NEW SYMMETRIES FROM OLD: EXPLOITING LIE ALGEBRA STRUCTURE TO DETERMINE INFINITESIMAL SYMMETRIES OF DIFFERENTIAL EQUATIONS

by

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Abstract

We give a method for using explicitly known Lie symmetries of a system of differential equations to help find more symmetries of the system. A Lie (or infinitesimal) symmetry of a system of differential equations is a transformation of its dependent and independent variables, depending on continuous parameters, which maps any solution of the system to another solution of the same system. Infinitesimal Lie symmetries of a system of differential equations arise as solutions of a related system of linear homogeneous partial differential equations called infinitesimal determining equations. The importance of symmetries in applications has prompted the development of many software packages to derive and attempt to integrate infinitesimal determining equations. For a given system of differential equations we usually have a priori explicit knowledge of many symmetries of the system because of their simple form or the physical origin of the system. Current methods for finding symmetries do not incorporate this a priori information. Our method uses such information to simplify the problem of finding the remaining unknown symmetries by exploiting the Lie algebra structure of the solution space of the infinitesimal determining equations.

We illustrate our method for simplifying infinitesimal determining systems by applying it to three well known test problems: the linear heat equation; Laplace’s equation; and a class of nonlinear diffusion equations. Our method uses the inspectional symmetries of each of these differential equations to determine the equation’s remaining symmetries. In these cases the method is so effective that the simplified determining systems for the unknown symmetries are just linear (algebraic) equations.
Finally we indicate that it is possible to obtain determining equations for the infinitesimal generators of various subalgebras of a Lie symmetry algebra specified by infinitesimal determining equations. In particular we prove that the determining equations for the centre of the algebra can always be obtained.
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Dedication

For professor John Ockendon.
Chapter 1

Introduction

In this thesis we give a method which uses explicitly known infinitesimal symmetries of a system of differential equations to help find more infinitesimal symmetries of the system. Recently [15, 16] it has been shown that the structure of the Lie symmetry algebra of such systems can be calculated without integrating the determining equations for the infinitesimal symmetries. Our method exploits this known structure as additional differential equations to considerably simplify the task of finding infinitesimal symmetries.

A Lie symmetry of a differential equation (or system) is a transformation of its dependent and independent variables, depending on continuous parameters, which maps every solution of the equation to another solution of the same equation. The Lie symmetries of a differential equation naturally form a group: since the composition of any two symmetries is also a symmetry, there is an identity transformation (the one that leaves all solutions fixed), composition of symmetries is obviously associative and any symmetry has an inverse. The theory of Lie symmetry groups of differential equations was developed by a Norwegian mathematician Sophus Lie in the late nineteenth century and is now a well studied topic. Such groups (now called Lie groups) are invertible point transformations of the dependent and independent variables of the differential equation which depend on continuous parameters. Lie showed that this group is of great assistance in understanding and constructing solutions of differential equations. Many integration techniques can be unified and extended by considering symmetry groups acting on differential equations. Some of the applications of Lie groups to differential equations include:
reduction of order for ordinary differential equations, mapping solutions to other solutions, reduction of the number of independent variables of partial differential equations, construction of invariant solutions to ordinary and partial differential equations, construction of invariant solutions to boundary value problems, construction of conservation laws, detection of linearizing transformations of differential equations, and many other applications [2, 10]. The last twenty years has seen a huge resurgence in interest of the applications of Lie groups to differential equations. Indeed, Daniel Zwillinger [26] in his Handbook of Differential Equations (1992) remarks: “Lie group analysis is the most useful and general of all the techniques presented in this book”.

To use any of the above methods for a given set of differential equations one must first be able to find symmetries of the equations. A naïve method for finding point symmetries of such systems is to make a general change of variables and then enforce the new variables to satisfy the same set of differential equations. This approach leads to complicated nonlinear overdetermined systems of differential equations for the transformations. Lie gave an infinitesimal formulation of the problem of finding the symmetry group of a set of differential equations, which replaces these intractable nonlinear equations by tractable linear overdetermined systems of partial differential equations for the infinitesimal form of the symmetries. The solutions of these so called infinitesimal determining equations can be used to determine symmetry transformations.

Systems of differential equations which model real problems often have some symmetries which are obvious from physical considerations. For example, Newton’s equations are invariant under the Galilean group of rotations, spatial and temporal translations and the transformations representing uniformly moving frames of reference. Consequently we usually have a priori explicit knowledge of many (obvious) solutions of the infinitesimal determining equations for symmetries and we wish to find the remaining (non-obvious)
Chapter 1. Introduction

solutions of the determining equations. Present methods for finding solutions of infinitesimal determining equations do not incorporate such a \textit{a priori} information and laboriously recalculate these obvious solutions during the search for non-obvious solutions. It has been an open problem as to how to exploit such obvious solutions to find the non-obvious ones, or equivalently, factor them out of the solution process. The commutator reduction method which is presented in this thesis addresses this problem. This method can often produce reduced and simplified determining equations which represent only the non-obvious solutions (infinitesimal symmetries) that we are interested in. In the examples of chapter 3 we will show that the commutator reduction method constructs extra differential equations which, when added to the determining equations, filter out their obvious solutions and allow only their non-obvious solutions to remain.

Our method depends on exploiting the special algebraic structure of the solution space of the infinitesimal determining equations due to their symmetry origin. It cannot be applied to arbitrary systems of partial differential equations. In particular the solution space of infinitesimal determining equations has not only the structure of a vector space, but also admits a bilinear operator called a \textit{commutator bracket}. That is, the commutator of any two solutions (of the determining equations) is again a solution. In addition the solution space has the structure of a \textit{Lie algebra} under this bracket. The interaction between solutions given by their commutator is usually different for each solution. For example, if \(X\), \(\hat{X}\) and \(Y\) are solutions of the infinitesimal determining equations then the commutator of \(X\) with \(Y\) (written \([X,Y]\)) is in general different than the commutator of \(\hat{X}\) with \(Y\) (written \([\hat{X},Y]\)). In general, solution \(X\) will interact with other solutions differently than the way \(\hat{X}\) does. Thus \(X\) and \(\hat{X}\) are, to an extent, characterised by relations expressing the results of taking their commutators with other solutions. Later we shall see that this characterisation of solutions by their commutators will enable us to produce extra differential equations (as mentioned in the previous paragraph) as
constraints on the infinitesimal determining equations. This information is then used to factor out the known (obvious) solutions of such systems.

Another key ingredient in our method is a collection of recent algorithms called Standard Form, Initial Data and Commutation Relations which we shall refer to as the Reid algorithms [15, 16, 17, 18]. Application of the Reid algorithms to infinitesimal determining equations yields information about their solutions without needing to integrate such equations. In a process only involving the application of a finite number of differentiations and linear eliminations to the determining equations, these algorithms can explicitly obtain the structure of the Lie symmetry algebra these equations represent. In particular, the commutation relations representing interactions between known and unknown solutions are obtained in a form which is parametrised by explicitly given structure constants. These equations are then realised as differential equations which the unknown solutions must satisfy in addition to the original determining equations. By appending these extra differential equations to the system of determining equations we can often obtain a more constrained system of partial differential equations for the remaining unknown solutions and thus simplify the task of integrating such systems. In all examples considered in this thesis the reduced problem for the remaining unknowns has proven to be much simpler to solve than the original set of determining equations. In each case the solution of the reduced problem was equivalent to the easy task of solving systems of linear algebraic equations.

In Chapter 2 we motivate and describe a minimal list of material from Lie group theory necessary to prepare the reader for our commutator method in Chapter 3. We begin Chapter 2 with a description of Lie symmetries of differential equations and their applications. We show how Lie symmetry transformations can be naturally represented as differential operators. Visual examples are given to provide the reader with an intuitive understanding of certain elementary symmetries which appear throughout the
thesis. We conclude §2.1 with a sketch of the derivation of the determining equations of infinitesimal symmetries for our three main examples: the linear heat equation; the 3D Laplace equation; and the nonlinear heat equation. In §2.2 we cover some basic algebraic aspects of symmetry groups: the commutator bracket of a Lie group; the Lie algebra of a Lie group; and commutation relations of a Lie algebra. In §2.3 we describe the Reid algorithms Standard Form, Initial Data and Commutation Relations. We are concerned with using the algorithms and do not discuss their inner mechanisms. We show how a system in standard form can be used as a substitution list to canonically simplify differential expressions involving derivatives of the dependent variables of the system. This leads to the important concept of parametric derivatives for a system of differential equations. Computation of parametric derivatives are key aspects of algorithms Initial Data and Commutation Relations. Specification of the values of the parametric derivatives at a point uniquely determines a Taylor series solution of the determining equations system. This one to one correspondence between the parametric initial conditions and Taylor series solutions is used in the algorithm Commutation Relations [18] to obtain the dimension and structure constants of Lie symmetry algebras of differential equations without having to integrate their determining equations.

In Chapter 3 we describe the commutator reduction method for simplifying infinitesimal determining equations by using the structure of their associated Lie algebras. Realising commutation relations as differential equations is the crucial ingredient of this method. We give a brief description of the commutator reduction method and for illustration apply it to the Lie symmetry analysis of the linear heat equation. In this example we have four explicitly known solutions (since the linear heat equation is invariant under the obvious symmetries of space and time translation, scalings of space and time and scaling of the temperature) of the determining equations. The Standard Form and Initial Data algorithms predict that the full algebra is six dimensional (without solving the
determining equations) and so we have two unknown symmetries to find. We choose one of these unknown symmetries and then use our method to obtain reduced determining equations for this particular unknown symmetry. These reduced determining equations do not even contain derivatives and directly yield the unknown solution without integration! In §3.2 we concisely outline the commutation reduction method and proceed to apply it to three examples. In each case we get complete reduction which yields the unknown symmetries without needing to integrate any differential equations.

In Chapter 4, in the Discussion, we discuss three areas of application for our method: iteratively calculating higher order symmetries (symmetries whose infinitesimals depend on derivatives); group classification problems (finding symmetries of differential equations with unspecified modelling functions in them); and problems of physical origin which are invariant under physically obvious symmetries (e.g. Euclidean rigid motions). In §4.3 we discuss generalisations and heuristic aspects of the method. These points indicate that there is scope for further research in extending and modifying our method.

In Appendix A we give an outline of the Standard Form algorithm.

In Appendix B we show that from the infinitesimal determining equations for a Lie symmetry algebra $\mathcal{L}$ it is possible to obtain determining equations for various subalgebras of $\mathcal{L}$. In particular we prove that determining equations for the centre of $\mathcal{L}$ can be obtained by adapting our commutator reduction method. Analogous results are stated, but not proved, for other subalgebras including: the centraliser and normaliser of a subalgebra whose determining equations are known; the derived subalgebras constituting the upper and lower central series of a Lie algebra; the radical of the killing form of $\mathcal{L}$; and the homomorphic image of $\mathcal{L}$ under the adjoint action of some explicitly known infinitesimal symmetry.
Chapter 2

Background material for the commutator reduction method

To understand the commutator reduction method we first give an introduction to the basic material from the area of symmetry analysis and its applications to differential equations. The Lie point symmetries used in this thesis are point transformations on the space of dependent and independent variables of a system of differential equations. Under the action of a symmetry transformation the image of a point traces out a path which is analogous to the trajectory of a particle. The collection of such trajectories (or group orbits) can be viewed as a fluid flow. The tangents to this flow represent its velocity vector field which may be regarded as a differential operator (or infinitesimal generator). This differential operator has the interpretation of ‘the rate of change following the fluid’. Representing symmetry transformations by infinitesimal generators is fundamental to Lie symmetry analysis. In §2.1 we aim to help the reader gain familiarity with the notation and develop an understanding of the connection between infinitesimal generators and their corresponding symmetry groups.

Finding symmetries of a given set of differential equations involves setting up and solving an associated system of linear homogeneous partial differential equations called determining equations. We discuss how determining equations arise from symmetry problems and illustrate this by outlining the derivation of such equations for three examples which will be used later in the thesis: the linear heat equation: the 3D Laplace equation; and the nonlinear heat equation.

The set of symmetries of a given problem have a rich algebraic structure, called a Lie
algebra, which we will exploit in our method. In §2.2 we introduce the necessary algebraic concepts — the commutator bracket — to specify the Lie algebra structure. The meaning of the commutator bracket as a measure of interaction between symmetries is emphasised and illustrated by examples. Commutation relations are defined and explicitly calculated for Lie algebras of vector fields.

Finally, in §2.3 we introduce the Reid algorithms Standard Form, Initial Data and Commutation Relations which are used extensively in our method. Using these algorithms we can obtain structural information about the symmetry algebra of a differential equation merely by algorithmically manipulating its determining equations without having to integrate them.

2.1 Lie symmetries of differential equations and their determining equations

2.1.1 Lie symmetries of differential equations

A symmetry of a system of partial differential equations (PDEs) is a transformation which maps every solution of the system to another solution of the same equation (i.e. it maps the solution set of the equation into itself.). The next example illustrates that sometimes symmetries are obvious.

Example 2.1.1 Consider Laplace’s equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$  \hspace{2cm} (2.1)

for $u = u(x, y, z)$, which arises in connection with many physical situations such as hydrodynamics, electromagnetism, gravitation, etc. It is to be expected that the physical laws underlying such situations are invariant with respect to the geometric symmetries of rotation and translation of $(x, y, z)$–space, which transform points $(x, y, z)$ to points
Chapter 2. Background material for the commutator reduction method

via the rule:

\[ x^* = R(\theta, \phi, \psi)x + a \]  \hspace{1cm} (2.2)

where \( x = (x, y, z) \), \( x^* = (x^*, y^*, z^*) \), \( a = (a, b, c) \) and \( R(\theta, \phi, \psi) \) is a rotation matrix representing a composite rotation of \( \theta \) radians about the \( x \)-axis, \( \phi \) radians about the \( y \)-axis and then \( \psi \) radians about the \( z \)-axis. For any choice of the parameters \( (\theta, \phi, \psi, a, b, c) \) the transformation \( x \mapsto x^* \) given by (2.2) is a symmetry of the Laplace equation (2.1). Furthermore the six-parameter family of transformations (2.2) forms a group: since the composition of any two transformations of type (2.2) is again of type (2.2), there is an identity transformation (given by setting all the parameters \( (\theta, \phi, \psi, a, b, c) \) to zero in (2.2)), for every transformation in the family (2.2) there exists an inverse transformation in the family and composition of transformations is associative. Later we shall see that there are many other symmetries of the Laplace equation (2.1) which are not in the six-parameter inspectional symmetry group (2.2).

Symmetries of differential equations are not always obvious, as the next example demonstrates.

**Example 2.1.2** The linear heat equation

\[ u_t = \kappa u_{xx} \]  \hspace{1cm} (2.3)

can be used to model one-dimensional heat flow in a homogeneous medium. In this equation \( u \), representing the temperature, depends on time \( t \) and a single spatial variable \( x \), and the constant \( \kappa \) is the diffusivity of the medium.

By a suitable scaling we can consider, without loss of generality, the equation

\[ u_t = u_{xx}. \]  \hspace{1cm} (2.4)
Chapter 2. Background material for the commutator reduction method

The linear heat equation (2.4) above has a (nonobvious) symmetry \((x, t, u) \mapsto (x^*, t^*, u^*)\) given by

\[
\begin{align*}
x^* &= \frac{x}{1 - \varepsilon t} \\
t^* &= \frac{t}{1 - \varepsilon t} \\
u^* &= u \sqrt{1 - \varepsilon t} \exp \left(\frac{-\varepsilon x^2}{4(1 - \varepsilon t)}\right)
\end{align*}
\] (2.5)

for \(\varepsilon \in \mathbb{R}\). By calculating how \(u_t\) and \(u_{xx}\) transform under (2.5) and demonstrating that \(u^*_t = u^*_{x,x}\) when \(u_t = u_{xx}\), it is straightforward, but algebraically tedious, to verify that (2.5) is a symmetry of (2.4). There is a systematic procedure due to Lie for obtaining related linear partial differential equations which, if solved fully, yield all infinitesimal symmetries. This procedure is illustrated by the examples in §2.1.4. These associated PDEs are the so called determining equations (DQs) for the Lie symmetry algebra.

Symmetries can be used to generate non-obvious solutions from obvious ones, a property which can be exploited in applications [10]. Take the trivial solution, \(u = 1\) (of 2.4); a plane in \((x, t, u)\)-space. The symmetry (2.5) maps \(u = 1\) into a family of surfaces in \((x, t, u)\)-space, representing solutions \(u^* = \psi^*(x^*, t^*)\) of \(u^*_t = u^*_{x,x}\). (see Figure 2.1.2).

We can find these surfaces explicitly as follows.
Figure 2.1.2 The solution surface of the trivial solution \( u = 1 \) of the linear heat equation \( u_t = u_{xx} \) transforms into a one-parameter family of solution surfaces of the linear heat equation using the symmetry (2.5).

Inverting (2.5) we obtain

\[

t^* = \frac{t}{1 + \varepsilon t^*}, \\
\varepsilon = \frac{\varepsilon}{t},
\]

(2.6)

\[

u = u^* \sqrt{1 + \varepsilon t^*} \exp \left( \frac{-\varepsilon x^*^2}{4(1 + \varepsilon t^*)} \right).
\]

So under the action of the above symmetry the solution \( u = 1 \) maps to

\[

u^* \sqrt{1 + \varepsilon t^*} \exp \left( \frac{-\varepsilon x^*^2}{4(1 + \varepsilon t^*)} \right) = 1.
\]

or equivalently to

\[

u^* = \frac{1}{\sqrt{1 + \varepsilon t^*}} \exp \left( \frac{-\varepsilon x^*^2}{4(1 + \varepsilon t^*)} \right).
\]

(2.7)

(2.8)

Summarising, from the known solution, \( u = 1 \), the symmetry (2.5) was used to generate the one-parameter \((\varepsilon)\) family of solutions\(^1\) (2.8).

\(^1\)Note also that scaling \( x \), translating \( t \), and putting \( \varepsilon = 1 \) yields the fundamental solution \( u = \frac{1}{\sqrt{2\pi t}} \exp \left( \frac{-x^2}{2t} \right) \) of the heat equation.
Next we will show how continuous point symmetries can be represented.

2.1.2 Lie symmetries as vector field flows

Consider a particle moving in a fluid. If the velocity vector field of the fluid is known explicitly then the trajectory of the particle can be calculated by integrating the following autonomous ordinary differential equation system

\[ \dot{x} = \xi(x), \quad x(0) = x_0, \]  

(2.9)

Here \( x = x(t), \xi(x), \) and \( x_0 \) are vectors in \( \mathbb{R}^n \) representing the position of the particle at time \( t, \) the velocity vector field of the fluid and the initial position of the particle, respectively. The collection of all trajectories for different initial points \( x_0 \) gives rise to a (one-parameter) transformation group \( g^\varepsilon: \mathbb{R}^n \to \mathbb{R}^n \) defined by \( g^\varepsilon(x_0) := x(\varepsilon). \) \( g^\varepsilon \) is called the flow of the vector field. Conversely if the flow \( g^\varepsilon \) is known explicitly then the velocity vector field can be calculated as tangents to the flow:

\[ \xi(x_0) = \left. \frac{d}{d\varepsilon} g^\varepsilon \right|_{\varepsilon=0}. \]  

(2.10)
Example 2.1.3 Flows.

The Vector field
\[ \xi(x, y) = (1, 0). \]

The corresponding flow
\[ g^\varepsilon(x_0, y_0) = (x_0 + \varepsilon, y_0) \]

of the vector field \( \xi(x, y) = (1, 0) \)

for various initial points \((x_0, y_0)\).

We see that there is a one to one correspondence between flows and their vector fields. By representing symmetry transformations as vector fields the problem of finding symmetries of a system of differential equations amounts to solving a set of linear homogeneous partial differential equations (called the infinitesimal determining equations) for the components \( \xi^1, \ldots, \xi^n \) of a vector field. In §2.1.4 we will give examples illustrating the derivation of infinitesimal determining equations.
2.1.3 Infinitesimal differential operators of vector field flows

The vector field $\mathbf{\xi}(\mathbf{x})$ can be represented by its associated differential operator (the infinitesimal generator) $\mathbf{X}$ given by $\mathbf{X} = \xi^i \partial_i$, where $\partial_i \equiv \frac{\partial}{\partial x^i}$ (we have used the convention of summing over repeated indices. Thus $\xi^i \partial_i \equiv \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}$. These notations will be used without comment throughout the sequel, unless otherwise indicated). The utility of this operator notation becomes clearer when we consider the physical interpretation of the action of this operator on some scalar field $f(\mathbf{x})$:

$$
\mathbf{X} f = \xi^i \partial_i f
$$

$$
= \frac{\partial f}{\partial x^i} \quad \text{by (2.9)}
$$

$$
= \frac{df}{dt}
$$

$$
= \text{rate of change of } f(\mathbf{x}) \text{ following particle at } \mathbf{x}(t)
$$

In fluid mechanics the operator $\mathbf{X}$ is called the material derivative and is used to calculate the rate of change of quantities which are flowing with a fluid.

We can see from the above that the operators $\frac{d}{dt}$ and $\mathbf{X}$ operate equivalently on scalar functions $f(\mathbf{x})$ and this yields an elegant expression for the Taylor series expansion of $f(\mathbf{x}(t))$ about $t = 0$:

$$
f(\mathbf{x}(t)) = f|_{t=0} + t \frac{df}{dt}|_{t=0} + \frac{t^2}{2!} \frac{d^2 f}{dt^2}|_{t=0} + \cdots
$$

$$
= f(\mathbf{x}_0) + t(\mathbf{X} f)(\mathbf{x}_0) + \frac{t^2}{2!}(\mathbf{X}^2 f)(\mathbf{x}_0) + \cdots \quad (2.11)
$$

$$
= (e^{t\mathbf{X}} f)(\mathbf{x}_0).
$$

Example 2.1.4 Putting $f(\mathbf{x}) = x^i$ and $t = \varepsilon$ into (2.11) above we obtain

$$
x^i(\varepsilon) = (e^{\varepsilon \mathbf{X}} x^i)(\mathbf{x}_0)
$$

$$
= x^i_0 + \varepsilon (\mathbf{X} x^i)(\mathbf{x}_0) + O(\varepsilon^2)
$$

$$
= x^i_0 + \varepsilon \xi^i(\mathbf{x}_0) + O(\varepsilon^2) \quad (2.12)
$$
for $\varepsilon \to 0$ and $i = 1 \ldots n$. For this reason $X = \xi \partial_x$ is called the *infinitesimal generator* of the flow $g^\varepsilon: x_0 \mapsto x(\varepsilon)$ and $x(\varepsilon)$ is said to be generated by exponentiating $X$. As a more convenient notation, henceforth we will denote the flow $g^\varepsilon$ by $x^*(\varepsilon) = X(x; \varepsilon)$, to highlight the $x$ and $\varepsilon$ dependence. (i.e. $g^\varepsilon: x \mapsto x^*(\varepsilon) = X(x; \varepsilon)$.)

**Example 2.1.5 Examples of infinitesimal generators in $(x, y)$–space.** In Example 2.1.3 we have the infinitesimal generator $X = \partial_x$ corresponding to the vector field $\xi(x, y) = (1, 0)$ (see the Figure in Example 2.1.3). The flow generated is the group of translations in the positive $x$ direction

$$g^\varepsilon: (x_0, y_0) \mapsto (x_0 + \varepsilon, y_0),$$

which, in the new notation above is

$$x^*(\varepsilon) = x + \varepsilon$$

$$y^*(\varepsilon) = y.$$

Some other commonly occurring two dimensional infinitesimal generators, and their corresponding flows, for several initial points, are depicted below.

**Generator 2:** $X_2 = \partial_y$

**Flow 2:** $((x^*, y^*)) = (x, y + \varepsilon)$
Chapter 2. Background material for the commutator reduction method

Generator 3: \( X_3 = x \partial_x + y \partial_y \)

Flow 3: \( (x^*, y^*) = (e^\epsilon x, e^\epsilon y) \)

Generator 4: \( X_4 = -y \partial_x - x \partial_y \)

Flow 4: \( (x^*, y^*) = (x \cos \epsilon + y \sin \epsilon, -x \sin \epsilon + y \cos \epsilon) \)

In summary there is an explicit one to one correspondence between an infinitesimal generator \( X = \xi^i(x) \partial_x \) and a flow \( x^*(\epsilon) = X(x; \epsilon) \) which is as follows:
Given $X$ we obtain $x^*(\varepsilon)$ by solving the initial value problem

\[ \dot{x}^* = \xi(x^*), \quad x^*(0) = x. \quad (2.14) \]

Conversely, given $x^*(\varepsilon) = X(x; \varepsilon)$, $X$ is obtained by differentiation:

\[ \xi(x) = \left. \frac{\partial X(x; \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}. \]

Both directions of this correspondence are illustrated in the following two examples

**Example 2.1.6** (Using infinitesimal generators to determine flows.)

Given $X = t \partial_x - \frac{1}{2} xu \partial_u$ then $\xi$ is a vector field in $(x, t, u)$-space and so the initial value problem (2.14) becomes

\[
\begin{align*}
\frac{dx^*}{dz} &= t^*, & x^*(0) &= x, \\
\frac{dt^*}{dz} &= 0, & t^*(0) &= t, \\
\frac{du^*}{dz} &= -\frac{1}{2} v^* u^*. & u^*(0) &= u.
\end{align*}
\]

Integrating this system gives the flow

\[
\begin{align*}
x^*(\varepsilon) &= x + \varepsilon t \\
t^*(\varepsilon) &= t \\
a^*(\varepsilon) &= u \exp \left( -\frac{\varepsilon t}{2} - \frac{\varepsilon^2 t}{4} \right).
\end{align*}
\]

**Example 2.1.7** (Using flows to determine infinitesimal operators.)

From Example 2.1.2 we have the flow

\[
x^*(\varepsilon) = X(x; \varepsilon) = \left( \frac{x}{1 - \varepsilon t}, \frac{t}{1 - \varepsilon t}, u \sqrt{1 - \varepsilon t} \exp \left( \frac{-\varepsilon x^2}{4(1 - \varepsilon t)} \right) \right)
\]

We can obtain the vector field $\xi$ by using

\[ \xi(x) = \left. \frac{\partial X}{\partial \varepsilon} \right|_{\varepsilon=0}. \]
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For example

\[
\xi^1 = \frac{\partial X^1}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \left( \frac{x}{1 - \varepsilon t} \right) \bigg|_{\varepsilon=0} = \frac{xt}{(1 - \varepsilon t)^2} \bigg|_{\varepsilon=0} = xt.
\]

Similarly

\[
\xi^2 = t^2,
\]

\[
\xi^3 = -\left( \frac{x^2}{4} + \frac{t}{2} \right) u.
\]

Thus the infinitesimal generator \( X \) of the flow is given by

\[
X = xt \partial_x + t^2 \partial_t - \left( \frac{x^2}{4} + \frac{t}{2} \right) u \partial_u.
\]

2.1.4 Determining equations for Lie’s infinitesimals — how they arise

The requirement that \( x \mapsto x^\varepsilon(\varepsilon) \) is a Lie symmetry of a given differential equation can be efficiently calculated by solving determining equations (DQs) for the components \( \xi^i \) of the infinitesimal generator \( X \) corresponding to \( x^\varepsilon(\varepsilon) \). In the following examples we will outline the process for setting up these determining equations. We should emphasise that the following derivations are given merely for the purpose of illustrating the underlying ideas. In practice one usually uses one of the various computer algebra packages [22, 4, 19] that have been written which can automatically setup and even partially solve the determining equations.

Example 2.1.8 Determining equations for the linear heat equation

We wish to set up the determining equations for the Lie symmetries of the linear heat equation

\[
u_t = u_{xx},
\]

\[ (2.15) \]
Consider the one-parameter family of point transformations

\begin{align*}
  x^*(\varepsilon) &= X(x, t, u; \varepsilon), \\
  t^*(\varepsilon) &= T(x, t, u; \varepsilon), \\
  u^*(\varepsilon) &= U(x, t, u; \varepsilon),
\end{align*}

which are symmetries of the heat equation (2.15). Expanding the above transformation in a Taylor series about the identity transformation (given by \( \varepsilon = 0 \)) we obtain infinitesimals \( \xi, \tau, \eta \) of the transformation

\begin{align*}
  x^*(\varepsilon) &= x + \varepsilon \xi + O(\varepsilon^2), \\
  t^*(\varepsilon) &= t + \varepsilon \tau + O(\varepsilon^2), \\
  u^*(\varepsilon) &= u + \varepsilon \eta + O(\varepsilon^2),
\end{align*}

and the corresponding infinitesimal generator \( X = \xi \partial_x + \tau \partial_t + \eta \partial_u \) induced by the flow (2.16). Using theorem 4.2.3-7 [2] then without loss of generality we can consider operators \( X \) with \( \xi = \xi(x, t), \tau = \tau(x, t) \) and \( \eta(x, t, u) = f(x, t)u + g(x, t) \).

To apply the transformation (2.17) to the heat equation (2.15) we have to extend its action to the derivatives \( u_t \) and \( u_{xx} \). Using the chain rule leads to general formulae for the prolonged action\(^2\) and these are given in [2, 10]. Application of these formulae gives

\begin{align*}
  u^*_{t^*} &= u_t + \varepsilon \eta^{(t)} + O(\varepsilon^2), \\
  u^*_{x^*,x^*} &= u_{xx} + \varepsilon \eta^{(xx)} + O(\varepsilon^2),
\end{align*}

where \( \eta^{(t)} \) and \( \eta^{(xx)} \) are given by

\begin{align*}
  \eta^{(t)} &= g_t + f_t u - \xi_t u_x + (f - \tau_t) u_t, \\
  \eta^{(xx)} &= g_{xx} + f_{xx} u + (2 f_x - \xi_{xx}) u_x - \tau_{xx} u_t + (f - 2 \xi_x) u_{xx} - 2 \tau_x u_{xt}.
\end{align*}

\(^2\)It is an important result of Lie theory that a one-parameter family of point transformations acting on a space \((x, u)\) of independent and dependent variables can prolong to give a one-parameter family of point transformations acting on the extended space \((x, u, u_x, u_{xx}, \ldots)\), where \( u \) denotes the set of all \( j \)-th derivatives of all the components \( u^\sigma \) of \( u \), for \( k = 1, 2, \ldots \).
The condition that \( x \mapsto x^\ast(\varepsilon) \) is a symmetry of the heat equation (2.15) is that

\[
u_{x^\ast} = u_{x^\ast x^\ast},
\]

whenever

\[
u_t = u_{xx}.
\]

Substitution of (2.19) into (2.18), and (2.18) into (2.20), followed by elimination of \( u_{xx} \) using (2.15) yields

\[
u_t + \varepsilon \left\{ g_t + f_t u - \xi_t u_x + (f - \tau_t) u_{xx} \right\} = u_t + \varepsilon \left\{ g_{xx} + f_x u + (2 f_x - \xi_x x) u_x - \tau_{xx} u_t + (f - 2 \xi_x) u_{t} - 2 \tau_x u_{xt} \right\} + O(\varepsilon^2).
\]

Equation (2.21) must hold for all solutions of \( u_t = u_{xx} \) and thus \( u, u_x, u_t, u_{xt} \), are functionally independent quantities. Hence we can decompose the \( O(\varepsilon) \) term in (2.21) to obtain the following determining equations (DQs) for \( (\xi, \tau, f, g) \)

\[
\begin{align*}
\text{DQs} & \begin{cases} 
\tau_x = 0, \\
\tau_t - \tau_{xx} - 2 \xi_x = 0, \\
\xi_t - \xi_{xx} + 2 f_x = 0, \\
f_t - f_{xx} = 0, \\
g_t - g_{xx} = 0.
\end{cases}
\end{align*}
\]

Notice that the DQs (2.22) are a linear, homogeneous system of PDEs. They characterise the infinitesimal generator \( \mathbf{X} = \xi(x,t) \partial_x + \tau(x,t) \partial_t + (f(x,t)u + g(x,t)) \partial_u \) corresponding to a one-parameter symmetry group admitted by the linear heat equation (2.15). For example, we can verify that \( \mathbf{X} = \partial_x \) yields a symmetry of (2.15) since \( \xi = 1, \tau = f = g = 0 \) satisfies the DQs (2.22).
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Example 2.1.9 Determining equations for the Laplace equation in 3-D.

Suppose that
\[ x^*(\varepsilon) = X(x, y, z, u; \varepsilon) = x + \varepsilon \xi^1 + O(\varepsilon^2) \]
\[ y^*(\varepsilon) = Y(x, y, z, u; \varepsilon) = y + \varepsilon \xi^2 + O(\varepsilon^2) \]
\[ z^*(\varepsilon) = Z(x, y, z, u; \varepsilon) = z + \varepsilon \xi^3 + O(\varepsilon^2) \]
\[ u^*(\varepsilon) = U(x, y, z, u; \varepsilon) = u + \varepsilon \eta + O(\varepsilon^2) , \]

is a one-parameter point symmetry group of Laplace's equation
\[ u_{xx} + u_{yy} + u_{zz} = 0, \tag{2.24} \]

with infinitesimal symmetry generator \( X = \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_z + \eta \partial_u \). Theorem 4.2.3-7 [2] shows there is no loss in considering symmetry generators with \( \xi^1, \xi^2, \xi^3 \) depending on only \( (x, y, z) \) and with \( \eta = f(x, y, z)u + g(x, y, z) \).

The transformation (2.23) induces
\[ u^*_{x^*} (\varepsilon) = u_{xx} + \varepsilon \eta^{(xx)} + O(\varepsilon^2) \]
\[ u^*_{y^*} (\varepsilon) = u_{yy} + \varepsilon \eta^{(yy)} + O(\varepsilon^2) \]
\[ u^*_{z^*} (\varepsilon) = u_{zz} + \varepsilon \eta^{(zz)} + O(\varepsilon^2) \], \hspace{1cm} \tag{2.25}

with \( \eta^{(xx)}, \eta^{(yy)}, \eta^{(zz)} \) given by complicated expressions similar to (2.19) (see [2] for details).

The condition that \( x^*(\varepsilon) \) is a symmetry of (2.24) is
\[ u^*_{y^*} + u^*_{y^*} + u^*_{z^*} = 0 \]
\[ u_{xx} + u_{yy} + u_{zz} = 0. \tag{2.26} \]

whenever
\[ u_{xx} + u_{yy} + u_{zz} = 0. \]

Substituting (2.25) into (2.26) and eliminating \( u_{xx} \) (say) by (2.24) we obtain equations which are polynomial in \( u, u_{xx}, u_{yy}, u_{zz}, \ldots \). These must be identically satisfied for all
values of \( u, u_{xx}, u_{yy}, u_{zz}, \ldots \). Hence we can decompose them into the following determining equations

\[
\begin{align*}
\xi_y^3 + \xi_z^2 &= 0, \quad \xi_x^3 + \xi_z^1 = 0, \quad \xi_x^2 + \xi_y^1 = 0, \\
\xi_y^2 - \xi_x^1 &= 0, \quad \xi_z^3 - \xi_x^1 = 0, \\
\xi_x^1 + \xi_y^1 + \xi_z^1 - 2f_x &= 0, \\
\xi_{xx}^2 + \xi_{yy}^2 + \xi_{zz}^2 - 2f_y &= 0, \\
\xi_{xx}^3 + \xi_{yy}^3 + \xi_{zz}^3 - 2f_z &= 0, \\
f_{xx} + f_{yy} + f_{zz} &= 0, \\
g_{xx} + g_{yy} + g_{zz} &= 0.
\end{align*}
\] (2.27)

Again, these DQs are a linear, homogeneous system of PDEs. They characterize the symmetry generator \( X = \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_z + (fu + g) \partial_u \) of (2.24).

**Example 2.1.10 Determining equations for the nonlinear heat equation.**

Consider the nonlinear heat equation

\[ u_t = (D(u)u_x)_x. \] (2.28)

where \( D(u) \) is an arbitrary function. The condition that \( \mathbf{x} \mapsto \mathbf{x}^*(\mathbf{z}) \), where \( \mathbf{x} = (x, t, u) \), is a symmetry of the nonlinear heat equation (2.28) is that

\[ u^*_{t} = (D(u^*)u^*_{x})_{x}. \] (2.29)

whenever

\[ u_t = (D(u)u_x)_x. \]
An analogous process to Examples (2.1.8), (2.1.9) leads to

\[
\begin{align*}
\xi_u = 0, \quad \tau_u = 0, \quad \tau_x = 0, \\
\xi_t + 2D'(u)\eta_x + D(u)(2\eta_{xx} - \xi_{xx}) = 0, \\
D(u)(\tau_t - 2\xi_x) + D'(u)\eta = 0, \\
D(u)\eta_{uu} + D'(u)(\tau_t - 2\xi_x + \eta_u) + D''(u)\eta = 0, \\
D(u)\eta_{xx} - \eta_t = 0,
\end{align*}
\]

for the infinitesimals \(\xi(x), \tau(x), \eta(x)\) (where \(x = (x,t,u)\)) of the point symmetries of (2.28) Any solution \((\xi, \tau, \eta)\) of the DQs (2.30) yields an infinitesimal generator \(X = \xi \partial_x + \tau \partial_t + \eta \partial_u\) whose corresponding \(x^*(\varepsilon) = e^\varepsilon X(x)\) is a point symmetry of (2.28).

### 2.2 The structure and commutation relations of a Lie symmetry algebra

Since we are interested in using known symmetries of a given system of differential equations to give us information about the remaining unknown symmetries, it is natural to investigate how symmetry transformations interact or interfere with each other. A measure of such interaction is the commutator or Lie bracket, \([\ , \ ]\), which we shall describe in this section. Given two group transformations \(g, h\) the degree to which \(g^{-1} \circ h^{-1} \circ g \circ h\) differs from the identity transformation is a measure of how much they interfere with each other. If \(g\) and \(h\) have infinitesimal generators \(X\) and \(Y\), respectively, then their commutator \(g^{-1} \circ h^{-1} \circ g \circ h\) has infinitesimal generator \(XY - YX\) (for details see [10]). We denote this expression by \([X, Y]\), called the commutator of \(X\) and \(Y\). Thus

\[
[X, Y] := XY - YX.
\]

In component form \([X, Y] = \zeta^i \partial_{x^i}\) is equivalent to

\[
\zeta^i = \xi^i \partial_{x^i} + \eta^j \partial_{x^j} \xi^i
\]

where \(X = \xi^i \partial_{x^i}\) and \(Y = \eta^j \partial_{x^j}\).
Example 2.2.1 Consider $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x \partial_x + y \partial_y$, and $X_4 = y \partial_x - x \partial_y$, representing translations in $x$ and $y$, a scaling, and a rotation respectively (see Example 2.1.5). Then $[X_1, X_2] = 0$ indicates that the transformations of translation in the $x$ and $y$ directions can be composed in any order with the same effect. Similarly, $[X_3, X_4] = 0$ shows that the order of composition of scaling and rotation is irrelevant. In contrast, $[X_1, X_3] = [\partial_x, x \partial_x + y \partial_y] = \partial_x \neq 0$ indicates that translation and scaling interfere with each other (i.e. the order of composition of the transformations is important).

The commutator bracket gives rise to a rich structure called a Lie algebra, defined below.

Definition 2.2.2 A Lie algebra $\mathcal{L}$ is a vector space (over a field $\mathbb{F}$) with an additional binary operator, the commutator bracket, $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, which is

- Skew symmetric
  \[ [X, Y] = -[Y, X], \quad \forall X, Y \in \mathcal{L} \]

- Bilinear
  \[ [\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z], \quad \forall \lambda, \mu \in \mathbb{F}, \]
  (thus $[X, \lambda Y + \mu Z] = \lambda[X, Y] + \mu[X, Z]$).

and satisfies the

- Jacobi identity
  \[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad \forall X, Y, Z \in \mathcal{L}. \]

Example 2.2.3 Well known examples of Lie algebras are:

(a) $\mathbb{R}^3$ over the field of real numbers, with the commutator given by the vector product:
\[ [x, y] := x \wedge y \text{, for } x, y \in \mathbb{R}^3. \]

(b) The set of \( n \times n \) matrices over any field, with the commutator of two matrices \( A \) and \( B \) given by \([A, B] := AB - BA\).

(c) The set of first order differential operators over the field of real or complex numbers, with the commutator of two operators \( X \) and \( Y \) given by \([X, Y] := XY - YX\).

For our purposes, the most important example of a Lie algebra is the set of infinitesimal generators of the symmetries of a differential equation. There is a huge literature on the theory of abstract Lie algebras which can be brought to bear on symmetry problems. To analyse the structure of a Lie algebra we must first have some way of describing the algebra. In the next section we show how this can be encoded by a set of equations called commutation relations.

In this thesis we confine ourselves to Lie algebras of finite dimension (viewing \( \mathcal{L} \) as a vector space). Crucial to our methods will be the commutation relations and structure constants of such algebras.

**Definition 2.2.4** Let \( \mathcal{L} \) be a finite dimensional Lie algebra, with a basis \( \{L_1, \ldots, L_n\} \). Then for \( L_i, L_j \in \mathcal{L} \), \([L_i, L_j] \) must be expressible as a linear combination of \( L_1, \ldots, L_n \). In particular we define the **commutation relations**

\[ [L_i, L_j] = C^k_{ij} L_k \]

where the \( C^k_{ij} \in \mathbb{F} \) (\( i, j, k = 1 \ldots n \)) are called the **structure constants** of the algebra.

The commutation relations between basis elements are sufficient to completely characterise the commutator of any pair of elements of the Lie algebra. Notice that the structure constants depend on the choice of basis for \( \mathcal{L} \). In particular if \( \hat{C}^k_{ij} \) are structure constants with respect to the basis \( \{\hat{L}_1, \ldots, \hat{L}_n\} \) with change of basis matrix \( A \) (i.e. \( \hat{L}_i = A_i^k L_k \)) then \( \hat{C}^k_{ij} = A_i^l A_j^m (A^{-1})^l_p C^k_{mn} \). Conversely, given Lie algebras \( \mathcal{L}, \hat{\mathcal{L}} \) with structure constants...
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$C_{ij}^k$, $\hat{C}_{ij}^k$ respectively, related by $\hat{C}_{ij}^k = A_i^m A_j^l (A^{-1})_m^n C_{lm}^n$ for some matrix $A$ then $\mathcal{L}$ and $\hat{\mathcal{L}}$ are isomorphic (i.e. there exists a bijective linear map $\rho: \mathcal{L} \to \hat{\mathcal{L}}$ which preserves the commutator operator of each algebra: $\rho[X, Y]_\mathcal{L} = [\rho X, \rho Y]_{\hat{\mathcal{L}}}, \forall X, Y \in \mathcal{L}$).

**Example 2.2.5** The operators $\{X_1, \ldots, X_6\}$ given below form a Lie algebra

\[
\begin{align*}
X_1 &= \partial_x, \\
X_2 &= \partial_t, \\
X_3 &= x \partial_x + 2t \partial_t, \\
X_4 &= u \partial_u, \\
X_5 &= t \partial_x - \frac{1}{2} xu \partial_u, \\
X_6 &= xt \partial_x + t^2 \partial_t - \left(\frac{x^2}{4} + \frac{t}{2}\right) u \partial_u.
\end{align*}
\]

Any commutator $[X_i, X_j]$ is expressible as a linear combination of $\{X_1, \ldots, X_6\}$. For example,

\[
[X_1, X_6] = \left[ \partial_x, xt \partial_x + t^2 \partial_t - \left(\frac{x^2}{4} + \frac{t}{2}\right) u \partial_u \right] = t \partial_x - \frac{1}{2} xu \partial_u = X_5.
\]

Similarly, $[X_2, X_6] = x \partial_x + 2t \partial_t - \frac{1}{2} xu \partial_u = X_3 - \frac{1}{2} X_4$ so that $C_{26}^3 = 1, C_{26}^4 = -\frac{1}{2}$ and $C_{26}^1 = C_{26}^2 = C_{26}^5 = C_{26}^6 = 0$.

The commutator $[X_i, X_j]$ can be calculated for each pair $X_i, X_j$ and the results are conveniently displayed in a commutator table, see Table 2.1, page 27. We shall see later that $\{X_1, \ldots, X_6\}$ is a basis for the Lie algebra represented by the determining equations (2.33).

### 2.3 The Reid algorithms — obtaining structural information from determining equations without solving them

In Example 2.2.5 above we calculated commutation relations by using the explicit form of the operators. In this section we will demonstrate that if the determining equations
representing a Lie symmetry algebra of a system of PDEs are known then we do not need the explicit form of the operators in order to find the structure constants of the algebra. In particular the constants can be calculated by "manipulating" the determining equations without solving them. These manipulations are carried out by the algorithms Standard Form, Initial Data and Commutation Relations. We will give a brief description of each algorithm. A more detailed account will follow later.

Standard Form [16] is an algorithm which reduces any system of linear PDEs to a simplified canonical linear system, whose properties will be exploited in our method. The Standard Form algorithm does not depend on the heuristic process of integration. Execution of the algorithm only involves a finite number of differentiations and linear eliminations. For a system in a Standard Form the algorithm Initial Data [16] produces a list of initial data which is in one to one correspondence with the solutions of the system.

Algorithm Commutation Relations [18] is used to find the commutation relations for a Lie algebra of symmetry operators given the determining equations. Commutation Relations exploits the one to one correspondence mentioned above by calculating commutators in terms of initial data.

We now give a more detailed introduction to the Reid algorithms.

Table 2.1: Commutator table for the symmetry operators \([X_1, \ldots, X_6]\) of Example 2.2.5.
2.3.1 Using algorithm Standard Form to reduce infinitesimal determining systems to a canonical form

An outline of a simplified Standard Form\(^3\) algorithm is given in Appendix A. We will exploit certain properties of the Standard Form of systems of determining equations. A detailed understanding of this algorithm will not be required for understanding the commutation reduction method.

Given an input system of a finite number of linear PDEs with dependent variables \( u = (u^1, \ldots, u^m) \) and independent variables \( x = (x^1, \ldots, x^n) \) the Standard Form algorithm gives an output system of a finite number \((k\ \text{say})\) of linear PDEs in the same variables and with the following properties

PROPERTY 1 The input and output systems are equivalent, in the sense that they have the same set of solutions.

PROPERTY 2 Each equation in the output system is in solved form for its highest (or leading) derivative term with respect to an admissible ordering on derivatives (see Appendix A). By convention we will place the leading derivatives on the LHS (left hand side) of such systems, so that they have the form

\[
\begin{align*}
\text{leading derivative}_1 & = F_1 \\
\text{leading derivative}_2 & = F_1 \\
\vdots & \quad \vdots \\
\text{leading derivative}_k & = F_k
\end{align*}
\]

where \( F_1, \ldots, F_k \) are explicitly given functions of \( x, u \), derivatives of \( u \), and do not depend on any leading derivatives.

\(^3\)Throughout this thesis the words Standard Form will be used in reference to the algorithm Standard Form described in [16] or to the output of that algorithm.
PROPERTY 3 No derivatives appearing in $F_1, \ldots, F_k$ are derivatives of leading derivatives (we will regard the dependent variables as zeroth order derivatives).

PROPERTY 4 The integrability conditions of the system are identically satisfied modulo the system\(^4\).

Example 2.3.1 Standard Form of the determining equations for the linear heat equation. Putting $g \equiv 0$ in the system of determining equations (2.22) for the linear heat equation and then reducing these to a Standard Form gives:

$$\begin{align*}
\text{SFDQs} &= \left\{ \begin{array}{ll}
  i. & \tau_x = 0, \\
  ii. & \tau_t = 2\xi_x, \\
  iii. & f_x = -\frac{1}{2}\xi_t, \\
  iv. & f_t = -\frac{1}{2}\xi_{xt}, \\
  v. & \xi_{xx} = 0, \\
  vi. & \xi_{tt} = 0.
\end{array} \right.
\end{align*}$$

(SFDQs is an abbreviation for Standard Form Determining Equations.) The leading derivatives are $\tau_x, \tau_t, f_x, f_t, \xi_{xx}, \xi_{tt}$ and $F_1 = 0$, $F_2 = 2\xi_x$, $F_3 = -\frac{1}{2}\xi_t$, $F_4 = -\frac{1}{2}\xi_{xt}$, $F_5 = F_6 = 0$. Note that none of the derivatives on the RHS $(\xi_x, \xi_t, \xi_{xt})$ are derivatives of any leading derivative. Note also that the integrability conditions of (2.33), are satisfied, hence property 4 holds. For example $\tau_{xt} - \tau_{tx} = \partial_x \tau_x - \partial_x \tau_t = \partial_t 0 - \partial_x 2\xi_x = -2\xi_{xx} = 0$ etc.

Although equation (2.33)iv seems to violate property 2 this is not the case: see Appendix A for details of the ordering on derivatives used here.

\(^4\)See Appendix A for definition of integrability conditions.
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Reduction modulo a Standard Form

One of the most important uses of the standard form of a system of PDEs is to simplify functions of the variables and derivatives of the system. By using the standard form of such a system as a substitution list we can eliminate from such functions any derivatives which are derivatives of the leading derivatives of the standard form. A property of this reduction is that it is canonical, i.e. the reduced expression is unique. Uniqueness of the reduction is not guaranteed if we don’t have a standard form. For example, using the system \( \{ \xi_x = \eta, \xi_t = \eta_x \} \) then \( \xi_{xt} \) can simplify to \( \eta_t \) or \( \eta_{xx} \) depending on which equation is used. In a standard form, for example, using the SFDQs (2.33) above, \( \tau_{st} \) reduces to 0 (from i or ii,iii), \( f_{xx} \) reduces to \(-\frac{1}{2}\xi_{st}\) (from iii), \( \xi_{xt} \) and \( f \) do not reduce any further, and \( f_{xt} \) reduces to 0 (from iii,vi or iv,v).

2.3.2 Algorithmic determination of initial data and Taylor series solutions for systems in standard form

In the previous section we saw that given a standard form of a system of differential equations there are some derivatives which are not reduced by the standard form. We call such derivatives parametric. Equivalently they can be defined as follows

Definition 2.3.2 Given a system of PDEs in \( m \) dependent variables \( u^1, \ldots, u^m \) and \( n \) independent variables \( x^1, \ldots, x^n \) in a Standard Form, a parametric derivative is a derivative \( D^\alpha u^\sigma \), which is not the derivative (possibly zeroth order) of any leading derivative of the standard form.

We use the abbreviated notation for derivatives

\[
(\partial_{x^1})^{n_1}(\partial_{x^2})^{n_2} \cdots (\partial_{x^n})^{n_n} u^\sigma := D^\alpha u^\sigma
\]
where $\sigma = 1 \ldots m$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{W}^n$, $\mathbf{W} \equiv \{0, 1, 2, \ldots\}$. This enables us to visually represent derivatives $D^\mathbf{a} u^\sigma$ by their corresponding vectors $\mathbf{a} \in \mathbf{W}^n$ (see Figure 2.3.2).

\[ \frac{\partial^5 u}{\partial x \partial t^4} := D^{(1,4)} u \]

\[ \frac{\partial^3 u}{\partial x^2 \partial t} := D^{(2,1)} u \]

**Figure 2.3.2** Pictorial representation of derivatives of $u(x, t)$.

Thus we can find the parametric derivatives of a system in a Standard Form by deleting from the set $\{D^\mathbf{a} u^\sigma: \mathbf{a} \in \mathbf{W}^n\}$ all those elements which are derivatives of the leading derivatives of the system. The remaining terms in the set will be the parametric derivatives of the system.

We can visualise the parametric derivatives of system (2.33) using the graphical representation of derivatives given above.
Figure 2.3.2a The leading derivatives $\xi_{xx}$ and $\xi_{tt}$.

For example, the $\xi$ leading derivatives are $\{\xi_{xx}, \xi_{tt}\}$ as shown in Figure 2.3.2a. The set of all derivatives of these leading derivatives is thus shown in Figure 2.3.2b. Hence the $\xi$ parametric derivatives are the complement (in $W^2$) of these derivatives, namely $\xi, \xi_x, \xi_t, \xi_{xt}$ as shown in Figure 2.3.2c. In a similar fashion we find that the remaining parametric derivatives are $\tau$ and $f$.

Having found all the parametric derivatives, we know that we can reduce any expression modulo the SFDQs (2.33) and obtain an expression only involving the terms $\{\tau, f, \xi, \xi_x, \xi_t, \xi_{xt}\}$ and the independent variables $x$ and $t$.

A solution $\xi(x) = (\xi(x,t), \tau(x,t), f(x,t))$ of system (2.33) which is analytic about
some point $x_0 = (x_0, t_0)$, is uniquely specified by its Taylor series

$$
\xi(x) = \sum_{i+j=0}^{\infty} \frac{(x - x_0)^i}{i!} \frac{(t - t_0)^j}{j!} \partial_x^i \partial_t^j \xi \bigg|_{x=x_0}.
$$

Every derivative term in the expansion (2.34) can be reduced modulo the SFDQs (2.33) to an expression only involving the parametric derivatives. Thus, by specifying the values of the parametric derivatives at $x = x_0$, the values of all the derivatives of $\xi$ at $x_0$ are determined, and $\xi(x)$ is uniquely specified by (2.34). For the system (2.33) the parametric initial conditions are

$$
\tau(x_0) = a_1, \quad f(x_0) = a_2, \quad \xi(x_0) = a_3,
$$

$$
\xi_x(x_0) = a_4, \quad \xi_t(x_0) = a_5, \quad \xi_{xt}(x_0) = a_6,
$$

where $a_1, \ldots, a_6$ are arbitrary constants.

The initial conditions (2.36) are in one to one correspondence with the arbitrary constants in the general solution of the SFDQ system (2.33). Thus system (2.33) has six arbitrary parameters in its solution space. Consequently the Lie symmetry algebra represented by the system (2.33) is six dimensional. We will later exploit the one to one correspondence between solutions and their initial data to label the symmetry operators by their initial data. Thus we will label the operator corresponding to the unique solution of system (2.33) with initial data (2.36) by $L_{(a_1, \ldots, a_6)}$. And, for example, $\partial_x$ will be denoted by $L_{(0,0,1,0,0,0)}$ and $\partial_t$ by $L_{(1,0,0,0,0,0)}$ if we choose $x_0 = (0,0)$.

2.3.3 The vector space of initial data for determining systems

We mentioned in the previous section that the full set of parametric initial conditions for a given system of SFDQs is in one to one correspondence with local analytic (Taylor series) solutions of that system. We shall represent this correspondence with a bijective map called the initial data map.
Definition 2.3.3 (initial data map, ID)

Consider a system of SFDQs \(\Omega\) with dependent variables \(\xi = (\xi^1, \ldots, \xi^n)\), independent variables \(x = (x^1, \ldots, x^n)\) and with solution space \(S\) of finite dimension \(d\). Let \(P^1(\xi)(x), P^2(\xi)(x), \ldots, P^d(\xi)(x)\), be the \(d\) parametric derivatives evaluated at a point \(x\). Further, let the values of the parametric derivatives at a particular point \(x = x_0 \in \mathbb{R}^n\) be \(a_1, \ldots, a_d\), i.e.

\[
P^i(\xi)(x_0) = a_i, \quad i = 1, \ldots, d.
\]

The initial data map \(ID: S \to \mathbb{R}^d\) which maps solutions \(\xi(x)\) of \(\Omega\) to their corresponding initial data \((a_1, \ldots, a_d)\) at the point \(x_0\) is defined by

\[
ID(\xi) := (P^1(\xi)(x_0), P^2(\xi)(x_0), \ldots, P^d(\xi)(x_0)).
\]

Notes:

1. We shall abuse notation slightly and identify solutions \(\xi\) with their corresponding infinitesimal symmetry generators \(X = \xi^i \partial_{x^i}\). Thus we write \(ID(X) = (a_1, \ldots, a_d)\). If the initial data of a given operator \(X\) is known to be \((a_1, \ldots, a_d)\) then we shall denote \(X\) by \(L_{(a_1, \ldots, a_d)}\). Hence \(X = L_{ID(X)}\).

2. Notice that \(ID: S \to \mathbb{R}^d\) is a linear map. Thus the initial data space, \(I := ID(S)\), is a vector space. The linearity of the initial data map will be of use in later calculations. Furthermore the bijection \(ID\) induces a Lie algebra structure on the space \(I\) which is isomorphic to the Lie algebra on the solution space \(S\).

For systems with more than 3 independent variables or for infinite dimensional systems it is no longer easy to determine or enumerate parametric initial conditions by hand so we shall use Reid’s algorithm Initial Data [16, 17] which generates initial data sets (such as (2.36)). We shall use algorithm Initial Data as a ‘black box’: given an input system of SFDQs (with solution space of finite dimension \(d\)) algorithm Initial Data will output a set
Chapter 2. Background material for the commutator reduction method

of parametric initial conditions sufficient to uniquely determine a Taylor series solution
of the system.

Example 2.3.4 Standard form and initial data of the DQs of the 3-D Laplace
equation. Putting \( g \equiv 0 \) into the DQs (2.27) for symmetry operators \( \xi_1 \partial_x + \xi_2 \partial_y + \xi_3 \partial_z + (fu + g) \partial_u \) of the Laplace equation and then reducing using Reid's computer
implementation (in the symbolic language MAPLE) of the Standard Form algorithm we
obtain the SFDQs (2.36) below.

\[
\begin{aligned}
& f_{xx} = 0, \quad f_{xy} = 0, \quad f_{xz} = 0, \quad \xi_{yy}^1 = 2f_x, \quad f_{yy} = 0, \quad \xi_{yz}^1 = 0, \\
& f_{yz} = 0, \quad \xi_{zz}^1 = 2f_x, \quad \xi_{zz}^2 = 2f_y, \quad \xi_{zz}^3 = -2f_z, \quad f_{zz} = 0, \\
& \xi_x^1 = \xi_z^3, \quad \xi_x^2 = -\xi_y^1, \quad \xi_x^3 = -\xi_z^1, \quad \xi_y^2 = \xi_z^3, \quad \xi_y^3 = -\xi_z^2.
\end{aligned}
\]  

(2.36)

The leading derivatives are \( \{f_{xx}, f_{xy}, f_{xz}, \xi_{yy}^1, \ldots, \xi_y^3\} \). Using algorithm Initial Data we
find that the following parametric initial conditions are sufficient to uniquely determine
a Taylor series solution of the SFDQs (2.36)

\[
\begin{aligned}
\xi_1(x_0) &= a_1, \quad \xi_2(x_0) = a_2, \quad \xi_3(x_0) = a_3, \quad \xi_4(x_0) = a_4, \quad \xi_5(x_0) = a_5, \quad \xi_6(x_0) = a_6, \\
\xi_7(x_0) &= a_7, \quad f(x_0) = a_8, \quad f_z(x_0) = a_9, \quad f_y(x_0) = a_{10}, \quad f_x(x_0) = a_{11}.
\end{aligned}
\]  

(2.37)

for some initial point \( x_0 = (x_0, y_0, z_0) \) Notice that, as expected, none of the parametric
derivatives are derivatives of any of the leading derivatives of system (2.36). Thus the
system of SFDQs (2.36) has 11 arbitrary constants in its general solution and we can
unambiguously denote by \( L(a_1, \ldots, a_{11}) \) the unique symmetry operator corresponding to the
solution with initial data (2.37) above. If, for example, we choose \( x_0 = (x_0, y_0, z_0) = \\
(0, 0, 0) \) then \( \partial_x = L_{(1,0,0,0,0,0,0,0,0,0,0,0)} \) and \( \partial_y = L_{(0,0,1,0,0,0,0,0,0,0,0,0)} \).
2.3.4 Algorithm Commutation Relations for finding the structure of determining equations of Lie symmetry algebras of DEs

Given a system of SFDQs we can use algorithm Initial Data to produce a set of parametric initial conditions sufficient to uniquely determine its Taylor series solution. Solutions of the SFDQs are in one to one correspondence with their initial data. If the system is of finite dimension $d$ then we can denote by $L_{(a_1,\ldots,a_d)}$ any symmetry operator of the Lie algebra $\mathcal{L}$ represented by the SFDQs. Suppose $L_a$ and $L_{\dot{a}}$ are two symmetry generators of the SFDQs uniquely specified by their initial data $a = (a_1, a_2, \ldots, a_d)$ and $\dot{a} = (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_d)$ Algorithm Commutation Relations generates the commutation relations (and structure constants) of $\mathcal{L}$ by returning a function $J(a, \dot{a})$ such that

$$L_{J(a, \dot{a})} = [L_a, L_{\dot{a}}]$$

(2.38)

for any choice of $a$ and $\dot{a}$.

Note: Executing the Commutation Relations algorithm only involves the operations of differentiation, linear elimination and evaluation at a point. The algorithm does not require solution of the system of SFDQs.

Outline of a simplified Commutation Relations algorithm

Input: A system of standard form determining equations (SFDQs) with solution space of finite dimension $d$. An initial point $x_0 \in \mathbb{R}^n$

Output: The commutation relations, in the neighbourhood of $x$, corresponding to the SFDQs, in the form (2.38).

Method: We produce the commutation relations corresponding to the input SFDQs by obtaining an explicit expression for the function $J(a, \dot{a})$ in (2.38). Writing $L_J = \zeta_i \partial_{x^i}$, $L_a = \xi_i \partial_{x^i}$, and $L_{\dot{a}} = \dot{\xi}_i \partial_{x^i}$ then from (2.38) we have

$$[\xi^i \partial_{x^i}, \dot{\xi}^i \partial_{x^i}] = \zeta^i \partial_{x^i}.$$

(2.39)
Chapter 2. Background material for the commutator reduction method

Expanding the commutator and equating components of \( \partial_x \), we obtain

\[
\zeta^i = \xi^j \partial_{x^j} \xi^i - \dot{\xi}^j \partial_{x^j} \xi^i. \tag{2.40}
\]

Denote by \( P^1(\xi)(x), P^2(\xi)(x), \ldots, P^d(\xi)(x) \), the parametric derivatives, evaluated at a point \( x \), for the solution \( \xi \) of the SFDQs. Evaluating these parametric derivatives at \( x = x_0 \) gives \( J = ID(\xi) = (P^1(\xi)(x_0), P^2(\xi)(x_0), \ldots, P^d(\xi)(x_0)) \). To find the components of \( J \) we must calculate the parametric derivatives of \( \zeta \) using (2.40), and evaluate them at the initial point \( x_0 \). By differentiating (2.40) appropriately we can obtain expressions for \( P^i(\xi), i = 1, \ldots, d \) in terms of derivatives of \( \xi \) and \( \dot{\xi} \). Simplifying these expressions modulo the SFDQs we are left with equations for \( P^i(\xi), i = 1, \ldots, d \) which only involve the parametric derivatives \( P^1(\xi)(x), P^2(\xi)(x), \ldots, P^d(\xi)(x), P^1(\dot{\xi})(x), P^2(\dot{\xi})(x), \ldots, P^d(\dot{\xi})(x) \). Evaluating these equations at the initial point \( x = x_0 \) and using \( J = ID(\xi), a = ID(\xi), \dot{a} = ID(\dot{\xi}) \) yields expressions for \( J^i, i = 1, \ldots, d \) in terms of \( \alpha_1, \ldots, \alpha_d \) as required.

Example 2.3.5 The linear heat equation. Let \( L_\alpha = \xi \partial_x + \tau \partial_t + f u \partial_u, L_{\dot{\alpha}} = \dot{\xi} \partial_x + \dot{\tau} \partial_t + \dot{f} u \partial_u \) be two symmetry generators of the linear heat equation, with initial data \( \alpha \) and \( \dot{\alpha} \) respectively. We wish to calculate \( J \) as a function of \( \alpha, \dot{\alpha} \) so that \( L_J(\alpha, \dot{\alpha}) = [L_\alpha, L_{\dot{\alpha}}] \). Let \( L_J = \Xi \partial_x + T \partial_t + F u \partial_u \), where \( \{\xi, \tau, f, \xi, \dot{\xi}, \dot{\tau}, \dot{f}, \Xi, T, F\} \) are functions of \((x, t)\). From (2.38) taking commutators of \( L_\alpha \) and \( L_{\dot{\alpha}} \) yields

\[
\Xi \partial_x + T \partial_t + F u \partial_u = [\xi \partial_x + \tau \partial_t + f u \partial_u, \dot{\xi} \partial_x + \dot{\tau} \partial_t + \dot{f} u \partial_u]
= (\xi \xi_x + \dot{\xi} \xi_x + \tau \dot{\xi}_t - \dot{\tau} \xi_t) \partial_x
+ (\xi \dot{\tau}_x - \dot{\xi} \tau_x + \tau \dot{\tau}_t - \dot{\tau} \tau_t) \partial_t
+ (\xi \dot{f}_x - \dot{\xi} f_x + \tau \dot{f}_t - \dot{\tau} f_t) u \partial_u. \tag{2.41}
\]

We need to calculate the parametric derivatives \( \{T, F, \Xi, \xi, \dot{\xi}, \dot{\tau}, \dot{f}, \xi_t, \Xi_t, \Xi_x, \xi_x, \dot{\xi}_x, \dot{\tau}_x, \dot{f}_x, \xi_t, \Xi_t, \Xi_x, \xi_x, \dot{\xi}_x, \dot{\tau}_x, \dot{f}_x, \} \) and then simplify these derivatives subject to the SFDQs (2.33) (see page 29). Thus, after some calculation,
we obtain,

\[ T = 2(\dot{\xi}_x - \dot{\tau}_x), \]
\[ F = -\frac{1}{2}(\ddot{\xi}_l - \ddot{\xi}_x + \tau \ddot{\xi}_t - \dot{\tau}_t), \]
\[ \Xi = \dot{\xi}_x - \dot{\xi}_x + \tau \dot{\xi}_l - \dot{\tau}_l, \]  \hspace{1cm} (2.42)
\[ \Xi_x = \tau \ddot{\xi}_t - \ddot{\tau}_t, \]
\[ \Xi_t = \xi_x \dot{\xi}_l - \dot{\xi}_x \xi_l + \xi \ddot{\xi}_x - \ddot{\xi}_x, \]
\[ \Xi_{xt} = 2(\xi_x \ddot{\xi}_x - \ddot{\xi}_x). \]

These equations express the \( \xi \) parametric derivatives \( \{T, F, \Xi, \Xi_x, \Xi_t, \Xi_{xt}\} \) in terms of the \( \xi \) and \( \dot{\xi} \) parametric derivatives \( \{\tau, f, \xi, \xi_x, \xi_l, \xi_{xt}, \dot{\tau}, \dot{f}, \dot{\xi}, \dot{\xi}_l, \dot{\xi}_{xt}, \dot{\tau}_t\} \). We evaluate the above equations at \( x = x_0 \) and use \( T(x_0) = J_1, F(x_0) = J_2, \ldots, \tau(x_0) = a_1, f(x_0) = a_2, \ldots, \dot{\tau}(x_0) = \dot{a}_1, \dot{f}(x_0) = \dot{a}_2, \ldots, \) etc to obtain

\[ J_1 = 2(a_1 \dot{a}_4 - \dot{a}_1 a_4), \]
\[ J_2 = -\frac{1}{2}(a_3 \dot{a}_5 - \dot{a}_3 a_5 + a_1 \dot{a}_6 - \dot{a}_1 a_6), \]
\[ J_3 = a_3 \dot{a}_4 - \dot{a}_3 a_4 + a_1 \dot{a}_5 - \dot{a}_1 a_5, \]
\[ J_4 = a_1 \dot{a}_6 - \dot{a}_1 a_6, \]
\[ J_5 = a_4 \dot{a}_5 - \dot{a}_4 a_5 + a_3 \dot{a}_6 - \dot{a}_3 a_6, \]
\[ J_6 = 2(a_4 \dot{a}_6 - \dot{a}_4 a_6). \]  \hspace{1cm} (2.43)

Notice that we did not need to solve the SFDQs (2.33) to obtain (2.43).

In this case we can verify the commutation relations encoded in (2.43) since we already know the full set of solutions to SFDQs (2.33), namely the operators \( \{X_1, \ldots, X_6\} \) given in Example 2.2.5, page 26. If we choose \( x_0 = (x_0, t_0) = (0, 0) \) then we get the following
correspondences,

\[ X_1 = L(0,0,1,0,0,0), \]
\[ X_2 = L(1,0,0,0,0,0), \]
\[ X_3 = L(0,0,0,1,0,0), \]
\[ X_4 = L(0,1,0,0,0,0), \]
\[ X_5 = L(0,0,0,0,1,0), \]
\[ X_6 = L(0,0,0,0,0,1). \] (2.44)

Now using (2.43) we can find the commutators of any two basis elements from the algebra. For example,

\[
\begin{align*}
[X_2, X_6] &= [L(1,0,0,0,0,0), L(0,0,0,0,0,1)], \\
&= L \frac{\partial}{\partial x_1} \bigg|_{(a_1, \ldots, a_6) = (1,0,0,0,0,0), (\bar{a}_1, \ldots, \bar{a}_6) = (0,0,0,0,0,1)}, \\
&= L_{(0,-\frac{1}{2},0,0,0,0)}, \\
&= L_{(0,0,0,1,0,0)} - \frac{1}{2} L_{(0,1,0,0,0,0)}, \\
&= X_3 - \frac{1}{2} X_4.
\end{align*}
\] (2.45)

In this manner we can reconstruct the complete commutator Table 2.1 of Example 2.2.5, page 26.

**2.4 Chapter summary**

In this chapter we showed how the symmetries of a given set of differential equations can be represented as infinitesimal generators whose components satisfy a linear homogeneous system of partial differential equations called the infinitesimal determining equations. We showed how the solution space for a set of determining equations has the structure of a Lie algebra. Furthermore we outlined the Reid algorithms Standard Form, Initial Data and Commutation Relations which enable us to completely characterise the Lie algebra.
structure of a given symmetry problem through a finite number of linear eliminations and differentiations of the corresponding determining equations. No integration of the determining equations is required to obtain this information.

These algorithms will serve as vital ingredients of the main topic of this thesis — the commutator reduction method — which is introduced in the next chapter.
Chapter 3

The commutator reduction method — new symmetries from old

The commutator reduction method is a method for using known infinitesimal symmetries of a given system of differential equations to help find the remaining symmetries of the system. The aim of this section is to describe the steps of our commutator reduction method and give supporting examples. Many of the steps involved in our method have already been illustrated in Chapter 2. However the step of exploiting commutator relations as differential equations, which is at the heart of our method, has not yet been described. After describing this step in §3.1 we give an outline of the commutator reduction method and then illustrate the method by applying it to the linear heat equation. In this example we assume that four symmetries of the linear heat equation are known explicitly and we use these to simplify the problem of finding the remaining symmetries. A reduced system of determining equations is produced which is simply a system of linear algebraic equations for the unknown symmetries.

In §3.2 we give a concise statement of the commutator reduction method and then illustrate it by applying it to three well known test examples: the linear heat equation; the Laplace equation in 3D; and the nonlinear heat equation. In each case we find the reduced system of determining equations for the unknown symmetries to be linear algebraic equations for the remaining symmetries. In other words, the unknown symmetries are obtained as explicit functions of the known symmetries without recourse to integration.
Chapter 3. The commutator reduction method — new symmetries from old

3.1 Exploiting commutation relations as differential equations

In this section we introduce a new perspective which is crucial to our method. Instead of viewing commutation relations as being determined by symmetries we reverse the picture and view symmetries as being determined by commutation relations. In the following example we will show how a commutation relation can translate directly into a set of differential equations for the components of an unknown symmetry generator.

Example 3.1.1 From Table 2.1, Example 2.2.5 we see that \([X_2, X_5] = X_1\). Suppose that

\[ X_5 = \xi(x,t) \partial_x + \tau(x,t) \partial_t + f(x,t) u \partial_u \]

is unknown and that \(X_1 = \partial_x, X_2 = \partial_t\). Then the above commutation relation can be rewritten as

\[ [a, \xi(x,t) \partial_x + \tau(x,t) \partial_t + f(x,t) u \partial_u] = \partial_x \]

which (by expanding the commutator bracket and equating components of \(\partial_x, \partial_t\) and \(\partial_u\)) is equivalent to the differential equations

\[ \xi_t = 1, \quad \tau_t = 0, \quad f_t = 0. \]

Notice that these differential equations are specific to the unknown symmetry \(X_5\). For example, the infinitesimals of the symmetries \(\partial_x\) and \(\partial_t\) do not satisfy them. The essence of our commutation reduction method is that such differential equations give new information which should be exploited. In particular we could add these equations to the relevant system of DQs and the resulting system would be satisfied by \(X_5\) but not by either \(\partial_x\) or \(\partial_t\). Thus we would have succeeded in filtering out the symmetries \(\partial_x\) and \(\partial_t\) from the solution space of the DQs.

Definition 3.1.2 Useable commutation relations. In general, if we have explicitly known symmetries \(K\) and \(K\) and an unknown symmetry \(\hat{X} = \xi^1 \partial_x^1 + \cdots + \xi^n \partial_x^n\) then a commutation relation of the form

\[ [K, \hat{X}] = \hat{K} + \alpha \hat{X}, \]

(3.46)
(for some constant $\alpha$) is equivalent to $n$ first order linear PDEs for the components $\xi^1, \ldots, \xi^n$ of the unknown symmetry. We shall call relations of the form (3.46) **useable** commutation relations.

**Example 3.1.3** Consider again the operators $\{ X_1, \ldots, X_6 \}$ of Example 2.2.5. Suppose that $X_5$ and $X_6$ are unknown and that $\{ X_1, X_2, X_3, X_4 \}$ are explicitly known. Then $X_5$ has four useable commutation relations:

\[
\begin{align*}
\{ X_1, X_5 \} &= -\frac{1}{2} X_4, \\
\{ X_2, X_5 \} &= X_1, \\
\{ X_3, X_5 \} &= X_5, \\
\{ X_4, X_5 \} &= 0.
\end{align*}
\]

(3.47)

But $X_6$ has only three useable commutation relations:

\[
\begin{align*}
\{ X_2, X_6 \} &= X_3 - \frac{1}{2} X_4, \\
\{ X_3, X_6 \} &= 2X_6, \\
\{ X_4, X_6 \} &= 0.
\end{align*}
\]

(3.48)

Choosing $\mathbf{X} = X_5 = \xi \partial_x + \tau \partial_t + f u \partial_u$ then the commutation relations (3.47) are equivalent to the following DEs,

\[
\begin{align*}
\xi_x &= 0, & \tau_x &= 0, & f_x &= -\frac{1}{2}, \\
\xi_t &= 1, & \tau_t &= 0, & f_t &= 0, \\
x\xi_x + 2\tau \xi_t - \xi &= \xi, & x\tau_x + 2\tau \tau_t - 2\tau &= \tau, & xf_x u + 2tf_t u &= fu.
\end{align*}
\]

(3.49)

These differential equations can be added to the relevant determining equations and will yield a reduced system of determining equations which is satisfied by $\mathbf{X}$. We shall carry this process out later in this chapter.
The power of the above process is that the resulting reduced determining equations are specific to the unknown symmetry \( \hat{X} \) and are not (normally) satisfied by all the symmetries of the algebra. Therefore some of the symmetries have been filtered out by this process. In all of our examples in this thesis we shall see that everything except the unknown symmetry is filtered out and the reduced system is merely a statement of the solution for the unknown symmetries in terms of the known symmetries.

We are now ready to describe our commutator reduction method. Essentially the method consists of the following steps. Use algorithm Standard Form to reduce the DQs to standard form determining equations (SFDQs). Then use algorithm Commutation Relations to find the commutation relations between each known symmetry and a chosen unknown symmetry \( \hat{X} \). Some of these commutation relations (the useable ones) may be written as further differential equations which the unknown symmetry \( \hat{X} \) must satisfy. We add these equations to the DQs and reduce this combined system to Standard Form to obtain a system of reduced standard form determining equations (RSFDQs) for the unknown symmetry \( \hat{X} \).

We illustrate the commutator reduction method by an example.

**Example 3.1.4 The linear heat equation** \( (u_t = u_{xx}) \). Through the examples in Chapter 2 and §3.1 we have already executed all of the necessary steps involved in the method. We now put them in order. In Example 2.1.8 we set up the determining equations for the Lie point symmetries of the linear heat equation, with generators of the form \( X = \xi(x,t) \partial_x + \tau(x,t) \partial_t + \eta(x,t,u) \partial_u \) with \( \eta(x,t,u) = f(x,t)u + g(x,t) \). Putting
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\[ g(x, t) \equiv 0 \] we obtain\(^1\) the following DQs

\[
\begin{align*}
\tau_x &= 0, \\
\tau_t - \tau_{xx} - 2\xi_x &= 0, \\
2f_x - \xi_{xx} + \xi_t &= 0, \\
f_{xx} - f_t &= 0.
\end{align*}
\] (3.50)

On page 29 in Example (2.33) we showed that this system in a Standard Form is

\[
\begin{align*}
\tau_x &= 0, \\
\tau_t &= 2\xi_x, \\
f_x &= -\frac{1}{2}\xi_t, \\
f_t &= -\frac{1}{2}\xi_{xx}, \\
\xi_{xx} &= 0, \\
\xi_{tt} &= 0.
\end{align*}
\] (3.51)

In addition, we noted on page 32 that the parametric derivatives are \{\xi, \tau, f, \xi_x, \xi_t, \xi_{xx}\}. Thus the Lie symmetry algebra represented by SFDQs (3.51) is six dimensional. We use \( L_{(a_1, \ldots, a_6)} \) to denote the unique symmetry operator \( \xi \partial_x + \tau \partial_t + f u \partial_u \) satisfying SFDQS (3.51) with the initial data

\[
\begin{align*}
\tau(x_0) &= a_1, \\
f(x_0) &= a_2, \\
\xi(x_0) &= a_3, \\
\xi_x(x_0) &= a_4, \\
\xi_t(x_0) &= a_5, \\
\xi_{xx}(x_0) &= a_6, \\
\xi_{tt}(x_0) &= 0,
\end{align*}
\] (3.52)

posed at an arbitrarily chosen initial point \( x_0 = (x_0, t_0) \).

It is not necessary to use the full machinery of Lie theory to determine several simple symmetries of the linear heat equation. Indeed since the heat equation does not involve any explicit \( x \) or \( t \) dependency it is obvious that it will be invariant under translations in

\(^1\)By doing this we are eliminating the \( \infty \) dimensional superpositional symmetry algebra consisting of generators of the form \( g(x, t) \partial_t \) with \( g_t = g_{xx} \). We are only considering finite dimensional symmetry algebras in this thesis.
Chapter 3. The commutator reduction method – new symmetries from old

$x, t$ with corresponding infinitesimal generators $\mathbf{X}_1 = \partial_x$, $\mathbf{X}_2 = \partial_t$. Further because of its physical origin we might expect the linear heat equation to be invariant under changes of scale. Indeed if we substitute $x = ax^*, t = bt^*, u = cu^*$ into $u_t = u_{xx}$ and demand that $u^*_t = u^*_{xx}$, it immediately follows that $b = a^2$ with $a, c$ being arbitrary constants. The corresponding infinitesimal generators of these symmetries are $\mathbf{X}_3 = x \partial_x + 2t \partial_t$ and $\mathbf{X}_6 = u \partial_u$. It can also be verified that each of these symmetries satisfies the SFDQs (3.51). Consequently four of the six symmetries of the Lie symmetry algebra represented by system (3.51) can be found by elementary or inspectional means. Summarising, we have the following explicitly known symmetries

$$\mathcal{K} \text{ (known)} \left\{ \begin{array}{l}
\mathbf{X}_1 = \partial_x, \\
\mathbf{X}_2 = \partial_t,
\mathbf{X}_3 = x \partial_x + 2t \partial_t, \\
\mathbf{X}_4 = u \partial_u.
\end{array} \right. \quad (3.53)$$

Choosing $\mathbf{x}_0 = (x_0, t_0) = (0, 0)$ then we get the following correspondences

$$\text{I.D.} \left\{ \begin{array}{l}
\mathbf{X}_1 = L_{(0,0,1,0,0,0)}, \\
\mathbf{X}_2 = L_{(1,0,0,0,0,0)}, \\
\mathbf{X}_3 = L_{(0,0,1,0,0,0)}, \\
\mathbf{X}_4 = L_{(0,1,0,0,0,0)}.
\end{array} \right. \quad (3.54)$$

For example for the known symmetry $\mathbf{X}_3 = x \partial_x + 2t \partial_t$, $\xi = x$, $\tau = 2t$ and $f = 0$. So its initial data is $(a_1, a_2, a_3, a_4, a_5, a_6) = (\tau(0, 0), f(0, 0), \xi(0, 0), \xi_x(0, 0), \xi_t(0, 0), \xi_{xt}(0, 0)) = (0, 0, 0, 0, 1, 0, 0)$. Thus there remain two unknown symmetries, $\mathbf{X}_5$ and $\mathbf{X}_6$ say, with initial data

$$\mathbf{X}_5 = L_{(0,0,0,0,1,0)}, \quad \mathbf{X}_6 = L_{(0,0,0,0,0,1)}. \quad (3.55)$$
We shall choose $\hat{X} = X_5$ as our first unknown and proceed to calculate its useable commutation relations with $X_1, X_2, X_3,$ and $X_4$.

In Example 2.3.5 we used the Standard Form (SFDQ) and Reid’s Commutation Relations algorithm to calculate the commutator between two general operators and we obtain:

\[[L(a_1, a_4), L(\hat{a}_1, a_4)] = LJ,\]

where

\[
J = \begin{bmatrix}
2(a_1\hat{a}_4 - \hat{a}_1a_4), \\
-\frac{1}{2}(a_3\hat{a}_5 - \hat{a}_3a_5 + a_1\hat{a}_6 - \hat{a}_1a_6), \\
a_3\hat{a}_4 - \hat{a}_3a_4 + a_1\hat{a}_5 - \hat{a}_1a_5, \\
a_1\hat{a}_6 - \hat{a}_1a_6, \\
a_4\hat{a}_5 - \hat{a}_4a_5 + a_3\hat{a}_6 - \hat{a}_3a_6, \\
2(a_4\hat{a}_6 - \hat{a}_4a_6) \end{bmatrix}.
\]

On page 43 in Example 3.1.3 we found useable commutation relations for the unknown $\hat{X} = X_5$

\[
\mathcal{R} \begin{cases}
[X_1, \hat{X}] = -\frac{1}{2}X_4, \\
[X_2, \hat{X}] = X_4, \\
[X_3, \hat{X}] = \hat{X}, \\
[X_4, \hat{X}] = 0,
\end{cases}
\tag{3.56}
\]

and we rewrote these as differential equations

\[
\mathcal{D} \begin{cases}
\xi_x = 0, & \tau_x = 0, & f_x = -\frac{1}{2}, \\
\xi_t = 1, & \tau_t = 0, & f_t = 0, \\
x\xi_x + 2t\xi_t - \xi = \xi, & x\tau_x + 2t\tau_t - 2\tau = \tau, & x f_x u + 2 t f_t u = f u.
\end{cases}
\tag{3.57}
\]
Adding these equations to the SFDQs and reducing to a Standard Form we obtain the RSFDQs

\[
\begin{align*}
\text{RSFDQs} &\quad \begin{cases} 
\xi = t, \\
\tau = 0, \\
f = -\frac{1}{2}x.
\end{cases}
\end{align*}
\] (3.58)

These are the reduced determining equations for the unknown \( \hat{X} \). Thus \( X_5 = \hat{X} \) is completely determined and is given by

\[
X_5 = t \partial_x - \frac{1}{2} xu \partial_u.
\] (3.59)

Note: It is not always the case that we get such dramatic simplification. In general there will be differential equations in the RSFDQs which will require integration.

We have just executed the method. Let’s survey what has been achieved here. We have determined a new symmetry, \( X_5 = \hat{X} \) by using the known symmetries \( K = \{X_1, X_2, X_3, X_4\} \) and also by applying algebraic and differential operations to the DQs. We will review the essential steps of the process just executed in the following section.

### 3.2 The commutator reduction method

Use known solutions of a system of DQs to help generate new\(^2\) solutions by following steps 1–5 below.

1. Reduce the DQs to a Standard Form (the SFDQs) and calculate its associated initial data.

2. Represent each known symmetry \( K \) by a generator labelled by its initial data:

\[ L_{ID(K)}. \]

\(^2\)I.e. linearly independent of the known solutions.
3. Choose an unknown solution of the form $\tilde{X} = Lb$ where $b$ is in a complement of the vector space of the known initial data$^3$.

4. Calculate the useable commutation relations $R$ between knowns, $\mathcal{K}$, and a single unknown solution, $\tilde{X}$, of the SFDQs, using the Commutation Relations algorithm.

5. Write $R$ as differential equations $D$.

6. Append $D$ to the system of SFDQs and reduce to a standard form to obtain the Reduced Standard Form Determining eQuations (RSFDQs) for the unknown $\tilde{X}$.

The remainder of this chapter consists of examples illustrating the above method.

**Example 3.2.1 Application to the linear heat equation continued...**

The system of SFDQs (2.33) has a six dimensional solution space and we now know five symmetries. Hence there is a remaining (linearly independent) unknown symmetry generator ($\tilde{X}$) to be determined. We will apply the method to produce RSFDQs for $\tilde{X} = X_6$.

Step 1. Reduce the DQs to a Standard Form (the SFDQs), and calculate its associated initial data.

This is already done: see the SFDQs (3.51) with corresponding initial data (3.52).

Step 2. Represent each known symmetry $K$ by a generator labelled by its initial data: $L_{ID(K)}$.

From (3.53) we have the original known symmetries $\{X_1, X_2, X_3, X_4\}$. In the previous Example 3.1.4 we used these symmetries to find the explicit form (3.59) of one of the unknown symmetries $X_5$. Regarding the known symmetries now as $^3$Regarded as a subspace of the vector space of initial data.
\[ \mathcal{K} = \{ \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5 \} \], from (3.54), (3.55) they are labelled by their initial data as:

\[
\begin{align*}
\mathbf{X}_1 &= L_{(0,0,1,0,0,0)}, \\
\mathbf{X}_2 &= L_{(1,0,0,0,0,0)}, \\
\mathbf{X}_3 &= L_{(0,0,0,1,0,0)}, \\
\mathbf{X}_4 &= L_{(0,1,0,0,0,0)}, \\
\mathbf{X}_5 &= L_{(0,0,0,0,1,0)}.
\end{align*}
\]

(3.60)

Step 3. Choose an unknown solution of the form \( \mathbf{X} = L_b \) where \( b \) is in a complement of the vector space of the known initial data.

Choose the remaining unknown symmetry to be \( \mathbf{X} = \mathbf{X}_6 \) with initial data \( L_{(0,0,0,0,0,1)} \).

Step 4. Calculate useable commutation relations \( \mathcal{R} \) between knowns, \( \mathcal{K} \), and a single unknown solution, \( \hat{\mathbf{X}} \), of the DQs, using the Commutation Relations algorithm.

The commutation relations between \( \hat{\mathbf{X}} = \mathbf{X}_6 \) and the known symmetries (3.60) can be calculated using the expression for \( J(a, \dot{a}) \) given in (2.43) page 38. They are all useable and are:

\[
\mathcal{R} = \begin{cases}
(i) & [\mathbf{X}_1, \hat{\mathbf{X}}] = \mathbf{X}_5 \\
(ii) & [\mathbf{X}_2, \hat{\mathbf{X}}] = \mathbf{X}_3 - \frac{1}{2} \mathbf{X}_4 \\
(iii) & [\mathbf{X}_3, \hat{\mathbf{X}}] = 2 \hat{\mathbf{X}} \\
(iv) & [\mathbf{X}_4, \hat{\mathbf{X}}] = 0 \\
v) & [\mathbf{X}_5, \hat{\mathbf{X}}] = 0.
\end{cases}
\]

(3.61)

Step 5. Write \( \mathcal{R} \) as differential equations \( \mathcal{D} \).

Writing \( \mathbf{X}_4 = \xi \partial_x + \tau \partial_t + fu \partial_u \) then the commutation relations \( \mathcal{R} \) above yield the following differential equations.

\[
(i) \quad \Rightarrow \quad [\partial_x, \xi \partial_x + \tau \partial_t + fu \partial_u] = t \partial_x - \frac{1}{2} x u \partial_u
\]

\[
\Rightarrow \quad \xi_x = t, \quad \tau_x = 0, \quad f_x = -\frac{1}{2} x.
\]

(3.62)
(ii) \[ [\partial_t, \xi \partial_x + \tau \partial_t + fu \partial_u] = (x \partial_x + 2t \partial_t) - \frac{1}{2}(u \partial_u) \]
\[ \Rightarrow \xi_t = x, \quad \tau_t = 2t, \quad f_t = -\frac{1}{2}. \] (3.63)

(iii) \[ [x \partial_x + 2t \partial_t, \xi \partial_x + \tau \partial_t + fu \partial_u] = 2\xi \partial_x + 2\tau \partial_t + 2fu \partial_u \]
\[ \Rightarrow x\xi_x + 2t\xi_t - \xi = 2\xi, \quad x\tau_x + 2t\tau_t - 2\tau = 2\tau. \]
\[ xf_xu + 2tf_xu = 2fu. \] (3.64)

(iv) \[ [u \partial_u, \xi \partial_x + \tau \partial_t + fu \partial_u] = 0 \]
\[ \Rightarrow \text{(identically satisfied).} \] (3.65)

(v) \[ [t \partial_x - \frac{1}{2} xu \partial_u, \xi \partial_x + \tau \partial_t + fu \partial_u] = 0 \]
\[ \Rightarrow t\xi_x - \tau = 0, \quad t\tau_x = 0, \quad -\xi\frac{1}{2}u - tf_xu = 0. \] (3.66)

Summarising equations (3.62-3.66) above we have the following set of differential equations \( \mathcal{D} \):

\[ \mathcal{D} \left\{ \begin{array}{llll}
\xi_x &= t, & \tau_x &= 0, & f_x &= -\frac{1}{2}x,
\xi_t &= x, & \tau_t &= 2t, & f_t &= -\frac{1}{2},
2\xi &= x\xi_x + 2t\xi_t - \xi, & 2\tau &= x\tau_x + 2t\tau_t - 2\tau, & 2fu &= xf_xu + 2tf_xu,
0 &= t\xi_x - \tau, & 0 &= t\tau_x, & 0 &= -\xi\frac{1}{2}u - tf_xu.
\end{array} \right. \] (3.67)

Step 6. Append \( \mathcal{D} \) to the system of SFDQs and reduce to a Standard Form to obtain RSFDQs for the unknown \( \hat{X} \).

In this case the calculations can be done easily by hand. Using only \( \mathcal{D} \) we obtain the RSFDQs

\[ \text{RSFDQs } \left\{ \begin{array}{l}
\xi = xt, \\
\tau = t^2, \\
f = -\left(\frac{j^2}{4} + \frac{t}{2}\right).
\end{array} \right. \] (3.68)
Chapter 3. The commutator reduction method – new symmetries from old

Again we were fortunate: The system of RSFDQs (3.68) does not require integrating, but is merely a statement of the explicit form of the unknown symmetry

\[ \dot{X} = X_6 = xt \partial_x + t^2 \partial_t - \left( \frac{x^2}{4} + \frac{t}{2} \right) u \partial_u. \]  

(3.69)

The explicit form of the full symmetry algebra \( \mathcal{L} = \{X_1, \ldots, X_6\} \) corresponding to the determining equations (2.22) is thus given by (3.53), (3.59) and (3.69). \( \mathcal{L} \) constitutes a six dimensional subalgebra of the full (infinite dimensional) symmetry algebra obtained in 1881 by Lie [7].

We now review what has been achieved here. From the previous Example 3.1.4 we explicitly knew five out of six symmetries for the linear heat equation. We calculated the useable commutation relations for the remaining (unknown) symmetry operator \( \dot{X} \) and rewrote them into differential equations which were then used to simplify the determining equations (2.33). The resulting simplified system yielded \( \dot{X} \) explicitly without needing to integrate any differential equations.

Example 3.2.2 Application to the Laplace Equation in 3D.

We will next consider the larger example of an 11 dimensional symmetry algebra of the 3D Laplace equation. In this example there are eight symmetries which are known explicitly and by closely following the steps of the method we obtain the three remaining unknown symmetries.

Step 1. Reduce the DQs to a Standard Form (the SFDQs), and calculate its associated initial data.
From Example 2.1.9 we have the following DQs.

\[
\begin{align*}
\xi_y^1 + \xi_z^2 &= 0, & \xi_x^3 + \xi_z^1 &= 0, & \xi_x^2 + \xi_y^1 &= 0, \\
\xi_y^2 - \xi_x^1 &= 0, & \xi_z^3 - \xi_x^1 &= 0, \\
\xi_x^1 &+ \xi_y^1 + \xi_z^1 - 2 f_x = 0, \\
\xi_x^2 &+ \xi_y^2 + \xi_z^2 - 2 f_y = 0, \\
\xi_x^3 &+ \xi_y^3 + \xi_z^3 - 2 f_z = 0, \\
f_{xx} + f_{yy} + f_{zz} &= 0.
\end{align*}
\] 
(3.70)

Putting \( g \equiv 0 \) and reducing to standard form using Reid's implementation of the Standard Form algorithm (in the symbolic language Maple) we obtain the SFDQs

\[
\begin{align*}
f_{xx} &= 0, & f_{xy} &= 0, & f_{xz} &= 0, & \xi_y^1 &= 2 f_x, & f_{yy} &= 0, & \xi_y^1 &= 0, \\
\xi_y^2 &= 0, & \xi_z^2 &= 2 f_x, & \xi_z^3 &= 2 f_y, & \xi_z^3 &= -2 f_z, & f_{zz} &= 0, \\
\xi_x^1 &= \xi_z^3, & \xi_x^2 &= -\xi_y^1, & \xi_x^3 &= -\xi_z^1, & \xi_y^3 &= \xi_z^3, & \xi_y^3 &= -\xi_z^2
\end{align*}
\] 
(3.71)

and associated initial data

\[
\begin{align*}
\xi_x^1(x_0) &= a_1, & \xi_y^1(x_0) &= a_2, & \xi_z^1(x_0) &= a_3, & \xi_x^2(x_0) &= a_4, & \xi_y^2(x_0) &= a_5, & \xi_z^2(x_0) &= a_6, \\
\xi_z^3(x_0) &= a_7, & f(x_0) &= a_8, & f_z(x_0) &= a_9, & f_y(x_0) &= a_{10}, & f_x(x_0) &= a_{11}
\end{align*}
\] 
(3.72)

for an initial point \( x_0 \in \mathbb{R}^3 \).

Step 2. Represent each known symmetry \( K \) by a generator labelled by its initial data: \( L_{ID(K)} \).

We noted in Example 2.1.1 that the 3D Laplace equation has rotation and translation symmetries due to its physical nature. In addition there are two scaling symmetries which are easy to spot by inspection. Thus the 3D Laplace equation
has the following inspectional symmetries:

\[ \mathcal{K} (\text{known}) \]

\[
\begin{align*}
\text{translations: } & T^x = \partial_x, \quad T^y = \partial_y, \quad T^z = \partial_z, \\
\text{rotations: } & R^{xy} = y \partial_x - x \partial_y, \quad R^{xz} = z \partial_x - x \partial_z, \quad R^{yz} = z \partial_y - y \partial_z, \\
\text{scalings: } & S = x \partial_x + y \partial_y + z \partial_z, \quad S^u = u \partial_u.
\end{align*}
\]

So there are three unknown symmetries. Denote by \( L_{(a_1, \ldots, a_{11})} \) the symmetry generator \( X = \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_z + \eta \partial_u \) satisfying SFDQs (3.71) with initial data (3.72). Choosing \( x_0 = (x_0, y_0, z_0) = (0, 0, 0) \) we have

\[
\begin{align*}
T^x &= L_{(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)}, \\
T^y &= L_{(0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)}, \\
T^z &= L_{(0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)}, \\
R^{xy} &= L_{(0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)}, \\
R^{xz} &= L_{(0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)}, \\
R^{yz} &= L_{(0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0)}, \\
S &= L_{(0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)}, \\
S^u &= L_{(0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0)}.
\end{align*}
\]  

(3.74)

Step 3. Choose an unknown solution of the form \( \bar{X} = L_b \) where \( b \) is in a complement of the vector space of the known initial data.

To complete the basis we can select the three remaining unknown symmetries to be \( \bar{X}_1, \bar{X}_2, \) and \( \bar{X}_3 \) with the following initial data:

\[
\begin{align*}
\bar{X}_1 &= L_{(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)}, \\
\bar{X}_2 &= L_{(0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0)}, \\
\bar{X}_3 &= L_{(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)}.
\end{align*}
\]

Choose \( \bar{X} \) to be a general unknown given by \( \bar{X} = A\bar{X}_1 + B\bar{X}_2 + C\bar{X}_3 = L_{(0, 0, 0, 0, 0, 0, 0, 0, A, B, C)} \) for some constants \( A, B \) and \( C \).
Step 4. Calculate the useable commutation relations $\mathcal{R}$ between knowns, $\mathcal{K}$, and a single unknown solution, $\hat{X}$, of the SFDQs, using the Commutation Relations algorithm. Using the algorithm Commutation Relations we generate the commutator relations for the whole algebra $[L_{(a_1, \ldots, a_{11})}, L_{(\tilde{a}_1, \ldots, \tilde{a}_{11})}] = L_J$, where

$$J = \begin{pmatrix}
(a_1 \hat{a}_7 - a_1 a_7 + a_4 \hat{a}_3 - \hat{a}_4 a_3 + a_6 \hat{a}_2 - \hat{a}_6 a_2, \\
2 \hat{a}_1 a_9 - 2 a_1 \hat{a}_9 + a_5 \hat{a}_3 - \hat{a}_5 a_3 + 2 a_6 \hat{a}_{11} - 2 \hat{a}_6 a_{11}, \\
2 \hat{a}_1 a_{10} - 2 a_1 \hat{a}_{10} + 2 a_4 \hat{a}_{11} - 2 \hat{a}_4 a_{11} - a_5 \hat{a}_2 + \hat{a}_5 a_2, \\
\hat{a}_1 a_3 - a_1 \hat{a}_3 + a_4 \hat{a}_7 - \hat{a}_4 a_7 + a_6 \hat{a}_5 - \hat{a}_6 a_5, \\
\hat{a}_2 a_3 - a_2 \hat{a}_3 - 2 a_4 \hat{a}_9 + 2 \hat{a}_4 a_9 + 2 a_6 \hat{a}_{10} - 2 \hat{a}_6 a_{10}, \\
\hat{a}_1 a_2 - a_1 \hat{a}_2 - a_4 \hat{a}_5 + \hat{a}_4 a_5 + a_6 \hat{a}_7 - \hat{a}_6 a_7, \\
2 \hat{a}_1 a_{11} - 2 a_1 \hat{a}_{11} - 2 a_4 \hat{a}_{10} + 2 \hat{a}_4 a_{10} - 2 a_6 \hat{a}_9 + 2 \hat{a}_6 a_9, \\
a_1 \hat{a}_{11} - a_1 a_{11} + a_4 \hat{a}_{10} - \hat{a}_4 a_{10} + a_6 \hat{a}_9 - \hat{a}_6 a_9, \\
a_2 \hat{a}_{11} - a_2 a_{11} + a_5 \hat{a}_{10} - \hat{a}_5 a_{10} + a_7 \hat{a}_9 - \hat{a}_7 a_9, \\
a_3 \hat{a}_{11} - a_3 a_{11} + a_7 \hat{a}_{10} - \hat{a}_7 a_{10} - a_5 \hat{a}_9 + \hat{a}_5 a_9, \\
a_7 \hat{a}_{11} - a_7 a_{11} - a_3 \hat{a}_{10} + \hat{a}_3 a_{10} - a_2 \hat{a}_9 + \hat{a}_2 a_9.
\end{pmatrix}$$

(3.75)
Substituting various values for $a_1, \ldots, a_{11}, \tilde{a}_1, \ldots, \tilde{a}_{11}$ into (3.75) we get the following commutation relations for $\hat{X}$.

$$
\begin{align*}
(i) \quad [T^x, \hat{X}] &= L(0, -2A, -2B, 0, 0, 0, -2C, C, 0, 0, 0) = -2AR^x - 2BR^y - 2CS + CS^u, \\
(ii) \quad [T^y, \hat{X}] &= L(0, 2C, 0, -2A, -2B, 0, 0, 0, 0, 0) = 2CR^y - 2AR^z - 2BS + BS^u, \\
(iii) \quad [T^z, \hat{X}] &= L(0, 0, 2B, 0, -2A, A, 0, 0, 0, 0) = 2CR^z + 2BR^y - 2AS + AS^u, \\
(iv) \quad [R^y, \hat{X}] &= L(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
(v) \quad [R^z, \hat{X}] &= L(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -A), \\
(vi) \quad [R^x, \hat{X}] &= L(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, B), \\
(vii) \quad [S, \hat{X}] &= L(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, A, H, C) = \hat{X}, \\
(viii) \quad [S^u, \hat{X}] &= L(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) = 0.
\end{align*}
$$

(3.76)

Note that these are all *useable* commutation relations, with the possible exception of (iv), (v) and (vi) depending on the values of $A$, $B$ and $C$. For example, if $A \neq B = C = 0$ then (iv) is *useable* but (v) and (vi) are not.

**Step 5. Write $\mathcal{R}$ as differential equations $D$.**

Writing $\hat{X} = \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_z + fu \partial_u$ then the useable commutation relations $\mathcal{R}$ yield the following DEs.

$$(i) \quad \Rightarrow \quad [\partial_x, \xi^1 \partial_x + \xi^2 \partial_y + \xi^3 \partial_z + fu \partial_u] = (-2Az - 2By - 2Cx) \partial_x + (-2Cy + 2Bx) \partial_y + (-2Cz + 2Ax) \partial_z + Cu \partial_u,$$

$$
\Rightarrow \quad \xi^1_x = -2Az - 2By - 2Cx, \quad \xi^2_x = -2Cy + 2Bx, \quad \xi^3_x = -2Cz + 2Ax, \quad f_x = C.
$$

(3.77)

Similarly,

$$(ii) \quad \Rightarrow \quad \xi^1_y = 2Cy - 2Bx, \quad \xi^2_y = -2Cx - 2Az - 2By.$$

Note that these are all *useable* commutation relations, with the possible exception of (iv), (v) and (vi) depending on the values of $A$, $B$ and $C$. For example, if $A \neq B = C = 0$ then (iv) is *useable* but (v) and (vi) are not.
\( \xi_y^3 = 2Ay - 2Bz, \quad f_y = B, \) (3.78)

\( (iii) \quad \Rightarrow \quad \xi_z^1 = -2Ax + 2Cz, \quad \xi_z^2 = -2Ay + 2Bz. \)
\( (vii) \quad \Rightarrow \quad \xi_z^3 = -2Az - 2Cx - 2By, \quad f_z = A, \) (3.79)

\[
\begin{align*}
\xi_x^1 + y\xi_y^1 + z\xi_z^1 - \xi^1 &= \xi^1, \\
\xi_x^2 + y\xi_y^2 + z\xi_z^2 - \xi^2 &= \xi^2, \\
\xi_x^3 + y\xi_y^3 + z\xi_z^3 - \xi^3 &= \xi^3, \\
f_x + yf_y + zf_z &= f. \quad (3.80)
\end{align*}
\]

Thus \( D \) comprises equations (3.77-3.80).

**Note:** We have missed out relations (3.2.2)(iv), (v) and (vi) since they are not useable for arbitrary \( A, B \) and \( C \). Also the relation (3.2.2)(viii) is omitted because it is identically satisfied.

Step 6. Append \( D \) to the system of SFDQs and reduce to a Standard Form to obtain RSFDQs for the unknown \( \dot{X} \).

**In principle:** We append the equations \( D \) (3.77-3.80) to the system of SFDQs (3.71) and apply the Standard Form algorithm to produce the RSFDQs.

**In practice:** The calculations can be dispensed with simply by hand. We can merely substitute (3.77),(3.78),(3.79) into (3.80), solve for \( \xi^1, \xi^2, \xi^3, \eta \) and obtain

\[
\text{RSFDQs } \left\{ \begin{array}{l}
\xi^1 = -(2xz)A - (2xy)B - (x^2 - y^2 - z^2)C, \\
\xi^2 = -(2yz)A - (x^2 + y^2 - z^2)B - (2xy)C, \\
\xi^3 = -(x^2 - y^2 + z^2)A - (2yz)B - (2xz)C, \\
f = zA + yB + xC.
\end{array} \right. \) (3.81)

Yet again we were fortunate; the system of RSFDQs (3.81) does not require integrating, it is merely a statement of the solution.

\[ \dot{X} = -(2xz \partial_x + 2yz \partial_y + (-x^2 - y^2 + z^2) \partial_z - zu \partial_u)A \\
+ (2x\dot{y} \partial_x + (-x^2 + y^2 - z^2) \partial_y + 2y\dot{z} \partial_z - yu \partial_u)B \]
Chapter 3. The commutator reduction method — new symmetries from old

\[-((x^2 - y^2 - z^2) \partial_x + 2xy \partial_y + 2xz \partial_z - xu \partial_u)C.\]

Summarising — we have obtained the general solution of the 11 dimensional system of SFDQs (3.71) by using eight known solutions and a finite number of differentiations and algebraic operations. These results agree with those obtained in [1].

The third and final example, the nonlinear heat equation \( u_t = (D(u)u_x)_x \), is of a different type to those already encountered in the sense that it is not just one symmetry problem but is a whole class of problems. The symmetry algebra is different for different forms of the diffusivity function \( D(u) \) and thus we must consider various cases. These cases arise naturally during the process of reducing the relevant DQs to a standard form, as we shall see below.

Example 3.2.3 Application to the nonlinear heat equation classification problem.

The determining equations for a symmetry operator \( X = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \eta(x, t, u) \partial_u \) of

\[ u_t = (D(u)u_x)_x \] (3.82)

are (see Example 2.1.10, page 22)

\[
\begin{align*}
\xi_t + 2D'(u)\eta_x + D(u)(2\eta_{xx} - \xi_{xx}) &= 0, \\
D(u)(\tau_t - 2\xi_x) + D'(u)\eta &= 0, \\
D(u)\eta_{uu} + D'(u)(\tau_t - 2\xi_x + \eta_u) + D''(u)\eta &= 0, \\
D(u)\eta_{xx} - \eta_t &= 0.
\end{align*}
\] (3.83)

In applying the Standard Form algorithm to the DQs (3.83) above we encounter case splitting depending on the form of \( D(u) \). For example, we expect that the linear heat
equation \(((3.82) \text{ with } D(u) = 1)\) has different symmetries to the genuinely nonlinear equation given by \((3.82) \text{ with } D(u) = u\). The various cases (or branches) arise in Reid’s algorithm via dividing an equation by a pivot\(^4\) which may be zero or non-zero, and different results can arise depending on which branch is taken. Reid’s Standard Form algorithm now makes the classification problem an algorithmic one. The various case splittings all arise automatically during the process of reducing the determining equations to a Standard Form. The zero versus non-zero decisions appear as pivots which can be enumerated automatically\(^5\). The pivots which arise in this example are nonlinear expressions involving \(D(u)\) and its derivatives \(\dot{D}(u), \ddot{D}(u)\) etc. For convenience we have introduced the functions \(I(u) := \dot{D}/D\) and \(J := \dot{I}/I^2\). By using these the pivots, which occur when executing the Standard Form algorithm, are \(\dot{D}, J, J - \frac{3}{4} \) and \(\dot{J}\). Different SFDQs result, depending on whether \(\dot{D}\) is zero or non-zero, \(J\) is zero or non-zero, etc. By systematically considering each case we can determine the dimension of the symmetry algebra for each branch and obtain the classification tree given in Figure 3.1. This tree is preliminary in the sense that we still have to determine the explicit forms of the symmetries for each branch.

We will use the commutator reduction method to determine the symmetries for each classification for the nonlinear heat equation.

**Case I.** (Generic -- \(D(u)\) arbitrary)

Step I.1 Reduce the DQs to a Standard Form (the SFDQs), and calculate its associated initial data.

\(^4\)Analogous to the Gaussian elimination and simplex algorithms.

\(^5\)There are still problems with this approach: often there are many branches which ultimately produce identical symmetry groups. Recent work by Lisle [8] demonstrates an elegant and efficient manner in which these spurious case splittings can be reduced or eliminated.
Chapter 3. The commutator reduction method — new symmetries from old

\[
D \neq 0
\]

\[
I := \dot{D}/D
\]

\[
J := \dot{I}/I^2
\]

Linear
see page 49

Figure 3.1: Preliminary classification tree showing the dimension of the Lie symmetry algebra of the nonlinear heat equation \( u_t = (D(u)u_x)_x \) for different forms of diffusivity function \( D(u) \).

Assuming that none of the pivots which occur (in the process of reducing DQs (3.83) to a Standard Form) are zero, we get the following, so called generic, Standard Form

\[
\tau_{tt} = 0, \quad \xi_x = \frac{1}{2} \tau_t, \quad \tau_x = 0,
\]

\[
\xi_t = 0, \quad \xi_u = 0, \quad \tau_u = 0, \quad \eta = 0.
\]

(3.84)

Step 1.2 Represent each known symmetry \( K \) by a generator labelled by its initial data: \( L_{ID(K)} \).

The parametric derivatives of SFDQs (3.84) are \{\( \xi, \tau, \tau_t \)\} and thus the system is three dimensional. This case has the obvious (affine) symmetries

\[
\mathcal{K}(\text{known}) \left\{ \begin{array}{l}
\text{translations: } K_1 = \partial_x, \quad K_2 = \partial_t, \\
\text{scaling: } K_3 = x \partial_x + 2t \partial_t.
\end{array} \right.
\]

(3.85)

The method terminates here for this branch since \( \text{Dim}(\mathcal{K}) = \text{Dim}(\text{SFDQs}) \), i.e. there are no remaining unknown symmetries.
Case II. \( \dot{D}(u) = 0 \) This case is degenerate if \( D = 0 \) and is equivalent to the linear heat equation otherwise.

Case III. \( \dot{J}(u) = 0 \) (where \( J := \hat{I}/I^2, I := \hat{D}/D \))

Step III.1 Reduce the DQs to a Standard Form (the SFDQs), and calculate its associated initial data.

From DQs (3.83) we obtain

\[
\text{SFDQs} \begin{cases}
\tau_\iota = 0, & \xi_\iota = \frac{1}{2} \tau_\iota + \frac{1}{2} \eta, & \tau_x = 0, & \eta_x = 0, \\
\xi_\iota = 0, & \eta_\iota = 0, & \xi_\iota = 0, & \tau_u = 0, \eta_u = -I \tau \eta.
\end{cases}
\] (3.86)

with corresponding initial data

\[
\xi(x_0) = a_1, \quad \tau(x_0) = a_2, \quad \xi_t(x_0) = a_3, \quad \eta(x_0) = a_4,
\] (3.87)

(Where \( x_0 \in \mathbb{R}^3 \) is some initial point). Thus the system of SFDQs (3.86) above has parametric derivatives \( \{\xi, \tau, \tau_\iota, \eta\} \) and hence has a four dimensional solution space.

Step III.2 Represent each known symmetry \( K \) by a generator labelled by its initial data: \( L_{ID(K)} \).

We have three known solutions (\( K = \{K_1, K_2, K_3\} \) inherited from the generic case I) so there remains one unknown symmetry to be found in this case. If we denote by \( L(a_1, a_2, a_3, a_4) \) the symmetry operator \( X = \xi \partial_x + \tau \partial_\iota + \eta \partial_u \) where \( \xi, \tau, \eta \) satisfy the SFDQs (3.86) with initial data (3.87), then, regardless of the choice of \( x_0 = (x_0, t_0, u_0) \), we obtain the following initial data for the known symmetries \( K \).

\[
\text{I.D.} \begin{cases}
K_1 = L_{(1,0,0,0)}, \\
K_2 = L_{(0,1,0,0)}, \\
K_3 = L_{(0,0,2,0)}.
\end{cases}
\] (3.88)

Step III.3 Choose an unknown solution of the form \( \tilde{X} = Lb \) where \( b \) is in a complement of the vector space of the known initial data.
Choose the unknown to be \( \overline{X} = L_{(a,b,c,d)} \) for arbitrary constants \( a, b, c, d \) with \( d \neq 0 \).

Step III.4 Calculate the *useable* commutation relations \( \mathcal{R} \) between knowns, \( \mathcal{K} \), and a single unknown solution, \( \overline{X} \), of the SFDQs, using the Commutation Relations algorithm.

Applying algorithm Commutation Relations to the SFDQs (3.86) we obtain

\[
[L_{(a_1, a_2, a_3, a_4)}, L_{(b_1, b_2, b_3, b_4)}] = L_{\mathcal{J}}
\]

where

\[
\mathcal{J} = \left[ \frac{a_1}{2} (b_3 + I_0 b_4) - \frac{b_1}{2} (a_3 + I_0 a_4), a_2 b_3 - b_2 a_3, 0, 0 \right].
\]

(\( I_0 = I(x_0), \dot{I} = \dot{D}/D. \))

Thus we can calculate the following commutation relations

\[
[K_1, \overline{X}] = [L_{(1,0,0,0)}, L_{(a,b,c,d)}] = L_{\left( \frac{1}{2} (c + I_0 d), 0, 0, 0 \right)}
\]

\[
= \frac{1}{2} (c + I_0 d) K_1,
\]

\[
[K_2, \overline{X}] = [L_{(0,1,0,0)}, L_{(a,b,c,d)}] = L_{(0, c, 0, 0)}
\]

\[
= c K_2,
\]

\[
[K_3, \overline{X}] = [L_{(0,0,2,0)}, L_{(a,b,c,d)}] = L_{(-a, -2b, 0, 0)}
\]

\[
= -a K_1 - 2b K_2.
\]

Note that these are all useable commutation relations. Putting \( a = b = c = 0, d = 2/I_0 \) for simplicity, these reduce to

\[
\mathcal{R} \left\{ \begin{array}{l} (i) \quad [K_1, \overline{X}] = K_1. \\ (ii) \quad [K_2, \overline{X}] = 0. \\ (iii) \quad [K_3, \overline{X}] = 0. \end{array} \right. \quad (3.89)
\]

Step III.5 Write \( \mathcal{R} \) as differential equations \( \mathcal{D} \).
Writing $\dot{X} = \xi \partial_x + \tau \partial_t + \eta \partial_u$ then the commutation relations $\mathcal{R}$ above yield the following DEs

\begin{align}
(i) \quad & \Rightarrow \quad [\partial_x, \xi \partial_x + \tau \partial_t + \eta \partial_u] = \partial_x \\
& \Rightarrow \quad \xi_x = 1, \quad \tau_t = 0, \quad \eta_x = 0. \tag{3.90}
\end{align}

\begin{align}
(ii) \quad & \Rightarrow \quad [\partial_t, \xi \partial_x + \tau \partial_t + \eta \partial_u] = 0 \\
& \Rightarrow \quad \xi_t = 0, \quad \tau_t = 0, \quad \eta_t = 0. \tag{3.91}
\end{align}

\begin{align}
(iii) \quad & \Rightarrow \quad [x \partial_x + 2t \partial_t, \xi \partial_x + \tau \partial_t + \eta \partial_u] = 2\xi \partial_x + 2\tau \partial_t + 2fu \partial_u \\
& \Rightarrow \quad x\xi_x + 2t\xi_t - \xi = 0, \quad x\tau_x + 2t\tau_t - 2\tau = 0, \quad x\eta_x + 2t\eta_t = 0. \tag{3.92}
\end{align}

\begin{align}
& \Rightarrow \quad \eta_x = 0.
\end{align} \tag{3.93}

Step III.6 Append $D$ to SFDQs and reduce to a Standard Form to obtain RSFDQs for the unknown $\dot{X}$.

Simplifying equations (3.90),(3.91),(3.92) alone gives $\xi = x, \quad \tau = 0, \quad \eta_x = \eta_t = 0$.

Adding these equations to SFDQs (3.86) and reducing to a Standard Form yields

\begin{align}
\text{RSFDQs} \quad \left\{ \begin{array}{l}
\xi = x, \\
\tau = 0, \\
\eta = \frac{2}{I}.
\end{array} \right. \tag{3.94}
\end{align}

Thus we now have the entire four dimensional symmetry algebra $\mathcal{L} = \{K_1, K_2, K_3, K_4\}$ for this branch ($\dot{J} = 0$) with

$$K_4 = \dot{X} = x\partial_x + \frac{2}{I} \partial_u.$$ 

Note: Integrating the classifying equations $\dot{J} = 0, \quad J := \dot{I}/I^2, \quad I := \dot{D}/D$ we obtain the general solution $D(u) = \lambda (u + \kappa)^\nu$ for constants $\kappa, \lambda, \nu$ and thus $K_4$ =
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\[ x \partial_x + \frac{2}{\nu} (u + \kappa) \partial_u. \] (For the limiting case \( D(u) = \lambda e^{\nu u} \) we obtain \( K_4 = x \partial_x + \frac{2}{\nu} \partial_u. \))

Hence \( K_4 \) is merely a scaling symmetry and could have been spotted by inspection had we integrated the classifying equations first.

**Case IV.** \( J(u) = 0 \) (where \( J := \frac{\dot{I}}{I^2}, I := \frac{\dot{D}}{D} \))

Step IV.1 Reduce the DQs to a Standard Form (the SFDQs), and calculate its associated initial data.

In this case we get

\[
\begin{align*}
\tau_{tt} &= 0, \quad \xi_x = \frac{1}{2} \tau_t + \frac{I}{2} \eta, \quad \tau_x = 0, \quad \eta_x = 0, \\
\xi_t &= 0, \quad \eta_t = 0, \quad \xi_u = 0, \quad \tau_u = 0, \quad \eta_u = 0.
\end{align*}
\]

This is identical to system (3.86) with \( J = 0 \). The parametric derivatives of SFDQs (3.95) are \( \{\xi, \tau, \tau_t, \eta\} \) as in Case III (\( \dot{J} = 0 \)) thus the solution space is four dimensional.

Step IV.2 Represent each known symmetry \( K \) by a generator labelled by its initial data: \( L_{ID(K)} \).

This \( J = 0 \) branch inherits the symmetries \( \mathcal{K} = \{K_1, K_2, K_3, K_4\} \) arising on the \( \dot{J} = 0 \) branch (since \( J = 0 \Rightarrow \dot{J} = 0 \)). Hence \( \text{Dim}(\mathcal{K}) = \text{Dim(} \text{SFDQs} \text{)} = 4 \) so there are no unknowns and the method terminates at this stage of this branch.

**Case V.** \( J(u) = 3/4 \) (where \( J := \frac{\dot{I}}{I^2}, I := \frac{\dot{D}}{D} \))

Step V.1 Reduce the DQs to a Standard Form (the SFDQs), and calculate its associated initial data.

For this case we obtain the following determining equations in a standard form.

\[
\begin{align*}
\eta_{xx} &= 0, \quad \tau_{tt} = 0, \quad \xi_x = \frac{1}{2} \tau_t + \frac{I}{2} \eta, \quad \tau_x = 0, \\
\xi_t &= 0, \quad \eta_t = 0, \quad \xi_u = 0, \quad \tau_u = 0, \quad \eta_u = -\frac{3}{4} \dot{I} \eta.
\end{align*}
\]

The parametric derivatives of SFDQs (3.96) are \( \{\xi, \tau, \tau_t, \eta, \eta_t\} \) thus the solution space is five dimensional.
Step V.2 Represent each known symmetry $K$ by a generator labelled by its initial data: $L_{ID(K)}$.

This $J = 3/4$ branch inherits the symmetries $K = \{K_1, K_2, K_3, K_4\}$ arising on the $\dot{J} = 0$ branch (since $J = 3/4 \Rightarrow \dot{J} = 0$) so there remains one unknown symmetry to be found in this case.

With the usual notation we have the following initial data for the four known symmetries.

\[
\begin{aligned}
I.D. \quad & \begin{cases} 
K_1 = L_{(1,0,0,0,0)}, \\
K_2 = L_{(0,1,0,0,0)}, \\
K_3 = L_{(0,0,2,0,0)}, \\
K_4 = L_{(0,0,0,0,0)}.
\end{cases}
\end{aligned}
\]  

(3.97)

Step V.3 Choose an unknown solution of the form $\tilde{X} = L b$ where $b$ is in a complement of the vector space of the known initial data.

Choose the unknown to be $\tilde{X} = L_{(a,b,c,d,e)}$, for arbitrary constants $a, b, c, d, e$ with $e \neq 0$.

Step V.3 Calculate useable commutation relations $\mathcal{R}$ between knowns $\mathcal{K}$ and a single unknown solution $\tilde{X}$, of the SFDQs.

Using algorithm Commutation Relations the commutation relations are

\[
[L_{(a_1,\ldots,a_5)}, L_{(\tilde{a}_1,\ldots,\tilde{a}_5)}] = L_{J} \quad \text{where}
\]

\[
J = \left( \frac{a_1}{2}(\tilde{a}_3 + I_0 \tilde{a}_4) - \frac{a_1}{2}(a_3 + I_0 a_4), a_2 \tilde{a}_3 - \tilde{a}_2 a_3, \right.
\]

\[
0, a_1 \tilde{a}_5 - \tilde{a}_1 a_5, \frac{a_5}{2}(a_3 + I_0 a_4) - \frac{a_5}{2}(\tilde{a}_3 + I_0 \tilde{a}_4) \bigg).
\]

In particular

\[
[K_1, \tilde{X}] = L_{(\frac{a}{2} + I_0 \frac{a}{2}, 0, 0, c, 0)}.
\]
Expressing the right hand sides of the above equations (3.98) with respect to the basis \( K_1, K_2, K_3, K_4, \dot{X} \) gives,

\[
\begin{align*}
[K_1, \dot{X}] &= \frac{1}{2}(c + I_0 d) K_1 + \frac{1}{2} I_0 e K_4, \\
[K_2, \dot{X}] &= cK_2, \\
[K_3, \dot{X}] &= -2aK_1 - 3bK_2 - \frac{1}{2} cK_3 - \frac{1}{2} I_0 d K_4 + \dot{X}, \\
[K_4, \dot{X}] &= -2aK_1 - bK_2 - \frac{1}{2} cK_3 - \frac{1}{2} I_0 d K_4 + \dot{X}.
\end{align*}
\]

Putting \( a = b = c = d = 0, e = 2/10 \) for convenience, gives us,

\[
\begin{align*}
(i) \quad [K_1, \dot{X}] &= K_4, \\
(ii) \quad [K_2, \dot{X}] &= 0, \\
(iii) \quad [K_3, \dot{X}] &= \dot{X}, \\
(iv) \quad [K_4, \dot{X}] &= \dot{X}.
\end{align*}
\]

Step V.5 Write \( \mathcal{R} \) as differential equations \( \mathcal{D} \).

Write \( \dot{X} = \xi \partial_x + \tau \partial_t + \eta \partial_u \). Then

\[
\begin{align*}
(i) \quad &\Rightarrow [\partial_x, \xi \partial_x + \tau \partial_t + \eta \partial_u] = x \partial_x + \frac{2}{l} \partial_a \\
&\Rightarrow \xi_x = x, \quad \tau_x = 0, \quad \eta_x = \frac{2}{l}, \\
(ii) \quad &\Rightarrow [\partial_t, \xi \partial_x + \tau \partial_t + \eta \partial_u] = 0 \\
&\Rightarrow \xi_t = 0, \quad \tau_t = 0, \quad \eta_t = 0, \\
(iii) \quad &\Rightarrow [x \partial_x + 2\tau \partial_t, \xi \partial_x + \tau \partial_t + \eta \partial_u] = \xi \partial_x + \tau \partial_t + \eta \partial_u
\end{align*}
\]

(3.100)
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\( \Rightarrow x\xi_x + 2t\xi_l - \xi = 0, \quad x\tau_x + 2t\tau_l - 2\tau = 0, \)

\( x\eta_x + 2t\eta_l = 0, \) \hspace{1cm} (3.102)

(iv) \( \Rightarrow [x \partial_x + \frac{2}{l} \partial_u, \xi \partial_x + \tau \partial_l + \eta \partial_u] = \xi \partial_x + \tau \partial_l + \eta \partial_u \)

\( \Rightarrow x\xi_x + \frac{2}{l}\xi_u - \xi = \xi, \quad x\tau_x + \frac{2}{l}\tau_u = \tau, \)

\( x\eta_x + \frac{2}{l}\eta_u + 2I\eta = \eta. \) \hspace{1cm} (3.103)

Thus \( \mathcal{D} \) is given by equations (3.100-3.103).

Step V.6 Append \( \mathcal{D} \) to SFDQs and reduce to a Standard Form to obtain RSFDQs for the unknown \( \bar{X} \).

Again theory and practice diverge, we can merely simplify \( \mathcal{D} \) alone and obtain

\[
\begin{aligned}
\text{RSFDQs} \quad & \begin{cases}
\xi = \frac{x^2}{2}, \\
\tau = 0, \\
\eta = \frac{2x}{l}.
\end{cases}
\end{aligned}
\] \hspace{1cm} (3.104)

Thus, again, the RSFDQs do not require integrating; they’re merely a statement of the new solution

\( K_5 := \bar{X} = \frac{x^2}{2} \partial_x + \frac{2x}{l} \partial_u. \)

The results are summarised in Figure 3.2, and correspond to those obtained in [1, 11].

3.3 Chapter summary

We began this chapter with a discussion on the importance of commutation relations. We showed how commutation relations can be rewritten as differential equations and, if they are useable, used to produce a reduced system of determining equations. We proceeded by giving a brief outline of the commutator reduction method, followed by Example 3.1.4, the linear heat equation. This example demonstrated how we could
use four known symmetries to dramatically simplify the problem of determining a new symmetry. In §3.2 the steps of the commutator reduction method were simply stated and the remainder of the chapter consisted of examples closely following these steps. In Example 3.2.2, the 3D Laplace equation, we had eight known symmetries and we found the remaining three symmetries using the method. Finally in Example 3.2.3, the classification problem for the nonlinear heat equation \( u_t = (D(u)u_x)_x \), we recursively applied the method to determine the full symmetry algebra for each form of the function \( D(u) \).
Chapter 4

Discussion

4.1 Thesis summary

In this thesis we have been concerned with improving the process of finding symmetries for systems of differential equations. Lie gave a systematic method for characterising symmetries of differential equations by the solution of associated sets of linear homogeneous partial differential equations called infinitesimal determining equations. By fully integrating the infinitesimal determining equations one obtains the Lie algebra of infinitesimal symmetries characterising the Lie symmetry group of the given differential equations. These infinitesimal symmetries play a crucial role in many of the applications of Lie groups to differential equations. Such applications include: reduction of order of ordinary differential equations, mapping solutions to other solutions, reduction of the number of independent variables of partial differential equations, construction of invariant solutions to ordinary and partial differential equations and the detection of linearising transformations of differential equations. The importance of symmetries has led to the development of many software packages to determine them [5].

The standard method for determining symmetries of differential equations has a serious drawback: it is not able to utilise a priori information about the symmetry group. In addition it is coordinate dependent. Our method, the commutator reduction method, is unique in its ability to make use of known symmetries of systems of differential equations, to help simplify the problem of finding the remaining symmetries of such systems.
The commutator reduction method involves using the Reid algorithms [15, 16, 17, 18] to calculate commutation relations. These relations when recast as differential equations are used to simplify the determining equations.

Our method demonstrates that it possible to use explicitly known symmetries to help determine the remaining symmetries of a given system of differential equations. In our method the Lie algebra structure of a symmetry group is intimately\(^1\) involved with the solution of its associated determining equations. This has the exciting implication that the structural theory of abstract Lie algebras can be of direct assistance in the analytical task of integrating determining equations. A first step in this direction is given in Appendix B where a method for obtaining determining equations for the centre of a Lie algebra is given. Our method illustrates that knowledge of some part of an algebra gives information about the rest of the algebra.

There is a new perspective provided by this thesis: Commutator relations can be viewed as differential constraints on their associated infinitesimal determining equations.

The commutator reduction method exploits the relationship between Lie symmetries and their infinitesimal determining equations; the Lie algebra structure of infinitesimal symmetries; and the Reid algorithms (Standard Form, Initial Data and Commutation Relations) for obtaining structural information from determining equations without solving them. We provided an introduction to these methods in Chapter 2. In Chapter 3 we described how commutation relations can be used as differential equation constraints which can help filter out the obvious solutions of a set of determining equations. We next pieced together the illustrative examples of the previous chapter to form an outline of the method which was clarified with further examples.

\(^1\)In so far as equations can be intimate.
4.2 Applications of the commutator reduction method

The commutator reduction method provides a new technique which can be incorporated into existing symbolic packages for automatically determining symmetries. In particular any symmetry package that has either a standard form algorithm or an implementation of the Riquier–Janet algorithm (e.g. Schwarz [21, 24, 23], Topunov [25]) coupled with an implementation of the Commutation Relations algorithm [18] can use the method. Many symmetry problems are ideally suited to the method: any area of symmetry analysis where we have some explicitly known symmetries in hand before setting up and solving the determining equations for the full algebra of infinitesimal symmetries. Two areas of symmetry analysis which fall into this category are given below.

II Group classification problems In applications one often wants to consider a class of differential equations, that is a set of differential equations where each member is characterised by arbitrary functions and parameters. The symmetry analysis of the nonlinear heat equation (Example 2.1.10) is an example of a classification problem. The symmetries found are classified into different algebras depending on the form of the function $D(u)$. There is an extensive theory about classification problems, see [8].

The results of a group classification can be conveniently expressed in the form of a tree, e.g. for the Nonlinear Heat equation, Example 3.2.3 we obtain Figure 3.2, page 68. This clearly demonstrates that some case splittings are subcases of others. Thus symmetries are inherited from "parent nodes" higher up the tree. From a practical viewpoint this means that we can calculate the symmetries for the top node (the generic case) first and then these will be considered as knowns $\mathcal{K}$ for any subcases.
In Example 3.2.3 we see that $J(u) \equiv \frac{3}{4}$ is a subcase of $\dot{J}(u) \equiv 0$. Hence all the $\dot{J}(u) \equiv 0$ symmetries are inherited by the $J(u) \equiv \frac{3}{4}$ branch and may be used as knowns when applying the method to this branch.

Thus the method leads to a more efficient way of recursively calculating the symmetry groups of the branches of group classification problems.

**III Physical problems** If a differential equation (possibly a system of differential equations) has a physical origin (i.e. it models some real problem) then it often possess many symmetries which are trivial to discover. Two reasons why physical problems often have many obvious symmetries are the appearance of certain symmetry groups in fundamental physical laws and the association of fundamental dimensions with physical measurable quantities. Many models in physics are based on fundamental laws and principles which are assumed invariant under some symmetry group. For example Maxwell’s equations of Electromagnetism are invariant under the Lorentz group, and the equations of Newtonian mechanics are invariant under the Galilean group. Thus any model derived from these laws should also be invariant under this group. For example, Laplace’s equation (3.73) possesses the rotational and translational symmetries (2.2) — a fact which we can trivially see when we consider some physical background for this equation.

Another reason for the frequent possession of obvious symmetries of physical problems is that such problems involve measurable quantities (variables and parameters) with associated fundamental dimensions. It can be shown [2] that the presence of these associated dimensions implies that the problem is invariant under a scaling group (change of units). Thus dimensional analysis is a special case of symmetry analysis, and the appearance of associated dimensions in a problem is another
source of obvious or inspectional symmetries. We can conclude that physical problems modelled by differential equations are prime candidates for the application of the commutator reduction method.

4.3 Suggestions for future research

For simplicity and clarity we only applied our commutator reduction method to systems of differential equations with finite dimensional Lie symmetry algebras. Many aspects of our technique can be extended to the important case where the Lie symmetry algebra is infinite dimensional.

We have only considered cases with a single unknown $\dot{X}$ and we have looked for useable commutation relations of the form $[K, \dot{X}] = \dot{K} + \alpha \dot{X}$. Partly we have done this because the equations produced are linear and thus can be exploited by many of the existing symmetry packages. If we allow two or more unknowns $\dot{X}_1, \dot{X}_2, \ldots$ then we can extend the definition of useable commutation relations. For the case of two unknowns we would define useable commutation relations to be relations of the form $[K, \dot{X}_i] = \dot{K} + \alpha_{ij} \dot{X}_j$, $(i, j = 1, 2)$ or $[\dot{X}_1, \dot{X}_2] = \dot{K} + \beta_k \dot{X}_k$, $(k = 1, 2)$. Notice that when these are written as differential equations then they are nonlinear\(^1\). For an example of this see [18]. The methods of Mansfield [9], and Schü et al [20] for bringing systems of nonlinear partial differential equations to the form of a Differential Gröbner Basis (Mansfield) or involutive form (Schü et al) may be useful here.

Higher order commutation relations also offer a useful generalisation of our method. We have seen that useable commutation relations can be rewritten as first order differential equations. This can be generalised to equations involving repeated commutators which yield higher order differential equations. For example, the relation $[\partial_z, [\partial_z, \xi \partial_z]] =$

\(^1\)More precisely, they are bilinear.
0 is equivalent the second order differential equation $\xi_{xx} = 0$. In cases when no useable commutation relations can be found, finding higher order relations may be useful.

In the examples given in this thesis we have been fortunate that the reduced forms of determining equations in standard form have always turned out to be simple linear algebraic equations. However we should caution the reader that the commutation reduction method is not always successful. In particular it can be shown that if the set $\mathcal{K}$ of known symmetries of a Lie symmetry algebra $\mathcal{L}$ is a subalgebra of its center (the elements of $\mathcal{L}$ which commute with all of $\mathcal{L}$) then we get no reduction (i.e. $\dim(\text{RSFDQs}) = \dim(\text{SFDQs})$). This happens, for example, in the case when $\mathcal{L}$ is abelian (i.e. the centre is $\mathcal{L}$). An important question is: how do we detect in advance whether we will get some reduction by using our method?

Even if the dimension of the solution space of the reduced system (the RSFDQs) is smaller than the dimension of the solution space of the original system (the DQs) there is no guarantee that it is simpler to solve (e.g. consider $\xi_{xxxxxxx} = 0$ versus $\xi_x = A(x,t)\xi_t + B(x,t)\xi + C(x,t)$). Consequently, measuring the degree of complexity of the reduced system is an important question.

More work needs to be done to investigate the effects and exploit the freedom of choosing different values for the initial point $x_0$. One cautionary remark can be made however: $x_0$ should not be a point at which the rank of the Lie algebra of symmetry operators degenerates or the standard form of the determining equations becomes singular.

In cases where there is more than one unknown symmetry remaining to be found, we are faced with the problem of deciding which unknown $\mathbf{X}$ to select for the method. Some choices of $\mathbf{X}$ may yield simple calculations whilst others may yield no useable commutation relations at all. In Example 3.1.4 the choice was critical. If we had selected $\mathbf{X} = X_4$ then there would be only three useable commutation relations and the resulting reduced system would turn out to be one dimensional hence some (simple) integration
would be necessary to find $\tilde{X}$. In Example 3.2.2 it was not necessary to make a choice for the unknown $\tilde{X}$ and so we left it arbitrary. In Example 3.2.3 there was no choice available. More research is required to discover a practical algorithm for making wise selections.

Finally we mention the further results listed in Appendix B showing that it is possible to obtain determining equations for certain subalgebras of a Lie algebra given the determining equations of the Lie algebra. These results, showing that there are subsystems corresponding to subalgebras, could be used to implement Lie algebra classification methods such as those given in [14] which work entirely with the determining equations without having to solve them. In addition, many of the results in this list apply to the difficult area of infinite dimensional Lie algebras.
Appendix A

A simplified Standard Form algorithm

We give a description of a simplified Standard Form algorithm chosen for clarity of exposition rather than for its efficiency (which is poor). The algorithm given below is a simplified mixture of aspects of Reid’s Standard Form algorithm[15, 16] and Mansfield’s differential Gröbner basis algorithm[9].

For an input system of linear partial differential equations in independent variables \((u^1, u^2, \ldots, u^m)\) and \(n\) independent variables \((x^1, x^2, \ldots, x^n)\) and an input admissible ordering on derivatives we can reduce the system to a simplified standard form system using the algorithm given below.

Algorithm Standard Form (simplified)

1. For each equation in the system choose a highest derivative with respect to an admissible ordering on derivatives (see note 1 below), solve for it, obtaining the equation in the solved form \(D^a u^\sigma = f\), where \(f\) is some function of derivatives smaller than \(D^a u^\sigma\) in the given ordering.

2. Eliminate every derivative in the right hand side (RHS) of the system which is a (possibly zeroth order) derivative of some derivative in the LHS of the system obtained in step 1. Do this for every equation in the system until no derivative in the RHS is a derivative of any derivative in the LHS of the system.

3. Take integrability conditions: for every pair of equations in the system of the form \(D^a u^\sigma = f\), \(D^b u^\sigma = g\), append the equation \(D^{c-a} f = D^{c-b} g\) to the system, where
Appendix A. A simplified Standard Form algorithm

\( c = \max(a, b) \), i.e. \( c_i = \max(a_i, b_i), i = 1 \ldots n \).

4. Repeat steps 1, 2, 3 until (a) every equation is in solved form with respect to the highest derivative, (b) no explicit substitutions from the LHS of the system into the RHS of the system are possible, and (c) every integrability condition \( D^{c-a}f = D^{c-b}g \) is identically satisfied subject to the reductions outlined in 2.

For more details and examples see [15, 16].

Notes:

1. In choosing a highest derivative in step 1 we require a complete ordering \( \succ \) on the set of derivatives which satisfies certain properties. In particular the ordering must be compatible with differentiation: \( D^{a_k}u \succ D^{b_l}w \Rightarrow D^{a_k+b_l}u \succ D^{b_l+c_k}u \) and must satisfy \( D^{a_k}u \succ u \). For simplicity the standard ordering given by \( \succ \) if the first non-zero member of the list \( \{a_1 - b_1, \ldots, a_n - b_n\} \) is positive, has been used in this thesis, except in the linear heat equation example. For the linear heat equation the determining equations have dependent variables \( (u_1, u_2, u_3), (e, f, T) \) and independent variables \( (x, t) \). Here the ordering is given by \( D^{a_k}u \succ D^{b_l}w \) iff the first non-zero member of the list \( \{k - l, \sum a_j - \sum b_j, a_1 - b_1, \ldots, a_m - b_m\} \) is positive.

2. It is possible to show that the above algorithm will yield a standard form of any linear system in a finite number of steps. The resulting system is formally integrable in the sense of Pommaret[13], a differential Gröbner basis in the sense of Mansfield[9], and it also allows the determination of initial data determining locally unique analytic solutions (when the ordering is graded first by total degree).

3. We discourage the reader from using the above outline as a basis for an efficient computer implementation of a standard form algorithm. Reducing a system of PDEs to a standard form is computationally intensive and the above algorithm is very inefficient compared to Reid’s algorithm[15, 16]. For example, for each dependent variable \( u^\sigma \), step
Appendix A. A simplified Standard Form algorithm

3 above requires the computation of \( \frac{N(\sigma)(N(\sigma)-1)}{2} \) \((= O(N(\sigma)^2))\) integrability conditions, where \( N(\sigma) \) is the number of equations in the system with left hand sides derivatives of \( u^\sigma \), but the algorithm of Reid has reduced this step to an \( O(N(\sigma)) \) process. (There is a certain price to be payed for this reduction: the process is no longer simple and a special elimination method is required for simplifying the integrability conditions.)
Appendix B

The determining equations of the centre of a Lie symmetry algebra

One direction to develop the work of this thesis is to find the determining equations for various subalgebras of infinitesimal Lie symmetry algebras. In particular the following results hold.

Given the determining equations (DQs) for a Lie algebra $\mathcal{L}$ (possibly infinite dimensional) and a subalgebra $\mathcal{K}$ of $\mathcal{L}$, it is possible to produce,

1. The DQs for $\mathcal{C}(\mathcal{L}):= \{X \in \mathcal{L}: [X, \dot{X}] = 0, \forall \dot{X} \in \mathcal{L}\}$, the center of $\mathcal{L}$.

2. The DQs for $\mathcal{C}_\mathcal{L}(\mathcal{K}):= \{X \in \mathcal{L}: [X, K] = 0, \forall K \in \mathcal{K}\}$, the centralizer of $\mathcal{K}$ in $\mathcal{L}$.

3. The DQs for the derived subalgebras $\mathcal{L}^{(i)}$ and $\mathcal{L}^i$, defined by

$$\mathcal{L}^{(1)} = \mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{(i+1)} = [\mathcal{L}^{(i)}, \mathcal{L}^{(i)}]$$

$$\mathcal{L}^{i+1} = [\mathcal{L}, \mathcal{L}^i]$$

where $i = 1, 2, \ldots$ and $[\mathcal{L}, \dot{\mathcal{L}}] := \{[X, \dot{X}]: X \in \mathcal{L}, \dot{X} \in \dot{\mathcal{L}}\}$.

4. The DQs for $\mathcal{N}_\mathcal{L}(\mathcal{K}):= \{X \in \mathcal{L}: [X, K] \in \mathcal{K}, \forall K \in \mathcal{K}\}$, the normalizer of $\mathcal{K}$ in $\mathcal{L}$.

5. The DQs for $\text{rad}(B):= \{X \in \mathcal{L}: \text{Trace}((\text{Ad} X)(\text{Ad} \dot{X})) = 0, \forall \dot{X} \in \mathcal{L}\}$, the radical of the killing form $B$ of the algebra $\mathcal{L}$. (When $\mathcal{L}$ is finite dimensional.) Note: $\text{Ad} X$ is the linear operator on $\mathcal{L}$ defined by $(\text{Ad} X)X = [X, \dot{X}]$.

6. The DQs for $\text{(Ad } K\text{)}\mathcal{L}:= \{[K, X]: X \in \mathcal{L}\}$ where $K$ is an explicitly known infinitesimal symmetry in the algebra $\mathcal{L}$. 

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We shall give the proof of the first result to lend credibility to the fact that the remaining results can be proved using related ideas.

**Proof of 1.**

The centre $\mathcal{C}(\mathcal{L})$ is the set of $X \in \mathcal{L}$ such that $[X, \tilde{X}] = 0$ for all $\tilde{X} \in \mathcal{L}$. We will show that we can convert this commutator equation into a set of differential equations which characterise $X$.

First the determining equations for the Lie algebra $\mathcal{L}$ are reduced to a standard form, obtaining $\Omega$. We write $\Omega(\xi) = 0$ to indicate that the components $\xi$ of the infinitesimal generator $X = \xi^i \partial_{x^i}$ solve the system $\Omega$, i.e. $\Omega(\xi) = 0 \Leftrightarrow \xi^i \partial_{x^i} \in \mathcal{L}$. We will use $X = \xi^i \partial_{x^i}$ to denote an element of the centre and $\tilde{X} = \tilde{\xi}^i \partial_{x^i}$ to denote an arbitrary element of the Lie algebra $\mathcal{L}$. The arbitrariness of $\tilde{X}$ and the commutator equation $[X, \tilde{X}] = 0$ will be exploited to produce DQs for $X$. Writing this equation in terms of its components yields the following differential equations.

\[ \xi^i \partial_{x^j} \tilde{\xi}^i - \tilde{\xi}^i \partial_{x^j} \xi^i = 0 \quad i = 1, \ldots, n. \quad (B.105) \]

We next reduce the differential equations (B.105) modulo $\Omega$, i.e. we use the equations $\Omega(\xi) = 0, \Omega(\tilde{\xi}) = 0$ as substitution rules to eliminate all nonparametric derivatives. Thus (B.105) reduces to a set of equations only involving a finite number ($k$) of parametric derivatives of $\xi$ and $\tilde{\xi}$. Denote these parametric derivatives by $P^1(\xi), P^2(\xi), \ldots, P^k(\xi), P^1(\tilde{\xi}), P^2(\tilde{\xi}), \ldots, P^k(\tilde{\xi})$. Note that there are only finitely many parametric derivatives although the set of all parametric derivatives for the system of SFDQs $\Omega$ may be infinite.

The reduced expression for (B.105) can be written as

\[ a_{ij}(x) P^i(\xi) P^j(\tilde{\xi}) = 0. \quad (B.106) \]

(summing over $i, j = 1, \ldots, k$) where $a_{ij}(x)$ is explicitly known and $a_{ij} \equiv -a_{ji}$ (because of the skew-symmetry of (B.105)). Remember that (B.106) holds for all solutions $\tilde{\xi}$ of...
the system $\Omega$ and this can be exploited by choosing the following initial data for $\tilde{\xi}$. For an arbitrary initial point $x_0$, choose $P^l(\tilde{\xi})(x_0) = 1$, for $l \in \{1, \ldots, k\}$, and set all other parametric initial conditions to zero. Then (B.106) gives $a_{il}(x_0)P^l(\tilde{\xi})(x_0) = 0$. But $x_0$ is arbitrary and repetition of this argument for each value of $l = 1 \ldots k$ yields the following differential equations

$$a_{il}(x)P^l(\xi) = 0, \quad l = 1 \ldots k.$$  \hspace{1cm} (B.107)

These differential equations, when added to the system $\Omega(\xi) = 0$, give the DQs for the Lie algebra $\mathcal{C}(\mathcal{L})$, as required. Conversely, if (B.107) are satisfied then it is trivial to see that (B.106) are satisfied and hence so is (B.105) and thus $X \in \mathcal{C}(\mathcal{L})$. $\square$
Bibliography


