An Illustrative Orbit of the Second Genus.

by. Walter Henry Gage

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS.

by

WALTER HENRY GAGE

A Thesis submitted for the Degree of

MASTER OF ARTS

in the Department

of

MATHEMATICS

THE UNIVERSITY OF BRITISH COLUMBIA

APRIL 1926.

Accepted

D. Buchanan

April 1926

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS.

Contents.

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS.

1. The Problem

The object of this paper is to construct algebraically and graphically a periodic orbit of the second genus.

In 1772 Lagrange[1] showed that if two finite spheres revolve in circles about their common centre of mass, then there are three points on the line joining their centres at which an infinitesimal body would remain if given an initial projection so as to be instantaneously fixed with respect to the moving bodies.

When the infinitesimal body is given an initial displacement from the point of equilibrium, it is possible, by proper choice of initial conditions, to construct closed orbits relative to the moving system.

Orbits known as periodic orbits of the first genus have been discussed by Poincaré[2], Darwin, Plummer, and others, including Moulton, whose "Oscillating Satellite[3]" (Chapter V, "Periodic Orbits") is quoted in this paper.

Second genus periodic orbits are defined by Poincaré. A discussion of such orbits, a proof of their existence, and the theory for their construction, are given by Dr. D. Buchanan in a paper entitled "Periodic Orbits of the Second Genus Near the Straight-Line Equilibrium Points."

Our construction is made following the method suggested by this latter dissertation.

2. The Equations of Motion.

Let the motion of the infinitesimal body be referred to a set of rotating rectangular co-ordinates ξ, η, ζ of which the origin is at the centre of mass of the finite bodies, the ξ axis is the line joining the finite bodies, and the ξ-η- plane is the plane of their motion. The units of length, mass, and time will be taken so that the distance between the finite bodies, the sum of their masses, and the constant of proportionality respectively shall each be unity. Let the masses of the finite bodies be denoted by μ and $1-\mu$, $(0<\mu \leqq \frac{1}{2})$. On denoting the co-ordinates of the infinitesimal body by ξ, η, ζ and differentiation with respect to t by primes, the differential equations of motion are[4]

$$
\begin{aligned}
\xi'' - 2\eta' &= \frac{\partial u}{\partial \xi}, \\
\eta'' + 2\xi' &= \frac{\partial u}{\partial \eta}, \\
\zeta'' &= \frac{\partial u}{\partial \zeta}, \\
2u &= \xi^2 + \eta^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}, \\
r_1 &= \left((\xi+\mu)^2 + \eta^2 + \zeta^2\right)^{\frac{1}{2}}, \\
r_2 &= \left((\xi-1+\mu)^2 + \eta^2 + \zeta^2\right)^{\frac{1}{2}}.
\end{aligned}
\qquad (1)
$$

The equilibrium points are given by the solutions of the equations[4]

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial \zeta}.$$

Of the two sets of points satisfying these equations we are concerned in this discussion only with those points which lie on the straight line joining the finite bodies. In this set there are three points, which are denoted by (a), (b), (c), where (a) lies between $+\infty$ and the finite mass μ, (b) between μ and $1-\mu$, and C between $1-\mu$ and $-\infty$. The co-ordinates of these points are represented by ξ_0, 0, 0 where the particular value of ξ_0 depends upon the point in question.

3. The First Genus Periodic Orbits.

To obtain the first genus periodic orbits we displace the infinitesimal body so that

$$\xi = \xi_0 + \varepsilon x, \quad \eta = 0 + \varepsilon y, \quad \zeta = 0 + \varepsilon z, \qquad (2)$$

where ε is an arbitrary parameter, and transform the time by putting

$$t - t_0 = (1+\delta)\tau,$$

where δ is determined as a function of ε so that the solutions in x, y and z shall be periodic with period 2π in τ. On denoting differentiation with respect to τ by a dot over the variable, equations[3] become

$$\begin{aligned} \ddot{x} - 2(1+\delta)\dot{y} &= (1+\delta)^2\{X_1 + X_2\varepsilon + \cdots + X_k\varepsilon^{k-1} + \cdots\}, \\ \ddot{y} + 2(1+\delta)\dot{x} &= (1+\delta)^2\{Y_1 + Y_2\varepsilon + \cdots + Y_k\varepsilon^{k-1} + \cdots\}, \qquad (3) \\ \ddot{z} &= (1+\delta)^2\{Z_1 + Z_2\varepsilon + \cdots + Z_k\varepsilon^{k-1} + \cdots\}, \end{aligned}$$

where X , Y , and Z , are homogeneous polynomials of degree K in x,y and z. The values needed in the practical constructions are[3]

$$
\begin{aligned}
X_1 &= (1+2A)x, \quad X_2 = \tfrac{3}{2} B(-2x^2+y^2+z^2), \\
X_3 &= 2C(2x^3-3xy^2-3xz^2), \\
Y_1 &= (1-A)y, \quad Y_2 = 3Bxy, \\
Y_3 &= \tfrac{3}{2} C(-4x^2y+y^3+yz^2), \\
Z_1 &= -Az, \quad Z_2 = 3Bxz, \\
Z_3 &= \tfrac{3}{2} c(-4x^2z+y^2z+z^3), \\
A &= \frac{1-\mu}{r_1^{(0)2}} + \frac{\mu}{r_2^{(0)2}}, \quad r_1^{(0)} = +\sqrt{(\xi_0+\mu)^2} \\
B &= \begin{smallmatrix}+\\+\\-\end{smallmatrix} \frac{1-\mu}{r_1^{(0)4}} \begin{smallmatrix}+\\-\\-\end{smallmatrix} \frac{\mu}{r_2^{(0)4}}, \quad r_2^{(0)} = +\sqrt{(\xi_0-1+\mu)^2} \\
C &= \frac{1-\mu}{r_1^{(0)5}} + \frac{\mu}{r_2^{(0)5}}.
\end{aligned}
\tag{4}
$$

The upper, middle or lower signs are taken in B, according as the equilibrium point is (a), (b), or (c) respectively.

The periods of the periodic solutions of (3) are determined from the solutions of the linear terms. These solutions[3] are

$$
\begin{aligned}
x &= K_1 e^{i\sigma\tau} + K_2 e^{-i\sigma\tau} + K_3 e^{\rho\tau} + K_4 e^{-\rho\tau}, \quad i = \sqrt{-1}, \\
y &= in(K_1 e^{i\sigma\tau} - K_2 e^{-i\sigma\tau}) + m(K_3 e^{\rho\tau} - K_4 e^{-\rho\tau}), \\
z &= K_5 \cos\sqrt{A}\,\tau + K_6 \sin\sqrt{A}\,\tau, \\
n &= \frac{\sigma^2+1+2A}{2\sigma}, \qquad m = \frac{\rho^2-1-2A}{\rho},
\end{aligned}
$$

where K_1, ------ K_6, are constants of integration, and σ^2 and ρ^2 are the negative and positive roots of the quadratic in λ^2: -

$$\lambda^4 + (2-A)\lambda^2 + (1-A)(1+2A) = 0$$

One real period (for which $K_1 = K_2 = 0$) is $2\pi/\sqrt{A}$, and orbits with this period are called orbits of Class A. Our construction concerns this class only. The periodic solutions of Class A are

$$\begin{aligned}
\varepsilon x_1 &= 0\varepsilon + (a_1 + b_1 \cos 2\sqrt{A}\,\tau)\varepsilon^2 + 0\varepsilon^3 + \cdots, \\
\varepsilon y_1 &= 0\varepsilon + (c_1 \sin 2\sqrt{A}\,\tau)\varepsilon^2 + 0\varepsilon^3 + \cdots, \qquad (5)\\
\varepsilon z_1 &= (1/\sqrt{A} \sin \sqrt{A}\,\tau)\varepsilon + d_1(3\sin\sqrt{A}t - \sin 3\sqrt{A}t)\varepsilon^3 + \cdots, \\
\delta_1 &= 0\varepsilon + \delta_1^{(2)}\varepsilon^2 + \cdots\cdots + \delta_1^{2s}\varepsilon^{2s},
\end{aligned}$$

where
$$a_1 = \frac{-3B}{4A(1+2A)}, \quad b_1 = \frac{3B(1+3A)}{4A(1-7A+18A^2)};$$
$$c_1 = \frac{-3B}{\sqrt{A}(1-7A+18A^2)}, \quad d_1 = \frac{3}{64A^{\frac{5}{2}}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2} - C\right],$$
$$\delta^{(2)} = -\frac{9}{16A^2}\left[\frac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)} - C\right].$$

The initial conditions chosen are

$$\varepsilon z(0) = 0, \quad \varepsilon \dot{z}(0) = \varepsilon.$$

4. Periodic Orbits of the Second Genus.

Following the method in the "Periodic Orbits of the Second Genus,"[5] we return to equations (1) and make the same transformations as before with the exception of putting

$$t - t_0 = (1+\delta)(1+\gamma)\tau, \tag{6}$$

where δ and γ are as yet undefined. We next substitute $\varepsilon_0+\lambda$ for ε, where ε_0 and λ are arbitrary parameters. Equations similar to (3) result, except that $(1+\delta)$ is replaced by $(1+\delta)(1+\gamma)$ and ε by $(\varepsilon_0+\lambda)$.

A displacement of the infinitesimal body from the periodic orbit of the first genus is then given, so that

$$x = x_1 + p, \qquad y = y_1 + q, \qquad z = z_1 + r, \tag{7}$$

where $\varepsilon_0 x_1$, $\varepsilon_0 y_1$, and $\varepsilon_0 z_1$ define orbits whose solutions (along with δ) are defined by (5), where $\varepsilon_0+\lambda$ replaces ε.

When $\gamma = \lambda = 0$, we obviously have equations (3), and consequently the equations defining p, q and r are

$$\begin{aligned}
\ddot{p} - 2(1+\delta)\dot{q} &+ (1+\delta)^2\left[P_{11}p + P_{12}q + P_{13}r\right] \\
= (1+\delta)^2 P_1 &+ 2(1+\delta)(\dot{y}_1+\dot{q})\gamma \\
&+ (1+\delta)^2(2\gamma+\gamma^2)\Big[(1+2A)(x_1+p) + (\varepsilon_0+\lambda)\tfrac{3}{2}B\{-2(x_1+p)^2 \\
&+ (y_1+q)^2 + (z_1+r)^2\} + (\varepsilon_0+\lambda)^2 2C\{2(x_1+p)^3 - \\
&3(x_1+p)(y_1+q)^2 - 3(x_1+p)(z_1+r)^2 + \cdots\cdots\Big] \\
&+ (1+\delta)^2\lambda\Big[\tfrac{3}{2}B\{-2(x_1+p)^2 + (y_1+q)^2 + (z_1+r)^2\} \\
&+ 4\varepsilon_0 C\{2(x_1+p)^3 - 3(x_1+p)(y_1+q)^2 - 3(x_1+p)(z_1+r)^2\} + \cdots\Big],
\end{aligned}$$

$$+(1+\delta)^2\lambda^2\left[2C\left\{2(x_1+p)^3-3(x_1+p)(y_1+q)^2-3(x_1+p)(z_1+r)^2\right\}+\cdots\right]+\cdots,$$

$$\ddot{q}+2(1+\delta)\dot{p}+(1+\delta)^2\left[Q_{11}p+Q_{12}q+Q_{13}r\right]$$
$$=(1+\delta)^2Q_1-2(1+\delta)\gamma(\dot{x}_1+\dot{p}) \qquad (8)$$
$$+(1+\delta)^2(2\gamma+\gamma^2)\Big[(1-A)(y_1+q)+(\varepsilon_0+\lambda)3B(x_1+p)(y_1+q)$$
$$+(\varepsilon_0+\lambda)^2\tfrac{3}{2}C\left\{-4(x_1+p)^2(y_1+q)+(y_1+q)^3+(y_1+q)(z_1+r)^2\right\}+\cdots\Big]$$
$$+(1+\delta)^2\lambda\Big[3B(x_1+p)(y_1+q)+3\varepsilon_0C\left\{-4(x_1+p)^2(y_1+q)\right.$$
$$\left.+(y_1+q)^3+(y_1+q)(z_1+r)^2\right\}+\cdots\Big]$$
$$+(1+\delta)^2\lambda^2\Big[\tfrac{3}{2}C\left\{-4(x_1+p)^2(y_1+q)+(y_1+q)^3\right.$$
$$\left.+(y_1+q)(z_1+r)^2\right\}+\cdots\cdots\Big]+\cdots,$$

$$\ddot{r}+(1+\delta)^2\left[R_{11}p+R_{12}q+R_{13}r\right]$$
$$=(1+\delta)^2R_1$$
$$+(1+\delta)^2(2\gamma+\gamma^2)\Big[-A(z_1+r)+(\varepsilon_0+\lambda)3B(x_1+p)(z_1+r)$$
$$+(\varepsilon_0+\lambda)^2\tfrac{3}{2}C\left\{-4(x_1+p)^2(z_1+r)+(y_1+q)^2(z_1+r)\right.$$
$$\left.+(z_1+r)^3+\cdots\right\}+\cdots\Big]$$
$$+(1+\delta)^2\lambda\Big[3B(x_1+p)(z_1+r)+3\varepsilon_0C\left\{-4(x_1+p)^2(z_1+r)\right.$$
$$\left.+(y_1+q)^2(z_1+r)+(z_1+r)^3\right\}+\cdots\Big]$$
$$+(1+\delta)^2\lambda^2\Big[\tfrac{3}{2}C\left\{-4(x_1+p)^2(z_1+r)+(y_1+q)^2(z_1+r)\right.$$
$$\left.+(z_1+r)^3+\cdots\right.\Big]+\cdots,$$

where [6]

$$P_{11} = -1-2A+6B\varepsilon_0 x_1 - 6C\varepsilon_0^2(2x_1^2-y_1^2-z_1^2)+\cdots,$$

$$P_{12}=Q_{11}=-3B\varepsilon_0 y_1 + 12C\varepsilon_0^2 x_1 y_1 + \cdots,$$

$$P_{13}=R_{11}=-3B\varepsilon_0 z_1 + 12C\varepsilon_0^2 x_1 z_1 + \cdots,$$

$$Q_{12} = -1+A-3B\varepsilon_0 x_1 + \frac{3}{2}C\varepsilon_0^2(4x_1^2-3y_1^2-z_1^2)+\cdots,$$

$$Q_{13}=R_{12}=-3C\varepsilon_0^2 y_1 z_1 + \cdots,$$

$$R_{13} = A-3B\varepsilon_0 x_1 + \frac{3}{2}C\varepsilon_0^2(4x_1^2-y_1^2-3z_1^2)+\cdots,$$

$$P_1 = \frac{3}{2}B\varepsilon_0(-2p^2+q^2+r^2)+\cdots,$$

$$Q_1 = 3B\varepsilon_0 pq + \cdots,$$

$$R_1 = 3B\varepsilon_0 pr + \cdots.$$

5. The Equations of Variation.

If the linear terms of (8) only are considered we obtain the "equations of variation", whose solutions[6] are shown to be

$$p = K_1 e^{i\sigma_1\tau}u_1 + K_2 e^{-i\sigma_1\tau}u_2 + K_3 e^{\rho_1\tau}u_3 + K_4 e^{-\rho_1\tau}u_4 + K_5 u_5 + K_6(u_6 + K\tau u_5),$$

$$q = i(K_1 e^{i\sigma_1\tau}v_1 - K_2 e^{-i\sigma_1\tau}v_2) + K_3 e^{\rho_1\tau}v_3 - K_4 e^{-\rho_1\tau}v_4 + K_5 v_5 + K_6(v_6 + K\tau v_5),$$

$$r = i(K_1 e^{i\sigma_1\tau}w_1 - K_2 e^{-i\sigma_1\tau}w_2) + K_3 e^{\rho_1\tau}w_3 - K_4 e^{-\rho_1\tau}w_4 + K_5 w_5 + K_6(w_6 + K\tau w_5).$$

where

$$u_1 = 1 + (\ ) \varepsilon_0^2 + \cdots\cdots,$$
$$v_1 = \lambda + (\ ) \varepsilon_0^2 + \cdots\cdots,$$
$$w_1 = 3B/(4A - \sigma^2)\left[2/\sigma \cos\sqrt{A}\,\tau - i/\sqrt{A} \sin\sqrt{A}\,\tau\right]\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots,$$
$$\sigma_1 = \sigma + \varepsilon_0^2\left[\frac{9B^2\{\sigma^2(1-13A) - (3+7A-22A^2)\}}{16A\sigma^3(4A-\sigma^3)} - \frac{\delta_1^{(2)}(1-A)(1+2A)}{\sigma^3}\right] + \cdots$$

u_2, v_2, w_2, differ from u_1, v_1, w_1 in the sign of i only.

$$u_3 = 1 + (\ ) \varepsilon_0^2 + \cdots,$$
$$v_3 = m + (\ ) \varepsilon_0^2 + \cdots,$$
$$w_3 = -3B/(4A+\rho^2)\left[2/\rho \cos\sqrt{A}\,\tau - \frac{1}{\sqrt{A}} \sin\sqrt{A}\,\tau\right]\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots,$$
$$\rho_1 = \rho + \varepsilon_0^2\left[\frac{9B^2\{\rho^2(1-13A)+3+7A-22A^2\}}{16A\rho^3(4A+\rho^2)} + \frac{\delta_1^{(2)}(1-A)(1+2A)}{\rho^3}\right],$$
$$u_4(\tau) = u_3(-\tau), \quad v_4(\tau) = v_3(-\tau), \quad w_4(\tau) = w_3(-\tau),$$

$$u_5 = (-2b_1\sqrt{A} \sin\sqrt{A}\,\tau)\varepsilon_0^2 + (\ )\varepsilon_0^4 + \cdots,$$
$$v_5 = (2c_1\sqrt{A} \cos 2\sqrt{A}\,\tau)\varepsilon_0^2 + (\ )\varepsilon_0^4 + \cdots,$$
$$w_5 = (\cos\sqrt{A}\,\tau)\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots,$$
$$u_6 = 2(a_1 + b_1 \cos 2\sqrt{A}\,\tau)\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots,$$
$$v_6 = 2(c_1 \sin 2\sqrt{A}\,\tau)\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots,$$
$$w_6 = 1/\sqrt{A} \sin\sqrt{A}\,\tau + (\ )\varepsilon_0^2 + \cdots,$$

$$K = -1/(1+\delta_1)\left[2\delta_1^{(2)}\varepsilon_0 + \cdots\cdots(\ )\varepsilon_0^3 + \cdots\right].$$

6. The Complete Equations.

In order to solve the complete equations we put[5]

$$p = \sum_{J=1}^{\infty} p_J \lambda^J, \quad q = \sum_{J=1}^{\infty} q_J \lambda^J, \quad r = \sum_{J=1}^{\infty} r_J \lambda^J, \quad \gamma = \sum_{J=1}^{\infty} \gamma_J \lambda^J.$$

We now solve the differential equations obtained by equating the same powers of λ in equations (8) after the above transformation is made. Since λ can be taken arbitrarily small the constructions are made for the first power of λ only, and for the second in ε_0. Certain terms, however, in the equations for λ^2 are considered in the determination of γ_1.

The Coefficients of λ.

The equations involving the coefficients of λ in equations (8) are

$$\begin{aligned}
&\ddot{p}_1 - 2(1+\delta)\dot{q}_1 + (1+\delta)^2[P_{11}p_1 + P_{12}q_1 + P_{13}r_1] \\
&\quad = \gamma_1\{\varepsilon_0(S_{11} + S_{12}\cos 2\sqrt{A}\tau) + (\ )\varepsilon_0^3 + \cdots\} \\
&\qquad + \{(S_{13} - S_{13}\cos 2\sqrt{A}\tau) + (S_{14} + S_{15}\cos 2\sqrt{A}\tau + S_{16}\cos 4\sqrt{A}\tau)\varepsilon_0^2\} + , \\
&\ddot{q}_1 + 2(1+\delta)\dot{p}_1 + (1+\delta)^2[Q_{11}p_1 + Q_{12}q_1 + Q_{13}r_1] \\
&\quad = \gamma_1\{(S_{21}\sin 2\sqrt{A}\tau)\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots\} \\
&\qquad + \{S_{22}\sin 2\sqrt{A}\tau + S_{23}\sin 4\sqrt{A}\tau\}\varepsilon_0^2 + \cdots, \\
&\ddot{r}_1 + (1+\delta)^2[R_{11}p_1 + R_{12}q_1 + R_{13}r_1] \\
&\quad = \gamma_1\{S_{31}\sin\sqrt{A}\tau + (S_{32}\sin\sqrt{A}\tau + S_{33}\sin 3\sqrt{A}\tau)\varepsilon_0^2 + \cdots\} \\
&\qquad + \{S_{34}\sin\sqrt{A}\tau + S_{35}\sin 3\sqrt{A}\tau\}\varepsilon_0 + (\ )\varepsilon_0^3 + \cdots,
\end{aligned} \tag{10}$$

where

$$S_{11} = 2(1+2A)a_1 + \frac{3B}{2A},$$

$$S_{12} = 4c_1\sqrt{A} - \frac{3B}{2A} + 2(1+2A)b_1,$$

$$S_{13} = \frac{3B}{4A},$$

$$S_{14} = \frac{3b_1C}{A} - \frac{6a_1C}{A} + \frac{9Bd_1}{2\sqrt{A}} + \frac{3B\delta_1^{(2)}}{2A} + \frac{3Bc_1^2}{4} - \frac{3Bb_1^2}{2} - 3Ba_1^2,$$

$$S_{15} = \frac{6a_1C}{A} - \frac{6b_1C}{A} - \frac{3Bd_1}{2\sqrt{A}} - \frac{9Bd_1}{2\sqrt{A}} - \frac{3B\delta_1^{(2)}}{2A} - 6a_1b_1B,$$

$$S_{16} = \frac{3b_1C}{A} + \frac{3}{2}\frac{Bd_1}{\sqrt{A}} - \frac{3Bc_1^2}{4} - \frac{3Bb_1^2}{2},$$

$$S_{22} = 3Ba_1c_1 + \frac{3Cc_1}{2A},$$

$$S_{21} = 4b_1\sqrt{A} + 2(1-A)c_1,$$

$$S_{23} = \frac{3Bb_1c_1}{2} - \frac{3Cc_1}{4A},$$

$$S_{31} = -2\sqrt{A},$$

$$S_{32} = -6Ad_1 + \frac{9C}{4A^{\frac{3}{2}}} + 6\frac{Ba_1}{\sqrt{A}} - \frac{3Bb_1}{\sqrt{A}} - 4A\delta_1^{(2)},$$

$$S_{33} = 2Ad_1 - \frac{3C}{4A^{\frac{3}{2}}} + 3\frac{Bb_1}{\sqrt{A}},$$

$$S_{34} = \frac{3Ba_1}{\sqrt{A}} - \frac{3b_1}{2\sqrt{A}} + \frac{9C}{4A^{\frac{3}{2}}},$$

$$S_{35} = \frac{3b_1}{2\sqrt{A}} - \frac{3C}{4A^{\frac{3}{2}}}.$$

The complementary functions of these differential equations are similar to (9) with, of course, different arbitrary multipliers. The particular integrals are obtained by the well known method of the "variation of parameters". The steps in the solution for the particular integrals are explained fully in the "Periodic Orbits of the Second Genus"[5], and therefore the results only are given in this paper.

As explained in the aforementioned paper the arbitrary multipliers (denoted by $k_1^{(1)}$ .... $k_6^{(1)}$) can be dealt with in pairs; and from each pair there results a part of the particular integral.

$k_1^{(1)}$ and $k_2^{(1)}$ give:-

$$p_1 = N\Bigg[\Big\{\frac{2F_4}{\sigma_1} + \frac{2F_4\sigma_1}{4-\sigma_1^2}\cos 2\sqrt{A}\,\tau\Big\} + \Big\{\frac{2F_7}{\sigma_1} - \frac{2F_8\sigma_1+4F_{11}}{4-\sigma_1^2}\cos 2\sqrt{A}\,\tau$$

$$- \frac{2F_9\sigma_1+8F_{12}}{16-\sigma_1^2}\cos 2\sqrt{A}\,\tau - \frac{2F_{10}\sigma_1+12F_{13}}{36-\sigma_1^2}\cos 6\sqrt{A}\,\tau\Big\}\xi_0^2 + \cdots$$

$$+ \gamma_1\Big\{\frac{2F_1}{\sigma_1} - \frac{2F_2\sigma_1+4F_5}{4-\sigma_1^2}\cos 2\sqrt{A}\,\tau - \frac{2F_3\sigma_1+8F_6}{16-\sigma_1^2}\cos 4\sqrt{A}\,\tau\Big\}\xi + \cdots\Bigg]$$

$$q_1 = Nn\Bigg[\Big\{-\frac{4F_4}{4-\sigma_1^2}\sin 2\sqrt{A}\,\tau\Big\} + \Big\{\frac{4F_8+2\sigma_1F_{11}}{4-\sigma_1^2}\sin 2\sqrt{A}\,\tau$$

$$+ \frac{8F_9+2\sigma_1F_{12}}{16-\sigma_1^2}\sin 4\sqrt{A}\,\tau + \frac{12F_{10}+2\sigma_1F_{13}}{36-\sigma_1^2}\sin 6\sqrt{A}\,\tau\Big\}\xi_0^2 + \cdots$$

$$+ \gamma_1\Big\{\frac{4F_2+2\sigma_1F_5}{4-\sigma_1^2}\sin 2\sqrt{A}\,\tau + \frac{8F_3+2\sigma_1F_6}{16-\sigma_1^2}\sin 4\sqrt{A}\,\tau\Big\}\xi_0 + \cdots \qquad (11)$$

$$r_1 = \frac{3NB}{4A-\sigma^2}\Bigg[\Bigg\{\left(\frac{2F_4}{\sigma_1\sqrt{A}} - \frac{4F_4}{\sigma_1(4-\sigma_1^2)} - \frac{F_4\sigma_1}{\sqrt{A}(4-\sigma_1^2)}\right)\sin\sqrt{A}\tau$$
$$+\left(\frac{F_4\sigma_1}{\sqrt{A}(4-\sigma_1^2)} - \frac{4F_4}{\sigma_1(4-\sigma_1^2)}\right)\sin 3\sqrt{A}\tau\Bigg\}\xi_0 + \cdots$$
$$+\gamma_1\Bigg\{\left(\frac{2F_1}{\sigma_1\sqrt{A}} + \frac{F_2\sigma_1+2F_5}{\sqrt{A}(4-\sigma_1^2)} + \frac{4F_2+2\sigma_1F_5}{\sigma_1(4-\sigma_1^2)}\right)\sin\sqrt{A}\tau$$
$$+\left(\frac{4F_2+2\sigma_1F_5}{\sigma_1(4-\sigma_1^2)} + \frac{8F_3+2\sigma_1F_6}{\sigma_1(16-\sigma_1^2)} + \frac{F_3\sigma_1+4F_6}{\sqrt{A}(16-\sigma_1^2)} - \frac{F_2\sigma_1+2F_5}{\sqrt{A}(4-\sigma_1^2)}\right)\sin 3\sqrt{A}\tau$$
$$+\left(\frac{8F_3+2\sigma_1F_6}{\sigma_1(16-\sigma_1^2)} - \frac{F_3\sigma_1+4F_6}{\sqrt{A}(16-\sigma_1^2)}\right)\sin 5\sqrt{A}\tau\Bigg\}\xi_0^2 + \cdots\Bigg].$$

$k_3^{(1)}$ and $k_4^{(1)}$ give:-

$$p_1 = -N\Bigg[\left\{\frac{2g_4}{\rho_1} - \frac{2\rho_1 g_4}{4+\rho_1^2}\cos 2\sqrt{A}\tau\right\} + \Bigg\{\frac{2g_7}{\rho_1} + \frac{2\rho_1 g_8+4g_{11}}{4+\rho_1^2}\cos 2\sqrt{A}\tau$$
$$+\frac{2\rho_1 g_9+8g_{12}}{\rho_1^2+16}\cos 4\sqrt{A}\tau + \frac{2\rho_1 g_{10}+12g_{13}}{\rho_1^2+36}\cos 6\sqrt{A}\tau\Bigg\}\xi_0^2 + \cdots$$
$$+\gamma_1\left\{\frac{2g_1}{\rho_1} + \frac{2\rho_1 g_2+4g_5}{\rho_1^2+4}\cos 2\sqrt{A}\tau + \frac{2g_3+8g_6}{\rho_1^2+16}\cos 4\sqrt{A}\tau\right\}\xi + \cdots\Bigg]$$

$$q_1 = Nm\Bigg[\left\{-\frac{4g_4}{\rho_1^2+4}\right\} + \Bigg\{\frac{4g_8-2\rho_1 g_{11}}{\rho_1^2+4}\sin 2\sqrt{A}\tau + \frac{8g_9-2\rho_1 g_{12}}{\rho_1^2+16}\sin 4\sqrt{A}\tau$$
$$+\frac{12g_{10}-2\rho g_{13}}{\rho_1^2+36}\sin 6\sqrt{A}\tau\Bigg\}\xi_0^2 + \cdots$$
$$+\gamma_1\left\{\frac{4g_2-2\rho g_5}{\rho_1^2+4}\sin 2\sqrt{A}\tau + \frac{8g_3-2\rho_1 g_6}{\rho_1^2+16}\sin 4\sqrt{A}\tau\right\}\xi + \cdots\Bigg]$$

$$r_1 = -\frac{3NB}{4A+\rho^2}\left[\left\{\left(\frac{2g_4}{\rho_1\sqrt{A}} - \frac{4g_4}{\rho_1(\rho_1^2+4)} + \frac{\rho_1 g_4}{\sqrt{A}(\rho_1^2+4)}\right)\sin\sqrt{A}\tau \right.\right. \tag{12}$$

$$\left. - \left(\frac{\rho_1 g_4}{\sqrt{A}(\rho_1^2+4)} + \frac{4g_4}{\rho_1(\rho_1^2+4)}\right)\sin 3\sqrt{A}\tau\right\}\xi_0 + \cdots$$

$$+ \gamma_1\left\{\left(\frac{2g_1}{\rho_1\sqrt{A}} + \frac{4g_2 - 2\rho_1 g_5}{\rho_1(\rho_1^2+4)} + \frac{\rho_1 g_2 + 2g_5}{\sqrt{A}(\rho_1^2+4)}\right)\sin\sqrt{A}\tau\right.$$

$$+ \left(\frac{\rho_1 g_2 + 2g_5}{\sqrt{A}(\rho_1^2+4)} - \frac{\rho_1 g_3 + 4g_6}{\sqrt{A}(\rho_1^2+16)} + \frac{4g_2 - 2\rho_1 g_5}{\rho_1(\rho_1^2+4)} + \frac{8g_3 - 2\rho_1 g_6}{\rho_1(\rho_1^2+16)}\right)\sin 3\sqrt{A}\tau$$

$$\left.\left. + \left(\frac{\rho_1 g_3 + 4g_6}{\sqrt{A}(\rho_1^2+16)} + \frac{8g_3 - 2\rho_1 g_6}{\rho_1(\rho_1^2+16)}\right)\sin 5\sqrt{A}\tau\right\}\xi_0^2 + \cdots\right]$$

$k_5^{(1)}$ and $k_6^{(1)}$ give:-

$$p_1 = N\left[\gamma_1\left\{\left(\frac{H_1 b_1}{2} - \frac{H_8 b_1}{4\delta_1^{(4)}\sqrt{A}}\right) - \frac{H_8 a_1}{2\delta_1^{(4)}\sqrt{A}}\cos 2\sqrt{A}\tau - \left(\frac{H_1 b_1}{2} + \frac{H_8}{4\delta_1^{(4)}\sqrt{A}}\right)\cos 4\sqrt{A}\tau\right\}\xi_0 - - \right.$$

$$- \left\{\left(\frac{H_6 b_1}{2} + \frac{H_5 b_1}{4\delta_1^{(4)}\sqrt{A}}\right) + \left(\frac{H_7 b_1}{4} + \frac{H_5 a_1}{2\delta_1^{(4)}\sqrt{A}} + \frac{H_2 b_1}{8\delta_1^{(4)}\sqrt{A}}\right)\cos 2\sqrt{A}\tau\right.$$

$$\left. + \left(\frac{H_5 b_1}{4\delta_1^{(4)}\sqrt{A}} - \frac{H_6 b_1}{2} + \frac{H_2 a_1}{4\delta_1^{(4)}\sqrt{A}}\right)\cos 4\sqrt{A}\tau + \left(\frac{H_2 b_1}{8\delta_1^{(4)}\sqrt{A}} - \frac{H_7 b_1}{4}\right)\cos 6\sqrt{A}\tau\right\}\xi_0^2$$

$$\left. + \frac{1}{\varepsilon_0}\left\{\gamma_1(H_1 + H_{11}\varepsilon_0^2) + H_{12}\varepsilon_0\right\}\tau u_5 + \cdots\right]$$

$$
\begin{aligned}
q_1 = N\Big[&\gamma_1\Big\{-\frac{H_1 C_1}{2} - \frac{H_8 C_1}{4\delta_1^{(4)}\sqrt{A}}\Big\}\varepsilon_0 \sin 4\sqrt{A}\tau + \Big\{\Big(\frac{H_7 C_1}{4} + \frac{H_2 C_1}{8\delta_1^{(4)}\sqrt{A}}\Big)\sin 2\sqrt{A}\tau \\
&+ \Big(\frac{H_6 C_1}{2} - \frac{H_5 C_1}{4\delta_1^{(4)}\sqrt{A}}\Big)\sin 4\sqrt{A}\tau + \Big(\frac{H_7 C_1}{4} - \frac{H_2 C_1}{8\delta_1^{(4)}\sqrt{A}}\Big)\sin 6\sqrt{A}\tau\Big\}\varepsilon_0^2 + \cdots \\
&+ \frac{1}{\varepsilon_0}\Big\{\gamma_1(H_1 + H_{11}\varepsilon_0^2) + H_{12}\varepsilon_0\Big\}\tau\omega_5 + \cdots\Big]
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
r_1 = N\Big[&\gamma_1\Big\{\Big(-\frac{H_1}{4\sqrt{A}} + \frac{H_8}{8\delta_1^{(4)}\sqrt{A}}\Big)\sin\sqrt{A}\tau - \Big(\frac{H_1}{4\sqrt{A}} + \frac{H_8}{8\delta_1^{(4)}\sqrt{A}}\Big)\sin 3\sqrt{A}\tau\Big\} \\
&+ \gamma_1\Big\{\Big(\frac{H_3}{4\sqrt{A}} + \frac{H_8}{8A} + \frac{H_9}{8\delta_1^{(4)}A}\Big)\sin\sqrt{A}\tau + \Big(\frac{H_3}{4\sqrt{A}} + \frac{H_4}{8\sqrt{A}} - \frac{H_9}{8\delta_1^{(4)}\sqrt{A}} \\
&+ \frac{H_{10}}{16\delta_1^{(4)}A} + \frac{H_{11}}{8A}\Big)\sin 3\sqrt{A}\tau + \Big(\frac{H_4}{8\sqrt{A}} - \frac{H_{10}}{16\delta_1^{(4)}A}\Big)\sin 5\sqrt{A}\tau\Big\}\varepsilon_0^2 + \cdots \\
&+ \Big\{\Big(\frac{H_6}{4\sqrt{A}} + \frac{H_5}{8A\delta_1^{(2)}}\Big)\sin\sqrt{A}\tau + \Big(\frac{H_6}{4\sqrt{A}} + \frac{H_7}{8\sqrt{A}} - \frac{H_5}{8\delta_1^{(4)}A} + \frac{H_2}{16\delta_1^{(4)}A}\Big)\sin 3\sqrt{A}\tau \\
&+ \Big(\frac{H_7}{8\sqrt{A}} - \frac{H_2}{16\delta_1^{(4)}A}\Big)\sin 5\sqrt{A}\tau\Big\}\varepsilon_0 + \cdots \\
&+ \frac{1}{\varepsilon_0}\Big\{\gamma_1(H_1 + H_{11}\varepsilon_0^2) + H_{12}\varepsilon_0\Big\}\tau\omega_5 + \cdots\Big].
\end{aligned}
$$

where

$$F_1 = E_1 S_{11} + \frac{E_2 S_{31}}{2}, \qquad E_1 = -2m(\sigma n + \rho m),$$

$$F_2 = E_1 S_{12}, \qquad E_2 = -(\sigma n + \rho m)(2\rho c_1 + 4b_1 m\sqrt{A}),$$

$$F_3 = -\frac{E_2 S_{31}}{2}, \qquad E_3 = 2(m\sigma - n\rho),$$

$$F_4 = E_1 S_{13}, \qquad E_4 = (m\sigma - n\rho)(2m\rho b_1 - 4c_1\sqrt{A}),$$

$$F_5 = E_3 S_{21} + \frac{E_5 S_{31}}{2} - \frac{E_4 S_{31}}{2},$$

$$F_6 = \frac{E_4 S_{31}}{2}, \qquad E_5 = (m\sigma - n\rho)(2m\rho b_1 + 4m\rho a_1 - 4c_1\sqrt{A})$$

$$F_7 = E_1 S_{14} + \frac{E_2 S_{35}}{2} + \frac{E_2 S_{34}}{2},$$

$$F_8 = E_1 S_{15} + \frac{E_2 S_{35}}{2},$$

$$F_9 = E_1 S_{16} - \frac{E_2 S_{34}}{2} - \frac{E_2 S_{35}}{2},$$

$$F_{10} = -\frac{E_2 S_{35}}{2}, \qquad F_{11} = E_3 S_{22} + \frac{E_5 S_{34}}{2} + \frac{E_5 S_{35}}{2} - \frac{E_4 S_{34}}{2},$$

$$F_{12} = E_3 S_{23} + \frac{E_4 S_{34}}{2} + \frac{E_5 S_{35}}{2}, \qquad F_{13} = \frac{E_4 S_{35}}{2},$$

$$G_1 = E_6 S_{11} + \frac{E_7 S_{31}}{2}, \quad E_6 = 2n(\sigma n + \rho m),$$

$$G_2 = E_6 S_{12}, \qquad G_3 = -\frac{E_7 S_{31}}{2}, \quad G_4 = E_6 S_{13}$$

$$G_5 = E_8 S_{21} + \frac{E_9 S_{31}}{2} - \frac{E_{10} S_{31}}{2}, \quad E_7 = (\sigma n + \rho m)(2\sigma c_1 + 4b_1 n\sqrt{A}),$$

$$G_6 = \frac{E_{10} S_{31}}{2}, \qquad E_8 = 2(n\sigma - m\rho),$$

$$G_7 = E_6 S_{14} + \frac{E_7 S_{34}}{2} + \frac{E_7 S_{35}}{2}, \quad E_9 = 4(m\sigma - n\rho)(n\sigma a_1 + c\sqrt{A} + \frac{n\sigma b_1}{2}),$$

$$G_8 = E_6 S_{15} + \frac{E_7 S_{35}}{2}, \quad E_{10} = 4(m\sigma - n\rho)(\sqrt{A}\, c_1 + \frac{n\sigma b_1}{2}),$$

$$G_9 = E_6 S_{16} - \frac{E_7 S_{35}}{2} - \frac{E_7 S_{34}}{2}, \quad G_{10} = -\frac{E_7 S_{35}}{2},$$

$$G_{11} = E_8 S_{22} + \frac{E_9 S_{34}}{2} - \frac{E_{10} S_{34}}{2} + \frac{E_9 S_{35}}{2},$$

$$H_1 = \frac{E_{11} S_{31}}{2}, \qquad E_{11} = \frac{4(\sigma n + \rho m)(m\sigma - n\rho)}{\sqrt{A}},$$

$$H_2 = \frac{E_{12} S_{35}}{2}, \qquad E_{12} = -8\delta_1^{(2)}(\sigma n + \rho m)(m\sigma - n\rho),$$

$$H_3 = \frac{E_{13}}{2} S_{12} + E_{14} S_{11} - \frac{E_{11} S_{32}}{2} + \frac{E_{11} S_{33}}{3} - \frac{E_{15} S_{31}}{2} + \frac{E_{16} S_{31}}{2},$$

$$H_4 = \frac{E_{14}S_{12}}{2} - \frac{E_{11}S_{33}}{2} - \frac{E_{16}S_{31}}{2} - \frac{E_{17}S_{31}}{2},$$

$$H_5 = \frac{E_{12}S_{34}}{2} + \frac{E_{12}S_{35}}{2}, \quad H_6 = E_{14}S_{13} - E_{13}S_{13} - \frac{E_{11}S_{34}}{2} + \frac{E_{11}S_{35}}{2},$$

$$H_7 = -\frac{E_{14}S_{13}}{2} - \frac{E_{11}S_{35}}{2}, \quad H_8 = \frac{E_{12}S_{31}}{2},$$

$$H_9 = \frac{E_{12}S_{32}}{2} + \frac{E_{12}S_{33}}{2},$$

$$H_{10} = \frac{E_{12}S_{33}}{2}, \quad H_{12} = E_{13}S_{13} + \frac{E_{11}S_{34}}{2} - \frac{E_{14}S_{13}}{2},$$

$$H_{11} = E_{13}S_{11} + \frac{E_{17}S_{21}}{2} + \frac{E_{11}S_{32}}{2} + \frac{E_{15}S_{11}}{2} + \frac{E_{14}S_{12}}{2},$$

$$E_{13} = 6B(\sigma n + \rho m)\left\{\frac{4n}{\rho(4A+\rho^2)} - \frac{4m}{\sigma(4A-\sigma^2)} + \frac{n\rho}{A(4A+\rho^2)} + \frac{m\sigma}{A(4A-\sigma^2)}\right\},$$

$$E_{14} = -6B(\sigma n + \rho m)\left\{\frac{\rho n}{A(4A+\rho^2)} + \frac{\sigma n}{\rho(4A-\sigma^2)}\right\},$$

$$E_{15} = 24B\Bigg[\frac{(\sigma n + \rho m)}{4}\left\{\frac{8b_1 n\sqrt{A} + 4\sigma c_1}{\rho(4A+\rho^2)} + \frac{8b_1 n\sqrt{A} + 4\sigma c_1}{\sigma(4A-\sigma^2)}\right\}$$
$$+\left(a_1 - \frac{b_1}{2}\right)\frac{(\sigma n + \rho m)(\rho n - \sigma m)}{\sqrt{A}(4A+\rho^2)}$$
$$-(\rho n - \sigma m)\left\{\frac{1}{\sqrt{A}(4A+\rho^2)} - \frac{1}{\sqrt{A}(4A-\sigma^2)}\right\}\left\{\sqrt{A}c_1 - \frac{\rho m b_1}{2} + \rho m a_1\right\}\Bigg],$$

$$E_{16} = 24B\Bigg[\frac{\sigma n + \rho m}{4}\left\{\frac{8b_1 n\sqrt{A}}{\rho(4A+\rho^2)} + \frac{8b_1 n\sqrt{A} + 4\sigma c_1}{\sigma(4A-\sigma^2)}\right\}$$
$$+\frac{b_1}{2}\frac{(\sigma n + \rho m)(n\rho - \sigma m)}{\sqrt{A}(4A+\rho^2)}$$
$$+(n\rho - \sigma m)\left\{\frac{1}{\sqrt{A}(4A+\rho^2)} - \frac{1}{\sqrt{A}(4A-\sigma^2)}\right\}\left\{\sqrt{A}c_1 - \frac{\rho m b_1}{2}\right\}\Bigg],$$

$$E_{17} = \frac{12B(\sigma n - \rho m)(\rho^2 + \sigma^2)}{\sqrt{A}\,(4A + \rho^2)(4A - \sigma^2)},$$

$$N = \frac{1}{4(m\rho + n\sigma)(m\sigma - n\rho)}.$$

Thus the particular integrals of the equations in p_1, q_1, and r_1 are found by summing the various p,'s, q,'s, or r,'s, in (11), (12), (13). The complementary functions are similar to the solutions (9), where we denote the arbitrary constants by $K_1^{(1)}$, $K_2^{(1)}$, ------- $K_6^{(1)}$

It will be noticed that the particular integrals of the complete p_1, q_1, and r_1 equations are made up of periodic terms, some of which are multiplied by γ_1, and alos non-periodic terms (i.e. terms multiplied by τ), both with and without γ_1. Let us write the particular integrals of the p_1, q_1 and r_1 equations as

$$\gamma_1 C_1 + C_2 + (e_1\gamma_1 + e_2)\tau\, u_5,$$

$$\gamma_1 S_1 + S_2 + (e_1\gamma_1 + e_2)\tau\, v_5,$$

$$\gamma_1 S_3 + S_4 + (e_1\gamma_1 + e_2)\tau\, w_5,$$

respectively. C_1, C_2, S_1, S_2, S_3 and S_4 denote simply cosine or sine series, as the letter indicates, while the subscript means the particular series, determined from (11), (12) and (13).

Combining the particular integral and the complementary function we can now write the complete solutions for p_1, q_1, and r_1, as

$$p_1 = K_1^{(1)} e^{i\sigma_1\tau} u_1 + K_2 e^{-i\sigma_1\tau} u_2 + K_3 e^{\rho_1\tau} u_3 + K_4 e^{-\rho_1\tau} u_4$$

$$+ K_5^{(1)} u_5 + K_6^{(1)} (u_6 + K\tau u_5) + \gamma_1 C_1 + C_2 + (e_1\gamma_1 + e_2)\tau u_5,$$

$$q_1 = i(K_1^{(1)} e^{i\sigma_1\tau} v_1 - K_2^{(1)} e^{-i\sigma_1\tau} v_2) + K_3^{(1)} e^{\rho_1\tau} v_3 - K_4^{(1)} e^{-\rho_1\tau} v_4$$
$$+ K_5^{(1)} v_5 + K_6^{(1)} (v_6 + K\tau v_5) + \gamma_1 S_1 + S_2 + (e_1\gamma_1 + e_2)\tau v_5,$$

$$r_1 = i(K_1^{(1)} e^{i\sigma_1\tau} w_1 - K_2 e^{-i\sigma_1\tau} w_2) + K_3 e^{\rho_1\tau} w_3 - K_4 e^{-\rho_1\tau} w_4$$
$$+ K_5^{(1)} w_5 + K_6^{(1)} (w_6 + K\tau w_6) + \gamma_1 S_3 + S_4 + (e_1\gamma_1 + e_2)\tau w_5.$$

In order to obtain periodic solutions it is necessary to choose $K_3^{(1)} = K_4^{(1)} = 0$, and conditions for symmetrical orbits ($\dot{p}_1(o) = q_1(o) = r_1(o)$) demand that $K_5^{(1)} = 0$, and $K_1^{(1)} = K_2^{(1)}$. The solutions are not yet periodic unless

$$K_6^{(1)} K + e_1\gamma_1 + e_2 = 0 \qquad (14)$$

The solutions at this step become

$$\begin{aligned} p_1 &= K_1^{(1)} (e^{i\sigma_1\tau} u_1 + e^{-i\sigma_1\tau} u_2) + K_6^{(1)} u_6 + \gamma_1 C_1 + C_2, \\ q_1 &= iK_1^{(1)} (e^{i\sigma_1\tau} v_1 - e^{-i\sigma_1\tau} v_2) + K_6^{(1)} v_6 + \gamma_1 S_1 + S_2, \\ r_1 &= iK_1^{(1)} (e^{i\sigma_1\tau} w_1 - e^{-i\sigma_1\tau} w_2) + K_6^{(1)} w_6 + \gamma_1 S_3 + S_4, \end{aligned} \qquad (15)$$

where C_1, C_2, S_1, S_2, S_3, and S_4 are the periodic terms of equations (11), (12) and (13).

At this step we have the condition (14) which leaves $K_1^{(1)}$ arbitrary and gives one relation connecting K_6 and γ_1. In order to find a second relation we proceed to consider the terms in λ^2

Coefficients of λ^2

The complementary functions of this set of equations are similar to those for p_1, q_1 r_1, with different arbitrary constants $K_1^{(2)}, --- K_6^{(2)}$

We consider the particular integrals, which are again found by the "variation of parameters". According to this method we suppose $k_1^{(2)}$, $k_2^{(2)}$ $k_6^{(2)}$ to be the arbitrary multipliers. The differential equations[5] for the multipliers $k_1^{(2)}$ and $k_2^{(2)}$ are of the form

$$\dot{k}_1^{(2)} = e^{-i\sigma_1\tau}\left[\ D_1\ \right],$$
$$\dot{k}_2^{(2)} = e^{+i\sigma_1\tau}\left[\ D_2\ \right],$$

where D_1 and D_2 denote the totality of terms in the equations.

Since the differential equations for p_2, q_2 and r_2 involve p_1, q_1, and r_1 and their powers it is easily seen that D_1 contains terms of the form $\beta e^{i\sigma_1\tau}$, where β is a constant. Consequently the integration for $k_1^{(2)}$ yields non-periodic terms. These terms, which involve γ_1 and $K_6^{(1)}$, must be equated to zero in order to have periodic solutions, and we have a second relation in $K_6^{(1)}$ and γ_1, namely

$$-K_1^{(1)}\left[K_6^{(1)} d_1\,\varepsilon_0^{\,2} + \gamma_1 d_2 \;+ d_3\,\varepsilon_0\right] = 0 \qquad (16)$$

if $K_1^{(1)}$, which is arbitrary is taken different from zero. In (16)

$$d_1 = \left[\left\{\frac{9B}{2A(4A-\sigma^2)} - 12Ba_1\right\}E_1 - 6Ba_1 n E_3 + \frac{3E_2B}{2\sqrt{A}}\right]$$

$$d_2 = \Big[\Big\{ 2(1+2A) - 2\sigma_1 n \Big\} E_1 - \Big\{ 2(1-A)n - 2\sigma_1 \Big\} E_3 \Big],$$

$$d_3 = \Big[\Big\{ 6B\left(\frac{2NG_4}{\rho_1} - \frac{2NF_4}{\sigma_1} \right) - \frac{3}{2} B \left(4a_1 + \frac{3B}{A(4A-\sigma^2)} \right) - \frac{6C}{A} \Big\} E_1$$

$$+ \Big\{ 3Bn\left(\frac{2NG_4}{\rho_1} - \frac{2NF_4}{\sigma_1} \right) - 3\left(Bna_1 + \frac{nC}{2A} \right) \Big\} E_3 + \frac{3B}{2A} E_2 \Big].$$

The equation in $k_2^{(2)}$ gives the same condition with the sign of $K_1^{(1)}$ changed.

When equations (14) and (16) are solved we have $K_6^{(1)}$ and determined.

An approximation higher than the first power in ε_0 for $K_6^{(1)}$ and γ_1 . involves the construction of solutions for the third and higher powers of ε_0 in all the equations. This would make an exceedingly large amount of computation which would be unnecessary on account of the value of ε_0 and λ .

The complete solutions of the second genus orbits involve two periods: $2\pi/\sqrt{A}$, and $2\pi/\sigma_1$. The value of σ_1 is given by [6]

$$\sigma_1 = \sigma + \varepsilon_0^2\left[\frac{9B^2\{\sigma^2(1-13A)-(3+7A-22A^2)\}}{16A\sigma^3(4A-\sigma^2)} - \frac{\delta_1^{(2)}(1-A)(1+2A)}{\sigma^3}\right] + \cdots$$

It is possible, by the choice of ε_0 , to make σ_1 very nearly equal to σ . Moulton states [3] that there are infinitely many values of μ for which $\sqrt{A}$ and σ are commensurable.

We can thus construct solutions in which the above periods are commensurable, or very nearly so.

7. Numerical Example for Equilibrium Point (a).

The following is a numerical example of second genus orbits. The construction is carried out to the second power of ε_0^2 for the first genus orbit, and to the first power in ε_0 and λ for the second genus.

Table I gives the values of the constants computed in this or other papers.[3,6] Tables II and IV give the values for τ of the first genus and second genus orbits respectively.

$K_1^{(1)}$, the arbitrary constant, is chosen equal to unity, while the values of ε_0 and λ are chosen as 0.1 and 0.01 respectively. This value of ε_0 makes the ratio $\sigma_1/\sqrt{A}$ very nearly equal to $31/28$.

The illustration pictures the first genus orbit in dotted lines, and the second genus in full lines. The second genus orbit makes several revolutions before re-entering, owing to the complexity of its period. Our computation is carried out for one revolution. Stroke lines, however, indicate the path of the infinitesimal body as its period approaches the close.

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS

TABLE I.

Constant	Value	Constant	Value
A	2.548	γ_1	$-0.849 \times \varepsilon_0$
σ^2	2.811	$K_6^{(1)}$	$0.924 \times 1/\varepsilon_0$
ρ^2	3.359	g_4	31.604
n	2.657	F_4	3.885
m	- 0.747	H_1	75.561
B	6.548	H_2	- 44.902
C	18.283	H_5	39.681
a_1	- 0.316	H_6	29.242
b_1	0.151	H_7	67.746
c_1	- 0.112	H_8	- 44.385
a_1	- 0.037	H_{12}	38.504
$\delta_1^{(2)}$	0.184	d_1	- 260.723
N	- 0.013	d_2	- 131.070
σ_1	1.767	d_3	129.467
ρ_1	1.830	μ	1/11

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS

TABLE II.

$\varepsilon_0' = .1$

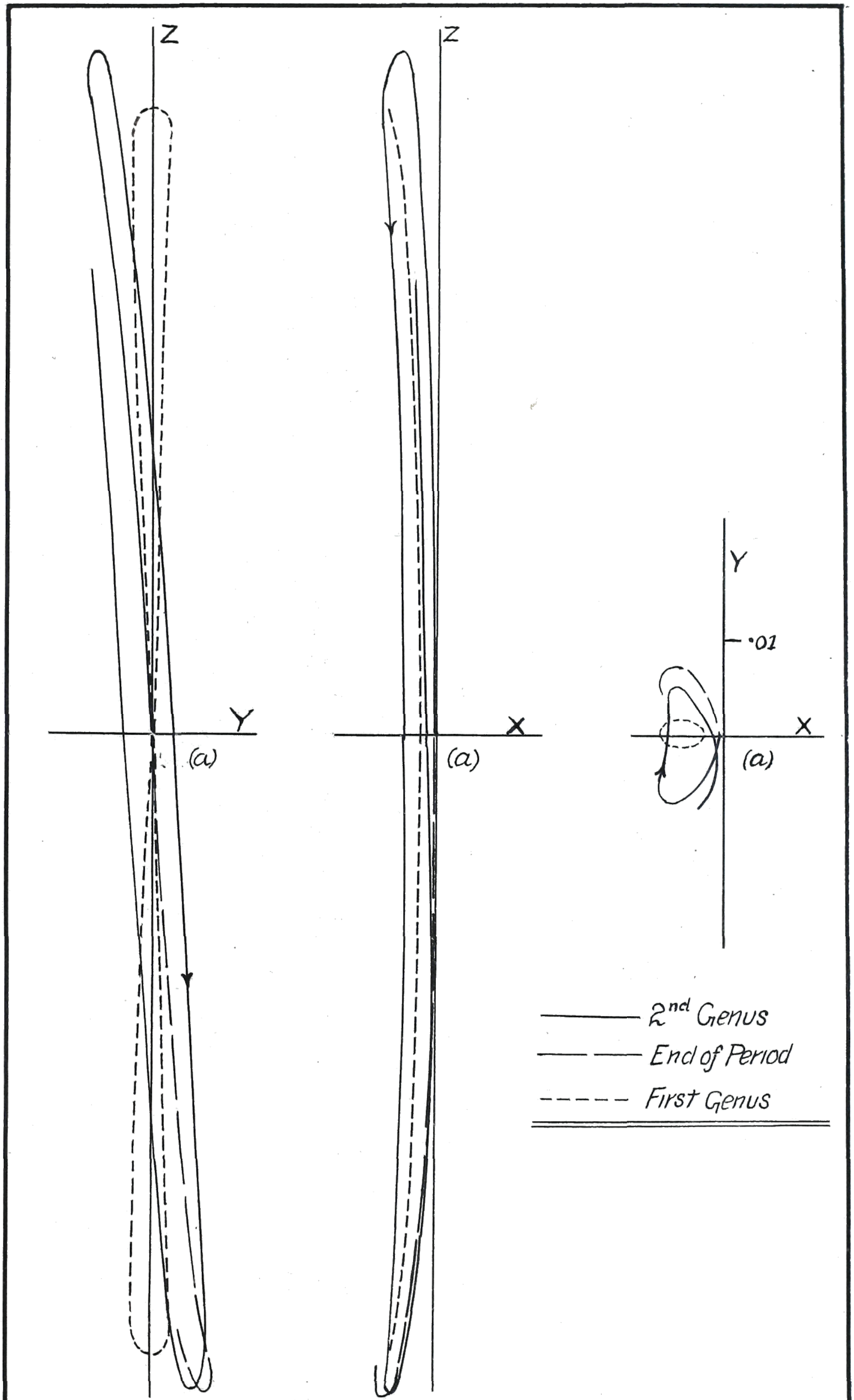
t	$\varepsilon_0 X_1$	$\varepsilon_0 Y_1$	$\varepsilon_0 Z_1$	t	$\varepsilon_0 X_1$	$\varepsilon_0 Y_1$	$\varepsilon_0 Z_1$
.0	-.00165	-00000	.00000	1.6	-.00258	.00103	.0347
.1	-.00173	-.00035	.00995	1.8	-.00186	.00057	.0166
.2	-.00195	- 00067	.01962	2.0	-.00166	-.00011	-.0032
.3	-.00229	-.00092	.02881	2.4	-.00287	-.00110	- 0398
.4	-.00272	-.00107	.0373	2.8	-.00449	-.00052	-.0607
.5	-.00320	-.00112	.0448	3.2	-.00422	.00080	-.0577
.6	-.00367	-.00105	.0511	3.4	-.00337	.00111	-.0473
.7	-.00409	-.00088	.0562	3.6	-.00244	-.00099	-.0320
.8	-.00442	-.00062	.0599	3.8	-.00179	.00047	-.0136
.9	-.00462	- 00030	.0620	4.0	- 00168	-.00022	.0063
1.0	-.00467	00006	.0625	4.2	-.00215	-.00083	.0255
1.2	-.00433	00071	.0589	4.4	-.00302	- 00112	.0421
1.4	- 00352	00109	.0493				

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS.

TABLE III.

$K_1^{(1)} = 1$, $\varepsilon_0 = .1$, $\lambda = .01$, $\sigma_1/\sqrt{A} = 31/28$.

τ	$\varepsilon_0(x+p)$	$\varepsilon_0(y+p)$	$\varepsilon_0(z+q)$	τ	$\varepsilon_0(x+p)$	$\varepsilon_0(y+q)$	$\varepsilon_0(z+r)$
.0	-.00031	-.00000	.00000	2.8	-.00393	.00514	-.0666
.1	-.00040	-.00095	.0109	3.2	-.00258	00318	-.0632
.2	-.00066	- 00189	.0214	3.4	- 00165	00152	-.0519
.3	-00106	-00275	.0314	3.6	-.00090	-.00034	-.0353
.4	-00158	-00352	.0407	3.8	-.00060	-.00219	-.0152
.5	-00218	-00418	.0488	4.0	-.00092	- 00376	.0065
.6	-00282	-00470	.0558	4.2	-.00184	- 00487	.0275
.7	-00345	-00508	.0613	4.4	- 00316	-.00537	.0457
.8	-00403	-00529	.0653				
.9	-00453	-00533	.0676				
1.0	-00492	-00520	.0682	109.6	- 00304	.00486	-.0576
1.2	-00532	-00449	.0644	109.8	- 00178	.00378	- 0433
1.4	-00525	-00322	.0539	110.0	-.00077	.00221	-.0246
1.6	-00490	-00158	.0380	110.2	-.00032	.00036	-.0033
1.8	-00446	00024	.0181	110.4	-.00055	-.00155	.0182
2.0	-.00417	.00203	-.0037				
2.4	- 00412	.00466	-.0439				


Z
Y
(a)
Z
X
(a)
Y
·01
X
(a)
2nd Genus
End of Period
First Genus

AN ILLUSTRATIVE PERIODIC ORBIT OF THE SECOND GENUS.

References.

1. Lagrange, "Collected Works," Vol VI pp 229-324.

2. Poincaré, "Les Méthodes Nouvelles de la Mécanique Céleste," Vol I (1892) p 159.
Sir George H. Darwin, "Acta Mathematica" Vol. XXI (1897) p.99
Plummer, "Monthly Notices, Royal Astronomical Society," Vol. LXIII (1903) p.436, and Vol. LXIV (1903)p 98.

3. Moulton, "Periodic Orbits", Chapter V.

4. Moulton, "Introduction to Celestial Mechanics".

5. Buchanan "Periodic Orbits of the Second Genus Near the Straight-Line Equilibrium Points." (In Press)

6. Buchanan, "Asymptotic Satellites Near the Straight-Line Equilibrium Points."
American Journal of Mathematics, Vol. XLI, No. 2 April,1919.