NONCOMMUTATIVE PRÜFER RINGS AND SOME GENERALIZATIONS

by

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Abstract

Noncommutative Prüfer rings appear naturally when one wants to transfer the known results for rings which arise in algebraic geometry (such as Dedekind, Krull and Prüfer, valuation rings ...) to noncommutative rings. We remove the left-right symmetry condition of the noncommutative Prüfer rings introduced by Alajbegovic and Dubrovin, and introduce three natural generalizations, semi-Prüfer rings, right w-semi-Prüfer rings, and right w-Prüfer rings. We study the relations between the four concepts, and present the various properties that characterize them. We formulate and prove the basic facts for those rings (decompositions of such rings; Morita invariants of these notions; relations with some other notions). A new module-theoretic characterization of semiprime right Goldie rings is achieved by using the newly-defined concept of strongly compressible modules. The result is used to provide new characterizations of semiprime Goldie (prime right Goldie, or prime Goldie) rings, and right w-semi-Prüfer (semi-Prüfer, right w-Prüfer, or Prüfer) rings. In particular, the characterization of semiprime Goldie rings of Lopez-Permouth, Rizvi, and Yousif using weakly-injective modules is an easy corollary of our results. We also study modules over noncommutative Prüfer rings. It is shown that a module over a noncommutative Prüfer ring has projective dimension at most one if and only if it is the union of a well-ordered continuous chain of submodules with each factor of the chain a finitely presented cyclic module. The result is used to present a characterization of divisible modules with projective dimension at most one over noncommutative Prüfer rings, which generalizes a known result of L.Fuchs.

Contents

Notations

Notations in this manuscript are fairly standard, and may be found in most graduate level texts on Algebra and Ring Theory. To keep the reader on track, we will introduce them as required. The following two books are our main references:

1) *Rings and Categories of Modules* by F.W. Anderson and K.R. Fuller, and

2) *An Introduction to Noncommutative Noetherian Rings* by K.R. Goodearl and R.B. Warfield. Jr.

We will feel free to use the results in the two books whenever we have such a demand.

Throughout this manuscript, a ring R will mean a nonzero associative noncommutative ring with an identity. And all modules are unitary. The notation M_R (or ${}_RM$) indicates that M is a right (or left) module over a ring R. Given a module M_R, we will denote by $E(M_R)$ the injective hull of the module M_R. For a subset X of a right R-module M, the annihilator right ideal of X in R is denoted by $X^\perp$, i.e., $X^\perp = \{r \in R : xr = 0 \text{ for all } x \in X\}$. Similarly, for a left R-module ${}_RN$ and a subset Y of N, we denote the annihilator left ideal of Y in R by ${}^\perp Y$. In particular, we write $x^\perp$ (or ${}^\perp y$) to indicate $\{x\}^\perp$ (or ${}^\perp\{y\}$).

Mod-R	the category of all right R-modules
R-Mod	the category of all left R-modules
$\subset$	proper inclusion

$\mathbf{N}$	the set of positive integers
$\mathbf{Z}$	the set of integers
$\mathrm{End}(M)$	the ring of all module endomorphisms of a module M
$Z(M_R)$	the singular submodule of a module M_R
$\tau(M)$	the torsion submodule of a module M
$T(M)$	the trace ideal of a module M
M^*	the dual module of a module M
$dim(M)$	the Goldie dimension of a module M
$Pd(M)$	the projective dimension of a module M
$M^{(I)}$	the direct sum of I copies of M
$M^{(n)}$	the direct sum of n copies of M
$M_n(R)$	the n by n matrix ring over a ring R
$Rad(R)$	the Jacobson radical of a ring R
$\mathcal{C}_R(0)$	the set of all regular elements of R
$Q^r_{cl}(R)$	the classical right quotient ring of a ring R (if it exists)
$Q^l_{cl}(R)$	the classical left quotient ring of a ring R (if it exists)
$Q_{cl}(R)$	the classical quotient ring of a ring R (if it exists)

ACC	the ascending chain condition
$\otimes$	tensor product
Ext	the extension functor

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Introduction

Prüfer domains form an important and much-studied class of integral domains in Commutative Algebra. With Dedekind domains, valuation domains and Krull domains, they constitute the main objects of study in the Multiplicative Theory of Ideals. The importance of the class of Prüfer domains lies mainly in: 1) Prüfer domains have an origin in Algebraic Number Theory. The rings of integers of finite algebraic number fields, which are the main objects of study in Algebraic Number Theory, are Prüfer domains. 2) Prüfer domains have tight connections with Dedekind domains and valuation domains. In fact, the class of Dedekind domains is precisely the class of Noetherian Prüfer domains; and a Prüfer domain can be characterized as an integral domain such that the localization of it at any prime (or maximal) ideal is a valuation domain. 3) The lattice of all ideals of a Prüfer domain possesses many beautiful arithmetics. For example, an integral domain is a Prüfer domain if and only if $A(B \cap C) = AB \cap AC$ for all ideals A, B, C of R if and only if $A \cap (B + C) = A \cap B + A \cap C$ for all ideals A, B, C of R.

In the past twenty years, the study of the noncommutative analogues of Dedekind domains, valuation domains and Krull domains has been a fascinating area of study in ring theory. And many results have been obtained on the various generalizations of them to noncommutative cases, e.g., Asano orders, Dedekind prime rings, hereditary Noetherian prime rings, chain rings, Dubrovin valuation rings, Chamerie Krull rings, Marubaynshi Krull rings, Ω-Krull rings, and others. It is from the abundance of the study of these objects and the close relations between Prüfer domains, Dedekind domains, valuation domains and Krull domains that one sees the need for an introduction of noncommutative analogues of Prüfer domains.

Let R be an integral domain with field of quotients Q. An R-submodule I of Q is said to be a fractional ideal of R if $dI \subseteq R$ for some $0 \neq d \in R$. For each fractional ideal I of R, define $I^{-} = \{q \in Q : qI \subseteq R\}$. Then I is said to be invertible if $I^{-}I = R$. We call an integral domain R a Prüfer domain if every nonzero finitely generated (f.g. for short) ideal of R is invertible. Prüfer domains can be characterized as any commutative rings with the property that each nonzero f.g. ideal is a progenerator, or is projective, or is a generator. This large selection of attributes suggests many possible generalizations to noncommutative cases, and at the same time raises the difficulty of the best choice among such numerous generalizations. In 1990, Alajbegovic and Dubrovin defined a noncommutative (right) Prüfer ring as a prime Goldie ring such that $I^{-1}I = R$ and $II^{-1} = O_l(I)$ for every f.g. fractional right ideal I of R, where $O_l(I) = \{q \in Q_{cl}(I) : qI \subseteq I\}$ and $I^{-1} = \{q \in Q_{cl}(R) : IqI \subseteq I\}$. Among the properties of noncommutative Prüfer rings, they show that the concept of a noncommutative Prüfer ring is a left-right symmetric concept; the notion is a Morita invariant, and every noncommutative Prüfer ring is Morita equivalent to a (noncommutative) Prüfer domain. They also note that the class of noncommutative Prüfer rings contains the classes of prime Dedekind rings, Dubrovin valuation rings, and commutative Prüfer domains.

The present manuscript is devoted to continuing the study of noncommutative Prüfer rings. We first observe that a noncommutative Prüfer ring can be characterized as a prime Goldie ring R such that every nonzero f.g.

submodule of a progenerator of Mod-R is a progenerator. The nature of the characterization brought our interests to noncommutative Prüfer rings. We note that the generalized discrete valuation ring of H.H.Brungs (see [5]) and the skew polynomial rings (see the example in §2.1, of Chapter 2) provide examples of prime right (but not left) Goldie rings satisfying the same property as above. The observation leads us to remove the left-right symmetric condition of noncommutative Prüfer rings and to consider more general definitions where the conditions of being a prime ring, being a Goldie ring are replaced by a semiprime ring, by a right Goldie ring respectively.

The manuscript is organized into four chapters. Chapter 1 summarizes certain basic concepts and theorems in ring theory which are needed in the sequel. Since they are all well-known and easy to find for reference, the proofs of most of them are omitted.

In Chapter 2, we introduce three generalizations of noncommutative Prüfer rings, semi-Prüfer rings, right w-semi-Prüfer rings, and right w-Prüfer rings. We study the relations between the four concepts, and present the various properties that characterize them. We formulate and prove the basic facts for those rings (decompositions of such rings; Morita invariants of these notions; relations with some other notions).

In Chapter 3, a new module-theoretic characterization of semiprime right Goldie rings is achieved by using the newly-defined concept of strongly compressible modules. The result is used to provide new characterizations of semiprime Goldie (prime right Goldie, or prime Goldie) rings, and right w-semi-Prüfer (semi-Prüfer, right w-Prüfer, or Prüfer) rings. In particular, the characterization of semiprime Goldie rings of Lopez-Permouth, Rizvi, and

Yousif using weakly-injective modules is an easy corollary of our results.

Chapter 4 is provided to study modules over noncommutative Prüfer rings. The study is motivated by the work of L.Fuchs on modules over valuation (or Prüfer) domains (see [13]). We give a characterization of modules of projective dimension at most one over noncommutative Prüfer rings, and present a structure theorem of divisible modules with projective dimension at most one over noncommutative Prüfer rings, which generalizes a known result of L.Fuchs.

1 The Preliminaries

This chapter is provided to review a number of basic concepts and some important results from ring theory, which will be used throughout the sequel. The proofs for most results are omitted, since they can be found in the standard texts in ring theory, such as [2] and [18].

Essential extensions and singular submodules

An essential submodule of a module M is any submodule N which has nonzero intersection with every nonzero submodule of M. We write $N \leq_e M$ to denote this situation, and we also say that M is an essential extension of N.

Proposition 1.1 *(a) Let N be a submodule of a module M, and let $f : P \longrightarrow M$ be a homomorphism. If $N \leq_e M$, then $f^{-1}(N) \leq_e P$.*

(b) Let N be a submodule of a module M, and P a submodule of M which is maximal with respect to the property $P \cap N = 0$. Then $N \oplus P \leq_e M$ and $(N \oplus P)/P \leq_e M/P$. $\square$

The singular submodule of a module M_R is defined by $Z(M_R) = \{x \in M : x^\perp \leq_e R_R\}$. Since $Z(M_R)$ is a fully invariant submodule of M, the right singular ideal $Z(R_R)$ is an ideal of R. If $Z(M_R) = 0$ then M is called a non-singular module. The ring R is called a right non-singular ring if $Z(R_R) = 0$. A right and left non-singular ring is called a non-singular ring.

Orders and quotient rings

A regular element in a ring R is any non-zero-divisor, i.e., any element $x \in R$ such that $x^{\perp} = {}^{\perp}x = 0$. We will denote by $\mathcal{C}_R(0)$ the set of all regular elements of R.

Definition 1.1 *Let Q be a ring. A right order in Q is any subring $R \subseteq Q$ such that*

(a) every regular element of R is invertible in Q;

(b) every element of Q has the form ab^{-1} for some $a \in R$ and some $b \in \mathcal{C}_R(0)$.

A left order is defined analogously, and a left and right order is called an order.

Definition 1.2 *Let R be a ring. A classical right quotient ring, denoted by $Q^r_{cl}(R)$ if it exists, is any overring $Q \supseteq R$ such that R is a right order in Q. A classical left quotient ring is defined analogously, and a classical left and right quotient ring is called a classical quotient ring.*

In Asano [3] it is shown that $Q^r_{cl}(R)$ exists if and only if R satisfies the right Ore condition, i.e., for any $a \in R$ and any $c \in \mathcal{C}_R(0)$ there exist $b \in R$ and $d \in \mathcal{C}_R(0)$ such that $ad = cb$ (a right (or left) Ore ring is any ring satisfying the right (or left) Ore condition). When both $Q^r_{cl}(R)$ and $Q^l_{cl}(R)$ exist, we have $Q^r_{cl}(R) \cong Q^l_{cl}(R)$. This occurs only when R is an order. We will denote by $Q_{cl}(R)$ the classical quotient ring of R (if it exists). Another basic fact is that the classical right quotient ring (if it exists) is unique, up to isomorphism (see [18, Cor.9.5, P146]).

Lemma 1.1 *Let R be a right order with $Q = Q^r_{cl}(R)$ and let S be an overring of R, i.e., $R \subseteq S \subseteq Q$. If I is a right S-submodule of Q such that I contains a regular element of R, then $Hom_S(I_S, S_S) = \{\sigma_q : q \in Q, qI \subseteq S\}$, where $\sigma_q : I \longrightarrow S$ is a S-homomorphism defined by $\sigma_q(x) = qx$.*

Proof. For each $q \in Q$ with $qI \subseteq S$, it is easy to see that σ_q is a S-homomorphism. Suppose $\phi : I \longrightarrow S$ is a S-homomorphism. Let $s \in I$ be a regular element of R. For each $x \in I$ there exists a regular element t of R such that $xt \in R$. Now, by the right Ore condition, there exist $a \in R$ and $u \in \mathcal{C}_R(0)$ such that $sa = xtu$. Then $\phi(x)tu = \phi(xtu) = \phi(sa) = \phi(s)a = \phi(s)s^{-1}sa = \phi(s)s^{-1}xtu$, which implies $\phi(x) = \phi(s)s^{-1}x = \sigma_q(x)$ with $q = \phi(s)s^{-1}$ satisfying $qI \subseteq S$. □

Goldie rings and Goldie Theorems

A right annihilator in a ring R is any right ideal I of R such that $I = X^\perp$ for some $X \subseteq R$. Left annihilators are defined in a similar way. Note that a right ideal I is a right annihilator if and only if $I = ({}^\perp I)^\perp$.

A module M_R is called finite-dimensional (or in other words, M_R has finite Goldie dimension) if M does not contain an infinite direct sum of nonzero submodules. In this case, there exists a nonnegative integer n such that M contains a direct sum of n nonzero submodules, but no direct sum of $n+1$ nonzero submodules. Such an n is uniquely determined by M. We shall call this integer the Goldie dimension of M_R, and denote it by $\dim(M_R)$.

Definition 1.3 *A right Goldie ring is any ring R such that R_R is finite-dimensional and R has ACC on right annihilators.*

Proposition 1.2 [*Goldie*]. *Let R be a semiprime right Goldie ring, and let I be a right ideal of R. Then I is an essential right ideal if and only if I contains a regular element.* $\square$

Theorem 1.1 [*Goldie*]. *Let R be a ring.*

(a) R is a right order in a semi-simple ring if and only if R is a semiprime right Goldie ring;

(b) R is a right order in a simple Artinian ring if and only if R is a prime right Goldie ring. $\square$

Theorem 1.2 *Let R be semiprime. Then R is a right Goldie ring if and only if $Z(R_R) = 0$, and R_R is finite-dimensional.* $\square$

Torsion modules and torsionfree modules

Given a module M_R, let $\tau(M) = \{x \in M : xr = 0 \text{ for some } r \in \mathcal{C}_R(0)\}$. If R is a right order, then $\tau(M)$ is a submodule of M. In fact, for $x, y \in \tau(M)$ and $r \in R$, we have $xs = 0 = yt$ for some $s, t \in \mathcal{C}_R(0)$. By the right Ore condition, there exist $c, d \in \mathcal{C}_R(0)$, and $a, b \in R$, such that $sc = ta$ and $ad = rb$. Then we have $(x - y)sc = xsc - ytd = 0$ and $(xa)d = xrb = 0$. When $\tau(M)$ is a submodule, it is called the torsion submodule of M. If $\tau(M) = M$, then M is called a torsion module, and if $\tau(M) = 0$, then M is called a torsionfree module. Clearly $M/\tau(M)$ is torsionfree for every module M_R. If R is a semiprime Goldie ring, then, because of Proposition 1.2, $Z(M_R) = \tau(M_R)$ for every module M_R.

Theorem 1.3 [*Gentile, Levy*]. *If R is a semiprime Goldie ring and M is a f.g. torsionfree right R-module, then M can be embedded in a f.g. free right R-module.* $\square$

Morita equivalences

Given a right R-module M. We let $M^* = \mathrm{Hom}(M, R)$. The trace of M, written $T(M_R)$, is defined by $T(M) = \Sigma\{f(M) : f \in M^*\}$. It is clear that $T(M)$ is an ideal of R. We now call a right R-module X a generator of the category Mod-R if the trace ideal $T(M) = R$. The concept of generator plays a central role in the study of equivalences between categories of modules. The following proposition gives a number of important characterizations of generators.

Proposition 1.3 *The following are equivalent for a module $X \in$ Mod-R:*

(a) X is a generator;

(b) For every $M \in$ Mod-R, there is an index set I such that M is a homomorphism image of $X^{(I)}$, where $X^{(I)}$ is the direct sum of I copies of X;

(c) There exists an n such that R is a homomorphism image of $X^{(n)}$. $\square$

A module P is called projective if given an epimorphism $p : M \longrightarrow N$, then any homomorphism $f : P \longrightarrow N$ can be factored as $f = p \circ g$ for some $g : P \longrightarrow N$. It is well-known that a module P_R is projective if and only if P is a direct summand of a free module if and only if any short exact sequence $0 \to M \to N \to P \to 0$ splits. A very useful criterion for projectivity is the following proposition which is often called the "dual basis lemma" for projective modules.

Proposition 1.4 *A R-module P_R is projective if and only if there exist a set $\{x_\alpha : \alpha \in I\}$ of elements in P and a set $\{f_\alpha : \alpha \in I\}$ of elements in $P^* =$ Hom (P, R), such that for any $x \in P$, $f_\alpha(x) = 0$ for all but finite number of the f_α, and $x = \Sigma_{\alpha \in I} x_\alpha f_\alpha(x)$.* $\Box$

A module X is a progenerator of Mod-R if and only if X is f.g. projective and X is a generator of Mod-R.

Definition 1.4 *Let F, G be functors from Mod-R to Mod-S. We say there is a natural isomorphism from F to G, written $F \cong G$, if there exists a map that assigns to every module $M \in$ Mod-R an isomorphism $\phi_M \in Hom_S(F(M), G(M))$ such that for any $M, N \in$ Mod-R and any $f \in Hom_R(M, N)$ the following diagram:*

$$\begin{array}{ccc} F(M) & \xrightarrow{\phi_M} & G(M) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(N) & \xrightarrow{\phi_N} & G(N) \end{array}$$

is commutative.

Definition 1.5 *Two rings R and S are said to be Morita equivalent , written $R \sim S$, if there exist functors F : Mod-R $\longrightarrow$ Mod-S and G : Mod-S $\longrightarrow$ Mod-R such that $GF \cong 1_{Mod-R}$, $FG \cong 1_{Mod-S}$. In this case, F is called a Morita equivalence and G an inverse equivalence between Mod-R and Mod-S.*

Any two rings which are isomorphic are of course Morita equivalent. The fact that $R \sim M_n(R)$, where $M_n(R)$ is the ring of n by n matrices with entries in R, shows simply that a noncommutative ring may be Morita equivalent to a commutative ring.

Theorem 1.4 *For two rings R and S, then $R \sim S$ if and only if $S \cong End(M)$ for some progenerator M of Mod-R. And in this case, $Hom_R(M, -) : N_R \longmapsto Hom_R({}_SM_R, N_R)$ defines a Morita equivalence between Mod-R and Mod-S with inverse equivalence $- \otimes_S M : P_S \longmapsto P \otimes_S M$.* $\square$

Theorem 1.5 *For two rings R and S, then $R \sim S$ if and only if $S \cong eM_n(R)e$ for some n and some idempotent e of $M_n(R)$ with $M_n(R)eM_n(R) = M_n(R)$.* $\square$

Any ring property which is preserved under Morita equivalence is called a Morita invariant. For example, being a semiprime right Goldie ring is a Morita invariant because any ring Morita equivalent to a semiprime right Goldie ring is semiprime right Goldie [29, Propo.5.10].

Semihereditary rings

A ring is right (or left) semihereditary if every f.g. right (or left) ideal is projective. A right and left semihereditary ring is called a semihereditary ring. An example of a ring which is right but not left semihereditary was given by Chase [6]. In the following, we introduce a theorem of Small which presents certain classes of rings for which right semihereditary implies left semihereditary.

Theorem 1.6 *[Small]. Let R be a ring in which every principal right ideal is projective and in which there is no infinite set of orthogonal idempotents. Then every right and every left annihilator is generated by an idempotent. In particular, every principal left ideal is projective.*

Proof. Suppose $0 \neq T = S^{\perp}$. If $s \in S$, then $T \subseteq s^{\perp}$. Thus, $T \subseteq hR$ where h is an idempotent. Now let L be an arbitrary (nonzero) left annihilator. $L^{\perp} \subseteq gR$ where $g^2 = g$. But then $L = {}^{\perp}(L^{\perp}) \supseteq {}^{\perp}(gR) = R(1-g)$. Hence, any left annihilator L contains a nontrivial idempotent. By [2, Ex. 10.11], we can choose an idempotent $e \in L$ such that ${}^{\perp}e$ is minimal amongst the left annihilators of idempotents in L. We claim ${}^{\perp}e \cap L = 0$. Suppose not. Then ${}^{\perp}e \cap L$ is a nonzero left annihilator which contains an idempotent $f \neq 0$. Now $e^* = e + f - ef$ is an idempotent in L. Since $e^*e = e$, $e^* \neq 0$ and ${}^{\perp}e^* \subseteq {}^{\perp} e$. However, $fe = 0$ and $fe^* = f \neq 0$. Thus, ${}^{\perp}e^* \not\subseteq {}^{\perp}e$, which contradicts the minimality of ${}^{\perp}e$. Hence ${}^{\perp}e \cap L = 0$. Now if $x \in L$, then $x - xe \in L$ and $(x - xe)e = 0$. Therefore $x - xe = 0$ and $L = Re$. Finally, if K is a right annihilator, then ${}^{\perp}K = Re$ where $e^2 = e$. But, $K = ({}^{\perp}K)^{\perp} = (1-e)R$. □

Proposition 1.5 *A ring R is right (left) semihereditary if and only if $M_n(R)$, for all n, has principal right (left) ideals projective.*

Proof. It is well known that if R is right (left) semihereditary, then so is $M_n(R)$.

In the other direction, we must show that any f.g. right ideal, say $I = a_1R + \cdots + a_nR$, is projective. In $M_n(R)$ let x be the matrix (c_{ij}) where $c_{1i} = a_i$ and all other entries are zero. Then $xM_n(R)$ is projective as a right $M_n(R)$-module. But, $xM_n(R)$ considered as a right R-module (R embedded in $M_n(R)$ in the usual way) is isomorphic to $I \oplus \cdots \oplus I$ (n times). Thus, since $M_n(R)$ is R-free, $I \oplus \cdots \oplus I$ is R-projective and I is R-projective. □

Combining Theorem 1.6 and Proposition 1.5, we immediately obtain

Theorem 1.7 [*Small*]. *Suppose R is a ring which is right semihereditary and such that $M_n(R)$, for all n, does not possess an infinite set of orthogonal idempotents, then R is left semihereditary.* $\square$

2 Noncommutative Prüfer rings and some generalizations

(Noncommutative) Prüfer rings were introduced and studied by Alajbegovic and Dubrovin [1]. Examples of Prüfer rings include prime Dedekind rings, commutative Prüfer domains and prime Goldie right (or left) Bezout rings (cf.[1]. Examples 1.13 and 1.15). Some important properties of Prüfer rings have been demonstrated in the paper of Alajbegovic and Dubrovin (cf.[1] for details). An observation is that a ring R is a Prüfer ring if and only if R is a prime Goldie ring with the following property (see Proposition 2.1.1):
(P): Every finitely generated essential right ideal of R is a progenerator of Mod-R. Replacing 'prime Goldie' by 'semiprime Goldie', 'prime right Goldie', and 'semiprime right Goldie', respectively, in the above condition, we introduce three natural generalizations of Prüfer ring which are to be called (right) semi-Prüfer ring, right w-semi-Prüfer ring, and right w-Prüfer ring respectively (see section 1 for the precise definitions). The main object of this chapter is to study the relationship between all these rings and to establish various properties of them.

In section 1, we first give the definitions of three generalizations of Prüfer rings. The four concepts, especially their implication relations, are further explained by using a known example. The rest of section 1 is used to present the various properties and characterizations of all these rings. In section 2, we will present a structure theorem of right w-semi-Prüfer rings. " A ring is a right w-semi-Prüfer ring if and only if it is a finite direct sum of right

w-Prüfer rings". Section 3 is devoted to studying Prüfer rings and semi-Prüfer rings. We will show that the right semi-Prüfer rings are exactly the left semi-Prüfer rings. A structure theorem states that a ring is a semi-Prüfer ring if and only if it is a finite direct sum of Prüfer rings. We will pay special attention to the cases where the Prüfer ring R is a Noetherian, bounded, semiperfect ring respectively. It was proved in [1] that every Prüfer ring is Morita equivalent to a Prüfer domain. We will give a stronger result here which says that every Prüfer ring R can be decomposed as a finite direct sum of uniform submodules such that the endomorphism ring of each of these uniform submodules is a Prüfer domain which is Morita equivalent to R. The last result can be used to give a characterization of f.g. torsionfree modules over a semi-Prüfer ring.

2.1 Definitions and properties

Let ring R be a right order with $Q = Q^r_{cl}(R)$. Given a subset I of Q, we set

$$O_r(I) = \{q \in Q : I \supseteq Iq\}; \quad O_l(I) = \{q \in Q : I \supseteq qI\};$$
$$[R : I]_r = \{q \in Q : R \supseteq Iq\}; \quad [R : I]_l = \{q \in Q : R \supseteq qI\};$$
$$\text{and} \qquad I^{-1} = \{q \in Q : I \supseteq IqI\}.$$

A submodule I of Q_R is called a fractional right ideal of R if I contains a regular element of Q, and there exists a regular element d of Q with $R \supseteq dI$.

Definition 2.1.1 *A semiprime Goldie (semiprime right Goldie or prime right Goldie or prime Goldie) ring R is called a right semi-Prüfer (right w-semi-Prüfer or right w-Prüfer, or right Prüfer) ring if every finitely generated (f.g. for short) fractional right ideal I of R satisfies:*

$$I^{-1}I = R, \quad II^{-1} = O_l(I).$$

The left-sided versions can be defined in a similar way. Clearly every right Prüfer ring is a right semi-Prüfer (right w-semi-Prüfer or right w-Prüfer) ring, and every right w-Prüfer ring is a right w-semi-Prüfer ring.

Remark 2.1.1 *The definition of a right Prüfer ring is due to Alajbegovic and Dubrovin [1].*

Lemma 2.1.1 [32]. *If I is a fractional right ideal of a right order R, then the following are equivalent:*

(a) $II^{-1} = O_l(I)$;

(b) I is a projective right $O_r(I)$-module.

Proof. First we note that given a fractional right ideal I of a right order R, $O_r(I)$ is an overring of R and I is a right $O_r(I)$-module.

$(a) \Leftarrow (b)$. By Lemma 1.1, $\mathrm{Hom}_{O_r(I)}(I, O_r(I)) = \{\sigma_q : q \in Q, qI \subseteq O_r(I)\} = \{\sigma_q : q \in I^{-1}\}$, where for each $q \in I^{-1}$, $\sigma_q : I \longrightarrow O_r(I)$ is the $O_r(I)$-homomorphism defined by $\sigma_q(a) = qa$. Suppose that I is a projective right $O_r(I)$-module. Then, by the dual basis lemma, there exist $\{a_\alpha : \alpha \in X\} \subseteq I$ and $\{\sigma_{q_\alpha} : \alpha \in X\} \subseteq \mathrm{Hom}_{O_r(I)}(I, O_r(I))$ such that for any $a \in I$, $\sigma_{q_\alpha}(a) = q_\alpha(a) = 0$ for all but a finite number of the σ_{q_α}, and $a = \Sigma_{\alpha\in X} a_\alpha \sigma_{q_\alpha}(a)$. Choosing a to be regular shows that $q_\alpha = 0$ for all but a finite number of α. Letting a be arbitrary again, we see that $a = \Sigma_\alpha a_\alpha q_\alpha a = (\Sigma_\alpha a_\alpha q_\alpha)a$. Thus $\Sigma_\alpha a_\alpha q_\alpha = 1 \in II^{-1}$ and hence $II^{-1} = O_l(I)$.

$(a) \Rightarrow (b)$. Suppose that $O_l(I) = II^{-1}$. Then there exist finite sets $\{a_\alpha\} \subseteq I$ and $\{q_\alpha\} \subseteq I^{-1}$ such that $\Sigma_\alpha a_\alpha q_\alpha = 1$. Hence $\Sigma_\alpha a_\alpha q_\alpha a = a$. Then

$\Sigma_\alpha a_\alpha \sigma_{q_\alpha}(a) = a$ with each $\sigma_{q_\alpha} \in \mathrm{Hom}_{O_r(I)}(I, O_r(I))$. Therefore, by the dual basis lemma, I is a projective right $O_r(I)$-module. □

Lemma 2.1.2 [*1*]. *If I is a fractional right ideal of a right order R, and $L = [R : I]_l$, then the following are equivalent:*

(a) $LI = R$;

(b) I_R is a generator of Mod-R;

(c) $I^{-1}I = R$.

Any of these conditions implies that $O_r(I) = R$.

Proof. $(a) \Leftrightarrow (b)$. By definition, I_R is a generator of Mod-R if and only if $R = T(I_R) = \Sigma\{f(I) : f \in \mathrm{Hom}_R(I, R)\}$. By Lemma 1.1, $\mathrm{Hom}_R(I, R) = \{\sigma_q : q \in L\}$. Therefore we have that I_R is a generator of Mod-R if and only if $R = \Sigma\{\sigma_q(I) : q \in L\} = \Sigma\{qI : q \in L\} = LI$.

Before proving $(b) \Leftrightarrow (c)$, we note the fact that if $KI = R$ for some subset K of Q, then $O_r(I) = R$. In fact, for $q \in Q$ with $Iq \subseteq I$, we have $KIq \subseteq KI$, i.e., $Rq \subseteq R$, and thus $q \in R$. In both cases (a) and (c) we therefore can use the equality $O_r(I) = R$.

$(c) \Rightarrow (a)$. Now we can take I^{-1} as K. Using (c), the inclusion $O_r(I)R \subseteq O_r(I)$ can be written in an equivalent form $O_r(I)I^{-1}I \subseteq O_r(I)$. By the definition of L it follows that $O_r(I)I^{-1} \subseteq L$, and thus $O_r(I) \subseteq LI$, i.e., $R \subseteq LI$. Consequently $R = LI$, and (a) holds.

$(a) \Rightarrow (c)$. This time we can put $K = L$. Also, from the definitions of I^{-1}, $O_r(I)$, and L it follows that $LI \subseteq I^{-1}I \subseteq O_r(I)$. Hence $R \subseteq I^{-1}I \subseteq R$, i.e., (c) holds. Finally, the remark above shows that either of (a), (b), or (c) implies $R = O_r(I)$. □

Proposition 2.1.1 *The following are equivalent for a ring R:*

(a) R is a right semi-Prüfer (right w-semi-Prüfer, or right w-Prüfer, or right Prüfer) ring;

(b) R is a semiprime Goldie (semiprime right Goldie, or prime right Goldie, or prime Goldie) ring, and every f.g. fractional right ideal of R is a progenerator of Mod-R;

(c) R is a semiprime Goldie (semiprime right Goldie, or prime right Goldie, or prime Goldie) ring, and R has property **(P)**.

Proof. We give a proof only for the case where R is a semiprime right Goldie ring.

$(a) \Leftrightarrow (b)$. By Lemma 2.1.1 and Lemma 2.1.2.

$(b) \Rightarrow (c)$. Since R is a semiprime right Goldie ring, every essential right ideal of R contains a regular element of R by Proposition 1.2. Therefore every f.g. essential right ideal of R is a fractional right ideal.

$(c) \Rightarrow (b)$. Let I be a f.g. fractional right ideal of R. From the definition of a fractional right ideal, we know that there exist regular elements c and d of Q such that $c \in I$ and $dI \subseteq R$. Then $dc \in dI$, and dc is a regular element of R. Hence dI is a f.g. essential right ideal of R by Proposition 1.2. Then (c) implies that dI is a progenerator of Mod-R. But we have $I_R \cong (dI)_R$, so I_R is a progenerator of Mod-R. □

Example 2.1.1 *Let F be a field such that there exists an isomorphism λ of F onto a proper subfield of F. Let R be the abelian group consisting of all polynomials in x with coefficients from F, with coefficients written on the right. Define a multiplication in R by using the rule $ax^n = x^n(\lambda^n a)$ for all*

$a \in F$ and all n. Then the ring R is a principal right ideal domain, and R is right Ore but not left Ore [17, Ex.1, P101]. Hence R is right Goldie but not left Goldie. Therefore we have (a) R is a right w-Prüfer ring; (b) R is not a left w-semi-Prüfer ring; (c) R is not a right semi-Prüfer ring.

The example also tells us that being a w-Prüfer ring (or a w-semi-Prüfer ring) is not a left-right symmetric concept. Since it will be shown that a Prüfer ring or semi-Prüfer ring is left-right symmetric and a ring is a semi-Prüfer ring if and only if it is a finite direct sum of Prüfer rings, we have the following implication diagram:

$$\begin{array}{ccccc} & & \text{semi-Prüfer} & & \\ & \not\swarrow\ \nearrow & & \searrow\ \not\nwarrow & \\ \text{Prüfer} & & \not\uparrow\ \not\downarrow & & \text{right } w\text{-semi-Prüfer} \\ & \not\nwarrow\ \searrow & & \nearrow\ \not\swarrow & \\ & & \text{right } w\text{-Prüfer} & & \end{array}$$

Proposition 2.1.2 *Every right w-semi-Prüfer ring is a right and left semi-hereditary ring.*

Proof. Suppose R is a right w-semi-Prüfer ring and I_R a f.g. right ideal of R. We have a right ideal J of R which is maximal with respect to $I \cap J = 0$. And $I + J = I \oplus J \leq_e R_R$. Since R is a semiprime right Goldie ring, $I \oplus J$ contains a regular element r of R by Proposition 1.2. Write $r = a + b$, $a \in I$ and $b \in J$, and let $K = I \oplus bJ$. Then K is a f.g. essential right ideal of R. By Proposition 2.1.1 K_R is projective, and so is I_R. We have shown that R is a right semihereditary ring. Because the property of being a semiprime right Goldie ring is Morita invariant, $M_n(R) \cong \mathrm{End}(R^n)$ is a semiprime right Goldie ring for all n, and thus $M_n(R)$ does not possess an infinite set of orthogonal idempotents. Hence R is left semihereditary by Theorem 1.7. □

Lemma 2.1.3 *If a ring R is a right w-semi-Prüfer ring, then $M_n(R)$ is a right w-semi-Prüfer ring for every n.*

Proof. Since the property of being a semiprime right Goldie ring is Morita invariant, $M_n(R)$ is a semiprime right Goldie ring. It is also clear that $M_n(R)$ is a semihereditary ring because of Proposition 2.1.2 and Theorem 1.7. So it suffices to show that every f.g. essential right ideal L of $M_n(R)$ is a generator of Mod-$M_n(R)$. We need some notation: if A is a subset of R, set $A[k] = \{(a_{ij}) \in M_n(R) : a_{ij} = 0 \ \ \forall i \neq k; a_{kj} \in A\}$. It is easy to see that

$$L_{M_n(R)} = (e_{11}L)_{M_n(R)} \oplus (e_{22}L)_{M_n(R)} \oplus \cdots \oplus (e_{nn}L)_{M_n(R)},$$

where e_{kk} is the matrix having a lone 1 as its (k,k)-entry and all other entries 0, and for each k ($1 \leq k \leq n$), there exists a right ideal I_k of R such that $(e_{kk}L)_{M_n(R)} = I_k[k]$. If I is a nonzero right ideal of R, then $I[1]$ is a nonzero right ideal of $M_n(R)$, so $L \cap I[1] = (I \cap I_1)[1] \neq 0$. This implies that $I \cap I_1 \neq 0$. Hence I_1 is a f.g. essential right ideal of R. By Proposition 2.1.1, I_1 is a generator of Mod-R. We know R is Morita equivalent to $M_n(R)$ via the Morita equivalence $G = (- - \otimes_{M_n(R)} R^n)_R : \text{Mod-}M_n(R) \longrightarrow \text{Mod-}R$. In particular, $G((e_{11}L)_{M_n(R)}) = (e_{11}L \otimes_{M_n(R)} R^n)_R$. But we have a R-homomorphism $\phi : (e_{11}L \otimes_{M_n(R)} R^n)_R \to (I_1)_R$ which is defined by

$$\begin{pmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto a_1x_1 + \cdots + a_nx_n.$$

Obviously ϕ is onto. Since $(I_1)_R$ is a generator of Mod-R, we infer that $(e_{11}L \otimes_{M_n(R)} R^n)_R$ is a generator of Mod-R. Hence $(e_{11}L)_{M_n(R)}$ is a generator of Mod-$M_n(R)$ by [2, Prop.21.6]. Thus we have L is a generator of Mod-$M_n(R)$ because $e_{11}L$ is an image of L as right $M_n(R)$-modules. □

Lemma 2.1.4 *Let R be a right w-semi-Prüfer ring, e an idempotent of R with $ReR = R$. Then eRe is a right w-semi-Prüfer ring.*

Proof. Clearly eRe is a semiprime right Goldie ring. Suppose L is a f.g. essential right ideal of eRe, we want to show that L is a generator of Mod-eRe. Write $L = \sum_i (ex_ie)eRe$. Then $L = LeRe = Te$, where $T = LeR$ is a f.g. right ideal of R. Clearly $T \subseteq eR$. We claim that $T_R \leq_e (eR)_R$. In fact, if $0 \neq er \in eR$, then $erRe \neq 0$ since R is a semiprime ring. Hence $erRe$ is a nonzero right ideal of eRe. Thus $erRe \cap L \neq 0$, i.e., $0 \neq erxe \in L$ for some $x \in R$. So $0 \neq erxeR \subseteq T$. Next we show that T_R is a generator of Mod-R. We know $((1-e)R)_R$ has finite Goldie dimension, and so there exist nonzero uniform right ideals U_i of R such that

$$U_1 + \cdots + U_n = U_1 \oplus \cdots \oplus U_n \leq_e ((1-e)R)_R.$$

We claim $U_ieR \neq 0$ $\forall i$. Otherwise $eR \subseteq U_i^{\perp} = fR$ for some idempotent $f \in R$ by Proposition 2.1.2 and Theorem 1.6. Since fR is an ideal, we have $Rf \subseteq fR$ and so $(1-f)Rf = 0$. Since $R = Rf + R(1-f)$, it follows that $R(1-f)$ is a two-sided ideal, and hence $fR(1-f)$ is a right ideal. Now $[fR(1-f)]^2 = 0$, and R has no nonzero nilpotent right ideals, hence $fR(1-f) = 0$. Given any $r \in R$, we thus have $fr(1-f) = 0$ as well as $(1-f)rf = 0$, whence $fr = frf = rf$. Then $R = ReR = RfR = Rf$, and this implies that $f = 1$. Therefore $U_i = U_iR = 0$. The contradiction shows that $U_ieR \neq 0$ $\forall i$. Thus $eRU_i \neq 0$ $\forall i$ since R is a semiprime ring. Since $T_R \leq_e (eR)_R$, we have $T \cap eRU_i \neq 0$. Then $0 \neq (T \cap eRU_i)^2 \subseteq eRU_iT$. So $U_iT \neq 0$ $\forall i$. For each i, choose an $a_i \in U_i$ such that $a_iT \neq 0$. Then

$$a_1T + \cdots + a_nT = a_1T \oplus \cdots \oplus a_nT \leq_e ((1-e)R)_R.$$

Therefore

$$T \oplus a_1T \oplus \cdots \oplus a_nT \leq_e eR \oplus (1-e)R = R_R.$$

By Proposition 2.1.1, $T \oplus a_1T \oplus \cdots \oplus a_nT$ is a generator of Mod-R. Since each a_iT is an image of T_R, we conclude that T is a generator of Mod-R. To see L is a generator of Mod-eRe, we use the Morita equivalence $\mathrm{Hom}_R(eR, --) : \text{Mod-}R \longrightarrow \text{Mod-}eRe$. Since T is a generator of Mod-R, we have $L_{eRe} = (Te)_{eRe} \cong (\mathrm{Hom}_R(eR, T_R))_{eRe}$ (by [2, Prop.4.6]) is a generator of Mod-eRe by [2, Prop. 21.6]. Finally, since R is a semihereditary ring, every f.g. submodule of $(eR)_R$ is projective. Therefore $eRe = \mathrm{Hom}_R(eR, eR)$ is a right semihereditary ring by [2, Prop.21.6; Prop.21.8]. □

Theorem 2.1.1 *The property of being a right w-semi-Prüfer (right semi-Prüfer, right w-Prüfer, or right Prüfer) ring is a Morita invariant.*

Proof. Suppose R is a right w-semi-Prüfer ring which is Morita equivalent to ring S. Then $S \cong eM_n(R)e$ for some n and some idempotent $e \in M_n(R)$

with $M_n(R)eM_n(R) = M_n(R)$ by Theorem 1.5. By Lemma 2.1.3 and Lemma 2.1.4, S is a right w-semi-Prüfer ring. Since the properties semiprime Goldie, prime right Goldie, and prime Goldie are all Morita invariants, the other parts follow immediately. □

Proposition 2.1.3 *The ring R is a right semi-Prüfer (right w-semi-Prüfer, or right w-Prüfer, or right Prüfer) ring if and only if R is a semiprime Goldie (semiprime right Goldie, or prime right Goldie, or prime Goldie) ring and every f.g. essential submodule of each progenerator of Mod-R is a progenerator of Mod-R.*

Proof. One direction is clear by Proposition 2.1.1. Suppose that R is a right w-semi-Prüfer ring. Let P_R be a progenerator and N_R a f.g. essential submodule of P_R. And set $S = \mathrm{End}(P_R)$. Then we have the Morita equivalence $F = \mathrm{Hom}_R({}_SP_R, --) : \text{Mod-}R \longrightarrow \text{Mod-}S$. By [2, Prop.21.6; Prop.21.8], $F(N)_S$ is a f.g. essential submodule of $F(P)_S = S_S$. We know S is a right w-semi-Prüfer ring from Theorem 2.1.1. Hence it follows that $F(N)_S$ is a progenerator of Mod-S from Proposition 2.1.1. Therefore N_R is a progenerator of Mod-R by [2, Prop.21.6; Prop.21.8]. □

Proposition 2.1.4 *The following are equivalent for a ring R:*

(a) R is a right w-Prüfer (or right Prüfer) ring;

(b) R is a right Goldie (or Goldie) ring and every f.g. nonzero right ideal of R is a progenerator of Mod-R;

(c) R is a right Goldie (or Goldie) ring and every f.g. nonzero submodule of each progenerator of Mod-R is a progenerator of Mod-R.

Proof. $(a) \Rightarrow (b)$. By Proposition 2.1.2, it is enough to show that every f.g. nonzero right ideal I of R is a generator. We can find a right ideal J of R such that $I + J = I \oplus J \leq_e R_R$. Since R is a right Goldie ring, there exist uniform submodules $J_1, \ldots, J_t$ of J_R such that $J_1 \oplus \cdots \oplus J_t \leq_e J_R$. Hence $I \oplus J_1 \oplus \cdots \oplus J_t \leq_e R_R$. Since R is prime, $J_i I \neq 0$ for each i. So we can choose some $a_i \in J_i$ with $a_i I \neq 0$. Then $I \oplus a_1 I \oplus \cdots \oplus a_t I \leq_e R_R$. By Proposition 2.1.1, $I \oplus a_1 I \oplus \cdots \oplus a_t I$ is a generator of Mod-R. Therefore I is a generator of Mod-R.

$(b) \Rightarrow (a)$. That every f.g. nonzero ideal of R is a generator implies that R is a prime ring.

$(b) \Rightarrow (c)$. Similar to the proof of Proposition 2.1.3.

$(c) \Rightarrow (a)$. By Proposition 2.1.1. □

Proposition 2.1.5 *The ring R is a right w-semi-Prüfer ring if and only if $Z(R_R) = 0$, R_R is finite-dimensional and R has Property* **(P)**.

Proof. One direction is clear. Suppose that $Z(R_R) = 0$, R_R is finite-dimensional, and R has Property **(P)**. We only need to show that R is semiprime right Goldie. Suppose $I^2 = 0$ for an ideal I of R. We have a right ideal J of R such that $I + J = I \oplus J \leq_e R_R$. Then $(I + J)I \subseteq JI \subseteq I \cap J = 0$. Since R_R is finite-dimensional, there exist f.g. right ideals I_1,J_1 of R such that $I_1 \leq_e I_R$, $J_1 \leq_e J_R$. Therefore we have $I_1 \oplus J_1 \leq_e R_R$. Since R has Property **(P)**, $I_1 \oplus J_1$ is a generator of Mod-R. Thus R_R is an R-homomorphic image of $(I_1 \oplus J_1)^{(n)}$ for some n. Noting that $(I_1 \oplus J_1)I = 0$, we have $I = RI = 0$. Therefore R is semiprime. By Theorem 1.2, R is a right Goldie ring. □

Proposition 2.1.6 *The ring R is a right w-Prüfer ring if and only if R_R is finite-dimensional and every f.g. nonzero right ideal of R is a progenerator of Mod-R.*

Proof. The necessity follows from Proposition 2.1.4. For the converse, it is easy to see that R is a prime ring. Suppose $Z(R_R) \neq 0$. We can choose a f.g. right ideal I of R such that $I \subseteq Z(R_R)$. By our assumption, I is a generator of Mod-R, and thus R_R is an epimorphic image of $I^{(n)}$ for some n by Proposition 1.3. Since $I^{(n)}$ is singular, we have that R_R is singular. This is a contradiction since $1 \notin Z(R_R)$. Therefore $Z(R_R) = 0$, and thus R is a prime right Goldie ring by Theorem 1.2. □

Some other characterizations of right w-semi-Prüfer (right semi-Prüfer, right w-Prüfer, or right Prüfer) rings will be presented in the next chapter.

2.2 A structure theorem and further properties of right w-semi-Prüfer rings

Lemma 2.2.1 *Let R be a right w-semi-Prüfer ring, and $Q = Q^r_{cl}(R)$. If e is a central idempotent of Q, then $(eR)_R$ is a projective R-module.*

Proof. Write $e = u_1v^{-1}$, $1 - e = u_2v^{-1}$, where $u_i \in R$ and $v \in \mathcal{C}_R(0)$. Define a map $\phi : eR \oplus (1-e)R \longrightarrow R$ by $\phi(ex + (1-e)y) = u_1x + u_2y$ $\forall x, y \in R$. Suppose $ex + (1-e)y = ex' + (1-e)y'$. Then $ex = ex'$, i.e., $(u_1v^{-1})x = (u_1v^{-1})x'$. So $u_1x = v[(u_1v^{-1})x] = v[(u_1v^{-1})x'] = u_1x'$. Similarly $u_2y = u_2y'$. Hence $u_1x + u_2y = u_1x' + u_2y'$. Thus ϕ is well defined. Clearly ϕ is a right R-module homomorphism. If $u_1x + u_2y = 0$, then $0 =$

$(u_1v^{-1})vx + (u_2v^{-1})vy = v[(u_1v^{-1})x + (u_2v^{-1})y]$, and then $ex + (1-e)y = 0$. So ϕ is one to one. Therefore we have $(eR \oplus (1-e)R)_R \cong Im\phi$. But $Im\phi$ is a f.g. right ideal of R, and so it is projective by Proposition 2.1.2. Hence $(eR)_R$ is projective. □

Proposition 2.2.1 *Let R, Q be as above, e any central idempotent of Q. Then $e \in R$.*

Proof. Since $(1-e)R$ is a right projective R-module, the exact sequence $0 \to eR \cap R \to R \to (1-e)R \to 0$ splits. Then $eR \cap R$ is a direct summand of R_R. So we have $eR \cap R = fR$ for an idempotent $f \in R$. Then $fQ \subseteq eQ$. If $0 \neq e\xi \in eQ$, write $\xi = ac^{-1}$ for some $a \in R$ and $c \in \mathcal{C}_R(0)$. Then $0 \neq (e\xi)c = ea \in eR$. Write $ea = uv^{-1}$ for some $u \in R$ and $v \in \mathcal{C}_R(0)$. We have $0 \neq (e\xi)cv = eav = u \in eR \cap R = fR$, and so $0 \neq (e\xi)cv \in fQ$. Therefore $(fQ)_Q \leq_e (eQ)_Q$. Since Q is a semi-simple Artinian ring, $(fQ)_Q$ is a direct summand of $(eQ)_Q$. It must be that $fQ = eQ$. Then $e = fe = ef = f$ since e is central. □

Proposition 2.2.2 *Let R be a right w-semi-Prüfer ring, $Q = Q^r_{cl}(R) = Q_1 \oplus \cdots \oplus Q_n$, where each Q_i is a simple Artinian ring. Then $R = (R \cap Q_1) \oplus \cdots \oplus (R \cap Q_n)$, each $R \cap Q_i$ is a right w-Prüfer ring and $Q^r_{cl}(R \cap Q_i) = Q_i$.*

Proof. By Theorem 1.1, $Q^r_{cl}(R)$ is a semi-simple Artinian ring. Hence the Wedderburn-Artin theorem asserts that $Q^r_{cl}(R)$ is a finite direct sum of simple Artinian rings: $Q^r_{cl}(R) = Q_1 \oplus \cdots \oplus Q_n$, with each Q_i being a simple Artinian ring. We have $1_R = 1_{Q^r_{cl}(R)} = 1_{Q_1} + \cdots + 1_{Q_n}$, where 1_{Q_i} is the identity of Q_i. Set $R_i = R \cap Q_i$. Then $1_{Q_i} \in R_i$ by Proposition 2.2.1. Hence R_i is a subring of Q_i. It is straightforward to check that each R_i is a right order of Q_i. So R_i is a prime right Goldie ring. Each R_i is obviously an ideal of R, and for every $x \in R$, $x = x1_{Q_1} + \cdots + x1_{Q_n} \in R_1 \oplus \cdots \oplus R_n$. Therefore we have $R = R_1 \oplus \cdots \oplus R_n$. To see each R_i is a right w-Prüfer ring, we only need to show that R_i has property **(P)** by Proposition 2.1.1. Let I_i be a f.g. essential right ideal of R_i and let $I = R_1 + \cdots + R_{i-1} + I_i + R_{i+1} + \cdots + R_n = R_1 \oplus \cdots \oplus I_i \oplus \cdots \oplus R_n$. Then I is a f.g. essential right ideal of R, and so I_R is a progenerator of Mod-R by Proposition 2.1.1. Hence $(I_i)_R$ is projective, and this implies that $(I_i)_{R_i}$ is projective. On the other hand, if $f \in \mathrm{Hom}(I_R, R_R)$, we have $f(R_j) \subseteq R_j$, if $j \neq i$, and $f(I_i) \subseteq R_i$. Since I_R is a generator of Mod-R,

$$R = \textstyle\sum_{f\in Hom(I_R,R_R)} Imf = (\sum_{f\in Hom(I_R,R_R)} f(R_1)) \oplus \cdots$$

$$\cdots \oplus (\textstyle\sum_{f\in Hom(I_R,R_R)} f(I_i)) \oplus \cdots \oplus (\sum_{f\in Hom(I_R,R_R)} f(R_n)).$$

So we have

$$R_i = \textstyle\sum_{f\in Hom(I_R,R_R)} f(I_i) = \sum_{f\in Hom(I_{iR},R_{iR})} f(I_i) = \sum_{f\in Hom(I_{iR_i},R_{iR_i})} f(I_i).$$

It follows that I_i is a generator of Mod-R_i. Hence I_i is a progenerator of Mod-R_i. We can conclude that each R_i is a right w-Prüfer ring. □

Theorem 2.2.1 *A ring R is a right w-semi-Prüfer ring if and only if it is a finite direct sum of right w-Prüfer rings.*

Proof. The necessity follows from Proposition 2.2.2.

Suppose $R = \bigoplus_{i=1}^n R_i$ be a direct sum of right w-Prüfer rings R_i. Then $Q_{cl}^r(R) \cong \bigoplus_{i=1}^n Q_{cl}^r(R_i)$ which is a semi-simple Artinian ring. Hence R is a semiprime right Goldie ring. Suppose I is a f.g. essential right ideal of R. Let π_i be the i^{th} projection of R onto R_i. We have $0 \neq I \cap R_i \subseteq \pi_i(I_R)$, and this implies that $\pi_i(I)$ is a f.g. essential right ideal of R_i. By Proposition 2.1.1, $\pi_i(I)$ generates R_i as a right R_i-module, and thus $\pi_i(I)$ generates R_i as a right R-module. Therefore we have shown that I_R is a generator of Mod-R. Next instead of proving I_R is projective, we show R is a right semihereditary ring. For each m, let $M_m(R) = M_m(R_1) \oplus \cdots \oplus M_m(R_n)$. Given $x \in M_m(R)$, write $x = x_1 + \cdots + x_n$, with each $x_i \in M_m(R_i)$. We want to show that $xM_m(R)$ is a projective right $M_m(R)$-module. Since each $M_m(R_i)$ is still a right w-Prüfer ring, we can assume $m = 1$. Since x_iR_i is a projective right R_i-module, we have $(R_i)_{R_i} \cong (x_iR_i)_{R_i} \oplus U_i$ for some right R_i-module U_i. We know U_i can be regarded as a right R-module canonically. Thus as right R-modules we still have $(R_i)_R \cong (x_iR_i)_R \oplus U_i$. Therefore $(x_iR_i)_R$ is projective since $(R_i)_R$ is. Then $xR = x_1R + \cdots + x_nR = x_1R_1 \oplus \cdots \oplus x_nR_n$ is a projective right R-module. We have actually shown that $xM_m(R)$ is a projective right $M_m(R)$-module for every $x \in M_m(R)$. By Proposition 1.5, R is a right semihereditary ring. □

Proposition 2.2.3 *If R is a right w-Prüfer ring, e a nonzero idempotent, then eRe is a right w-Prüfer ring.*

Proof. Since eR is a progenerator of Mod-R by Proposition 2.1.4, then $eRe \cong \mathrm{End}(eR)$ is Morita equivalent to R, thus is a right w-Prüfer ring by Theorem 2.1.1. □

Corollary 2.2.1 *If R is a right w-semi-Prüfer ring, e a nonzero idempotent, then eRe is a right w-semi-Prüfer ring.*

Proof. By Theorem 2.2.1 and Proposition 2.2.3. □

By a complete set of idempotents of a ring we mean a set of pairwise orthogonal idempotents: $\{e_1, \cdots, e_t\}$ with $\Sigma_{i=1}^t e_i = 1$.

Proposition 2.2.4 *If R is a right w-Prüfer ring, then there exists a complete set of idempotents $e_1,\cdots,e_n$, such that $R = e_1R\oplus\cdots\oplus e_nR$ and for each i, e_iRe_i is a right w-Prüfer domain which is Morita equivalent to R.*

Proof. Since R is a prime right Goldie ring, R_R has the ascending chain condition (and the desending chain condition) on the set of direct summands of R_R (see [2, Ex.§10.11]). By [2, Prop.10.14; Prop.7.2], there exists a complete set $e_1, \cdots, e_m$ of idempotents in R such that $R = e_1R \oplus \cdots \oplus e_mR$ and each e_iR is indecomposable as a right R-module. We know e_iR is a progenerator by Proposition 2.1.4. Therefore R is Morita equivalent to $e_iRe_i \cong \mathrm{End}_R(e_iR)$. By Proposition 2.2.3, e_iRe_i is also a right w-Prüfer ring. Now let $0 \neq x \in e_iRe_i$, then xe_iRe_i is a projective e_iRe_i-module and it follows that $x^\perp = fe_iRe_i$ for some $f^2 = f \in e_iRe_i$. But the ring e_iRe_i has exactly one nonzero idempotent, namely e_i. It follows that $f = 0$ or e_i. Since $x \neq 0$, it follows $f = 0$, i.e., $x^\perp = 0$ for all $0 \neq x \in e_iRe_i$. Therefore e_iRe_i is a domain. □

Lemma 2.2.2 *Let $M_1 \oplus M_2 \oplus \cdots \oplus M_n = A \oplus B$ be a decomposition in Mod-R such that End(A_R) is a local ring. Then there exists $i, 1 \leq i \leq n$, and an isomorphism $M_i \cong A \oplus X$ for some $X \in$ Mod-R.*

Proof. See [9, P39–40]. □

A module M_R is called a quasi-injective module if for each submodule N of M, every R-homomorphism from N into M can be extended to an R-homomorphism from M into M.

Proposition 2.2.5 *Let R be a right w-Prüfer ring. If there exists a nonzero f.g. quasi-injective projective right R-module, then R is a simple Artinian ring.*

Proof. As in Proposition 2.2.4, $R = e_1R \oplus \cdots \oplus e_nR$, where each e_iR is an indecomposable R-module. Let M_R be a f.g. quasi-injective projective module. Since e_1R is a generator, there exist an integer $m > 0$ and some R-module X such that $(e_1R)^m \cong M \oplus X$. Since R is finite-dimensional, e_1R, hence $(e_1R)^m$ has finite Goldie dimension. So M has finite Goldie dimension. Write $M = M_1 \oplus \cdots \oplus M_k$, where each M_i is an indecomposable submodule of M. Now, if M is a quasi-injective module, then each M_i is a quasi-injective module. Therefore $\mathrm{End}(M_i)$ is a local ring. Thus Lemma 2.2.2 implies that $e_1R \cong M_1 \oplus U$, for some U. As we know e_1R is an indecomposable R-module, we have $e_1R \cong M_1$ is quasi-injective. We can also show that each $e_iR \cong M_1$. Therefore $R_R \cong M_1^n$ is a quasi-injective module. Now Baer's Criterion implies that R is a right self-injective ring. Then $R = E(R_R)$ is a semi-simple ring by [18, Th.4.28]. Hence R is a simple Artinian ring. □

We know that $\mathbf{Z}$, the ring of integers, is a Prüfer ring, but not a simple Artinian ring. We also know that $\mathbf{Q}$, the field of rational numbers, cannot be embedded in $\mathbf{Z}^{(I)}$ for any index set I. The following is one way to see this:

Corollary 2.2.2 *If R is a right w-Prüfer ring, but not simple Artinian, then, for any f.g. right R-module M, the injective hull $E(M)$ of M cannot* *be embedded in a free R-module.*

Proof. Suppose M is a f.g. R-module, and $0 \longrightarrow E(M) \xrightarrow{l} R^{(I)}$ is exact for some I. Since M_R is finitely generated, $l(M) \subseteq R^{(F)}$, where F is a finite subset of I. Let $p : R^{(I)} \longrightarrow R^{(F)}$ be the canonical projection. We consider $Ker(p \circ l)$. Since $Ker(p \circ l) \cap M = 0$, and $M \leq_e E(M)$, we conclude that $Ker(p \circ l) = 0$. Thus $E(M)$ is embedded in $R^{(F)}$. But $E(M)$ is injective, so it is a direct summand of $R^{(F)}$, and therefore finitely generated. Now the previous proposition implies that R is a simple Artinian ring. □

2.3 Prüfer rings and semi-Prüfer rings

Proposition 2.3.1 *Let R be a right semi-Prüfer ring, and $Q^r_{cl}(R) = Q_1 \oplus \cdots \oplus Q_n$, where each Q_i is a simple Artinian ring. Then $R = (R \cap Q_1) \oplus \cdots \oplus (R \cap Q_n)$, where each $R \cap Q_i$ is a right Prüfer ring and $Q^r_{cl}(R \cap Q_i) = Q_i$.*

Proof. Similar to the proof of Proposition 2.2.2. □

Theorem 2.3.1 *A ring R is a right semi-Prüfer ring if and only if R is a finite direct sum of right Prüfer rings.*

Proof. Similar to the proof of Theorem 2.2.1. □

Next we turn to the left-right symmetry of Prüfer rings and semi-Prüfer rings.

Theorem 2.3.2 [1, Prop. 1.12]. *A ring R is a right Prüfer ring if and only if R is a left Prüfer ring.*

Proof. Suppose R is a right Prüfer ring. We want to show R is left Prüfer. We know R is a Goldie and left semihereditary ring by Proposition 2.1.2. So, to show R is a left Prüfer ring, it suffices to show that for any f.g. nonzero left ideal J of R, J is a generator of R-Mod by using the left version of Proposition 2.1.4. Since R is left semihereditary, J is a projective left R-module. So we may assume that $R^n = J \oplus N$ for some n and some $N \in R$-Mod. Therefore we have ${}_RJ = R^n f$ for some idempotent $f \in \mathrm{End}_R(R^n) = M_n(R)$. Since R^n is a progenerator of R-Mod and $M_n(R) = \mathrm{End}_R(R^n)$, we have a Morita equivalence $R^n \otimes_{M_n(R)} -- : M_n(R)\text{−Mod} \longrightarrow R\text{-Mod}$. As left R-modules, $R^n \otimes_{M_n(R)} M_n(R)f \cong_R (R^n f) =_R J$ (via $a \otimes b \to ab$). So ${}_RJ$ is a generator of R-Mod if and only if ${}_{M_n(R)}(M_n(R)f)$ is a generator of $M_n(R)$-Mod by [2, Prop.21.6]. Also, we know that $M_n(R)$ is a right Prüfer ring from Theorem 2.1.1. Therefore, without lose of generality, we may assume that $J = Re$ for some idempotent e of R. Since R is a prime ring, $ReR \leq_e R_R$. Then $ReR \cap \mathcal{C}_R(0)$ is not empty by Proposition 1.2. Thus there exist elements $r_i, t_i \in R$ $(i = 1, \cdots, m)$ such that $x = r_1et_1 + \cdots + r_met_m \in \mathcal{C}_R(0)$. Consider right ideals $I = r_1eR + \cdots + r_meR$ and $P = fR + xR$. Then I is a f.g. fractional right ideal of R, and so $I^{-1}I = R$, since R is right Prüfer. On the other hand, $P \subseteq ReR$, and $I = r_1eR + \cdots + r_meR \subseteq r_1eReR + \cdots + r_meReR = IeR \subseteq TP$. From $P \subseteq ReR$ and $I \subseteq IP$, it follows that $R = I^{-1}I \subseteq I^{-1}IP \subseteq I^{-1}IReR = RReR \subseteq ReR$. Hence $R = ReR$. Because the trace ideal $T({}_RRe)$ is a two-sided ideal of R and $T({}_RRe) = \Sigma\{\phi(Re) : \phi \in \mathrm{Hom}({}_RRe, {}_RR)\} \supseteq Re$, we have $R = ReR \subseteq T({}_RRe) \subseteq R$. Therefore $R = T({}_RRe)$. By (the left version of) Proposition 1.3, $J = Re$ is a generator of R-Mod. □

Corollary 2.3.1 *The ring R is a right semi-Prüfer ring if and only if R is left semi-Prüfer.*

Proof. By Theorems 2.3.1, 2.3.2. □

From now on we will use the terms Prüfer ring and semi-Prüfer ring instead of right (or left) Prüfer ring and right (or left) semi-Prüfer ring respectively.

Proposition 2.3.2 *If R is a Prüfer ring and e is a nonzero idempotent, then eRe is a Prüfer ring.*

Proof. By Proposition 2.1.4, eR is a progenerator of Mod-R, and so $eRe \cong \mathrm{End}(eR_R)$ is Morita equivalent to R. It follows that eRe is a Prüfer ring from Theorem 2.1.1. □

Corollary 2.3.2 *If R is a semi-Prüfer ring and e is a nonzero idempotent, then eRe is a semi-Prüfer ring.*

Proof. We may assume that $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ with each R_i a Prüfer ring and $e = e_1 + \cdots + e_t$ with $t \leq n$ and each e_i a nonzero idempotent of R_i. Then $eRe = e_1R_1e_1 \oplus \cdots \oplus e_tR_te_t$. The previous proposition implies that each $e_iR_ie_i$ is a Prüfer ring. Hence Theorem 2.3.1 implies that eRe is a semi-Prüfer ring. □

Proposition 2.3.3 [*1*]. *Each overring of a Prüfer ring is a Prüfer ring.*

Proof. Let R be a Prüfer ring, $Q = Q_{cl}(R)$, and let S be an overring of R, i.e., S is a subring of Q such that $R \subseteq S \subseteq Q$. Clearly S is right and left order in Q. Hence S is a prime Goldie ring with $Q_{cl}(S) = Q$. Suppose that J

is a f.g. fractional right ideal of S, e.g., $J = a_1S + \cdots + a_nS$. We may assume that a_1 is a unit of Q. Consider $I = a_1R + \cdots + a_nR$. Then I is a fractional right ideal of R, and thus $I^{-1}I = R$ and $II^{-1} = O_l(I)$. Now we have $IS = J$ and $S = RS = (I^{-1}I)S = I^{-1}J$. The last equality implies that $T(J_S) = S$, i.e., J is a generator of Mod-S. Therefore, by Lemma 2.1.2, $J^{-1}J = S$. On the other hand, $JI^{-1}J = JI^{-1}IS = JRS = J$. This implies that $I^{-1} \subseteq J^{-1}$. Hence $1 \in O_l(I) = II^{-1} \subseteq JJ^{-1}$. Then $O_l(J) \subseteq (O_l(J)J)J^{-1} \subseteq JJ^{-1}$. Therefore $JJ^{-1} = O_l(J)$. It follows that S is a Prüfer ring. □

Corollary 2.3.3 *Each overring of a semi-Prüfer ring is a semi-Prüfer ring.*

Proof. Let R be a semi-Prüfer ring, and $Q_{cl}(R) = Q_1 \oplus \cdots \oplus Q_n$, where each Q_i is a simple Artinian ring. Then by Proposition 2.3.1, $R = (R \cap Q_1) \oplus \cdots \oplus (R \cap Q_n)$, and each $R \cap Q_i$ is a Prüfer ring with $Q^r_{cl}(R \cap Q_i) = Q_i$. Now if S is an overring of R, then $S \cap Q_i$ is an overring of $R \cap Q_i$. By Proposition 2.3.3, $S \cap Q_i$ is a Prüfer ring. But it is easy to see $S = (S \cap Q_1) \oplus \cdots \oplus (S \cap Q_n)$. It follows from Theorem 2.3.1 that S is a semi-Prüfer ring. □

The concept of a prime Dedekind ring was first introduced by Robson in [32] by the term "maximal order". An important characterization of the prime Dedekind rings of Robson is stated as follows: A ring R is a prime Dedekind ring if and only if every nonzero submodule of a (left or right) progenerator is also a progenerator [29, Th.2.10, P.140]. It was proved in [1] that a ring is a Prüfer ring and a bounded Krull ring if and only if it is a prime Dedekind ring, where a bounded Krull ring is defined in the sense of Marubayashi (cf.[27] Sec.1).

Theorem 2.3.3 *The following are equivalent for a ring R:*

(a) R is a Prüfer and (both sides) Noetherian ring;

(b) R is a prime Dedekind ring.

Proof. $(b) \Rightarrow (a)$. Every one-sided ideal of R is a progenerator, and hence a f.g. R-module. It follows that R is a Noetherian ring. Thus Proposition 2.1.4 implies that R is a Prüfer ring.

$(a) \Rightarrow (b)$. Let P_R be a progenerator and N_R nonzero submodule of P_R. We want to show that N_R is also a progenerator. Let $S = \mathrm{End} P_R$. Then R is Morita equivalent to S via the Morita equivalence $F = \mathrm{Hom}_R({}_S P_R, --) :$ Mod-$R \longrightarrow$ Mod-S, and $F(N)_S$ is a nonzero right ideal of S. S is also a Prüfer ring. Since the property of being a one-sided Noetherian ring is a Morita invariant, S is a right Noetherian ring. So $F(N)_S$ is a nonzero f.g. right ideal of S. By Proposition 2.1.4, $F(N)_S$ is a progenerator of Mod-S. Hence N_S is a progenerator of Mod-R. So R is a prime Dedekind ring. □

Corollary 2.3.4 *A ring is a semi-Prüfer Noetherian ring if and only if it is a finite direct sum of prime Dedekind rings.*

Proof. This follows from Theorems 2.3.1, 2.3.3. □

A ring R is right bounded if every essential right ideal of R contains an ideal which is essential as a right ideal. Note that a prime ring R is right bounded if and only if every essential right ideal of R contains a nonzero ideal. A right and left bounded ring is called a bounded ring.

Proposition 2.3.4 *The ring R is a right bounded semi-Prüfer ring if and only if R is a finite direct sum of right bounded Prüfer rings.*

Proof. ($\Rightarrow$). By Theorem 2.3.1, $R = R_1 \oplus \cdots \oplus R_n$, with each R_i being a Prüfer ring. Given an essential right ideal I_i of R_i. Then $I = R_1 \oplus \cdots \oplus R_{i-1} \oplus I_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ is an essential right ideal of R. Hence there exists an ideal J of R such that $J \subseteq I$ and $J_R \leq_e R_R$. $J \cap R_i$ is an ideal of R_i, and we have $0 \neq J \cap R_i \subseteq I \cap R_i \subseteq I_i$. Therefore R_i is right bounded.

($\Leftarrow$). Let $R = R_1 \oplus \cdots \oplus R_n$, where each R_i is a right bounded Prüfer ring. Then R is a semi-Prüfer ring by Theorem 2.3.1. Suppose I_R is an essential right ideal of R. We need to show that I_R contains an ideal of R which is essential as a right ideal. It is easy to see that $I \cap R_i \leq_e (R_i)_{R_i}$. Hence for each i there exists a nonzero ideal K_i of R_i such that $K_i \subseteq I \cap R_i$ and $(K_i)_{R_i} \leq_e (R_i)_{R_i}$. Hence $(K_i)_R \leq_e (R_i)_R$. Set $K = K_1 \oplus \cdots \oplus K_n$. Then K is an ideal of R, $K \subseteq I$ and $K_R \leq_e R_R$. $\square$

A module M_R is faithful if for every $0 \neq r \in R$, $Mr \neq 0$. Every generator is faithful. But the converse is not true. We call a ring a right FPF ring if every f.g. right faithful module is a generator. An FPF ring is defined to be a left and right FPF ring. There are some known relations between bounded prime Dedekind rings and prime FPF rings. In fact a bounded prime Dedekind ring can be characterized as a Noetherian prime right (or left) FPF ring [10, Th.4.6]. In the following we point out how a bounded Prüfer (or semi-Prüfer) ring is related to an FPF ring.

Theorem 2.3.4 *For a ring R, the following are equivalent:*

(a) R is a prime right FPF right semihereditary ring;

(b) R is a right bounded Prüfer ring;

(c) R is a prime right FPF left semihereditary ring.

Proof. $(b) \Rightarrow (a)\&(c)$. If R is a right bounded Prüfer ring , then R is prime Goldie semihereditary ring. Moreover, every f.g. nonzero right ideal is a generator by Proposition 2.1.4. Now, by [10, Th.4.7], R is a prime right FPF ring. So (b) implies (a) and (c).

$(a) \Rightarrow (b)$. If R is a prime FPF right semihereditary ring, then, by [10, Th.4.7], R is a right bounded Goldie (both sides) ring and every nonzero f.g. right ideal is a generator. Now, since R is also a right semihereditary ring, it follows that every nonzero f.g. right ideal of R is a progenerator. By Proposition 2.1.4, R is a Prüfer ring.

$(a) \Leftarrow (c)$. From the proof above, we know any ring R which possesses (c) must be a right bounded prime Goldie ring for which every f.g. nonzero right ideal is a generator. Since the property of being a prime Goldie ring is Morita invariant, $M_n(R)$ is prime Goldie ring for all n. In particular, $M_n(R)$ does not possess an infinite set of orthogonal idempotents. So Theorem 1.7 implies that R is right semihereditary ring, and so (a) holds. □

Corollary 2.3.5 *For a ring R, the following are equivalent:*

(a) R is a prime FPF left semihereditary ring;

(b) R is a bounded Prüfer ring;

(c) R is a prime FPF right semihereditary ring. □

Corollary 2.3.6 *Every right bounded semi-Prüfer ring R is a semiprime semihereditary right FPF ring; The converse is true if R also has ACC on annihilators.*

Proof. For the first part, it is enough to show that R is a right FPF ring. By Proposition 2.3.4, $R = R_1 \oplus \cdots \oplus R_n$, with each R_i being a right bounded

Prüfer ring. Then each R_i is a right FPF ring by Theorem 2.3.4. Therefore we have R is a right FPF ring by [10, Th.3.4]. For the second part, we first note that R is a Goldie ring by [10, Cor.3.16C]. Then [10, Th.3.4(1)] implies that $R = R_1 \oplus \cdots \oplus R_n$, where each R_i is a prime right FPF ring. Since R is a semihereditary ring, it is easy to show that each R_i is also a semihereditary ring. By Theorem 2.3.4, R_i is a right bounded Prüfer ring. Now Proposition 2.3.4 implies that R is a right bounded semi-Prüfer ring. □

It was proved that every Prüfer ring is Morita equivalent to a Prüfer domain in [1, Th.2.3]. We give the following stronger result:

Theorem 2.3.5 *Let R be a Prüfer ring. Then there exists a complete set of idempotents $e_1, \cdots, e_n$ such that $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where for each i, e_iR is a uniform R-module, e_iRe_i is a Prüfer domain and R is Morita equivalent to e_iRe_i.*

Proof. By Proposition 2.2.4, it is enough to show that eR is a uniform right R-module for each indecomposable module eR. Let N be a nonzero R-submodule of eR. We want to show that N is an essential submodule of eR. Suppose $N \cap K = 0$ for a submodule K of eR. We know that there exists a submodule L of eR which is maximal with respect to $K \subseteq L$ and $N \cap L = 0$. By Proposition 1.1, N is embedded in eR/L as an essential submodule. From Theorem 1.2, we have $Z(R_R) = 0$, and thus $Z(N_R) = 0$. It follows that $Z(eR/L) = 0$. By noting Proposition 1.2, we have that eR/L is a f.g. torsionfree right R-module. Therefore, it follows from Theorem 1.3 that eR/L is embedded in a f.g. free right R-module. Since R is Prüfer, eR/L is a projective right R-module. Thus $eR \cong L \oplus (eR/L)$. Now the indecomposablity of eR implies that $L = 0$. Hence $K = 0$, and N is essential in eR. $\square$

Lemma 2.3.1 *Let $R = R_1 \oplus \cdots \oplus R_n$ and $S = S_1 \oplus \cdots \oplus S_n$. If R_i is Morita equivalent to S_i $(i = 1, \cdots, n)$, then R is Morita equivalent to S.*

Proof. Well-known. $\square$

Corollary 2.3.7 *Every semi-Prüfer ring is Morita equivalent to a finite direct sum of Prüfer domains.*

Proof. By Theorem 2.3.1 , Lemma 2.3.1 and Theorem 2.3.5. $\square$

In the final part of this section, we consider semiperfect Prüfer rings. A ring R is semiperfect if $R/Rad(R)$ is semi-simple, and idempotents of $R/Rad(R)$ lift. By a theorem of Bass, the ring R is semiperfect if and only if there exists a complete set of primitive idempotents $e_1, \cdots, e_n$ such that $R = e_1R \oplus \cdots \oplus e_nR$ and each e_iRe_i is local, where a primitive idempotent is any idempotent which cannot be written as the sum of two nontrivial orthogonal idempotents (see [2,Th.27.6]). For any semiperfect ring R, there exists a basic set of orthogonal primitive idempotents $\{e_1, \cdots, e_t\}$ in the sense that for every primitive idempotent f we have $Rf \cong Re_i$ for exactly one e_i, $1 \leq i \leq t$. In this case $e = \Sigma_{i=1}^t e_i$ is called a basic idempotent and eRe is called the basic ring of R. A module is uniserial if its submodules are linearly ordered with respect to inclusion. A ring R is right serial if R_R is a direct sum of uniserial modules. The ring is serial if it is both left and right serial. A local serial ring is called a valuation ring.

Lemma 2.3.2 *Let R be a Prüfer ring, $e^2 = e \in R$. If eRe is a local ring, then R is a semiperfect ring, and every indecomposable projective right R-module is isomorphic to eR, eRe is the basic ring of R, and $R \cong M_n(eRe)$, where n is the Goldie dimension of R_R (or ${}_RR$).*

Proof. By Theorem 2.3.5, $R = e_1R \oplus \cdots \oplus e_nR$, where n is the Goldie dimension of R_R, and e_iR is indecomposable for all i. For each i, e_iR is a generator and eR is projective module, so we have $(e_iR)^m \cong eR \oplus X$ for some $m > 0$ and some R-module X. Because eRe is a local ring, Lemma 2.2.2 implies $e_iR \cong eR \oplus Y$ for some Y. Hence $e_iR \cong eR$ since e_iR is indecomposable. And so $e_iRe_i \cong \mathrm{End}(e_iR) \cong \mathrm{End}(eR) \cong eRe$ is local ring. Hence R is a semiperfect ring, and $R_R \cong (eR)^n$. So $R \cong \mathrm{End}((eR)^n) \cong M_n(eRe)$. The other assertions follow from [2, Prop.27.10]. □

Lemma 2.3.3 *Let R be a Prüfer ring, $e^2 = e \in R$. The following are equivalent:*

(a) $(eR)_R$ is a uniserial module;

(b) eRe is a local ring;

(c) ${}_R(Re)$ is a uniserial module.

Proof. $(a) \Rightarrow (b)$. Let $J = Rad(R)$. We know $Rad(eR) = eJ$ is the intersection of all the maximal submodules of eR. Hence eJ is the unique maximal submodule of eR. By [2, Cor.17.20], eRe is a local ring.

$(a) \Leftarrow (b)$. If eRe is a local ring, then R is a semiperfect ring by Lemma 2.3.2. By noting a result of Warfield which says a semiperfect semiprime Goldie ring is left semihereditary if and only if it is right serial [38, Cor.4.7], we have that R is a right serial ring. Lemma 2.3.2 implies that every uniserial summand of R_R is isomorphic to eR. So eR is a uniserial module.

$(b) \Leftrightarrow (c)$. Similarly. $\square$

Corollary 2.3.8 *Let R be a Prüfer ring, then R is a left (or right) serial ring if and only if R is a semiperfect ring.*

Proof. By Lemma 2.3.3. $\square$

Theorem 2.3.6 *Let R be a Prüfer ring.*

(a) If R is a local ring, then R is a valuation Prüfer domain with both ${}_RR$ and R_R uniserial modules.

(b) If R is a semiperfect ring, then R is a serial ring and $R \cong M_n(B)$, where n is the Goldie dimension of R and B, its basic ring, is a valuation Prüfer domain.

Proof. (a). By Lemma 2.3.3.

(b). This follows from Lemmas 2.3.2, 2.3.3 and Corollary 2.3.8. $\square$

Corollary 2.3.9 *The ring R is a semiperfect semi-Prüfer ring if and only if it is a finite direct sum of matrix rings over valuation Prüfer domains.*

Proof. $(\Rightarrow)$. By Theorem 2.3.1, $R = R_1 \oplus \cdots \oplus R_n$, each R_i is a Prüfer ring. If R is a semiperfect ring, then every R_i is semiperfect by [2, Coro.27.9]. Therefore we have $R_i \cong M_{n_i}(B_i)$ for some valuation Prüfer domain B_i by Theorem 2.3.6.

$(\Leftarrow)$. A finite direct sum of matrix rings over valuation Prüfer domains is clearly a semiperfect ring, and is also a semi-Prüfer ring by Theorem 2.3.1. $\square$

Finally, we give a characterization of f.g. torsionfree modules over a semi-Prüfer ring. A module M_R is flat if whenever $f : {}_RN_1 \longrightarrow {}_RN_2$ is a monomorphism, we have $1 \otimes f : M \otimes_R N_1 \longrightarrow M \otimes_R N_2$ is a monomorphism.

Proposition 2.3.5 *The following are equivalent for a module M_R over a semi-Prüfer ring R:*

(a) M is f.g. torsionfree;

(b) M is f.g. flat;

(c) M is f.g. projective;

(d) M is projective with finite Goldie dimension;

(e) M is a finite direct sum of f.g. uniform right ideals of R.

Proof. $(a) \Rightarrow (e)$. By Theorem 1.3, M is a submodule of a f.g. free module F_R. Because of Theorem 2.3.1, we may assume $R = R_1 \oplus \cdots \oplus R_n$, where each R_i is a Prüfer ring. By Theorem 2.3.5, each R_i is a finite direct sum of f.g. uniform right ideals of R_i. Since every f.g. uniform right ideal of R_i is clearly a f.g. uniform right ideal of R, we have $F = \sum_{i=1}^{m} \bigoplus I_i$, where each I_i is a f.g. uniform right ideal of R. Since R is a semihereditary ring, every f.g. R-submodule of I_i is projective. By [24, Prop.8, P85], $M \cong \sum_{i=1}^{m} \bigoplus N_i$, with each $N_i \subseteq I_i$.

$(d) \Leftarrow (e)$. This is because R is a semihereditary Goldie ring.

$(c) \Leftarrow (d)$. Sandomierski showed in [32, Th.2.1] that if R is a ring such that $Z(R_R) = 0$ and P_R is a projective module containing a f.g. essential submodule, then P is finitely generated [33, Th.2.1]. Our claim follows.

$(b) \Leftarrow (c)$. Well-known.

$(a) \Leftarrow (b)$. Suppose M_R is flat and let $s \in \mathcal{C}_R(0)$. Define $\sigma_s : R \longrightarrow R$ by $\sigma_s(a) = as$, which is a monomorphism as left R-modules. This gives rise to a commutative diagram

$$\begin{array}{ccc} M \otimes_R R & \xrightarrow{1\otimes\sigma_s} & M \otimes_R R \\ \downarrow\cong & & \downarrow\cong \\ M & \xrightarrow{f} & M \end{array}$$

where $f(x) = xs$. Since $1 \otimes \sigma_s$ is a monomorphism, so is f, and thus $x \neq 0$ implies $xs \neq 0$. Therefore M is torsionfree. $\square$

3 Strongly compressible modules

Semiprime right Goldie rings constitute a much studied and well known family of rings, and satisfy one of basic conditions satisfied by right w-semi-Prüfer (right w-Prüfer, semi-Prüfer, or Prüfer) rings which were defined in chapter 2. Recently López-Permouth, Rizvi and Yousif [26] provided some interesting characterizations of semiprime Goldie rings in terms of their right ideals and of their nonsingular right modules. It was shown that a ring R is semiprime Goldie if and only if every right ideal of R is weakly-injective if and only if R is right nonsingular and every nonsingular right R-module is weakly-injective [26, Th.3.9]. This motivates us to look for module-theoretic characterizations of semiprime right Goldie rings. Once such characterizations are established, it can be expected that one can present some new characterizations of right w-semi-Prüfer (right w-Prüfer, semi-Prüfer, or Prüfer) rings. In this chapter, we give the definition of strongly compressible modules. It turns out that the concept of strongly compressible modules is closely related to that of weakly-injective modules and is precisely what we want for our purposes. In fact the connection between strongly compressible modules and weakly-injective modules is similar to that between compressible modules and tight modules (Proposition 3.2.1). We show that a ring R is semiprime right Goldie if and only if R_R is strongly compressible if and only if every right ideal of R is strongly compressible if and only if every submodule of each progenerator of Mod-R is strongly compressible (Theorem 3.1.1). As a corollary of this result, it is shown that a ring R is semiprime Goldie if and only if every f.g. submodule of the injective hull of R_R is strongly compressible if and only if R is right nonsingular and every f.g. nonsingular right R-module is strongly compressible. This characterization theorem can easily imply the above-mentioned characterization theorem of López-Permouth, Rizvi and Yousif because of the strong connection between strongly compressible modules and weakly-injective modules. In the latter part of the chapter, we apply our results to obtain some new module-theoretic characterizations of prime Goldie (prime right Goldie) rings, and right w-semi-Prüfer (right w-Prüfer, semi-Prüfer, or Prüfer) rings, respectively.

3.1 New characterizations of semiprime right Goldie rings

Following Jain and López-Permouth [20], a module M is weakly-injective if and only if for every f.g. submodule N of $E(M)$ there exists $X \subseteq E(M)$ such that $N \subseteq X \cong M$. In [23] a module M is said to be compressible if it is embeddable in each of its essential submodules.

Definition 3.1.1 *A module M_R is said to be strongly compressible if for every essential submodule N of M there exists $X \subseteq E(M)$ such that $M \subseteq X \cong N$.*

Every essential submodule of a strongly compressible module is strongly compressible. Every strongly compressible module is clearly compressible. After Theorem 3.1.5, we will give an example of a compressible module which is not strongly compressible.

Lemma 3.1.1 *Every f.g. strongly compressible right module has finite Goldie dimension.*

Proof. Let M_R be a f.g. strongly compressible module. Suppose M_R is not finite-dimensional. Then there exists an essential submodule N of M such that $N = \bigoplus_{i=1}^{\infty} N_i$, where each $N_i \neq 0$. Since M_R is strongly compressible, there exists a submodule X of $E(M_R)$ such that $M \subseteq X \cong N$. Then $X = \bigoplus_{i=1}^{\infty} X_i$ and $(X_i)_R \cong (N_i)_R$ for all i. Clearly $M \subseteq \bigoplus_{i=1}^{k} X_i$ for some k. Thus $M \cap X_i = 0$ for all $i > k$, contradicting the essentiality of M in $E(M_R)$. $\square$

Lemma 3.1.2 *Let P_R be a progenerator of Mod-R. If P_R is strongly compressible, then R is semiprime.*

Proof. Since P_R is a progenerator of Mod-R, we can assume that $P^n = R \oplus X$ and $R^m = P \oplus Y$ for some positive integers n, m and some $X, Y \in$ Mod-R. If $I^2 = 0$ for some ideal I of R, then $I \subseteq {}^{\perp}I$. There exists a right ideal J of R maximal with respect to ${}^{\perp}I \cap J = 0$. Then ${}^{\perp}I \oplus J \leq_e R_R$. $JI \subseteq J \cap I \subseteq J \cap {}^{\perp}I = 0$. Then $J \subseteq J \cap {}^{\perp}I = 0$. Hence ${}^{\perp}I \leq_e R_R$. Therefore we have $({}^{\perp}I)^m \leq_e R^m$, and thus $({}^{\perp}I)^m \cap P \leq_e P$. Since P_R is strongly compressible, there exists $Z_R \subseteq E(P_R)$ such that $P \subseteq Z \cong ({}^{\perp}I)^m \cap P$. Then $PI \subseteq ZI \cong (({}^{\perp}I)^m \cap P)I = 0$. So $PI = 0$. Then $P^n I = 0$. It implies that $RI = 0$. Therefore $I = 0$. $\square$

Lemma 3.1.3 *Let P_R be a progenerator of Mod-R. If P_R is strongly compressible, then $Z(P_R) = 0$. In particular, $Z(R_R) = 0$.*

Proof. We can assume that $R^m = P \oplus X$ for some positive integer m and some $X \in$ Mod-R. There exists a submodule N of P such that $Z(P_R) \oplus N \leq_e P$.

Since P is strongly compressible, there exists $Y_R \subseteq E(P_R)$ such that $P \subseteq Y \cong Z(P_R) \oplus N$. Write $Y = Y_1 \oplus Y_2$ with $Y_1 \cong Z(P_R)$ and $Y_2 \cong N$ as right R-modules. For each i $(1 \leq i \leq m)$, let e_i be the element of R^m with i^{th} component 1 and all others 0. Write $e_i = a_i + b_i$ for some $a_i \in Y_1$ and $b_i \in Y_2 \oplus X$. Since Y_1 is right singular, $a_i^{\perp} \leq_e R_R$. And $e_i a_i^{\perp} = b_i a_i^{\perp} \subseteq Y_2 \oplus X$ for $i = 1, 2, \cdots, m$. It is easy to see that $(e_i a_i^{\perp})_R \cong (a_i^{\perp})_R$ and $\sum_1^m e_i a_i^{\perp}$ is a direct sum. Noting that P_R, and hence R_R has finite Goldie dimension by Lemma 3.1.1, we have that $dim(R^m) = dim(P \oplus X) = dim(P) + dim(X) = dim(Y) + dim(X) = dim(Y_1) + dim(Y_2) + dim(X) = dim(Y_1) + dim(Y_2 \oplus X) \geq dim(Y_1) + dim(\sum_1^m \oplus e_i a_i^{\perp}) = dim(Y_1) + \sum dim(e_i a_i^{\perp}) = dim(Y_1) + \sum dim(a_i^{\perp}) = dim(Y_1) + m \cdot dim(R) = dim(Y_1) + dim(R^m)$. Thus $dim(Y_1) = 0$, i.e., $Y_1 = 0$. Therefore $Z(P_R) = 0$. □

Lemma 3.1.4 [*Jategaonkar*]. *Let R be a semiprime right Goldie ring. Then any submodule of a f.g. free right R-module is compressible.*

Proof. Since R_R is finite-dimensional, there exist f.g.uniform right ideals of R whose sum, say K, is direct and essential in R. By Proposition 1.2, K contains a regular element r of R. Clearly, the map $a \mapsto ra$, $a \in R$, embeds R_R in K. It follows that any f.g. free right R-module can be embedded in a finite direct sum of f.g. uniform right ideals of R. Then, if M is a submodule of a f.g. free right R-module, there exist f.g. uniform right ideals of R: $I_1, \cdots, I_n$ such that $M \subseteq \oplus_{i=1}^n I_i$. If $M \cap I_j = 0$ for some j, then the restriction of the obvious map $\oplus_{i=1}^n I_i \longrightarrow \oplus_{i \neq j} I_i$ embeds M in $\oplus_{i \neq j} I_i$. Thus, after omitting some of the modules I_i and then reindexing, we may assume that $M \cap I_i \neq 0$ for all i. It follows that M is essential in $\oplus_{i=1}^n I_i$. Let N be an essential submodule of M. Then $N \cap I_i \neq 0$ for all i. Since R is semiprime, we have $(N \cap I_i)I_i \neq 0$. Thus, $tI_i \neq 0$ for some $t \in N \cap I_i$. Now, consider the R-homomorphism $f : I_i \longrightarrow N \cap I_i$ defined by $f(b) = tb$. If $Ker(f) \neq 0$, then $Ker(f) \leq_e I_i$, and so $N \cap I_i \cong I_i/Ker(f)$ is torsion by Proposition 1.2. This is impossible because $N \cap I_i$ is torsionfree. So f is a monomorphism. Clearly, the map $\oplus f_i : \oplus I_i \longrightarrow \oplus(N \cap I_i)$ provides an embedding of M into N. $\square$

Now we can characterize semiprime right Goldie rings as follows.

Theorem 3.1.1 *The following are equivalent for a ring R:*

(a) R is semiprime right Goldie;

(b) R_R is strongly compressible;

(c) Every cyclic right ideal of R is strongly compressible;

(c') Every cyclic essential right ideal of R is strongly compressible;

(d) Every f.g. right ideal of R is strongly compressible;

(d') Every f.g. essential right ideal of R is strongly compressible;

(e) Every right ideal of R is strongly compressible;

(e') Every essential right ideal of R is strongly compressible;

(f) Every cyclic submodule of each progenerator of Mod-R is strongly compressible;

(f') Every cyclic essential submodule of each progenerator of Mod-R is strongly compressible;

(g) Every f.g. submodule of each progenerator of Mod-R is strongly compressible;

(g') Every f.g. essential submodule of each progenerator of Mod-R is strongly compressible;

(h) Every submodule of each progenerator of Mod-R is strongly compressible;

(h') Every essential submodule of each progenerator of Mod-R is strongly compressible.

Proof. $(e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b)$ and $(e) \Rightarrow (e') \Rightarrow (d') \Rightarrow (c') \Rightarrow (b)$. Obviously.

$(b) \Rightarrow (a)$. By Lemmas 3.1.1, 3.1.2 and 3.1.3.

$(a) \Rightarrow (e)$. Let I be a right ideal of R and $K_R \leq_e I_R$. There exists $J \subseteq R_R$ such that $I \cap J = 0$ and $I \oplus J \leq_e R_R$. Then $K \oplus J \leq_e R_R$. By Proposition 1.2, $K \oplus J$ contains a regular element r of R. Then the map $f : (rI)_R \to I_R$, which is defined by $f(rx) = x$ for all $x \in I$, is an isomorphism. Since $K_R \leq_e I_R$, we have $(rK)_R \leq_e (rI)_R$ by Proposition 1.1. Since R is semiprime right Goldie, as a submodule of R_R, $(rI)_R$ is compressible by Lemma 3.1.4. Hence there exists a monomorphism $g : (rI)_R \to (rK)_R$. Since $E(I)$ is an injective module, there exists $h : (rK)_R \to E(I)$, such that $h \circ g = f$. Since $(rK)_R \leq_e (rI)_R$, we have $dim(rK)_R = dim(rI)_R = dim(g(rI))$, and thus $g(rI) \leq_e (rK)_R$. Then h is one to one since f is an isomorphism and g is one to one. Let $X = h(rK)$. Then $I = f(rI) = h \circ g(rI) \subseteq h(rK) = X \subseteq E(I)$, and $X_R \cong (rK)_R \cong K_R$. Therefore I_R is strongly compressible.

$(h) \Rightarrow (g) \Rightarrow (f) \Rightarrow (b)$ and $(h) \Rightarrow (h') \Rightarrow (g') \Rightarrow (f') \Rightarrow (b)$. Obviously.

$(a) \Rightarrow (h)$. Suppose that P_R is a progenerator of Mod-R, N a submodule of P and $K_R \leq_e N_R$. Set $S = \text{End}(P_R)$. Then we have the Morita equivalence $F = \mathrm{Hom}_R({}_SP_R, --)$: Mod-$R \longrightarrow$ Mod-S with inverse equivalence $G = (- - \otimes_S P)_R$: Mod-$S \longleftarrow$ Mod-R. By [2, Prop.21.6], we have $F(K)_S \leq_e F(N)_S \subseteq F(P)_S = S_S$. We know that the property of being a semiprime right Goldie ring is Morita invariant, and thus S is a semiprime right Goldie ring. By the equivalence of (a) and (e), we have $F(N)$ is a right strongly compressible S-module. Hence there exists $Y_S \subseteq E(F(N)_S)$ such that $F(N)_S \subseteq Y_S \cong F(K)_S$. Then $GF(N) \subseteq G(Y) \cong GF(K) \cong K_R$ and $G(Y) \subseteq G(E(F(N)))$. Noting that $F(N) \subseteq F(E(N))$ and $F(E(N))$ is injective [2, Prop.21.6], we have $E(F(N)) \subseteq F(E(N))$. Hence $E(F(N)) = F(E(N))$, since $dimE(F(N))_S = dimF(N)_S = dimN_R = dimE(N)_R = dimF(E(N))_S < \infty$ by [2, Prop.21.7]. So $G(E(F(N))) = GF(E(N)$ and $GF(N)_S \subseteq G(Y) \subseteq GF(E(N))$. If $\eta : GF \longrightarrow 1_{Mod-R}$ is the natural isomorphism, then $N \subseteq \eta(G(Y)) \subseteq E(N)$ and $\eta(G(Y) \cong G(Y) \cong K_R$. Therefore N is strongly compressible. □

Example 3.1.1 *An example of a compressible module which is not strongly compressible can be given as follows: Let R be a domain such that $R^2 \cong R^3$ as right R-modules. Such a ring R exists by J.D.O'Neill [31]. Clearly R_R is compressible, and $dim(R_R) = \infty$. By Theorem 3.1.1, R_R is not strongly compressible.*

Corollary 3.1.1 *The following are equivalent for a ring R:*

(a) R is semiprime right Goldie;

(b) P_R is strongly compressible for some progenerator P_R of Mod-R;

(c) Every cyclic submodule of some progenerator of Mod-R is strongly compressible;

(c') Every cyclic essential submodule of some progenerator of Mod-R is strongly compressible;

(d) Every f.g. submodule of some progenerator of Mod-R is strongly compressible;

(d') Every f.g. essential submodule of some progenerator of Mod-R is strongly compressible;

(e) Every submodule of some progenerator of Mod-R is strongly compressible;

(e') Every essential submodule of some progenerator of Mod-R is strongly compressible.

Proof. By Theorem 3.1.1 and Lemmas 3.1.1, 3.1.2 and 3.1.3. □

Corollary 3.1.2 *A ring R is semiprime right Goldie if and only if $R = End(P_S)$, where P_S is a strongly compressible progenerator of Mod-S for some ring S.* □

3.2 Some applications

In this section, using the notion of strongly compressible modules we will present many module-theoretic characterizations of semiprime Goldie (prime right Goldie, or prime Goldie) rings, right w-semi-Prüfer (right w-Prüfer, semi-Prüfer, or Prüfer) rings as corollaries of Theorem 3.1.1.

Theorem 3.2.1 *The following are equivalent for a ring R:*

(a) R is semiprime Goldie;

(b) Every f.g. essential submodule of $E(R_R)$ is strongly compressible;

(c) Every f.g. submodule of $E(R_R)$ is strongly compressible;

(d) $Z(R_R) = 0$, and every f.g. nonsingular right R-module is strongly compressible.

(e) Every f.g. essential submodule of $E(P_R)$ is strongly compressible for each progenerator P of Mod-R;

(f) Every f.g. submodule of $E(P_R)$ is strongly compressible for each progenerator P of Mod-R.

Proof. $(a) \Rightarrow (d)$. Clearly $Z(R_R) = 0$. If M_R is f.g. nonsingular, then M is embeddable in a f.g. right free R-module by Theorem 1.3. Then M is strongly compressible by Theorem 3.1.1.

$(d) \Rightarrow (f)$. This is because for each progenerator P of Mod-R, every f.g. submodule of $E(P_R)$ is nonsingular when R is right nonsingular.

$(f) \Rightarrow (e) \Rightarrow (b)$ and $(f) \Rightarrow (c) \Rightarrow (b)$. Obviously.

$(b) \Rightarrow (a)$. By noting that every f.g. essential right ideal of R is essential in $E(R_R)$, we have that R is a semiprime right Goldie ring by Theorem 3.1.1. It is enough to show that R is left Goldie. Let $Q = E(R_R)$. It is well known that Q is a semi-simple Artinian ring and R is a right order of Q. Let $x \in Q$. Then $R_R \leq_e R + xR \subseteq E(R_R)$. Since $R + xR$ is essential in $E(R_R)$, $R + xR$ is strongly compressible, and thus there exists $Y \subseteq E(R_R)$ such that $(R + xR)_R \subseteq Y_R \overset{f}{\cong} R_R$. Let $y = f^{-1}(1)$. Then $Y = yR$ and $y^{\perp} = 0$, and thus y is a regular element of Q. Write $x = yr_1$, $1 = yr_2$ for some $r_i \in R$. Then $x = r_2^{-1} r_1$. Hence R is also a left order of Q, showing that R is left Goldie. $\square$

Next we show that the characterization theorem of semiprime Goldie rings of López-Permouth, Rizvi and Yousif, which we mentioned in the beginning of this chapter, is a corollary of the previous theorem. To see this we set up a connection between strongly compressible modules and weakly-injective modules which is given by the following proposition. (Comparing it with [26, Prop.3.7].)

Proposition 3.2.1 *The following are equivalent for an injective right R-module E:*

(a) Every submodule of E is weakly-injective;

(b) Every f.g. submodule of E is strongly compressible.

Proof. $(a) \Rightarrow (b)$. Let N be a f.g. submodule of E and A an essential submodule of N. Then $E(A) = E(N)$. Since A is weakly-injective, there exists $X \subseteq E(A) = E(N)$ such that $N \subseteq X \cong A$. Thus N is strongly compressible.

$(b) \Rightarrow (a)$. Suppose that M is a submodule of E. Let A be a f.g. submodule of $E(M)$. Then $M \cap A$ is essential in A. Since A is strongly compressible, there exists a submodule Y of $E(A)$ such that $A \subseteq Y \overset{f}{\cong} M \cap A$. Then f induces an isomorphism $E(Y) \overset{f}{\cong} E(M \cap A)$. Because $M \cap A$ is essential in A and $A \subseteq Y \subseteq E(A)$, we have $E(Y) = E(A) = E(M \cap A)$. There exists $B \subseteq E(M)$ such that $E(M) = E(A) \oplus B$. If we define $g : E(M) \longrightarrow E(M)$ by $g(x+b) = f(x)+b$ for all $x \in E(A)$ and $b \in B$, then g is an R-isomorphism and $g|_{E(A)} = f$. Let $X = g^{-1}(M)$. Since $f(A) \subseteq M$, we have $A \subseteq X \cong M$ and $X \subseteq E(M)$. Therefore M is weakly-injective. $\square$

Remark 3.2.1 *A ring R is called right weakly-semisimple if every right R-module is weakly-injective [21]. From the previous proposition, it follows immediately that a ring R is right weakly-semisimple if and only if every f.g. right R-module is strongly compressible.*

Corollary 3.2.1 [*26, Theorem 3.9*]. *The following are equivalent for a ring R:*

(a) R is semiprime Goldie;

(b) Every right ideal of R is weakly-injective;

(c) $Z(R_R) = 0$ and every nonsingular right R-module is weakly-injective.

Proof. Because the class of weakly-injective modules is closed under taking essential extensions, Proposition 3.2.1 implies that (b) is equivalent to (c) of Theorem 3.2.1, and (c) is equivalent to (d) of Theorem 3.2.1. □

Proposition 3.2.2 *The following are equivalent for a ring R:*

(a) R is prime right Goldie;

(b) Every nonzero cyclic right ideal of R is strongly compressible and faithful;

(c) Every nonzero right ideal of R is strongly compressible and faithful;

(d) Every nonzero cyclic submodule of each progenerator of Mod-R is strongly compressible and faithful;

(e) Every nonzero submodule of each progenerator of Mod-R is strongly compressible and faithful.

Proof. By Theorem 3.1.1. □

Proposition 3.2.3 *The following are equivalent for a ring R:*

(a) R is prime Goldie;

(b) Every f.g. nonzero submodule of $E(R_R)$ is strongly compressible and faithful;

(c) $Z(R_R) = 0$ and every f.g. nonsingular right R-module is strongly compressible and faithful.

Proof. By Theorem 3.2.1. □

The following are some new characterizations of right w-semi-Prüfer (right w-Prüfer, semi-Prüfer, or Prüfer) rings.

Proposition 3.2.4 *The following are equivalent for a ring R:*

(a) R is a right w-semi-Prüfer ring;

(b) Every f.g. essential right ideal of R is a strongly compressible progenerator;

(c) Every f.g. essential submodule of each progenerator of Mod-R is a strongly compressible progenerator.

Proof. By Theorem 3.1.1, Proposition 2.1.1, and Proposition 2.1.3. □

Proposition 3.2.5 *The following are equivalent for a ring R:*

(a) R is a right w-Prüfer ring;

(b) Every f.g. nonzero right ideal of R is a strongly compressible progenerator;

(c) Every f.g. nonzero submodule of each progenerator of Mod-R is a strongly compressible progenerator.

Proof. By Theorem 3.1.1, and Proposition 2.1.4. □

Proposition 3.2.6 *The following are equivalent for a ring R:*

(a) R is a semi-Prüfer ring;

(b) Every f.g. essential submodule of $E(R_R)$ is a strongly compressible progenerator;

(c) Every f.g. essential submodule of $E(P_R)$ is a strongly compressible progenerator for each progenerator P of Mod-R.

Proof. $(c) \Rightarrow (b)$. Obviously.

$(b) \Rightarrow (a)$. By Theorem 3.2.1, Proposition 2.1.1, and noting that every f.g. essential right ideal of R_R is essential in $E(R_R)$.

$(a) \Rightarrow (c)$. Let P be a progenerator of Mod-R, and N a f.g. essential submodule of $E(P_R)$. By Theorem 3.2.1, N is strongly compressible. Note that $N \cap P \leq_e N$. Thus, there exists $X \subseteq E(N)$ such that $N \subseteq X \cong N \cap P$. Since both N and P have the same finite Goldie dimension, it follows that N can embed in P as an essential submodule. Then N is a progenerator of Mod-R by Proposition 2.1.4. □

Proposition 3.2.7 *The following are equivalent for a ring R:*

(a) R is a Prüfer ring;

(b) Every f.g. nonzero submodule of $E(R_R)$ is a strongly compressible progenerator;

(c) Every f.g. nonzero submodule of $E(P_R)$ is a strongly compressible progenerator for each progenerator P of Mod-R;

(d) $Z(R_R) = 0$ and every f.g. nonsingular right R-module is a strongly compressible progenerator.

Proof. $(d) \Rightarrow (c) \Rightarrow (b)$. Clearly.

$(b) \Rightarrow (a)$. By Theorem 3.2.1 and Proposition 2.1.4.

$(a) \Rightarrow (d)$. By Theorem 3.2.1, Proposition 2.1.9, and the fact that every f.g. nonsingular right R-module can be embedded in a f.g. free R-module. $\square$

Proposition 3.2.8 *The following are equivalent for a ring R:*

(a) R is semi-simple;

(b) Every (right) R-module is strongly compressible;

(c) Every (right) injective R-module is strongly compressible;

(d) $E(R_R)$ is strongly compressible.

Proof. $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. Clearly.

$(d) \Rightarrow (a)$. Since every essential submodule of a strongly compressible module is strongly compressible, it follows from (d) that every f.g. essential submodule of $E(R_R)$ is strongly compressible. Then R is semiprime Goldie by Theorem 3.2.1. On the other hand, condition (d) implies easily that $R \cong E(R_R)$. Thus $R \cong E(R)$ is semi-simple by [18, Th.4.28]. $\square$

4 Modules over Prüfer rings

Given a ring R, we know that a module M_R is projective if and only if M_R is a direct summand of some direct sum of copies of R. Simply from this, we see that there is a special projective module R which determines the structure of all projective modules. For a commutative Prüfer domain R, Fuchs [12] constructed a divisible module ∂ with projective dimension at most one which functions as R in the sense that a module M_R is divisible with projective dimension at most one if and only if M is a direct summand of some direct sum of copies of ∂. In this chapter, we will extend this result to a noncommutative Prüfer ring. This work is carried out in Section 2. In Section 1, we establish a structure theorem for modules of projective dimension one over a noncommutative Prüfer ring. Besides its own interest, the structure theorem is also needed for the proof of the above-mentioned result.

4.1 Modules of projective dimension at most one

First let us recall some concepts in Module Theory. For a fixed module M_R, $\mathrm{Ext}^n(M,-)$ is the nth right derived functor of $\mathrm{Hom}(M,-)$. If $0 \to A \to B \to C \to 0$ is a short exact sequence of right R-modules, then we have the long exact sequence in the second variable

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom(M,A) & \longrightarrow & Hom(M,B) & \longrightarrow & Hom(M,C) & & \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & Ext^0(M,A) & \longrightarrow & Ext^0(M,B) & \longrightarrow & Ext^0(M,C) & \longrightarrow & \\ & & Ext^1(M,A) & \longrightarrow & Ext^1(M,B) & \longrightarrow & \cdots\cdots & & \end{array}$$

Similarly, $\mathrm{Ext}^n(-, M)$ is the nth right derived functor of $\mathrm{Hom}(-, M)$. And it induces the long exact sequence in the first variable. A basic fact of the Ext functor is that $\mathrm{Ext}^1_R(M, N) = 0$ if and only if any exact sequence $0 \to N \to D \to M \to 0$ splits.

The projective dimension of a module M_R, denoted by $Pd(M_R)$ or simply by $Pd(M)$, is the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(M, N) = 0$ for all $N \in$ Mod-R, if such an integer n exists. If no such n exists, then $Pd(M_R) = \infty$. Also, $Pd(M_R) = n$ if and only if for any projective resolution of M_R:

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0,$$

$Im(d_n)$ is projective [22, P90]. Clearly, $Pd(M_R) = 0$ if and only if M is projective. If $0 \to A \to B \to C \to 0$ is a short exact sequence of right R-modules with B projective, then, by examining the induced long exact sequence in the second variable, we have $Pd(A) = Pd(C) - 1$.

Lemma 4.1.1 *If M_R is finitely generated and R is a Prüfer ring, then $M \cong \tau(M) \oplus M/\tau(M)$.*

Proof. Since $M/\tau(M)$ is f.g. torsionfree, then it is projective by Proposition 2.3.5. Therefore the short exact sequence $0 \to \tau(M) \to M \to M/\tau(M) \to 0$ splits, and so $M \cong \tau(M) \oplus M/\tau(M)$. □

For some ordinal ρ, let

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\rho = M \quad (\alpha < \rho) \qquad (1)$$

be a well-ordered ascending chain of submodules of a module M_R. The chain (1) is said to be continuous if $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for every limit ordinal $\beta \leq \rho$.

Lemma 4.1.2 [*Auslander*]. *For an ordinal ρ, let*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \; (\alpha < \rho) \tag{1}$$

be a well-ordered ascending chain of submodules of a module M_R such that

(a) $\bigcup_{\alpha<\rho} M_\alpha = M$;

(b) (1) *is a continuous chain;*

(c) $Pd(M_{\alpha+1}/M_\alpha) \leq n$ *for some fixed integer n and all* $1 \leq \alpha + 1 < \rho$.

Then $Pd(M) \leq n$.

Proof. If $n = 0$, then, since (b), $M_\alpha/(\cup_{\sigma<\alpha} M_\sigma)$ is projective for all $\alpha < \rho$. It follows that $M_\alpha = (\cup_{\sigma<\alpha} M_\sigma) \oplus M_\alpha'$ for some projective submodule M_α' of M. Therefore $M = \cup_{\alpha<\rho} M_\alpha = \oplus_{\alpha<\rho} M_\alpha'$ is projective, thus $Pd(M) = 0$. Now assume $n > 0$. Let $M_\alpha' = M_\alpha/(\cup_{\sigma<\alpha} M_\sigma)$, and F_α' be a free right R-module mapping onto M_α' with kernel K_α'. If α is a limit ordinal, then $M_\alpha' = 0$. In this case we choose 0 as F_α'. Therefore we have $Pd(K_\alpha') = Pd(M_\alpha') - 1 \leq n - 1$. Let $F_\alpha = \oplus_{\sigma\leq\alpha} F_\sigma'$. Since F_α' is free, there is a map $F_\alpha' \to M_\alpha$ which lifts the map $F_\alpha' \to M_\alpha'$. By transfinite induction, the map $F_\alpha' \to M_\alpha$ can be extended to a map $F_\alpha \to M_\alpha$ such that if K_α is the kernel, then $K_\sigma \subseteq K_\alpha$ for $\sigma < \alpha$ and $K_\alpha' \cong K_\alpha/(\cup_{\sigma<\alpha} K_\sigma)$. Thus, $Pd(K_\alpha/(\cup_{\sigma<\alpha} K_\sigma)) = Pd(K_\alpha') \leq n - 1$. Note that $0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\alpha \subseteq \cdots$ is a continuous chain. By the induction hypothesis, $Pd(\cup_{\alpha<\rho} K_\alpha) \leq n - 1$. Since $\cup_{\alpha<\rho} K_\alpha$ is the kernel of $\cup_{\alpha<\rho} F_\alpha \to M$, and $\cup_{\alpha<\rho} F_\alpha = \oplus_{\alpha<\rho} F_\alpha'$ is projective, we obtain $Pd(M) \leq n$. □

A module M_R is finitely presented if there is an exact sequence $0 \to K \to R^n \to M \to 0$, where n is a positive integer and K is finitely generated. This is equivalent to the requirement that there exist f.g. modules K_R and P_R

such that $0 \to K \to P \to M \to 0$ is exact (see [2, Ex.17, P233]).

Lemma 4.1.3 *Let R be a Prüfer ring. A f.g. module M_R is finitely presented if and only if $Pd(M) \leq 1$.*

Proof. Let $0 \to H \to F \to M \to 0$ be an exact sequence with F f.g. free. If M is finitely presented, then H is finitely generated. Hence H is f.g. torsionfree. By Proposition 2.3.5, H is projective. Therefore $Pd(M) \leq 1$.

Conversely, if $Pd(M) \leq 1$, then H is projective. Since $H \subseteq F$ and F has finite Goldie dimension, H is of finite Goldie dimension. By Proposition 2.3.5, H is finitely generated. □

Lemma 4.1.4 *Let R be a Prüfer ring, and H a projective submodule of a torsionfree module F_R. If F/H is finitely generated, then F is projective and F/H is finitely presented.*

Proof. **Step 1.** First we assume R is a Prüfer domain (noncommutative), H is free and F/H is f.g. torsion. Write $H = \bigoplus\{yR : y \in Y\}$. Let $Q = Q_{cl}(R)$. Then Q is a division ring. Since F is torsionfree, the map $\psi: F \to F \otimes_R Q$ which is defined by $\psi(x) = x \otimes 1$ is one to one. Since F/H is torsion, and Y is a basis for H, $\{y \otimes 1 : y \in Y\}$ becomes a basis for the Q-vector space $(F \otimes_R Q)_Q$. Suppose $F/H = \bar{x}_1 R + \cdots + \bar{x}_m R$, where $\bar{x}_i = x_i + H$, $i = 1, \ldots, m$. Clearly $F = x_1 R + \cdots + x_m R + H$. For each i, there exists a nonzero r_i of R such that $x_i r_i \in y_1 R + \cdots + y_k R$, where k is a fixed positive integer. Let $H_0 = y_1 R + \cdots + y_k R$, $F_0 = x_1 R + \cdots + x_m R + H_0$. Then the map $\varphi: F_0/H_0 \to F/H$ defined by $\varphi(\xi + H_0) = \xi + H$ is onto.

Claim: $F_0 \cap H = H_0$. Let $\xi = x_1 a_1 + \cdots + x_m a_m \in H$, where each $a_i \in R$. Write $\xi = y_{i_1} b_1 + \cdots + y_{i_n} b_n$ for some $y_{i_j} \in Y$ and $0 \neq b_i \in R$.

Then $\xi \otimes 1 = (y_{i_1} \otimes 1)b_1 + \cdots + (y_{i_n} \otimes 1)b_n$. On the other hand, $\xi \otimes 1 = (x_1 \otimes 1)a_1 + \cdots + (x_m \otimes 1)a_m = (x_1 r_1 \otimes 1)r_1^{-1}a_1 + \cdots + (x_m r_m \otimes 1)r_m^{-1}a_m = (y_1 \otimes 1)u_1 + \cdots + (y_k \otimes 1)u_k$ for some $u_k \in Q$, since each $x_i r_i \in y_1 R + \cdots + y_k R$. By noting that each $b_i \neq 0$, and $\{y \otimes 1 : y \in Y\}$ is a basis of $(F \otimes_R Q)_Q$, we have $\{y_{i_1}, \ldots, y_{i_n}\} \subseteq \{y_1, \ldots, y_k\}$. Therefore $\xi \in H_0$. Consequently $F_0 \cap H = H_0$, implying that φ is an isomorphism. Since both H_0 and F_0 are f.g. torsionfree, they are f.g. projective by Proposition 2.3.5. Therefore F_0/H_0, and F/H is finitely presented. We note that H_0 is a direct summand of H_R, hence H/H_0 is projective. Since $F = x_1 R + \cdots + x_m R + H$ and $F_0 \cap H = H_0$, we have an R-module isomorphism θ: $H/H_0 \to F/F_0$ (via $\theta(x + H_0) = x + F_0$). Therefore F/F_0 is projective. So we have $F \cong F_0 \oplus (F/F_0)$ and F is projective.

Step 2. Assume R is a Prüfer domain, H is projective and F/H is f.g. torsion. Then $H \oplus X$ is free for some $X \in$ Mod-R. X is, of course, torsionfree. Therefore $F \oplus X$ is torsionfree and

$$0 \to H \oplus X \to F \oplus X \to (F \oplus X)/(H \oplus X) \cong F/H \to 0$$

is exact. Step 1 implies that $(F \oplus X)/(H \oplus X)$ is finitely presented and $F \oplus X$ is projective. Consequently F/H is finitely presented and F is projective.

Step 3. We assume R is a Prüfer domain, H is projective and F/H is finitely generated. By Lemma 4.1.1, we may assume $F/H = (U/H) \oplus (V/H)$ where $U/H = \tau(F/H)$ is f.g. torsion, and $V/H \cong (F/H)/\tau(F/H)$ is f.g. torsionfree. By Proposition 2.3.5, V/H is projective. Therefore we have $V \cong H \oplus (V/H)$, and so V is projective. In the following short exact sequence:

$$0 \to V \to F \to F/V \cong U/H \to 0$$

U/H is f.g. torsion, and V is projective. Therefore Step 2 implies that F is projective and U/H is finitely presented. As a direct sum of two finitely presented modules, F/H is of course finitely presented.

Step 4. The general case: R is a Prüfer ring. We know that R is Morita equivalent to a Prüfer domain S by Theorem 2.3.5. There exists a Morita equivalence G : Mod-R $\longrightarrow$ Mod-S, and G induces an exact sequence in Mod-S:

$$0 \to G(H) \to G(F) \to G(F/H) \to 0.$$

By [2, Prop.21.6], $G(H)_S$ is projective, and $G(F/H)_S$ is finitely generated. It is well-known that the singularity of modules is preserved under Morita equivalences (e.g., see [17, P43]). Then the torsionfreeness of F_R implies that $G(F)_S$ is torsionfree. Therefore Step 3 implies that $G(F)_S$ is projective and $G(F/H)_S$ is finitely presented. Hence F_R is projective and $(F/H)_R$ is finitely presented by [2, Ex.11, P262]. □

A right R-module is called coherent, if every f.g. submodule is finitely presented.

Proposition 4.1.1 *Every module M_R of projective dimension 1 over a Prüfer ring R is coherent, and for any submodule N of M, $Pd(N) \leq 1$ and $Pd(M/N) \leq 1$.*

Proof. Let N be a f.g. submodule of M_R where $Pd(M) = 1$. Then we can write $M \cong F/H$ with F free and H projective. There exists a submodule G of F_R such that $H \subseteq G$ and $N \cong G/H$. Clearly G is torsionfree. Therefore N is finitely presented, and G is projective by Lemma 4.1.4. Since $0 \to G \to F \to F/G \cong M/N \to 0$ is exact, we have $Pd(M/N) \leq 1$. □

Theorem 4.1.1 *Over a Prüfer ring, a countably generated right module has projective dimension* ≤ 1 *if and only if it is the union of a countable ascending chain of finitely presented right modules.*

Proof. Given a countably generated module M_R, then M_R is a union of a countable ascending chain of f.g. submodules. If $Pd(M) = 0$, i.e., M is projective, then every f.g. submodule of M is torsionfree, and hence is projective by Proposition 2.3.5, and hence finitely presented by Lemma 4.1.3. If $Pd(M) = 1$, then, by Proposition 4.1.1, every f.g. submodule of M is finitely presented. For the converse, we suppose M_R is the union of a chain of right finitely presented R-modules:

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots.$$

By [2, Ex.17, P233], all M_{n+1}/M_n are finitely presented. Then Lemma 4.1.3 implies that $Pd(M_{n+1}/M_n) \leq 1$ for all n. By Lemma 4.1.2, $Pd(M) \leq 1$. □

Let R be a Prüfer domain, and $0 \to H \to F \xrightarrow{\phi} M \to 0$ be an exact sequence of right R-modules such that $F_R = \bigoplus\{xR : x \in X\}$ is free on X and H is projective. By [2, Cor.26.2], $H_R = \bigoplus\{H_y : y \in Y\}$, where the H_y's are countably generated projective right R-modules. Consider all pairs (X_i, Y_i) of subsets $X_i \subseteq X$, $Y_i \subseteq Y$ such that $F_i = \bigoplus\{xR : x \in X_i\}$ and $H_i = \bigoplus\{H_y : y \in Y_i\}$ satisfy $H_i = H \cap F_i$. Let i run over an index set I. Note that $H = H_i \oplus H_i^*$ and $F_i + H = F_i \oplus H_i^*$, where $H_i^* = \bigoplus\{H_y : y \in Y \setminus Y_i\}$. Therefore each $F_i + H$ is projective. Set $\mathcal{T} = \{M_i : i \in I\}$, where $M_i = (F_i + H)/H$. Then, clearly, $(0), M \in \mathcal{T}$, and for $M_i, M_j \in \mathcal{T}$ with $M_i \subset M_j$, $M_j/M_i \cong (F_j + H)/(F_i + H)$ has projective dimension at most one.

Lemma 4.1.5 [*Fuchs*]. *Let R, M_R, and $\mathcal{T}$ be as above. Then for any countable subset $\triangle$ of M, there exists some $M_i \in \mathcal{T}$ with M_i countably generated such that $\langle\triangle\rangle \subseteq M_i$, where $\langle\triangle\rangle$ indicates the submodule of M generated by $\triangle$.*

Proof. Given a countable subset $\triangle$ of M, there is a countable subset $X^{(1)}$ of X such that $\phi\langle X^{(1)}\rangle$ contains $\triangle$. Let $Q = Q_{cl}(R)$. Then Q is a division ring. Since $\langle X^{(1)}\rangle$ is torsionfree, we have that $f : \langle X^{(1)}\rangle \longrightarrow \langle X^{(1)}\rangle \otimes_R Q$ which is defined by $f(a) = a \otimes 1$ is one to one. Similarly, $g : \langle X^{(1)}\rangle \cap H \longrightarrow (\langle X^{(1)}\rangle \cap H) \otimes_R Q$ $(g(b) = b \otimes 1)$ is one to one. Since Q is a flat left R-module, the map $l \otimes 1 : (\langle X^{(1)}\rangle \cap H) \otimes_R Q \longrightarrow \langle X^{(1)}\rangle \otimes_R Q$ is a monomorphism, where l is the inclusion of $\langle X^{(1)}\rangle \cap H$ into $\langle X^{(1)}\rangle$. Therefore we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & \langle X^{(1)}\rangle \cap H & \xrightarrow{l} & \langle X^{(1)}\rangle \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
0 & \longrightarrow & (\langle X^{(1)}\rangle \cap H) \otimes_R Q & \xrightarrow{l\otimes 1} & \langle X^{(1)}\rangle \otimes_R Q
\end{array}$$

Clearly, $l \otimes 1$ is a Q-homomorphism. Since $\langle X^{(1)}\rangle$ is free with a basis $X^{(1)}$, $\langle X^{(1)}\rangle \otimes_R Q$ is a Q-vector space with a basis $\{x \otimes 1 : x \in X^{(1)}\}$. Thus, as a Q-subspace, $(\langle X^{(1)}\rangle \cap H) \otimes_R Q$ has a countable basis which, we may assume, is $\{z_i \otimes 1 : i \in \mathbf{N}\}$ with all $z_i \in \langle X^{(1)}\rangle \cap H$. There is a countable subset $Y^{(1)}$ of Y such that all $z_i \in \oplus_{y\in Y^{(1)}} H_y$. We claim that $\langle X^{(1)}\rangle \cap H \subseteq \oplus_{y\in Y^{(1)}} H_y$. In fact, if not, then we can find an $h = h_a + h_b \in \langle X^{(1)}\rangle \cap H$ with $0 \neq h_a \in$

$\oplus_{y\in Y\setminus Y^{(1)}}H_y$, and $h_b \in \oplus_{y\in Y^{(1)}}H_y$. But since $h\otimes 1 \in (\langle X^{(1)}\rangle \cap H)\otimes_R Q$, $h\otimes 1 = \Sigma_{i=1}^n (z_i \otimes 1)q_i$ for some $q_i \in Q$. There exist $a_i \in R$ and $c \in \mathcal{C}_R(0)$ such that $q_i = a_i c^{-1}$ for $i = 1, \cdots, n$. Then $hc\otimes 1 = (h\otimes 1)c = (\Sigma_{i=1}^n z_i a_i)\otimes 1$. It follows that $hc = \Sigma_{i=1}^n z_i a_i$. This implies that $h_a c = 0$, contradicting the fact that H is torsionfree. Hence the claim is true. We can select a countable subset $X^{(2)}$ of X that contains $X^{(1)}$ and satisfies $\oplus_{y\in Y^{(1)}}H_y \subseteq \langle X^{(2)}\rangle$. Repeating this process, we obtain ascending chains of countable subsets

$$X^{(1)} \subseteq X^{(2)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots$$

and

$$Y^{(1)} \subseteq Y^{(2)} \subseteq \cdots \subseteq Y^{(n)} \subseteq \cdots$$

of X and Y, respectively, such that

$$\langle X^{(n)}\rangle \cap H \subseteq \oplus_{y\in Y^{(n)}}H_y \subseteq \langle X^{(n+1)}\rangle$$

for each $n \leq 1$. Let $X^* = \cup_n X^{(n)}$, $Y^* = \cup_n Y^{(n)}$, $F^* = \oplus\{Rx : x \in X^*\}$, and $H^* = \oplus\{H_y : y \in Y^*\}$. Then $F^* \cap H = H^*$. Thus $M^* = (F^* + H)/H \in \mathcal{T}$. It is clear that $\langle \triangle \rangle \subseteq M^*$, and M^* is countably generated. □

Lemma 4.1.6 *Let R, M_R, and $\mathcal{T}$ be as above. Given $M_i = (F_i + H)/H \in \mathcal{T}$ and a countable subset $\triangle$ of M, there exists some $M_j = (F_j + H)/H \in \mathcal{T}$ such that $\langle \triangle, M_i\rangle \subseteq M_j$, M_j/M_i is countably generated, and $F_i \subseteq F_j$.*

Note. The required condition $F_i \subseteq F_j$ is really indispensable for the proof of the next lemma.

Proof. We consider the following short exact sequence:

$$0 \to (F_i + H)/F_i \to F/F_i \to (F/F_i)/[(F_i + H)/F_i] \to 0.$$

Clearly

$$(F/F_i)/[(F_i + H)/F_i] \cong F/(F_i + H)$$

has projective dimension at most one,

$$F/F_i = \bigoplus\{\bar{x}R : x \in X \setminus X_i\}$$

is free, where

$$\bar{x} = x + F_i \in F/F_i,$$

and

$$(F_i + H)/F_i = \bigoplus\{(H_y + F_i)/F_i : y \in Y \setminus Y_i\}(\cong H_i^*)$$

is projective with each

$$(H_y + F_i)/F_i \cong H_y$$

countably generated. By Lemma 4.1.5, there exist

$$X' \subseteq X \setminus X_i,\ Y' \subseteq Y \setminus Y_i$$

such that:

$$(a) \quad \bigoplus\{\bar{x}R : x \in X'\} \cap ((F_i + H)/F_i) = \bigoplus\{(H_y + F_i)/F_i : y \in Y'\},$$

i.e.,

$$(\bigoplus\{xR : x \in X'\} \cap (F_i + H)) + F_i = \bigoplus\{H_y : y \in Y'\} + F_i; \qquad (*)$$

and

$$(b) \quad [\bigoplus\{\bar{x}R : x \in X'\} + ((F_i + H)/F_i)]/[(F_i + H)/F_i]$$

is countably generated; and

$$(c) \quad [\bigoplus\{\bar{x}R : x \in X'\} + ((F_i + H)/F_i)]/[(F_i + H)/F_i] \supseteq \langle\ \overline{\overline{\triangle_F}}\ \rangle,$$

where $\triangle_F$ is a countable subset of F such that

$$\triangle = \{u + H : u \in \triangle_F\},\ \overline{\triangle_F} = \{v + F_i : v \in \triangle_F\},$$

and

$$\overline{\overline{\triangle_F}} = \{f + [(F_i + H)/F_i] : f \in \overline{\triangle_{\mathrm{F}}}\}.$$

It is easy to see that condition (c) is equivalent to

$$\bigoplus\{xR : x \in X'\} + F_i + H \supseteq \textstyle\sum_{u \in \triangle_F} uR + F_i + H. \qquad (**)$$

Let

$$F'' = \bigoplus\{xR : x \in X' \cup X_i\},$$
$$H'' = \bigoplus\{H_y : y \in Y' \cup Y_i\},$$
$$M'' = (F'' + H)/H.$$

Then, by $(*)$,

$$F'' \cap H = (\bigoplus\{xR : x \in X'\} \oplus F_i) \cap H \supseteq (\bigoplus\{H_y : y \in Y'\}) \oplus H_i = H''.$$

On the other hand, if

$$b \in F'' \cap H, \text{ i.e., } b = b_1 + b_2 \in H$$

for some

$$b_1 \in \bigoplus\{xR : x \in X'\},\, b_2 \in F_i,$$

then

$$b_1 = b - b_2 \in \bigoplus\{xR : x \in X'\} \cap (F_i + H).$$

By $(*)$, $b - b_2 = a_1 + a_2$ for some

$$a_1 \in \bigoplus\{H_y : y \in Y'\},\, a_2 \in F_i.$$

Then

$$b - a_1 = a_2 + b_2 \in F_i \cap H = H_i.$$

Therefore

$$b = a_1 + (a_2 + b_2) \in \{H_y : y \in Y'\} + H_i = H''.$$

Consequently we have

$$F'' \cap H = H'', \text{ and hence } M'' \in \mathcal{T}.$$

Also, by $(**)$,

$$\langle \triangle, M_i\rangle = (\textstyle\sum_{u\in\triangle_F} uR + H)/H + (F_i + H)/H =$$
$$(\textstyle\sum_{u\in\triangle_F} uR + F_i + H)/H \subseteq (F'' + H)/H = M''.$$

Clearly $F_i \subseteq F''$. Finally

$$M''/M_i \cong [\bigoplus\{\bar{x}R : x \in X'\} + ((F_i + H)/F_i)]/[(F_i + H)/F_i]$$

is countably generated by (b). The proof is complete. □

Now we can prove the following lemma:

Lemma 4.1.7 *Let R be a Prüfer ring and $Pd(M) \leq 1$. Then there exists a well-ordered continuous chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \subset M_\rho = M \quad (\alpha < \rho)$$

such that for each $\alpha < \rho$, $M_{\alpha+1}/M_\alpha$ is finitely presented.

Proof. **Step 1.** We assume R is a Prüfer domain. Then we can set up $\mathcal{T}$ as in the above discussion. Choose $M_0 = (0) \in \mathcal{T}$. Suppose we have already chosen all $M_\alpha = (F_\alpha + H)/H$ for all $\alpha < \sigma$ with $M_\alpha \in \mathcal{T}$ such that $0 \neq M_{\alpha+1}/M_\alpha$ is countably generated and $F_\alpha \subseteq F_{\alpha+1}$ for all $\alpha + 1 < \sigma$.

(i) σ is not a limit ordinal. We are done if $M = M_{\sigma-1}$. If $M \neq M_{\sigma-1}$, then, by Lemma 4.1.6, there exists some $M_\sigma \in \mathcal{T}$ such that $M_{\sigma-1} \subset M_\sigma$, $M_\sigma/M_{\sigma-1}$ is countably generated, and $F_{\sigma-1} \subseteq F_\sigma$.

(ii) σ is a limit ordinal. We can define $M_\sigma = \bigcup_{\alpha<\sigma} M_\alpha$. Let $F_\sigma = \sum_{\alpha<\sigma} F_\alpha$, $H_\sigma = \bigoplus\{H_y : y \in \bigcup_{\alpha<\sigma} Y_\alpha\}$. Then $F_\sigma = \bigoplus\{xR : x \in \bigcup_{\alpha<\sigma} X_\alpha\}$, and $F_\sigma \cap H = H_\sigma$ since $\{F_\alpha\}$ is a chain. Therefore $M_\sigma = \sum_{\alpha<\sigma} M_\alpha = (F_\sigma + H)/H \in \mathcal{T}$. Note that $0 \neq M_{\alpha+1}/M_\alpha$ is countably generated for all $\alpha < \sigma$. By transfinite induction, we can get a continuous chain of submodules of M_R from $\mathcal{T}$:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \subset M_\rho = M$$

such that $M_{\alpha+1}/M_\alpha$ is countably generated for all $\alpha < \rho$. From the notes before Lemma 4.1.5, each $Pd(M_{\alpha+1}/M_\alpha) \leq 1$. Then, for each α, Theorem 4.1.1 ensures that there exists a chain of submodules

$$M_\alpha = M_\alpha^0 \subset M_\alpha^1 \subset \cdots \subset M_\alpha^{\aleph^0} = M_{\alpha+1}$$

such that $M_\alpha^{i+1}/M_\alpha^i$ is finitely presented for all i. Therefore, without loss of generality, we may assume each $M_{\alpha+1}/M_\alpha$ is finitely presented.

Step 2. Let R be a Prüfer ring. Then, by Theorem 2.3.5, R is Morita equivalent to a Prüfer domain S via an equivalence F : Mod-$R \to$ Mod-S with inverse G : Mod-$S \to$ Mod-R. Since $Pd(M) \leq 1$, then $Pd(F(M)) \leq 1$. By Step 1, there exists a continuous chain of submodules of $F(M)_S$:

$$0 = N_0 \subset N_1 \subset \cdots \subset N_\alpha \subset \cdots \subset N_\rho = F(M)$$

such that $N_{\alpha+1}/N_\alpha$ is finitely presented for all $\alpha < \rho$. Since Morita equivalence preserves exactness [2, Prop.21.4], we have $G(N_{\alpha+1})/G(N_\alpha) \cong G(N_{\alpha+1}/N_\alpha)$. It follows from [2, Ex.11, P262] that $G(N_{\alpha+1})/G(N_\alpha)$ is finitely presented for each $\alpha < \rho$. If $N_\sigma = \bigcup_{\alpha<\sigma} N_\sigma$, then $G(N_\alpha) = \bigcup_{\alpha<\sigma} G(N_\sigma)$ by [2, Prop.21.7]. Therefore we have shown

$$0 = G(N_0) \subset G(N_1) \subset \cdots \subset G(N_\alpha) \subset \cdots \subset G(N_\rho) = GF(M)_R$$

is a continuous chain of submodules of $GF(M)_R$ such that $G(N_{\alpha+1})/G(N_\alpha)$ is finitely presented for all $\alpha < \rho$. Since $M_R \cong GF(M)_R$, we can get such a similar chain for M_R. □

Now we can prove the main theorem of this section.

Theorem 4.1.2 *Let M_R be a module over a Prüfer ring R. Then $Pd(M) \leq 1$ if and only if M is the union of a well-ordered continuous chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \subset M_\rho = M$$

such that $M_{\alpha+1}/M_\alpha$ is finitely presented cyclic for all $\alpha < \rho$.

Proof. The sufficiency follows from Lemmas 4.1.2, 4.1.3. For the necessity, we know that there is a well-ordered continuous chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \subset M_\rho = M$$

such that $M_{\alpha+1}/M_\alpha$ is finitely presented for all $\alpha < \rho$, by Lemma 4.1.7. Therefore, to complete the proof, it suffices to show the fact that for every finitely presented module N_R, there exists a finite chain of submodules of N_R such that each factor of this chain is finitely presented cyclic. To see this, let $N = x_1R + \cdots + x_nR$ be a finitely presented module, and $P = x_1R + \cdots + x_{n-1}R$. Then N/P is finitely presented cyclic by [2, Ex.17, P233]. If $Pd(N) = 0$, then P is f.g. torsionfree, and hence projective by Proposition 2.3.5. If $Pd(N) = 1$, then P is a finitely presented module by Proposition 4.1.1. Therefore P is a finitely presented module with $n-1$ generators. Thus, the induction hypothesis implies that there is a chain of submodules of P:

$$0 = P_0 \subset P_1 \subset \cdots \subset P_k = P$$

such that P_{i+1}/P_i are finitely presented cyclic for all $i = 0, 1, \cdots, k-1$. Hence

$$0 = P_0 \subset P_1 \subset \cdots \subset P_k = P \subseteq N$$

is the required chain for N □

4.2 Divisible modules of projective dimension at most one

Given a Prüfer ring R, we construct a special divisible module ∂ with projective dimension at most one by following Fuchs, and then we characterize all divisible right R-modules with projective dimension at most one by using the module ∂.

Lemma 4.2.1 [*Fuchs*]. *Let* $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots (\alpha < \rho)$ *be a well-ordered continuous ascending chain of submodules of* M_R. *Suppose that* $Ext^1_R(M_{\alpha+1}/M_\alpha, X) = 0$ *for all* $\alpha + 1 < \rho$, *and some* $X \in$ *Mod-R. Then* $Ext^1_R(\bigcup_{\alpha<\beta} M_\alpha, X) = 0$ *for every* $\beta \leq \rho$.

Proof. We can assume $\cup M_\alpha = M$. Let $0 \to X \to E \to M \to 0$ be an extension of X by M. We want to show that it splits by constructing a module A such that $E = X \oplus A$.

Let $0 \to X \to E_\alpha \to M_\alpha \to 0$ be the exact sequence induced by the inclusion $M_\alpha \to M$. Obviously, this splits for $\alpha = 0$. Regard E as the union of the ascending chain

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_\alpha \subseteq \cdots \ (\alpha < \rho),$$

and suppose that we have found R-submodules A_β of E_β for each $\beta < \alpha$ such that

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\beta \subseteq \cdots \ (\beta < \alpha),$$

is a well-ordered continuous ascending chain satisfying $E_\beta = X \oplus A_\beta$ $(\beta < \alpha)$. If α is a limit ordinal, then set $A_\alpha = \cup_{\beta<\alpha} A_\beta$. This will satisfy $E_\alpha = X \oplus A_\alpha$. If $\alpha - 1$ exists, then $E_\alpha/A_{\alpha-1}$ is an extension of $E_{\alpha-1}/A_{\alpha-1} \cong X$ by $E_\alpha/E_{\alpha-1} \cong M_\alpha/M_{\alpha-1}$. By our hypothesis, this splits, i.e., $E_\alpha/A_{\alpha-1} = (E_{\alpha-1}/A_{\alpha-1}) \oplus (A_\alpha/A_{\alpha-1})$ for some $A_\alpha \supseteq A_{\alpha-1}$. Evidently, $E_\alpha = X + A_\alpha$. On the other hand, $X \cap A_\alpha = X \cap E_{\alpha-1} \cap A_\alpha = X \cap A_{\alpha-1} = 0$, thus $E_\alpha = X \oplus A_\alpha$. Therefore, there is a well-ordered continuous ascending chain

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots \ (\alpha < \rho),$$

such that $E_\alpha = X \oplus A_\alpha$ for all $\alpha < \rho$. Set $A = \cup_{\alpha<\rho} A_\alpha$. Then $E = X \oplus A$. $\square$

The module ∂ was first constructed by Fuchs. Facchini used a slight modification of ∂ to study divisible modules over a commutative domain [7]. Here, we follow Facchini for the construction of ∂.

Given a ring R, for every positive integer k let

$$X_k = \{(r_1, \cdots, r_k) : r_i \in \mathcal{C}_R(0), i = 1, \ldots, k\} \text{ and } X_0 = \{w\}.$$

Set $X = \bigcup_{j \geq 0} X_j$. For $(r_1, \cdots, r_k)$, $(r_1', \cdots, r_l')$, both in X, we define

$$(r_1, \cdots, r_k) = (r_1', \cdots, r_l') \Leftrightarrow k = l \text{ and } r_i = r_i' \text{ for } i = 1, \cdots, k.$$

Let U be the free right R-module with basis X, i.e.,

$$U = wR \oplus [\oplus_{(r_1) \in X_1} (r_1)R] \oplus [\oplus_{(r_1,r_2) \in X_2} (r_1, r_2)R] \oplus \cdots.$$

Set

$$Y = \{(r_1, \cdots, r_k)r_k - (r_1, \cdots, r_{k-1}) : (r_1, \cdots, r_k) \in X_k, k > 0\}$$

(note $(r_1, \cdots, r_{k-1}) = w$ if $k = 1$), and let V be the submodule of U generated by Y. We define $\partial = U/V$.

An element a of R is called left invertible if $ab = 1$ for some $b \in R$. And such a b is called a right inverse of a. Some basic facts about ∂ are included in the following proposition.

Proposition 4.2.1 *Let ∂_k be the submodule of ∂ generated by $\{\xi + V : \xi \in \bigcup_{i \leq k} X_i\}$. Then*

(a) $0 \subset \partial_0 \subseteq \partial_1 \subseteq \cdots \subseteq \partial_k \subseteq \cdots$, and $\partial = \bigcup_{k \geq 0} \partial_k$. If every element in $\mathcal{C}_R(0)$ is left invertible, then $\partial = \partial_0$; if some element in $\mathcal{C}_R(0)$ is not left invertible, then $\partial_k \subset \partial_{k+1}$ for all k;

(b) $\partial_0 = \bar{w}R(\bar{w} = w + V) \cong R_R$; And ∂/∂_0 is torsion if R is a right order;

(c) For each $k \geq 0$, either $\partial_{k+1}/\partial_k \neq 0$, or there exists a non-empty subset Z_k of X_{k+1} such that $\partial_{k+1}/\partial_k = \bigoplus_{\xi \in Z_k}(\bar{\xi} + \partial_k)R$ with $(\bar{\xi} + \partial_k)^{\perp} = r_{k+1}R$, where $\bar{\xi} = \xi + V$ and $\xi = (r_1, \cdots, r_{k+1})$;

(d) $Pd(\partial_{k+1}/\partial_k) \leq 1$ for every $k \geq 0$, and $Pd(\partial) \leq 1$;

(e) A module D_R is called divisible if $Dr = D$ for every $r \in \mathcal{C}_R(0)$. Let D_R be a divisible module, and $a \in D$. Then there exists a homomorphism $f : \partial \longrightarrow D$ with $f(\bar{w}) = a$;

(f) If the ring R is an order, then ∂ is divisible;

(g) Let the ring R be an order. For every divisible module M_R, there exists an exact sequence $0 \to N \to D \to M \to 0$ of divisible right R-modules such that D is a direct sum of modules each of which is isomorphic to $\partial/\bar{w}r_iR$ for some $r_i \in \mathcal{C}_R(0) \cup \{0\}$; if M is divisible torsion then we can choose every such r_i in $\mathcal{C}_R(0)$.

Proof. (a). Directly from the constructions of ∂ and ∂_k, we have $0 \subset \partial_0 \subseteq \partial_1 \subseteq \cdots \subseteq \partial_k \subseteq \cdots$, and $\partial = \bigcup_{k \geq 0} \partial_k$. Moreover, $w \notin V$ implies that $\partial_0 \neq 0$. If $r \in \mathcal{C}_R(0)$ is not left invertible, then, for each k, $(r_1, \cdots, r_k) + V \in \partial_k \setminus \partial_{k-1}$, where $r_1 = \cdots = r_k = r$. If every element in $\mathcal{C}_R(0)$ is left invertible, then for each $(r_1, \cdots, r_k) \in X_k$, $(r_1, \cdots, r_k) + V = ((r_1, \cdots, r_{k-1}) + V)s_k \in \partial_{k-1}$, where s_k is a right inverse of r_k. It follows that $\partial_k = \partial_{k-1} = \cdots$.

(b). For any $0 \neq a \in R$, $wa \notin V$. This implies that $\partial_0 = \bar{w}R$ is a free R-module with a single element basis set $\{\bar{w}\}$. So $\partial_0 \cong R_R$. Since R is a right order, $\tau(\partial/\partial_0)$ is a submodule. From the construction of ∂, we see $\tau(\partial/\partial_0)$ contains a set of generators of ∂/∂_0. It follows that $\partial/\partial_0 = \tau(\partial/\partial_0)$.

(c). If $\partial_{k+1}/\partial_k \neq 0$, then, by (a), $\mathcal{C}_R(0)$ contains an element which is not left invertible. Set $Z_k = \{(r_1, \cdots, r_{k+1}) \in X_{k+1} : r_{k+1}$ is not left invertible $\}$. Note that if $\xi = (r_1, \cdots, r_{k+1}) \in X_{k+1}$ and r_{k+1} is left invertible, then $\bar{\xi} \in \partial_k$. Thus $\bar{\xi} + \partial_k = 0$. From the constructions of ∂, ∂_k, and ∂_{k+1}, we have $\partial_{k+1}/\partial_k = \bigoplus_{\xi \in Z_k} (\bar{\xi} + \partial_k)R$, and for each $\xi = (r_1, \cdots, r_{r_{k+1}}) \in Z_k$, $(\bar{\xi} + \partial_k)^{\perp} = r_{k+1}R$.

(d). By (c), $\partial_{k+1}/\partial_k = \bigoplus_{\xi \in Z_k \subseteq X_{k+1}} (\bar{\xi} + \partial_k)R$. By defining a well-ordering on Z_k, we can write $\partial_{k+1}/\partial_k$ as the union of a well-ordered continuous chain of submodules with each factor of the chain isomorphic to some $(\bar{\xi} + \partial_k)R$. Since $0 \to r_{k+1}R \to R \to (\bar{\xi} + \partial_k)R \to 0$ is exact for $\xi = (r_1, \cdots, r_{k+1}) \in Z_k$, we have $Pd(\partial_{k+1}/\partial_k) \leq 1$ by Lemma 4.1.2. Therefore, by Lemma 4.1.2, $Pd(\partial) \leq 1$.

(e). We construct a map $\eta : \cup_{0 \leq k} X_k \longrightarrow D$ as follows:

Let $\eta(w) = a$. For $(r) \in X_1$, choose one $x \in D$ with $xr = a$ and let $\eta((r)) = x$. Suppose for each element ξ of X_{k-1}, $\eta(\xi)$ has been defined. For $(r_1, \cdots, r_k) \in X_k$, we choose one $x \in D$ with $xr_k = \eta((r_1, \cdots, r_{k-1}))$ and let $\eta((r_1, \cdots, r_k) = x$. In this manner, we define a map $\eta : \cup_{0 \leq k} X_k \longrightarrow D$. Since U is a free R-module with a basis $\cup_{0 \leq k} X_k$, the map η determines uniquely a homomorphism $\eta : U \to D$. From the construction of η, we see $Y \subseteq Ker(\eta)$, and so $V \subseteq Ker(\eta)$. Therefore there is a natural epimorphism $\partial = U/V \xrightarrow{\phi} U/Ker(\eta)$. η induces a monomorphism $U/Ker(\eta) \xrightarrow{\bar{\eta}} D$. Then $\partial \xrightarrow{\bar{\eta} \circ \phi} D$ is a homomorphism such that $(\bar{\eta} \circ \phi)(\bar{w}) = a$.

(f). Let $Q = Q_{cl}(R)$, $\xi = \sum_{i,k} ((r^i_{1k}, r^i_{2k}, \cdots, r^i_{kk}) + V)a_{ik} \in \partial$, and $t \in \mathcal{C}_R(0)$. Then $ta_{ik}t^{-1} \in Q$. Write $ta_{ik}t^{-1} = p_{ik}^{-1} q_{ik}$ for some $q_{ik} \in R$ and some $p_{ik} \in \mathcal{C}_R(0)$. By [18, Lemma 5.1, P87], $p_{ik}^{-1} = r^{-1}a_i$ for some $a_i \in R$

and $r \in \mathcal{C}_R(0)$. Then $ta_{ik}t^{-1} = r^{-1}a_iq_{ik}$, and thus $a_iq_{ik}t = rta_{ik}$. Therefore $\xi = \sum_{i,k}((r^i_{1k}, \cdots, r^i_{kk}) + V)a_{ik} = \sum_{i,k}((r^i_{1k}, \cdots, r^i_{kk}, t, r) + V)rta_{ik} = [\sum_{i,k}((r^i_{1k}, \cdots, r^i_{kk}, t, r) + V)a_iq_{ik}]t$.

(g). Given a divisible module M_R. For any nonzero element a in M, if $ar \neq 0$ for any $r \in \mathcal{C}_R(0)$, then we let $I_a = \{(a, 0)\}$; otherwise, we set $I_a = \{(a, r) : r \in \mathcal{C}_R(0) \text{ with } ar = 0\}$. For each $(a, r) \in I_a$, we choose an $f_{a,r} \in \mathrm{Hom}(\partial/\bar{w}rR, M)$ satisfying $f_{a,r}(\bar{w} + \bar{w}rR) = a$. Such an $f_{a,r}$ exists by (e). Let $D = \oplus_a \oplus_{r \in I_a} (\partial/\bar{w}rR)$. Then $\{f_{a,r}\}$ induces a homomorphism $f = \oplus f_{a,r} : D \longrightarrow M$, and f is clearly onto. Also, f induces an exact sequence:

$$0 \longrightarrow N \longrightarrow D \longrightarrow M \longrightarrow 0, \quad \text{where } N = ker(f). \tag{2}$$

By (f), D is divisible. To see N is divisible, let $x \in N$ and $t \in \mathcal{C}_R(0)$. Since D is divisible, $x = yt$ for some $y \in D$. Let $z = f(y) \in M$. Then $zt = f(y)t = f(yt) = f(x) = 0$. Therefore the map $g : R/tR \longrightarrow M$ defined by $g(\bar{b}) = zb$ is a well-defined homomorphism. Define $h : R \to D$ by $h(b) = (\bar{w} + \bar{w}tR)b$. Then $h(t) = 0$, and thus h induces a homomorphism $\bar{h} : R/tR \to D$. Directly from the definition of D and the map f, we have $f \circ h = g$. Let $u = y - \bar{h}(\bar{1}) \in D$. Then $f(u) = f(y) - f \circ \bar{h}(\bar{1}) = z - g(\bar{1}) = 0$, and $ut = yt - \bar{h}(\bar{1})t = x$. Therefore N is divisible. The last part of (g) is now clear from the proof above. □

A short exact sequence of right R-modules: $0 \to M' \to M \to M'' \to 0$ is called pure if $M' \otimes_R L \longrightarrow M \otimes_R L$ is a monomorphism for every left R-module L. A module N_R is called absolutely pure (or FP-injective) if every exact sequence $0 \to N_R \to M_R \to P_R \to 0$ is pure.

Proposition 4.2.2 *Let M_R be a module over a Prüfer ring R. Then the following are equivalent:*

(a) M_R is divisible;

(b) $Ext^1_R(R/rR, M) = 0$, for every $r \in \mathcal{C}_R(0)$;

(c) $Ext^1_R(R/I, M) = 0$, for every f.g. right ideal I of R;

(d) M_R is absolutely pure.

Proof. $(a) \Leftrightarrow (b)$. From the exact sequence $0 \to rR \to R \to R/rR \to 0$, we have an exact sequence $\mathrm{Hom}(R, M) \to \mathrm{Hom}(rR, M) \to \mathrm{Ext}^1(R/rR, M) \to 0$. Therefore, $\mathrm{Ext}^1(R/rR, M) = 0$ if and only if for every homomorphism $rR \xrightarrow{f} M$, there exists a homomorphism $R \xrightarrow{g} M$ such that g extends f. If M_R is divisible, $r \in \mathcal{C}_R(0)$ and $rR \xrightarrow{f} M$ is a homomorphism, then $f(r) = yr$ for some $y \in M$. Define $g : R \to M$ by $g(1) = y$. Then g extends f, and so $\mathrm{Ext}^1(R/rR, M) = 0$. Conversely, let $x \in M$, $r \in \mathcal{C}_R(0)$. Clearly $f : rR \to M$ via $f(ra) = rx$ is a homomorphism. Since f can be extended to a homomorphism $R \xrightarrow{g} M$, then $x = rg(1)$. Therefore D is divisible.

$(b) \Leftarrow (c)$. Trivial.

$(b) \Rightarrow (c)$. Let I be a f.g. right ideal of R. From the exact sequence $0 \to I_R \xrightarrow{i} R_R \to (R/I)_R \to 0$, we have the exact sequence $Hom(R, M) \xrightarrow{Hom(i,M)} Hom(I, M) \longrightarrow Ext^1_R(R/I, M) \longrightarrow 0$. Therefore $Ext^1_R(R/I, M) = 0$ if and only if $\mathrm{Hom}(i, M)$ is onto if and only if each homomorphism $f : I_R \to M$ can be extended to R. We can find a right ideal J of R which is maximal with respect to $I \cap J = 0$. Then $I + J = I \oplus J$ is an essential right ideal of R. By Proposition 1.2, $I + J$ contains an element $r \in \mathcal{C}_R(0)$. Write $r = r_1 + r_2$, for some $r_1 \in I$ and some $r_2 \in J$, and let $K = I + r_2R = I \oplus r_2R$. Obviously

$f : I \to M$ can be extended to $\bar{f} : K \to M$. K is a f.g. right ideal of R, hence K is projective, since R is semihereditary. By Proposition 1.4, there exist $\{a_i\} \subseteq K$ and $\{f_i\} \subseteq \mathrm{Hom}(K, R)$, such that for any $a \in K$, $f_i(x) = 0$ for all but a finite number of the f_i, and $a = \Sigma a_i f_i(a)$. Since $K \cap \mathcal{C}_R(0) \neq 0$, there exists, for each i, a $q_i \in Q_{cl}(R)$ satisfying $q_i K \subseteq R$ such that $f_i(a) = q_i a$ for all $a \in K$. For $s \in K \cap \mathcal{C}_R(0)$, we have $s = \Sigma a_i f_i(s) = (\Sigma a_i q_i)s$. This implies that $\Sigma a_i q_i = 1$. Since R is also a left order in $Q_{cl}(R)$, there exists $t \in \mathcal{C}_R(0)$ such that all $tq_i \in R$. Now the divisibility of M implies that we can write $f(a_i) = x_i t$ with all $x_i \in M$. Then for any $a \in K$ we obtain $f(a) = f(\Sigma a_i q_i a) = \Sigma f(a_i)(q_i a) = \Sigma x_i(tq_i)a = xa$ with $x = \Sigma x_i t q_i \in M$. Hence the map $a \mapsto xa$ from R to M is a R-homomorphism that extends $\bar{f}$.

$(c) \Leftrightarrow (d)$. Megibben and Stenström proved, independently, that $(c) \Leftrightarrow (d)$ for an arbitrary ring R (see [30, Prop.1] or [35, Prop.2.6]). $\square$

The concept of a semicompact module was defined by Matlis in [28], where it was shown that a module over a commutative Prüfer domain is injective if and only if it is divisible and semicompact. The same result holds in a noncommutative Prüfer ring.

For a module M_R, let $R(M)$ denote the set of subsets of M of the form $\{x \in M : xI = 0\}$ for a right ideal I of R. M will be called semicompact if every finitely solvable set of congruences

$$x \equiv x_\alpha \pmod{M_\alpha}$$

where $x_\alpha \in M$ and $M_\alpha \in R(M)$, has a solution in M [28]. If we note a result of Stenström [35, Prop.2.5] that an absolutely pure module is injective if and only if it is semicompact, then the following is immediate:

Corollary 4.2.1 *Let M_R be a module over a Prüfer ring R. Then M is injective if and only if it is divisible and semicompact.* □

Proposition 4.2.3 *Let M_R be a module over a Prüfer ring R. If $Pd(M) = m \geq 1$, then $Ext_R^m(M, D) = 0$ for all divisible module D_R.*

Proof. We induct on m. If $m = 1$, then, by Theorem 4.1.2, M_R is the union of a well-ordered continuous chain of submodules:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \subset M_\rho = M \qquad (\alpha < \rho)$$

such that $M_{\alpha+1}/M_\alpha$ is finitely presented cyclic for all $\alpha < \rho$. Thus, for each $\alpha < \rho$, $M_{\alpha+1}/M_\alpha \cong R/I_\alpha$ for some f.g. right ideal I_R. Since D_R is divisible, Proposition 4.2.2 implies that $\mathrm{Ext}_R^1(M_{\alpha+1}/M_\alpha, D) = 0$, for every $\alpha < \rho$. By Lemma 4.2.1, $\mathrm{Ext}_R^1(M, D) = 0$.

For $m > 1$, let $0 \to N_R \to F_R \to M_R \to 0$ be an exact sequence with F projective. Then $Pd(N) = Pd(M) - 1 = m - 1$. Now the induction hypothesis implies that $\mathrm{Ext}_R^{m-1}(N, D) = 0$ for all divisible module D_R. From the exact sequence $0 \to N \to F \to M \to 0$, we have $\mathrm{Ext}_R^k(N, D) \cong \mathrm{Ext}_R^{k+1}(M, D)$ for all $k \geq 1$. Therefore $\mathrm{Ext}_R^m(M, D) = 0$ for every divisible module D_R. □

Remark 4.2.1 *Proposition 4.2.3 generalizes a result of L.Fuchs [13, Prop.3.9, P126].*

We now can give the following characterization of divisible modules of projective dimension at most one:

Proposition 4.2.4 *Let M_R be a module over a Prüfer ring R. Then M is divisible with $Pd(M) \leq 1$ if and only if it is a summand of a direct sum of modules of the form $\partial/\bar{w}r_iR$, where every $r_i \in \mathcal{C}_R(0) \cup \{0\}$.*

Proof. ($\Rightarrow$). By Proposition 4.2.1 (g), there exists an exact sequence $0 \to N \to D \to M \to 0$, where N is divisible, and $D \cong \bigoplus_{r \in S} \partial/\bar{w}rR$ for a subset S of $\mathcal{C}_R(0) \cup \{0\}$. If $Pd(M) = 0$, then $\mathrm{Ext}^1_R(M, N) = 0$ from the definition of projective dimension. If $Pd(M) = 1$, then Proposition 4.2.3 implies $\mathrm{Ext}^1_R(M, N) = 0$. Hence, $0 \to N \to D \to M \to 0$ splits. It follows that M is a summand of D.

($\Leftarrow$). Let D be as above, and M be a summand of D. Then Proposition 4.2.1 (f) implies M is divisible. We know $Pd(\partial) \leq 1$ from (d) of Proposition 4.2.1. Suppose $Pd(\partial) = 1$, we have $Pd(\partial/\bar{w}rR) \leq 1$ for all $r \in R$ by Proposition 4.1.1. Then, a similar proof of Proposition 4.2.1 (d) shows that $Pd(D) \leq 1$. If $Pd(D) = 1$, then we have $Pd(M) = 1$ by Proposition 4.1.1. On the other hand, $Pd(D) = 0$ implies M is projective and hence $Pd(M) = 0$. Therefore $Pd(M) \leq 1$ holds if $Pd(\partial) = 1$. Suppose $Pd(\partial) = 0$, i.e., ∂ is projective, then ∂ is torsionfree. Therefore $\bar{w}rR$ is f.g. torsionfree. It follows from Proposition 2.3.5 that $\bar{w}rR$ is projective. Therefore we still have $Pd(\partial/\bar{w}rR) \leq 1$. Repeating the argument above, we have $Pd(M) \leq 1$. $\square$

Corollary 4.2.2 *Let M_R be a module over a Prüfer ring R. Then M_R is divisible torsion with $Pd(M) \leq 1$ if and only if it is a summand of a direct sum of modules of the form $\partial/\bar{w}r_iR$, where each $r_i \in \mathcal{C}_R(0)$.*

Proof. It follows from the last part of (g) of Proposition 4.2.1 and the proof of Proposition 4.2.4. $\square$

Let $C(R)$ denote the center of a ring R, and $r \in \mathcal{C}_R(0)$. Suppose $1 \neq s \in C(R) \cap \mathcal{C}_R(0)$. We define two maps as follows:

$$\phi : \cup_{0\le k} X_k \longrightarrow \partial$$

by $\phi(w) = \overline{(r)} - \overline{(rs)}s$, and

$$\phi((r_1, \cdots, r_k)) = \overline{(r, r_1, \cdots, r_k)} - \overline{(rs, r_1, \cdots, r_k)}s \text{ for } k \ge 1.$$

And

$$\psi : \cup_{0\le k} X_k \longrightarrow \partial/\bar{w}rR$$

by $\psi(w) = \bar{0}$, $\psi((r)) = \bar{w} + \bar{w}rR$, and

$$\psi((r_1, r_2, \cdots, r_k)) = \overline{(r_2, \cdots, r_k)} + \bar{w}rR \text{ if } r_1 = r; \text{ or } \bar{0} \text{ if } r_1 \ne r.$$

Then ϕ determines uniquely a homomorphism $U \xrightarrow{\phi} \partial$, and ψ defines a homomorphism $U \xrightarrow{\psi} \partial/\bar{w}rR$. It is straightforward to check that $Y + wrR \subseteq Ker(\phi)$ and $V \subseteq Ker(\psi)$. Therefore ϕ and ψ induce canonically two homomorphisms

$$U/(wrR + V) \xrightarrow{\Phi} U/Ker(\phi) \text{ and } U/V \xrightarrow{\Psi} U/Ker(\psi).$$

Note that

$$\partial/\bar{w}rR \cong U/(wrR + V) \text{ and } \partial = U/V.$$

Then the homomorphism

$$\Phi : \partial/\bar{w}rR \to \partial$$

satisfies

$$\Phi(\bar{w} + \bar{w}rR) = \overline{(r)} - \overline{(rs)}s$$

and

$$\Phi(\overline{(r_1, \cdots, r_k)} + \bar{w}rR) = \overline{(r, r_1, \cdots, r_k)} - \overline{(rs, r_1, \cdots, r_k)}s \text{ for } k \ge 1;$$

and the homomorphism

$$\Psi : \partial \to \partial/\bar{w}rR$$

satisfies

$$\Psi(\bar{w}) = \bar{0},\ \Psi(\bar{(r)}) = \bar{w} + \bar{w}rR,$$

and

$$\Psi(\overline{(r, r_2, \cdots, r_k)}) = \overline{(r_2, \cdots, r_k)} + \bar{w}rR,$$

and

$$\Psi(\overline{(r_1, r_2, \cdots, r_k)}) = \bar{0} \text{ if } r_1 \neq r.$$

Let $\partial \xrightarrow{n} \partial/\partial_0$ be the natural homomorphism, and $\Phi_1 = n \circ \Phi$. Since $\Psi(\bar{w}) = \bar{0}$, Ψ induces a homomorphism $\Psi_1 : \partial/\partial_0 \to \partial/\bar{w}rR$.

Lemma 4.2.2 *Let Φ,Ψ, Φ_1, and Ψ_1 be the same as above.*

(a) $\Psi \circ \Phi = 1_{\partial/\bar{w}rR}$. In particular, $\partial/\bar{w}rR$ is a summand of ∂;

(b) $\Psi_1 \circ \Phi_1 = 1_{\partial/\bar{w}rR}$. In particular, $\partial/\bar{w}rR$ is a summand of ∂/∂_0.

Proof. (a). Since $Z = \{\bar{\xi} + \bar{w}rR : \xi \in \cup X_k\}$ is a set of generators of $\partial/\bar{w}rR$, it suffices to check that $\Psi \circ \Phi$ fixs every element of Z. However, the verification is straightforward.

(b). Similarly. □

Theorem 4.2.1 *If R is a Prüfer ring, and $C(R) \neq \{0, 1\}$ (e.g., if the characteristic of $R \neq 2$), then M_R is divisible with $Pd(M) \leq 1$ if and only if it is a summand of a direct sum of copies of ∂ .*

Proof. Note that if R is a Prüfer ring, then $C(R) \subseteq \mathcal{C}_R(0)$. Now apply Proposition 4.2.4 and Lemma 4.2.2. □

Theorem 4.2.2 *If R is a Prüfer ring, and $C(R) \neq \{0, 1\}$, then M_R is divisible torsion with $Pd(M) \leq 1$ if and only if it is a summand of a direct sum of copies of ∂/∂_0 .*

Proof. By Corollary 4.2.2, and Lemma 4.2.2. □

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