $t > e$, $t \in G$. Given $x \in G$ and $n \in \mathbb{Z}^+$, there exists $m \in \mathbb{Z}$ (m dependent on x, n, t) such that $t^m \leq x^{2^n} < t^{m+1}$. So for each x, for every n, define $\zeta_n(x) = \frac{m}{2^n} \in \mathbb{R}$ so that $t^m \leq x^{2^n} < t^{m+1}$.

Claim: $\lim_{n\to\infty} \zeta_n(x)$ exists (for each x).

To prove the claim, it suffices to show that $\{\zeta_n\}$ is a Cauchy sequence, that is, it suffices to show that

$$\zeta_n \leq \zeta_{n+1} \leq \zeta_n + \frac{1}{2^n}.$$

Lemma: For any four elements a, b, c, d of a bi-orderable group with $a < b$ and $c < d$, we have $ac < bd$ (and so, in particular, $a^n < b^n$ for every $n \in \mathbb{N}$.

Proof of Lemma: $ac < bc < bd$.

Now, if $t^m \leq x^{2^n} < t^{m+1}$, then it follows from the lemma that $t^{2m} \leq x^{2^{n+1}} < t^{2(m+1)}$. This implies that

$$\zeta_n = \frac{m}{2^n} = \frac{2m}{2^{n+1}} \leq \zeta_{n+1} < \frac{2(m+1)}{2^{n+1}} = \frac{m+1}{2^n} = \zeta_n + \frac{1}{2^n},$$

as desired. Thus the sequence is Cauchy, hence convergent.

We are therefore in a position to define a map $\phi : G \to (\mathbb{R}, +)$, $\phi(x) = \lim_{n\to\infty} \zeta_n(x)$.

Claim: ϕ is order-preserving (that is, $x \leq y \Rightarrow \phi(x) \leq \phi(y)$).

Proof of claim.

Let $x, y \in G, x \leq y$. Now we already saw that $\zeta_n(x)$and $\zeta_n(y)$ are monotonic increasing sequences. Thus to show that $\phi(x) \leq \phi(y)$, it suffices to show that $\zeta_n(x) \leq \zeta_n(y)$ for every n. But for every n we have $x \leq y \Rightarrow x^{2^n} \leq y^{2^n}$ by the lemma. If m_x and m_y are the largest possible integers satisfying $t^{m_x} \leq x^{2^n}$ and $t^{m_y} \leq y^{2^n}$, then the fact that $x^{2^n} \leq y^{2^n}$ implies that $m_x \leq m_y$. It follows that

$$\zeta_n(x) = \frac{m_x}{2^n} \leq \frac{m_y}{2^n} = \zeta_n(y).$$

Thus ϕ is order-preserving.

Claim: ϕ is a homomorphism.

Proof of claim.

We we must show that $\phi(xy) = \phi(x) + \phi(y)$. As usual, there exist $m_{n,x}$, $m_{n,y}$ for every $n \in \mathbb{N}$ such that

$$t^{m_{n,x}} \leq x^{2^n} < t^{m_{n,x}+1}$$

$$t^{m_{n,y}} \leq y^{2^n} < t^{m_{n,y}+1}$$

These, together with the lemma and the fact that G is abelian, imply that

$$\begin{aligned} t^{m_{x,n}+m_{y,n}} &\leq x^{2^n} y^{2^n} \\ &= (xy)^{2^n} \\ &< t^{m_{x,n}+1+m_{y,n}+1}. \end{aligned}$$

So $t^{m_{x,n}+m_{y,n}} \leq (xy)^{2^n} < t^{m_{x,n}+m_{y,n}+2}$, and therefore $\zeta_n(xy)$ lies between $\frac{m_{x,n}+m_{y,n}}{2^n} = \zeta_n(x) + \zeta_n(y)$ and $\frac{m_{x,n}+m_{y,n}+2}{2^n} = \zeta_n(x) + \zeta_n(y) + 2^{(1-n)}$. Then taking $n \to \infty$ we have

$\phi(xy) = \phi(x) + \phi(y)$. So ϕ is a homomorphism.

Claim: ϕ is injective.

Proof of Claim.

We show $ker\phi = e$ as follows. Assume $x \in P$.

Case 1: $t \leq x$. There exists m_n such that $t^{m_n} \leq x^{2^n} < t^{m_n+1}$ for every n, implying that $m_n + 1 > 2^n \; \forall n$ (as $t \leq x$). It follows that $m_n > 2^n - 1 \Rightarrow \frac{m_n}{2^n} > 1 - \frac{1}{2^n} > 0 \; \forall n \geq 1 \Rightarrow \lim_{n\to\infty} \zeta_n(x) \neq 0$ as $\zeta_n(x)$ is monotonically increasing. Thus $\phi(x) \neq 0$.

Case 2: $x < t$. Here there exists $N \in \mathbb{N}$ such that $t < x^{2^N}$. Again m_n satisfies $t^{m_n} \leq x^{2^n} < t^{m_n+1} \; \forall n \in \mathbb{N}$, implying that $t^{m_n} \leq x^{2^{N+(n-N)}} < t^{m_n+1}$. But $x^{2^N} > t \Rightarrow m_n + 1 > 2^{n-N} \Rightarrow m_n + 1 > 2^n 2^{-N} \Rightarrow m_n > 2^n 2^{-N} - 1 \Rightarrow \frac{m_n}{2^n} > \frac{1}{2^N} - \frac{1}{2^n} \Rightarrow \lim_{n\to\infty} \zeta_n(x) > 0$.

Finally, $x \notin P, x \neq e \Rightarrow x^{-1} \in P \Rightarrow \phi(x^{-1}) = -\phi(x) \neq 0$ (as $x^{-1} \notin ker\phi$ by the above) $\Rightarrow \phi(x) \neq 0$. □

This, then, is the proof of Hölder's Theorem. Paul Conrad [9] added a remarkable footnote in 1959:

Theorem 2 (Conrad) *If G is right orderable and archimedean, then G is orderable. Thus by Hölder's Theorem, G is order-isomorphic to a subgroup of* $(\mathbb{R}, +)$.

Conrad's proof is short and simple, and depends upon the fact, easily verified, that G is orderable if and only if P is normal in G.

Chapter 3

Some Group Theory and Satan's Parkade

In attempting to understand right orderable groups it becomes important to situate them in relation to other classes of groups. Within this context the class of locally indicable groups assumes particular prominence, as it has been shown that locally indicable implies right orderable, but right orderable does not imply locally indicable. We now turn to these results with a view to understanding a remarkable topological application, the universal covering group of $\mathbf{SL}(2, \mathbb{R})$, which we will denote by $\widetilde{\mathbf{SL}}$.

Definition 5 *A group G is said to be* **locally indicable** *if each of its non-trivial finitely generated subgroups can be mapped homomorphically onto a non-trivial subgroup of $\mathbb{Z}$. More generally, if X is a class of groups closed under forming isomorphic images then G is* **locally X-indicable** *if every non-trivial finitely generated subgroup admits a non-trivial homomorphism onto a group in X.*

The following results are stated without proof ([9], [8]):

Theorem 3 (Conrad) *A group G is right orderable if and only if for every finite subset $\{x_1, ..., x_n\} \subset G$ that does not contain e, there exist $\epsilon_i = \pm 1$, $1 \leq i \leq n$, such that e does not belong to the subsemigroup of G that is generated by $\{x_1^{\epsilon_1}, ..., x_n^{\epsilon_n}\}$.*

Theorem 4 (Burns and Hale) *If G is locally RO-indicable, then it is right orderable.*

Corollary: If G is locally indicable, then G is right orderable, as $\mathbb{Z}$ is right orderable.

An example of a right orderable group that is not locally indicable will be found in the universal covering group of $\mathbf{SL}(2, \mathbb{R})$. But this is not a simple construct and will require some additional work to understand. We begin with some definitions.

Definition 6 *A* **topological group** *G is a group that is also a topological space, satisfying the requirements that the multiplication map $m : G \times G \longrightarrow G$ sending (x, y) to $x \cdot y$ and the inversion map sending x to x^{-1} are continuous.*

An important example: the set $M_n(\mathbb{R})$ of n by n matrices is a euclidean space of dimension n^2. As the determinant function $det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous, $det^{-1}(0)$ is a closed set, and its complement, the group $\mathbf{GL}(n, \mathbb{R})$, is thus an open subset of $\mathbb{R}^{n^2}$. Note that matrix multiplication, which is given by polynomials in the coefficients, is continuous, and the inversion map, which by Cramer's rule is a rational function of the coefficients, is also continuous.

Definition 7 *Let G be a topological group with operation $\cdot$, and assume that $p : \tilde{G} \longrightarrow G$ is a simply connected covering space of G. Then if G is locally path-connected, there exists a multiplication map $\tilde{m}$ on $\tilde{G}$ relative to which $\tilde{G}$ is a topological group and p is a homomorphism. The group $\tilde{G}$ is called the* **universal covering group** *of G.*

Let $\mathcal{P} : \tilde{G} \times \tilde{G} \rightarrow G \times G$ be the map $p \times p$, and let $\tilde{e} \in p^{-1}(e)$ be selected, where $e = 1_G$. For the existence of $\tilde{m}$ required by the definition, we note that by the lifting theorem $m\mathcal{P}$ lifts to $\tilde{m}$ with $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$ if and only if $(m\mathcal{P})_*(\pi_1(\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e}))) = m_*\mathcal{P}_*(\pi_1(\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e}))) \subset p_*(\pi_1(\tilde{G}, \tilde{e}))$. Since $\tilde{G} \times \tilde{G}$ is simply connected the containment is automatic and we are guaranteed the lift $\tilde{m}$, which as the lift of a continuous map is continuous. For the condition that p is a homomorphism, the multiplication map $\tilde{m}$ is defined as follows. Let $\tilde{x}, \tilde{y} \in \tilde{G}$. Choose paths $\tilde{\alpha}$ from $\tilde{e}$ to $\tilde{x}$ and $\tilde{\beta}$ from $\tilde{e}$ to $\tilde{y}$. Let $\alpha(t) = p\tilde{\alpha}(t)$, and $\beta(t) = p\tilde{\beta}(t)$. Then take the path $\gamma(t) = \alpha(t) \cdot \beta(t)$ (where $\cdot$ is the group operation in G, not a path product) and lift it to a path $\tilde{\gamma}$ in $\tilde{G}$ that begins at $\tilde{e}$. Then $\tilde{x} \cdot \tilde{y}$ is defined to be the endpoint of $\tilde{\gamma}$. Note that we have $p(\tilde{x} \cdot \tilde{y}) = p(\tilde{\gamma}(1)) = \gamma(1) = \alpha(1) \cdot \beta(1) = p\tilde{\alpha}(1) \cdot p\tilde{\beta}(1) = p(\tilde{x})p(\tilde{y})$.

Now $\mathbf{SL}(2, \mathbb{R})$ is not its own universal covering group, as we establish in the following claim.

Claim: $\pi_1(\mathbf{SL}(2, \mathbb{R}), I) \cong \mathbb{Z}$ (where I is the identity matrix).

Proof of Claim.

Recall the following definitions.

Definition 8 *An* **orthogonal matrix** *is a matrix A for which $AA^t = I$. The set $\mathbf{O(n)}$ of orthogonal (real) matrices forms a subgroup of $\mathbf{GL}(n, \mathbb{R})$, and $\mathbf{S0(n)} = \{A \in \mathbf{O(n)} \mid det(A) = 1\}$, the special orthogonal group, is a subgroup of $\mathbf{SL}(n, \mathbb{R})$.*

It is a standard result that $\mathbf{O(n)}$ is compact.

Definition 9 *A subspace Y of a space X is called a* **deformation retract** *if there exists a continuous retract $r : X \longrightarrow Y$ such that the identity map on X is homotopic to the map $i \circ r$, where i is the inclusion of Y in X. A subspace $Y \subset X$ is a* **strong deformation retract** *of X if there exists a continuous map $H : X \times I \longrightarrow X$ such that $H(x, 0) = x$ for every $x \in X, H(x, 1) \in Y\ \forall x \in X$, and $H(s, t) = s\ \forall s \in Y$ and $\forall t \in I$.*

Recall that if Y is a strong deformation retract of X, with $y_0 \in Y$, then the inclusion map $i : (Y, y_0) \longrightarrow (X, y_0)$ induces an isomorphism of fundamental groups. With this artillery we are ready to prove that $\mathbf{SL}(2, \mathbb{R})$ is not simply connected, by showing first that $\pi_1(\mathbf{SL}(2, \mathbb{R})) = \pi_1(\mathbf{SO(2)})$, and then that $\pi_1(\mathbf{SO(2)}) = \pi_1(S^1)$.

Subclaim #1: $\mathbf{SO(2)} \hookrightarrow \mathbf{SL}(2, \mathbb{R})$ is a strong deformation retract.

Proof of Subclaim.

We construct the homotopy as follows. Let $A \in \mathbf{SL}(2, \mathbb{R})$, let $e_1 = \binom{1}{0}, e_2 = \binom{0}{1}$. Let $v_1 = Ae_1, v_2 = Ae_2$. Since $det(A) = 1, A$ is invertible, v_1 and v_2 are linearly independent. We use the Gram Schmidt process on v_1 and v_2 to orthnormalize via three homotopies:

1). Set $h_1(t) = t\frac{v_1}{||v_1||} + (1-t)v_1$, sending $v_1 \longrightarrow \frac{v_1}{||v_1||} = v_1''$.

2. Replace v_2 by $v_2' = v_2 - \frac{(v_2 \cdot v_1'')}{||v_1''||} v_1$ via the map $h_2(t) = v_2 - t(v_2 \cdot v_1'')v_1$.

3. Replace v_2' by $v_2'' = \frac{v_2'}{||v_2||}$ via $h_3(t) = t\frac{v_2'}{||v_2'||} + (1-t)v_2'$.

After reparametrizing these three homotopies we obtain the single homotopy desired.

Thus to prove that $\pi_1(\mathbf{SL}(2, \mathbb{R})) \cong \mathbb{Z}$, it suffices to show that $\mathbf{SO(2)}$ is homeomorphic to S^1.

Subclaim #2: $\mathbf{SO(2)} \cong S^1$.

Proof of Subclaim.

We will use the fact that for X compact and Y Hausdorff, if $f : X \longrightarrow Y$ is continuous, one-to-one, and onto, then f is a homeomorphism. Since $det : \mathbf{0(2)} \longrightarrow \mathbb{R}$ is continuous, $det^{-1}(1) = \mathbf{SO(2)}$ is closed, hence compact. Moreover S^1 is Hausdorff. We define $f : \mathbf{SO(2)} \longrightarrow S^1, f(A) = A\binom{1}{0}$. Then f takes a matrix to where it moves the point e_1, and since $\mathbf{SO(2)} \subset \mathbf{0(2)}$ preserves lengths of vectors f maps S^1 into itself. Note that f is clearly onto, as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$\in \mathbf{SO(2)}$ for every $\theta \in [0, 2\pi)$. Moreover the invertibility of elements in $\mathbf{SO(2)}$ gives $A\binom{1}{0} = B\binom{1}{0} \Leftrightarrow I\binom{1}{0} = A^{-1}B\binom{1}{0} \Leftrightarrow B = A$. Continuity is also immediate from the continuity of the sine and cosine functions and the fact that elements of $\mathbf{SO(2)}$ acts as rotations on S^1: for if $R_\theta \in \mathbf{SO(2)}$ is rotation by angle θ with image $R_\theta {e_1}^t = (\cos\theta, \sin\theta)$, we can always choose rotation by angle ϕ, R_ϕ, with $\theta < \phi < \theta + \epsilon$ to obtain images that are arbitrarily close.

Summarizing: we have seen that $\mathbf{SL}(2, \mathbb{R})$ is not simply connected; and in the ensuing discussion we will investigate more closely the geometry associated with its covering group.

Definition 10 *A* **(right) group action** *is a map $X \times G \longrightarrow X, (x, g) \mapsto x \cdot g$ satisfying $x \cdot e = x$ and $x \cdot (hg) = (x \cdot h) \cdot g$ for all $g, h \in G, x \in X$. The* **kernel** *of the action is the set $\{g \in G \mid x \cdot g = x \ \forall x \in X\}$. An action is* **faithful** *provided that the kernel is trivial. In equivalent terminology that is often used: an action is* **effective** *if $(x \cdot g = x \ \forall x \in X) \Rightarrow g = e$.*

Of course, a left action is defined in the obvious way. If a set X admits a right (respectively, left) group action, X is said to be a right (respectively, left) G-space. It is easy to check that if X is a left G-space and if we define $x \cdot g = (g^{-1}) \cdot x$ for all $x \in X$ and $g \in G$, then X is a right G-space.

To show that G is right orderable, we will need the following characterization of right orderable groups:

Proposition: G is right orderable if and only if G acts effectively on an ordered set by order-preserving bijections.

Proof:

($\Rightarrow$) Let G act on itself by right multiplication. Then right invariance gives precisely the order-preserving property desired.

($\Leftarrow$) Assume now that G acts effectively on an ordered set X by order-preserving bijections. Assuming the Well Ordering Principle, let X be well-ordered by some order $(X, \prec)$. The idea is to define an order $(G, <)$ in the following way. Since G acts effectively on X, if $g, h \in G$ are such that $g \neq h$, then there exists $x \in X$ such that $x \cdot g \neq x \cdot h$. Under the well-ordering, there exists a minimal such x at which g and h differ: denote this $x_{(g,h)}$. Say $g < h$ provided $x_{(g,h)} \cdot g < x_{(g,h)} \cdot h$. For right invariance assume that $g < h$, that is, $x_{(g,h)} \cdot g < x_{(g,h)} \cdot h$; then for any $f \in G$ we require $gf < hf$, but in fact $x_{(gf,hf)} \cdot gf < x_{(gf,hf)} \cdot hf$ follows at once from the assumption that G acts in an order-preserving manner. Finally we show that $g < h, h < k \Rightarrow g < k$. There are a couple of cases to consider. If $x_{(g,h)} = x_{(h,k)}$, then easily $x_{(g,h)} \cdot g < x_{(g,h)} \cdot h = x_{(h,k)} \cdot h < x_{(h,k)} \cdot k = x_{(g,h)} \cdot k$, so $g < k$. Otherwise assume that $x_{(g,h)} \neq x_{(h,k)}$; let us first suppose that $x_{(g,h)} \prec x_{(h,k)}$. Now g and h differ at $x_{(g,h)}$, but h and k must agree at $x_{(g,h)} \prec x_{(h,k)}$, so necessarily g and k disagree there, that is, $x_{(g,h)} \cdot g < x_{(g,h)} \cdot h = x_{(g,h)} \cdot k$. At no point less than $x_{(g,h)}$ do g and k differ, since $k = h$ and $h = g$ prior to $x_{(g,h)}$. Thus we obtain $g < k$. If on the other hand $x_{(h,k)} < x_{(g,h)}$, then $g = h$ at $x_{(h,k)}$ and therefore $x_{(h,k)} \cdot g = x_{(h,k)} \cdot h < x_{(h,k)} \cdot k$. Since $x_{(h,k)}$ is the first point at which g, h, and k differ, we again have $g < k$. $\square$

Now it is evident that $\mathbf{SL}(2, \mathbb{R})$ acts by matrix multiplication on the set of rays through 0 in the plane: if $A \in \mathbf{SL}(2, \mathbb{R})$, and $\mathbf{x} = (x, y)$ is a point on a ray ρ through the origin, $A(t\mathbf{x}) = tA(\mathbf{x})$ for every $t \in \mathbb{R}$ by linearity, so rays are indeed taken to rays. The kernel of this action is trivial, since if Ω denotes the set of rays through the origin and $A \in \mathbf{SL}(2,\mathbb{R})$ fixes every ray through the origin, then certainly $A\binom{1}{0} = \binom{\lambda}{0}$ for some $\lambda > 0$, and $A\binom{0}{1} = \binom{0}{\beta}$, some $\beta > 0$; but then $\det(A) = 1 \Rightarrow \beta = \lambda^{-1}$. Now A fixes every ray, so the point $\bar{x} = \binom{1}{1}$ is mapped to $\binom{\lambda}{\lambda^{-1}} = t\binom{1}{1}$ for some $t \in \mathbb{R}^+ \Rightarrow \lambda = 1$ and $A = I$. In the action just described, the angle through which a ray is moved is specified only modulo 2π for $A \in \mathbf{SL}(2,\mathbb{R})$, but an element of the covering group $\widetilde{\mathbf{SL}}$ which projects down into $\mathbf{SL}(2,\mathbb{R})$ may be thought of as a linear transformation of the plane in which the angle a ray is moved is specified continuously in the real number s: if A moves ρ by angle s, there exist elements of $\widetilde{\mathbf{SL}}$ that move ρ by angles $s + 2\pi k$ for every $k \in \mathbb{Z}$. Thus the action of $\widetilde{\mathbf{SL}}$ on Ω is certainly not faithful (there are infinitely many elements that project to $I \in \mathbf{SL}(2,\mathbb{R})$), but if we "unwind" the circle of rays through the origin and identify them with $\mathbb{R}$, then $\widetilde{\mathbf{SL}}$ indeed acts faithfully on this set. If one is mathematically inclined to lurid geometrical visions, one can visualize this set–which is the set of rays through $\mathbf{0}$ in the infinite-sheeted branched covering of the plane with branchpoint $\mathbf{0}$–as a sort of infinite parkade spiralling above the plane (in which parking space $2\pi k$ is directly overhead parking space $2\pi(k-1)$). Hence the name Satan's Parkade.

The claim is that $\widetilde{\mathbf{SL}}$ is right orderable, preserving the usual ordering of the line $\mathbb{R}$. We will use the proposition above, showing that $\widetilde{\mathbf{SL}}$ acts effectively on $\mathbb{R}$ by order-preserving bijections. Note that $\mathbf{SL}(2,\mathbb{R})$, in acting on the rays through the origin of the plane, can also be considered as acting on S^1 by orientation-preserving homeomorphisms, where $\bar{x} \in S^1$ is taken by $A \in \mathbf{SL}(2,\mathbb{R})$ to $\frac{A\bar{x}}{||A\bar{x}||}$. So we will prove the more general result that if $\tilde{A} : \mathbb{R} \to \mathbb{R}$ is a lift of any homeomorphism $A : S^1 \to S^1$, then $\tilde{A}$ is also a homeomorphism; and that if A preserves orientation on the circle, then $\tilde{A}$ preserves the ordering on the line.

For given A a homeomorphism from S^1 to itself, $\tilde{A}$ will be a lift of the map Ap and is thus continuous:

$$\begin{array}{ccc} \mathbb{R}, \tilde{x_0} & \xrightarrow{\tilde{A}} & \mathbb{R}, \tilde{y_0} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1, x_0 & \xrightarrow[A]{} & S^1, y_0 \end{array}$$

To see that $\tilde{A}$ has a (continuous) inverse, let $\tilde{B} = \widetilde{A^{-1}}$. That is, $\tilde{B}$ is the lift of the map $A^{-1}p$, and we obtain $p\tilde{B}\tilde{A} = A^{-1}Ap = Ip$. Since $Ip\tilde{x_0} = Ix_0 = x_0$, $p\tilde{B}\tilde{A}\tilde{x_0} = x_0$; but also $p\tilde{I}\tilde{x_0} = p\tilde{x_0} = x_0$, so $\tilde{B}\tilde{A} = \tilde{I}$ as maps. Analogous reasoning yields $\tilde{A}\tilde{B} = \tilde{I}$, and we can conclude that the inverse of $\tilde{A}$ is $\widetilde{A^{-1}}$; note that $\widetilde{A^{-1}}$ is continuous. Thus homeomorphisms of S^1 lift to homeomorphisms of $\mathbb{R}$.

We now assume that $A : S^1 \to S^1$ preserves orientation, and show that A lifts to an order-preserving homeomorphism on $\mathbb{R}$. Consider the points $\binom{1}{0}$ and $A\binom{1}{0} = \binom{\cos\theta}{\sin\theta}$ (we will write $(\cos\theta, \sin\theta)$ for convenience). Let R_θ be rotation on S^1 by angle $-\theta$. Note that R_θ is an orientation-preserving homeomorphism and that $(R_\theta \circ A)\binom{1}{0} = \binom{1}{0}$. Let $g : S^1 \to S^1$ be this map $R_\theta \circ A$. Then it suffices to show that $\tilde{g} : \mathbb{R} \to \mathbb{R}$ preserves order, where $(\mathbb{R}, p)$ is a covering map (that is, $p\tilde{g} = gp$; we take $p(t) = (\cos t, \sin t)$). We already know that homeomorphisms of the circle lift to homeomorphisms of the line; so as $\tilde{g}$ is either an increasing or a decreasing function it suffices to show that $\tilde{g}$ maps $\mathbb{R}^+$ to $\mathbb{R}^+$.

Since p is a covering map there exists an open neighbourhood $U \subseteq S^1$ of $\binom{1}{0}$ such that each component of $p^{-1}(U)$ is mapped topologically onto U. So there exists an interval $V = (-\epsilon, \epsilon) \subseteq \mathbb{R}$ such that p restricted to V is a homeomorphism. Since $\tilde{g}$ is continuous there exists $\delta_1 > 0$ (take $\delta_1 \leq \epsilon$) such that $d(0, y) < \delta_1 \Rightarrow d(\tilde{g}(0), \tilde{g}(y)) < \epsilon$. Then putting $U_1 = p(-\delta_1, \delta_1)$ we have $U_1 \subseteq U$. Moreover since g is continuous we can choose a neighbourhood $U_2 \subseteq U_1$ small enough that $g(U_2) \subseteq U_1$. Let $V' = p^{-1}(U_2) = (-\delta, \delta)$ for some $\delta > 0$. To show that $\tilde{g}$ takes $\mathbb{R}^+$ to itself we need only show that $\tilde{g}(\frac{\delta}{2}) > 0$. Recall that g fixes $\binom{1}{0}$ on S^1, so that on V' we have $\tilde{g}(0) = 0$. To see why $\tilde{g}(\frac{\delta}{2}) > 0$, consider $p\tilde{g}(\frac{\delta}{2}) = gp(\frac{\delta}{2}) = g(\cos\frac{\delta}{2}, \sin\frac{\delta}{2})$. Since $\sin t$ is an increasing function at 0 and g preserves orientation, $g(\cos\frac{\delta}{2}, \sin\frac{\delta}{2})$ will have positive angle measure. It follows that $\tilde{g}(\frac{\delta}{2}) > 0$ on $V' \subset \mathbb{R}$.

Thus orientation-preserving homeomorphisms on S^{-1} lift to order-preserving homeomorphisms on $\mathbb{R}$; so finally we can deduce that $\widetilde{SL}$ is a a right-orderable group.

We obtain additional information about $\widetilde{SL}$ in the following

Lemma: For any $A \in \mathbf{SL}(2, \mathbb{R})$, with t_0 and $t_f \in \mathbb{R}$ and $t_0 \leq t_f$, if $|t_0 - t_f| \leq 2\pi$, then $|\tilde{A}(t_0) - \tilde{A}(t_f)| \leq 2\pi$ for every $\tilde{A} \in \widetilde{\mathbf{SL}}$ projecting to A.

Proof of Lemma: Suppose for a contradiction that $|\tilde{A}(t_f) - \tilde{A}(t_0)| > 2\pi$; let's first assume that $\tilde{A}(t_0) < \tilde{A}(t_f)$. Then since $\tilde{A}$ is continuous, there exists $t_1 \in (t_0, t_f)$ such that $\tilde{A}(t_1) = \tilde{A}(t_0) + 2\pi$. Letting $\rho_0 \in \Omega$ denote the ray of angle t_0, ρ_f the ray of angle t_f, and so on, it follows that $A(\rho_1) = A(\rho_0) + 2\pi = A(\rho_0)$. So we have $t_0 < t_1$ but $A(t_0) = A(t_1)$, contradicting that $\det(A) = 1$. (If we assume $\tilde{A}(t_0) > \tilde{A}(t_f)$, then there exists $t_1 \in (t_0, t_f)$ with $\tilde{A}(t_1) = \tilde{A}(t_f) + 2\pi$ and we have $t_1 < t_f$ but $A(t_1) = A(t_f)$, contradiction.) □

Now suppose $\tilde{A}$ takes the point $t_0 \in \mathbb{R}$ to r_0. Then from the lemma it follows that $|\tilde{A}(t_0+2\pi) - \tilde{A}(t_0)| \leq 2\pi$, and since $\tilde{A}$ is order-preserving, as well as a bijection, $\tilde{A}(t_0+2\pi)$ must be $\tilde{A}(t_o) + 2\pi = r_0 + 2\pi$. In fact this observation generalizes to the following proposition, of which it is the base case:

Proposition: If $\tilde{A} : \mathbb{R} \longrightarrow \mathbb{R}$ is an order-preserving homeomorphism such that $\tilde{A}(\mathbb{Z}) = \mathbb{Z}$, then if $\tilde{A}(0) = 0$, we also have $\tilde{A}(n) = (n)\ \forall n \in \mathbb{Z}$.

Proof: Since $\tilde{A}(-n) = -\tilde{A}(n)$ for every n, it suffices to show that the Proposition is true for $n \in \mathbb{N}$. Assume that the assertion is true for all $k \leq n - 1$, but suppose that $\tilde{A}(n) = m, m > n$. Then since $\tilde{A}$ is onto, there exists $s \in \mathbb{N}, s > n$, such that $\tilde{A}(s) = n$, contradicting that $\tilde{A}$ is order-preserving. □

Thus we now know that $\widetilde{\mathbf{SL}}$ acts on 2π-multiples of vectors in Ω in predictable fashion, rather than shuffling them around.

Our goal is to define a nontrivial subgroup of the covering group of $\mathbf{SL}(2,\mathbb{R})$ that will map homomorphically to the identity, thus establishing that our group is not locally indicable. To do this we begin by specifying some matrices of interest in $\mathbf{SL}(2,\mathbb{R})$. Let

$$a^r = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}, \quad b^r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad c_s = \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix}$$

where $r \in \mathbb{R}, s \in \mathbb{R}^+$. In fact a^r is a horizontal shear, while b^r is a vertical shear. One easily computes that a^r has repeated eigenvalue 1 and eigenspace $\{\binom{t}{0} \mid t \in \mathbb{R}\}$, while b^r also has repeated eigenvalue 1 though with eigenspace $\{\binom{0}{t} \mid t \in \mathbb{R}\}$, and c_s has eigenvalues $\sqrt{s}$ and $\frac{1}{\sqrt{s}}$, both positive by the condition on s with eigenspaces $\{\binom{t}{0} : t \in \mathbb{R}\}$ and $\{\binom{0}{s} : s \in \mathbb{R}\}$. One also obtains the following identities:

$$a^r a^{r\prime} = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r\prime \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -r\prime - r \\ 0 & 1 \end{pmatrix} = a^{r+r\prime}$$

$$b^r b^{r\prime} = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r\prime & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r + r\prime & 1 \end{pmatrix} = b^{r+r\prime}$$

$$c_s c_{s\prime} = \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix} \begin{pmatrix} \sqrt{s\prime} & 0 \\ 0 & \frac{1}{\sqrt{s\prime}} \end{pmatrix} = \begin{pmatrix} \sqrt{ss\prime} & 0 \\ 0 & \frac{1}{\sqrt{ss\prime}} \end{pmatrix} = c_{ss\prime}$$

The following easy calculations are also very much to our purpose:

$$c_s a^r c_{s^{-1}} = \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix} \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{s}} & 0 \\ 0 & \sqrt{s} \end{pmatrix} = \begin{pmatrix} 1 & -sr \\ 0 & 1 \end{pmatrix} = a^{sr}$$

$$c_s^{-1} b^r c_s = \begin{pmatrix} \frac{1}{\sqrt{s}} & 0 \\ 0 & \sqrt{s} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ rs & 1 \end{pmatrix} = b^{sr}.$$

Finally, letting

$$d = aba = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we note that

$$d^{-1} c_s d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{s}} & 0 \\ 0 & \sqrt{s} \end{pmatrix} = c_s^{-1}$$

$$d^{-1} a^r d = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} = b^r$$

$$d^{-1} b^r d = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} = a^r.$$

Now $a, b, c, d \in \mathbf{SL}(2, \mathbb{R})$ satisfy these relations, but the immediate goal is to show that certain elements in their fibres satisfy the same relations in $\widetilde{\mathbf{SL}}$; for once this is established it will not be too difficult to define a finitely generated subgroup of $\widetilde{\mathbf{SL}}$ which maps homomorphically to the identity in $\mathbb{Z}$, thus showing that $\widetilde{\mathbf{SL}}$ is not locally indicable.

Suppose $h \in \mathbf{SL}(2,\mathbb{R})$ has a positive eigenvalue $\lambda > 0$ (relevant aside: a, b, c above all do). Then there exists $\bar{v} \in \mathbb{R}^2$ with $h\bar{v} = \lambda\bar{v}$, so in terms of the action of $\mathbf{SL}(2,\mathbb{R})$ on Ω, h fixes a ray $\vec{v}$, and in terms of the action of $\widetilde{\mathbf{SL}}$ on Satan's Parkade, each element of $p^{-1}(h)$ must move $\vec{v}$ by $2\pi k$, some $k \in \mathbb{Z}$ (p here is the projection map in the definition of the universal covering group above). We define A^r, B^r, C_s to be the liftings of a^r, b^r, and c_s that actually fix the rays in the Parkade (rather than moving them by $2\pi k$). So in order for this definition to even make sense we need to check that only one of h's infinitely many liftings to $\widetilde{\mathbf{SL}}$ fixes the rays that map into the eigenspace of λ. Moreover, since c_s has two distinct positive eigenvalues, it is necessary to check that the lift that fixes the rays that map into one eigenspace is the same lift that fixes the rays that map into the other. To see why this latter claim is so, we claim that if $f \in \mathbf{SL}(2,\mathbb{R})$ has at least one positive eigenvalue, then f moves every ray through an angle strictly less than π. For let $\lambda > 0$ be an eigenvalue, with $\bar{v}$ its eigenvector. Suppose $\bar{w}$ is a ray moved by angle $\geq \pi$. Note that $\bar{w} \neq -\bar{v}$, as then $\bar{w}$ is also fixed by f. Thus $\bar{v}$ and $\bar{w}$ are linearly independent and form an ordered basis for $\mathbb{R}^2$, but if f moves $\bar{w}$ by an angle greater than or equal to π, then f is in fact orientation-reversing, contradicting that $det(f) = 1$. But since f moves every ray through an angle of magnitude less than π, it cannot be lifted to a map $\tilde{f}$ that fixes a ray $\vec{v_1}$ of angle ν_1 and moves some ray $\vec{v_2}$ of angle ν_2 by some $2\pi k$, where $k \geq 1$. For (assuming without loss of generality that $\nu_1 < \nu_2$) $\tilde{f}$ is continuous, and by the Intermediate Value Theorem there exists $c \in \mathbb{R}$, with associated vector $\vec{w}$ of angle c with $\nu_1 < c < \nu_2$ such that $\tilde{f}(\vec{w}) = \pi$, so that f moves $\vec{w}$ by π, a contradiction.

Now to see that only one of h's infinitely many liftings to $\widetilde{\mathbf{SL}}$ fixes the rays that map into the eigenspace of λ, suppose that there exist $\tilde{h}_1, \tilde{h}_2 \in p^{-1}(h)$, and $\tilde{h}_1$ and $\tilde{h}_2$ both fix the eigenvector $\bar{v}$ of h in the Parkade. We must show that $\tilde{h}_1 = \tilde{h}_2$. Let $\tilde{h} = \tilde{h}_1\tilde{h}_2^{-1}$. Then $p\tilde{h} = p(\tilde{h}_1\tilde{h}_2^{-1}) = p(\tilde{h}_1)p(\tilde{h}_2)^{-1} = hh^{-1} = I$. Consider $w \in \Omega, w \neq v$ and chosen to be linearly independent from v. Then of course $p\tilde{h}(w) = w$. But then $\tilde{h}$ also fixes w in the Parkade, since we saw above that I cannot lift to a map $\tilde{h}$ that moves $\bar{v}$ by 0 degrees and w by $2\pi k, k \neq 0$ (else there exists $\bar{u}$ between $\bar{v}$ and $\bar{w}$ that is moved by π. We conclude that $\tilde{h}$ is the identity, so $\tilde{h}_1 = \tilde{h}_2$.

Thus it makes sense to speak of the unique liftings A^r, B^r, and C_s as defined above.

Claim: A^r, B^r, and C_s satisfy the corresponding identities in $\widetilde{\mathbf{SL}}$, namely:

$$A^r A^{r\prime} = A^{r+r\prime},\ B^r B^{r\prime} = B^{r+r\prime},\ C_s C_{s\prime} = C_{ss\prime},$$

$$C_s A^r C_s^{-1} = A^{rs},$$

$$C_s^{-1} B^r C_s = B^{rs}.$$

Proof of Claim.

We already know that in each case, both sides the equation project to the same element in $\mathbf{SL}(2,\mathbb{R})$. So by the preceding arguments we simply need to show that both sides of each equation have fixed rays, rather than moving them by some $2\pi k$. But by definition the right hand side of each equation has fixed rays. Now consider $A^r A^{r\prime}$. Recall that a^r has eigenspace $\{span\binom{1}{0}\}$ for every $r \in \mathbb{R}$. Since matrix multiplication is composition of transformations the eigenvectors stay fixed. So the same holds true for the chosen lift. This argument applies to $B^r B^{r\prime}$ and $C_s C_{s\prime}$ as well. For the last two equations the left hand sides are conjugates of elements having fixed rays, hence have fixed rays. If we define $D = ABA$ we obtain in similar fashion the identities:

$$D^{-1} C_s D = C_s^{-1}$$

$$D^{-1}A^r D = B^r$$

$$D^{-1}B^r D = A^r.$$

Theorem: $\widetilde{\mathbf{SL}}$ is not locally indicable.

Proof.

Let $n \in \mathbb{N} > 1$ and let $H = \langle A, B, C_n \rangle < \widetilde{\mathbf{SL}}$. Let $\Phi : H \longrightarrow (\mathbb{Z}, +)$ be any homomorphism. The identities above give that $C_n A C_n^{-1} = A^n, C_n^{-1} B C_n = B^n$, and $D^{-1} C_n D = C_n^{-1}$. So A, B, and C_n are conjugate in this group to powers of themselves. Then $\Phi(C_n A C_n^{-1}) = \Phi(A^n) \Rightarrow \Phi(C_n) + \Phi(A) - \Phi(C_n) = \Phi(A^n) \Rightarrow \Phi(A) = n\Phi(A)$ for some $n > 1$, implying that $\Phi(A) = 0$, with similar arguments holding for the other generators. $\square$

Chapter 4

Knot Theory and Orderable Groups

Knot (and braid) theory crop up notoriously in all sorts of different mathematical contexts, so it is hardly surprising that orderable groups should intersect knot theory in nontrivial ways. This chapter undertakes an investigation of a result of N. Smythe which exploits order considerations to generalize a classical theorem of Alexander's pertaining to knots to all surfaces. Once again we begin with some definitions.

Definition 11 *A* **surface** *is a connected 2-manifold (that is, a connected Hausdorff space such that each point P has an open neighbourhood homeomorphic to the open disk $D^0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$). A* **surface with boundary** *is a Hausdorff space such that every point P has a neighbourhood homeomorphic either to the open disk or to the half-disk $\{(x, y) \in D^0 \mid y \geq 0\}$. If the boundary is empty and the surface is compact, then it is said to be* **closed**.

Definition 12 *A closed non-self-intersecting polygonal line in $\mathbb{R}^3$ is called a* **polygonal knot**. *A* **smooth knot** *is the image of an infinitely differentiable embedding $f : S^1 \longrightarrow \mathbb{R}^3, f(t) = (x(t), y(t), z(t))$, with $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \neq (0, 0, 0)$.*

Thus we can envisage a knot as an entwined polygon in $\mathbb{R}^3$ (with finitely many edges), or we can envisage it as an entwined circle in that space. Two polygonal knots K_0 and K_1 are said to be equivalent provided K_1 can be obtained from K_0 via a finite sequence of "elementary moves" (or their inverses), where these are as follows: let E_i and E_j be adjacent edges of the polygonal knot K_0, and assume that the triangle spanned by E_i and E_j does not intersect K_0 in any other points. Then an elementary move is simply the replacement of E_i and E_j by the third leg of the triangle spanned. On the other hand, two smooth knots K_0 and K_1 are equivalent if there exists a one-parameter family $f_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, t \in [0,1]$ of diffeomorphisms smoothly depending on the parameter t such that $f_0(K_0) = K_0$ and $f_1(K_0) = K_1$. Here, "smoothly depending" means that the map $F : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3, F(x,t) = f_t(x)$ is differentiable. The family of diffeomorphisms f_t is called an isotopy, and equivalent knots are sometimes referred to as isotopic, or ambient isotopic. Note that if K_0 and K_1 are equivalent, there is a homeomorphism, namely, f_1, between $(\mathbb{R}^3, K_0)$ and $(\mathbb{R}^3, K_1)$. One might well ask whether the double-barreled definition of knots given above–polygonal versus smooth–reflects two different irreconcilable approaches to knots, or whether the definitions are interchangeable. It is the latter that is true: there is a process called "smoothing" taking equivalence classes of polygonal knots to equivalence classes of smooth knots (for details see [7]). The upshot is that we can choose either definition we prefer to work with.

Definition 13 *A knot which bounds a disc in* $\mathbb{R}^3$ *is* **trivial**.

A further technicality that must be broached in any discussion of knots is the issue of projection and knot diagrams, which we treat briefly and informally here. In essence what is required in choosing a "regular" projection for a (smooth) knot is a plane such that:

- no more than two distinct points of the knot are projected onto the same point on the plane
- the set of such crossing points is finite, and at any such crossing point the projections of two tangent lines to the knot do not coincide

- the tangent lines to the knot project onto lines on the plane, not to points.

With polygonal knots some adjustment in vocabulary is necessary: we require that a vertex of the knot is never mapped onto a double point, as well as the conditions above that no more than two points are in the pre-image of any projection point (with at most finitely many such double points).

Now there is an algorithm known as "laying down the rope", known at least as far back as Alexander and sometimes attributed to him, demonstrating that for any regular projection $p(k)$ of a polygonal knot k, there is a trivial knot with the same projection (see, for instance, [26]). The procedure is simply to orient $p(k)$, choose a starting point x on $p(k)$, and at each double point P of $p(k)$ with itself, in which arcs $\alpha(P)$ and $\beta(P)$ cross, to designate $\alpha(P)$ as an undercrossing if in the chosen orientation $\alpha(P)$ lies between x and arc $\beta(P)$. The knot that is obtained will be trivial. The question is whether this result generalizes to all surfaces. One generalization, to $S^2 \times I$, is immediate: that is, if k is embedded in $S^2 \times I$ and $p(k)$ is its projection onto $S^2 \times \{0\}$, Alexander's Algorithm applies since $p(k)$ is contractible in the complement of a point Q of S^2. Since $S^2 - \{Q\} \cong \mathbb{R}^2$, the algorithm proceeds exactly as above, and the post-algorithmic $p(k)$ will still bound a disc. Here is the theorem in full generality:

Theorem 5 (N. Smythe) *Let S be a surface, orientable or not, compact or not. Let k be a polygonal knot contained in, and contractible in, the interior of $S \times I$, with regular projection $p(k)$ in $S \times \{0\}$. Then there exists a knot $k' \subseteq S \times [\frac{1}{4}, \frac{1}{2}]$ which has the same regular projection, and which bounds a disc in $S \times [\frac{1}{4}, \frac{1}{2}]$.*

What is extraordinary is that the theorem hinges upon the left-invariant orderability of the fundamental group $\pi_1(S)$, so that a necessary preliminary involves the (non-trivial) verification that such orderings exist for every surface other than the projective plane $\mathbb{R}P^2$ (which has fundamental group $\mathbb{Z}/2\mathbb{Z}$, clearly non-orderable, and will require separate treatment). First we observe that the free group of finite or countable rank has a left-invariant ordering. Various proofs exists: see for instance [12] (another slick approach uses the Magnus map). Note that this result will take care of finite surfaces with boundary (whose canonical polygons have perforations or "holes" bounded by the boundary curves), since any such is homeomorphic to a disc with strips attached (double strips corresponding to handles, Möbius strips to crosscaps, and single strips for any extra perforations); and such surfaces deformation retract onto bouquets of circles passing through the strips, hence have fundamental groups free of rank equal to the number of strips. In addition this will take care of the case in which the surface is infinite (that is to say, non-compact), since the fundamental group of an infinite surface is free. For the details of this result, see, for instance [32]. The basic idea is to represent an infinite surface $\mathcal{F}$ as a nested union $\bigcup_n \mathcal{F}_n$ of finite surfaces with boundary; with some care one inductively establishes that the free generators of $\pi_1(\mathcal{F}_n)$ remain free in $\pi_1(\mathcal{F}_{n+1})$; then, since any closed path based at $x \in \mathcal{F}$ is compact, it must lie in some $\mathcal{F}_n$.

We have as remaining cases the closed orientable surfaces which have fundamental group $G_h = \langle a_1, b_1, ..., a_h, b_h \mid \prod a_i b_i a_i{}^{-1} b_i{}^{-1} = 1\rangle, h \geq 0$, and the closed non-orientable $H_k = \langle a_1, ..., a_k \mid a_1{}^2 a_2{}^2 \cdots a_k{}^2 = 1\rangle$, for $k \geq 1$. Note that $\langle a_1 \mid a_1{}^2 = 1\rangle$ is the fundamental group of the projective plane, to be discussed separately, while if $h = 0$ the group is trivial, corresponding to the surface S^2, and is thus orderable. Turning to the orientable case first we note that if $h \geq 1$ we can define a homomorphism $\Psi_h : G_h \to \mathbb{Z}$ given by $\Psi_h(a_1) = 1, \Psi_h(b_1) = 0, \Psi_h(a_i) = \Psi_h(b_i) = 0\,\forall i > 1$. Note that the kernel K_h contains b_1 and a_i, b_i for all $i \geq 2$. In the non-orientable case we define the homomorphism $\Psi_k : H_k \to \mathbb{Z}$ given by $\Psi_k(a_1) = 1, \Psi_k(a_2) = -1, \Psi_k(a_i) = 0\ \forall i > 2$. Note that if $k > 2$, the kernel contains all powers of at least one a_i. In case $k = 2$ we observe that the kernel contains $\{a_1{}^{i_1} a_2{}^{j_1} \cdots a_1{}^{i_n} a_2{}^{j_n} \mid \sum_{s=1}^{s=n} i_s = \sum_{s=1}^{s=n} j_s\}$.

We are interested in the covering spaces corresponding to these (normal) subgroups of G_h and H_k. Recall that it is possible to define an action of the group $\pi_1(S, x)$ on the fibre set $p^{-1}(x)$ for any x: for chosen $\tilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi_1(S, x)$, define $\tilde{x} \cdot \alpha \in p^{-1}(x)$ to be the terminal point of the unique path class $\tilde{\alpha}$ in $\tilde{S}$ such that $p_*(\tilde{\alpha}) = \alpha$ and the initial point of $\tilde{\alpha}$ is the point $\tilde{x}$. Of course the actions by G_h and H_k, restricted to their respective kernels, are still group actions. The claim is that in each instance $p^{-1}(x)$ has infinitely many sheets. To see that this is so, consider first the case $k = 2$, that is, $H_k = \langle a, b \mid a^2b^2 = 1 \rangle$. Then the kernel of the homomorphism to $\mathbb{Z}$ contains such elements of the fundamental group as $a^3b^3, a^3b^3a^3b^3$, and $(a^3b^3)^j$ for any $j > 1$; for chosen $\tilde{x}_0 \in p^{-1}(x)$ we will denote by $\tilde{x}_j = \tilde{x}_0 \cdot (a^3b^3)^j$ the terminal point of the lift. If $\tilde{x}_j = \tilde{x}_k$ for some j, k, where $j \neq k$, then the respective lifts are identical, so $p_*(\widetilde{(a^3b^3)^j}) = p_*(\widetilde{(a^3b^3)^k})$, implying, since p_* is a monomorphism, that $(a^3b^3)^j = (a^3b^3)^k$ in H_2. Since $j \neq k$ this implies that we have $(a^3b^3)^m = 1$ for some $m \in \mathbb{Z}$, $m \neq 0$. But H_2 is not a commutative group, hence $(a^3b^3)^m \neq (a^2b^2)^n$ for any n; so this is a contradiction. If $k > 2$, then there exists a_i, $i > 2$, and all powers of a_i, in the kernel, and similar reasoning leads us to conclude that $p^{-1}(x)$ has infinitely many sheets, and if $h \geq 1$, the fact that all powers of b_1 are in the kernel gives the same result. In all three cases, therefore, the covering space corresponding to the kernel is an open infinite surface, so the kernel is a free group, hence left-orderable. Since G/K is isomorphic to a subgroup of $\mathbb{Z}$ and left-orderable, it follows that H_k and G_h are left-orderable, since as mentioned earlier in this thesis, if $K \lhd G$, and both K and G/K have left-invariant orderings, then so does G.

With the left-orderability of the fundamental group of these surfaces now established, we turn to an investigation of the theorem in the case wherein S is any surface with the exception of S^2 or $\mathbb{R}P^2$. By assumption, k, a polygonal knot, is contained in and contractible within $M \times I$, where M denotes the interior of S. Let $(\tilde{M}, e)$ be the universal covering space of $M \times I$. Now it is a standard result of covering space theory that if

$e : X \to Y$ is a covering map with X simply connected, then $Aut(X) \cong \pi_1(Y, yo)$. We shall also make use of the fact that the universal covering surfaces are the sphere (for the sphere itself, and the projective plane), and the plane, which follows from every non-orientable surface having an orientable surface as a 2-sheeted cover [32]. Thus $\tilde{M} \cong \mathbb{R}^2 \times I$, and $Aut(\tilde{M}) \cong \pi_1(M \times I) \cong \pi_1(S)$, which we know is left orderable. Let $\tilde{x}$ be chosen as a basepoint in $\tilde{M}$ and let k_1 be the lift of $p(k)$ through $\tilde{x}$. If we let T_u denote the covering translation corresponding to the element $u \in \pi_1(M \times I)$, then $p(k)$ is covered by $T_u(k_1)$, which we denote by k_u. A chosen orientation for $p(k) \in M \times \{0\}$ will induce an orientation in $k_u \in \mathbb{R}^2 \times 0$. Let $\tilde{p} : \mathbb{R}^2 \times I \to \mathbb{R}^2 \times \{0\}$ be the projection over p, that is, $e\tilde{p} = p$.

Alexander's algorithm of "laying down the rope" can be generalized as follows. Choose neighbourhoods around the double points of the collection $\{k_u\}$ sufficiently small to be disjoint and contain only the two simple arcs of the singularity. Then focussing attention on k_1, we notice that k_1 might cross itself, or k_1 might cross some other k_u, where $u \neq 1$. If k_1 crosses itself at the point P, let $\alpha(P)$ denote whichever of the two arcs precedes with respect to $\tilde{x}$ in the orientation of k_1; let $\beta(P)$ denote the arc that succeeds. If k_1 crosses k_v, $v \neq 1$, let $\gamma(P)$ denote the arc at the crossing that belongs to k_1, and $\nu(P)$ be the arc belonging to k_v. Then a simple closed curve $\tilde{k}_1 \in \mathbb{R}^2 \times I$ is constructed in the following manner. For any point $(r, s, 0) \in k_1$, if $(r, s, 0)$ does not belong to any of the singularity neighbourhoods, or if $(r, s, 0) \in \alpha(P)$, or if $(r, s, 0) \in \gamma(P)$ where P is a crossing of k_1 and k_u such that $u > 1$, then let $(r, s, \frac{1}{4})$ be a point of $\tilde{k}_1$. If on the other hand $(r, s, 0) \in \beta(P)$, or if $(r, s, 0) \in \gamma(P)$, where P is a crossing of k_1 and k_v such that $v < 1$, then let $(r, s, \frac{1}{4} + t), 0 < t \leq \frac{1}{4}$, be a point of $\tilde{k}_1$, where $t = \frac{1}{4}$ at P itself (giving $(r, s, \frac{1}{2})$ as the point above P on $\tilde{k}_1$), and t varies continuously to approach 0 as $(r, s, 0)$ approaches the boundary of the neighbourhood at P. Thus we have fashioned a closed curve $\tilde{k}_1 \in \mathbb{R}^2 \times I$ such that $\tilde{p}(\tilde{k}_1) = k_1$. Note that $\alpha(P)$ is sent to an undercrossing on

$\tilde{k}_1$, and $\beta(P)$ becomes an overcrossing. Finally we define $\tilde{k}_u = T_u(\tilde{k}_1)$. The algorithm is rather dauntingly technical, but roughly speaking (and we return to make this more precise below), we are to think of the curves $\{\tilde{k_w}\}$ as suspended in $\mathbb{R}^2 \times I$ with their "altitudes" on the interval dependent on the group element which indexes them: higher up for those whose group element is positive in the group ordering, lower down if the group element is negative.

Claim: $\tilde{k}_u$ and $\tilde{k}_v$ are disjoint.

Indeed, let $Q \in \tilde{k}_u \cap \tilde{k}_v$. Consider $T_{u^{-1}}(Q) = P$. Since $\tilde{k}_u = T_u(\tilde{k}_1)$, we have $P = T_{u^{-1}}(Q) \subseteq T_{u^{-1}}T_u(\tilde{k}_1) \in \tilde{k}_1$, but also $P \subseteq T_{u^{-1}}T_v(\tilde{k}_1) \in \tilde{k}_{u^{-1}v}$. Thus $P \in \tilde{k}_1 \cap \tilde{k}_{u^{-1}v}$. Consider also $P' = T_{v^{-1}}(Q)$. Since $\tilde{k}_v = T_v(\tilde{k}_1)$, we have $P' = T_{v^{-1}}(Q) \subseteq T_{v^{-1}}T_v(\tilde{k}_1) \in \tilde{k}_1$ and $P' \subseteq T_{v^{-1}}T_u(\tilde{k}_1) \in \tilde{k}_{v^{-1}u}$, so $P' \in \tilde{k}_1 \cap \tilde{k}_{v^{-1}u}$. Without loss of generality, take $u > v$, i.e. $u^{-1}v < 1$. Then we have $\tilde{p}(P)$ a double point of k_1 and $k_{u^{-1}v}$ by definition of $\tilde{p}$, so by construction P must be in $\mathbb{R}^2 \times \frac{1}{2}$. But turning to consider $\tilde{p}(P')$ a double point of k_1 and $k_{v^{-1}u}$, we must have $P' \in \mathbb{R}^2 \times \frac{1}{4}$, as $v^{-1}u > 1$. But this a contradiction, since $T_u(P) = T_v(P') = Q$, and covering translations do not affect the third coordinate. Thus $\tilde{k}_u$ and $\tilde{k}_v$ are disjoint for all $u, v, u \neq v$.

Moreover since by the very definition of $\tilde{k}_u$ we have $u > v$ implying that any crossing of $\tilde{k}_u$ lies above $\tilde{k}_v$ in $\mathbb{R}^2 \times I$, $u > v$ implies that $\tilde{k}_u$ lies above $\tilde{k}_v$; hence we can isotope the curves $\{\tilde{k}_w\}$ such that $\tilde{k}_u \subseteq \mathbb{R}^2 \times (0, \frac{1}{4})$ if $u < 1$, $\tilde{k}_1$ remains in $\mathbb{R}^2 \times [\frac{1}{4}, \frac{1}{2}]$, and $\tilde{k}_v \subseteq \mathbb{R}^2 \times (\frac{1}{2}, 1)$ if $v > 1$; and in all cases $\tilde{p}(\tilde{k}_w) = k_w$ remains a regular projection for all w. Note that $\tilde{k}_1$ is isolated in $\mathbb{R}^2 \times [\frac{1}{4}, \frac{1}{2}]$, and that the construction of $\tilde{k}_1$ with specific reference to the double points of $\tilde{k}_1$ with itself, coincides exactly with the "laying down the rope" algorithm. Thus by Alexander, $\tilde{k}_1$ is contractible in $\left(\cup_{u\neq 1} \tilde{k}_u\right)^C$, and bounds a non-singular disc D in $\mathbb{R}^2 \times [\frac{1}{4}, \frac{1}{2}] \subseteq \tilde{M}$, the universal cover. Consider $e(\tilde{k}_1) \subseteq M \times I : e(\tilde{k}_1)$ bounds $e(D)$. Since $D \in \mathbb{R}^2 \times [\frac{1}{4}, \frac{1}{2}]$ is isolated from the other curves $\{\tilde{k}_u\}$, if $e(D)$ has singularities, these will be points of self-intersection. We invoke now a celebrated result:

Theorem 6 (Dehn's Lemma) *Let M be a PL 3-manifold, compact or not, with boundary which may be empty, and in M let D be a two-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve C and such that there exists a closed neighbourhood of C in D which is an annulus (that is, no point of C is singular). Then there exists a two-cell D_0 with boundary C semi-linearly embedded in M.*

By Dehn's Lemma, $e(D)$ may be modified in a neighbourhood of these singularities, and thus within $M \times [\frac{1}{4}, \frac{1}{2}]$, so that $e(\tilde{k}_1)$ bounds a non-singular disc.

There remains one final surface to attend to. As noted before, the fundamental group of the projective plane is not orderable, so this case requires an extra step that will reduce the argument to that of the Möbius band, which has fundamental group $\mathbb{Z}$ (the Möbius band deformation retracts to S^1). If $p(k)$ is the regular projection onto $S \times \{0\}$ of a knot in $S \times I$, where S is now the projective plane, one can transform the closed curve $p(k)$ into a finite number of simple closed curves $C_1, C_2, ..., C_m$, by choosing open neighbourhoods of each crossing point small enough to contain no other crossing point and "splicing" the crossing in the following way: if P is the crossing point of arcs α and β, and $N(P)$ is the selected small disc neighbourhood, one can choose a point $A \in \alpha \cap N(P)$ preceding P in the orientation of $p(k)$, and a point B on β succeeding P in the orientation. Then one joins A and B by an arc γ within $N(P)$ that doesn't elsewhere intersect $p(k)$, and by deleting the open arcs AP and PB one obtains disjoint closed curves.

If we choose a basepoint x on $p(k)$, $p(k)$ is homotopic to a product of conjugates of the C_i, namely $w_i C_i w_i^{-1}, i \leq i \leq m$, where w_i is an approach path (the trivial path in the case $i = 1$) from the basepoint x that connects to C_i across the disc neighbourhood and coincides exactly with $p(k)$ elsewhere. It will now be expedient to transcribe this set-up into the language of homology: note that as a closed path $p(k)$ can be considered as a 1-cycle, that is, as a (formal) sum of oriented edges whose boundary is 0. So we have $p(k)$ homotopic to a path product $w_1C_1w_1^{-1}w_2C_2w_2^{-1}\cdots w_mC_mw_m^{-1}$, but this latter is the same in homology as $\sum_{i=1}^m C_i$, since the formal sum of edges is unchanged under conjugation by the $\{w_i\}$. Thus since $p(k)$ is contractible, $w_1C_1w_1^{-1}w_2C_2w_2^{-1}\cdots w_mC_mw_m^{-1}$ is contractible, whence $\sum_{i=1}^m C_i$ is homologous to 0. It will follow that each C_i is null-homologous and thus contractible: for suppose that some C_i is not null-homologous. Since the fundamental group of $\mathbb{R}P^2$ is already abelian, $H_1(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ implies that an even number of the C_i must be non-bounding as $\sum_{i=1}^m C_i \sim 0$. Assume that C_1 and C_2 are not null-homologous. Recall that $\mathbb{R}P^2 \cong N \cup_\partial D^2$, where N is a Möbius band attached along the boundary of the disc D^2. If C_1 does not bound a disc, it circles around the Möbius band. So $C_2 \subseteq S - C_1$ is contained in the interior of D^2, hence is contractible, contradicting the assumption that C_2 is non-bounding. Thus each C_i bounds a disc.

Since each C_i is bounding, each C_i separates $\mathbb{R}P^2$ into two connected regions, a disc D_i and the complement of the disc, which contains a Möbius band. Since no C_i intersects any C_j, $i \neq j$, their corresponding discs will either be disjoint or one will be contained in the other. If D_i contains C_j, then D_i contains D_j (as D_i contains no Möbius band); so we can take $D_1, D_2, ..., D_n$, $n \in \{1, ..., m\}$ to be the outermost discs not contained in any other discs, and set $\hat{S} = \bigcup_{i=1}^n D_i \cup N(P)$, where $N(P)$ is the disc neighbourhood for each crossing P of $p(k)$; $\hat{S}$ is a proper connected subspace of $\mathbb{R}P^2$ which contains $p(k)$. Since within $\hat{S}$ we have $p(k)$ homotopic to a product of conjugates of C_i, each of which is contractible in $\hat{S}$, it follows that $p(k)$ is contractible in $\hat{S}$, hence contractible in the complement of a small disc in S, that is, within a Möbius band contained in S. With this, the argument is returned to the earlier case of the proof, and we are done. □

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