REAL FLAG MANIFOLDS AND A CONSTRUCTION OF SPACES OVER A POLYHEDRON
Mathematical Investigations arising from the Jahn-Teller Effect

by

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Abstract

We examine a construction of topological spaces over an arbitrary polyhedron and show that it subsumes the lattice construction of R. R. Douglas and A. R. Rutherford. A Simplicial Approximation Theorem is proven for the general construction, for maps from a polyhedron to one of our spaces lying over another polyhedron. A special case of our construction (a slight generalization of the lattice construction) is examined and a class of locally trivial bundles is constructed. These are used to examine neighbourhood structure in the special case. We also enumerate exactly which spheres can be constructed by a lattice construction on a product of real orthogonal, complex unitary or quaternionic symplectic groups.

The fundamental group of the real complete flag manifolds is determined following a detailed exposition of Clifford algebras. Appendices are provided on the diagonalization of quaternionic Hermitian matrices and on a generalized mapping cylinder that can be regarded as an endofunctor on the category of locally trivial bundles over a fixed locally compact base.
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Introduction

This work expands on some topological aspects of recent work by R. R. Douglas and A. R. Rutherford on the Jahn-Teller effect of molecular physics. It is not, however, a work on physics and can be read independently of any understanding of this subject.

The Jahn-Teller effect is the name given to the physical phenomenon in which a polyatomic molecule distorts from a symmetric configuration so as to remove degeneracy in its electronic spectrum. Its study was initiated by the 1937 paper [JT] of H. A. Jahn and E. Teller, in which the authors studied the effects of symmetry on the electronic degeneracy of polyatomic molecules. They showed that, except for linear molecules, electronic degeneracy is unstable and is removed by distortion of the atomic nuclei within linear subspaces classified by the irreducible representations of the symmetry group of the molecular Hamiltonian; representations which also classify the electronic degeneracy. Since 1937, the physics community has produced an enormous volume of literature on the Jahn-Teller effect and its consequences, as evidenced by the equally enormous bibliographic review [B].

Few people have succeeded in predicting spectra for molecules exhibiting a Jahn-Teller effect and, until recently, the only such predictions have been for electronic doublet states in triangular molecules. In their papers [DR3] and [DR4], Douglas and Rutherford announce their prediction of spectra for triplet states in molecules exhibiting octahedral symmetry and quadruplet states in molecules exhibiting icosahedral symmetry. Their methods are highly geometric and involve a detailed dissection of Euclidean spaces of real symmetric and complex Hermitean matrices into regions of similar degeneracy, on which the eigenspace associated with a selected eigenvalue has constant dimension. This dissection quite naturally produces a decomposition of certain spheres above standard simplices, with the subset lying above an open simplex decomposing as a product of the open simplex with a flag manifold. The decomposition of spheres in turn generalizes readily to a construction which can be applied to any topological group together with a list of subgroups. Douglas coined the term lattice construction and studied some examples in [D].

The lattice construction can itself be viewed as a special case of a construction in which one builds a space over a polyhedron by gluing together spaces lying above the faces of the polyhedron. Chapter 2, 3 and 4 are devoted to different aspects of this construction.

In chapter 2 we introduce the lattice construction and construct the real and complex spheres considered by Douglas and Rutherford, as well as quaternionic spheres. We then present our general construction as a functor on an appropriate category and show that the lattice construction is a special case.

In chapter 3 we show that the classical Simplicial Approximation Theorem has a natural generalization to a statement about maps from a polyhedron to a class of our spaces lying
above another polyhedron. We apply this theorem to a construction of a space from a map between two arbitrary compact Hausdorff spaces.

In chapter 4 we construct a class of bundles, using a locally compact Hausdorff group to shape a special case of our construction. We then use these bundles to examine the local structure in the case that the group is compact and the underlying simplicial complex is locally finite, giving a necessary condition for the construction of a topological manifold. Re-examining the spheres constructed in chapter 2, we show that these are not the only spheres constructed by the lattice construction. Indeed, in each of the real, complex and quaternionic cases, all but a finite number of spheres can be constructed from products of orthogonal, unitary or symplectic groups respectively.

The real and complex flag manifolds are ubiquitous in the analyses of Douglas and Rutherford and some algebraic topology of these spaces can be found in [R]. In chapter 1 we determine the fundamental group of the real complete flags by examining their universal coverings by the spin groups, having been unable to find a reference in the literature. We also use covering space methods to identify a class of trivial bundles over the real projective space $\mathbb{R}P^3$. This class contains an $\mathbb{R}P^2$-bundle whose triviality is required in [DR4].
Chapter 0
Preliminaries

We begin by summarizing some basic notational conventions that will be used throughout.

- The ring of integers is denoted by \( \mathbb{Z} \). The set of positive (strictly greater than zero) integers, or natural numbers, is denoted by \( \mathbb{N} \). The set of non-negative integers is denoted by \( \mathbb{N}_0 \). If \( n \in \mathbb{N} \), the additive group of integers modulo \( n \) is denoted by \( \mathbb{Z}_n \).

- The real and complex fields are denoted by \( \mathbb{R} \) and \( \mathbb{C} \) respectively. The division algebra of quaternions is denoted by \( \mathbb{H} \). The conjugate of an element \( x \) lying in either \( \mathbb{C} \) or \( \mathbb{H} \) is denoted by \( \bar{x} \).

- The standard basis vectors in \( \mathbb{R}^n \) are denoted by \( \hat{e}_1, \ldots, \hat{e}_n \); \( \hat{e}_i \) is the tuple whose only non-zero component is a 1 in the \( i \)th position. In chapter 1 we use \( e_1, \ldots, e_n \) when regarding these as elements of a Clifford algebra on \( \mathbb{R}^n \).

- Let \( n \in \mathbb{N} \). If \( \vec{v} = (v_1, \ldots, v_n) \) and \( \vec{w} = (w_1, \ldots, w_n) \) are two elements of either \( \mathbb{C}^n \) or \( \mathbb{H}^n \) then their inner product is \( \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^n v_i \bar{w}_i \). We use the dot product notation \( \vec{v} \cdot \vec{w} \) to denote the inner product in \( \mathbb{R}^n \). We denote the inner product preserving groups of \( n \times n \) real orthogonal, complex unitary and quaternionic symplectic matrices by \( O(n) \), \( U(n) \) and \( Sp(n) \) respectively.

- The norm of an element \( x \) in \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \) is denoted by \( |x| \). If \( n > 1 \), the norm of an element \( \vec{v} \) in \( \mathbb{R}^n \), \( \mathbb{C}^n \) or \( \mathbb{H}^n \) is denoted by \( ||\vec{v}|| \).

- If \( R \) is a ring and \( n \) is a non-negative integer then \( R(n) \) is the \( R \)-algebra of \( n \times n \) matrices having entries in \( R \). If \( A \in R(n) \) then \( A^T \) denotes the transpose matrix. \( A \) is symmetric if \( A^T = A \). \( A \) is antisymmetric if \( A^T = -A \).

- If \( A \in \mathbb{C}(n) \) or \( \mathbb{H}(n) \) then \( A^\dagger \) denotes the conjugate transpose of \( A \). If \( A \in \mathbb{R}(n) \) we let \( A^\dagger = A^T \). (Equivalently, we can endow \( \mathbb{R} \) with the identity conjugation.)

- If \( K = \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \) and \( A \in K(n) \) then \( A \) is Hermitean if \( A^\dagger = A \). We denote the \( K \)-algebra of Hermitean matrices by \( \text{Herm}(n, K) \).
• The relation \( \approx \) denotes homeomorphism of topological spaces. The relation \( \simeq \) denotes homotopy.

• The relation \( \cong \) is used to denote equivalence in several algebraic categories: Isomorphism of groups, vector spaces, algebras, etc.

• The term *neighbourhood* is used synonymously with *open set* in a topological space. That is, the elements of a topology are neighbourhoods.\(^1\) A function between two topological spaces is called a *map* if it is continuous. We will use the word *function* in this context only if continuity is either not assumed or not evident.

• The relation \( \equiv \) is occasionally used to introduce notation. For example, if \( b_1 \) and \( b_2 \) are two previously defined quantities and we write "... \( b_1 \equiv b_2 \equiv b \ldots \)" then the symbol \( b \) is being introduced as a synonym for the common value of \( b_1 \) and \( b_2 \). In chapter 1, \( \equiv \) is also used to denote congruence modulo an integer, as in \( a \equiv b \mod n \).

• The relation \( \times \) is used to denote the product in various categories. The relation \( \oplus \) is used to denote the coproduct. (In particular, the disjoint union of topological spaces and the direct sum of real algebras.)

• In addition to denoting the \( i^{\text{th}} \) homotopy group functor, the symbol \( \pi_i \) is often used to denote a generic projection map from a cartesian product of spaces onto the \( i^{\text{th}} \) factor.

• The symbol • is used to denote the end of a proof. It is also used to mark the end of a statement whose proof is immediate from preceding discussion. The symbol ◐ is occasionally used within a proof to terminate the verification of some claim made during the course of the proof.

We will construct many maps using the universal property of the identification. Recall that if \( f : X \to Y \) is a surjective map then \( Y \) has the *identification topology* by \( f \) if its topology consists of exactly those subsets of \( Y \) whose pre-image by \( f \) is an open subset of \( X \), and in this case \( f \) is called an *identification* or *identification map*.

An identification map \( f : X \to Y \) satisfies a universal property: If \( g : X \to Z \) is a map for which \( g(x) = g(x') \) whenever \( f(x) = f(x') \) then there is a unique map \( h : Y \to Z \) such that \( h \circ f = g \). That is, there is a unique map \( h \) which makes the following diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
\downarrow{f} & & \downarrow{h} \\
Y & \downarrow & \\
& & \\
\end{array}
\]

Given an equivalence relation \( R \) on a space \( X \), the projection \( p : X \to X/R \) is an identification map when \( X/R \) is given the quotient topology. Thus, if \( f : X \to Y \) is an identification and \( R \)

\(^1\) This differs from the use in [Br] for example.
is the equivalence relation on $X$ defined by

$$(x, x') \in R \text{ if and only if } f(x) = f(x')$$

then the unique map $Y \to X/R$ which makes the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{X/R} \\
Y & \rightarrow & X/R \\
\end{array}
\]

commute is a homeomorphism. If the equivalence relation $R$ is the smallest equivalence relation containing some relation $A \subseteq X \times X$ then there is the an alternate version of the universal property of the identification $p : X \to X/R$. For this, if we write $R = \langle A \rangle$ for this smallest equivalence containing $A$ then $\langle A \rangle$ has the following characterization.

0.1 Proposition. Let $x, x' \in X$. Then $(x, x') \in \langle A \rangle$ if and only if either $x' = x$ or there are $n \in \mathbb{N}, (x_1, x'_1), \ldots, (x_n, x'_n) \in A$ such that the following hold.

1. $x \in \{x_1, x'_1\}$.
2. If $n > 1$ then $\{x_i, x'_i\} \cap \{x_{i+1}, x'_{i+1}\} \neq \emptyset$ for each $i \in \{1, \ldots, n - 1\}$.
3. $x' \in \{x_n, x'_n\}$.

Proof. Let $R \subseteq X \times X$ be the set of pairs $(x, x')$ such that either $x' = x$ or there are $n \in \mathbb{N}, (x_1, x'_1), \ldots, (x_n, x'_n) \in A$ such that 1, 2 and 3 hold. Then $R$ is an equivalence relation and $A \subseteq R$. Thus, $\langle A \rangle \subseteq R$. However, any equivalence relation containing $A$ must also contain $R$ so that $R \subseteq \langle A \rangle$ as well. □

An easy consequence is then the following universal property for $p : X \to X/\langle A \rangle$.

0.2 Proposition. Let $X$ be a topological space and let $A \subseteq X \times X$ be a relation on $X$. If a map $g : X \to Z$ has $g(x) = g(x')$ whenever $(x, x') \in A$ then there is a unique map $h : X/\langle A \rangle \to Z$ such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
\downarrow{p} & & \downarrow{h} \\
X/\langle A \rangle & & \\
\end{array}
\]

That is, $R$ is the intersection of all equivalence relations containing $A$. 2
Proof. Continuity is clear, so we need only demonstrate the existence of \( h \). However, by Proposition 0.1, it is clear that the hypothesis implies that \( g(x) = g(x') \) whenever \((x, x') \in \langle A \rangle \). Thus, the existence of \( h \) follows from the standard universal property for \( p \).

We will often employ the following result without extensive commentary.

**0.3 Proposition.** Let \( f : X \to Y \) be an identification and let \( Y_0 \subseteq Y \). If either (1) \( Y_0 \subseteq Y \) is an open or closed subset or (2) \( f \) is an open or closed map then the subspace topology on \( Y_0 \) equals the identification topology by the map \( x \mapsto f(x) : f^{-1}(Y_0) \to Y_0 \).

In other words, the map \( x \mapsto f(x) : f^{-1}(Y_0) \to Y_0 \) is an identification.

**Proof.** [Du], chapter VI, section 2.1. ■

We will say that a space \( X \) is *locally compact* if, given a point \( x \in X \) and a neighbourhood \( U \) of \( x \), there is a neighbourhood \( V \) of \( x \) whose closure is a compact subset of \( U \). The following very useful result can be found in [Br] as proposition 4.3.2. Since the author there states the definition of local compactness slightly differently than do we (although our definition is equivalent to theirs), we include the proof.

**0.4 Proposition.** Let \( f : X \to Y \) be an identification and let \( B \) be locally compact. Then \( f \times 1 : X \times B \to Y \times B \) is an identification.

**Proof.** Let us write \( h = f \times 1 \) and let \( U \subseteq Y \times B \) be a subset for which \( h^{-1}(U) \subseteq X \times B \) is open. We must show that \( U \) is open in \( Y \times B \).

To see this, let \((y_0, b_0)\) be any point in \( U \), and let \( x_0 \in X \) such that \( f(x_0) = y_0 \). If \( B_0 \subseteq B \) is defined by

\[
B_0 = \{ b \in B \mid (x_0, b) \in h^{-1}(U) \}
\]

then \( B_0 \) is open in \( B \). For if \( b' \in B \) then since \( h^{-1}(U) \subseteq X \times B \) is an open subset, there are open neighbourhoods \( V \) of \( x_0 \) and \( W \) of \( b' \) such that \( V \times W \subseteq h^{-1}(U) \), giving \( W \subseteq B_0 \). Now, since \( B \) is locally compact, there is a neighbourhood \( V \) of \( b_0 \) whose closure \( \bar{V} \) is a compact subset of \( B_0 \). Let

\[
W = \{ x \in X \mid \{x\} \times \bar{V} \subseteq h^{-1}(U) \}
\]

and note that \( W \) is the largest subset of \( X \) having \( W \times \bar{V} \subseteq h^{-1}(U) \). Since

\[
f^{-1}(f(W)) \times \bar{V} = h^{-1}(h(W \times \bar{V})) \subseteq h^{-1}(h(h^{-1}(U))) = h^{-1}(U),
\]

we have \( f^{-1}(f(W)) \subseteq W \) so that \( f^{-1}(f(W)) = W \), \( W \subseteq f^{-1}(f(W)) \) being the case for any \( W \). Now, since

\[
(y_0, b_0) \in f(W) \times V = h(W \times V) \subseteq h(W \times \bar{V}) \subseteq U,
\]

it suffices to show that \( f(W) \) is an open subset of \( Y \) (since then \( f(W) \times V \) is a neighbourhood of \((y_0, b_0)\) contained in \( U \)), and for this we must show that \( f^{-1}(f(W)) = W \) is an open subset of \( X \).
However, if $x \in W$ then $\{x\} \times \bar{V} \subset h^{-1}(U)$ and, since $\bar{V}$ is compact, there is a neighbourhood $W'$ of $x$ such that $W' \times \bar{V} \subset h^{-1}(U)$.

Many sources define locally compact to mean that each point has a neighbourhood whose closure is compact. If $X$ is Hausdorff then the two definitions coincide and, except for in appendix C (the statement of Proposition C.4 in particular), all the locally compact spaces we consider will also be Hausdorff.

As a general reference for topological facts we refer the reader to [Du]. Note that a locally compact space is assumed to be Hausdorff in this reference.

We use the language of Category Theory throughout, although no specialized knowledge is required. Let us provide a brief introduction to the basic terminology.

A category $C$ consists of a class of objects and, for any two objects $X$ and $Y$, a class of morphisms, or arrows, having source $X$ and target $Y$. We write $f : X \to Y$ if $f$ is a morphism with source $X$ and target $Y$. It is also required that there be a distinguished identity morphism $1_X : X \to X$ for each object $X$, as well as a rule of composition of morphisms which assigns to morphisms $f : X \to Y$ and $g : Y \to Z$ a morphism $g \circ f : X \to Z$ satisfying the following two properties.

- If $f : X \to Y$ then $f$ coincides with both $1_Y \circ f$ and $f \circ 1_X$.
- If $f : W \to X$, $g : X \to Y$ and $h : Y \to X$ then the compositions $f \circ (g \circ h)$ and $(f \circ g) \circ h$ coincide.

The first rule of Category Theory is that objects are unnecessary: A category is completely determined by its morphisms.

Any set $A$ with preordering $\prec$ (a reflexive, transitive relation) is the set of objects of a category in which there is a unique arrow with source $a$ and target $b$ if and only if $a \prec b$. We will use this in chapter 2 to regard the face containment preordering as defining a category structure on a simplicial complex.

A functor $F$ from a category $C$ to a category $D$ assigns to each object $X$ of $C$ an object $F(X)$ of $D$, and to each morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$, in such a way that the following hold.

- For each object $X$ of $C$, $F(1_X)$ coincides with $1_{F(X)}$.

---

3 For each point $b \in \bar{V}$, choose neighbourhoods $W_b''$ of $x$ and $B'_b$ of $b$ such that $W_b'' \times B'_b \subset h^{-1}(U)$. Let $b_1, \ldots, b_n \in B$ be such that $\bigcup_{i=1}^n B_{b_i} \supset \bar{V}$ and set $W' = \bigcap_{i=1}^n W_{b_i}''$.

4 Still, it is common that one refer to well known categories by naming their objects rather than their morphisms. Thus, one speaks of the category Set of sets. A more precise way of referring to Set would be as the category of functions.
• If $f : X \to Y$ and $g : Y \to Z$ in $C$ then $F(f \circ g)$ coincides with $F(f) \circ F(g)$.

We write $F : C \to D$ if $F$ is a functor from $C$ to $D$.\footnote{Some authors refer to such an $F$ as a covariant functor from $C$ to $D$, and also define contravariant functors from $C$ to $D$. These are just functors from $C$ to $D^{\text{op}}$, where $D^{\text{op}}$ is the opposite category of $D$, defined by letting the morphisms with source $X$ and target $Y$ be the morphisms of $D$ with source $Y$ and target $X.}$

Functors from $C$ to $D$ are objects of the functor category $D^C$, the morphisms in which are natural transformations. A natural transformation $\Phi$ from $F : C \to D$ to $G : C \to D$ assigns to each object $X$ of $C$ an arrow $\Phi(X) : F(X) \to G(X)$ of $D$ such that if $f : X \to Y$ is an arrow of $C$ then the following diagram commutes.

\[
\begin{array}{ccc}
\Phi(X) & \frac{F(f)}{\downarrow} & \Phi(Y) \\
G(X) & \frac{\downarrow}{G(f)} & G(Y)
\end{array}
\]

The universal property of the identification is but one instance of a general phenomenon. If $C$ is a category and $F : C \to \text{Set}$ is a functor to the category of sets and functions then a universal element of $F$ consists of an object $c$ of $C$ and an element $x \in F(c)$ such that if $c'$ is some other object of $C$ and $x' \in F(c')$ then there is a unique arrow $f : c \to c'$ such that $F(f)(x) = x'$. For example, if $f : X \to Y$ is an identification map then there is a functor $F : \text{Top} \to \text{Set}$ from the category of topological spaces and maps which assigns to each space $Z$ the set of maps $g : X \to Z$ such that $g(x) = g(x')$ whenever $f(x) = f(x')$. The space $Y$ and map $f$ then define a universal element of $F$. As another example, if $V$ is a real vector spaces then the tensor product $V \otimes V$ and the bilinear map $(v, v') \mapsto v \otimes v' : V \times V \to V \otimes V$ define a universal element of the functor which assigns to each real vector space $W$ the set of all bilinear maps from $V \times V$ to $W$.

More generally still, given a functor $F : C \to D$ and an object $d$ of $D$, a universal arrow from $d$ to $F$ consists of an object $c$ of $C$ and an arrow $f : d \to F(c)$ such that if $c'$ and $f' : d \to F(c')$ is any other such pair then there is a unique arrow $h : c \to c'$ such that the following diagram commutes.

\[
\begin{array}{ccc}
d & \frac{f}{\downarrow} & f' \\
F(c) & \frac{\downarrow}{h} & F(c')
\end{array}
\]

If we identify an element of a set $X$ with a function from the one-point set into $X$ then a universal element of a functor $F : C \to \text{Set}$ is exactly a universal arrow from the one-point
set to $F$. Colimits are another example of universal arrows from an object to a functor. Let $J$ and $C$ be categories and let $\Delta : C \to C^J$ be the diagonal functor that sends an object of $C$ to the constant functor to that object, and an arrow $f : c \to d$ to the natural transformation from $\Delta(c)$ to $\Delta(d)$ whose component at every object of $J$ is $f$. If $F : J \to C$ is an object of $C^J$ then a universal arrow from $F$ to $\Delta$ is a colimit of $F$. By reversing arrows in (0.1), one can also define a universal arrow from $F$ to $d$. By considering the diagonal functor $\Delta : C \to C^J$ one is then lead to the notion of limit.

We refer the reader to [M] for an in-depth exposition of Category Theory.

---

6 That is, $\Delta(c) : J \to C$ is the functor which sends every morphism $j \to j'$ in $J$ to the identity morphism $1_c : c \to c$ in $C$. 

9
Chapter 1
Two Computations

Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $n, \ell \geq 2$ and $n_1, \ldots, n_\ell \geq 1$ be integers with $n_1 + \cdots + n_\ell = n$. Let $F_K(n_1, \ldots, n_\ell)$ denote the collection of tuples $(V_1, \ldots, V_\ell)$ of mutually orthogonal subspaces of $K^n$ having $\dim_K V_i = n_i$.\footnote{Here $\dim_K V$ denotes the dimension of the subspace $V$ over the division ring $K$.} We can identify these with the homogeneous spaces $O(n)/\left(O(n_1) \times \cdots \times O(n_\ell)\right)$, $U(n)/(U(n_1) \times \cdots \times U(n_\ell))$ and $Sp(n)/(Sp(n_1) \times \cdots \times Sp(n_\ell))$ respectively, so the $F_K(n_1, \ldots, n_\ell)$ can be given the structure of real manifolds. These are the real, complex and quaternionic flag manifolds. If $n_1 = \cdots = n_\ell = 1$ we refer to $F_K(1, \ldots, 1)$ as a complete flag manifold.

Let us write $F(n_1, \ldots, n_\ell) = F_R(n_1, \ldots, n_\ell)$ for the real flag manifolds and $F(n) = F(1, \ldots, 1)$ for the real complete flag manifold of 1-dimensional subspaces of $\mathbb{R}^n$. Since $F(2) \cong \mathbb{R}P^1 \cong S^1$, we have $\pi_1 F(2) \cong \mathbb{Z}$. We compute $\pi_1 F(n)$ for $n \geq 3$ in section A by examining the universal covering of $F(n)$ by $\text{Spin}(n)$ after giving a detailed exposition of the definition of the spin groups via Clifford algebras. In section B we construct the universal covering of $F(4)$ using the universal covering of $SO(3)$ by $SU(2)$ and identify a class of trivial bundles over $\mathbb{R}P^3$.

A. Fundamental Group of the Complete Flags

For $n \geq 3$, consider the universal covering of $F(n)$ by $\text{Spin}(n)$.

$$\begin{align*}
\text{Spin}(n) & \xrightarrow{2:1} \text{SO}(n) \\
\text{SO}(n) & \xrightarrow{2^{n-1}:1} F(n) \cong O(n)/O(1)^n = SO(n)/(O(1)^n \cap SO(n))
\end{align*}$$
If Spin'(n) is the finite subgroup of Spin(n) which maps onto the subgroup \( O(1)^n \cap SO(n) \subset SO(n) \), then we have a continuous bijection from the orbit space Spin(n)/Spin'(n) to \( F(n) \) defined by the universal property of the quotient.

\[
\begin{array}{rcl}
\text{Spin}(n) & \rightarrow & SO(n) \\
\downarrow & & \downarrow \\
\text{Spin}(n)/\text{Spin}'(n) & \rightarrow & SO(n)/(O(1)^n \cap SO(n)) \cong F(n)
\end{array}
\]

This map is a homeomorphism, being a continuous bijection from a compact space to a Hausdorff space. By standard results on proper group actions, which we review in the following subsection, it then follows that \( \pi_1 F(n) \cong \text{Spin}'(n) \).\(^2\) We need thus only examine the definition of the universal covering of \( SO(n) \) by \( \text{Spin}(n) \) to determine the subgroup \( \text{Spin}'(n) \).

Let us note for completeness that the situation for the complex and quaternionic complete flags is simpler than the real case. Indeed, we can compute the fundamental group of all the remaining flag manifolds by diagram chasing using the long exact homotopy sequences of various fibrations. These results, among others, are collected in appendix D.

1. Proper Group Actions and the Fundamental Group

Suppose a topological group \( H \) acts continuously on a space \( X \). A neighbourhood \( U \subset X \) is called proper if \( hU \cap U = \emptyset \) for all \( h \in H - \{1\} \). The group action is called proper if each point in \( X \) has a proper neighbourhood. Note, in particular, that a proper action must be free: \( h \cdot x = x \) if and only if \( h = 1 \).

1.1 Proposition. Suppose a topological group \( H \) acts continuously on a space \( X \). If \( H \) acts properly on \( X \) then the projection \( p : X \rightarrow X/H \) is a covering map.

Proof. Since the map \( x \mapsto hx : X \rightarrow X \) is a homeomorphism for each \( h \in H \), if \( U \subset X \) is a proper neighbourhood then

\[
p^{-1}(p(U)) = \bigcup_{h \in H} hU
\]

is a disjoint union of open sets. For each \( h \in H \), the homeomorphism \( x \mapsto h^{-1}x : X \rightarrow X \) is a covering transformation. Thus, to show that \( p|_{hU} : hU \approx p(U) \), it suffices to show this for \( h = 1 \). However, this is clear since \( p|_U \) is a continuous, open bijection.

A finite group \( G \) acting freely on a Hausdorff space \( X \) acts properly. For if \( x \in X \) then for each \( h \in H - \{1\} \) we can choose neighbourhoods \( U_h \) of \( x \) and \( V_h \) of \( hx \) such that \( U_h \cap V_h = \emptyset \), and

\[
\bigcap_{h \in H - \{1\}} (U_h \cap h^{-1}V_h)
\]

\(^2\) Of course, \( \pi_i F(n) \cong \pi_i \text{Spin}(n) \cong \pi_i SO(n) \) for all \( i \geq 2 \).
is then a proper neighbourhood of \( x \).

Given a topological group \( G \), a discrete subgroup \( H \subset G \) acts properly on \( G \) by multiplication. To see this, choose an open neighbourhood \( W \) of the identity element such that \( W \cap H = \{1\} \). If \( x \in G \) then, since the map \((g', g) \mapsto g'g^{-1} : G \times G \to G \) is continuous, there is a neighbourhood \( V \) of \( x \) such that \( VV^{-1} = \{g'g^{-1} \mid g, g' \in V\} \subseteq W \). \( V \) is necessarily a proper neighbourhood of \( x \) since \( hV \cap V \neq \emptyset \) if and only if \( h \in V V^{-1} \).

Assume now that \( H \) acts properly on \( X \). Given a covering transformation \( f \in \text{Cov}(X, X/H) \) then, since the action of \( H \) is free, there is a function \( \phi_f : X \to H \) defined by the condition \( f(x) = \phi_f(x) \cdot x \). In the special case of \( X = G \), a group, and \( H \) a subgroup acting by multiplication then \( \phi_f : G \to H \) is defined by \( \phi_f(x) = f(x) \cdot x^{-1} \). If, further, \( H \) is discrete then \( \phi_f \) is the constant map \( \phi_f(x) = \phi_f(1) = f(1) \), and each covering transformation is multiplication by an element of \( H \). This function is necessarily the inverse of the obvious homomorphism \( H \to \text{Cov}(G, G/H) \).

1.2 **Proposition.** Let \( G \) be a group and let \( H \subset G \) be a discrete subgroup. Then the map

\[
\phi \mapsto f(1) : \text{Cov}(G, G/H) \to H
\]

is an isomorphism. ■

1.3 **Corollary.** Let \( G \) be a simply connected topological group and let \( H \subset G \) be a discrete subgroup. Then \( \pi_1(G/H) \cong H \).

**Proof.** \( p : G \to G/H \) is a universal covering so \( \pi_1(G/H) \cong \text{Cov}(G, G/H) \). ■

2. **Clifford Algebras**

Let us now examine the definition of Spin\(^\prime(\mathfrak{g})\) via Clifford Algebras so as to identify Spin\(^\prime(\mathfrak{g})\).

The **Clifford algebra** of a pair \((V, q)\) consisting of a real vector space \( V \) and a symmetric bilinear map \( q : V \times V \to \mathbb{R} \) is the universal element of the functor from the category of real, associative, unitary algebras to the category of sets which assigns to each algebra \( A \) the set of those linear transformations \( \phi \in \text{hom}(V, A) \) satisfying \( \phi(v)^2 = q(v, v) \cdot 1 \). In other words, the Clifford algebra of \( V \) is a real algebra \( C(V, q) \) together with a linear map \( \theta : V \to C(V, q) \) satisfying \( \theta(v)^2 = q(v, v) \cdot 1 \) for each \( v \in V \), such that if \( \phi : V \to A \) is another such map then there is a unique algebra homomorphism \( f : C(V, q) \to A \) such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & C(V, q) \\
\downarrow{\theta} & & \downarrow{f} \\
 & A
\end{array}
\]

commutes.
Other examples of universal algebras:

- The tensor algebra of a real vector space,
  \[ T(V) = \bigoplus_{i=0}^{\infty} \otimes^i V \]
  where \( \otimes^0 V = \mathbb{R} \) and \( \otimes^1 V = V \), is universal for all linear maps \( f : V \to A \).
- The exterior algebra of a real vector space,
  \[ \Lambda(V) = T(V) / I \]
  where \( I \) is the ideal generated by the elements \( v \otimes v \) with \( v \in V \), is universal for all linear maps \( f : V \to A \) satisfying \( f(v)^2 = 0 \). In fact, the exterior algebra is a Clifford algebra: Take \( q \) identically zero.

Let us write \( \alpha \cdot 1 = \alpha \) for a multiple of the unit element of an algebra. We can construct the Clifford algebra as we do the exterior algebra: Let \( I \) be the ideal of \( T(V) \) generated by elements of the form \( v \otimes v - q(v, v) \). Set \( C(V) = T(V)/I \) and define \( \theta : V \to C(V, q) \) by composing the inclusion \( V \hookrightarrow T(V) \) with the projection \( T(V) \to C(V, q) \).

\[
\begin{array}{ccc}
V & \hookrightarrow & T(V) \\
\downarrow \theta & & \downarrow \\
C(V, q) & & \\
\end{array}
\]

It is an easy exercise to show that \( \theta : V \to C(V, q) \) satisfies the universal property.

We will write \( v_1 \cdots v_k \) for the image in \( C(V, q) \) of an element \( v_1 \otimes \cdots \otimes v_k \in T(V) \). In the finite-dimensional case, a basis for the Clifford algebra is given by the following result, which we state without proof.\(^3\)

1.4 Theorem. Suppose that \( \dim V = n \) and that \( e_1, \ldots, e_n \) is a basis for \( V \) having \( q(e_i, e_j) = 0 \) whenever \( i \neq j \). Then the unit element \( 1 \) together with the elements \( e_{i_1} \cdots e_{i_k} \in C(V, q) \), with \( i_1 < \cdots < i_k \) and \( 1 \leq k \leq n \), form a basis for \( C(V, q) \). In particular, \( \dim C(V, q) = 2^n \).

Of course, the basis \( e_1, \ldots, e_n \) always exists since \( q \) is symmetric: Its matrix representation with respect to any basis can be diagonalized. Let us refer to such a basis as being orthogonal with respect to \( q \) or simply orthogonal when \( q \) is understood.

We will identify \( \mathbb{R} \) with the subalgebra \( \mathbb{R} \cdot 1 \subseteq C(V, q) \) and \( V \) with its isomorphic image \( \theta(V) \subseteq C(V, q) \).

---

\(^3\) See, for example, [L]. (Chapter XIV, section 8.)
1.5 Proposition. If \(v, w \in V \subseteq C(V, q)\) then \(vw + wv = 2q(v, w)\). In particular, \(vw + wv = 0\) whenever \(q(v, w) = 0\).

**Proof.**
\[
(v + w)^2 = v^2 + vw + wv + w^2 = q(v, v) + vw + wv + q(w, w)
\]
\[
= q(v + w, v + w) = q(v, v) + 2q(v, w) + q(w, w)
\]

1.6 Corollary. If \(v, w \in V \subseteq C(V, q)\) then \(vww = 2q(v, w)v - q(v, v)w\). In particular, \(vww \in V\).

**Proof.**
\[
vww = (2q(v, w) - wv)v = 2q(v, w)v - wvv = 2q(v, w)v - q(v, v)w
\]

If \(v \in V\), let \(R_{q, v} : V \to V\) be the linear transformation defined by \(R_{q, v}w = 2q(v, w)v - q(v, v)w\). Thus, \(vww = R_{q, v}w\) in \(C(V, q)\). We will write \(R_{q, v} = R_v\) when \(q\) is understood. The following properties are immediate.

- \(R_vv = q(v, v)v\)
- \(R_vw = -q(v, v)w\) whenever \(q(v, w) = 0\)
- \(R^2_v = q(v, v)^2I\)
- \(R_{av} = a^2R_v\) for all \(a \in \mathbb{R}\)

The intuition for \(R_v\) comes from the following.

1.7 Proposition. Suppose \(V\) is a real inner product space with inner product \(q\). If \(a \in \mathbb{R} - \{0\}\) and \(v, w \in V\) with \(q(v, v) = 1\) then \(R_{aq, v}w = -aw - 2g(v, w)v\). That is, \(R_{aq, v}\) is the multiple \(-a\) of reflection across the plane perpendicular to \(v\). 

1.8 Proposition. If \(v, w, w' \in V\) then \(q(R_vw, R_vw') = q(v, v)^2q(w, w')\).

**Proof.**
\[
q(R_vw, R_vw') = q(2q(v, w)v - q(v, v)w, 2q(v, w)v - q(v, v)w')
\]
\[
= 2q(v, w) \cdot 2q(v, w') \cdot q(v, v) - 2q(v, w) \cdot q(v, v) \cdot q(v, w')
\]
\[
- q(v, v) \cdot 2q(v, w') \cdot q(w, v) + q(v, v)^2q(w, w')
\]
\[
= q(v, v)^2q(w, w')
\]

1.9 Corollary. If \(v, w, w' \in V\) and \(q(v, v) \in \{1, -1\}\) then \(q(R_vw, R_vw') = q(w, w')\).

The corollary to the following identity will become useful.

1.10 Proposition. If \(v, w \in V\) and \(a \in \mathbb{R}\) then \(R_{v+aw}w = 2q(v + aw, w)v + [a^2q(w, w) - q(v, v)]w\).
Proof. \( R_{v+aw}w = 2q(v + aw, w)(v + aw) - q(v + aw, v + aw)w \)
\[
= 2q(v + aw, w)v + [2aq(v + aw, w) - q(v + aw, v + aw)]w \\
= 2q(v + aw, w)v + [q(v + aw, aw) + q(v + aw, aw) - q(v + aw, v + aw)]w \\
= 2q(v + aw, w)v + [q(v + aw, aw) + q(v + aw, aw - (v + aw))]w \\
= 2q(v + aw, w)v + q(v + aw, aw - v)w \\
= 2q(v + aw, w)v + [a^2q(w, w) - q(v, v)]w 
\]

1.11 Corollary. If \( a \in \{1, -1\} \) and \( v, w \in V \) with \( q(v, v) = q(w, w) \) then \( R_{v+aw}w = aq(v + aw, v + aw)v \).

Proof. \( R_{v+aw}w = 2q(v + aw, w)v = 2aq(v + aw, aw)v = aq(v + aw, v + aw)v \)

Let us henceforth assume that the vector space \( V \) is finite dimensional. We will find several occasions to use the fact that an epimorphism from one finite dimensional vector space to another of the same dimension is an isomorphism.

There is an anti-automorphism of the tensor algebra \( T(V) \) which maps a product \( v_1 \otimes \cdots \otimes v_k \) to the reversed product \( v_k \otimes \cdots \otimes v_1 \). Since an element of the form \( v \otimes v - q(v, v) \) is mapped to itself by this morphism, we get an anti-endomorphism of the Clifford algebra which makes the following diagram commute.

\[
\begin{array}{ccc}
T(V) & \longrightarrow & T(V) \\
\downarrow \theta & & \downarrow \theta \\
C(V, q) & \longrightarrow & C(V, q) \\
\theta \quad \downarrow \beta & & \quad \downarrow \theta \\
v_1 \cdots v_k & \longrightarrow & v_k \cdots v_1 \equiv (v_1 \cdots v_k)^* \\
\end{array}
\]

Since this anti-endomorphism is onto, it is in fact an anti-automorphism.

Next, let us define an endomorphism \( \beta : C(V, q) \rightarrow C(V, q) \) by applying the universal property of the Clifford algebra to the linear map \( -\theta : V \rightarrow C(V, q) \).

\[
\begin{array}{ccc}
V & \overset{\theta}{\longrightarrow} & C(V, q) \\
\downarrow \beta & & \downarrow \theta \\
C(V, q) & \overset{\beta}{\longrightarrow} & C(V, q) \\
\end{array}
\]

Again, \( \beta \) is an isomorphism as it is clearly an epimorphism. In fact, since \( \beta^2 \circ \theta = \theta \) we have \( \beta^2 = 1 \) by uniqueness of the algebra homomorphism in the universal property.
The automorphism $\beta$ allows us to define a decomposition of $C(V, q)$ as the sum of two vector subspaces: If $C_e(V, q) = \ker (\beta - 1)$ and $C_o(V, q) = \ker (\beta + 1)$ then $C(V, q) = C_e(V, q) + C_o(V, q)$ and $C_e(V, q) \cap C_o(V, q) = \{0\}$. Also, the correct relationships between products hold to make the decomposition a $\mathbb{Z}_2$ grading.

\[
C_e(V, q) \cdot C_e(V, q) \subset C_e(V, q) \quad \quad \quad \quad \quad C_o(V, q) \cdot C_o(V, q) \subset C_o(V, q)
\]

\[
C_e(V, q) \cdot C_o(V, q) \subset C_o(V, q) \quad \quad \quad \quad \quad C_o(V, q) \cdot C_e(V, q) \subset C_e(V, q)
\]

In particular, $C_e(V, q) \subset C(V, q)$ is a subalgebra, the even Clifford algebra for the pair $(V, q)$. The following follows easily from the corresponding result for the full Clifford algebra.

1.12 Proposition. Suppose that $\dim V = n$ and that $e_1, \ldots, e_n$ is a basis for $V$ having $q(e_i, e_j) = 0$ whenever $i \neq j$. Then the unit element 1 together with the elements $e_{i_1} \cdots e_{i_k} \in C(V, q)$, with $i_1 < \cdots < i_k$, $1 \leq k \leq n$ and $k$ even, form a basis for $C_e(V, q)$, and $\dim C_e(V, q) = 2^{n-1}$. Also, the elements $e_{i_1} \cdots e_{i_k} \in C(V, q)$, with $i_1 < \cdots < i_k$, $1 \leq k \leq n$ and $k$ odd, form a basis for $C_o(V, q)$, and $\dim C_o(V, q) = 2^{n-1}$. ■

Note that the products $e_i e_j$, $i < j$, generate the even Clifford algebra.

Let us now identify those elements of the Clifford algebra which either commute with each element of $V \subset C(V, q)$ or which anticommute with each of $V$. Since $C(V, q)$ is generated as an algebra by the unit element 1 and the elements of $V$, the former is the center subalgebra, $Z(C(V, q))$, while the latter is a vector subspace of $C(V, q)$ which we will denote by $\bar{Z}(C(V, q))$.

If $e_1, \ldots, e_n$ is an orthogonal basis for $V$ and if $i_1, \ldots, i_k \in \{1, \ldots, n\}$ are distinct then, since the $e_i$ anticommute in $C(V, q)$, we have the following.

\[
e_{i_1} \cdots e_{i_k} e_{i_\ell} = \begin{cases} (-1)^k e_{i_\ell} e_{i_1} \cdots e_{i_k} & \text{if } \ell \notin \{i_1, \ldots, i_k\} \\ (-1)^{k-1} e_{i_\ell} e_{i_1} \cdots e_{i_k} & \text{if } \ell \in \{i_1, \ldots, i_k\} \end{cases}
\]

This identity has several immediate consequences.

- If $k < n$ then $e_{i_1} \cdots e_{i_k} \in Z(C(V, q))$ if and only if $k$ is even and $q(e_{i_1}, e_{i_1}) = \cdots = q(e_{i_k}, e_{i_k}) = 0$
- If $k < n$ then $e_{i_1} \cdots e_{i_k} \in \bar{Z}(C(V, q))$ if and only if $k$ is odd and $q(e_{i_1}, e_{i_1}) = \cdots = q(e_{i_k}, e_{i_k}) = 0$
- $e_1 \cdots e_n \in Z(C(V, q))$ if and only if $n$ is odd or $q = 0$
- $e_1 \cdots e_n \in \bar{Z}(C(V, q))$ if and only if $n$ is even or $q = 0$

These properties lead immediately to the equalities in the following result.

---

4 See the addendum to the current section (page 26) for the counting argument.
1.13 **Proposition.** Let \( e_1, \ldots, e_n \) be an orthogonal basis for \( V \).

1. If \( q \) is nondegenerate then the following hold.

\[
Z(C(V, q)) = \begin{cases} 
\mathbb{R} \cdot 1 & \text{if } n \text{ is even} \\
\mathbb{R} \cdot 1 + \mathbb{R} \cdot e_1 \cdots e_n & \text{if } n \text{ is odd}
\end{cases}
\]

\[
\bar{Z}(C(V, q)) = \begin{cases} 
\mathbb{R} \cdot e_1 \cdots e_n & \text{if } n \text{ is even} \\
\{0\} & \text{if } n \text{ is odd}
\end{cases}
\]

2. If \( q \) is degenerate then the following hold, where \( V_0 \subset V \) is the subspace spanned by those \( e_i \) having \( q(e_i, e_i) = 0 \).

\[
Z(C(V, q)) = \begin{cases} 
C_e(V_0, q) & \text{if } n \text{ is even} \\
C_e(V_0, q) + \mathbb{R} \cdot e_1 \cdots e_n & \text{if } n \text{ is odd}
\end{cases}
\]

\[
\bar{Z}(C(V, q)) = \begin{cases} 
C_o(V_0, q) + \mathbb{R} \cdot e_1 \cdots e_n & \text{if } n \text{ is even} \\
C_o(V_0, q) & \text{if } n \text{ is odd}
\end{cases}
\]

Consequently, the subspace of \( C(V, q) \) spanned by \( e_1 \cdots e_n \) is independent of the particular choice of the orthogonal basis.

**Proof.** We need only demonstrate the uniqueness of the subspace spanned by \( e_1 \cdots e_n \). Suppose \( u_1, \ldots, u_n \) is another orthogonal basis for \( V \). Let us consider the case when \( q \) is nondegenerate. If \( n \) is even and then

\[
\bar{Z}(C(V, q)) = \mathbb{R} \cdot e_1 \cdots e_n = \mathbb{R} \cdot u_1 \cdots u_n
\]

and we are done. If \( n \) is odd then

\[
Z(C(V, q)) = \mathbb{R} \cdot 1 + \mathbb{R} \cdot e_1 \cdots e_n = \mathbb{R} \cdot 1 + \mathbb{R} \cdot u_1 \cdots u_n
\]

so that \( u_1 \cdots u_n = a \cdot 1 + b \cdot e_1 \cdots e_n \) for some \( a, b \in \mathbb{R} \). Thus, \( a \cdot 1 = u_1 \cdots u_n - b \cdot e_1 \cdots e_n \in C_o(V, q) \), implying that \( a = 0 \) since \( C_o(V, q) \cap \mathbb{R} \cdot 1 = \{0\} \). Consequently, \( u_1 \cdots u_n = b \cdot e_1 \cdots e_n \), and \( b \neq 0 \) is necessary since \( u_1 \cdots u_n \) is a basis element. Thus, \( \mathbb{R} \cdot e_1 \cdots e_n = \mathbb{R} \cdot u_1 \cdots u_n \) as claimed. Similarly, the case when \( q \) is degenerate is a consequence of the facts \( C_e(V_0, q) \cap C_o(V, q) = \{0\} \) and \( C_o(V_0, q) \cap C_e(V, q) = \{0\} \). \( \blacksquare \)

**The Nondegenerate Case**

Suppose now that the symmetric, bilinear map \( q : V \times V \to \mathbb{R} \) is nondegenerate. Then \( q \) is either positive definite or negative definite and we can choose \( \varepsilon \in \{1, -1\} \) so that \( \varepsilon q \) is an inner
product on $V$. We will use the terms orthogonal and orthonormal in relation to this inner product. Let $S_q = \{ v \in V \mid q(v, v) = \varepsilon \}$ be the unit sphere in $V$. Since every element of $V$ is a multiple of an element of $S_q$, this set generates $C(V, q)$. Let $C(V, q)^*$ be the group of multiplicative units in $C(V, q)$. We have $V - \{ 0 \} \subset C(V, q)^*$ since $v^2 = q(v, v) \neq 0$ for each $v \in V$.

Let $U(V, q)$ be the subgroup of $C(V, q)^*$ generated by $S_q$. Note that $-1 \in U(V, q)$ since for any $x \in S_q$ we have $x \cdot x = \varepsilon$ and $x \cdot -x = -\varepsilon$ both in $U(V, q)$. $U(V, q)$ necessarily contains all products $x_1 \cdots x_k$ with $x_1, \ldots, x_k \in S_q$ and $k \geq 1$. However, since $x_1 \cdots x_k(x_1 \cdots x_k)^* = \varepsilon^k \in \{1, -1\}$, all elements of $U(V, q)$ are of this form. Thus,

$$U(V, q) = \{ x_1 \cdots x_k \mid x_1, \ldots, x_k \in S_q, k \geq 1 \}.$$ 

Now set $U_e(V, q) = U(V, q) \cap C_e(V, q)$, the subgroup of $U(V, q)$ consisting of products of even numbers of elements from $S_q$. Clearly $U(V, q) - U_e(V, q) = U(V, q) \cap C_o(V, q)$ so that $C_e(V, q)$ and $C_o(V, q)$ partition $U(V, q)$.

We have already seen that if $e_1, \ldots, e_n$ and $u_1, \ldots, u_n$ are two orthogonal bases for $V$ then $e_1 \cdots e_n$ and $u_1 \cdots u_n$ span the same subspace of $C(V, q)$ so that $e_1 \cdots e_n = au_1 \cdots u_n$ for some non-zero $a \in \mathbb{R}$. If these bases are both orthonormal then the equalities

$$(e_1 \cdots e_n)(e_1 \cdots e_n)^* = \varepsilon^n = (au_1 \cdots u_n)(au_1 \cdots u_n)^* = a^2 \varepsilon^n$$

give $a^2 = 1$.

**1.14 Proposition.** Let $e_1, \ldots, e_n$ be an orthonormal basis for $V$. Then

$$Z(C(V, q)) \cap U(V, q) = \begin{cases} \{ 1, -1 \} & \text{if } n \text{ is even} \\ \{ 1, -1, e, -e \} & \text{if } n \text{ is odd} \end{cases}$$

$$\bar{Z}(C(V, q)) \cap U(V, q) = \begin{cases} \{ e, -e \} & \text{if } n \text{ is even} \\ \emptyset & \text{if } n \text{ is odd} \end{cases}$$

where $e = e_1 \cdots e_n$.

**Proof.** We have already determined, in Proposition 1.13, that $Z(C(V, q))$ is spanned by 1 if $n$ is even and 1 and $e$ if $n$ is odd. Thus, the first equality is clear for even $n$. For odd $n$, no element $a \cdot 1 + b \cdot e$ with $a$ and $b$ both non-zero can lie in $U(V, q)$ since $1 \in C_e(V, q)$ and $e \in C_o(V, q)$: Each element of $U(V, q)$ lies in either $C_e(V, q)$ or $C_o(V, q)$ and such a sum lies in neither of these.

The second equality follows immediately from our determination that $\bar{Z}(C(V, q))$ is spanned by $e$ if $n$ is even and is the trivial subspace if $n$ is odd. ■
Now let $\text{Aut}_q(V)$ consist of all those endomorphisms of $V$ which preserve the norm (and, hence, also the inner product).

$$\text{Aut}_q(V) = \{ T \in \text{hom}(V,V) \mid q(Tv, Tv) = q(v, v) \text{ for all } v \in V \}$$

Since every such endomorphism is necessarily an isomorphism, $\text{Aut}_q(V)$ is a subgroup of the group $\text{Aut}(V)$ of automorphisms of $V$.

Recall that if $v \in V$ then the endomorphism $R_v : w \mapsto 2q(v, w)v - q(v, v)w : V \to V$ was defined so that $vuw = R_v(w)$ in $C(V, q)$. By Corollary 1.9, $R_v \in \text{Aut}_q(V)$ whenever $v \in S_q$. Thus, there is a representation $\gamma : U(V, q) \to \text{Aut}_q(V)$ defined by $\gamma(u)v = uvu^*$. If $u = x_1 \cdots x_k$ then $\gamma(u) = R_{x_1} \circ \cdots \circ R_{x_k}$.

For $v \in V - \{0\}$, let us write $F_v = R_{v/r(v)}$ where $r : V - \{0\} \to S_q$ is defined by $r(v) = v/\sqrt{q(v, v)}$. Thus, $F_v = R_v$ whenever $v \in S_q$. The following properties of the function $v \mapsto F_v$ are easy consequences of the four properties of $v \mapsto R_v$ stated on page 14.

- $F_vv = \varepsilon v$
- $F_vw = -\varepsilon w$ whenever $w$ is orthogonal to $v$
- $F_v^2 = I$
- $F_a v = F_v$ for any $a \in \mathbb{R} - \{0\}$

In addition, there is the following as a consequence of Corollary 1.11.

- $F_{v+aw} = a\varepsilon v$ whenever $v, w \in S_q$, $a \in \{1, -1\}$ and $v + aw \neq 0$

The following proposition will aid us in identifying the image of $\gamma$.

1.15 Proposition. Suppose that $q$ is negative definite, or that $q$ is positive definite and $V$ is of even dimension. If $v, w \in S_q$ then there exists an automorphism $T \in \text{im}\gamma$ such that $Tv = w$ and $Tz = z$ whenever $z$ is orthogonal to both $v$ and $w$.

Proof. If $v = w$ we can take $T = I$ so let us assume $v \neq w$. Of course, $F_v = \gamma(v/r(v)) \in \text{im}\gamma$ for any $v \in V - \{0\}$.

1. $q$ negative definite. In this case $\varepsilon = -1$ and $F_{v-w}$ has desired properties.

2. $q$ is positive definite and $V$ is of even dimension. In this case $\varepsilon = 1$ and $F_{v-w}$ has the following properties.

- $F_{v-w}w = -w$
- $F_{v-w}z = -z$ whenever $z$ is orthogonal to $v$ and $w$

Thus, we can take $T = -F_{v-w}$ once we show $-I \in \text{im}\gamma$. To show this, let $e_1, \ldots, e_n$ be an orthonormal basis for $V$. Then, since $n$ is even, $\gamma(e_1 \cdots e_n) = F_{e_n} \circ \cdots \circ F_{e_1} = -I$ and we are done.
1.16 Proposition. Suppose that $q$ is negative definite, or that $q$ is positive definite and $V$ is of even dimension. Then the representation $\gamma : U(V, q) \to \text{Aut}_q(V)$ is onto.

Note that $\gamma$ cannot be onto if $q$ is positive definite and $V$ is of odd dimension. In this case it is easy to see that $\gamma(x) = F_x$ has determinant 1 for all $x \in S_q$. In particular, $-I \notin \gamma$.

Proof. Suppose $A \in \text{Aut}_q(V) - \{I\}$. We will find $T_1, \ldots, T_\ell \in \text{im} \gamma$ such that $A = T_\ell \circ \cdots \circ T_1$.

Let $i_1$ to be the least integer such that $Ae_{i_1} \neq e_{i_1}$. There is an automorphism $T_1 \in \text{im} \gamma$ such that $T_1 e_{i_1} = Ae_{i_1}$ and $T_1 z = z$ whenever $z$ is orthogonal to both $e_{i_1}$ and $Ae_{i_1}$. Since, for $i < i_1$, $e_i = Ae_i$ is orthogonal to both $e_{i_1}$ and $Ae_{i_1}$, $T_1 e_i = Ae_i$ for all $i \leq i_1$.

If $i_1 = n$ we are done. Otherwise let us suppose we have chosen integers $1 \leq i_1 < \cdots < i_k < n$ and automorphisms $T_1, \ldots, T_k \in \text{im} \gamma$ such that, $(T_k \circ \cdots \circ T_1)e_i = Ae_i$ for all $i \leq i_k$. Let $i_{k+1}$ be the least integer such that $(T_k \circ \cdots \circ T_1)e_{i_{k+1}} \neq Ae_{i_{k+1}}$. Necessarily $i_{k+1} > i_k$. There is an automorphism $T_{k+1} \in \text{im} \gamma$ such that $T_{k+1}((T_k \circ \cdots \circ T_1)e_{i_{k+1}}) = Ae_{i_{k+1}}$ and $T_{k+1} z = z$ whenever $z$ is orthogonal to both $(T_k \circ \cdots \circ T_1)e_{i_{k+1}}$ and $Ae_{i_{k+1}}$. Since, for $i < i_{k+1}$, $(T_k \circ \cdots \circ T_1)e_i = Ae_i$ is orthogonal to both $(T_k \circ \cdots \circ T_1)e_{i_{k+1}}$ and $Ae_{i_{k+1}}$, $T_{k+1}((T_k \circ \cdots \circ T_1)e_i) = (T_{k+1} \circ \cdots \circ T_1)e_i = Ae_i$ for all $i \leq i_{k+1}$.

Continuing in this way, we eventually get $T_1, \ldots, T_\ell$ as desired. □

1.17 Proposition. Let $e_1, \ldots, e_n$ be an orthonormal basis for $V$ and let $e = e_1 \cdots e_n$. Then the kernel of the representation $\gamma : U(V, q) \to \text{Aut}_q(V)$ is given by the following.

\[
\ker \gamma = \begin{cases} 
\{1, -1, e, -e\} & \text{if $q$ is positive definite and $V$ has odd dimension} \\
\{1, -1\} & \text{otherwise}
\end{cases}
\]

Proof. If $u \in U(V, q)$ then $\gamma(u) = I$ if and only if $ux(u^*u) = xu$ for all $x \in S_q$. Let us now consider the negative and positive definite cases separately. The results in both cases follow easily from our determination of $Z(C(V, q)) \cap U(V, q)$ and $\bar{Z}(C(V, q)) \cap U(V, q)$ in Proposition 1.14.

1. $q$ is negative definite. In this case $u^*u = 1$ for all $u \in U_e(V, q)$ and $u^*u = -1$ for all $u \in U(V, q) - U_e(V, q)$. Thus,

\[
\ker \gamma \cap U_e(V, q) = Z(C(V, q)) \cap U_e(V, q) = \{1, -1\}
\]

and

---

5 Choose an orthonormal basis $z_1, \ldots, z_n$ of $V$ with $z_1 = x$. Then $F_zx_1 = x_1$ and $F_zx_i = -x_i$ for all $1 < i \leq n$ so that $\det F_z = (-1)^{n-1} = 1$. 20
\[
\ker \gamma \cap (U(V, q) - U_e(V, q)) = Z(C(V, q)) \cap (U(V, q) - U_e(V, q)) = \emptyset.
\]

2. \( q \) is positive definite. In this case \( u^*u = 1 \) for all \( u \in U(V, q) \) so that \( \ker \gamma = Z(C(V, q)) \cap U(V, q) \). \( \blacksquare \)

**Clifford Algebras on Euclidean Space**

Let us now specialize to the cases \((V, q) = (\mathbb{R}^n, g)\) and \((V, q) = (\mathbb{R}^n, -g)\), where \( g(v, w) = v \cdot w \) is the usual inner product on \( \mathbb{R}^n \). We will write \( \Gamma(\mathbb{R}^n, +) = C(\mathbb{R}^n, g) \) and \( \Gamma(\mathbb{R}^n, -) = C(\mathbb{R}^n, -g) \) as well as \( \Gamma_e(\mathbb{R}^n, +) = C_e(\mathbb{R}^n, g) \) and \( \Gamma_e(\mathbb{R}^n, -) = C_e(\mathbb{R}^n, -g) \).

Let \( e_1, \ldots, e_n \) now denote the standard basis for \( \mathbb{R}^n \).

The following result shows that the algebras \( \Gamma(\mathbb{R}^n, +) \) and \( \Gamma(\mathbb{R}^n, -) \) are completely determined by \( \Gamma(\mathbb{R}^1, +) \), \( \Gamma(\mathbb{R}^1, -) \), \( \Gamma(\mathbb{R}^2, +) \) and \( \Gamma(\mathbb{R}^2, -) \).

1.18 Proposition. For every \( n \geq 1 \), there are the following algebra isomorphisms.

\[
\begin{align*}
\Gamma(\mathbb{R}^{n+2}, +) &\cong \Gamma(\mathbb{R}^n, -) \otimes \Gamma(\mathbb{R}^2, +) \\
\Gamma(\mathbb{R}^{n+2}, -) &\cong \Gamma(\mathbb{R}^n, +) \otimes \Gamma(\mathbb{R}^2, -)
\end{align*}
\]

Proof. We construct both isomorphisms using the universal property of Clifford algebras. The first isomorphism will be defined by the following diagram.

\[
\begin{array}{ccc}
\mathbb{R}^{n+2} & \xrightarrow{\theta} & \Gamma(\mathbb{R}^{n+2}, +) \\
\phi \downarrow & & \downarrow \tilde{f} \\
\Gamma(\mathbb{R}^n, -) \otimes \Gamma(\mathbb{R}^2, +) & \xrightarrow{f} & \Gamma(\mathbb{R}^n, -) \otimes \Gamma(\mathbb{R}^2, +)
\end{array}
\]

To show that the above diagram does indeed define an algebra homomorphism \( f \), which we will soon see to be an isomorphism, we need only show that the linear map \( \phi \) satisfies \( \phi(v)^2 = v \cdot v \) for each \( v \in \mathbb{R}^{n+2} \). It is easy to check that each \( \phi(e_i)^2 = 1 \) for each \( i \) and that \( \phi(e_i)\phi(e_j) = -\phi(e_j)\phi(e_i) \) whenever \( i \neq j \). It follows that \( \phi(v)^2 = v \cdot v \) for all \( v \in \mathbb{R}^{n+2} \) since cross terms in the square of \( f \) applied to a linear combination of the \( e_i \) will cancel. Thus, \( f \) is uniquely defined by the universal property. Moreover, \( f \) is onto since \( f(e_{n+1}e_n) = e_i \otimes 1 \) for \( 1 \leq i \leq n \). Thus, since the domain and range of \( f \) have the same dimension, \( f \) is an isomorphism.

The second isomorphism is defined similarly.
Verification that this diagram defines a homomorphism of algebras is, in this case, an easy exercise in checking that $\phi(v)^2 = -v \cdot v$ for each $v \in \mathbb{R}^{n+2}$ using the same reasoning as above. Again, we get an isomorphism by reason of surjectivity and dimension.

Identical reasoning to that of the above can be used to construct several additional algebra isomorphisms. In each case we will require knowledge of the square of the element $e_1 \cdots e_n$ in either $\Gamma(\mathbb{R}^n, +)$ or $\Gamma(\mathbb{R}^n, -)$. Indeed, if $i_1, \ldots, i_\ell \in \{1, \ldots, n\}$ are distinct integers then $(e_{i_1} \cdots e_{i_\ell})^2$ is easily computed using anticommutation of the $e_i$ in both Clifford algebras plus the fact that $e_i^2 = 1$ in $\Gamma(\mathbb{R}, +)$ and $e_i^2 = -1$ in $\Gamma(\mathbb{R}, -)$.

\[
\begin{align*}
\Gamma(\mathbb{R}^n, +) : \quad (e_{i_1} \cdots e_{i_\ell})^2 &= (-1)^{\frac{1}{2}(\ell-1)} = \\
&= \begin{cases} 
1 & \text{if \( \ell \equiv 0 \mod 4 \) or \( \ell \equiv 1 \mod 4 \)} \\
-1 & \text{if \( \ell \equiv 2 \mod 4 \) or \( \ell \equiv 3 \mod 4 \)}
\end{cases} \quad (1.1) \\
\Gamma(\mathbb{R}^n, -) : \quad (e_{i_1} \cdots e_{i_\ell})^2 &= (-1)^{\frac{1}{2}(\ell+1)} = \\
&= \begin{cases} 
1 & \text{if \( \ell \equiv 0 \mod 4 \) or \( \ell \equiv 3 \mod 4 \)} \\
-1 & \text{if \( \ell \equiv 1 \mod 4 \) or \( \ell \equiv 2 \mod 4 \)}
\end{cases} \quad (1.2)
\end{align*}
\]

Thus, the following diagrams are seen to define algebra isomorphisms providing the stated condition is satisfied.

\[
\begin{align*}
\text{n \equiv 0 mod 4} & \quad \mathbb{R}^{n+1} \quad \Gamma(\mathbb{R}^{n+1}, +) \longrightarrow \Gamma(\mathbb{R}^n, \otimes \Gamma(\mathbb{R}^1, +)) \\
& \quad e_i \mapsto e_i \otimes 1, \quad 1 \leq i \leq n \\
& \quad e_{n+1} \mapsto e_1 \cdots e_n \otimes e_1 \\
\text{n \equiv 2 mod 4} & \quad \mathbb{R}^{n+1} \quad \Gamma(\mathbb{R}^{n+1}, +) \longrightarrow \Gamma(\mathbb{R}^n, \otimes \Gamma(\mathbb{R}^1, -)) \\
& \quad e_i \mapsto e_i \otimes 1, \quad 1 \leq i \leq n \\
& \quad e_{n+1} \mapsto e_1 \cdots e_n \otimes e_1 \\
\text{n \equiv 2 mod 4} & \quad \mathbb{R}^{n+1} \quad \Gamma(\mathbb{R}^{n+1}, -) \longrightarrow \Gamma(\mathbb{R}^n, \otimes \Gamma(\mathbb{R}^1, +)) \\
& \quad e_i \mapsto e_i \otimes 1, \quad 1 \leq i \leq n \\
& \quad e_{n+1} \mapsto e_1 \cdots e_n \otimes e_1 \\
\text{n \equiv 0 mod 4} & \quad \mathbb{R}^{n+1} \quad \Gamma(\mathbb{R}^{n+1}, -) \longrightarrow \Gamma(\mathbb{R}^n, \otimes \Gamma(\mathbb{R}^1, -)) \\
& \quad e_i \mapsto e_i \otimes 1, \quad 1 \leq i \leq n \\
& \quad e_{n+1} \mapsto e_1 \cdots e_n \otimes e_1
\end{align*}
\]
The Clifford algebras \( \Gamma(R^1,+) \), \( \Gamma(R^1,-) \), \( \Gamma(R^2,+)+ \) and \( \Gamma(R^2,-) \) are identified by isomorphisms constructed by the following diagrams. Here \( R(2) \) is the algebra of real \( 2 \times 2 \) matrices and \( H \) is the quaternionic algebra spanned by 1, \( i, j \) and \( k \) with \( i^2 = j^2 = k^2 = -1 \), \( ij = -ji = k, \) \( jk = -kj = i \) and \( ki = -ik = j \).

With these identifications, let us restate the isomorphisms of Proposition 1.18 and (1.3) through (1.8).

\[
\Gamma(R^{n+2},+) \cong \Gamma(R^n,-) \otimes R(2)
\]
\[
\Gamma(R^{n+2},-) \cong \Gamma(R^n,+) \otimes H
\]
\[
\begin{align*}
\Gamma(R^{n+1}, +) &\cong \Gamma(R^n, +) \otimes (R \oplus R) \cong \Gamma(R^n, +) \oplus \Gamma(R^n, +) \\
\Gamma(R^{n+1}, -) &\cong \Gamma(R^n, -) \otimes C \\
\Gamma(R^{n+1}, -) &\cong \Gamma(R^n, -) \otimes (R \oplus R) \cong \Gamma(R^n, -) \oplus \Gamma(R^n, -) \\
\Gamma(R^{n+1}, +) &\cong \Gamma(R^n, +) \otimes C \\
\Gamma(R^{n+2}, +) &\cong \Gamma(R^n, +) \otimes H \\
\Gamma(R^{n+2}, -) &\cong \Gamma(R^n, -) \otimes R(2)
\end{align*}
\]

\( n \equiv 0 \text{ mod } 4 \) 
\( n \equiv 2 \text{ mod } 4 \)

The center subalgebras \( Z(\Gamma(R^n, +)) \) and \( Z(\Gamma(R^n, -)) \) can be identified using the order of the element \( e_1 \cdots e_n \) since we already know by Proposition 1.13 that each algebra is spanned by 1 if \( n \) is even and 1 and \( e_1 \cdots e_n \) if \( n \) is odd.

For odd \( n \), we have an isomorphism from either center to \( R \oplus R \) or \( C \), depending on whether \( e_1 \cdots e_n \) has order 2 or 4. The isomorphism sends \( e_1 \cdots e_n \) to either \( (1, -1) \in R \oplus R \) or \( i \in C \), as appropriate.

\[
Z(\Gamma(R^n, +)) \cong \begin{cases} 
R & \text{if } n \text{ is even} \\
R \oplus R & \text{if } n \equiv 1 \text{ mod } 4 \\
C & \text{if } n \equiv 3 \text{ mod } 4 \\
R & \text{if } n \text{ is even} 
\end{cases}
\]

\[
Z(\Gamma(R^n, -)) \cong \begin{cases} 
C & \text{if } n \equiv 1 \text{ mod } 4 \\
R \oplus R & \text{if } n \equiv 3 \text{ mod } 4 
\end{cases}
\]

For the even Clifford algebras, the elements \( e_i e_{n+1} \), with \( i \leq n \), generate \( \Gamma_c(R^{n+1}, +) \) and \( \Gamma_c(R^{n+1}, -) \) since \( e_i e_{n+1} e_j e_{n+1} = e_i e_j - e_{n+1}^2 \). The following diagrams define algebra isomorphisms.

\[
\begin{align*}
\Gamma(R^n, -) &\longrightarrow \Gamma_c(R^{n+1}, +) \\
\Gamma(R^n, -) &\longrightarrow \Gamma_c(R^{n+1}, -)
\end{align*}
\]
Spin Groups

Most of the work has now been done in the more general setting. Let \( \text{pin}(n) = U(R^n, -g) \) and \( \text{Spin}(n) = U(e(R^n, -g)) \). We have \( \text{Aut}_g(R^n) = O(n) \) and the restriction of the representation \( \gamma : \text{pin}(n) \to O(n) \) to \( \text{Spin}(n) \) defines a representation \( \gamma_e : \text{Spin}(n) \to SO(n) \) with kernel \( \{1, -1\} \). This representation double covers \( SO(n) \) and is a universal covering when \( n \geq 3 \).

3. Clifford Groups

We are now in a position to identify the subgroup \( \text{Spin}'(n) = \gamma^{-1}(SO(n) \cap O(1)^n) = \text{Spin}(n) \cap 
\gamma^{-1}(O(1)^n) \subset \text{Spin}(n) \).

Set \( \text{pin}'(n) = \gamma^{-1}(O(1)^n) \). Since \( \gamma \) maps a standard basis vector \( e_i \) to \( F_{e_i} \), which is reflection across the plane perpendicular to \( e_i \), \( \text{pin}'(n) \) is exactly the subgroup of \( \text{pin}(n) \) generated by the standard basis vectors. Thus, \( \text{pin}'(n) \) is generated by \( e_1, \ldots, e_n \) with relations \( e_1^2 = \cdots = e_n^2 = -1 \), and \( e_i e_j = -e_j e_i \) whenever \( i \neq j \). This group is well-known: It is the Clifford group on \( n \) generators, \( C_n \).

Note that \( C_n \) has order \( 2^{n+1} \). Indeed, each element of \( C_n \) has a unique representation as a product \( \varepsilon e_{i_1} \cdots e_{i_k} \) with \( \varepsilon \in \{1, -1\} \) and \( 1 \leq i_1 < \cdots < i_k \leq n \).

Notice that Proposition 1.14 tells us that the center of \( C_n \) is \( \{1, -1, e, -e\} \) if \( n = 2m+1 \) is odd, where \( e = e_1 \cdots e_n \). In this case (1.2) tells us that \( e \) has order 2 if \( n \equiv 3 \mod 4 \) and order 4 if \( n \equiv 1 \mod 4 \), giving \( Z(C_n) \cong Z_2 \times Z_2 \) and \( Z(C_n) \cong Z_4 \) respectively. Since Proposition 1.14 also tells us that the center of \( C_n \) is \( \{1, -1\} \) if \( n \) is even, the claim on page 156 of \([G]\) that there are isomorphisms \( C_n \cong \mathbb{Z}_4 \) if \( n \) is even and \( C_n \cong \mathbb{Z}_{2m} \) if \( n \) is odd, where \( H_n \) are finite Heisenberg groups over \( Z_2 \), is false. Indeed, the author there first states that \( Z(C_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) for all odd \( n \) and this, as we have just seen, is also false. We have not verified which of the claimed isomorphisms do exist, but certainly we cannot have both \( C_n \cong \mathbb{Z}_4 \) and \( C_n \cong \mathbb{Z}_8 \) regardless of the identity of the group \( G \) since \( Z(C_{n+1}) \cong \mathbb{Z}_4 \) while \( Z(C_n \times Z_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Instead, regarding \( C_n \) as the subgroup of \( C_{n+1} \) generated by \( e_1, \ldots, e_n \in C_{n+1} \), there is a homomorphism

\[
(\alpha, (-1)^k) \mapsto \alpha e^k : C_{2m} \times \mathbb{Z}_2 \to C_{2m+1}
\]

if \( m \) is odd and a homomorphism

\[
(\alpha, i^k) \mapsto \alpha e^k : C_{2m} \times \mathbb{Z}_4 \to C_{2m+1}
\]

if \( m \) is even, regarding \( \mathbb{Z}_2 \) and \( \mathbb{Z}_4 \) as the multiplicative subgroups \( \{1, -1\} \) and \( \{1, -1, i, -i\} \) of \( S^1 \subset C \). Both of these homomorphisms are onto since \( (e_n \cdots e_1) e = (-1)^n e_{n+1} \), and their kernels are \( \{(1, 1)\} \) and \( K = \{(1, 1), (-1, 1)\} \) respectively. Thus, \( C_{2m+1} \cong C_{2m} \times \mathbb{Z}_2 \) if \( m \) is odd and \( C_{2m+1} \cong (C_{2m} \times \mathbb{Z}_4)/K \) if \( m \) is even.

Note that \( C_1 \) and \( C_2 \) are thus easily recognizable as \( \mathbb{Z}_4 \) and \( Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset H \) respectively.
We can now identify Spin'(n) by examining the Clifford algebra isomorphism (1.10).

\[ \begin{array}{ccc}
\mathbb{R}^n \\
\leftrightarrow \\
\Gamma(\mathbb{R}^n, -) \rightarrow \Gamma_c(\mathbb{R}^{n+1}, -) \\
e_i \mapsto e_i e_{n+1}
\end{array} \]

Since this isomorphism maps pin'(n) \subset \Gamma(\mathbb{R}^n, -) onto Spin'(n + 1) we have Spin'(n + 1) \cong pin'(n) = C_n for all n ≥ 1. Thus, Spin'(n) \cong C_{n-1} for all n ≥ 2 and we have found the fundamental group of all the real complete flags.

1.19 Proposition. \[ \pi_1 F(2) \cong \mathbb{Z} \text{ and } \pi_1 F(n) \cong C_{n-1} \text{ for all } n \geq 3. \]

4. Addendum: Counting by Twos

Given an integer n ≥ 1, define integers \( d_e(n) \) and \( d_o(n) \) as follows.

\[
d_e(n) = \begin{cases} 
\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n} & \text{if } n \text{ is even} \\
\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1} & \text{if } n \text{ is odd}
\end{cases}
\]

\[
d_o(n) = \begin{cases} 
\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n-1} & \text{if } n \text{ is even} \\
\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n} & \text{if } n \text{ is odd}
\end{cases}
\]

If \( V \) is a vector space of dimension \( n \) then any Clifford algebra \( C(V, q) \) has \( \dim C_e(V, q) = d_e(n) \) and \( \dim C_o(V, q) = d_o(n) \). If \( n \) is odd then, since \( \binom{n}{k} = \binom{n}{n-k} \), we have

\[ 2d_e(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n \]

so that \( d_e(n) = 2^{n-1} \). If \( n \geq 4 \) is even, apply the identity \( \binom{m}{\ell} + \binom{m}{\ell+1} = \binom{m+1}{\ell+1} \) with \( m + 1 = n \) and \( \ell + 1 = 2, \ldots, n - 2 \) to get

\[ d_e(n) = \binom{n}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-3} + \binom{n-1}{n-2} + \binom{n}{n} \]

so that \( d_e(n) = 2^{n-1} \) in this case as well. Clearly this also holds when \( n = 2 \). Since \( d_e(n) + d_o(n) = 2^n \), we also have \( d_o(n) = 2^{n-1} \) for all \( n \).
B. Triviality of Certain Quotients of $O(4)$ over $\mathbb{RP}^3$

In addition to their coverings by Spin(3) and Spin(4) respectively, universal coverings of $F(3)$ and $F(4)$ can be constructed using the universal covering of $SO(3)$ by $SU(2)$.

$$SU(2) \approx S^3$$
$$\downarrow 2:1$$
$$SO(3)$$
$$\downarrow 4:1$$

$F(3) \approx O(3) / O(1)^3 = SO(3) / (O(1)^3 \cap SO(3))$

$$S^3 \times SU(2) \approx S^3 \times S^3$$
$$\downarrow 2:1$$

$F(4) \approx O(4) / O(1)^4 = SO(4) / (O(1)^4 \cap SO(4))$

Let us examine these universal coverings to determine the action of the group of covering transformations on $S^3$ and $S^3 \times S^3$ respectively.

First, recall the construction of the covering of $SO(3)$ by $SU(2)$. We have

$$SU(2) = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

as well as an isomorphism

$$\left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) \mapsto \alpha + \beta j : SU(2) \to S^3$$

of topological groups, where we regard $S^3 \subset H$ as the unit quaternions and $\mathbb{C}$ as a subalgebra of $H$ in the usual way. Embedding $\mathbb{R}^3$ in $H$ as the imaginary quaternions, conjugation by an element of $S^3$ defines an element of $SO(3)$ and the homomorphism $\gamma : S^3 \to SO(3)$ which maps $u \in S^3$ to conjugation by $u$ is a 2-fold universal covering.

It is clear that $\gamma^{-1}(O(1)^3 \cap SO(3)) = Q_8 \subset S^3$ where $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \subset H$ is the order 8 quaternionic group. Since the covering $\gamma$ is a homomorphism of groups the following diagram
is easily seen to define a map $S^3/Q_8 \to F(3)$ by the universal property of quotients.

$$
\begin{array}{ccc}
S^3 & \xrightarrow{\gamma} & SO(3) \\
\downarrow & & \downarrow \\
S^3 / Q_8 & \longrightarrow & SO(3) / (O(1)^3 \cap SO(3)) \approx F(3)
\end{array}
$$

Further, this map is a homeomorphism being a continuous bijection from a compact space to a Hausdorff space. Thus, $F(3) \approx S^3/Q_8$, $\pi_1 F(3) \cong Q_8$ and each covering transformation of $S^3$ over $F(3)$ is multiplication by an element of $Q_8$.

To examine the covering of $F(4)$ let us first construct a homeomorphism $SO(4) \to S^3 \times SO(3)$. Identifying the underlying vector space of the quaternionic algebra $H$ with $\mathbb{R}^4$ in the usual way ($1 \leftrightarrow \hat{e}_1$, $i \leftrightarrow \hat{e}_2$, $j \leftrightarrow \hat{e}_3$, $k \leftrightarrow \hat{e}_4$) and $SO(3)$ with the subgroup of $SO(4)$ consisting of those transformations which leave $\hat{e}_1$ fixed, the principal bundle

$$
\begin{array}{ccc}
SO(3) & \longrightarrow & SO(4) \\
\downarrow & & \downarrow \\
S^3 & \longrightarrow & S^3 \times SO(3)
\end{array}
$$

has $s : x \mapsto (x \mid xi \mid xj \mid xk) : S^3 \to SO(4)$ as a section. It is easily checked that $(x \mid xi \mid xj \mid xk)$ is indeed an element of $SO(4)$: The columns of this matrix are orthonormal and it has determinant equal to $|x|^2 = 1$. Thus, the map

$$f : A \mapsto (A \hat{e}_1, s(A \hat{e}_1)^{-1} A) : SO(4) \to S^3 \times SO(3)$$

is a homeomorphism with inverse $(x, R) \mapsto s(x) R$.

Notice that $s(-x) = -s(x)$ for all $x \in S^3$.

1.20 Proposition. Let $H$ be any subgroup of $O(3)$ which contains $-I$. Then there is a homeomorphism

$$\tilde{f} : SO(4) / (O(1) \times H) \cap SO(4) \to S^3 \times SO(3) / \{1, -1\} \times (H \cap SO(3))$$

which makes the diagram

$$
\begin{array}{ccc}
SO(4) & \longrightarrow & f \longrightarrow S^3 \times SO(3) \\
\downarrow & & \downarrow \\
SO(4) / (O(1) \times H) \cap SO(4) & \longrightarrow & S^3 \times SO(3) / \{1, -1\} \times (H \cap SO(3)) \\
\| & & \| \\
O(4) / (O(1) \times H) & \longrightarrow & \mathbb{R}P^3 \times (O(3) / H) \\
\downarrow & & \downarrow \\
\mathbb{R}P^3 & \longrightarrow & \pi_1 \\
\pi & &
\end{array}
$$

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commute, where $\pi : O(4) / O(1) \times H \to \mathbb{RP}^3 : [A] \mapsto [A \hat{e}_1]$ and $\pi_1$ is projection onto the first factor.

**Proof.** To demonstrate the existence of the homeomorphism $\hat{f}$, we let our subgroups of $SO(4)$ and $S^3 \times SO(3)$ act by multiplication on the right.

1. $\hat{f}$ is well-defined.

If $A, B \in SO(4)$ lie in the same $(O(1) \times H) \cap SO(4)$ orbit then $A = BD$ for some $D \in (O(1) \times H) \cap SO(4)$. Now, either $D\hat{e}_1 = \hat{e}_1$ or $D\hat{e}_1 = -\hat{e}_1$. In the first case we have $D \in H \cap SO(3)$ and

$$f(A) = f(BD) = (BD\hat{e}_1, s(BD\hat{e}_1)^{-1}BD) = (\hat{e}_1, s(\hat{e}_1)^{-1}BD) = f(B) \cdot (1, D)$$

so that $f(A)$ and $f(B)$ lie in the same $\{1, -1\} \times (H \cap SO(3))$ orbit of $S^3 \times SO(3)$. In the second case we have $-D \in H \cap SO(3)$ and

$$f(A) = f(BD) = (BD\hat{e}_1, s(BD\hat{e}_1)^{-1}BD) = (-B\hat{e}_1, s(-B\hat{e}_1)^{-1}BD) = (-B(\hat{e}_1), -s(B\hat{e}_1)^{-1}BD) = f(B) \cdot (-1, -D)$$

so that again $f(A)$ and $f(B)$ lie in the same orbit. $\Box$

2. $\hat{f}$ is injective.

Suppose $f(A) = f(B) \cdot (\varepsilon, D)$ for some $(\varepsilon, D) \in \{1, -1\} \times (H \cap SO(3))$. Applying $f^{-1}$ we have

$$A = f^{-1}(f(B) \cdot (\varepsilon, D)) = f^{-1}(\varepsilon B\hat{e}_1, s(B\hat{e}_1)^{-1}BD) = s(\varepsilon B\hat{e}_1)s(B\hat{e}_1)^{-1}BD = \varepsilon BD = B \cdot \varepsilon D$$

so that $A$ and $B$ lie in the same $(O(1) \times H) \cap SO(4)$ orbit of $SO(4)$. $\Box$

Clearly $\hat{f}$ is surjective. Thus, $\hat{f}$ is homeomorphism being a continuous bijection from a compact space to a Hausdorff space.

Commutativity of the triangle is clear since $f$ maps $A \in SO(4)$ onto a pair whose first component is $A\hat{e}_1$. $\blacksquare$

**1.21 Corollary.** The following bundles are trivial.

\[
\begin{array}{cccc}
F(1, 1, 1) & \hookrightarrow & F(1, 1, 1, 1) & [A] & \mathbb{RP}^2 & \hookrightarrow & F(1, 1, 2) & [A] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{RP}^3 & \hookrightarrow & [A\hat{e}_1] & & \mathbb{RP}^3 & \hookrightarrow & [A\hat{e}_1] & \\
\end{array}
\]
1.22 Corollary. \( F(4) \cong \mathbb{RP}^3 \times F(3) \cong S^3 \times S^3 / \{1, -1\} \times Q_8 \). Thus, \( \pi_1 F(4) \cong \mathbb{Z}_2 \times Q_8 \) and each covering transformation of \( S^3 \times S^3 \) over \( F(4) \) is multiplication by an element of \( \{1, -1\} \times Q_8 \).

**Proof.** We have just demonstrated \( F(4) \cong \mathbb{RP}^3 \times F(3) \).

Since the universal covering \( 1 \times \gamma : S^3 \times S^3 \to S^3 \times SO(3) \) is a group homomorphism the following diagram is easily seen to define a mapping between the coset spaces in its bottom row by the universal property of quotients.

\[
\begin{array}{ccc}
S^3 \times S^3 & \xrightarrow{1 \times \gamma} & S^3 \times SO(3) \\
\downarrow & & \downarrow \\
S^3 \times S^3 / \{1, -1\} \times Q_8 & \longrightarrow & S^3 \times SO(3) / \{1, -1\} \times (O(1)^3 \cap SO(3)) \cong \mathbb{RP}^3 \times F(3)
\end{array}
\]

Being a continuous bijection from a compact space to a Hausdorff space, this map is a homeomorphism. \( \blacksquare \)
In this chapter we construct topological spaces lying over an arbitrary polyhedron. The motivation for the spaces we construct here is the lattice construction of R. R. Douglas and A. R. Rutherford, whose interest stems primarily from the novel decompositions of certain spheres of real and complex matrices obtained by Douglas and Rutherford in their work on the Jahn-Teller effect.\(^1\) The lattice construction is presented in section A and spheres are constructed in this manner in section B.

Intuitively, the construction presented in section C captures the idea of replacing the faces of a polyhedron by spaces lying above these faces. We define a category on which our construction is realized as a functor to the topological category and show that this construction subsumes the lattice construction. The construction is presented in a slightly more general form than we actually require in the following chapters since doing so simplifies much of the required notation.

A. The Lattice Construction

Let \( \Delta_n \) denote the standard \( n \)-simplex in \( \mathbb{R}^{n+1} \).

\[
\Delta_n = \{ (b_0, \ldots, b_n) \in \mathbb{R}^{n+1} \mid b_0, \ldots, b_n \geq 0, b_0 + \cdots + b_n = 1 \}
\]

As defined in [D], the lattice construction defines a topological space \( X \) and a map \( \beta : X \to \Delta_n \) from a list \( G_0, \ldots, G_n \) of closed subgroups of a compact Lie group \( G \). The space \( X \) is defined as the quotient of \( G \times \Delta_n \) by the equivalence relation that identifies \( (g, b) \) with \( (g', b') \) if and

\(^1\) We recently learned of the 1974 paper [Ma], in which the author constructs imbeddings of real, complex, quaternionic and Cayley projective planes in spheres of dimension 4, 7, 13 and 25 (respectively) using methods similar to those of Douglas and Rutherford. The decomposition there corresponds to the lattice construction over a 1-simplex.
only if \( b' = b \equiv (b_0, \ldots, b_n) \) and \( g^{-1}g' \in \bigcap_{b_i > 0} G_i \), and \( \beta \) is the map induced by the projection \( G \times \Delta_n \to \Delta_n \) onto the second factor.

Certain spheres can be constructed in this manner by examining the conjugation action of a real orthogonal, complex unitary or quaternionic symplectic group on, respectively, a Euclidean space of real symmetric, complex Hermitean or quaternionic Hermitean matrices. We will see in chapter 4 that many more spheres can be constructed as a lattice construction on products of orthogonal, unitary or symplectic matrices.

### B. Constructing Spheres

By the well known 1877 theorem of Frobenius, there are only three finite dimensional, associative division algebras over \( \mathbb{R} \): \( \mathbb{R} \) itself, the complex field \( \mathbb{C} \) and the quaternion algebra \( \mathbb{H} \).\(^2\) Let \( \mathbb{K} \) denote any of these three and let \( \mathcal{O}(n) \subset \mathbb{K}(n) \) denote the corresponding group of inner product preserving matrices. That is, \( \mathcal{O}(n) \) is either the orthogonal group \( \mathcal{O}(n) \), the unitary group \( U(n) \) or the symplectic group \( Sp(n) \). Of course, \( \mathcal{O}(n) \) acts on \( \text{Herm}(n, \mathbb{K}) \) by conjugation: \( P \cdot A = PAP^\dagger \).

Matrices in \( \text{Herm}(n, \mathbb{K}) \) can be diagonalized over \( \mathbb{R} \): If \( A \in \text{Herm}(n, \mathbb{K}) \) then there is a real diagonal matrix \( D \) and a \( P \in \mathcal{O}(n) \) such that \( A = PDP^\dagger \).\(^3\) Moreover, the diagonal entries of \( D \) are unique up to a permutation. Let us endow \( \text{Herm}(n, \mathbb{K}) \) with the Hilbert-Schmidt inner product, \( (A, B) = \text{tr} AB \). Let \( \text{Herm}_0(n, \mathbb{K}) \subset \text{Herm}(n, \mathbb{K}) \) denote the real (codimension 1) subspace of trace zero matrices.

**2.1 Proposition.** The fixed point set of the conjugation action of \( \mathcal{O}(n) \) on \( \text{Herm}(n, \mathbb{K}) \) is \( \mathbb{R}I \), the set of real multiples of the identity matrix.

**Proof.** Let \( A \in \text{Herm}(n, \mathbb{K}) \) be a fixed point of the action.

If \( A \) is singular then there is a non-zero unit vector \( x \in \ker A \). Since \( APx = PAx = 0 \) for all \( P \in \mathcal{O}(n) \) and since \( Px \) ranges over all elements of the unit sphere in \( \mathbb{K}^n \) as \( P \in \mathcal{O}(n) \), we have \( \ker A = \mathbb{K}^n \).

If \( A \) is invertible, choose an eigenvector \( \lambda \in \mathbb{R} \) and consider \( A - \lambda I \). \( \blacksquare \)

For \( n \geq 2 \), let us define a function \( \delta : \text{Herm}(n, \mathbb{K}) - \mathbb{R}I \to \Delta_{n-2} \) as follows. If \( A \in \text{Herm}(n, \mathbb{K}) - \mathbb{R}I \) has eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) then

\[
\delta(A) = \frac{1}{\lambda_n - \lambda_1} (\lambda_2 - \lambda_1, \ldots, \lambda_n - \lambda_{n-1}).
\]  

\(^2\) See, for example, \[E\] for a modern exposition. (Chapter 8, section 2.)

\(^3\) The quaternionic case is treated in appendix A.
Notice that $\delta$ has the following properties.

- $\delta(P A P^t) = \delta(A)$ for each $P \in O(n)$.
- $\delta(a_0 A + a_1 I) = \delta(A)$ whenever $a_0, a_1 \in \mathbb{R}$ with $a_0 \neq 0$.

Now, each element of $\text{Herm}(n, \mathbb{K}) - RI$ lies in the orbit of a real diagonal matrix. Let $D = (\lambda_1 \delta_1 | \cdots | \lambda_n \delta_n)$ be the unique such matrix such that $\lambda_1 \leq \cdots \leq \lambda_n$, having

$$\lambda_1 = \cdots = \lambda_{i_1} < \cdots < \lambda_{i_1 + \cdots + i_k + 1} = \cdots = \lambda_n$$

and $\lambda_n - \lambda_1 > 0$. If $P \in O(n)$ and $\delta(D) = (b_0, \ldots, b_{n-2})$ then $PDP^t = D$ if and only if

$$P \in O(i_1) \times \cdots \times O(i_k) \times O(n - i_1 - \cdots - i_k) = \bigcap_{\lambda_{i+1} > \lambda_i} O(\ell) \times O(n - \ell)$$

$$= \bigcap_{\ell > 0} O(\ell + 1) \times O(n - \ell - 1) \quad (2.2)$$

so that the stabilizer of each element of $\text{Herm}(n, \mathbb{K}) - RI$ is conjugate to one of these subgroups.

There is a homeomorphism

$$A \mapsto (A - (\text{tr} A) I / n, \text{tr} A) : \text{Herm}(n, \mathbb{K}) \to \text{Herm}_0(n, \mathbb{K}) \times \mathbb{R}$$

with inverse $(A, \tau) \mapsto A + \tau I / n$ under which the fixed point set of the $O(n)$ action is mapped to the line $\{0\} \times \mathbb{R}$. There is also a homeomorphism

$$A \mapsto (A/\|A\|, \|A\|) : \text{Herm}_0(n, \mathbb{K}) - \{0\} \to S(n, \mathbb{K}) \times (0, \infty)$$

where $S(n, \mathbb{K}) \subset \text{Herm}_0(n, \mathbb{K})$ is the unit sphere with respect to the Hilbert-Schmidt norm.

Since (counting components of a real basis for $\mathbb{K}$ along diagonals of matrices in $\text{Herm}(n, \mathbb{K})$)

$$\dim \text{Herm}(n, \mathbb{R}) = n + (n - 1) + \cdots + 1 = n(n + 1)/2$$

$$\dim \text{Herm}(n, \mathbb{C}) = n + 2((n - 1) + \cdots + 1) = n + n(n - 1) = n^2$$

and

$$\dim \text{Herm}(n, \mathbb{H}) = n + 4((n - 1) + \cdots + 1) = n + 2n(n - 1) = n(2n - 1)$$

as real vector spaces, and since $\text{Herm}_0(n, \mathbb{K}) \subset \text{Herm}(n, \mathbb{K})$ has codimension 1, the spheres $S(n, \mathbb{R})$, $S(n, \mathbb{C})$ and $S(n, \mathbb{H})$ have dimensions $n(n + 1)/2 - 2$, $n^2 - 2$ and $n(2n - 1) - 2$ respectively.\(^4\)

---

\(^4\) To elucidate only the real case, the isometry

$$A = (a_{ij}) \mapsto \sum_{i=1}^n a_{ii} \delta_i + \sqrt{2} \sum_{i=1}^{n-1} \sum_{j > i} a_{ij} \delta_{i+1} \delta_{i+2} \cdots \delta_{i+j-1} \delta_{i+j} / 2^{j-i} \cdot i / 2^{j-i} \cdot j / 2^{j-i+1} : \text{Herm}(n, \mathbb{R}) \to \mathbb{R}^{n(n+1)/2}$$

maps $S(n, \mathbb{R})$ onto an ellipsoid in the $n(n + 1)/2 - 1$ dimensional image of $\text{Herm}_0(n, \mathbb{R})$. 

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Let $\mu : \text{Herm}(n, K) - \mathcal{R}I \to S(n, K)$ denote the retraction defined by the first factor of the composition homeomorphism $\text{Herm}(n, K) - \mathcal{R}I \to S(n, K) \times (0, \infty) \times \mathcal{R}$.

$$\mu(A) = \frac{A - (\text{tr} A) I/n}{\|A - (\text{tr} A) I/n\|} \quad (2.3)$$

Notice that $\mu$ has the following properties.

- $\mu(PAP^\dagger) = P\mu(A)P^\dagger$ for each $P \in \mathcal{O}(n)$.
- $\mu(a_0 A + a_1 I) = \mu(A)$ whenever $a_0, a_1 \in \mathcal{R}$ with $a_0 \neq 0$.
- $\delta(A) = \delta(\mu(A))$.
- $\mu(A) = A$ if $A \in S(n, K)$.

2.2 Proposition. Let $A, B \in \text{Herm}(n, K) - \mathcal{R}I$. Then $\mu(A)$ and $\mu(B)$ lie in the same $\mathcal{O}(n)$ orbit of $S(n, K)$ if and only if $\delta(A) = \delta(B)$.

Before proving this statement, define a map $\varsigma : \Delta_{n-2} \to \text{Herm}(n, K) - \mathcal{R}I$ as follows.

$$\varsigma(b_0, \ldots, b_{n-2}) = (0 \hat{e}_1 | b_0 \hat{e}_2 | (b_0 + b_1) \hat{e}_3 | \cdots | (b_0 + \cdots + b_{n-2}) \hat{e}_n = 1 \hat{e}_n) \quad (2.4)$$

If $A \in \text{Herm}(n, K) - \mathcal{R}I$ has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ then

$$(\lambda_n - \lambda_1)\varsigma(\delta(A)) + \lambda_1 I = (\lambda_1 \hat{e}_1 | \cdots | \lambda_n \hat{e}_n)$$

so that this matrix lies in the same $\mathcal{O}(n)$ orbit as $A$. Thus, $\mu(A)$ and $\mu((\lambda_n - \lambda_1)\varsigma(\delta(A)) + \lambda_1 I) = \mu(\varsigma(\delta(A)))$ lie in the same $\mathcal{O}(n)$ orbit of $S(n, K)$. Note that $\delta \circ \varsigma = 1$.

Proof of 2.2. If $\mu(A) = P\mu(B)P^\dagger$ for some $P \in \mathcal{O}(n)$ then $\delta(A) = \delta(\mu(A)) = \delta(P\mu(B)P^\dagger) = \delta(\mu(B)) = \delta(B)$.

If $\delta(A) = \delta(B)$ then $\mu(\varsigma(\delta(A))) = \mu(\varsigma(\delta(B)))$, which lies in the same $\mathcal{O}(n)$ orbit of $S(n, K)$ as both $\mu(A)$ and $\mu(B)$.

2.3 Corollary. Let $A, B \in S(n, K)$. Then $A$ and $B$ lie in the same $\mathcal{O}(n)$ orbit if and only if $\delta(A) = \delta(B)$.

We can now exhibit $S(n, K)$ as a lattice construction. Consider $G = \mathcal{O}(n)$ together with the $n - 1$ closed subgroups

$$G_0 = \mathcal{O}(1) \times \mathcal{O}(n - 1), \ldots, G_{k-1} = \mathcal{O}(k) \times \mathcal{O}(n - k), \ldots, G_{n-2} = \mathcal{O}(n - 1) \times \mathcal{O}(1),$$

where $\mathcal{O}(k) \times \mathcal{O}(n - k)$ is identified with the subgroup of $\mathcal{O}(n)$ consisting of those matrices whose only non-zero entries lie in the upper left $k \times k$ and lower right $(n - k) \times (n - k)$ blocks.
Let the space \( X \) and the map \( \beta : X \to \Delta_{n-2} \) be the result of the lattice construction on \( O(n) \) and the aforementioned \( n-1 \) subgroups. There is the following commutative triangle.

\[
\begin{array}{ccc}
(P, b) & \xrightarrow{\mu} & \mu(P\zeta(b)P^t) = P\mu(\zeta(b))P^T \\
O(n) \times \Delta_{n-2} & \xrightarrow{\pi_2} & S(n, K) \\
\Delta_{n-2} & \xrightarrow{\delta} & 
\end{array}
\]

By definition, two pairs \((P, b), (Q, b) \in O(n) \times \Delta_{n-2}\) represent the same point in \( X \) if and only if \( Q^tP \in \bigcap_{b_i > 0} G_i \), and this holds if and only if \( Q^tP\mu(\zeta(b))P^tQ = \mu(\zeta(b)) \) by (2.2) since \( \delta(\mu(\zeta(b))) = \delta(\zeta(b)) = b \). Thus, the map \((P, b) \mapsto \mu(P\zeta(b)P^t)\) induces an injective map \( \xi : X \to S(n, K) \). Moreover, \( \xi \) is surjective since if \( D = (\lambda_1 \varepsilon_1 | \cdots | \lambda_n \varepsilon_n) \) is a diagonal element of \( S(n, K) \) then

\[
D = (\lambda_n - \lambda_1)\zeta(\delta(D)) + \lambda_1 I \\
= \mu(D) = \mu(\zeta(\delta(D))) \\
= \xi[1, \delta(D)].
\]

Since \( X \) is compact and \( S(n, K) \) is Hausdorff, the continuous bijection \( \xi \) is a homeomorphism. As an interesting consequence, notice that the function \( \delta : \text{Herm}(n, K) \to RI \to \Delta_{n-2} \) is continuous because of the following commutative diagram.

\[
\begin{array}{ccc}
\text{Herm}(n, K) & \xrightarrow{\mu} & S(n, K) \\
\delta & \downarrow & \downarrow \xi^{-1} \\
\Delta_{n-2} & \xrightarrow{\beta} & X \\
\end{array}
\]

C. A More General Construction

Let \( K \) be a simplicial complex: A set of finite non-empty subsets, called simplices, of some set \( \text{Vert}(K) \) of vertices, having the property that (1) every subset consisting of a single vertex is a simplex and (2) any non-empty subset of a simplex is itself a simplex. We will often identify a vertex with the one-element simplex containing it.

Let \( |K| \) denote the set of all functions \( b : \text{Vert}(K) \to [0, 1] \) such that \( b^{-1}(0, 1] \in K \) and \( \sum_{v \in \text{Vert}(K)} b(v) = 1 \). For each \( s \in K \), the closed simplex

\[
|s| = \{ b \in |K| \mid b^{-1}(0, 1] \subseteq s \}
\]

has the metric topology of a standard Euclidean simplex. The realization of \( K \) is the set \( |K| \) endowed with the largest topology for which the inclusions \( |s| \hookrightarrow |K| \) are continuous. We refer the reader to chapter 3 of [S] for introductory material on simplicial complexes.
If \( u \subset \text{Vert}(K) \) is any non-empty subset, let \( |u| = \{ b \in |K| \mid b^{-1}(0,1) \subset u \} \). The subset \( u \) generates a subcomplex of \( K \), \( \bar{u} = \{ s \in K \mid s \subset u \} \). Note that \( |u| = \{ b \in |K| \mid b^{-1}(0,1) \in \bar{u} \} = \bigcup_{s \in \bar{u}} |s| \).

We want to replace the faces of \( |K| \) by spaces lying over them. In order to show that our definition constructs a space homeomorphic to the lattice construction in the appropriate situation, it will be convenient to define our spaces in terms of an order-preserving function \( \sigma \) from a partially ordered set \( A \) to the set of non-empty subsets of \( \text{Vert}(K) \) partially ordered by containment (rather than inclusion). Given such a function, we let \( |a| = |\sigma(a)| \) and \( \bar{a} = \{ s \in K \mid s \subset \sigma(a) \} \). Note that \( |a| = \{ b \in |K| \mid b^{-1}(0,1) \in \bar{a} \} = \bigcup_{s \in \bar{a}} |s| \). We also require that \( \sigma \) satisfy the following property.

If \( s \in K \) then the set \( \{ a \in A \mid s \in \bar{a} \} \subset A \) is nonempty and contains a lower bound.

We will denote this (necessarily unique) lower bound by \( T(s) \). Further, we will write \( a \to a' \) when \( a \) precedes \( a' \) in \( A \), viewing the partial ordering as a category whenever convenient.

Now, suppose we are given the following data.

1. A collection \( X = \{ X_a \mid a \in A \} \) of spaces.

2. A collection \( \beta = \{ \beta_a : X_a \to |a| \mid a \in A \} \) of surjective maps such that, for each \( a \in A \), the topology on \( X_a \) is that generated by the collection \( \{ \beta_a^{-1}|s| \mid s \in \bar{a} \} \) of closed subsets.\(^5\)

3. A collection \( f = \{ f_{a \to a'} : \beta_a^{-1}|a'| \to X_{a'} \mid a, a' \in A; a \to a' \} \) of maps such that \( f_{a \to a} \) is the identity map on \( X_a \) and the diagrams

\[
\begin{array}{ccc}
\beta_a^{-1}|a'| & \xrightarrow{f_{a \to a'}} & X_{a'} \\
\beta_a & \downarrow & \beta_{a'} \\
|a'| & \xrightarrow{f_{a \to a'}} & X_{a'}
\end{array}
\quad
\begin{array}{ccc}
\beta_a^{-1}|a''| & \xrightarrow{f_{a \to a'}} & \beta_{a'}^{-1}|a''| \\
\beta_a & \downarrow & \beta_{a'} \\
|a''| & \xrightarrow{f_{a \to a'}} & \beta_{a'}^{-1}|a''|
\end{array}
\]

commute whenever \( a \to a' \) and \( a \to a' \to a'' \) respectively.

Note that 2 is automatic when \( \bar{a} \) is a locally finite complex. In particular, 2 holds when \( \bar{a} \) is finite, and this is the case whenever \( \sigma(a) \in K \). Note also that the assignment of the space \( X_a \) to the element \( a \in A \) and of the map \( f_{a \to a'} \) to the arrow \( a \to a' \) defines a functor from \( A \) to the category \( \text{Par} \) of spaces and partial maps. Indeed, the triple \( (X, \beta, f) \) is an object in a category which we denote by \( \text{Glue}_A \)\(^6\) and the assignment of the aforementioned functor to

\[^{5}\text{That is, } U \subset X_a \text{ is open if and only if } U \cap \beta_a^{-1}|s| \text{ is open in } \beta_a^{-1}|s| \text{ for each } s \in \bar{a}. \text{ Appendix B collects the results we require on this topology.}\]

\[^{6}\text{This might more properly be denoted } \text{Glue}_\sigma, \text{ but we prefer to emphasize the role of the partially ordered set } A \text{ since we are always dealing with a single } \sigma.\]
this pair defines a functor from $\text{Glue}_A$ into the functor category $\text{Par}^A$. For the morphisms from $(X, \beta, f)$ to $(Y, \gamma, g)$ in $\text{Glue}_A$, we take collections $\phi = \{ \phi_a : X_a \to Y_a \mid a \in A \}$ such that the diagrams

\[
\begin{array}{ccc}
X_a & \xrightarrow{\phi_a} & Y_a \\
\downarrow{\beta_a} & & \downarrow{\gamma_a} \\
|a| & \xrightarrow{\gamma_a^{-1}[a']} & Y_{a'}
\end{array}
\quad \quad \quad \quad
\begin{array}{ccc}
\beta_a^{-1}[a'] & \xrightarrow{f_{a \to a'}} & X_{a'} \\
\downarrow{\phi_a} & & \downarrow{\phi_{a'}} \\
\gamma_a^{-1}[a'] & \xrightarrow{g_{a \to a'}} & Y_{a'}
\end{array}
\]

commute whenever $a \in A$ and $a \to a'$ respectively.

In the case that $A \subset K$, partially ordered by face containment, and $\sigma(a) = a$, the category $\text{Glue}_A$ contains an embedded copy of the functor category $\text{Top}^A$. We can define a functor from $\text{Top}^A$ to $\text{Glue}_A$ by sending $F \in \text{Top}^A$ to the triple $(X, \beta, f)$, where $X_s = F(s) \times |s|$, $\beta_x(x, b) = b$ and $f_{s 	o t}(x, b) = (F(s \to t)(x), b)$, and sending a natural transformation $\Phi : F \to G$ to the collection $\{ \Phi_s \times 1 : F(s) \times |s| \to G(s) \times |s| \mid s \in A \}$. This may not work for an arbitrary $A$ because of the restriction of property 2 in the definition of $\text{Glue}_A$.

Given an object $X = (X, \beta, f)$ in $\text{Glue}_A$, define $G_A X$ to be the quotient of the coproduct $\bigoplus_{a \in A} X_a$ by the smallest equivalence relation that identifies $x \in \beta_a^{-1}[a']$ with $f_{a \to a'}(x)$ whenever $a \to a'$.\(^7\) The maps $\beta_a : X_a \to |a|$ and the universal property of the coproduct provides an obvious map $\bigoplus_{a \in A} X_a \to |K|$ which we can use to construct a map $\beta_X : G_A X \to |K|$ by the following diagram.

The assignment $X \mapsto G_A X$ is the object map of a functor $G_A : \text{Glue}_A \to \text{Top}$. For if $\phi : X \to Y = (Y, \gamma, g)$ is a morphism in $\text{Glue}_A$ then the map $G_A \phi : G_A X \to G_A Y$ is uniquely defined by the following diagram.

\[\begin{array}{ccc}
\bigoplus_{a \in A} X_a & \xrightarrow{\bigoplus \phi_a} & \bigoplus_{a \in A} Y_a \\
\downarrow & & \downarrow \\
G_A X & \xrightarrow{G_A \phi} & G_A Y
\end{array}\]

\(^7\) Thus, $G_A X$ is the colimit space in $\text{Par}^A$ of the aforementioned functor determined by $X$ and $f$.  

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Since $\beta_Y \circ G_A \phi = \beta_X$, the assignment $X \mapsto \beta_X$ is the object map of a functor from $\text{Glue}_A$ to the category $\textbf{Top} \downarrow |K|$ of spaces over $|K|$. 

If $b \in |K|$, let us denote the carrier of $b$ by $c(b)$ or just $cb$:

$$cb = \{ v \in \text{Vert}(K) \mid b(v) > 0 \}$$

is the unique smallest simplex of $K$ having $b \in |cb|$. Note that if $a \in A$ then $|a| = \{ b \in |K| \mid cb \in a \}$. Thus, if $b \in |a|$ then $a \rightarrow T(cb)$. The open simplex determined by $s \in K$ is the subset $\langle s \rangle = \{ b \in |K| \mid cb = s \} \subset |K|$. It is an open subset of $|s|$ but not necessarily of $|K|$.

2.4 Proposition. Let $X = (X, \beta, f)$ be an object in $\text{Glue}_A$. Two points $x \in X_a$ and $x' \in X_{a'}$ represent the same point in $G_A X$ if and only if $\beta_a(x) = \beta_{a'}(x') \equiv b$ and $f_{a \rightarrow T(cb)}(x) = f_{a' \rightarrow T(cb)}(x')$.

Proof. If $f_{a \rightarrow T(cb)}(x) = f_{a' \rightarrow T(cb)}(x')$ then $x$ and $x'$ represent the same point since these represent the same point as $f_{a \rightarrow T(cb)}(x)$ and $f_{a' \rightarrow T(cb)}(x')$ respectively.

Conversely, suppose $x$ and $x'$ represent the same point in $G_A X$. Applying $\beta_X$ to this common point immediately gives $\beta_a(x) = \beta_{a'}(x')$. Since they represent the same point in $G_A X$ as $x$ and $x'$ respectively, $f_{a \rightarrow T(cb)}(x)$ and $f_{a' \rightarrow T(cb)}(x')$ represent this same point. If these points are distinct then, by Proposition 0.1, we necessarily have sequences $a_1 \rightarrow a'_1, \ldots, a_n \rightarrow a'_n$ and $x_1 \in \beta_{a_1}^{-1} |a_1|, \ldots, x_1 \in \beta_{a_n}^{-1} |a'_n|$ such that the following hold.

1. $f_{a \rightarrow T(cb)}(x) \in \{ x_1, f_{a_1 \rightarrow a'_1}(x_1) \}$.
2. $\{ x_i, f_{a_i \rightarrow a'_i}(x_i) \} \cap \{ x_i, f_{a_{i+1} \rightarrow a'_{i+1}}(x_{i+1}) \} \neq \emptyset$ for each $i \in \{ 1, \ldots, n - 1 \}$.
3. $f_{a' \rightarrow T(cb)}(x') \in \{ x_n, f_{a_n \rightarrow a'_n}(x_n) \}$.

Note that $a'_i \rightarrow T(cb)$ for each $i$ since $b = \beta_a(x_i) \in |a'_i|$. Let us show that the assumption that $f_{a \rightarrow T(cb)}(x)$ and $f_{a' \rightarrow T(cb')}(x')$ are distinct leads to a contradiction.

First, we prove by induction that $f_{a_i \rightarrow T(cb)}(x_i) = f_{a_i \rightarrow T(cb)}(x)$ for all $i \in \{ 1, \ldots, n \}$. For $i = 1$, necessarily $a'_1 = T(cb)$ and $f_{a_1 \rightarrow a'_1}(x_1) = f_{a \rightarrow T(cb)}(x)$ by (1), for if $x_1 = f_{a \rightarrow T(cb)}(x)$ then $a_1 = T(cb)$ so that $a'_1 = T(cb)$ as well. Similarly $a'_n = T(cb)$ and $f_{a_n \rightarrow T(cb)}(x_n) = f_{a' \rightarrow T(cb)}(x')$ by (3).

Now, suppose $f_{a_i \rightarrow T(cb)}(x_i) = f_{a_i \rightarrow T(cb)}(x)$ for some $i \geq 1$. By (2) there are four possibilities: $a_i = a_{i+1}$ and $x_i = x_{i+1}, a_i = a'_i + 1$ and $x_i = f_{a_{i+1} \rightarrow a'_{i+1}}(x_{i+1}), a_i = a_{i+1}$ and $f_{a_i \rightarrow a'_i}(x_i) = x_{i+1}$, or $a'_i = a'_{i+1}$ and $f_{a_i \rightarrow a'_i}(x_i) = f_{a_{i+1} \rightarrow a'_{i+1}}(x_{i+1})$. In each case, $f_{a_i \rightarrow T(cb)}(x_{i+1}) = f_{a \rightarrow T(cb)}(x)$ follows easily using the composition property of the maps in the collection $f$.

Thus, we have, in particular, $f_{a_n \rightarrow T(cb)}(x_n) = f_{a \rightarrow T(cb)}(x)$. Since, as already noted, $a'_n = T(cb)$ and $f_{a_n \rightarrow T(cb)}(x_n) = f_{a' \rightarrow T(cb)}(x')$, we therefore have $f_{a \rightarrow T(cb)}(x) = f_{a' \rightarrow T(cb)}(x')$, contradicting the assumption that these pairs be distinct. \[ \blacksquare \]
Because of the lower bound condition we have imposed on $A$, there is a functor $F_A: \text{Glue}_A \rightarrow \text{Glue}_K$ which is defined on objects by $F_A(X, \beta, f) = (\bar{X}, \bar{\beta}, \bar{f})$, where $\bar{X}_s, \bar{\beta}_s: \bar{X}_s \rightarrow |s|$ and $\bar{f}_{s\rightarrow t}: \bar{X}_s \rightarrow \bar{X}_t$ are defined as follows.

$$
\bar{X}_s = \beta_{T(s)}^{-1}|s| \subset X_{T(s)} \\
\bar{\beta}_s(y) = \beta_{T(s)}(y) \\
\bar{f}_{s\rightarrow t}(y) = f_{T(s)\rightarrow T(t)}(y)
$$

Now, if $(X, \beta, f) \equiv X$ then a simple diagram chase reveals that the following diagram defines the component at $X$ of a natural transformation $G_K \circ F_A \rightarrow G_A$.

$$
\begin{array}{ccc}
\bar{X}_s & \xrightarrow{i} & X_{T(s)} \\
\oplus_{s \in K} \bar{X}_s & \xrightarrow{p} & \oplus_{a \in A} X_a \\
\end{array}
$$

We claim that $i$ is a homeomorphism. It is surjective since a point $x \in \beta_a^{-1}|s|$ represents the same point in $G_A X$ as $f_{a \rightarrow T(s)}(x) \in \bar{X}_s$, and an application of Proposition 2.4 shows that it is injective. We need only show that $i$ is a closed map. Since the topology on $X_a$ is generated by the collection $\{\beta_a^{-1}|s| \mid s \in a\}$ of closed subsets, this is a consequence of the following.

2.5 Claim. If $s \in a$ and $U \subset G_K(F_A X)$ then $p_a^{-1}(i(U)) \cap \beta_a^{-1}|s| = f_{a \rightarrow T(s)}^{-1}(p^{-1}(U) \cap \bar{X}_s)$.

Proof. Suppose $x \in p_a^{-1}(i(U)) \cap \beta_a^{-1}|s|$. Then $p_A(x) = i(p(y))$ for some $y \in p^{-1}(U)$. Since $x$ represents the same point in $G_A X$ as $f_{a \rightarrow T(s)}(x) \in \bar{X}_s$,

$$
i(p(f_{a \rightarrow T(s)}(x))) = p_A(f_{a \rightarrow T(s)}(x)) \\
= p_A(x) \\
= i(p(y))
$$

which implies that $p(f_{a \rightarrow T(s)}(x)) = p(y) \in U$ since $i$ is injective. Thus, $f_{a \rightarrow T(s)}(x) \in p^{-1}(U) \cap \bar{X}_s$ so that $x \in f_{a \rightarrow T(s)}^{-1}(p^{-1}(U) \cap \bar{X}_s)$.

Conversely, suppose $x \in f_{a \rightarrow T(s)}^{-1}(p^{-1}(U) \cap \bar{X}_s)$. Since $x$ represents the same point in $G_A X$ as $f_{a \rightarrow T(s)}(x)$,

$$
p_A(x) = p_A(f_{a \rightarrow T(s)}(x)) \\
= i(p(f_{a \rightarrow T(s)}(x))) \in i(U)
$$
so that $x \in p^{-1}_A(i(U))$. $x \in \beta^{-1}_a|s|$ is clear since $\beta_a(x) = \beta_{\tau(s)}(f_{a\rightarrow\tau(s)}(x))$ and $f_{a\rightarrow\tau(s)}(x) \in X_s$. ■

Thus, we have proven the following.

**2.6 Proposition.** The functors $G_A$ and $G_K \circ F_A$ are equivalent. ■

As alluded to earlier, the point of introducing $A$ is to show that under the appropriate circumstances the space $G_KX$ is a lattice construction. For this we will apply the previous proposition with $A = K_0$, where $K_0$ is the partially ordered set constructed from $K$ by adjoining a single element, call it 0, as an initial object. That is we set $K_0 = K \cup \{0\}$ and require that $0 \rightarrow s$ for each $s \in K$. We define the function $\sigma$ from $K_0$ to the collection of non-empty subsets of $Vert(K)$ by $\sigma(s) = s$ for all $s \in K$ and $\sigma(0) = Vert(K)$ (so that $|0| = |K|$).

Given an arbitrary functor $F : K_0 \rightarrow \text{Top}$, the definition of an object $X = (X, \beta, f)$ of $\text{Glue}_{K_0}$ by $X_s = F(s) \times |s|$, $\beta_s(x,b) = b$ and $f_{s\rightarrow t}(x,b) = (F(s \rightarrow t)(x), b)$ is flawed unless the topology of space $F(0) \times |K|$ is generated by the collection $\{ F(0) \times |s| \mid s \in K \}$ of closed subsets. This is certainly the case when $K$ is locally finite. By Proposition B.5, $F(0)$ locally compact Hausdorff is also sufficient. Now, if $G$ is a topological group and $/G/K = \{ G_s \subset G \mid s \in K \}$ is a lattice of subgroups, indexed by $K$ and having $G_s \subset G_t$ whenever $s \rightarrow t$, there is an obvious functor $F$ defined by letting its application to a non-identity arrow $s \rightarrow t$ in $K_0$ be one of the left coset projections $G \rightarrow G/G_t$ or $G/G_s \rightarrow G/G_t$, respective of whether or not $s$ is the initial object in $K_0$. Thus, the pair $(G, /G/K)$ defines an object $X$ in $\text{Glue}_{K_0}$ as above when either $K$ is locally finite or $G$ is locally compact Hausdorff. For the lattice construction, the lattice $/G/K$ is constructed from a collection $\{ H_v \mid v \in Vert(K) \}$ of subgroups of $G$, indexed by the vertices of $K$, by setting $G_s = \bigcap_{v \in s} H_v$. The space defined by the lattice construction is then the application to the object $X$ of the functor $L_K : \text{Glue}_{K_0} \rightarrow \text{Top}$ defined as follows. If $X = (X, \beta, f)$ is any object in $\text{Glue}_{K_0}$ then the space $L_KX$ is defined as the quotient of $X_0$ by the equivalence relation which identifies $x$ and $x'$ if and only if $\beta_0(x) = \beta_0(x') \equiv b$ and $f_{0\rightarrow b}(x) = f_{0\rightarrow b}(x')$.

**2.7 Proposition.** The functors $L_K$ and $G_{K_0}$ are equivalent on the full subcategory of $\text{Glue}_{K_0}$ consisting of those triples $(X, \beta, f)$ satisfying the following two properties.

1. $f_{0\rightarrow s} : \beta^{-1}_0|s| \rightarrow X_s$ is either a closed map for each $s \in K$ or an open map for each $s \in K$.

2. The map $x \mapsto f_{0\rightarrow s}(x) : \beta^{-1}_0\langle s \rangle \rightarrow \beta^{-1}_s\langle s \rangle$ is surjective for each $s \in K$.

Consequently, $L_K$ is equivalent to $G_K \circ F_{K_0}$ on the same subcategory.

Note that both properties 1 and 2 are satisfied for the objects of $\text{Glue}_{K_0}$ defined, as above, by a group and lattice of subgroups indexed by $K$. In this case, the map $f_{0\rightarrow s} : G \times |s| \rightarrow G/G_s \times |s|$ is the product of two open maps: The coset projection $G \rightarrow G/G_s$ and the identity map on the closed simplex $|s|$.
Proof. The component of the desired equivalence at $X = (X, \beta, f)$ is defined by the following diagram.

\[
\begin{array}{ccc}
X_0 & \longrightarrow & \bigoplus_{s \in K_0} X_s \\
q & & \downarrow p_0 \\
\mathcal{L}_K X & \longrightarrow & \mathcal{G}_K X
\end{array}
\]

That $j$ is indeed the component of a natural transformation is a simple diagram chase. An application of Proposition 2.4 shows that $j$ is injective. The hypothesis that $x \mapsto f_{0 \to s}(x) : \beta_0^{-1}(s) \rightarrow \beta_s^{-1}(s)$ is surjective for each $s \in K$ implies that $j$ is surjective since a point $x \in \beta_0^{-1}(s)$ represents the same point in $\mathcal{G}_K X$ as $f_{0 \to s}(x) \in \beta_s^{-1}(s)$. Since $f_{0 \to s} : \beta_0^{-1}|s| \rightarrow X_s$ is either a closed map for each $s \in K$ or an open map for each $s \in K$, the following identity shows that $j$ is closed or open (respectively).

\[
p_0^{-1}(j(U)) \cap \beta_a^{-1}|s| = \bigcup_{t \in s} f_{a \to t}^{-1} \left( f_{0 \to t}(q^{-1}(U) \cap \beta_0^{-1}|t|) \right)
\]

\[(a \in K_0, s \in K, a \to s)\]

To see this, suppose $x \in p_0^{-1}(j(U)) \cap \beta_a^{-1}|s|$. Let $b = \beta_a(x)$ and consider $f_{a \to cb}(x) \in \beta_{cb}^{-1}(cb)$. By hypothesis, $f_{a \to cb}(x) = f_{0 \to cb}(x_0)$ for some $x_0 \in \beta_0^{-1}|s|$. We claim that $x_0 \in q^{-1}(U)$. Since $p_0(x) = p_0(x_0) \in j(U)$, there is a point $z \in q^{-1}(U)$ such that $p_0(x_0) = j(q(z)) = p_0(z)$. Thus, $f_{0 \to cb}(x_0) = f_{0 \to cb}(z)$ by Proposition 2.4 so that $q(x_0) = q(z) \in U$, giving $x_0 \in q^{-1}(U)$. Thus, $x \in f_{a \to cb}^{-1} \left( f_{0 \to cb}(q^{-1}(U) \cap \beta_0^{-1}|cb|) \right)$.

Conversely, if $x \in f_{a \to t}^{-1} \left( f_{0 \to t}(q^{-1}(U) \cap \beta_0^{-1}|t|) \right)$ for some $t \in s$ then $f_{a \to t}(x) = f_{0 \to t}(x_0)$ for some $x_0 \in q^{-1}(U) \cap \beta_0^{-1}|t|$. Thus, $\beta_a(x) = \beta_0(x_0)$ so that $x \in \beta_a^{-1}|t| \subset \beta_a^{-1}|s|$. Also, $j(q(x_0)) = p_0(x_0) = p_0(x)$ so that $x \in p_0^{-1}(j(U)) \cap \beta_a^{-1}|s|$. \[\square\]

In the remaining two chapters we will examine two aspects of the spaces $\mathcal{G}_K X$. In chapter 3 we work directly with the construction of $\mathcal{G}_K X$. In chapter 4 we work instead with the construction of $\mathcal{L}_K X$ since Proposition 2.7 guarantees that $\mathcal{L}_K X \approx \mathcal{G}_K X$ in the situation we place ourselves in. Thus, the next two results collect a few properties of the identification maps $p : \bigoplus_{s \in K} X_s \rightarrow \mathcal{G}_K X$ and $q : X_0 \rightarrow \mathcal{L}_K X$.

2.8 Proposition. Let $X = (X, \beta, f)$ be an object in Glue$_K$ and let $p : \bigoplus_{s \in K} X_s \rightarrow \mathcal{G}_K X$ be the identification map.

1. If $U \subseteq \bigoplus_{s \in K} X_s$ and $s \in K$ then there is the following equality.

\[
p^{-1}(p(U)) \cap X_s = \bigcup_{t \in K, t' \in s \cap t \neq \emptyset} f_{s \to t'}^{-1} \left( f_{t \to t'}(\beta_t^{-1}|t'| \cap U) \right)
\]

2. If each member of $f$ is an open map then $p$ is an open map.
3. If $K$ is locally finite and each member of $f$ is a closed map then $p$ is a closed map.

4. If $p$ is either an open map or a closed map then the map $x \mapsto p(x) : \beta_{s}^{-1}(s) \rightarrow \beta_{X}^{-1}(s)$ is a homeomorphism for each $s \in K$.

5. If $U \subset G_{K}X$ and $t \in s$ then $p^{-1}(U) \cap \beta_{s}^{-1}|t| = f_{s\rightarrow t}^{-1}(p^{-1}(U) \cap X_{t})$.

6. If $s \in K$ then the map $x \mapsto p(x) : \bigoplus_{t \in \delta} X_{t} \rightarrow \beta_{X}^{-1}|s|$ is an identification.

**Proof.** 1. This is an almost immediate consequence of Proposition 2.4. If $x$ lies in the left hand side of (2.5) then $p(x) = p(x_{0})$ for some $x_{0} \in U$. Suppose $x \in U \cap X_{t}$. Then we necessarily have $\beta_{s}(x) = \beta_{t}(x_{0}) \equiv b \in |s \cap t|$ and $f_{s\rightarrow cb}(x) = f_{t\rightarrow cb}(x_{0})$. In particular, $x \in f_{s\rightarrow cb}(f_{t\rightarrow cb}(\beta_{t}^{-1}|cb| \cup U))$, which is a subset of the right hand side of (2.5) since $cb$ is a face of both $s$ and $t$. Conversely, if $x$ is an element of the right hand side of (2.5) then $f_{s\rightarrow t'}(x) = f_{t\rightarrow t'}(x_{0})$ for some $x_{0} \in \beta_{t}^{-1}|t'| \cup U$, where $t \in K$ and $t' \in \delta \cap t$. Thus, $p(x) = p(x_{0})$ so that $x \in p^{-1}(p(U)) \cap X_{s}$.

2. If $U \subset \bigoplus_{t \in K} X_{s}$ is an open set then $p^{-1}(p(U)) \cap X_{s}$ is a union of open sets by (2.5).

3. If $U \subset \bigoplus_{t \in K} X_{s}$ is a closed set and $K$ is locally finite then the union in (2.5) is finite, so $p^{-1}(p(U)) \cap X_{s}$ is a finite union of closed sets.

4. If $p$ is an open or closed map then $x \mapsto p(x) : p^{-1}(A) \rightarrow A$ is an identification for any $A \subset G_{K}X$. Taking $A = \beta_{X}^{-1}(s)$, we have $p^{-1}(A) = \beta_{s}^{-1}(s)$. Since $p$ is injective on this subset by Proposition 2.4, the map $x \mapsto p(x) : \beta_{s}^{-1}(s) \rightarrow \beta_{X}^{-1}(s)$ is a homeomorphism.

5. If $x \in p^{-1}(U) \cap \beta_{s}^{-1}|t|$ then $x$ represents the same point in $U$ as $f_{s\rightarrow t}(x) \in X_{t}$. Thus, $p^{-1}(U) \cap \beta_{s}^{-1}|t| \subset f_{s\rightarrow t}^{-1}(p^{-1}(U) \cap X_{t})$. Conversely, if $x \in f_{s\rightarrow t}^{-1}(p^{-1}(U) \cap X_{t})$ then $x \in \beta_{s}^{-1}|t|$ and $x = f_{s\rightarrow t}(x_{0})$ for some $x_{0} \in p^{-1}(U) \cap X_{t}$. Since $x$ and $x_{0}$ represent the same point in $G_{K}X$, necessarily $x \in p^{-1}(U) \cap \beta_{s}^{-1}|t|$. Thus, $f_{s\rightarrow t}^{-1}(p^{-1}(U) \cap X_{t}) \subset p^{-1}(U) \cap \beta_{s}^{-1}|t|$.

6. Since $\beta_{X}^{-1}|s|$ is a closed subset of $G_{K}X$, the maps $x \mapsto p(x) : p^{-1}(\beta_{X}^{-1}|s|) \rightarrow \beta_{X}^{-1}|s|$ is an identification. We have

$$p^{-1}(\beta_{X}^{-1}|s|) = \bigoplus_{t \in K} \bigoplus_{s \cap t \neq \emptyset} \bigoplus_{s \cap t \neq \emptyset} \beta_{t}^{-1}|s \cap t|$$

and, if $U \subset \beta_{X}^{-1}|s|$ and $t \in K$ with $s \cap t \neq \emptyset$ and $t \notin \delta$,

$$p^{-1}(U) \cap \beta_{t}^{-1}|s \cap t| = f_{t\rightarrow s\cap t}^{-1}(p^{-1}(U) \cap X_{s\cap t})$$

by 5. Thus, $p^{-1}(U)$ is open in $p^{-1}(\beta_{X}^{-1}|s|)$ if and only if $p^{-1}(U) \cap \bigoplus_{t \in \delta} X_{t}$ is open in $\bigoplus_{t \in \delta} X_{t}$. It follows that the map $x \mapsto p(x) : \bigoplus_{t \in \delta} X_{t} \rightarrow \beta_{X}^{-1}|s|$ is an identification. ■
2.9 Proposition. Let \( X = (X, \beta, f) \) be an object in \( \text{Glue}_{K_0} \) and let \( q : X_0 \to \mathcal{L}_K X \) be the identification map.

1. If \( U \subset X_0 \) and \( s \in K \) then there is the following equality.
\[
q^{-1}(q(U)) \cap \beta_0^{-1}|s| = \bigcup_{t \in \mathfrak{s}} f_{0-t}^{-1}(f_{0-t}(\beta_0^{-1}|t| \cap U)) \tag{2.6}
\]

2. If the map \( f_{0-s} : \beta_0^{-1}|s| \to X_s \) is an open map for each \( s \in K \) then \( q \) is an open map.

3. If the map \( f_{0-s} : \beta_0^{-1}|s| \to X_s \) is a closed map for each \( s \in K \) then \( q \) is a closed map.

4. If \( q \) is either an open or closed map, \( s \in K \) and the map \( x \mapsto f_{0-s}(x) : \beta_0^{-1}\langle s \rangle \to \beta_s^{-1}\langle s \rangle \) is an identification then the map \( x \mapsto q(x) : \beta_0^{-1}\langle s \rangle \to \beta_X^{-1}\langle s \rangle \) induces a homeomorphism \( \beta_X^{-1}\langle s \rangle \to \beta_s^{-1}\langle s \rangle \).

5. If \( s \in K \) then the map \( x \mapsto g(x) : f_{0-s}(x) \to f_{0-s}(x) \) is an identification.

Proof. 1. If \( x \) lies in the left hand side of (2.6) then \( q(x) = q(x_0) \) for some \( x_0 \in U \). We then have \( \beta_0(x) = \beta_0(x_0) \equiv b \in |s| \) and \( f_{0-cb}(x) = f_{0-cb}(x_0) \). In particular, \( x \in f_{0-cb}^{-1}(f_{0-cb}(\beta_0^{-1}|cb| \cap U)) \), which is a subset of the right hand side of (2.6) since \( cb \in cs \). Conversely, if \( x \) is an element of the right hand side of (2.6) then \( f_{0-t}(x) = f_{0-t}(x_0) \) for some \( x_0 \in \beta_0^{-1}|t| \cap U \), where \( t \in \mathfrak{s} \). Thus, \( \beta_0(x) \in \beta_0^{-1}|s| \) and \( q(x) = q(x_0) \) so that \( x \in q^{-1}(q(U)) \cap \beta_0^{-1}|s| \).

2 and 3 follow immediately from (2.6).

4. Since \( q \) is either an open or closed map, the map \( x \mapsto q(x) : q^{-1}(\beta_X^{-1}\langle s \rangle) \to \beta^{-1}s\langle s \rangle \) is an identification. We have \( q^{-1}(\beta_X^{-1}\langle s \rangle) = \beta_0^{-1}\langle s \rangle \) and two point \( x, x' \in \beta_0^{-1}\langle s \rangle \) represent the same point in \( \beta_X^{-1}\langle s \rangle \) if and only if \( f_{0-s}(x) = f_{0-s}(x') \). Thus, the map \( x \mapsto f_{0-s}(x) : \beta_0^{-1}\langle s \rangle \to \beta_s^{-1}\langle s \rangle \) induces a bijection \( \beta_X^{-1}\langle s \rangle \to \beta_s^{-1}\langle s \rangle \).

\[
\begin{array}{ccc}
\beta_0^{-1}\langle s \rangle & \xrightarrow{q} & \beta_X^{-1}\langle s \rangle \\
& f_{0-s} & \\
& \beta_s^{-1}\langle s \rangle & \xrightarrow{\beta_s^{-1}\langle s \rangle}
\end{array}
\]

Since \( x \mapsto f_{0-s}(x) : \beta_0^{-1}\langle s \rangle \to \beta_s^{-1}\langle s \rangle \) is an identification, this bijection is a homeomorphism.

5. \( \beta_X^{-1}|s| \) is a closed subset of \( \mathcal{L}_K X \). \( \blacksquare \)

Finally, let us introduce some new notation which we will use in the next two chapters to discuss the spheres constructed earlier. Let \( \delta_n \) be the simplicial complex consisting of all non-empty subsets of the vertex set \( \text{Vert}(\delta_n) = \{0, 1, \ldots, n\} \). That is, \( \delta_n \) is the simplicial complex underlying the standard simplex \( \Delta_n \). Let \( \mathcal{O}(n) \) be either the dimension \( n(n-1)/2 \) orthogonal
group $O(n)$, the dimension $n^2$ unitary group $U(n)$ or the dimension $n(2n+1)$ symplectic group $Sp(n)$.

For each integer $i$ with $0 \leq i \leq n - 2$, let $$O(n, i) = O(i + 1) \times O(n - i - 1) \subset O(n)$$

be the subgroup of matrices whose only non-zero entries lie in the upper $(i + 1) \times (i + 1)$ and lower $(n - i - 1) \times (n - i - 1)$ blocks. For each $s \in \delta_{n-2}$ let $$O_s(n) = \bigcap \{ O(n, i) \subset O(n) \mid i \in \text{Vert}(\delta_{n-2}) - s \},$$

where an empty intersection is taken to be $O(n)$. Define a lattice

$$/O_s(n)/\hat{s} = \{ O_s, t(n) \mid t \in \hat{s} \}$$

of subgroups of $O_s(n)$ by $O_{s, t}(n) = O_s(n) \cap \bigcap_{i \in t} O(n, i)$. Notice that $O_{s, t_1 \cup t_2}(n) = O_{s, t_1, t_2}(n)$ whenever $t_1$ and $t_2$ are disjoint faces of $s$.

If $s$ is the $(n-2)$-simplex of $\delta_{n-2}$ then $\hat{s} = \delta_{n-2}$, $O_s(n) = O(n)$ and we will write $/O_s(n)/\hat{s} = /O(n)/\delta_{n-2}$. The space $L_{\delta_{n-2}}(O(n), /O(n)/\delta_{n-2})$ is exactly the lattice construction of spheres presented earlier.
Chapter 3
Lifting Simplicial Approximation

Let \( \text{CH} \) be the category of compact Hausdorff spaces. If we restrict our attention to the embedded subcategory \( \text{CH}^K \) of \( \text{Glue}_K \), the spaces \( G_K X \) retain enough of the structure of the underlying polyhedron that we can prove a generalization of the classical Simplicial Approximation Theorem, for maps from a polyhedron. After presenting this result, we apply it to a space constructed over an \( n \)-simplex from an arbitrary map between compact Hausdorff spaces, resulting in a statement on the connectivity of the construction. We then show that this result can be improved if the map in question is locally trivial.

Throughout this chapter we use the embedding of \( \text{Top}^K \rightarrow \text{Glue}_K \) defined in the previous chapter (page 37) to identify \( \text{Top}^K \) with its embedded image in \( \text{Glue}_K \), writing \( G_K X \) whenever \( X \) is a functor in \( \text{Top}^K \).

A. Lifting a Homotopy

Let us begin by making the following observation.

3.1 Proposition. Let \( p : X \rightarrow Y \) and \( f : Z \rightarrow Y \) be any maps with \( p \) surjective. Consider the following pullback square.

\[
\begin{array}{ccc}
P & \rightarrow & X \\
\downarrow & & \downarrow p \\
Z & \rightarrow & Y \\
\end{array}
\]

\( P = \{(z,x) \in Z \times X \mid f(z) = p(x)\} \)

Let \( \pi : P \rightarrow Y \) be the diagonal composition in this diagram. If \( X \) and \( Z \) are compact and \( Y \) is Hausdorff then the map \( (z,x) \mapsto f(z) = p(x) : \pi^{-1}(f(Z)) \rightarrow f(Z) \) is an identification.

Proof. The map in question is a continuous surjection from a compact space to a Hausdorff space.
We want to apply this result when $p$ the identification $\bigoplus_{s \in K} X(s) \times |s| \to \mathcal{G}_K X = \mathcal{G}_K(\mathcal{T}_K X)$, so let us first show that $\mathcal{G}_K X$ is Hausdorff whenever $X$ is an object in $\text{CH}^K$.

3.2 Proposition. Let $X$ be an object in $\text{Top}^K$. Then $\mathcal{G}_K X$ is a Hausdorff space if and only if $X(s)$ is Hausdorff for each $s \in K$.

To prove this, we employ the following.

3.3 Proposition. Let $X$ be any space and suppose there is a map $\beta : X \to Y$ with $Y$ Hausdorff. The following are equivalent.

1. $X$ is Hausdorff.
2. For any two distinct points $x, x' \in X$ with $\beta(x) = \beta(x') = y$, there is an open neighbourhood $U$ of $y$, a Hausdorff space $Z$ and a map $h : \beta^{-1}(U) \to Z$ such that $h(x) \neq h(x')$.

Proof. If $X$ is Hausdorff then for any $x, x' \in X$ we can choose $U = Y$ and $h = 1 : X \to X$, so only the opposite direction requires proof. Thus, assume 2 holds and let $x, x' \in X$ be distinct points.

If $\beta(x) \neq \beta(x')$ then, since $Y$ is Hausdorff, there are disjoint neighbourhoods $V, V' \subset Y$ of $\beta(x)$ and $\beta(x')$ respectively. Thus, $\beta^{-1}(V)$ and $\beta^{-1}(V')$ are disjoint neighbourhoods of $x$ and $x'$ respectively.

If $\beta(x) = \beta(x')$ then, by the hypothesis, there is an open neighbourhood $U$ of this point, a Hausdorff space $Z$ and a map $h : \beta^{-1}(U) \to Z$ such that $h(x) \neq h(x')$. Since $Z$ is Hausdorff, there are disjoint neighbourhoods $V, V' \subset Z$ of $h(x)$ and $h(x')$ respectively. Since $\beta^{-1}(U) \subset X$ is open, $h^{-1}(V)$ and $h^{-1}(V')$ are the required disjoint neighbourhoods of $x$ and $x'$.

We require more notation before proving Proposition 3.2. The open star of a vertex $v$ of $K$ is the open subset

$$\text{St}(v) = \{ b \in |K| \mid b(v) > 0 \}$$

of $|K|$. If $s$ is a simplex of $K$, let $W_s = \bigcap_{v \in s} \text{St}(v)$.

Proof of 3.2. If $\mathcal{G}_K X$ is Hausdorff and $s \in K$, choose a point $b \in \langle s \rangle$. Then we have a map $h : X(s) \to \mathcal{G}_K X$ defined by the following diagram.

$$
\begin{array}{ccc}
X(s) & \leftarrow & \bigoplus_{s \in K} X(s) \times |s| \\
\downarrow h & & \downarrow \mathcal{G}_K X \\
(x, b) & \mapsto & \bigoplus_{s \in K} X(s) \times |s|
\end{array}
$$

Since $b \in \langle s \rangle$, $h$ is an injection by Proposition 2.4. Thus, $\mathcal{G}_K X$ Hausdorff implies the same for $X(s)$.
Conversely, suppose each $X(s)$ is Hausdorff. Denote the identification $\bigoplus_{s \in K} X(s) \times |s| \to G_K X$ by $p$. Since $|K|$ is a Hausdorff space,$^1$ we can apply Proposition 3.3 with $\beta : G_K X \to |K|$ the canonical map onto the underlying polyhedron. Thus, let $(x, b) \in X(s) \times |s|$ and $(x', b) \in X(s') \times |s'|$ represent distinct points in $G_K X$ which are mapped to the same point by $\beta$. We need to exhibit a neighbourhood $U$ of $b$, a Hausdorff space $Z$ and a map $h : \beta^{-1}(U) \to Z$ such that $h(p(x, b)) \neq h(p(x', b))$.

If $u$ is any simplex of $K$ then, since $\beta^{-1}(W_u)$ is an open subset of $G_K X$, the map $(y, a) \mapsto p(y, a) : p^{-1}(\beta^{-1}(W_u)) \to \beta^{-1}(W_u)$ is an identification. Also, $|t| \cap W_u \neq \emptyset$ if and only if $t \rightarrow u$. Thus,

$$p^{-1}(\beta^{-1}(W_u)) = \bigoplus_{t \in K \atop t \rightarrow u} X(t) \times (|t| \cap W_u).$$

We can now define a map $h_u : \beta^{-1}(W_u) \to X_s$ by the following diagram.

$$\begin{array}{ccc}
X(t) \times (|t| \cap W_u) & \xrightarrow{(y, a)} & (y, a) \\
\bigoplus_{t \in K} X(t) \times (|t| \cap W_u) & \xrightarrow{h_u} & X(u) \\
\beta^{-1}(W_u) & \xrightarrow{h_u} & X(t \rightarrow u)(y)
\end{array}$$

To complete the proof, take $U = W_c b$, $Z = X_c b$ and $h = h_c b : \beta^{-1}(W_c b) \to X(c b)$. Since $p(x, b)$ and $p(x', b)$ are distinct points in $G_K X$, $X(s \rightarrow c b)(x)$ and $X(s' \rightarrow c b)(x')$ are distinct by Proposition 2.4. Thus, $h_c b(p(x, b)) \neq h_c b(p(x', b))$ as required. □

Now, suppose we are given a map $f : Z \to G_K X$, where $Z$ is compact, $X$ is an object in $\text{CH}^K$ and $K$ is finite. The pullback square

$$\begin{array}{ccc}
P & \xrightarrow{\bigoplus_{s \in K} X(s) \times |s|} & X(s) \\
\downarrow \pi & & \downarrow p \\
Z & \xrightarrow{f} & G_K X
\end{array}$$

then satisfies the hypotheses of Proposition 3.1. Let $h : Z \times I \to |K|$ be a homotopy satisfying $h(z, 0) = \beta(f(z))$ and $h(z, t) \in |c(\beta(f(z)))|$ for all $z \in Z$ and $t \in I$. Since $I$ is locally compact, the map $(z, x, b, t) \mapsto (f(z), t) = (p(z, b), t) : \pi^{-1}(f(Z)) \times I \to f(Z) \times I$ is an identification.

$^1$ [S] page 111.
Thus, the following diagram defines a deformation $D : f(Z) \times I \to \mathcal{G}_K X$ of $f(Z)$ in $\mathcal{G}_K X$ and, then, a homotopy $H : Z \times I \to \mathcal{G}_K X$.

\[
\begin{array}{cccc}
(z, z, b, t) & \xrightarrow{\quad} & (z, h(z, t)) \\
\pi^{-1}(f(Z)) \times I & \xrightarrow{\quad} & \bigoplus_{s \in K} X(s) \times |s| \\
\pi \times 1 & \downarrow & p \\
f(Z) \times I & \xrightarrow{D} & \mathcal{G}_K X \\
f \times 1 & \downarrow & \downarrow H \\
Z \times I & & & \\
\end{array}
\]

Note that $\beta \circ H = h$ and that $H(z, 0) = f(z)$ for all $z \in Z$. We have thus proven the following.

3.4 Proposition. Let $f : Z \to \mathcal{G}_K X$, where $Z$ is compact, $X$ is an object in $\mathbf{CH}^K$ and $K$ is finite. Let $h : Z \times I \to |K|$ be a homotopy with $h(z, 0) = \beta(f(z))$ and $h(z, t) \in |c(\beta(f(z)))|$ for all $z \in Z$ and $t \in I$. Then there is homotopy $H : Z \times I \to \mathcal{G}_K X$ such that $\beta \circ H = h$ and $H(z, 0) = f(z)$ for all $z \in Z$, and having $H(z, t) = p(x, h(z, t))$ whenever $f(z) = p(x, b)$. \(\blacksquare\)

In other words, we have verified the existence of a homotopy $H$ which makes the following diagram commute.

\[
\begin{array}{ccc}
z & \xrightarrow{f} & \mathcal{G}_K X \\
(z, 0) & \xrightarrow{h} & |K| \\
\end{array}
\]

Of course, the map $\beta$ is almost certainly not a fibration. The existence of the lifting $H$ depends on the hypothesis that the path $t \mapsto h(z, t) : I \to |K|$ remains inside the closed simplex $|c(\beta(f(z)))|$. We can easily remove the restriction that $K$ be finite in Proposition 3.4.

3.5 Proposition. Let $f : Z \to \mathcal{G}_K X$, where $Z$ is compact and $X$ is an object in $\mathbf{CH}^K$. Let $h : Z \times I \to |K|$ be a homotopy with $h(z, 0) = \beta(f(z))$ and $h(z, t) \in |c(\beta(f(z)))|$ for all $z \in Z$ and $t \in I$. Then there is homotopy $H : Z \times I \to \mathcal{G}_K X$ such that $\beta \circ H = h$ and $H(z, 0) = f(z)$ for all $z \in Z$, and having $H(z, t) = p(x, h(z, t))$ whenever $f(z) = p(x, b)$. \(\blacksquare\)

Proof. Since $\beta(f(Z)) \subset |K|$ is compact, $\beta(f(Z)) \subset |K'|$ for some finite subcomplex $K' \subset K$. \(\blacksquare\)

If $X'$ is the restriction of $X$ to $K'$ then consider the map $i : \mathcal{G}_{K'} X' \to \mathcal{G}_K X$ defined by the

---

2 The fibres over different points in $|K|$ may have different homotopy types: See Proposition 2.8.

3 [S], page 113.
An application of Proposition 2.4 shows that \( i \) is injective. Thus, since \( \mathcal{G}_K'X' \) is compact and \( \mathcal{G}_KX \) is Hausdorff, \( i \) is an embedding onto its image (which is \( \beta^{-1}[K'] \)). Apply Proposition 3.4 to the map \( z \mapsto i^{-1}(f(z)) : Z \to \mathcal{G}_K'X' \) and homotopy \( (z, t) \mapsto h(z, t) : Z \times I \to |K'| \) to get a lifting \( H' : Z \times I \to \mathcal{G}_K'X' \), and set \( H = i \circ H \). □

B. Approximation by Simplicial Maps

Let \( f : |L| \to |K| \), where \( L \) and \( K \) are simplicial complexes. A simplicial map \( \phi : L \to K \) is called a simplicial approximation to \( f \) if the induced map \( |\phi| : |L| \to |K| \) satisfies the condition that \( |\phi|(\alpha) \in |s| \) whenever \( f(\alpha) \in |s| \). Because of this condition, if \( \phi \) exists then the map

\[
(\alpha, t) \mapsto (1 - t)f(\alpha) + t|\phi|(\alpha)
\]  

(3.1)

is a well-defined homotopy \( h : |L| \times I \to |K| \). Moreover, this homotopy has \( h(\alpha, t) \in |s| \) whenever \( f(\alpha) \in |s| \) or, equivalently, \( h(\alpha, t) \in |c(f(\alpha))| \). Note also that \( h \) is constant on the set of points where \( f \) and \( |\phi| \) agree.

Recall that a subdivision of \( L \) is a simplicial complex \( L' \) such that (1) the vertices of \( L' \) are points in \( |L| \), (2) if \( s' \) is a simplex of \( L' \) then there is a simplex \( s \) of \( L \) such that \( s' \subset |s| \) and (3) the linear map sending \( \alpha \in |L'| \) to the point \( \sum_{v \in L'} \alpha(v)v \in |L| \) is a homeomorphism. We will identify \( |L'| \) with \( |L| \) by the homeomorphism of (3), implicitly composing with either this map or its inverse whenever context demands. The classical Simplicial Approximation Theorem\(^5\) states that there exists a subdivision \( L' \) of \( L \) and a simplicial approximation \( \phi : L' \to K \) to \( f : |L'| \to |K| \). As noted above, this simplicial approximation is necessarily homotopic to \( f \) by a homotopy \( h \) satisfying \( h(\alpha, t) \in |c(f(\alpha))| \).

Let us call a map \( \psi : |L| \to \mathcal{G}_KX \) simplicial if there is a simplicial map \( \phi : L \to K \) such that \( \beta \circ \psi = |\phi| \), where \( \beta : \mathcal{G}_KX \to |K| \) is the map onto the underlying polyhedron. Call it a simplicial approximation to a map \( f : |L| \to \mathcal{G}_KX \) if \( \phi \) is a simplicial approximation to \( \beta \circ f \).

3.6 Proposition. Let \( K \) and \( L \) be simplicial complexes and suppose \( f : |L| \to \mathcal{G}_KX \), where \( X \) is an object in \( \text{CH}^K \). Then there is a subdivision \( L' \) of \( L \) and a simplicial approximation

---

\(^4\) Continuity follows from the fact that \( h_{|s| \times I} \) is continuous for each \( s \in L \) since, by Proposition B.5, the topology of \( |L| \times I \) is generated by the collection \( \{ |s| \times I \mid s \in L \} \).

\(^5\) See [S], page 128, or [Mu], page 95 (Theorem 16.5).
\[ \psi : |L'| \to G_K X \text{ to } f. \] Moreover, if \( A \) is the set of points in \(|L'|\) where \( \beta \circ f \) and \( \beta \circ \psi \) agree then \( f \simeq \psi \text{ rel } \beta^{-1}(A) \).

**Proof.** By the Simplicial Approximation Theorem, there is a subdivision \( L' \) of \( L \) and a simplicial approximation \( \phi : L' \to K \) to \( \beta \circ f \). Let \( h : |L'| \times I \to |K| \) be the homotopy defined, as in (3.1), by \( h(\alpha, t) = (1 - t)\beta(f(\alpha)) + t\phi(\alpha) \). If \( s \) is a simplex of \( L' \), apply Proposition 3.5 to the map \( f|_s \) and the homotopy \( h|_s \) to get a homotopy \( H_s : |s| \times I \to G_K X \) such that \( \beta(H_s(\alpha, t)) = h(\alpha, t) \) and \( H_s(\alpha, 0) = f(\alpha) \), and having \( H_s(\alpha, t) = p(x, h(\alpha, t)) \) whenever \( f(\alpha) = p(x, b) \). Because of this last property, the function \( H : |L'| \times I \to G_K X \) defined by \( H|_s \times I = H_s \) is well defined. It is continuous since each \( H|_s \times I \) is and the topology on \(|L'| \times I\) is generated by the collection \( \{ |s| \times I \mid s \in L' \} \) (by Proposition B.5).

Since \( \beta(\psi(\alpha)) = h(\alpha, 1) = \phi(\alpha) \), \( H \) is a homotopy from \( f \) to a simplicial map. The homotopy is relative \( \beta^{-1}(A) \) since \( H(\alpha, t) = p(x, h(\alpha, t)) \) whenever \( f(\alpha) = p(x, b) \) and since the homotopy \( h \) is relative the set \( A \).

As with ordinary simplicial approximation, Proposition 3.6 can be used to infer results on the vanishing of the homotopy groups of \( G_K X \). A map \( f : S^n \to G_K X \) is homotopic to a map whose image is contained in \( \beta^{-1}[K(n)] \), where \( K(n) \) is the \( n \)-skeleton of \( K \). If we can determine that \( \pi_n A = 0 \) for some \( A \subseteq G_K X \) containing \( \beta^{-1}[K(n)] \) then necessarily \( \pi_n G_K X = 0 \). We examine an example where we can do this in the next section.

Let us try to replace the polyhedron \(|L|\) in Proposition 3.6 by a space \( G_L Y \) lying above it, where \( Y \) is an object in \( CH^L \). Call a map \( \psi : G_L Y \to G_K X \) simplicial if there is a simplicial map \( \phi : L \to K \) such that the following diagram commutes, the vertical maps being the canonical maps onto the underlying polyhedron.

\[
\begin{array}{ccc}
G_L Y & \xrightarrow{\psi} & G_K X \\
\beta_Y \downarrow & & \beta_X \\
|L| & \xrightarrow{|\phi|} & |K|
\end{array}
\]

Call \( \psi \) a simplicial approximation to a map \( f : G_L Y \to G_K X \) if \( \beta_X(\psi(\alpha)) = |\phi|(|\beta_Y(\alpha)|) \in |s| \) whenever \( \beta_X(f(\alpha)) \in |s| \). As above, a simplicial approximation is homotopic to the map it is approximating.

The notion of subdivision can also be extended to \( G_L Y \). Suppose \( L' \) is a subdivision of \( L \). If \( s' \) is a simplex of \( L' \) then, by the definition of subdivision, there is a simplex \( s \) of \( L \) such that \( s' \subseteq |s| \). Since any non-empty intersection of simplices is itself a simplex, there is a smallest such simplex of \( L \). Denote this simplex by \([s']\). We can then define a functor \( Y' : L' \to CH \) by \( Y'(s \to t) = Y([s] \to [t]) \). The identity maps \( Y'(s) \to Y([s]) \) then induce a homeomorphism from \( G_L Y' \to G_L Y \), and we refer to \( G_L Y' \) as the subdivision of \( G_L Y \) over \( L' \).

Now, one can try to prove the statement that given a map \( f : G_L Y \to G_K X \), where \( Y \) is an object of \( CH^L \) and \( X \) is an object in \( CH^K \), then there is a subdivision \( L' \) of \( L \) and a simplicial
approximation \( \psi : G_L Y' \to G_K X \) to \( f \). However, this is false in general as the following example shows.

As we saw in the previous chapter, \( S^4 \) can be constructed as a lattice construction over the standard 1-simplex. Consider the maps from \( S^4 \) over a standard 1-simplex to \( S^2 \), triangulated as the boundary of a standard 3-simplex.6

A simplicial (in our newly defined sense) map \( S^4 \to S^2 \) necessarily maps into the 1-skeleton of \( S^2 \). In particular, such a map is not onto and is therefore null-homotopic. If every map from \( S^4 \) to \( S^2 \) had a simplicial approximation (to which it would be homotopic) then every map from \( S^4 \) to \( S^2 \) would be null-homotopic, and this is certainly not true as \( \pi_4 S^2 \cong \mathbb{Z}_2 \).

C. An Example

Let \( D^n \) denote the closed \( n \)-ball with interior \( B^n \) and bounding sphere \( S^{n-1} \). Let \( f : X \to Y \) be any map with \( X \) and \( Y \) compact Hausdorff. Define the space \( D_n f \) to be the quotient of the disjoint union \( (X \times D^n) \oplus (Y \times S^{n-1}) \) by the smallest equivalence relation that identifies \( (x, b) \) and \( (f(x), b) \) whenever \( b \in S^{n-1} \). Let \( p_f \) denote the quotient map.

There is a map \( q_f : D_n f \to Y \) induced by the maps \( (x, b) \mapsto f(x) : X \times D^n \to Y \) and \( (y, b) \mapsto y : Y \times S^{n-1} \to Y \). Similarly, there is a map \( \beta_f : D_n f \to D^n \) induced by the maps \( (x, b) \mapsto b : X \times D^n \to D^n \) and \( (y, b) \mapsto b : Y \times S^{n-1} \to D^n \). Moreover, an application of Proposition 3.3 to this map shows that \( D_n f \) is a Hausdorff space. To see this, suppose \( z \) and \( z' \) are two distinct points in \( D_n f \) with \( \beta_f(z) = \beta_f(z') \equiv b \). If \( b \in S^{n-1} \subset D^n \) then the map \( q : D_n f \to Y \) has \( q_f(z) \neq q_f(z') \) since \( z = p_f(q_f(z), b) \) and \( z' = p_f(q_f(z'), b) \). If \( b \in B^n \) then the map \( \alpha \mapsto p(\alpha) : p^{-1}(\beta_f^{-1}(B^n)) \to \beta_f^{-1}(B^n) \) is an identification since \( \beta_f^{-1}(B^n) \) is an open subset of \( D_n f \). Thus, since \( p^{-1}(\beta_f^{-1}(B^n)) = X \times B^n \), the map on \( (x, b) \mapsto x : X \times B^n \to X \) induces a map \( q'_f : \beta_f^{-1}(B^n) \to X \times B^n \) having \( q'_f(p_f(x, b)) = x \). Necessarily \( q'_f(z) \neq q'_f(z') \) since \( z = p_f(q'_f(z), b) \) and \( z' = p_f(q'_f(z'), b) \).

Note that if \( f \) is surjective then \( D_n f \) is homeomorphic to the quotient of \( X \times D^n \) by the equivalence relation that identifies \( (x, b) \) with \( (x', b') \) if and only if \( b' = b \in S^{n-1} \) and \( f(x') = \ldots \)

6 Note that \( G_K X \approx |K| \) whenever \( K \) is locally finite and \( X \) is the constant functor to a 1-point space.
The composition

\[ X \times D^n \longrightarrow (X \times D^n) \oplus (Y \times S^{n-1}) \longrightarrow D_n f \]

factors through a bijection from this compact quotient onto the Hausdorff space \( D_n f \).

If \( * \) is the constant map onto a one point space, \( D_n * \) is homeomorphic to the join \( S^{n-1} \ast X \).

To see this, first recall that the join \( Y \ast Z \) of two spaces \( Y \) and \( Z \) is the quotient of \( Y \times I \times Z \) by the smallest equivalence relation that identifies each of the subspaces \( Y \times \{0\} \times \{z\} \) \( (z \in Z) \) and \( \{y\} \times \{1\} \times Z \( (y \in Y) \) to distinct points. The map

\[(b, t, x) \mapsto (tb, x) : S^{n-1} \times I \times X \to D^n \times X\]

then induces a homeomorphism from \( S^{n-1} \ast X \) to the quotient of \( D^n \times X \) by the smallest equivalence relation that identifies \( \{b\} \times X \) to a point for each \( b \in S^{n-1} \), and this latter quotient is homeomorphic to \( D_n * \) as previously indicated. Since \( S^{n-1} \ast X \) is homeomorphic to the \( n \)-fold suspension of \( X \), \( D_n * \) is \( (n-1) \)-connected.

It is not hard to see that \( D_n f \) is homeomorphic to a space in the image of \( G_{\delta_n} \). Indeed, if \( \partial \delta_n \) denotes the subcomplex of \( \delta_n \) consisting only of the proper faces of the \( n \)-simplex in \( \delta_n \), choose a triangulation \( \xi : (|\delta_n|, |\partial \delta_n|) \to (B^n, S^{n-1}) \). Define a functor \( Z : \delta_n \to \text{CH} \) by the following.

\[
Z(s \to t) = \begin{cases}
1 : X \to X & \text{if } t \in \delta_n - \partial \delta_n \\
1 : Y \to Y & \text{if } s \in \partial \delta_n \\
f : X \to Y & \text{otherwise}
\end{cases}
\]

The maps \((x, b) \mapsto (x, \xi(b)) : X \times |\delta_n| \to X \times B^n \) and \((y, b) \mapsto (y, \xi(b)) : Y \times |s| \to Y \times S^{n-1} \) induce a map \( \bigoplus_{s \in \delta_n} Z(s) \times |s| \to (X \times B^n) \oplus (Y \times S^{n-1}) \) which, in turn, induces a bijection \( i : G_{\delta_n} Z \to D_n f \).

\[
\begin{array}{ccc}
\bigoplus_{s \in \delta_n} Z(s) \times |s| & \longrightarrow & (X \times B^n) \oplus (Y \times S^{n-1}) \\
\downarrow & & \downarrow p \\
G_{\delta_n} Z & \longrightarrow & D_n f \\
i & &
\end{array}
\]

Since \( G_{\delta_n} Z \) is compact and \( D_n f \) is Hausdorff, \( i \) is a homeomorphism.

Now, consider maps from a \( k \)-sphere into \( D_n f \). Each map from \( S^k \) into \( D_n f \) is homotopic to a map into \( \beta_X^{-1}|\delta_n^{(k)}| \). Since the homeomorphism \( i : G_{\delta_n} Z \to D_n f \) maps \( \beta_X^{-1}|\delta_n^{(n-1)}| \) onto \( \beta_f^{-1}(S^{n-1}) \approx Y \times S^{n-1} \), and since \( S^{n-1} \) is \( (n-2) \)-connected, the following result is immediate.

**3.7 Proposition.** For any \( k \leq n - 2 \), if \( \pi_k Y = 0 \) then \( \pi_k D_n f = 0 \).
Additional Structure

Proposition 3.7 imposes no constraints on the map \( f \) other than that it be a map between compact Hausdorff spaces. Let us now assume that \( f \) is locally trivial with fibre \( F \).

3.8 Proposition. Let \(*\) denote the constant map from \( F \) to a 1-point space. Then the map \( q_f : D_n f \rightarrow Y \) is locally trivial with fibre \( D_n * \).

Proof. Given a trivialization \( \phi = (f, \psi) : f^{-1}(U) \rightarrow U \times F \), let us construct a trivialization \( q_f^{-1}(U) \rightarrow U \times D_n * \). First, choose any point \( a \in F \) and define a map \( p_f^{-1}(q_f^{-1}(U)) \rightarrow U \times D_n * \) by the following diagram

\[
\begin{array}{ccc}
(x, b) & \rightarrow & (f(x), \psi(x), b) \\
f^{-1}(U) \times D^n & \rightarrow & U \times F \times D^n \\
\downarrow & & \downarrow \\
p_f^{-1}(q_f^{-1}(U)) = (f^{-1}(U) \times D^n) \oplus (U \times S^{n-1}) & \rightarrow & U \times D_n * \\
\downarrow & & \downarrow \\
U \times S^{n-1} & \rightarrow & U \times F \times D^n \\
(y, b) & \leftarrow & (y, a, b)
\end{array}
\]

Since \( q_f^{-1}(U) \) is an open subset of \( D_n f \), the map \( \alpha \mapsto p_f(\alpha) : p_f^{-1}(q_f^{-1}(U)) \rightarrow q_f^{-1}(U) \) is an identification. Thus, since any two pairs in \( F \times S^{n-1} \) represent the same point in \( D_n * \), the above map factors through this identification to define a map \( \zeta : q_f^{-1}(U) \rightarrow U \times D_n * \).

\[
\begin{array}{ccc}
(f^{-1}(U) \times D^n) \oplus (U \times S^{n-1}) & \rightarrow & U \times D_n * \\
\downarrow & & \downarrow \\
q_f^{-1}(U) & \rightarrow & U \times D_n *
\end{array}
\]

As noted previously, since \( * \) is a surjective map, \( D_n * \) is homeomorphic to a quotient of \( F \times D^n \). Identifying \( D_n * \) with this quotient, the inverse of \( \zeta \) is defined by the following diagram.

\[
\begin{array}{ccc}
(y, a, b) & \rightarrow & (\phi^{-1}(y, a), b) \\
U \times F \times D^n & \rightarrow & f^{-1}(U) \times D^n \\
\downarrow 1 \times p_* & & \downarrow \\
U \times D_n * & \rightarrow & q_f^{-1}(U)
\end{array}
\]
Here the map $1 \times p_\ast$ is an identification since $U$ is locally compact. Thus, the existence of $\zeta^{-1}$ follows by the universal property of the identification. ■

3.9 Corollary. The map $q : D_n f \to Y$ induces isomorphisms $\pi_k D_n f \cong \pi_k Y$ for all $k \leq n - 1$ and an epimorphism $\pi_n D_n f \twoheadrightarrow \pi_n Y$.

In particular, if $\pi_k Y = 0$ for some $k \leq n - 1$ then $\pi_k D_n f = 0$, improving the result of Proposition 3.7.\footnote{Is there an example (for any $n$) of a map $f$ such that $\pi_{n-1} Y = 0$ but $\pi_{n-1} D_n f \neq 0$?}

Proof. The $n$-fold suspension of $F$ is $(n - 1)$-connected. Apply the long exact homotopy sequence of the fibration. ■
Chapter 4

Bundle Structure with a Topological Group

As discussed in chapter 2, a group $G$ together with a lattice $\mathcal{L} = \{ G_s \mid s \in K \}$ of subgroups determines an object in $\text{Glue}_{\mathcal{L}_0}$ whenever either $G$ is locally compact Hausdorff or $K$ is locally finite. If this object is the triple $\mathcal{G} = (X, \beta, f)$ then, setting $G_0 = \{1\}$, the spaces $X_s$ and the maps $\beta_s : X_s \to |K|$ and $f_{s-t} : \beta_s^{-1}[t] \to X_t$ are defined by $X_s = G/G_s \times |s|$, $\beta_s(gG_s, b) = b$ and $f_{s-t}(gG_s, b) = (gG_t, b)$. Pairs $(G, /G/K)$ are the objects of a category in which a morphism from $(G, /G/K)$ to $(H, /H/K)$ is a homomorphism $f : G \to H$ such that $f(G_s) \subset H_s$ for each $s \in K$, and the construction of $\mathcal{G}$ from the pair $(G, /G/K)$ defines an embedding of this category into the category $\text{Glue}_{\mathcal{L}_0}$. Let us identify the embedded image with the category itself and write $\mathcal{G}_{K_0} = \mathcal{G}_{K_0}(G, /G/K)$ and $\mathcal{L}_K \mathcal{G} = \mathcal{L}_K(G, /G/K)$.

We see in the proof of Proposition 2.7 that the inclusion $G \times |K| \hookrightarrow \bigoplus_{s \in K_0} G/G_s \times |s|$ induces a homeomorphism $\mathcal{L}_K(G, /G/K) \to \mathcal{G}_{K_0}(G, /G/K)$. Now, if $H$ is another subgroup of $G$, let us write $/G/K \cap H = \{ G_s \cap H \mid s \in K \}$. In this chapter we examine the space $\mathcal{L}_K(G, /G/K \cap H)$ in the case that (1) $G$ is locally compact Hausdorff, (2) each of the members of $/G/K$ is a closed subgroup and (3) $H \subset G$ is a closed subgroup for which the coset projection $G \to G/H$ is locally trivial. There is then a locally trivial map from $\mathcal{L}_K(G, /G/K \cap H)$ to $G/H$. This will provide us with examples of lattice construction in which the space constructed is not simply connected. It will also give information on the structure of neighbourhoods in $\mathcal{L}_K(G, /G/K)$ in the case that $G$ is compact and $K$ is locally finite.

In section C, the spheres originally constructed by R. R. Douglas and A. R. Rutherford are re-examined, and many more spheres are constructed. In each of the real, complex and quaternionic cases we construct a finite number of spheres over a simplex of a given dimension, but all but a finite number of spheres in totality. Exactly which spheres are constructed is enumerated.

A. A Class of Bundles

We begin with the following fact.

55
4.1 Proposition. Let $G$ be a locally compact Hausdorff group and let $H', H \subset G$ be subgroups with $H' \subset H$ and $H$ closed. Let $p : G/H' \to G/H$ be the coset projection. If the coset projection $G \to G/H$ has a local section $\varsigma : U \to G$ on the neighbourhood $U \subset G/H$ then the map

$$gH' \mapsto (gH, \varsigma(gH)^{-1}gH') : p^{-1}(U) \to U \times H/H'$$

(4.1)

is a homeomorphism. In particular, taking $H' = \{1\}$, the map

$$g \mapsto (gH, \varsigma(gH)^{-1}g) : p^{-1}(U) \to U \times H$$

is a homeomorphism.

Proof. Let us denote the coset projection $G \to G/H$ by $p_H$. Since $U$ is an open subset of $G/H$, the identification $G \to G/H'$ restricts to an identification $p^{-1}_H(U) \to p^{-1}(U)$. Thus, the following diagram defines the map (4.1).

$$
\begin{array}{c}
g \\
p^{-1}_H(U) \\
p^{-1}(U)
\end{array} \longrightarrow \begin{array}{c}
(gH, \varsigma(gH)^{-1}g) \\
U \times H \\
U \times H/H'
\end{array}
$$

To construct the inverse map, since $G$ is locally compact Hausdorff and $H \subset G$ is a closed subgroup, the coset space $G/H$ is locally compact Hausdorff.\(^1\) Since an open subset of a locally compact space is itself locally compact, the product of the identity map on $U$ with the identification $H \to H/H'$ is an identification. Thus, the following diagram defines a map $U \times H/H' \to p^{-1}(U)$.

$$
\begin{array}{c}
(gH, h) \\
U \times H \\
U \times H/H'
\end{array} \longrightarrow \begin{array}{c}
\varsigma(gH)h \\
p^{-1}_H(U) \\
p^{-1}(U)
\end{array}
$$

It is easy to check that this map is the inverse of (4.1). \(\blacksquare\)

We can now apply the above result to spaces of the form $\mathcal{L}_K(G, /G/K \cap H)$. In this situation, the map $(g, b) \mapsto gH : G \times |K| \to G/H$ then induces a map $\pi : \mathcal{L}_K(G, /G/K \cap H) \to G/H$.

\(^1\) See [MZ], page 52. Note that the authors here assume that a topological space satisfies the $T_0$ separation axiom by definition. As discussed on pages 20 and 25 of [MZ], this implies the Hausdorff property in a topological group.
4.2 Proposition. Let $G$ be a locally compact Hausdorff group, $H \subset G$ a closed subgroup such that the coset projection $p_H : G \to G/H$ is locally trivial, and $/G/K = \{ G_s \mid s \in K \}$ a lattice of closed subgroups of $G$. Then the map $\pi : \mathcal{L}_K(G, /G/K \cap H) \to G/H$ is locally trivial with fibre $\mathcal{L}_K(H, /G/K \cap H)$.

Proof. Let $q_G : G \times |K| \to \mathcal{L}_K(G, /G/K \cap H)$ and $q_H : H \times |K| \to \mathcal{L}_K(H, /G/K \cap H)$ be the identification maps, and let $U \subset G/H$ be any neighbourhood on which $p_H : G \to G/H$ has a section $\xi : U \to G$. Let us also write $H_s = G_s \cap H$ and $/H/K = /G/K \cap H$. Since $\pi^{-1}(U)$ is an open subset of $\mathcal{L}_K(H, /H/K)$, the map $(g, b) \to q_G(g, b) : q_G^{-1}(\pi^{-1}(U)) \to \pi^{-1}(U)$ is an identification. Now, if $s \in K$ then two pairs $(g, b)$ and $(g', b')$ in $p_H^{-1}(U) \times s \subset p_H^{-1}(H) \times |K| = q_G^{-1}(\pi^{-1}(U))$ represent the same point in $\pi^{-1}(U)$ if and only if $b' = b$ and $g H_s = g'H_s$. Thus, the pairs $(\xi(gH)g, b)$ and $(\xi(g'H)g', b')$ in $H \times |K|$ represent the same point in $\mathcal{L}_K(H, /H/K)$, and the following diagram defines a map $\pi^{-1}(U) \to U \times \mathcal{L}_K(H, /H/K)$.

$$
\begin{array}{ccc}
(g, b) & \longrightarrow & (gH, \xi(gH)^{-1}g, b) \\
p_H^{-1}(U) \times |K| & \longrightarrow & U \times H \times |K| \\
1 \times q_H & & 1 \times q_H \\
\pi^{-1}(U) & \longrightarrow & U \times \mathcal{L}_K(H, /H/K) \\
\end{array}
$$

(4.2)

The topmost map in this diagram is just the product of the identity map on $U$ with the homeomorphism $p_H^{-1}(U) \to U \times H$ provided by Proposition 4.1. In fact, it is clear that two pairs $(g, b)$ and $(g', b')$ represent the same point in $\pi^{-1}(U)$ if and only if $(\xi(gH)g, b)$ and $(g'H, \xi(g'H)^{-1}g', b')$ represent the same point in $U \times \mathcal{L}_K(H, /H/K)$. Thus, since $U$ is locally compact, the product of the identity map on $U$ with the identification $q_H$ is again an identification, and we can reverse the topmost horizontal map in the previous diagram to define the inverse of our map $\pi^{-1}(U) \to U \times \mathcal{L}_K(H, /H/K)$.

$$
\begin{array}{ccc}
(gH, h, b) & \longrightarrow & (\xi(gH)h, b) \\
U \times H \times |K| & \longrightarrow & p_H^{-1}(U) \times |K| \\
1 \times q_H & & 1 \times q_H \\
U \times \mathcal{L}_K(H, /H/K) & \longrightarrow & \pi^{-1}(U) \\
\end{array}
$$

Of course, the previous result leads to a long exact homotopy sequence. Let us continue to write $H_s = G_s \cap H$ and $/H/K = /G/K \cap H$ as in the just-completed proof of Proposition 4.2.

$$
\cdots \longrightarrow \pi_k \mathcal{L}_K(H, /H/K) \longrightarrow \pi_k \mathcal{L}_K(G, /H/K) \xrightarrow{\pi_k} \pi_k G/H \longrightarrow \pi_{k-1} \mathcal{L}_K(H, /H/K) \longrightarrow \cdots
$$

If $\mathcal{L}_K(H, /H/K)$ is contractible then it follows that $\pi : \mathcal{L}_K(G, /H/K) \to G/H$ is a weak homotopy equivalence; that is, that it induces isomorphisms of homotopy groups.
Consider the case where \( K = \delta_n \) is the simplicial complex of all subsets of the vertex set \( \{0, \ldots, n\} \). If \( v \) is a vertex of \( \delta_n \) then there is a retraction \( r : \delta_n \times I \to \delta_n \) defined by sliding a point in \( \delta_n \) along the line segment joining it to \( v \).

\[
r(\alpha, t) = (1 - t)\alpha + tv
\]

Here we are identifying the vertex \( v \) of the simplicial complex \( \delta_n \) with the obvious element of the polyhedron \( \delta_n \). In general the map \( 1 \times r : H \times \delta_n \to H \times \delta_n \) will not induce a map \( \mathcal{L}_{\delta_n}(H, H/\delta_n) \to \mathcal{L}_{\delta_n}(H, H/\delta_n) \) for the reason that if \( s \in \delta_n \) is a simplex not containing \( v \) then two pairs \((h, b)\) and \((h', b)\) in \( H \times \langle s \rangle \) represent the same point in \( \mathcal{L}_{\delta_n}(H, H/\delta_n) \) if and only if \( hH_s = h'H_s \) while the pairs \((h, r(b, 1/2))\) and \((h', r(b, 1/2))\) represent the same point if and only if \( hH_s U \{ v \} = h'H_s U \{ v \} \). Since \( S U \{ v \} = s \) implies that \( hH_s U \{ v \} \subseteq H_s \), a necessary and sufficient condition for the map \( 1 \times r : H \times \delta_n \to H \times \delta_n \) to induce a map \( \mathcal{L}_{\delta_n}(H, H/\delta_n) \to \mathcal{L}_{\delta_n}(H, H/\delta_n) \) is the following: If \( s \) is a simplex of \( \delta_n \) not containing \( v \) then \( H s U \{ v \} = H_s \). In this case, the induced map is a retraction onto \( \beta^{-1}(v) \approx H/H_{\{v\}} \).

4.3 Proposition. Let \( G \) be a locally compact Hausdorff group, \( /G/\delta_n = \{ G_s \mid s \in \delta_n \} \) a lattice of closed subgroups of \( G \), and \( H \subset G \) a closed subgroup such that the coset projection \( G \to G/H \) is locally trivial and \( H \subset G_{\{v\}} \) for some vertex \( v \) of \( \delta_n \) and. The following are true.

1. If \( \mathcal{L}_{\delta_n}(H, /G/\delta_n \cap H) \) is contractible then the map \( \pi : \mathcal{L}_{\delta_n}(G, /G/\delta_n \cap H) \to G/H \) is a weak homotopy equivalence.

2. If \( G_{s U \{ v \}} \cap H = G_s \cap H \) for each simplex \( s \) not containing \( v \) then \( \mathcal{L}_{\delta_n}(H, /G/\delta_n \cap H) \) is contractible.

3. If \( G_s \cap H = \bigcap_{\omega \in s} G_{\{\omega\}} \cap H \) for each \( s \in \delta_n \) then \( G_{s U \{ v \}} \cap H = G_s \cap H \) for each simplex \( s \) not containing \( v \).

Note \( G_s = \bigcap_{\omega \in s} G_{\{\omega\}} \) is always the case for the lattice construction so that the hypothesis of 3 and, thus, 2 and 1 are satisfied in this case.

Proof. 1 follows from the long exact homotopy sequence of the bundle \( \mathcal{L}_{\delta_n}(H, /G/\delta_n \cap H) \to \mathcal{L}_{\delta_n}(G, /G/\delta_n \cap H) \to G/H \) as noted above.

For 2, the preceding discussion showed that there is a retraction of \( \mathcal{L}_{\delta_n}(H, /G/\delta_n \cap H) \) onto \( \beta^{-1}(v) \approx H/(G_{\{v\}} \cap H) = H/H_{\{v\}} \), a one point space.

3 is clear since \( h \subset G_{\{v\}} \).

Let us state the conclusion of Proposition 4.3 in the case \( G = O(n) \) and \( /G/\delta_{n-2} = /O(n)/\delta_{n-2} \).

If \( H \subset O(n) \) is a closed subgroup then with \( H \subset O(n, i) = O(i+1) \times O(n-i-1) \) for some \( 0 \leq i \leq n-2 \) then there is a weak homotopy equivalence \( \mathcal{L}_{\delta_{n-2}}(O(n), /O(n)/\delta_{n-2} \cap H) \to O(n)/H \).

Notice that if \( H = O(n_1) \times \cdots \times O(n_k) \), with \( k \geq 2 \) and \( n_1 + \cdots + n_k = n \), then \( O(n)/H \) is a real, complex or quaternionic flag manifold.
B. Local Structure with a Compact Group

Let us now use Proposition 4.2 to decompose certain neighbourhoods in $L_K(G, G/K)$ in the case that $G$ is compact, $K$ is locally finite, and each of the coset projections $G \to G/G_s$ is locally trivial. This is the case when, for example, $G$ is a Lie group.

Let $q : G \times |K| \to L_K(G, G/K)$ be the identification map and let $\beta : L_K(G, G/K) \to |K|$ be the map induced by the projection $G \times |K| \to |K|$. Note that $q$ is an open map by statement 2 of Proposition 2.9. Thus, the map $\alpha \mapsto q(\alpha) : q^{-1}(A) \to \mathcal{A}$ is an identification for any subset $A \subset L_K(G, G/K)$. We will use this fact repeatedly (and sometimes without mention) to define maps on subsets of $L_K(G, G/K)$.

1. Strong Deformation Retractions above the Polyhedron

Let us start by defining two open subsets of $|K|$ for each $s \in K$.

$$U_s = \bigcup_{v \in s} St(v) \quad \quad W_s = \bigcap_{v \in s} St(v)$$

Define a strong deformation retraction $h_s : U_s \times I \to U_s$ as follows.

$$h_s(b, t) = \frac{t}{\sum_{v \in s} b(v)} \sum_{v \in s} b(v)v + (1 - t)b$$

Here we use the same letter to denote a vertex of $K$ and the function in $|K|$ which is 1 at this vertex and 0 at all others.

Since $h_s((U_s \cap t) \times I) \subset U_s \cap t$ for any $t \in K$, $h_s$ induces a map $H_s : \beta^{-1}(U_s) \times I \to \beta^{-1}(U_s)$.

$$\begin{array}{ccc}
G \times U_s \times I & \xrightarrow{1 \times h_s} & G \times U_s \\
\downarrow q \times 1 & & \downarrow q \\
\beta^{-1}(U_s) \times I & \xrightarrow{H_s} & \beta^{-1}(U_s)
\end{array}$$

Notice that $H_s$ maps $\beta^{-1}(U_s) \times \{1\}$ and $\beta^{-1}(W_s) \times \{1\}$ onto $\beta^{-1}|s|$ and $\beta^{-1}\langle s \rangle$ respectively since $h_s(U_s \times \{1\}) = |s|$ and $h_s(W_s \times \{1\}) = \langle s \rangle$.

Let $r_s : U_s \to |s|$ and $R_s : \beta^{-1}(U_s) \to \beta^{-1}|s|$ denote the retracts associated with the retractions $h_s$ and $H_s$ respectively.

$$r_s(b) = h_s(b, 1)$$
$$R_s(\alpha) = H_s(\alpha, 1)$$

We will show that the map $\alpha \mapsto R_s(\alpha) : \beta^{-1}(W_s) \to \beta^{-1}\langle s \rangle$ is locally trivial. Notice that statement 4 of Proposition 2.9 gives a homeomorphism $q(g, b) \mapsto (gG_s, b) : \beta^{-1}\langle s \rangle \to G/G_s \times \langle s \rangle$.
2. Fibre Structure of the Retracts

The simplex $s$ determines a collection of complementary simplices.

$$s^c = \{ t \in K \mid s \cap t = \emptyset, s \cup t \in K \}$$

This collection is either empty or a finite (since $K$ is locally finite) subcomplex of $K$. The former holds if and only if $s$ is not a proper face of any simplex or, equivalently, $W_s = \langle s \rangle$. In this case we have $\beta^{-1}(W_s) = \beta^{-1}(s) \approx \langle s \rangle \times G/G_s$. Let us now assume that $s^c$ is not empty.

Define a lattice of subgroups of $G$, indexed by $s^c$, by $\langle G \rangle_s = \{ G_a \mid t \in s^c \}$. Note that $G_{s \cup t} \subset G_s$ since $s \cup t \rightarrow s$ in $K$. Thus, by Proposition 4.2, the map $(g, b) \mapsto gG_s : G \times |s^c| \rightarrow G/G_s$ induces a locally trivial map $\pi_s : \mathcal{L}_{s^c}(G, /G_s/s^c) \rightarrow G/G_s$, having fibre $\mathcal{L}_{s^c}(G_s, /G_s/s^c)$.

There is an embedding of $\mathcal{L}_{s^c}(G, /G_s/s^c)$ in $\mathcal{L}_K(G, /G/K)$. To construct this embedding, define two subsets of $|K|$.

$$F^p_s = \{ b \in |K| \mid \sum_{v \in s} b(v) = 1/2 \} \subset U_s$$
$$F^\circ_s = \{ b \in U_s \mid r_s(b) = b_s \} \subset W_s$$

Here $b_s = 1/||s|| \cdot \sum_{v \in s} v$ is the barycenter of the closed face $|s| \subset |K|$, where $||s||$ is the number of vertices contained in $s$. Now, because the union of any member of $s^c$ and any face of $s$ is a member of $K$, the function $b/2 + b_s/2 : \text{Vert}(K) \rightarrow [0, 1]$ is a member of $|K|$ for any $b \in |s^c|$. That is, the inverse image, by this function, of the half-open interval $(0, 1]$ is a simplex of $K$. Moreover, it is clear that this function lies in the subset $F^p_s \cap F_s^\circ \subset |K|$ and that the map $b \mapsto b/2 + b_s/2 : |s^c| \rightarrow F^p_s \cap F_s^\circ$ is a homeomorphism. Also, $b \in \langle t \rangle$ if and only if $b/2 + b_s/2 \in \langle s \cup t \rangle$. Thus, the map $(g, b) \mapsto (g, b/2 + b_s/2) : G \times |s^c| \rightarrow G \times (F^p_s \cap F^\circ_s)$ induces a bijective map $\pi : \mathcal{L}_{s^c}(G, /G_s/s^c) \rightarrow \beta^{-1}(F^p_s \cap F^\circ_s)$.

Since the rightmost map in the above diagram is an identification as well, we can simply reverse the topmost arrow in this diagram to define the inverse map $\beta^{-1}(F^p_s \cap F^\circ_s) \rightarrow \mathcal{L}_{s^c}(G, /G_s/s^c)$.

Now, since $s$ is a face of the carrier of $b$ for each $b \in W_s \rightarrow cb \rightarrow s$ in symbols — the map $(g, b) \mapsto gG_s : G \times W_s \rightarrow G/G_s$ induces a map $\pi_s : \beta^{-1}(W_s) \rightarrow G/G_s$, and this map clearly
makes the following diagram commute.

\[
\begin{array}{ccc}
q : (g, b) & \mapsto & (g, b/2 + b_2/2) \\
\mathcal{L}_s^e(G, /G_s/, \pi_s^e) & \xrightarrow{\sim} & \beta^{-1}(F^p_s \cap F^0_s) \\
\pi_s^e & \downarrow & \\
G/G_s & \xrightarrow{\pi_s} & \\
\end{array}
\]

Thus, \(\pi_s |_{\beta^{-1}(F^p_s \cap F^0_s)}\) is locally trivial with fibre \(\mathcal{L}_s^e(G_s, /G_s/)\) since \(\pi_s^e\) is.

Note that if \(b \in F^p_s\) then \(\sum_{v \in \mathcal{V}} b(v) v = r_s(b)/2\). Thus, the function \(b - r_s(b)/2 + b_2/2\) is a member of \(|K|\) if and only of \(b^{-1}(0, 1] \cup s \in K\), and in this case it is clearly a member of the subset \(F^p_s \cap F^0_s\). If \(b \in F^p_s \cap W_s\) then \(s \subseteq b^{-1}(0, 1]\), and there is a map \(\gamma : F^p_s \cap W_s \rightarrow (F^p_s \cap F^0_s) \times \langle s \rangle\) defined by \(\gamma = (\gamma_1, r_s)\), where \(\gamma_1(b) = b - r_s(b)/2 + b_2/2\). It is clear that \(\gamma\) is a homeomorphism, the inverse map being \((a, b) \mapsto b + a/2 - b_2/2\).

We can now use \(\gamma\) to define a homeomorphism on \(\beta^{-1}(F^p_s \cap W_s)\). If \(b \in F^p_s \cap W_s\) then \(\gamma_1(b) \in \langle cb \rangle\). Thus, the following diagram defines a bijective map \(\Gamma : \beta^{-1}(F^p_s \cap W_s) \rightarrow \beta^{-1}(F^p_s \cap F^0_s) \times \langle s \rangle\).

\[
\begin{array}{ccc}
(g, b) & \mapsto & (g, \gamma_1(b), r_s(b)) \\
G \times (F^p_s \cap W_s) & \xrightarrow{1 \times \gamma} & G \times (F^p_s \cap F^0_s) \times \langle s \rangle \\
q & \downarrow & q \times 1 \\
\beta^{-1}(F^p_s \cap W_s) & \xrightarrow{\Gamma} & \beta^{-1}(F^p_s \cap F^0_s) \times \langle s \rangle \\
\end{array}
\]

(4.4)

Again, since \(\langle s \rangle\) is locally compact, the rightmost map above is an identification so we can reverse the arrows in the above diagram to define an inverse map, showing that \(\Gamma\) is a homeomorphism. This homeomorphism evidently makes the following diagram commute.

\[
\begin{array}{ccc}
q(g, b) & \longrightarrow & (q(g, \gamma_1(b)), r_s(b)) \\
\beta^{-1}(F^p_s \cap W_s) & \xrightarrow{\beta^{-1}(F^p_s \cap F^0_s) \times \langle s \rangle} & \Gamma \downarrow \pi_s \times 1 \\
R_s & \downarrow \beta^{-1}(s) \xrightarrow{\approx} & G/G_s \times \langle s \rangle \\
q(g, r_s(b)) & \xrightarrow{\approx} & (gG_s, r_s(b)) \\
\end{array}
\]

Thus, the map \(\alpha \mapsto R_s(\alpha) : \beta^{-1}(F^p_s \cap W_s) \rightarrow \beta^{-1}(s)\) is locally trivial with fibre \(\mathcal{L}_s^e(G, /G_s/)\) since \(\pi_s |_{\beta^{-1}(F^p_s \cap F^0_s)}\) is. Denote this map by \(\hat{R}_s\).

We now want to exhibit \(\beta^{-1}(W_s)\) as the quotient of \(\beta^{-1}(F^p_s \cap W_s) \times [0, 1]\) by the equivalence relation that identifies \((\alpha, t)\) and \((\alpha', t')\) if and only if \(t' = t = 0\) and \(R_s(\alpha) = R_s(\alpha')\). We
denote this quotient space by \( Q\hat{R}_s \). Since \( \hat{R}_s \) is locally trivial with fibre \( \mathbb{L}_{s'}(G_s, G_s/s') \), it follows that the map \( Q\hat{R}_s \to \beta^{-1}(s) \) induced by the composition of \( \hat{R}_s \) with the projection \( \beta^{-1}(F^p \cap W_s) \times [0, 1] \to \beta^{-1}(F^p \cap W_s) \) onto the first factor is locally trivial with fibre the open cone on \( \mathbb{L}_{s'}(G_s, G_s/s') \). A rather more general version of this statement can be found in appendix C.

To exhibit \( \beta^{-1}(W_s) \) as desired, let us first define two more subcomplexes of \( K \) determined by \( s \). The simplicial neighbourhood of \( s \) is the subcomplex

\[
\tilde{s} = \{ t \in K \mid t \cup s' \in K \text{ for some } s' \in \tilde{s} \}
\]

of all simplices which are faces of a simplex intersecting \( s \). Note that, because \( K \) is locally finite, \( \tilde{s} \) is a finite subcomplex of \( K \). The link of \( s \) is the subcomplex

\[
s^\perp = \{ t \in \tilde{s} \mid t \cap s = 0 \}
\]

of those simplices in \( \tilde{s} \) which do not intersect \( s \). Note that \( |\tilde{s}| = U_s \cup |s^\perp| \) and \( U_s \cap |s^\perp| = \emptyset \).

There is a surjective map \( \phi : F^p \times I \to |s| \) defined by sliding \( b \in F^p \) along the line segment joining \( r_s(b) \in |s| \) and \( 2b - r_s(b) \in |s^\perp| \).

\[
\phi(b, t) = (1 - t)r_s(b) + t(2b - r_s(b)) = (1 - t) \cdot 2 \sum_{v \in s} b(v)v + t \cdot 2 \sum_{v \in \text{Vert}(s^\perp)} b(v)v
\]

Notice that \( \phi^{-1}(W_s) = (F^p \cap W_s) \times [0, 1] \) and that \( r_s(\phi(b, t)) = r_s(b) \) for each \( t \in [0, 1] \). Since \( \phi(b, t) \in |cb| \) for all \( b \in F^p \) and \( t \in I \), \( \phi \) induces a surjective map \( \Phi : \beta^{-1}(F^p) \times I \to \beta^{-1}|s| \).

\[
\begin{array}{ccc}
G \times F^p \times I & \xrightarrow{\phi} & G \times |\tilde{s}| \\
q \times 1 \downarrow & & \downarrow q \\
\beta^{-1}(F^p) \times I & \xrightarrow{\Phi} & \beta^{-1}|\tilde{s}|
\end{array}
\]

Because \( \phi \) has the corresponding property, we have \( \Phi^{-1}(\beta^{-1}(W_s)) = \beta^{-1}(F^p \cap W_s) \times [0, 1] \).

Now, the map \( (\alpha, t) \mapsto \Phi(\alpha, t) : \beta^{-1}(F^p \cap W_s) \times [0, 1] \to \beta^{-1}(W_s) \) factors through the projection \( \beta^{-1}(F^p \cap W_s) \times [0, 1] \to Q\hat{R}_s \) to define a map \( \Theta : Q\hat{R}_s \to \beta^{-1}(W_s) \).

\[
(\alpha, t) \quad \beta^{-1}(F^p \cap W_s) \times [0, 1] \\
\downarrow \quad \downarrow \Phi \\
[\alpha, t] \quad Q\hat{R}_s \xrightarrow{\Theta} \beta^{-1}(W_s)
\]

2 The open cone of a space \( X \) is here the quotient of \( X \times [0, 1) \) which identifies the subspace \( X \times \{0\} \) to a point.
We claim that \( \Theta \) is a homeomorphism and show this in two steps.

1. \( \Theta \) is a bijection.

It is clearly a surjection since \( \Phi \) is. We can see it is an injection by considering separately the disjoint images of \( \beta^{-1}(F^p_s \cap W_s) \times \{0\} \) and \( \beta^{-1}(F^p_s \cap W_s) \times (0,1) \) in \( Q \hat{R}_s \). On the first set, two pairs \((a, 0)\) and \((a', 0)\) represent the same point in \( Q \hat{R}_s \) if and only if \( \Theta(a, 0) = R_s(a) = R_s(a') = \Theta(a', 0) \). On the second set, we can construct a partial inverse to \( \Theta \). For this, first define a partial inverse \( \psi : U_s \to F^p_s \times (0,1) \) to \( \phi : F^p_s \times I \to |\hat{s}|. \) If \( b \in U_s - |\hat{s}| \) then we can write \( b = (1 - \varepsilon) \cdot r_s(b) + \varepsilon \cdot (b - (1 - \varepsilon)r_s(b)) / \varepsilon, \) where \( \varepsilon = 1 - \sum_{v \in s} b(v) \in (0,1). \) Thus, if we define

\[
\psi(b) = \left( \frac{r_s(b)}{2} + \frac{(1 - \varepsilon) r_s(b)}{2 \varepsilon}, 1 - \varepsilon \right)
\]

then \( \psi \circ \phi = 1 \) and \( \phi \circ \psi = 1. \) Now, since \( \psi(b) \in |t| \times (0,1) \) whenever \( b \in |t| \), we can define a partial inverse \( \Psi : \beta^{-1}(U_s - |s|) \to \beta^{-1}(F^p_s) \times (0,1) \) to \( \Phi : \beta^{-1}(F^p_s) \times I \to \beta^{-1}|\hat{s}| \) by the following diagram.

\[
\begin{array}{ccc}
G \times (U_s - |s|) & \xrightarrow{1 \times \psi} & G \times F^p_s \times (0,1) \\
q \downarrow & & \downarrow q \times 1 \\
\beta^{-1}(U_s - |s|) & \xrightarrow{\Psi} & \beta^{-1}(F^p_s) \times (0,1)
\end{array}
\]

Composing the map \( \alpha \mapsto \Psi(\alpha) : \beta^{-1}(W_s - \langle s \rangle) \to \beta^{-1}(F^p_s \cap W_s) \times (0,1) \) with the projection into \( Q \hat{R}_s \) gives the desired partial inverse to \( \Theta \). \( \Box \)

2. \( \Theta \) is a closed map.

Because \( \hat{s} \) is finite, so that \(|\hat{s}|\) is compact, the closed subset \( F^p_s \subset |\hat{s}| \) is compact. Thus, \( \beta^{-1}(F^p_s) = q(G \times F^p_s) \) is compact since \( G \) is. It follows that \( \Phi : \beta^{-1}(F^p_s) \times I \to \beta^{-1}|\hat{s}| \) is a closed map since \( \beta^{-1}|\hat{s}| \subset L_K(G, G/K) \) is Hausdorff.\(^3\) Given any closed map \( f : X \to Y \) and a subset \( B \subset Y \), the map \( x \mapsto f(x) : f^{-1}(B) \to B \) is also closed since \( f(A \cap f^{-1}(B)) = f(A) \cap B \). Thus, the map \( (\alpha, t) \mapsto \Phi(\alpha, t) : \beta^{-1}(F^p_s \cap W_s) \times [0,1) \to \beta^{-1}(W_s) \) is a closed map. That \( \Theta \) is a closed map now follows from (4.5): The image by \( \Theta \) of a closed set is equal to the image by \( (\alpha, t) \mapsto \Phi(\alpha, t) : \beta^{-1}(F^p_s \cap W_s) \times [0,1) \to \beta^{-1}(W_s) \) of a closed set. \( \Box \)

Finally, as already noted, the composition \( \hat{R}_s : \beta^{-1}(W_s) \to \beta^{-1} \langle s \rangle \) with the projection \( \beta^{-1}(F^p_s \cap W_s) \times [0,1) \to \beta^{-1}(F^p_s \cap W_s) \) onto the first factor induces a map \( Q \hat{R}_s \to \beta^{-1} \langle s \rangle \) which is locally trivial with fibre the open cone on \( L_K(G_s, G_s/s) \). This map makes the

\(^3\) That \( L_K(G, G/K) \) is a Hausdorff space follows from Proposition 3.2 since \( G_s \subset G \) is closed for each \( s \in K \) : \( G/G_s \) and, thus, \( G/G_s \times |s| \) is Hausdorff for each \( s \in K \).
following diagram commute.

\[
\begin{align*}
\text{q}(g, b, t) & \xrightarrow{\Theta} \text{q}(\phi(g, b), t) \\
\text{Q}_{\tilde{R}_s} & \xrightarrow{\cong} \beta^{-1}(W_s) \\
\text{R}_s & \xrightarrow{\beta^{-1}(s)} \\
\text{q}(g, r_s(b)) &= \text{q}(g, r_s(\phi(g, b), t))
\end{align*}
\]

Thus, the map \( \alpha \mapsto R_s(\alpha) : \beta^{-1}(W_s) \to \beta^{-1}(s) \) is locally trivial with fibre the open cone on \( L_{s^c}(G_s, G_s/s^c) \). Let us explicitly write down a trivialization. If \( \xi = (\tilde{R}_s, \xi_2) : R_s^{-1}(V) \to V \times L_{s^c}(G_s, G_s/s^c) \) is a trivialization of \( \tilde{R}_s : \beta^{-1}(F_s \cap W_s) \to \beta^{-1}(s) \) then, as in Appendix C, the map \( [\alpha, t] \mapsto (\tilde{R}_s(\alpha), [\xi_2(\alpha), t]) \) is a trivialization of the induced map \( Q\tilde{R}_s \to Y \), where square brackets denote the appropriate equivalence classes.

\[
\begin{align*}
[\Theta(\alpha), t] & \mapsto (R_s(\alpha), [\xi_2(\alpha), t]) \\
\Theta^{-1}(R_s^{-1}(V)) & \to V \times CL_{s^c}(G_s, G_s/s^c)
\end{align*}
\]

If \( U \subset G/G_s \) is a neighbourhood on which we have a section \( \varsigma : U \to G \) of the coset projection \( G \to G/G_s \), the form of the map \( \xi_2 \) can be extracted from the following diagram, which summarizes (4.2), (4.3) and (4.4).

\[
\begin{align*}
q(g, b) & \quad \beta^{-1}(F_s \cap W_s) \\
\Gamma & \quad \beta^{-1}(s) \\
\text{q}(g, b - r_s(b)/2 + b_s/2, r_s(b)) & \quad \beta^{-1}(F_s \cap F_s^\circ) \times \langle s \rangle \\
\text{L}_{s^c}(G_s, G_s/s^c) \times \langle s \rangle & \quad G/G_s \times \langle s \rangle \\
\text{U} \times \langle s \rangle & \quad \text{U} \times \langle s \rangle \\
\pi_{s^c}^{-1}(U) \times \langle s \rangle & \quad \text{U} \times \text{L}_{s^c}(G_s, G_s/s^c) \times \langle s \rangle \\
(gG_s, b) & \quad (gG_s, g \varsigma^{-1}_s gG_s) \cdot b' \\
(gG_s, b) & \quad \text{gG}_s \cdot q_{s^c}(\varsigma(gG_s)^{-1}g, b', b')
\end{align*}
\]
Thus, if \( p \) denotes the homeomorphism \( q(g, b) \mapsto (gG_s, b) : \beta^{-1}(s) \to G/G_s \times (s) \), we define \( \xi \) above the open subset \( V = \rho^{-1}(U \times (s)) \subset \beta^{-1}(s) \).

\[
\xi(q(g, b)) = \left(q(g, r_s(b)), q_s(\varsigma(gG_s)^{-1}g, 2b - r_s(b))\right)
\]

3. Locally Euclidean Structure with a Compact Lie Group

When \( G \) is a compact group, we have shown that the neighbourhood \( \beta^{-1}(W_s) \) in \( L_K(G, G/K) \) either decomposes as a product or is the total space of a bundle over \( \beta^{-1}(s) \).

\[
s^c = 0 : \quad \beta^{-1}(W_s) = \beta^{-1}(s) \approx G/G_s \times (s)
\]

\[
s^c \neq 0 : \quad C \mathcal{L}_{s^c}(G_s, G_s/s^c) \xrightarrow{R_s} \beta^{-1}(W_s) \approx G/G_s \times (s)
\]

4.4 Proposition. Let \( G \) be a compact Lie group and let \( G/K = \{ G_s \mid s \in K \} \) be a lattice of closed subgroups indexed by a locally finite simplicial complex \( K \). Let \( s \in K \) and \( G_s/s^c = \{ G_s \cup t \mid t \in s^c \} \).

1. If \( s^c = 0 \) then each point in \( \beta^{-1}(s) \) has a Euclidean neighbourhood of dimension \( \dim G - \dim G_s + \dim s \).

2. If \( s^c \neq 0 \) then a point in \( \beta^{-1}(s) \) has a Euclidean neighbourhood only if \( \mathcal{L}_{s^c}(G_s, G_s/s^c) \) has the reduced homology of a sphere. Moreover, the dimension of such a neighbourhood must be \( \dim G - \dim G_s + \dim s + d + 1 \), where \( d \) is the dimension of the homology sphere.

To prove this we need the following fact.

4.5 Proposition. Let \( X \) be any space and let \( x_0 \in X \) be a closed point. Then

\[
H_k(X \times \mathbb{R}, X \times \mathbb{R} - (x_0, 0)) \cong H_{k-1}(X, X - x_0).
\]

Consequently, \( H_k(X \times \mathbb{R}^n, X \times \mathbb{R}^n - (x_0, 0)) \cong H_{k-n}(X, X - x_0) \).

Proof. The second statement follows inductively from the first. For the first statement, consider the long exact homology sequence of the triple \( (X \times \mathbb{R}, X \times \mathbb{R} - (x_0, 0), X \times \mathbb{R} - x_0 \times [0, \infty)) \).

\[
\cdots \to H_k(X \times \mathbb{R} - (x_0, 0), X \times \mathbb{R} - x_0 \times [0, \infty)) \to H_k(X \times \mathbb{R}, X \times \mathbb{R} - x_0 \times [0, \infty)) \to H_k(X \times \mathbb{R}, X \times \mathbb{R} - (x_0, 0)) \to \cdots
\]

\[\text{(4.6)}\]
Since \( X \times (-\infty, 0] - (x_0, 0) \subset X \times \mathbb{R} - x_0 \times [0, \infty) \) is closed in \( X \times \mathbb{R} - (x_0, 0) \) (a consequence of \( x \) being closed in \( X \)) and \( X \times \mathbb{R} - x_0 \times [0, \infty) \) is open, we can excise \( X \times (-\infty, 0] - (x_0, 0) \) from the pair \( (X \times \mathbb{R} - (x_0, 0), X \times \mathbb{R} - x_0 \times [0, \infty)) \) to get

\[
H_k(X \times \mathbb{R} - (x_0, 0), X \times \mathbb{R} - x_0 \times [0, \infty)) \cong H_k(X \times (0, \infty), X \times (0, \infty) - x_0 \times (0, \infty))
\]

\[
\cong H_k(X, X - x_0).
\]

Also, the projection \( X \times \mathbb{R} \to X \) onto the first factor is a homotopy equivalence \( (X \times \mathbb{R}, X \times \mathbb{R} - x_0 \times [0, \infty)) \to (X, X) \), the map \( x \mapsto (x, -1) \) being an inverse. Thus, (4.6) gives the desired isomorphism.

**Proof of 4.4.** 1. We have \( \beta^{-1}(s) = \beta^{-1}(W_s) \approx G/G_s \times \langle s \rangle \), which is locally Euclidean since \( \langle s \rangle \) is homeomorphic to a Euclidean space and the homogeneous space \( G/G_s \) is locally Euclidean.

2. As previously, let \( \rho : \beta^{-1}(s) \to G/G_s \times \langle s \rangle \) be the homeomorphism \( q(g, b) \mapsto (gG_s, b) \). If \( x \in \beta^{-1}(s) \), we've seen there is a neighbourhood \( U \subset G/G_s \) such that \( x \in \rho^{-1}(U \times \langle s \rangle) \) and such that \( \rho^{-1}(U \times \langle s \rangle) \) is a trivialization neighbourhood for \( \alpha \mapsto R_s(\alpha) : \beta^{-1}(W_s) \to \beta^{-1}(s) \).

\[
\begin{array}{c}
\Theta(q(g, b), t) \quad \mapsto \quad (gG_s, r_s(b), [\xi_2(q(g, b)), t])
\\
R_s^{-1}(\rho^{-1}(U \times \langle s \rangle)) \cap \beta^{-1}(W_s) \quad \cong \quad U \times \langle s \rangle \times \mathcal{L}_{s^c}(G_s, /G_s//s^c)
\\
\rho^{-1}(U \times \langle s \rangle) \quad \mapsto \quad U \times \langle s \rangle
\end{array}
\]

Since \( x \in \beta^{-1}(s) \), the third component of its image under the above trivialization is the vertex, \( * \), of the open cone. Now, since \( G/G_s \) is locally Euclidean, we can choose \( U \) homeomorphic to a Euclidean space. Suppose \( U \) and \( \langle s \rangle \) have dimensions \( n = \dim G - \dim G_s \) and \( m = \dim s \) respectively so that \( U \times \langle s \rangle \approx \mathbb{R}^{n+m} \). Moreover, we can choose the homeomorphism so that it maps the appropriate components of the image of \( x \) (under the above trivialization) to the origin in \( \mathbb{R}^{n+m} \).

\[
R_s^{-1}(\rho^{-1}(U \times \langle s \rangle)) \cap \beta^{-1}(W_s) \approx \mathbb{R}^{n+m} \times \mathcal{L}_{s^c}(G_s, /G_s//s^c)
\]

\[
(x) \quad \mapsto \quad (0, *)
\]

We now compute the local homology of \( \mathcal{L}_K(G, /G/K) \) at \( x \).

Since \( \mathcal{L}_K(G, /G/K) \) is a Hausdorff space, \( H_k(\mathcal{L}_K(G, /G/K), \mathcal{L}_K(G, /G/K) - x) \cong H_k(V, V - x) \) for any neighbourhood \( V \) of \( x \). If we choose \( V = R_s^{-1}(\rho^{-1}(U \times \langle s \rangle)) \cap \beta^{-1}(W_s) \) then by (4.7) and Proposition 4.5 we have

\[
H_k(\mathcal{L}_K(G, /G/K), \mathcal{L}_K(G, /G/K) - x) \cong H_{k-n-m}(\mathcal{L}_{s^c}(G_s, /G_s//s^c), \mathcal{L}_{s^c}(G_s, /G_s//s^c) - *)
\]

\[
\cong \tilde{H}_{k-n-m}(\mathcal{L}_{s^c}(G_s, /G_s//s^c),
\]

66
the second isomorphism following from the long exact homology sequence of the pair since the open cone is contractible and \( C \mathcal{L}_{x^e}(G_s, /G_s/x^e) - \approx \mathcal{L}_{x^e}(G_s, /G_s/x^e) \times (0, 1) \). On the other hand, if we choose the neighbourhood \( V \) to be homeomorphic to an open ball in some Euclidean space then we have \( H_k(V, V - x) \cong H_k(B^N, B^N - 0) \cong \tilde{H}_{k-1}S^{N-1} \), where \( N \) is the dimension of the Euclidean space. Thus,

\[
\tilde{H}_{k-n-m-1} \mathcal{L}_{x^e}(G_s, /G_s/x^e) \cong \tilde{H}_{k-1}S^{N-1}
\]

so that \( \mathcal{L}_{x^e}(G_s, /G_s/x^e) \) is a reduced homology sphere of dimension \( d = N - n - m - 1 = N - \dim G + \dim G_s - \dim s - 1 \), giving \( N = \dim G - \dim G_s + \dim s + d + 1 \) as required. 

It is unclear whether or not the homology spheres of Proposition 4.4 are necessarily homeomorphic to spheres. In the case of the spheres \( \mathcal{L}_{\delta_{n-2}}(O(n), /O(n)/\delta_{n-2}) \) this is indeed the case as we will now proceed to show.

C. More Spheres

Let \( K \) and \( L \) be any two simplicial complexes, not necessarily locally finite. Without loss of generality, the vertex sets of these two are disjoint. The join of \( K \) and \( L \) is the simplicial complex \( K \ast L \) defined as follows.

\[
K \ast L = K \cup L \cup \{ s \cup t \mid s \in K, t \in L \}
\]

Thus, \( K \ast L \) consists of all nonempty subsets \( u \subseteq \text{Vert}(K) \cup \text{Vert}(L) \) such that \( u \cap \text{Vert}(K) \) and \( u \cap \text{Vert}(L) \) are either empty or simplices of \( K \) and \( L \) respectively. Notice that the union \( (4.8) \) is disjoint: A simplex of \( u \in K \ast L \) is either a simplex of \( K \), a simplex of \( L \) or can be written as \( s \cup t \) where \( s = u \cap \text{Vert}(K) \in K \) and \( t = u \cap \text{Vert}(L) \in L \).

Suppose now that we have two topological groups \( G \) and \( H \) together with lattices \( /G_K = \{ G_s \subset G \mid s \in K \} \) and \( /H_L = \{ H_t \subset H \mid t \in L \} \) indexed by \( K \) and \( L \) respectively. Define the join of \( /G_K \) and \( /H_L \) to be the lattice \( /G_K \ast /H_L = \{ A_u \subset G \times H \mid u \in K \ast L \} \), where

\[
A_u = \begin{cases} 
G_u \times H & \text{if } u \in K, \\
G \times H_u & \text{if } u \in L, \\
G_s \times H_t & \text{if } s = u \cap \text{Vert}(K) \neq \emptyset \text{ and } t = u \cap \text{Vert}(L) \neq \emptyset.
\end{cases}
\]

We can embed the realizations \(|K|\) and \(|L|\) as subsets of \(|K \ast L|\) in the obvious way, as subspaces whose member functions are identically zero on the vertices of \( L \) and \( K \) respectively. With these embeddings in mind, we define a map \( \theta : |K| \times I \times |L| \to |K \ast L| \) by \( \theta(b, t, c) = tb + (1 - t)c \). This map is indeed a well-defined member of \(|K \ast L|\) since \( \theta(b, t, c)^{-1}(0, 1) \in K \ast L \) for all \( t \in I \), the union of a simplex of \( K \) with a simplex of \( L \) being a simplex of \( K \ast L \). Notice that \( \theta \) has the following properties.

\[
\theta(|K| \times \{0\} \times |L|) = |L| \quad (4.10)
\]
\[
\theta(|K| \times \{1\} \times |L|) = |K| \quad (4.11)
\]
\[
\theta(|K| \times (0, 1) \times |L|) = |K \ast L| - (|K| \cup |L|) \quad (4.12)
\]
(4.10) and (4.11) are clear. For (4.12), if \( \alpha \in |K \ast L| - (|K| \cup |L|) \) then we have \( \alpha = tb + (1-t)c \) where \( t = \sum_{v \in K} \alpha(v) \in (0,1), \delta = 1/t \cdot \sum_{v \in K} \alpha(v)v \in |K| \) and \( c = 1/(1-t) \cdot \sum_{v \in L} \alpha(v)v \in |L| \). This shows that \( \theta \) is injective as well as surjective on \( |K| \times (0,1) \times |L| \).

4.6 Proposition. Let \( K \) and \( L \) be finite simplicial complexes. Let \( G \) and \( H \) be two compact groups with lattices \( /G/K = \{ G_s \subset G \mid s \in K \} \) and \( /H/L = \{ H_t \subset H \mid t \in L \} \) of closed subgroups. Then the map

\[
(g, b, t, h, c) \mapsto (g, h, \theta(b, t, c)) : G \times |K| \times I \times H \times |L| \to G \times H \times |K \ast L|
\]

induces a homeomorphism \( \mathcal{L}_K(G, /G/K) \ast \mathcal{L}_L(H, /H/L) \to \mathcal{L}_{K \ast L}(G \times H, /G/K \ast /H/L) \).

**Proof.** Let \( q_K, q_L \) and \( q \) denote the identifications \( G \times |K| \to \mathcal{L}_K(G, /G/K), H \times |L| \to \mathcal{L}_L(H, /H/L) \) and \( G \times H \times |K \ast L| \to \mathcal{L}_{K \ast L}(G \times H, /G/K \ast /H/L) \) respectively.

Since \( \mathcal{L}_K(G, /G/K) \ast \mathcal{L}_L(H, /H/L) \) is compact and \( \mathcal{L}_{K \ast L}(G \times H, /G/K \ast /H/L) \) is Hausdorff, we need only demonstrate that (4.13) induces a continuous bijection.

\[
\begin{align*}
\mathcal{L}_K(G, /G/K) \ast \mathcal{L}_L(H, /H/L) &\to \mathcal{L}_{K \ast L}(G \times H, /G/K \ast /H/L) \\
(g, b, t, h, c) &\mapsto (g, h, \theta(b, t, c)) \\
G \times |K| \times I \times H \times |L| &\to G \times H \times |K \ast L|
\end{align*}
\]

Since \( G \times |K| \times I \times H \times |L| \) is compact and \( \mathcal{L}_K(G, /G/K) \ast \mathcal{L}_L(H, /H/L) \) is Hausdorff (being a join of Hausdorff spaces), the composition of the product map \( q_K \times q_L \) with the quotient map into the join are closed maps. Thus, it is an identification and we need only show that two tuples \( (g, b, t, h, c) \) and \( (g', b', t', h', c') \) in \( G \times |K| \times I \times H \times |L| \) represent the same point in \( \mathcal{L}_K(G, /G/K) \ast \mathcal{L}_L(H, /H/L) \) if and only if \( (g, h, \theta(b, t, c)) \) and \( (g', h', \theta(b', t', c')) \) represent the same point in \( \mathcal{L}_{K \ast L}(G \times H, /G/K \ast /H/L) \).

Writing \( /G/K \ast /H/L = \{ A_u \mid u \in K \ast L \} \) as in (4.9), \( (g, h, \theta(b, t, c)) \) and \( (g', h', \theta(b', t', c')) \) represent the same point in \( \mathcal{L}_{K \ast L}(G \times H, /G/K \ast /H/L) \) if and only if the following holds.

1. \( \theta(b, t, c) = \theta(b', t', c') \equiv \alpha \) and \( (g'g^{-1}, h'h^{-1}) \in A_{\alpha} \).

By (4.10), (4.11) and (4.12), if \( \theta(b, t, c) = \theta(b', t', c') \) then either \( t = t' = 0 \), \( t = t' = 1 \) or \( t, t' \in (0, 1) \). Thus, by the definition of \( \theta \) and since, as previously remarked, \( \theta \) is injective on \( |K| \times (0,1) \times |L| \), \( \theta(b, t, c) = \theta(b', t', c') \) if and only if one of the following hold.

2. \( t = t' = 0 \) and \( c = c' \).
(3') \( t = t' = 1 \) and \( b = b' \).

(4') \( t = t' \in (0,1), b = b' \) and \( c = c' \).

Therefore, by the definition of \( A_{cc} \) and since \( c(\theta(b,t,c)) = cb \cup cc \) whenever \( t \in (0,1) \), (1) holds if and only if one of the following hold.

(2) \( t = t' = 0, c = c' \) and \( h'h^{-1} \in H_{cc} \).

(3) \( t = t' = 1 \) and \( b = b' \) and \( g'g^{-1} \in G_{cb} \).

(4) \( t = t' \in (0,1), b = b', c = c', g'g^{-1} \in G_{cb} \) and \( h'h^{-1} \in H_{cc} \).

These are exactly the conditions under which \((g,b,t,h,c)\) and \((g',b',t',h',c')\) represent the same point in \( L_K(G,G/K) \times L_L(H,H/L) \).

We now apply Proposition 4.6 to the spaces \( L_\delta(O_\delta(n),/O_\delta(n)/\delta) \), \( s \in \delta_{n-2} \).

4.7 Proposition. Let \( n \geq 2 \) and let \( s \in \delta_{n-2} \). Then \( L_\delta(O_\delta(n),/O_\delta(n)/\delta) \) is homeomorphic to a sphere of dimension \( \dim O_\delta(n) - n \cdot \dim O(1) + \dim s \).

Proof. It will be convenient to extend our notation slightly. If \( k \) is an integer and lying between 0 and \( n-2 \) inclusive and if \( \sigma : \{0,\ldots,k\} \to \{0,\ldots,n-2\} \) is an injective function with which we embed \( \delta_k \) in \( \delta_{n-2} \), let \( O_\sigma(n) = O_{\infty \sigma}(n) \) and \( /O_\sigma(n)/\delta_k = \{ O_\sigma,t(n) \mid t \in \delta_k \} \), where \( O_\sigma,t(n) = O_\sigma(n) \cap \bigcap_{i \in t} O(n,\sigma(i)) \). We will show that \( L_\delta(O_\delta(n),/O_\delta(n)/\delta_k) \) is homeomorphic to a sphere of dimension \( \dim O_\delta(n) - n \cdot \dim O(1) + k \) for each \( n \geq 2 \) and \( 0 \leq k \leq n-2 \).

We can easily construct a permutation \( \tau \) of \( \{0,\ldots,k\} \) for which the composition \( \sigma \circ \tau \) is order-preserving. This permutation is then a simplicial isomorphism \( \tau : \delta_k \to \delta_k \). Since \( O_{\sigma \circ \tau}(n) = O_\sigma(n) \) we then have a homeomorphism

\[
1 \times |\tau| : O_{\sigma \circ \tau}(n) \times |\delta_k| \to O_\sigma(n) \times |\tau|,
\]

and this map induces a homeomorphism \( L_\delta(O_{\sigma \circ \tau}(n),/O_{\sigma \circ \tau}(n)/\delta_k) \to L_\delta(O_\sigma(n),/O_\sigma(n)/\delta_k) \).

We can therefore assume that \( \sigma \) is order-preserving. Thus, \( O(n,\sigma(0)),\ldots,O(n,\sigma(k)) \) is just a subsequence of \( O(n,0),\ldots,O(n,n-2) \).

The proof is now by induction on \( n \). If \( n = 2 \) then \( k = 0 \) and \( L_\delta(O_\delta(n),/O_\delta(n)/\delta_k) = \). \( O(2)/(O(1) \times O(1)) \), which is either \( O(2)/(O(1) \times O(1)) = RP^1 \approx S^1 \), \( U(2)/(U(1) \times U(1)) = CP^1 \approx S^2 \) or \( Sp(2)/(Sp(1) \times Sp(1)) = HP^1 \approx S^4 \). In the first case, \( 4 \dim O_\sigma(n) - n \cdot \dim O(1) + k = \dim O(2) = 1 \). In the second case, \( \dim O_\sigma(n) - n \cdot \dim O(1) + k = \dim U(2) - 2 = 2 \). In the third case, \( \dim O_\sigma(n) - n \cdot \dim O(1) + k = \dim Sp(2) - 6 = 4 \). Thus, the dimensions are as claimed.

\[\text{4 Recall that } \dim O(n) = n(n-1)/2, \dim U(n) = n^2 \text{ and } \dim Sp(n) = n(2n+1).\]
Assume now that \( n > 2 \) and that \( \mathcal{L}_{\delta_k}(O_{\sigma}(n'), /O_{\sigma}(n')/\delta_k) \) is homeomorphic to a sphere of dimension \( \dim O_{\sigma}(n') - n \cdot \dim O(1) + k \) for each \( n' < n, \ 0 \leq k \leq n' - 2 \) and injective \( \sigma : \{0, \ldots, k\} \rightarrow \{0, \ldots, n' - 2\} \).

If \( k = n - 2 \) then \( O_{\sigma}(n) = O(n), \) and \( \mathcal{L}_{\delta_{n-2}}(O(n), /O(n)/\delta_{n-2}) \) is one of the spheres constructed in chapter 2. Since \( \dim O(n) - n \cdot \dim O(1) + k = n(n - 1)/2 - n \cdot 0 + n - 2 = n(n + 1)/2 - 2, \) \( \dim U(n) - n \cdot \dim U(1) + k = n^2 - n \cdot 1 + n - 2 = n^2 - 2 \) and \( \dim S_p(n) - n \cdot \dim S_p(1) + k = n(2n + 1) - n \cdot 3 + n - 2 = n(2n - 1) - 2, \) the formula for the dimension of the sphere is again correct. (See page 33.)

If \( k < n - 2, \) write \( \{0, \ldots, n - 2\} \setminus \text{im} \sigma = \{i_1, \ldots, i_{n-2-k}\}, \) where \( i_1 < \cdots < i_{n-2-k} \). Then

\[
O_{\sigma}(n) = O(i_1 + 1) \times O(i_2 - i_1) \times \cdots \times O(i_{n-2-k} - i_{n-2-k-1}) \times O(n - i_{n-2-k} - 1),
\]

(4.14)

with the obvious interpretation if \( n - 2 - k = 1 \) or 2. We have three cases to consider.

1. \( i_1 = 0. \)

We then have \( O_{\sigma}(n) = O(1) \times O_{\sigma'}(n - 1), \) where \( \sigma' : \{0, \ldots, k\} \rightarrow \{0, \ldots, n - 3\} \) is defined by \( \sigma'(j) = \sigma(j) - 1. \) Since each element of the lattice \( /O_{\sigma}(n)/\delta_k \) is of the form \( O(1) \times H, \) the product of the projection \( O(1) \times O_{\sigma'}(n - 1) \rightarrow O_{\sigma'}(n - 1) \) with the identity map on \( \delta_k \) induces a homeomorphism between \( \mathcal{L}_{\delta_k}(O_{\sigma}(n), /O_{\sigma}(n)/\delta_k) \) and \( \mathcal{L}_{\delta_k}(O_{\sigma'}(n - 1), /O_{\sigma'}(n - 1)/\delta_k) \). By the inductive hypothesis, the latter is a sphere of dimension \( \dim O_{\sigma'}(n - 1) - (n - 1) \cdot \dim O(1) + k = \dim O_{\sigma}(n) - n \cdot \dim O(1) + k. \)

2. \( i_{n-k-2} = n - 2. \)

In this case \( O_{\sigma}(n) = O_{\sigma'}(n - 1) \times O(1), \) where \( \sigma' : \{0, \ldots, k\} \rightarrow \{0, \ldots, n - 3\} \) is defined by \( \sigma'(j) = \sigma(j). \) As in the previous case, we can use projection \( O_{\sigma}(n) \rightarrow O_{\sigma'}(n - 1) \) to define a homeomorphism between \( \mathcal{L}_{\delta_k}(O_{\sigma}(n), /O_{\sigma}(n)/\delta_k) \) and \( \mathcal{L}_{\delta_k}(O_{\sigma'}(n - 1), /O_{\sigma'}(n - 1)/\delta_k). \) This is a again sphere of the claimed dimension by the inductive hypothesis.

3. \( i_1 > 0 \) and \( i_{n-2-k} < n - 2. \)

Define two simplices of \( \delta_k \) by \( s_1 = \sigma^{-1}\{0, \ldots, i_1\} \) and \( s_2 = \sigma^{-1}\{i_1, \ldots, n - 2\}. \) These are indeed nonempty sets since \( \{0, n-2\} \subset \text{im} \sigma. \) Moreover, since \( i_1 \notin \text{im} \sigma, \) \( s_1 \) and \( s_2 \) are disjoint. Since their union is \( \text{Vert}(\delta_k) = \{0, \ldots, k\}, \) it follows that \( \delta_1 \star \delta_2 = \delta_k. \) Note that \( s_1 \) is an \((i_1 - 1)\)-simplex and \( s_2 \) is a \((k - i_1)\)-simplex.

Define \( \sigma_1 : \{0, \ldots, i_1 - 1\} \rightarrow \{0, \ldots, i_1 + 1\} \) and \( \sigma_2 : \{0, \ldots, k - i_1\} \rightarrow \{0, \ldots, n - i_1 - 1\} \) by \( \sigma_1(j) = \sigma(j) \) and \( \sigma_2(j) = \sigma(i_1 + j) - i_1 - 1. \) Then

\[
O_{\sigma}(n) = O_{\sigma_1}(i_1 + 1) \times O_{\sigma_2}(n - i_1 - 1)
= O(i_1 + 1) \times O_{\sigma_2}(n - i_1 - 1)
\]

and, if we let \( O_{\sigma_1, \emptyset}(i_1 + 1) = O_{\sigma_1, \emptyset}(i_1 + 1) \) and \( O_{n-i_1-1}(\sigma_2, \emptyset) = O_{\sigma_2}(n - i_1 - 1), \)

\[
O_{\sigma, \emptyset}(n) = O_{\sigma_2, \emptyset}(i_1 + 1) \times O_{\sigma_2, \emptyset}(n - i_1 - 1)
\]

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for each \( t \in \delta_k \). Thus, \( O_\sigma(n)/\delta_k = O_{\sigma_1}(i_1 + 1)/\delta_{i_1 - 1} \ast O_{\sigma_2}(n - i_1 - 1)/\delta_{k - i_1} \), and we get

\[
L_{\delta_k}(O_\sigma(n), O_\sigma(n)/\delta_k) = L_{\delta_1 \ast \delta_2}(O_{\sigma_1}(i_1 + 1) \times O_{\sigma_2}(n - i_1 - 1),
\]

\[
O_{\sigma_1}(i_1 + 1)/\delta_{i_1 - 1} \ast O_{\sigma_2}(n - i_1 - 1)/\delta_{k - i_1}
\]

\[
\approx L_{\delta_1}(O_{\sigma_1}(i_1 + 1), O_{\sigma_1}(i_1 + 1)/\delta_{i_1 - 1})
\]

\[
\ast L_{\delta_2}(O_{\sigma_2}(n - i_1 - 1), O_{\sigma_2}(n - i_1 - 1)/\delta_{k - i_1})
\]

by Proposition 4.6. By our inductive hypothesis, the two factors of the above join are spheres of dimension \( \dim O_{\sigma_1}(i_1 + 1) - (i_1 + 1) \cdot \dim O(1) + \dim s_1 \) and \( \dim O_{\sigma_2}(n - i_1 - 1) - (n - i_1 - 1) \cdot \dim O(1) + \dim s_2 \) respectively. Thus, since \( S^{m_1} \ast S^{m_2} \approx S^{m_1 + m_2 + 1} \), \( \dim O_\sigma(n) = \dim O_{\sigma_1}(i_1 + 1) + \dim O_{\sigma_2}(n - i_1 - 1) \) and \( \dim s_1 + \dim s_2 = k - 1 \), their join is a sphere of dimension

\[
\left( \dim O_{\sigma_1}(i_1 + 1) - (i_1 + 1) \cdot \dim O(1) + \dim s_1 \right) +
\left( \dim O_{\sigma_2}(n - i_1 - 1) - (n - i_1 - 1) \cdot \dim O(1) + \dim s_2 \right) + 1
\]

\[
= \dim O_\sigma(n) - n \cdot \dim O(1) + k.
\]

Let us apply 4.7 to the homology spheres in the second statement of Proposition 4.4 in the case \( K = u \) for some \( u \in \delta_{n-2} \), \( G = O_u(n) \) and \( /G/\mathring{u} = /O_u(n)/\mathring{u} \). If \( s \in \mathring{u} \) is a proper face of \( u \) then

\[
G_s = O_u(n) \cap \bigcap_{i \in s} O(n,i) = O_{u,s}(n)
\]

and, letting \( w = u - s \),

\[
/G_s/\mathring{s} = \{ G_{st} \mid t \in s^c \}
\]

\[
= \{ O_{u,s \cup i}(n) \mid t \in s^c \}
\]

\[
= \{ O_{w,t}(n) \mid t \in \mathring{w} \}
\]

\[
= /O_w(n)/\mathring{w}
\]

so that \( L_{s^c}(G, /G_s/\mathring{s}) = L_{\mathring{w}}(O(n), /O_w(n)/\mathring{w}) \) and \( L_{s^c}(G_s, /G_s/\mathring{s}) = L_{\mathring{w}}(O_w(n), /O_w(n)/\mathring{w}) \). Thus, in this case the homology spheres \( L_{s^c}(G_s, /G_s/\mathring{s}) \) are in fact homeomorphic to spheres.

**Counting Dimensions**

We can enumerate exactly which spheres are constructed in Proposition 4.7. Let \( D_R \), \( D_C \) and \( D_H \) be the sets of dimensions of the spheres constructed in the real, complex and quaternionic cases respectively. We then have

\[
D_R = \{ \dim O_s(n) + \dim s \mid s \in \delta_{n-2}, n \geq 2 \},
\]

\[
D_C = \{ \dim U_s(n) - n + \dim s \mid s \in \delta_{n-2}, n \geq 2 \},
\]

and

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\[ D_H = \{ \dim S_p(n) - 3n + \dim s \mid s \in \delta_{n-2}, \ n \geq 2 \} \]

since \( \dim O(1) = 0 \), \( \dim U(1) = 1 \) and \( \dim S^p(1) = 3 \). Let us also consider those dimensions which are constructed over an \( N \)-simplex for some \( N \in \mathbb{N}_0 \).

\[ D_R(N) = \{ \dim O_s(n) + N \mid s \in \delta_{n-2}, \ n \geq 2, \ \dim s = N \} \]
\[ D_C(N) = \{ \dim U_s(n) - n + N \mid s \in \delta_{n-2}, \ n \geq 2, \ \dim s = N \} \]
\[ D_H(N) = \{ \dim S_p(n) - 3n + N \mid s \in \delta_{n-2}, \ n \geq 2, \ \dim s = N \} \]

Of course, \( D_K = \bigcup_{N=0}^{\infty} D_K(N) \) for \( K = R, C \) and \( H \).

4.8 Proposition. \( D_R(N), D_C(N) \) and \( D_H(N) \) are finite sets for each \( N \in \mathbb{N}_0 \), and there are the following equalities.

\[
\begin{align*}
\min D_R(N) &= 2N + 1 & \max D_R(N) &= (N + 2)(N + 1)/2 + N \\
\min D_C(N) &= 3N + 2 & \max D_C(N) &= (N + 2)(N + 1) + N \\
\min D_H(N) &= 5N + 4 & \max D_H(N) &= 2(N + 2)(N + 1) + N
\end{align*}
\]

4.9 Proposition. There are the following equalities.

1. \( D_R = N - \{2\} \).
2. \( D_C = N - \{1, 3, 4, 6, 9, 12\} \).
3. \( D_H = N - \{1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 20, 21, 22, 25, 30, 35\} \).

To prove 4.8 and 4.9, note that for each integer \( 2 \leq k \leq n \), there is the following one to one correspondence between the \( (k - 2) \)-simplices of \( \delta_{n-2} \) and the collection of \( k \)-tuples of positive integers summing to \( n \).

\[
\begin{align*}
\{i_1, \ldots, i_{k-1}\} &\xrightarrow{i_1 < \cdots < i_{k-1}} (i_1 + 1, i_2 - i_1, \ldots, i_{k-1} - i_{k-2}, n - i_{k-1} - 1) \\
\delta_{n-2} &\leftrightarrow \{(n_1, \ldots, n_k) \in \mathbb{N}^k \mid k \geq 2, \ n_1 + \cdots + n_k = n \} \\
\{n_1 - 1, \ldots, n_1 + \cdots + n_{k-1} - 1\} &\leftrightarrow (n_1, \ldots, n_k)
\end{align*}
\]

If we let the empty set correspond to the "1-tuple" \( n \) we then have a one to one correspondence between \( \delta_{n-2} \cup \{\emptyset\} \) and the set of tuples as above, but with \( k \geq 1 \) rather than \( k \geq 2 \). Now, if \( s \in \delta_{n-2} \) then the complement of \( s \) in the vertex set of \( \delta_{n-2} \) is either empty or a proper face in \( \delta_{n-2} \), and this corresponds to some \( k \)-tuple \( (n_1, \ldots, n_k) \neq (1, \ldots, 1) \equiv \bar{1} \). Moreover,

\[ O_s(n) = O(n_1) \times \cdots \times O(n_k) \]
(with the obvious interpretation if $k = 1$) and $\dim s = n - k - 1 = \sum_{j=1}^k n_j - k - 1$.\footnote{If $t$ is the complement of $s$ in the vertex set of $\delta_{n-2}$ then $\dim t = k - 2$. Since $\dim s + \dim t = \dim \delta_{n-2} - 1$, $\dim s = (n - 2) - 1 - (k - 2) = n - k - 1.$}

**Proof of 4.8.** Since $\dim O(\ell) = \ell(\ell - 1)/2$, $\dim U(\ell) = \ell^2$ and $\dim Sp(\ell) = \ell(2\ell + 1)$, and because of the aforementioned correspondence, we have the following equalities.

\[
D_R(N) = \{ \dim O_s(n) + N \mid s \in \delta_{n-2}, n \geq 2, \dim s = N \}
\]
\[
= \left\{ \sum_{j=1}^k n_j(n_j - 1)/2 + N \right\}
\]
\[
k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}^k - \{1\}, \sum_{j=1}^k n_j - k - 1 = N \} \quad (4.15)
\]

\[
D_C(N) = \{ \dim U_s(n) - n + N \mid s \in \delta_{n-2}, n \geq 2, \dim s = N \}
\]
\[
= \left\{ \sum_{j=1}^k (n_j^2 - n_j) + N \right\}
\]
\[
k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}^k - \{1\}, \sum_{j=1}^k n_j - k - 1 = N \} \quad (4.16)
\]

\[
D_H(N) = \{ \dim Sp_s(n) - 3n + N \mid s \in \delta_{n-2}, n \geq 2, \dim s = N \}
\]
\[
= \left\{ \sum_{j=1}^k (n_j(2n_j + 1) - 3n_j) + N \right\}
\]
\[
k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}^k - \{1\}, \sum_{j=1}^k n_j - k - 1 = N \} \quad (4.17)
\]

Notice that any $n_j = 1$ does not contribute to the sums in (4.15), (4.16) and (4.17). Since the constraint $\sum_{j=1}^k n_j - k - 1 = N$ is still satisfied if we omit these (the value of $k$ decreasing by 1 for each $n_j$ omitted), it suffices to consider sums with each $n_j \in \mathbb{N}_2$, where $\mathbb{N}_2 = \{ n \in \mathbb{N} \mid n \geq 2 \}$.

\[
D_R(N) = \left\{ \sum_{j=1}^k n_j(n_j - 1)/2 + N \right\}
\]
\[
k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k, \sum_{j=1}^k n_j - k - 1 = N \} \quad (4.18)
\]
\[D_C(N) = \left\{ \sum_{j=1}^{k} n_j(n_j - 1) + N \right\}
\quad \text{where } k \in \mathbb{N}, \ (n_1, \ldots, n_k) \in \mathbb{N}_2^k, \ \sum_{j=1}^{k} n_j - k - 1 = N \right\} \quad (4.19)\]

\[D_H(N) = \left\{ \sum_{j=1}^{k} 2n_j(n_j - 1) + N \right\}
\quad \text{where } k \in \mathbb{N}, \ (n_1, \ldots, n_k) \in \mathbb{N}_2^k, \ \sum_{j=1}^{k} n_j - k - 1 = N \right\} \quad (4.20)\]

We need now only find the minimum and maximum integers occurring in (4.18), (4.19) and (4.20).

Let us write \(K\) for any of \(R, C\) or \(H\) and suppose \(k \in \mathbb{N}, \ (n_1, \ldots, n_k) \in \mathbb{N}_2^k\) and \(\sum_{j=1}^{k} n_j - k - 1 = N\) so that \((n_1, \ldots, n_k)\) determines an element of \(D_K(N)\). If \(i \in \{1, \ldots, k\}\) and \(n_i \geq 3\), let \(a \in \mathbb{N}_2\) and \(b \in \mathbb{N}\) be any two elements whose sum is \(n_i\). Then the \((k + 1)\)-tuple \((m_1, \ldots, m_{k+1}) = (n_1, \ldots, n_{i-1}, a, b+1, n_{i+1}, \ldots, n_k)\) (with the obvious interpretation if \(i = 1\) or \(k\)) determines another element of \(D_K\). Moreover, starting with the 1-tuple \(n_1 = N + 2\), every \((n_1, \ldots, n_k) \in \mathbb{N}_2^k\) satisfying \(\sum_{j=1}^{k} n_j - k - 1 = N\) can be obtained in this manner, and from such a tuple the \((N + 1)\)-tuple \((2, \ldots, 2)\) can be obtained. Thus, since

\[
\sum_{j=1}^{k} n_j(n_j - 1) - \sum_{j=1}^{k+1} m_j(m_j - 1) = n_i(n_i - 1) - [a(a - 1) + (b + 1)b]
= (a + b)(a + b - 1) - (a^2 - a + b^2 + b)
= (a^2 + 2ab + b^2 - a - b) - a^2 + a - b^2 - b
= 2ab - 2b
= 2b(a - 1)
> 0
\]

for any \(a \in \mathbb{N}_2\) and \(b \in \mathbb{N}\), the minimum values of the sums in (4.18), (4.19) and (4.20) are attained when \(k = N + 1\) and \(n_1 = \cdots = n_{N+1} = 2\), and the maximum values are attained when \(k = 1\) and \(n_1 = N + 2\). Thus, (4.18), (4.19) and (4.20) give the following minimum and maximum values.

\[
\begin{align*}
\min D_R(N) &= (N + 1) + N = 2N + 1 \\
\max D_R(N) &= (N + 2)(N + 1)/2 + N \\
\min D_C(N) &= 2(N + 1) + N = 3N + 2 \\
\max D_C(N) &= (N + 2)(N + 1) + N \\
\min D_H(N) &= 4(N + 1) + N = 5N + 4 \\
\max D_H(N) &= 2(N + 2)(N + 1) + N
\end{align*}
\]
Proof of 4.9. Since $\mathcal{D}_K = \bigcup_{N=0}^{\infty} \mathcal{D}_K(N)$ for $K = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, the following equalities are a direct consequence of (4.18), (4.19) and (4.20).

$$\mathcal{D}_R = \left\{ \sum_{j=1}^{k} n_j(n_j - 1)/2 + \sum_{j=1}^{k} n_j - k - 1 \bigg| k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k \right\}$$

$$= \left\{ \sum_{j=1}^{k} n_j(n_j + 1)/2 - k - 1 \bigg| k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k \right\} \quad (4.21)$$

$$\mathcal{D}_C = \left\{ \sum_{j=1}^{k} n_j(n_j - 1) + \sum_{j=1}^{k} n_j - k - 1 \bigg| k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k \right\}$$

$$= \left\{ \sum_{j=1}^{k} n_j^2 - k - 1 \bigg| k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k \right\} \quad (4.22)$$

$$\mathcal{D}_H = \left\{ \sum_{j=1}^{k} 2n_j(n_j - 1) + \sum_{j=1}^{k} n_j - k - 1 \bigg| k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k \right\}$$

$$= \left\{ \sum_{j=1}^{k} n_j(2n_j - 1) - k - 1 \bigg| k \in \mathbb{N}, (n_1, \ldots, n_k) \in \mathbb{N}_2^k \right\} \quad (4.23)$$

We now consider our three cases separately.

1. $\mathcal{D}_R = \mathbb{N} - \{2\}$.

   Taking $k = 1$ and $n_1 = 2$, we get $1 \in \mathcal{D}_R$. Taking $k = 2$ and $n_1 = n_2 = 2$, we get $4 \in \mathcal{D}_R$. Now, if $a \in \mathcal{D}_R$ write $a = \sum_{j=1}^{k} n_j(n_j)/2 - k - 1$, where $k \in \mathbb{N}$ and $(n_1, \ldots, n_k) \in \mathbb{N}_2^k$. Setting $n_{k+1} = 2$, we then have

$$\sum_{j=1}^{k+1} n_j(n_j + 1)/2 - (k + 1) - 1 = \sum_{j=1}^{k} n_j(n_j + 1)/2 + 3 - (k + 1) - 1$$

$$= \sum_{j=1}^{k} n_j(n_j + 1)/2 - k - 1 + 2 = a + 2$$

so that $a + 2 \in \mathcal{D}_R$. Thus, $\{1, 3, 5, \ldots\} \cup \{4, 6, 8, \ldots\} = \mathbb{N} - \{2\} \subset \mathcal{D}_R$.

   To see that $2 \notin \mathcal{D}_R$, if $(n_1, \ldots, n_k) \in \mathbb{N}_2^k$ then $N \equiv \sum_{j=1}^{k} n_j(n_j + 1)/2 - k - 1 \geq 2k - 1$. Thus, $N = 2$ only if $k = 1$. In this case $N = n_1(n_1 + 1)/2 - 2$, and this sum is equal to $1$ if $n_1 = 2$ and greater than or equal to $4$ if $n_1 \geq 3$. □

2. $\mathcal{D}_C = \mathbb{N} - \{1, 3, 4, 6, 9, 12\}$.

   Taking $k = 1$ and $n_1 = 2$, we get $2 \in \mathcal{D}_C$. Taking $k = 1$ and $n_1 = 3$, we get $7 \in \mathcal{D}_C$. Taking $k = 2$ and $n_1 = n_2 = 3$, we get $15 \in \mathcal{D}_C$. Now, if $a \in \mathcal{D}_C$ write $a = \sum_{j=1}^{k} n_j^2 - k - 1$, where
\( k \in \mathbb{N} \) and \((n_1, \ldots, n_k) \in \mathbb{N}_2^k \). Setting \( n_{k+1} = 2 \), we then have
\[
\sum_{j=1}^{k+1} n_j^2 - (k + 1) - 1 = \sum_{j=1}^{k} n_j^2 + 4 - (k + 1) - 1
\]
\[
= \sum_{j=1}^{k} n_j^2 - k - 1 + 3 = a + 3
\]
so that \( a + 3 \in \mathcal{D}_G \). Thus, \( \{2, 5, 8, \ldots\} \cup \{7, 10, 13, \ldots\} \cup \{15, 18, 21, \ldots\} = \mathbb{N} - \{1, 3, 4, 6, 9, 12\} \subseteq \mathcal{D}_G \).

To see that equality holds, if \((n_1, \ldots, n_k) \in \mathbb{N}_2^k \) then \( N \equiv \sum_{j=1}^{k} n_j^2 - k - 1 = 4k - k - 1 \geq 3k - 1 \). Thus, \( N \leq 12 \) only if \( k \leq 4 \). Now, we can clearly assume \( n_1 \leq \cdots \leq n_k \) for every tuple \((n_1, \ldots, n_k) \in \mathbb{N}_2^k \) defining a sum in (4.22), and there are only a finite number of such tuples for which the corresponding element of \( \mathcal{D}_G \) is at most 12. We enumerate these below.

<table>
<thead>
<tr>
<th>( k )</th>
<th>((n_1, \ldots, n_k))</th>
<th>( \sum_{j=1}^{k} n_j^2 - k - 1 )</th>
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<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>(2, 2)</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>(2, 3)</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>(2, 2, 2)</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>(2, 2, 2, 2)</td>
<td>11</td>
</tr>
</tbody>
</table>

3. \( \mathcal{D}_H = \mathbb{N} - \{1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 20, 21, 22, 25, 30, 35\} \).

Taking \( k = 1 \) and \( n_1 = 2, 3 \) or 4, we get 4, 13, 26 \( \in \mathcal{D}_H \). Taking \( k = 2 \) and \( n_1 = n_2 = 3 \), we get 27 \( \in \mathcal{D}_H \). Taking \( k = 2, n_1 = 3 \) and \( n_2 = 4 \), we get 40 \( \in \mathcal{D}_H \). Now, if \( a \in \mathcal{D}_H \) write \( a = \sum_{j=1}^{k} (2n_j^2 - n_j) - k - 1 \), where \( k \in \mathbb{N} \) and \((n_1, \ldots, n_k) \in \mathbb{N}_2^k \). Setting \( n_{k+1} = 2 \), we then have
\[
\sum_{j=1}^{k+1} n_j(2n_j - 1) - (k + 1) - 1 = \sum_{j=1}^{k} n_j(2n_j - 1) + 6 - (k + 1) - 1
\]
\[
= \sum_{j=1}^{k} n_j(2n_j - 1) - k - 1 + 5 = a + 5
\]
so that \( a + 5 \in \mathcal{D}_H \). Thus, \( \{4, 9, 14, \ldots\} \cup \{13, 18, 23, \ldots\} \cup \{26, 31, 36, \ldots\} \cup \{27, 32, 37, \ldots\} \cup \{40, 45, 50, \ldots\} = \mathbb{N} - \{1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 15, 16, 17, 20, 21, 22, 25, 30, 35\} \subseteq \mathcal{D}_H \).

To see that equality holds, if \((n_1, \ldots, n_k) \in \mathbb{N}_2^k \) then \( N \equiv \sum_{j=1}^{k} n_j(2n_j - 1) - k - 1 \geq 6k - k - 1 = 5k - 1 \). Thus, \( N \leq 35 \) only if \( k \leq 7 \). As in the previous case, we can assume \( n_1 \leq \cdots \leq n_k \) for
every tuple \((n_1, \ldots, n_k) \in \mathbb{N}_2^k\) defining a sum in (4.23), and there are only a finite number of such tuples for which the corresponding element of \(D_H\) is at most 35.

<table>
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<th>(k)</th>
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<td>4</td>
</tr>
<tr>
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</tr>
<tr>
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<td>(2, 4)</td>
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<td>32</td>
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<td>7</td>
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<td>34</td>
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</tbody>
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Appendix A

Diagonalization in Herm(n, \mathcal{H})

The Principal Axis Theorem states that every symmetric matrix with entries lying in the real number field is orthogonally diagonalizable. That is, if $A \in \text{Herm}(n, \mathbb{R})$ then there is an orthogonal matrix $P \in O(n)$ and a diagonal matrix $D$ such that $A = PDP^T$, where $P^T$ is the transpose of $P$. Of course, the diagonal elements of $D$ are then just the (real) eigenvalues of $A$, repeated with the dimension of the corresponding eigenspace, and the columns of $P$ are eigenvectors of $A$, forming a basis for $\mathbb{R}^n$. The corresponding statement for complex Hermitian matrices is also well known: If $A \in \text{Herm}(n, \mathbb{C})$ then there is a unitary matrix $P \in U(n)$ and a diagonal matrix $D$ such that $A = PDP^\ast$, where $P^\ast$ is the conjugate transpose of $P$. Less well known is the fact that one can also diagonalize Hermitian matrices having entries in the quaternion algebra $\mathcal{H}$: If $A \in \text{Herm}(n, \mathcal{H})$ then there is a symplectic matrix $P \in Sp(n)$ and a real diagonal matrix $D$ such that $A = PDP^\ast$. This is a consequence of Theorem 13 in the 1939 paper [J], but the approach we take here differs from that of this reference. We will show that diagonalizability in $\text{Herm}(n, \mathcal{H})$ is a consequence of diagonalizability in $\text{Herm}(n, \mathbb{R})$. The method of proof can be adapted to show that diagonalizability in $\text{Herm}(n, \mathbb{C})$ is also a consequence of diagonalizability in $\text{Herm}(n, \mathbb{R})$ but we will restrict our attention to $\text{Herm}(n, \mathcal{H})$.

Recall that $\mathcal{H}$ has a basis consisting a multiplicative unit 1 and the three imaginary quaternions $i$, $j$ and $k$, with 1 generating (as a real subspace) the center subalgebra $\mathbb{R} \subset \mathcal{H}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Also, the symplectic group $Sp(n)$ consists of those matrices $P \in \mathcal{H}(n)$ satisfying $PP^\ast = P^\ast P = I$, where $P^\ast$ is now the conjugate transpose in $\mathcal{H}(n)$. Equivalently, $Sp(n)$ consists of those $P$ satisfying $\langle P\tilde{v}, P\tilde{w} \rangle = \langle \tilde{v}, \tilde{w} \rangle$ for all $\tilde{v}, \tilde{w} \in \mathcal{H}^n$, where the inner product $\langle \cdot, \cdot \rangle$ is defined by $\langle \tilde{v}, \tilde{w} \rangle = \sum_{i=1}^n \overline{v}_i w_i$, when $\tilde{v} = (v_1, \ldots, v_n)$ and $\tilde{w} = (w_1, \ldots, w_n)$, $\overline{v}_i$ being the conjugate in $\mathcal{H}$.

Suppose now that $A \in \mathcal{H}(n)$ and that $A = PDP^\ast$ for some $P \in Sp(n)$ and real diagonal $D \in \mathcal{H}(n)$. If $\tilde{v}_i \in \mathcal{H}^n$ is the $i^{th}$ column of $P$ and $\lambda_i \in \mathbb{R}$ is the entry of $D$ lying in row and column $i$ then $A\tilde{v}_i = PDP^\ast\tilde{v}_i = PD\tilde{e}_i = P \cdot \lambda_i \tilde{e}_i = \tilde{v}_i \lambda_i$. That is, the column $\tilde{v}_i$ of
$P$ is an eigenvector of $A$ corresponding to the right eigenvalue $\lambda$. Thus, the problem of diagonalization in $\mathbb{H}(n)$ is one of finding solutions to the equation $A\tilde{v} = \tilde{v}\lambda$.

As in the complex case, it is easy to see that $\langle A\tilde{v}, \tilde{w} \rangle = \langle \tilde{v}, A^\dagger \tilde{w} \rangle$, for any $A \in \mathbb{H}(n)$ and $\tilde{v}, \tilde{w} \in \mathbb{H}^n$. Moreover, $\langle \tilde{v}\alpha, \tilde{w} \rangle = \tilde{v} \cdot \langle \tilde{v}, \tilde{w} \rangle$ and $\langle \tilde{v}, \tilde{w}\alpha \rangle = \langle \tilde{v}, \tilde{w} \rangle \cdot \alpha$ for any $\alpha \in \mathbb{H}$. Thus, there is the following.

**A.1 Proposition.** The right eigenvalues of a quaternionic Hermitean matrix are real.

Of course, any real right eigenvalue is also a left eigenvalue and vice versa.

**Proof.** If $A \in \text{Herm}(n, \mathbb{H})$ and $A\tilde{v} = \tilde{v}\lambda$ then

$$
\langle A\tilde{v}, \tilde{v} \rangle = \langle \tilde{v}\lambda, \tilde{v} \rangle = \lambda \cdot \langle \tilde{v}, \tilde{v} \rangle = \langle \tilde{v}, A\tilde{v} \rangle = \langle \tilde{v}, \tilde{v}\lambda \rangle = \langle \tilde{v}, \tilde{v} \rangle \cdot \lambda,
$$

so that $(\lambda - \lambda) \cdot \langle \tilde{v}, \tilde{v} \rangle = 0$, $\langle \tilde{v}, \tilde{v} \rangle = ||\tilde{v}||^2$ being a real number (which commutes with $\lambda$). Thus, if $\tilde{v} \neq 0$ then $\lambda = \lambda$. 

The situation is different for left eigenvalues. For example, the matrix

$$
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \in \text{Herm}(2, \mathbb{H})
$$

has

$$
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \begin{pmatrix} k \\
i
\end{pmatrix} = \begin{pmatrix} -j \\
i
\end{pmatrix} = -k \cdot \begin{pmatrix} k \\
i
\end{pmatrix},
$$

so that $-k$ is a left eigenvalue.

**A.2 Proposition.** Let $A \in \text{Herm}(n, \mathbb{H})$ and $P \in \text{Sp}(n)$. The following are equivalent.

1. The columns of $P$ are eigenvectors of $A$ corresponding to real eigenvalues.

2. $P^\dagger AP$ is a diagonal matrix.

3. $P^\dagger AP$ is a real diagonal matrix.

**Proof.** 2 and 3 are equivalent since $P^\dagger AP$ is Hermitean. That 3 follows from 1 is clear.

If 3 holds, let $D = P^\dagger AP$. Then $AP = PD$, and this says exactly that the columns of $P$ are eigenvectors of $A$ with the eigenvalues being the diagonal entries of $D$. 

We find the real eigenvalues of an $A \in \text{Herm}(n, \mathbb{H})$ by constructing a real symmetric matrix in $\text{Herm}(4n, \mathbb{R})$ which has the same real eigenvalues as $A$. Let us write $\varepsilon_1 = 1, \varepsilon_2 = i, \varepsilon_3 = j$.

---

1 Since the multiplication in $\mathbb{H}$ is not commutative, we must distinguish between right eigenvalues satisfying $A\tilde{v} = \tilde{v}\lambda$ and left eigenvalues satisfying $A\tilde{v} = \lambda\tilde{v}$.
and $\varepsilon_4 = k$. Then the elements $\varepsilon_i \hat{e}_m$, where $i \in \{1, 2, 3, 4\}$ and $m \in \{1, \ldots, n\}$, form a basis for $\mathbb{H}^n$ as a real vector space, and we define a real vector space isomorphism $\Phi : \mathbb{H}^n \to \mathbb{R}^{4n}$ by $\Phi(\varepsilon_i \hat{e}_m) = \hat{e}_{4(m-1)+i}$. Use this isomorphism to define another real vector space isomorphism $\Psi : \mathbb{H}(n) \to \mathbb{R}(4n)$ by $\Psi(A) \cdot \bar{v} = \Phi(A \cdot \Phi^{-1}(\bar{v}))$. It is clear that an element $\bar{v} \in \mathbb{R}^{4n}$ is an eigenvector of $\Psi(A)$ corresponding to an eigenvalue $\lambda \in \mathbb{R}$ if and only if $\Phi^{-1}(\bar{v}) \in \mathbb{H}^n$ is an eigenvector of $A$ corresponding to the same real eigenvalue. Thus, the existence of real eigenvalues of quaternionic Hermitian matrices is a consequence of their existence for real symmetric matrices and the following.

A.3 Proposition. Let $A \in \mathbb{H}(n)$. Then $\Psi(A) \in \text{Herm}(4n, \mathbb{R})$ if and only if $A \in \text{Herm}(n, \mathbb{H})$.

Proof. Let $A[k, k'] \in \mathbb{H}$ be the entry of $A$ lying in the row $k$ and column $k'$. If $i$ and $j$ are two integers lying between 1 and $4n$ inclusive, then the entry of $\Psi(A)$ lying in row $i$ and column $j$ is the dot product $\hat{e}_i \cdot \Psi(A) \hat{e}_j$. If we write $i = 4(m - 1) + \ell$ and $j = 4(m' - 1) + \ell'$ respectively, where $\ell, \ell' \in \{1, 2, 3, 4\}$ and $m, m' \in \{1, \ldots, n\}$, and if $A[k, k'] = \sum_{r=1}^{4} a[k, k', r] \varepsilon_r$ with $a[k, k', r] \in \mathbb{R}$, we then have the following equalities.

$$
\Psi(A) \hat{e}_j = \Psi(A) \hat{e}_{4(m-1)+\ell} = \Phi(A \cdot \varepsilon_{\ell'} \hat{e}_{m'}) = \Phi\left(\sum_k A[k, m'] \varepsilon_{\ell'} \hat{e}_k\right) \\
= \sum_k \Phi(A[k, m'] \varepsilon_{\ell'} \hat{e}_k) \\
= \sum_{k, r} \Phi(a[k, m', r] \varepsilon_r \varepsilon_{\ell'} \hat{e}_k) \\
= \sum_{k, r} a[k, m', r] \Phi(\varepsilon_r \varepsilon_{\ell'} \hat{e}_k) \quad (A.1)
$$

Now, since $i = 4(m - 1) + \ell$ and $\Phi(\varepsilon_i \hat{e}_k) = \hat{e}_{4(k-1)+\ell}$, $\hat{e}_i \cdot \Phi(\varepsilon_{r} \varepsilon_{\ell'} \hat{e}_k) \neq 0$ if and only if $k = m$ and $r \in \{1, 2, 3, 4\}$ is the unique integer such that $\varepsilon_r \varepsilon_{\ell'} \in \{\varepsilon_\ell, -\varepsilon_\ell\}$. Indeed, if we write $\varepsilon_r \varepsilon_{\ell'} = \mu \varepsilon_\ell$, where $\mu \in \{-1, 1\}$, then $\hat{e}_i \cdot A \hat{e}_j = a[m, m', r] \cdot \mu$. Interchanging the roles of $m$ and $m'$ and of $\ell$ and $\ell'$, we also have $\hat{e}_j \cdot A \hat{e}_i = a[m', m, r'] \cdot \mu'$. Thus, $\mu' \in \{-1, 1\}$ are the unique integers such that $\varepsilon_r \varepsilon_{\ell'} = \mu \varepsilon_{\ell'}$. Of course, since we already have $\varepsilon_r \varepsilon_{\ell'} = \mu \varepsilon_\ell$, necessarily $\varepsilon_r \varepsilon_{\ell} = \mu \varepsilon_{\ell'} \varepsilon_r$ so that $r' = r$ and $\mu' = \mu$ or $-\mu$ respective of whether $\varepsilon^2_r = \varepsilon_1$ or $-\varepsilon_1$; that is, of whether $r = 1$ or $r \in \{2, 3, 4\}$. Thus, we have

$$
\hat{e}_i \cdot A \hat{e}_j = \mu \cdot a[m, m', r] \\
\hat{e}_j \cdot A \hat{e}_i = \begin{cases} 
\mu \cdot a[m', m, 1] & \text{if } r = 1, \\
-\mu \cdot a[m', m, r] & \text{if } r \in \{2, 3, 4\}.
\end{cases}
$$

It is clear then that $\Psi(A) \in \text{Herm}(4n, \mathbb{R})$ if and only if $A \in \text{Herm}(n, \mathbb{H})$. ■

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Although we won’t use this fact, it is not difficult to verify that (A.1) is equivalent to the statement that the matrix $\Psi(A)$ is obtained from $A \in \mathbb{H}(n)$ by replacing each entry $a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 + d\varepsilon_4 = a + bi + cj + dk \in \mathbb{H}$ by the antisymmetric matrix

$$
\begin{pmatrix}
a & -b & -c & -d \\
b & a & -d & c \\
c & d & a & -b \\
d & -c & b & a
\end{pmatrix}
\in \mathbb{R}(4).
$$

The notion of dimension is well defined for $R$-modules in the case that $R$ is a division ring, the dimension of a submodule being the number of elements in a basis.\(^2\) Since $\mathbb{H}$ is a division ring and since $\mathbb{H}^n$ has the structure of a right $\mathbb{H}$-module we therefore can speak of the dimension of a submodule $W \subset \mathbb{H}^n$ and we write $\dim_{\mathbb{H}} W$ for this integer. For a right eigenvalue of an matrix $A \in \mathbb{H}(n)$ the eigenspace $E_\lambda = \{ \tilde{v} \in \mathbb{H}^n \mid A\tilde{v} = \tilde{v}\lambda \}$ is a submodule of $\mathbb{H}^n$ and we can therefore speak of $\dim_{\mathbb{H}} E_\lambda$.\(^3\) Of course, $\dim_{\mathbb{H}} \mathbb{H}^n = n$.

**A.4 Proposition.** Let $A \in \text{Herm}(n, \mathbb{H})$. Then there is a symplectic matrix $P \in \text{Sp}(n)$ and a real diagonal matrix $D$ such that $A = PDP^t$. Moreover, $D$ is unique up to a permutation of its diagonal entries and every real eigenvalue $\lambda$ of $A$ occurs as one of these entries with multiplicity $\dim_{\mathbb{H}} E_\lambda$.

**Proof.** If $P \in \text{Sp}(n)$ exists as claimed then its columns form a basis for $\mathbb{H}^n$ as a right $\mathbb{H}$-module. For these columns are mutually orthogonal and therefore linearly independent, and there are $n = \dim_{\mathbb{H}} \mathbb{H}^n$ of them. Denote these columns by $\tilde{v}_1, \ldots, \tilde{v}_n$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the respective eigenvalues of $A$ to which they correspond, these being the diagonal entries of $D$. If $\lambda$ is any real eigenvalue of $A$ and $\tilde{v} \in \mathbb{H}^n$ is a corresponding eigenvector then we have $\tilde{v} = \tilde{v}_1\alpha_1 + \cdots + \tilde{v}_n\alpha_n$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{H}$, not all zero. Since

$$
A\tilde{v} = \tilde{v}\lambda = (\tilde{v}_1\alpha_1 + \cdots + \tilde{v}_n\alpha_n)\lambda
= \tilde{v}_1\lambda_1\alpha_1 + \cdots + \tilde{v}_n\lambda_n\alpha_n
$$

we have $\tilde{v}_1(\lambda - \lambda_1)\alpha_1 + \cdots + \tilde{v}_n(\lambda - \lambda_n)\alpha_n = 0$. Thus, $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$ since at least one of $\alpha_1, \ldots, \alpha_n$ is non-zero. Moreover, $\dim_{\mathbb{H}} E_\lambda = \dim_{\mathbb{H}} \ker (A - \lambda I) = \dim_{\mathbb{H}} \ker (D - \lambda I)$ and this number is exactly the number of times that $\lambda$ occurs on the diagonal of $D$. It remains only to demonstrate the existence of $P$.

Let $\lambda_1, \ldots, \lambda_\ell \in \mathbb{R}$ be the distinct eigenvalues of $\Psi(A) \in \text{Herm}(4n, \mathbb{R})$ and let $E_1, \ldots, E_\ell \subset \mathbb{R}^{4n}$ be the corresponding eigenspaces, which are mutually orthogonal and whose sum is all of $\mathbb{R}^{4n}$.

\(^2\) See, for example, [H]. (Chapter IV, section 2.)

\(^3\) Note that the eigenspace of a left eigenvalue need not be a submodule (since multiplication in $\mathbb{H}$ is not commutative) although it is a subspace of $\mathbb{H}^n$ when this is regarded as a real vector space.

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As we noted previously, an element \( \tilde{v} \in \mathbb{R}^{4n} \) is an eigenvector of \( \Psi(A) \) corresponding to an eigenvalue \( \lambda \in \mathbb{R} \) if and only if \( \Phi^{-1}(\tilde{v}) \in \mathbb{H}^n \) is an eigenvector of \( A \) corresponding to the same real eigenvalue. Thus, \( E'_1 = \Phi^{-1}(E_1), \ldots, E'_\ell = \Phi^{-1}(E_\ell) \) are eigenspaces of \( A \). Moreover, as in the real and complex cases, these eigenspaces are mutually orthogonal since if \( \tilde{v}, \tilde{v}' \in \mathbb{H}^n \) are eigenvectors corresponding to distinct real eigenvalues \( \lambda, \lambda' \in \mathbb{R} \) respectively then

\[
\langle A\tilde{v}, \tilde{v}' \rangle = \lambda \cdot \langle \tilde{v}, \tilde{v}' \rangle = \langle \tilde{v}, A\tilde{v}' \rangle = \lambda' \cdot \langle \tilde{v}, \tilde{v}' \rangle
\]

so that necessarily \( \langle \tilde{v}, \tilde{v}' \rangle = 0 \).

Choose \( \tilde{v}_1 \in \mathbb{H}^n \) to be any eigenvector of \( A \) corresponding to one of \( \lambda_1, \ldots, \lambda_\ell \). Suppose now that there are mutually orthogonal eigenvectors \( \tilde{v}_1, \ldots, \tilde{v}_m \in \mathbb{H}^n \), with \( m < n \) and each eigenvector corresponding to one of the eigenvalues \( \lambda_1, \ldots, \lambda_\ell \). Let \( W \subset \mathbb{R}^{4n} \) be the subspace spanned by the elements \( \Phi(\tilde{v}_r, \varepsilon) \in \mathbb{R}^{4n} \), where \( r \in \{1, \ldots, m\} \) and \( \varepsilon \in \{1, i, j, k\} \). \( W \) is a proper subspace since \( m < n \). Since \( E_1 + \cdots + E_\ell = \mathbb{R}^{4n} \) we can choose a non-zero element \( \tilde{w} \in E_1 \cup \cdots \cup E_\ell \) which does not lie in \( W \). If \( \tilde{v} = \Phi^{-1}(\tilde{w}) \) then \( \tilde{v} \) is an eigenvector of \( A \) and we can define \( \tilde{v}_{m+1} \in \mathbb{H}^n \) by subtracting off the components along each of \( \tilde{v}_1, \ldots, \tilde{v}_m \).

\[
\tilde{v}_{m+1} = \tilde{v} - \tilde{v}_1 \langle \tilde{v}_1, \tilde{v} \rangle / \| \tilde{v}_1 \|^2 - \cdots - \tilde{v}_m \langle \tilde{v}_m, \tilde{v} \rangle / \| \tilde{v}_m \|^2
\]

Note that \( \tilde{v}_{m+1} \neq 0 \) since otherwise \( \Phi(v) \in W \). Note also that \( \tilde{v}_{m+1} \) is an eigenvector of \( A \) since \( \langle \tilde{v}_r, \tilde{v} \rangle = 0 \) unless \( \tilde{v}_r \) corresponds to the same real eigenvalue of \( A \) as does \( \tilde{v} \). Thus, we have mutually orthogonal eigenvectors \( \tilde{v}_1, \ldots, \tilde{v}_{m+1} \in \mathbb{H}^n \) of \( A \), each eigenvector corresponding to one of the eigenvalues \( \lambda_1, \ldots, \lambda_\ell \). Starting with \( \tilde{v}_1 \) we can therefore construct mutually orthogonal eigenvectors \( \tilde{v}_1, \ldots, \tilde{v}_n \in \mathbb{H}^n \). If we normalize each of these eigenvectors through by their norms and if we let \( P \) be a matrix having the normalized eigenvectors as columns then \( P \in Sp(n) \) and \( D = P^t A P \) is real diagonal.
Appendix B

Topologizing with Subsets

In this appendix we collect some results on the construction of a topology on a set from a collection of topologized subsets. These have the misfortune of having often been saddled with the rather nondescriptive and overused adjective “weak”. Our goal is Proposition B.5, which we employ in chapters 2 and 3.

Given a topology $\mathcal{T}$ on a set $X$ and a subset $Y \subseteq X$, let $Y \cap \mathcal{T}$ denote the subspace topology induced on $Y$: $Y \cap \mathcal{T} = \{Y \cap U \mid U \in \mathcal{T}\}$. Let $\mathcal{T}^c$ denote the collection of closed sets: $\mathcal{T}^c = \{U^c = X - U \mid U \in \mathcal{T}\}$.

Now, let $X$ be any set and suppose $\mathcal{A}$ is a collection of topologized subsets. If $A \in \mathcal{A}$, we will write $\mathcal{T}_A$ for the topology on $A$. The collection

$$\mathcal{I} = \{U \subseteq X \mid A \cap U \in \mathcal{T}_A \text{ for all } A \in \mathcal{A}\}$$

is a topology on $X$, the largest topology for which the inclusion $A \hookrightarrow X$ is a continuous function for each $A \in \mathcal{A}$. We will refer to $\mathcal{I}$ as the topology generated by $\mathcal{A}$. We clearly have $A \cap \mathcal{I} \subseteq \mathcal{T}_A$ for any $A \in \mathcal{A}$ but the converse need not hold. For example, $A \cap B \in \mathcal{T}_A$ may fail for some $B \in \mathcal{A}$ distinct from $A$.

**B.1 Lemma.** The following are equivalent.

1. For all $A, B \in \mathcal{A}$, $(A \cap B) \cap \mathcal{T}_A = (A \cap B) \cap \mathcal{T}_B$ and $A \cap B \in \mathcal{T}_B$.

2. For all $A \in \mathcal{A}$, $A \in \mathcal{I}$ and $A \cap \mathcal{I} = \mathcal{T}_A$.

**Proof.** (1 $\Rightarrow$ 2) Let $A \in \mathcal{A}$. $A \in \mathcal{I}$ is clear from the definition of $\mathcal{I}$ since $B \cap A \in \mathcal{T}_B$ for all $B \in \mathcal{A}$. As already noted, we always have $A \cap \mathcal{I} \subseteq \mathcal{T}_A$. Suppose $U \in \mathcal{T}_A$. Then for any $B \in \mathcal{A}$ we have $B \cap U = (A \cap B) \cap U \in (A \cap B) \cap \mathcal{T}_A = (A \cap B) \cap \mathcal{T}_B$. Since $A \cap B \in \mathcal{T}_B$, necessarily $B \cap U \in \mathcal{T}_B$. Thus, $U \in \mathcal{I}$ as required.

---

1 Take $X = \mathbb{R}$ and $\mathcal{A} = \{A, B\}$, where $A = (-1, 1)$ and $B = [0, 1)$ are given Euclidean topologies.
Let $A, B \in \mathcal{A}$. Since $A \cap \mathcal{T} = \mathcal{T}_A$ and $B \in \mathcal{T}$, necessarily $A \cap B \in \mathcal{T}_A$. Also, $A \cap \mathcal{T} = \mathcal{T}_A$ gives $(A \cap B) \cap \mathcal{T}_A = (A \cap B) \cap \mathcal{T}$. Similarly, $(B \cap A) \cap \mathcal{T}_B = (B \cap A) \cap \mathcal{T}$. Thus, $(A \cap B) \cap \mathcal{T}_A = (A \cap B) \cap \mathcal{T}_B$ as required.

**B.2 Lemma.** The following are equivalent.

1. For all $A, B \in \mathcal{A}$, $(A \cap B) \cap \mathcal{T}_A = (A \cap B) \cap \mathcal{T}_B$ and $A \cap B \in \mathcal{T}^c$.

2. For all $A \in \mathcal{A}$, $A \in \mathcal{T}^c$ and $A \cap \mathcal{T}^c = \mathcal{T}_A^c$.

**Proof.** Work with closed sets rather than open sets, replacing each occurrence of a topology in the proof of Lemma B.1 by its complement.

Suppose now that the set $X$ is a topological space with topology $\mathcal{T}_X$ and that, for each $A \in \mathcal{A}$, the topology $\mathcal{T}_A$ is the subspace topology: $\mathcal{T}_A = A \cap \mathcal{T}_X$. As before, $\mathcal{A}$ generates a topology $\mathcal{T} = \{ U \subset X \mid A \cap U \in \mathcal{T}_A \text{ for all } A \in \mathcal{A} \}$ on $X$. We clearly have $\mathcal{T}_X \subset \mathcal{T}$, but the converse need not hold. For example, take $X = \{1, 2, 3\}$, $\mathcal{T}_X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and $\mathcal{A} = \{A\}$, where $A = \{1, 2\}$. Since $A$ has the discrete topology, so does $X$.

Recall that a collection of subsets of a space is called *locally finite* if each point in the space has a neighbourhood that intersects at most finitely many members of the collection. Recall also that the union of a locally finite collection of closed sets is closed.

**B.3 Lemma.** The following statements hold.

1. If $\mathcal{A}$ generates $\mathcal{T}_X$ then either $\cup \mathcal{A} = X$ or any subset of $X - \cup \mathcal{A}$ is open in $X$.

2. If each $A \in \mathcal{A}$ is open in $X$ and either $\cup \mathcal{A} = X$ or any subset of $X - \cup \mathcal{A}$ is open in $X$ then $\mathcal{A}$ generates $\mathcal{T}_X$.

3. If $\mathcal{A}$ is locally finite, each $A \in \mathcal{A}$ is closed in $X$ and $\cup \mathcal{A} = X$ then $\mathcal{A}$ generates $\mathcal{T}_X$. Moreover, if $Y$ is any space then the collection $\{Y \times A \mid A \in \mathcal{A}\}$ generates the product topology on $Y \times X$.

**Proof.** 1. If $\cup \mathcal{A} \neq X$ then any subset which does not intersect $\cup \mathcal{A}$ lies in $\mathcal{T}$.

2. Let $U \in \mathcal{T}$ and write $U = (U \cap \cup \mathcal{A}) \cup (U - \cup \mathcal{A})$. By hypothesis, the second term in this union is open in $X$. As for the first, for any $A \in \mathcal{A}$, $A \cap U \in \mathcal{T}_A$ since $U \in \mathcal{T}$, and $A \in \mathcal{T}_X$ by hypothesis. Thus, $A \cap U \in \mathcal{T}_X$ for all $A \in \mathcal{A}$ so that $\cup_{A \in \mathcal{A}} (U \cap A) = U \cap \cup \mathcal{A}$ is also open in $X$.

3. Let $K \in \mathcal{T}^c$. Since $K \in \mathcal{T}^c$, we have $A \cap K \in \mathcal{T}_A^c$ for each $A \in \mathcal{A}$. Since each $A \in \mathcal{A}$ is closed in $X$, we therefore have $A \cap K$ closed in $X$ for each $A \in \mathcal{A}$. Thus, since the collection $\{ A \cap K \mid A \in \mathcal{A} \}$ is locally finite, $K = \cup_{A \in \mathcal{A}} (A \cap K)$ is closed in $X$.

The second statement follows similarly since $\{Y \times A \mid A \in \mathcal{A}\}$ is also locally finite.
Let us now suppose that our space $X$ is Hausdorff. If $\mathcal{K}$ is the collection of all compact subsets of $X$, define yet another topology on $X$.

$$\mathcal{T}_{CG} = \{U \subset X \mid U^c \cap K \in \mathcal{T}_K^c \text{ for all } K \in \mathcal{K}\}$$

We clearly have $\mathcal{T}_X \subset \mathcal{T}_{CG}$. $X$ is said to be compactly generated if $\mathcal{K}$ generates $\mathcal{T}_X$.

Note that if $\mathcal{A} \subset \mathcal{K}$ then $\mathcal{T}_{CG} \subset \mathcal{T}$ so that $\mathcal{T}_X \subset \mathcal{T}_{CG} \subset \mathcal{T}$. In particular, if $\mathcal{A}$ generates $\mathcal{T}_X$ then $X$ is compactly generated. We are now in a position to prove our desired result.

**B.4 Proposition.** Suppose the topology of a Hausdorff space $X$ is generated by a collection $\mathcal{A}$ of compact subsets. Suppose further that any compact subset of $X$ is contained in a finite union of members of $\mathcal{A}$. If $Y$ is locally compact Hausdorff then the product topology on $X \times Y$ is generated by $\{A \times Y \mid A \in \mathcal{A}\}$.

**Proof.** As noted above, $X$ is compactly generated since $\mathcal{A}$ generates $\mathcal{T}_X$. Thus, since $Y$ is locally compact Hausdorff, Theorem 4.3 in [St2] tells us that $X \times Y$ is compactly generated. Let $\mathcal{T}$ be the topology on $X \times Y$ generated by $\mathcal{A} \times Y = \{A \times Y \mid A \in \mathcal{A}\}$. As always, $\mathcal{T}_{X \times Y} \subset \mathcal{T}$. To demonstrate the reverse inclusion, let $C \in \mathcal{T}^c$. We will show that $C \in \mathcal{T}_{X \times Y}^c$ by showing that $C \cap K$ is closed in $K$ for each compact $K \subset X \times Y$.

Given such a $K$, consider its image under the projection $X \times Y \rightarrow X$ onto the first factor. Since this image is compact, our hypotheses imply that it is contained in a finite union of members of $\mathcal{A}$. Thus, $K$ is contained in a finite union of members of $\mathcal{A} \times Y$, say $K \subset \bigcup_{i=1}^k A_i \times Y$. Since $C \in \mathcal{T}^c$, necessarily $C \cap (A \times Y)$ is closed in $A \times Y$ for each $A \in \mathcal{A}$. Since $A \times Y$ is itself closed in $X \times Y$, $C \cap (A \times Y)$ is closed in $X \times Y$ for each $A \in \mathcal{A}$. Thus, $\bigcup_{i=1}^k C \cap (A_i \times Y)$ is closed in $X \times Y$ so that $\bigcup_{i=1}^k C \cap (A_i \times Y) \cap K = C \cap K$ is closed in $K$. 

**B.5 Proposition.** Let $K$ be a simplicial complex. If either $K$ is locally finite or $Z$ is locally compact Hausdorff then the product topology on $Z \times |K|$ is generated by $\{|s| \mid s \in K\}$.

**Proof.** If $K$ is locally finite then $\{|s| \mid s \in K\}$ is locally finite since the open star of any vertex intersects only finitely many closed simplices in a nonempty set, and the result follows from Lemma B.3.

If $Z$ is locally compact Hausdorff, apply Proposition B.4. By definition, the topology of $|K|$ is generated by $\{|s| \mid s \in K\}$. It is Hausdorff by Theorem 17 on page 111 of [S]. Every compact subset of $|K|$ is contained in a finite union of closed simplices by Corollary 19 on page 113 of the same reference. 

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Appendix C

A Generalized Mapping Cylinder

In this appendix we define a generalization of the ordinary mapping cylinder and show that this generalization is an endofunctor on the category of locally trivial bundles over a fixed locally compact space.

Given a collection $\mathcal{X}$ of topological spaces, let us denote their disjoint union by $\bigoplus \mathcal{X}$. If $\mathcal{X} = \{X_1, \ldots, X_n\}$ is a finite collection, we will write $\bigoplus \mathcal{X} = X_1 \oplus \cdots \oplus X_n$. If $X_1 = \cdots = X_n \equiv X$, we may write $X_1 \oplus \cdots \oplus X_n \equiv \oplus^n X$. Of course, the disjoint union is the coproduct in the topological category and there is the usual universal property. If $\mathbb{I}$ denotes the discrete space $\{1, \ldots, n\}$ then it is easy to see that $X \times \mathbb{I}$ satisfies the universal property of $\oplus^n X$ so that $\oplus^n X \approx X \times \mathbb{I}$.

Fix a tuple $(K; J_1, \ldots, J_n)$ of spaces with $J_1, \ldots, J_n \subset K$. For any map $f : X \to Y$, define the mapping cylinder of $f$ on $(K; J_1, \ldots, J_n)$ to be the colimit space of the following diagram.

$$
\begin{array}{ccc}
(f \circ \pi_1, 1) & X \times J_1 & \cdots \\
\downarrow & \downarrow & \downarrow \\
Y \times \mathbb{I} & \cdots & X \times K \\
(f \circ \pi_1, n) & X \times J_n
\end{array}
$$

We will denote this colimit by $M(K; J_1, \ldots, J_n)f$ and write $M(K; J_1, \ldots, J_n)f = Mf$ when the tuple $(K; J_1, \ldots, J_n)$ is understood. Explicitly, let us take mapping cylinder to be the quotient of $(X \times K) \oplus (Y \times \mathbb{I})$ by the smallest equivalence relation that identifies points $(x, s) \in X \times K$ and $(f(x), i) \in Y \times \mathbb{I}$ whenever $s \in J_i$. Note that the ordinary mapping cylinder is the mapping cylinder on $(I; \{0\})$.

Let us also define the stunted mapping cylinder of $f$ on $(K; J_1, \ldots, J_n)$ to be the identification space $Q(K; J_1, \ldots, J_n)f = Qf$ of $X \times K$ by the smallest equivalence relation which identifies pairs $(x, s)$ and $(x', s')$ whenever $f(x) = f(x')$ and $s, s' \in J_i$ for some $i$. Thus, by Proposition
0.2, a map \( g : X \times K \to Z \) has a unique factorization through \( Qf \) if and only if \( g(x, s) = g(x', s') \) for all \( x, x' \in X \) and \( s, s' \in J_i, i = 1, \ldots, n \).

By the universal property of the identification, the map \( (x, s) \mapsto [x, s] : X \times K \to Mf \) factors through the identification \( X \times K \to Qf \) to give a map \( i : [x, s] \mapsto [x, s] : Qf \to Mf \). This is clearly an injection.

**C.1 Proposition.** If \( f : X \to Y \) is an identification then \( i : Qf \to Mf \) is a homeomorphism.

**Proof.** For each \( i = 1, \ldots, n \), choose an element \( a_i \in J_i \) and define a map \( h_i : Y \to Qf \) by the universal property of the identification \( f \) and the following diagram.

\[
\begin{array}{c}
X \times K \\
\downarrow f \\
Y
\end{array} \quad \begin{array}{c}
\rightarrow [x, a_i] \\
Qf
\end{array} \quad \begin{array}{c}
h_i \\
\end{array}
\]

Define \( h : Y \times n \to Qf \) by \( h(y, i) = h_i(y) \). Then, since \( h(f(x), i) = [x, a_i] \in Qf \) for each \( i \), the universal property of the colimit gives a map \( k : Mf \to Qf \) which makes the following diagram commute.

\[
\begin{array}{c}
X \times J_1 \\
\downarrow h \\
Y \times n
\end{array} \quad \begin{array}{c}
\rightarrow [x, a_i] \\
Qf
\end{array} \quad \begin{array}{c}
\leftarrow k \\
\leftarrow \{[x, s] \} \leftarrow \{[x, s] \}
\end{array}
\]

Since \( k([x, s]) = [x, s] \) whenever \( (x, s) \in X \times K \), \( k \) is the inverse of \( i \).

**C.2 Corollary.** If \( p : E \to B \) is a bundle map then \( Qp \approx Mp \).

Next, define the *generalized cone* of a space \( X \) on \((K; J_1, \ldots, J_n)\) to be \( C(K; J_1, \ldots, J_n)X = CX = Q* \), where \(*\) is a map to a one point space. That is, \( CX \) is the quotient of \( X \times K \) by the smallest equivalence relation that identifies points \((x, s)\) and \((x', s')\) whenever \( s, s' \in J_i \) for some \( i \). Note that the ordinary cone is the generalized cone on \((I; \{0\})\) and that the suspension is the generalized cone on \((I; \{0\}, \{1\})\).
If \( \psi = (p, \phi) : E \to B \times F \) is a bundle trivialization then, by the universal property of the identification, we have a maps \( \tilde{p} : Qp \to B \) and \( \tilde{\psi} : Qp \to B \times CF \) which make the following diagram commute.

\[
\begin{array}{ccc}
E \times K & \xrightarrow{\psi \times 1} & B \times F \times K \\
\downarrow{p \circ \pi_1} & & \downarrow{\pi_1} \\
Qp & \xrightarrow{\tilde{p}} & B \times CF \\
\end{array}
\]

\( (b, f, s) \)

C.3 Proposition. Suppose \( p : E \to B \) is the bundle map of a trivial bundle with fibre \( F \). If \( B \) is locally compact then \( \tilde{p} : Qp \to B \) is the bundle map of a trivial bundle with fibre \( CF \).

Proof. Since \( B \) is locally compact, the map \( (b, f, s) \mapsto (b, [f, s]) : B \times F \times K \to B \times CF \) in the previous diagram is an identification. Thus, since \( \psi : E \to B \times F \) is a homeomorphism and since two points in \( E \times K \) project to the same point in \( Qp \) if and only if their images (by \( \psi \times 1 \)) in \( B \times F \times K \) project to the same point in \( B \times CF \), we can construct a map \( B \times CF \to Qp \) from \( \psi^{-1} \times 1 \) using the universal property of the identification.

\[
\begin{array}{ccc}
E \times K & \xrightarrow{\psi^{-1} \times 1} & B \times F \times K \\
\downarrow & & \downarrow \\
Qp & \xrightarrow{\tilde{p}} & B \times CF \\
\end{array}
\]

The constructed map is clearly the inverse of \( \tilde{\psi} \).

For non-trivial bundles, we use the above result on a trivialization neighbourhood.

C.4 Proposition. Let \( p : E \to B \) be locally trivial with fibre \( F \). If \( B \) is locally compact then \( \tilde{p} : Qp \to B \) is locally trivial with fibre \( CF \).

Before proving this, let us first recall a topological result.

C.5 Proposition. Let \( R \) be an equivalence relation on a space \( X \) and let \( \pi : X \to X/R \) be the identification. If \( B \subset X/R \) is either open or closed then the map \([x] \mapsto \pi(x) : \pi^{-1}(B)/R_0 \to B\) is a homeomorphism, where \( R_0 \) is the equivalence relation on \( \pi^{-1}(B) \) induced by \( R \).

Proof. [Du], chapter VI, section 2.

Proof of Proposition C.4. Let \( U \subset B \) be a trivialization neighbourhood and let \( \psi : p^{-1}(U) \to U \times F \) be a local trivialization. If \( p_U : x \mapsto p(x) : p^{-1}(U) \to U \) then, since an open subset of a locally compact space is locally compact, Proposition C.3 gives a trivialization
\[ \tilde{\psi} : Q_{pU} \rightarrow U \times CF \text{ over } U. \]

To complete the proof, note that Proposition C.5 provides a homeomorphism \( Q_{pU} \rightarrow \tilde{p}^{-1}(U) \) over \( U \) since \( \tilde{p}^{-1}(U) \subset Q_p \) is open, \( \pi^{-1}(\tilde{p}^{-1}(U)) = p^{-1}(U) \times K \) where \( \pi : E \times K \rightarrow Q_p \) is the identification, and \( Q_{pU} \) is exactly \( p^{-1}(U) \times K \) modulo the equivalence relation induced by that on \( E \times K \). \qed
Appendix D
Homotopy Groups of Flag Manifolds

The fundamental group of the real complete flag manifolds was computed in chapter 1, where we showed that \( \pi_1 F(n) \) was a Clifford group on \( n - 1 \) generators whenever \( n \geq 3 \). Diagram chasing with the long exact homotopy sequence of various fibrations can be used to compute the fundamental group of the remaining real flag manifolds as well as the complex and quaternionic flag manifolds. Many higher homotopy groups can also be computed in the same manner, and we collect these results here. As in the real case, we write \( F_C(n) = F_C(1, \ldots, 1) \) and \( F_H(n) = F_H(1, \ldots, 1) \) for the complex and quaternionic complete flag manifolds.

As previously, we will use the notation \( O(n) \) to denote \( O(n), U(n) \) and \( Sp(n) \) in the real, complex and quaternionic cases respectively. For any given \( n \) we can include \( O(n) \) in \( O(n + 1) \) as the subgroup of matrices which leave the standard basis vector \( e_{n+1} \) fixed. We can thus regard these groups as forming an infinite chain \( O(1) \subset O(2) \subset \cdots \) of inclusions. We will also use the notation \( V(n, k) = O(n + k)/O(k) \) for the Stiefel Manifolds \( V_R(n, k) = O(n + k)/O(k) \), \( V_C(n, k) = U(n + k)/U(n) \) and \( V_H(n, k) = Sp(n + k)/Sp(k) \), as well as \( F(n_1, \ldots, n_\ell) \) for the respective flag manifolds.

Stability of \( \pi_k O(n) \)

Each of the principal bundles

\[
\begin{align*}
&O(n) \longrightarrow O(n + 1) & U(n) \longrightarrow U(n + 1) & Sp(n) \longrightarrow Sp(n + 1) \\
&\downarrow \quad \downarrow \quad \downarrow \\
&S^n \quad S^{2n+1} \quad S^{4n+3}
\end{align*}
\]

leads to a stability theorem of homotopy groups when we apply the long exact homotopy sequence of the fibration. Indeed, these three sequences have the following portions respectively.

\[
\begin{align*}
\pi_{k+1}S^n &\longrightarrow \pi_k O(n) \longrightarrow \pi_k O(n + 1) \longrightarrow \pi_k S^n \\
\pi_{k+1}S^{2n+1} &\longrightarrow \pi_k U(n) \longrightarrow \pi_k U(n + 1) \longrightarrow \pi_k S^{2n+1} \\
\pi_{k+1}S^{4n+3} &\longrightarrow \pi_k Sp(n) \longrightarrow \pi_k Sp(n + 1) \longrightarrow \pi_k S^{4n+3}
\end{align*}
\]
The connectivity of the spheres then leads immediately to the following three statements.

**D.1 Proposition.** The following are true.

1. The inclusion \( O(n) \hookrightarrow O(n + 1) \) induces an epimorphism \( \pi_k O(n) \twoheadrightarrow \pi_k O(n + 1) \) if \( k = n - 1 \) and an isomorphism \( \pi_k O(n) \cong \pi_k O(n + 1) \) if \( k < n - 1 \).

2. The inclusion \( U(n) \hookrightarrow U(n + 1) \) induces an epimorphism \( \pi_k U(n) \twoheadrightarrow \pi_k U(n + 1) \) if \( k = 2n \) and an isomorphism \( \pi_k U(n) \cong \pi_k U(n + 1) \) if \( k < 2n \).

3. The inclusion \( Sp(n) \hookrightarrow Sp(n + 1) \) induces an epimorphism \( \pi_k Sp(n) \twoheadrightarrow \pi_k Sp(n + 1) \) if \( k = 4n + 2 \) and an isomorphism \( \pi_k Sp(n) \cong \pi_k Sp(n + 1) \) whenever \( k < 4n + 2 \).

Thus, for any given \( k \), the groups \( \pi_k O(n) \), \( \pi_k U(n) \) and \( \pi_k Sp(n) \) become stable for sufficiently large \( n \).

**Connectedness of the Stiefel Manifolds**

Stability of the orthogonal, unitary and symplectic groups leads to statements about the connectivity of the Stiefel Manifolds \( V(n, k) \). Indeed, the principal bundle

\[
\begin{array}{ccc}
\mathcal{O}(k) & \longrightarrow & \mathcal{O}(n + k) \\
\downarrow & & \downarrow \\
V(n, k) & \longrightarrow & \mathcal{O}(n + k)
\end{array}
\]

gives rise to the following portion of the long exact homotopy sequence of the fibration.

\[
\pi_{\ell+1} \mathcal{O}(k) \longrightarrow \pi_{\ell+1} \mathcal{O}(n + k) \longrightarrow \pi_{\ell+1} V(n, k) \longrightarrow \pi_{\ell} \mathcal{O}(k) \longrightarrow \pi_{\ell} \mathcal{O}(n + k)
\]

The leftmost homomorphism of this sequence being an epimorphism and the rightmost being a monomorphism is then a necessary and sufficient condition for \( \pi_{\ell+1} V(n, k) = 0 \). By Proposition D.1, this is the case when \( \ell + 1 \leq k - 1 \) in the real case, \( \ell + 1 \leq 2k \) in the complex case and \( \ell + 1 \leq 4k + 2 \) in the quaternionic case. Thus, we have the following results.

**D.2 Proposition.** The following are true.

1. \( V_R(n, k) \) is \((k - 1)\)-connected.

2. \( V_C(n, k) \) is \((2k)\)-connected.

3. \( V_H(n, k) \) is \((4k + 2)\)-connected. □
Homotopy Groups of Flags

Connectedness of the Stiefel Manifolds now leads to the homotopy groups of many flag mani-

folds by consideration of the long exact homotopy sequence of the following principal bundle.

\[
O(n_1) \times \cdots \times O(n_\ell) \to V(n_1 + \cdots + n_\ell, k) \nonumber
\]

\[
\mathcal{F}(n_1, \ldots, n_\ell, k)
\]

The following result is an immediate consequence of Proposition D.2.

**D.3 Proposition.** \(F_\mathcal{R}(n_1, \ldots, n_\ell, k), F_\mathcal{C}(n_1, \ldots, n_\ell, k)\) and \(F_\mathcal{H}(n_1, \ldots, n_\ell, k)\) are all path con-

nected and the following are true.

1. If \(1 \leq i \leq k - 1\) then \(\pi_i F_\mathcal{R}(n_1, \ldots, n_\ell, k) \cong \pi_{i-1}O(n_1) \times \cdots \times \pi_{i-1}O(n_\ell)\).
2. If \(1 \leq i \leq 2k\) then \(\pi_i F_\mathcal{C}(n_1, \ldots, n_\ell, k) \cong \pi_{i-1}U(n_1) \times \cdots \times \pi_{i-1}U(n_\ell)\).
3. If \(1 \leq i \leq 4k + 2\) then \(\pi_i F_\mathcal{H}(n_1, \ldots, n_\ell, k) \cong \pi_{i-1}Sp(n_1) \times \cdots \pi_{i-1}Sp(n_\ell)\). &

Note that we do indeed have a group isomorphism when \(i = 1\): Since the bundle (D.1) is

principal, the function

\[
\pi_1 \mathcal{F}(n_1, \ldots, n_\ell, k) \to \pi_0 O(n_1) \times \cdots \pi_0 O(n_\ell)
\]

in the long exact homotopy sequence is a group homomorphism. (See [St1], page 93.)

Of course, the position of the integer \(k\) in Proposition D.3 is immaterial and we can make

corresponding statements about isomorphisms between \(\pi_i \mathcal{F}(n_1, \ldots, n_\ell)\) and a product of all

but one of \(\pi_{i-1}O(n_1), \ldots, \pi_{i-1}O(n_\ell)\) when \(\ell \geq 2\). Since we are guaranteed that at least one of

\(n_1, \ldots, n_\ell\) be greater than or equal to 2 if \(n_1 + \cdots + n_\ell > \ell\), there is the following consequence

for the real flags.

**D.4 Corollary.** If \(n_1 + \cdots + n_\ell > \ell\) then \(\pi_1 F_\mathcal{R}(n_1, \ldots, n_\ell) \cong \bigoplus^{\ell-1} \mathbb{Z}_2\). &

Since \(U(1) \cong S^1\) and \(Sp(1) \cong S^3\), there are also the following two consequences of Proposition

D.3 for the complex and quaternionic complete flags.

**D.5 Corollary.** \(F_\mathcal{C}(n)\) is 1-connected and \(\pi_2 F_\mathcal{C}(n) \cong \bigoplus^{n-1} \mathbb{Z}\) for all \(n \geq 2\). &

**D.6 Corollary.** \(F_\mathcal{H}(n)\) is 3-connected and \(\pi_4 F_\mathcal{H}(n) \cong \bigoplus^{n-1} \mathbb{Z}\) for all \(n \geq 2\). &
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