MELNIKOV'S METHOD WITH APPLICATIONS

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Abstract

This thesis gives a detailed discussion of Melnikov's method, which is an analytical tool to study global bifurcations that occur in homoclinic or heteroclinic loops, or in one-parameter families of periodic orbits of a perturbed system. Basic results of the Melnikov theory relating the number, positions and multiplicities of the limit cycles by the number, positions and multiplicities of the zeros of the Melnikov function are proved. We then give several examples to illustrate the theory. In particular, we use the Melnikov theory to study the exact number of limit cycles in the Bogdanov-Takens system with reflection symmetry. We then extend the first-order Melnikov theory to higher-order and establish some results relating the number, positions and multiplicities of the limit cycles by the number, positions and multiplicities of the zeros of the first non-vanishing Melnikov function. Next, we derive a formula for the second-order Melnikov function for certain perturbed Hamiltonian systems using Françoise's recursive algorithm. Finally, this formula is applied to an example.

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Introduction

Bifurcation analysis is the study of the changes that occur in the structure of solutions of differential equations as parameters are varied. For example, consider parameterized families of ordinary differential equations

$$\dot{x} = f(x, \mu), \qquad x \in \Re^n, \tag{1}$$

depending on a parameter $\mu \in \Re$ (or on several parameters $\mu \in \Re^n$), where f depends smoothly on x and μ. Let us denote the solution to the initial value problem consisting of (1) and the initial condition $x(0) = x_0$ by $x(t) = \phi(t, x_0, \mu)$. As we vary μ, phase portraits (collections of solution curves $x(t)$ in $\Re^n$) of (1) 'look pretty much the same' except perhaps at certain values of μ where there is a qualitative change in the phase portrait. These values are called bifurcation values. There are mainly two types of bifurcation: local bifurcation and global bifurcation. Local bifurcation is the study of qualitative changes in phase portraits that take place in a neighbourhood of a point. For example, saddle-node, transcritical, pitchfork and Hopf bifurcations are local bifurcations. Bifurcations that are not local are called global. Examples of global bifurcations are heteroclinic (or homoclinic) bifurcations and saddle-node bifurcations of cycles. Global bifurcations can be more complicated than local bifurcations, and some are not completely understood at present (e.g. global bifurcations in the Lorenz system).

Before going further, we introduce some important types of orbits that can occur in systems (1): equilibrium points, periodic orbits, homoclinic orbits and heteroclinic orbits.

A point x_0 is an *equilibrium point* of (1) if $\phi(t, x_0, \mu) = x_0$ for all t. In other words, x_0 corresponds to the intersections of (1) with $\dot{x} = 0$.

A *periodic orbit* of (1) is any closed solution curve which is not an equilibrium point. Closed solution curves correspond to periodic solutions, since $\phi(\cdot, x_0, \mu)$ defines a closed solution curve if and only if for all $t \in \Re$, $\phi(t + T, x_0, \mu) = \phi(t, x_0, \mu)$ for some $T > 0$. The smallest T for which the above equality holds is called the *period* of the periodic orbit $\phi(\cdot, x_0, \mu)$.

We call an equilibrium point or a periodic orbit Γ *stable* if for each $\epsilon > 0$ there is a neighbourhood U of Γ such that for all $x_0 \in U$ and $t \geq 0$, we have $dist(\phi(t, x_0, \mu), \Gamma) < \epsilon$, where $x(t) = \phi(t, x_0, \mu)$ is the solution of (1) satisfying the initial condition $x(0) = x_0$. A periodic orbit Γ is called *unstable* if it is not stable. We call Γ *asymptotically stable* if it is stable, and for all points x_0 in some neighbourhood V of Γ we have

$$\lim_{t \to \infty} dist(\phi(t, x_0, \mu), \Gamma) = 0.$$

Before talking about homoclinic and heteroclinic orbits, we define the stable and unstable manifolds of an equilibrium point. The *stable manifold* of an equilibrium point p_0 is the set

$$W^s(p_0) = \{x_0 |\ \phi(t, x_0, \mu) \to p_0 \quad as\ t \to \infty\},$$

and the *unstable manifold* of p_0 is the set

$$W^u(p_0) = \{x_0 |\ \phi(t, x_0, \mu) \to p_0 \quad as\ t \to -\infty\}.$$

A point q is a *homoclinic point* for (1) if there is an equilibrium point $p_0 \neq q$ such that $q \in W^s(p_0) \cap W^u(p_0)$, where $W^s(p_0)$ and $W^u(p_0)$ are the stable and unstable manifolds of p_0. The orbit of a homoclinic point is called a *homoclinic orbit.* Thus, a solution curve Γ is a homoclinic orbit if and only if $\Gamma \subset W^s(p_0) \cap W^u(p_0)$. In other words, the solution curve Γ approaches p_0 both as $t \to \infty$ and as $t \to -\infty$.

A point r is a *heteroclinic point* for (1) if there are two distinct equilibrium points $p_0 \neq r$, $q_0 \neq r$ such that $r \in W^s(p_0) \cap W^u(q_0)$ (or $W^s(q_0) \cap W^u(p_0)$). The orbit of a heteroclinic point is called a *heteroclinic orbit.* A solution curve Γ is a heteroclinic orbit if $\Gamma \subset W^s(p_0) \cap W^u(q_0)$, i.e., the solution curve Γ approaches p_0 as $t \to \infty$ and approaches q_0 as $t \to -\infty$.

Periodic, homoclinic and heteroclinic orbits play an important role in the thesis. We study perturbed Hamiltonian systems in the plane, of the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} + \epsilon f(x, y, \epsilon, \mu) \\ \dot{y} &= -\frac{\partial H}{\partial x} + \epsilon g(x, y, \epsilon, \mu), \end{aligned} \tag{2}$$

where $H(x,y)$ is the Hamiltonian function, $f(x,y,\epsilon,\mu)$ and $g(x,y,\epsilon,\mu)$ are functions depending smoothly on x,y,ϵ,μ, $\mu = (\mu_1,\ldots,\mu_n) \in \Re^n$ and ϵ is a small perturbation parameter. We make two assumptions. The first assumption is that for $\epsilon = 0$, the system (2) has a continuous one-parameter family of periodic orbits Γ_h of period T_h with parameter h belonging to an interval $I \subset \Re$ equal to the total energy along the orbit. The second assumption is that for $\epsilon = 0$, the system (2) has a homoclinic orbit Γ_0 to a hyperbolic saddle point x_0. If we add some small perturbations (i.e. make $\epsilon \neq 0$ but small), what is the behaviour? For example, will the homoclinic orbit persist under perturbation and if so, for which parameter values? Will the homoclinic orbit break as we pass through some parameter values? How many of the limit cycles from the continuous family of periodic orbits are still preserved under perturbation? To answer these questions, we employ Melnikov's method, a global perturbation method due to Melnikov [15] and others, which gives us an excellent tool to study the global bifurcations that occur at homoclinic (or heteroclinic) loops or in a one-parameter family of periodic orbits of perturbed system.

To apply Melnikov's method, we need to compute two quantities:

1. The Melnikov function along a periodic orbit, which is used to prove the existence of periodic orbits in system (2) for small ϵ.

2. The Melnikov function along a homoclinic (or heteroclinic) orbit, which is used to prove the existence of homoclinic (or heteroclinic) orbits in system (2) for small ϵ.

The Melnikov function along a periodic orbit is very similar to the Melnikov function along a homoclinic orbit. You may ask: Is it always possible to compute the Melnikov function (along the periodic orbit) for system (2)? The answer is no but in many useful cases (as in our examples), the Melnikov function can be either computed explicitly or expressed as a linear combination of complete elliptic integrals. Although the computation of the Melnikov function can be somewhat technical, the benefits are great: we can determine the number, positions and multiplicities of limit cycles of (2) by the number, positions and multiplicities of the zeros of Melnikov function. If the Melnikov function of (2) is identically equal to zero across the continuous band of periodic orbits, a higher-order analysis is necessary.

The Melnikov type functions first appear in the 1890 paper [16] by H. Poincaré, in the 1963 paper by V. Melnikov [15], in the 1964 paper by V.I. Arnold [2], in the book by A.A. Androdov et al. [1]. It also is given in textbooks, such as the one by Guckenheimer & Holmes [11]. It is therefore difficult to pin down the origins of Melnikov's method. It suffices to say that the idea of computing the displacement function, as well as its partial derivatives with respect to parameters in terms of certain functions along periodic orbits, was used by many mathematicians working on the theory of dynamical systems at various times during the past 100 years.

The thesis is organized as follows: In Chapter 1, we derive the (first-order) Melnikov function along a periodic orbit. We then prove several theorems concerning the exact number, positions and multiplicities of limit cycles of (2). We then derive the Melnikov function along a homoclinic orbit and give a theorem which guarantees that a unique homoclinic orbit for (2) exists for some parameters μ. In Chapter 2, we give several examples to illustrate the versatility and power of Melnikov's method. The examples are worked in the context of normal forms [11, p. 365-376] [8, p. 54-83]. In some of the examples, the Melnikov function can be computed explicitly, while in other examples the Melnikov functions are expressed as linear combinations of complete elliptic integrals. We analyze the two-parameter system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 x + \mu_2 y \pm x^3 - x^2 y, \end{aligned}$$

where $(x, y) \in \Re^2$, $(\mu_1, \mu_2) \in \Re^2$, studied by Takens [23] in his well known 1974 paper. In Chapter 3, we extend the first-order theory to higher-order. In particular, we use Françoise's recursive algorithm [10] to give us a formula for the second-order Melnikov function for certain perturbed Hamiltonian systems in $\Re^2$. Finally, in Chapter 4, I give an example in which the first-order Melnikov function is identically equal to zero and so a second-order analysis is required to determine the number and positions of limit cycles.

Chapter 1
First-order Melnikov theory

1.1. Introduction

We introduce Melnikov theory in this chapter. With this theory we are able to determine global bifurcations that occur in perturbations of one-parameter families of periodic orbits, or of homoclinic or heteroclinic orbits. Assuming that when $\epsilon = 0$, the system (2) has a one-parameter family of periodic orbits, can we determine which periodic orbits are still preserved under perturbation? Finding the zeros of the Melnikov function, which is defined in Section 2, can allow us to do that. Therefore, a difficult global bifurcation problem is reduced to a reasonably straightforward computational problem. In Section 2, we derive the Melnikov function for (2) and show that in simple cases the periodic orbits correspond to zeros of the Melnikov function. This is a standard result. See, for example, Guckenheimer & Holmes [11] or Andronov et al. [1]. We then present a more general theory relating the multiplicities of periodic orbits to the multiplicities of the zeros of the Melnikov function, using the Implicit Function Theorem [19] or the Weierstrass Preparation Theorem [1, p. 388].

1.2. Derivation of the first-order Melnikov function

In this section, we derive a formula for the first-order Melnikov function for system (2). We assume that the functions $f(x, y, \epsilon, \mu)$ and $g(x, y, \epsilon, \mu)$ in (2) depend smoothly on x, y, ϵ, μ. Following Guckenheimer and Holmes [11, p. 184–188], we make the assumptions:

Assumption 1.1 *For $\epsilon = 0$, the system (2) has a homoclinic orbit $\Gamma_0 : x = \gamma_0(t)$, $-\infty < t <$*

∞ *at a hyperbolic saddle point* x_0.

Assumption 1.2 *For* $\epsilon = 0$, *the system (2) has a continuous one-parameter family of periodic orbits* $\Gamma_h : (x, y) = (x_h(t), y_h(t)) = \gamma_h(t)$, $0 \leq t \leq T_h$ *of period* T_h *with parameter* $h \in I \subset \Re$ *equal to the total energy along the orbit.*

We are interested in knowing which periodic solutions give rise to periodic solutions after perturbations. For that purpose we consider a curve Σ normal to the family Γ_h, parameterized by the value of h for the level curve of the Hamiltonian function $H(x, y) = h$, and we calculate the Poincaré return map[1] $P(h, \epsilon, \mu)$ for initial values h in an open set of Σ. We assume that the return map $P(h, \epsilon, \mu)$ is well-defined on an open set U containing $I \times \{0\} \times \Re^n$. Then the displacement function

$$d(h, \epsilon, \mu) = P(h, \epsilon, \mu) - h \tag{1.1}$$

is also well-defined on U, and zeros of the displacement function $d(h, \epsilon, \mu)$ correspond to periodic solutions (see Figure 1.1). Let us call $\Gamma_{h,\epsilon,\mu} : (x_{h,\epsilon,\mu}(t), y_{h,\epsilon,\mu}(t)) = \gamma_{h,\epsilon,\mu}(t)$ the trajectory of the system (2) starting at h on Σ. Then

$$\begin{aligned}
d(h, \epsilon, \mu) &= \int_{\Gamma_{h,\epsilon,\mu}} \dot{H}\, dt \\
&= \int_{\Gamma_{h,\epsilon,\mu}} \left(\frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \right) dt \\
&= \int_{\Gamma_{h,\epsilon,\mu}} \left[\frac{\partial H}{\partial x} \left(\frac{\partial H}{\partial y} + \epsilon f(x, y, \epsilon, \mu) \right) \right. \\
&\quad \left. + \frac{\partial H}{\partial y} \left(-\frac{\partial H}{\partial x} + \epsilon g(x, y, \epsilon, \mu) \right) \right] dt \\
&= \epsilon \int_{\Gamma_{h,\epsilon,\mu}} \left[f(x, y, 0, \mu) \frac{\partial H}{\partial x}(x, y) + g(x, y, 0, \mu) \frac{\partial H}{\partial y}(x, y) \right] dt + O(\epsilon^2) \\
&= \epsilon \int_0^{T(h,\epsilon,\mu)} \left[f(x_{h,\epsilon,\mu}(t), y_{h,\epsilon,\mu}(t), 0, \mu) \frac{\partial H}{\partial x}(x_{h,\epsilon,\mu}(t), y_{h,\epsilon,\mu}(t)) \right. \\
&\quad \left. + g(x_{h,\epsilon,\mu}(t), y_{h,\epsilon,\mu}(t), 0, \mu) \frac{\partial H}{\partial y}(x_{h,\epsilon,\mu}(t), y_{h,\epsilon,\mu}(t)) \right] dt + O(\epsilon^2),
\end{aligned} \tag{1.2}$$

[1]The Poincaré return map $P(h, \epsilon, \mu)$ is a mapping from Σ into itself. More explicitly, $P(h, \epsilon, \mu)$ is the value of the Hamiltonian function at the point where the trajectory of (2) first returns to Σ after starting from the point on Σ where the value of the Hamiltonian function is h. A periodic orbit corresponds to a value of h such that $P(h, \epsilon, \mu) = h$.

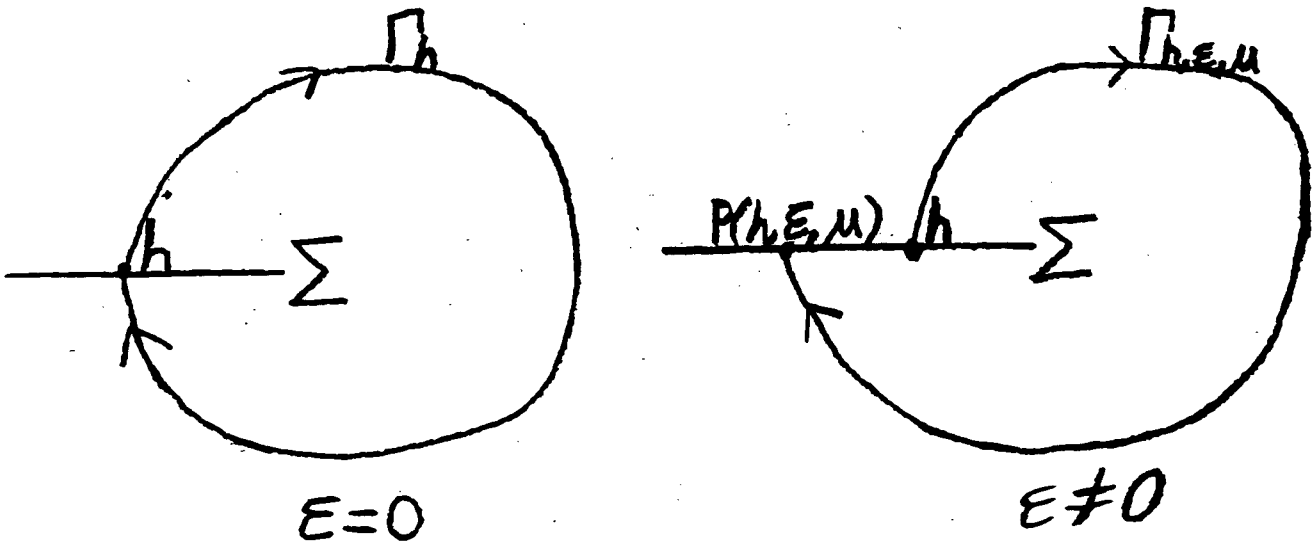

Figure 1.1: Illustration of the Poincaré map.

where we have expanded f and g in their Taylor series in ϵ, and $T(h,\epsilon,\mu)$ is time it takes the trajectory $\Gamma_{h,\epsilon,\mu}$ to first return to Σ. Formula (1.2) is exact but involves the orbit $\Gamma_{h,\epsilon,\mu}$ of the perturbed Hamiltonian system (2), which we do not know explicitly. However, for small $\epsilon > 0$, the orbit $\Gamma_{h,\epsilon,\mu}$ of (2) differs only slightly from the closed orbit Γ_h (for (2) when $\epsilon = 0$). Therefore, we approximate $\Gamma_{h,\epsilon,\mu}$ by the closed orbit Γ_h in order to find an approximation to formula (1.2). Using the smoothness of $\Gamma_{h,\epsilon,\mu}$ with respect to the parameter ϵ, we have

$$T(h,\epsilon,\mu) = T_h + O(\epsilon),$$

where T_h is the period of the unperturbed closed orbit Γ_h, and

$$x_{h,\epsilon,\mu}(t) = x_h(t) + O(\epsilon), \quad y_{h,\epsilon,\mu}(t) = y_h(t) + O(\epsilon),$$

where $(x_h(t), y_h(t))$ are the coordinates of Γ_h. Thus we see that $\Gamma_{h,\epsilon,\mu}$ lies in an $O(\epsilon)$ neighbourhood of Γ_h and therefore tends to the closed orbit Γ_h as $\epsilon \to 0$. This justifies the following

first-order approximation to formula (1.2):

$$\begin{aligned} d(h,\epsilon,\mu) &= \epsilon \int_0^{T_h} \left[f(x_h(t), y_h(t), 0, \mu) \frac{\partial H}{\partial x}(x_h(t), y_h(t)) \right. \\ &\quad \left. + g(x_h(t), y_h(t), 0, \mu) \frac{\partial H}{\partial y}(x_h(t), y_h(t)) \right] dt + O(\epsilon^2) \\ &= \epsilon \int_0^{T_h} [-f(x_h(t), y_h(t), 0, \mu)\, \dot{y_h}(t) \\ &\quad + g(x_h(t), y_h(t), 0, \mu)\, \dot{x_h}(t)]\, dt + O(\epsilon^2), \end{aligned}$$

since $\dot{x_h}(t) = \partial H/\partial y(x_h(t), y_h(t))$ and $\dot{y_h}(t) = -\partial H/\partial x(x_h(t), y_h(t))$. Now, making change of variables

$$dx_h(t) = \dot{x_h}(t)\, dt, \quad dy_h(t) = \dot{y_h}(t)\, dt,$$

and using our earlier notation

$$\Gamma_h : (x, y) = (x_h(t), y_h(t)) = \gamma_h(t), \quad 0 \le t \le T_h$$

for the closed orbit of the unperturbed Hamiltonian system (2), we have

$$d(h,\epsilon,\mu) = \epsilon \oint_{\Gamma_h} (g(x,y,0,\mu)\, dx - f(x,y,0,\mu)\, dy) + O(\epsilon^2).$$

This leads to the following definition of the Melnikov function $M(h,\mu)$, which determines the displacement function $d(h,\epsilon,\mu)$ to $O(\epsilon)$.

Definition 1.1 *Assume that Assumption 1.2 holds for all $h \in I$. Then*

$$d(h,\epsilon,\mu) = \epsilon\, M(h,\mu) + O(\epsilon^2)$$

as $\epsilon \to 0$, where $M(h,\mu)$ is the (first-order) Melnikov function for (2) given by

$$M(h,\mu) = \oint_{\Gamma_h} (g(x,y,0,\mu)\, dx - f(x,y,0,\mu)\, dy). \tag{1.3}$$

Remark 1.1 *For computational purposes, it is useful to write the Melnikov function $M(h,\mu)$ as*

$$M(h,\mu) = \int_0^{T_h} \left[f(x_h(t), y_h(t), 0, \mu) \frac{\partial H}{\partial x}(x_h(t), y_h(t)) + g(x_h(t), y_h(t), 0, \mu) \frac{\partial H}{\partial y}(x_h(t), y_h(t)) \right] dt. \tag{1.4}$$

1.3. First-order Melnikov theory

In Section 1, we know that the limit cycles correspond to the zeros of the displacement function, and that the first-order contribution (in ϵ) to the displacement function is essentially the Melnikov function. In order to show that limit cycles correspond to the zeros of the Melnikov function, and to know more 'information' about the limit cycles (i.e. uniqueness, etc.), we need to assume some non-degeneracy condition. For example, the simplest is $\frac{\partial M}{\partial h}(h_0, \mu_0) \neq 0$, in which case we can apply the implicit function theorem to obtain a unique, hyperbolic limit cycle for small ϵ. This result will be proved in this section. But before doing that, we need to define hyperbolicity and multiplicity of limit cycles.

A limit cycle Γ_h is *hyperbolic* if

$$\frac{\partial P}{\partial h}(h, \epsilon, \mu) \neq 1,$$

where $P(h, \epsilon, \mu)$ is the return map in Section 1.2, and h is the point on Σ where Γ_h intersects.

Hyperbolicity is important since it tells us about the stability of the limit cycles. For example, if $0 < \frac{\partial P}{\partial h}(h, \epsilon, \mu) < 1$, then the limit cycle is stable. If $\frac{\partial P}{\partial h}(h, \epsilon, \mu) > 1$, then the limit cycle is unstable. We call a limit cycle *non-hyperbolic* if it is not hyperbolic. A non-hyperbolic limit cycle tells us nothing about the stability of the limit cycle (since linearization is not sufficient to determine stability).

A limit cycle Γ_h is called a limit cycle of *multiplicity* k if

$$d(h, \epsilon, \mu) = \frac{\partial d}{\partial h}(h, \epsilon, \mu) = \cdots = \frac{\partial^{(k-1)} d}{\partial h^{(k-1)}}(h, \epsilon, \mu) = 0,$$

and

$$\frac{\partial^{(k)} d}{\partial h^{(k)}}(h, \epsilon, \mu) \neq 0.$$

If $k = 1$, then Γ_h is called a simple limit cycle. We now prove a Theorem [1] giving the simplest conditions under which the perturbed Hamiltonian system (2) has a unique, hyperbolic limit cycle.

Theorem 1.1 *Assume that Assumption 1.2 holds for all $h \in I$. If there exists a $h_0 \in I$ and a $\mu_0 \in \Re^n$ such that*

$$M(h_0, \mu_0) = 0 \quad \textit{and} \quad \frac{\partial M}{\partial h}(h_0, \mu_0) \neq 0,$$

then for all sufficiently small $\epsilon \neq 0$, the system (2) has a unique, hyperbolic limit cycle Γ_ϵ which tends to the periodic orbit Γ_{h_0} when $\epsilon \to 0$.

Proof: Under Assumption 1.2, $d(h, 0, \mu) \equiv 0$ for all $h \in I$ and $\mu \in \Re^n$. Define the function

$$F(h, \epsilon) = \begin{cases} \frac{d(h,\epsilon,\mu_0)}{\epsilon} & \text{if } \epsilon \neq 0 \\ \frac{\partial d}{\partial \epsilon}(h, 0, \mu_0) & \text{if } \epsilon = 0 \end{cases}$$

so that

$$d(h, \epsilon, \mu_0) = \epsilon F(h, \epsilon). \tag{1.5}$$

By Definition 1.1, we have

$$F(h, \epsilon) = M(h, \mu_0) + O(\epsilon).$$

Thus,

$$F(h_0, 0) = M(h_0, \mu_0) = 0,$$

and

$$\frac{\partial F}{\partial h}(h_0, 0) = \frac{\partial M}{\partial h}(h_0, \mu_0) \neq 0.$$

Then, by the implicit function theorem [19], there exists a $\delta > 0$ and a unique function $h = h(\epsilon)$, defined for $|\epsilon| < \delta$, such that $h(0) = h_0$ and $F(h(\epsilon), \epsilon) = 0$ for all $|\epsilon| < \delta$. It follows from the above definition of $F(h, \epsilon)$ that for sufficiently small ϵ, $d(h(\epsilon), \epsilon, \mu_0) = 0$ and for sufficiently small $\epsilon \neq 0$, $\partial d/\partial h(h(\epsilon), \epsilon, \mu_0) \neq 0$. Therefore, for sufficiently small $\epsilon \neq 0$, there is a unique isolated limit cycle Γ_ϵ of (2) cutting the section Σ at the point $h = h(\epsilon)$. Using (1.1), and the fact that $\partial d/\partial h(h(\epsilon), \epsilon, \mu_0) \neq 0$, it follows that $\partial P/\partial h(h(\epsilon), \epsilon, \mu_0) \neq 1$ and so the limit cycle Γ_ϵ is hyperbolic. Since $h(\epsilon) = h_0 + O(\epsilon)$, this limit cycle tends to the cycle Γ_{h_0} as $\epsilon \to 0$. $\square$

In a more degenerate situation, for example, if $M(h_0, \mu_0) = \partial M/\partial h(h_0, \mu_0) = 0$, then assuming $\partial^2 M/\partial h^2(h_0, \mu_0) \neq 0$ and $\partial M/\partial \mu_1(h_0, \mu_0) \neq 0$, we can use the Weierstrass preparation theorem to obtain a unique non-hyperbolic limit cycle of multiplicity two for $\epsilon \neq 0$. This result will be proved next.

Theorem 1.2 *Assume that Assumption 1.2 holds for all $h \in I$. Then if there exists a $h_0 \in I$ and a $\mu_0 \in \Re^n$ such that*

$$M(h_0, \mu_0) = \frac{\partial M}{\partial h}(h_0, \mu_0) = 0,$$

$$\frac{\partial^2 M}{\partial h^2}(h_0, \mu_0) \neq 0, \frac{\partial M}{\partial \mu_j}(h_0, \mu_0) \neq 0$$

for some $j = 1, \ldots, n$, it follows that for all sufficiently small ϵ, there are functions $h(\epsilon) = h_0 + O(\epsilon)$, $\mu(\epsilon) = \mu_0 + O(\epsilon)$ such that for sufficiently small $\epsilon \neq 0$, the system (2) has a unique non-hyperbolic limit cycle of multiplicity two which tends to the periodic orbit Γ_{h_0} as $\epsilon \to 0$.

Proof: First we let $\mu_0 = (\mu_1^0, \mu_2^0, \ldots, \mu_n^0)$. Without loss of generality, we assume that $j = 1$ (i.e. $\partial M/\partial \mu_1(h_0, \mu_0) \neq 0$). Under Assumption 1.2, $d(h, 0, \mu) \equiv 0$ for all $h \in I$ and $\mu \in \Re^n$. Define the function

$$F(h, \epsilon, \mu) = \begin{cases} \frac{d(h,\epsilon,\mu)}{\epsilon} & \text{if } \epsilon \neq 0 \\ \frac{\partial d}{\partial \epsilon}(h, 0, \mu) & \text{if } \epsilon = 0 \end{cases}$$

so that

$$d(h, \epsilon, \mu) = \epsilon F(h, \epsilon, \mu). \tag{1.6}$$

Also,

$$F(h_0, 0, \mu_0) = M(h_0, \mu_0) = 0,$$

$$\frac{\partial F}{\partial h}(h_0, 0, \mu_0) = \frac{\partial M}{\partial h}(h_0, \mu_0) = 0,$$

$$\frac{\partial^2 F}{\partial h^2}(h_0, 0, \mu_0) = \frac{\partial^2 M}{\partial h^2}(h_0, \mu_0) \neq 0,$$

$$\frac{\partial F}{\partial \mu_1}(h_0, 0, \mu_0) = \frac{\partial M}{\partial \mu_1}(h_0, \mu_0) \neq 0.$$

Therefore, by the Weierstrass preparation theorem [1, p. 388, Theorem 69], there exists a $\delta > 0$ such that

$$F(h, \epsilon, \mu) = [(h - h_0)^2 + A_1(\epsilon, \mu)(h - h_0) + A_2(\epsilon, \mu)]\Phi(h, \epsilon, \mu), \tag{1.7}$$

where $A_1(\epsilon, \mu)$, $A_2(\epsilon, \mu)$, and $\Phi(h, \epsilon, \mu)$ are defined for $|\epsilon| < \delta$, $|h - h_0| < \delta$, $|\mu - \mu_0| < \delta$; $A_1(0, \mu_0) = A_2(0, \mu_0) = 0$, $\Phi(h_0, 0, \mu_0) \neq 0$, and $\frac{\partial A_2}{\partial \mu_1}(0, \mu_0) \neq 0$ since $\frac{\partial F}{\partial \mu_1}(h_0, 0, \mu_0) \neq 0$. It follows from (1.7) that

$$\begin{aligned} \frac{\partial F}{\partial h}(h, \epsilon, \mu) &= [2(h - h_0) + A_1(\epsilon, \mu)]\Phi(h, \epsilon, \mu) \\ &\quad + [(h - h_0)^2 + A_1(\epsilon, \mu)(h - h_0) + A_2(\epsilon, \mu)]\frac{\partial \Phi}{\partial h}(h, \epsilon, \mu), \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \frac{\partial^2 F}{\partial h^2}(h, \epsilon, \mu) &= 2\Phi(h, \epsilon, \mu) + 2[2(h - h_0) + A_1(\epsilon, \mu)]\frac{\partial \Phi}{\partial h}(h, \epsilon, \mu) \\ &\quad + [(h - h_0)^2 + A_1(\epsilon, \mu)(h - h_0) + A_2(\epsilon, \mu)]\frac{\partial^2 \Phi}{\partial h^2}(h, \epsilon, \mu). \end{aligned} \tag{1.9}$$

If $2(h - h_0) + A_1(\epsilon, \mu) = 0$ and $(h - h_0)^2 + A_1(\epsilon, \mu)(h - h_0) + A_2(\epsilon, \mu) = 0$, it follows from (1.7), (1.8) and (1.9) that (2) has a multiplicity two limit cycle. Therefore, we set $h = h_0 - A_1(\epsilon, \mu)/2$ and find from (1.7) that $F(h_0 - A_1(\epsilon, \mu)/2, \epsilon, \mu) = 0$ if and only if the function

$$B(\epsilon, \mu) \stackrel{\text{def}}{=} -\frac{1}{4}A_1^2(\epsilon, \mu) + A_2(\epsilon, \mu) = 0$$

(since, by continuity, $\Phi(h, \epsilon, \mu) \neq 0$ for small $|\epsilon|$, $|h - h_0|$, $|\mu - \mu_0|$). Now,

$$B(0, \mu_0) = -\frac{1}{4}A_1^2(0, \mu_0) + A_2(0, \mu_0) = 0$$

(since $A_1(0, \mu_0) = A_2(0, \mu_0) = 0$) and

$$\frac{\partial B}{\partial \mu_1}(0, \mu_0) = \frac{\partial A_2}{\partial \mu_1}(0, \mu_0) \neq 0$$

(since $\partial F/\partial \mu_1(h_0, 0, \mu_0) \neq 0$). By the implicit function theorem, there exists a $\delta > 0$ and a unique function $\mu_1 = \mu_1(\epsilon, \mu_2, \ldots, \mu_n)$, defined for $|\epsilon| < \delta$, $|\mu_2 - \mu_2^0| < \delta, \ldots, |\mu_n - \mu_n^0| < \delta$, such

that $\mu_1^0 = \mu_1(0, \mu_2^0, \ldots, \mu_n^0)$ and $B(\epsilon, \mu_1(\epsilon, \mu_2, \ldots, \mu_n), \mu_2, \ldots, \mu_n) = 0$ for $|\epsilon| < \delta$, $|\mu_2 - \mu_2^0| < \delta$, $\ldots$, $|\mu_n - \mu_n^0| < \delta$. For $|\epsilon| < \delta$, we define $\mu(\epsilon) = (\mu_1(\epsilon, \mu_2^0, \ldots, \mu_n^0), \mu_2^0, \ldots, \mu_n^0)$. Then, $\mu(\epsilon) = \mu_0 + O(\epsilon)$ and (2) has a unique multiplicity two limit cycle Γ_ϵ through the point

$$h(\epsilon) = h_0 - A_1(\epsilon, \mu(\epsilon))/2 \tag{1.10}$$

on Σ. Using (1.6) and the fact that $\partial F/\partial h(h(\epsilon), \epsilon, \mu(\epsilon)) = 0$ with $h(\epsilon)$, $\mu(\epsilon)$ defined above, we have $\partial d/\partial h(h(\epsilon), \epsilon, \mu(\epsilon)) = 0$. Now, using (1.1), we immediately get $\partial P/\partial h(h(\epsilon), \epsilon, \mu(\epsilon)) = 1$ and so Γ_ϵ is non-hyperbolic. Finally, by continuity with respect to initial conditions and parameters, it follows that Γ_ϵ tends to the cycle Γ_{h_0} as $\epsilon \to 0$ since $A_1(0, \mu_0) = 0$. $\square$

Remarks 1.1 *1. The proof of Theorem 1.2 in fact establishes that there is an n-dimensional surface $\mu_1 = \mu_1(\epsilon, \mu_2, \ldots, \mu_n)$ through the point $(0, \mu_0) \in \Re^{n+1}$ on which (2) has a non-hyperbolic multiplicity-two limit cycle for sufficiently small $\epsilon \neq 0$. On one side of the surface where $B(\epsilon, \mu) < 0$, the equation $F(h(\epsilon), \epsilon, \mu) = 0$ with $h(\epsilon)$ given by (1.10) has two real solutions given by*

$$h^{\pm}(\epsilon, \mu) = h_0 - \frac{A_1(\epsilon, \mu)}{2} \pm \sqrt{\frac{A_1^2(\epsilon, \mu)}{4} - A_2(\epsilon, \mu)} \tag{1.11}$$

and on the other side where $B(\epsilon, \mu) > 0$, $F(h(\epsilon), \epsilon, \mu)$ has no real solution; i.e., system (2) has two limit cycles if $B(\epsilon, \mu) < 0$ and no limit cycle if $B(\epsilon, \mu) > 0$. It follows from (1.8) that if $h^{\pm}(\epsilon, \mu)$ are given by (1.11), then $\partial F/\partial h(h^{\pm}(\epsilon, \mu), \epsilon, \mu) \gtrless 0$ respectively. Using (1.6), we have $\partial d/\partial h(h^{\pm}(\epsilon, \mu), \epsilon, \mu) \gtrless 0$. Now, using (1.1), we immediately get $\partial P/\partial h(\alpha^{\pm}(\epsilon, \mu), \epsilon, \mu) \gtrless 1$; i.e. system (2) has two hyperbolic (unstable or stable) limit cycles if $B(\epsilon, \mu) < 0$. Therefore, the system (2) experiences a saddle-node bifurcation of periodic orbits as we cross this surface (see p. 22 for a description of saddle-node bifurcation).

2. If $M(h_0, \mu_0) = \partial M/\partial h(h_0, \mu_0) = \cdots = \partial^{(k-1)} M/\partial h^{(k-1)}(h_0, \mu_0) = 0$, $\partial^{(k)} M/\partial h^{(k)}(h_0, \mu_0) \neq 0$, $\partial M/\partial \mu_j(h_0, \mu_0) \neq 0$ for some $j = 1, \ldots, n$, then it can be shown that for sufficiently small $\epsilon \neq 0$, the system (2) has a unique non-hyperbolic limit cycle of multiplicity k which tends to the periodic orbit Γ_{h_0} as $\epsilon \to 0$.

3. *If $\partial^{(k)}M/\partial h^{(k)}(h_0,\mu_0) = 0$ for all $k = 0,1,2,\ldots$, then $\partial d/\partial\epsilon(h,0,\mu_0) \equiv 0$ for all $h \in I$ and a higher-order analysis in ϵ is necessary in order to determine the number, positions, and multiplicities of the limit cycles for small $\epsilon \neq 0$. This type of higher-order analysis is discussed in Chapter 3.*

Besides the global bifurcation of periodic orbits from a continuous band of cycles, there is another type of global bifurcation that occurs in systems in $\Re^2$, namely, the homoclinic loop (or heteroclinic loop) bifurcation [11, 24]. The Melnikov theory for (2) also gives us explicit information on this bifurcation.

Similar to what was done before, it can be shown (see Wiggins [24] for more details) that the distance $d(\epsilon,\mu)$ between the saddle separatrices $\Gamma^s_{\epsilon,\mu}$ and $\Gamma^u_{\epsilon,\mu}$ of (2) along a section Σ to the homoclinic orbit Γ_0 (in Assumption 1.1 above) at the point $\gamma_0(0) = a_0$ satisfies (see Figure 1.2)

$$d(\epsilon,\mu) = \epsilon\, M(\mu) + O(\epsilon^2),$$

where

$$M(\mu) = \int_{-\infty}^{\infty} \left(g(\gamma_0(t),0,\mu)\frac{\partial H}{\partial y}(\gamma_0(t)) + f(\gamma_0(t),0,\mu)\frac{\partial H}{\partial x}(\gamma_0(t)) \right) dt \tag{1.12}$$

is the Melnikov function for (2) along the homoclinic orbit Γ_0: $(x_0(t), y_0(t)) = \gamma_0(t)$, $-\infty < t < \infty$. From (1.4) and (1.12), we see that the Melnikov function along the periodic orbit Γ_h is very similar to the Melnikov function along the homoclinic orbit Γ_0. The following theorem [24] gives us conditions under which the system (2) has a unique homoclinic orbit.

Theorem 1.3 *Under Assumption 1.1, if there exists a $\mu_0 \in \Re^n$ such that $M(\mu_0) = 0$ and $\frac{\partial M}{\partial \mu_1}(\mu_0) \neq 0$, then for sufficiently small $\epsilon \neq 0$, there is a function $\mu(\epsilon) = \mu_0 + O(\epsilon)$ such that the system (2) has a unique homoclinic orbit Γ_ϵ which tends to the homoclinic orbit Γ_0 as $\epsilon \to 0$.*

Proof: Under Assumption 1.1, $d(0,\mu) \equiv 0$ for all $\mu \in \Re^n$. Define the function

$$F(\epsilon,\mu) = \begin{cases} \frac{d(\epsilon,\mu)}{\epsilon} & \text{if } \epsilon \neq 0 \\ \frac{\partial d}{\partial \epsilon}(0,\mu) & \text{if } \epsilon = 0. \end{cases}$$

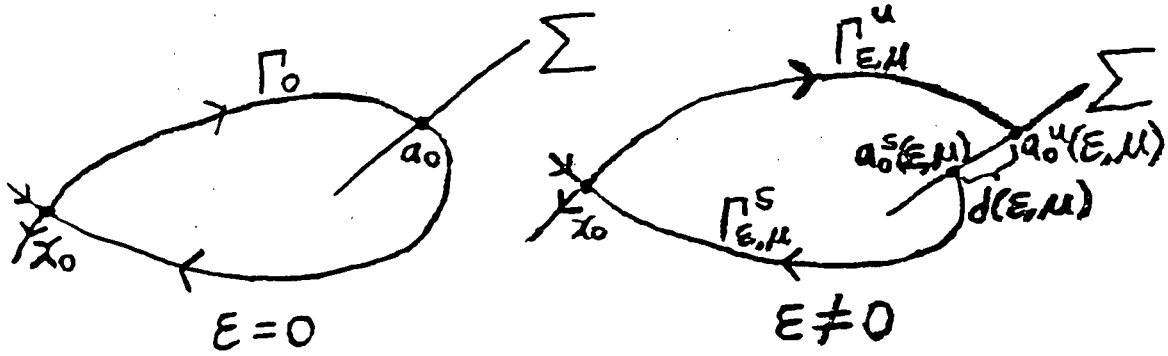

Figure 1.2: The displacement function $d(\epsilon, \mu)$ defined for ϵ near 0, near a homoclinic orbit.

Also,

$$F(0, \mu_0) = M(\mu_0) = 0,$$

$$\frac{\partial F}{\partial \mu_1}(0, \mu_0) = \frac{\partial M}{\partial \mu_1}(\mu_0) \neq 0.$$

By the implicit function theorem, there exists a $\delta > 0$ and a unique function $\mu_1 = \mu_1(\epsilon, \mu_2, \ldots, \mu_n)$ such that $\mu_1^0 = \mu_1(0, \mu_2^0, \ldots, \mu_n^0)$ and $F(\epsilon, \mu_1(\epsilon, \mu_2, \ldots, \mu_n), \mu_2, \ldots, \mu_n) = 0$ for all $|\epsilon| < \delta$, $|\mu_2 - \mu_2^0| < \delta, \ldots, |\mu_n - \mu_n^0| < \delta$. For $|\epsilon| < \delta$, we define $\mu(\epsilon) = (\mu_1(\epsilon, \mu_2^0, \ldots, \mu_n^0), \mu_2^0, \ldots, \mu_n^0)$. Then, $\mu(\epsilon) = \mu_0 + O(\epsilon)$ and (2) has a unique, homoclinic orbit Γ_ϵ. It then follows from the uniqueness of solutions and from the continuity of solutions with respect to initial conditions that the homoclinic orbit Γ_ϵ tends to Γ_0 as $\epsilon \to 0$. $\square$

Chapter 2
Applications of first-order Melnikov theory

In this chapter, we give two examples to illustrate the usefulness of the Melnikov theory developed in Chapter 1. The first example is the $(2n+1)$-th degree perturbed non-hyperbolic linear centre. The problem is determine the maximum number of limit cycles. To study this analytically, we compute the Melnikov function. The number, positions and multiplicities of the zeros of the Melnikov function is related to the number, positions and multiplicities of the limit cycles using Theorem 1.1 or 1.2. This example generalizes Theorem 76 in [1, p. 414]. The second example is the Bogdanov-Takens bifurcation with reflection symmetry. This occurs in the family of vector fields equivariant with respect to rotation by π in the plane which have an equilibrium with a double zero eigenvalue. The normal form of this bifurcation is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 x + \mu_2 y + a x^3 + b x^2 y, \qquad (a \neq 0,\ b \neq 0). \end{aligned}$$

This was first proved independently by Bogdanov [6] and Takens [23]. We study the system in a neighbourhood of parameter space near $(\mu_1, \mu_2) = (0,0)$, with emphasis on proving the existence of periodic orbits, homoclinic or heteroclinic orbits corresponding to different values of the parameters.

2.1. Perturbed linear centre

In this section, we use the Melnikov theory developed in Chapter 1 to study the number, positions and multiplicities of the limit cycles that occur in a $(2n+1)$-th degree perturbed non-hyperbolic linear centre of the form (2.2). One reason this example is of particular interest

is that it is useful in the study of Hilbert's 16th problem, which asks for an upper bound on the number of limit cycles for nth degree polynomial systems in terms of the degree n [12]. For $n = 1$, the maximum number of limit cycles is 0 since linear systems do not have any limit cycles. However, even for the simplest class of nonlinear systems (i.e. $n = 2$), the maximum number of limit cycles has not yet been determined [4, p. 283–284]. There are quadratic systems with as many as four limit cycles [20]. However, for a perturbed non-hyperbolic linear centre of the form

$$\begin{aligned} \dot{x} &= y + \epsilon f(x, y, \epsilon) \\ \dot{y} &= -x + \epsilon g(x, y, \epsilon), \end{aligned} \tag{2.1}$$

where f and g are polynomials of x, y with coefficients depending smoothly on small ϵ, some useful results concerning the maximum number of limit cycles have been obtained [5, 21, 26]. Bautin [5] proved that the maximum number of limit cycles in a quadratically perturbed non-hyperbolic linear centre is three. He also proved that this maximum number can only be attained in a sixth- or higher-order analysis in ϵ. That is, the kth-order Melnikov function, where $k \geq 6$, has at most three zeros. However, when $n = 3$ (i.e. a cubically perturbed linear centre), we still don't know the maximum number of limit cycles. We just know that this number is at least 11 [26]. One reason that the determination of the maximum number of limit cycles in system (2.1) is difficult is to find at which order k the maximum number of limit cycles will 'stabilize'. Therefore, one way to make this problem easier to study is fix the order k. Since we know the Melnikov function when $k = 1$, we can determine the maximum number of limit cycles from a first-order analysis of an n-th degree perturbed non-hyperbolic linear centre. This motivates us to consider the perturbed system

$$\begin{aligned} \dot{x} &= y + \epsilon(\mu_1 x + \mu_2 x^2 + \cdots + \mu_{2n+1} x^{2n+1}) \\ \dot{y} &= -x, \end{aligned} \tag{2.2}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_{2n+1}) \in \Re^{2n+1}$. For $\epsilon = 0$, (2.2) is a Hamiltonian system with $H(x, y) = (x^2 + y^2)/2$; it has a one-parameter family of periodic orbits

$$x_h(t) = \sqrt{2h}\cos t, \qquad y_h(t) = -\sqrt{2h}\sin t,$$

with the parameter $h \in (0, \infty)$ being the energy level along the periodic orbit. The Melnikov function is given by

$$\begin{aligned} M(h,\mu) &= \int_0^{2\pi} \left[f(x_h(t), y_h(t), 0, \mu) \frac{\partial H}{\partial x}(x_h(t), y_h(t)) + g(x_h(t), y_h(t), 0, \mu) \frac{\partial H}{\partial y}(x_h(t), y_h(t)) \right] dt \\ &= \int_0^{2\pi} x_h(t)(\mu_1 x_h(t) + \mu_2 x_h^2(t) + \cdots + \mu_{2n+1} x_h^{2n+1}(t))\, dt. \end{aligned}$$

Using the fact that $\int_0^{2\pi} x_h^k(t)\, dt = (2h)^{k/2} \int_0^{2\pi} \cos^k t\, dt = 0$ for k is odd, we get

$$\begin{aligned} M(h,\mu) &= 2h \int_0^{2\pi} (\mu_1 \cos^2 t + 2\mu_3 h \cos^4 t + \cdots + \mu_{2n+1}(2h)^n \cos^{2n+2} t)\, dt \\ &= 4\pi h \left(\frac{\mu_1}{2} + \frac{3}{4}\mu_3 h + \cdots + \frac{\mu_{2n+1}}{2^{2n+2}} \binom{2n+2}{n+1} (2h)^n \right), \end{aligned} \tag{2.3}$$

where $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Since $h = \alpha^2/2$, where $\alpha \in (0, \infty)$ is the positive x-axis intercept of the unperturbed periodic orbit, (2.3) becomes

$$M(\alpha, \mu) = 2\pi\alpha^2 \left(\frac{\mu_1}{2} + \frac{3}{8}\mu_3 \alpha^2 + \cdots + \frac{\mu_{2n+1}}{2^{2n+2}} \binom{2n+2}{n+1} \alpha^{2n} \right). \tag{2.4}$$

Using Theorems 1.1 and 1.2 we obtain the following result:

Result 2.1 *For sufficiently small $\epsilon \neq 0$, the above system has at most n limit cycles. Furthermore, for $\epsilon \neq 0$, it has exactly n hyperbolic limit cycles asymptotic to circles of radii r_j, $j = 1, \ldots, n$ as $\epsilon \to 0$ if and only if the nth degree equation in α^2*

$$\frac{\mu_1}{2} + \frac{3}{8}\mu_3 \alpha^2 + \cdots + \frac{\mu_{2n+1}}{2^{2n+2}} \binom{2n+2}{n+1} \alpha^{2n} = 0 \tag{2.5}$$

has n positive roots $\alpha^2 = r_j^2$, $j = 1, \ldots, n$.

Now, we illustrate with a concrete example with $n = 2$. Consider the system

$$\begin{aligned} \dot{x} &= y + \epsilon(\mu_1 x - 2x^3 + 3x^5) \\ \dot{y} &= -x. \end{aligned} \tag{2.6}$$

Using (2.4) with $\mu_1 = \mu_1$, $\mu_3 = -2$, $\mu_5 = 3$ and all other $\mu_i = 0$, we get

$$M(\alpha, \mu) = 2\pi\alpha^2 \left(\frac{\mu_1}{2} - \frac{3}{4}\alpha^2 + \frac{15}{16}\alpha^4 \right).$$

We need to solve

$$\frac{15}{16}\alpha^4 - \frac{3}{4}\alpha^2 + \frac{\mu_1}{2} = 0.$$

Solving, we have

$$\alpha^2 = \frac{2}{5}\left(1 \pm \sqrt{1 - \frac{10}{3}\mu_1}\right).$$

So we have the following results:

Result 2.2 *1. If $0 < \mu_1 < 0.3$ and $\epsilon \neq 0$ is sufficiently small, (2.6) has exactly two hyperbolic limit cycles asymptotic to circles of radii*

$$r = \sqrt{\frac{2}{5}\left(1 \pm \sqrt{1 - \frac{10}{3}\mu_1}\right)}$$

as $\epsilon \to 0$.

2. For all sufficiently small ϵ, there is a function $\mu_1(\epsilon) = 0.3 + O(\epsilon)$ such that (2.6) has a unique limit cycle of multiplicity two, asymptotic to the circle of radius $r = \sqrt{2/5}$ as $\epsilon \to 0$.

3. If $\mu_1 > 0.3$ and $\epsilon \neq 0$ is sufficiently small, (2.6) has no limit cycles.

The results above are illustrated by the numerical computations shown in Figures 2.1, 2.2, 2.3.

The result obtained above (c.f. Result 2.1) allows us to construct systems with as many limit cycles as we like. For example, suppose that we wish to find a system of the form (2.2) with exactly four limit cycles asymptotic to circles of radius $r = 1$, $r = 2$, $r = 3$ and $r = 4$. To do this, we simply set the polynomial $(\alpha^2 - 1)(\alpha^2 - 4)(\alpha^2 - 9)(\alpha^2 - 16)$ equal to the polynomial in (2.5) with $n = 4$. So we have

$$(\alpha^2 - 1)(\alpha^2 - 4)(\alpha^2 - 9)(\alpha^2 - 16) = \frac{\mu_1}{2} + \frac{3}{8}\mu_3\alpha^2 + \frac{5}{16}\mu_5\alpha^4 + \frac{35}{128}\mu_7\alpha^6 + \frac{63}{256}\mu_9\alpha^8,$$

or

$$\alpha^8 - 30\alpha^6 + 273\alpha^4 - 820\alpha^2 + 576 = \frac{\mu_1}{2} + \frac{3}{8}\mu_3\alpha^2 + \frac{5}{16}\mu_5\alpha^4 + \frac{35}{128}\mu_7\alpha^6 + \frac{63}{256}\mu_9\alpha^8.$$

Equating coefficients, we have $\mu_9 = 256/63$, $\mu_7 = -768/7$, $\mu_5 = 4368/5$, $\mu_3 = -6560/3$ and $\mu_1 = 1152$. For $\epsilon \neq 0$ sufficiently small, Result 2.1 implies that the system

$$\begin{aligned} \dot{x} &= y + \epsilon\left(1152x - \tfrac{6560}{3}x^3 + \tfrac{4368}{5}x^5 - \tfrac{768}{7}x^7 + \tfrac{256}{63}x^9\right) \\ \dot{y} &= -x \end{aligned} \tag{2.7}$$

has exactly four limit cycles asymptotic to r=1, 2, 3, 4 as $\epsilon \to 0$. The four limit cycles for this system with $\epsilon = 0.00005$ are shown in Figure 2.4.

2.2. Application to the Bogdanov-Takens bifurcation with reflection symmetry

We investigate the parameterized family of vector fields

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 x + \mu_2 y \pm x^3 - x^2 y, \end{aligned} \tag{2.8}$$

where $\mu = (\mu_1, \mu_2)$ are parameters. This system possesses a reflection symmetry under $(x, y) \mapsto (-x, -y)$. In other words, the vector field is equivariant with respect to a rotation in the plane by π. This bifurcation with reflection symmetry occurs frequently in applications. In general, the symmetry occurs due to the geometry of the system or to assumptions made for the model. When such a system undergoes a bifurcation, it is a general principle that the presence of symmetry in the system modifies the generic behaviour that would be expected in a system with no symmetry present (Arnold [3]). Moreover, people have studied the effect of small deviations from symmetry in the system (2.8), which appear in some models of chemical reactors [7, 17]. This is necessary in order to give a theoretical interpretation of experimental data like those of [17].

We will focus on the most difficult parts of the analysis of the system:

1. The determination of the exact number and positions of the limit cycles in the system.

2. The determination of the parameter values for which a homoclinic (or heteroclinic) orbit exists.

However, in order to apply the Melnikov theory developed in Chapter 1, we need to reduce the system (2.8) to a perturbed system in the form of (2) by rescaling the variables and parameters. This will be done in detail in this section.

The Bogdanov-Takens bifurcation with reflection symmetry results from unfoldings[1] of the normal form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= ax^3 + bx^2y. \end{aligned} \tag{2.9}$$

In the Appendix, we see that this normal form is obtained from a generic system with reflection symmetry at an equilibrium point with a double zero eigenvalue. Without loss of generality, we may assume the equilibrium point is the origin, and the vector field is of the form

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + F_3(x) + O(|x|^5),$$

where $F_3(x)$ is a homogeneous polynomial of degree (exactly) 3. In [23], Takens proves that all possible types of dynamical behaviour that can occur in C^∞ perturbations of the system (2.9) with reflection symmetry are determined by the unfolding of (2.9) given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 x + \mu_2 y + ax^3 + bx^2y. \end{aligned} \tag{2.10}$$

By rescaling the variables (i.e. $x \to (-|a|^{\frac{1}{2}}/b)x$, $y \to (|a|^{\frac{3}{2}}/b^2)y$, $\mu_1 \to (|a|^2/b^2)\mu_1$, $\mu_2 \to (-|a|/b)\mu_2$, $t \to (-b/|a|)t$), the system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 x + \mu_2 y \pm x^3 - x^2y. \end{aligned} \tag{2.11}$$

[1]system (1) is called an *unfolding* of the vector field $f_0(x)$ if it is embedded in a parameterized family of vector fields (1) with $f(x, \mu_0) = f_0(x)$.

Note that the case with the plus sign is only possible if $a > 0$, and the case with the minus sign is only possible if $a < 0$. Also, note that time reversal occurs if $b > 0$ (that is why we do not need to consider the system (2.10) when $b > 0$).

Before going on any further, we need a dictionary that establishes 'names' for the local bifurcations which occur in the Bogdanov-Takens system. The local bifurcations are well known and discussed more fully in [11, 24]. In the following, we give the list of these (with a brief description of each) which occur in the system (2.11).

1. Saddle-Node Bifurcation: The normal form for a saddle-node bifurcation in $\Re^2$ is

$$\begin{aligned} \dot{x} &= \alpha - x^2 \\ \dot{y} &= ay, \quad (a \neq 0). \end{aligned}$$

   This bifurcation describes the simultaneous creation or annihilation of a pair of equilibrium points as the parameter α varies. One equilibrium is a saddle and the other is a stable or unstable node. The saddle-node bifurcation is sometimes called the fold bifurcation. Although this bifurcation does not occur in the system (2.11), we need this notion in order to understand the saddle-node bifurcation of periodic orbits, which does occur in system (2.11).

2. Pitchfork Bifurcation: The normal form for a pitchfork bifurcation in $\Re^2$ is

$$\begin{aligned} \dot{x} &= \alpha x \pm x^3 \\ \dot{y} &= ay, \quad (a \neq 0). \end{aligned}$$

   Note that the above system possess a reflection symmetry under $(x, y) \mapsto (-x, -y)$ (thus the origin must always be an equilibrium point). This bifurcation describes the creation or annihilation of a pair of new equilibria (for the parameter α on one side of 0) from an existing one (i.e. the origin). By symmetry, each equilibrium of the pair of new equilibria must be of the same type.

3. Hopf Bifurcation: The normal form for Hopf bifurcation in $\Re^2$ is

$$\begin{aligned} \dot{x} &= -y + x(\alpha \pm (x^2 + y^2)) \\ \dot{y} &= x + y(\alpha \pm (x^2 + y^2)), \end{aligned}$$

or in polar coordinates

$$\begin{aligned} \dot{r} &= \alpha r \pm r^3, \quad (r \geq 0) \\ \dot{\theta} &= 1. \end{aligned}$$

We see that a family of periodic orbits emerges from an equilibrium (i.e. the origin) as the eigenvalues of the equilibrium cross the imaginary axis at $\alpha = 0$.

We are now in good shape to study system (2.11) in detail. Since this thesis focuses on global bifurcations and Melnikov theory, we just state (without proofs) the local bifurcations which occur in system (2.11). Those who are interested in the details of the local bifurcations (e.g. how to prove the existence of Hopf bifurcation) are encouraged to look at [11, 24]. First, consider the case with the minus sign in system (2.11). For $\mu_1 < 0$ there is only one equilibrium point at the origin, and it is a hyperbolic sink for $\mu_2 < 0$, a hyperbolic source for $\mu_2 > 0$ and a non-hyperbolic linear centre for $\mu_2 = 0$. For $\mu_1 > 0$ there are three equilibrium points, a hyperbolic saddle at the origin, and hyperbolic sources at $(\pm\sqrt{\mu_1}, 0)$ for $\mu_2 > \mu_1$, hyperbolic sinks at $(\pm\sqrt{\mu_1}, 0)$ for $\mu_2 < \mu_1$, and non-hyperbolic linear centres at $(\pm\sqrt{\mu_1}, 0)$ for $\mu_2 = \mu_1$. There is a pitchfork bifurcation at points on the μ_2-axis with $\mu_2 \neq 0$. For $\mu_1 < 0$ there is a supercritical Hopf bifurcation at points on the μ_1-axis, and for $\mu_1 > 0$ there is a subcritical Hopf bifurcation at points on $\mu_2 = \mu_1$. The partial bifurcation set and phase portraits is shown in Figure 2.5.

We know that for all $\mu_1 > 0$, the unstable limit cycles are generated (via Hopf bifurcation) at $\mu_2 < \mu_1$ (μ_2 close to μ_1). What happens to the unstable limit cycles as μ_2 continually decreases from μ_1 ($\mu_1 > 0$ fixed)? Do the limit cycles exist forever? Later in this chapter, we prove that this is actually not the case. In fact, the unstable limit cycles expand monotonically as μ_2 decreases from μ_1 until they intersect the saddle point at the origin and form a 'double' homoclinic orbit at some parameter value $\mu_2 = h(\mu_1)$, i.e., there exists a homoclinic loop bifurcation curve in the $\mu_1\mu_2$-plane given by $\mu_2 = h(\mu_1)$ for $\mu_1 > 0$. To prove the existence of the homoclinic-loop bifurcation curve requires the use of the Melnikov theory developed in Chapter 1. However, since the system (2.11) is not of the form (2), we use the rescaling of

variables and parameters

$$x = \epsilon u, \quad y = \epsilon^2 v, \quad \mu_1 = \epsilon^2 \gamma, \quad \mu_2 = \epsilon^2 \lambda, \quad t \to t/\epsilon, \tag{2.12}$$

given by Takens in [23], in order to reduce the system (2.11) to a perturbed system (i.e. (2)) to which the Melnikov theory developed in Chapter 1 applies. Substituting the rescaling transformation (2.12) into the system (2.11), we get

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= u - u^3 + \epsilon(\lambda v - u^2 v), \end{aligned} \tag{2.13}$$

where we have set $\gamma = +1$ (so that $\mu_1 > 0$ as on p. 373 of [11]). The system has three equilibria: $(0,0)$, $(1,0)$ and $(-1,0)$. Linearization shows that $(0,0)$ is a hyperbolic saddle point, and that $(\pm 1, 0)$ are hyperbolic sinks if $\lambda < 1$, hyperbolic sources if $\lambda > 1$ and non-hyperbolic linear centres if $\lambda = 1$. For $\epsilon > 0$, there is a subcritical Hopf bifurcation at these points on $\lambda = 1$.

For $\epsilon = 0$, (2.13) is a Hamiltonian system with Hamiltonian $H(u,v) = v^2/2 - u^2/2 + u^4/4$. The phase portrait with $\epsilon = 0$ is given in Figure 2.6. Note that the 'double' homoclinic orbits $\Gamma_0^{\pm}$ correspond to the level set $H(u,v) = 0$. Due to the reflection symmetry, we only need to compute the Melnikov function $M(\lambda)$ along Γ_0^+. Before doing so, we need to find an expression for the homoclinic solution $\gamma_0^+(t)$ (i.e. Γ_0^+: $(u_0(t), v_0(t)) = \gamma_0^+(t)$). Since Γ_0^+ corresponds to the level set $H(u,v) = 0$, this is represented by

$$v = \pm u \sqrt{1 - \frac{u^2}{2}},$$

so

$$\frac{du}{dt} = \pm u \sqrt{1 - \frac{u^2}{2}}.$$

Solving the above differential equation, we get

$$u_0^+(t) = \frac{\sqrt{2}}{\cosh(t)}.$$

Also,

$$v = \dot{u} = v_0^+(t) = -\sqrt{2}\frac{\tanh(t)}{\cosh(t)},$$

therefore

$$\gamma_0^+(t) = \left(\frac{\sqrt{2}}{\cosh(t)}, -\sqrt{2}\frac{\tanh(t)}{\cosh(t)} \right).$$

We now use (1.12) to compute the Melnikov function.

$$\begin{aligned} M(\lambda) &= \int_{-\infty}^{\infty} \left(g(\gamma_0^+(t), 0, \lambda)\frac{\partial H}{\partial v}(\gamma_0^+(t)) + f(\gamma_0^+(t), 0, \lambda)\frac{\partial H}{\partial u}(\gamma_0^+(t)) \right) dt \\ &= \int_{-\infty}^{\infty} (\lambda v_0^2(t) - u_0^2(t)v_0^2(t))\, dt \\ &= \int_{-\infty}^{\infty} \left(2\lambda\frac{\tanh^2(t)}{\cosh^2(t)} - 4\frac{\tanh^2(t)}{\cosh^4(t)} \right) dt \\ &= \frac{4}{3}\lambda - \frac{16}{15}, \end{aligned}$$

where the integral above can be evaluated using an integral table or a symbolic computation program such as Maple. We see that $M(\lambda) = 0$ if and only if

$$\lambda = \frac{4}{5}, \tag{2.14}$$

therefore, according to Theorem 1.3, for all sufficiently small $\epsilon \neq 0$, there is a function $\lambda(\epsilon) = 4/5 + O(\epsilon)$ such that the system (2.13) with $\lambda = \lambda(\epsilon)$ has two homoclinic orbits $\Gamma_\epsilon^\pm$ which tend to $\Gamma_0^\pm$ as $\epsilon \to 0$.

We next consider the difficult question of the exact number of periodic orbits that persist for sufficiently small $\epsilon \neq 0$. The proofs for these are essentially due to Carr [8] and Cushman and Sanders [9]. The main ideas are to compute the Melnikov function, which is expressed in terms of elliptic integrals, and then use Picard-Fuchs analysis to determine the number of zeros of the Melnikov function (e.g. the number of limit cycles) corresponding to different values of the parameters.

Now, using (1.3), the Melnikov function along the periodic orbit $\gamma_h(t)$ for system (2.13) is given

by

$$\begin{aligned} M(h,\lambda) &= \oint_{\Gamma_h} (g(u,v,0,\lambda)\,du - f(u,v,0,\lambda)\,dv) \\ &= \oint_{\Gamma_h} (\lambda v - u^2 v) du \\ &= \lambda \oint_{\Gamma_h} v\,du - \oint_{\Gamma_h} u^2 v\,du \\ &= \lambda I_1(h) - I_2(h), \end{aligned}$$

where

$$I_1(h) = \oint_{\Gamma_h} v\,du,$$

$$I_2(h) = \oint_{\Gamma_h} u^2 v\,du.$$

Let $R(h) = I_2(h)/I_1(h)$, then $M(h,\lambda) = 0$ if and only if the function

$$\bar{M}(h,\lambda) = \lambda - R(h) \tag{2.15}$$

vanishes (since by Green's Theorem,

$$I_1(h) = \oint_{\Gamma_h} v\,du = \iint_{Q(h)} du\,dv > 0, \quad h > -\frac{1}{4},$$

where $Q(h)$ is the region surrounded by Γ_h).

In what follows, we determine the number of zeros of the function $\bar{M}(h,\lambda)$ for $h \in (-1/4, 0)$ and $h \in (0,\infty)$ respectively. In order to do so, we derive and analyze the Picard-Fuchs system (the system of differential equations that $I_1(h)$ and $I_2(h)$ satisfies) and the Riccati equation (the differential equation that $R(h)$ satisfies). The analysis is similar to the ones given in Carr [8] and Cushman and Sanders [9], but is given in more detail. Note that the interior periodic orbits correspond to the level sets of the Hamiltonian $H(u,v) = h$ for $h \in (-1/4, 0)$ and the exterior periodic orbits correspond to $H(u,v) = h$ for $h \in (0,\infty)$. The "double" homoclinic loops correspond to $h = 0$. Considering v as a function of u and h, and differentiating $H(u,v) = h$ with respect to h, one obtains

$$\frac{\partial v}{\partial h} = \frac{1}{v}. \tag{2.16}$$

Also, differentiating $H(u,v) = h$ with respect to u, one obtains

$$v\frac{\partial v}{\partial u} - u + u^3 = 0. \tag{2.17}$$

Multiplying (2.17) by u^k/v and integrating by parts along Γ_h, we get the following identity:

$$-\oint_{\Gamma_h} \frac{u^{k+1}}{v} du + \oint_{\Gamma_h} \frac{u^{k+3}}{v} du = k \oint_{\Gamma_h} u^{k-1} v\, du. \tag{2.18}$$

Now, using (2.16) and the identity (2.18) with $k = 1$, we get

$$\begin{aligned} h\frac{dI_1}{dh} &= h \oint_{\Gamma_h} \frac{du}{v} \\ &= \frac{1}{2}\oint_{\Gamma_h} v\, du - \frac{1}{2}\oint_{\Gamma_h} \frac{u^2}{v} du + \frac{1}{4}\oint_{\Gamma_h} \frac{u^4}{v} du \\ &= \frac{1}{2}I_1 - \frac{1}{2}\frac{dI_2}{dh} + \frac{1}{4}\oint_{\Gamma_h} \frac{u^4}{v} du \\ &= \frac{1}{2}I_1 - \frac{1}{2}\frac{dI_2}{dh} + \frac{1}{4}\left(\oint_{\Gamma_h} v\, du + \oint_{\Gamma_h} \frac{u^2}{v} du\right) \\ &= \frac{3I_1}{4} - \frac{1}{4}\frac{dI_2}{dh}. \end{aligned} \tag{2.19}$$

Similarly, using (2.16) and the identity (2.18) when $k = 1$ and 3, we get

$$\begin{aligned} h\frac{dI_2}{dh} &= h \oint_{\Gamma_h} \frac{u^2}{v} du \\ &= \frac{1}{2}\oint_{\Gamma_h} u^2 v\, du - \frac{1}{2}\oint_{\Gamma_h} \frac{u^4}{v} du + \frac{1}{4}\oint_{\Gamma_h} \frac{u^6}{v} du \\ &= \frac{1}{2}I_2 - \frac{1}{2}\left(\oint_{\Gamma_h} v\, du + \oint_{\Gamma_h} \frac{u^2}{v} du\right) + \frac{1}{4}\left(3\oint_{\Gamma_h} u^2 v\, du + \oint_{\Gamma_h} \frac{u^4}{v} du\right) \\ &= \frac{1}{2}I_2 - \frac{1}{2}\left(I_1 + \frac{dI_2}{dh}\right) + \frac{1}{4}\left(3I_2 + \oint_{\Gamma_h} v\, du + \oint_{\Gamma_h} \frac{u^2}{v} du\right) \\ &= \frac{5}{4}I_2 - \frac{1}{4}I_1 - \frac{1}{4}\frac{dI_2}{dh}. \end{aligned} \tag{2.20}$$

Taking into account (2.19) and (2.20), we arrive at the following lemma:

Lemma 2.1 (Picard-Fuchs system) *The integrals $I_1(h)$ and $I_2(h)$ satisfy the following system of differential equations:*

$$\begin{aligned} h(h+\tfrac{1}{4})I_1' &= (\tfrac{3}{4}h + \tfrac{1}{4})I_1 - \tfrac{5}{16}I_2 \\ h(h+\tfrac{1}{4})I_2' &= -\tfrac{1}{4}hI_1 + \tfrac{5}{4}hI_2. \end{aligned} \tag{2.21}$$

Now we can easily derive the Riccati equation that $R(h)$ satisfies. Using (2.21),

$$\begin{aligned} h\left(h+\frac{1}{4}\right)R' &= h\left(h+\frac{1}{4}\right)\left(\frac{I_2'}{I_1}-\frac{I_2}{I_1^2}I_1'\right) \\ &= \left(-\frac{1}{4}h+\frac{5}{4}h\frac{I_2}{I_1}\right)-\left(\frac{3}{4}h+\frac{1}{4}\right)\frac{I_2}{I_1}+\frac{5}{16}\frac{I_2^2}{I_1^2} \\ &= \left(-\frac{1}{4}h+\frac{5}{4}hR\right)-\left(\frac{3}{4}h+\frac{1}{4}\right)R+\frac{5}{16}R^2 \\ &= \frac{5}{16}R^2+\left(\frac{1}{2}h-\frac{1}{4}\right)R-\frac{1}{4}h. \end{aligned}$$

Therefore, we have the following lemma:

Lemma 2.2 (Riccati equation) *The function $R(h)$ satisfies the Riccati equation*

$$4h(4h+1)R' = 5R^2+(8h-4)R-4h. \tag{2.22}$$

We can also write (2.22) as a system with respect to (h, R):

$$\begin{aligned} \dot{h} &= 4h(4h+1) \\ \dot{R} &= 5R^2+(8h-4)R-4h. \end{aligned} \tag{2.23}$$

The vector field is sketched in Figure 2.7. The system (2.23) has 4 equilibrium points: $S_0 = (0,0)$, $N_0 = (-1/4, 1/5)$, $S_1 = (-1/4, 1)$, $N_1 = (0, 4/5)$. Linearization at the equilibrium points shows that S_0 and S_1 are hyperbolic saddle points, while N_0 and N_1 are hyperbolic (degenerate) nodes, respectively stable and unstable. Note that the sets $\{h = -1/4\}$ and $\{h = 0\}$ are invariant sets. Now, we need to determine which 'points' $R(h)$ actually passes through. This is done in the following lemma.

Lemma 2.3 *(a)* $\lim_{h\to -\frac{1}{4}} R(h) = 1$.

(b) $R(0) = \frac{4}{5}$.

Proof:

(a) Recall that $R(h) = I_2(h)/I_1(h)$, where

$$I_1(h) = \oint_{\Gamma_h} v\,du, \quad I_2(h) = \oint_{\Gamma_h} u^2 v\,du.$$

It is easy to see that

$$\lim_{h\to-\frac{1}{4}} I_1(h) = \lim_{h\to-\frac{1}{4}} I_2(h) = 0.$$

Now, using the Green's Theorem, we have

$$\begin{aligned}
\lim_{h\to-\frac{1}{4}} R(h) &= \lim_{h\to-\frac{1}{4}} \frac{I_2(h)}{I_1(h)} \\
&= \lim_{h\to-\frac{1}{4}} \frac{\oint_{\Gamma_h} u^2 v\, du}{\oint_{\Gamma_h} v\, du} \\
&= \lim_{h\to-\frac{1}{4}} \frac{\iint_{Q(h)} u^2\, du\, dv}{\iint_{Q(h)} du\, dv}. \qquad (2.24)
\end{aligned}$$

By applying the Mean Value Theorem for double integrals to (2.24), we have

$$\lim_{h\to-\frac{1}{4}} R(h) = \lim_{h\to-\frac{1}{4}} u_0^2(h)$$

for some points $(u_0(h), v_0(h))$ in $Q(h)$. As $h \to -1/4$, $Q(h)$ will 'shrink' to the points $(\pm 1, 0)$ and so

$$\lim_{h\to-\frac{1}{4}} R(h) = 1.$$

(b) Recall that $H = 0$ corresponds to "double" homoclinic orbits for system (2.13) when $\epsilon = 0$. Now, using (2.15) with $h = 0$, we have

$$\bar{M}(0, \lambda) = \lambda - R(0).$$

We see that $\bar{M}(0, \lambda)$ vanishes if and only if $\lambda = R(0)$. Using the fact that $\lambda = 4/5$ (i.e. (2.14)), the result follows. □

Using the above lemma, it follows that the graph of $R(h)$ is the stable manifold of S_1. It joins S_1 to the node N_1 (see Figure 2.7). We need to find the properties that $R(h)$ has. This is summarized in the following lemma.

Lemma 2.4 *The function $R(h)$ has the following properties:*

(a) $R'(h) \to -1/2$ as $h \to -1/4$, and $R'(h) \to -\infty$ as $h \to 0$.

(b) $R(h) \to +\infty$ *as* $h \to +\infty$.

(c) $R(h) > 1/2$ *for* $h > 0$.

(d) $R'(h)$ *has a unique zero for* $h = h^* > 0$, *is negative for* $-\frac{1}{4} \le h < h^*$ *and positive for* $h > h^*$.

(e) $R''(h^*) > 0$. *That is,* $R(h)$ *attains the minimum value at* $h = h^*$.

Proof: Some of the proofs given below are similar to the ones given in Carr [8], but with more explanations and clarity.

(a) Consider just the interval $h \in [-1/4, 0]$. Since system (2.23) is symmetric with respect to the point $(-1/8, 1/2)$, it is sufficient to show that $R'(h) \to -1/2$ as $h \to 0$ with $R(0) = 0$. Let $R(h) = Ch + O(h^2)$. Substituting $R(h)$ and its derivative into (2.22), we easily get $C = -1/2$ and the result follows. The second part of proof is straightforward.

(b) Without loss of generality, we assume $h > 0$. Recall that

$$\begin{aligned} I_1(h) &= \oint_{\Gamma_h} v\, du \\ &= 4\int_0^{\xi} \sqrt{2h + u^2 - \frac{u^4}{2}}\, du \end{aligned} \tag{2.25}$$

(by symmetry), where $\xi = \xi(h)$ is the unique positive root of

$$2h + u^2 - \frac{u^4}{2} = 0 \tag{2.26}$$

(see Figure 2.8). Let $u = \xi z$ in (2.25), we get

$$\begin{aligned} I_1 &= 4\xi \int_0^1 \sqrt{2h + \xi^2 z^2 - \frac{\xi^4 z^4}{2}}\, dz \\ &= 4\xi^2 \int_0^1 \sqrt{\frac{\xi^2}{2}(1 - z^4) + (z^2 - 1)}\, dz. \end{aligned}$$

Similarly,

$$I_2 = 4\xi^4 \int_0^1 z^2 \sqrt{\frac{\xi^2}{2}(1 - z^4) + (z^2 - 1)}\, dz.$$

Now, let

$$g(z) = \sqrt{\frac{\xi^2}{2}(1 - z^4) + (z^2 - 1)},$$

for $0 \leq z \leq 1$. The function $g(z)$ defined above has two critical points $z = 0$ and $z = 1/\xi$. It is easy to see that $z = 0$ is the (local) minimum and $z = 1/\xi$ is the global maximum. Therefore, $g(z) \leq g(1/\xi)$ for $0 \leq z \leq 1$. Since $h \to \infty$ if and only if $\xi \to \infty$ (see (2.26)), we have $I_1 \leq k_1\xi^3$ for some positive constant k_1. It is easy to obtain that $I_2 \geq k_2\xi^5$ for some positive constant k_2. So we have

$$\lim_{h\to\infty} R(h) = \lim_{h\to\infty} \frac{I_2(h)}{I_1(h)} = +\infty.$$

(c) Recall that system (2.23) has four nullclines: $h = 0$, $h = -1/4$ and the two branches of the hyperbola $5R^2 + (8h - 4)R - 4h = 0$. Let us denote these two branches by $\bar{R}(h)$ and $\hat{R}(h)$ (see Figure 2.9). Note that the phase plane is divided into nine different regions by the invariant sets $\{h = 0\}$, $\{h = -1/4\}$ and the two branches $R = \bar{R}(h)$, $R = \hat{R}(h)$. The vector field is vertical on the two invariant sets and the vector field is horizontal on the two branches. Using the fact that $\bar{R}(-1/4) = 1$ and differentiating the equation $5\bar{R}^2 + (8h - 4)\bar{R} - 4h = 0$ with respect to h and set $h = -1/4$, we easily find that $\bar{R}'(-1/4) = -1$. Similarly, using the fact that $\bar{R}(0) = 4/5$ and differentiating the equation $5\bar{R}^2 + (8h - 4)\bar{R} - 4h = 0$ with respect to h and set $h = 0$, we find that $\bar{R}'(0) = -3/5$. Using these two results together with part (a), we see that the graph of $R(h)$ must stay in Region 1 for $-1/4 < h < 0$ and must go to Region 2 for $0 < h \ll 1$. In Regions 1 and 2, $dR/dh < 0$. But $R(h) \to +\infty$ as $h \to +\infty$ (part (b)) and $\bar{R}(h) \to 1/2$ as $h \to +\infty$. Therefore, there exists $h^* > 0$ such that $R(h^*) = \bar{R}(h^*)$, that is, $R'(h^*) = 0$ and $R'(h) > 0$ for $h > h^*$. Also, since $\bar{R}'(h) < 0$ and $\bar{R}(h) \to 1/2$ as $h \to +\infty$, we have $R(h^*) = \bar{R}(h^*) > 1/2$. Hence $R(h) > 1/2$ for $h > 0$. Finally, note that $R(h^*)$ $(= R_{min})$ is determined numerically to be $0.752\cdots$.

(d) Proved in part (c).

(e) Using Lemma 2.2 to compute the second derivative of $R(h)$, we have

$$4h(4h+1)R'' = R'(10R - 24h - 8) + (8R - 4).$$

Substitute $h = h^*$ into the above equation and using part (c) and (d), we immediately get

$$R''(h^*) > 0. \qquad \square$$

Using the above lemma, we see that $R(h)$ has the form shown in Figure 2.9. Recalling that $\bar{M}(h, \lambda)$ vanishes when $\lambda = R(h)$, we have the following results:

Result 2.3 *1. If $\lambda > 1$, then $\lambda - R(h) > 0$ $\forall h \in (-1/4, 0)$. This implies $d(h, \epsilon, \lambda) > 0$ and all orbits spiral outward.*

2. If $\lambda < 4/5$, then $\lambda - R(h) < 0$ $\forall h \in (-1/4, 0)$. This implies $d(h, \epsilon, \lambda) < 0$ and all orbits spiral inward.

3. If $4/5 < \lambda_0 < 1$, then $\lambda_0 - R(h) = 0$ for some unique $h_0 \in (-1/4, 0)$. This implies $d(h, \epsilon, \lambda) = 0$ for unique $\lambda(\epsilon) = \lambda_0 + O(\epsilon)$ and $h(\epsilon) = h_0 + O(\epsilon)$. Using Theorem 1.1, the system (2.13) has a unique, hyperbolic periodic orbit Γ_ϵ which tends to Γ_{h_0} as $\epsilon \to 0$. Now, if $h > h(\epsilon)$, then $d(h, \epsilon, \lambda) > 0$. That means orbits spiral outward. If $h < h(\epsilon)$, then $d(h, \epsilon, \lambda) < 0$. That means orbits spiral inward. Therefore, the periodic orbit is unstable.

4. If $\lambda_0 > 4/5$, then $\lambda_0 - R(h) = 0$ for some unique $h_0 \in (0, \infty)$. This implies $d(h, \epsilon, \lambda) = 0$ for unique $\lambda(\epsilon) = \lambda_0 + O(\epsilon)$ and $h(\epsilon) = h_0 + O(\epsilon)$. Using Theorem 1.1, the system (2.13) has a unique, hyperbolic periodic orbit Γ_ϵ which tends to Γ_{h_0} as $\epsilon \to 0$. Now, if $h > h(\epsilon)$, then $d(h, \epsilon, \lambda) < 0$. That means orbits spiral inward. If $h < h(\epsilon)$, then $d(h, \epsilon, \lambda) > 0$. That means orbits spiral outward. Therefore, the periodic orbit is stable.

5. If $R_{min} < \lambda_0 < 4/5$, then there are 2 values of h, h_{10} and h_{20} $(0 < h_{10} < h_{20})$ such that $\lambda_0 - R(h) = 0$. This implies there are 2 periodic orbits for $h_1(\epsilon) = h_{10} + O(\epsilon)$, $h_2(\epsilon) = h_{20} + O(\epsilon)$, respectively, and $\lambda(\epsilon) = \lambda_0 + O(\epsilon)$. Now, if $h < h_1(\epsilon)$, then $d(h, \epsilon, \lambda) < 0$. That means orbits spiral inward. If $h_1(\epsilon) < h < h_2(\epsilon)$, then $d(h, \epsilon, \lambda) > 0$. That means

orbits spiral outward. If $h > h_2(\epsilon)$, then $d(h, \epsilon, \lambda) < 0$. That means orbits spiral inward. Therefore, the periodic orbit is unstable at $h = h_1(\epsilon)$ and the periodic orbit is stable at $h = h_2(\epsilon)$.

6. *If $\lambda_0 = R_{min} \approx 0.752$, then $\lambda_0 - R(h) = 0$ for unique $h = h^* > 0$. This implies $d(h, \epsilon, \lambda) = 0$ for unique $\lambda(\epsilon) = \lambda_0 + O(\epsilon)$ and $h(\epsilon) = h^* + O(\epsilon)$. Using Theorem 1.2, the system (2.13) has a unique, non-hyperbolic periodic orbit of multiplicity two which tends to Γ_{h^*} as $\epsilon \to 0$. Now, if $h > h(\epsilon)$, then $d(h, \epsilon, \lambda) < 0$. That means orbits spiral inward. If $h < h(\epsilon)$, then $d(h, \epsilon, \lambda) < 0$. That means orbits spiral inward. Therefore, the periodic orbit is "semi-stable" (i.e. saddle-node bifurcation of periodic orbits).*

7. *If $\lambda < R_{min}$, then $\lambda - R(h) < 0$ $\forall h \in (0, \infty)$. This implies $d(h, \epsilon, \lambda) < 0$ and all orbits spiral inward.*

We see that, even though the computation of the Melnikov function is somewhat technical, the benefits are great: we determine the exact number, positions, and multiplicities of the limit cycles from the zeros of the Melnikov function.

It is time now for us to return to system (2.11) with the minus sign. The homoclinic loop bifurcation, which occurs at $\lambda = 4/5 + O(\epsilon)$ for (2.13), corresponds to

$$\mu_2 = \frac{4\mu_1}{5} + O(\mu_1^{\frac{3}{2}}) \quad (\mu_1 > 0)$$

for (2.11) with the minus sign. Also, the saddle-node bifurcation of limit cycles, which occurs at $\lambda \approx 0.752 + O(\epsilon)$ for (2.13), corresponds to

$$\mu_2 \approx 0.752\mu_1 + O(\mu_1^{\frac{3}{2}}) \quad (\mu_1 > 0)$$

for (2.11) with the minus sign. Using Result 2.3, we can determine the exact number of periodic orbits, corresponding to different values of the parameters in system (2.11) with the minus sign. For $\mu_2 > \mu_1$, there is a unique (stable) limit cycle around the three equilibria. For $4\mu_1/5 + O(\mu_1^{\frac{3}{2}}) < \mu_2 < \mu_1$, there are two (unstable) limit cycles around the two equilibria $(\pm\sqrt{\mu_1}, 0)$ respectively, while there is a unique (stable) limit cycle surrounding the two limit cycles and

the three equilibria. For $\mu_2 = 4\mu_1/5 + O(\mu_1^{\frac{3}{2}})$, there is a pair of symmetric homoclinic orbits, while there is a unique (stable) limit cycle surrounding the homoclinic orbits and the three equilibria. For $R_{min}\mu_1 + O(\mu_1^{\frac{3}{2}}) < \mu_2 < 4\mu_1/5 + O(\mu_1^{\frac{3}{2}})$, there are two limit cycles (smaller one unstable, larger one stable) surrounding the three equilibria. For $\mu_2 = R_{min}\mu_1 + O(\mu_1^{\frac{3}{2}})$, there is a unique multiplicity two (semi-stable) limit cycle. For $\mu_2 < R_{min}\mu_1 + O(\mu_1^{\frac{3}{2}})$, there is no limit cycle. As μ_2 increases past $R_{min}\mu_1 + O(\mu_1^{\frac{3}{2}})$, the system (2.11) with the minus sign undergoes a saddle-node bifurcation of periodic orbits. The bifurcation set and the corresponding phase portraits for the system (2.11) with the minus sign is shown in Figure 2.10.

Next, we consider the system (2.11) with the plus sign. This case is actually simpler. For $\mu_1 > 0$ there is only one equilibrium point at the origin, and it is a hyperbolic saddle. For $\mu_1 < 0$ there are three equilibria, a hyperbolic source at the origin for $\mu_2 > 0$, a hyperbolic sink at the origin for $\mu_2 < 0$, a non-hyperbolic linear centre at the origin for $\mu_2 = 0$, and two hyperbolic saddle points at $(\pm\sqrt{-\mu_1}, 0)$. There is a pitchfork bifurcation at points on the μ_2-axis with $\mu_2 \neq 0$. There is a supercritical Hopf bifurcation at points on the μ_1-axis with $\mu_1 < 0$. The partial bifurcation set and phase portraits is shown in Figure 2.11.

We know that for all $\mu_1 < 0$, an asymptotically stable limit cycle is generated (via Hopf bifurcation) at $\mu_2 > 0$ (μ_2 close to 0). What happens to the stable limit cycle as μ_2 increases from 0 ($\mu_1 < 0$ fixed)? Do the limit cycles exist forever? In fact, the asymptotically stable limit cycle expands monotonically as μ_2 increases from 0 until it intersects the saddle points at $(\pm\sqrt{-\mu_1}, 0)$ and forms a 'double' heteroclinic orbit at some parameter value $\mu_2 = h(\mu_1) = -\mu_1/5 + O(|\mu_1|^{\frac{3}{2}})$ ($\mu_1 < 0$), i.e., there exists a heteroclinic loop bifurcation curve in the $\mu_1\mu_2$-plane given by $\mu_2 = h(\mu_1)$ for $\mu_1 < 0$. The proof is similar to the case with the minus sign treated above, and is omitted. Similarly, we can prove that the asymptotically stable limit cycle is unique when $\mu_1 < 0$, $0 < \mu_2 < h(\mu_1)$. The bifurcation set and the corresponding phase portraits for the system (2.11) with the plus sign is shown in Figure 2.12.

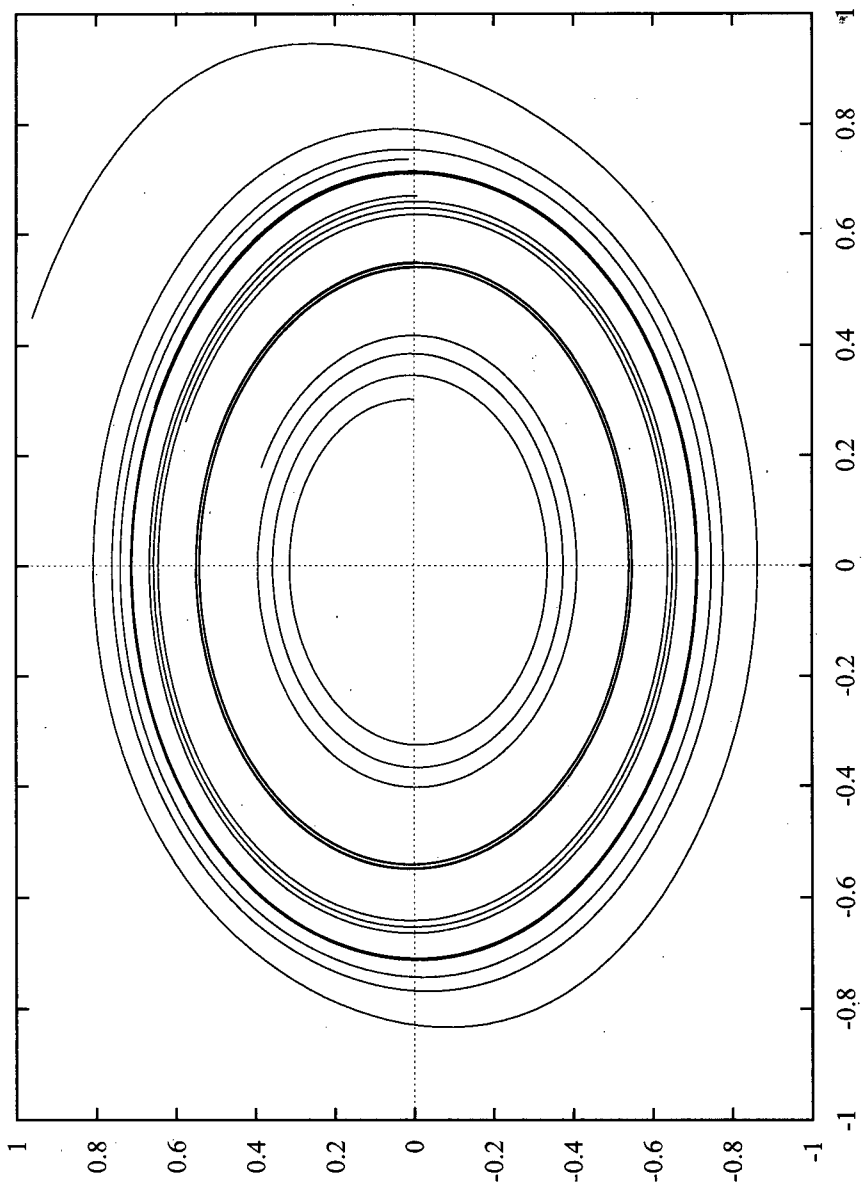

Figure 2.1: Two hyperbolic limit cycles for system (2.6) with $\epsilon = 0.3$ and $\mu_1 = 0.28$.

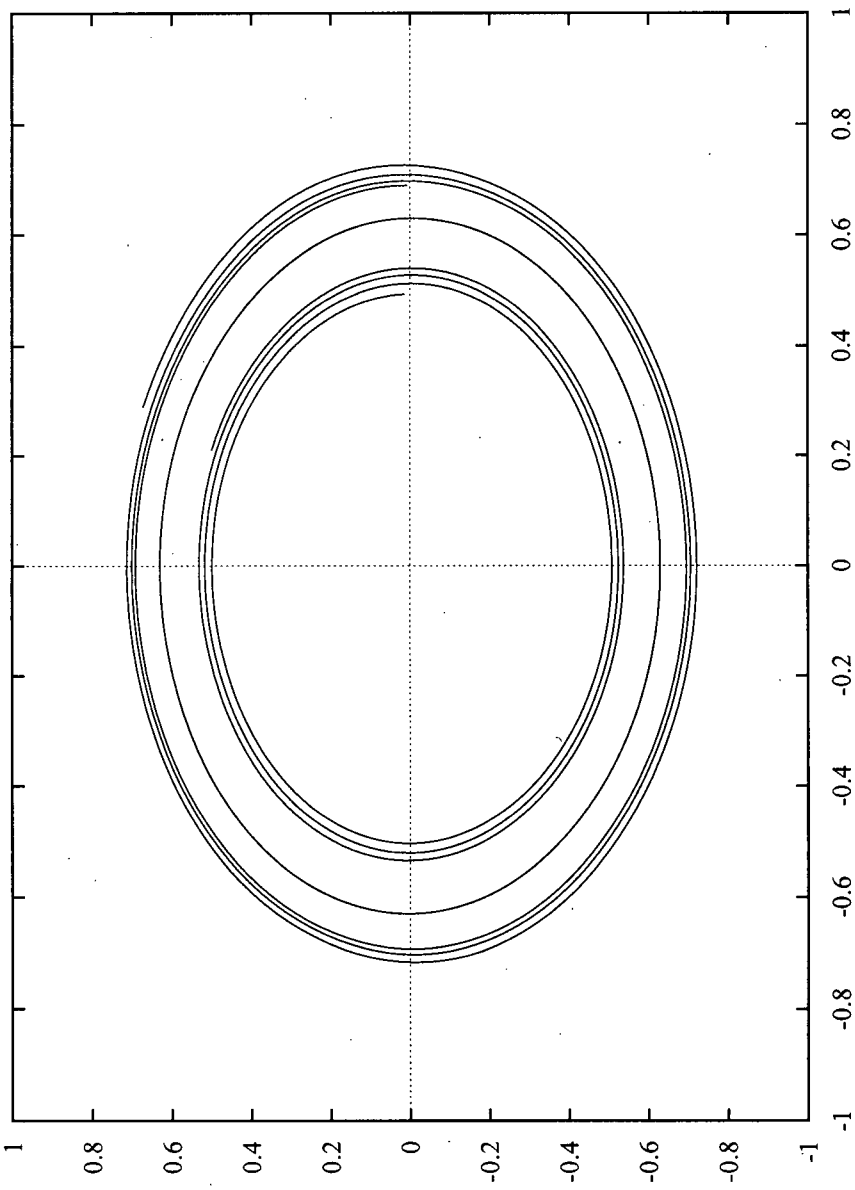

Figure 2.2: A unique non-hyperbolic limit cycle for system (2.6) with $\epsilon = 0.3$ and $\mu_1 = 0.3$.

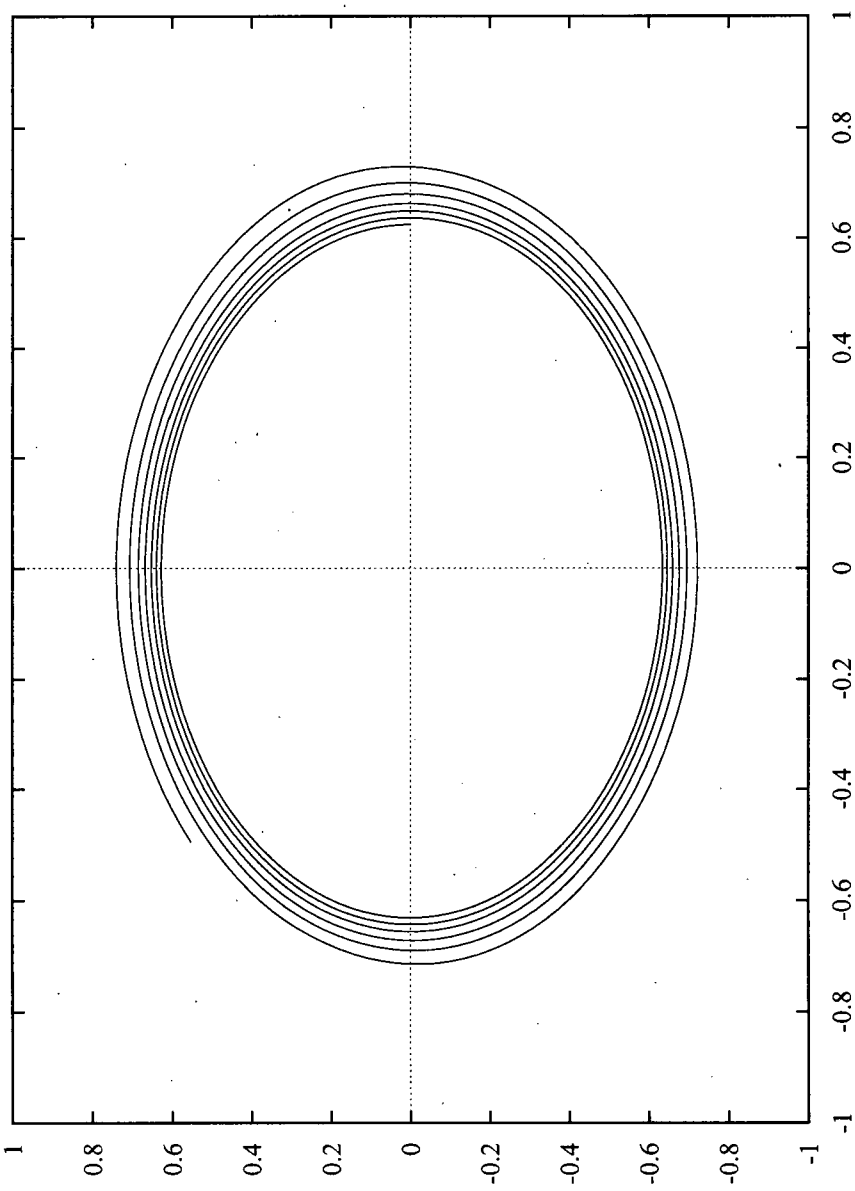

Figure 2.3: No limit cycles for system (2.6) with $\epsilon = 0.3$ and $\mu_1 = 0.32$.

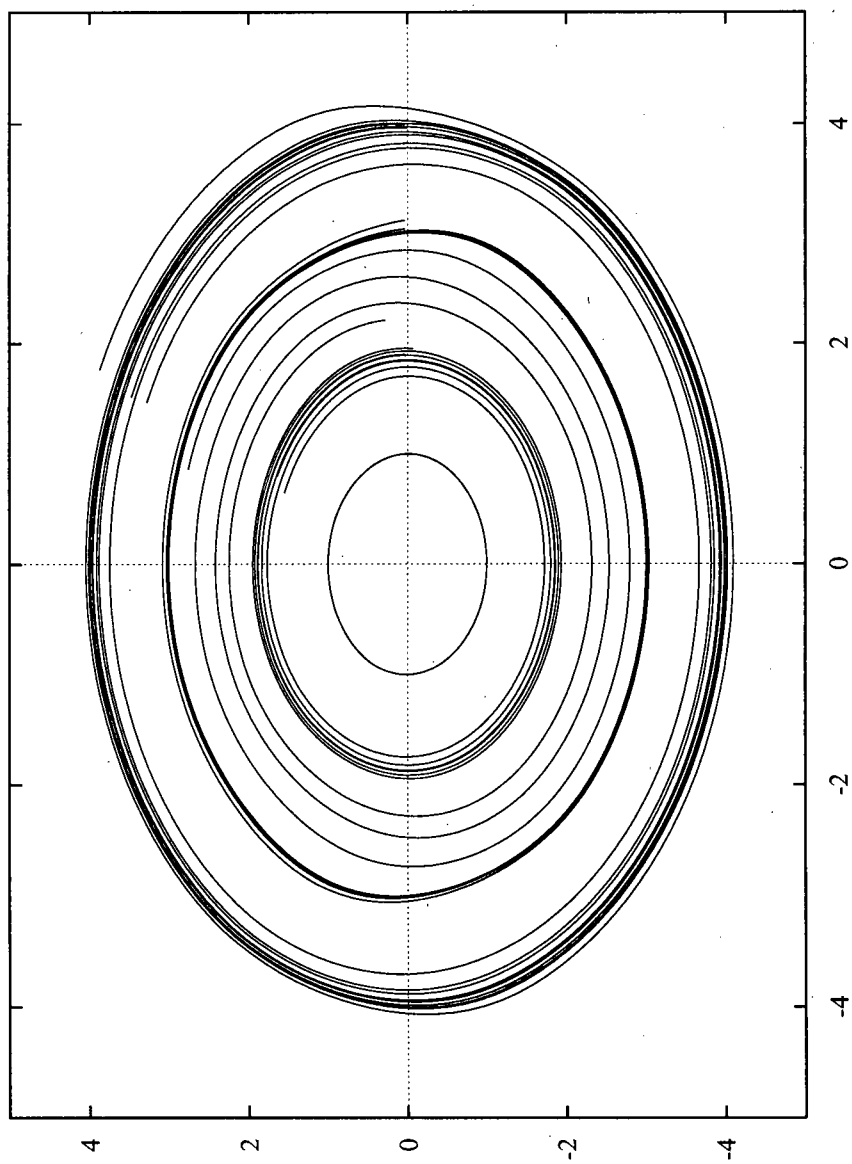

Figure 2.4: Four limit cycles for system (2.7) with $\epsilon = 0.00005$.

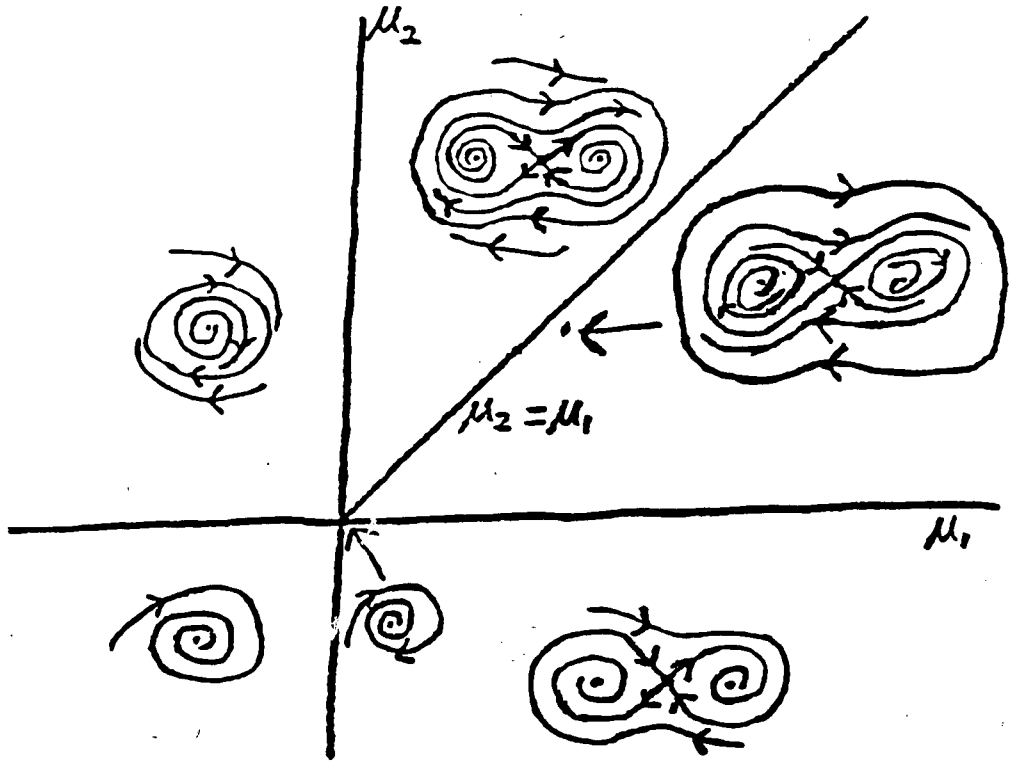

Figure 2.5: The partial bifurcation set and the corresponding phase portraits for the system (2.11) with the minus sign.

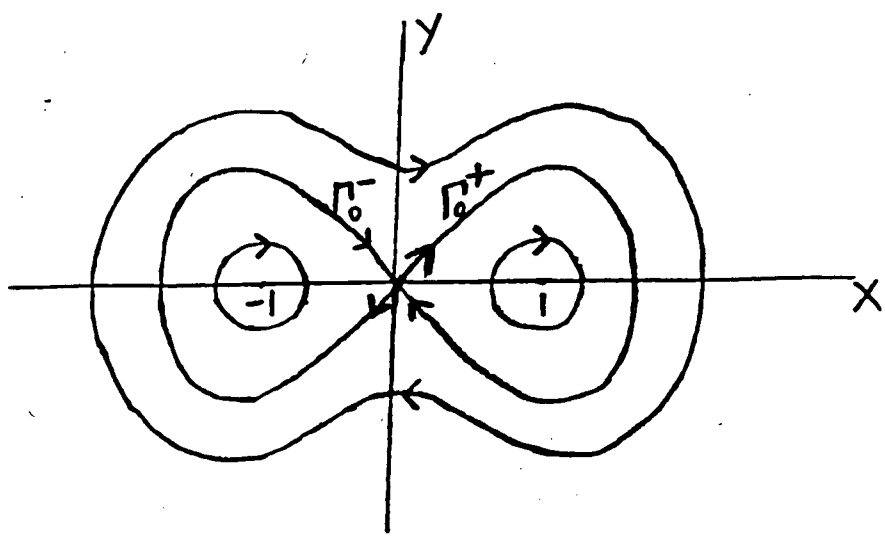

Figure 2.6: Phase portrait of the system (2.13) with $\epsilon = 0$.

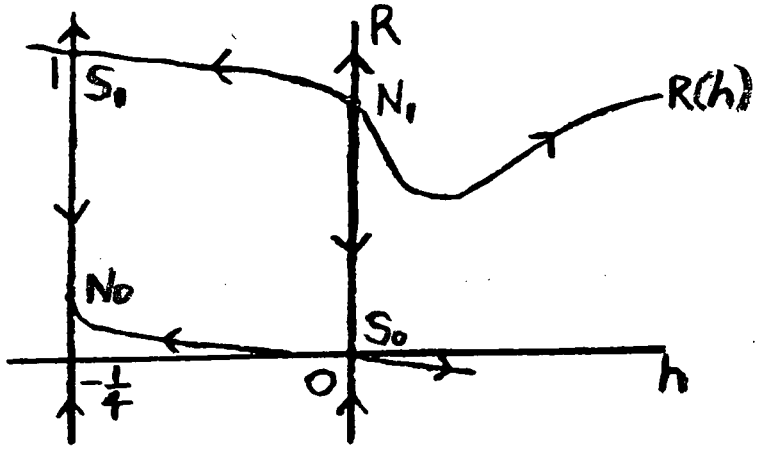

Figure 2.7: Phase portrait of the system (2.23).

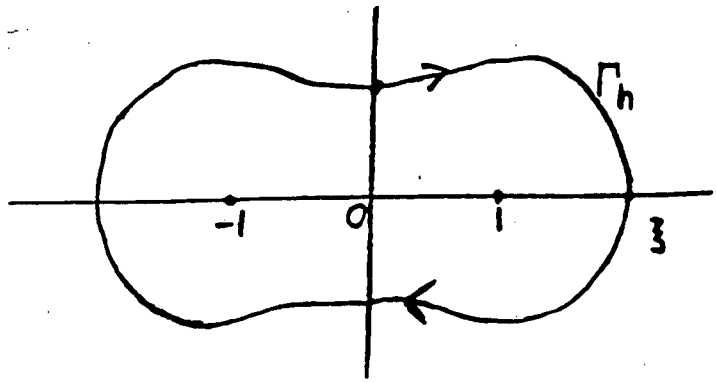

Figure 2.8: Trajectory when $h > 0$.

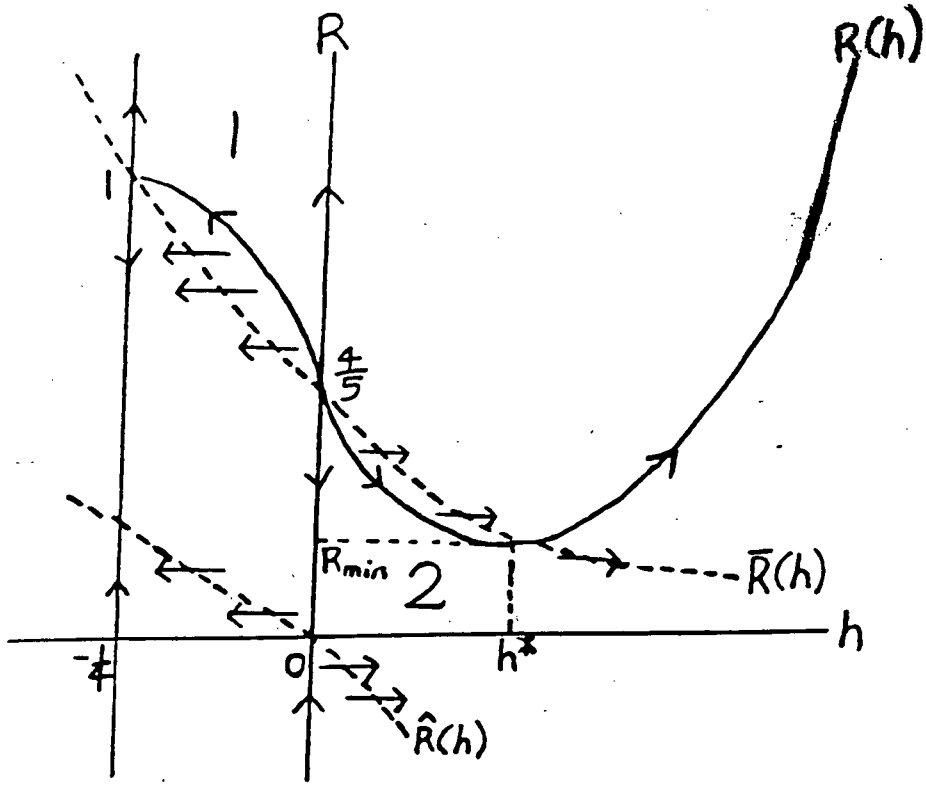

Figure 2.9: Detailed phase portrait for system (2.23).

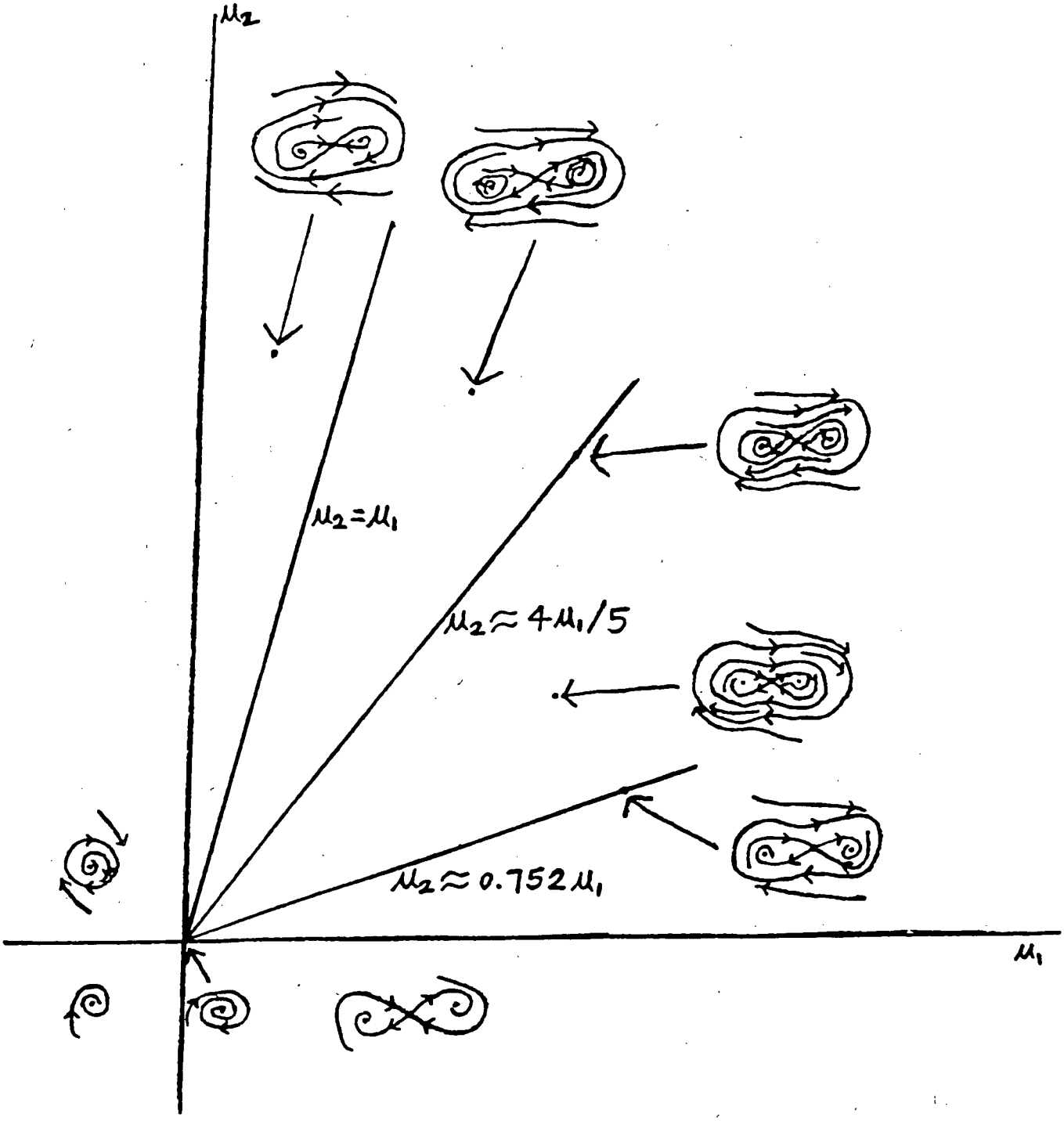

Figure 2.10: Bifurcation set and the corresponding phase portraits for the system (2.11) with the minus sign.

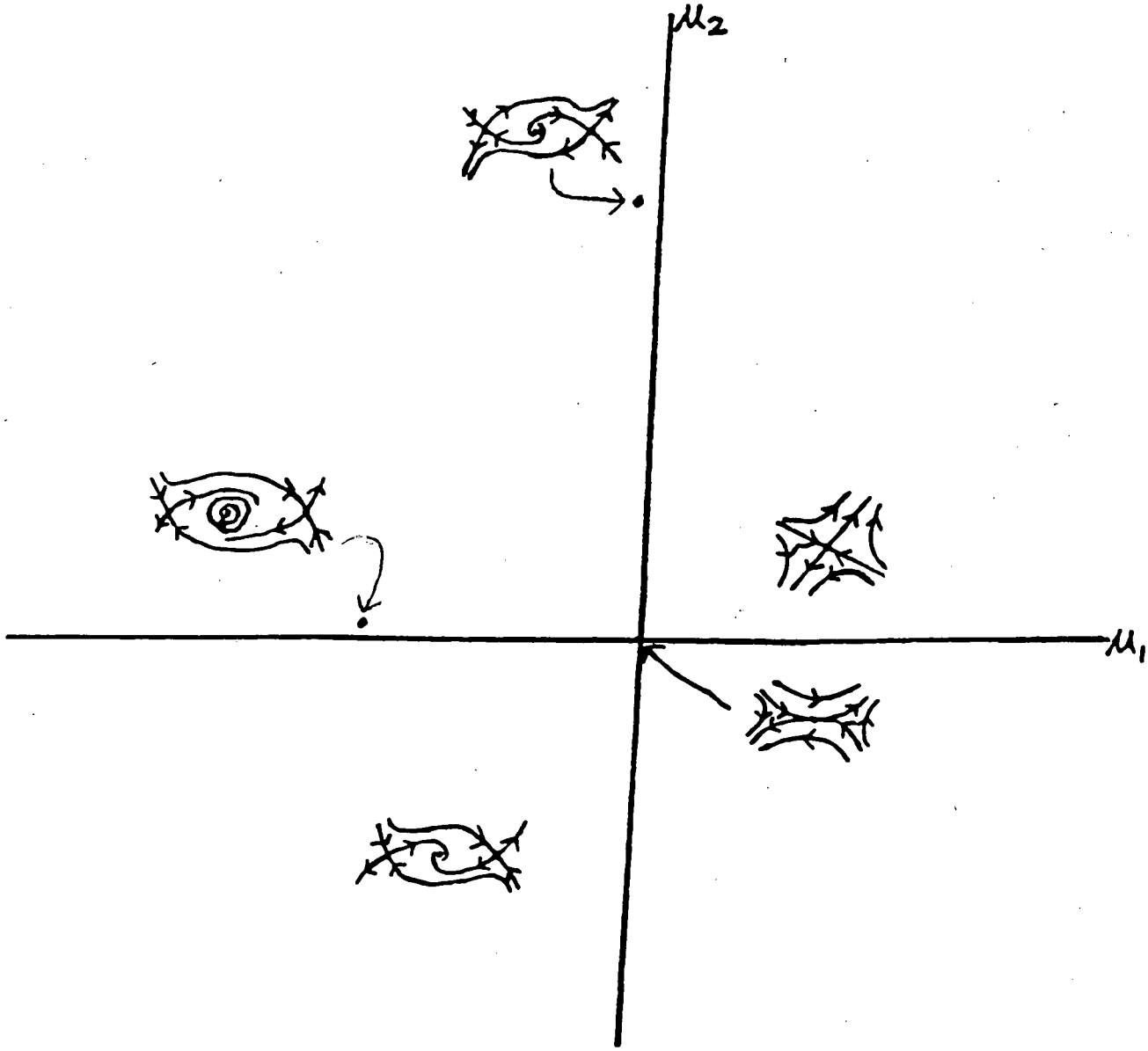

Figure 2.11: The partial bifurcation set and the corresponding phase portraits for the system (2.11) with the plus sign.

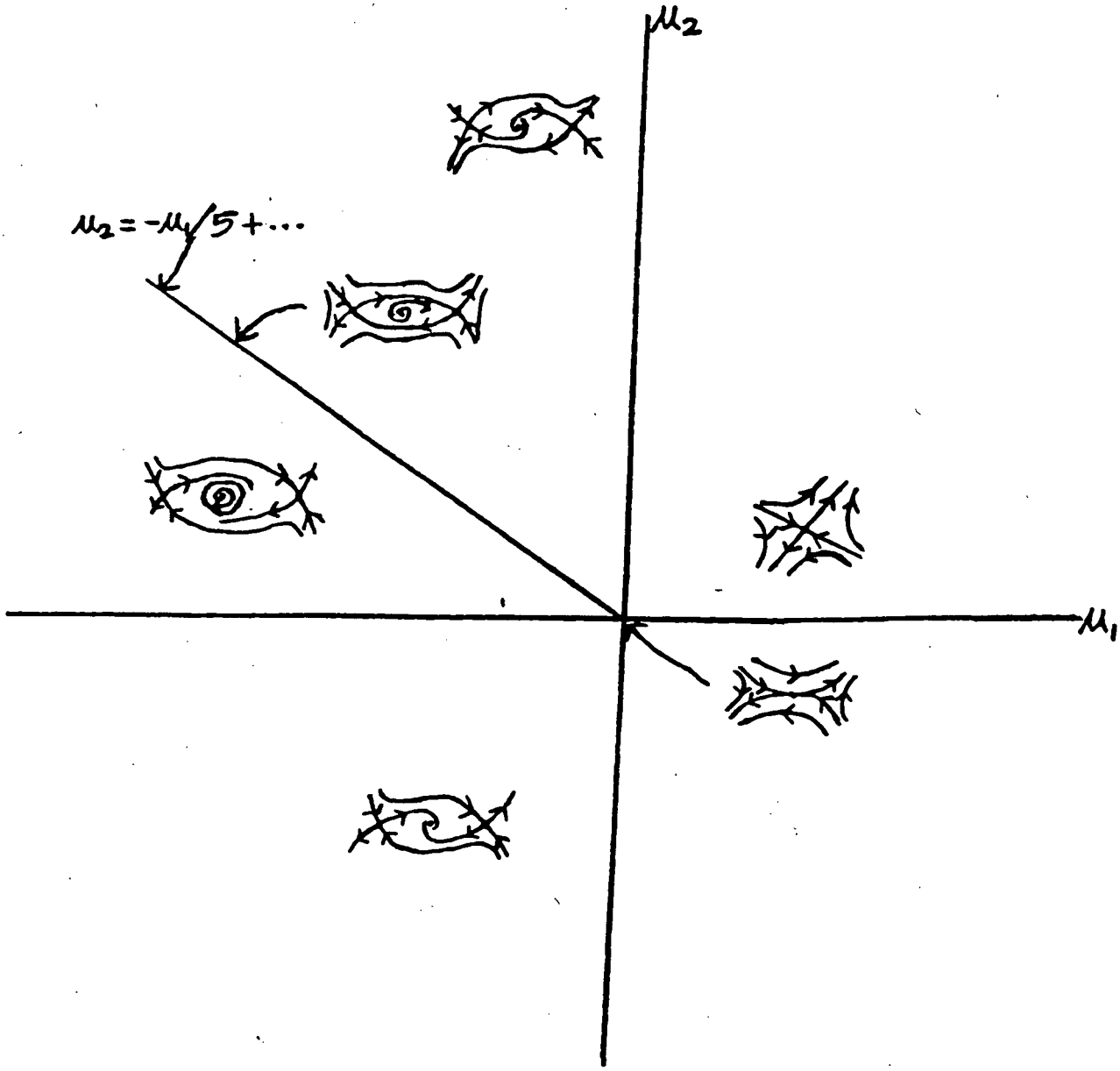

Figure 2.12: Bifurcation set and the corresponding phase portraits for the system (2.11) with the plus sign.

Chapter 3
Higher-order Melnikov theory

3.1. Introduction

In Chapter 1 we saw that the number, positions and multiplicities of the limit cycles of (2) are determined by the number, positions and multiplicities of the zeros of the Melnikov function $M(h, \mu)$. But what if $M(h, \mu_0) \equiv 0$ for some $\mu_0 \in \Re^n$? In this case a higher-order analysis in ϵ is necessary. For instance, if the first-order Melnikov function that we derived in Chapter 1 is identically equal to 0, then we need to look at the second-order Melnikov function, which we hope is not identically equal to 0, and relate the number, positions and multiplicities of the zeros of it to the number, positions and multiplicities of the limit cycles. Similarly, if the first- and second-order Melnikov functions are both identically equal to 0, then we need to look at the third-order Melnikov function. So, in general, we suppose that if the first $k-1$ Melnikov functions are all identically equal to zero, then we need to look at the kth-order Melnikov function. You may ask: How to relate the number, positions and multiplicities of the zeros of the kth-order Melnikov function to the number, positions and multiplicities of the limit cycles? The answer is that this relationship is the same as the relationship between the number, positions and multiplicities of the zeros of the first-order Melnikov function and the number, positions and multiplicities of the limit cycles. For example, if the kth-order Melnikov function $M_k(h, \mu)$ is equal to zero and satisfies some non-degeneracy condition (i.e. $\partial M_k / \partial h(h, \mu) \neq 0$), we can apply the implicit function theorem to obtain a unique hyperbolic limit cycle for small ϵ. This kind of result, which is an extension of the results obtained in Chapter 1, is presented in Section 2. In Section 3, we use a recursive algorithm of Françoise [10] to derive a second-order

Melnikov function to which the theory developed in Section 2 applies.

3.2. Higher-order theory

Although the theorems in Chapter 1 are sufficient and convenient for a first-order analysis, they do not apply to a higher-order analysis of (2). In this section, I present the general results concerning higher-order Melnikov theory, which extend Theorems 1.1, 1.2.

The following theorem, which is an extension of Theorem 1.1, gives us conditions under which the system has a unique, hyperbolic limit cycle for small $\epsilon \neq 0$ when a higher-order analysis is required. Note that Theorem 1.1 is equivalent to the following theorem with $k = 1$.

Theorem 3.1 *Assume that Assumption 1.2 holds for all $h \in I$. If there exists a $\mu_0 \in \Re^n$ such that*

$$d(h, 0, \mu_0) = \frac{\partial d}{\partial \epsilon}(h, 0, \mu_0) = \cdots = \frac{\partial^{(k-1)} d}{\partial \epsilon^{(k-1)}}(h, 0, \mu_0) \equiv 0 \quad \textit{for all } h \in I$$

and if there exists a $h_0 \in I$ such that

$$\frac{\partial^{(k)} d}{\partial \epsilon^{(k)}}(h_0, 0, \mu_0) = 0 \quad \textit{and} \quad \frac{\partial^{(k+1)} d}{\partial \epsilon^{(k)} \partial h}(h_0, 0, \mu_0) \neq 0,$$

then for all sufficiently small $\epsilon \neq 0$, the system (2) has a unique, hyperbolic limit cycle Γ_ϵ which tends to the periodic orbit Γ_{h_0} as $\epsilon \to 0$.

Proof: Define the function

$$F(h, \epsilon) = \begin{cases} \frac{d(h, \epsilon, \mu_0)}{\epsilon^k} & \text{if } \epsilon \neq 0 \\ \frac{1}{k!} \frac{\partial^{(k)} d}{\partial \epsilon^{(k)}}(h, 0, \mu_0) & \text{if } \epsilon = 0 \end{cases}$$

so that

$$d(h, \epsilon, \mu_0) = \epsilon^k F(h, \epsilon),$$

where

$$F(h, \epsilon) = \frac{1}{k!} \frac{\partial^{(k)} d}{\partial \epsilon^{(k)}}(h, 0, \mu_0) + O(\epsilon).$$

Also,

$$F(h_0, 0) = \frac{1}{k!}\frac{\partial^{(k)} d}{\partial \epsilon^{(k)}}(h_0, 0, \mu_0) = 0,$$

$$\frac{\partial F}{\partial h}(h_0, 0) = \frac{1}{k!}\frac{\partial^{(k+1)} d}{\partial \epsilon^{(k)} \partial h}(h_0, 0, \mu_0) \neq 0.$$

Then, by the implicit function theorem [19], there exists a $\delta > 0$ and a unique function $h = h(\epsilon)$, defined for $|\epsilon| < \delta$, such that $h(0) = h_0$ and $F(h(\epsilon), \epsilon) = 0$ for all $|\epsilon| < \delta$. The rest of the proof is exactly the same as of Theorem 1.1. $\square$

The next theorem gives us an extension of Theorem 1.2 for determining a unique, non-hyperbolic limit cycle of multiplicity two from system (2) when a higher-order analysis is required. Theorem 1.2 is equivalent to the following theorem with $k = 1$.

Theorem 3.2 *Assume that Assumption 1.2 holds for all $h \in I$. If there exists a $\mu_0 \in \Re^n$ such that*

$$d(h, 0, \mu_0) = \frac{\partial d}{\partial \epsilon}(h, 0, \mu_0) = \cdots = \frac{\partial^{(k-1)} d}{\partial \epsilon^{(k-1)}}(h, 0, \mu_0) \equiv 0 \quad \textit{for all } h \in I$$

and if there exists a $h_0 \in I$ such that

$$\frac{\partial^{(k)} d}{\partial \epsilon^{(k)}}(h_0, 0, \mu_0) = 0, \ \frac{\partial^{(k+1)} d}{\partial \epsilon^{(k)} \partial h}(h_0, 0, \mu_0) = 0,$$

$$\frac{\partial^{(k+2)} d}{\partial \epsilon^{(k)} \partial h^2}(h_0, 0, \mu_0) \neq 0, \ \frac{\partial^{(k+1)} d}{\partial \epsilon^{(k)} \partial \mu_j}(h_0, 0, \mu_0) \neq 0 \quad \textit{for some } j = 1, \ldots, n,$$

then for all sufficiently small ϵ, there are functions $h(\epsilon) = h_0 + O(\epsilon)$, $\mu(\epsilon) = \mu_0 + O(\epsilon)$ such that for sufficiently small $\epsilon \neq 0$, the system (2) has a unique non-hyperbolic limit cycle of multiplicity two which tends to the periodic orbit Γ_{h_0} as $\epsilon \to 0$.

Proof: First we let $\mu_0 = (\mu_1^0, \mu_2^0, \ldots, \mu_n^0)$. Without loss of generality, we assume that $j = 1$ (i.e. $\frac{\partial^{(k+1)} d}{\partial \epsilon^{(k)} \partial \mu_1}(h_0, 0, \mu_0) \neq 0$). Define the function

$$F(h, \epsilon, \mu) = \begin{cases} \frac{d(h, \epsilon, \mu)}{\epsilon^k} & \text{if } \epsilon \neq 0 \\ \frac{1}{k!}\frac{\partial^{(k)} d}{\partial \epsilon^{(k)}}(h, 0, \mu) & \text{if } \epsilon = 0 \end{cases}$$

so that

$$d(h,\epsilon,\mu) = \epsilon^k F(h,\epsilon,\mu),$$

where

$$F(h,\epsilon,\mu) = \frac{1}{k!}\frac{\partial^{(k)}d}{\partial\epsilon^{(k)}}(h,0,\mu) + O(\epsilon).$$

Also,

$$F(h_0,0,\mu_0) = \frac{1}{k!}\frac{\partial^{(k)}d}{\partial\epsilon^{(k)}}(h_0,0,\mu_0) = 0,$$

$$\frac{\partial F}{\partial h}(h_0,0,\mu_0) = \frac{1}{k!}\frac{\partial^{(k+1)}d}{\partial\epsilon^{(k)}\partial h}(h_0,0,\mu_0) = 0,$$

$$\frac{\partial^2 F}{\partial h^2}(h_0,0,\mu_0) = \frac{1}{k!}\frac{\partial^{(k+2)}d}{\partial\epsilon^{(k)}\partial h^2}(h_0,0,\mu_0) \neq 0,$$

$$\frac{\partial F}{\partial \mu_1}(h_0,0,\mu_0) = \frac{1}{k!}\frac{\partial^{(k+1)}d}{\partial\epsilon^{(k)}\partial \mu_1}(h_0,0,\mu_0) \neq 0.$$

Therefore, by the Weierstrass preparation theorem [1, p. 388, Theorem 69], there exists a $\delta > 0$ such that $F(h,\epsilon,\mu) = [(h-h_0)^2 + A_1(\epsilon,\mu)(h-h_0) + A_2(\epsilon,\mu)]\Phi(h,\epsilon,\mu)$, where $A_1(\epsilon,\mu)$, $A_2(\epsilon,\mu)$, and $\Phi(h,\epsilon,\mu)$ are defined for $|\epsilon| < \delta$, $|h-h_0| < \delta$, $|\mu-\mu_0| < \delta$; $A_1(0,\mu_0) = A_2(0,\mu_0) = 0$, $\Phi(h_0,0,\mu_0) \neq 0$, and $\frac{\partial A_2}{\partial \mu_1}(0,\mu_0) \neq 0$ since $\frac{\partial F}{\partial \mu_1}(h_0,0,\mu_0) \neq 0$. The remainder of the proof is exactly the same as of Theorem 1.2. $\square$

Remark 3.1 *If* $d(h,0,\mu_0) = \frac{\partial d}{\partial \epsilon}(h,0,\mu_0) = \cdots = \frac{\partial^{(k-1)}d}{\partial\epsilon^{(k-1)}}(h,0,\mu_0) \equiv 0$,

$$\frac{\partial^{(k)}d}{\partial\epsilon^{(k)}}(h_0,0,\mu_0) = \frac{\partial^{(k+1)}d}{\partial\epsilon^{(k)}\partial h}(h_0,0,\mu_0) = \cdots = \frac{\partial^{(k+m-1)}d}{\partial\epsilon^{(k)}\partial h^{(m-1)}}(h_0,0,\mu_0) = 0,$$

$$\frac{\partial^{(k+m)}d}{\partial\epsilon^{(k)}\partial h^{(m)}}(h_0,0,\mu_0) \neq 0, \quad \frac{\partial^{(k+1)}d}{\partial\epsilon^{(k)}\partial \mu_j}(h_0,0,\mu_0) \neq 0 \quad \textit{for some } j = 1,\ldots,n,$$

then it can be shown that for sufficiently small $\epsilon \neq 0$, *the system (2) has a unique non-hyperbolic limit cycle of multiplicity* m *which tends to* Γ_{h_0} *as* $\epsilon \to 0$.

3.3. Derivation of the second-order Melnikov function

In Section 2, we have seen that limit cycles correspond to the zeros of the displacement function. The kth-order contribution (in ϵ) to the displacement function is essentially the kth-order Melnikov function (if the first $k-1$ Melnikov functions are identically equal to zero). However, this kind of result can be useful in applications only if we know the formula for the second (or higher) order Melnikov function. In this section I will use Françoise's recursive algorithm [10] to derive an explicit formula for the second-order Melnikov function $M_2(h,\mu)$ in terms of certain integrals along the periodic orbits Γ_h in Assumption 1.2. This formula applies to perturbed Hamiltonian systems of the form

$$\begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial y} + \epsilon f(x,y,\epsilon,\mu), \\ \dot{y} &= -\tfrac{\partial H}{\partial x} + \epsilon g(x,y,\epsilon,\mu), \end{aligned} \tag{3.1}$$

where $\frac{\partial H}{\partial y} = y$, $\frac{\partial H}{\partial x} = -U'(x)$, $U'(x)$ is a polynomial of degree one or higher, f and g are functions depending smoothly on x, y, ϵ, μ, and ϵ is a small perturbation parameter. Note that for $\epsilon = 0$, the system (3.1) is a Hamiltonian system with

$$H(x,y) = \frac{y^2}{2} - U(x).$$

Assume that Assumption 1.2 holds for all $h \in I$. The first-order Melnikov function $M(h,\mu)$, which we denote by $M_1(h,\mu)$ in this section, can be written as

$$\begin{aligned} M_1(h,\mu) &= \oint_{\Gamma_h} g(x,y,0,\mu)dx - f(x,y,0,\mu)dy \\ &= -\iint_{Q(h)} \left(\frac{\partial f}{\partial x}(x,y,0,\mu) + \frac{\partial g}{\partial y}(x,y,0,\mu) \right) dx\, dy, \\ &\quad (by\ Green's\ Theorem) \end{aligned}$$

where $Q(h)$ is the simply connected region surrounded by Γ_h. If $M_1(h,\mu) \equiv 0$ for all $h \in I$ and $\mu \in \Re^n$ (this occurs, for example, when $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is an odd function with respect to y), we need to look at the next term $M_2(h,\mu)$ in the expansion of the displacement function $d(h,\epsilon,\mu)$:

$$\begin{aligned} d(h,\epsilon,\mu) &= \epsilon^2 \frac{d_{\epsilon\epsilon}(h,0,\mu)}{2!} + O(\epsilon^3) \\ &= \epsilon^2 M_2(h,\mu) + O(\epsilon^3). \end{aligned}$$

In order to compute the second-order Melnikov function $M_2(h,\mu)$, provided that the first-order Melnikov function $M_1(h,\mu) \equiv 0$, we develop the idea of [10]. Given a perturbation $(f,g)^T$ in (3.1), we let $\omega = g(x,y,\epsilon,\mu)dx - f(x,y,\epsilon,\mu)dy$. Expand f and g in Taylor series in ϵ and thus

$$\begin{aligned}
\omega &= g(x,y,0,\mu)dx - f(x,y,0,\mu)dy + \epsilon\left(\frac{\partial g}{\partial \epsilon}(x,y,0,\mu)dx\right. \\
&\quad \left. - \frac{\partial f}{\partial \epsilon}(x,y,0,\mu)dy\right) + \cdots \\
&= \omega_0 + \epsilon\,\omega_1 + \cdots,
\end{aligned}$$

and the system (3.1) can be rewritten as $dH - \epsilon\,\omega = 0$, since

$$\begin{aligned}
dH - \epsilon\,\omega &= \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy - \epsilon(g\,dx - f\,dy) \\
&= \left(\frac{\partial H}{\partial x} - \epsilon g\right)dx + \left(\frac{\partial H}{\partial y} + \epsilon f\right)dy \\
&= 0
\end{aligned}$$

implies that

$$\left(\frac{\partial H}{\partial y} + \epsilon f\right)\frac{dy}{dt} = \left(-\frac{\partial H}{\partial x} + \epsilon g\right)\frac{dx}{dt},$$

and therefore

$$\begin{aligned}
\tfrac{dx}{dt} &= \tfrac{\partial H}{\partial y} + \epsilon f, \\
\tfrac{dy}{dt} &= -\tfrac{\partial H}{\partial x} + \epsilon g.
\end{aligned}$$

With this notation, the first-order Melnikov function $M_1(h,\mu)$ can be written as

$$M_1(h,\mu) = \oint_{\Gamma_h} \omega_0.$$

In the following, I will state and prove the theorem which gives us a formula for the second-order Melnikov function for (3.1).

Theorem 3.3 *Under Assumption 1.2, if $M_1(h,\mu) \equiv 0$ for all $h \in I$ and $\mu \in \Re^n$, then the displacement function for the system (3.1) is*

$$d(h,\epsilon,\mu) = \epsilon^2 M_2(h,\mu) + O(\epsilon^3), \tag{3.2}$$

where the second-order Melnikov function $M_2(h,\mu)$ is given by

$$\begin{aligned} M_2(h,\mu) &= \oint_{\Gamma_h} [G_{1h}(x,y,\mu)P_2(x,h,\mu) - G_1(x,y,\mu)P_{2h}(x,h,\mu)]dx \\ &\quad - \oint_{\Gamma_h} \frac{F(x,y,\mu)}{y}\left[\frac{\partial f}{\partial x}(x,y,0,\mu) + \frac{\partial g}{\partial y}(x,y,0,\mu)\right]dx \\ &\quad + \oint_{\Gamma_h}\left[\frac{\partial g}{\partial \epsilon}(x,y,0,\mu)dx - \frac{\partial f}{\partial \epsilon}(x,y,0,\mu)dy\right], \end{aligned} \tag{3.3}$$

where

$$F(x,y,\mu) = \int_0^y f(x,s,0,\mu)ds - \int_0^x g(s,0,0,\mu)ds, \tag{3.4}$$

$$G(x,y,\mu) = g(x,y,0,\mu) + \frac{\partial F}{\partial x}(x,y,\mu), \tag{3.5}$$

$G_1(x,y,\mu)$ denotes the odd part of $G(x,y,\mu)$ with respect to y, $G_2(x,y,\mu)$ denotes the even part of $G(x,y,\mu)$ with respect to y, $G(x,y,\mu) = G_1(x,y,\mu)+G_2(x,y,\mu)$, $G_1(x,y,\mu) = y\tilde{G}_1(x,y^2,\mu)$, $G_2(x,y,\mu) = \tilde{G}_2(x,y^2,\mu)$,

$$\begin{aligned} G_{1h}(x,y,\mu) &= \frac{1}{y}\frac{\partial G_1}{\partial y}(x,y,\mu), \\ P_2(x,h,\mu) &= \int_0^x \tilde{G}_2(s,2h+2U(s),\mu)ds, \end{aligned}$$

and $P_{2h}(x,h,\mu)$ denotes the partial derivative of $P_2(x,h,\mu)$ with respect to h.

Before proving the above theorem, let us introduce some of the main ideas from [10]. The following definition is taken, with some modifications, from [10].

Definition 3.1 *We say that the Hamiltonian function H satisfies condition $(*)$ if for all polynomial one-forms*[1] *ω such that $\oint_{\Gamma_h}\omega \equiv 0$, there exists a continuous function g and a locally Lipschitz continuous function R such that $\omega = g\,dH + dR$.* $(*)$

If H satisfies the condition $(*)$, we can compute, using the algorithm in [10], the first derivative of the displacement function which is not identically 0. In [10], Françoise showed that the

[1] ω is a polynomial one-form in $\Re^2$ if ω can be expressed as $f\,dx + g\,dy$, where f and g are polynomials in x, y.

Hamiltonian function $H(x,y) = (x^2+y^2)/2$ satisfies the condition $(*)$. We now want to extend this specific Hamiltonian to a larger class so that any Hamiltonian of the form $H(x,y) = y^2/2 - U(x)$ satisfies condition $(*)$. This result is proved in the following lemma. Now, for convenience of notation, we write $M_2(h) = M_2(h,\mu)$, $G_{1h}(x,y) = G_{1h}(x,y,\mu)$ and so on (since μ is regarded as a "constant" in the formula for the second-order Melnikov function).

Lemma 3.1 *Assume that Assumption 1.2 holds for all $h \in I$ and that $M_1(h) \equiv 0$. Then H satisfies the condition $(*)$. That is, there exists a continuous function $g_0(x,y)$ and a locally Lipschitz continuous function $R_0(x,y)$ such that ω_0 can be expressed as*

$$\omega_0 = g_0\, dH + dR_0.$$

Proof: Recall that

$$\begin{aligned} \omega_0 &= g(x,y,0)dx - f(x,y,0)dy \\ &= G\,dx - dF, \end{aligned}$$

where F, G are defined by (3.4) and (3.5) respectively. We need to show that $G\,dx$ can be expressed as $g_0\,dH + dR_0$. Given a point (x,y), let Γ_H be a closed smooth curve passing through (x,y). Suppose x_{min} and x_{max} are the minimal and the maximal values of x respectively on Γ_H. Now we let $x_{min} = a(H)$ and $x_{max} = b(H)$, where $a(H)$ and $b(H)$ are the roots of $H + U(x) = 0$ (i.e. $H(a(H),0) = H(b(H),0) = H$). Also, let $B(x,H) = |y| = \sqrt{2H + 2U(x)}$. Take a path $\Gamma_{(x,y)} \subset \Gamma_H$ which begins at $(a(H),0)$ and ends at (x,y). Also, $\Gamma_{(x,y)}$ has the same orientation as Γ_H. Let

$$R_0(x,y) \stackrel{\text{def}}{=} \int_{\Gamma_{(x,y)}} G(\xi,\eta)\, d\xi. \tag{3.6}$$

First, assume that $y \geq 0$. R_0, which is defined in (3.6), can be written as

$$R_0 = \int_{a(H)}^{x} G(\xi, B(\xi,H))\, d\xi. \tag{3.7}$$

Therefore, R_0 can be considered as a function of x and H. Now,

$$dR_0(x,H) = \frac{\partial R_0}{\partial x}(x,H)\, dx + \frac{\partial R_0}{\partial H}(x,H)\, dH,$$

where

$$\frac{\partial R_0}{\partial x}(x,H) = G(x,B(x,H)) = G(x,y), \tag{3.8}$$

and (using the Chain Rule),

$$\frac{\partial R_0}{\partial H}(x,H) = -G(a(H),B(a(H),H))a'(H) + \int_{a(H)}^{x} \frac{\partial G/\partial y(\xi,B(\xi,H))}{B(\xi,H)} d\xi. \tag{3.9}$$

As $B(a(H),H) = 0$ and $G(x,0) = 0$ (by definition) , (3.9) becomes

$$\begin{aligned} \frac{\partial R_0}{\partial H}(x,H) &= \int_{a(H)}^{x} \frac{\partial G/\partial y(\xi,B(\xi,H))}{B(\xi,H)} d\xi \\ &= \int_{a(H)}^{x} \frac{\partial g/\partial y(\xi,B(\xi,H),0) + \partial f/\partial x(\xi,B(\xi,H),0)}{B(\xi,H)} d\xi, \end{aligned}$$

which can be written as

$$\frac{\partial R_0}{\partial H} = \int_{\Gamma_{(x,y)}} \frac{\partial f/\partial x(\xi,\eta,0) + \partial g/\partial y(\xi,\eta,0)}{\eta} d\xi \stackrel{\text{def}}{=} -g_0(x,y). \tag{3.10}$$

Similarly, for $y < 0$, R_0 can be written as

$$R_0 = \int_{a(H)}^{b(H)} G(\xi,B(\xi,H))\, d\xi + \int_{b(H)}^{x} G(\xi,-B(\xi,H))\, d\xi, \tag{3.11}$$

which is a function of x and H. Now,

$$dR_0(x,H) = \frac{\partial R_0}{\partial x}(x,H)\, dx + \frac{\partial R_0}{\partial H}(x,H)\, dH,$$

where

$$\frac{\partial R_0}{\partial x}(x,H) = G(x,-B(x,H)) = G(x,y), \tag{3.12}$$

and

$$\begin{aligned} \frac{\partial R_0}{\partial H}(x,H) &= -G(a(H),B(a(H),H))a'(H) + G(b(H),B(b(H),H))b'(H) \\ &\quad + \int_{a(H)}^{b(H)} \frac{\partial G/\partial y(\xi,B(\xi,H))}{B(\xi,H)} d\xi - G(b(H),-B(b(H),H))b'(H) \\ &\quad - \int_{b(H)}^{x} \frac{\partial G/\partial y(\xi,-B(\xi,H))}{B(\xi,H)} d\xi. \end{aligned} \tag{3.13}$$

As $B(a(H),H)=0$, $B(b(H),H)=0$ and $G(x,0)=0$, (3.13) becomes

$$\begin{aligned}\frac{\partial R_0}{\partial H}(x,H) &= \int_{a(H)}^{b(H)} \frac{\partial G/\partial y(\xi,B(\xi,H))}{B(\xi,H)}d\xi - \int_{b(H)}^{x} \frac{\partial G/\partial y(\xi,-B(\xi,H))}{B(\xi,H)}d\xi \\ &= \int_{a(H)}^{b(H)} \frac{\partial g/\partial y(\xi,B(\xi,H),0)+\partial f/\partial x(\xi,B(\xi,H),0)}{B(\xi,H)}d\xi \\ &\quad - \int_{b(H)}^{x} \frac{\partial g/\partial y(\xi,-B(\xi,H),0)+\partial f/\partial x(\xi,-B(\xi,H),0)}{B(\xi,H)}d\xi,\end{aligned}$$

which can be written as

$$\frac{\partial R_0}{\partial H} = \int_{\Gamma_{(x,y)}} \frac{\partial f/\partial x(\xi,\eta,0)+\partial g/\partial y(\xi,\eta,0)}{\eta}d\xi \stackrel{\text{def}}{=} -g_0(x,y). \tag{3.14}$$

So in any case we have

$$dR_0 = G(x,y)dx - g_0(x,y)dH.$$

Since

$$\begin{aligned}M_1(h) &= \oint_{\Gamma_h} \omega_0 \\ &= \oint_{\Gamma_h} G dx,\end{aligned}$$

the hypothesis $M_1(h)=M_1'(h)\equiv 0$ implies that

$$\lim_{y\to 0^-} R_0(a(h),y) = \lim_{y\to 0^+} R_0(a(h),y) = 0,$$

$$\lim_{y\to 0^-} g_0(a(h),y) = \lim_{y\to 0^+} g_0(a(h),y) = 0.$$

Thus, $R_0(x,y)$ and $g_0(x,y)$ are single-valued functions. This ensures that they have the required continuity. The proof is now complete. □

In Françoise's paper [10], he derives an algorithm for computing the higher-order Melnikov function for the system

$$\begin{aligned}\dot{x} &= y+\epsilon f(x,y) \\ \dot{y} &= U'(x)+\epsilon g(x,y)\end{aligned} \tag{3.15}$$

Note that in (3.15), f and g are independent of ϵ. But in general, f and g may depend on ϵ (i.e. system (3.1)). In the following, I will use [10] to derive an algorithm for computing the higher-order Melnikov function for system (3.1) (i.e. $f = f(x, y, \epsilon)$ and $g = g(x, y, \epsilon)$). But before doing that, let us first derive a formula for the second-order Melnikov function. Using the above lemma, we are able to prove the following theorem which gives a formula for the second-order Melnikov function (c.f. [10]).

Theorem 3.4 *Assume that Assumption 1.2 holds for all $h \in I$ and that $M_1(h) \equiv 0$. Then the second-order Melnikov function is given by*

$$M_2(h) = \oint_{\Gamma_h} (g_0\omega_0 + \omega_1). \tag{3.16}$$

Proof: We recall the construction from [10]. Fix $h \in I$ and denote by α the smallest solution of the equation $H(\alpha, 0) = h$. Let $P_0 = (\alpha, 0)$ and choose a line segment Σ containing P_0 that is normal to the trajectory of (3.1) at P_0. For sufficiently small $\epsilon \neq 0$, let $P_1 = (\alpha_1, 0) \in \Sigma$ be the point of the Poincaré first return map (see Figure 3.1). Let $d(h, \epsilon) = H(\alpha_1, 0) - H(\alpha, 0)$ be the displacement function. Now, using Lemma 3.1, $\omega_0 = g_0\, dH + dR_0$ yields

$$\begin{aligned}
(1 + \epsilon g_0)(dH - \epsilon\omega) &= (1 + \epsilon g_0)(dH - \epsilon\omega_0 - \epsilon^2\omega_1 + O(\epsilon^3)) \\
&= dH - \epsilon(\omega_0 - g_0\, dH) - \epsilon^2(g_0\omega_0 + \omega_1) + O(\epsilon^3) \\
&= dH - \epsilon\, dR_0 - \epsilon^2(g_0\omega_0 + \omega_1) + O(\epsilon^3).
\end{aligned} \tag{3.17}$$

Integrating (3.17) along Γ_ϵ, where Γ_ϵ $(= \Gamma(\epsilon, h))$ is the trajectory of (3.1) connecting P_0 and P_1 (see Figure 3.1), and recalling that $dH - \epsilon\omega = 0$ on the trajectory Γ_ϵ, we get

$$0 = \int_{\Gamma_\epsilon} dH - \epsilon \int_{\Gamma_\epsilon} dR_0 - \epsilon^2 \int_{\Gamma_\epsilon} (g_0\omega_0 + \omega_1) + O(\epsilon^3). \tag{3.18}$$

Now, since $d(h, \epsilon) = H(P_1) - H(P_0)$, (3.18) becomes

$$d(h, \epsilon) = \epsilon \int_{\Gamma_\epsilon} dR_0 + \epsilon^2 \int_{\Gamma_\epsilon} (g_0\omega_0 + \omega_1) + O(\epsilon^3). \tag{3.19}$$

We want to find a bound for $\int_{\Gamma_\epsilon} dR_0$. For ϵ sufficiently small and since $R_0(x, y)$ is locally Lipschitz continuous, there exists a constant $c > 0$ such that

$$\left| \int_{\Gamma_\epsilon} dR_0 \right| = |R_0(\alpha_1, 0) - R_0(\alpha, 0)| \leq c|\alpha_1 - \alpha|. \tag{3.20}$$

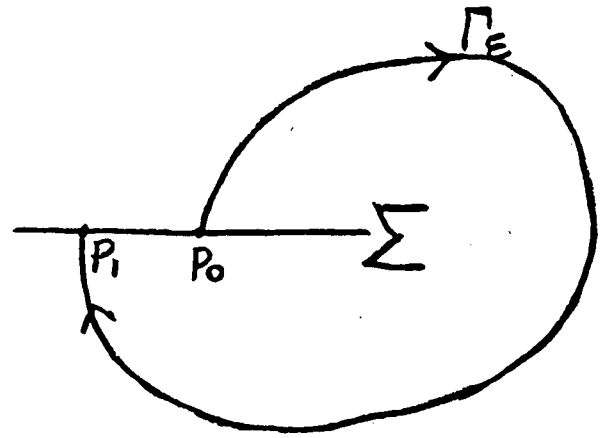

Figure 3.1: Illustration of the Poincaré map.

Now, by the Mean Value Theorem, there exists a P^* between P_0 and P_1 such that

$$|H(\alpha_1, 0) - H(\alpha, 0)| = \left|\frac{\partial H}{\partial x}(P^*)\right| |\alpha_1 - \alpha| \tag{3.21}$$

for ϵ sufficiently small. Therefore,

$$\begin{aligned} \left|\epsilon \int_{\Gamma_\epsilon} dR_0\right| &= |\epsilon| \left|\int_{\Gamma_\epsilon} dR_0\right| \\ &\leq c|\epsilon||\alpha_1 - \alpha| \quad (by\ (3.20)) \\ &= \frac{c|\epsilon||d(h,\epsilon)|}{|\frac{\partial H}{\partial x}(P^*)|} \quad (by\ (3.21)) \\ &= O(\epsilon^3), \end{aligned} \tag{3.22}$$

because $d(h, \epsilon) = O(\epsilon^2)$ by hypothesis. Finally, using the fact that Γ_ϵ tends to the closed orbit Γ_h as $\epsilon \to 0$ and using (3.22), (3.19) becomes

$$d(h, \epsilon) = \epsilon^2 \oint_{\Gamma_h} (g_0\omega_0 + \omega_1) + O(\epsilon^3). \qquad \square$$

Next, we derive a more general version of Françoise's recursive algorithm which applies for systems of the form (3.1). Note that Theorem 3.4 is equivalent to the following theorem with $k = 1$.

Theorem 3.5 *Assume that Assumption 1.2 holds for all $h \in I$ and that $M_1(h) = \cdots = M_k(h) \equiv 0$ for some $k \geq 1$. Then*

$$M_{k+1}(h) = \oint_{\Gamma_h} \Omega_k,$$

where

$$\Omega_0 = \omega_0, \tag{3.23}$$

$$\Omega_n = \omega_n + \sum_{i+j=n-1} g_i \omega_j, \qquad 1 \leq n \leq k \tag{3.24}$$

and the functions g_i, $0 \leq i \leq k-1$, are determined successively from the representations $\Omega_i = g_i dH + dR_i$ with g_i, R_i as in Lemma 3.1.

Proof: First, by integration of the equation

$$dH - \epsilon\omega = dH - \epsilon\omega_0 + O(\epsilon^2)$$

along Γ_ϵ and noting that $dH - \epsilon\omega = 0$ on Γ_ϵ, we immediately obtain

$$\int_{\Gamma_\epsilon} dH = \epsilon \int_{\Gamma_\epsilon} \omega_0 + O(\epsilon^2).$$

Using the fact that $\int_{\Gamma_\epsilon} dH = d(h, \epsilon)$ and Γ_ϵ tends to the closed orbit Γ_h as $\epsilon \to 0$, we have

$$d(h, \epsilon) = \epsilon \oint_{\Gamma_h} \omega_0 + O(\epsilon^2),$$

and therefore $M_1(h) = \oint_{\Gamma_h} \omega_0$ (the usual first-order Melnikov function).

We now make the following induction hypothesis (c.f. [10]): there exist continuous functions $g_0, g_1, \ldots, g_{k-2}$ such that for all $n = 1, \ldots, k$,

$$M_n(h) = \oint_{\Gamma_h} \Omega_{n-1} \equiv 0,$$

where

$$\Omega_0 = \omega_0,$$

and

$$\Omega_{n-1} = \omega_{n-1} + \sum_{i+j=n-2} g_i \omega_j$$

for $n = 1, \ldots, k$. Using this relation for $n = k$, and applying Lemma 3.1, there exist a continuous function g_{k-1} and a locally Lipschitz continuous function R_{k-1} such that

$$\Omega_{k-1} = g_{k-1} dH + dR_{k-1} = \omega_{k-1} + \sum_{i+j=k-2} g_i \omega_j. \tag{3.25}$$

This proves that the functions g_{k-1}, R_{k-1} can be constructed by recurrence.

Next, we multiply $dH - \epsilon\omega$ by $1 + \epsilon g_0 + \epsilon^2 g_1 + \cdots + \epsilon^k g_{k-1}$, which gives

$$\begin{aligned} dH &- \epsilon(\omega_0 - g_0 dH) - \epsilon^2(\omega_1 + g_0\omega_0 - g_1 dH) - \cdots \\ &- \epsilon^k(\omega_{k-1} + g_0\omega_{k-2} + \cdots + g_{k-2}\omega_0 - g_{k-1} dH) \\ &- \epsilon^{k+1}(\omega_k + g_0\omega_{k-1} + \cdots + g_{k-1}\omega_0) + O(\epsilon^{k+2}), \end{aligned}$$

which can be written as (using (3.25))

$$dH - \epsilon\, dR_0 - \epsilon^2\, dR_1 - \cdots - \epsilon^k\, dR_{k-1} - \epsilon^{k+1}\Omega_k + O(\epsilon^{k+2}).$$

Therefore,

$$\begin{aligned} &(1 + \epsilon g_0 + \cdots + \epsilon^k g_{k-1})(dH - \epsilon\omega) = \\ &\quad dH - \epsilon\, dR_0 - \epsilon^2\, dR_1 - \cdots - \epsilon^k\, dR_{k-1} - \epsilon^{k+1}\Omega_k + O(\epsilon^{k+2}). \end{aligned}$$

Integrating the above equation along Γ_ϵ, where Γ_ϵ is the trajectory of $dH - \epsilon\omega = 0$, and noting that

$$\int_{\Gamma_\epsilon} dH = d(h, \epsilon),$$

$$\int_{\Gamma_\epsilon} (\epsilon\, dR_0 + \epsilon^2\, dR_1 + \cdots + \epsilon^k\, dR_{k-1}) = O(\epsilon^{k+2})$$

(the second estimate follows from the fact that $d(h, \epsilon) = O(\epsilon^{k+1})$), we get

$$d(h, \epsilon) = \epsilon^{k+1} \int_{\Gamma_\epsilon} \Omega_k + O(\epsilon^{k+2}).$$

Finally, Γ_ϵ coincides with the closed curve Γ_h (up to $O(\epsilon)$) and therefore we obtain

$$d(h,\epsilon) = \epsilon^{k+1} \oint_{\Gamma_h} \Omega_k + O(\epsilon^{k+2}),$$

which proves the theorem. $\square$

We derived a formula for the second-order Melnikov function (c.f. (3.16)). Is this formula useful in practice? The answer is no since $g_0(x,y)$, which is given by (3.10), is an elliptic integral in general (and so the calculation of g_0 is very messy, which we do not want to go into it). However, if $\frac{\partial f}{\partial x}(x,y,0) + \frac{\partial g}{\partial y}(x,y,0)$ is an odd function of y, then $g_0(x,y)$ is a polynomial, so the complexity is due to the even part of $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$. Therefore, we need to express the second-order Melnikov function $M_2(h)$ in a more convenient form (i.e. Theorem 3.3) which will be useful in any case. In the following, I will derive Theorem 3.3 using (3.16).

From Lemma 3.1, ω_0 can be written as $\omega_0 = g_0\, dH + d(R_0 - F)$. Substituting the above expression for ω_0 into (3.16) and using the fact that $dH = 0$ on Γ_h, we get

$$\begin{aligned} M_2(h) &= \oint_{\Gamma_h} [g_0\, d(R_0 - F) + \omega_1] \\ &= \oint_{\Gamma_h} g_0\, dR_0 - \oint_{\Gamma_h} g_0\, dF + \oint_{\Gamma_h} \omega_1. \end{aligned} \tag{3.26}$$

Applying integration by parts to the second integral on the right-hand side of (3.26), we get

$$M_2(h) = \oint_{\Gamma_h} g_0\, dR_0 + \oint_{\Gamma_h} F\, dg_0 + \oint_{\Gamma_h} \omega_1. \tag{3.27}$$

Using the definition of g_0, we see that on Γ_h,

$$dg_0 = -\frac{(\partial f/\partial x(x,y,0) + \partial g/\partial y(x,y,0))}{y} dx. \tag{3.28}$$

Also, we know that $\omega_1 = \frac{\partial g}{\partial \epsilon}(x,y,0)\, dx - \frac{\partial f}{\partial \epsilon}(x,y,0)\, dy$. Therefore, it remains to find a convenient form for the first integral on the right-hand side of (3.27). In order to do so, we split the function $G(x,y)$ into odd and even parts with respect to y: $G(x,y) = G_1(x,y) + G_2(x,y)$ (i.e. $G_1(x,-y) =$

$-G_1(x,y)$, $G_2(x,-y) = G_2(x,y)$). We write $G_1(x,y) = y\tilde{G}_1(x,y^2)$, $G_2(x,y) = \tilde{G}_2(x,y^2)$. Now,

$$\begin{aligned}
R_0 &= \int_{\Gamma_{(x,y)}} G(\xi,\eta)\,d\xi \\
&= \int_{\Gamma_{(x,y)}} (G_1(\xi,\eta) + G_2(\xi,\eta))d\xi \\
&= \int_{\Gamma_{(x,y)}} \eta\tilde{G}_1(\xi,\eta^2)\,d\xi + \int_{\Gamma_{(x,y)}} \tilde{G}_2(\xi,\eta^2)\,d\xi \\
&= \int_{\Gamma_{(x,y)}} \eta\tilde{G}_1(\xi,\eta^2)\,d\xi + \int_{a(h)}^{x} \tilde{G}_2(\xi,2h+2U(\xi))\,d\xi \\
&\stackrel{\text{def}}{=} \rho_1(x,y) + P_2(x,h). \qquad (3.29)
\end{aligned}$$

Also,

$$\begin{aligned}
g_0 &\stackrel{\text{def}}{=} -\frac{\partial R_0}{\partial h} \\
&= -\frac{\partial \rho_1}{\partial h}(x,y) - \frac{\partial P_2}{\partial h}(x,h). \qquad (3.30)
\end{aligned}$$

Note that $\rho_1(x,y)$ is an elliptic integral and $P_2(x,h)$ is a polynomial in x (and hence continuous with respect to x). Using the hypothesis that $M_1(h) = M_1'(h) \equiv 0$, we obtain

$$\lim_{y\to 0^-} \rho_1(a(h),y) = \lim_{y\to 0^+} \rho_1(a(h),y) = 0 = \rho_1(b(h),0)$$

and

$$\lim_{y\to 0^-} \frac{\partial \rho_1}{\partial h}(a(h),y) = \lim_{y\to 0^+} \frac{\partial \rho_1}{\partial h}(a(h),y) = 0 = \frac{\partial \rho_1}{\partial h}(b(h),0).$$

The above imply that

$$\oint_{\Gamma_h} \frac{\partial \rho_1}{\partial h}(x,y)\,d[\rho_1(x,y)] = 0, \qquad (3.31)$$

$$\oint_{\Gamma_h} \frac{\partial \rho_1}{\partial h}(x,y)\,d[P_2(x,h)] + P_2(x,h)\,d\left[\frac{\partial \rho_1}{\partial h}(x,y)\right] = 0. \qquad (3.32)$$

Also, it is easy to see that

$$\oint_{\Gamma_h} \frac{\partial P_2}{\partial h}(x,h)\,d[P_2(x,h)] = 0. \qquad (3.33)$$

Using (3.29), (3.30), (3.31), (3.32) and (3.33), we can now find a convenient form for $\oint_{\Gamma_h} g_0 dR_0$ easily. We have

$$\begin{aligned}
\oint_{\Gamma_h} g_0\, dR_0 &= -\oint_{\Gamma_h} \left[\frac{\partial \rho_1}{\partial h}(x,y) + \frac{\partial P_2}{\partial h}(x,h)\right] d[\rho_1(x,y) + P_2(x,h)] \\
&= -\oint_{\Gamma_h} \left[\frac{\partial \rho_1}{\partial h}(x,y)\, d[\rho_1(x,y)] + \frac{\partial \rho_1}{\partial h}(x,y)\, d[P_2(x,h)]\right. \\
&\quad \left. + \frac{\partial P_2}{\partial h}(x,h)\, d[\rho_1(x,y)] + \frac{\partial P_2}{\partial h}(x,h)\, d[P_2(x,h)]\right] \\
&= \oint_{\Gamma_h} \left[P_2(x,h)\, d\left[\frac{\partial \rho_1}{\partial h}(x,y)\right] - \frac{\partial P_2}{\partial h}(x,h)\, d[\rho_1(x,y)]\right] \qquad (3.34)
\end{aligned}$$

Now, using the definition of $\rho_1(x,y)$, we see that on Γ_h,

$$d[\rho_1(x,y)] = G_1(x,y)\, dx, \qquad (3.35)$$

$$d\left[\frac{\partial \rho_1}{\partial h}(x,y)\right] = \frac{\partial G_1}{\partial h}(x,y)\, dx = \frac{1}{y}\frac{\partial G_1}{\partial y}(x,y)\, dx. \qquad (3.36)$$

Substituting (3.35) and (3.36) into (3.34), we immediately get

$$\oint_{\Gamma_h} g_0\, dR_0 = \oint_{\Gamma_h} [G_{1h}(x,y)P_2(x,h) - G_1(x,y)P_{2h}(x,h)]dx,$$

so that Theorem 3.3 is proved. Finally, note that without loss of generality in (3.3), we can take a primitive with $P_2(0,h) = 0$.

Chapter 4
An application of higher-order Melnikov theory

In this chapter, we apply the theory described in the last chapter to an example. We consider the quadratically perturbed non-hyperbolic linear centre studied by Bautin [5]. As in the examples of Chapter 2, the problem is determine the maximum number of limit cycles. In order to do so, we compute the first-order Melnikov function. However, if the first-order Melnikov function is identically equal to zero, then we need to compute the second-order Melnikov function. The number, positions and multiplicities of the zeros of the second-order Melnikov function is related to the number, positions and multiplicities of the limit cycles using Theorem 3.1 or 3.2.

4.1. Perturbed linear centre

In this section, we use the Melnikov theory developed in Chapters 1 and 3 to study the number and positions of the limit cycles that occur in a quadratically perturbed non-hyperbolic linear centre of the form (4.1). This example is of particular interest since it concerns Hilbert's 16th problem, which asks for a bound for the number of limit cycles of a polynomial system in terms of the degrees of the polynomials that define the system. This problem is not solved, even for quadratic systems [4]. Hilbert's 16th problem was studied intensively by many authors, see, for example, [5, 18, 25]. In [5, 18, 25], they study the determination of an exact upper bound of limit cycles under perturbations of certain specific polynomial systems. The strongest theoretical result obtained so far is the following deep theorem of Il'yashenko [13].

Theorem 4.1 (Il'yashenko) *A polynomial system has at most a finite number of limit cycles.*

The above discussion motivates us to study the 6-parameter family

$$\begin{aligned} \dot{x} &= y + \lambda_1 x - \lambda_3 x^2 - (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2 \\ \dot{y} &= -x + \lambda_1 y - \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy + \lambda_2 y^2, \end{aligned} \tag{4.1}$$

where $\lambda_i(\epsilon) = \sum_{j=1}^{\infty} \lambda_{ij}\epsilon^j$, $i = 1, \ldots, 6$ and ϵ is a small perturbation parameter. This is the same system studied by Bautin [5], who proved that there are at most three limit cycles in (4.1) and that a sixth- or higher-order analysis is required to produce that number. The hardest part in his proof is to determine at which order k (the order of the Melnikov function) the maximum number of limit cycles will 'stabilize'. That is, to determine the order of the Melnikov function M_k for which the next Melnikov functions will have the same maximum number of zeros. Therefore, one way to make this problem easier to study is fix the order k. Since we know the Melnikov functions when $k = 1$ and 2, we can determine the maximum number of limit cycles obtained from a first- and a second-order analysis of a quadratically perturbed non-hyperbolic linear centre of the form (4.1). In order to do so, we compute the first- and second-order Melnikov functions using the formulas derived in Chapters 1 and 3. We show that there are no limit cycles from a first-order analysis and that there is at most one hyperbolic limit cycle from a second-order analysis. Our results are consistent with those obtained by Bautin [5]. Bautin, however, goes on to a sixth-order analysis.

For $\epsilon = 0$, (4.1) is a Hamiltonian system with $H(x, y) = (x^2 + y^2)/2$, and it has a one-parameter family of periodic orbits

$$x_h(t) = \sqrt{2h}\cos t, \qquad y_h(t) = -\sqrt{2h}\sin t, \tag{4.2}$$

with the parameter $h \in (0, \infty)$ being the total energy along the orbit. We compute the first- and second-order Melnikov functions of system (4.1). The first-order Melnikov function is given by

$$M_1(h, \lambda) = \oint_{\Gamma_h} g(x, y, 0, \lambda)\, dx - f(x, y, 0, \lambda)\, dy. \tag{4.3}$$

From (4.1), we find that

$$f(x, y, 0, \lambda) = \lambda_{11}x - \lambda_{31}x^2 - (2\lambda_{21} + \lambda_{51})xy + \lambda_{61}y^2,$$

$$g(x,y,0,\lambda) = \lambda_{11}y - \lambda_{21}x^2 + (2\lambda_{31} + \lambda_{41})xy + \lambda_{21}y^2.$$

Substituting into (4.3), we have

$$\begin{aligned}
M_1(h,\lambda) &= \oint_{\Gamma_h} [(\lambda_{11}y - \lambda_{21}x^2 + (2\lambda_{31} + \lambda_{41})xy + \lambda_{21}y^2)\,dx \\
&\quad - (\lambda_{11}x - \lambda_{31}x^2 - (2\lambda_{21} + \lambda_{51})xy + \lambda_{61}y^2)\,dy] \\
&= \lambda_{11}\oint_{\Gamma_h} y\,dx - \lambda_{11}\oint_{\Gamma_h} x\,dy \\
&\quad (\textit{All other integrals vanish}) \\
&= \lambda_{11}\int_0^{2\pi} y_h^2(t)\,dt + \lambda_{11}\int_0^{2\pi} x_h^2(t)\,dt \\
&= 2\lambda_{11}h\int_0^{2\pi} \sin^2 t\,dt + 2\lambda_{11}h\int_0^{2\pi} \cos^2 t\,dt \\
&= 4\pi\lambda_{11}h. \qquad (4.4)
\end{aligned}$$

Since $h = \alpha^2/2$, where $\alpha \in (0,\infty)$ is the positive x-axis intercept of the unperturbed periodic orbit, (4.4) becomes

$$M_1(\alpha,\lambda) = 2\pi\lambda_{11}\alpha^2. \qquad (4.5)$$

We see from (4.5) that the system (4.1) has no limit cycles from a first-order analysis. Note that $M_1(\alpha,\lambda) \equiv 0$ for all $\alpha > 0$ if and only if $\lambda_{11} = 0$. Therefore, for a second-order analysis it suffices to consider the following system

$$\begin{aligned}
\dot{x} &= y + \epsilon[\epsilon\lambda_{12}x - \lambda_{31}x^2 - (2\lambda_{21} + \lambda_{51})xy + \lambda_{61}y^2] \\
\dot{y} &= -x + \epsilon[\epsilon\lambda_{12}y - \lambda_{21}x^2 + (2\lambda_{31} + \lambda_{41})xy + \lambda_{21}y^2].
\end{aligned} \qquad (4.6)$$

From (4.6), we find that

$$f(x,y,\epsilon,\lambda) = \epsilon\lambda_{12}x - \lambda_{31}x^2 - (2\lambda_{21} + \lambda_{51})xy + \lambda_{61}y^2,$$

$$g(x,y,\epsilon,\lambda) = \epsilon\lambda_{12}y - \lambda_{21}x^2 + (2\lambda_{31} + \lambda_{41})xy + \lambda_{21}y^2,$$

$$F(x,y,\lambda) = -\lambda_{31}x^2y - \frac{1}{2}(2\lambda_{21} + \lambda_{51})xy^2 + \frac{1}{3}\lambda_{61}y^3 + \frac{1}{3}\lambda_{21}x^3,$$

$$G(x,y,\lambda) = \lambda_{41}xy - \frac{1}{2}\lambda_{51}y^2,$$

$$G_1(x,y,\lambda) = \lambda_{41}xy, \qquad G_2(x,y,\lambda) = -\frac{1}{2}\lambda_{51}y^2, \qquad G_{1h}(x,y,\lambda) = \frac{\lambda_{41}x}{y},$$

$$P_2(x,h,\lambda) = -\lambda_{51}(hx - \frac{x^3}{6}), \quad P_{2h}(x,h,\lambda) = -\lambda_{51}x.$$

Now, using Theorem 3.3 to compute the second-order Melnikov function, we have

$$\begin{aligned} M_2(h,\lambda) &= -\lambda_{41}\lambda_{51}\oint_{\Gamma_h}\left[\frac{x}{y}(hx - \frac{x^3}{6}) - x^2y\right]dx \\ &\quad - \oint_{\Gamma_h}(-\lambda_{31}x^2y - \frac{1}{2}(2\lambda_{21}+\lambda_{51})xy^2 + \frac{1}{3}\lambda_{61}y^3 + \frac{1}{3}\lambda_{21}x^3) \\ &\quad (-\lambda_{51}y + \lambda_{41}x)\frac{dx}{y} + \lambda_{12}\oint_{\Gamma_h}(y\,dx - x\,dy) \qquad (4.7) \\ &= -\lambda_{41}\lambda_{51}\int_0^{2\pi}(hx_h^2(t) - \frac{1}{6}x_h^4(t) - x_h^2(t)y_h^2(t))\,dt \\ &\quad - \int_0^{2\pi}(\lambda_{31}\lambda_{51}x_h^2(t)y_h^2(t) - \frac{1}{2}(2\lambda_{21}+\lambda_{51})\lambda_{41}x_h^2(t)y_h^2(t) \\ &\quad - \frac{1}{3}\lambda_{51}\lambda_{61}y_h^4(t) + \frac{1}{3}\lambda_{21}\lambda_{41}x_h^4(t))\,dt \\ &\quad + \lambda_{12}\int_0^{2\pi}(y_h^2(t) + x_h^2(t))\,dt \qquad (4.8) \\ &\quad (\textit{All other integrals in } (4.7) \textit{ vanish}). \end{aligned}$$

Substituting $x_h(t) = \sqrt{2h}\cos t$, $y_h(t) = -\sqrt{2h}\sin t$ into (4.8), we get

$$\begin{aligned} M_2(h,\lambda) &= -\lambda_{41}\lambda_{51}\int_0^{2\pi}(2h^2\cos^2 t - \frac{2}{3}h^2\cos^4 t - 4h^2\cos^2 t\sin^2 t)\,dt \\ &\quad - \int_0^{2\pi}(4\lambda_{31}\lambda_{51}h^2\cos^2 t\sin^2 t - 2(2\lambda_{21}+\lambda_{51})\lambda_{41}h^2\cos^2 t\sin^2 t \\ &\quad - \frac{4}{3}\lambda_{51}\lambda_{61}h^2\sin^4 t + \frac{4}{3}\lambda_{21}\lambda_{41}h^2\cos^4 t)\,dt \\ &\quad + \lambda_{12}\int_0^{2\pi}(2h\sin^2 t + 2h\cos^2 t)\,dt \\ &= 4\pi\lambda_{12}h - \pi\lambda_{51}(\lambda_{31}-\lambda_{61})h^2. \qquad (4.9) \end{aligned}$$

Since $h = \alpha^2/2$, where $\alpha \in (0,\infty)$ is the positive x-axis intercept of the unperturbed periodic orbit, (4.9) becomes

$$M_2(\alpha,\lambda) = 2\pi\lambda_{12}\alpha^2 - \frac{\pi}{4}\lambda_{51}(\lambda_{31}-\lambda_{61})\alpha^4. \qquad (4.10)$$

We see from (4.10) that the system (4.1) has at most one hyperbolic limit cycle from a second-order analysis. Finally, note that if the second-order Melnikov function is identically equal to

zero, then we need to consider the third-order Melnikov function. For details on the derivation of the third- or higher-order Melnikov function, see, for example, [5, 18, 25].

Chapter 5
Conclusions and future work

In the Introduction we asked the question: What is the exact number, positions and multiplicities of limit cycles in a perturbed planar system of the form (2), assuming that Assumption 1.2 is satisfied? We then showed that they can be determined by the number, positions and multiplicities of the zeros of the (first-order) Melnikov function, which was derived in Chapter 1. In Chapter 1, we established some non-degeneracy conditions which guarantee that a first-order Melnikov analysis is valid. Also, we established a more degenerate condition which guarantees that a unique non-hyperbolic limit cycle of multiplicity two exists in (2). Chapter 2 contains some examples which illustrate the first-order Melnikov theory. In particular, we analyzed the Bogdanov-Takens bifurcation with reflection symmetry, with the emphasis on the determination of the number of limit cycles corresponding to different values of the parameters. In Chapter 3, the first-order theory was extended to higher-order. In particular, we derived a formula for the second-order Melnikov function for certain perturbed Hamiltonian systems. This formula is useful if the first-order Melnikov function is identically equal to zero. This formula is then applied to a quadratically perturbed non-hyperbolic linear centre in Chapter 4, the results of which agree with those obtained previously by Bautin [5].

Although the Melnikov function can be computed for any system of the form (2), finding the zeros of the Melnikov function is sometimes formidable, especially when a higher-order analysis is required. For example, consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 + \lambda_1 y + \lambda_2 x^2 y, \end{aligned} \tag{5.1}$$

where $\lambda_i(\epsilon) = \sum_{j=1}^{\infty} \lambda_{ij}\epsilon^j$, $i = 1, 2$ and ϵ is a small perturbation parameter. If $\lambda_{11} = \lambda_{21} = 0$, then the first-order Melnikov function is identically equal to zero and a higher-order analysis of (5.1) will be necessary to determine the number of limit cycles that are still preserved under perturbation. In fact, it is possible, using the second-order Melnikov function derived in Chapter 3, to determine the number of limit cycles from a second-order analysis of (5.1). Again, the non-trivial part is to determine the number of zeros of the second-order Melnikov function $M_2(h, \lambda)$. This determination of the number of zeros is beyond the scope of this thesis.

The other problem that may arise is that both the first- and second-order Melnikov functions are identically equal to zero. It is possible to use Françoise's resursive algorithm (e.g. Theorem 3.5) to compute the higher-order Melnikov functions, but since Françoise's method requires calculations of growing complexity at each successive step, in practice only the first few Melnikov functions can be derived by using Theorem 3.5. Other approaches to higher-order Melnikov functions share similar problems.

Much work has been done on the determination of the number of limit cycles in a quadratically perturbed non-hyperbolic linear centre. In [5], Bautin shows that a quadratically perturbed non-hyperbolic linear centre has at most three limit cycles, and that a sixth- or higher-order analysis is required to produce that number. His proof is to derive a 'structural' result about the form of the displacement function $d(\alpha, \epsilon)$ for small α, ϵ, and based on this structure, determine what order analysis in ϵ (i.e. what order Melnikov function) is required to produce that number. On the other hand, much less is known about the number of limit cycles in a cubically perturbed non-hyperbolic linear centre. For example, what is the maximum number of limit cycles in a cubically perturbed non-hyperbolic linear centre, and what order Melnikov function is required to produce that number? These questions are still not completely answered. It is known that the maximum number of limit cycles is greater than or equal to 11 [26]. Li and others [14] in fact constructed an example of a cubic system with eleven limit cycles. Much mathematical research has been devoted to Hilbert's 16th problem in the past and will probably continue to be in the future.

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Appendix A
Derivation of (2.9) by the method of normal forms

In this Appendix, we use the method of Poincaré normal forms to derive (2.9) (see Wiggins [24] for more details). Consider the system

$$\dot{x} = Ax + F_3(x) + O(|x|^5), \quad x = (x, y) \in \Re^2, \tag{A.1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $F_3(x)$ is a vector field consisting entirely of third-degree terms. Note that in order for (A.1) to preserve reflection symmetry, the higher-order terms need to be odd (i.e. fifth-order terms). The idea of Poincaré normal form is to introduce a near-identity coordinate change

$$x = u + h_3(u), \quad u = (u, v) \in \Re^2, \tag{A.2}$$

where $h_3(u) = O(|u|^3)$, to reduce the system (A.1) to its 'normal form'. Substituting the transformation (A.2) into (A.1), we get

$$(I + Dh_3(u))\dot{u} = Au + Ah_3(u) + F_3(u) + O(|u|^5)$$

Now, since

$$(I + Dh_3(u))^{-1} = I - Dh_3(u) + O(|u|^4),$$

it follows that

$$\begin{aligned}\dot{u} &= Au + Ah_3(u) - Dh_3(u)Au + F_3(u) + O(|u|^5) \\ &= Au + \hat{F}_3(u) + O(|u|^5),\end{aligned}$$

where

$$\hat{F}_3(u) = Ah_3(u) - Dh_3(u)Au + F_3(u). \tag{A.3}$$

The point is that we can choose $h_3(u)$ to make $\hat{F}_3(u)$ as simple as possible. Ideally, this would mean choosing $h_3(u)$ such that $\hat{F}_3(u) = 0$; however, this is not always possible.

In order to get a clear understanding of how $h_3(u)$ is chosen, let us view the function

$$L_A[h_3(u)] \equiv Ah_3(u) - Dh_3(u)Au \tag{A.4}$$

as a linear transformation on the space H_3 of all third-degree (homogeneous) polynomials. Since $x = (x, y) \in \Re^2$, we consider L_A as a linear operator on the eight-dimensional vector space

$$\begin{aligned}H_3 &= Span\left\{\begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \end{pmatrix},\right. \\ &\quad \left.\begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \end{pmatrix}\right\}.\end{aligned}$$

Then, $h_3 \in H_3$ is given by

$$h_3(u) = \begin{pmatrix} f_{30}u^3 + f_{21}u^2v + f_{12}uv^2 + f_{03}v^3 \\ g_{30}u^3 + g_{21}u^2v + g_{12}uv^2 + g_{03}v^3 \end{pmatrix}. \tag{A.5}$$

We need to compute $L_A[h_3(u)]$. We get

$$L_A[h_3(u)] = \begin{pmatrix} g_{30}u^3 + (g_{21} - 3f_{30})u^2v + (g_{12} - 2f_{21})uv^2 + (g_{03} - f_{12})v^3 \\ -3g_{30}u^2v - 2g_{21}uv^2 - g_{12}v^3 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}L_A(H_3) &= Span\left\{\begin{pmatrix} u^3 \\ -3u^2v \end{pmatrix}, \begin{pmatrix} u^2v \\ 0 \end{pmatrix}, \begin{pmatrix} uv^2 \\ 0 \end{pmatrix}, \begin{pmatrix} v^3 \\ 0 \end{pmatrix},\right. \\ &\quad \left.\begin{pmatrix} 0 \\ uv^2 \end{pmatrix}, \begin{pmatrix} 0 \\ v^3 \end{pmatrix}\right\},\end{aligned}$$

and we see that

$$H_3 = L_A(H_3) \oplus C_3,$$

where

$$C_3 = Span\left\{\begin{pmatrix} 0 \\ u^3 \end{pmatrix}, \begin{pmatrix} 0 \\ u^2 v \end{pmatrix}\right\}.$$

This shows that any system of the form

$$\dot{x} = Ax + F_3(x) + O(|x|^5), \qquad x = (x, y) \in \Re^2$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $F_3 \in H_3$ can be reduced, by a nonlinear transformation of coordinates $x = u + h_3(u)$ with $h_3 \in H_3$ given by (A.5), to the normal form

$$\begin{aligned} \dot{u} &= v + O(|u, v|^5) \\ \dot{v} &= au^3 + bu^2 v + O(|u, v|^5). \end{aligned} \tag{A.6}$$

It can be shown that if $a \neq 0$ and $b \neq 0$, then the higher-order terms (i.e. fifth-order terms) do not affect the qualitative nature of the non-hyperbolic equilibrium point at the origin provided that they respect the reflection symmetry. Therefore, we can delete these terms in studying the bifurcations that take place in a neighbourhood of this non-hyperbolic equilibrium point. That is, we can just consider the truncated normal form

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= au^3 + bu^2 v. \end{aligned} \tag{A.7}$$