PATH PROPERTIES AND CONVERGENCE OF INTERACTING SUPERPROCESSES

by

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Abstract

Dawson-Watanabe superprocesses are stochastic models for populations undergoing spatial migration and random reproduction. Recently E. Perkins (1993, 1995) introduced an infinite dimensional stochastic calculus in order to characterize superprocesses in which both the reproduction mechanism and the spatial motion of each individual are allowed to depend on the state of the entire population, i.e. superprocesses with interactions.

This work consists of three independent chapters. In the first chapter we show that interactive superprocesses arise as diffusion approximations of interacting particle systems. We construct an approximating system of interacting particles and show that it converges (weakly) to a limit which is exactly the superprocess with interactions. This result depends very intimately upon the structure of the particle systems.

In the second chapter we study some path properties of a class of one-dimensional interactive superprocesses. These are random measures in the real line that evolve in time. We employ the aforementioned stochastic calculus to show that they have a density with respect to the Lebesgue measure. We also show that this density function is jointly continuous in space and time and compute its modulus of continuity. Along with the proof we develop a technique that can be used to solve some related problems. As an application we investigate path properties of a one-dimensional super-Brownian motion in a random environment.

In the third chapter we investigate the local time of a very general class of one-dimensional interactive superprocesses. We apply Perkins' stochastic calculus to show that the local time exists and possesses a jointly continuous version.

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Chapter 0
Introduction

This work is devoted to the study of superprocesses with interactions. Superprocesses (or measure-valued branching diffusions) are measure-valued processes that model populations undergoing random branching and spatial motion. By a population we mean a system involving a number of similar particles. We are interested in the approximations that are possible when the number of particles is large. Consider for example a large population of goats. Individual goats reproduce, die (branching) and wander around (spatial motion). They also interact with each other in many ways. For example they live in clusters or clans. They also have memory and tend not to return to fields they have grazed until some time has passed. Suppose we are asked to implement a computer simulation of the evolution in time of the population. We are interested not only in the total number of goats, but in their geographical locations as well. To this end, we would place a square grid over the region of interest and assign to each square (or pixel) a height (or color) proportional to the number of goats within. Due to the births-deaths (i.e. branching) and spatial motion of each animal the color of each pixel would change as time goes on. We are looking at the evolution in time of the density of goats. Note that the value of such random process at any given instant is not a number but a colored map. Therefore we must give a mathematical interpretation to "colored map"-valued processes. One way to do it is to regard each map as a measure. This example allows us to see why measure-valued processes may arise naturally and are not a mere technical device. We wish to know if after appropriate rescaling, the density of goats can be well approximated by some diffusion process. That is, we are after a limit theorem. We quote the great mathematician A. N. Kolmogorov:

> *The epistemological value of the theory of probability is revealed only by limit theorems. Moreover, without limit theorems it is impossible to understand the real content of the primary concept of all of our sciences – the concept of probability.*

Suppose that we have established the fact that as the number of goats goes to infinity a limit process does exist. We call it the goat superprocess. It is then natural to ask, how does it look like? More specifically, do the measures corresponding to the values of the goat superprocess have density functions? If so, are they continuous? Since continuous functions can oscillate wildly, how continuous? Of course, we require a rigorous formulation of all these questions.

The subject of superprocesses is a rapidly developing field. It has been stimulated from several different directions including population genetics, branching processes, interacting particle systems and stochastic partial differential equations. For example the so-called Fleming-Viot superprocess is a generalization of the Wright-Fischer model (for random replication with er-

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rors) and has been studied in connection with population genetics (Hartl and Clark 1989). Rogers and Williams (1986) wrote in the preface of their book:

> *Here are some guidelines on what you might move on when your reading of our book is done.*
>
> *. . .*
>
> *(vi) Measure-valued diffusions, random media, etc.. Durrett (1985) and Dawson and Gärtner (1987) can be your 'open sesame' to what is sure to be one of the richest of Aladdin's caves.*

We believe that the monograph of Dawson (1993) proves that Rogers and Williams were correct.

In this thesis we will give a rigorous description of a general class of models that hopefully includes what our intuition says the goat superprocess should be. We will then answer some specific questions about certain subsets of the class of models. In Section 0.1 we survey some of the theory of superprocesses. In Section 0.2 we explain our results.

0.1 Review

In this Section we review the most basic concepts needed to understand both the meaning and relevance of our results. Most of the background information is very recent and known only by a handful of experts. The ideal preparation is furnished by a thorough reading of the papers Perkins (1993, 1995). These in turn can be well understood after reading either of our favorites Walsh (1986) or Dawson (1993). Our notation is consistent with that of Perkins (1995), and a reader familiar with this material can safely skip the rest of this Section. We have made an effort to give an intuitive understanding of the ideas presented below. However many of them are accompanied by heavy technical baggage, and these technicalities are important for a careful examination of this thesis. They cannot be avoided. Some of these ideas may seem obscure at a first glance, but they contain large amounts of valuable information. This is hardly a surprise since we will be studying some fairly complex objects. We encourage the reader to refer to the excellent references when in doubt or need. We assume a basic knowledge of Probability theory on the part of the reader. We expect the reader to be acquainted with

1. Martingales, Brownian motion, weak convergence of probability measures on metric spaces, critical branching processes (the standard Galton-Watson process will suffice), the Poisson process.

2. Martingale inequalities (Doob's, Burkholder's and maximal), stopping times, stochastic integration (Itô's lemma), local times (Tanaka's formula).

Most of these topics (certainly those in 1) are covered in a first graduate course in Probability theory. Those not familiar with the topics listed in 2 should still be able to understand the statement and meaning of the most important theorems. They should also be able to read the remainder of this Section. The subjects listed in 1 and 2 are the absolute minimum required to understand most of the proofs. Familiarity with superprocesses and/or stochastic partial differential equations (abbreviated SPDE) is highly recommended.

By a measure-valued process we mean a random process whose state space is $M_F(E)$, the space of finite measures on some complete, separable, metrizable topological space $(E, \mathcal{B}(E))$. As usual, $\mathcal{B}(E)$ denotes the Borel σ-field and $M_F(E)$ is endowed with the topology of weak convergence. Superprocesses are related to branching processes, population genetics models, stochastic partial differential equations and interactive particle systems. The canonical example is d-dimensional super Brownian motion, which we now describe. Fix a positive integer N, a positive real number γ and a finite measure m on $\mathbb{R}^d$. At time t=0, N particles are located in $\mathbb{R}^d$ with law $m(\cdot)/m(\mathbb{R}^d)$. They move independently according to d-dimensional Brownian motions. If a particle is located at position x at time t, then let the probability that it dies before time $t + dt$ be $\gamma N dt + o(dt)$. If it dies, a fair coin is tossed and the particle is replaced by 0 or 2 identical particles situated at the position of death of their parent. The new particles then start undergoing independent Brownian motions and the process continues in the same fashion, with particles moving, dying and branching ad infinitum. In this model we want to keep track of the number of particles as well as their locations. Let $I(N,t)$ be the total number of particles alive at time t, and let Z_t^i, $i = 1, \ldots, I(N,t)$, label their locations. Consider the rescaled measure-valued process

$$X^N(t) = \frac{1}{N} \sum_{i=1}^{I(N,t)} \delta_{Z_t^i},$$ (0.1)

where δ_e denotes a unit mass at e. When the number of particles is large, i.e. as $N \to \infty$, the particle system X^N is approximated by a measure-valued diffusion X which we call super Brownian motion with branching rate γ. In fact, the sequence of probability measures $(\mathbb{P}(X^N \in \cdot))_N$ converges weakly to a law $\mathbb{P}_m$ on $C([0, \infty), M_F(\mathbb{R}^d))$ (Dawson 1993). Following the usual convention, we endow $C([0, \infty), E)$, the set of continuous paths $t \mapsto x_t \in E$, with the compact-open topology and $D([0, \infty), E)$, the space of càdlàg paths $t \mapsto x_t \in E$, with Skorohod's J_1-topology. Super Brownian motion X can be characterized through a martingale problem (Dawson 1993). A typical example of a martingale problem is Lévy's characterization of Brownian motion. It says that if an $\mathbb{R}^d$-valued random process B is a continuous martingale with square function $\langle B^i, B^j \rangle_t = t\delta_{ij}$ (here δ_{ij} denotes Kronecker's delta), then B is a d-dimensional Brownian motion. The following is a martingale problem that uniquely characterizes the law $\mathbb{P}_m$. If μ is a measure we write $\mu(\phi) \equiv \int \phi \, d\mu$.

$(MP)^m_{\gamma,1,0,0}$ There is a continuous $M_F(\mathbb{R}^d)$ valued, adapted process X_t defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that

(i) $\mathbb{P}(X_0 = m) = 1$

(ii) If $\phi \in C_b^2(\mathbb{R}^d)$, then

$$X_t(\phi) = X_0(\phi) + \int_0^t X_s\left(\frac{\Delta}{2}\phi\right) ds + Z_t(\phi),$$ (0.2)

where $Z_t(\phi)$ is a continuous square integrable $(\mathcal{F}_t)$-martingale null at zero with square function

$$\langle Z(\phi) \rangle_t = \int_0^t X_s(\gamma \phi^2) ds.$$

Moreover the law $\mathbb{Q}^m$ of the canonical process $X_t(\omega) = \omega_t$ on $\Omega \equiv C([0,\infty), M_F(\mathbb{R}^d))$ is uniquely determined by (i) and (ii) (Dawson 1993).

Recall that there are two independent sources of noise, namely the Brownian noise and the branching noise. For each test function $\phi \in C_b^2(\mathbb{R}^d)$ (0.2) provides the semimartingale decomposition of $(X_t(\phi) : t \geq 0)$. The martingale part $Z.(\phi)$ comes entirely from the branching, while the drift part $\int_0^t X_s\left(\frac{\Delta}{2}\phi\right)ds$ comes exclusively from the spatial motions. Note that if there is no branching (i.e. $\gamma = 0$), then (0.2) becomes the deterministic equation

$$X_t(\phi) = X_0(\phi) + \int_0^t X_s\left(\frac{\Delta}{2}\phi\right)ds,$$

a weak form of the heat equation. This is in fact a version of the law of large numbers. On the other hand, it can be shown that if we set $\phi \equiv 1$, that is, if we disregard the spatial motions and look only at the total mass, $(MP)^m_{\gamma,1,0,0}$ gives a continuous-time continuous-state space branching process studied by Feller (1951). This process is a diffusion approximation for the classical Bienaymé-Galton-Watson critical branching process.

The construction of super Brownian motion can be greatly generalized to yield more general superprocesses. One can tinker with the branching mechanisms or the space motions. For example it can be proved that the recipe used to obtain super Brownian motion may be adjusted to obtain super Feller processes. Just replace the Brownian motions by a Feller process with infinitesimal generator A and locally compact state space. The result will be a limit superprocess characterized by a martingale problem analogous to $(MP)^m_{\gamma,0,1,0}$ but with $(A, Domain(A))$ instead of $(\frac{\Delta}{2}, C_b^2(\mathbb{R}^d))$ (Dawson 1993).

When $d = 1$ we can recast $(MP)^m_{\gamma,1,0,0}$ in stochastic partial differential equation (SPDE) form: $X_t(dx) = u(t,x)dx \; \mathbb{P}_m - a.s.$, where u is the unique (in law) solution of

$$\frac{\partial u}{\partial t} = \frac{\Delta}{2}u + \sqrt{\gamma u}\,\dot{W},\tag{0.3}$$

and W is a space time white noise (Reimers 1989, Konno and Shiga 1988). We interpret equation (0.3) in a weak sense, as is usually done in the theory of Partial Differential Equations. That is, we multiply (0.3) by a smooth test function ϕ with compact support and integrate over space-time:

$$\int \phi(x)u(t,x)dx = \int_0^t \int \frac{\Delta}{2}u(s,x)\phi(x)dxds + \int_0^t \int \sqrt{\gamma u(s,x)}\phi(x)\dot{W}(x,s)dxds.$$

Integrate by parts twice on the r.h.s. to obtain

$$X_t(\phi) = \int_0^t X_s\left(\frac{\Delta}{2}\phi\right)ds + \int_0^t \int \sqrt{\gamma u(s,x)}\phi(x)\dot{W}(x,s)dxds$$
$$\equiv \int_0^t X_s\left(\frac{\Delta}{2}\phi\right)ds + Z_t(\phi).$$

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Note that

$$\langle Z(\phi) \rangle_t = \left\langle \int_0^\cdot \int \sqrt{\gamma u(s,x)} \phi(x) \dot{W}(x,s) dx ds \right\rangle_t$$

$$= \int_0^t \int \gamma u(s,x) \phi(x)^2 dx ds$$

$$= \int_0^t X_s(\gamma \phi^2) ds.$$

By analogy with the PDE (partial differential equation) setting in which $\dot{W}$ is replaced by a smooth function, it is possible to write a Green's function (or inverse Laplacian) representation of the solution of (0.3):

$$u(t,x) = \int p(t, x-z) u(0,z) dz + \int_0^t \int p(t-s, x-z) \sqrt{\gamma u(s,z)} \dot{W}(z,s) dz ds, \qquad (0.4)$$

where $p(t,x)$ is the one-dimensional Brownian transition density. Equation 0.4 has proved to be extremely useful to compute the moments of u (Konno and Shiga 1988).

The study of super Brownian motion (among other things) revealed the need to introduce a related object called the historical process or historical super Brownian motion. While analyzing fine path properties Dawson and Perkins (1991) realized that the genealogy (or family structure) of the particles is of great importance. We give an informal example to motivate this last statement. Pick a "typical" particle in the population at time $t = 1$. That is, choose a point x in the set $Supp(X_1)$ (the support of X_1) according to the measure X_1. Suppose we are investigating the asymptotic behaviour of the total mass in a ball of radius r about x as $r \to 0$ (very useful when studying path properties). To estimate this quantity trace the trajectory of our particle backwards in time until time $t = 0$. It turns out that in two or more dimensions "most" of the mass in the ball will come from particles that branched off this trajectory between $t = 0$ and $t = 1$. This property stems from the fact that due to the criticality of the branching, only a finite number of particles alive at time $t = 0$ have (an infinite number of) descendants alive at time $t = 1$. This type of argument can be made rigorous leading to very precise computations of Hausdorff functions, etc. (Perkins 1988). To describe the historical process consider the tree of branching Brownian motions described above. Let H_t^N be the random measure on the space $C([0,\infty), \mathbb{R}^d)$ which assigns mass N^{-1} to the trajectory of each particle alive at time t. If x^t denotes a path x stopped at time t $(x^t(\cdot) = x(t \wedge \cdot))$ then

$$H^N(t) = \frac{1}{N} \sum_{i=1}^{I(N,t)} \delta_{(Z^i)^t}. \qquad (0.5)$$

Equation (0.5) generalizes (0.1). As $N \to \infty$ $\mathbb{P}(H^N \in \cdot)$ converges weakly to a law $\mathbb{Q}_m$ on $C([0,\infty), M_F(C([0,\infty), \mathbb{R}^d)))$ (Dawson 1993). The historical process H is the canonical process $H_t(\omega) = \omega_t$ in the probability space $\Omega = (\mathbb{Q}^m, C([0,\infty), M_F(C([0,\infty), \mathbb{R}^d))))$. It can also be characterized by a martingale problem which generalizes $(MP)_{\gamma,1,,0,0}^m$.

From a modelling perspective, it is natural to try to introduce measure-valued processes in which the particles interact. Three of the most important tools to analyze finite dimensional

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diffusions are the change of measure (Cameron-Martin-Girsanov theorem), the martingale problem of Stroock and Varadhan and Itô's theory of stochastic differential equations. These three approaches admit highly non-trivial generalizations to the infinite dimensional setting of superprocesses, thus allowing the introduction of interactions (Dawson (1993) and Perkins (1993, 1995)).

Perkins (1993, 1995) carried out a program to construct interactive historical superprocesses $K_t(dy)$ in which a trajectory y^t in a population K_t is subject to a drift $b(t, K_t, y)$, diffusion matrix $\sigma(t, K_t, y)$ and branching rate $\gamma(t, K_t, y)$. As an application one can generalize $(MP)^m_{\gamma,1,0,0}$ to $(MP)_{\gamma,\sigma,b,0}$. That is, the Laplacian (infinitesimal generator of Brownian motion) can be replaced by an elliptic operator of the form $A(t, X_t, x) = \frac{1}{2}(\sigma\sigma^*)_{ij}(t, X_t, x)\partial_i\partial_j + b_i(t, X_t, x)\partial_i$ (sum over repeated indices), and the constant γ by a function $\gamma(t, X_t, x)$. These coefficients depend on time, the state of the population and the position of the particle. Of course some constraints are required on these coefficients. A very interesting example of an interactive historical process due to Adler (Perkins 1995) is the following.

Example. Let $p_s(x) = (2\pi)^{-1/2}\exp(-\frac{x^2}{2s})$, $\varepsilon > 0$ (small), and suppose $b(t, K, y)$ and $\gamma(t, K, y)$ are given by the following

$$b(t, K, y) = \int_0^t \int \nabla p_\varepsilon(y_s' - y_t)K_s(dy')e^{-\lambda(t-s)}ds$$

$$\gamma(t, K, y) = \exp\left(-\int_0^t \int p_\varepsilon(y_s' - y_t)K_s(dy')e^{-\lambda(t-s)}ds\right).$$

Here the goat-like particles tend to drift away from regions where the population has already grazed (and consumed the resources). They also reproduce at a lower rate on those regions. The parameter λ represents the rate of recovery of the environment. □

Perkins has developed a theory of stochastic integration along Brownian trees (Perkins 1993, 1995) and has used it to characterize a broad class of interactive measure-valued branching diffusions as the unique solution of a stochastic equation driven by a historical Brownian motion. We shall explain Perkins' idea shortly. As a prerequisite let us recall some facts about stochastic differential equations (SDE's). Write $C^d = C([0,\infty), \mathbb{R}^d)$ and $C^d_t = \sigma(y_s : s \leq t)$, let S^d be the set of $d \times d$ symmetric positive definite matrices, and let

$$b(t, y) : [0,\infty) \times C^d \to \mathbb{R}^d,$$
$$a(t, y) : [0,\infty) \times C^d \to S^d,$$

be predictable path functionals. The intent is to give meaning to the notion of a locally Gaussian d-dimensional process Y satisfying the following

$$E[Y^i_{t+dt} - Y^i_t | Y^t] = dt\, b_i(t, Y) + o(dt),$$
$$E[(Y^i_{t+dt} - Y^i_t)(Y^j_{t+dt} - Y^j_t) | Y^t] = dt\, a_{ij}(t, Y) + o(dt). \tag{0.6}$$

This models the motion of a particle in a velocity field b which is subject to a random thermal motion of covariance a. Itô's brilliant idea was to give meaning to (0.6) by rewriting it in the form

$$dY^i_t = b_i(t, Y)dt + \sum_{j=1}^d \sigma_{ij}(t, Y)dB^j_t,$$

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$i = 1, \ldots, d$, or for short

$$dY_t = b(t, Y)dt + \sigma(t, Y)dB_t,$$

where B is a d-dimensional Brownian motion and σ is the square root of a. The last equation should be interpreted as an integral equation. Itô gave meaning to the integral $\int_0^t \sigma(t, Y)dB_s$, which is named after him. This is not straightforward since B is of unbounded variation. Therefore $\int_0^t \sigma(t, Y)dB_s$ cannot be a Stieltjes integral.

A *strong* solution of the SDE

$$Y_t = Y_0 + \int_0^t b(s, Y)ds + \int_0^t \sigma(s, Y)dB_s, \tag{0.7}$$

on a given probability space $(\Omega, \mathcal{F}, P)$ and with respect to a fixed Brownian motion $B = (B, (\mathcal{F}_t))$ and initial condition ξ is a process $Y = \{Y_t : 0 \leq t < \infty\}$ with continuous sample paths which satisfies the following properties:

- Y is adapted to the filtration $(\mathcal{F}_t)$.

- $P[Y_0 = \xi] = 1$.

- $P[\int_0^t (\|b(s, Y)\| + \|\sigma\sigma^*(s, Y)\|)ds < \infty] = 1$ for all $t \geq 0$. ($\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{R}^{d \times d}$).

- $Y_t = Y_0 + \int_0^t b(s, Y)ds + \int_0^t \sigma(s, Y)dB_s$ holds $P - a.s.$ The second integral on th r.h.s. must be interpreted in the sense of Itô.

If Y is a strong solution of (0.7) there is a (measurable) map $h : \mathbb{R}^d \times C^d \to C^d$ such that $Y = h(\xi, B)$ $P - a.s.$

We are now in a position to elucidate Perkins' idea. Let

$$b : [0, \infty) \times C([0, \infty), M_F(C^d)) \times C^d \longrightarrow \mathbb{R}^d,$$
$$\sigma : [0, \infty) \times C([0, \infty), M_F(C^d)) \times C^d \longrightarrow \mathbb{R}^{d \times d},$$
$$\gamma : [0, \infty) \times C([0, \infty), M_F(C^d)) \times C^d \longrightarrow (0, \infty).$$

Recall that H denotes a historical Brownian motion. Intuitively, a typical path y chosen according to H_t is a Brownian path stopped at time t. This is certainly true if H is replaced by H^N. Perkins (1993) found a way to prove it. This allowed him to define an essentially unique version of the Itô integral $\int_0^t \sigma(s, K, y)dy(s)$. He then considered the following simultaneous equations

$$Y_t = y_0 + \int_0^t b(s, K, Y)ds + \int_0^t \sigma(s, K, Y)dy(s), \quad t \geq 0 \tag{0.8}$$

$$K_t(A) = \int 1(Y^t(y) \in A)\gamma(t, K, Y)H_t(dy), \quad t \geq 0, \quad A \in \mathcal{C}^d \tag{0.9}$$

(Perkins (1993, 1995)). By (0.8), Y solves an Itô equation along the branch y with drift b and diffusion σ. If the coefficients σ, b, γ are Lipschitz in an appropriate sense then (0.8) has a

unique strong solution of the form $Y = h(y_0, y)$, and we write $Y = Y(y)$ for short. Hence, in principle at least, (0.9) makes sense. γ can still be interpreted as a branching rate, although some additional work is needed (Perkins 1995). If $\gamma \equiv 1$, then Y^t is a typical path in the population K_t. Y is an auxiliary process and K is the desired interactive historical process. Other intuitive explanations are found in the introductions to the papers Perkins (1993) and Perkins (1995). With suitable hypotheses the system (0.8-0.9) has a pathwise unique solution (Perkins 1995). These hypotheses are satisfied by the coefficients b, γ in the previous Example and by $\sigma \equiv Id$. If $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d$, $\bar{\sigma} : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, are Lipschitz continuous (with respect to the Euclidean norm), $\gamma \equiv 1$ and $b(t, K, y)$ and $\sigma(t, K, y)$ are defined by

$$b(t, K, y) := \int \bar{b}(y_t - y_t') K_t(dy')$$

and

$$\sigma(t, K, y) := \int \bar{\sigma}(y_t - y_t') K_t(dy')$$

respectively, then $(0.8 - 0.9)$ has a unique strong solution. Moreover, the process $X_t(A) = K_t(y : y_t \in A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$ solves the following martingale problem

$(MP)_{1,\sigma,b,0}^m$ If $\phi \in C_b^2(\mathbb{R}^d)$, then

$$X_t(\phi) = m(\phi) + \int_0^t \int \left(\frac{1}{2}(\sigma\sigma^*)_{ij}(s, X_s, x)\partial_i\partial_j\phi(x) + b_i(s, X_s, x)\partial_i\phi(x) \right) X_s(dx)ds + Z_t(\phi),$$

where $Z_t(\phi)$ is a continuous square integrable $(\mathcal{F}_t)$-martingale null at zero with square function

$$\langle Z(\phi) \rangle_t = \int_0^t X_s(\phi^2)ds.$$

Here $b(t, X_t, x) = \int \bar{b}(x - z)X_t(dz)$ and $\sigma(t, X_t, x) = \int \bar{\sigma}(x - z)X_t(dz)$. As usual, integrals of vector valued functions are computed componentwise.

0.2 Summary of the Main Results

After pummelling the reader with a lengthy review let us finally enunciate our results. The exposition is divided into three independent chapters that can be read separately. However we suggest to the reader not familiar with the subject to read them in order.

The thesis is organized as follows: Chapter one presents the weak convergence of interacting particle systems to superprocesses with interactions. The interactive historical process K solution of (0.8-0.9) is expected to arise as a diffusion approximation for a sequence (K^N) of properly rescaled interacting branching particle systems. Dawson & Perkins (1996) claim that this fact is proven in this thesis so we are compelled to demonstrate it. The proof requires a good understanding of the particle systems and is not a corollary of some well known general convergence theorem.

In Chapter 2 we study interactive measure valued diffusions of the form

$$Y_t = y_t + \int_0^t b(s, X_s, Y_s) ds,$$

$$X_t(A) = \int 1(Y_t(y) \in A) \gamma(t, X, Y) H_t(dy), \quad A \in \mathcal{B}(\mathbb{R}).$$

Here $d = 1$. We show that $X_t(dx) = u(t, x) dx$, where $(t, x) \mapsto u(t, x)$ is Hölder continuous. The proof is an application of historical stochastic calculus. This result was proved for super Brownian motion relatively recently by Reimers (1989) and independently by Konno & Shiga (1988). As an example we show that the above result leads to the existence of approximate solutions to the following version of Burgers' equation with noise. Let W be a space time white noise and consider

$$\frac{\partial u}{\partial t} = \frac{\Delta}{2} u - \frac{\partial}{\partial x}(u^2) + \sqrt{u}\, \dot{W}. \tag{0.10}$$

In fact, for any arbitrary $\varepsilon > 0$ we solve a smoothed version of (0.10), namely

$$\frac{\partial u}{\partial t} = \frac{\Delta}{2} u - \frac{\partial}{\partial x}\left(u \cdot p_\varepsilon * u\right) + \sqrt{u}\, \dot{W},$$

where p_s is the heat kernel and the convolution is in the x variable. We also show that the method of proof of the main theorem can be employed to investigate some related problems. In particular, we study a super Brownian motion in a random environment.

In Chapter 3 we look at the local times of one-dimensional interacting superprocesses which arise as solutions of

$$Y_t = y_0 + \int_0^t \sigma(X_s, Y_s) dy(s) + \int_0^t b(X_s, Y_s) ds,$$

$$X_t(A) = \int 1(Y_t(y) \in A) H_t(dy), \quad A \in \mathcal{B}(\mathbb{R}).$$

Once again $d = 1$ and the notation is the same as above. We show that there exists a jointly continuous process $(a, t) \mapsto L_t^a(X)$ such that

$$\int_0^t X_s(\phi) ds = \int_{-\infty}^{\infty} \phi(a) L_t^a(X) da$$

for any positive Borel function ϕ. The process $L_t^a(X)$ is called the local time of X. The proof of this result relies on a Tanaka-like formula of Perkins (1995). The proof uses many of the tools of the infinite dimensional stochastic analysis developed by Perkins as well as some abstract tools like the predictable section theorem to obtain concrete estimates. The mere existence of the local time is a path property of X (Geman and Horowitz 1980). We can also see directly that the local time is an interesting quantity. As Adler (1992) has pointed out, the superprocess discussed in the Example of Section 0.1 exhibits waves of local time. This is also evident in computer simulations.

Most of the topics treated in this work have been studied in the non-interacting case, and sharper answers have been given in this case. However none of the methods of proof can be

translated to the interactive case as they rely very heavily on the independence assumptions. In this work we introduce new methodologies. However, by considering rather general coefficients we renounce to the possibility of carrying out very precise computations such as those in Perkins (1988) or Perkins-Le Gall-Taylor (1995).

Chapter 1

Weak Convergence of Interacting Branching Particle Systems

1.1 Introduction

Super-Brownian motion arises as a diffusion approximation for Bienaymé-Galton-Watson (BGW) trees of branching Brownian motions (Walsh 1986.) A BGW tree of branching Brownian motions with parameters (N, γ) can be described as follows: Fix $\gamma > 0$. At time $t = 0$, N particles are located in $\mathbb{R}^d$ according to some fixed initial distribution. To be concrete, let us place all of them at the origin. The particles start moving independently, undergoing Brownian motions. During a time interval $(t, t + dt)$ any given particle has a probability $\gamma N\, dt$ of dying. If it does die, it either splits into 2 identical particles or it goes instantaneously to an isolated point ∂ (a cemetery), with probability 1/2 each. If it branches, the two offspring start their lifetime at the point of death of their parent. They continue moving independently according to d-dimensional Brownian motions until they either die or branch and so on. In this model we want to keep track of the locations of the particles as well as their numbers. One way to describe the state of the system of particles is to identify the position of the particles with the position of the atoms of a purely atomic measure, say X_t^N. If each particle is assigned mass 1 then $X_t^N(A)$ is the number of particles in the set A at time t. We are interested in the high density limit when $N \to \infty$. Of course some renormalization is required. It turns out that the properly renormalized object to study is $N^{-1}X_t^N$. In fact $N^{-1}X^N \Rightarrow X$, where X is a measure-valued process which we call super-Brownian motion with branching rate γ. X can be characterized by a martingale problem. Recall that if μ is a measure, $\mu(\phi) \equiv \int \phi\, d\mu$ and $M_F(E)$ denotes the set of finite Borel measures on a metric space E and it is endowed with the weak topology.

Theorem 1.1 (Super-Brownian Motion with Branching Rate γ). *Let δ_0 denote a unit mass at 0. There is a continuous $M_F(\mathbb{R}^d)$ valued, adapted process X_t defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ such that*

1. $\mathbb{P}(X_0 = \delta_0) = 1$

2. If $\phi \in C_b^2(\mathbb{R}^d)$, then

$$X_t(\phi) = X_0(\phi) + \int_0^t X_s\left(\frac{\Delta}{2}\phi\right)ds + Z_t(\phi), \qquad (MP)_{\Delta,\gamma}$$

where $Z_t(\phi)$ is a continuous square integrable $(\mathcal{F}_t)$-martingale null at zero with square function

$$\langle Z(\phi) \rangle_t = \int_0^t X_s(\gamma \phi^2) ds.$$

Moreover the law, $\mathbb{Q}^{\delta_0}$, of X on $C([0,\infty), M_F(\mathbb{R}^d))$ is uniquely determined by 1 and 2.

Observe that $(MP)_{\Delta,\gamma}$ gives a semimartingale decomposition for $X.(\phi)$ when $\phi \in C_b^2(\mathbb{R}^d)$. As we shall see later, the drift part comes from the spatial motions while the martingale part comes from the branching. More generally one can replace the space motions (i.e. the Brownian motions) by a Feller process ζ taking values in a locally compact space. Call L its infinitesimal generator and $D(L)$ its domain. ζ could be, for example, a stable process. In this more general setting Theorem (1.1) holds if $\Delta/2$ is replaced by L and $C_b^2(\mathbb{R}^d)$ by $D(L)$. That is, the martingale problem $(MP)_{L,\gamma}$ is well posed (Dawson 1993).

The branching particle systems contain genealogical information that is not evident in the limiting super-Brownian motion. This information has proven to be extremely valuable when analyzing path properties of super-Brownian motion (Perkins 1988, Dawson and Perkins 1991). A process called historical super-Brownian motion, which records the past histories of all individuals in the population, can be defined as follows. Let $C^d = C([0,\infty), \mathbb{R}^d)$ and $\mathcal{C}_t^d = \sigma(y_s : s \leq t)$, the canonical σ-field of C^d. Let H_t^N be the random measure on the space C^d of $\mathbb{R}^d$-valued continuous paths which assigns mass N^{-1} to the *trajectory* of each particle alive (in a super-BM) at time t. It can be shown (Dawson and Perkins 1991) that H^N converges weakly to a $M_F(C^d)$-valued continuous process H. This limit process is called the historical process and its law may also be characterized by a martingale problem.

Theorem 1.2 (Historical Super-Brownian Motion). *Let D_0 denote the set of $f \in C^d$ such that $f(y) = g(y(t_1), \dots, y(t_n))$ for some $0 \leq t_1 \leq \dots \leq t_n$ and a smooth function g which is constant outside some compact set. For $f \in D_0$ let*

$$\frac{1}{2}\bar{\Delta}f(y,t) = \frac{1}{2}\sum_{i=1}^d \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} 1(t < t_{k+1} \wedge t_{l+1}) \frac{\partial^2 g}{\partial x_{kd+i} \partial x_{ld+i}}(y(t \wedge t_1), \dots, y(t \wedge t_n)).$$

The law $\mathbb{P}^{\delta_0}$ of historical Brownian motion on $C([0,\infty), M_F(C^d))$ is uniquely determined by the following martingale problem.

$$\forall f \in D_0 \qquad H_t(f) = \delta_0(f) + \int_0^t H_s\left(\frac{1}{2}\bar{\Delta}f(\cdot, s)\right) ds + Z_t(f),$$

where $Z_t(f)$ is a continuous square integrable martingale null at zero with square function

$$\langle Z(f) \rangle_t = \int_0^t H_s(\gamma f^2) ds.$$

Note that super Brownian motion can be obtained as a projection of historical Brownian motion, $X_t(\cdot) = H_t(\{y \in C^d : y_t \in \cdot\})$.

If one is interested in modeling it is natural to try to introduce some sort of dependence into the superprocess. For example one may want to consider space motions in which individuals are attracted to each other, or branching rates that reflect the fact that lonely individuals tend to die faster. With this in mind, one is tempted to replace the Laplacian in Theorem (1.1) by an elliptic operator of the form

$$A(t, X, x) = \frac{1}{2} \sum_{i,j=1}^{d} a^{i,j}(t, X, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(t, X, x) \frac{\partial}{\partial x_i}. \tag{1.1}$$

Similarly, one may want to consider $\gamma = \gamma(t, X, x)$. These coefficients depend on time, the state and past history of the population and the current position of a particle. For example (in $d=1$) if $\bar{b} : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, then we could define $b(t, X, x) = \int_{\mathbb{R}} \bar{b}(t, z - x) X_t(dz)$. In this case, a particle located at x at time t would feel a drift (i.e. pull or push) of magnitude $\bar{b}(t, z-x) X_t(dz)$ coming from the particles located in a cube of side dz and centered at z. With appropriate assumptions on the coefficients a, b, γ the martingale problem $(MP)_{A,\gamma}$ has a solution (Perkins 1995, Roelly and Meleard 1990). Roelly and Meleard showed that a solution of the martingale problem can be obtained as a limit of a renormalized system of interacting particles.

More generally, one is led to consider interactions in the enriched historical setting. This can be very useful for example in the modeling of non-Markovian superprocesses (such as the goat superprocess introduced in Chapter 0). We shall need a generalization of (1.1). To this end we present some notation. Fix $0 \le t_1 < t_2 < ... < t_n$ and $\psi \in C_b^2(\mathbb{R}^{nd})$. Let

$$\bar{\psi}(y) \equiv \bar{\psi}(t_1, t_2, ..., t_n)(y) = \psi(y(t_1), ..., y(t_n))$$

ψ_i and ψ_{ij} denote the first and second order partials of ψ. $\nabla \bar{\psi}(t, y) : [0, \infty) \times C^d \to \mathbb{R}^d$ is the $(\mathcal{C}_t)$ predictable process whose j-th component at (t, y) is

$$\sum_{i=0}^{n-1} 1(t < t_{i+1}) \psi_{id+j}(y(t \wedge t_1), ..., y(t \wedge t_n))$$

If $1 \le i, j \le d$, $\bar{\psi}_{ij} : [0, \infty) \times C^d \to \mathbb{R}$ is the $(\mathcal{C}_t)$ predictable process defined by

$$\bar{\psi}_{ij}(t, y) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} 1(t \le t_{k+1} \wedge t_{l+1}) \psi_{kd+i\, ld+j}(y(t \wedge t_1), ..., y(t \wedge t_n))$$

Let

$$D_0 \equiv \bigcup_{m=1}^{\infty} \{ \bar{\Psi}(t_1, t_2, ..., t_n) : 0 \le t_1 < t_2 < ... < t_n, \Psi \in C_0^\infty(\mathbb{R}^m) \} \cup \{1\}$$

We will work on filtered probability space $\bar{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

The following definition is motivated by Theorem (1.2) and by the preceding discussion.

Definition 1.3 (Historical Super-Brownian Motion with Interactions). Let S^d denote the space of symmetric positive definite $d \times d$ matrices. Suppose

$$a : [0, \infty) \times D([0, \infty), M_F(C^d)) \times C^d \longrightarrow S^d,$$
$$b : [0, \infty) \times D([0, \infty), M_F(C^d)) \times C^d \longrightarrow \mathbb{R}^d,$$
$$\gamma : [0, \infty) \times D([0, \infty), M_F(C^d)) \times C^d \longrightarrow (0, \infty).$$

A predictable process $K \in C([0, \infty), M_F(C^d))$ on $\bar{\Omega}$ satisfies $(MP)_{a,b,\gamma}$ (with initial condition δ_0) if

$$\forall \bar{\psi} \in D_0 \quad K_t(\bar{\psi}) = \delta_0(\bar{\psi}) + Z_t(\bar{\psi}) + \int_0^t \int \Big[\langle \nabla \bar{\psi}(s,y), b(s,K,y) \rangle$$

$$+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \bar{\psi}_{i,j}(s,y) a_{i,j}(s,K,y) \Big] K_s(dy) ds \quad \forall t \geq 0,$$

where $Z_t(\bar{\psi})$ is a continuous square integrable martingale with square function

$$\langle Z(\bar{\psi}) \rangle_t = \int_0^t \int \gamma(s,K,y) \bar{\psi}(y)^2 K_s(dy) ds \quad \forall t \geq 0 \quad a.s. \qquad \square$$

Perkins (1995) has shown that (under suitable hypothesis) the above martingale problem is well posed. We expect the solution to be the limit point of a sequence of renormalized interacting BGW branching Brownian motions, just as in the non-interacting case. In this Chapter we introduce such a sequence of systems of particles. We also show that they are tight and that their limit points satisfy the martingale problem $(MP)_{a,b,\gamma}$. Our results are a non-trivial extension of Meleard and Roelly (1990). The main difference is that by working in the historical setting the state space for the particle motions becomes C^d. Therefore the convergence of an approximating sequence K^N is no longer completely determined by the convergence of the projections $f \mapsto \int f \, dK_t^N$.

1.2 Interactive Branching Particle Systems

1.2.1 The Particle Picture

In this Subsection we define a sequence of processes K^N which converges weakly to a solution of the martingale problem $(MP)_{a,b,\gamma}$ mentioned in the introduction (see Definition 1.3).

We begin by introducing a set of labels I. Let $I := \bigcup_{i=0}^\infty \mathbf{N} \times \{1,2\}^i$, where by convention $\{1,2\}^0 = \emptyset$. For any $\alpha = (\alpha_0, \ldots, \alpha_k) \in I$ we write $|\alpha| = k$ and $\alpha|i = (\alpha_0, \ldots, \alpha_i)$ for $i \leq k$. If $\alpha = (\alpha_0, \ldots, \alpha_j) \in I$ we denote $\alpha i = (\alpha_0, \ldots, \alpha_j, i)$, $i = 1, 2$, and $\pi(\alpha) = (\alpha_0, \ldots, \alpha_{j-1})$, $j \geq 1$.

Let $\{\xi_\alpha : \alpha \in I\}$ be i.i.d. random variables with $\mathbb{P}[\xi_\alpha = 0] = \mathbb{P}[\xi_\alpha = 2] = 1/2$. We say that a subset $\mathcal{A}$ of I is a Bienaymé-Galton-Watson tree (or BGW tree) with N roots iff

(i) $\{\alpha \in I : |\alpha| = 0, \alpha|0 = i$ for some $i \leq N\} \subset \mathcal{A}$ and if $\alpha|0 > N$ then $\alpha \notin \mathcal{A}$.

(ii) for any $\alpha \in \mathcal{A}$ such that $|\alpha| \geq 1$, $\prod_{i=0}^{|\alpha|-1} \xi_{\alpha|i} > 0$.

The family $\{\xi_\alpha : \alpha \in I\}$ induces a unique probability distribution Π_N on the set of all trees with N roots.

Remark 1.4. The random variable $card(\mathcal{A})$ is the total number of individuals that ever lived in a critical Bienaymé-Galton-Watson process. Therefore $card(\mathcal{A}) < \infty$ a.s. $\qquad \square$

The next ingredient we need is a collection $\{B^\alpha : \alpha \in I\}$ of independent d-dimensional Brownian motions. Define also a family $\{e_\alpha : \alpha \in I\}$ of i.i.d. exponential(1) random variables. We

assume that the three collections of Bernoullis, Brownian motions and exponentials are mutually independent. We also assume that they are carried by the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We shall also need drift, diffusion and branching rate coefficients. *For the sake of simplicity, throughout the chapter we shall assume that at time $t = 0$ all the particles are located at the origin.* We could have started distributing the initial particles according to any probability distribution, but we want to keep the notation as simple as possible. All the proofs can be easily modified to cover such general initial condition.

Notation 1.5. If E is a topological space and $x \in C([0, \infty), E)$, x^t denotes the path x stopped at t: $x^t = x(t \wedge \cdot)$. If $\{X(t, \omega) : t \geq 0\}$ is a process taking values in a normed linear space $(L, \|\ \ \|)$ then $X_t^* = \sup\{\|X_s\| : 0 \leq s \leq t\}$. If $(M, \mathcal{M})$ is a measurable space, $b\mathcal{M}$ denotes the space of bounded real-valued $\mathcal{M}$-measurable functions, and $\mathcal{M}^*$ denotes the universal completion of $\mathcal{M}$. $C^d = C([0, \infty), \mathbb{R}^d)$ is endowed with the sup metric ρ, $(\mathcal{C}_t^d)$ is its canonical filtration, $M_F(C^d)$ is the space of finite Borel measures on C^d with the weak topology. It will also be convenient to append an isolated point ∂ to $\mathbb{R}^d$. $\quad\Box$

Let

$$Lip(C^d) = \{\phi : C^d \to \mathbb{R} : \|\phi\|_\infty \leq 1, |\phi(x) - \phi(y)| \leq \rho(x, y) \quad \forall x, y \in C^d\}.$$

The Vasershtein metric $d = d_\rho$ on $M_F(C^d)$ is given by

$$d(\mu, \mu') = \sup\{|\mu(\phi) - \mu'(\phi)| : \phi \in Lip(C^d)\}.$$

This metric induces the weak topology on $M_F(C^d)$ (Ethier and Kurtz 1986, p. 150, Ex. 2). Suppose that $\varsigma > 0$. Let

$$\sigma : [0, \infty) \times M_F(C^d) \times C^d \longrightarrow \mathbb{R}^{d \times d},$$
$$b : [0, \infty) \times M_F(C^d) \times C^d \longrightarrow \mathbb{R}^d,$$
$$\gamma : [0, \infty) \times D([0, \infty), M_F(C^d)) \times C^d \longrightarrow [\varsigma, \infty).$$

Suppose that $v, \hat{\Gamma} : [0, \infty) \to [1, \infty)$ are non-decreasing functions. p is an arbitrary but otherwise fixed positive integer. $\Upsilon : \mathbb{N} \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined by $(p, t, x) \mapsto v(t) x^p$. We will assume that the maps σ, b, γ have the following properties:

Boundedness by the total mass

$$\sup_{y \in C^d} \|b(t, K_t, y)\| + \sup_{y \in C^d} \|\sigma(t, K_t, y)\| \leq \Upsilon(p, t, K_t^*(1)) \quad \forall K \in D([0, \infty), M_F(C^d)), \quad (1.2)$$

$$\sup_{y \in C^d} \gamma(t, K, y) \leq \hat{\Gamma}(t)\left(1 + \int_0^t K_s^*(1)ds\right) \quad \forall K \in D([0, \infty), M_F(C^d)). \quad (1.3)$$

Lipschitz condition

$$\|\sigma(t, \mu, y) - \sigma(t, \mu', y')\| + \|b(t, \mu, y) - b(t, \mu', y')\| \leq$$
$$\Upsilon(p, t, \mu(1) \vee \mu'(1))(\rho(y, y') + d(\mu, \mu')) \quad \forall \mu, \mu' \in M_F(C^d). \quad (1.4)$$

Finally, for any $T > 0$ there is a finite constant c_T such that $\|b(s, 0, 0)\| + \|\sigma(s, 0, 0)\| \leq c_T$ for all $s \leq T$. Note that if $y \in C^d$, $K \in D([0, \infty), M_F(C^d))$ then

$$\|b(s, K_s, y^s)\| \leq \|b(s, 0, 0)\| + \Upsilon(p, s, K_s^*(1))(\|y^s\|_\infty + K_s(1))$$
$$\leq \|b(s, 0, 0)\| + (K_s(1) \vee \Upsilon(p, s, K_s^*(1)))(1 + \|y^s\|_\infty).$$

Similarly,

$$\|\sigma(s, K_s, y^s)\| \leq \|\sigma(s, 0, 0)\| + (K_s(1) \vee \Upsilon(p, s, K_s^*(1)))(1 + \|y^s\|_\infty).$$

Remark 1.6. (a) Notice that the second argument of b and σ is less general than the second argument of γ. This stems from the fact that we wanted to keep the notation as simple as possible, but also wanted to show the reader how to generalize the proof and to cover some interesting examples. It will be easy for the reader to modify our proof to more general coefficients of the form

$$\sigma : [0, \infty) \times D([0, \infty) \times M_F(C^d) \times C^d \longrightarrow \mathbb{R}^{d \times d},$$
$$b : [0, \infty) \times D([0, \infty) \times M_F(C^d) \times C^d \longrightarrow \mathbb{R}^d.$$

(b) The condition that γ be bounded away from 0 can be weakened to

$$\int_0^\infty \gamma(t, K, y) dt = \infty \quad \forall K \in D([0, \infty), M_F(C^d)), \quad \forall y \in C^d$$

Unfortunately any further generalization is beyond the scope of the techniques employed in the chapter. □

Examples. The reader will find many interesting examples in Perkins (1995). Here is a sample of several types of coefficients that satisfy our hypotheses. See Perkins (1995) for more details.

(a) Assume $\lambda, \delta, \varepsilon > 0$. Let $p_s(x)$ be the d-dimensional heat kernel. Set

$$\gamma(t, K, y) = \delta + \exp\left(-\int_0^t \int p_\varepsilon(y_s' - y_t) K_s(dy') e^{-\lambda(t-s)} ds\right).$$

We can interpret a superprocess as a biological population of say goat-like particles. The branching rate of a particle located at y_t becomes smaller if many of the particles (goats) have spent too much time near $y(t)$ in the past, thus depleting the food supplies and making local conditions less favorable for reproduction. The parameter λ represents the rate of replenishment of the environment. We would like to consider $\varepsilon = \delta = 0$, but for technical reasons are obliged to assume that both parameters are strictly positive. Note that γ is bounded and bounded away from 0, so it satisfies trivially the hypothesis.

(b) Let p, ε be as in Example 1. Define

$$b(t, K_t, y) = \int_0^t \int \nabla p_\varepsilon(y_s' - y_t) K_t(dy') ds.$$

In this case the particles drift away from the places they or their living fellow particles have visited in the past.

(c) Let d_e be the Vasershtein metric on $M_F(\mathbb{R}^d)$ associated with the Euclidean metric on $\mathbb{R}^d$. Assume $\bar{\sigma} : M_F(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ and $\bar{b} : M_F(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy

$$\|\bar{\sigma}(\mu, x) - \bar{\sigma}(\mu', x')\| + \|\bar{b}(\mu, x) - \bar{b}(\mu', x')\| \leq const.(1 + \mu(1) \vee \mu'(1))[d_e(\mu, \mu') + \|x - x'\|],$$

and

$$\|\bar{\sigma}(\mu, x)\| + \|\bar{b}(\mu, x)\| \leq const.(1 + \mu(1)) \quad \forall \mu, \mu' \in M_F(\mathbb{R}^d), \quad \forall x, x' \in \mathbb{R}^d.$$

Define $\Pi_t : M_F(C^d) \to M_F(\mathbb{R}^d)$ by $\Pi_t(\mu)(A) = \mu(\{y : y_t \in A\})$, and σ, b by $\sigma(t, K_t, y) = \bar{\sigma}(t, \Pi_t(K_t), y)$, $b(t, K_t, y) = \bar{b}(t, \Pi_t(K_t), y)$. This example shows that the case of $M_F(\mathbb{R}^d)$-valued interacting superprocesses follows as an instance of the historical superprocesses.

(d) This is a continuation of Example (c). Let $\hat{\sigma} : \mathbb{R}^d \to \mathbb{R}^{d \times n}$ and $\hat{b} : \mathbb{R}^d \to \mathbb{R}^d$ be bounded Lipschitz continuous functions, and define $\bar{\sigma}$ and $\bar{b}$ by

$$\bar{\sigma}(\mu, x) = \int \hat{\sigma}(x - y)\mu(dy),$$

$$\bar{b}(\mu, x) = \int \hat{b}(x - y)\mu(dy).$$

More generally if $b_k : \mathbb{R}^{d(k+1)} \to \mathbb{R}^d$ and $\sigma_k : \mathbb{R}^{d(k+1)} \to \mathbb{R}^{d \times n}$, $k = 1, \dots, j$, are bounded Lipschitz functions then we can define coefficients

$$\bar{\sigma}(\mu, x) = \sum_{k=1}^{j} \int \sigma_k(x, x_1, \dots, x_k)\mu(dx_1) \dots \mu(dx_k),$$

$$\bar{b}(\mu, x) = \sum_{k=1}^{j} \int b_k(x, x_1, \dots, x_k)\mu(dx_1) \dots \mu(dx_k). \qquad \square$$

We proceed to define the particle systems. For each N we define a system of age dependent branching processes as follows.

Step 0. Pick a BGW tree $\mathcal{A}$ with N roots at random, i.e. according to the law Π_N.

Step 1. Define predictable path functionals $\bar{b} : \mathbf{N} \times [0, \infty) \times C^{Nd} \longrightarrow \mathbb{R}^d$ and $\bar{\sigma} : \mathbf{N} \times [0, \infty) \times C^{Nd} \longrightarrow \mathbb{R}^{n \times d}$ by

$$\bar{b}_i(s, y^1, \dots, y^N) := b\left(s, \frac{1}{N} \sum_{j=1}^{N} \delta_{(y^j)^s}, (y^i)^s\right)$$

$$\bar{\sigma}_i(s, y^1, \dots, y^N) := \sigma\left(s, \frac{1}{N} \sum_{j=1}^{N} \delta_{(y^j)^s}, (y^i)^s\right).$$

Here each $y^i \in C^d$. Let $\hat{b} := (\bar{b}_1, \dots, \bar{b}_N)$ and let $\hat{\sigma}$ be the $Nd \times Nn$ matrix whose j-th diagonal $d \times n$ block is $\bar{\sigma}_j$, $j = 1, \dots, N$, and any other entry is equal to 0. Note that by the assumptions on b and σ

$$\|\bar{b}_i(s, y^1, \dots, y^N)\| + \|\bar{\sigma}_i(s, y^1, \dots, y^N)\| \leq \|\bar{b}_i(s, 0, \dots, 0)\| + \|\bar{\sigma}_i(s, 0, \dots, 0)\|$$
$$+ \Upsilon(p, s, card(\mathcal{A})/N)(1 + y_s^*). \quad (1.5)$$

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Moreover, both $\hat{b}$ and $\hat{\sigma}$ are Lipschitz:

$$\|\hat{b}(s,y^1,\ldots,y^N) - \hat{b}(s,x^1,\ldots,x^N)\|$$

$$\leq \sqrt{\|\bar{b}_1(s,y^1,\ldots,y^N) - \bar{b}_1(s,x^1,\ldots,x^N)\|^2 + \ldots + \|\bar{b}_N(s,y^1,\ldots,y^N) - \bar{b}_N(s,x^1,\ldots,x^N)\|^2}$$

$$\leq \Upsilon(p,s,card(\mathcal{A})/N)\sqrt{(y^1-x^1)_s^{*2} + \ldots + (y^N-x^N)_s^{*2}}$$

$$\leq \Upsilon(p,s,card(\mathcal{A})/N)((y^1,\ldots,y^d) - (x^1,\ldots,x^d))_s^*. \tag{1.6}$$

Similarly

$$\|\hat{\sigma}(s,y^1,\ldots,y^d) - \hat{\sigma}(s,x^1,\ldots,x^d)\| \leq \Upsilon(p,s,card(\mathcal{A})/N)((y^1,\ldots,y^d) - (x^1,\ldots,x^d))_s^*. \tag{1.7}$$

Note that once $\mathcal{A}$ has been chosen the term $\Upsilon(p,s,card(\mathcal{A})/N)$ can be regarded as a function of s, bounded on bounded intervals. We shall need also an index set for the particles "alive" at a given time. For this purpose need to define an auxiliary set

$$J(1,t) := \{\alpha \in \mathcal{A} : |\alpha| = 0\} \subset \mathcal{A}, \quad \text{for all } t \geq 0.$$

Note that $J(1,t)$ does not depend on t. We shall see later the reason for this notation. For each $\alpha \in J(1,t)$ set $\beta(\alpha) = 0$. This is the birth time of α.

Consider the following system of N d-dimensional SDE's:

$$Y_t^\alpha = \int_0^t \bar{b}_\alpha(s,Y^1,\ldots,Y^N)ds + \int_0^t \bar{\sigma}_\alpha(s,Y^1,\ldots,Y^N) \cdot dB_s^\alpha, \tag{1.8}$$
$$\alpha \in J(1,t).$$

By (1.5), (1.6), (1.7) and the well known theorem for existence and uniqueness of SDE's with Lipschitz coefficients that grow linearly (e.g. Rogers & Williams p. 128), the system (1.8) possesses a unique, global (i.e. non-exploding) strong solution. Define stopping times

$$T_1^\alpha := \inf\left\{t > 0 : N\int_{\beta(\alpha)}^t \gamma\left(s,\left(\frac{1}{N}\sum_{\theta \in J(1,\cdot)} \delta_{(Y^\theta)_\cdot}\right)^s, (Y^\alpha)^s\right) ds > e_\alpha\right\},$$
$$\alpha \in J(1,t).$$

Recall that $\left(\frac{1}{N}\sum_{\theta \in J(1,\cdot)} \delta_{(Y^\theta)_\cdot}\right)^s$ denotes the path $u \mapsto \frac{1}{N}\sum_{\theta \in J(1,u)} \delta_{(Y^\theta)^u}$ stopped at time s. All of these stopping times are a.s. finite since γ is bounded away from zero, and a.s. different from each other since the exponentials are independent and independent of the Y's. Now define

$$T_1 = \min\{T_1^\alpha : \alpha \in J(1,t)\}$$
$$\bar{\alpha}^1 = \arg\min\{T_1^\alpha : \alpha \in J(1,t)\},$$
$$\delta(\bar{\alpha}^1) = T_1 \quad \text{(the death time of } \bar{\alpha}^1),$$

and

$$K_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{(Y^\alpha)^t}, \quad 0 \leq t < T_1.$$

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Step 2. $\bar{\alpha}^1$ has just died. There are two possibilities. It had either zero or two offspring. Let's examine each possibility separately.

Case 1. $\xi_{\bar{\alpha}^1} = 0$. Define

$$J(2,t) = \begin{cases} J(1,t) & \text{if } t < T_1, \\ J(1,t) - \{\bar{\alpha}^1\} & \text{if } t \geq T_1. \end{cases}$$

Set $Y_t^{\bar{\alpha}^1} = \partial$ for $t \geq T_1$. Now solve the system of $card(J(2,T_1))$ d-dimensional SDE's

$$Y_t^\alpha = Y_{T_1}^\alpha + \int_{T_1}^t b\left(s, \frac{1}{N} \sum_{\theta \in J(2,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s\right) ds + \int_{T_1}^t \sigma\left(s, \frac{1}{N} \sum_{\theta \in J(2,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s\right) \cdot dB_s^\alpha, \quad (1.9)$$

$$\alpha \in J(2,t), \quad t \geq T_1.$$

Just as before, we interpret the coefficients b, σ in (1.9) as being some predictable Lipschitz path functionals with linear growth. Therefore (1.9) has a unique global strong solution for all times $t \geq T_1$. Proceed as before. Define stopping times

$$T_2^\alpha := \inf\left\{ t > T_1 : N \int_{\beta(\alpha)}^t \gamma\left(s, \left(\frac{1}{N} \sum_{\theta \in J(2,\cdot)} \delta_{(Y^\theta)\cdot}\right)^s, (Y^\alpha)^s\right) ds > e_\alpha \right\},$$

$$\alpha \in J(2,t).$$

These are also a.s. finite and different. Set

$$T_2 = \min\{T_2^\alpha : \alpha \in J(2,t)\}$$
$$\bar{\alpha}^2 = \arg\min\{T_2^\alpha : \alpha \in J(2,t)\},$$
$$\delta(\bar{\alpha}^2) = T_2$$

and

$$K_t^N = \frac{1}{N} \sum_{\alpha \in J(2,t)} \delta_{(Y^\alpha)^t}, \quad T_1 \leq t < T_2.$$

Case 2. $\xi_{\bar{\alpha}^1} = 2$. Define

$$J(2,t) = \begin{cases} J(1,t) & \text{if } t < T_1, \\ J(1,t) \cup \{\bar{\alpha}^1 1, \bar{\alpha}^1 2\} - \{\bar{\alpha}^1\} & \text{if } t \geq T_1. \end{cases}$$

We set T_1 to be the birth time of $\bar{\alpha}^1 1$, $\bar{\alpha}^1 2$:

$$\beta(\bar{\alpha}^1 1) = \beta(\bar{\alpha}^1 2) := T_1.$$

Set $Y_t^{\bar{\alpha}^1} = \partial$ for all $t \geq T_1$ and define $(Y^{\bar{\alpha}^1 1})^{T_1} = (Y^{\bar{\alpha}^1 2})^{T_1} := (Y^{\bar{\alpha}^1})^{T_1-}$. Now solve the system of $card(J(2,T_1))$ d-dimensional SDE's

$$Y_t^\alpha = Y_{T_1}^\alpha + \int_{T_1}^t b\left(s, \frac{1}{N} \sum_{\theta \in J(2,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s\right) ds + \int_{T_1}^t \sigma\left(s, \frac{1}{N} \sum_{\theta \in J(2,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s\right) \cdot dB_s^\alpha,$$

$$(1.10)$$

$$\alpha \in J(1,t), \quad t \geq T_1.$$

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Once more we interpret the coefficients b, σ in (1.10) as being some predictable Lipschitz path functionals with linear growth. Notice that even though the $\bar{b}$, $\bar{\sigma}$ in this step are possibly different from those in step 1, the Lipschitz "constant" is the same: $\Upsilon(p, s, card(\mathcal{A})/N)$. Hence (1.10) has a unique global strong solution for all times $t \geq T_1$. Define stopping times

$$T_2^\alpha := \inf\left\{ t > T_1 : N \int_{\beta(\alpha)}^t \gamma\left(s, \left(\frac{1}{N}\sum_{\theta \in J(2,\cdot)} \delta_{(Y^\theta)}\right)^s, (Y^\alpha)^s \right) ds > e_\alpha \right\},$$

$$\alpha \in J(2, t).$$

These are a.s. finite and different. Set

$$T_2 = \min\{T_2^\alpha : \alpha \in J(2, t)\}$$
$$\bar{\alpha}^2 = \arg\min\{T_2^\alpha : \alpha \in J(2, t)\},$$
$$\delta(\bar{\alpha}^2) = T_2$$

and

$$K_t^N = \frac{1}{N}\sum_{\alpha \in J(2, t)} \delta_{(Y^\alpha)^t}, \qquad T_1 \leq t < T_2.$$

Step (n+1). Suppose that we have already defined $J(n, t)$, $T_1 < \ldots < T_n$. $\bar{\alpha}^n$ died at time T_n.
Case 1. $\xi_{\bar{\alpha}^n} = 0$. Define

$$J(n + 1, t) = \begin{cases} J(n, t) & \text{if } t < T_n, \\ J(n, t) - \{\bar{\alpha}^n\} & \text{if } t \geq T_n. \end{cases}$$

Stop if $J(n + 1, T_n)$ is the empty set. Define $K_t^N \equiv 0$ for $t \geq T_n$. Otherwise, define $Y_t^{\bar{\alpha}^n} = \partial$ for $t \geq T_n$ and solve the system of $card(J(n + 1, T_n))$ d-dimensional SDE's

$$Y_t^\alpha = Y_{T_n}^\alpha + \int_{T_n}^t b(s, \frac{1}{N}\sum_{\theta \in J(n+1,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s)ds + \int_{T_n}^t \sigma(s, \frac{1}{N}\sum_{\theta \in J(n+1,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s) \cdot dB_s^\alpha,$$

$$\tag{1.11}$$

$$\alpha \in J(n + 1, t), \quad t \geq T_n.$$

As before (1.11) has a unique global strong solution for all times $t \geq T_n$. Define stopping times

$$T_{n+1}^\alpha := \inf\left\{ t > T_n : N \int_{\beta(\alpha)}^t \gamma\left(s, \left(\frac{1}{N}\sum_{\theta \in J(n+1,\cdot)} \delta_{(Y^\theta)}\right)^s, (Y^\alpha)^s \right) ds > e_\alpha \right\}$$

$$\alpha \in J(n + 1, t).$$

These are a.s. finite and different. Set

$$T_{n+1} = \min\{T_{n+1}^\alpha : \alpha \in J(n + 1, t)\}$$
$$\bar{\alpha}^{n+1} = \arg\min\{T_{n+1}^\alpha : \alpha \in J(n + 1, t)\},$$
$$\delta(\bar{\alpha}^{n+1}) = T_{n+1}$$

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and

$$K_t^N = \frac{1}{N} \sum_{\alpha \in J(n+1,t)} \delta_{(Y^\alpha)^t}, \qquad T_n \le t < T_{n+1}.$$

Case 2. $\xi_{\bar{\alpha}^n} = 2$. Define

$$J(n+1,t) = \begin{cases} J(n,t) & \text{if } t < T_n, \\ J(n,t) \cup \{\bar{\alpha}^n 1, \bar{\alpha}^n 2\} - \{\bar{\alpha}^n\} & \text{if } t \ge T_n. \end{cases}$$

We also set T_n to be the birth time of $\bar{\alpha}^n 1$, $\bar{\alpha}^n 2$:

$$\beta(\bar{\alpha}^n 1) = \beta(\bar{\alpha}^n 2) := T_n.$$

Set $Y_t^{\bar{\alpha}^n} = \partial$ for all $t \ge T_n$ and define $(Y^{\bar{\alpha}^n 1})^{T_n} = (Y^{\bar{\alpha}^n 2})^{T_n} := (Y^\alpha)^{T_n-}$. Now solve the system of $card(J(n+1,T_n))$ d-dimensional SDE's

$$Y_t^\alpha = Y_{T_n}^\alpha + \int_{T_n}^t b\left(s, \frac{1}{N} \sum_{\theta \in J(n+1,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s\right) ds + \int_{T_n}^t \sigma\left(s, \frac{1}{N} \sum_{\theta \in J(n+1,s)} \delta_{(Y^\theta)^s}, (Y^\alpha)^s\right) \cdot dB_s^\alpha,$$

$$\tag{1.12}$$

$$\alpha \in J(n+1,t), \quad t \ge T_n.$$

(1.12) has a unique global strong solution for all times $t \ge T_n$. Define stopping times

$$T_{n+1}^\alpha := \inf\left\{t > T_n : N \int_{\beta(\alpha)}^t \gamma\left(s, \left(\frac{1}{N} \sum_{\theta \in J(n+1,\cdot)} \delta_{(Y^\theta)\cdot}\right)^s, Y^\alpha\right) ds > e_\alpha\right\},$$

$$\alpha \in J(n+1,t).$$

These a.s. finite and different. Set

$$T_{n+1} = \min\{T_{n+1}^\alpha : \alpha \in J(n+1,t)\}$$
$$\bar{\alpha}^{n+1} = \arg\min\{T_{n+1}^\alpha : \alpha \in J(n+1,t)\},$$
$$\delta(\bar{\alpha}^{n+1}) = T_{n+1}$$

and

$$K_t^N = \frac{1}{N} \sum_{\alpha \in J(n+1,t)} \delta_{(Y^\alpha)^t}, \qquad T_n \le t < T_{n+1}.$$

Remark 1.7. This procedure ends with step $card(\mathcal{A})$. Notice that with probability one it takes only a finite amount of time for the procedure to finish. This is due to the fact that the branching rate is bounded below. In fact, $T_k - T_{k-1} \le e_{\bar{\alpha}^k}/(N\varsigma)$. □

We shall need some terminology.

Definition 1.8. Remark 1.7 implies that

$$\mathcal{A} = \{\bar{\alpha}^n, n = 1, \dots, card(\mathcal{A})\}.$$

Therefore we assign a birth and a death time to each particle in $\mathcal{A}$. By convention $Y^\alpha \equiv \partial$, $\beta(\alpha) = \delta(\alpha) = \infty$ if $\alpha \notin \mathcal{A}$. Set $h^\alpha(s) = 1(\beta(\alpha) \leq s < \delta(\alpha))$, the indicator of the lifespan of α, and

$$\alpha \sim t \Leftrightarrow h^\alpha(t) = 1,$$
$$\alpha \preceq t \Leftrightarrow \delta(\alpha) \leq t,$$
$$t \prec \alpha \Leftrightarrow t < \beta(\alpha).$$

We shall adopt the convention that for any $\phi : \mathbb{R}^k \to \mathbb{R}$ and any $1 \leq j \leq k$, $(x_1, \dots, x_k) \in \mathbb{R}^k$ we have $\phi(x_1, \dots, x_{j-1}, \partial, x_{j+1}, \dots, x_k) = 0$. Finally, we define some very important σ-fields. Let

$$\mathcal{F}_t^{N,o} := \sigma\Big(\Big[J(n,s), \{B_s^\alpha : \alpha \in J(n,s)\}, \{e_\alpha : \alpha \in J(n,s), \ \delta(\alpha) \leq s\},$$

$$\{\xi_\alpha : \alpha \in J(n,s), \ \delta(\alpha) \leq s\}; n = 1, \dots, \infty\Big] : s \leq t\Big), \quad t \geq 0.$$

and

$$\mathcal{F}_t^N := \bigcap_{u > t} \mathcal{F}_u^{N,o}, \quad t \geq 0. \qquad \square$$

Remark 1.9. (a) K^N is right continuous.

(b) From the definition of $\delta(\alpha)$ and since $\gamma > 0$ it follows that

$$\delta(\alpha) = \inf\Big\{t > \beta(\alpha) : N \int_{\beta(\alpha)}^t \gamma(s, (K^N)^s, (Y^\alpha)^s)ds > e_\alpha\Big\}.$$

In fact one easily sees that

$$N \int_{\beta(\alpha)}^{\delta(\alpha)} \gamma(s, (K^N)^s, (Y^\alpha)^s)ds = e_\alpha. \tag{1.13}$$

Moreover,

$$Y_t^\alpha = Y_{T_n}^\alpha + \int_{T_n}^t b(s, K_{s-}^N, (Y^\alpha)^s)ds + \int_{T_n}^t \sigma(s, K_{s-}^N, (Y^\alpha)^s) \cdot dB_s^\alpha, \tag{1.14}$$

$$\alpha \in J(n+1, t), \quad T_n \leq t \leq T_{n+1}. \tag{1.15}$$

(c) At this point a discussion of what we have accomplished so far will be helpful. The procedure to construct the interacting particle system K^N is the following. First, a BGW tree is chosen at random. This tree contains the genealogical information and nothing else. N particles are placed at the origin and start moving according to some SDE in which the drift and diffusion depend in their own trajectories and (in a symmetric way) in the trajectories of all other particles. This SDE possesses a unique strong solution because of the Lipschitz nature of

the coefficients. Each particle carries an alarm clock which is a Poisson point process with a random adapted intensity (also called doubly stochastic Poisson process). This intensity is strictly increasing until the first arrival and then it is zero (so it's a degenerate Poisson point process with only one arrival). When the first bell rings we decide whether to kill that particle or to replace it with two offspring. At this instant of time, say t_0, we also update $K_{t_0}^N$, which will then be concentrated in the paths of the $N-1$ or $N+1$ particles alive. We also know the value of the exponential random variable driving the Poisson process and the number of offspring. This information is also included in the filtration $\mathcal{F}_{t_0}^N$. Observe that the driving Brownian motions are adapted to this filtration and are in fact still Brownian motions, since the information contained in $(\mathcal{F}_t^N)$ and not contained in their own filtration concerns some other independent Brownian motions or some independent exponentials or some independent Bernoulli random variables. The fact that the intensities of the clocks are adapted is therefore essential, and it is embodied in equation (1.13). Note also that since the system of SDE's has a strong solution, the trajectories Y^α are $(\mathcal{F}_t^N)$-adapted and therefore K^N itself is $(\mathcal{F}_t^N)$-adapted. The process continues in the same fashion until the next alarm clock goes off and so on. □

1.2.2 An Equation for K^N

Observe that as a consequence of (1.14)

$$Y_t^\alpha = Y_{\beta(\alpha)}^\alpha + \int_{\beta(\alpha)}^t b(s, K_{s-}^N, (Y^\alpha)^s)ds + \int_{\beta(\alpha)}^t \sigma(s, K_{s-}^N, (Y^\alpha)^s) \cdot dB_s^\alpha, \qquad \beta(\alpha) \le t < \delta(\alpha).$$
(1.16)

Indeed, if $\beta(\alpha) = T_k$, $\delta(\alpha) = T_{k+j}$ and $t \in [T_k, T_{k+j})$ then (1.14) implies

$$Y_t^\alpha = Y_{\beta(\alpha)}^\alpha + \sum_{i=k}^j \int_{t \wedge T_i}^{t \wedge T_{i+1}} b(s, K_{s-}^N, (Y^\alpha)^s)ds + \int_{t \wedge T_i}^{t \wedge T_{i+1}} \sigma(s, K_{s-}^N, (Y^\alpha)^s) \cdot dB_s^\alpha$$

This is equivalent to (1.16). For any $t \in [\beta(\alpha), \delta(\alpha))$, $r \in \mathbf{N}$, $0 \le t_1 < \ldots < t_r$ and $\Psi \in C_b^2(\mathbb{R}^{dr})$, Itô's lemma and (1.16) yield

$$\Psi(t_1, \ldots, t_r)(t, Y^\alpha) = \Psi(t_1, \ldots, t_r)(\beta(\alpha), Y^\alpha) + \int_0^t \langle \nabla \bar{\Psi}(s, Y^\alpha), b(s, K_{s-}^N, (Y^\alpha)^s) \rangle_d h^\alpha(s)ds$$

$$+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \bar{\Psi}_{ij}(s, Y^\alpha)(\sigma(s, K_{s-}^N, (Y^\alpha)^s)\sigma(s, K_{s-}^N, (Y^\alpha)^s)^*)_{ij} h^\alpha(s)ds$$

$$+ \int_0^t \langle \nabla \bar{\Psi}(s, Y^\alpha)\sigma(s, K_{s-}^N, (Y^\alpha)^s), h^\alpha(s)dB_s^\alpha \rangle_n,$$
(1.17)

where $\nabla \bar{\Psi}(s, Y^\alpha)$ is considered as a row vector for matrix multiplication and $\langle \cdot, \cdot \rangle_k$ denotes the usual inner product in $\mathbb{R}^k$.

Notation 1.10. Let $t \geq 0$, $y \in C^d$, $r \in \mathbf{N}$, $0 \leq t_1 < \ldots < t_r$ and $\Psi \in C_b^2(\mathbb{R}^{dr})$, $K \in M_F(C^d)$. We write

$$A(K)(\bar{\Psi})(t,y) = \langle \nabla \bar{\Psi}(t,y), b(t,K_t,y^t) \rangle_d + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \bar{\Psi}_{ij}(t,y)(\sigma(t,K_t,y^t)\sigma(t,K_t,y^t)^*)_{ij}$$

$$Z_t^N(\bar{\Psi}) = \frac{1}{N} \sum_{\alpha \preceq t} \left[\bar{\Psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \leq t)(\xi_\alpha - 1) \right]$$

$$W_t^N(\bar{\Psi}) = \frac{1}{N} \sum_{\alpha \preceq t} \left[\int_0^t \langle \nabla \bar{\Psi}(s,Y^\alpha)\sigma(s,K_{s-}^N,(Y^\alpha)^s), h^\alpha(s) dB_s^\alpha \rangle_r \right] \qquad \square$$

The following is the main result of this Subsection.

Proposition 1.11. *For any* $\Psi \in D_0$

$$K_t^N(\Psi) = K_0^N(\Psi) + \int_0^t \int A(K_{s-}^N)(\Psi)(s,y) K_s^N(dy) ds + W_t^N(\Psi) + Z_t^N(\Psi) \qquad (1.18)$$

In particular, $K_t^N(1) = 1 + Z_t^N(1)$.

Proof. We compute

$$K_t^N(\bar{\Psi}) = \frac{1}{N} \sum_{\alpha \sim t} \bar{\Psi}(t,Y^\alpha)$$

$$= \frac{1}{N} \sum_{\alpha \sim t} \left[\bar{\Psi}(\beta(\alpha), Y^\alpha) + \int_0^t A(K)(\bar{\Psi})(s, Y^\alpha) h^\alpha(s) ds \right.$$

$$\left. + \int_0^t \langle \nabla \bar{\Psi}(s,Y^\alpha)\sigma(s,K_{s-}^N,(Y^\alpha)^s), h^\alpha(s) dB_s^\alpha \rangle_r \right]$$

$$(\text{by } 1.17)$$

$$= \frac{1}{N} \sum_{\alpha \preceq t} \left[\bar{\Psi}(\beta(\alpha), Y^\alpha) - \bar{\Psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \leq t) + \int_0^t A(K)(\bar{\Psi})(s, Y^\alpha) h^\alpha(s) ds \right.$$

$$\left. + \int_0^t \langle \nabla \bar{\Psi}(s,Y^\alpha)\sigma(s,K_{s-}^N,(Y^\alpha)^s), h^\alpha(s) dB_s^\alpha \rangle_r \right]$$

$$(\text{since the term inside the square brackets is 0 if } \delta(\alpha) \leq t)$$

$$= \frac{1}{N} \sum_{\alpha \preceq t} \left[\int_0^t A(K)(\bar{\Psi})(s, Y^\alpha) h^\alpha(s) ds \right]$$

$$+ \frac{1}{N} \sum_{\alpha \preceq t} \left[\int_0^t \langle \nabla \bar{\Psi}(s,Y^\alpha)\sigma(s,K_{s-}^N,(Y^\alpha)^s), h^\alpha(s) dB_s^\alpha \rangle_r \right]$$

$$+ \frac{1}{N} \sum_{\alpha \preceq t} \left[\bar{\Psi}(\beta(\alpha), Y^\alpha) - \bar{\Psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \leq t) \right]$$

$$= \int_0^t \int A(K)(\bar{\Psi})(s, y) K_s^N(dy) ds + W_t^N(\bar{\Psi})$$

$$+ \frac{1}{N} \sum_{\alpha \preceq t} \left[\bar{\Psi}(\beta(\alpha), Y^\alpha) - \bar{\Psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \leq t) \right]$$

$$= \int_0^t \int A(K)(\bar{\Psi})(s, y) K_s^N(dy) ds + W_t^N(\bar{\Psi}) + K_0^N(\bar{\Psi})$$

$$+ \frac{1}{N} \sum_{\alpha \preceq t} \left[\bar{\Psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \leq t)(\xi_\alpha - 1) \right]$$

$$= K_0^N(\bar{\Psi}) + \int_0^t \int A(K)(\bar{\Psi})(s, y) K_s^N(dy) ds + W_t^N(\bar{\Psi}) + Z_t^N(\bar{\Psi}).$$

This proves (1.18). To finish the proof of the proposition observe that if $\Psi \equiv 1$ then $\nabla \bar{\Psi} = 0$, $A(\cdot)(\bar{\Psi}) = 0$. ∎

Lemma 1.12. *For any $\bar{\Psi} \in D_0$, $\alpha \in I$, let $M_t^\alpha = M_t^\alpha(\bar{\Psi}) := (\xi_\alpha - 1) 1(\delta(\alpha) \leq t) \bar{\Psi}(\delta(\alpha)-, Y^\alpha)$. Then*

(a) M^α *is an $(\mathcal{F}_t^N)$-martingale.*

(b) $\langle M^\alpha \rangle_t = \int_0^t h^\alpha(s) \gamma(s, (K^N)^s, (Y^\alpha)^s) \bar{\Psi}(s, Y^\alpha)^2 ds.$

(c) *If $\alpha_1 \neq \alpha_2$ then $\langle M^{\alpha_1}, M^{\alpha_2} \rangle_t = 0$.*

Proof (a) By stochastic integration it suffices to prove the result for $\bar{\Psi} \equiv 1$ (see the proof of part (b) below for a similar argument). Let

$$\mathcal{J}_t := \sigma(1(\delta(\alpha) \leq s) : s \leq t).$$

The only information contained in $\mathcal{J}_\infty$ is the death time of α. Therefore ξ_α is independent of $\mathcal{J}_\infty$. We must show that

$$E\left[(\xi_\alpha - 1)\left(1(\delta(\alpha) \leq t) - 1(\delta(\alpha) \leq s) \right) | \mathcal{F}_s^N \right] = E\left[(\xi_\alpha - 1)\left(1(s < \delta(\alpha) \leq t) \right) | \mathcal{F}_s^N \right]$$
$$= 0. \tag{1.19}$$

Let b, q, be real numbers, $r \in \{0, 2\}$ and

$$A_1(\eta, u, b) = \{ B_u^\eta < b : u \leq s, \eta \in J(n, s) \text{ for some } n \}$$
$$A_2(\nu, q) = \{ e_\nu < q : \nu \in J(n, s) \text{ for some } n, \ \delta(\nu) \leq s \}$$
$$A_3(\theta, r) = \{ \xi_\theta = r, \theta \in J(n, s) \text{ for some } n, \ \delta(\theta) \leq s \}.$$

The σ-field $\mathcal{F}_s^N$ is generated by sets of the form

$$A = A_1(\eta_1, u_1, b_1) \cap \ldots A_1(\eta_i, u_i, b_i) \cap A_2(\nu_1, q_1) \cap \ldots \cap A_2(\nu_j, q_j) \cap A_3(\theta_1, r_1) \ldots \cap A_3(\theta_k, r_k),$$

where $\delta(\nu_l), \delta(\theta_m) \leq s$. $l = 1, \ldots, j$, $m = 1 \ldots, k$. Hence ξ_α is conditionally independent of A given $\delta(\alpha) > s$. Therefore

$$\mathbb{P}[\xi_\alpha = r, A, \delta(\alpha) > s, \delta(\alpha) \leq t] = \mathbb{P}[\xi_\alpha = r] \mathbb{P}[A, \delta(\alpha) > s, \delta(\alpha) \leq t].$$

Since $(\xi - 1)$ has mean zero, the above equation implies (1.19).

(b) We claim that it suffices to show that

$$\langle 1(\delta(\alpha) \le \cdot) \rangle_t = \int_0^t h^\alpha(s) \gamma(s, (K^N)^s, (Y^\alpha)^s) ds \qquad (1.20)$$

where $\langle 1(\delta(\alpha) \le \cdot) \rangle_t$ denotes the dual predictable projection of $1(\delta(\alpha) \le t)$. The claim follows from the theory of stochastic integration. Put $H_t := \bar{\Psi}((\delta(\alpha) \wedge t)-, Y^\alpha)$. H is a bounded predictable process. Moreover $X_t := (\xi_\alpha - 1) 1(\delta(\alpha) \le t)$ is a semimartingale. Clearly

$$(H \cdot X)_t = (\xi_\alpha - 1)\bar{\Psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \le t).$$

Hence, if (1.20) holds then

$$\begin{aligned}
\langle M^\alpha \rangle_t &= \langle H \cdot X \rangle_t \\
&= \int_0^t H_s^2 d\langle X \rangle_s \\
&= \int_0^t \bar{\Psi}((\delta(\alpha) \wedge s)-, Y^\alpha)^2 d\langle (\xi_\alpha - 1) 1(\delta(\alpha) \le \cdot) \rangle_s \\
&= \int_0^t \bar{\Psi}((\delta(\alpha) \wedge s)-, Y^\alpha)^2 (\xi_\alpha - 1)^2 d\langle 1(\delta(\alpha) \le \cdot) \rangle_s \\
&\qquad \text{(by the independence of } \xi_\alpha) \\
&= \int_0^t \bar{\Psi}(s-, Y^\alpha)^2 h^\alpha(s) \gamma(s, (K^N)^s, (Y^\alpha)^s) ds \\
&\qquad \text{(by (1.20))}.
\end{aligned}$$

To prove (1.20) we will need the following fact from the theory of semimartingales: Let $(A_t, \mathcal{F}_t)$ be a locally integrable strictly increasing adapted process such that $\lim_{t \to \infty} A_t = \infty$ a.s. Let $u \mapsto C_u$ be the inverse function of $t \mapsto A_t$. Note that for every u, C_u is a $(\mathcal{F}_t)$-stopping time. Set $\mathcal{G}_t = \mathcal{F}_{C_t}$. For any (progressively measurable) process X we can define the time changed process $X_t^A = X(C_t)$. If $(X, (\mathcal{F}_t))$ is a semimartingale and its semimartingale decomposition is given by $X_t = X_0 + M_t + V_t$ then $(X^A, (\mathcal{G}_t))$ is also a semimartingale and a semimartingale decomposition is given by $X_t^A = X_0 + M_t^A + V_t^A$. Let $A_t := N \int_{\beta(\alpha)}^t \gamma(s, (K^N)^s, (Y^\alpha)^s) ds$. With this notation Remark 1.9(b) implies $\delta(\alpha) = \inf\{r > \beta(\alpha) : A_r > e_\alpha\}$. We now apply the time change result to the process $(A_t, \mathcal{F}_t^N)$. Let C_u be the inverse of A_t (on $\{t > \beta(\alpha)\}$). That is, C_t is defined by $\int_0^{C_t} \gamma(s, (K^N)^s, (Y^\alpha)^s) ds = t$. Let $\mathcal{G}_t = \mathcal{F}_{C_t}^N$. Set $X_t = 1(\delta(\alpha) \le t)$. Then X^A is a $(\mathcal{G}_t)$-semimartingale and $\langle X \rangle_t = \langle X^A \rangle_{A_t}$. We compute the compensator of the increasing process $(X^A, (\mathcal{G}_t))$. One easily guesses $\langle X^A \rangle_t = A_{\delta(\alpha)} \wedge t$. We must check that

$$E[1(\delta(\alpha) \le C_t) - 1(\delta(\alpha) \le C_s) | \mathcal{G}_s] = E[A_{\delta(\alpha)} \wedge t - A_{\delta(\alpha)} \wedge s | \mathcal{G}_s]. \qquad (1.21)$$

To avoid trivialities we assume $t > s > \beta(\alpha)$. On the set $\{\delta(\alpha) \le C_s\}$ (1.21) is trivially true. So let's assume that $\delta(\alpha) > C_s$. We have then

$$\begin{aligned}
E[1(\delta(\alpha) \le C_t) | \mathcal{G}_s, \delta(\alpha) > C_s] &= E[1(A_{\delta(\alpha)} \le t) | \mathcal{G}_s, \delta(\alpha) > C_s] \\
&= E[1(e_\alpha \le t) | \mathcal{G}_s, e_\alpha > s] \\
&\qquad \text{(by the definition of } A, C).
\end{aligned}$$

$$= 1 - \exp(-(t-s)).$$

In the last line we have used the fact that e_α is an exponential r.v. independent of $\mathcal{G}_s$ given $e_\alpha > s$. We also compute

$$
\begin{aligned}
E[A_{\delta(\alpha)} \wedge t - s | \mathcal{G}_s, \delta(\alpha) > C_s] &= E\left[\int_s^t 1(\tau \leq A_{\delta(\alpha)}) d\tau | \mathcal{G}_s, \delta(\alpha) > C_s \right] \\
&= E\left[\int_s^t 1(\tau \leq e_\alpha) d\tau | \mathcal{G}_s, e_\alpha > s \right] \\
&= E\left[\int_s^t 1(\tau \leq e_\alpha) d\tau | e_\alpha > s \right] \\
&= \int_0^{t-s} P[\tau \leq e_\alpha] d\tau \\
&= 1 - \exp(-(t-s)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle X \rangle_t &= \langle X^A \rangle_{A_t} \\
&= A_{\delta(\alpha)} \wedge A_t \\
&= A_{\delta(\alpha) \wedge t} \quad \text{(since } A \text{ is increasing)}
\end{aligned}
$$

(c) Arguing as in part (b) we can assume $\bar{\Psi} \equiv 1$. Since M^{α_1} and M^{α_2} are jump martingales whose jumps can't happen simultaneously *a.s.*,

$$[M^{\alpha_1}, M^{\alpha_2}]_t = \sum_{s \leq t} \Delta M_s^{\alpha_1} \Delta M_s^{\alpha_2} = 0$$

The desired conclusion follows since the angle bracket is the dual predictable projection of the square bracket. ∎

Corollary 1.13. *For any $\bar{\Psi} \in D_0$, $Z^N(\bar{\Psi})$ is a right continuous $(\mathcal{F}_t^N)$-martingale with square function*

$$\langle Z^N(\bar{\Psi}) \rangle_t = \int_0^t \int \bar{\Psi}(s, y)^2 \gamma(s, (K^N)^s, y^s) K_s^N(dy) ds.$$

Moreover $|\Delta Z^N(\bar{\Psi})| \leq \frac{1}{N} \|\Psi\|_\infty.$

Proof. Clearly $Z_t^N(\bar{\Psi})$ is an r.c.l.l. martingale, since each of the M_t^α is. Its square function is easily calculated:

$$\langle Z_t^N(\bar{\Psi}) \rangle_t = \frac{1}{N^2} \left\langle \sum_{\alpha \preceq t} M^\alpha \right\rangle_t$$

$$(M^\alpha \text{ was defined in Lemma 1.12})$$

$$= \frac{1}{N^2} \sum_{\alpha \preceq t} \langle M^\alpha \rangle_t$$

(by Lemma 1.12 (c))

$$= \frac{1}{N^2} \sum_{\alpha \preceq t} N \int_0^t h^\alpha(s) \gamma(s, (K^N)^s, (Y^\alpha)^s) \bar{\Psi}(s, Y^\alpha)^2 ds$$

(by Lemma 1.12 (b))

$$= \int_0^t \int \gamma(s, (K^N)^s, y^s) \bar{\Psi}(s, y)^2 K_s^N(dy) ds.$$

The deterministic bound on the jumps of $Z^N(\bar{\Psi})$ comes from the fact that $|\Delta M^\alpha| \leq \frac{1}{N} \|\Psi\|_\infty$ and two jumps can't happen at the same time a.s. ∎

Our next step is to show that $Z_t^N(\bar{\Psi})$ is in L^p, for $p \geq 1$. In fact, we shall proof a stronger result. We will require the following well known lemma.

Lemma 1.14 (Gronwall's lemma). *Assume $f, g : [0, \infty) \to [0, \infty)$ where g is non-decreasing, f is bounded on compacts and*

$$f(t) \leq c \left(g(t) + \int_0^t f(s) ds \right).$$

Then $f(t) \leq c e^{ct} g(t)$ for all $t > 0$.

Proof. See for example (Perkins 1993, Lemma 4.6.) ∎

Lemma 1.15. *There is a function $\Gamma : \mathbf{N} \times [0, \infty) \longrightarrow [0, \infty)$ such that*

$$\sup_N \mathbb{E}[K_t^{N*}(1)^p + \langle Z^N(1) \rangle_t^{p/2}] \leq \Gamma_p(t) \equiv \Gamma(p, t).$$

Proof We replicate the proof of Lemma 2.1 (Perkins 1995). By Proposition 1.11 and Corollary 1.13 $K^N(1) = 1 + Z^N(1)$ is a non-negative martingale. Therefore

$$\mathbb{P}[K_t^{N*}(1) > \lambda] \leq \frac{\mathbb{E}[K_t^N(1)]}{\lambda}$$

$$= \frac{1}{\lambda}. \tag{1.22}$$

Define a sequence of stopping times by $\tau_n := \inf\{t > 0 : K_t^N(1) \geq n\} \wedge n$. (1.22) implies that $\tau_n \to \infty$ a.s. Now,

$$\langle Z^N(1) \rangle_{t \wedge \tau_n} = \int_0^{t \wedge \tau_n} \int \gamma(s, (K^N)^s, y) K_s^N(dy) ds$$

$$\leq \int_0^{t \wedge \tau_n} \hat{\Gamma}(s) \left(1 + \int_0^s K_s^{N*}(1) \right) K_s^N(1) ds$$

(by the assumption on γ)

$$\leq \hat{\Gamma}(t) \left(1 + \int_0^{t \wedge \tau_n} K_s^{N*}(1) ds^2 \right).$$

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Burkholder's inequality yields

$$\mathbb{E}[K^{N*}(1)_{t\wedge\tau_n}^p] \le c_1(p)\left[1 + \hat{\Gamma}(t)^{p/2}\mathbb{E}\left(1 + \int_0^{t\wedge\tau_n} K_s^{N*}(1)ds^p\right)\right]$$

$$\le c_2(p,t)\left(1 + \int_0^t \mathbb{E}[K^{N*}(1)_{s\wedge\tau_n}^p ds]\right).$$

Notice that $\mathbb{E}[K^{N*}(1)_{t\wedge\tau_n}^p] \le n^p$. Therefore we can apply Gronwall's lemma to obtain

$$\mathbb{E}[K^{N*}(1)_{t\wedge\tau_n}^p] \le \Gamma_p(t) \equiv c_2(p,t)e^{c_2(p,t)} \tag{1.23}$$

Hence

$$\mathbb{E}[K^{N*}(1)_t^p] \le \liminf_{n\to\infty} \mathbb{E}[K^{N*}(1)_{t\wedge\tau_n}^p]$$

$$\text{(by Fatou's lemma)}$$

$$\le \Gamma_p(t).$$

Note that $\Gamma_p(t)$ is independent of N to complete the proof of the lemma. ∎

Corollary 1.16. *For any $\bar{\psi} \in D_0$, $Z^N(\bar{\psi})$ is a square integrable martingale.*

Remark 1.17. Both W^N and Z^N can be shown to be orthogonal martingale measures. We won't prove this result because we won't need it. We can recast (1.18) is SPDE form. We shall restrict ourselves to $d = 1$ and to the Markovian case. That is, we consider only $M_F(\mathbb{R})$-valued superprocesses. For simplicity we assume $\gamma \equiv 1$. Then solution of (1.18) corresponds formally to

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(t,u(t,\cdot),x)u(t,x)\right) - \frac{\partial}{\partial x}\left(b(t,u(t,\cdot),x)u(t,x)\right) + \dot{Z}^N + \nabla \cdot \dot{W}^N.$$

We shall see that $W^N \to 0$ in a strong sense. Taking into account Corollary (1.13) and letting $N \to \infty$ we see that this equation becomes (formally)

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(t,u(t,\cdot),x)u(t,x)\right) - \frac{\partial}{\partial x}\left(b(t,u(t,\cdot),x)u(t,x)\right) + \sqrt{u(t,x)}\dot{W}(t,x),$$

where W is a space time white noise. □

1.3 Tightness of the Normalized Branching Particle Systems

In this section we show that the sequence $\{K^N\}$ defined in Section 1.2 possesses at least one limit point. To this end we shall employ the following theorem (Dawson 1993, p. 48).

Theorem 1.18 (Jakubowski's Criterion). *Let (E,d) be a Polish space. Let $\mathbf{F}$ be a family of real continuous functions on E that separates points in E and is closed under addition. Given $f \in \mathbf{F}$, define $\tilde{f} : D_E \longrightarrow D([0,\infty),\mathbb{R})$ by $(\tilde{f}x)(t) := f(x(t))$. A sequence $\{P_n\}$ of probability measures on D_E is tight iff the following two conditions hold:*

(a)(Convergence of projections.) *The family $\{P_n\}$ is $\mathbf{F}$-weakly tight, i.e. for each $f \in \mathbf{F}$ the sequence $\{P_n \circ (\tilde{f})^{-1}\}$ of probability measures in $D([0, \infty), \mathbb{R})$ is tight.*

(b)(Compact containment condition.) *For each $T > 0$ and $\varepsilon > 0$ there is a compact set $K_{T,\varepsilon} \subset E$ such that*

$$\inf_n P_n(D([0, T], K_{T,\varepsilon})) > 1 - \varepsilon.$$

If the sequence $\{P_n\}$ is tight, then it is relatively compact in the weak topology on $M_1(D_E)$.

We will apply Jakubowski's criterion to the sequence $\{Law(K^N)\}$. Here $E = M_F(C^d)$. We tackle steps (i) and (ii) in the next two Subsections.

1.3.1 Convergence of the Projections

We begin this Subsection by quoting some useful theorems.

Theorem 1.19. *Suppose that (X^n) is a sequence of locally square integrable martingales. Then for (X^n) to be tight with continuous limit points it is sufficient that*

1. The sequence (X_0^n) is tight in $\mathbb{R}$.

2. The sequence $\langle X^n \rangle$ is tight and its limit points are continuous.

3. For all $M \in \mathbf{N}$, $\varepsilon > 0$

$$\lim_{n \to \infty} P^n\left[\sup_{t \leq M} |\Delta X_t^n| > \varepsilon\right] = 0.$$

Proof. See Theorems VI.3.26 and VI.4.13 of (Jacod & Shiryaev 1987). ∎

Theorem 1.20. (a) *Let (M^n), (Y^n) be sequences of real-valued processes. If the sequence (M^n) is such that*

$$\forall T > 0, \quad \forall \varepsilon > 0, \quad \lim_{n \to \infty} P^n\left[\sup_{t \leq T} |M_t^n| > \varepsilon\right] = 0,$$

and if the sequence (Y^n) is tight (resp. converges in law to Y), then the sequence $(Y^n + M^n)$ is tight (resp. converges in law to Y).

(b) *Let (M^n) be a tight sequence of processes with values in Polish space E_1, and let (Y^n) be a tight sequence of processes with values in a (possibly different) Polish space E_2. Suppose that the limit points of both sequences are continuous. Then the sequence (Y^n, M^n) is tight and its limit points are continuous.*

Proof. See Lemma VI.3.31 and Corollary VI.3.33 of (Jacod & Shiryaev 1987). ∎

Notation 1.21. Let's write $\Delta(\delta, T, y)$ for the oscillation of the path y:

$$\Delta(\delta, T, y) := \sup\{|y(t) - y(s)| : s, t \in [0, T], |s - t| < \delta\}.$$

Proposition 1.22. *We keep the notation of the preceding Section. Fix* $\bar{\Psi} \in D_0$.

(a) *For any* $t > 0$,
$$\lim_{N \to \infty} \mathbb{E}[W_t^{N*}(\bar{\Psi})] = 0.$$

(b) *The sequence* $(\langle Z^N(\bar{\Psi}) \rangle)$ *is tight and its limit points are continuous.*

(c) *The sequence* $(Z^N(\bar{\Psi}))$ *is tight and its limit points are continuous.*

(d) *The sequence* $\left(\int_0^\cdot \int A(K^N)(\bar{\Psi})(s,y) K_s^N(dy) ds \right)$ *is tight and its limit points are continuous.*

Proof (a) Let

$$V_t^\alpha = \int_0^t \langle \nabla \bar{\Psi}(s, Y^\alpha) \sigma(s, K_{s-}^N, Y^\alpha), h^\alpha(s) dB_s^\alpha \rangle_n$$

This is a continuous local $(\mathcal{F}_t^N)$-martingale (a martingale indeed!), since B^α is a $(\mathcal{F}_t^N)$-Brownian motion (see Remark 1.9(c)). Note that $\langle V^\alpha, V^{\alpha'} \rangle_t = 0$ if $\alpha \neq \alpha'$ because of the independence of the Brownian motions B^α and $B^{\alpha'}$. Moreover

$$\langle V^\alpha \rangle_t \leq \int_0^t \| \nabla \bar{\Psi}(s, Y^\alpha) \sigma(s, K_{s-}^N, Y^\alpha) \|^2 h_s^\alpha ds$$

$$\leq \| \nabla \Psi \|_\infty^2 \int_0^t \Upsilon(p, s, K_s^{N*}(1))^2 h_s^\alpha ds$$

(by the hypothesis on σ).

Hence

$$\mathbb{E}[\langle W^{N*}(\bar{\Psi}) \rangle_t] = \mathbb{E}\left[\frac{1}{N^2} \left\langle \sum_{\alpha \preceq t} V^\alpha \right\rangle_t \right]$$

$$= \frac{1}{N} \mathbb{E}\left[\frac{1}{N} \sum_{\alpha \preceq t} \langle V^\alpha \rangle_t \right]$$

(since the cross-covariations are 0)

$$\leq \frac{1}{N} \mathbb{E}\left[\frac{1}{N} \sum_{\alpha \preceq t} \| \nabla \Psi \|_\infty^2 \int_0^t \Upsilon(p, s, K_s^{N*}(1))^2 h_s^\alpha ds \right]$$

$$= \frac{\| \nabla \Psi \|_\infty^2}{N} \mathbb{E}\left[\int_0^t K_s^N (\Upsilon(p, s, K_s^{N*}(1))^2) ds \right]$$

$$= \frac{\| \nabla \Psi \|_\infty^2}{N} \int_0^t \mathbb{E}[K_s^N(1) \Upsilon(p, s, K_s^{N*}(1))^2] ds$$

$$\leq \frac{t \| \nabla \Psi \|_\infty^2}{N} \mathbb{E}[\Upsilon(p+1, s, K_s^{N*}(1))^2]$$

$$\longrightarrow 0 \quad \text{as } N \to \infty.$$

This concludes the proof of (a).

(b) Fix $T > 0$ and choose $0 \leq s < t \leq T$.

$$\langle Z^N(\bar{\Psi})\rangle_t - \langle Z^N(\bar{\Psi})\rangle_s = \int_s^t \int \bar{\Psi}(u,y)^2 \gamma(u, (K^N)^u, y^u) K_u^N(dy) du$$

(by Corollary 1.13).

Recall that our hypothesis imply that $\gamma(u, (K^N)^u, y^u) \leq \hat{\Gamma}(t)(1 + tK_t^{N*}(1))$ for $u \leq t$. Thus

$$\langle Z^N(\bar{\Psi})\rangle_t - \langle Z^N(\bar{\Psi})\rangle_s \leq \parallel \Psi \parallel_\infty^2 \hat{\Gamma}(t)(1 + tK_t^{N*}(1)) \int_s^t K_u^N(1)ds$$

$$\leq (t-s) \parallel \Psi \parallel_\infty^2 \hat{\Gamma}(T)(1 + TK_T^{N*}(1))K_T^{N*}(1).$$

By Lemma (1.15), there is a finite constant $A = A(T)$ such that

$$\sup_N \{\mathbb{E}[((1 + TK_T^{N*}(1))K_T^{N*}(1))^2]\} \leq A.$$

Hence, for any $\varepsilon > 0$

$$\mathbb{P}[\Delta(\delta, T, \langle Z^N(\bar{\Psi})\rangle.) > \varepsilon] \leq \varepsilon^{-2} \mathbb{E}[\Delta(\delta, T, \langle Z^N(\bar{\Psi})\rangle.)^2]$$

$$\leq \left(\frac{\delta}{\varepsilon}\right)^2 \parallel \Psi \parallel_\infty^4 \hat{\Gamma}(T)^2 A$$

Therefore

$$\lim_{\delta \downarrow 0} \sup_N \mathbb{P}[\Delta(\delta, T, \langle Z^N(\bar{\Psi})\rangle.) > \varepsilon] = 0.$$

This shows that $(\langle Z^N(\bar{\Psi})\rangle.)$ is tight with continuous limit points. This completes the proof of (b).

(c) We shall use Theorem 1.19. Use part (b) and note that condition 3 in Theorem 1.19 is satisfied since $|\Delta Z^N(\bar{\Psi})| \leq \frac{\parallel \Psi \parallel_\infty}{N}$ by Corollary 1.13.

(d) The proof is very similar to that of part (b). Let's call

$$S_t^N := \int_0^t \int A(K^N)(\bar{\Psi})(s,y)K_s^N(dy)ds. \tag{1.24}$$

From the assumptions on b, σ and Ψ it follows that

$$|A(K^N)(\bar{\Psi})(s,y)| \leq \parallel \nabla\Psi \parallel_\infty \Upsilon(p, s, K_s^{N*}(1)) + \parallel \frac{\Delta}{2}\Psi \parallel_\infty \Upsilon(p, s, K_s^{N*}(1))^2.$$

Therefore, for any $T > 0$, $0 \leq s < t \leq T$

$$S_t^N - S_s^N \leq constant_T \cdot (t-s)\Upsilon(p+1, s, K_T^{N*}(1))^2.$$

We know that there is a finite constant $A = A(T)$ such that

$$\sup_N \mathbb{E}[\Upsilon(p+1, s, K_T^{N*}(1))^4] < A.$$

Use Arzela-Ascoli as in (b) to finish the proof of (d). ∎

Corollary 1.23. *For any $\bar{\Psi} \in D_0$ the sequence $(K^N(\bar{\Psi}).)_N$ is tight and its limit points are continuous.*

Proof. The Corollary follows immediately from Propositions 1.11, 1.22 and Theorem 1.20. ∎

1.3.2 Compact Containment Condition

Proposition 1.24. *Given T, $\varepsilon > 0$ there exists a compact set $F_{T,\varepsilon} \in M_F(C^d)$ such that*

$$\inf_N \mathbb{P}\{K^N \in D([0,T], F_{T,\varepsilon})\} \geq 1 - \varepsilon. \tag{1.25}$$

Proof Suppose that for all n, $T > 0$ there is a compact subset $G_{T,n}$ of C^d such that

$$\sup_N \mathbb{P}\{\sup_{0 \leq t \leq T} K_t^N(G_{T,n}^c) > 2^{-n}\} \leq 2^{-n}. \tag{1.26}$$

We claim that (1.26) implies (1.25). Indeed,

$$\sup_N \sum_n \mathbb{P}\{\sup_{0 \leq t \leq T} K_t^N(G_{T,n}^c) > 2^{-n}\} \leq 1.$$

The Borel-Cantelli lemma implies that

$$\sup_N \mathbb{P}\{\sup_{0 \leq t \leq T} K_t^N(G_{T,n}^c) \leq 2^{-n} \quad \text{eventually}\} = 1$$

That is,

$$\lim_{m \to \infty} \sup_N \mathbb{P}\{\sup_{0 \leq t \leq T} K_t^N(G_{T,n}^c) \leq 2^{-n} \quad \forall n \geq m\} = 1.$$

Hence there is an $m = m(\varepsilon)$ such that

$$\sup_N \mathbb{P}\{\sup_{0 \leq t \leq T} K_t^N(G_{T,n}^c) \leq 2^{-n} \quad \forall n \geq m\} \geq 1 - \varepsilon/2.$$

By relabeling we can assume $m = 1$. Thus (1.26) implies

$$\sup_N \mathbb{P}\{\forall n \quad \sup_{0 \leq t \leq T} K_t^N(G_{T,n}^c) \leq 2^{-n}\} \geq 1 - \varepsilon/2. \tag{1.27}$$

Recall Prohorov's theorem. A set $F_{T,\varepsilon} \in M_F(C^d)$ is compact if it is closed and the conditions below are met.

(P1) For all n there exists a compact set $G_n \subset C^d$ such that $\sup_{\mu \in F_{T,\varepsilon}}\{\mu(G_n^c)\} \leq 2^{-n}$.

(P2) $\sup_{\mu \in F_{T,\varepsilon}}\{\mu(1)\} < \infty$.

Let us suppose that there is an $M = M(\varepsilon)$ such that

$$\sup_N \mathbb{P}\{K_T^{N*}(1) > M(\varepsilon)\} \leq \varepsilon/2. \tag{1.28}$$

Set

$$F_{T,\varepsilon} := \{\mu \in M_F(C^d) : \mu(1) \leq M(\varepsilon), \quad \mu(G_n^c) \leq 2^{-n} \quad \forall n\}.$$

With this choice of $F_{T,\varepsilon}$ we see that (1.27), (1.28) together with (P1–P2) imply (1.25). We now show that (1.28) holds. Note that since $K^N(1)$ is a martingale,

$$\mathbb{P}\{K_T^{N*}(1) > M\} \leq \frac{\mathbb{E}[K_T^N(1)]}{M}. \tag{1.29}$$

By Lemma 1.15 there is a function Γ_1 independent of N such that $E[K_T^N(1)] \leq \Gamma_1(T)$. Put $M = \varepsilon/2\Gamma_1(T)$ to see that (1.29) implies (1.28).

Our next task is to exhibit a set $G_{T,n}$ such that (1.26) holds. For this purpose we shall need the following two lemmas.

Lemma 1.25. *For any $\eta > 0$ there is a compact set $F = F(\eta, T) \in C^d$ such that*

$$\sup_N \mathbb{E}[K_T^N(F^c)] < \eta \qquad (1.30)$$

Proof Recall from Section 1.2 that conditional on $\mathcal{A}$, the law of a path $(Y^\alpha)^T$ is obtained as the unique solution of a system of SDE's with coefficients uniformly bounded by $\Upsilon(p, T, K_T^{N*}(1))$. So for any Y^α in the support of K_T^N a standard argument (e.g. Rogers and Williams (1987) p. 166) shows that there is a constant $c > 0$ such that for any $\varepsilon > 0$, $k \in \mathbf{N}$, $T > 0$, (k is allowed to depend on ε)

$$P[\Delta(k^{-1}, T, Y^\alpha) > \varepsilon | \mathcal{A}] \le \frac{cT}{\varepsilon^4} \Big(\frac{\Upsilon(p, T, K_T^{N*}(1))^2}{k} + \frac{\Upsilon(p, T, K_T^{N*}(1))^4}{k^3} \Big).$$

Define $F_{k,\varepsilon} = \{y \in C^d : \Delta(k^{-1}, T, y) \le \varepsilon\}$. We compute with $F = F_{k,\varepsilon}$

$$
\begin{aligned}
\mathbb{E}[K_T^N(F^c)] &= \mathbb{E}\left[\frac{1}{N} \sum_{\alpha \sim T} 1((Y^\alpha)^T \notin F) \right] \\
&= \mathbb{E}\left[E\left[\frac{1}{N} \sum_{\alpha \sim T} 1((Y^\alpha)^T \notin F) | \mathcal{A} \right] \right] \\
&= \mathbb{E}\left[\frac{1}{N} \sum_{\alpha \sim T} E[1(Y^\alpha)^T \notin F) | \mathcal{A}] \right] \\
&\le \frac{cT}{\varepsilon^4} \mathbb{E}\left[\frac{1}{N} \sum_{\alpha \sim T} \frac{\Upsilon(p, T, K_T^{N*}(1))^2}{k} + \frac{\Upsilon(p, T, K_T^{N*}(1))^4}{k^3} \right] \\
&\le \frac{cT}{\varepsilon^4} \mathbb{E}\left[K_T^{N*}(1) \left(k^{-1}\Upsilon(p, T, K_T^{N*}(1))^2 + k^{-3}\Upsilon(p, T, K_T^{N*}(1))^4 \right) \right].
\end{aligned}
$$

Lemma 1.15 implies that there is a constant A such that

$$\mathbb{E}[K_T^{N*}(1)\Upsilon(p, T, K_T^{N*}(1))^2] \vee \mathbb{E}[K_T^{N*}(1)\Upsilon(p, T, K_T^{N*}(1))^4] < A$$

since the random terms inside the expectations involve only polynomials on $K_T^{N*}(1)$. To finish the proof set $\varepsilon = k^{-1/8}$, choose k large enough so that $cTA(k^{-1/2} + k^{-5/2}) < \eta$. This gives

$$\mathbb{E}[K_T^N(F^c)] \le \frac{cTA(k^{-1} + k^{-3})}{\varepsilon^4}$$

$$\le \eta \qquad \blacksquare$$

Lemma 1.26. *Let $\phi \in bC^d$, τ a bounded $(\mathcal{F}_t^N)$-stopping time. We write $\phi^t(y) := \phi(y^t)$. Then there exists an $(\mathcal{F}_t^N)$-martingale $m = m^{\phi, \tau}$ such that $m_t = 0$ if $t \le \tau$ and*

$$K_t^N(\phi^\tau) = K_\tau^N(\phi) + m_t \quad \forall t \ge \tau \quad a.s. \qquad (1.31)$$

Proof. For any ϕ, τ satisfying the hypothesis of the lemma we define

$$m_t^{\psi, \tau} = \begin{cases} \frac{1}{N} \sum_{\tau \prec \alpha \preceq t} \left[\phi((Y^\alpha)^{\delta(\alpha)-})1(\delta(\alpha) \le t)(\xi_\alpha - 1) \right], & \text{if } t > \tau; \\ 0 & \text{otherwise.} \end{cases}$$

Let's assume first that $\phi \in bC_u$ and that ϕ is of the form $\phi = \bar{\psi} \in D_0$. That is, for some $r \in \mathbb{N}$, $0 \le t_1 \le t_2 \le, \dots, t_r \le u$, $\psi \in C_0^\infty(\mathbb{R}^{rd})$, $\phi = \bar{\psi}(t_1, \dots, t_n)$. Assume further that $\tau \equiv u$. Then $\nabla \bar{\psi}(s, y) = 0$ and $\bar{\psi}_{ij} = 0$ for $s \ge u$. Therefore Proposition (1.11) implies

$$
\begin{aligned}
K_t^N(\bar{\psi}) - K_u^N(\bar{\psi}) &= Z_t^N(\bar{\psi}) - Z_u^N(\bar{\psi}) \\
&= \frac{1}{N} \sum_{u \prec \alpha \preceq t} \left[\bar{\psi}(\delta(\alpha)-, Y^\alpha) 1(\delta(\alpha) \le t)(\xi_\alpha - 1) \right] \\
&= m_t^{\bar{\psi}, \tau} \quad \forall t \ge \tau \quad a.s.
\end{aligned}
\tag{1.32}
$$

Clearly $m_t^{\bar{\psi}, \tau}$ is a martingale. Assume next that $\phi \in bC_u$ and $\tau \equiv u$. Any such function ϕ is the bp-limit of a sequence (ϕ_n) where each $\phi_n = \in bC_u \cap D_0$. The bounded convergence theorem and (1.32) imply that

$$
\begin{aligned}
K_t^N(\phi) - K_u^N(\phi) &= K_t^N(\phi^\tau) - K_\tau^N(\phi) \\
&= m_t^{\phi, \tau} \quad \forall t \ge \tau \quad a.s.
\end{aligned}
\tag{1.33}
$$

Now suppose that $\phi \in bC$ and τ takes on only finitely many values, that is $\tau \in \{u_1, \dots, u_m\}$. Then $\phi(y^\tau) = \sum_{i=1}^m 1(\tau = u_i) \phi(y^{u_i})$. We see that $K_t^N(\phi^\tau) = K_\tau^N(\phi) + m_t^{\phi, \tau} \, \forall t \ge \tau$ a.s. as a consequence of (1.33). Finally, we consider a general bounded τ, $\phi \in bC$. Take a decreasing sequence of stopping times $\{\tau_n\}$ taking only finitely many values and converging a.s. to τ. Note that $\phi^{\tau_n} \xrightarrow{bp} \phi^\tau$ a.s. We know that $K_t^N(\phi^{\tau_n}) = K_{\tau_n}^N(\phi) + m_t^{\tau_n, \phi}$. Note that $K_t^N(\phi^{\tau_n}) \longrightarrow K_t^N(\phi^\tau)$ a.s. by bounded convergence, $K_{\tau_n}^N(\phi) \longrightarrow K_\tau^N(\phi)$ a.s. by the right continuity of K^N. Moreover it is clear from the definition of m that $m^{\tau_n, \phi} \longrightarrow m^{\tau, \phi}$ a.s. This shows that (1.31) holds. Finally note that we always took a.s. limits of martingales bounded in L^2 so that $m^{\tau, \phi}$ is indeed a martingale. $\blacksquare$

Let's continue the proof of Proposition 1.24. By Lemma 1.25 we can find a compact set $F = F_{T, n}$ such that $\sup_N \mathbb{E}[K_T^N(F^c)] < 2^{-2n}$. Define $G := \bigcup_{s \le T} F^s$. Note that by the theorem of Arzela-Ascoli G is also compact. We now show that this is in fact the desired set that satisfies (1.26). Let

$$
\tau^N := \inf\{s > 0 : K_s^N(G^c) > 2^{-n}\} \wedge T.
$$

Set $\phi(y) := 1(y \in G^c)$. From Lemma 1.26 and the right continuity of $K^N(G^c)$ we see that

$$
K_T^N((G^{\tau^N})^c) = 2^{-n} + m_T^{\phi, \tau^N}
\tag{1.34}
$$

on the set $\{\tau^N < T\}$. The second term on the r.h.s. of (1.34) is a martingale null at zero and on the set $\{\tau^N \ge T\}$. Thus

$$
1(\tau^N < T) 2^{-n} + m_T^{\phi, \tau^N} = 1(\tau^N < T) K_T^N[\{y \in C^d : y^{\tau^N} \notin G\}].
$$

Take expectations to obtain

$$
\begin{aligned}
2^{-n} \mathbb{P}[\tau^N < T] &= \mathbb{E}[1(\tau^N < T) K_T^N[\{y \in C^d : y^{\tau^N} \notin G\}] \\
&\le \mathbb{E}[K_T^N(F^c)] \quad (\text{since } \{y : y^{\tau^N} \notin G\} \subset \{y : y \notin F\} \text{ because } \tau^N \le T) \\
&\le 2^{-2n} \quad (\text{by the definition of the set F}).
\end{aligned}
$$

Therefore

$$\sup_N \mathbb{P}[\tau^N < T] \le 2^{-n} \qquad (1.35)$$

Since $\{\sup_{0 \le t \le T} K_t^N(G^c) > 2^{-n}\} \subset \{\tau^N < T\}$, (1.26) follows from (1.35). ∎

1.3.3 Relative Compactness of (K^N)

We are now ready to begin reaping the rewards of our effort.

Theorem 1.27. *The sequence* $(\mathbb{P}[K^N \in \cdot])_{N \in \mathbf{N}}$ *is relatively compact in* $\mathcal{M}_1(D([0, \infty), M_F(C^d)))$.

Proof. Corollary 1.23 and Proposition 1.24 show that the conditions of Jakubowski's Theorem 1.18 are satisfied. ∎

1.4 Identification and Uniqueness of the Limit

Typically, it is more difficult to prove uniqueness than existence. This is yet another instance. Fortunately, the uniqueness of the limit points has already been studied (Perkins 1995).

In this Section we find a martingale problem satisfied by the limit points of the sequence (K^N). Uniqueness, with some restrictions on the branching rate γ, follows from results of (Perkins 1995.) In light of Propositions 1.11, 1.22 and Corollary 1.13 it is easy to guess that the limit points satisfy a martingale problem of the form 1.3. To prove it we will need a number of results.

In addition to our hypothesis (1.3) we will assume that γ is Lipschitz continuous. Recall that ρ denotes the sup metric on C^d, $d = d_\rho$ denotes the Vasershtein metric on $M_F(C^d)$. Let d' be the metric in the space $D([0, \infty), M_F(C^d))$ defined in equation (5.2) p. 153 of Ethier and Kurtz (1986). This metric induces the usual Skorohod J_1 topology. Recall that as a subspace of $D([0, \infty), M_F(C^d))$ the space $(C([0, \infty), E), d')$ is closed and is in fact the usual space of continuous paths with the compact open topology.

The following hypothesis will be in force throughout this Section.

$$|\gamma(t, K, y) - \gamma(t, K', y')| \le \Upsilon(p, t, K_t^*(1) \vee K_t'^*(1))[\rho(y^t, y'^t) + d'(K^t, K'^t)]. \qquad (Lip_\gamma)$$

Fix $\phi \in D_0$. Let S^N be as in (1.24). Let's assume that there are some continuous real-valued random processes $S(\phi)$, $Z(\phi)$, $V(\phi)$ and an r.c.l.l. $M_F(C^d)$-valued process K such that

$$Z^N(\phi) \underset{N \to \infty}{\Longrightarrow} Z(\phi),$$

$$\langle Z^N(\phi) \rangle \underset{N \to \infty}{\Longrightarrow} V(\phi),$$

$$S^N(\phi) \underset{N \to \infty}{\Longrightarrow} S(\phi),$$

$$K^N \underset{N \to \infty}{\Longrightarrow} K.$$

The results of Section 1.3 ensure that these are valid assumptions upon taking an appropriate subsequence. By Skorohod's representation theorem and taking a subsequence if necessary we

may further assume that the convergence above takes place simultaneously and almost surely. That is,

$$(K^N, Z^N(\phi), \langle Z^N(\phi) \rangle, S^N(\phi)) \xrightarrow[N \to \infty]{} (K, Z(\phi), V(\phi), S(\phi)) \quad a.s. \tag{1.36}$$

All the 4-tuples in (1.36) are considered as $D([0, \infty), M_F(C^d)) \times D([0, \infty), \mathbb{R})^3$-valued random variables defined on a common probability space (possibly different from the original one.) Let's call $\mathcal{F}^N$ (resp. $\mathcal{F}$) the filtration generated by K^N (resp. K).

Proposition 1.28. $Z.(\phi)$ *is a continuous square integrable* $(\mathcal{F}_t)$-*martingale with square function*

$$\langle Z(\phi) \rangle_t = V(\phi)_t$$
$$= \lim_{N \to \infty} \langle Z^N(\phi) \rangle_t.$$

Proof. We already know that $Z.(\phi)$ is continuous. To prove that it is martingale it suffices to show that for $s \le t$, $r \in \mathbb{N}$, $0 \le s_1 \le \ldots \le s_r \le s$, (the s_i's may be restricted to belong to some dense set), and $f : M_F(C^d)^r \longrightarrow \mathbb{R}$ bounded continuous

$$\mathbb{E}[f(K_{s_1}, \ldots, K_{s_r}) Z_s(\phi)] = \mathbb{E}[f(K_{s_1}, \ldots, K_{s_r}) Z_t(\phi)]. \tag{1.37}$$

Note that by Lemma 1.15

$$\sup_N \mathbb{E}[(f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_t^{N*}(\phi))^2] \le \|f\|_\infty^2 \|\phi\|_\infty^2 \Gamma(2, t).$$

Hence the families $\{f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_t^N(\phi) : N \in \mathbb{N}\}$ and $\{f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_s^N(\phi) : N \in \mathbb{N}\}$ are both uniformly integrable. Moreover $f(K_{s_1}^N, \ldots, K_{s_r}^N) \to f(K_{s_1}, \ldots, K_{s_r})$ *a.s.* since almost sure convergence in the Skorohod space $D([0, \infty), M_F(C^d))$ implies almost sure convergence at the points of continuity of $t \mapsto K_t$, i.e. everywhere with the possible exception of a countable set. And we are allowed to choose the s_i's outside such exceptional set. Also, $Z_t^N(\phi) \to Z_t(\phi)$ *a.s.* and $Z_s^N(\phi) \to Z_s(\phi)$ *a.s.* Thus a.s. convergence plus uniform integrability imply

$$\lim_{N \to \infty} \mathbb{E}[f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_t^N(\phi)] = \mathbb{E}[f(K_{s_1}, \ldots, K_{s_r}) Z_t(\phi)],$$
$$\lim_{N \to \infty} \mathbb{E}[f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_s^N(\phi)] = \mathbb{E}[f(K_{s_1}, \ldots, K_{s_r}) Z_s(\phi)]$$

But $Z^N(\phi)$ is a $\mathcal{F}^N$-martingale. Therefore

$$\mathbb{E}[f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_t^N(\phi)] = \mathbb{E}[f(K_{s_1}^N, \ldots, K_{s_r}^N) Z_s^N(\phi)].$$

The desired result (1.37) follows. We must now compute the square function of $Z_t(\phi)$. We know that $\langle Z^N(\phi) \rangle_t$ converges almost surely. Note that by Lemma 1.15 the sequences

$$\{f(K_{s_1}^N, \ldots, K_{s_r}^N)(Z_t^N(\phi)^2 - \langle Z^N(\phi) \rangle_t) : N \in \mathbb{N}\},$$
$$\{f(K_{s_1}^N, \ldots, K_{s_r}^N)(Z_s^N(\phi)^2 - \langle Z^N(\phi) \rangle_s) : N \in \mathbb{N}\},$$

are uniformly integrable. Argue as before to prove that $(Z_t(\phi)^2 - \langle Z(\phi) \rangle_t)_{t \ge 0}$ is a $\mathcal{F}^N$-martingale. This concludes the proof of the Proposition. ∎

Proposition 1.29.

$$\langle Z(\phi)\rangle_t = \int_0^t \gamma(s,K,y)\phi(y)^2 K_s(dy)ds, \tag{1.38}$$

$$S_t(\phi) = \int_0^t \int A(K)(\bar{\Psi})(s,y)K_s(dy)ds. \quad \forall t \geq 0 \quad a.s. \tag{1.39}$$

(S^N, S were defined at the beginning of the section.)

Proof Let

$$T_j := \inf\{s > 0 : \sup_N K_s^{N*}(1) \wedge \int_0^s (1 + K_u^{N*}(1))du > j\}.$$

Note that by Lemma 1.15 $\lim_{j \to \infty} T_j = \infty$ a.s. Moreover $\gamma(\cdot,\cdot,\cdot)1(\cdot \leq T_j)$ is bounded. Therefore, for any $m, m', n, n' \in \mathbf{N}$

$$\int_0^{t \wedge T_j} \left| K_s^m(\gamma(s,K^n,\cdot)\phi(\cdot)^2) - K_s^{m'}(\gamma(s,K^{n'},\cdot)\phi(\cdot)^2) \right| ds$$

$$\leq \int_0^{t \wedge T_j} \left| K_s^m((\gamma(s,K^n,\cdot) - \gamma(s,K^{n'},\cdot))\phi(\cdot)^2) \right| ds$$

$$+ \int_0^{t \wedge T_j} \left| K_s^m(\gamma(s,K^{n'},\cdot)\phi(\cdot)^2) - K_s^{m'}(\gamma(s,K^{n'},\cdot)\phi(\cdot)^2) \right| ds$$

We estimate

$$\int_0^{t \wedge T_j} \left| K_s^m((\gamma(s,K^n,\cdot) - \gamma(s,K^{n'},\cdot))\phi(\cdot)^2) \right| ds$$

$$\leq \|\phi\|_\infty^2 d((K^n)^t,(K^{n'})^t)\Upsilon_p(K_{t \wedge T_j}^{m*}) \int_0^{t \wedge T_j} K_s^m(1)ds$$

$$= c_1(j)d((K^n)^t,(K^{n'})^t).$$

In addition,

$$\int_0^{t \wedge T_j} \left| K_s^m(\gamma(s,K^{n'},\cdot)\phi(\cdot)^2) - K_s^{m'}(\gamma(s,K^{n'},\cdot)\phi(\cdot)^2) \right| ds$$

$$\leq \|\phi\|_\infty^2 \Upsilon_p(K_{t \wedge T_j}^{n'*}) \int_0^{t \wedge T_j} |K_s^m(1) - K_s^{m'}(1)| ds$$

$$= c_2(j) \int_0^{t \wedge T_j} |K_s^m(1) - K_s^{m'}(1)| ds.$$

Let $n' \to \infty$. Use the continuity of γ and dominated convergence to obtain

$$\int_0^{t \wedge T_j} \left| K_s^m(\gamma(s,K^n,\cdot)\phi(\cdot)^2) - K_s^{m'}(\gamma(s,K,\cdot)\phi(\cdot)^2) \right| ds$$

$$\leq c_1(j)d((K^n)^t,K^t) + c_2(j) \int_0^{t \wedge T_j} |K_s^m(1) - K_s^{m'}(1)| ds \quad a.s.$$

Now let $m' \to \infty$. Use the fact that $y \mapsto \gamma(s, K, y)\phi(y)^2$ is bounded continuous and dominated convergence to get

$$\int_0^{t \wedge T_j} \left| K_s^m(\gamma(s, K^n, \cdot)\phi(\cdot)^2) - K_s(\gamma(s, K, \cdot)\phi(\cdot)^2) \right| ds$$

$$\leq c_1(j)d\left((K^n)^t, K^t\right) + c_2(j) \int_0^{t \wedge T_j} |K_s^m(1) - K_s(1)| ds \quad a.s.$$

Now put $n = m$. Dominated convergence yields

$$\int_0^{t \wedge T_j} \left| K_s^n(\gamma(s, K^n, \cdot)\phi(\cdot)^2) - K_s(\gamma(s, K, \cdot)\phi(\cdot)^2) \right| ds \xrightarrow[n \to \infty]{} 0 \quad a.s. \tag{1.40}$$

For every integer t there is a $j = j(t)$ such that $t \wedge T_j = t$. Moreover, the integral in (1.40) is increasing as a function of t. Therefore

$$\int_0^t \left| K_s^n(\gamma(s, K^n, \cdot)\phi(\cdot)^2) - K_s(\gamma(s, K, \cdot)\phi(\cdot)^2) \right| ds \xrightarrow[n \to \infty]{} 0 \quad \forall t > 0 \quad a.s.$$

The validity of (1.38) follows from the above and Proposition 1.28. The proof of (1.39) is almost identical and will be omitted. ∎

We end this chapter introducing some notation and stating the main theorem of the chapter.

Notation 1.30. Let

$$M_F(C^d)^t = \{\mu \in M_F(C^d) : y = y^t \quad \mu - a.a. \quad y\}, \quad t \geq 0.$$
$$\Omega_H = \{K \in C([0, \infty), M_F(C^d)) : K_t \in M_F(C^d)^t \quad \forall t \geq 0\},$$
$$H_t(\omega) = \omega(t) \quad \text{for} \quad \omega \in \Omega_H,$$
$$\mathcal{H}_t = \sigma(H_u : 0 \leq s \leq t), \quad \mathcal{H} = \sigma(H_s : s \geq 0).$$

and

$$(\widehat{\Omega}_H, \widehat{\mathcal{H}}_t) = (\Omega_H \times C^d, \mathcal{H}_t \times \mathcal{C}_t^d). \qquad \square$$

Theorem 1.31. *Any limit point K of the sequence (K^N) satisfies the martingale problem:*

$$\forall \phi \in D_0 \quad Z_t(\phi) = K_t(\phi) - \phi(\vec{0}) - \int_0^t \int_{C^d} A(K)(\phi)(s, y)K_s(dy)ds, \quad t \geq 0,$$

is a continuous square summable $(\mathcal{F}_t)$ martingale such that

$$\langle Z(\phi) \rangle_t = \int_0^t \int_{C^d} \gamma(s, K, y)\phi(y)^2 ds \quad \forall t \geq 0 \quad a.s.$$

If in addition we assume that the branching rate γ satisfies the following condition:

- *There are $h : [0, \infty) \times \widehat{\Omega}_H \to \mathbb{R}^d$ and $f : [0, \infty) \times \widehat{\Omega}_H \to \mathbb{R}$ which are $(\widehat{\mathcal{H}}_t)$-predictable and satisfy the same Lipschitz condition (1.4) as b, σ and $\|h(t, K, y)\| + |f(t, K, y)| \leq \Upsilon(p, K_t^*(1))$. Moreover, if $\bar{\Omega}' = (\Omega', \mathcal{F}', (\mathcal{F}_t'), \mathbb{Q})$ is a filtered probability space carrying on*

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$(\mathcal{F}'_t)$-*predictable processes* $(K_t : t \geq 0)$ *with sample paths in* Ω_H *a.s. and an* $\mathbb{R}^d$-*valued* $(\mathcal{F}'_t)$-*predictable process* $(Y_t : t \geq 0)$ *such that for* $t \geq 0$ $Y_t = Y_0 + M_t + \int_0^t b(s, K, Y)ds$ *where* M *is a continuous local martingale such that* $\langle M^i, M^j \rangle_t = \int_0^t (\sigma \sigma^*)(s, K, Y)_{ij}ds$ *a.s., then*

$$\gamma(t, K, Y) = \gamma(0, K, Y) + \int_0^t h(s, K, Y)dY(s) + \int_0^t f(s, K, Y)ds \quad \forall t \geq 0 \quad \mathbb{Q} - a.s.$$

then the martingale problem is well posed.

Proof. The first part of the theorem follows from the fact that $Z_t^N(\phi) = K_t^N(\phi) - \phi(\vec{0}) - S_t^N(\phi)$ together with Propositions 1.28, 1.29 and equation (1.36). The uniqueness part is a consequence of Theorem 5.6 (Perkins 1995.) ∎

Remark 1.32. (a) The additional technical condition needed in Theorem 1.31 to ensure that the martingale problem is well posed is restrictive. However it is satisfied by a large number of interesting examples (Perkins 1995, p. 50). In particular it holds for the branching rate given in Example (a) (page 16).

(b) In Theorem 1.31 we don't need the full strength of condition (Lip_γ) to prove that the limit points of the sequence (K^N) satisfy the martingale problem. Continuity (as opposed to Lipschitz continuity) and boundedness by some power $p \geq 1$ of the total mass suffice. □

Chapter 2

Path Properties of a One-Dimensional Superdiffusion with Interactions

2.1 Introduction and Statement of Results

In this Chapter we study some path properties of the solution of the historical stochastic equation (Perkins 1995, p. 47)

$$
Y_t = y_t + \int_0^t b(s, X_s, Y_s)\, ds
$$
$$
X_t(\phi) = \int \phi(Y_t)\gamma(t, X, Y)\, H_t(dy)
$$

(2.1)

in dimension $d = 1$. H is a one-dimensional historical Brownian motion and X is a superprocess with interactions. (2.1) shows that X and H have the same family structure. However, the path y (a Brownian path) is replaced by a path $Y(y)$ which is a Brownian motion with a drift depending on X. The term γ is a mass factor, but under suitable hypothesis (see condition (R_γ) in page 44) it can be interpreted as a branching rate (Remark 5.2 of Perkins 1995). An intuitive explanation of (2.1) is given in Chapter 0.

The present chapter is organized as follows. In this Section we review some basic results concerning historical stochastic analysis (Perkins 1993, 1995). We define what we mean by a solution of (2.1) and state Theorem 2.9, the main result of the chapter. The reader is referred to Chapter 0 for additional comments and motivation. In Section 2.2 we introduce and examine some auxiliary processes. These results, as well as their proofs, will be needed in the proof of the main theorem. In Section 2.3 we find a generalized Green's function representation of a localized version of the process X analog to (0.4). With this formula we shall be able to estimate the moments of X. In Section 2.4 we prove Theorem 2.9 using the results of the previous sections together with Kolmogorov's criterion. Finally, in Section 2.5 we give a non-trivial example in which the techniques developed in this Chapter can be successfully applied to discover some path properties of a one-dimensional superdiffusion in a random environment.

2.1.1 Main Result

We begin by introducing some basic notation and recalling the definition of historical Brownian motion.

If μ is a measure, $\mu(\phi) \equiv \int \phi \, d\mu$. Throughout this chapter $C \equiv C^1 = C([0,\infty), \mathbb{R})$ and $\mathcal{C}_t \equiv \mathcal{C}_t^1 = \sigma(y_s : s \le t)$, the canonical σ-field of C. If $y \in C$ we write y^t for the path y stopped at t, i.e. $y^t = y(t \wedge \cdot)$. Let

$$M_F(C)^t = \{\mu \in M_F(C) : y = y^t \quad \mu - a.a. \quad y\}, \quad t \ge 0.$$
$$\Omega_H = \{K \in C([0,\infty), M_F(C)) : K_t \in M_F(C)^t \quad \forall t \ge 0\},$$
$$H_t(\omega) = \omega(t) \quad \text{for} \quad \omega \in \Omega_H,$$
$$\mathcal{H}_t = \sigma(H_s : 0 \le s \le t), \quad \mathcal{H} = \sigma(H_s : s \ge 0).$$

$\bar{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ will denote a filtered probability space satisfying the usual hypotheses. $\mathcal{P}(\mathcal{F}_t)$ denotes the σ-field of $(\mathcal{F}_t)$-predictable sets in $[0,\infty) \times \Omega$ and

$$(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)) = (\Omega \times C, \mathcal{F} \times \mathcal{C}, (\mathcal{F}_t \times \mathcal{C}_t)).$$

Fix $0 \le t_1 < t_2 < ... < t_k$ and $\Psi \in C_b^2(\mathbb{R}^k)$. If $y \in C$, let

$$\bar{\Psi}(y) \equiv \bar{\Psi}(t_1, t_2, ..., t_k)(y) = \Psi(y(t_1), ..., y(t_k)).$$

Ψ_i and Ψ_{ij} denote the first and second order partials of Ψ.

$$\nabla \bar{\Psi}(t, y) \equiv \sum_{i=0}^{k-1} 1(t < t_{i+1}) \Psi_{i+1}(y(t \wedge t_1), ..., y(t \wedge t_k));$$

$$\frac{\bar{\Delta}}{2} \bar{\Psi}(t, y) \equiv \sum_{m=0}^{k-1} \sum_{l=0}^{k-1} 1(t < t_{m+1} \wedge t_{l+1}) \Psi_{m+1 \, l+1}(y(t \wedge t_1), ..., y(t \wedge t_k)).$$

Let

$$D_0 \equiv \bigcup_{m=1}^{\infty} \{\bar{\Psi}(t_1, t_2, ..., t_m) : 0 \le t_1 < t_2 < ... < t_m, \Psi \in C_0^\infty(\mathbb{R}^m)\} \cup \{1\}.$$

Definition 2.1 (One-dimensional Historical Brownian Motion). Let m be a finite Borel measure on $\mathbb{R}$. A predictable process $(H_t : t \ge 0)$ on $\bar{\Omega}$ with sample paths on Ω_H is a one-dimensional historical Brownian motion starting at m iff (H_t) satisfies the following martingale problem:

$(MP)_{1,1,0,0}^m$

$$\forall \bar{\psi} \in D_0 \qquad Z_t^H(\bar{\psi}) = H_t(\bar{\psi}) - m(\bar{\psi}) - \int_0^t \int \frac{1}{2} \bar{\Delta} \psi(s, y) H_s(dy) ds$$

is a continuous square integrable $\mathcal{F}_t$-martingale such that

$$\langle Z_t^H(\bar{\psi}) \rangle_t = \int_0^t \int \bar{\psi}(y)^2 H_s(dy) ds \quad \forall t \ge 0 \quad a.s. \qquad \square$$

There exists a process H satisfying the conditions of Definition 2.1 and it is unique in law. (Dawson & Perkins 1991). We picture H as an infinitesimal tree of branching one-dimensional Brownian motions.

Definition 2.2. Let $(K_t : t \geq 0)$ be a process on $\bar{\Omega}$ with paths on Ω_H. A set $\Lambda \subset [0, \infty) \times \widehat{\Omega}$ is $(K, \mathbb{P})$ evanescent (or K evanescent) iff $\Lambda \subset \Lambda_1$ where Λ_1 is $(\widehat{\mathcal{F}}_t^*)$-predictable and

$$\sup_{0 \leq u \leq t} 1_{\Lambda_1}(u, \omega, y) = 0 \quad K_t - a.e. \quad y \quad \forall t \geq 0 \quad \mathbb{P} - a.s.$$

A property holds $(K, \mathbb{P}) - a.e.$ (or $K - a.e.$) if it holds off a K-evanescent set. □

Definition 2.3. Let $(K_t : t \geq 0)$ be a process on $\bar{\Omega}$ with paths on Ω_H. A map $b : [0, \infty) \times \bar{\Omega} \to \mathbb{R}$ is K-integrable (respectively K-locally integrable) iff it is $(\widehat{\mathcal{F}}_t^*)$-predictable and $\int_0^t K_s(|b(s)|)ds < \infty$ (respectively, $\int_0^t |b(s)|ds < \infty$ $K_t - a.a.$ y) $\forall t \geq 0$ $\mathbb{P} - a.s.$ □

For technical reasons that will become apparent later we shall not look directly at (2.1) but rather at a historical version of it. We shall call such version $(HSE)_{b,\gamma}$ and it is defined below.

Definition 2.4. Suppose that H is a one-dimensional historical Brownian motion starting at $m \in M_F(\mathbb{R})$. Let

$$b : [0, \infty) \times M_F(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R},$$
$$\gamma : [0, \infty) \times M_F(\mathbb{R}) \times C \longrightarrow (0, \infty).$$

Define the projection $\bar{\Pi}_t : \Omega_H \to M_F(\mathbb{R})$ by $\bar{\Pi}_i(K)(A) = K_t(\{y : y(t) \in A\})$. We say that the pair (K, Y) solves the historical stochastic equation

$$\text{(a)} \qquad Y_t = y_t + \int_0^t b(s, \bar{\Pi}_s(K), Y_s)ds, \quad t \geq 0,$$

$$\text{(b)} \qquad K_t(\phi) = \int \phi(Y^t)\gamma(t, \bar{\Pi}.(K), Y)H_t(dy), \quad \phi \in b\mathcal{C}.$$

$(HSE)_{b,\gamma}$

iff

- $Y : [0, \infty) \times \widehat{\Omega} \to \mathbb{R}$ is $(\widehat{\mathcal{F}}_t^*)$-predictable and $K : [0, \infty) \times \Omega \to M_F(\mathbb{R})$ is $(\mathcal{F}_t)$-predictable.

- The map $|\bar{b}(s, \omega, y)| = |b(s, \bar{\Pi}_s(K)(\omega), Y_s(\omega, y))|$ is H-locally integrable.

- $(HSE)_{b,\gamma}$(a) holds up to an H-evanescent set in $[0, \infty) \times \widehat{\Omega}$, $(HSE)_{b,\gamma}$(b) holds for all $\phi \in b\mathcal{C}$ $\forall t \geq 0$ $\mathbb{P} - a.s.$

Pathwise uniqueness holds in $(HSE)_{b,\gamma}$ if whenever (K, Y) and $(\widetilde{X}, \widetilde{Y})$ are solutions of (HSE) then $Y^t = \widetilde{Y}^t$ except on an H-evanescent set in $[0, \infty) \times \widehat{\Omega}$ and $K = \widetilde{K}$ except on a $\mathbb{P}$-evanescent set in $[0, \infty) \times \Omega$.

To each K solution of $(HSE)_{b,\gamma}$ we can associate its one dimensional projection X defined by $X. \equiv \bar{\Pi}.(K)$. □

Remark 2.5. It is easy to modify the above definition in order to give a rigorous description of (2.1). It is then straightforward to verify that if K solves $(HSE)_{b,\gamma}$ then its projection $X. \equiv \bar{\Pi}.(K)$ solves (2.1).

The existence and uniqueness of solutions to $(HSE)_{\gamma,b}$ requires some Lipschitz conditions on the coefficients b, γ. Since their arguments include measures we need to introduce an appropriate metric.

Definition 2.6 (Vasershtein Metric). If (S, δ) is a metric space let

$$Lip(S) = \{\phi : S \to \mathbb{R} : \|\phi\|_\infty \le 1, \quad |\phi(x) - \phi(z)| \le \delta(x, z) \quad \forall x, z \in S\}.$$

The Vasershtein metric d_δ is the metric on $M_F(S)$ defined by

$$d_\delta(\mu, \nu) = \sup\{|\mu(\phi) - \nu(\phi)| : \phi \in Lip(S)\}.$$

This metric induces the weak topology on $M_F(S)$ (e.g. Ethier and Kurtz 1986, p. 150 Exercise 2).　□

The following conditions on the coefficients appearing in $(HSE)_{b,\gamma}$ will be frequently required.

(IC) $m(dx) \equiv \rho(x)dx$, where $\rho \in C_K(\mathbb{R})$ is Hölder-α for any $\alpha < \frac{1}{2}$, $\int_\mathbb{R} \rho(x)dx = 1$.

There is a nondecreasing function $\Upsilon : [0, \infty) \to [1, \infty)$ such that

(Lip) For any $\mu \in M_F(\mathbb{R})$

$$|b(t, \mu, x)| \le \Upsilon(t \vee \mu(1)) \quad \forall x \in \mathbb{R}. \tag{2.2}$$

Let d_e denote the Vasershtein metric on $M_F(\mathbb{R})$.

$$|b(t, \mu, x) - b(t, \nu, z)| \le \Upsilon(t \vee \mu(1) \vee \nu(1))(d_e(\mu, \nu) + |x - z|).$$

(B_γ) For any $X \in C([0, \infty), M_F(\mathbb{R}))$ and any $\forall y \in C$

$$\gamma(t, X, y)^{-1} \le \Upsilon(t \vee X_t^*(1)).$$

Moreover, there is a nondecreasing function $\Gamma : [0, \infty) \to [1, \infty)$ such that

$$\gamma(t, X, y) \le \Gamma(t)\left(1 + \int_0^t X_s^*(1)ds\right), \quad \forall X \in C([0, \infty), M_F(\mathbb{R})) \quad \forall y \in C.$$

The following Lipschitz condition on γ is more restrictive than (Lip).

(R_γ) Suppose that there are two functions $h, f : [0, \infty) \times M_F(\mathbb{R}) \times C \to \mathbb{R}$ such that

$$|h(t, \mu, y)| + |f(t, \mu, y)| \le \Upsilon(t \vee \mu(1)) \quad \forall y \in C, \tag{2.3}$$

$$|h(t, \mu, y) - h(t, \mu', y')| \le \Upsilon(t \vee \mu(1) \vee \mu'(1))(d_e(\mu, \mu') + \sup_{s \le t}|y(s) - y'(s)|). \tag{2.4}$$

$$|f(t, \mu, y) - f(t, \mu', y')| \le \Upsilon(t \vee \mu(1) \vee \mu'(1))(d_e(\mu, \mu') + \sup_{s \le t}|y(s) - y'(s)|). \tag{2.5}$$

and with the following property. Recall that $\bar{\Pi}_t : \Omega_H \to M_F(\mathbb{R})$ be given by

$$\bar{\Pi}_t(K)(A) = K_t(\{y : y(t) \in A\}).$$

If a space $\bar{\Omega}' = (\Omega', \mathcal{F}', (\mathcal{F}'_t), \mathbb{P}')$ carries an $(\mathcal{F}'_t)$-predictable process (K_t) with sample paths in Ω_H a.s. and an $\mathbb{R}$-valued $(\mathcal{F}'_t)$-predictable process (Y_t) such that

$$M(t) = Y(t) - Y(0) - \int_0^t b(s, \bar{\Pi}_s(K), Y_s)ds, \quad t \ge 0$$

is a Brownian motion then

$$\gamma(t, \bar{\Pi}(K), Y) = 1 + \int_0^t h(s, \bar{\Pi}(K), Y)dY(s) + \int_0^t f(s, \bar{\Pi}(K), Y)ds \quad \forall t \ge 0 \quad \mathbb{P}' - a.s.$$

Remark 2.7. Condition (R_γ) seems strange at a first glance. It ensures that γ can be interpreted as a branching rate (Remark 5.2 of Perkins 1995). □

The next Proposition is fundamental.

Proposition 2.8. *Assume b satisfies (Lip) and γ satisfies (B_γ), (R_γ). Then $(HSE_{b,\gamma})$ has a pathwise unique solution.*

Proof. See Theorem 4.10 of Perkins (1995). ∎

Before stating the main result of this Chapter we give some examples of coefficients b and γ satisfying the basic hypothesis.

Example. (a) $\gamma \equiv 1$ satisfies (R_γ) and (B_γ).

(b) Assume $\bar{\gamma} \in C_b^{1,2}([0,\infty))$ is a non-negative function such that $\frac{\partial \bar{\gamma}}{\partial s}(s,\cdot)$ and $\frac{\partial^2 \bar{\gamma}}{\partial x^2}(s,\cdot)$ are Lipschitz continuous with a uniform Lipschitz constant for s in compacts. Then $\gamma(t,X,y) = \bar{\gamma}(t,y(t))$ satisfies (R_γ) and (B_γ). This is proved in Example 4.4 Perkins (1995). In this case the mass of each particle depends only on time and the position of the particle, and not on the rest of the population.

(c) Let $p_s(x)$ be the one dimensional heat kernel. Fix $\varepsilon, \alpha > 0$ and set

$$\gamma(t,X,y) = \exp\left(-\int_0^t \int p_\varepsilon(y_t - x)X_s(dx)e^{-\alpha(t-s)}ds\right).$$

It is not obvious that γ satisfies (R_γ). This is proved in p. 51 of Perkins(1995).

(d) Let $\bar{b} : \mathbb{R} \to \mathbb{R}$ be bounded Lipschitz continuous. Then

$$b(\mu, x) = \int \bar{b}(x-z)\mu(dz)$$

satisfies (Lip). See Example 4.2 Perkins (1995) for a proof. In this model a particle at z in a population μ exerts a drift $\bar{b}(x-z)\mu(dz)$ on a particle at x.

(e) More examples can be found in Perkins (1995, p.49). □

The following is the main result of this Chapter.

Theorem 2.9. *Let H be a one dimensional historical Brownian motion starting at m. Assume m satisfies (IC), b satisfies (2.2) and γ satisfies (B_γ) and (R_γ). Let K be a solution of $(HSE)_{b,\gamma}$ and suppose that X is its one-dimensional projection, i.e. $X. = \bar{\Pi}.(K)$. (See Definition 2.4.) Then*

$$X_t(dx) = u(t,x)dx \quad for\ all \quad t \geq 0 \quad \mathbb{P} - a.s. \tag{2.6}$$

where $u(\cdot,\cdot)$ is an a.s. jointly continuous (adapted) function. In addition, u is Hölder continuous in t with any exponent $\alpha < 1/4$ and in x with any exponent $\alpha < 1/2$ a.s.

Remark 2.10. (a) The case $b = 0$ and $\gamma \equiv 1$, i.e. when X is one-dimensional super Brownian motion, of Theorem 2.9 was proved independently by Reimers (1989) and Konno and Shiga (1988).

(b) Note that b is not assumed to satisfy (Lip) (we only assume the boundedness condition (2.2)) so pathwise uniqueness may not hold. □

2.1.2. Historical Stochastic Calculus

The results in this Subsection will be used repeatedly. The basic reference is Chapter 2 of Perkins (1995).

Definition 2.11. Let $(E, \|\ \|)$ be a normed linear space. Let $f : [0, \infty) \times \widehat{\Omega} \to E$. A bounded $(\mathcal{F}_t)$-stopping time T is a reducing time for f iff $1(t \leq T)\|f(t, \omega, y)\|$ is uniformly bounded. The sequence $\{T_n\}$ reduces f iff each T_n reduces f and $T_n \uparrow \infty$ $\mathbb{P} - a.s.$ If such a sequence exists, we say that f is locally bounded. $\qquad\square$

Definition 2.12. Let m be a finite Borel measure on $\mathbb{R}$ and let

$$\hat{\gamma} : [0, \infty) \times \widehat{\Omega} \to (0, \infty),$$
$$\hat{b} : [0, \infty) \times \widehat{\Omega} \to \mathbb{R},$$
$$\hat{g} : [0, \infty) \times \widehat{\Omega} \to \mathbb{R},$$

be $(\widehat{\mathcal{F}}_t^*)$-predictable. Assume that $\Lambda = (\hat{\gamma}, \hat{b}, \hat{g}\hat{\gamma}^{-1}1(\hat{g} \neq 0))$ is locally bounded. A predictable process $(K_t : t \geq 0)$ satisfies $(MP)^m_{\hat{\gamma}, 1, \hat{b}, \hat{g}}$ on $\widehat{\Omega}$ if and only if $K_t \in M_F(C)^t$ for all $t \geq 0$ $a.s.$ and

$$\forall \bar{\psi} \in D_0 \qquad Z_t^K(\bar{\psi}) = K_t(\bar{\psi}) - m(\bar{\psi})$$
$$- \int_0^t \int \left[\bar{\psi}(y)\hat{g}(s, \omega, y) + \nabla\bar{\psi}(s, y)\hat{b}(s, \omega, y) + \frac{1}{2}\bar{\Delta}\psi(s, y) \right] K_s(dy)ds$$

is a continuous square integrable $\mathcal{F}_t$-martingale such that

$$\langle Z_t^K(\bar{\psi})\rangle_t = \int_0^t \int \hat{\gamma}(s, \omega, y)\bar{\psi}(y)^2 K_s(dy)ds \quad \forall t \geq 0 \quad a.s. \qquad\square$$

Assume that K satisfies $(MP)^m_{\hat{\gamma}, 1, \hat{b}, \hat{g}}$. Let (T_k) be a reducing sequence for Λ. If $\bar{\phi}_n \in D_0$, $\bar{\phi}_n \xrightarrow{bp} \phi$ (bounded pointwise convergence) then dominated convergence and local boundedness of $\hat{\gamma}$ imply

$$\langle Z^K(\bar{\phi}_n - \bar{\phi}_m)\rangle_t \longrightarrow 0,$$
$$\text{as} \quad n, m \to \infty \quad \forall t \geq 0 \quad a.s.$$

Therefore there is an *a.s.* continuous adapted process $(Z_t^K(\phi) : t \geq 0)$ such that

$$\sup_{0 \leq s \leq k} |Z_s^K(\bar{\phi}_n) - Z_s^K(\phi)| \xrightarrow{\mathbf{P}} 0$$
$$\text{as} \quad n \to \infty \quad \forall k > 0.$$

and

$$\sup_{0 \leq s \leq T_k} |Z_s^K(\bar{\phi}_n) - Z_s^K(\phi)| \xrightarrow{L^2} 0$$
$$\text{as} \quad n \to \infty \quad \forall k \in \mathbf{N}.$$

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See Perkins (1993, 1995) for details. $(Z_t^K(\phi))_t$ is a continuous local $(\mathcal{F}_t)$-martingale with square function

$$\langle Z^K(\psi)\rangle_t = \int_0^t K_s(\gamma_s \psi^2)ds. \tag{2.7}$$

Since D_0 is bp-dense in $b\mathcal{C}$, Z_t^K can be extended to an orthogonal martingale measure $\{Z_t^K(\phi) : t \geq 0, \; \phi \in b\mathcal{C}\}$ (Perkins 1993, 1995). This implies that if $\psi : [0,\infty) \times \Omega \times C \to \mathbb{R}$ is $\mathcal{P}(\mathcal{F}_t) \times \mathcal{C}$-measurable and satisfies

$$\mathbb{P}\left[\int_0^t \int \gamma(s,\omega,y)\psi(s,\omega,y)^2 K_s(dy)ds\right] < \infty \quad \forall t > 0$$

then we may define the stochastic integral

$$Z_t^K(\psi) \equiv \int_0^t \int_C \psi(s,\omega,y)Z^K(ds,dy).$$

$Z_t^K(\psi)$ is a continuous square-summable martingale with square function (2.7). More generally, if

$$D(Z) = \Big\{\psi : [0,\infty) \times \Omega \times C \to \mathbb{R} : \psi \;\; \text{is } \mathcal{P}(\mathcal{F}_t) \times \mathcal{C}\text{-measurable and}$$

$$\int_0^t \int \gamma(s,\omega,y)\psi(s,\omega,y)^2 K_s(dy)ds < \infty \quad \forall t > 0 \quad \mathbb{P} - a.s.\Big\}$$

then $Z_t^K(\psi)$ can still be defined for $\psi \in D(Z)$. In this case it is a continuous local martingale with square function (2.7). For a definition of martingale measures and the construction of the associated infinite dimensional stochastic integrals the reader is referred to the excellent book Walsh (1986). A word of warning to the reader familiar with the applications of stochastic calculus in finance. What we call martingale measures are L^2-valued measures on C. They are totally different from the "martingale measures" usually employed in finance (these are just Girsanov transformations). The fact to be remembered while reading this chapter is that $Z_t^K(\phi)$ is a (local) martingale for a large class of integrands $\phi = \phi(s,\omega,y)$.

Definition 2.13. If $(K_t : t \geq 0)$ is a r.c.l.l. $(\mathcal{F}_t)$-adapted $M_F(C)$-valued process such that $(K_t(1) : t \geq 0)$ is an $(\mathcal{F}_t)$-martingale, and T is a bounded $(\mathcal{F}_t)$-stopping time, define a probability $\widehat{\mathbb{P}}_T^K$ on $\widehat{\Omega}$ by

$$\widehat{\mathbb{P}}_T^K[A \times B] = \frac{\mathbb{P}[1_A K_T(B)]}{\mathbb{P}[K_0(1)]}, \quad A \in \mathcal{F}, \quad B \in \mathcal{C}.$$

We call $\widehat{\mathbb{P}}_T^K$ the Campbell measure associated with K and T, or simply Campbell measure when K and T are clear from the context. $\qquad \qquad \square$

Proposition 2.14. *Let K be a solution of $(MP)_{\hat{\gamma},1,\hat{b},0}^m$. (Note that $\hat{g} \equiv 0$). Then*

(a) *$K \in \Omega_H$ $\mathbb{P} - a.s.$ and K is $(\mathcal{F}_t)$-adapted.*

(b) *If T reduces $(\hat{\gamma}, \hat{b})$, then under the Campbell measure $\widehat{\mathbb{P}}_T^K$,*

$$n(t,\omega,y) = y(t) - y(0) - \int_0^{t \wedge T} \hat{b}(s,\omega,y)ds, \quad t \geq 0,$$

is a $(\widehat{\mathcal{F}}_t^*)$-Brownian motion stopped at T.

Proof. See Theorem 2.6 of Perkins (1995). ∎

Let K, T be as in Proposition 2.14. Under $\widehat{\mathbb{P}}_T^K$, y is an $(\widehat{\mathcal{F}}_t^*)$-semimartingale. Therefore

$$\int_0^t \sigma(s, \omega, y) dy(s)$$

may be defined for the class

$$D(I, K) = \left\{ \sigma : [0, \infty) \times \Omega \times C \to \mathbb{R} : \quad \sigma \quad \text{is } (\widehat{\mathcal{F}}_t^*)\text{-predictable},\right.$$

$$\left. \int_0^t \sigma(s, \omega, y)^2 \, ds < \infty \quad K_t - a.a. \quad y \quad \forall t > 0 \quad a.s. \right\}$$

Proposition 2.15. *Let K be a solution of* $(MP)^m_{\hat{\gamma}, 1, \hat{b}, \hat{g}}$.

(a) *If $\sigma \in D(I, K)$ there is an $\mathbb{R}$-valued $(\widehat{\mathcal{F}}_t^*)$-predictable process $I_K(\sigma, t, \omega, y)$ such that for all reducing $(\mathcal{F}_t)$-stopping times T*

$$I_K(\sigma, t, \omega, y) = \int_0^t \sigma(s, \omega, y) dy(s) \quad \forall t \geq 0 \quad \widehat{\mathbb{P}}_T^K - a.s.$$

We sometimes write $I(\sigma, t)$ instead of $I_K(\sigma, t, \omega, y)$.

(b) *If $\tilde{I}(\sigma, s, \omega, y)$ is another such process, then*

$$\tilde{I}(\sigma, s, \omega, y) = I_K(\sigma, s, \omega, y) \quad \forall s \leq t \quad K_t - a.a. \quad y \quad \forall t \geq 0 \quad \mathbb{P} - a.s.$$

Proof. This is a special case of Theorem 2.12 (Perkins 1995). ∎

Notation 2.16. If $b : [0, \infty) \times \widehat{\Omega} \to \mathbb{R}$ is universally measurable, let

$$V(b, t) = \begin{cases} \int_0^t b(s, \omega, y) ds & \text{if} \quad \int_0^t |b(s, \omega, y)| ds < \infty. \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.17 (Itô's lemma). *Assume K satisfies $(MP)^m_{\hat{\gamma}, 1, \hat{b}, 0}$, $\sigma : [0, \infty) \times \widehat{\Omega} \to \mathbb{R}$ and $b : [0, \infty) \times \widehat{\Omega} \to \mathbb{R}$ are $(\widehat{\mathcal{F}}_t^*)$-predictable, $|\sigma|^2 + |b|$ is K-locally integrable and $Y_0 : \widehat{\Omega} \to \mathbb{R}$ is $\widehat{\mathcal{F}}_0^*$-measurable. Let $Y_t = Y_0 + I(\sigma, t) + V(b, t)$ (using the notation of Proposition 2.15 and Notation 2.16) and $\vec{Y}_t = (t, Y_t)$. If $\psi \in C_b^{1,2}([0, \infty) \times \mathbb{R})$, the space of bounded continuous functions $\psi(t, x)$ with a bounded continuous derivative in t and two bounded continuous derivatives in x, then*

$$\psi(\vec{Y}_t) = \psi(\vec{Y}_0) + I\left(\frac{\partial \psi}{\partial x}(\vec{Y}) \cdot \sigma, t\right) + V\left(\frac{\partial \psi}{\partial t}(\vec{Y}) + \frac{\partial \psi}{\partial x}(\vec{Y}) \cdot b + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}(\vec{Y}) \cdot \sigma^2, t\right)$$

up to a K-evanescent set of (t, ω, y).

Proof. This proposition follows from the usual Itô's lemma and Proposition 2.15. See Lemma 3.18 of Perkins (1993). ∎

Proposition 2.18 (Historical Itô's lemma). *Suppose K satisfies $(MP)^m_{\hat{\gamma},1,\hat{b},\hat{g}}$. Assume $\sigma \in D(I,K)$, $b \in \mathbb{R}$ are $(\widehat{\mathcal{F}}^*_t)$-predictable and $|\sigma|^2 + |b|$ is K-locally integrable. Let $Y_0 : \widehat{\Omega} \to \mathbb{R}$ be $\widehat{\mathcal{F}}^*_0$-measurable, $Y_t = Y_0 + I(\sigma,t) + V(b,t)$ (using the notation of Proposition 2.15 and Notation 2.16) and $\vec{Y}_t = (t, Y_t)$. If $\psi \in C^{1,2}_b([0,\infty) \times \mathbb{R})$, then $K_t(\psi(\vec{Y}_t))$ is an a.s. continuous $(\mathcal{F}_t)$-semimartingale and satisfies*

$$K_t(\psi(\vec{Y}_t)) = K_0(\psi(\vec{Y}_0)) + \int_0^t \int \psi(\vec{Y}_s) Z^K(ds, dy)$$

$$+ \int_0^t \int \left[\psi(\vec{Y}_s)\hat{g}(s) + \frac{\partial \psi}{\partial s}(\vec{Y}_s) + \frac{\partial \psi}{\partial x}(\vec{Y}_s)(b(s) + \sigma(s)\hat{b}(s)) \right.$$

$$\left. + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2}(\vec{Y}_s) \cdot \sigma(s)^2 \right] K_s(dy) ds \quad \forall t \geq 0 \quad a.s.$$

Proof. This is a special case of Theorem 2.14 in Perkins (1995). ∎

We shall also need the following version of Proposition 2.18.

Proposition 2.19. *Suppose K satisfies $(MP)^m_{\hat{\gamma},1,\hat{b},0}$. Let $\sigma \in D(I,K)$, $b \in \mathbb{R}$ be $(\widehat{\mathcal{F}}^*_t)$-predictable and assume $|\sigma|^2 + |b|$ is K-locally integrable. Assume also*

$$\sup_{0 \leq s \leq t} K_s(V^*(b,s)^2)) < \infty \quad \forall t > 0 \quad a.s.$$

Then $I(\sigma,t) \in D(Z)$ and

$$K_t(V(b,t) + I(\sigma,t)) = \int_0^t \int (V(b,s) + I(\sigma,s)) Z^K(ds,dy) + \int_0^t \int \left(\sigma(s)\hat{b}(s) + b(s) \right) K_s(dy) ds$$

$$\forall t \geq 0 \quad a.s.$$

Proof. See Theorem 2.8 and Proposition 2.13 of Perkins (1995) for the proof. ∎

Proposition 2.20 (Lévy's modulus of continuity). *Let $h(t) = \sqrt{t \log^+(1/t)}$. Define*

$$L(\delta,c) = \{ y \in C : |y(t) - y(s)| \leq c h(t-s) \quad \forall s,t \geq 0 \quad \text{satisfying} \quad 0 \leq t - s \leq \delta \}.$$

Assume K solves $(MP)^m_{\hat{\gamma},1,\hat{b},\hat{g}}$. Let $\sigma \in D(I,K)$ be $(\widehat{\mathcal{F}}^*_t)$-predictable and locally bounded. Define $\widehat{K}_t(\omega) \in M_F(C)$ by $\widehat{K}_t(\cdot) = K_t(\{ y : I(\sigma,\cdot,\omega,y)^t \in \cdot \})$. Assume T reduces $(\hat{\gamma}, \hat{\gamma}^{-1}, \hat{b}, \sigma)$ and let

$$\Theta = \sup\{\sigma^2(s,\omega,y) : \quad 0 < s < T, \quad (\omega, y) \in \widehat{\Omega}\}.$$

(a) *If $c > 2\sqrt{\Theta}$, there is a $\delta(\omega,c) > 0$ a.s. such that*

$$\widehat{K}_t(L(\delta,c)^c) = 0 \quad \forall t \in [0,T].$$

(b) *For each $\gamma, \Theta_0, b, q \in (0,\infty)$ and $c > 2\sqrt{\Theta_0}$ there is a non-decreasing function $p : [0,\infty) \to [0,1]$ such that $\lim_{\lambda \to 0} p(\lambda) = 0$ and if*

$$\gamma \leq \inf\{\hat{\gamma}(t,\omega,y) : 0 < t < T(\omega), \quad (\omega, y) \in \widehat{\Omega}\}, \qquad m(1) \leq q,$$

$$\sup\{|\sigma \cdot \hat{b}(t,\omega,y)| : 0 < t < T(\omega), \quad (\omega, y) \in \widehat{\Omega}\} \leq b, \qquad \Theta \leq \Theta_0,$$

*then we may choose $\delta(\omega,c)$ in **(a)** so that*

$$\mathbb{P}(\delta \leq \lambda) \leq p(\lambda).$$

Proof. This is a special case of Corollary 3.3 of Perkins (1995). ∎

We end this Subsection stating some useful estimates.

Proposition 2.21. *Recall that H denotes a one-dimensional historical Brownian motion starting at m. There is a one-dimensional Brownian motion (B_t) such that*

$$H_t(1) = m(1) + \int_0^t \sqrt{H_s(1)} \, dB_s.$$

$(H_t(1) : t \geq 0)$ is the continuous time continuous state space branching process studied by Feller (1951). Hence, for any $p \in \mathbf{N}$

$$\mathbb{P}[H_t^*(1)^p] < \infty \quad \forall t > 0.$$

More generally, assume that $\hat{g}$ is bounded above and let K be a solution of $(MP)^m_{\hat{\gamma},1,\hat{b},\hat{g}}$. Then, for any $p \geq 1$

$$\mathbb{P}[K_t^*(1)^p] < \infty \quad \forall t > 0.$$

Proof. See Lemma 2.1 of Perkins (1995). ∎

2.2 Some Auxiliary Processes

Proposition 2.22. *Let b and γ be as in Theorem 2.9, and h, f as in (R_γ). If we define*

$$\bar{b}(t,\omega,y) = b(t,\bar{\Pi}_t(K)(\omega),y) + \frac{h(t,\bar{\Pi}.(K)(\omega),y)}{\gamma(t,\bar{\Pi}.(K)(\omega),Y(\omega,y))},$$

$$\bar{g}(t,\omega,y) = \frac{f(t,\bar{\Pi}.(K)(\omega),y)}{\gamma(t,\bar{\Pi}.(K)(\omega),y)} + \frac{h(t,\bar{\Pi}.(K)(\omega),y) \cdot b(t,\bar{\Pi}_t(K)(\omega),y)}{\gamma(t,\bar{\Pi}.(K)(\omega),y)},$$

$$\hat{\gamma}(t,\omega,y) = \gamma(t,\bar{\Pi}.(K)(\omega),y)$$

then the unique solution of the historical strong equation $(HSE)_{b,\gamma}$ solves the martingale problem $(MP)^m_{\hat{\gamma},1,\bar{b},\bar{g}}$ defined in page 46.

Proof. The proof is essentially a lengthy application of historical Itô's lemma. See Theorem 5.1 of Perkins (1995) for the details. ∎

Remark 2.23. In the remainder of the chapter we will only assume that K solves the martingale problem $(MP)^m_{\hat{\gamma},1,\bar{b},\bar{g}}$. *In fact we shall abuse the notation and suppose that K is a solution of the martingale problem (not necessarily obtained as a solution of the strong equation $(HSE)_{b,\gamma}$).*

Proposition 2.24. *If we define*

$$\widetilde{K}_t(\phi) = \int \frac{\phi(y)}{\hat{\gamma}(t,\omega,y)} K_t(dy), \quad t \geq 0, \quad \phi \in b\mathcal{C},$$

then $\widetilde{K}$ satisfies $(MP)^m_{1,1,\hat{b},0}$, where $\hat{b}(s,\omega,y) = b(s,\bar{\Pi}_s(K)(\omega),y)$.

Proof. This follows from Proposition 2.22 and it is shown during the proof of Theorem 5.6 on p. 70 of Perkins (1995). ∎

The following hypothesis will be in force through Sections 2.2 and 2.3.

(UB) There is a constant $c_{UB} > 0$ such that

$$|\hat{b}(s,\omega,y)| \leq c_{UB} \quad \forall s \geq 0 \quad \forall y \in C \quad \mathbb{P} - a.s.$$

We need to introduce yet one more historical process. By Proposition 2.24 the results of Section 2.1 concerning stochastic integration are at our disposal. In particular we may define the stochastic integrals $I_{\widetilde{K}}$.

Definition 2.25. Note that (UB) implies $\hat{b} \in D(I, \widetilde{K})$. Let

$$\mathcal{E}_t(1,\omega,y) \equiv \exp\left(-I_{\widetilde{K}}(\hat{b},t,\omega,y) + \frac{1}{2}\int_0^t \hat{b}(s,\omega,y)^2 ds \right)$$

$$= \exp\left(-\int_0^t \hat{b}(s,\omega,y)dy(s) + \frac{1}{2}\int_0^t \hat{b}(s,\omega,y)^2 ds \right),$$

(observe that $\mathcal{E}_t(1,\omega,y)$ is unique up to $(\widetilde{K},\mathbb{P})$-evanescent sets) and define

$$J_t(\phi) = \int \phi(y)\mathcal{E}_t(1,\omega,y)\widetilde{K}_t(\omega)(dy), \quad t \geq 0, \quad \phi \in b\mathcal{C}, \tag{2.8}$$

$$j_t(\psi) = \int \psi(y_t)\mathcal{E}_t(1,\omega,y)\widetilde{K}_t(\omega)(dy), \quad t \geq 0, \quad \psi \in b\mathcal{B}, \tag{2.9}$$

$$= \Pi_t(J)(\psi). \quad \square$$

The usefulness of this auxiliary processes depends in part on the following lemmas. The reader may want to look ahead at Proposition 2.42 to see why the process J is relevant.

Lemma 2.26. *There is a $(\widehat{\mathcal{F}}_t^*)$-predictable $\widetilde{K}$-version of $\mathcal{E}_t(1,\omega,y)$ which is locally bounded. We call such a version $\tilde{\mathcal{E}}_t(1,\omega,y)^{-1}$.*

Proof. Apply Proposition 2.20 with $(K,\sigma) = (\widetilde{K},\hat{b})$. Note that since $\widetilde{K}$ solves $(MP)_{1,1,\hat{b},0}^m$ and $\hat{b}$ is uniformly bounded, the hypotheses of Proposition 2.20 are satisfied with $\Theta = 1$, $T = \infty$, $c = 3$. Hence

$$\widetilde{K}_t(\{y : I_{\widetilde{K}}(\hat{b},\cdot)^t \in L(\delta,3)^c\}) = 0 \quad \forall t > 0.$$

Therefore

$$I_{\widetilde{K}}(\hat{b},\cdot)^t \in L(\delta,3) \quad \widetilde{K}_t - a.e.y \quad \forall t > 0. \tag{2.10}$$

Define

$$V_k = \inf\{s > 0 : \widetilde{K}_s(\{y : I_{\widetilde{K}}(\hat{b},\cdot)^s \in L(k^{-1},3)^c\}) > 0\} \wedge k.$$

Note that $V_k \uparrow \infty$ $\mathbb{P} - a.s.$ Indeed, since $\delta > 0$ $a.s.$ $V_k(\omega) = k$ for all $k > \delta(\omega)^{-1}$. Hence (2.10) implies

$$1(t < V_k)\mathcal{E}_t(1, \omega, y) \leq \exp\left(3(kt+1)\sqrt{\frac{1}{k}\log^+(k)} + (kt+1)\frac{c_{UB}^2}{2}\right) \quad \widetilde{K}_t - a.e.y \quad \forall t > 0 \quad a.s.$$

$$(2.11)$$

Recall that evanescent sets are predictable and that we work with the universally complete σ-field $(\widehat{\mathcal{F}}_t^*)$. Modify $\mathcal{E}_t(1, \omega, y)$ on a $(\widetilde{K}, \mathbb{P})$-evanescent set to obtain the desired result. ∎

Remark 2.27. Since $\widetilde{K}$ and J are equivalent, (2.11) gives also a J-version of $\mathcal{E}_t(1, \omega, y)$ which is $(\widehat{\mathcal{F}}_t^*)$-predictable and locally bounded. □

We are now able to give a martingale problem characterization of J.

Proposition 2.28. *The historical process J defined in (2.8) satisfies the martingale problem* $(MP)_{\widetilde{\mathcal{E}}^{-1}, 1, 0, 0}^m$. *That is,*

$$\forall \bar{\Psi} \in D_0 \quad Z_t^J(\bar{\Psi}) = J_t(\bar{\Psi}) - m(\bar{\Psi}) - \int_0^t \int \frac{\bar{\Delta}}{2}\bar{\Psi}(s, y)J_s(dy)ds$$

is a continuous square integrable $(\mathcal{F}_t)$-martingale starting at zero with

$$\langle Z^J(\bar{\Psi})\rangle_t = \int_0^t \int \tilde{\mathcal{E}}_s(1, \omega, y)^{-1}\bar{\Psi}(y)^2 J_s(dy)ds \quad \forall t \geq 0 \quad \mathbb{P} - a.s.$$

Proof. Pick $\psi_1, ..., \psi_k \in C_0^\infty(\mathbb{R})$, $0 \leq t_1 < t_2 < ... < t_k$ and set $\bar{\Psi} = \prod_{i=1}^k \psi_i$. We sometimes write $\mathcal{E}_t(1) \equiv \mathcal{E}_t(1, \omega, y)$ and $\hat{b}(s) = \hat{b}(s, \omega, y)$. By Itô's lemma 2.17 and since $y_s - y_0 - \int_0^s \hat{b}(r, \omega, y)dr$ is a Brownian motion on $(\widehat{\Omega}, \widehat{\mathbb{P}}_t^{\widetilde{K}})$,

$$\psi_i(y(t \wedge t_i)) = \psi_i(y_0) + \int_0^t \left[\psi_i'(y(s \wedge t_i))1(s < t_i)dy(s) + \frac{1}{2}\psi_i''(y(s \wedge t_i))1(s < t_i)\right]ds$$

$$(\widetilde{K}, \mathbb{P}) - a.e.$$

and

$$\mathcal{E}_t(1) = 1 - \int_0^t \hat{b}(s)\mathcal{E}_s(1)dy(s) + \int_0^t \hat{b}(s)^2\mathcal{E}_s(1)ds \quad (\widetilde{K}, \mathbb{P}) - a.e.$$

Integration by parts (also justified by Itô's lemma 2.17) yields

$$\mathcal{E}_t(1, \omega, y)\bar{\Psi}(y^t) = \bar{\Psi}(y^0) - \int_0^t \bar{\Psi}(y^s)\hat{b}(s, \omega, s)\mathcal{E}_s(1, \omega, y)dy(s)$$

$$+ \int_0^t \bar{\Psi}(y^s)\hat{b}(s, \omega, s)^2\mathcal{E}_s(1, \omega, y)ds$$

$$+ \sum_{i=1}^k \int_0^t \prod_{\substack{j=1 \\ j \neq i}}^k \psi_j(y(s \wedge t_j))\psi_i'(y(s \wedge t_i))1(s < t_i)\mathcal{E}_s(1, \omega, y)dy(s)$$

$$+ \sum_{i=1}^{k} \int_0^t \prod_{\substack{j=1 \\ j \neq i}}^{k} \psi_j(y(s \wedge t_j)) \frac{1}{2} \psi_i''(y(s \wedge t_i)) 1(s < t_i) \mathcal{E}_s(1, \omega, y) ds$$

$$+ \frac{1}{2} \sum_{i=1}^{k} \sum_{l=1}^{k} \int_0^t \prod_{\substack{j=1 \\ j \neq i,l}}^{k} \psi_j(y(s \wedge t_j)) \psi_i'(y(s \wedge t_i)) \psi_l'(y(s \wedge t_l)) 1(s < t_i) 1(s < t_l) \mathcal{E}_s(1, \omega, y) ds$$

$$- \sum_{i=1}^{k} \int_0^t \prod_{\substack{j=1 \\ j \neq i}}^{k} \psi_j(y(s \wedge t_j)) \psi_i'(y(s \wedge t_i)) 1(s < t_i) \hat{b}(s, \omega, y) \mathcal{E}_s(1, \omega, y) ds$$

$$= \bar{\Psi}(y^0) + \int_0^t \left[\nabla \bar{\Psi}(s, y) - \bar{\Psi}(y^s) \hat{b}(s, \omega, y) \right] \mathcal{E}_s(1, \omega, y) dy(s)$$

$$+ \int_0^t \left[\bar{\Psi}(y^s) \hat{b}(s, \omega, y)^2 - \nabla \bar{\Psi}(y, s) \hat{b}(s, \omega, y) \right] \mathcal{E}_s(1, \omega, y) ds$$

$$+ \int_0^t \frac{\bar{\Delta}}{2} \bar{\Psi}(s, y) \mathcal{E}_s(1, \omega, y) ds \quad (\widetilde{K}, \mathbb{P}) - a.e.$$

Hence, by historical Itô's lemma 2.19

$$J_t(\bar{\Psi}) \equiv \widetilde{K}_t(\mathcal{E}_t(1) \bar{\Psi})$$

$$= \widetilde{K}_0(\bar{\Psi}) + \int_0^t \int \mathcal{E}_s(1, \omega, y) \bar{\Psi}(y^s) Z^{\widetilde{K}}(ds, dy)$$

$$+ \int_0^t \int \left[\left(\nabla \bar{\Psi}(s, y) - \bar{\Psi}(y^s) \hat{b}(s, \omega, y) \right) \hat{b}(s, \omega, y) \right.$$

$$+ \left. \left(\bar{\Psi}(y^s) \hat{b}(s, \omega, y)^2 - \nabla \bar{\Psi}(y, s) \hat{b}(s, \omega, y) \right) \right] \mathcal{E}_s(1, \omega, y) \widetilde{K}_s(dy) ds$$

$$+ \int_0^t \int \mathcal{E}_s(1, \omega, y) \frac{\bar{\Delta}}{2} \bar{\Psi}(s, y) \widetilde{K}_s(dy) ds$$

$$= J_0(\bar{\Psi}) + Z_t^J(\bar{\Psi}) + \int_0^t \int \frac{\bar{\Delta}}{2} \bar{\Psi}(s, y) J_s(dy) ds \quad \forall t \geq 0 \ a.s.$$

where

$$Z_t^J(\bar{\Psi}) = \int_0^t \int \mathcal{E}_s(1, \omega, y) \bar{\Psi}(y^s) Z^{\widetilde{K}}(ds, dy)$$

is a square integrable martingale with

$$\langle Z^J(\bar{\Psi}) \rangle_t = \left\langle \int_0^t \int \mathcal{E}_s(1, \omega, y) \bar{\Psi}(y^s) Z^{\widetilde{K}}(ds, dy) \right\rangle_t$$

$$= \int_0^t \int \mathcal{E}_s(1, \omega, y)^2 \bar{\Psi}(y^s)^2 \widetilde{K}_s(dy) ds$$

$$= \int_0^t \int \widetilde{\mathcal{E}}_s(1, \omega, y)^{-1} \bar{\Psi}(y^s)^2 J_s(dy) ds.$$

∎

Remark 2.29. Proposition 2.28 allows us to use J-historical stochastic calculus. In particular, Proposition 2.14 implies that $y_{\cdot \wedge t}$ is a $(\widehat{\mathcal{F}}_t^*)$-Brownian motion under the law $\widehat{\mathbb{P}}_t^J$. □

Lemma 2.30. *There is a J-version of $\tilde{\mathcal{E}}_t(1, \omega, y)$ such that*

$$\tilde{\mathcal{E}}_t(1, \omega, y) = \exp\left(\int_0^t \hat{b}(s, \omega, y) dy(s) - \frac{1}{2}\int_0^t \hat{b}(s, \omega, y)^2 ds\right) \quad (J, \mathbb{P}) - a.e.$$

We abuse the notation and call also that version $\tilde{\mathcal{E}}_t(1, \omega, y)$.

Proof. Let T be a bounded predictable $(\mathcal{F}_t)$-stopping time. By the invariance of the stochastic integral under change of law (Dellacherie and Meyer, Theorem VIII.12) the stochastic integral $\int_0^{\cdot} \hat{b}(s, \omega, y) dy(s)$ calculated under the law $\widehat{\mathbb{P}}_T^{\widetilde{K}}$ is $\widehat{\mathbb{P}}_T^J$ indistinguishable from the stochastic integral $\int_0^{\cdot} \hat{b}(s, \omega, y) dy(s)$ calculated under the law $\widehat{\mathbb{P}}_T^J$. Therefore

$$I_{\widetilde{K}}(\hat{b}, t, \omega, y) = I_J(\hat{b}, t, \omega, y) \quad \forall t \geq 0 \quad \widehat{\mathbb{P}}_T^J - a.s.$$

The desired conclusion now follows from Proposition 2.15(b). ∎

The last result in this Section is the proof that the random measure j defined by (2.9) possesses a density with respect to Lebesgue measure. We also estimate its moments. Some of the key ideas and techniques needed to prove Theorem 2.9 are introduced along with the proofs.

Notation 2.31. $p_t(x)$ denotes the one-dimensional heat kernel and $P_t\phi(x) \equiv \int p_t(x-y)\phi(y)dy$. We sometimes write $p_t^x(y) = p_t(x - y) = p(t, x - y)$.

We begin by giving a Green's function representation for j.

Proposition 2.32. *For any $\phi \in C_0^2(\mathbb{R})$*

$$j_t(\phi) = \int P_t\phi(x)\rho(x)dx + \int_0^t \int P_{t-s}\phi(y_s)\mathcal{E}_s(1, \omega, y) Z^{\widetilde{K}}(ds, dy) \quad (2.12)$$

for all $t \in [0, 1]$ $\mathbb{P} - a.s.$

Proof. Fix $t \leq 1$ and set $\psi(s, x) \equiv P_{t-s}\phi(x)$. Applying Itô's lemma 2.17 with $Y_t = I(1, t) = y_t$ we get

$$\psi(t, y_t) = \psi(0, Y_0) + \int_0^t \left(\frac{\partial \psi}{\partial s}(s, y_s) + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}(s, y_s)\right) ds + \int_0^t \frac{\partial \psi}{\partial x}(s, y_s) dy(s)$$

$$= \psi(0, Y_0) + \int_0^t \frac{\partial \psi}{\partial x}(s, y_s) dy(s)$$

for $\widetilde{K}_t$-a.e. y $\forall t \in [0, 1]$ a.s., (since $\frac{\partial \psi}{\partial s}(s, x) = -\frac{\partial^2 \psi}{\partial x^2}(s, x)$).

Moreover, by the same lemma

$$\mathcal{E}_t(1, \omega, y) = 1 - \int_0^t \mathcal{E}_s(1, \omega, y)\hat{b}(s, \omega, y) dy(s) + \frac{1}{2}\int_0^t \hat{b}(s, \omega, y)^2 \mathcal{E}_s(1, \omega, y) ds$$

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for $\widetilde{K}_t$-a.e.-y $\forall t \in [0,1]$ a.s. Integration by parts, justified once more by Itô's lemma 2.17 yields

$$\mathcal{E}_t(1,\omega,y)\psi(t,y_t) = \psi(0,y_0) + \int_0^t \left(\mathcal{E}_s(1)\frac{\partial\psi}{\partial x}(s,y_s) - \psi(s,y_s)\mathcal{E}_s(1)\hat{b}(s) \right) dy(s) +$$
$$\int_0^t \left(\hat{b}(s)^2 \mathcal{E}_s(1)\psi(s,y_s) - \hat{b}(s)\mathcal{E}_s(1)\frac{\partial\psi}{\partial x}(s,y_s) \right) ds$$

for $\widetilde{K}_t$-a.e. y $\forall t \in [0,1]$ a.s. Now apply historical Itô's lemma 2.18 to obtain

$$\int \mathcal{E}_t(1)\psi(t,y_t)\widetilde{K}_t(dy) = \int \psi(0,x)\rho(x)dx +$$
$$\int_0^t \int \mathcal{E}_s(1,\omega,y)\psi(s,y_s)Z^{\widetilde{K}}(ds,dy) \quad \forall t \in [0,1] \text{ a.s.}$$

Recalling the definition of ψ and j^n, we see that this last equation is exactly (2.12). ∎

Remark 2.33. If $\hat{b} \equiv 0$ and $\gamma \equiv 1$ then Proposition 2.32 provides a very simple proof for the usual Green's function representation of super Brownian motion (compare with that of Konno and Shiga 1988, p 212).

Proposition 2.34. (a) *For each $\theta > 0$ let*

$$\mathcal{E}_t(\theta,\omega,y) = \exp\left(-\theta I_{\widetilde{K}}(\hat{b},t,\omega,y) + \left(\theta - \frac{\theta^2}{2}\right)\int_0^t \hat{b}(s,\omega,y)^2 ds \right).$$

Then for any $\theta > 0$, $\mathcal{E}_{s\wedge\cdot}(\theta)$ is a $\widehat{\mathbb{P}}_s^{\widetilde{K}}$-martingale starting at 1.

(b) *There is a function $\aleph : [1,\infty) \times [0,\infty) \to \mathbb{R}_+$, non-decreasing in each variable such that*

$$\mathcal{E}_s(1)^p \leq \aleph(p,s)\mathcal{E}_s(p)$$

(c) *The second term on the r.h.s. of (2.12) is a square integrable martingale null at zero.*

Proof. (a) Recall that by Proposition 2.14 $n_t = y_t - y_0 - \int_0^t \hat{b}(s,\omega,y)$ is a $\widehat{\mathbb{P}}_s^{\widetilde{K}}$-Brownian motion. Since $\hat{b}$ is bounded, $\mathcal{E}_{s\wedge\cdot}(\theta)$ is an exponential martingale.

(b) We estimate

$$\mathcal{E}_s(1)^p = \mathcal{E}_s(p)e^{(\frac{p^2}{2}-p)\int_0^s \hat{b}(u)^2 du}$$
$$\leq \mathcal{E}_s(p)e^{s(\frac{p^2}{2}-p)c_{BU}^2}$$
$$\equiv \aleph(p,s)\mathcal{E}_s(p).$$

(c) Note that $\|P_{t-s}\phi\|_\infty < \infty$. Moreover

$$\mathbb{P}\left[\int_0^t \widetilde{K}_s(\mathcal{E}_s(1)^2)ds\right] \leq \int_0^t \widehat{\mathbb{P}}_s^{\widetilde{K}}[\mathcal{E}_s(2)\aleph(2,s)]ds$$
$$\leq t\aleph(2,t)$$

(by parts (a) and (b) of this Proposition).

This implies that the stochastic integral on the l.h.s. of (2.12) is an L^2 martingale null at 0. ∎

We now employ equation (2.12) to show that j has a density. We state first a technical lemma.

Lemma 2.35. *There exists a constant $C > 0$ such that*

$$\int_0^{t \vee t'} \int_{-\infty}^{\infty} \left(p(t-s, x-z) - p(t'-s, x'-z) \right)^2 dz ds \le C(\sqrt{|t - t'|} + |x - x'|),$$

$$\forall t, t' \ge 0, \quad x, x' \in \mathbb{R}.$$

Proof. See for example Lemma 6.2 of Shiga (1994). ∎

Proposition 2.36. *There exists a (measurable) function $g = g(\omega, s, x)$ such that for any $\Phi \in C_K(\mathbb{R})$ and any $t \in [0, 1]$*

$$\int_0^t \int \Phi(x) j_s(dx) ds = \int_0^t \int \Phi(x) g(s, x) dx ds \quad \mathbb{P}\text{-}a.s. \tag{2.13}$$

Proof. First, we prove that for any $\epsilon \in (0, 1)$

$$\int_0^1 \int_{-\infty}^{\infty} \mathbb{P}[j_s(p_\epsilon^x)^2] dx ds < A \tag{2.14}$$

where $A < \infty$ is a constant independent of ϵ. By Proposition 2.32

$$\mathbb{P}[j_s(p_\epsilon^x)^2] = \int P_s p_\epsilon^x(z) \rho(z) dz^2 + \mathbb{P}\left[\int_0^s \int P_{s-r} p_\epsilon^x(y_r) \mathcal{E}_r(1, \omega, y) Z^{\widetilde{K}}(dr, dy)^2 \right]$$

$$= \int p_{s+\epsilon}^x(z) \rho(z) dz^2 + \mathbb{P}\left[\int_0^s \int p_{s-r+\epsilon}^x(y_r)^2 \mathcal{E}_r(1)^2 \widetilde{K}_r(dy) dr \right]$$

(by Chapman-Kolmogorov)

$$= \int p_{s+\epsilon}^x(z) \rho(z) dz^2 + \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [p_{s-r+\epsilon}^x(y_r)^2 \mathcal{E}_r(1)^2] dr$$

(by Fubini's theorem)

$$\le \int p_{s+\epsilon}^x(z) \rho(z) dz^2 +$$

$$\aleph(2, s) \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [p_{s-r+\epsilon}^x(y_r)^2 \mathcal{E}_r(2, \omega, y)] dr$$

(by Proposition 2.34(b)).

Now integrate over space-time:

$$\int_0^1 \int_{-\infty}^{\infty} \mathbb{P}[j_s^n(p_\epsilon^x)^2] dx ds \le \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{s+\epsilon}^x(z) \rho(z) dz^2 dx ds$$

$$+ \aleph(2, s) \int_0^1 \int_{-\infty}^{\infty} \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [p_{s-r+\epsilon}^x(y_r)^2 \mathcal{E}_r(2)] dr dx ds.$$

We estimate

$$\int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{s+\epsilon}^x(z) \rho(z) dz^2 dx ds$$

$$= \int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty p_{s+\epsilon}^x(z_1) p_{s+\epsilon}^x(z_2) \rho(z_1)\rho(z_2) dz_1 dz_2 dx ds$$

$$\leq \int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi(s+\epsilon)}} \rho(z_1) dz_1 \int_{-\infty}^\infty p_{s+\epsilon}^x(z_2)\rho(z_2) dz_2 dx ds$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi(s+\epsilon)}} \int_{-\infty}^\infty \rho(z_1) dz_1 \int_{-\infty}^\infty \int_{-\infty}^\infty p_{s+\epsilon}^x(z_2)\rho(z_2) dx dz_2 ds$$

$$= \int_0^1 \frac{1}{\sqrt{2\pi(s+\epsilon)}} ds, \quad (\text{since } \int \rho = 1)$$

$$\leq \frac{\sqrt{2}}{\sqrt{\pi}}.$$

Furthermore,

$$\int_0^1 \int_{-\infty}^\infty \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [p_{s-r+\epsilon}^x(y_r)^2 \mathcal{E}_r(2)] dr dx ds$$

$$= \int_0^1 \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} \left[\int_{-\infty}^\infty p_{s-r+\epsilon}^x(y_r)^2 dx \mathcal{E}_r(2) \right] dr ds$$
(by Fubini)

$$\leq \int_0^1 \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [\mathcal{E}_r(2)] \frac{1}{\sqrt{2\pi(s-r+\epsilon)}} dr ds$$

$$\leq \frac{2}{3\sqrt{\pi}}, \quad \text{since } \widehat{\mathbb{P}}_r^{\widetilde{K}}[\mathcal{E}_r(2)] = 1.$$

Take $A = \frac{\sqrt{2}}{\sqrt{\pi}} + \aleph(2,1)\frac{2}{3\sqrt{\pi}}$ to conclude the proof of (2.14).

Our next step is to prove that

$$\lim_{\epsilon,\delta \to 0} \int_0^1 \int_{-\infty}^\infty \mathbb{P}[j_s(p_\epsilon^x - p_\delta^x)^2] dx ds = 0. \tag{2.15}$$

Apply Proposition 2.32 again. Just as before

$$\mathbb{P}[j_s(p_\epsilon^x - p_\delta^x)^2] = \int_{-\infty}^\infty (p_{s+\epsilon}^x(z) - p_{s+\delta}^x(z))\rho(z) dz^2 +$$

$$\int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [(p_{s-r+\epsilon}^x(y_r) - p_{s-r+\delta}^x(y_r))^2 \mathcal{E}_r(1)^2] dr.$$

Integrate over space-time to obtain

$$\int_0^1 \int_{-\infty}^\infty \mathbb{P}[j_s(p_\epsilon^x - p_\delta^x)^2] dx ds \leq \int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty (p_{s+\epsilon}^x(z) - p_{s+\delta}^x(z))\rho(z) dz^2 dx ds$$

$$+ \aleph(2,1) \int_0^1 \int_{-\infty}^\infty \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [(p_{s-r+\epsilon}^x(y_r) - p_{s-r+\delta}^x(y_r))^2 \mathcal{E}_r(2)] dr dx ds$$

The two terms on the r.h.s. are readily estimated:

$$\int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty (p_{s+\epsilon}^x(z) - p_{s+\delta}^x(z))\rho(z) dz^2 dx ds$$

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$$\leq \int_0^1 \int_{-\infty}^\infty \int_{-\infty}^\infty (p_{s+\varepsilon}^x(z) - p_{s+\delta}^x(z))^2 \rho(z) dz dx ds$$

(by Jensen)

$$= \int_{-\infty}^\infty \int_0^1 \int_{-\infty}^\infty (p_{s+\varepsilon}^x(z) - p_{s+\delta}^x(z))^2 dx ds \rho(z) dz$$

$$\leq c_{2.36.1} \sqrt{|\varepsilon - \delta|} \int_{-\infty}^\infty \rho(z) dz = c_{2.36.1} \sqrt{|\varepsilon - \delta|}$$

(by Lemma 2.35).

To estimate the second term note that for any $\eta > 0$

$$\int (p_{s+\eta}^x(z) - p_s^x(z))^2 dx = \int (p_{s+\eta}^x(z)^2 + p_s^x(z)^2 - 2p_{s+\eta}^x(z) p_s^x(z)) dx$$

$$= \frac{1}{\sqrt{2\pi(2s+2\eta)}} + \frac{1}{\sqrt{2\pi 2s}} - \frac{2}{\sqrt{2\pi(2s+\eta)}}$$

$$\leq \frac{\eta}{(s+\eta)\sqrt{4\pi s}}.$$

Hence, assuming w.l.o.g. $\varepsilon > \delta$,

$$\int_0^1 \int_{-\infty}^\infty \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} [(p_{s-r+\varepsilon}^x(y_r) - p_{s-r+\delta}^x(y_r))^2 \mathcal{E}_r(2)] dr dx ds$$

$$= \int_0^1 \int_0^s \widehat{\mathbb{P}}_r^{\widetilde{K}} \Big[\int_{-\infty}^\infty (p_{s-r+\varepsilon}^x(y_r) - p_{s-r+\delta}^x(y_r))^2 dx \mathcal{E}_r(2) \Big] dr ds$$

$$= \int_0^1 \int_0^s \frac{(\varepsilon - \delta)}{(s-r+(\varepsilon-\delta))\sqrt{4\pi(s-r)}} \widehat{\mathbb{P}}_r^{\widetilde{K}} [\mathcal{E}_r(2)] dr ds$$

$$= \int_0^1 \int_0^s \frac{(\varepsilon - \delta)}{(s-r+(\varepsilon-\delta))\sqrt{4\pi(s-r)}} dr ds$$

(since $\mathcal{E}_{r\wedge\cdot}(2)$ is a $\widehat{\mathbb{P}}_r^{\widetilde{K}}$-martingale)

$$= \int_0^1 \frac{2\sqrt{\varepsilon - \delta}}{\sqrt{4\pi}} \arctan(\frac{\sqrt{s}}{\sqrt{\varepsilon - \delta}}) ds$$

$$\leq c_{2.36.2} \sqrt{|\varepsilon - \delta|}.$$

Therefore

$$\int_0^1 \int_{-\infty}^\infty \mathbb{P}[j_s(p_\varepsilon^x - p_\delta^x)^2] dx ds \leq c_{2.36.1} \sqrt{|\varepsilon - \delta|} + c_{2.36.2} \aleph(2,1) \sqrt{|\varepsilon - \delta|}$$

$$\equiv c_{2.36.3} \sqrt{|\varepsilon - \delta|},$$

which proves (2.15).

Thus (2.14) and (2.15) show the existence of an $L^2(\Omega \times [0,1] \times \mathbb{R})$ density function $g = g(\omega, s, x)$ for j_s defined by

$$\lim_{\varepsilon \to 0} \int_0^1 \int_{-\infty}^\infty \mathbb{P}\left[(j_s(p_\varepsilon^x) - g(s,x))^2 \right] dx ds = 0 \qquad (2.16)$$

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To finish the proof of the proposition we verify that g is in fact the desired density. Pick a nonnegative, nonidentically zero $\Phi \in C_K(\mathbb{R})$, $t \in [0,1]$ and compute

$$\int_0^t \mathbb{P}\left[\left(j_s(\Phi) - \int \Phi(x)g(s,x)dx \right)^2 \right] ds$$

$$= \lim_{\varepsilon \to 0} \int_0^t \mathbb{P}\left[\left(\int \Phi(x)j_s(p_\varepsilon^x)dx - \int \Phi(x)g(s,x)dx \right)^2 \right] ds$$

(by bounded convergence)

$$\leq \left(\int \Phi(x)dx \right)^{-1} \lim_{\varepsilon \to 0} \int_0^t \mathbb{P}\left[(j_s(p_\varepsilon^x) - g(s,x))^2 \Phi(x)dx \right] ds$$

(by Jensen)

$$\leq \left(\int \Phi(x)dx \right)^{-1} \| \Phi \|_\infty \lim_{\varepsilon \to 0} \int_0^t \int_{-\infty}^{\infty} \mathbb{P}[(j_s(p_\varepsilon^x) - g(s,x))^2]dx ds$$

$$= 0 \text{ (by (2.16))}.$$

This implies (2.13) and concludes the proof of the proposition. ∎

Remark 2.37. By considering functions of the form $1_{[a,b)}(s)\Phi(x)$ a monotone class argument shows that (2.13) is equivalent to

$$\int_0^t \int F(s,x)j_s(dx)ds = \int_0^t \int F(s,x)g(s,x)dxds \quad \mathbb{P}\text{-a.s.}$$

for (integrable) nonrandom functions $F = F(s,x)$. This fact will be needed in the proof of the main theorem. □

Having proved the existence of g, we proceed to estimate its moments. These bounds will also be needed in the proof of the main theorem of this chapter.

Proposition 2.38. *For any* $k \geq 2$, $t \in [0,1]$ *there is a constant* $C_{2.38}$ *depending on* k *and* ρ *(but not on* x*) such that*

$$\mathbb{P}[g(t,x)^k] \leq C_{2.38} \tag{2.17}$$

Proof. Write $g(\varepsilon,t,x) \equiv j_t(p_\varepsilon^x)$. Using Proposition 2.32 we estimate

$$\mathbb{P}[g(\varepsilon,t,x)^k] \leq 2^k \int p_{t+\varepsilon}^x(z)\rho(z)dz^k + 2^k \mathbb{P}\int_0^t \int p_{t-s+\varepsilon}^x(y_s)\mathcal{E}_s(1)Z^H(ds,dy)^k$$

(since $(a+b)^k \leq 2^k(a^k + b^k)$).

The first term on the r.h.s. is easily estimated:

$$2^k \int p_{t+\varepsilon}^x(z)\rho(z)dz^k \leq 2^k \| \rho \|_\infty^k \int p_{t+\varepsilon}^x(z)dz^k$$

$$= 2^k \| \rho \|_\infty^k . \tag{2.18}$$

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The other term is estimated as follows:

$$\mathbb{P}\Big[\int_0^t \int p_{t-s+\varepsilon}^x(y_s)\mathcal{E}_s(1)Z^{\widetilde{K}}(ds,dy)^k\Big]$$

$$\leq c_{2.38.1}(k)\mathbb{P}\Big[\int_0^t \widetilde{K}_s\big[p_{t-s+\varepsilon}^x(y_s)^2\mathcal{E}_s(1)^2\big]\,ds^{k/2}\Big]$$

(by Burkholder)

$$\leq c_{2.38.1}(k)\mathbb{P}\Big[\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)^2\mathcal{E}_s(1)^{1/(2k-2)}]ds^{(k-1)/2}\times$$

$$\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)^2\mathcal{E}_s(1)^{2k-\frac{1}{2}}]ds^{1/2}\Big]$$

(by Jensen's inequality)

$$\leq c_{2.38.2}(k)\sqrt{\mathbb{P}\Big[\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)^2\mathcal{E}_s(1)^{1/(2k-2)}]ds^{k-1}\Big]}\times$$

$$\sqrt{\mathbb{P}\Big[\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)^2\mathcal{E}_s(1)^{2k-\frac{1}{2}}]ds\Big]}$$

(by Schwarz)

$$\equiv c_{2.38.2}(k)\sqrt{I_1}\sqrt{I_2}. \tag{2.19}$$

Notice that since $y_\cdot - y_0 - \int_0^\cdot \hat{b}(s,\omega,y)$ is a Brownian motion under $\widehat{\mathbb{P}}_t^{\widetilde{K}}$, then the definition of $\mathcal{E}(1)$ and Girsanov's theorem imply that y is a Brownian motion (with initial distribution ρ) under the law $\widehat{\mathbb{Q}}^{\widetilde{K}}[\cdot] \equiv \widehat{\mathbb{P}}^{\widetilde{K}}[\mathcal{E}(1)\cdot]$. We use this fact to estimate I_1.

$$I_1 = \mathbb{P}\Big[\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)^{\frac{4k-5}{2k-2}}p_{t-s+\varepsilon}^x(y_s)^{\frac{1}{2k-2}}\mathcal{E}_s(1)^{\frac{1}{2k-2}}ds^{k-1}]\Big]$$

$$\leq \mathbb{P}\Big[\int_0^t \widetilde{K}_s[(t-s+\varepsilon)^{-\frac{4k-5}{4k-4}}p_{t-s+\varepsilon}^x(y_s)^{\frac{1}{2k-2}}\mathcal{E}_s(1)^{\frac{1}{2k-2}}ds^{2(k-1)/2}]\Big]$$

$$\leq \mathbb{P}\Big[\int_0^t (t-s+\varepsilon)^{-\frac{4k-5}{4k-4}}\widetilde{K}_s(1)ds^{(2k-3)/2}\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)\mathcal{E}_s(1)]ds^{1/2}\Big]$$

(by Jensen)

$$\leq \sqrt{\mathbb{P}\Big[\int_0^t (t-s+\varepsilon)^{-\frac{4k-5}{4k-4}}\widetilde{K}_s(1)ds^{2k-3}\Big]}\sqrt{\mathbb{P}\Big[\int_0^t \widetilde{K}_s[p_{t-s+\varepsilon}^x(y_s)\mathcal{E}_s(1)]ds\Big]}$$

(by Schwarz)

$$\leq \sqrt{\mathbb{P}[\widetilde{K}_t^*(1)^{2k-3}\int_0^t (t-s+\varepsilon)^{-\frac{4k-5}{4k-4}}ds^{2k-3}]}\sqrt{\int_0^t \widehat{\mathbb{Q}}_s^{\widetilde{K}}[p_{t-s+\varepsilon}^x(y_s)]ds}$$

$$= c_{2.38.3}(k)t^{\frac{2k-3}{8k-8}}\sqrt{\mathbb{P}[\widetilde{K}_t^*(1)^{2k-3}]}\sqrt{\int\int p_{t-s+\varepsilon}(x-y)p_s(y-z)dy\rho(z)dz}$$

Since $\mathbb{P}[\widetilde{K}_t^*(1)^{2k-3}] \leq c_{2.38.4}$ by Proposition 2.21 and $\int\int p_{t-s+\varepsilon}(x-y)p_s(y-z)dy\rho(z)dz =$

$\int p_{t+\varepsilon}(x-z)\rho(z)dz \leq \| \rho \|_\infty$, it follows that by defining

$$c_{2.38.5}(k, N, \rho) \equiv c_{2.38.3}(k)\sqrt{c_{2.38.4} \| \rho \|_\infty}$$

we obtain

$$I_1 \leq c_{2.38.5}(k, \rho). \tag{2.20}$$

Finally we estimate I_2:

$$I_2 = \int_0^t \widehat{\mathbb{P}}_s^{\widetilde{K}} \left[p_{t-s+\varepsilon}^x(y_s)^2 \mathcal{E}_s(1)^{2k-\frac{1}{2}} \right] ds$$

$$\leq \int_0^t \widehat{\mathbb{P}}_s^K \left[p_{t-s+\varepsilon}^x(y_s)^{\frac{5}{2}} \mathcal{E}_s(1) \right]^{\frac{4}{5}} \widehat{\mathbb{P}}_s^{\widetilde{K}} \left[\mathcal{E}_s(1)^{\frac{20k-13}{2}} \right]^{\frac{1}{5}} ds$$

(by Hölder with $p = 5/4$, $q = 5$)

$$\leq \aleph((20k-13)/2, 1) \int_0^t \widehat{\mathbb{Q}}_s^{\widetilde{K}} \left[p_{t-s+\varepsilon}^x(y_s)^{\frac{5}{2}} \right]^{\frac{4}{5}} \widehat{\mathbb{P}}_s^{\widetilde{K}} \left[\mathcal{E}_s\left(\frac{20k-13}{2}\right) \right]^{\frac{1}{5}} ds$$

(by Proposition 2.34(b))

$$= \aleph((20k-13)/2, 1) \int_0^t \int_{-\infty}^\infty \int_{-\infty}^\infty p_{t-s+\varepsilon}(x-z)^{\frac{5}{2}} p_s(z-w) dz \rho(w) dw^{\frac{4}{5}} ds$$

$$= \aleph((20k-13)/2, 1) \int_0^t (2\pi(t-s))^{-3/5} \int_{-\infty}^\infty \int_{-\infty}^\infty p_{\frac{2}{5}(t-s+\varepsilon)}(x-z) p_s(z-w) dz \rho(w) dw^{\frac{4}{5}} ds$$

$$= \aleph((20k-13)/2, 1) \int_0^t (2\pi(t-s))^{-3/5} \int_{-\infty}^\infty p_{\frac{2t}{5}+\frac{3s}{5}+\frac{2\varepsilon}{5}}(x-w) \rho(w) dw^{\frac{4}{5}} ds$$

$$\leq c_{2.38.6}(k) \int_0^t (t-s))^{-3/5} t^{-2/5} \int_{-\infty}^\infty \rho(w) dw^{\frac{4}{5}} ds$$

$$= \frac{2}{5} c_{2.38.6}(k)$$

$$\equiv c_{2.38.7}(k)$$

That is,

$$I_2 \leq c_{2.38.7}(k) \tag{2.21}$$

If we define $C_{2.38} \equiv 2^k \| \rho \|_\infty + 2^k c_{2.38.2}(k) c_{2.38.5}(k, \rho)^2 c_{2.38.7}(k)^2$ then

$$\mathbb{P}[g(\varepsilon, t, x)^k] \leq C_{2.38} \tag{2.22}$$

follows from (2.18), (2.19), (2.20) and (2.21). Since L^2 convergence implies pointwise convergence along some subsequence, by choosing appropriately a sequence $\varepsilon_n \to 0$ and using Fatou's lemma we obtain (2.17) from (2.22). ∎

2.3 A Generalized Green's Function Representation for X

In this section we give a new representation of X as a J-integral. This is the main ingredient in the proof of Theorem 2.9.

Recall that we assume (UB) throughout this Section.

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Definition 2.39. For any $\theta > 0$ let

$$\tilde{\mathcal{E}}_t(\theta, \omega, y) = \exp\left(\theta \int_0^t \hat{b}(s, \omega, y) dy(s) - \frac{\theta^2}{2} \int_0^t \hat{b}(s, \omega, y)^2 ds\right) \quad (J, \mathbb{P}) - a.e. \qquad \square$$

Moreover we denote $p'(t, x) = D_x p(t, x)$.

Remark 2.40. Arguing as in the proof of Proposition 2.34 we see that for any $\theta > 0$, $\tilde{\mathcal{E}}_{t \wedge \cdot}(\theta)$ is a $\widehat{\mathbb{P}}_t^J$-martingale and that there is a function $\aleph : [0, \infty) \times \mathbb{R}_+$, nondecreasing in each variable such that $\tilde{\mathcal{E}}_t(1)^p \leq \aleph(p, t) \tilde{\mathcal{E}}_t(p)$.

Definition 2.41. For any $\varepsilon > 0$ set

$$u(t, x, \varepsilon) \equiv X_t(p_\varepsilon^x).$$

Proposition 2.42. *Let $u(t, x, \varepsilon)$ be as in the above definition and $p'(t, x) = D_x p(t, x)$. We write*

$$\hat{\gamma}(s, \omega, y) = \gamma(s, \bar{\Pi}_s(K)(\omega), y)$$
$$\hat{f}(s, \omega, y) = f(s, \bar{\Pi}_s(K)(\omega), y)$$
$$\hat{h}(s, \omega, y) = h(s, \bar{\Pi}_s(K)(\omega), y).$$

Then the following representation holds for any $\varepsilon \in (0, 1]$

$$u(t, x, \varepsilon) = \int_{\mathbf{R}} p_{t+\varepsilon}^x(z) \rho(z) dz + \int_0^t \int p_{t-s+\varepsilon}^x(y(s)) \tilde{\mathcal{E}}_s(1) \hat{\gamma}_s Z^J(ds, dy)$$

$$+ \int_0^t \int \left(p_{t-s+\varepsilon}^x(y(s)) \tilde{\mathcal{E}}_s(1) \hat{f}_s + \hat{\gamma}_s \tilde{\mathcal{E}}_s(1) \hat{b}_s \hat{h}_s + \tilde{\mathcal{E}}_s(1) p_{t-s+\varepsilon}'(y_s - x) \hat{h}_s \right. \tag{2.23}$$

$$\left. + \hat{\gamma}_s p_{t-s+\varepsilon}'(y_s - x) \tilde{\mathcal{E}}_s(1) \hat{b}_s \right) J_s(dy) ds \quad \forall t \in [0, 1] \quad \mathbb{P} - a.s.$$

Proof. Recall that $y_{t \wedge \cdot}$ is a Brownian motion under $\widehat{\mathbb{P}}_t^J$. Since $D_s p_{t-s+\varepsilon}^x = -D_x^2 p_{t-s+\varepsilon}^x$ it follows from Ito's lemma 2.17 applied to $B_s^t = p_{t-s+\varepsilon}^x(y(s))$, $0 \leq s \leq t$, that B^t is a martingale,

$$B_s^t = B_0^t + \int_0^s p_{t-r+\varepsilon}'(y_r - x) dy(r) \quad (J, \mathbb{P}) - a.e.$$

The familiar section theorem argument and (R_γ) yield

$$\tilde{\mathcal{E}}_t(1) = 1 + \int_0^t \tilde{\mathcal{E}}_s(1) \hat{b}_s dy(s)$$

$$\hat{\gamma}_t = 1 + \int_0^t \hat{h}_s dy(s) + \int_0^t \hat{f}_s ds \quad J - a.e.$$

Integrate by parts (justified by Proposition 2.17).

$$p_\varepsilon^x(y_t)\tilde{\mathcal{E}}_t(1)\hat{\gamma}_t = p_{t+\varepsilon}^x(y_0) + \int_0^t \left(\tilde{\mathcal{E}}_s(1)\hat{\gamma}_s p_{t-s+\varepsilon}'(y_s - x)\right.$$

$$\left. + p_{t-s+\varepsilon}^x(y_s)\hat{\gamma}_s\tilde{\mathcal{E}}_s(1)\hat{b}_s + p_{t-s+\varepsilon}^x(y_s)\tilde{\mathcal{E}}_s(1)\hat{h}_s\right) dy(s)$$

$$+ \int_0^t \left(\hat{\gamma}_s\tilde{\mathcal{E}}_s(1)\hat{b}_s\hat{h}_s + \tilde{\mathcal{E}}_s(1)p_{t-s+\varepsilon}'(y_s - x)\hat{h}_s\right.$$

$$\left. + \hat{\gamma}_s p_{t-s+\varepsilon}'(y_s - x)\tilde{\mathcal{E}}_s(1)\hat{b}_s + p_{t-s+\varepsilon}^x(y_s)\tilde{\mathcal{E}}_s(1)\hat{f}_s\right) ds \quad (J, \mathbb{P}) - a.e.$$

(2.24)

Moreover, for any $\phi \in C_b^2(\mathbb{R})$, $t \in [0,1]$

$$X_t(\phi) = \int \phi(y_t) K_t(dy)$$

$$= \int \phi(y_t)\hat{\gamma}_t \widetilde{K}_t(dy)$$

$$= \int \phi(y_t)\hat{\gamma}_t\tilde{\mathcal{E}}_t(1) J_t(dy) \quad \forall t \geq 0 \quad \mathbb{P} - a.s.$$

(2.25)

Using (2.24), (2.25) and Ito's lemma for historical integrals 2.18 we obtain (2.23) ∎

2.4 Proof of the Main Result

In this section we put together the results from Sections 2.2, and 2.3 to prove Theorem 2.9. We begin with some technical lemmas.

Lemma 2.43 (Kolmogorov). (a) *Let* $(B(x) : x \in \mathbb{R}^d)$ *be a family of random variables indexed by* $x \in \mathbb{R}^d$. *Suppose that there exists a real* $p > 0$ *and two constants* $C_0, \beta > 0$ *such that*

$$\forall x, \tilde{x} \in \mathbb{R}^d, \qquad \mathbb{E}[|B(x) - B(\tilde{x})|^p] \leq C_0\|x - \tilde{x}\|^{d+\beta}.$$

Then the process $(B(x) : x \in \mathbb{R}^d)$ *has a continuous version which is globally Hölder with exponent* α, *for any* $\alpha < \beta/p$.

(b) *Let* $I \subset \mathbb{R}^3$ *be the product of 3 intervals (either closed, open or semi-open). Let* $(B(x) : x \in I)$ *be a 3-dimensional random field. Suppose that for any* $k > 13$ *there is a constant* $C_0 > 0$ *such that*

$$\mathbb{E}[|B(x_1, x_2, x_3) - B(\tilde{x}_1, \tilde{x}_3, \tilde{x}_3)|^k] \leq C_0\left(|x_1 - \tilde{x}_1|^{\frac{k-1}{4}} + |x_2 - \tilde{x}_2|^{\frac{k-1}{2}} + |x_3 - \tilde{x}_3|^{\frac{k-1}{4}}\right).$$

for any (x_1, x_2, x_3), $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in I$ *such that* $0 \leq |x_1 - \tilde{x}_1|, |x_2 - \tilde{x}_2|, |x_3 - \tilde{x}_3| \leq 1$. *Then the process* $(B(x) : x \in I)$ *has a continuous version which is Hölder with exponent* α, *for any* $\alpha < 1/4$. *Moreover, for any* x_2, x_3 *fixed, the map* $x_1 \mapsto B(x_1, x_2, x_3)$ *is Hölder-*α *for any* $\alpha < 1/4$ *and the map* $x_2 \mapsto B(x_1, x_2, x_3)$ *is Hölder-*α *for any* $\alpha < 1/2$.

If we know that the process $(B(x) : x \in \mathbb{R}^d)$ *is continuous to begin with then there is no need to take a version.*

Proof. Although Kolmogorov's theorem is not generally stated in this form, the standard proof (Revuz and Yor 1991) works equally well. ∎

Lemma 2.44. (a) *Denote $p'(t,x) = D_x p(t,x)$. For any $0 < \varepsilon < 1$, $t \in [0,1]$*

$$\int_0^t \int_{-\infty}^\infty |p'(t-s, x+\varepsilon) - p'(t-s,x)| dx ds \leq c_{2.44.1} \sqrt{\varepsilon}.$$

(b) *Let $0 < s < t \leq 1$. Then*

$$\int_0^s \int_{-\infty}^\infty |p'(t-r,x) - p'(s-r,x)| dx dr \leq c_{2.44.2} \sqrt{t-s}.$$

(c) *Let*

$$T(n) := \inf\{t > 0 : H_t(1) \geq n\} \wedge 1.$$

There is a function $\Theta : \mathbf{N} \times [0,\infty) \to \mathbb{R}_+$, non-decreasing in each variable such that

$$X_s^*(1) \leq \Theta(n,s) \quad on \quad \{T(n) \geq s\}.$$

Notice also that $K_s^*(1) = X_s^*(1)$.

Proof. (a) Estimates (a) and (b) should be well known. We prove them since we don't know a reference. We need the following elementary estimate (e.g. Ladyženskaja 1968 p. 274)

$$|D_t^n D_x^m p(x,t)| \leq C_{n,m} t^{-\frac{1}{2} - n - \frac{m}{2}} \exp\left(-c_{n,m} \frac{x^2}{t}\right). \tag{2.26}$$

Let $\delta > 0$, $\delta < t$. We estimate

$$\int_0^t \int_{-\infty}^\infty |p'(t-s,x+\varepsilon) - p'(t-s,x)| dx ds = \int_0^{t-\delta} \int_{-\infty}^\infty |p'(t-s,x+\varepsilon) - p'(t-s,x)| dx ds$$

$$+ \int_{t-\delta}^t \int_{-\infty}^\infty |p'(t-s,x+\varepsilon) - p'(t-s,x)| dx ds$$

$$\leq \int_0^{t-\delta} \int_{-\infty}^\infty |p'(t-s,x+\varepsilon) - p'(t-s,x)| dx ds$$

$$+ 2 \int_{t-\delta}^t \int_{-\infty}^\infty |p'(t-s,x)| dx ds$$

$$\equiv I_1 + I_2.$$

Use estimate (2.26) to check that

$$I_2 \leq C \int_{t-\delta}^t \frac{1}{\sqrt{t-s}} ds$$

$$\leq C_1 \sqrt{\delta}.$$

To estimate I_1 we use the fundamental theorem of calculus followed by a linear change of variables,

$$I_1 = \int_0^{t-\delta} \int_{-\infty}^{\infty} \left| \varepsilon \int_0^1 D_x^2 p(t-s, x+z\varepsilon) dz \right| dx ds$$

$$\leq \varepsilon \int_0^{t-\delta} \int_0^1 \int_{-\infty}^{\infty} |D_x^2 p(t-s, x+z\varepsilon)| dx dz ds$$

(by Fubini)

$$\leq \varepsilon \int_0^{t-\delta} \int_0^1 \frac{C_2}{t-s} dz ds$$

(using estimate (2.26))

$$= C_2 \varepsilon \log \left(\frac{t}{\delta} \right)$$

$$\leq C_2 \varepsilon \log \left(\frac{1}{\delta} \right).$$

If $t \leq \delta$ the estimate (2.26) yields

$$\int_0^t \int_{-\infty}^{\infty} |p'(t-s, x+\varepsilon) - p'(t-s, x)| dx ds \leq 2 \int_0^t \int_{-\infty}^{\infty} |p'(t-s, x)| dx ds$$

$$\leq C_3 \sqrt{t}$$

$$\leq C_3 \sqrt{\delta}.$$

Thus, for any $\delta > 0$,

$$\int_0^t \int_{-\infty}^{\infty} |p'(t-s, x+\varepsilon) - p'(t-s, x)| dx ds \leq C_1 \sqrt{\delta} + C_2 \varepsilon \log \left(\frac{1}{\delta} \right) + C_3 \sqrt{\delta}.$$

Put $\delta = \varepsilon$ to obtain the desired result.

(b) We compute

$$\int_0^s \int_{-\infty}^{\infty} |p'(t-r, x) - p'(s-r, x)| dx dr$$

$$\leq \int_0^s \int_{-\infty}^{\infty} \int_0^{t-s} |D_\tau D_x p(\tau + s - r, x)| d\tau dx dr$$

$$\leq C_{1,1} \int_0^s \int_0^{t-s} \int_{-\infty}^{\infty} (\tau + s - r)^{-2} \exp \left(-c_{1,1} \frac{x^2}{\tau + s - r} \right) dx d\tau dr$$

(estimate (2.26)

$$= C \int_0^s \int_0^{t-s} (\tau + s - r)^{-3/2} d\tau dr$$

$$= 4C(\sqrt{t-s} - (\sqrt{t} - \sqrt{s}))$$

$$\leq c_{2.44.2} \sqrt{t-s}.$$

(c) A more general result is proved in p.61 of Perkins (1995). ∎

Proof of Theorem 2.9.

We claim that it suffices to consider $t \in [0,1]$.

Exercise. We do not wish to rob the reader of the pleasure of doing some things for herself, so we leave the proof of the claim as an exercise.

The proof rests on the following lemma.

Lemma 2.45. *Let u be as in Definition 2.41. Suppose that there is a constant c_{UB} such that*

$$|\hat{\gamma}(s,\omega,y)| \leq c_{UB}$$
$$|\hat{h}(s,\omega,y)| \leq c_{UB}$$
$$|\hat{f}(s,\omega,y)| \leq c_{UB}$$
$$|\hat{b}(s,\omega,y)| \leq c_{UB} \quad \forall s \geq 0 \quad \forall y \in C \quad \mathbb{P} - a.s.$$

Then for each $k \geq 1$ there is a constant $C_{2.45}(k)$ such that for any $\varepsilon, \delta \in (0,1]$, $t, s \in [0,1]$, $x, \tilde{x} \in \mathbb{R}$, $|x - \tilde{x}| \leq 1$,

$$P[|u(t,x,\varepsilon) - u(s,\tilde{x},\delta)|^k] \leq C_{2.45}(|t-s|^{\frac{k-1}{4}} + |x-\tilde{x}|^{\frac{k-1}{2}} + |\varepsilon - \delta|^{\frac{k-1}{4}}). \qquad (2.27)$$

We defer the proof of Lemma 2.45 and show how to prove the theorem.

Case 1. Suppose that $\hat{b}, \hat{\gamma}, \hat{f}, \hat{h}$ are uniformly bounded. Set $R > 0$ and consider the function

$$u : [0,1] \times [-R, R] \times (0,1] \longrightarrow \mathbb{R}.$$

Lemmas 2.45 and 2.43 show that this function is Hölder-α for any $\alpha < 1/4$ and therefore uniformly continuous. It follows that it has a unique continuous extension to $[0,1] \times [-R,R] \times [0,1]$. Define

$$u(t,x) = u(t,x,0) = \lim_{\varepsilon \to 0} u(t,x,\varepsilon).$$

By considering a sequence $R_n \uparrow \infty$ this definition is extended to any $x \in \mathbb{R}$. Note that the function $(t,x) \longmapsto u(t,x)$ is Hölder-α for $\alpha < 1/4$. This gives the joint continuity of u. Lemma 2.45 also gives the desired exponents in x and in t. It remains to show that u is indeed the density of X. Let $\phi \in C_K(\mathbb{R})$, $\phi > 0$. We compute

$$X_t(\phi) = \lim_{\varepsilon \to 0} \int \phi * p_\varepsilon(z) X(dz)$$

$$= \lim_{\varepsilon \to 0} \int \int \phi(x) p_\varepsilon(x - z) dx X_t(dz)$$

$$= \lim_{\varepsilon \to 0} \int \int p_\varepsilon^x(z) X_t(dz) \phi(x) dx$$

(by Fubini)

$$= \lim_{\varepsilon \to 0} \int u(x,t,\varepsilon) \phi(x) dx$$

$$= \int u(t,x) \phi(x) dx$$

(by dominated convergence)

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Case 2. Suppose that the coefficients are not uniformly bounded. Recall that $T(n)$ was defined in Lemma 2.44(c). Note that Lemma 2.44(c), $(B)_\gamma$, (2.2), and (2.3) show that for each $n \in \mathbb{N}$

$$(\hat{\gamma}_n, \hat{\gamma}_n^{-1}, \hat{b}_n, \hat{f}_n, \hat{h}_n) = (\hat{\gamma}, \hat{\gamma}^{-1}, \hat{b}, \hat{f}, \hat{h})1(0 \leq t \leq T(n))$$

is uniformly bounded. Hence $K_{\cdot \wedge T_n}$ solves $(MP)^m_{\hat{\gamma}_n, 1, \bar{b}_n, \bar{g}_n}$ (see Remark 2.4 of Perkins (1995)), where

$$(\bar{b}_n, \bar{g}_n) = (\bar{b}, \bar{g})1(0 \leq t \leq T(n))$$
$$= (\hat{b}_n + \hat{h}_n \hat{\gamma}_n^{-1}, (\hat{f}_n + \hat{h}_n \hat{b}_n)\hat{\gamma}^{-1}).$$

Therefore $X_{\cdot \wedge T(n)}$ satisfies the conditions of Case 1. (see Remark 2.23). To finish the proof note that Proposition 2.21 implies that for $\mathbb{P}$-almost every ω there is an $n_0 = n_0(\omega)$ such that $T(m) = 1$ for any $m \geq n_0$. ∎

Proof of Lemma 2.45. Note that since $(A+B+C)^k \leq 3^k(A^k + B^k + C^k)$ it suffices to estimate $\mathbb{P}[|u(t, x, \varepsilon) - u(s, x, \varepsilon)|^k]$, $\mathbb{P}[|u(t, x, \varepsilon) - u(t, \tilde{x}, \varepsilon)|^k]$ and $\mathbb{P}[|u(t, x, \varepsilon) - u(s, x, \delta)|^k]$ separately. Throughout this proof we assume w.l.o.g. that $t > s$ and k is an even integer. We start with the first term. By Proposition 2.42 (generalized Green's function representation)

$$6^{-k}\mathbb{P}[|u(t, x, \varepsilon) - u(s, x, \varepsilon)|^k] \leq \int_{\mathbb{R}} (p^x_{t+\varepsilon}(z) - p^x_{s+\varepsilon}(z))\rho(z)dz^k$$

$$+ \mathbb{P}\left[\int_0^t \int (p^x_{t-r+\varepsilon}(y_r) - p^x_{s-r+\varepsilon}(y_r))\tilde{\mathcal{E}}_r(1)\hat{\gamma}_r Z^J(dr, dy)^k\right]$$

$$+ \mathbb{P}\left[\int_0^t \int (p^x_{t-r+\varepsilon}(y_r) - p^x_{s-r+\varepsilon}(y_r))\tilde{\mathcal{E}}_r(1)\hat{f}_r J_r(dy)dr^k\right] \qquad (2.28)$$

$$+ \mathbb{P}\left[\int_0^t \int (p'_{t-r+\varepsilon}(y_r - x) - p'_{s-r+\varepsilon}(y_r - x))\tilde{\mathcal{E}}_r(1)\hat{h}_r J_r(dy)dr^k\right]$$

$$+ \mathbb{P}\left[\int_0^t \int (p'_{t-r+\varepsilon}(y_r - x) - p'_{s-r+\varepsilon}(y_r - x))\tilde{\mathcal{E}}_r(1)\hat{\gamma}_r \hat{b}_r J_r(dy)dr^k\right]$$

$$+ \mathbb{P}\left[\int_s^t \tilde{\mathcal{E}}_r(1)\hat{\gamma}_r \hat{b}_r \hat{h}_r J_r(dy)dr^k\right]$$

$$\equiv \sum_{i=1}^6 I_i.$$

In (2.28) we have adopted the (bizarre) convention $p_r(x) = 0$ for $r \leq \varepsilon$.

We compute

$$I_1 = (P_{t+\varepsilon}\rho(x) - P_{s+\varepsilon}\rho(x))^k$$
$$\leq \| P_{t+\varepsilon}\rho - P_{s+\varepsilon}\rho \|_\infty^k$$
$$\leq \| P_{s+\varepsilon}(P_{t-s} - 1)\rho \|_\infty^k$$
$$\leq \| (P_{t-s} - 1)\rho \|_\infty^k$$
(since P is a contractive semigroup)
$$= \sup_z |\mathbb{E}^z[\rho(B_{t-s}) - \rho(z)]|^k$$

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(Here $(\mathbb{E}^z, B)$ is a Brownian motion started at $z \in \mathbb{R}$)

$$\leq c_{2.45.1} \sup_z \mathbb{E}^z[|(B_{t-s} - z)|^{\frac{k-1}{2k}}]^k$$

(by (IC))

$$\leq c_{2.45.1} \mathbb{E}^0[(\sqrt{t-s}|B_1|)^{\frac{k-1}{2k}}]^k$$

$$= c_{2.45.2}(t-s)^{\frac{k-1}{4}} \tag{2.29}$$

We estimate

$$I_2 \leq \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^2 \hat{\gamma}_r^2 \tilde{\mathcal{E}}_r(1)^{-1} J_r(dy) dr^{k/2}\right]$$

(by Burkholder)

$$\leq \mathbb{P}\left[\left(\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 J_r(dy) dr^{k-1}\right.\right.$$

$$\times \left.\left.\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^k \hat{\gamma}_r^{2k} J_r(dy) dr\right)^{1/2}\right]$$

(by Jensen)

$$\leq \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 J_r(dy) dr^{k-1}\right]^{1/2}$$

$$\times \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^k \hat{\gamma}_r^{2k} J_r(dy) dr\right]^{1/2}$$

(by Schwarz)

$$= \mathbb{P}\left[\int_0^t \int_{\mathbb{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 j_r(da) dr^{k-1}\right]^{1/2}$$

$$\times \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^k \hat{\gamma}_r^{2k} J_r(dy) dr\right]^{1/2}$$

(by Definition 2.25)

$$= \mathbb{P}\left[\int_0^t \int_{\mathbb{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 g(r,a) da \, dr^{k-1}\right]^{1/2}$$

$$\times \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^k \hat{\gamma}_r^{2k} J_r(dy) dr\right]^{1/2}$$

(by Remark 2.37)

$$\leq \mathbb{P}\left[\int_0^t \int_{\mathbb{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 g(r,a) da \, dr^{k-1}\right]^{1/2}$$

$$\times \aleph(k,1)^{1/2} c_{UB}^k \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(k) J_r(dy) dr\right]^{1/2}$$

(by (UB) and Proposition 2.34)

$$\leq \mathbb{P}\left[\int_0^t \int_{\mathbb{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 da \, dr^{k-2}\right.$$

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$$\times \left. \int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 g(r,a)^{k-1} da dr \right]^{1/2}$$

$$\times \aleph(k,1)^{1/2} c_{UB}^k \int_0^t \widehat{\mathbb{P}}_r^J \left[(p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(k) \right] dr^{1/2}$$

(by Jensen)

$$\leq c_{2.45.3}(k)(t-s)^{\frac{k-2}{4}} \int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 \mathbb{P}[g(r,a)^{k-1}] da dr^{1/2}$$

$$\times \int_0^t \widehat{\mathbb{P}}_r^J \left[(p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(k) \right] dr^{1/2}$$

(by Lemma 2.35)

$$\leq c_{2.45.4}(k)(t-s)^{\frac{k-1}{4}} \int_0^t \widehat{\mathbb{P}}_r^J \left[(p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(k) \right] dr^{1/2}$$

(by Proposition 2.38 and Lemma 2.35)

$$\equiv c_{2.45.5}(k)(t-s)^{\frac{k-1}{4}} I_{2.1}. \tag{2.30}$$

$I_{2.1}$ is easily estimated (recall that $y_{. \wedge t}$ is a Brownian motion under $\widehat{\mathbb{P}}_t^J$):

$$I_{2.1} \leq 2 \int_0^t \widehat{\mathbb{P}}_r^J [p_{t-r+\varepsilon}^x(y_r)^2 \tilde{\mathcal{E}}_r(k)] dr^{1/2}$$

$$\leq 2 \int_0^t \widehat{\mathbb{P}}_r^J [p_{t-r+\varepsilon}^x(y_r)^{5/2}]^{4/5} \widehat{\mathbb{P}}_r^J [\tilde{\mathcal{E}}_r(k)^5]^{1/5} dr^{1/2}$$

(by Hölder)

$$\leq c_{2.45.6}(k) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} p_{t-r+\varepsilon}(x-z)^{5/2} p_r(z-w) dz \rho(w) dw^{4/5} \widehat{\mathbb{P}}_r^J [\tilde{\mathcal{E}}_r(5k)]^{1/5} dr^{1/2}$$

$$= c_{2.45.6}(k) \int_0^t \int_{\mathbf{R}} \int_{\mathbf{R}} p_{t-r+\varepsilon}(x-z)^{5/2} p_r(z-w) dz \rho(w) dw^{4/5} dr^{1/2}$$

$$\leq c_{2.45.7}(k) \tag{2.31}$$

(the integral was estimated during the proof of Proposition 2.38).

(2.30) and (2.31) yield

$$I_2 \leq c_{2.45.8}(k)(t-s)^{\frac{k-1}{4}}. \tag{2.32}$$

I_3 can be estimated in a similar manner.

$$I_3 \leq \mathbb{P} \left[\left(\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 J_r(dy) dr^{2k-1} \right. \right.$$

$$\left. \left. \times \int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^{2k} \hat{f}_r^{2k} J_r(dy) dr \right)^{1/2} \right]$$

(by Jensen)

$$\leq \mathbb{P} \left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 J_r(dy) dr^{2k-1} \right]^{1/2}$$

$$\times \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^{2k} \hat{f}_r^{2k} J_r(dy) dr\right]^{1/2}$$

(by Schwarz)

$$= \mathbb{P}\left[\int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 g(r,a) da dr^{2k-1}\right]^{1/2}$$
$$\times \int_0^t \widehat{\mathbb{P}}_r^J\left[(p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(1)^{2k} \hat{f}_r^{2k}\right] dr^{1/2}$$

$$\leq \int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 da dr^{k-1}$$
$$\times \int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{s-r+\varepsilon}^x(a))^2 \mathbb{P}[g(r,a)^{2k-1}] da dr^{1/2}$$

$$c_{UB}^k \aleph(2k,1)^{1/2} \times \int_0^t \widehat{\mathbb{P}}_r^J\left[(p_{t-r+\varepsilon}^x(y_r) - p_{s-r+\varepsilon}^x(y_r))^2 \tilde{\mathcal{E}}_r(2k)\right] dr^{1/2}$$

(by Jensen)

$$\leq c_{2.45.9}(k)(t-s)^{\frac{2k-1}{4}} I_{2.1}$$

(by (2.31), Lemma 2.35 and Proposition 2.38)

$$\leq c_{2.45.10}(k)(t-s)^{\frac{2k-1}{4}} \tag{2.33}$$

By the same token

$$I_4 \leq \mathbb{P}\left[\left(\int_0^t \int |p_{t-r+\varepsilon}'(y_r - x) - p_{s-r+\varepsilon}'(y_r - x)| J_r(dy) dr^{2k-1}\right.\right.$$
$$\left.\left.\times \int_0^t \int |p_{t-r+\varepsilon}'(y_r - x) - p_{s-r+\varepsilon}'(y_r - x)| \tilde{\mathcal{E}}_r(1)^{2k} \hat{h}_r^{2k} J_r(dy) dr\right)^{1/2}\right]$$

(by Jensen)

$$\leq \mathbb{P}\left[\int_0^t \int |p_{t-r+\varepsilon}'(y_r - x) - p_{s-r+\varepsilon}'(y_r - x)| J_r(dy) dr^{2k-1}\right]^{1/2}$$
$$\times \mathbb{P}\left[\int_0^t \int |p_{t-r+\varepsilon}'(y_r - x) - p_{s-r+\varepsilon}'(y_r - x)| \tilde{\mathcal{E}}_r(1)^{2k} \hat{h}_r^{2k} J_r(dy) dr\right]^{1/2}$$

(by Schwarz)

$$= \mathbb{P}\left[\int_0^t \int_{\mathbf{R}} |p_{t-r+\varepsilon}'(a - x) - p_{s-r+\varepsilon}'(a - x)| g(r,a) da dr^{2k-1}\right]^{1/2}$$
$$\times \int_0^t \widehat{\mathbb{P}}_r^J\left[|p_{t-r+\varepsilon}'(y_r - x) - p_{s-r+\varepsilon}'(y_r - x)| \tilde{\mathcal{E}}_r(1)^{2k} \hat{h}_r^{2k}\right] dr^{1/2}$$

$$\leq \int_0^t \int_{\mathbf{R}} |p_{t-r+\varepsilon}'(a - x) - p_{s-r+\varepsilon}'(a - x)| da dr^{k-1}$$
$$\times \int_0^t \int_{\mathbf{R}} |p_{t-r+\varepsilon}'(a - x) - p_{s-r+\varepsilon}'(a - x)| \mathbb{P}[g(r,a)^{2k-1}] da dr^{1/2}$$

$$c_{UB}^k \aleph(2k,1)^{1/2} \times \int_0^t \widehat{\mathbb{P}}_r^J\left[|p_{t-r+\varepsilon}'(y_r - x) - p_{s-r+\varepsilon}'(y_r - x)| \tilde{\mathcal{E}}_r(2k)\right] dr^{1/2}$$

(by Jensen)

$$= c_{2.45.11}(k)(t-s)^{\frac{2k-1}{4}} \int_0^t \widehat{\mathbb{P}}_r^J \left[|p'_{t-r+\varepsilon}(y_r - x) - p'_{s-r+\varepsilon}(y_r - x)|\tilde{\mathcal{E}}_r(2k) \right] dr^{1/2}$$

(by Lemma (2.44)(b))

$$\equiv c_{2.45.12}(n,k,N)(t-s)^{\frac{2k-1}{4}} I_{4.1} \qquad (2.34)$$

We readily estimate

$$I_{4.1} \leq 2 \int_0^t \widehat{\mathbb{P}}_r^J \left[|p'_{t-r+\varepsilon}(y_r - x)|\tilde{\mathcal{E}}_r(2k) \right] dr^{1/2}$$

$$\leq 2 \int_0^t \widehat{\mathbb{P}}_r^J \left[|p'_{t-r+\varepsilon}(y_r - x)|^{5/4} \right]^{4/5} \widehat{\mathbb{P}}_r^J \left[\tilde{\mathcal{E}}_r(2k)^5 \right]^{1/5} dr^{1/2}$$

(by Hölder)

$$\leq 2c_{2.45.13}(k) \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p'_{t-r+\varepsilon}(z - x)|^{\frac{5}{4}} p_r(z - w) dz \rho(w) dw^{\frac{4}{5}} \widehat{\mathbb{P}}_r^J \left[\tilde{\mathcal{E}}_r(10k) \right]^{\frac{1}{5}} dr^{\frac{1}{2}}$$

$$\leq 2c_{2.45.13}(k)\|\rho\|_{\infty}^{2/5} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |p'_{t-r+\varepsilon}(z - x)|^{5/4} p_r(z - w) dz dw^{4/5} dr^{1/2}$$

$$= \leq 2c_{2.45.13}(k)\|\rho\|_{\infty}^{2/5} \int_0^t \int_{\mathbb{R}} |p'_{t-r+\varepsilon}(z - x)|^{5/4} \int_{\mathbb{R}} p_r(z - w) dw dz^{4/5} dr^{1/2}$$

(by Fubini)

$$= 2c_{2.45.13}(k)\|\rho\|_{\infty}^{2/5} \int_0^t \int_{\mathbb{R}} |p'_{t-r+\varepsilon}(z - x)|^{5/4} dz^{4/5} dr^{1/2}$$

$$\leq c_{2.45.14}(k) \int_0^t \frac{1}{t-r+\varepsilon} \int \exp\left(- c_{0,1} \frac{(z-x)^2}{t-r+\varepsilon} \right) dz^{4/5} dr^{1/2}$$

(by estimate (2.26)

$$= c_{2.45.15}(k) \int_0^t \frac{1}{(t-r+\varepsilon)^{3/5}} dr^{1/2}$$

$$\leq c_{2.45.16}(k). \qquad (2.35)$$

From (2.34) and (2.35) it follows that

$$I_4 \leq c_{2.45.17}(k)(t-s)^{\frac{2k'-1}{4}}. \qquad (2.36)$$

The estimation of I_5 is identical to that of I_4. We obtain

$$I_5 \leq c_{2.45.18}(n,k,N)(t-s)^{\frac{2k-1}{4}}. \qquad (2.37)$$

We now estimate I_6

$$I_6 \leq c_{UB}^{3k} \mathbb{P} \left[\int_s^t \tilde{\mathcal{E}}_r(1) J_r(dy) dr^k \right]$$

$$\leq c_{UB}^{3k} \mathbb{P} \left[\left(\int_s^t J_r(1) dr^{2k-1} \int_s^t \int \tilde{\mathcal{E}}_r(1)^{2k} J_r(dy) dr \right)^{1/2} \right]$$

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(by Jensen)

$$\leq c_{UB}^{3k} \aleph(2k,1)^{1/2} \mathbb{P}\left[J_1^*(1)^{2k-1}\int_s^t dr^{2k-1}\right]^{1/2} \mathbb{P}\left[\int_s^t \tilde{\mathcal{E}}_r(2k)J_r(dy)dr\right]^{1/2}$$

$$\leq c_{2.45.19}(k)(t-s)^{\frac{2k-1}{2}}. \tag{2.38}$$

Combining the estimates (2.29), (2.32), (2.33), (2.36), (2.37), (2.38) with (2.28) we see that for $0 < t - s$, $t \in [0,1]$

$$\mathbb{P}[|u(t,x,\varepsilon) - u(s,x,\varepsilon)|^k] \leq c_{2.45.20}(n,k)(t-s)^{\frac{k-1}{4}}. \tag{2.39}$$

Next we estimate

$$5^{-k}\mathbb{P}[|u(t,x,\varepsilon) - u(t,\tilde{x},\varepsilon)|^k] \leq \int_{\mathbb{R}} (p_{t+\varepsilon}^x(z) - p_{t+\varepsilon}^{\tilde{x}}(z))\rho(z)dz^k$$

$$+ \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))\tilde{\mathcal{E}}_r(1)\hat{\gamma}_r Z^J(dr,dy)^k\right]$$

$$+ \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))\tilde{\mathcal{E}}_r(1)\hat{f}_r J_r(dy)dr^k\right] \tag{2.40}$$

$$+ \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}'(y_r - x) - p_{t-r+\varepsilon}'(y_r - \tilde{x}))\tilde{\mathcal{E}}_r(1)\hat{h}_r J_r(dy)dr^k\right]$$

$$+ \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}'(y_r - x) - p_{t-r+\varepsilon}'(y_r - \tilde{x}))\tilde{\mathcal{E}}_r(1)\hat{\gamma}_r\hat{b}_r J_r(dy)dr^k\right]$$

$$\equiv \sum_{i=1}^5 L_i$$

We compute

$$L_1 = (P_t\rho(x) - P_t\rho(\tilde{x}))^k$$
$$\leq \| P_t(\rho^x - \rho^{\tilde{x}}) \|_\infty^k$$
$$\quad (\text{here } \rho^x = \rho(x - \cdot))$$
$$\leq \| \rho^x - \rho^{\tilde{x}} \|_\infty^k$$
$$\leq c_{2.45.21}|x - \tilde{x}|^{\frac{k-1}{2}} \tag{2.41}$$
$$(\text{by } (IC)).$$

L_2 is estimated as follows:

$$L_2 \leq \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(1)^2\hat{\gamma}_r^2\tilde{\mathcal{E}}_r(1)^{-1} J_r(dy)dr^{k/2}\right]$$
$$(\text{by Burkholder})$$
$$\leq \mathbb{P}\left[\left(\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 J_r(dy)dr^{k-1}\right.\right.$$

$$\times \int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(1)^k \hat{\gamma}_r^{2k} J_r(dy) dr\Big)^{1/2}\Big]$$

(by Jensen)

$$\leq \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 J_r(dy) dr^{k-1}\right]^{1/2}$$

$$\times \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(1)^{2k} \hat{\gamma}_r^{2k} J_r(dy) dr\right]^{1/2}$$

(by Schwarz)

$$\leq \mathbb{P}\left[\int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{t-r+\varepsilon}^{\tilde{x}}(a))^2 g(r,a) da dr^{k-1}\right]^{1/2}$$

$$\times c_{UB}^k \aleph(k,1)^{1/2} \mathbb{P}\left[\int_0^t \int (p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(k) J_r(dy) dr\right]^{1/2}$$

$$\leq \mathbb{P}\left[\int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{t-r+\varepsilon}^{\tilde{x}}(a))^2 da dr^{k-2}\right.$$

$$\times \int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{t-r+\varepsilon}^{\tilde{x}}(a))^2 g(r,a)^{k-1} da dr\Big]^{1/2}$$

$$\times c_{UB}^k \aleph(k,1)^{1/2} \int_0^t \widehat{\mathbb{P}}_r^J\left[(p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(k)\right] dr^{1/2}$$

(by Jensen)

$$\leq c_{2.45.22}(k)|x-\tilde{x}|^{\frac{k-2}{2}} \int_0^t \int_{\mathbf{R}} (p_{t-r+\varepsilon}^x(a) - p_{t-r+\varepsilon}^{\tilde{x}}(a))^2 \mathbb{P}[g(r,a)^{k-1}] da dr^{1/2}$$

$$\times \int_0^t \widehat{\mathbb{P}}_r^J\left[(p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(k)\right] dr^{1/2}$$

(by Lemma 2.35)

$$\leq c_{2.45.23}(k)|x-\tilde{x}|^{\frac{k-1}{2}} \int_0^t \widehat{\mathbb{P}}_r^J\left[(p_{t-r+\varepsilon}^x(y_r) - p_{t-r+\varepsilon}^{\tilde{x}}(y_r))^2 \tilde{\mathcal{E}}_r(k)\right] dr^{1/2}$$

(by Proposition 2.38 and Lemma 2.35)

$$\equiv c_{2.45.24}(k)|x-\tilde{x}|^{\frac{k-1}{2}} L_{2.1}. \tag{2.42}$$

Note that in fact $L_{2.1} \leq I_{2.1}$.

$$L_2 \leq c_{2.45.25}(k)|x-\tilde{x}|^{\frac{k-1}{2}}. \tag{2.43}$$

The estimation of L_3 is analogous to that of I_3 and we obtain

$$L_3 \leq c_{2.45.26}|x-\tilde{x}|^{\frac{2k-1}{4}}. \tag{2.44}$$

We now proceed to estimate L_4.

$$L_4 \leq \mathbb{P}\left[\left(\int_0^t \int |p_{t-r+\varepsilon}'(y_r - x) - p_{t-r+\varepsilon}'(y_r - \tilde{x})| J_r(dy) dr^{2k-1}\right.\right.$$

$$\times \int_0^t \int |p'_{t-r+\varepsilon}(y_r - x) - p'_{t-r+\varepsilon}(y_r - \tilde{x})|\tilde{\mathcal{E}}_r(1)^{2k} \hat{h}_r^{2k} J_r(dy)dr \Big)^{1/2} \Bigg]$$

(by Jensen)

$$\leq \mathbb{P} \left[\left[\int_0^t \int |p'_{t-r+\varepsilon}(y_r - x) - p'_{t-r+\varepsilon}(y_r - \tilde{x})| J_r(dy)dr^{2k-1} \right]^{1/2} \right.$$
$$\times \mathbb{P} \left[\int_0^t \int |p'_{t-r+\varepsilon}(y_r - x) - p'_{t-r+\varepsilon}(y_r - \tilde{x})|\tilde{\mathcal{E}}_r(1)^{2k}\hat{h}_r^{2k} J_r(dy)dr \right]^{1/2}$$

(by Schwarz)

$$= \mathbb{P} \left[\left[\int_0^t \int_{\mathbf{R}} |p'_{t-r+\varepsilon}(a - x) - p'_{t-r+\varepsilon}(a - \tilde{x})| g(r,a)da dr^{2k-1} \right]^{1/2} \right.$$
$$\times \int_0^t \widehat{\mathbb{P}}_r^J \left[|p'_{t-r+\varepsilon}(y_r - x) - p'_{t-r+\varepsilon}(y_r - \tilde{x})|\tilde{\mathcal{E}}_r(1)^{2k}\hat{h}_r^{2k} \right] dr^{1/2}$$

$$\leq \int_0^t \int_{\mathbf{R}} |p'_{t-r+\varepsilon}(a - x) - p'_{t-r+\varepsilon}(a - \tilde{x})| da dr^{k-1}$$
$$\times \int_0^t \int_{\mathbf{R}} |p'_{t-r+\varepsilon}(a - \tilde{x}) - p'_{t-r+\varepsilon}(a - \tilde{x})| \mathbb{P}[g(r,a)^{2k-1}] da dr^{1/2}$$

$$c_{UB}^k \aleph(2k,1)^{1/2} \times \int_0^t \widehat{\mathbb{P}}_r^J \left[|p'_{t-r+\varepsilon}(y_r - x) - p'_{t-r+\varepsilon}(y_r - \tilde{x})|\tilde{\mathcal{E}}_r(2k) \right] dr^{1/2}$$

(by Jensen)

$$= c_{2.45.27}(k)|x - \tilde{x}|^{\frac{k-1}{2}} \int_0^t \widehat{\mathbb{P}}_r^J \left[|p'_{t-r+\varepsilon}(y_r - x) - p'_{t-r+\varepsilon}(y_r - \tilde{x})|\tilde{\mathcal{E}}_r(2k) \right] dr^{1/2}$$

(by Lemma 2.44(a))

$$\equiv c_{2.45.27}(k)|x - \tilde{x}|^{\frac{k-1}{2}} L_{4.1} \tag{2.45}$$

Arguing as in the estimation of $I_{4.1}$ we see that $L_{4.1}$ is bounded by a constant. Therefore

$$L_4 \leq c_{2.45.28}(k)|x - \tilde{x}|^{\frac{k-1}{2}} \tag{2.46}$$

The estimation of L_5 is identical to that of L_4. We obtain

$$L_5 \leq c_{2.45.29}(k)|x - \tilde{x}|^{\frac{k-1}{2}} \tag{2.47}$$

Combining the estimates (2.41), (2.43), (2.44), (2.46), (2.47) with (2.40) we see that for any $x, \tilde{x}$

$$\mathbb{P}[|u(t,x,\varepsilon) - u(t,\tilde{x},\varepsilon)|^k] \leq c_{2.45.30}(k)|x - \tilde{x}|^{\frac{k-1}{2}} \tag{2.48}$$

To estimate $\mathbb{P}[|u(t,x,\varepsilon) - u(t,x,\delta)|^k]$ we follow the procedure used to estimate $\mathbb{P}[|u(t,x,\varepsilon) - u(s,x,\varepsilon)|^k]$. Only a few trivial changes are required. We obtain

$$P[|u(t,x,\varepsilon) - u(t,x,\delta)|^k] \leq c_{2.45.31}|\varepsilon - \delta|^{\frac{k-1}{4}}. \tag{2.49}$$

Estimates (2.39), (2.48) and (2.49) show that the hypothesis of Lemma 2.43 (b) hold. $\blacksquare$

Remark 2.46. (a) In this chapter we focused on the projection X of a solution K of the strong equation $(HSE)_{b,\gamma}$. (See Definition 2.4.) However, we could have chosen the projection $X = \bar{\Pi}.(K)$ of a solution K of the following martingale problem as initial data. Let $m\, b,\, \gamma$ be as in Theorem 2.9. Suppose that for all $\phi \in D_0$

$$K_t(\phi) = m(\phi) + \int_0^t \int \left(\frac{1}{2} \bar{\Delta}\bar{\phi}(s,y) + b(s, \bar{\Pi}_s(K), y) \nabla \bar{\phi}(s,y) \right) K_s(dy)ds + Z_t(\phi),$$

where $Z_t(\phi)$ is a continuous square integrable martingale null at zero with square function

$$\langle Z(\phi) \rangle_t = \int_0^t K_s(\gamma_s \phi^2) ds.$$

With this definition of X the conclusions of Theorem 2.9 hold. The starting point of the proof would be a slight variation of Proposition 2.24, where we would need to employ Dawson's Girsanov theorem as in Remark 2.4 of Perkins (1995). The remainder of the proof would be essentially unchanged. If $\gamma \equiv 1$ then our proof works without any changes (see Remark 2.23), although some statements like the conclusion of Proposition 2.24 are trivial. $\quad\square$

2.5 Examples: Super Brownian Motions with Singular Interactions

This section is an application of the foregoing.

We begin by recalling an example of Sznitman (1989). Roughly speaking the model is the following: N particles are placed at random (independently) in the real line at time $t = 0$. They evolve in time according to the dynamics:

$$dX_t^i = dw_t^i + \frac{1}{N} \sum_{j=1}^N \delta_0(X_t^i - X_t^j)dt, \; i = 1...N, \tag{2.50}$$

where δ is Dirac's delta function, $(w^i)_{i=1}^N$ are independent one-dimensional Brownian motions and X_t^i represents the position of the i-th particle at time t. Thus the particles feel a "push" in the positive direction when they collide. Between collisions they follow independent Brownian motions. Sznitman showed (among other things) that when $N \to \infty$ a law of large numbers phenomenon occurs and the empirical distribution of the particles

$$X_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X_t^i} \tag{2.51}$$

converges to a non-random limit X^∞. In the limit each particle follows an independent copy of the "nonlinear" process $\bar{X}$:

$$d\bar{X}_t = dw_t + u(t, \bar{X}_t)dt \tag{2.52}$$
$$X_{t=0} = X_0,$$

where w is a one-dimensional Brownian motion and $u(t, x)$ is the density of the random variable $\bar{X}_t$. Moreover $X_t^\infty(dx) = u(t, x)dx$. Notice that the nonlinear process has marginals which satisfy in a weak sense Burgers' equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(u^2). \tag{2.53}$$

Indeed, for any $\phi \in C_b^2(\mathbb{R})$, (the classical version of) Itô's lemma and (2.52) yield

$$\phi(X_t) = \phi(X_0) + \int_0^t \phi'(X_s)dB_s + \int_0^t \left(\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(X_s) + u(s, X_s)\phi'(X_s)\right)ds.$$

Now take expectations to obtain

$$\mathbb{E}[\phi(X_t) - \phi(X_0)] = \int_0^t \mathbb{E}\left(\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(X_s) + u(s, X_s)\phi'(X_s)\right)ds.$$

Recall that $u(s, \cdot)$ is the density of X_s so that for example $E[\phi(X_t)] = \int \phi(x)u(t, x)dx$. Hence

$$\int \phi(x)(u(t, x) - u(0, x))dx = \int_0^t \int \left(\frac{1}{2}\phi(x)''u(s, x) + \phi(x)'u(s, x)^2\right)dxds.$$

Integrate by parts on the r.h.s.,

$$\int \phi(x)u((t, x) - u(0, x))dx = \int_0^t \int \left(\frac{1}{2}\phi(x)\frac{\partial^2 u}{\partial x^2}(s, x) - \phi(x)\frac{\partial}{\partial x}\left(u(s, x)^2\right)\right)dxds.$$

This last equation can also be obtained multiplying (2.53) by ϕ and integrating over space-time. Sznitman (1989) proved that $u(t, x)$ is in fact a classical solution of Burgers' equation (2.53). We refer the reader to the paper Sznitman (1989) for the proof of this and many more very interesting results.

Sznitman's work motivated us to look at the following examples.

Example 1 Let X^N be the critical Bienaymé-Galton-Watson tree of one-dimensional Brownian motions defined by (0.1) with $\gamma = 1$ (see page 3).

$$X_t^N = \frac{1}{N}\sum_{i=1}^{I(N,t)} \delta_{Z_t^i}.$$

Recall that $I(N, t)$ labels the particles alive at time t and Z_t^i, $i = 1, \ldots, I(N, t)$ labels their locations. Consider the following interacting particle system U^N. The family structure of U^N is identical to that of X^N. The particles belonging to U^N are labeled $U^{N,i}$ and obey the following dynamics.

$$dU_t^{N,i} = dZ_t^i + \frac{1}{N}\sum_{j=1}^{I(N,t)} \delta_0(U_t^{N,i} - U_t^{N,j})dt, \quad j = 1, \ldots, I(N, t). \tag{2.54}$$

(Note the similarity between equations (2.50) and (2.54)). U^N is defined by

$$U_t^N = \frac{1}{N}\sum_{i=1}^{I(N,t)} \delta_{U_t^{N,i}}.$$

This model is very similar to Sznitman's, but it has one extra ingredient. Instead of N independent Brownian motions, the driving noise is now a system of branching Brownian motions. Several natural questions arise. Does the sequence (U^N) converge weakly to an interacting superprocess? If so, is there a unique limit? What martingale problem is solved by the limit points? Note that since the interactions are singular, this model is not covered by the results of Chapter 1. We are currently working on these questions. We conjecture that the sequence (U^N) possesses at least one limit point which solves the stochastic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\Delta}{2} u - \frac{\partial}{\partial x}(u^2) + \sqrt{u}\, \dot{W}.$$

Here W is a space time white noise. Recall that a white noise W on $[0, \infty) \times \mathbb{R}$ is a stochastic process $(W(A) : A \in \mathcal{B}_f)$, where $\mathcal{B}_f$ denotes the Borel subsets of $[0, \infty) \times \mathbb{R}$ of finite Lebesgue measure ($|A| < \infty$), such that if A and B are disjoint subsets in $\mathcal{B}_f$, then $W(A)$ and $W(B)$ are independent mean zero Gaussian random variables with variances $|A|$ and $|B|$, respectively. However, if we replace the delta function by a smooth approximation then the model will in fact be covered by our results. Let $p_t(x)$ be the one dimensional Brownian transition density. Fix $\varepsilon > 0$ and consider the system $U^{\varepsilon,N}$ defined by

$$U_t^{\varepsilon,N,i} = Z_t^i + \frac{1}{N} \sum_{j=1}^{I(N,t)} p_\varepsilon(U_t^{\varepsilon,N,i} - U_t^{\varepsilon,N,j})dt, \quad j = 1, \ldots; I(N,t).$$

and

$$U_t^{\varepsilon,N} = \frac{1}{N} \sum_{i=1}^{I(N,t)} \delta_{U_t^{\varepsilon,N,i}}.$$

By Theorem 1.27 the sequence $(U^{\varepsilon,N})_N$ (ε is fixed) converges weakly to a unique (in law) superprocess U^ε characterized by the martingale problem $(MP)_{1,1,p_\varepsilon,0}^m$.

$$\forall \psi \in C_b^2(\mathbb{R}) \qquad Z_t^{U^\varepsilon}(\psi) = U_t^\varepsilon(\psi) - U_0^\varepsilon(\psi) - \int_0^t \int \left[\psi'(x)U_s^\varepsilon(p_\varepsilon(x - \cdot)) + \frac{1}{2}\psi''(x)\right]U_s^\varepsilon(dx)ds$$

is a continuous square integrable martingale such that

$$\langle Z_t^{U^\varepsilon}(\psi)\rangle_t = \int_0^t U^\varepsilon(\psi^2)ds \quad \forall t \geq 0 \quad a.s.$$

Theorem 5.6 of Perkins (1995) implies that U^ε solves the historical strong equation

$$Y_t = y_t + \int_0^t \int p_\varepsilon(Y_s - x)U_s(dx)ds$$

$$U_t(\cdot) = \int 1(Y_t \in \cdot)H_t(dy).$$

Theorem 2.9 shows that there is a jointly continuous function $u^\varepsilon(t,x)$ such that $U_t^\varepsilon(dx) = u^\varepsilon(t,x)dx$. After integration by parts (see page 3 for a similar computation) we see that u^ε solves the stochastic partial differential equation

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\Delta}{2} u^\varepsilon - \frac{\partial}{\partial x}\left(u^\varepsilon \cdot p_\varepsilon * u^\varepsilon\right) + \sqrt{u^\varepsilon}\, \dot{W},$$

where W is a space time white noise and $*$ denotes convolution in the space variable x. □

Example 2 In this example we look at a super Brownian motion in a super Brownian field. We work on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ rich enough to carry all the random processes defined below. Let X_1^N and X_2^N be two independent trees of critically branching one-dimensional Brownian motions (see page 3). The notation is as in the previous example. $I_1(N, t)$ (resp. $I_2(N, t)$) labels the set of particles belonging to X_1 (resp. X_2) alive at time t and Z_1^i (resp. Z_2^i) labels their locations.

$$X_1^N(t) = \frac{1}{N} \sum_{i=1}^{I_1(N,t)} \delta_{Z_1^i(t)},$$

$$X_2^N(t) = \frac{1}{N} \sum_{i=1}^{I_2(N,t)} \delta_{Z_2^i(t)}.$$

We define a process U^N as follows. The family structure of U^N is identical to that of X_1^N. The particles belonging to U^N are labeled $U^{N,i}$ and obey the dynamics

$$dU_t^{N,i} = dZ_1^i(t) + \frac{1}{N} \sum_{j=1}^{I_2(N,t)} \delta_0(U_t^{N,i} - Z_2^j(t))dt, \quad i = 1, \ldots, I_1(N, t). \tag{2.55}$$

(Note the similarity between equations (2.50), (2.54) and (2.55)). U^N is defined by

$$U_t^N = \frac{1}{N} \sum_{i=1}^{I_1(N,t)} \delta_{U_t^{N,i}}. \tag{2.56}$$

When $N \to \infty$, X_1^N converges to a super Brownian motion X_1, while X_2^N converges to a super Brownian motion X_2. Call H_1 and H_2 the historical processes corresponding to X_1 and X_2 respectively. We expect that the sequence (U^N) will converge to a superprocess U. In fact the system of equations (2.55)–(2.56) should converge to the historical strong equation

$$dY_t = dy_1(t) + \int \delta(Y_t - y_2(t))H_2(t)(dy_2)dt$$

$$U_t(\cdot) = \int 1(Y_t \in \cdot)H_1(t)(dy_1). \tag{2.57}$$

Note that (2.55)–(2.56) and (2.57) are equivalent if (U, H_1, H_2) are replaced by (U^N, H_1^N, H_2^N). The convergence does not follow from the results of Chapter 1. We do not prove that result here (although we hope to prove it in a future work). Instead we work directly with the limiting system (2.57), which we call super Brownian motion in a super Brownian field. Recall that one dimensional super Brownian motion has a jointly continuous density *a.s.* Therefore we may write $X_2(t)(dx) = v(t, x)dx$. *We now condition on X_2.* Therefore we consider v as a fixed function and so (2.2) holds. We may recast (2.57) in the form

$$Y_t = y_t + \int_0^t v(s, Y_s)ds$$

$$U_t(\cdot) = \int 1(Y_t \in \cdot)H_1(t)(dy). \tag{2.58}$$

We now show that (2.58) has a solution. Since v is uniformly bounded and measurable, the method of Zvonkin (1974) shows that for any bounded $(\mathcal{F}_t)$-stopping time the equation

$$Y_t = y_t + \int_0^t v(s, Y_s)ds$$

has a unique non-exploding strong solution on $(\widehat{\Omega}, \widehat{\mathbb{P}}_T^{H_1})$. We can use the section theorem and the procedure in the proof of Theorem 2.12 of Perkins (1995) to show that there is a universal version of $Y_t(\omega, y)$ such that

$$Y_t(\omega, y) = y_t + \int_0^t v(s, Y_s(\omega, y))ds, \quad H_1 - a.e.$$

We then use the second equation in (2.58) to define U ($\mathbb{P} - a.s.$). Apply Theorem 2.9 to the system (2.58). Therefore, given X_2, there is an $a.s.$ jointly continuous function $u(t, x)$ such that $U_t(dx) = u(t, x)dx$ $a.s.$ □

Remark 2.47. In fact it is not necessary to condition on X_2. But then we must deal with some lengthy technical details. We have decided to leave them out but hope to include them in some future work. □

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Chapter 3
Local Times for One-Dimensional Interacting Superprocesses

3.1 Introduction and Statement of Results

In this Chapter we study the local times for interacting superprocesses. We begin by describing the notions of superprocess occupation and local times. We consider first the case of super Brownian motion W. Throughout this Chapter we shall restrict ourselves to the case where the underlying particle motions are one-dimensional. The occupation time process Y is defined by

$$Y_t(\phi) = \int_0^t W_s(\phi)ds \tag{3.1}$$

for any Borel $\phi : \mathbb{R} \to \mathbb{R}_+$.

Formally, the local time $L_t^a(W)$ of W is obtained by replacing ϕ by δ_a (Dirac's delta) in (3.1). Thus, if $u = u(t,x)$ is the jointly continuous density of super Brownian motion with respect to Lebesgue measure then

$$L_t^a(W) = \int_0^t u(s,a)ds.$$

This implies the existence of a jointly continuous version of $L_t^a(W)$ which is continuously differentiable in t. (See Sugitani (1988) for more information on regularity properties of $L_t^a(W)$). Note that $L_t^a(W)$ satisfies the *fundamental* density of occupation formula:

$$\int_0^t W_s(\phi)ds = \int_{-\infty}^\infty \phi(a)L_t^a(W)da.$$

Indeed,

$$\int_0^t W_s(\phi)ds = \int_0^t \int_{-\infty}^\infty \phi(a)u(s,a)dads$$

$$= \int_{-\infty}^\infty \int_0^t \phi(a)u(s,a)dsda$$

$$= \int_{-\infty}^\infty \phi(a)L_t^a(W)da.$$

In general we define local times for superprocesses to be occupation densities. We have the following:

Definition 3.1. Let $(X, \Omega, (\mathcal{F}_t), \mathbb{P})$ be a $M_F(\mathbb{R})$-valued process, i.e. for each t, X_t is a finite random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that $L_t^a(X)$ is the local time of X if the following conditions are satisfied:

(i) The map $(a, t) \longmapsto L_t^a(X)$ is $\mathbb{P} - a.s.$ continuous.

(ii) For any nonnegative Borel function ϕ

$$\int_0^t X_s(\phi)ds = \int_{-\infty}^\infty \phi(a)L_t^a(X)da \qquad (3.2)$$

□

As an immediate consequence of Theorem 2.9 we obtain

Corollary 3.2. *The interacting superprocess X described in Chapter 2 possesses a local time $L_t^a(X)$ which is jointly continuous and continuously differentiable in t.*

Our goal in this chapter is to show that more general interacting one-dimensional superprocesses have local times. The measure-valued processes we will consider arise as solutions to strong historical equations (see the Introduction of Chaper 2). We must therefore introduce diffusion, drift and mass coefficients. We will also need some notation. In this Chapter $\bar{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ will denote a filtered probability space satisfying the usual hypothesis. We denote $C = C^1 = C([0, \infty), \mathbb{R})$ and call $(\mathcal{C}_t)$ (resp. $\mathcal{C}$) its canonical filtration (resp. its Borel σ-field). H is a one-dimensional historical Brownian motion on $\bar{\Omega}$ starting at $m \in M_F(\mathbb{R})$. $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)) = (\Omega \times C, \mathcal{F} \times C, (\mathcal{F}_t \times \mathcal{C}_t))$ and Z^H denotes the martingale measure associated with H (see p. 11 Perkins 1995). A set $\Lambda \subset [0, \infty) \times \widehat{\Omega}$ is $(H, \mathbb{P})$ evanescent (or H evanescent) iff $\Lambda \subset \Lambda_1$ where Λ_1 is $(\widehat{\mathcal{F}}_t^*)$-predictable and

$$\sup_{0 \leq u \leq t} 1_{\Lambda_1}(u, \omega, y) = 0 \quad H_t - a.e. \quad y \quad \forall t \geq 0 \quad \mathbb{P} - a.s.$$

A property holds $(H, \mathbb{P}) - a.e.$ (or $H - a.e.$) if it holds off an H-evanescent set. If T is a bounded $(\mathcal{F}_t)$-stopping time, $\widehat{\mathbb{P}}_T^H$ denotes the Campbell measure on $\widehat{\Omega}$ defined by $\widehat{\mathbb{P}}_T^H(A \times B) := \mathbb{P}(H_T(B)1_A)/\mathbb{P}(H_0(1))$ for $A \in \mathcal{F}$, $B \in \mathcal{C}$. Suppose that $\varepsilon > 0$. Let

$$\sigma : M_F(\mathbb{R}) \times \mathbb{R} \longrightarrow [\varepsilon, \infty),$$
$$b : M_F(\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R},$$
$$\gamma : [0, \infty) \times C([0, \infty), M_F(\mathbb{R})) \times C \longrightarrow (0, \infty).$$

The hypothesis on these coefficients are the following. Here $d = d_{|\cdot|}$ is the Vasershtein metric on $M_F(\mathbb{R})$ defined by

$$d(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| : f : \mathbb{R} \to \mathbb{R}, \|f\|_\infty \leq 1, |f(x) - f(y)| \leq |x - y| \,\forall x, y \in \mathbb{R}\}.$$

Boundedness by the total mass. There are non-decreasing functions $\Gamma, \Upsilon : [0, \infty) \to [1, \infty)$ such that

$$\sup_{x \in \mathbb{R}} |b(\mu, x)| + |\sigma(\mu, x)| \leq \Upsilon(\mu(1)) \quad \forall \mu \in M_F(\mathbb{R}), \tag{3.3}$$

$$\sup_{y \in C^1} \gamma(t, X, y) \leq \Gamma(t) \left(1 + \int_0^t X_s^*(1) \, ds \right) \quad \forall X \in C([0, \infty), M_F(\mathbb{R})). \tag{3.4}$$

Lipschitz condition.

$$|\sigma(\mu, x) - \sigma(\nu, z)| + |b(\mu, x) - b(\nu, z)| \leq \Upsilon(\mu(1) \vee \nu(1))(|x - z| + d(\mu, \nu)). \tag{3.5}$$

The reader will find many examples of coefficients b, σ, γ satisfying the conditions above in Chapter 1 and in Chapter 4 of (Perkins 1995).

We say that (X, Y) is a solution of

$(HSE)_{\sigma, b, \gamma}$

$$Y_t = Y_0 + \int_0^t \sigma(X_s, Y_s) dy(s) + \int_0^t b(X_s, Y_s) ds, \tag{i}$$

$$X_t(\phi) = \int \gamma(t, X, Y^t) \phi(Y_t) H_t(dy). \tag{ii}$$

iff

- $Y : [0, \infty) \times \widehat{\Omega} \to \mathbb{R}$ is $(\widehat{\mathcal{F}}_t^*)$-predictable, $X : [0, \infty) \times \Omega \to M_F(\mathbb{R})$ is $(\mathcal{F}_t)$-predictable, $X \in C([0, \infty), M_F(\mathbb{R}))$.

- $(HSE)_{\sigma, b, \gamma}$(ii) holds for all $\phi : \mathbb{R} \to \mathbb{R}$ bounded measurable for all $t \geq 0$ $\mathbb{P} - a.s.$ and $(HSE)_{\sigma, b, \gamma}$(i) holds $H - a.e.$

The stochastic integral in the r.h.s. of $(HSE)_{\sigma, b, \gamma}$(i) was defined in Proposition 2.15.

Here is the main result of this chapter:

Theorem 3.3. *Let X be a solution of $(HSE)_{\sigma, b, \gamma}$. Then X has a local time $L_t^a(X)$.*

The basic tool needed in the proof is a Tanaka-like formula of Perkins (1995). We shall not need its more general form. We will employ the following version.

Theorem 3.4. *Assume $L : [0, \infty) \times \widehat{\Omega} \to [0, \infty)$ is $(\widehat{\mathcal{F}}_t^*)$-predictable, $L(\cdot, \omega, y)$ is non-decreasing, left continuous for H_t-a.e. y for all $t \geq 0$ $\mathbb{P} - a.s.$ Also assume*

$$\int_0^t H_s(L_s^2) \, ds < \infty \quad \forall t > 0 \quad \mathbb{P} - a.s.$$

Then there is an a.s. non-decreasing, left continuous $[0, \infty)$-valued $(\mathcal{F}_t)$-predictable process $A_t(L) \equiv \int_0^t H_s(dL_s)$ such that $A_0(L) = 0$ and

$$H_t(L_t) = H_0(L_0) + \int_0^t \int L(s, \omega, y) Z^H(ds, dy) + \int_0^t H_s(dL_s) \quad \forall t > 0 \quad a.s.$$

Moreover if L is continuous H-a.e. then $A_t(L)$ is a.s. continuous.

Proof. This is a special case of Theorem 2.24 (Perkins 1995). ∎

We illustrate the idea of the proof of Theorem 3.3 with the particular example of super Brownian motion. Recall that if T is a bounded $(\mathcal{F}_t)$ stopping time, $B_t(\omega, y) = y_t - y_0$ is a Brownian motion stopped at T on $(\Omega \times C, \mathcal{F} \times C, \widehat{\mathbb{P}}_T^H)$ (see Proposition 2.14). Let $\mathcal{L}_t^a(\omega, y)$ be its local time. $\mathcal{L}$ is normalized so that $\int_0^t \phi(B_s)ds = \int_{\mathbb{R}} \phi(a)\mathcal{L}_t^a da$. Applying Theorem 3.4 with $L_s = \mathcal{L}_s^a$ we get

$$H_t(\mathcal{L}_t^a) = \int_0^t \int \mathcal{L}_s^a Z^H(ds, dy) + \int_0^t H_s(d\mathcal{L}_s^a). \tag{3.6}$$

The second term on the right hand side of (3.6) is precisely the local time of super Brownian motion. This is intuitively clear from the particle picture. A straightforward computation shows that $L_t^a(W) \equiv \int_0^t H_s(d\mathcal{L}_s^a)$ satisfies (3.2). Using the representation of the local time furnished by (3.6) and Kolmogorov's criterion we are able to prove that the local time is indeed jointly (Hölder) continuous.

Remark 3.5. (a) Another way to verify that $L_t^a(W)$ as defined above is in fact the density of occupation is the following. For simplicity we consider only the local time at 0. Let R^λ be the λ-Green's function

$$R^\lambda(x) = \int_0^\infty e^{-\lambda t} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \, dt.$$

By Ito's lemma, the local time of Brownian motion satisfies the Tanaka formula (see e.g. p.14 Adler 1992):

$$\mathcal{L}_s^0 = R^\lambda(B_0) - R^\lambda(B_t) + \int_0^t (R^\lambda)'(B_s)dB_s + \lambda \int_0^t R^\lambda(B_s)ds. \tag{3.7}$$

Now, $H_t(\mathcal{L}_t^0)$ may be computed using the representation (3.7) for $\mathcal{L}_t^0$ together with Ito's lemma for historical integrals (Proposition 2.18.) It yields (formally)

$$H_t(\mathcal{L}_t^0) = -H_t(R^\lambda(y_t) - R^\lambda(y_0)) + \int_0^t \int \int_0^s (R^\lambda)'(y_r)dy_r Z^H(ds, dy)$$

$$+ \lambda \int_0^t \int \int_0^s R^\lambda(y_r)dr Z^H(ds, dy) + \lambda \int_0^t \int R^\lambda(y_s)H_s(dy)ds$$

$$= -\int_0^t \int (R^\lambda(y_s) - R^\lambda(y_0))Z^H(ds, dy) - \frac{1}{2}\int_0^t \int (R^\lambda)''(y_s)H_s(dy)ds$$

$$+ \int_0^t \int \int_0^s (R^\lambda)'(y_r)dy_r Z^H(ds, dy) + \lambda \int_0^t \int \int_0^s R^\lambda(y_r)dr Z^H(ds, dy)$$

$$+ \lambda \int_0^t \int R^\lambda(y_s)H_s(dy)ds$$

$$= \int_0^t \int \left(R^\lambda(y_0) - R^\lambda(y_s) + \int_0^s (R^\lambda)'(y_r)dy_r + \lambda \int_0^s R^\lambda(y_r)dr \right) Z^H(ds, dy)$$

$$+ \int_0^t \int \left(\lambda R^\lambda(y_s) - \frac{1}{2}(R^\lambda)''(y_s) \right) H_s(dy)ds$$

$$= \int_0^t \int \mathcal{L}_s^0 Z^H(ds, dy) + \int_0^t \int \left(\lambda R^\lambda(y_s) - \frac{1}{2}(R^\lambda)''(y_s) \right) H_s(dy)ds$$

Hence

$$H_t(\mathcal{L}_t^0) = \int_0^t \int \mathcal{L}_s^0 Z^H(ds, dy) + \int_0^t H_s(\delta_0)ds \qquad (3.8)$$

$$\text{(since } R^\lambda - \frac{1}{2}(R^\lambda)'' + \lambda R^\lambda = \delta_0).$$

Comparing (3.6) and (3.8) we see that $\int_0^t H_s(d\mathcal{L}_s^0) = \int_0^t H_s(\delta)ds$.

b) All the arguments in this Subsection concerning super Brownian motion are easily made rigorous. However, all these results are elementary and their proofs may be found elsewhere.

c) Local time for super Brownian motion is known to exist in dimensions $d \leq 3$. The analogous result for interacting superprocesses is currently under investigation. The cases $d = 2, 3$ are harder than the case $d = 1$, since super Brownian motion does not have a density in dimensions greater than one. □

This chapter is organized as follows. Section 3.2 is devoted to the proof of Theorem (3.3). We define a family of random variables $\{L_t^a(X); a \in \mathbb{R}, t \in \mathbb{R}_+\}$ and show that it satisfies the conditions of Definition 3.1.

3.2 Existence and Regularity of Local Times

Recall that for any bounded $(\mathcal{F}_t)$-stopping time T the process Y defined by $(HSE)_{\sigma, b, \gamma}$ is a semimartingale on the space $\widehat{\Omega} = (\Omega \times C, (\mathcal{F}_t \times \mathcal{C}_t), \widehat{\mathbb{P}}_T^H)$. We wish to define an $(\mathcal{F}_t \times \mathcal{C}_t)$-predictable process $\mathcal{L}_t^a(\omega, y)$ which agrees with the semimartingale local time of Y in $\widehat{\Omega} \; \widehat{\mathbb{P}}_T^H - a.s.$ for all bounded $(\mathcal{F}_t)$-stopping times T. We use Tanaka's formula for semimartingales.

Definition 3.6. For any $a \in \mathbb{R}$ and $t \geq 0$ let

$$\mathcal{L}_s^a = |Y_t - a| - |Y_0 - a| - \int_0^t \mathrm{sgn}(Y_s - a)\sigma(X_s, Y_s)dy(s)$$

$$- \int_0^t \mathrm{sgn}(Y_s - a)b(X_s, Y_s)ds. \qquad (3.9)$$

Note that all the terms in the left hand side of (3.9) are defined up to $(H, \mathbb{P})$-evanescent sets in $[0, \infty) \times \widehat{\Omega}$. Also, $\int_0^t \phi(Y_s)d\langle Y \rangle_s = \int \phi(a)\mathcal{L}_t^a da \; H - a.e.$ in $[0, \infty) \times \widehat{\Omega}$ (see Section 1 of Chapter VI of Revuz-Yor (1991)). □

We shall need the following technical results.

Lemma 3.7. *Let*

$$T(n) = \inf\{s > 0 : H_s(1) \geq n\}. \qquad (3.10)$$

Then there is a function $\Theta : \mathbf{N} \times [0, \infty) \to \mathbb{R}_+$, non-decreasing in each variable such that

$$X_s^*(1) \leq \Theta(n, s) \quad on \quad \{T(n) \geq s\}. \qquad (3.11)$$

Proof. This statement is proved in p. 61 of (Perkins 1995). In the notation therein $\Theta(n,s) = \Gamma_1(n)e^{\Gamma_1(n)s}$. The hypothesis (3.4) on γ is needed here. ∎

Remark 3.8. In our present setting it is not necessarily true that for $p \in \mathbf{N}$, $\mathbb{E}[X_s^*(1)^p] < \infty$. □

Lemma 3.9. *Assume T is a bounded $(\mathcal{F}_t)$-stopping time and $\psi \in b\widehat{\mathcal{F}}_{T \wedge s}$. Then*

$$H_{T \wedge t}(\psi) = H_{T \wedge s}(\psi) + \int_{s \wedge T}^{t \wedge T} \int \psi \, Z^H(dr, dy) \quad \forall t \geq s \quad a.s.$$

Proof. This is a particular case of Proposition 2.7 of Perkins (1995). ∎

Lemma 3.10. *Let $L : [0, \infty) \times \Omega \to [0, \infty)$ be $(\mathcal{F}_t)$-optional. and let S be an $(\mathcal{F}_t)$-stopping time. Then*

$$\mathbb{P}[L^*(S) > \varepsilon] = \sup_{T \leq S} \mathbb{P}[L(T) > \varepsilon] \quad \forall \varepsilon > 0,$$

where the sup on the right is over all $(\mathcal{F}_t)$-stopping times T bounded by S.

Proof. This result is a consequence of the optional section theorem. See Lemma 3.7 of Perkins (1993). ∎

Suppose we could apply Theorem 3.4 to the process $L_s = \mathcal{L}_s^a$. We would obtain

$$\int \mathcal{L}_t^a H_t(dy) = \int_0^t \int \mathcal{L}_s^a Z^H(ds, dy) + \int_0^t H_s(d\mathcal{L}_s^a) \quad \forall t \quad \mathbb{P} - a.s. \tag{3.12}$$

The next argument shows that $\mathcal{L}_s^a$ satisfies the hypotheses of the Theorem 3.4 so that (3.12) holds. Indeed, for any bounded $(\mathcal{F}_t)$-stopping time T, $\mathcal{L}^a$ is non-decreasing on $[0, T]$ $\widehat{\mathbb{P}}_T^H - a.s.$ The section theorem shows that $\mathcal{L}^a$ is continuous, non-negative and non-decreasing $H - a.e.$ on $[0, \infty) \times \widehat{\Omega}$. To finish, we need to check that $\int_0^t H_s((\mathcal{L}_s^a)^2)ds < \infty$ $\mathbb{P} - a.s.$ Define stopping times $T(n)$ as in (3.10). We compute

$$\mathbb{P}\left[H_{s \wedge T(n)}(|Y_{s \wedge T(n)} - a|^2)\right]$$

$$= \widehat{\mathbb{P}}_{s \wedge T(n)}^H\left[\left|Y_0 - a + \int_0^{s \wedge T(n)} \sigma(X_r, Y_r)dy(r) + \int_0^{s \wedge T(n)} b(X_r, Y_r)ds\right|^2\right]$$

$$\leq 3\widehat{\mathbb{P}}_{s \wedge T(n)}^H[|Y_0 - a|^2] + 3c_2\widehat{\mathbb{P}}_{s \wedge T(n)}^H\left[\int_0^{s \wedge T(n)} \sigma(X_r, Y_r)^2 dr\right]$$

$$+ 3\widehat{\mathbb{P}}_{s \wedge T(n)}^H\left[\left|\int_0^{s \wedge T(n)} b(X_r, Y_r)dr\right|^2\right]$$

(by Burkholder)

$$\leq 3\left(\widehat{\mathbb{P}}_{s \wedge T(n)}^H[|Y_0 - a|^2] + c_2\widehat{\mathbb{P}}_{s \wedge T(n)}^H\left[\int_0^{s \wedge T(n)} \Upsilon(X_r(1))^2 dr\right]\right.$$

$$+ \widehat{\mathbb{P}}_{s \wedge T(n)}^H \left[\int_0^{s \wedge T(n)} \Upsilon(X_r(1)) dr^2 \right] \Bigg)$$

(by (3.3))

$$\leq 3 \left(\widehat{\mathbb{P}}_{s \wedge T(n)}^H [|Y_0 - a|^2] + c_2 \widehat{\mathbb{P}}_{s \wedge T(n)}^H \left[\int_0^{s \wedge T(n)} \Upsilon(\Theta(n, r))^2 dr \right] \right.$$

$$+ \widehat{\mathbb{P}}_{s \wedge T(n)}^H \left[\int_0^{s \wedge T(n)} \Upsilon(\Theta(n, r)) dr^2 \right] \Bigg)$$

(by Lemma 3.7)

$$\leq 3 \left(\widehat{\mathbb{P}}_{s \wedge T(n)}^H [|Y_0 - a|^2] + c_2 s \Upsilon(\Theta(n, s))^2 + s^2 \Upsilon(\Theta(n, s))^2 \right)$$

$$\equiv c_{3.12.1}(s, n). \tag{3.13}$$

Similarly,

$$\mathbb{P} \left[H_{s \wedge T(n)} \left(\left| \int_0^{s \wedge T(n)} \text{sgn}(Y_r - a) \sigma(X_r, Y_r) dy(r) \right|^2 \right) \right]$$

$$\leq \mathbb{P} \left[H_{s \wedge T(n)} \left(\int_0^{s \wedge T(n)} \sigma(X_r, Y_r)^2 dr \right) \right]$$

$$\equiv c_{3.12.2}(s, n). \tag{3.14}$$

The same reasoning also shows that

$$\mathbb{P} \left[H_{s \wedge T(n)} \left(\left| \int_0^{t \wedge T(n)} \text{sgn}(Y_s - a) b(X_s, Y_s) ds \right|^2 \right) \right] \leq c_{3.12.3}(s, n). \tag{3.15}$$

As a consequence of (3.9), (3.13), (3.14) and (3.15) we obtain

$$\sup_{0 \leq s \leq T(n)} \mathbb{P} \left[H_{s \wedge T(n)} (\mathcal{L}_{s \wedge T(n)}^a)^2 \right] < \infty \tag{3.16}$$

Hence $\int_0^t H_s((\mathcal{L}_s^a)^2) ds < \infty$ $\mathbb{P} - a.s.$ and (3.12) holds.

Definition 3.11. We set

$$\widetilde{L}_t^a(X) := \int_0^t H_s(d\mathcal{L}_s^a) \tag{3.17}$$

$$= \int \mathcal{L}_t^a H_t(dy) - \int_0^t \int \mathcal{L}_s^a Z^H(ds, dy).$$

(See equation (3.12)). By Theorem 3.4 this process is $a.s.$ increasing and continuous in t. Therefore it defines a measure $\widetilde{L}_{ds}^a(X)$ on $\mathbb{R}_+$. Let

$$L_t^a(X) := \int_0^t \sigma^{-2}(X_s, a) \widetilde{L}_{ds}^a(X). \tag{3.18}$$

As the notation indicates, the random process $L_t^a(X)$ defined above is indeed the desired local time of X. We must now show that it fulfills the conditions of Definition 3.1. Note that the Lipschitz condition (3.5) implies that the mapping $s \mapsto \sigma^{-2}(X_s, a)$ is continuous and therefore integrable with respect to the measure $\widetilde{L}_{ds}^a(X)$.

First, we study the continuity of $L_t^a(X)$. *Without loss of generality, we fix $N > 0$ and restrict ourselves to the time interval $[0, N]$.* As usual, some localization procedure is needed. We shall stop the processes H, X, Y and $\mathcal{L}^a$ at $T(n)$. Note that $\mathcal{L}_{T(n)\wedge\cdot}^a$ is the local time of $Y^{T(n)}$ (in Campbell space.) Also, $Z_{\cdot \wedge T(n)}^H$ is the orthogonal martingale measure associated with $H^{T(n)}$. We shall need the following estimates.

Lemma 3.12. *For any $p > 1$ there are constants $c_{3.12.1}(n, N, p)$, $c_{3.12.2}(n, N, p)$, $c_{3.12.3}(n, N, p)$ such that for any $(\mathcal{F}_t)$-stopping time $T \leq T(n) \wedge N$ and any $x, z \in \mathbb{R}$, $s, t \in [0, N]$*

$$\sup_{a \in \mathbb{R}} \widehat{\mathbb{P}}_T^H[(\mathcal{L}_T^a)^p] \leq c_{3.12.1}(n, N, p) \tag{3.19}$$

$$\widehat{\mathbb{P}}_T^H[|\mathcal{L}_T^x - \mathcal{L}_T^z|^p] \leq c_{3.12.2}(n, N, p)|x - z|^{\frac{p}{2}} \tag{3.20}$$

and

$$\widehat{\mathbb{P}}_{T \wedge t}^H[|\mathcal{L}_{T \wedge t}^x - \mathcal{L}_{T \wedge s}^x|^p] \leq c_{3.12.3}(n, N, p)|t - s|^{\frac{p}{2}}. \tag{3.21}$$

providing $|x - z| \leq 1$, $|t - s| \leq 1$.

Proof. Barlow & Yor (1981) show that there is a constant C_p such that

$$\sup_a \|\mathcal{L}_T^a\|_{L^p} \leq (C_p)^{1/p} \|Y_T\|_{H^p}.$$

That is,

$$\sup_{a \in \mathbb{R}} \widehat{\mathbb{P}}_T^H[(\mathcal{L}_T^a)^p] \leq C_p \widehat{\mathbb{P}}_T^H \left[\left| Y_0 + \int_0^T \sigma(X_s, Y_s)^2 ds^{1/2} + \int_0^T |b(X_s, Y_s)| ds \right|^p \right].$$

The quantity on the right hand side has already been estimated in the special case $p = 2$ (see the discussion preceeding Definition 3.11.) The case with arbitrary p is similar.

$$\sup_{a \in \mathbb{R}} \widehat{\mathbb{P}}_T^H[(\mathcal{L}_T^a)^p] \leq 3^{p-1} C_p \left(\widehat{\mathbb{P}}_T^H[|Y_0|^p] + \widehat{\mathbb{P}}_T^H \left[\int_0^T \sigma(X_s, Y_s)^2 ds^{p/2} \right] \right.$$

$$+ \widehat{\mathbb{P}}_T^H \left[\int_0^T |b(X_s, Y_s)| ds^p \right] \right)$$

$$\leq 3^{p-1} C_p \left(\widehat{\mathbb{P}}_T^H[|Y_0|^p] + \widehat{\mathbb{P}}_T^H \left[\int_0^T \Upsilon(X_s(1))^2 ds^{p/2} \right] \right.$$

$$+ \widehat{\mathbb{P}}_T^H \left[\int_0^T \Upsilon(X_s(1)) ds^p \right] \right)$$

$$\leq 3^{p-1} C_p \left(\widehat{\mathbb{P}}_T^H[|Y_0|^p] + \widehat{\mathbb{P}}_T^H \left[\int_0^T \Upsilon(\Theta(n, s))^2 ds^{p/2} \right] \right.$$

$$+ \widehat{\mathbb{P}}_T^H \left[\int_0^T \Upsilon(\Theta(n,s)) ds^p \right] \Bigg)$$

$$\text{(since } T \leq T(n))$$

$$\leq 3^{p-1} C_p \left(\widehat{\mathbb{P}}_T^H [|Y_0|^p] + N^{p/2} \Upsilon(\Theta(n,N))^p + N^p \Upsilon(\Theta(n,N))^p \right)$$

$$\equiv c_{3.12.1}(n, N, p).$$

This concludes the proof of (3.19).

We now show the validity of (3.20). Assume w.l.o.g. $x < z$.

$$\widehat{\mathbb{P}}_T^H[|\mathcal{L}_T^x - \mathcal{L}_T^z|^p] \leq 3^{p-1} \left(\widehat{\mathbb{P}}_T^H[|\,|Y_0 - x| - |Y_0 - z|\,|^p] + \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) \sigma(X_s, Y_s) dy(s)^p \right] \right.$$

$$\left. + \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) b(X_s, Y_s) ds^p \right] \right)$$

$$\text{(by 3.9)}$$

$$\leq 3^{p-1} \left(|x - z|^p + \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) \sigma(X_s, Y_s)^2 ds^{p/2} \right] \right.$$

$$\left. + \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) b(X_s, Y_s) ds^p \right] \right)$$

$$\text{(by Burkholder)}$$

$$\equiv 3^{p-1}(|x - z|^p + I_1 + I_2). \tag{3.22}$$

We estimate

$$I_1 = \widehat{\mathbb{P}}_T^H \left[\int_x^z \mathcal{L}_T^a da^{p/2} \right]$$

$$\text{(by the density of occupation formula)}$$

$$\leq |x - z|^{(p-2)/2} \sup_{a \in \mathbb{R}} \widehat{\mathbb{P}}_T^H [(\mathcal{L}_T^a)^{p/2}]$$

$$\leq c_{3.12.1}(n, N, p/2)|x - z|^{p/2} \tag{3.23}$$

$$\text{(by (3.19))}.$$

$$I_2 = \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) \frac{b(X_s, Y_s)}{\sigma(X_s, Y_s)^2} \sigma(X_s, Y_s)^2 ds^p \right]$$

$$\leq \varepsilon^{-2} \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) \Upsilon(X_s(1)) \sigma(X_s, Y_s)^2 ds^p \right]$$

$$\leq \varepsilon^{-2} \Upsilon(\Theta(n,N))^p \widehat{\mathbb{P}}_T^H \left[\int_0^T 1_{(x,z]}(Y_s) \sigma(X_s, Y_s)^2 ds^p \right]$$

$$= \varepsilon^{-2} \Upsilon(\Theta(n,N))^p \widehat{\mathbb{P}}_T^H \left[\int_x^z \mathcal{L}_T^a \, da^p \right]$$

$$\leq \varepsilon^{-2} \Upsilon(\Theta(n,N))^p \sup_{a \in \mathbb{R}} \widehat{\mathbb{P}}_T^H [(\mathcal{L}_T^a)^p]|x - z|^p$$

$$\leq \varepsilon^{-2} \Upsilon(\Theta(n,N))^p c_{3.12.1}(n, N, p)|x - z|^p. \tag{3.24}$$

Notice that (3.20) follows from (3.22), (3.23) and (3.24).

The proof of (3.21) is easier. Without loss of generality assume $t > s$.

$$\widehat{\mathbb{P}}_{T\wedge t}^{H}[|\mathcal{L}_{T\wedge t}^{x} - \mathcal{L}_{T\wedge s}^{x}|^{p}] \leq 2^{p-1}\left(\widehat{\mathbb{P}}_{T\wedge t}^{H}\Big[\Big|\int_{T\wedge s}^{T\wedge t}\mathrm{sgn}(Y_r - x)\sigma(X_r, Y_r)dy(r)^p\Big|\Big]\right.$$

$$+ \widehat{\mathbb{P}}_{T\wedge t}^{H}\Big[\Big|\int_{T\wedge s}^{T\wedge t}\mathrm{sgn}(Y_r - x)b(X_r, Y_r)dr^p\Big|\Big]\Big)$$

$$\leq 2^{p-1}\left(\widehat{\mathbb{P}}_{T\wedge t}^{H}\Big[\int_{T\wedge s}^{T\wedge t}\sigma(X_r, Y_r)^2 dr^{p/2}\Big]\right.$$

$$+ \widehat{\mathbb{P}}_{T\wedge t}^{H}\Big[\int_{T\wedge s}^{T\wedge t}|b(X_r, Y_r)|dr^p\Big]\Big)$$

$$\leq 2^{p-1}\left(\Upsilon(\Theta(n, N))^p|t-s|^{p/2} + \Upsilon(\Theta(n, N))^p|t-s|^p\right) \qquad (3.25)$$

Set $c_{3.12.3} = 2^p\Upsilon(\Theta(n, N))^p$. This proves (3.21). ∎

Lemma 3.13. *Fix $p \geq 1$. Assume $|x - z| \leq 1$, $|t - s| \leq 1$. There exist constants $c_{3.13.1}(N, n, p)$, $c_{3.13.2}(N, n, p)$, $c_{3.13.3}(N, n, p)$ such that*

$$\mathbb{P}\left[|\widetilde{L}_{t\wedge T(n)}^{x}(X)|^{p}\right] \leq c_{3.13.1}(N, n, p), \qquad (3.26)$$

$$\mathbb{P}\left[\sup_{s \leq T(n)}|\widetilde{L}_{s}^{x}(X) - \widetilde{L}_{s}^{z}(X)|^{p}\right] \leq c_{3.13.2}(N, n, p)|x - z|^{p/2} \qquad (3.27)$$

and

$$\mathbb{P}\left[|\widetilde{L}_{t\wedge T(n)}^{x}(X) - \widetilde{L}_{s\wedge T(n)}^{x}(X)|^{p}\right] \leq c_{3.13.3}(N, n, p)|t - s|^{p/2}. \qquad (3.28)$$

for any $t \in [0, N]$.

Proof. We estimate

$$\mathbb{P}\left[|\widetilde{L}_{t\wedge T(n)}^{x}(X)|^{p}\right] \leq 2^{p-1}\mathbb{P}[|H_{t\wedge T(n)}(\mathcal{L}_{t\wedge T(n)}^{x}(Y))|^{p}]$$

$$+ 2^{p-1}\mathbb{P}\left[\Big|\int_{0}^{t\wedge T(n)}\int \mathcal{L}_{s}^{x}(Y)Z^{H}(ds, dy)\Big|^{p}\right]$$

$$(\text{by } (3.12))$$

$$\equiv I_1 + I_2.$$

Now,

$$I_1 \leq 2^{p-1}\mathbb{P}[H_{t\wedge T(n)}(1)^{p-1}H_{t\wedge T(n)}((\mathcal{L}_{t\wedge T(n)}^{x})^{p})]$$

$$(\text{by Jensen})$$

$$\leq 2^{p-1}n^{p-1}\widehat{\mathbb{P}}_{t\wedge T(n)}^{H}[(\mathcal{L}_{t\wedge T(n)}^{x})^{p}]$$

$$\leq 2^{p-1}n^{p-1}c_{3.12.1}(n, N, p)$$

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(by Lemma 3.12)

By the same token,

$$I_2 \leq 2^{p-1} c_p \mathbb{P} \left[\int_0^{t \wedge T(n)} \int (\mathcal{L}^x_{s \wedge T(n)})^2 H_{s \wedge T(n)}(dy) ds^{p/2} \right]$$

(by Burkholder)

$$\leq 2^{p-1} c_p \mathbb{P} \left[\int_0^{t \wedge T(n)} H_s(1) ds^{\frac{p-2}{2}} \int_0^{t \wedge T(n)} \int (\mathcal{L}^x_{s \wedge T(n)})^p H_{s \wedge T(n)}(dy) ds \right]$$

(by Jensen)

$$\leq 2^{p-1} c_p n^{\frac{p-2}{2}} N^{\frac{p-2}{2}} \mathbb{P} \left[\int_0^{t \wedge T(n)} \int (\mathcal{L}^x_{s \wedge T(n)})^p H_{s \wedge T(n)}(dy) ds \right]$$

$$\leq 2^{p-1} c_p n^{\frac{p-2}{2}} N^{\frac{p-2}{2}} \int_0^N \mathbb{P} \left[\int (\mathcal{L}^x_{s \wedge T(n)})^p H_{s \wedge T(n)}(dy) \right] ds$$

$$= 2^{p-1} c_p n^{\frac{p-2}{2}} N^{\frac{p-2}{2}} \int_0^N \widehat{\mathbb{P}}^H_{s \wedge T(n)} \left[(\mathcal{L}^x_{s \wedge T(n)})^p \right] ds$$

$$\leq 2^{p-1} c_p n^{\frac{p-2}{2}} N^{\frac{p}{2}} c_{3.12.1}(n, N, p)$$

(by Lemma 3.12)

Put $c_{3.13.1}(N, n, p) = 2^{p-1} n^{p-1} c_{3.12.1}(n, N, p) + 2^{p-1} c_p n^{\frac{p-2}{2}} N^{\frac{p}{2}} c_{3.12.1}(n, N, p)$. This proves (3.26). Next, we prove (3.27).

$$\mathbb{P} \left[\left(\sup_{s \leq T(n)} |\widetilde{L}^x_s(X) - \widetilde{L}^z_s(X)| \right)^p \right] \leq 2^{p-1} \mathbb{P} \left[\left(\sup_{s \leq T(n)} H_s(|\mathcal{L}^x_s - \mathcal{L}^z_s|) \right)^p \right]$$

$$+ 2^{p-1} \mathbb{P} \left[\left(\sup_{s \leq T(n)} \left| \int_0^s \int (\mathcal{L}^x_r - \mathcal{L}^z_r) Z^H(dr, dy) \right| \right)^p \right]$$

$$\equiv J_1 + J_2.$$

Consider any $(\mathcal{F}_t)$-stopping time $S \leq T(n) \wedge N$. Then

$$\mathbb{P}[|H_S(\mathcal{L}^x_S - \mathcal{L}^z_S)|^p] \leq \mathbb{P}[H_S(1)^{p-1} H_S(|\mathcal{L}^x_S - \mathcal{L}^z_S|^p)]$$

(by Jensen)

$$\leq n^{p-1} \widehat{\mathbb{P}}^H_S [|\mathcal{L}^x_S - \mathcal{L}^z_S|^p]$$

$$\leq n^{p-1} c_{3.12.2}(n, N, p) |x - z|^{p/2} \tag{3.29}$$

(by Lemma 3.12.)

Therefore

$$\sup_{S \leq T(n) \wedge N} \{ \mathbb{P}[H_S(|\mathcal{L}^x_S - \mathcal{L}^z_S|^p)] \leq n^{p-1} c_{3.12.2}(n, N, p) |x - z|^{p/2}$$

$$\equiv k_p |x - z|^{p/2}$$

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where the sup is taken over all stopping times S bounded by $T(n) \wedge N$. Note that for any such stopping time and any $\theta > 0$, $q \geq 1$

$$\mathbb{P}[H_S(|\mathcal{L}_S^x - \mathcal{L}_S^z|) > \theta] \leq \frac{\mathbb{P}[H_S(|\mathcal{L}_S^x - \mathcal{L}_S^z|)^q]}{\theta^q} \wedge 1$$

(Chebyshev)

$$\leq \frac{k_q|x - z|^{q/2}}{\theta^q} \wedge 1.$$

But by Lemma 3.10

$$\mathbb{P}[\sup_{r \leq T_n \wedge N} H_r(|\mathcal{L}_r^x - \mathcal{L}_r^z|) > \theta] = \sup_{S \leq T(n) \wedge N} \mathbb{P}[H_S(|\mathcal{L}_S^x - \mathcal{L}_S^z|) > \theta].$$

Hence

$$\mathbb{P}\left[\left(\sup_{r \leq T_n \wedge N} H_r(|\mathcal{L}_r^x - \mathcal{L}_r^z|)\right)^p\right] = p \int_0^\infty \theta^{p-1} \mathbb{P}\left[\sup_{r \leq T_n \wedge N} H_r(|\mathcal{L}_r^x - \mathcal{L}_r^z|) > \theta\right] d\theta$$

$$= p \int_0^\infty \theta^{p-1} \sup_{S \leq T(n) \wedge N} \mathbb{P}[H_S(|\mathcal{L}_S^x - \mathcal{L}_S^z|) > \theta] d\theta$$

$$\leq p \int_0^\infty \theta^{p-1} \frac{k_{p+1}|x - z|^{\frac{p+1}{2}}}{\theta^{p+1}} \wedge 1 d\theta$$

$$= p \int_0^{k_{p+1}^{1/(p+1)} \sqrt{|x-z|}} \theta^{p-1} d\theta$$

$$+ pk_{p+1}|x - z|^{\frac{p+1}{2}} \int_{k_{p+1}^{1/(p+1)} \sqrt{|x-z|}}^\infty \theta^{-2} d\theta$$

$$= k_{p+1}^{\frac{p+1}{2}}|x - z|^{p/2} + pk_{p+1}^{\frac{p+1}{2}}|x - z|^{p/2}$$

$$= (p+1)k_{p+1}^{\frac{p+1}{2}}|x - z|^{p/2}.$$

Moreover,

$$J_2 \leq 2^{p-1}c_p \mathbb{P}\left[\int_0^{T(n)} \int |\mathcal{L}_{s \wedge T(n)}^x - \mathcal{L}_{s \wedge T(n)}^z|^2 H_{s \wedge T(n)} ds^{p/2}\right]$$

(by Burkholder)

$$\leq 2^{p-1}c_p \mathbb{P}\left[\int_0^{T(n)} H_s(1) ds^{\frac{p-2}{2}} \int_0^{T(n)} \int |\mathcal{L}_{s \wedge T(n)}^x - \mathcal{L}_{s \wedge T(n)}^z|^p H_{s \wedge T(n)} ds\right]$$

$$\leq 2^{p-1}c_p n^{\frac{p-2}{2}} N^{\frac{p-2}{2}} \int_0^N \widehat{\mathbb{P}}_{s \wedge T(n)}^H \left[|\mathcal{L}_{s \wedge T(n)}^x - \mathcal{L}_{s \wedge T(n)}^z|^p\right] ds$$

$$\leq 2^{p-1}c_p n^{\frac{p-2}{2}} N^{p/2} c_{3.12.2}(n, N, p)|x - z|^{p/2}$$

(by Lemma 3.12)

Define $\widetilde{c}_{3.13.2}(t, p, N) = (p+1)k_{p+1}^{\frac{p}{p+1}} + 2^{p-1}c_p n^{\frac{p-2}{2}} N^{p/2} c_{3.12.2}(n, N, p)$. This proves (3.27).

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To finish the proof we estimate

$$\mathbb{P}\left[|\widetilde{L}^x_{t\wedge T(n)}(X) - \widetilde{L}^x_{s\wedge T(n)}(X)|^p\right] \le 2^{p-1}\mathbb{P}[|H_{t\wedge T(n)}(\mathcal{L}^x_{t\wedge T(n)}) - H_{s\wedge T(n)}(\mathcal{L}^z_{s\wedge T(n)})|^p]$$

$$+ 2^{p-1}\mathbb{P}\left[\left|\int_{s\wedge T(n)}^{t\wedge T(n)} \mathcal{L}^x_{r\wedge T(n)} Z^H(dr,dy)\right|^p\right]$$

$$\equiv A_1 + A_2.$$

We now estimate A_1:

$$A_1 \le 2^{2p-2}\mathbb{P}[|H_{t\wedge T(n)}(\mathcal{L}^x_{t\wedge T(n)} - \mathcal{L}^x_{s\wedge T(n)})|^p]$$

$$+ 2^{2p-2}\mathbb{P}[|H_{t\wedge T(n)}(\mathcal{L}^x_{s\wedge T(n)}) - H_{s\wedge T(n)}(\mathcal{L}^x_{s\wedge T(n)})|^p]$$

$$\equiv A_{11} + A_{12}$$

$$A_{11} \le 2^{2p-2}\mathbb{P}\left[H_{t\wedge T(n)}(1)^{p-1} H_{t\wedge T(n)}|\mathcal{L}^x_{t\wedge T(n)} - \mathcal{L}^x_{s\wedge T(n)})|^p)\right]$$

$$\le 2^{2p-2}n^{p-1}\widehat{\mathbb{P}}^H_{t\wedge T(n)}[|\mathcal{L}^x_{t\wedge T(n)} - \mathcal{L}^x_{s\wedge T(n)})|^p]$$

$$\le 2^{2p-2}n^{p-1}c_{3.12.3}(n,N,p)|t-s|^{p/2}$$

(by Lemma 3.12).

By Lemma 3.9,

$$H_{t\wedge T(n)}[\mathcal{L}^x_{s\wedge T(n)}] = H_{s\wedge T(n)}[\mathcal{L}^x_{s\wedge T(n)}] + \int_{s\wedge T(n)}^{t\wedge T(n)}\int \mathcal{L}^x_r Z^H(dr,dy).$$

Therefore

$$A_{12} = 2^{2p-2}\mathbb{P}\left[\left|\int_{s\wedge T(n)}^{t\wedge T(n)}\int \mathcal{L}^x_r Z^H(dr,dy)\right|^p\right]$$

$$\le 2^{2p-2}c_p\mathbb{P}\left[\int_{s\wedge T(n)}^{t\wedge T(n)}\int (\mathcal{L}^x_r)^2 H_r(dy)dr^{p/2}\right]$$

$$\le 2^{2p-2}c_p\mathbb{P}\left[\int_{s\wedge T(n)}^{t\wedge T(n)} H_r(1)ds^{\frac{p-2}{2}}\int_{s\wedge T(n)}^{t\wedge T(n)}\int (\mathcal{L}^x_r)^p H_r(dy)dr\right]$$

$$\le 2^{2p-2}c_p n^{\frac{p-2}{2}}(t-s)^{\frac{p-2}{2}}\mathbb{P}\left[\int_{s\wedge T(n)}^{t\wedge T(n)}\int (\mathcal{L}^x_r)^p H_r(dy)dr\right]$$

$$\le 2^{p-1}c_p(t-s)^{\frac{p-2}{2}}n^{\frac{p-2}{2}}\int_s^t \widehat{\mathbb{P}}^H_{r\wedge T(n)}[(\mathcal{L}^x_{r\wedge T(n)})^p]dr$$

$$\le 2^{p-1}c_p(t-s)^{p/2}n^{\frac{p-2}{2}}c_{3.12.1}(n,N,p)$$

(by Lemma 3.12).

These computations also show that

$$A_2 \le 2^{p-1}c_p(t-s)^{p/2}n^{\frac{p-2}{2}}c_{3.12.1}(n,N,p).$$

Set $c_{3.13.3}(n,N,p) = 2^p c_p(t-s)^p n^{\frac{p-2}{2}}c_{3.12.1}(n,N,p) + 2^{2p-2}n^{p-1}c_{3.12.3}$. This proves (3.28). ∎

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Corollary 3.14. *The map* $(s, x) \mapsto \widetilde{L}_s^x(X)$ *has a continuous version.*

Proof. The estimate (3.27) and Kolmogorov's criterion (Theorem 2.43) show that $(s, x) \mapsto \widetilde{L}_s^x(X)$ has a continuous version on $[0, T(n)] \times \mathbb{R}$. The desired result follows from the fact that $T(n) \to \infty$ a.s. ∎

The following result will be needed in the proof of the next proposition.

Lemma 3.15. *There exists a non-decreasing function* $\Xi : [0, \infty) \to [0, \infty)$ *such that for any measure* μ *and any* $x, z \in \mathbb{R}$,

$$|\sigma^{-2}(\mu, x) - \sigma^{-2}(\mu, z)| \leq \Xi(\mu(1))|x - z|. \tag{3.30}$$

Proof. Recall that by (3.3)–(3.5)

$$|\sigma(\mu, x) - \sigma(\mu, z)| \leq \Upsilon(\mu(1))|x - z|$$

and

$$\sigma(\mu, x) \leq \Upsilon(\mu(1)).$$

Therefore

$$
\begin{aligned}
|\sigma(\mu, x)^{-2} - \sigma(\mu, z)^{-2}| &= \left| \frac{\sigma(\mu, x)^2 - \sigma(\mu, z)^2}{\sigma(\mu, x)^2 \sigma(\mu, z)^2} \right| \\
&\leq \varepsilon^{-4} |\sigma(\mu, x)^2 - \sigma(\mu, z)^2| \\
&\quad (\text{since } \sigma \geq \varepsilon.) \\
&= \varepsilon^{-4} |\sigma(\mu, x) + \sigma(\mu, z)| \, |\sigma(\mu, x) - \sigma(\mu, z)| \\
&\leq \varepsilon^{-4} 2\Upsilon(\mu(1))^2 |x - z|.
\end{aligned}
$$

Setting $\Xi(\cdot) = 2\Upsilon(\cdot)^2 \varepsilon^{-4}$ we obtain (3.30). ∎

Proposition 3.16. **(a)** *The map* $(t, a) \longmapsto (L_t^a(X))$ *is* $\mathbb{P} - a.s.$ *continuous.*

(b) *Suppose for each* $z \in \mathbb{R}$ $s \mapsto \sigma^{-2}(X_s, z)$ *is a semimartingale with canonical decomposition* $M^z + A^z$ *such that for any* $t > 0$ *and any* $k \geq 1$

$$\sup_z \left\{ \mathbb{P}[\langle M^z \rangle_t^k] + \mathbb{P}\left[\int_0^t |dA_s^z|^{2k} \right] \right\} < \infty. \tag{3.31}$$

Then there exists a version of $(t, a, \omega) \longmapsto (L_t^a(X))$ *which is (jointly)* ζ-*Hölder continuous in* (t, a) *for any* $\zeta < 1/2$ $\mathbb{P} - a.s.$

Remark 3.17. Condition (3.31) holds for all the examples in Chapter 4 of Perkins (1995, p. 49) with some additional smoothness on the coefficients. We now show one method to check it. For simplicity we concentrate on a particular example. Suppose $\gamma = 1$. Suppose also that there are positive constants K and i such that $b(\mu, x) \leq K(1 + \mu(1)^i)$ and let $\bar{\sigma} : \mathbb{R} \to \mathbb{R}_+$ be bounded continuous with two bounded continuous derivatives. Assume that K also bounds $\bar{\sigma}$ and its derivatives. Fix $\varepsilon > 0$ and define

$$\sigma(\mu, x) = \varepsilon + \int \bar{\sigma}(x - z)\mu(dz), \quad \forall \mu \in M_F(\mathbb{R}), \quad \forall x \in \mathbb{R}.$$

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We claim that with this choice of (γ, σ, b), the hypothesis of Proposition 3.16(b) is satisfied. We showed in Chapter 1 that $\sigma(\mu, x) - \varepsilon$ satisifies (3.5) and (3.3). It follows immediately that $\sigma(\mu, x)$ also satisfies (3.5) and (3.3). Theorem 5.6 of Perkins (1995) implies that X solves the martingale problem $(MP)^m_{1,\sigma,b,0}$.

$$\forall \psi \in C_b^2(\mathbb{R}) \qquad Z_t^X(\psi) = X_t(\psi) - X_0(\psi) - \int_0^t \int \left[\psi'(x) b(X_s, x) + \frac{1}{2} \sigma(X_s, x) \psi''(x) \right] X_s(dx) ds$$

is a continuous square integrable martingale such that

$$\langle Z_t^X(\psi) \rangle_t = \int_0^t X_s(\psi^2) ds \quad \forall t \geq 0 \quad a.s.$$

We write $\bar{\sigma}^z = \bar{\sigma}(x - z)$. From the above martingale problem we see that

$$X_s(\bar{\sigma}^z) = X_0(\bar{\sigma}^z) + Z_t^X(\bar{\sigma}^z) + \int_0^t \int \left[\bar{\sigma}_x^z(x) b(X_s, x) + \frac{1}{2} \sigma(X_s, x) \bar{\sigma}_{xx}^z \right] X_s(dx) ds$$

Applying Itô's lemma we obtain

$$\sigma(X_s, z)^{-2} = \frac{1}{(\varepsilon + X_0(\bar{\sigma}^z))^2} - \int_0^t \frac{2}{(\varepsilon + X_s(\bar{\sigma}^z))^2} dZ_s^X(\bar{\sigma}^z)$$

$$- \int_0^t \frac{2 \int \left[\bar{\sigma}_x^z(x) b(X_s, x) + \frac{1}{2} \sigma(X_s, x) \bar{\sigma}_{xx}^z \right] X_s(dx)}{(\varepsilon + X_s(\bar{\sigma}^z))^2} ds$$

$$+ \int_0^t \frac{3 X_s((\bar{\sigma}^z)^2)}{(\varepsilon + X_s(\bar{\sigma}^z))^3} ds$$

$$\equiv \frac{1}{(\varepsilon + X_0(\bar{\sigma}^z))^2} + M_t^z + A_t^z.$$

We compute

$$\sup_z \mathbb{P}[\langle M^z \rangle_t^k] = \mathbb{P}\left[\left\langle \int_0^{\cdot} \frac{2}{(\varepsilon + X_s(\bar{\sigma}^z))^2} dZ_s^X(\bar{\sigma}^z) \right\rangle_t^k \right]$$

$$= \mathbb{P}\left[\int_0^t \frac{4 X_s((\bar{\sigma}^z)^2)}{(\varepsilon + X_s(\bar{\sigma}^z))^4} ds^k \right]$$

$$\leq 4^k \varepsilon^{-4k} \mathbb{P}\left[\int_0^t K^2 X_s(1) ds^k \right]$$

$$\leq 4^k \varepsilon^{-4k} t^k K^{2k} \mathbb{P}[X_t^*(1)^k]$$

$$< \infty$$

(by Proposition 2.21).

Similarly,

$$\mathbb{P}\left[\int_0^t |dA_s^z|^{2k} \right] = \mathbb{P}\left[\int_0^t \left| -\frac{2 \int \left[\bar{\sigma}_x^z(x) b(X_s, x) + \frac{1}{2} \sigma(X_s, x) \bar{\sigma}_{xx}^z \right] X_s(dx)}{(\varepsilon + X_s(\bar{\sigma}^z))^2} + \frac{3 X_s((\bar{\sigma}^z)^2)}{(\varepsilon + X_s(\bar{\sigma}^z))^3} \right| ds^{2k} \right]$$

$$\leq 2^{2k}\mathbb{P}\left[\int_0^t \frac{(2K^2(1+X_s(1)^i)+K^2X_s(1))X_s(1)}{\varepsilon^2}ds^{2k}\right]$$

$$+2^{2k}P\left[\int_0^t \frac{3K^2X_s(1)}{\varepsilon^3}ds^{2k}\right]$$

$$\leq C(K,k,\varepsilon,t)<\infty$$

(by Proposition 2.21)

This concludes the proof of the claim. The same method can be employed to show that more general coefficents σ satisfy (3.31). □

Proof of Proposition 3.16. (a) Notice that the Lipschitz condition (3.5) and the continuity of $s\to X_s$ ensure the continuity of $(r,x)\mapsto\sigma(X_r,x)$. Therefore we can fix ω in a set of probability one such that the maps $(t,x,\omega)\mapsto\widetilde{L}_t^x$ and $(r,x,\omega)\mapsto\sigma^{-2}(X_r,x)$ are continuous. Note that

$$|L_t^x(X)-L_s^z(X)|=\left|\int_0^t\sigma^{-2}(X_s,x)\widetilde{L}_{dr}^x-\int_0^s\sigma^{-2}(X_s,z)\widetilde{L}_{dr}^z\right|$$

$$\leq\left|\int_s^t\sigma^{-2}(X_r,x)\widetilde{L}_{dr}^x\right|+\left|\int_0^s(\sigma^{-2}(X_r,x)-\sigma^{-2}(X_r,z))\widetilde{L}_{dr}^x\right|$$

$$+\left|\int_0^s\sigma^{-2}(X_s,z)\widetilde{L}_{dr}^x-\int_0^s\sigma^{-2}(X_s,z)\widetilde{L}_{dr}^z\right|$$

$$\equiv J_1+J_2+J_3.$$

We estimate

$$J_1\leq\varepsilon^{-2}|\widetilde{L}_t^x-\widetilde{L}_s^x|,$$

(since $\sigma\geq\varepsilon$)

$$J_2\leq\Xi(X_s^*(1))|x-z|\widetilde{L}_s^z$$

(by Lemma (3.15).

Observe that $\widetilde{L}_{dr}^x$ converges weakly to $\widetilde{L}_{dr}^z$ when $x\to z$ (these are finite Borel measures on $[0,s]$). Recall that the map $r\mapsto\sigma^{-2}(X_r,z)$ is continuous. Therefore $\int_0^s\sigma^{-2}(X_r,z)\widetilde{L}_{dr}^x\to\int_0^s\sigma^{-2}(X_r,z)\widetilde{L}_{dr}^z$ as $x\to z$. Hence

$$\lim_{(t,x)\to(s,z)}|L_t^x(X)-L_s^z(X)|=0.$$

(b) Fix $p\geq 2$. We compute

$$\mathbb{P}\left[|L_{t\wedge T(n)}^x(X)-L_{t\wedge T(n)}^z(X)|^p\right]=\mathbb{P}\left[\left|\int_0^{t\wedge T(n)}\sigma^{-2}(X_s,x)\widetilde{L}_{ds}^x-\int_0^{t\wedge T(n)}\sigma^{-2}(X_s,z)\widetilde{L}_{ds}^z\right|^p\right]$$

$$\leq 2^{p-1}\mathbb{P}\left[\left|\int_0^{t\wedge T(n)}\sigma^{-2}(X_s,x)-\sigma^{-2}(X_s,z)\widetilde{L}_{ds}^x\right|^p\right]$$

$$+2^{p-1}\mathbb{P}\left[\left|\int_0^{t\wedge T(n)}\sigma^{-2}(X_s,z)(\widetilde{L}_{ds}^x-\widetilde{L}_{ds}^z)\right|^p\right]$$

$$\equiv I_1 + I_2.$$

Now,

$$I_1 \le 2^{p-1}|x - z|^p \mathbb{P}\left[\left|\int_0^{t\wedge T(n)} \Xi(X_s(1))\widetilde{L}^x_{ds}\right|^p\right]$$

(by Lemma 3.15.)

$$\le 2^{p-1}|x - z|^p \mathbb{P}\left[\left|\int_0^{t\wedge T(n)} \Xi(\Theta(n,s))\widetilde{L}^x_{ds}\right|^p\right]$$

(by Lemma 3.7)

$$\le 2^{p-1}|x - z|^p \Xi(\Theta(n,N))^p \mathbb{P}\left[\left|\int_0^{t\wedge T(n)} \widetilde{L}^x_{ds}\right|^p\right]$$

$$= 2^{p-1}|x - z|^p \Xi(\Theta(n,N))^p \mathbb{P}\left[\left|\widetilde{L}^x_{t\wedge T(n)}\right|^p\right]$$

$$= 2^{p-1}|x - z|^p \Xi(\Theta(n,N))^p c_{3.13.1}(N,n,p)$$

(by Lemma 3.13)

$$\equiv c_{3.16.1}(n,N,p)|x - z|^p. \tag{3.32}$$

To estimate I_2 we need the fact that $\sigma^{-2}(X_s,z) = M_s^z + A_s^z$ is a semimartingale.

$$I_2 = 2^{p-1}\mathbb{P}\left[\left|\int_0^{t\wedge T(n)} \sigma^{-2}(X_s,z)(\widetilde{L}^x_{ds} - \widetilde{L}^z_{ds})\right|^p\right]$$

$$= 2^{p-1}\mathbb{P}\left[\left|\sigma^{-2}(X_{t\wedge T(n)},z)(\widetilde{L}^x_{t\wedge T(n)} - \widetilde{L}^z_{t\wedge T(n)}) - \int_0^{t\wedge T(n)} (\widetilde{L}^x_{s\wedge T(n)} - \widetilde{L}^z_{s\wedge T(n)})d\sigma(X_\cdot,z)_s\right|^p\right]$$

(integrating by parts and since $\langle M^z, \widetilde{L}^x\rangle_t = \langle M^z, \widetilde{L}^z\rangle_t = \widetilde{L}^x_0 = \widetilde{L}^z_0 = 0$).

$$\le 6^{2p-1}\varepsilon^{-2p}\mathbb{P}\left[\left|\widetilde{L}^x_{t\wedge T(n)} - \widetilde{L}^z_{t\wedge T(n)}\right|^p\right] + 6^{2p-1}\mathbb{P}\left[\left|\int_0^{t\wedge T(n)} (\widetilde{L}^x_{s\wedge T(n)} - \widetilde{L}^z_{s\wedge T(n)})dM_s^z\right|^p\right]$$

$$+ 6^{2p-1}\mathbb{P}\left[\left|\int_0^{t\wedge T(n)} (\widetilde{L}^x_{s\wedge T(n)} - \widetilde{L}^z_{s\wedge T(n)})dA_s^z\right|^p\right]$$

$$\le 6^{2p-1}\varepsilon^{-2p}c_{3.13.2}(N,n,p)|x - z|^{p/2} + 6^{2p-1}C_p\mathbb{P}\left[\int_0^{t\wedge T(n)} (\widetilde{L}^x_{s\wedge T(n)} - \widetilde{L}^z_{s\wedge T(n)})^2 d\langle M^z\rangle_s^{p/2}\right]$$

$$+ 6^{2p-1}\mathbb{P}\left[\left(\sup_{s\le t\wedge T(n)}|\widetilde{L}^x_s - \widetilde{L}^z_s|\right)^{2p/2}\int_0^{t\wedge T(n)} |dA_s^z|^{2p/2}\right]$$

(by Lemma 3.13 and Burkholder)

$$\le 6^{2p-1}\varepsilon^{-2p}c_{3.13.2}(N,n,p)|x - z|^{p/2} + 6^{2p-1}C_p\mathbb{P}\left[\left(\sup_{s\le t\wedge T(n)}|\widetilde{L}^x_s - \widetilde{L}^z_s|\right)^{2p/2}\int_0^{t\wedge T(n)} d\langle M^z\rangle_s^{p/2}\right]$$

$$+ 6^{2p-1}C_p \sqrt{\mathbb{P}\left[\left(\sup_{s \leq t \wedge T(n)} |\widetilde{L}_s^x - \widetilde{L}_s^z|\right)^{2p}\right]} \sqrt{\mathbb{P}\left[\int_0^t |dA_s^z|^{2p}\right]}$$

(by Schwarz)

$$\leq 6^{2p-1}\varepsilon^{-2p}c_{3.13.2}(N,n,p)|x-z|^{p/2} + 6^{2p-1}C_p\sqrt{\mathbb{P}\left[\left(\sup_{s \leq t \wedge T(n)} |\widetilde{L}_s^x - \widetilde{L}_s^z|\right)^p\right]}\sqrt{\mathbb{P}[\langle M^z\rangle_t^p]}$$

$$+ 6^{2p-1}c_{3.13.2}(N,n,p)c_{3.31.1}(p,N)|x-z|^{p/2}$$

(by Schwarz, Lemma 3.13 and (3.31))

$$\leq 6^{2p-1}\varepsilon^{-2p}c_{3.13.2}(N,n,p)|x-z|^{p/2} + 6^{2p-1}C_p c_{3.31.1}(p,N)c_{3.13.2}(N,n,p)|x-z|^{p/2}$$

$$+ 6^{2p-1}c_{3.13.2}(N,n,p)c_{3.31.1}(p,N)|x-z|^{p/2} \tag{3.33}$$

$$\equiv c_{3.16.2}(n,N,p)|x-z|^{p/2}. \tag{3.34}$$

From (3.32) and (3.33) it follows that there is a constant $c_{3.16.3}(n,N,p)$ such that

$$\mathbb{P}\left[|L_{t \wedge T(n)}^x(X) - L_{t \wedge T(n)}^z(X)|^p\right] \leq c_{3.16.3}(n,N,p)|x-z|^{p/2}. \tag{3.35}$$

Without loss of generality assume $s \leq t$. We estimate

$$\mathbb{P}\left[|L_{t \wedge T(n)}^x(X) - L_{s \wedge T(n)}^x(X)|^p\right] = \mathbb{P}\left[\left|\int_{s \wedge T(n)}^{t \wedge T(n)} \sigma^{-2}(X_r,x)\widetilde{L}_{dr}^x\right|^p\right]$$

$$\leq \varepsilon^{-2p}\mathbb{P}\left[\left|\widetilde{L}_{t \wedge T(n)}^x - \widetilde{L}_{s \wedge T(n)}^x\right|^p\right]$$

$$\leq \varepsilon^{-2p}c_{3.13.3}(n,N,p)|t-s|^{p/2}$$

(by Lemma 3.13)

$$\equiv c_{3.16.4}(n,N,p)|t-s|^{p/2}. \tag{3.36}$$

We shall prove the existence of a continuous modification of $L_{\cdot \wedge T(n)}^\cdot(X)$ by using Lemma 2.43 (Kolmogorov's criterion). According to the latter, it suffices to show that

$$\mathbb{P}[|L_{t \wedge T(n)}^x - L_{s \wedge T(n)}^z|^\alpha \leq C[(t-s)^2 + (x-z)^2]^{1+\beta} \tag{3.37}$$

for some positive constants α, β, C. Set $C = 2^\alpha c_{3.16.3}(n,N,p) \vee c_{3.16.4}(n,N,p)$. Observe that

$$\mathbb{P}[|L_{t \wedge T(n)}^x - L_{s \wedge T(n)}^z|^\alpha] \leq 2^\alpha \mathbb{P}[|L_{t \wedge T(n)}^x - L_{t \wedge T(n)}^z|^\alpha] + 2^\alpha \mathbb{P}[|L_{t \wedge T(n)}^z - L_{s \wedge T(n)}^z|^\alpha]$$

$$\leq C[(t-s)^{\alpha/2} + |x-z|^{\alpha/2}] \tag{3.38}$$

(by (3.35) and (3.36)).

Realizing that (3.38) is of the type (3.37) when $\alpha > 4$ we obtain a continuous version of $L_{\cdot \wedge T(n)}^\cdot(X)$. To finish the demonstration note that $L_\cdot^\cdot(X) = L_{\cdot \wedge T(n)}^\cdot(X)$ on $\{T(n) > N\}$ and $T(n) \uparrow \infty$ a.s. Therefore $L_\cdot^\cdot(X)$ also has a jointly Hölder continuous version of order ζ for any $\zeta < 1/2$. ∎

Our next step is to verify the occupation times formula (3.2).

Lemma 3.18. *For any nonnegative Borel function* ϕ

$$\int \phi(a) \widetilde{L}_t^a(X) da = \int_0^t \int \phi(a) \sigma^2(X_s, a) X_s(da) ds \quad a.s. \tag{3.39}$$

Proof To establish (3.39) it suffices to show that for every $n > 0$ and any non-negative Borel function ϕ in a countable family of functions with compact support

$$\int \phi(a) \widetilde{L}_{t \wedge T(n)}^a(X) da = \int_0^{t \wedge T(n)} \int \phi(a) \sigma(X_s, a)^2 X_s(da) ds \quad a.s. \tag{3.40}$$

The null set in (3.40) depends only on ϕ and t. But there are only countable many ϕ's and by Corollary 3.14 $\widetilde{L}(X)$ is continuous, so we may take the null set in (3.40) independent of t and ϕ.

Recall that

$$\widetilde{L}_{t \wedge T(n)}^a(X) = \int \mathcal{L}_{t \wedge T(n)}^a H_{t \wedge T(n)}(dy) - \int_0^{t \wedge T(n)} \int \mathcal{L}_s^a Z^H(ds, dy). \tag{3.41}$$

We multiply both sides of (3.41) by $\phi(a)$ and integrate:

$$\int \phi(a) \widetilde{L}_{t \wedge T(n)}^a da = \int \phi(a) \int \mathcal{L}_{t \wedge T(n)}^a H_{t \wedge T(n)}(dy) da$$
$$- \int \phi(a) \int_0^{t \wedge T(n)} \int \mathcal{L}_s^a Z^H(ds, dy) da. \tag{3.42}$$

Now,

$$\int \phi(a) \int \mathcal{L}_{t \wedge T(n)}^a H_{t \wedge T(n)}(dy) da = \int \int \phi(a) \mathcal{L}_{t \wedge T(n)}^a da H_{t \wedge T(n)}(dy)$$

$$\text{(by Fubini)}$$

$$= \int \int_0^{t \wedge T(n)} \phi(Y_s) \sigma^2(X_s, Y_s) ds H_{t \wedge T(n)}(dy)$$

$$\text{(by the density of occupation formula).}$$

We can apply Ito's lemma for historical integrals to this last H-integral to obtain:

$$\int \int_0^{t \wedge T(n)} \phi(Y_s) \sigma^2(X_s, Y_s) ds H_{t \wedge T(n)}(dy) = \int_0^{t \wedge T(n)} \int \int_0^s \phi(Y_r) \sigma^2(X_r, Y_r) dr Z^H(dsdy)$$

$$+ \int_0^{t \wedge T(n)} \int \phi(Y_s) \sigma^2(X_s, Y_s) H_s(dy) ds$$

$$= \int_0^{t \wedge T(n)} \int \int \phi(a) \mathcal{L}_s^a da Z^H(ds, dy)$$

$$+ \int_0^{t \wedge T(n)} \int \phi(Y_s) \sigma^2(X_s, Y_s) H_s(dy) ds$$

$$\text{(again by the density of occupation formula).}$$

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We claim that we can use Fubini's theorem for stochastic integrals (Theorem 2.6 Walsh (1986)) to interchange the order of integration in

$$\int_0^{t\wedge T(n)} \int\int \phi(a)\mathcal{L}_s^a\, da\, Z^H(ds, dy) = \int \phi(a)\int_0^{t\wedge T(n)}\int \mathcal{L}_s^a Z^H(ds, dy)\, da.$$

Hence

$$\int \phi(a)\int \mathcal{L}_{t\wedge T(n)}^a H_{t\wedge T(n)}(dy)\, da = \int \phi(a)\int_0^{t\wedge T(n)}\int \mathcal{L}_{s\wedge T(n)}^a\, da\, Z^H(ds, dy)\, da$$
$$+ \int_0^{t\wedge T(n)}\int \phi(Y_s)\sigma^2(X_s, Y_s) H_s(dy)\, ds \qquad (3.43)$$

We obtain (3.40) by substituting (3.43) into (3.42) and using $(HSE)_{\sigma,b,\gamma}$. It remains to show that Fubini's theorem may be applied. We set $\mu(da) := \phi(a)da$. We need to show that

$$\mathbb{P}\left[\int\int_0^{t\wedge T(n)}\int (\mathcal{L}_{s\wedge T(n)}^a)^2 H_{s\wedge T(n)}(dy)\, ds\, \mu(a)\right] < \infty.$$

This is easy to check:

$$\mathbb{P}\left[\int\int_0^{t\wedge T(n)}\int (\mathcal{L}_{s\wedge T(n)}^a)^2 H_{s\wedge T(n)}(dy)\, ds\, \mu(da)\right] \leq \int\int_0^N \widehat{\mathbb{P}}_{s\wedge T(n)}^H[(\mathcal{L}_{s\wedge T(n)}^a)^2]\, ds\, \mu(da)$$
$$\leq N\mu(\mathbb{R})c_{3.12.1}(n, N, 2)$$
$$< \infty$$
$$\text{(by Lemma 3.12)}.$$

Thus (3.39) holds. ∎

Proposition 3.19. $L_t^a(X)$ *satisfies the following density of occupation formula. For any nonnegative Borel function* ϕ,

$$\int_0^t X_s(\phi)\, ds = \int \phi(a)L_t^a(X)\, da \quad a.s. \qquad (3.44)$$

Proof. By considering functions of the form $\phi(a)1_{[s,t)}R(\omega)$ it follows easily from a monotone class argument and (3.39) that

$$\int\int_0^t \psi(s, a, \omega)\widetilde{L}_{ds}^a(X)\, da = \int_0^t\int \sigma^2(X_s, a)\psi(s, a, \omega)X_s(dx)\, ds \qquad (3.45)$$

for any measurable $\psi: \mathbb{R}_+ \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}_+$. Substituting $\psi(s, a, \omega) = \sigma^{-2}(X_s, a)\phi(a)$ into (3.45) we get

$$\int_0^t \phi(a)X_s(da)\, ds = \int \phi(a)\int_0^t \sigma^{-2}(X_s, a)\widetilde{L}_{ds}^a(X)\, da$$
$$= \int \phi(a)L_t^a(X)\, da.$$
$$\text{(by (3.18))} \qquad ∎$$

Proof of Theorem 3.4. Propositions 3.16 and 3.19 show that the family of random variables $(L_t^a(X))$ satisfies the conditions (i) and (ii) of Definition 3.1. ∎

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