C · 1

HOPF BIFURCATIONS IN MAGNETOCONVECTION IN THE PRESENCE OF SIDEWALLS

By

Hamid R. Z. Zangeneh

B. Sc.(Mathematics) Northeastern University, Boston, Massachussets, 1977.

M. Sc.(Mathematics) Northeastern University, Boston, Massachussets, 1980.

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

MATHEMATICS

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

1993

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Mathematics
The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date: July 7, 1993

Abstract

We study multiple Hopf bifurcations that occur in a model of a layer of a viscous, electrically conducting fluid that is heated from below in the presence of a magnetic field. We assume that the fluid flow is two-dimensional, and consider the effects of sidewalls with stress-free boundary conditions. Our model partial differential equations together with the boundary conditions have two reflection symmetries. We use center manifold theory to reduce the partial differential equations to a two-parameter family of four-dimensional ordinary differential equations. We show that two different normal forms are appropriate, depending on the sizes of certain magnetoconvection parameters for large aspect ratios. We denote the two normal forms by "Case I" and "Case II". In both cases we prove the primary Hopf bifurcation of standing wave (SW) solutions, and we prove the existence of secondary Hopf bifurcations of invariant tori from the SW solutions. We prove that the tori persist in 'wedges' in the parametric plane. In Case II we show that there are also secondary Bogdanov-Takens bifurcation points. Using this, we show there are additional secondary and tertiary bifurcations of periodic solutions and invariant tori, and also argue that generically, there exist transversal homoclinic and heteroclinc points, and consequently open regions of parameter space that correspond to chaos of chaotic regions, and show the existence of quasiperiodic saddle-node bifurcations of invariant tori. Also, we show that in this case the system is a small perturbation of a system with the symmetries of the square, as the aspect ratio approaches infinity.

Table of Contents

List of Tables

List of Figures

Acknowledgement

First of all, I would like to thank my parents, who were my first teachers. They fostered in me the love of knowledge and wisdom. They also taught me to always evoke reasoning and never to accept anything without questioning, and to value education and truth over any material achievement and success. To my brother Bijan, who helped me to choose mathematics over the other fields of study, while I had a chance to study in other fields. He helped and inspired me throughout my studies.

A very special thanks to my supervisor, Wayne Nagata, who taught me what I know about different aspects of the theory of differential equations and bifurcation theory during his very well-organized courses in differential equations and dynamical systems. He introduced my thesis topic to me, and during my research he provided me with constructive and insightful comments, from which I learned a lot. He carefully read my thesis several times, and proof read my thesis very thoroughly. Overall without his help and inspiration this work could not have been done.

I would like to thank Dr. Colin Sparrow, who gave me a lot of inspiration and created a lot of interest in me for chaotic dynamical systems during my short stay in Cambridge.

I would like to thank Dr. Robert Israel for his assistance in using the MG Graphics package for the graphs in this thesis. He answered my questions without any hesitation.

I would like to express my great appreciation to the entire Mathematics Department at U.B.C. for its invaluable and extensive support during my stay here and especially to those who taught the courses I took.

I would like to dedicate this thesis to my wife Afshan and my two wonderful children, Alireza and Maryam, and express my deep gratitude for their patience, support and understanding during my studies at U.B.C.; without their support this thesis would not have been possible.

Chapter 1

Introduction

To improve his living conditions and satisfy his curiosity, mankind needs to find, understand and study laws governing nature. In particular, differential equations modelling natural phenomena have been shown to be extremely useful. The study of differential equations themselves has produced an extensive theory, which in turn motivated more abstract mathematical theories such as the theory of Lie groups, differential geometry and functional analysis.

For one with little background in differential equations, the subject might be seen as a collection of tricks and hints for finding solutions. But with a little acquaintance with the theory it becomes clear that, apart from linear equations, it is rarely possible to integrate systems of differential equations and find the solutions explicitly, while theorems on existence and uniqueness of solutions do not convey much information about the behavior of solutions. This shows the importance of the ideas and methods used in the qualitative study of solutions, or dynamical systems.

The theory of dynamical systems has a rather short history. It can be considered to have been originated by Poincaré, who in the last decade of the nineteenth century revolutionized the study of nonlinear systems of differential equations, by combining techniques of geometry and topology with analytic methods to study qualitative properties of solutions. Around this time, Liapunov also made important contributions to the qualitative study of differential equations. The work of Poincaré and Liapunov was continued and furthered by Birkhoff in the first part of this century. Birkhoff realized the importance of the study of maps and emphasized discrete dynamics, since the qualitative study of differential equations can often be reduced to the study of the iterates of an associated map (the Poincaré map). Also, many problems and phenomena in the qualitative study of differential equations can be seen in their simplest form in the study of discrete dynamical systems. After Birkhoff, the study of dynamical systems was relatively inactive in the West. However, Soviet mathematicians such as Andronov and Pontriagin continued to study differential equations from the qualitative point of view [10].

In the early nineteen-sixties there began a great resurgence of interest in dynamical systems, mainly due to influence of Smale, Peixoto and Moser and in West, and Kolmogorov, Anosov and Arnold in the Soviet Union. In his important survey article, Smale [43] reviewed the concepts of dynamical systems developed by many mathematicians (such as Anosov, Peixoto and Smale himself) during this period, and outlined a program that was followed by many mathematicians, and which led to a good understanding of a class of dynamical systems known as Axiom A or hyperbolic systems. His study of Van der Pol differential equations motivated Smale to construct a two-dimensional map, with chaotic dynamics, which is now known as the Smale horseshoe. This example, studied with the help of differential topological techniques and symbolic dynamics, led to the study of chaotic dynamics in many other systems. In other significant mathematical work, Kolmogrov, Arnold and Moser used hard analysis to develop their celebrated K.A.M. theory on the persistence of certain solutions (invariant tori) under perturbations of integrable Hamiltonian systems. In addition, scientists studying nonlinear models of natural phenomena came to realize the power and beauty of the geometric and qualitative techniques developed during this period, and at the same time raised interesting problems of their own, which provided new sources of motivation for the theory beyond the traditional questions arising from mechanics. Lorenz [28], a meteorologist, presented an analysis of system of three quadratic ordinary differential equations which eventually created great interest in chaotic dynamical systems for mathematicians as well as scientists from other disciplines. The advancement of computer graphics has also contributed to great interest in dynamical systems among non-mathematicians. For more information in the history of dynamical systems see [1, 16, 10, 43, 2].

1.1 Basic concepts

A first step in the qualitative study of a system of differential equations is to study the dynamics of the system close to its fixed points or periodic orbits, since these represent stationary or repeating behavior. Since the theory of linear equations is well-developed, one can consider the linearization of the system about its fixed points or periodic orbits [16]. If the linearized system is hyperbolic (i.e., all the eigenvalues of the linearized system have non-zero real parts), then one can apply the Hartman-Grobman theorem [16] to show that the nonlinear system is topologically equivalent to the linearized system in a small neighborhood of the fixed point or periodic orbit. However if the linearized system is non-hyperbolic (i.e., the linearized system has at least one eigenvalue with zero real part), then the linearized system does not not give enough information about the nonlinear dynamics. In this case one uses center manifold theory [20, 5, 16] to establish the existence of a locally invariant (center) manifold of solutions for the original nonlinear system and then study the dynamics close to the fixed points or periodic orbits restricted to the center manifold. If the rest of eigenvalues of the linearized system have negative real parts, then the center manifold is exponentially attracting, and the product of the dynamics restricted to the center manifold with a linear exponential decay is locally topologically equivalent to the dynamics of the original system [5]. The center manifold can be approximated by its Taylor series to finite order, and this approximation is usually sufficient to determine the dynamics on the center manifold. To be more precise, consider a system of ordinary differential equations

$$\begin{aligned} \dot{x} &= Bx + f(x,y), \\ \dot{y} &= Cy + g(x,y), \end{aligned} \tag{1.1}$$

where $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$, and B and C are $n \times n$ and $m \times m$ matrices whose eigenvalues have zero real parts and negative real parts, respectively. We assume that the nonlinear functions f and g vanish along with their first derivatives, at the origin. Then the center manifold theorem implies that there is a locally invariant center manifold, which can be represented by a local graph

$$W^c = \{(x,y) : y = h(x),\ h(0) = Dh(0) = 0\},$$

where $h : U \to \mathbf{R}^n$ defined in some neighborhood $U \subset \mathbf{R}^m$ of origin. The dynamics of (1.1) at the origin is locally topologically equivalent to

$$\begin{aligned} \dot{x} &= Bx + f(x,h(x)), \\ \dot{y} &= Cy. \end{aligned} \tag{1.2}$$

Thus the local study of (1.1) is reduced to the study the n-dimensional system

$$\dot{x} = Bx + f(x,h(x)). \tag{1.3}$$

The center manifold function $h(x)$ satisfies

$$Dh(x)\,[Bx + f(x,h(x))] - Ch(x) - g(x,h(x)) = 0, \tag{1.4}$$

which just expresses the local invariance of the center manifold. Using the center manifold reduction, the dimensions of the problem can be reduced considerably. The relation (1.4) can be solved approximately for $h(x)$ by expanding in a Taylor series, collecting terms of like powers, and then solving term by term for each Taylor series coefficient of $h(x)$. For proofs of the above statements, see [5, §9.2].

We illustrate these ideas with a simple example [16]. Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= -y + \alpha x^2, \end{aligned} \tag{1.5}$$

and observe that one of the eigenvalues of the linearization about the origin of (1.5) is zero, while the other eigenvalue is negative. By the center manifold theorem, there exists a differentiable one-dimensional center manifold $y = h(x)$ such that $h(0) = 0,\ h'(0) = 0$. By substituting the Taylor series

$$h(x) = ax^2 + bx^3 + \dots,$$

into (1.4), i.e.,

$$h'(x)[xh(x)] + h(x) - \alpha x^2 = 0,$$

we obtain $h(x) = \alpha x^2 + O(x^4)$. Thus the reduced system representing the dynamics on the center manifold is

$$\dot{x} = xh(x) = \alpha x^3 + O(x^5). \tag{1.6}$$

It is easy to see that the fixed point $x = 0$ in (1.6) is asymptotically stable if $\alpha < 0$ and unstable if $\alpha > 0$. Therefore $(x, y) = (0, 0)$ in the system (1.5) is asymptotically stable if $\alpha < 0$ and unstable if $\alpha > 0$.

If after a center manifold reduction the reduced system has dimension greater than one, the system can be simplified further by using the method of Poincaré-Birkhoff normal forms. The basic idea in normal form reduction is to construct appropriate near-identity nonlinear coordinate transformations which annihilate certain nonlinear terms in the Taylor expansion of the system. The method of normal forms is of fundamental importance in local theory of differential equations. For a discussion of normal form theory, see [1, 16].

The methods of center manifold theory and Poincaré-Birkhoff normal forms are important not only in the study of a single system of differential equations, but also in bifurcation theory, where one attempts to analyze a parametrized family of systems of differential equations. One concentrates on bifurcation points, i.e., those parameters for which the system is structurally unstable (a dynamical system is structurally stable if under any small perturbation the perturbed system is still topologically equivalent to the original system). Thus arbitrarily small perturbations of parameters from a bifurcation point will produce topologically inequivalent dynamics. One attempts to find and classify all the topologically inequivalent dynamics possible when parameters are varied in a neighborhood of the bifurcation point. If the analysis is local in a neighborhood of a fixed point or periodic orbit, then one can use center manifold theory and normal forms to simplify the analysis. An illustrative example is that of Hopf bifurcation in a one-parameter family. This bifurcation is associated with pure imaginary eigenvalues for the linearization, and periodic solutions for the nonlinear system. See [16] for more information. If two or more parameters are varied, then more degenerate bifurcation points can be found, and this typically enables one to describe a wide range of behaviors using local analysis. This area of research has been very active in recent years, and there are still many open questions. See [16, Chapter 7] for a survey of two-parameter bifurcations. There exists a parallel theory for discrete dynamical systems, i.e., qualitative study of iterated maps.

Mathematical models of many physical problems have some sort of symmetry. The symmetry can be intrinsic to the physical system, or come from the idealization of an approximate symmetry. Symmetry leads to more degenerate behavior, yet at the same time the presence of symmetry can simplify the analysis. The books of Golubitsky et al. [13, 15] give a systematic treatment of bifurcation with symmetry from the group theory point of view (see also the references therein).

One can use dynamical system methods to study certain partial differential equations. For example, to study local bifurcations in a system of parabolic partial differential equations, one usually find the parameter values for which the linearized system has zero or pure imaginary eigenvalues about its steady state solution. Then by considering the system as an evolution equation in a Hilbert space, or more generally, a Banach space, one can then apply center manifold theory for infinite-dimensional systems. One then obtains a finite-dimensional system of ordinary differential equations, and then one can study the bifurcation of these reduced equations. If the system of partial differential equations have some symmetry, then the center manifold reduction can be done so that the reduced system of ordinary differential equations has the corresponding symmetry.

1.2 Oscillatory convection in fluids and Hopf bifurcations with symmetry

In this thesis, we study the nonlinear dynamics of a model of a horizontal layer of a viscous, electrically conducting fluid that is heated from below in the presence of a vertical magnetic field. Such situations arise in astrophysics, geophysics and in laboratory experiments. We consider two-dimensional motion near the onset of oscillatory (time periodic) convection. Unlike previous studies of magnetoconvection, we consider the effect of sidewalls, especially distant ones. The magnetoconvection model consists of a system of partial differential equations, together with boundary conditions. Several parameters occur naturally in the model, and these represent physical quantities.

In models of two-dimensional convection (e.g., magnetoconvection, convection in binary fluid mixtures) the symmetry group $O(2)$ of rotations and reflection of the circle is often present. This symmetry is due to equivariance of the model partial differential equations (e.g., Navier-Stokes equations) under spatial translations and reflections, and the use of periodic boundary conditions. The theory of Hopf bifurcation in the presence of $O(2)$ symmetry (e.g. [14, 29]) has been successful in accounting for a variety of phenomena observed in experiments, especially on binary fluid mixtures. As the fluid layer is heated from below with increasing intensity, in these experiments the motionless conduction state loses stability to oscillatory modes and appears to undergo Hopf bifurcations as time-dependent convection onsets. Spatio-temporal patterns such as standing waves and travelling waves have been observed. The corresponding experiments in magnetoconvection are more difficult, and we know of no experiments corresponding to the physical situation we study in this thesis. However, see [39, 40] for descriptions of related experiments in magnetoconvection.

In the dynamical system analysis of this phenomenon, the normal form describing this bifurcation has $O(2)$ symmetry (e.g., [15]). In this bifurcation, two branches of symmetry-breaking solutions, denoted by standing waves (SW) (a family of solutions with reflection symmetries) and travelling waves (TW) (solutions with spatio-temporal symmetries) are created [29]. While it can be hoped that the idealizations that are responsible for the $O(2)$ symmetry (infinite layer, periodic boundary conditions) will not qualitatively affect the dynamics much, it is interesting and useful to consider the effect of breaking the symmetry of the system, especially if the corresponding idealization is not satisfied by the real system. For example, periodic boundary conditions are an approximation to more realistic models with only reflection symmetry, due to presence of distant sidewalls which break the continuous translation symmetry of $O(2)$.

To consider the effects of sidewalls, one attempts to reduce the model to a simpler one that captures the dynamics of the original system, at least under certain restrictions. There are several ways to achieve this. One traditional approach has been to use the formal method of multiple scales which results in a simplified partial differential equation for a slowly varying envelope function [24, 9]. However, there is no rigorous explanation for validity of this formal method, and in fact [33] showed that the bifurcation results of [9] using this method are only valid for very small range of parameters. To account for the effect of sidewalls this model can be considered as symmetry breaking perturbation to the idealized system. Another commonly used method is a formal Galerkin reduction, by systematically using only finitely many modes of a Fourier expansion of the solutions. This method usually is justified by physical intuition but often can be made rigorous mathematically. To reduce the problem to a finite dimensional system of ordinary differential equations in a rigorous way, the original system of partial differential equations is considered as an evolution equation in a Hilbert space (e.g. [36, 37, 38]). At critical parameter values, the linearized partial differential equation has only finitely many eigenvalues with zero real parts. If the rest of eigenvalues at these parameter values have negative real parts, then using center manifold theory [20, 47], the existence of an attracting center manifold can be proved, and the evolution equation restricted to the center manifold leads to a finite-dimensional system of ordinary differential equations. The reduced system carries the symmetry of original system, and it can be considered as a system with broken $O(2)$ symmetry (e.g. [32]). The sidewalls destroy the translational symmetry $SO(2)$, but keep a reflection symmetry $\mathbf{Z}_2$. Such an approach has been taken by several authors, and in particular has been used to describe the effects of distant sidewalls on the onset of steady convection in the Rayleigh-Bénard problem [33]. For $O(2)$-equivarient Hopf bifurcations, the effects of various different symmetry breaking perturbations have been considered in [29, 34, 6].

1.3 Overview of the thesis

We consider our magnetoconvection problem in a rectangular region

$$\Omega_L = \{(x, y) : \ -L < x < L, \ 0 < y < 1\},$$

with aspect ratio $2L$, and we use boundary conditions which are extensions of the stress-free boundary conditions commonly used in the regular Rayleigh-Bénard convection problem [48]. With these boundary conditions the magnetoconvection problem has $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry. A parameter R (Rayleigh number) gives a measure of the intensity of heating from below. Using standard methods (e.g. [46]), we express the system as an evolution equation in a Hilbert space. We prove that the spectrum of the linearization $K(R, L)$ about the trivial solution of the evolution equation, consists entirely of isolated eigenvalues with finite multiplicities. Along a particular family of curves $R = R_m(L), m = 1, 2, 3, \ldots$ (curves of values of the Rayleigh number R as a function of half the aspect ratio L), the linearized operator has pure imaginary eigenvalues. Two consecutive curves $R_m(L)$, $R_{m+1}(L)$ intersect at a single point defining a particular value of $L = L_m$. At such a point of intersection K will have a double Hopf point, as two different spatial modes simultaneously become unstable. We prove that for large enough imposed magnetic fields, at $(R, L) = (R_m(L_m), L_m)$ all the rest of the eigenvalues of K have negative real parts (Chapter 2). Then using the center manifold theory for parabolic partial differential equations [20] we find a reduced parametrized family of four-dimensional ordinary differential equations which represents the dynamics on an exponentially attracting, locally invariant center manifold. By using normal form theory, we simplify the reduced equation further. We show that for large L, depending on the size of other parameters in magnetoconvection problem, two different normal forms, which we denote by Case I and Case II, will be appropriate. Case II corresponds to convection with very strong magnetic fields, in fluid with a very small ratio of magnetic diffusivity to thermal diffusivity. Both the normal forms have a double Hopf point near $1:1$ resonance (Chapter 3). After a long calculation, we find explicit expressions for the normal form coefficients and their asymptotic behavior for large L, in both Cases I and II. We then evaluate these coefficients numerically for some parameter values (Chapter 4, Appendices A and

B). In both Cases I and II we prove the existence of primary Hopf bifurcations of two families of standing wave solutions, which we denote by SW_0 and SW_π. Also, we prove the existence of a secondary Hopf bifurcation of invariant tori from the SW solutions, and the persistence of the tori in open regions ('wedges') in parameter space (Chapter 5). For large aspect ratios in Case II, we also find more complicated dynamics. We prove the existence of secondary Bogdanov-Takens bifurcations points at a particular parameter values, and the existence of such bifurcation points implies more complicated dynamics and leads to further bifurcations of invariant tori, existence of transversal homoclinic and heteroclinic points, quasiperiodic saddle-node bifurcations of invariant tori, and consequently the existence of open regions in parameter space for which the dynamics of the system is chaotic. Also, we show in this case that the system is a small perturbation of a system with D_4 symmetry, in the limit as L approaches infinity.

Chapter 2

Oscillatory instabilities of magnetoconvection equations

In this chapter, we describe the physical basis of our problem, and perform some preliminary analysis. In the first section, we present the partial differential equations and boundary conditions that describe magnetoconvection in a two-dimensional layer. Then in §2.2 we discuss the symmetry which the system enjoys. In §2.3 we discuss the linearized stability analysis of the trivial, motionless solution of the magnetoconvection equations and find that there are an infinite number of values of the aspect ratio $2L_m$ of the layer, $m = 1, 2, ...$, such that the linearized equation has pure imaginary eigenvalues and both "even" and "odd" eigenfunctions (oscillatory instabilities). In §2.4 we consider the adjoint problem to the linearized eigenvalue problem, and compute its eigenfunctions. Finally, in §2.5 we study some of the asymptotic behavior of the linearized system for large aspect ratios.

2.1 Magnetoconvection equations

In this section, we consider the partial differential equations that describe the state of an electrically conducting fluid, in the presence of an externally imposed vertical magnetic field. The electrical conductivity of the fluid and the presence of magnetic fields contribute to effects of two kinds. Due to the motion of the electrically conducting fluid across magnetic lines of force, electric currents are generated and the associated magnetic fields contribute to changes in the existing fields. In addition, fluid elements carrying currents transverse to magnetic lines of force contribute to additional forces acting on the fluid elements. The equations describing this situation are (Chandrasekhar [3]):

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &= (1/\rho_0)\left[\nu\Delta\mathbf{u} - \rho g \mathbf{e}_y - \nabla P + (1/\mu_0)(\nabla\times\mathbf{B})\times\mathbf{B}\right] - (\mathbf{u}\cdot\nabla)\,\mathbf{u}, \\
\frac{\partial T}{\partial t} &= \kappa\Delta T - \mathbf{u}\cdot\nabla T, \\
\frac{\partial \mathbf{B}}{\partial t} &= \eta\Delta\mathbf{B} + \nabla\times(\mathbf{u}\times\mathbf{B}), \\
\nabla\cdot\mathbf{u} &= 0, \\
\nabla\cdot\mathbf{B} &= 0,
\end{aligned}
\tag{2.1}
$$

where Δ is the Laplacian operator, ∇ is the the gradient operator, $\mathbf{u}$ is the fluid velocity, ρ is the density, T is the temperature, $\mathbf{B}$ is the magnetic field, P is the pressure, $\mathbf{e}_y$ is the unit vector in vertical direction, g is the acceleration due to gravity, ρ_0 is the density at some reference temperature T_0 , ν is the viscosity, κ is the coefficient of thermometric conductivity, μ_0 the magnetic permeability, and η is the magnetic resistivity. The parameters $g, \rho_0, \nu, \kappa, \mu_0, \eta$ are all assumed to be positive constants.

The first of the above equations is the equation of motion, and can be derived from the conservation of momentum, while the fourth equation is the equation of continuity, and can be derived from the conservation of the mass. We have used the Boussinesq approximation, which treats the density ρ as a constant ρ_0 except where it appears in the external force in the momentum balance. The second equation is the equation of heat conduction, and is obtained from the conservation of energy. The third and last equations, which express the interaction between the fluid motion and the magnetic fields, can be derived from Maxwell's equations. We assume that the density obeys an Oberbeck-Boussinesq equation of state

$$
\rho = \rho_0[1 - a(T - T_0)], \tag{2.2}
$$

where a is the coefficient of thermal expansion, assumed to be a positive constant.

We simplify this problem by assuming that $\mathbf{u}$ and $\mathbf{b}$ are constant in the z direction, and we write $\mathbf{u} = (u, v)$, where u is the horizontal component of the fluid velocity and v is the vertical component. Similarly, we write the magnetic field as $\mathbf{B} = (B_x, B_y)$. We assume that the fluid is confined between the two horizontal planes $y = 0$ and $y = h$ (> 0) and that the temperatures on these two planes are maintained constant at $T = T_0$ on $y = 0$, and at $T = T_1$ on $y = h$, with $T_0 > T_1$.

In the presence of a uniform, vertical magnetic field, the system (2.1)–(2.2) has the trivial motionless solution

$$
\begin{aligned}
\mathbf{u}^{(0)} &= (0, 0), \\
\mathbf{B}^{(0)} &= (0, B_0), \\
T^{(0)} &= T_0 - (T_0 - T_1)(y/h), \\
P^{(0)} &= P_0 - \rho_0 g[y + a(T_0 - T_1)(y^2/2h)].
\end{aligned}
\tag{2.3}
$$

Now we consider finite amplitude perturbations from the motionless solution defined by

$$
\begin{aligned}
\mathbf{B} &= \mathbf{B}^{(0)} + \mathbf{b}, \\
T &= T^{(0)} + \theta, \\
P &= P^{(0)} + \chi.
\end{aligned}
\tag{2.4}
$$

By rescaling the variables as

$$x = h\bar{x},\ y = h\bar{y},\ t = (h^2/\kappa)\bar{t},\ \mathbf{u} = (\kappa/h)\bar{\mathbf{u}},$$

$$\chi = (\rho_0 \nu\kappa/h^2)\bar{\chi},\ \theta = (T_0 - T_1)\bar{\theta},\ \mathbf{b} = B_0\bar{\mathbf{b}},$$

and then "dropping the bars", we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = \sigma\left[\Delta\mathbf{u} - \nabla\chi + R\theta\mathbf{e}_y + \zeta Q\left(\nabla \times \mathbf{b}\right) \times \left(\mathbf{e}_y + \mathbf{b}\right)\right] - \left(\mathbf{u}\cdot\nabla\right)\mathbf{u},$$

$$\begin{aligned}
\frac{\partial \theta}{\partial t} &= \Delta\theta + v - \mathbf{u}\cdot\nabla\theta, \qquad (2.5)\\
\frac{\partial \mathbf{b}}{\partial t} &= \zeta\Delta\mathbf{b} + \nabla\times\left[\mathbf{u}\times(\mathbf{e}_y+\mathbf{b})\right],\\
\nabla\cdot\mathbf{u} &= 0,\\
\nabla\cdot\mathbf{b} &= 0.
\end{aligned}$$

The parameters appearing in the convection equation (2.5) are all positive, and are defined by

$$\begin{aligned}
R &= \frac{ga(T_0-T_1)h^3}{\kappa\nu} \quad \text{(Rayleigh number)},\\
Q &= \frac{B_0^2h^2}{\mu_0\rho_0\eta_0\nu} \quad \text{(Chandrasekhar number)},\\
\sigma &= \nu/\kappa \quad \text{(Prandtl number)},\\
\zeta &= \eta/\kappa \quad \text{(magnetic Prandtl number)}.
\end{aligned}$$

Note that the Rayleigh number R is proportional to the temperature difference between the lower (warmer) and upper (cooler) boundaries, and Q increases with the strength of the imposed magnetic field.

Our system of equations is accompanied by boundary conditions. The simplest boundary conditions to work with analytically are the extensions to magnetohydrodynamics of the "stress-free" boundary conditions used for ordinary Rayleigh-Bénard convection [3]. We assume that the fluid is confined to the rectangular region

$$\Omega_L = \{(x,y):\ 0<y<1,\ -L<x<L\}, \qquad (2.6)$$

the temperature is kept constant at the upper and lower boundaries $y=0,1$, and the sidewalls at $x=L,-L$ are insulated. The total magnetic flux through the region remains constant, and the normal velocity, together with the tangential components of both the viscous and magnetic stresses vanishes on all boundaries. Thus

$$\frac{\partial u}{\partial y} = v = \theta = b_x = \frac{\partial b_y}{\partial y} = 0, \quad \text{on} \quad y = 0,1, \qquad (2.7)$$

$$u = \frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = b_x = \frac{\partial b_y}{\partial x} = 0, \quad \text{on } x = L, -L.$$

These boundary conditions do not correspond to a physical situation that is easily produced in a laboratory, but they are commonly used for computational convenience since the eigenfunctions of the linear problem yield sines and cosines. Gibson [12] shows that the criteria for the onset of instability are not substantially altered when more realistic boundary conditions are adopted.

2.2 Symmetry

Symmetry can play an important role in the bifurcations of systems of differential equations, and has received much attention in recent years (see, e.g., Golubitsky et al. [13, 14, 15]). System (2.5)-(2.7) possesses a $\mathbf{Z_2} \oplus \mathbf{Z_2}$ symmetry. To explain this fact, we define the action J on the dependent variables corresponding to the reflection $x \to -x$ through the vertical midline $x = 0$ of the layer:

$$\begin{aligned} J\phi(t,x,y) &= -\phi(t,-x,y) \qquad \text{if } \phi = u \text{ or } b_x, \\ J\phi(t,x,y) &= \phi(t,-x,y) \qquad \text{if } \phi = v, \chi, \theta \text{ or } b_y. \end{aligned} \tag{2.8}$$

There is an additional symmetry, due to our use of the Boussinesq approximation, with respect to the reflection $y \to 1-y$ about the horizontal midline $y = 1/2$ of the layer. We define the action β on the dependent variables corresponding to this symmetry by

$$\begin{aligned} \beta\phi(t,x,y) &= \phi(t,x,1-y) \qquad \text{if } \phi = u, b_y \text{ or } \chi \\ \beta\phi(t,x,y) &= -\phi(t,x,1-y) \qquad \text{if } \phi = v, \theta \text{ or } b_x. \end{aligned} \tag{2.9}$$

The transformations J, β generate a group of symmetries for equations (2.5) and (2.7) that is isomorphic to the group $\mathbf{Z_2} \oplus \mathbf{Z_2}$. We will exploit this symmetry in our treatment of the onset of convection.

2.3 Linear stability analysis

To analyze local bifurcations from the steady state solution of the system (2.5)–(2.7) one first considers the linearized equation as an approximation to the original system. We expand an arbitrary disturbance in terms of some suitable set of normal modes, and examine the stability of the system with respect to each of these modes. In our case we use Fourier modes which satisfy the boundary conditions. Then we seek solutions $\Phi = (u, v, \theta, b_x, b_y)$ in the form

$$\Phi(x, y, t) = \hat{\Phi}(x, y)e^{t\alpha}. \tag{2.10}$$

We will justify this formal stability analysis in Chapter 3. Using equations (2.5), (2.7), and (2.10) and then linearizing, it is easy to see that α satisfies the following eigenvalue problem with boundary conditions (2.7):

$$\begin{aligned}
\sigma\Delta\hat{u} + \sigma\zeta Q\left(\frac{\partial\hat{b}_x}{\partial y} - \frac{\partial\hat{b}_y}{\partial x}\right) - \sigma\frac{\partial\hat{\chi}}{\partial x} &= \alpha\hat{u}, \\
\sigma\Delta\hat{v} + \sigma R\hat{\theta} - \sigma\frac{\partial\hat{\chi}}{\partial y} &= \alpha\hat{v}, \\
\Delta\hat{\theta} + \hat{v} &= \alpha\hat{\theta}, \\
\zeta\Delta\hat{b}_x + \frac{\partial\hat{u}}{\partial y} &= \alpha\hat{b}_x, \\
\zeta\Delta\hat{b}_y - \frac{\partial\hat{u}}{\partial x} &= \alpha\hat{b}_y, \\
\frac{\partial\hat{u}}{\partial x} + \frac{\partial\hat{v}}{\partial y} &= 0, \\
\frac{\partial\hat{b}_x}{\partial x} + \frac{\partial\hat{b}_y}{\partial y} &= 0.
\end{aligned} \tag{2.11}$$

We will find two sets of solutions.

i) *Even solutions (m even):*

$$\hat{u} = c_1 \sin(m\pi x/2L)\cos(n\pi y),$$

$$
\begin{aligned}
\hat{v} &= c_2 \cos(m\pi x/2L)\sin(n\pi y), \\
\hat{\theta} &= c_3 \cos(m\pi x/2L)\sin(n\pi y), \\
\hat{b}_x &= c_4 \sin(m\pi x/2L)\sin(n\pi y), \\
\hat{b}_y &= c_5 \cos(m\pi x/2L)\cos(n\pi y), \\
\hat{\chi} &= c_6 \cos(m\pi x/2L)\cos(n\pi y).
\end{aligned} \tag{2.12}
$$

ii) *Odd solutions (m odd):*

$$
\begin{aligned}
\hat{u} &= c_1 \cos(m\pi x/2L)\cos(n\pi y), \\
\hat{v} &= c_2 \sin(m\pi x/2L)\sin(n\pi y), \\
\hat{\theta} &= c_3 \sin(m\pi x/2L)\sin(n\pi y), \\
\hat{b}_x &= c_4 \cos(m\pi x/2L)\sin(n\pi y), \\
\hat{b}_y &= c_5 \sin(m\pi x/2L)\cos(n\pi y), \\
\hat{\chi} &= c_6 \sin(m\pi x/2L)\cos(n\pi y).
\end{aligned} \tag{2.13}
$$

Substituting (2.12) or (2.13) into (2.11), we find in both cases that eigenvalues α must satisfy the cubic equation

$$
\begin{aligned}
\alpha^3 + (\sigma+\zeta+1)P_{mn}\alpha^2 &+ \left[P_{mn}^2(\sigma\zeta+\sigma+\zeta) + \sigma n^2\zeta Q\pi^2 - \frac{m^2\pi^2\sigma R}{4P_{mn}L^2}\right]\alpha \\
&+ \sigma n^2\zeta Q\pi^2 P_{mn} - \frac{m^2\pi^2\sigma R\zeta}{4L^2} + \sigma\zeta P_{mn}^3 = 0.
\end{aligned} \tag{2.14}
$$

where

$$
P_{mn} = \pi^2\left((m/2L)^2 + n^2\right). \tag{2.15}
$$

To simplify (2.14), we put

$$
\begin{aligned}
Q &= \frac{q_{mn}P_{mn}^2}{n^2\pi^2}, \\
R &= \frac{4r_{mn}L^2P_{mn}^3}{\pi^2 m^2}, \\
\alpha &= sP_{mn},
\end{aligned} \tag{2.16}
$$

so then s must satisfy

$$\begin{aligned} s^3+(\sigma+\zeta+1)s^2+s\left[\sigma(1+\zeta q_{mn}-r_{mn})+\zeta(\sigma+1)\right] & \\ +\sigma\zeta(1-r_{mn}+q_{mn}) &= 0. \end{aligned} \tag{2.17}$$

Since we are interested in Hopf bifurcation from the trivial solution, we look for pure imaginary roots. Equation (2.17) will have pure imaginary roots $s=\pm i\omega_{mn}$, and (2.11) will have pure imaginary eigenvalues $\alpha=\pm iP_{mn}\omega_{mn}$, if

$$r_{mn}=r^*_{mn}=(\sigma+\zeta)\left[\frac{\zeta q_{mn}}{\sigma+1}+\frac{\zeta+1}{\sigma}\right], \tag{2.18}$$

with

$$\begin{aligned} \omega^2_{mn} &= -\zeta^2+\frac{\sigma\zeta q_{mn}(1-\zeta)}{(\sigma+1)} \\ &= -\zeta^2+\frac{\sigma\zeta Q\pi^2n^2(1-\zeta)}{(\sigma+1)P^2_{mn}}. \end{aligned} \tag{2.19}$$

We require that $\omega^2_{mn}>0$, which is satisfied if $\zeta<1$ and

$$q_{mn}>q_0=\frac{\zeta(1+\sigma)}{\sigma(1-\zeta)}, \tag{2.20}$$

i.e.,

$$Q>\frac{P^2_{mn}q_0}{n^2\pi^2}.$$

Remark 2.1 *Equation (2.17) has a zero eigenvalue if*

$$r_{mn}=1+q_{mn}$$

which implies that (2.11) has a zero eigenvalue for parameters satisfying

$$R=R^0_m(L)=\frac{4L^2P^2_{mn}(P^2_{mn}+n^2\pi^2Q)}{\pi^2m^2}$$

(see Figure 2.2). However, if Q is sufficiently large, the imaginary eigenvalues occur at a lower value of the Rayleigh number, thus the onset of instability is through oscillatory modes.

Now by substituting $\alpha = iP_{mn}\omega_{mn}$ into (2.12) and (2.13) we find eigenfunctions of (2.11) for both cases (i) and (ii). In case (i) we denote the eigenfunction as the even eigenfunction Φ^E and in case (ii) we denote the eigenfunction as the odd eigenfunction Φ^O. We have

$$\Phi^E_{mn} = \begin{pmatrix} -\dfrac{2L}{m}\sin(m\pi x/2L)\cos(n\pi y) \\ \cos(m\pi x/2L)\sin(n\pi y) \\ \dfrac{1}{p_{mn}(1+i\omega_{mn})}\cos(m\pi x/2L)\sin(n\pi y) \\ \dfrac{2n^2\pi}{mP_{mn}(\zeta+i\omega_{mn})}\sin(m\pi x/2L)\sin(n\pi y) \\ \dfrac{n\pi}{P_{mn}(\zeta+i\omega_{mn})}\cos(m\pi x/2L)\cos(n\pi y) \end{pmatrix}, \tag{2.21}$$

for m even, and

$$\Phi^O_{mn} = \begin{pmatrix} \dfrac{2L}{m}\cos(m\pi x/2L)\cos(n\pi y) \\ \sin(m\pi x/2L)\sin(n\pi y) \\ \dfrac{1}{P_{mn}(1+i\omega_{mn})}\sin(m\pi x/2L)\sin(n\pi y) \\ -\dfrac{2n^2L\pi}{mP_{mn}(\zeta+i\omega_{mn})}\cos(m\pi x/2L)\sin(n\pi y) \\ \dfrac{n\pi}{P_{mn}(\zeta+i\omega_{mn})}\sin(m\pi x/2L)\cos(n\pi y) \end{pmatrix}, \tag{2.22}$$

for m odd, where $\Phi = (u, v, \theta, b_x, b_y)^T$. We will not include the pressure term χ, since it can be recovered using the velocity terms (u, v).

Now from (2.16) and (2.18) the critical Rayleigh numbers $R_{mn}(L)$, for which (2.11) has pure imaginary eigenvalues $\alpha = \pm iP_{mn}\omega_{mn}$, are

$$R_{mn}(L) = (\sigma+\zeta)\left[\frac{n^2\pi^2\zeta Q}{\sigma+1} + \frac{(\zeta+1)P_{mn}^2}{\sigma}\right]\frac{4L^2P_{mn}}{\pi^2m^2}. \tag{2.23}$$

To simplify the above expression, we temporarily fix m, and let $x = 4L^2/m^2, \bar{R}_n(x) = R_{mn}(L)$. Then the critical Rayleigh number R for each n is given by

$$\bar{R}_n(x) = n^2 Ax(n^2 + 1/x) + Bx(1/x + n^2)^3, \tag{2.24}$$

where

$$A = \frac{(\sigma + \zeta)\zeta Q \pi^2}{\sigma + 1} \quad \text{and} \quad B = \frac{(\zeta + 1)(\sigma + \zeta)\pi^4}{\sigma}. \tag{2.25}$$

It is clear that $\bar{R}_n(x)$ is minimized when $n = 1$, and

$$\lim_{x \to 0+} \bar{R}_1(x) = \infty, \qquad \lim_{x \to \infty} \bar{R}_1(x) = \infty.$$

Since

$$\bar{R}_1'(x) = \frac{(A+B)x^3 - 3Bx - 2B}{x^3}$$

has only one positive root $x = x^*$, and since $\bar{R}_1''(x) = 6B(x+1)/x^4 > 0 \quad$ for $\quad x > 0$, $\bar{R}_1$ will have its minimum at $x = x^*$. Since $x = (2L/m)^2$, we have infinitely many curves $R_{m1}(L) = \bar{R}_1(4L^2/m^2)$ depending on m, and each will have its minimum point at $L_m^* = m\sqrt{x^*}/2$, with minimum value $R_{m1}(L_m^*) = \bar{R}_1(x^*)$.

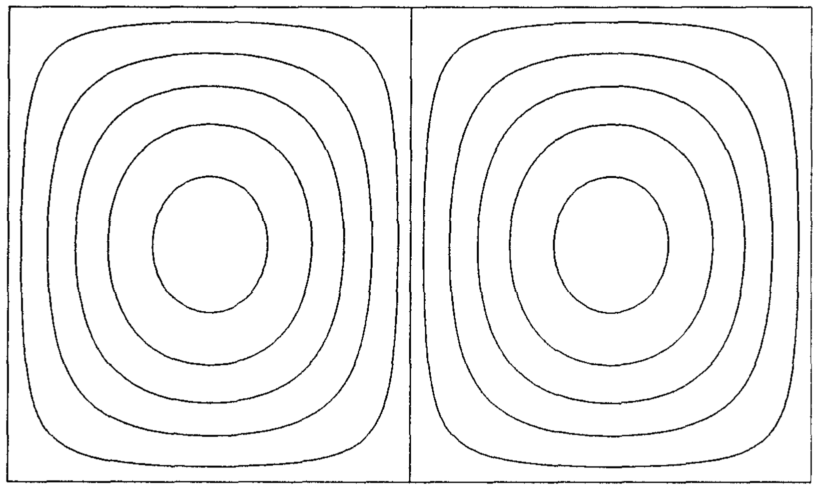
The number m in $R_{m1}(L)$ corresponds to the number of rolls in the region Ω_L, that we expect to bifurcate from the steady-state solution at the critical Rayleigh number $R = R_m(L)$ of the original convection problem. We refer to these solutions as "even" when m is "even" and as "odd" when m is odd (Figure 2.1).

We define L_m by the value of L where the curves of the critical Rayleigh numbers corresponding to odd and even solutions intersect, i.e., $L = L_m$ is the unique solution of $R_{m1}(L) = R_{m+1,1}(L)$. It is clear that L_m lies between L_m^* and L_{m+1}^*, i.e.,

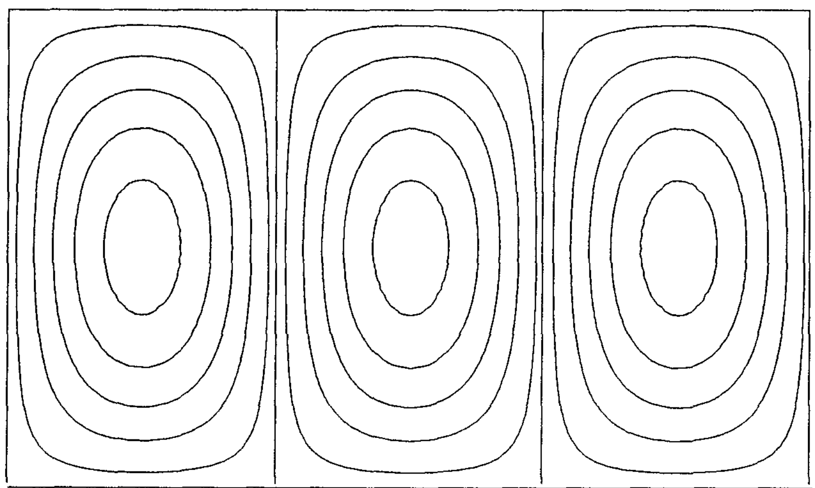
$$\frac{m\lambda}{2} < L_m < \frac{(m+1)\lambda}{2} \tag{2.26}$$

where $\lambda = \sqrt{x^*}$ satisfies

$$\frac{2}{\lambda^2} + \frac{3}{\lambda^4} - 1 - \frac{\sigma\zeta Q}{(\sigma+1)(1+\zeta)\pi^2} = 0. \tag{2.27}$$



(*a*)



(*b*)

Figure 2.1: Level curves of the stream function: (*a*) the even mode $m = 2$; (*b*) the odd mode $m = 3$.

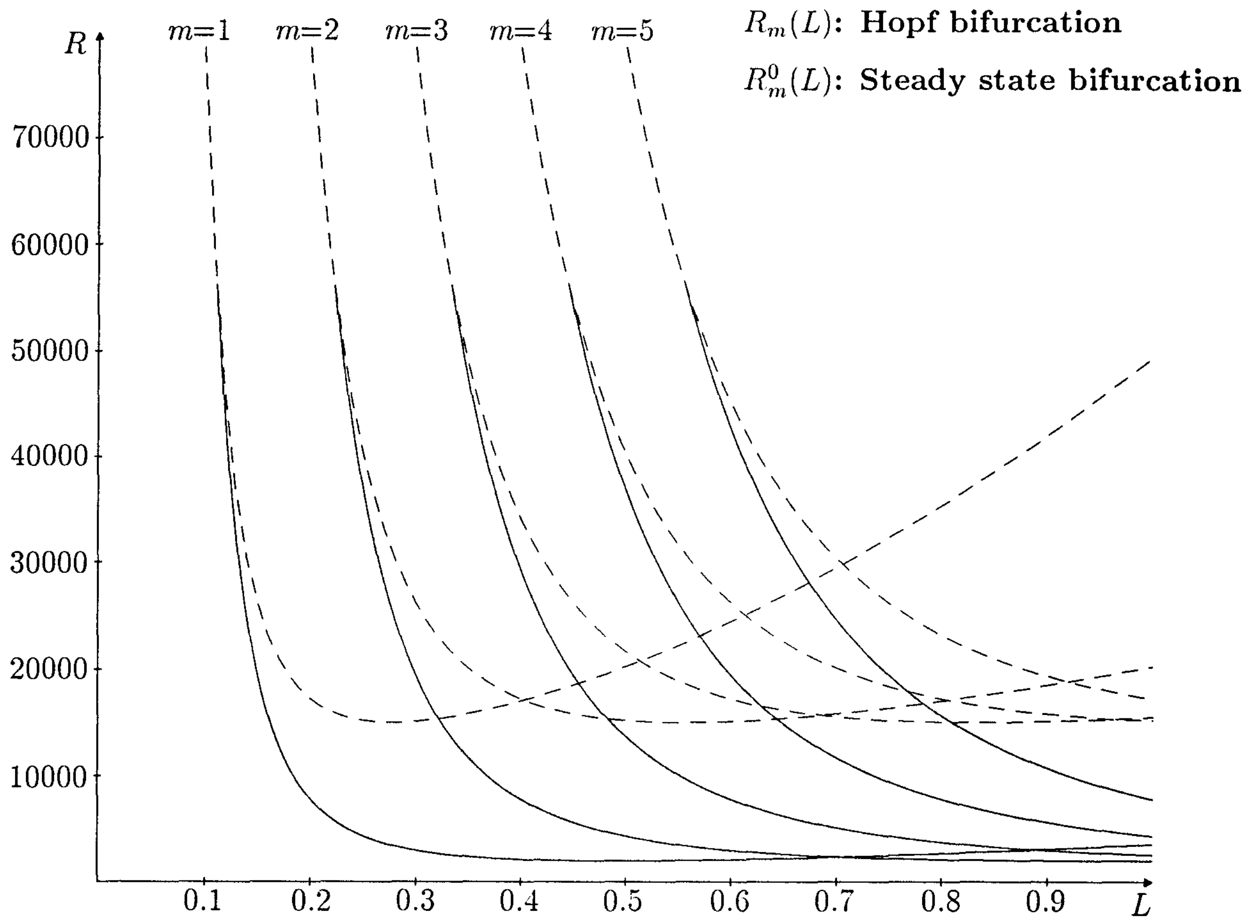


Figure 2.2: Graphs of critical Rayleigh numbers $R_m(L)$ and $R_m^0(L)$vs. L for $m = 1, ..., 5$, with $\sigma = 1$, $\zeta = .1$, $Q = 100\pi^2$. Solid curves represent the graphs of $R_m(L)$ and dashed curves show the graphs of $R_m^0(L)$. For a given m, the graphs of $R_m(L)$ and $R_m^0(L)$ intersect at $L = (.1112)m$ and $R \approx 56000$.

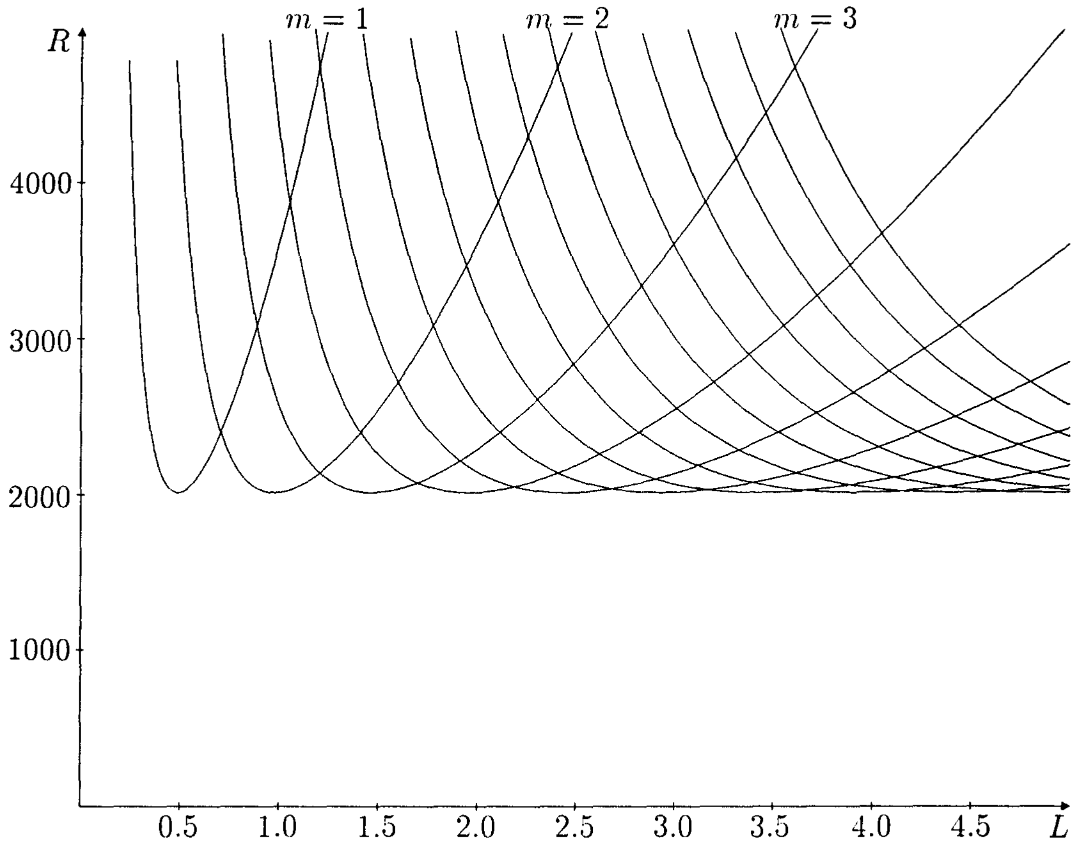


Figure 2.3: Graphs of critical Rayleigh numbers $R_m(L)$ vs. L, for $m = 1, ..., 15$ with $\sigma = 1,\ \ \zeta = .1,\ \ Q = 100\pi^2$.

We observe that (2.27) implies that λ is adecreasing function of Q, and $0 < \lambda < \sqrt{2}$. (see Figure (2.2)).

To simplify our notation further, we put

$$R_m = R_{m1}(L_m), \tag{2.28}$$

and for each fixed m, we put

$$P_1 = P_{m1}, \quad P_2 = P_{m+1,1}, \quad \omega_1 = \omega_{m1}, \quad \omega_2 = \omega_{m+1,1}. \tag{2.29}$$

Then if m is odd, the critical eigenfunctions become

$$\Phi_1 = \Phi^O_{m1} = \begin{pmatrix} \frac{2L_m}{m}\cos(m\pi x/2L_m)\cos(\pi y) \\ \sin(m\pi x/2L_m)\sin(\pi y) \\ \frac{1}{P_1(1+i\omega_1)}\sin(m\pi x/2L_m)\sin(\pi y) \\ -\frac{2L_m\pi}{mP_1(\zeta+i\omega_1)}\cos(m\pi x/2L_m)\sin(\pi y) \\ \frac{n\pi}{P_1(\zeta+i\omega_1)}\sin(m\pi x/2L_m)\cos(\pi y) \end{pmatrix}, \tag{2.30}$$

and

$$\Phi_2 = \Phi^E_{m+1,1} = \begin{pmatrix} -\frac{2L_m}{m+1}\sin((m+1)\pi x/2L_m)\cos(\pi y) \\ \cos((m+1)\pi x/2L_m)\sin(\pi y) \\ \frac{1}{P_2(1+i\omega_2)}\cos((m+1)\pi x/2L_m)\sin(\pi y) \\ \frac{2\pi L_m}{(m+1)P_2(\zeta+i\omega_2)}\sin((m+1)\pi x/2L_m)\sin(\pi y) \\ \frac{\pi}{P_2(\zeta+i\omega_2)}\cos((m+1)\pi x/2L_m)\cos(\pi y) \end{pmatrix}, \tag{2.31}$$

Remark 2.2 *Without loss of generality we assume m is odd, since when m is even only the roles of Φ_1 and Φ_2 are interchanged.*

Now we can prove the following proposition:

Proposition 2.1 *Fix $\sigma > 0$, $0 < \zeta < 1$, $Q > 0$. and fix a positive integer m. If Q is sufficiently large, then when $L = L_m$ and $R = R_m$ the linearized system (2.11) has eigenvalues $\pm p_j\omega_j$, $j = 1, 2$, and the rest of eigenvalues have negative real parts, uniformly bounded away from the imaginary axis.*

Proof: For fixed $\bar{m}$, by (2.14), when $L = L_{\bar{m}}$ and $R = R_{\bar{m}}$ the eigenvalues α satisfy

$$\alpha^3 + C\alpha^2 + D\alpha + E = 0, \tag{2.32}$$

where

$$\begin{aligned}
C &= (\sigma + \zeta + 1)P_{mn}, \\
D &= P_{mn}^2(\sigma\zeta + \sigma + \zeta) + \sigma n^2\zeta Q\pi^2 - \frac{m^2\pi^2\sigma R_{\bar{m}}}{4P_{mn}L_{\bar{m}}^2}, \\
E &= \sigma\zeta\left(n^2 Q\pi^2 P_{mn} - \frac{m^2\pi^2 R_{\bar{m}}}{4L_{\bar{m}}^2} + P_{mn}^3\right) = 0,
\end{aligned}$$

and

$$P_{mn} = \pi^2\left[(m/2L_{\bar{m}})^2 + n^2\right], \quad m, n = 1, 2, \ldots.$$

By the Hurwitz criteria for stability [17] roots of equation (2.32) will have negative real parts if $C, CD - E$ and E are all positive. Clearly $C > 0$ and is uniformly bounded away from zero for all m and n. Also it is clear that $CD - E$ and E are increasing in n, and therefore will be minimized at $n = 1$, so we consider their values when $n = 1$. Using (2.18) we obtain

$$\begin{aligned}
CD - E &= \frac{\sigma\zeta(\sigma + \zeta)\pi^2 Q\left(\bar{m}^2 P_{m1} - m^2 P_{\bar{m}1}\right)}{\bar{m}^2} \\
&\quad + \frac{(\sigma + 1)(\sigma + \zeta)(\zeta + 1)\left[\bar{m}^2 P_{m1}^3 - m^2 P_{\bar{m}1}^3\right]}{\bar{m}^2}
\end{aligned} \tag{2.33}$$

Substituting the definitions of P_{m1} and $P_{\bar{m}1}$ into (2.33) after some simplification we find that $CD - E$ is positive if

$$\Sigma(m) = \left(\bar{m}^2 - m^2\right)\left[\frac{\sigma\zeta Q}{\pi^2(\sigma + 1)(\zeta + 1)} + 1 - \frac{\bar{m}^2 m^2(\bar{m}^2 + m^2)}{(2L_{\bar{m}})^6} - \frac{3\bar{m}^2 m^2}{(2L_{\bar{m}})^4}\right] \tag{2.34}$$

is positive. Now using (2.23) and $R_{\bar{m}}(L_{\bar{m}}) = R_{\bar{m}+1}(L_{\bar{m}})$, after some simplification we get

$$\frac{\sigma\zeta Q}{\pi^2(\sigma+1)(\zeta+1)} + 1 = \frac{3(\bar{m}+1)^2\bar{m}^2}{(2L_{\bar{m}})^4} + \frac{\bar{m}^2(\bar{m}+1)^2[\bar{m}^2+(\bar{m}+1)^2]}{(2L_{\bar{m}})^6}. \tag{2.35}$$

We substitute (2.35) into (2.34) to get

$$\begin{aligned}\Sigma(m) &= \bar{m}^2(\bar{m}^2-m^2)\left\{\frac{3\left[(\bar{m}+1)^2-m^2\right]}{(2L_{\bar{m}})^4}+\right.\\ &\qquad \left.\frac{(\bar{m}+1)^2\left[\bar{m}^2+(\bar{m}+1)^2\right]-m^2(m^2+\bar{m}^2)}{(2L_{\bar{m}})^6}\right\}.\end{aligned} \tag{2.36}$$

From (2.36) it is evident that $\Sigma(m) = 0$ when $m = \bar{m}$ or $m = \bar{m}+1$ and we have two pairs of pure imaginary eigenvalues, while the third eigenvalues $-(\sigma+\zeta+1)$ are real and negative. For $m > \bar{m}+1$ or $m < \bar{m}$, $\Sigma(m) > 0$, and therefore $CD - E$ are positive for all values of m and n. To find a uniform bound on $\Sigma(m)$ we notice that for all $m \neq \bar{m}$ and $m \neq \bar{m}+1$, we have

$$\Sigma(m) \geq \min\{\Sigma(\bar{m}-1), \Sigma(\bar{m}+2)\} > 0, \tag{2.37}$$

where m is fixed.

Now we need to show that E is also positive uniformly bounded away from zero, for sufficiently large Q. For $n = 1$ we can rewrite E in the form

$$E = \sigma\zeta\left[P_{m1}^3 + P_{m1}(\pi^2 Q - R_{\bar{m}}) + \pi^2 R_{\bar{m}}\right]. \tag{2.38}$$

E is a cubic function in P_{m1}, by finding its minimum it is easy to show that

$$E \geq \pi^2 R_{\bar{m}} - (4/3\sqrt{3})\left(R_{\bar{m}} - \pi^2 Q\right)^{3/2}, \tag{2.39}$$

therefore $E > 0$ if

$$\left(R_{\bar{m}} - \pi^2 Q\right)^{3/2} < (3\sqrt{3}/4)\pi^2 R_{\bar{m}}. \tag{2.40}$$

For a given σ, ζ, Q and $\bar{m}$, one can always check whether (2.40) is satisfied. However, $R_{\bar{m}}$ depends on Q, so we proceed to show that (2.40) is always satisfied for sufficiently large

Q. It is clear that (2.40) is satisfied and E will be positive uniformly bounded away from zero if

$$\pi^2 Q - R_{\bar{m}} > 0. \tag{2.41}$$

We note that $R_{\bar{m}} = R_{\bar{m}+1}(L_{\bar{m}})$, and therefore

$$\pi^2 Q - R_{\bar{m}+1} = P^2_{\bar{m}+1,1}\left\{q_{\bar{m}+1,1} - r_{\bar{m}+1,1}\left[1 + \left(\frac{2L_{\bar{m}}}{\bar{m}+1}\right)^2\right]\right\}, \tag{2.42}$$

On the other hand, using (2.35), after some calculation we have

$$\frac{\pi^4\sigma\zeta Q}{(\sigma+1)(\zeta+1)} < 2P^3_{\bar{m}+1,1} - 3\pi^2 P^2_{\bar{m}+1,1}, \tag{2.43}$$

and it follows that

$$\left(\frac{2L_{\bar{m}}}{\bar{m}+1}\right)^2 < \frac{2(\sigma+1)(\zeta+1)}{\sigma\zeta q_{\bar{m}+1,1}}. \tag{2.44}$$

Then using (2.18) we have

$$\begin{aligned}\pi^2 Q - R_{\bar{m}} \;&>\; P^2_{\bar{m}+1,1}\left[\frac{(1-\zeta)(1+\sigma+\zeta)q_{\bar{m}+1,1}}{\sigma+1} - \frac{3(\sigma+\zeta)(1+\zeta)}{\sigma}\right. \\ &\quad\left. - \frac{2(1+\zeta)^2(1+\sigma)(\sigma+\zeta)}{\sigma^2\zeta q_{\bar{m}+1,1}}\right],\end{aligned} \tag{2.45}$$

It is clear that the expression inside the square brackets on the right hand side of (2.45) is increasing with respect to $q_{\bar{m}+1,1}$, and it is easy to check that it is positive for

$$q_{\bar{m}+1,1} > q_1 = \frac{3(1+\zeta)^2(1+\sigma)}{\sigma\zeta(1-\zeta)}. \tag{2.46}$$

i.e.,

$$Q > \frac{\pi^2}{P^2_{\bar{m}+1,1}} q_1. \tag{2.47}$$

From (2.26) and (2.27), it follows that for fixed σ, ζ and $\bar{m}$ the right hand side of (2.44) is $O(Q^{2/3})$ as $Q \to \infty$, hence (2.44) is satisfied for all sufficiently large Q. Q.E.D.

Remark 2.3 *For specific choices of parameter values, we can check (2.40) and (2.41) numerically (see Chapter 4).*

We briefly summarize the results of this chapter so far. We have considered the linearized stability of the magnetoconvection problem for fixed values of σ, ζ, Q, where $\sigma > 0$, $0 < \zeta < 1$ and $Q > Q_0$, as we increase the Rayleigh number R through the positive numbers. For given $L > 0$, when $R < R_{m1}(L)$ for all m, all the eigenvalues of the linearized eigenvalue problem (2.11) have negative real parts, and by the principle of linearized stability, the steady state solution (2.3) is asymptotically stable. If $L = L_m$, then when $R = R_m = R_m(L_m)$ the linearized eigenvalue problem will have two pairs of imaginary eigenvalues, while the other eigenvalues have negative real parts, and we expect a double Hopf bifurcation with reflection symmetry.

2.4 The adjoint problem

In this section we consider the adjoint problem to the linearized system (2.11), and calculate its eigenfunctions when $L = L_m$ and $R = R_m$. This information will be useful for our nonlinear analysis in the next chapter.

We define the adjoint of linearized eigenvalue problem with respect to the inner product

$$\langle \Phi^1, \Phi^2 \rangle_L = (2/L) \int_0^1 \int_{-L}^{L} \left(u^1 \bar{u}^2 + v^1 \bar{v}^2 + \theta^1 \bar{\theta}^2 + b_x^1 \bar{b}_x^2 + b_y^1 \bar{b}_y^2 \right) dx dy, \tag{2.48}$$

where

$$\Phi^j = \left(u^j, v^j, \theta^j, b_x^j, b_y^j\right)^T, \quad j = 1, 2,$$

and the overbars denote complex conjugation. Integration by parts yields the adjoint eigenvalue problem to (2.11),

$$\begin{aligned}
\sigma \left(\Delta u^* - \frac{\partial \chi^*}{\partial x} \right) - \sigma \zeta Q \left(\frac{\partial b_x{}^*}{\partial y} - \frac{\partial b_y{}^*}{\partial x} \right) &= \bar{\alpha} u^*, \\
\sigma \left(\Delta v^* - \frac{\partial \chi^*}{\partial y} \right) + \theta^* &= \bar{\alpha} v^*, \\
\Delta \theta^* + \sigma R v^* &= \bar{\alpha} \theta^*,
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
\zeta\Delta b_x{}^* - \sigma\zeta Q\frac{\partial u^*}{\partial y} &= \bar{\alpha} b_x{}^*,\\
\zeta\Delta b_y{}^* + \sigma\zeta Q\frac{\partial u^*}{\partial x} &= \bar{\alpha} b_y{}^*,\\
\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} &= 0,\\
\frac{\partial b_x{}^*}{\partial x} + \frac{\partial b_y{}^*}{\partial y} &= 0,
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\frac{\partial u^*}{\partial y} = v^* = \theta^* = b_x^* = \frac{\partial b_y^*}{\partial y} &= 0, \quad \text{on} \quad y = 0, 1,\\
u = \frac{\partial v^*}{\partial x} = \frac{\partial \theta^*}{\partial x} = b_x^* = \frac{\partial b_y^*}{\partial x} &= 0, \quad \text{on} \ x = L, -L.
\end{aligned} \tag{2.50}$$

Using Fourier series we find eigenfunctions $\Phi^* = (u^*, v^*, \theta^*, b_x^*, b_y^*)$ corresponding to the eigenvalues $\pm i P_j \omega_j$, $j = 1, 2$, of (2.49)-(2.50)

$$\Phi_1^* = \Phi_{m,1}^{*O} = C_m \begin{pmatrix} \dfrac{2L_m}{m}\cos(m\pi x/2L_m)\cos(\pi y) \\ \sin(m\pi x/2L_m)\sin(\pi y) \\ \dfrac{\sigma R_m}{P_1(1-i\omega_1)}\sin(m\pi x/2L_m)\sin(\pi y) \\ \dfrac{2\sigma\zeta Q\pi L_m}{P_1(\zeta - i\omega_1)}\cos(m\pi x/2L_m)\sin(\pi y) \\ \dfrac{-\pi\sigma\zeta Q}{P_1(\zeta - i\omega_1)}\sin(m\pi x/2L_m)\cos(\pi y) \end{pmatrix}, \tag{2.51}$$

and

$$\Phi_2^* = \Phi_{m+1,1}^{*E} = C_{m+1} \begin{pmatrix} -\dfrac{2L_m}{(m+1)}\sin((m+1)\pi x/2L_m)\cos(\pi y) \\ \cos((m+1)\pi x/2L_m)\sin(\pi y) \\ \dfrac{\sigma R_m}{P_2(1-i\omega_2)}\cos((m+1)\pi x/2L_m)\sin(\pi y) \\ \dfrac{-\sigma\zeta Q 2\pi L_m}{(m+1)P_2(\zeta - i\omega_2)}\sin((m+1)\pi x/2L_m)\sin(\pi y) \\ \dfrac{-\pi\sigma\zeta Q}{P_2(\zeta - i\omega_2)}\cos((m+1)\pi x/2L_m)\cos(\pi y) \end{pmatrix}, \tag{2.52}$$

where

$$\begin{aligned} C_m &= \left[\frac{4L_m^2 P_1}{\pi^2 m^2} - \frac{4L_m^2 \sigma\zeta Q}{m^2 P_1(\zeta - i\omega_1)^2} + \frac{\sigma R_m}{P_1^2(1-i\omega_1)^2}\right]^{-1} \\ \bar{C}_{m+1} &= \left[\frac{4L_m^2 P_2}{\pi^2 (m+1)^2} - \frac{4L_m^2 \sigma\zeta Q}{(m+1)^2 P_2(\zeta - i\omega_2)^2} + \frac{\sigma R_m}{P_2^2(1-i\omega_2)^2}\right]^{-1}. \end{aligned} \tag{2.53}$$

The normalization constants C_m, C_{m+1} are chosen so that

$$\langle \Phi_j, \Phi_k^* \rangle_L = \delta_{jk}, \qquad j,k = 1,2. \tag{2.54}$$

Using (2.19) and (2.33) and after some simplification we get

$$\begin{aligned} \bar{C}_m &= -\frac{m^2\pi^2(1+i\omega_1)(\zeta + i\omega_1)}{8L_m^2 P_1 \omega_1(\omega_1 - i\delta)}, \\ \bar{C}_{m+1} &= -\frac{(m+1)^2\pi^2(1+i\omega_2)(\zeta + i\omega_2)}{8L_m^2 P_2 \omega_2(\omega_2 - i\delta)} \end{aligned} \tag{2.55}$$

where $\delta = 1 + \sigma + \zeta$ and the overbar denotes complex conjugation.

2.5 Asymptotic results as $m \to \infty$

In this section we discuss some of the asymptotic behavior of the critical aspect ratios and related quantities as $m \to \infty$.

2.5.1 Case I (fixed ζ and Q)

We are interested in behaviour for large aspect ratios. Since (2.26) implies that $L_m = O(m)$ as $m \to \infty$, for fixed σ, ζ and Q we consider limiting behaviour as $m \to \infty$. We call these "Case I" limits.

Proposition 2.2 $L_m = O(m)$ *as* $m \to \infty$. *More precisely, we have*

$$\frac{2L_m}{m} = \lambda + \frac{\lambda}{2}m^{-1} - \frac{\lambda(1+3\lambda^2)}{24(\lambda^2+1)}m^{-2} + O(m^{-3}), \tag{2.56}$$

where $\lambda = \sqrt{x^*}$ *satisfies (2.27).*

Proof: By (2.26) it is clear that

$$\lim_{m\to\infty} 2L_m/m = \lambda. \tag{2.57}$$

expanding

$$2L_m/m = \lambda + \epsilon_1 m^{-1} + \epsilon_2 m^{-2} + O(m^{-3}).$$

and substituting this expression into the expression

$$R_{m1}(L_m) = R_{m+1,1}(L_m)$$

that defines L_m, after some calculation we solve for ϵ_1 and ϵ_2 to get (2.56). Q.E.D.

From Proposition 2.2, it follows immediately that

$$\lim_{m\to\infty} P_1 = \lim_{m\to\infty} P_2 = P, \tag{2.58}$$

where

$$P = \pi^2\left(\frac{1}{\lambda^2} + 1\right), \tag{2.59}$$

and

$$\omega = \lim_{m\to\infty} \omega_1 = \lim_{m\to\infty} \omega_2, \tag{2.60}$$

where

$$\omega = -\zeta^2 + \frac{\pi^2\sigma\zeta Q(1-\zeta)}{P^2(1+\sigma)}, \tag{2.61}$$

Remark 2.4 *There is a limiting relation between the eigenvalues and eigenfunctions and normalization constant in this problem and the infinite layer ($L = \infty$)problem with periodic boundary conditions ([35]). If*

$$\Phi_1^m = \frac{\Phi_1 + \Phi_2}{2}, \quad \Phi_2^m = \frac{\Phi_1 - \Phi_2}{2i},$$

and if

$$\Phi_1^{*m} = \frac{\Phi_1^* + \Phi_2^*}{2}, \quad \Phi_2^{*m} = \frac{\Phi_1^* - \Phi_2^*}{2i}.$$

then for fixed x, y, we have

$$\lim_{L_m \to \infty} \Phi_1^m(x,y) = \Phi_1(x,y) \quad \lim_{L_m \to \infty} \Phi_2^m(x,y) = \Phi_2(x,y),$$

$$\lim_{L_m \to \infty} \Phi_1^{*m}(x,y) = \Phi_1^*(x,y), \quad \lim_{L_m \to \infty} \Phi_2^{*m}(x,y) = \Phi_2^*(x,y),$$

where $\Phi_1, \Phi_2, \Phi_1^*, \Phi_2^*$ are eigenfunctions of the linearized equation, and Φ_1^*, Φ_2^* are adjoint eigenfunctions for the infinite layer with periodic boundary conditions. We also have

$$\lim_{L_m \to \infty} P_1\omega_1 = P\omega, \quad and \quad \lim_{L_m \to \infty} r^*{}_{m1} = r^{(0)},$$

and

$$\lim_{L_m \to \infty} C_m = C.$$

where $iP\omega$ is the imaginary eigenvalue, r_0 is the scaled critical Rayleigh number and C is the normalization constant used in the infinite layer case.

Let $\epsilon = m^{-1}$. Then we also have

$$P_1 = P - \frac{\pi^2}{\lambda^2}\epsilon + O(\epsilon^2), \tag{2.62}$$

$$P_2 = P + \frac{\pi^2}{\lambda^2}\epsilon + O(\epsilon^2), \tag{2.63}$$

$$\omega_1 = \omega + \frac{\pi^2(\omega^2+\zeta^2)}{P\omega\lambda^2}\epsilon + O(\epsilon^2), \tag{2.64}$$

$$\omega_2 = \omega - \frac{\pi^2(\omega^2+\zeta^2)}{P\omega\lambda^2}\epsilon + O(\epsilon^2), \tag{2.65}$$

$$R_m = R_0 + O(\epsilon) \tag{2.66}$$

where

$$R_0 = (\sigma+\zeta)\left[\frac{\pi^2\zeta Q}{\sigma+1} + \frac{(\zeta+1)P^2}{\sigma}\right]\left[\frac{\lambda^2 P}{\pi^2}\right], \tag{2.67}$$

Proposition 2.3 *For fixed $L = L_m$ there will be no other instability unless we increase R above R_m at least by $O(\epsilon^2)$.*

Proof: To prove this we need to look at the size of $R_{m+2}(L_m) - R_m(L_m)$, since clearly $R_{m+k}(L_m) - R_m(L_m) > R_{m+2}(L_m) - R_m(L_m)$, for all $k > 2$. Using equation (2.23) we have

$$\begin{aligned}
R_{m+2}(L_m) - R_m(L_m) &= 4L_m^2 \left[\frac{-4m-4}{m^2(m+2)^2}\right](A+B) \\
&\quad + B\left[\frac{(m+2)^4 - m^4}{16L_m^4} + \frac{3\left((m+2)^2 - m^2\right)}{4L_m^2}\right] && (2.68) \\
&= \frac{4B(m+1)}{2L_m(m+2)^2}\left\{\frac{3\left[(m+2)^2-(m+1)^2\right]}{2L_m}\right. && (2.69) \\
&\quad \left. + \frac{\left[(m+2)^2-(m+1)^2\right]\left[m^2+(m+1)^2+(m+2)^2\right]}{(2L_m)^3}\right\} \\
&= \frac{4B(m+1)(2m+3)}{(2L_m)(m+2)^2}\left\{3 + \frac{m^2+(m+1)^2+(m+2)^2}{(2L_m)^2}\right\} \\
&= \frac{24PB}{\lambda^2\pi^2}\epsilon^2 + O(\epsilon^3) \\
&= O(\epsilon^2), && (2.70)
\end{aligned}$$

since L_m and λ satisfies(2.35) and (2.27) respectively, where A and B are as in equation (2.25). Also since $B \neq 0$ this difference is not zero. Q.E.D.

2.5.2 Case II (decreasing ζ, increasing Q as $m \to \infty$)

In subsequent chapters, we will also consider behaviour for large aspect ratios, small ζ and large Q. Suppose we fix real positive numbers σ, k, $\hat{\zeta}$ and $\hat{Q}$, and consider the sequences given by

$$\zeta_{\hat{m}} = \hat{m}^{-k/2}\hat{\zeta}, \quad Q_{\hat{m}} = \hat{m}^{k/2}\hat{Q}, \tag{2.71}$$

for $\hat{m} = 1, 2, 3, \ldots$. For each $\zeta = \zeta_{\hat{m}}$, $Q = Q_{\hat{m}}$ we obtain a sequence of critical half aspect ratios $L_m^{\hat{m}}$ and corresponding critical Rayleigh numbers $R_m^{\hat{m}}$, $m = 1, 2, \ldots$. We consider limiting behaviour when $\hat{m} = m$ and $m \to \infty$. We call these "Case II" limits. Taking the limits in Case II change the values of λ, P, ω, R_0 of §2.5.1. We denote their corresponding values in Case II by $\check{\lambda}, \check{P}, \check{\omega}, \check{R}_0$ respectively; their corresponding equations are replaced by

$$
\begin{aligned}
0 &= \frac{2}{\check{\lambda}^6} + \frac{3}{\check{\lambda}^4} - 1 - \frac{\sigma\zeta Q}{\pi^2(\sigma+1)}, &(2.72)\\
\check{P} &= \pi^2\left(\frac{1}{\check{\lambda}^2} + 1\right), &(2.73)\\
\check{\omega}^2 &= \frac{\sigma\zeta Q\pi^2}{\check{P}^2(\sigma+1)}, &(2.74)\\
\check{R}_0 &= \left[\frac{\sigma\check{P}\check{\lambda}^2}{\pi^2}\right]\left[\frac{\pi^2\sigma Q}{\sigma+1} + \frac{\check{P}^2}{\sigma}\right]. &(2.75)
\end{aligned}
$$

The asymptotic expansions of the terms

$$\frac{2L_m}{m}, P_1, P_2, \omega_1, \omega_2$$

as $m \to \infty$ change, but the leading terms are of the same order in $\epsilon = m^{-1}$. Corresponding to equations (2.64)-(2.66) we have

$$
\begin{aligned}
\frac{2L_m^m}{m} &= \check{\lambda} + \check{\lambda}_1\epsilon^{k/2} + O(\epsilon^k + \epsilon) &(2.76)\\
P_1 &= \check{P} - \frac{2\pi^2\check{\lambda}_1}{\check{\lambda}^3}\epsilon^{k/2} + O(\epsilon^k + \epsilon), &(2.77)\\
P_2 &= \check{P} - \frac{2\pi^2\check{\lambda}_1}{\check{\lambda}^3}\epsilon^{k/2} + O(\epsilon^k + \epsilon), &(2.78)\\
\omega_1 &= \check{\omega} + \frac{\sigma\zeta Q[-\check{\zeta}\check{P} + (4\check{\lambda}_1/\check{\lambda}^3)\pi^2]}{2(\sigma+1)\check{\omega}\check{P}^3}\epsilon^{k/2} + O(\epsilon^k + \epsilon), &(2.79)\\
\omega_2 &= \check{\omega} + \frac{\sigma\zeta Q[-\check{\zeta}\check{P} + (4\check{\lambda}_1/\check{\lambda}^3)\pi^2]}{2(\sigma+1)\check{\omega}\check{P}^3}\epsilon^{k/2} + O(\epsilon^k + \epsilon), &(2.80)\\
R_m^m &= \check{R}_0 + O(\epsilon^{k/2}), &(2.81)
\end{aligned}
$$

where

$$\check{\lambda}_1 = -\frac{\sigma\hat{\zeta}^2\hat{Q}\hat{\lambda}^5}{12(\sigma+1)\hat{P}}.$$

The appearance of $R_{m+2}(L_m) - R_m(L_m)$ also changes, but by (2.69) it is of the same order as in Case I and equation (2.70) will be replaced by

$$R_{m+2}^m(L_m^m) - R_m^m(L_m^m) = \frac{24\check{P}\check{B}}{\check{\lambda}^2\pi^2}\epsilon^2 + O(\epsilon^{2+k/2} + \epsilon^3), \tag{2.82}$$

where

$$\breve{B} = \pi^4 + O(\epsilon^{k/2}).$$

These results will be useful when we calculate the coefficients of the reduced equation on the center manifold and the determine the dynamics of the related equations in Case II.

Chapter 3

Center manifold and normal form reductions

In this chapter we begin our nonlinear analysis of the magnetoconvection equations. In §3.1 we give an abstract formulation of the magnetoconvection equations as an evolution equation in a Hilbert space, and then prove the existence of a nonlinear analytic semiflow in that Hilbert space. Then in §3.2 we rescale variables so that L is explicitly introduced as a parameter into the evolution equations, while the domain becomes fixed. In §3.3 we establish the existence of a locally invariant attracting center manifold W^c for the evolution equation, which we will then use to study the dynamics of magnetoconvection. We represent the flow on W^c as a two-parameter family of four-dimensional ordinary differential equations with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry. In §3.4 we use the symmetry and near-identity coordinate transformations to simplify the family of ordinary differential equations by putting them into normal form. Finally, in §3.5 we discuss the choice of normal forms when the aspect ratios of the fluid layer are large. We use different normal forms for the Case I and Case II limiting situations introduced in Chapter 2.

3.1 Abstract formulation

In this section we reformulate the magnetoconvection equations (2.5), (2.7) as an equation in a Hilbert space, which generates an analytic semiflow. For fixed L, let

$$\Omega_L = \{(x, y) : \ 0 < y < 1, \ -L < x < L\},$$

be the rectangular region as in equation (2.6), and let Γ_L denote its boundary. Let $L^2(\Omega_L)$ be the space of all square-integrable functions on Ω_L, with norm

$$\|\phi\|_{L^2(\Omega_L)} = \left[\int_0^1 \int_{-L}^{L} |\phi(x,y)|^2 dxdy\right]^{1/2}, \tag{3.1}$$

and inner product

$$(\phi, \psi)_{L^2(\Omega_L)} = \int_0^1 \int_{-L}^{L} \phi(x,y)\overline{\psi(x,y)}dxdy, \tag{3.2}$$

where the overbar denotes complex conjugation. When $k \geq 0$ is an integer, the Sobolev space $W^{k,2}(\Omega_L)$ of order k is defined by

$$W^{k,2}(\Omega_L) = \left\{\phi : D^\alpha\phi \in L^2(\Omega_L) \ \ \forall\alpha, |\alpha| \leq k\right\}, \tag{3.3}$$

where

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}, \qquad \alpha = \{\alpha_1, \alpha_2\}, \qquad |\alpha| = \alpha_1 + \alpha_2.$$

and the derivatives $D^\alpha\phi$ are taken in the weak (distributional) sense. The space $W^{k,2}(\Omega_L)$, with the norm

$$\|\phi\|_k = \|\phi\|_{W^{k,2}(\Omega_L)} = \left[\sum_{|\alpha|\leq k} \|D^\alpha\phi\|^2{}_{L^2(\Omega_L)}\right]^{1/2}, \tag{3.4}$$

and scalar product

$$(\phi, \psi)_{W^{k,2}(\Omega_L)} = \sum_{|\alpha|\leq k} (D^\alpha\phi, D^\alpha\psi)_{L^2(\Omega_L)}, \tag{3.5}$$

is an Hilbert space [27]. When $k = 0, W^{0,2}(\Omega_L) = L^2(\Omega_L)$.

Let $\mathbf{L}^2(\Omega_L) = L^2(\Omega_L) \times L^2(\Omega_L)$ and $\mathbf{W}^{k,2}(\Omega_L) = W^{k,2}(\Omega_L) \times W^{k,2}(\Omega_L)$, with the norms and inner products inherited from the product structures. In order to introduce suitable function spaces for the magnetoconvection equations, we first consider the following boundary value problems:

(1) Given $\mathbf{f} \in \mathbf{L}^2(\Omega_L)$, find $\mathbf{u} = (u, v)$ and χ satisfying

$$-\Delta\mathbf{u} + \nabla\chi = \mathbf{f} \ \text{ in } \ \Omega_L,$$

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega_L, \qquad (3.6)\\
\mathbf{u} \cdot \mathbf{n} &= 0 \text{ on } \Gamma_L \ \text{ (i.e. } u = 0 \text{ on } x = \pm L,\ v = 0 \text{ on } y = 0, 1),\\
\nabla \times (\mathbf{u} \times \mathbf{n}) &= 0 \text{ on } \Gamma_L \ \text{ (i.e. } \partial v/\partial x = 0 \text{ on } x = \pm L,\ \partial u/\partial y = 0 \text{ on } y = 0, 1),
\end{aligned}
$$

where $\mathbf{n} = (n_x, n_y)$ is the unit outward normal vector on Γ_L. (This is the Stokes problem with "stress-free" boundary conditions.) Define the following Hilbert spaces

$$
H_{1L} = \left\{ \mathbf{u} \in \mathbf{L}^2(\Omega_L) : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_L, \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_L \right\}, \qquad (3.7)
$$

$$
V_{1L} = \left\{ \mathbf{u} \in \mathbf{W}^{1,2}(\Omega_L) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_L, \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma_L \right\}, \qquad (3.8)
$$

$$
\begin{aligned}
D_{1L} = \Big\{ \mathbf{u} \in \mathbf{W}^{2,2}(\Omega_L) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega_L, \text{ and } \ \mathbf{u} \cdot \mathbf{n} = 0,\\
\nabla \times (\mathbf{u} \times \mathbf{n}) = 0 \text{ on } \Gamma_L \Big\}. \qquad (3.9)
\end{aligned}
$$

Then one can define an unbounded self-adjoint operator A_1 in H_{1L} with domain D_{1L}, such that $A_1\mathbf{u} = -\Pi\Delta\mathbf{u}$ for all $\mathbf{u} \in D_{1L}$, where Π is the orthogonal projection of $\mathbf{L}^2(\Omega_L)$ onto H_{1L}. The problem (3.6) is equivalent to

$$
A_1\mathbf{u} = \Pi\mathbf{f}, \quad \mathbf{u} \in D_{1L}, \qquad (3.10)
$$

since the pressure term $\chi \in W^{1,2}(\Omega_L)$ can be recovered from $\mathbf{u}$ and $\mathbf{f}$. The inverse operator A_1^{-1} is a compact operator in H_{1L} [46, pp. 104–105 and Remark 2.4 on pp. 110–111].

(2) Given $f \in L^2(\Omega_L)$, find θ satisfying Laplace's equation with mixed boundary conditions:

$$
\begin{aligned}
-\Delta\theta &= f \text{ in } \Omega_L, \qquad (3.11)\\
\theta &= 0 \text{ on } y = 0, 1, \qquad (3.12)\\
\frac{\partial\theta}{\partial x} &= 0 \text{ on } x = \pm L. \qquad (3.13)
\end{aligned}
$$

Define

$$
\begin{aligned}
H_{2L} &= L^2(\Omega_L), &(3.14)\\
V_{2L} &= \left\{\theta \in W^{1,2}(\Omega_L) : \theta = 0 \ \text{ on } \ y = 0,1\right\}, &(3.15)\\
D_{2L} &= \left\{\theta \in W^{2,2}(\Omega_L) : \ \theta = 0 \text{ on } y = 0,1, \ \partial\theta/\partial x = 0 \text{ on } x = \pm L\right\}. &(3.16)
\end{aligned}
$$

Then one can define an unbounded self-adjoint operator A_2 in H_{2L}, with domain D_{2L}, such that $A_2\theta = -\Delta\theta$ for all $\theta \in D_{2L}$. The problem (3.11)–(3.13) is equivalent to

$$
A_2\theta = f, \quad \theta \in D_{2L}, \tag{3.17}
$$

and A_2^{-1} is a compact operator in H_{2L} [46].

(3) Given $\mathbf{f} = (f_x, f_y) \in \mathbf{L}^2(\Omega_L)$ with $\int_{\Omega_L} f_y = 0$, find $\mathbf{b} = (b_x, b_y)$ satisfying a mixed problem:

$$
\begin{aligned}
-\Delta\mathbf{b} &= \mathbf{f} \ \text{ in } \ \Omega_L, &(3.18)\\
b_x &= 0 \ \text{ on } \ \Gamma_L, &(3.19)\\
\partial b_y/\partial n &= 0 \ \text{ on } \ \Gamma_L, &(3.20)
\end{aligned}
$$

The boundary value problem (3.18)-(3.20) is just a pair of decoupled Dirichlet and Neumann problems. If $\nabla \cdot \mathbf{f} = 0$ we note that the solution automatically satisfies $\nabla \cdot \mathbf{b} = 0$. The Dirichlet and Neumann problems are classical, and solutions are given in almost every book on elliptic partial differential equations (e.g. [46]). Let us define the following Hilbert spaces

$$
\begin{aligned}
H_{3L} &= \left\{\mathbf{b} \in \mathbf{L}^2(\Omega_L) : \ \int_{\Omega_L} b_y = 0\right\}, &(3.21)\\
V_{3L} &= \left\{\mathbf{b} \in \mathbf{W}^{1,2}(\Omega_L) : \int_{\Omega_L} b_y = 0, \text{ and } b_x = 0 \ \text{ on } \ \Gamma_L\right\}. &(3.22)\\
D_{3L} &= \left\{\mathbf{b} \in \mathbf{W}^{2,2}(\Omega_L), \int_{\Omega_L} b_y = 0, \text{ and } b_x = 0, \ \partial b_y/\partial n = 0 \text{ on } \Gamma_L, \right\}. &(3.23)
\end{aligned}
$$

There is an unbounded self-adjoint operator A_3 on H_{3L} with domain D_{3L}, so that $A_3\mathbf{b} = -\Delta\mathbf{b}$ for all $\mathbf{b} = (b_x, b_y) \in D_{3L}$, and A_3^{-1} is a compact operator in H_{3L}.

Each of the above boundary value problems has been considered as an abstract equation of the form

$$A_i\phi = f, \quad i = 1,2,3$$

in a Hilbert space H_{iL}, $\quad i = 1,2,3$. The operators A_i are positive, linear unbounded operators in H_{iL} with domains D_{iL} and their inverses A_i^{-1} are as self-adjoint compact operators in H_{iL}. For fixed parameters σ, ζ, let $\Phi = (\mathbf{u}, \theta, \mathbf{b})^T$, define the Hilbert space

$$X_L = H_{1L} \times H_{2L} \times H_{3L},$$

with inner product as in equation (2.33), and define the unbounded linear operator A in X_L by

$$A\Phi = (\sigma A_1\mathbf{u}, A_2\theta, \zeta A_3\mathbf{b}), \quad \Phi \in D(A) = D_{1L} \times D_{2L} \times D_{3L}. \tag{3.24}$$

Then the operator A is a strictly positive self-adjoint operator in X_L with domain $D(A)$. The inverse operator A^{-1} is self-adjoint and compact, and the spectrum of A consists entirely of isolated eigenvalues with finite multiplicities, so its eigenfunctions are dense in X_L [46]. The eigenfunctions have the form

$$\Phi_{mn} = \begin{pmatrix} [A^1_{mn}\cos(m\pi x/2L) + B^1_{mn}\sin((m+1)\pi x/2L)]\cos(n\pi y) \\ [A^2_{mn}\sin(m\pi x/2L) + B^2_{mn}\cos((m+1)\pi x/2L)]\sin(n\pi y) \\ [A^3_{mn}\sin(m\pi x/2L) + B^3_{mn}\cos((m+1)\pi x/2L)]\sin(n\pi y) \\ [A^4_{mn}\cos(m\pi x/2L) + B^4_{mn}\sin((m+1)\pi x/2L)]\sin(n\pi y) \\ [A^5_{mn}\sin(m\pi x/2L) + B^5_{mn}\cos((m+1)\pi x/2L)]\cos(n\pi y) \end{pmatrix}, \tag{3.25}$$

where m is an odd positive integer and n is any positive integer. The coefficients A^1_{mn} and A^2_{mn}, B^1_{mn} and B^2_{mn}, A^4_{mn} and A^5_{mn}, B^4_{mn} and B^5_{mn} are related in such a way that the divergence of the first two and the last two components are zero, i.e.,

$$\begin{aligned}
A^2_{mn} &= \frac{m}{2Ln}A^1_{mn}, \quad A^5_{mn} = -\frac{m}{2Ln}A^4_{m,n}, \\
B^2_{mn} &= -\frac{m+1}{2Ln}B^1_{mn}, \quad B^4_{mn} = \frac{m+1}{2Ln}B^5_{mn}.
\end{aligned}$$

The operator A is sectorial in X_L [20], and for each $\gamma \in \mathbf{R}$ we may define the fractional powers A^γ, with domain $X_L^\gamma = D(A^\gamma)$ in X_L. The space $D(A^\gamma)$, endowed with the scalar product

$$(\Phi^1, \Phi^2)_{D(A^\gamma)} = (A^\gamma \Phi^1, A^\gamma \Phi^2)_{X_L}, \tag{3.26}$$

and the graph norm

$$|\Phi|_{D(A^\gamma)} = \left\{(\Phi, \Phi)_{D(A^\gamma)}\right\}^{1/2}, \tag{3.27}$$

is a Hilbert space, and A^γ is an isomorphism from $D(A^\gamma)$ onto X_L. For $\gamma = 1$, we recover $D(A)$. Since A is a sectorial operator with compact resolvent, the embeddings $D(A) \subset X_L^\gamma \subset X_L$ are continuous for $0 \le \gamma \le 1$ and are compact for $0 < \gamma < 1$ [20, Theorem 1.4.8]. Now for fixed σ, ζ, Q, we put

$$\begin{aligned}
B(R)\Phi &= (\sigma\Pi[R\theta\mathbf{e}_y + \zeta Q(\nabla \times \mathbf{b}) \times \mathbf{e}_y], v, \nabla \times [\mathbf{u} \times \mathbf{e}_y])^T, && (3.28) \\
M(\Phi, \Phi') &= (\Pi[\sigma\zeta Q(\nabla \times \mathbf{b}) \times \mathbf{b}' - (\mathbf{u} \cdot \nabla)\mathbf{u}'], -\mathbf{u} \cdot \nabla\theta', \nabla \times (\mathbf{u} \times \mathbf{b}'))^T, && (3.29)
\end{aligned}$$

where $\Phi = (\mathbf{u}, \theta, \mathbf{b})^T, \Phi' = (\mathbf{u}', \theta', \mathbf{b}')^T$. We now write the magnetoconvection equations (2.5) and (2.7) as an evolution equation in X_L,

$$\frac{d\Phi}{dt} = -A\Phi + B(R)\Phi + N(\Phi), \tag{3.30}$$

where $N(\Phi) = M(\Phi, \Phi)$. We observe that $B(R) : X_L^\gamma \longrightarrow X_L$ is a bounded linear operator and $M : X_L^\gamma \times X_L^\gamma \longrightarrow X_L$ is bounded bilinear operator for $1/2 < \gamma < 1$ [20, Theorem 1.6.1 and the arguments on pp.79-81], and so $B(R) + N(\cdot)$ is analytic from X_L^γ into X_L when $\gamma > 1/2$. By [20, Theorem 3.4.4 and Corollary 3.4.6] we have the following fact:

Proposition 3.1 *If $1/2 < \gamma < 1$, then the evolution equation (3.30) generates a local semiflow in X_L^γ that depends analytically on $t > 0$, on $R \in \mathbf{R}$ and on the initial condition $\Phi(0) \in X_L^\gamma$.*

We note that if $\nabla \cdot \mathbf{b} = 0$ when $t = 0$, then the solution $\Phi = (\mathbf{u}, \theta, \mathbf{b})^T$ satisfies $\nabla \cdot \mathbf{b} = 0$ automatically for $t > 0$.

Now let

$$K(R) = -A + B(R), \quad D(K(R)) = D(A), \tag{3.31}$$

denote the linearization of the vector field of (3.30) about $\Phi = 0$. By the above arguments and [46, p.54], it is clear that for each fixed R the spectrum of $K(R)$ consists entirely of isolated eigenvalues with finite multiplicities. To find them, we solve the eigenvalue problem

$$K(R)\Phi = \alpha\Phi, \quad \Phi \in D(A),$$

and we seek eigenfunctions in the form of (3.25). This is system (2.11) that we have already considered in §2.3, and we found all eigenfunctions in the form of (3.25). Since these eigenfunctions are dense in $D(A)$, our formal calculations in §2.3 are justified.

3.2 The rescaled problem

Since we would like to study the dynamics of magnetoconvection when L is near one of the L_m which were found in Chapter 2, we introduce L as parameter explicitly to the equations. For this, we use the rescaling

$$x = L\hat{x}, \quad u = L\hat{u}, \quad b_x = L\hat{b}_x. \tag{3.32}$$

Then system (2.5), after "dropping the hats", becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \sigma\left[\Delta^1 \mathbf{u} - \nabla^1 \chi + R\theta \mathbf{e}_y + \zeta Q\left(\nabla^1 \times \mathbf{b}\right) \times \left(\mathbf{e}_y + \mathbf{b}^1\right)\right] - (\mathbf{u} \cdot \nabla)\,\mathbf{u},$$

$$\begin{aligned}
\frac{\partial \theta}{\partial t} &= \Delta^1 \theta + v - \mathbf{u} \cdot \nabla \theta, \\
\frac{\partial \mathbf{b}}{\partial t} &= \zeta \Delta^1 \mathbf{b} + \nabla \times [\mathbf{u} \times (\mathbf{e}_y + \mathbf{b})], \\
\nabla \cdot \mathbf{u} &= 0, \\
\nabla \cdot \mathbf{b} &= 0,
\end{aligned} \tag{3.33}$$

in the fixed domain

$$\Omega = \{(x, y) : -1 < x < 1,\ 0 < y < 1\},$$

where

$$\Delta^1 = L^{-2}\partial^2/\partial x^2 + \partial^2/\partial y^2, \quad \nabla^1 = \left(L^{-2}\partial/\partial x, \partial/\partial y\right),$$

$$\mathbf{b}^1 = (L^2 b_x, b_y), \quad \mathbf{u} = (u, v), \quad \mathbf{b} = (b_x, b_y).$$

The boundary conditions (2.7) become

$$\begin{aligned}
\frac{\partial u}{\partial y} &= v = \theta = b_x = \frac{\partial b_y}{\partial y} = 0, \quad \text{on} \quad y = 0, 1, \\
u &= \frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = b_x = \frac{\partial b_y}{\partial x} = 0, \quad \text{on} \ x = \pm 1.
\end{aligned} \tag{3.34}$$

The Hilbert space X_L defined in §3.1 corresponds, under the rescaling, to a Hilbert space X, and for an inner product in X we may take

$$\langle \Phi^1, \Phi^2 \rangle = (1/2) \int_0^1 \int_{-1}^1 \left(L^2 u^1 \bar{u}^2 + v^1 \bar{v}^2 + \theta^1 \bar{\theta}^2 + L^2 b_x^1 \bar{b}_x^2 + b_y^1 \bar{b}_y^2 \right) dx\, dy, \tag{3.35}$$

where $\Phi^i = (u^i, v^i, \theta^i, b_x^i, b_y^i)$, $i = 1, 2$, and the overbars denote complex conjugation. The linearized operator $K(R)$ of the previous section corresponds, under the rescaling (3.32), to

$$K(R, L) = -A(L) + B(R, L), \tag{3.36}$$

where

$$A(L)\Phi = \left(-\sigma \Pi^1 \Delta^1 \mathbf{u}, -\Delta^1 \theta, -\zeta \Delta^1 \mathbf{b}\right)^T, \tag{3.37}$$

$$B(R, L)\Phi = \left(\sigma \Pi^1 [R\theta \mathbf{e}_y + \zeta Q(\nabla^1 \times \mathbf{b}) \times \mathbf{e}_y], v, \nabla^1 \times (\mathbf{u} \times \mathbf{e}_y)\right)^T, \tag{3.38}$$

and Π^1 is the orthogonal projection that corresponds to Π under the rescaling (3.32). For fixed R and L, $A(L)$ is a sectorial operator in X, and $B(R,L) : X^\gamma \to X$ is a bounded operator if $1/2 < \gamma < 1$. The eigenfunctions $\Phi_1, \Phi_2, \Phi_1^*, \Phi_2^*$ now take the form

$$\Phi_1 = \begin{pmatrix} \frac{2}{m}\cos(m\pi x/2)\cos(\pi y) \\ \sin(m\pi x/2)\sin(\pi y) \\ \frac{1}{P_1(1+i\omega_1)}\sin(m\pi x/2)\sin(\pi y) \\ -\frac{2\pi}{mP_1(\zeta+i\omega_1)}\cos(m\pi x/2)\sin(\pi y) \\ \frac{\pi}{P_1(\zeta+i\omega_1)}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{3.39}$$

$$\Phi_2 = \begin{pmatrix} -\frac{2}{m+1}\sin((m+1)\pi x/2)\cos(\pi y) \\ \cos((m+1)\pi x/2)\sin(\pi y) \\ \frac{1}{P_2(1+i\omega_2)}\cos((m+1)\pi x/2)\sin(\pi y) \\ \frac{2\pi}{(m+1)P_2(\zeta+i\omega_2)}\sin((m+1)\pi x/2)\sin(\pi y) \\ \frac{\pi}{P_2(\zeta+i\omega_2)}\cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{3.40}$$

$$\Phi_1^* = C_m\begin{pmatrix} \frac{2}{m}\cos(m\pi x/2)\cos(\pi y) \\ \sin(m\pi x/2)\sin(\pi y) \\ \frac{\sigma R}{P_1(1-i\omega_1)}\sin(m\pi x/2)\sin(\pi y) \\ \frac{2\sigma\zeta Q\pi}{P_2(\zeta-i\omega_2)}\cos(m\pi x/2)\sin(\pi y) \\ \frac{-\pi\sigma\zeta Q}{P_1(\zeta-i\omega_1)}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{3.41}$$

and

$$\Phi_2^* = C_{m+1} \begin{pmatrix} -\dfrac{2}{(m+1)}\sin((m+1)\pi x/2)\cos(\pi y) \\ \cos((m+1)\pi x/2)\sin(\pi y) \\ \dfrac{\sigma R}{P_2(1-i\omega_2)}\cos((m+1)\pi x/2)\sin(\pi y) \\ \dfrac{-\sigma\zeta Q 2\pi}{(m+1)P_1(\zeta - i\omega_1)}\sin((m+1)\pi x/2)\sin(\pi y) \\ \dfrac{-\pi\sigma\zeta Q}{P_2(\zeta - i\omega_2)}\cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{3.42}$$

with C_m, C_{m+1} as in equations (2.55). The evolution equation (3.29) becomes

$$\frac{d\Phi}{dt} = -A(L)\Phi + B(R,L)\Phi + N(\Phi, L), \tag{3.43}$$

where

$$N(\Phi, L) = M(\Phi,\Phi,L) = \left(\Pi[\sigma\zeta Q\nabla^1 \times \mathbf{b}^1) \times \mathbf{b} - (\mathbf{u}\cdot\nabla^1)\mathbf{u}], -\mathbf{u}\cdot\nabla^1\theta, \nabla^1 \times (\mathbf{u}\times\mathbf{b})\right)^T, \tag{3.44}$$

and Proposition 3.1 holds for (3.43).

3.3 Center manifold reduction

The study of bifurcation and stability in differential equations can often be greatly simplified by the use of center manifold theory. This theory allows one to reduce the dimension of the state space, while preserving the local behavior of solutions of differential equation. In this section we apply a suitable version of the center manifold theorem to reduce the parametrized family of evolution equations to a parameterized family of ordinary differential equations in a four dimensional phase space, the dimension four of the phase space being determined by the number of eigenvalues with zero real part at the critical parameter values.

When $L = L_m$, "odd" and "even" curves of the Rayleigh numbers $R_m(L)$ and $R_{m+1}(L)$ intersect at $R = R_m$ and modes corresponding to both odd and even numbers of rolls are marginally stable. (Recall that we assume m is odd). Therefore for R, L in a small neighborhood of R_m, L_m, we may see bifurcations of even solutions, odd solutions, or other solutions arising from nonlinear mode interactions.

When $(R, L) = (R_m, L_m)$ the center eigenspace E_c, corresponding to the eigenvalues of $K(R_m, L_m)$ with zero real part, is given by

$$E_c = \left\{ Z_1\Phi_1 + \bar{Z}_1\bar{\Phi}_1 + Z_2\Phi_2 + \bar{Z}_2\bar{\Phi}_2 : (Z_1, Z_2) \in \mathbf{C}^2 \right\}, \tag{3.45}$$

where $\Phi_j,\ j = 1, 2$ are given by (3.39),(3.40). Now we define the projection $\tilde{P}$ of the Hilbert space X onto E_c by

$$\tilde{P}\Phi = \langle \Phi, \Phi_1^* \rangle \Phi_1 + \overline{\langle \Phi, \Phi_1^* \rangle} \bar{\Phi}_1 + \langle \Phi, \Phi_2^* \rangle \Phi_2 + \overline{\langle \Phi, \Phi_2^* \rangle} \bar{\Phi}_2, \tag{3.46}$$

where the overbars denote complex conjugates, the inner product is given by (3.40) with $L = L_m$, and $\Phi_j^*,\ \ j = 1, 2$, are the eigenfunctions of the adjoint operator $K^*(R_m, L_m)$ for $K(R_m, L_m)$ given by (3.41)–(3.42).

Proposition 3.2 *When $L = L_m$ and $R = R_m$ the space X decomposes into a direct sum $X = E_c \bigoplus E_s$, where $E_c = \mathcal{R}(\tilde{P})$ and $E_s = \mathcal{N}(\tilde{P})$ are $K(R_m, L_m)$-invariant subspaces. The spectrum of $K(R_m, L_m)$ restricted to E_c is $\{\pm iP_1\omega_1, \pm iP_2\omega_2\}$, and the spectrum of $K(R_m, L_m)$ restricted to E_s is contained in the left complex half plane, with real parts uniformly bounded away from the imaginary axis. Thus if $\Sigma(K)$ denotes the spectrum of $K(R_m, L_m)$, we have $\Re(\Sigma(K)) \leq 0$, and $\Sigma(K) \cap i\mathbf{R}$ is a spectral set.*

Proof: Since any $\Phi \in D(A)$ can be expressed as a convergent series of eigenfunctions of $K = K(R_m, L_m)$, and since the eigenfunctions are orthogonal, it is clear that the projection $\tilde{P}$ commutes with operator K (i.e. $\tilde{P}K\Phi = K\tilde{P}\Phi \ \ \forall\ \Phi \in D(A(L_m))$), thus K leaves the subspaces

$$\mathcal{R}(\tilde{P}) = span\{\Phi_1, \bar{\Phi}_1, \Phi_2, \bar{\Phi}_2\} = E_c,$$

$$\mathcal{N}(\tilde{P}) = \left\{\Phi_1^*, \bar{\Phi}_1^*, \Phi_2{}^*, \bar{\Phi}_2^*\right\}^\perp = E_s$$

invariant with $X = E_c \oplus E_s$. We have already proved the second part of the proposition in §2.4. Q.E.D.

We now state two results on center manifolds due to Henry [20, Theorem 6.2.1 and Corollary 6.2.2, Theorem 6.2.3]:

Theorem 3.1 . *Consider the abstract differential equation*

$$\frac{d\Phi}{dt} = -A\Phi + f(\Phi), \tag{3.47}$$

where A is a sectorial operator in a Banach space X, $0 \leq \gamma < 1$, U is a neighborhood of the origin in X^γ, and $f : U \to X$ is C^1 with $f(0) = 0$ and f' Lipschitz continuous in U. Assume $K = A + f'(0)$ has $\Re(\Sigma(K)) \leq 0$ with $\Sigma(K) \cap i\mathbf{R}$ a spectral set. Let $X = E_c + E_s$ be the decomposition into K-invariant subspaces with $\Re(\Sigma(K|_{E_c})) = 0$ and $\Re(\Sigma(K|_{E_s})) < 0$. Then there exists a C^1 attracting locally invariant manifold (a center manifold)

$$W^c = \{\Phi = \Phi_c + \Phi_s : \ \Phi_s = \Psi(\Phi_c), \ \Phi_c \in E_c, ||\Phi_c|| < \delta\} \tag{3.48}$$

tangent to E_c at origin. The flow in W^c is represented by the ordinary differential equation

$$\frac{d\Phi_c}{dt} = \tilde{P}\left[K(\Phi_c + \Psi(\Phi_c)) + N(\Phi_c + \Psi(\Phi_c))\right], \tag{3.49}$$

where $N(\Phi) = f(\Phi) - f'(0)\Phi$ and $\tilde{P}$ is the projection of X onto E_c, along E_s.

If the nonlinear part f in Theorem 3.1 is smooth enough, the center manifold W^c is smooth and we can approximate $\Psi(\Phi_c)$ by the first finitely many terms in the Taylor series for $\Psi(\Phi_c)$:

Theorem 3.2 . *Assume the hypothesis of Theorem 3.1. and assume that $N : U \to X$ is C^p, where $N(\Phi) = f(\Phi) - f'(0)\Phi$. If there is a C^p function h with Lipschitzian derivative* *from a neighborhood of the origin in E_c into E_s^γ, with range in $D(K|_{E_s})$, such that*

$$\begin{aligned} &h'(\Phi_c)\tilde{P}\left[K(\Phi_c + h(\Phi_c)) + N(\Phi_c + h(\Phi_c))\right] \\ &-(I-\tilde{P})\left[K(\Phi_c + h(\Phi_c)) + N(\Phi_c + h(\Phi_c))\right] = O(\|\Phi_c\|^p) \end{aligned} \tag{3.50}$$

as $\Phi_c \to 0$ in E_c, then

$$|\Psi(\Phi_c) - h(\Phi_c)| = O(\|\Phi_c\|^p) \tag{3.51}$$

as $\Phi_c \to 0$ in X_c, where $\Psi(\Phi_c)$ defines the center manifold of Theorem 3.1. If N is C^p near the origin, there is a unique polynomial function h of order p satisfying the conditions above.

We have shown that for each fixed L the operator $A(L)$ is sectorial, and we can apply Theorems 3.1 and 3.2 to the system

$$\begin{aligned} \frac{d\Phi}{dt} &= -A(L_m)\Phi + B(R_m + \mu, L_m)\Phi + N(\Phi, L_m), \\ \frac{d\mu}{dt} &= 0, \end{aligned} \tag{3.52}$$

and obtain the existence of a center manifold for R near R_m and fixed $L = L_m$. But since the operator $A(L)$ depends on L, to also treat L varying near L_m we need to apply a parameter dependent version of Theorem 3.1 due to Vanderbauwhede and Iooss [47, section 2, comments on page 136]. Then the center manifold can be represented as

$$W^c = \left\{(\Phi_c, \Phi_s) \in E_c \oplus E_s{}^\gamma : \Phi_s = \Psi(\Phi_c, \mu, \nu), \|\Phi_c\| < \delta_1, |\mu| < \delta_2, |\nu| < \delta_3\right\}, \tag{3.53}$$

where $\mu = R - R_m$, $\nu = L - L_m$ and Ψ is a smooth function from a neighborhood of the origin in $E_c \times \mathbf{R}^2$ into $E_s{}^\gamma = E_s \cap X^\gamma$, with $\Psi(0,0,0) = 0$, $\Psi'(0,0,0) = 0$. Actually Ψ is C^p for any integer $p > 0$, since $N(\Phi, L)$ in X^γ is analytic (although Ψ is not necessarily analytic). The flow in W^c is represented by the ordinary differential equation

$$\frac{d\Phi_c}{dt} = \tilde{P}\left(\hat{K}(R_m + \mu, L_m + \nu)(\Phi_c + \Psi(\Phi_c, \mu, \nu)) + N(\Phi_c + \Psi(\Phi_c, \mu, \nu))\right). \tag{3.54}$$

The study of bifurcation and stability of solutions of the evolution equation (3.29) near the origin in X^γ, for R near R_m and L near L_m is now reduced to the study of equation (3.54).

The projection $\tilde{P}$ commutes with the action of the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ on X defined by equations (2.8) and (2.9), and hence the reduced equation (3.54) is equivariant with respect to the action of J and β restricted to E_c. If we identify E_c with $\mathbf{C}^2$ by

$$E_c = \left\{ \Phi_c = Z_1\Phi_1 + \bar{Z}_1\bar{\Phi}_1 + Z_2\Phi_2 + \bar{Z}_2\bar{\Phi}_2 : (Z_1, Z_2) \in \mathbf{C}^2 \right\},$$

then since

$$J\Phi_1 = \Phi_1, \quad J\bar{\Phi}_1 = \bar{\Phi}_1 \quad J\Phi_2 = -\Phi_2, \quad J\bar{\Phi}_2 = -\bar{\Phi}_2$$

and

$$\beta\Phi_i = -\Phi_i, \quad \beta\bar{\Phi}_i = -\bar{\Phi}_i, \quad i = 1, 2,$$

the action of the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ on $\mathbf{C}^2$ will be given by

$$\begin{aligned} (Z_1, \bar{Z}_1, Z_2, \bar{Z}_2) &\xrightarrow{J} (Z_1, \bar{Z}_1, -Z_2, -\bar{Z}_2), \\ (Z_1, \bar{Z}_1, Z_2, \bar{Z}_2) &\xrightarrow{\beta} (-Z_1, -\bar{Z}_1, -Z_2, -\bar{Z}_2). \end{aligned} \tag{3.55}$$

Using the definition of $\tilde{P}$, the reduced equation (3.54) becomes

$$\dot{Z}_j = \left\langle K(R_m + \mu, L_m + \nu)(\Phi_c + \Psi(\Phi_c, \mu, \nu) + N(\Phi_c + \Psi(\Phi_c, \mu, \nu), \Phi_j^*\right\rangle, \quad j = 1, 2. \tag{3.56}$$

where the overdot denotes differentiation with respect to t, $\Phi_c = Z_1\Phi_1 + \bar{Z}_1\bar{\Phi}_1 + Z_2\Phi_2 + \bar{Z}_2\bar{\Phi}_2$ and $(Z_1, Z_2) \in \mathbf{C}^2$. By our remarks above, (3.56) is equivariant with respect to the action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ given by (3.55). Note that $\dot{\bar{Z}}_j$, $j = 1, 2$ satisfy the complex conjugates of equations (3.56).

Thus, for L near L_m and R near R_m the trajectories of the magnetoconvection equation are attracted to a center manifold, where the time evolution is given by a parametrized family of systems (3.56) of the general form

$$\dot{Z} = F(Z, \mu, \nu), \tag{3.57}$$

near $Z = 0$, $\mu = 0$, $\nu = 0$, where $Z = \left(Z_1, \bar{Z}_1, Z_2, \bar{Z}_2\right)^T \in \mathbf{C}^4$, and μ, ν are real parameters. The vector field $F = \left(F_1, \bar{F}_1, F_2, \bar{F}_2\right)^T$ is C^p for any positive integer p in a neighborhood of $(Z, \mu, \nu) = (0, 0, 0)$, and

$$F(0, \mu, \nu) = 0,$$

for all (μ, ν) near $(0, 0)$. Also

$$A_0 = D_Z F(0,0,0) = \begin{pmatrix} iP_1\omega_1 & 0 & 0 & 0 \\ 0 & -iP_1\omega_1 & 0 & 0 \\ 0 & 0 & iP_2\omega_2 & 0 \\ 0 & 0 & 0 & -iP_2\omega_2 \end{pmatrix}, \tag{3.58}$$

where $P_1, \omega_1, P_2, \omega_2$ are as in equation (2.28). Furthermore, the family (3.57) is equivariant under the action of J and β described in (3.55), i.e.,

$$\begin{aligned} F(JZ, \mu, \nu) &= JF(Z, \mu, \nu), \\ F(\beta Z, \mu, \nu) &= \beta F(Z, \mu, \nu), \end{aligned} \tag{3.59}$$

for all (Z, μ, ν) near $(0, 0, 0)$. We have reduced our problem locally to that of a double Hopf bifurcation with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry.

3.4 Normal forms

Now we put the system (3.57) into Poincaré-Birkhoff normal form. This involves simplifying (3.57) by removing terms from the Taylor series expansion of F order by order, using near-identity changes of coordinates. We can demand that these coordinate changes respect the symmetry (3.59). Those terms that cannot be removed at a given order are those with an extra symmetry under

$$\psi : (Z_1, \bar{Z}_1, Z_2, \bar{Z}_2) \to \left(e^{i\psi} Z_1, e^{i\psi} \bar{Z}_1, e^{i\psi} Z_2, e^{i\psi} \bar{Z}_2\right).$$

We now outline the calculation required to put (3.57) in normal form. First, we note that the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry (3.59) implies that each component of $F(\cdot, \mu, \nu)$ is odd in the components of Z, thus the lowest order terms in the Taylor series expansion of F about $Z = 0$ are odd in the components of Z, and (3.57) has the form

$$\begin{aligned} \dot{Z}_1 &= Z_1 \left(iP_1\omega_1 + a_1\mu + b_1\nu + C_1|Z_1|^2 + C_2|Z_2|^2 + C_7 Z_1^2 + C_8 Z_2^2 \right. \\ &\quad \left. + C_9 \bar{Z}_2^2\right) + \bar{Z}_1 \left(C_3 Z_2^2 + C_{10}|Z_1|^2 + C_{11}|Z_2|^2 + C_{12}\bar{Z}_1^2 + C_{13}\bar{Z}_2^2\right) \\ &\quad + O(|Z|^5 + |\mu,\nu||Z|^3 + |\mu,\nu|^2|Z|), \qquad (3.60) \\ \dot{Z}_2 &= Z_2 \left(iP_2\omega_2 + a_2\mu + b_2\nu + C_4|Z_1|^2 + C_5|Z_2|^2 + C_{14} Z_2^2 + C_{15} Z_1^2 \right. \\ &\quad \left. + C_{16}\bar{Z}_1^2\right) + \bar{Z}_2 \left(C_6 Z_1^2 + C_{17}|Z_2|^2 + C_{18}|Z_1|^2 + C_{19}\bar{Z}_2^2 + C_{20}\bar{Z}_1^2\right) \\ &\quad + O(|Z|^5 + |\mu,\nu||Z|^3 + |\mu,\nu|^2|Z|), \end{aligned}$$

where $a_1, b_1, a_2, b_2, C_1, \ldots, C_{20}$ are complex numbers that can be determined from the magnetoconvection equations (2.15), but as we will see below we will not need to compute them all. (The equations for $\dot{\bar{Z}}_1$ and $\dot{\bar{Z}}_2$ are the complex conjugates of those for $\dot{Z}_1$ and $\dot{Z}_2$).

We temporarily fix $\mu = 0, \nu = 0$, and write (3.57) as

$$\dot{Z} = A_0 Z + O(|Z|^3). \qquad (3.61)$$

For any positive integer n, let H_n denote the finite-dimensional vector space of vector fields $P(Z)$ in $\mathbf{C}^4$, whose components are homogeneous polynomials in the components of Z, of degree n. Define the linear map $adA_0 : H_n \to H_n$ by

$$adA_0(P) = [P, A_0] = DA_0 P - DPA_0, \qquad (3.62)$$

where $[\cdot,\cdot]$ is the Lie bracket. Let G_n be a complementary subspace to the range $adA_0(H_n)$ in H_n so that $H_n = adA_0(H_n) \oplus G_n$. Then by normal form theory (e.g. [16, Theorem 3.3.1]) and using the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry, for any odd positive integer $k \geq 3$ there is a coordinate transformation

$$Z = W + \hat{P}(W), \tag{3.63}$$

where the components of $\hat{P}(W)$ are $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant polynomials in the components of W, of degree at least 3, such that (3.57) becomes

$$\dot{W} = A_0 W + g^{(3)}(W) + g^{(5)}(W) + \ldots + g^{(k)}(W) + R_k(W), \tag{3.64}$$

with $g^{(n)} \in G_n$ for $3 \leq n \leq k, n$ odd, and $R_k = O(|W|^{k+2})$. Furthermore (3.64) is still $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant. To characterize G_n explicitly, we observe that

$$H_n = span\{W_{1,jklm}, W_{2,jklm}, W_{3,jklm}, W_{4,jklm}\},$$

where

$$\begin{aligned} W_{1,jklm} &= \left(W_1^j \bar{W}_1^k W_2^l \bar{W}_2^m, 0, 0, 0\right)^T, \quad W_{2,jklm} = \left(0, W_1^j \bar{W}_1^k W_2^l \bar{W}_2^m, 0, 0\right)^T, \\ W_{3,jklm} &= \left(0, 0, W_1^j \bar{W}_1^k W_2^l \bar{W}_2^m, 0\right)^T, \quad W_{4,jklm} = \left(0, 0, 0, W_1^j \bar{W}_1^k W_2^l \bar{W}_2^m\right)^T, \end{aligned}$$

and j, k, l, m run over all positive integers such that, $j + k + l + m = n$. Applying (3.62) we find that

$$adA_0(W_{1,jklm}) = i[(k - j - 1)P_1\omega_1 + (m - l)P_2\omega_2]W_{1,jklm} \tag{3.65}$$

$$adA_0(W_{2,jklm}) = i[(k - j + 1)P_1\omega_1 + (m - l)P_2\omega_2]W_{2,jklm} \tag{3.66}$$

$$adA_0(W_{3,jklm}) = i[(k - j)P_1\omega_1 + (m - l - 1)P_2\omega_2]W_{3,jklm} \tag{3.67}$$

$$adA_0(W_{4,jklm}) = i[(k - j)P_1\omega_1 + (m - l + 1)P_2\omega_2]W_{4,jklm}. \tag{3.68}$$

Since the matrix of $adA_0(H_n)$ is diagonal in this basis, G_n can be found merely by locating the zero eigenvalues of $adA_0(H_n)$. We observe that the conditions on $adA_0(W_{i,jklm}) =$

$0, i = 3,4$ can be derived from those of $adA_0(H_{i,jklm}) = 0, i = 1,2$ simply by interchanging $(j, k, P_1\omega_1)$ with $(l, m, P_2\omega_2)$. Also, the conditions for $adA_0(W_{2,jklm} = 0$ can be derived from $adA_0(W_{1,jklm}) = 0$ simply by interchanging $(j, l, P_1\omega_1, P_2\omega_2)$ with $(k, m, -P_1\omega_1, -P_2\omega_2)$, so it is enough only to consider $adA_0(W_{1,jklm}) = 0$. To find the condition on cubic terms in G_3 we must find j, k, l, m with $j + k + l + m = 3$ such that

$$P_1\omega_1(1 - j + k) + P_2\omega_2(m - l) = 0. \tag{3.69}$$

The $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariance also implies that $j + k$ be odd while $m + l$ should be even. Since $P_1\omega_1$ and $P_2\omega_2$ are both positive, (3.69) will be satisfied, only if

$$j = k + 1 \quad \text{and} \quad m = l$$

or

$$1 - j + k = n_1 \quad \text{and} \quad m - l = n_2 \quad \text{with} \quad n_1 n_2 < 0.$$

The possiblities $j = k + 1$ and $m = l$ show that the vectors

$$\begin{pmatrix} W_1|W_1|^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \bar{W}_1|W_1|^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ W_2|W_2|^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{W}_2|W_2|^2 \end{pmatrix},$$

$$\begin{pmatrix} W_1|W_2|^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \bar{W}_1|W_2|^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ W_2|W_1|^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{W}_2|W_1|^2 \end{pmatrix},$$

are in G_3. To consider the only other possibilities first we notice that

$$-2 \le 1 - j \le n_1 \le 1 + k \le 4, \quad -3 \le -l \le n_2 \le 3 \quad \text{and also} \quad |n_1| + |n_2| \le 4.$$

But neither n_1 nor n_2 can be odd, since this implies that $j+k$ is even or $m+l$ is odd, respectively, which was ruled out by $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry. The only remaining possibility is when

$$1 - j + k = 2, \text{ and } m - l = -2,$$

which implies that $j = m = 0$. For any fixed L_m, however, $2(P_1\omega_1 - P_2\omega_2)$ is not zero, therefore the coefficients of the vectors

$$\begin{pmatrix} \bar{W}_1 W_2^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ W_1 \bar{W}_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \bar{W}_2 W_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ W_2 \bar{W}_1^2 \end{pmatrix}, \tag{3.70}$$

can be removed by near-identity normal form transformations. However we note that, by doing so, the term $(P_1\omega_1 - P_2\omega_2)^{-1}$ will be appear as a factor in the coefficients of some fifth order terms in the transformed equation.

Ignoring the possible implications of this last remark fo now, normal form theory and the results of our calculations imply that for $k = 3$, (3.64) takes the form

$$\begin{aligned} \dot{W}_1 &= W_1\left(iP_1\omega_1 + C_1|W_1|^2 + C_2|W_2|^2\right) + O(|W|^5) \\ \dot{W}_2 &= W_2\left(iP_2\omega_2 + W_5|W_2|^2 + C_4|W_1|^2\right) + O(|W|^5), \end{aligned} \tag{3.71}$$

(the equations for $\dot{\overline{W}}_1$ and $\dot{\overline{W}}_2$ are the complex conjugates of those for $\dot{W}_1$ and $\dot{W}_2$). Now to restore the parameter dependence, we apply normal form theory to

$$\dot{Z} = F(Z, \mu, \nu), \ \dot{\mu} = 0, \ \dot{\nu} = 0, \tag{3.72}$$

seeking coordinate changes of the form

$$Z = W + P(W, \tilde{\mu}, \tilde{\nu}), \ \mu = \tilde{\mu}, \ \nu = \tilde{\nu} \tag{3.73}$$

(see, for example, [16, p.145]). The calculations then reduce to those outlined above, and we obtain the following result.

Proposition 3.3 *There exists a* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$*-equivariant coordinate transformation* $Z = W + P(W, \mu, \nu)$ *that transforms (3.60) into*

$$\begin{aligned} \dot{W}_1 &= W_1 \left(iP_1\omega_1 + a_1\mu + b_1\nu + C_1|W_1|^2 + C_2|W_2|^2 \right) \\ &\quad + O(|\mu,\nu|^2|W| + |\mu,\nu||W|^2 + |W|^5), \\ \dot{W}_2 &= W_2 \left(iP_2\omega_2 + a_2\mu + b_2\nu + W_5|W_2|^2 + C_4|W_1|^2 \right) \\ &\quad + O(|\mu,\nu|^2|W| + |\mu,\nu||W|^2 + |W|^5), \end{aligned} \tag{3.74}$$

where $a_1, b_1, a_2, b_2, C_1, C_2, C_4, C_5$ *are the same complex numbers that appear in (3.60). Note that (3.74) is still* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$*-equivariant.*

3.5 Large aspect ratios

We would like to study the third order truncation of system (3.74) as a means to study the system itself, but this is reasonable only when the higher order terms are much smaller than third order terms. For fixed σ, ζ, Q and all L_m, the normal form (3.74) gives a valid description of the local dynamics of the magnetoconvection equation (2.5), for parameters (L, R) in some sufficiently small neighborhood of (L_m, R_m), $m = 1, 2, \ldots$. We are also interested in behavior when the aspect ratio is large. Therefore for fixed σ, ζ and Q, we consider the "Case I" limiting behavior of the coefficients in (3.74) as $m \to \infty$. As m (and L_m) increase to ∞, the domain of validity of (3.74) shrinks to zero, so in our application we are mainly interested in large, but finite, m and L_m. Since some of higher order terms in (3.74) are proportional to $(P_1\omega_1 - P_2\omega_2)^{-1}$, and $P_1\omega_1 - P_2\omega_2$ is small for large L_m, we should study the size of $P_1\omega_1 - P_2\omega_2$ further. For fixed σ, ζ, Q from §2.5 we have

$$\begin{aligned} P_1\omega_1 &= P\omega + \frac{\zeta^2\pi^2}{2\lambda\omega} L_m^{-1} + O(L_m^{-2}), \\ P_2\omega_2 &= P\omega - \frac{\zeta^2\pi^2}{2\lambda\omega} L_m^{-1} + O(L_m^{-2}), \end{aligned} \tag{3.75}$$

therefore

$$P_1\omega_1 - P_2\omega_2 = \frac{\zeta^2\pi^2}{\lambda\omega} L_m^{-1} + O(L_m^{-2}), \tag{3.76}$$

as $L_m \to \infty$. We can expect the center manifold reduction, used to obtain (3.74), to break down if μ and ν are large enough so that modes other than those corresponding to Φ_1 and Φ_2 have eigenvalues for the linearization $K(R_m + \mu, L_m + \nu)$ that are close to the imaginary axis. Proposition 2.3 suggests that for large m, we should restrict our analysis at least so that $|a_j\mu + b_j\nu| < \delta,\ j = 1, 2$ where $\delta = O(m^{-2})$ as $m \to \infty$. The solutions of interest from the third order truncation have size $|W|^2 \leq \text{constant} \cdot \delta$, and in this case the higher order terms in (3.74), even if some of the fifth order terms have coefficients proportional to $(P_1\omega_1 - P_2\omega_2)^{-1}$, can be expected to have size only up to $\sim O(m^{-4})$. Since the principal parts of (3.74) have size up to $\sim O(m^{-3})$, it is reasonable to neglect the higher-order terms of (3.74).

Now suppose we take limiting values in "Case II". Recall that we fix σ but take a sequence of small ζ and large Q,

$$\zeta = \hat{m}^{-k/2}\hat{\zeta}, \quad Q = \hat{m}^{k/2}\hat{Q}, \tag{3.77}$$

where $k, \hat{\zeta}, \hat{Q}$ are fixed positive quantities and $\hat{m} = 1, 2, \ldots$. For each $\hat{m}$ we get a sequence $(L_m^{\hat{m}}, R_m^{\hat{m}}),\ m = 1, 2, ..$ of critical parameter values giving double Hopf bifurcations. We consider our "Case II" limit, when $m = \hat{m}$, and $m \to \infty$. We recall that $L_m^m = O(m)$ as before, and from (3.76) we have

$$P_1\omega_1 - P_2\omega_2 = O(m^{-(k+1)} + m^{-2}) \tag{3.78}$$

as $m \to \infty$. In Case II, fifth order terms in (3.74) with coefficients proportional to $(P_1\omega_1 - P_2\omega_2)^{-1}$ will have size up to $\sim O(m^{k-4})$, and if $k \geq 1$, such terms could be comparable in size to the principal parts and should not be neglected, unless $|a_j\mu + b_j\nu| \ll \delta$. To avoid this last restriction, we can use only a near-identity normal form transformation that do not result in fifth order terms with coefficients proportional to $(P_1\omega_1 - P_2\omega_2)^{-1}$. As discussed above, this means the cubic terms corresponding to the vectors (3.70) are not removed. Thus the resulting normal form is

$$\begin{aligned} \dot{W}_1 &= W_1\left(iP_1\omega_1 + a_1\mu + b_1\nu + C_1|W_1|^2 + C_2|W_2|^2\right) + C_3\bar{W}_1 W_2^2 + \text{h.o.t.}, \\ \dot{W}_2 &= W_2\left(iP_2\omega_2 + a_2\mu + b_2\nu + C_4|W_1|^2 + C_5|W_2|^2\right) + C_6\bar{W}_2 W_1^2 + \text{h.o.t.}, \end{aligned} \quad (3.79)$$

where h.o.t. $= O(|\mu,\nu|^2|W| + |\mu,\nu||W|^2 + |W|^5)$. Since the matrix $adA_0(H_n)$ is diagonal, the coefficients a_j, b_j, $j = 1, 2$, $C_1, \ldots, C_6$ in (3.79) are not affected by the normal form transformation and are the same complex numbers as in (3.60).

We will study (3.79) in Chapter 6, expecting it to give useful information on the magnetoconvection problem if $\zeta \ll 1$ and $Q \gg 1$, for a wider range of parameter values (R, L) than the normal form (3.74).

Chapter 4

Evaluation of center manifold coefficients

So far, we have shown that the dynamics of the magnetoconvection system (2.5)–(2.7) can be reduced to the normal forms (3.74) or (3.79). Both are valid, (3.74) is simpler, but (3.79) will give additional information in the Case II limit of decreasing ζ, and increasing Q and L. However, to make specific predictions of dynamical behavior, we need more information about the coefficients in the normal forms.

In this chapter we evaluate the normal form coefficients of (3.74) and (3.79). In §4.1 we give explicit formulae for the coefficients in terms of the parameters of the original magnetoconvection system (2.5)–(2.7). Then in §4.2 we study the asymptotic behavior of these coefficients for large aspect ratios, in both the Case I and II limits.. Finally, in §4.3 we evaluate the coefficients numerically for some specific values of the parameters of the magnetoconvection system.

4.1 Center manifold coefficients

In §3.4, we showed that the coefficients in the normal forms (3.74) and (3.79) are the same as the corresponding coefficients that appear in the system (3.60) which represents the phase flow in the center manifold W^c. The relation between (3.60) and the magnetoconvection system is given by (3.56).

First, we determine the terms in (3.60) that depend on the parameters μ and ν. By comparing derivatives with respect to μ in (3.54) and in (3.56), we have

$$a_j = \frac{\partial}{\partial \mu} \left\langle K(R_m + \mu, L_m + \nu)\Phi_j, \Phi_j^* \right\rangle |_{(\mu,\nu)=(0,0)}, \quad j = 1, 2 \tag{4.1}$$

Using equation (2.55), after some simplification we get

$$a_1 = \frac{(\sigma+1)\sigma\pi^2\omega_1 - i\sigma\pi^2(\delta\zeta + \omega_1^2)}{4(2L_m/m)^2 P_1^2 \omega_1(\omega_1^2 + \delta^2)}, \tag{4.2}$$

$$a_2 = \frac{(\sigma+1)\sigma\pi^2\omega_2 - i\sigma\pi^2(\delta\zeta + \omega_2^2)}{4(2L_m/m)^2 P_1^2 \omega_2(\omega_2^2 + \delta^2)}, \tag{4.3}$$

where

$$\delta = 1 + \sigma + \zeta. \tag{4.4}$$

From equations (4.2) and (4.3) it is clear that $a_{jR}, j = 1, 2$ are positive. We now find expressions for b_1 and b_2. Since $K(\mu, \nu)$ is analytic in ν, and $iP_j\omega_j$ are simple eigenvalues for $K_0 = K(R_m, L_m)$, there exist analytic functions $\alpha_j(\nu)$, $j = 1, 2$, such that $\alpha_j(\nu)$ is an eigenvalue for $K(R_m, L_m + \nu)$, with $\alpha_j(0) = iP_j\omega_j$ and

$$b_j = \frac{d\alpha_j}{d\nu}(0), \quad j = 1, 2, \tag{4.5}$$

By equation (2.14), $\alpha_1(\nu)$ satisfies the cubic equation

$$\begin{aligned} 0 = {} & \alpha_1^3 + \delta\tilde{P}_1(\nu)\alpha_1^2 + \left(\tilde{P}_1^2(\nu)(\sigma\zeta + \sigma + \zeta) + \sigma\zeta Q\pi^2 - \frac{m^2\pi^2\sigma R_m}{4\tilde{P}_1(\nu)(L_m+\nu)^2}\right)\alpha_1 \\ & + \sigma\zeta Q\pi^2\tilde{P}_1(\nu) - \frac{m^2\pi^2\sigma R_m\zeta}{4(L_m+\nu)^2} + \sigma\zeta\tilde{P}_1^3(\nu), \end{aligned} \tag{4.6}$$

where

$$\tilde{P}_1(\nu) = \pi^2 \left\{ [m/2(L_m+\nu)]^2 + 1 \right\}.$$

But from (2.19) and (2.23), we have

$$\sigma\zeta Q = \frac{(\omega_1^2 + \zeta^2)\tilde{P}_1(0)^2(\sigma+1)}{\pi^2(1-\zeta)},$$

$$\sigma R_m = \frac{4L_m^2\tilde{P}_1(0)^3(\sigma+\zeta)(1+\omega_1^2)}{m^2\pi^2(1-\zeta)}. \tag{4.7}$$

therefore $s_1 = \alpha_1/\tilde{P}(\nu)$ satisfies

$$
\begin{aligned}
0 &= s_1^3 + (\sigma+\zeta+1)s_1^2 + \left[(\sigma\zeta+\sigma+\zeta) + \frac{(\sigma+1)(\omega_1^2+\zeta^2)P_1^2}{(1-\zeta)P_1^2(\nu)}\right. \\
&\quad \left. - \frac{L_m^2 \tilde{P}_1(0)^3(\sigma+\zeta)(1+\omega_1^2)}{(L_m+\nu)^2\tilde{P}_1^3(\nu)(1-\zeta)}\right] s_1 + \frac{(\omega_1^2+\zeta^2)(\sigma+1)\tilde{P}_1(0)^2}{(1-\zeta)\tilde{P}_1^2(\nu)} \\
&\quad - \frac{L_m^2 \tilde{P}_1(0)^3(\sigma+\zeta)\zeta(1+\omega_1^2)}{(L_m+\nu)^2\tilde{P}_1^3(\nu)(1-\zeta)} + \sigma\zeta \\
&= f_1(s_1,\nu),
\end{aligned}
\tag{4.8}
$$

Since $f_1(i\omega_1, 0) = 0$, and $\partial f_1/\partial\nu(i\omega_1,\nu) \neq 0$, by the implicit function theorem, for sufficiently small ν there exists a unique solution of $f_1(s_1,\nu) = 0$, $s_1 = \Omega_1(\nu) = i\omega_1 + (B_{1R} + iB_{1I})\nu + O(|\nu|^2)$. To find B_{1R}, B_{1I} explicitly, we substitute $s_1 = \Omega_1(\nu)$ into equation (4.8), and calculate

$$
B_{1R} = \frac{(\sigma+1)(\sigma+\zeta)\pi^2\left[m^2(1-\zeta^2) - 2L_m^2(1+\omega_1^2)\right]}{2(\omega_1^2+\delta^2)(1-\zeta)P_1L_m^3}, \tag{4.9}
$$

$$
\begin{aligned}
B_{1I} &= \frac{\pi^2 m^2(\omega_1^2+\zeta^2)(\sigma+1)}{2P_1\omega_1(1-\zeta)L_m^3} \\
&\quad + \frac{\pi^2(2L_m^2 - m^2)(\sigma+\zeta)(1+\omega_1^2)(\zeta\delta+\omega_1^2)}{2P_1\omega_1(1-\zeta)L_m^3(\omega_1^2+\delta^2)},
\end{aligned}
\tag{4.10}
$$

where P_1 is given by (2.28). Similarly, we find

$$
B_{2R} = \frac{(\sigma+1)(\sigma+\zeta)\pi^2\left[(m+1)^2(1-\zeta^2) - 2L_m^2(1+\omega_2^2)\right]}{2(\omega_2^2+\delta^2)(1-\zeta)P_2L_m^3}, \tag{4.11}
$$

$$
\begin{aligned}
B_{2I} &= \frac{\pi^2(m+1)^2(\omega_2^2+\zeta^2)(\sigma+1)}{2P_2\omega_2(1-\zeta)L_m^3} \\
&\quad + \frac{\pi^2(2L_m^2 - (m+1)^2)(\sigma+\zeta)(1+\omega_2^2)(\zeta\delta+\omega_2^2)}{2P_2\omega_2(1-\zeta)L_m^3(\omega_1^2+\delta^2)},
\end{aligned}
\tag{4.12}
$$

where P_2 is given by (2.28). Since

$$
b_j = \frac{d\alpha_j(0)}{d\nu} = i\frac{d\tilde{P}_j(0)(\nu)}{d\nu}\omega_j + P_j(B_{jR} + iB_{jI}), \quad j = 1,2,
$$

we have

$$
b_{jR} = P_j B_{jR}, \quad j = 1,2, \tag{4.13}
$$

$$b_{1I} = \frac{-\pi^2 m^2 \omega_1}{2L_m^3} + P_1 B_{1I}, \tag{4.14}$$

$$b_{1I} = \frac{-\pi^2 (m+1)^2 \omega_2}{2L_m^3} + P_2 B_{2I}. \tag{4.15}$$

Proposition 4.1 *For all odd m, we have $b_{1R} < 0$ and $b_{2R} > 0$.*

Proof: To determine the sign of b_{1R} we note from (4.10) that

$$A = \left[m^2(1-\zeta^2) - 2L_m^2(1+\omega_1^2)\right] \frac{P_1^2}{2\pi^4 L_m^2 (1-\zeta^2)}$$

and b_{1R} have the same sign. But by (2.28) we have

$$1 + \omega_1^2 = 1 - \zeta^2 + \frac{\sigma\zeta Q \pi^2 (1-\zeta)}{(\sigma+1)P_1^2}, \tag{4.16}$$

and by (2.35) we have

$$\frac{\sigma\zeta Q}{\pi^2(\sigma+1)(1+\zeta)} = \frac{3m^2(m+1)^2}{(2L_m)} + \frac{m^2(m+1)^2\left[m^2+(m+1)^2\right]}{(2L_m)^6} - 1. \tag{4.17}$$

Therefore using (4.16) and (4.17) we get

$$A = \frac{2m^6 - (m+1)^2 m^2 \left[m^2 + (m+1)^2\right]}{(2Lm)^6} + \frac{3m^2\left[m^2-(m+1)^2\right]}{(2L_m)^4} < 0. \tag{4.18}$$

The proof that $b_{2R} > 0$ is similar. Q.E.D.

We now calculate the coefficients $C_1, ..., C_6$ of system (3.79) in terms of the parameters of the original convection problem. To do this, we need to approximate the center manifold function $\Psi(\Phi_c, \mu, \nu)$ at $\mu = 0$, $\nu = 0$. We observe that by Theorem 3.2, $\Psi(\Phi_c, 0, 0)$ can be approximated by its Taylor series to any finite order. We expand

$$\begin{aligned}
\Psi(\Phi_c, 0, 0) &= Z_1^2\Psi_{2000} + Z_1\bar{Z}_1\Psi_{1100} + Z_1 Z_2 \Psi_{1010} + Z_1\bar{Z}_2\Psi_{1001} \\
&\quad + \bar{Z}_1^2\Psi_{0200} + \bar{Z}_1 Z_2\Psi_{0110} + \bar{Z}_1\bar{Z}_2\Psi_{0101} + Z_2^2\Psi_{0020} \\
&\quad + Z_2\bar{Z}_2\Psi_{0011} + \bar{Z}_2^2\Psi_{0002} + O(|Z|^3).
\end{aligned} \tag{4.19}$$

where

$$\Phi_c = Z_1\Phi_1 + \bar{Z}_1\bar{\Phi}_1 + Z_2\Phi_2 + \bar{Z}_2\bar{\Phi}_2.$$

Then using the chain rule and (4.19), we have

$$\begin{aligned}\dot{\Psi} &= iP_1\omega_1 Z_1\left(2\Psi_{2000}Z_1 + \bar{Z}_1\Psi_{1100} + Z_2\Psi_{1010} + \bar{Z}_2\Psi_{1001}\right)\\ &- iP_1\omega_1\bar{Z}_1\left(Z_1\Psi_{1100} + 2\bar{Z}_1\Psi_{0200} + Z_2\Psi_{0110} + \bar{Z}_2\Psi_{1010}\right)\\ &+ iP_2\omega_2 Z_2\left(Z_1\Psi_{1010} + \bar{Z}_1\Psi_{0110} + 2Z_2\Psi_{0020} + \bar{Z}_2\Psi_{0011}\right)\\ &- iP_2\omega_2\bar{Z}_2\left(Z_1\Psi_{1001} + \bar{Z}_1\Psi_{0101} + Z_2\Psi_{0011} + 2\bar{Z}_2\Psi_{0002}\right)\\ &+ O(|Z|^3). \end{aligned} \tag{4.20}$$

On the other hand, from (3.54) we have

$$\dot{\Psi} = (I - \tilde{P})\left[K_0(\Psi(\Phi_c, 0, 0) + M_0(\Phi_c, \Phi_c)\right]. \tag{4.21}$$

where $K_0 = K(R_m, L_m)$, $M_0(\Phi_1, \Phi_2) = M(\Phi_1, \Phi_2, L_m)$ for any Φ_1, Φ_2, $\Psi(\Phi_c, 0, 0)$ is given by (4.19), and

$$\begin{aligned} M_0(\Phi_c, \Phi_c) &= Z_1^2 M_0(\Phi_1, \Phi_1) + \bar{Z}_1^2 M_0(\bar{\Phi}_1, \bar{\Phi}_1) + Z_1\bar{Z}_1^2\left[M_0(\Phi_1, \bar{\Phi}_1) + M_0(\bar{\Phi}_1, \Phi_1)\right] +\\ &\quad Z_1Z_2\left[M_0(\Phi_1, \Phi_2) + M_0(\Phi_2, \Phi_1)\right] + \bar{Z}_1\bar{Z}_2\left[M_0(\bar{\Phi}_1, \bar{\Phi}_2) + M_0(\bar{\Phi}_2, \bar{\Phi}_1)\right]\\ &\quad + Z_1\bar{Z}_2\left[M_0(\Phi_1, \bar{\Phi}_2) + M_0(\bar{\Phi}_2, \Phi_1)\right] + \bar{Z}_1 Z_2\left[M_0(\Phi_2, \bar{\Phi}_1) + M_0(\bar{\Phi}_1, \Phi_2)\right]\\ &\quad + Z_2^2 M_0(\Phi_2, \Phi_2) + \bar{Z}_2^2 M_0(\bar{\Phi}_2, \bar{\Phi}_2) + Z_2\bar{Z}_2^2\left[M_0(\Phi_2, \bar{\Phi}_2) + M_0(\bar{\Phi}_2, \Phi_2)\right]. \end{aligned} \tag{4.22}$$

Then by identifying the coefficients of quadratic terms we have

$$(K_0 - 2iP_1\omega_1)\Psi_{2000} = -(I - \tilde{P})M_0(\Phi_1, \Phi_1), \tag{4.23}$$

$$(K_0 + 2iP_1\omega_1)\Psi_{0200} = -(I - \tilde{P})M(\bar{\Phi}_1, \bar{\Phi}_1). \tag{4.24}$$

$$(K_0 - 2iP_2\omega_2)\Psi_{0020} = -(I - \tilde{P})M_0(\Phi_2, \Phi_2), \tag{4.25}$$

$$
\begin{aligned}
(K_0 + 2iP_2\omega_2)\Psi_{0002} &= -(I-\tilde{P})M_0(\bar{\Phi}_2, \bar{\Phi}_2), && (4.26)\\
K_0\Psi_{1100} &= -(I-\tilde{P})\left[M_0(\Phi_1, \bar{\Phi}_1) + M_0(\bar{\Phi}_1, \Phi_1\right], && (4.27)\\
K_0\Psi_{0011} &= -(I-\tilde{P})\left[M_0(\Phi_2, \bar{\Phi}_2) + M_0(\bar{\Phi}_2, \Phi_2)\right], && (4.28)\\
(K_0 - iP_1\omega_1 - iP_2\omega_2)\Psi_{1010} &= -(I-\tilde{P})\left[M_0(\Phi_1, \Phi_2) + M_0(\Phi_2, \Phi_1)\right], && (4.29)\\
(K_0 + iP_1\omega_1 - iP_2\omega_2)\Psi_{0110} &= -(I-\tilde{P})\left[M_0(\bar{\Phi}_1, \Phi_2) + M_0(\Phi_2, \bar{\Phi}_1)\right], && (4.30)\\
(K_0 - iP_1\omega_1 + iP_2\omega_2)\Psi_{1001} &= -(I-\tilde{P})\left[M_0(\Phi_1, \bar{\Phi}_2) + M_0(\bar{\Phi}_2, \Phi_1)\right], && (4.31)\\
(K_0 + iP_1\omega_1 + iP_2\omega_2)\Psi_{0101} &= -(I-\tilde{P})\left[M_0(\bar{\Phi}_1, \bar{\Phi}_2) + M_0(\bar{\Phi}_2, \bar{\Phi}_1)\right]. && (4.32)
\end{aligned}
$$

Since $\overline{M_0(\Phi_1, \Phi_2)} = M_0(\bar{\Phi}_1, \bar{\Phi}_2)$, from the above equations it is clear that

$$
\Psi_{1010} = \bar{\Psi}_{0101}, \quad \Psi_{1001} = \bar{\Psi}_{0110}, \quad \Psi_{0200} = \bar{\Psi}_{2000}, \quad \Psi_{0020} = \bar{\Psi}_{0002}.
$$

Also, we note that Ψ_{0110}, Ψ_{1001} can be obtained from Ψ_{1010} under the changes $\omega_1 \to -\omega_1$ and $\omega_2 \to -\omega_2$, respectively. To find $C_1, ..., C_6$ we substitute (4.19) into (3.56) with $\mu = 0$, $\nu = 0$, and compare with (3.79). Identifying the coefficients of the cubic terms gives

$$
\begin{aligned}
C_1 &= \langle M_0(\Psi_{1100}, \Phi_1) + M_0(\Phi_1, \Psi_{1100}), \Phi_1^*\rangle + && (4.33)\\
&\quad \left\langle M_0(\Psi_{2000}, \bar{\Phi}_1) + M_0(\bar{\Phi}_1, \Psi_{2000}), \Phi_1^*\right\rangle, \\
C_2 &= \left\langle M_0(\Psi_{0011}, \Phi_1) + M_0(\Phi_1, \Psi_{0011}) + M_0(\Psi_{1001}, \bar{\Phi}_2), \Phi_1^*\right\rangle + \\
&\quad \langle M_0(\Phi_2, \Psi_{1001}) + M_0(\Psi_{1010}, \Phi_2) + M_0(\bar{\Phi}_2, \Psi_{1010}), \Phi_1^*\rangle, && (4.34)\\
C_3 &= \left\langle M_0(\Psi_{0020}, \bar{\Phi}_1) + M_0(\bar{\Phi}_1, \Psi_{0020}), \Phi_1^*\right\rangle + && (4.35)\\
&\quad \langle M_0(\Psi_{0110}, \Phi_2) + M_0(\Phi_2, \Psi_{0110}), \Phi_1^*\rangle, \\
C_4 &= \left\langle M_0(\Psi_{1100}, \Phi_2) + M_0(\Phi_2, \Psi_{1100}) + M_0(\Psi_{1010}, \bar{\Phi}_1), \Phi_2^*\right\rangle \\
&\quad + \left\langle M_0(\bar{\Phi}_1, \Psi_{1010}) + M_0(\Psi_{0110}, \Phi_1) + M_0(\Phi_1, \Psi_{0110}), \Phi_2^*\right\rangle, && (4.36)\\
C_5 &= \langle M_0(\Psi_{0011}, \Phi_2) + M_0(\Phi_2, \Psi_{0011}), \Phi_2^*\rangle + && (4.37)
\end{aligned}
$$

$$
\begin{aligned}
& \left\langle M_0(\Psi_{0020},\bar{\Phi}_2)+M_0(\bar{\Phi}_2,\Psi_{0020}),\Phi_2^*\right\rangle, \\
C_6 \;=\; & \langle M_0(\Psi_{1001},\Phi_1)+M_0(\Phi_1,\Psi_{1001}),\Phi_2^*\rangle + \qquad (4.38)\\
& \left\langle M_0(\Psi_{2000},\bar{\Phi}_2)+M_0(\bar{\Phi}_2,\Psi_{2000}),\Phi_2^*\right\rangle .
\end{aligned}
$$

After lengthy calculations, which we have checked using the symbolic computation program Maple, we find explicit formulae for $C_1, ..., C_6$. The calculations consist of three major steps:

1) Computation of the terms $M_0(\Phi_i, \Phi_j)$.

2) Using the results of Step 1 to compute the Taylor series coefficients Ψ_{ijkl} of the center manifolds by solving (4.23)-(4.32).

3) Computing $C_1, ..., C_6$ according to (4.33)-(4.38).

The results are given below; more details of the calculations are given in Appendix A. We have

$$
\begin{aligned}
C_1 \;=\; & (\bar{C}_m/2)\left\{\frac{\sigma R_m}{P_1(1+i\omega_1)}\left[-\frac{1}{4P_1(1+\omega_1^2)}-\frac{\pi^2}{2P_1^2(1+i\omega_1)(\varpi_1+2i\omega_1)}\right]\right. \qquad (4.39)\\
& \left.-\frac{i\sigma\zeta Q(2L_m/m)^4\pi^2\omega_2}{P_1^2(\zeta+i\omega_1)(\zeta^2+\omega_1^2)(\varpi_1\zeta+2i\omega_1(2L_m/m)^2)}\right\},\\
C_2 \;=\; & \bar{C}_m/2\left\{-\frac{\sigma R_m}{P_1(1+i\omega_1)}\left[\frac{\pi\left((2m+1)(c_2+d_2)+c_2'+d'_2\right)}{4(m+1)}+\right.\right. \qquad (4.40)\\
& \left.\frac{\pi\left((2m+1)c_1+c_1'\right)}{16P_2(1-i\omega_2)}+\frac{\pi\left((2m+1)d_1+d'_1\right)}{16P_2(1+i\omega_2)}+\frac{1}{4P_2(1+\omega_2^2)}\right]\\
& \left.+\frac{2L_m^2P_1}{\pi^2 m}\left[D_3+D_4+\hat{D}_3+\hat{D}_4+\frac{\sigma\zeta Q\pi}{(\zeta+i\omega_1)P_1}(E_3+E_4+\hat{E}_3+\hat{E}_4)\right]\right\},\\
C_3 \;=\; & (\bar{C}_m/2)\left\{\frac{2L_m^2P_1}{\pi^2 m}\left[\tilde{D}_3+\tilde{D}_4\right]+\frac{2L_m^2\sigma\zeta Q}{m\pi(\zeta+i\omega_1)}\left[\tilde{E}_3+\tilde{E}_4\right]\right. \qquad (4.41)\\
& -\frac{\sigma R_m}{P_1(1+i\omega_1)}\left[\frac{\pi^2}{2P_2^2(1+i\omega_2)(\varpi_2+2i\omega_2)}+\frac{\pi\left((2m+1)\bar{d}_1-\bar{d}'_1\right)}{16P_2(1+i\omega_2)}\right.\\
& \left.\left.+\frac{\pi\left((2m+1)\bar{d}_2+\bar{d}'_2\right)}{4(m+1)}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
C_4 &= \bar{C}_{m+1}/2\left\{\frac{\sigma R_m}{P_2(1+i\omega_2)}\left[-\frac{\pi\left((2m+1)(c_2+\bar{d}_2)+c_2'+\bar{d}'_2\right)}{4m}+\right.\right. &(4.42)\\
&\left.\frac{\pi\left((2m+1)c_1+c_1'\right)}{16P_1(1-i\omega_1)}+\frac{\pi\left((2m+1)\bar{d}_1+\bar{d}'_1\right)}{16P_1(1+i\omega_1)}-\frac{1}{4P_1(1+\omega_1^2)}\right]\\
&\left.-\frac{2L_m^2P_2}{(m+1)\pi^2}\left[D_1+D_2+\bar{\hat{D}}_1+\bar{\hat{D}}_2+\frac{\sigma\zeta Q\pi}{P_2(\zeta+i\omega_2)}(E_1+E_2+\bar{\hat{E}}_1+\bar{\hat{E}}_2)\right]\right\},\\
C_5 &= (\bar{C}_{m+1}/2)\left\{\frac{\sigma R_m}{P_2(1+i\omega_2)}\left[-\frac{1}{4P_2(1+\omega_2^2)}-\frac{\pi^2}{2P_2^2(1+i\omega_2)(\varpi_2+2i\omega_2)}\right]\right. &(4.43)\\
&\left.-\frac{i\sigma\zeta Q(2L_m/(m+1))^4\pi^2\omega_2}{P_2^2(\zeta+i\omega_2)(\zeta^2+\omega_2^2)(\varpi_2\zeta+2i\omega_2(2L_m/(m+1))^2)}\right\},\\
C_6 &= (\bar{C}_{m+1}/2)\left\{-\frac{2L_m^2P_2}{(m+1)\pi^2}\left[\tilde{D}_1+\tilde{D}_2\right]-\frac{2L_m^2\sigma\zeta Q}{\pi(m+1)(\zeta+i\omega_2)}\left[\tilde{E}_1+\tilde{E}_2\right]\right. &(4.44)\\
&-\frac{\sigma R_m}{P_2(1+i\omega_2)}\left[\frac{\pi^2}{2P_1^2(1+i\omega_1)(\varpi_1+2i\omega_1)}-\frac{\pi\left((2m+1)d_1-d_1'\right)}{16P_1(1+i\omega_1)}\right.\\
&\left.\left.+\frac{\pi\left((2m+1)d_2+d'_2\right)}{4m}\right]\right\},
\end{aligned}
$$

where $\bar{C}_m, \bar{C}_{m+1}$ are given by (2.55), $\varpi_j = \dfrac{4\pi^2}{P_1}$ for $j=1,2$,

$$
\begin{aligned}
D_1 &= \frac{\pi^2}{P_2}\left\{-\frac{\sigma\zeta Q\left((2m+1)c_3+c_3'+8(c_4+c_4')\right)}{8m(\zeta-i\omega_1)}\right.\\
&\left.+\frac{\pi\left((2m+1)c_1+c_1'\right)}{4m}+\frac{\pi(m+1)(2m+1)(c_1+c_1')}{32mL_m^2}\right\}, &(4.45)\\
D_2 &= \frac{\pi^2}{P_2}\left\{\frac{\pi^2\sigma\zeta Q\left((2m+1)c_3+c_3'\right)}{2mP_1(\zeta-i\omega_1)}+\frac{\pi^2\sigma\zeta Q(2m+1)c_3}{32mL_m^2P_1(\zeta-i\omega_1)}\right.\\
&+\frac{\pi^2\sigma\zeta Q\left[(2m+1)^3c_3'+8\left(c_4+(2m+1)^2c_4'\right)\right]}{32mL_m^2P_1(\zeta-i\omega_1)}\\
&\left.+\frac{\pi\left((2m+1)c_1+c_1'\right)}{8m}-\frac{\pi(m+1)\left((2m+1)c_1-c_1'\right)}{32L_m^2}\right\}, &(4.46)\\
D_3 &= \frac{\pi^2}{P_1}\left[\frac{\sigma\zeta Q\left(-(2m+1)c_3-c_3'+8(c_4+c_4')\right)}{8(m+1)(\zeta-i\omega_2)}\right.\\
&\left.+\frac{\pi\left((2m+1)c_1+c_1'\right)}{4(m+1)}-\frac{\pi m(2m+1)(c_1-c_1')}{32L_m^2(m+1)}\right], &(4.47)\\
D_4 &= \frac{\pi^2}{P_1}\left\{\frac{\pi^2\sigma\zeta Q\left((2m+1)c_3+c_3'\right)}{2(m+1)P_2(\zeta-i\omega_2)}+\frac{\pi^2\sigma\zeta Q(2m+1)c_3}{32(m+1)L_m^2P_2(\zeta-i\omega_2)}\right.
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{\pi^2\sigma\zeta Q\left[(2m+1)^3 c_3' - 8\left(c_4 + (2m+1)^2 c_4'\right)\right]}{32(m+1)L_m^2 P_2(\zeta - i\omega_2)} \\
& + \frac{\pi\left((2m+1)c_1 + c_1'\right)}{8(m+1)} - \frac{m\pi\left((2m+1)c_1 - c_1'\right)}{32L_m^2}\Bigg\}, \qquad (4.48)
\end{aligned}
$$

$$
E_1 = \frac{\pi\left((2m+1)c_3 + c_3'\right) - 8\pi(c_4 + c_4')}{8m}, \qquad (4.49)
$$

$$
E_2 = -\frac{\pi^2\left((2m+1)c_1 + c_1'\right)}{8mP_1(\zeta - i\omega_1)}, \qquad (4.50)
$$

$$
E_3 = \frac{\pi\left[(2m+1)c_3 + c_3' + 8(c_4 + c_4')\right]}{8(m+1)}. \qquad (4.51)
$$

$$
E_4 = -\frac{\pi^2\left[(2m+1)c_1 + c_1'\right]}{8P_2(m+1)(\zeta - i\omega_2)}, \qquad (4.52)
$$

$$
\begin{aligned}
\tilde{D}_1 = {} & \frac{\pi^2}{P_2}\Bigg\{-\frac{\sigma\zeta Q\left((2m+1)d_3 + d_3' + 8(d_4 + d_4')\right)}{8m(\zeta + i\omega_1)} \\
& + \frac{\pi\left((2m+1)d_1 + d_1'\right)}{4m} + \frac{\pi(m+1)(2m+1)(d_1 + d_1')}{32mL_m^2}\Bigg\}, \qquad (4.53)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}_2 = {} & \frac{\pi^2}{P_2}\Bigg\{\frac{\pi^2\sigma\zeta Q\left((2m+1)d_3 + d_3'\right)}{2mP_1(\zeta + i\omega_1)} + \frac{\pi^2\sigma\zeta Q(2m+1)d_3}{32mL_m^2 P_1(\zeta + i\omega_1)} \\
& + \frac{\pi^2\sigma\zeta Q\left[(2m+1)^3 d_3' + 8\left(d_4 + (2m+1)^2 d_4'\right)\right]}{32mL_m^2 P_1(\zeta + i\omega_1)} \\
& + \frac{\pi\left((2m+1)d_1 + d_1'\right)}{8m} - \frac{\pi(m+1)\left((2m+1)d_1 - d_1'\right)}{32L_m^2}\Bigg\}, \qquad (4.54)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}_3 = {} & \frac{\pi^2}{P_1}\Bigg[\frac{\sigma\zeta Q\left(-(2m+1)\bar{d}_3 - \bar{d}_3' + 8(\bar{d}_4 + \bar{d}_4')\right)}{8(m+1)(\zeta + i\omega_2)} \\
& + \frac{\pi\left((2m+1)\bar{d}_1 + \bar{d}_1'\right)}{4(m+1)} - \frac{m(2m+1)\pi(\bar{d}_1 - \bar{d}_1')}{32L_m^2(m+1)}\Bigg], \qquad (4.55)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}_4 = {} & \frac{\pi^2}{P_1}\Bigg\{\frac{\pi^2\sigma\zeta Q\left((2m+1)\bar{d}_3 + \bar{d}_3'\right)}{2(m+1)P_2(\zeta + i\omega_2)} + \frac{\pi^2\sigma\zeta Q(2m+1)\bar{d}_3}{32(m+1)L_m^2 P_2(\zeta + i\omega_2)} \\
& + \frac{\pi^2\sigma\zeta Q\left[(2m+1)^3\bar{d}_3' - 8\left(\bar{d}_4 + (2m+1)^2\bar{d}_4'\right)\right]}{32(m+1)L_m^2 P_2(\zeta + i\omega_2)} \\
& + \frac{\pi\left((2m+1)\bar{d}_1 + \bar{d}_1'\right)}{8(m+1)} - \frac{m\pi\left((2m+1)\bar{d}_1 - \bar{d}_1'\right)}{32L_m^2}\Bigg\}, \qquad (4.56)
\end{aligned}
$$

$$
\tilde{E}_1 = \frac{\pi\left((2m+1)d_3 + d_3'\right) - 8\pi(d_4 + d_4')}{8m}, \qquad (4.57)
$$

$$\tilde{E}_2 = -\frac{\pi^2\left((2m+1)d_1 + d_1'\right)}{8mP_1(\zeta + i\omega_1)}, \tag{4.58}$$

$$\tilde{E}_3 = \frac{\pi\left((2m+1)\bar{d}_3 + \bar{d}_3'\right) + 8\pi(\bar{d}_4 + \bar{d}_4')}{8(m+1)}, \tag{4.59}$$

$$\tilde{E}_4 = -\frac{\pi^2\left((2m+1)\bar{d}_1 + \bar{d}_1'\right)}{8P_2(m+1)(\zeta + i\omega_2)}, \tag{4.60}$$

and $\hat{D}_j, \hat{E}_j,\ j = 1, ..., 4$ are obtained from $D_j, E_j, j = 1, ..., 4$, respectively, by making the change $\omega_2 \to -\omega_2$. Also

$$c_1 = -\frac{(A_1/4) + (\sigma\zeta QA_0A_3/2\pi\eta_3) + (\sigma RA_2/4L_m^2\eta_1)}{(\eta_2A_0/4\pi^2) + (\sigma\zeta QA_0/\eta_3) - (\sigma R/16L_m^2\eta_1)}, \tag{4.61}$$

$$c_2 = (c_1/4\eta_1) - (A_2/\eta_1), \tag{4.62}$$

$$c_3 = -\frac{2\pi c_1 + A_3}{\eta_3}, \tag{4.63}$$

$$c_4 = -A_4/\eta_4, \tag{4.64}$$

$$c_1' = -\frac{(B_1/4) + (\sigma\zeta QB_0B_3/2\pi\psi_3) + ((2m+1)\sigma RB_2/4L_m^2\psi_1)}{(\psi_2B_0/4\pi^2) + (\sigma\zeta QB_0/\psi_3) - ((2m+1)^2\sigma R/16L_m^2\psi_1)}, \tag{4.65}$$

$$c_2' = (2m+1)(c_1'/4\psi_1) - (B_2/\psi_1), \tag{4.66}$$

$$c_3' = -\frac{2\pi c_1' + B_3}{\psi_3}, \tag{4.67}$$

$$c_4' = -B_4/\psi_4. \tag{4.68}$$

where

$$\eta_1 = A_0 + iP_1\omega_1 + iP_2\omega_2, \quad \eta_2 = \sigma A_0 + iP_1\omega_1 + iP_2\omega_2,$$

$$\eta_3 = \zeta A_0 + iP_1\omega_1 + iP_2\omega_2, \quad \eta_4 = \zeta(\pi/2L_m)^2 + iP_1\omega_1 + iP_2\omega_2,$$

$$\psi_1 = B_0 + iP_1\omega_1 + iP_2\omega_2, \quad \psi_2 = \sigma B_0 + iP_1\omega_1 + iP_2\omega_2,$$

$$\psi_3 = \zeta B_0 + iP_1\omega_1 + iP_2\omega_2, \quad \psi_4 = \zeta((2m+1)\pi/2L_m)^2 + iP_1\omega_1 + iP_2\omega_2,$$

$$A_0 = \pi^2(1/4L_m^2 + 4),$$

$$B_0 = \pi^2((2m+1/2L_m)^2 + 4),$$

$$\begin{aligned}
B_1 &= -\frac{\sigma\zeta Q\pi(P_2 - P_1)}{m(m+1)P_1P_2(\zeta + i\omega_1)(\zeta + i\omega_2)} - \frac{(2m+1)\pi}{4L_m^2 m(m+1)},\\
A_1 &= (2m+1)B_1,\\
A_2 &= -\frac{\pi(2m+1)\left[mP_2(1+i\omega_2) + (m+1)P_1(1+i\omega_1)\right]}{4m(m+1)P_1P_2(1+i\omega_1)(1+i\omega_2)},\\
B_2 &= -\frac{\pi\left[-mP_2(1+i\omega_2) + (m+1)P_1(1+i\omega_1)\right]}{4m(m+1)P_1P_2(1+i\omega_1)(1+i\omega_2)},\\
B_3 &= \frac{\pi^2\left[P_1(\zeta + i\omega_1) - P_2(\zeta + i\omega_2)\right]}{m(m+1)P_1P_2(\zeta + i\omega_1)(\zeta + i\omega_2)},\\
A_3 &= (2m+1)B_3,\\
A_4 &= -\frac{\pi^2\left[P_1(\zeta + i\omega_1) + P_2(\zeta + i\omega_2)\right]}{4m(m+1)P_1P_2(\zeta + i\omega_1)(\zeta + i\omega_2)},\\
B_4 &= (2m+1)^2A_4.
\end{aligned}$$

and $d_j, d'_j, j = 1, ..., 4$ are obtained from $c_j, c'_j, j = 1, ..., 4$ by making the change $\omega_2 \to -\omega_2$.

4.2 Asymptotic results

In this section we give explicit formulae for the limiting values of the coefficients

$$a_1, a_2, b_1, b_2, C_1, ..., C_6$$

in the normal forms (3.74) and (3.79), as $m \to \infty$. We consider Cases I and II discussed in §3.5. Details of these calculations are given in Appendix A. Throughout this chapter we assume

$$\epsilon = m^{-1}.$$

4.2.1 Case I

It is easy to see from (4.2) and (4.3) that

$$\lim_{m\to\infty} a_1 = \lim_{m\to\infty} a_2 = a,$$

where

$$a = \frac{(\sigma+1)\sigma\pi^2\omega - i\sigma\pi^2(\delta\zeta+\omega^2)}{4\lambda^2P^2\omega(\omega^2+\delta^2)}, \tag{4.69}$$

and λ, P, ω are given by (2.27), (2.59), (2.60) $\delta = 1+\sigma+\zeta$. It is clear from (4.13)–(4.15) that b_j, $j = 1, 2$, are $O(\epsilon)$ as $m \to \infty$, but we will show that actually the real parts b_{jR} are $O(\epsilon^2)$. By equation (4.18) and (4.9) we have

$$b_{1R} = \frac{-\pi^6(\sigma+1)(\sigma+\zeta)(1+\zeta)A}{L_m P_1^2(\omega_1^2+\delta^2)}, \tag{4.70}$$

where A is as in (4.18), after some simplification we get

$$b_{1R} = -\frac{12\pi^4(\sigma+\zeta)(\sigma+1)(\zeta+1)}{\lambda^5P(\omega^2+\delta^2)}\epsilon^2 + O(\epsilon^3), \tag{4.71}$$

and similarly we have

$$b_{2R} = \frac{12\pi^4(\sigma+\zeta)(\sigma+1)(\zeta+1)}{\lambda^5P(\omega^2+\delta^2)}\epsilon^2 + O(\epsilon^3), \tag{4.72}$$

Also, after some calculation we find asymptotic expansions of $b_{jI}, j = 1, 2$ and we get

$$\begin{aligned} b_{1I} &= \left\{\frac{4\pi^2\left[(\sigma+1)(\omega^2+\zeta^2) - \omega^2(1-\zeta)\right]}{\omega(1-\zeta)\lambda^3}\right. \\ &\quad \left. + \frac{2\pi^2\left(\lambda^2-2\right)\left(\sigma+\zeta\right)\left(1+\omega^2\right)\left(\zeta\delta+\omega^2\right)}{\omega(1-\zeta)\left(\delta^2+\omega^2\right)\lambda^3}\right\}\epsilon + O(\epsilon^2), \end{aligned} \tag{4.73}$$

$$\begin{aligned} b_{2I} &= \left\{\frac{4\pi^2\left[(\sigma+1)(\omega^2+\zeta^2) - \omega^2(1-\zeta)\right]}{\omega(1-\zeta)\lambda^3}\right. \\ &\quad \left. + \frac{2\pi^2\left(\lambda^2-2\right)\left(\sigma+\zeta\right)\left(1+\omega^2\right)\left(\zeta\delta+\omega^2\right)}{\omega(1-\zeta)\left(\delta^2+\omega^2\right)\lambda^3}\right\}\epsilon + O(\epsilon^2), \end{aligned} \tag{4.74}$$

Remark 4.1 *We notice that in the asymptotic expansions of a_1, b_1 and $a_2, , b_2$ the terms at order $O(\epsilon)$ are equal while the terms at order $O(\epsilon^2)$ are negatives of each other. This is due to the fact that the same relation holds for the asymptotic expansions of P_1, ω_1 and P_2, ω_2. We will find similar behavior for the center manifold coefficients.*

For the normal form coefficients $C_1, ..., C_6$ we have

$$\begin{aligned}
\lim_{m\to\infty} C_1 &= \lim_{m\to\infty} C_5 = A + B, \\
\lim_{m\to\infty} C_2 &= \lim_{m\to\infty} C_4 = A, \\
\lim_{m\to\infty} C_3 &= \lim_{m\to\infty} C_6 = C,
\end{aligned}$$

where

$$\begin{aligned}
A + B &= -\frac{\pi^2\sigma R_0(\zeta + i\omega)}{4\lambda^2 P^2\omega(\omega - i\delta)}\left[-\frac{1}{4P(1+\omega^2)} - \frac{\pi^2}{2P^2(1+i\omega)(\varpi + 2i\omega)}\right] \\
&\quad + \frac{i\sigma\zeta Q\lambda^2\pi^4(1+i\omega)}{4P^3(\zeta^2+\omega^2)(\varpi\zeta + 2i\omega\lambda^2)(\omega - i\delta)}, \qquad (4.75)\\
A &= \frac{-\pi^2(\zeta+i\omega)(1+i\omega)}{4\lambda^2 P\omega(\omega - i\delta)}\left\{-\frac{\sigma R_0}{P(1+i\omega)}\left[\frac{\pi(c_{21}+d_{21})}{2} + \frac{\pi d_{11}}{4\lambda P(1+i\omega)}\right.\right. \\
&\quad \left. + \frac{1}{4P(1+\omega^2)}\right] + \frac{\sigma Q\zeta^2\lambda^2 c'_{41}}{(\zeta^2+\omega^2)} - \frac{2\pi^2\sigma\zeta Q c'_{41}}{P(\zeta - i\omega)} + \frac{\sigma\zeta Q d'_{41}\lambda^2}{\zeta + i\omega} \\
&\quad \left. + \frac{\pi d_{11}(3\lambda^2 - 1)}{4\lambda} + \frac{\pi^2\sigma\zeta Q(\lambda d_{31} - 2d'_{41})}{P(\zeta+i\omega)} - \frac{\pi\sigma\zeta Q d_{11}}{4P(\zeta+i\omega)^2}\right\}, \qquad (4.76)\\
C &= \frac{-\pi^2(\zeta+i\omega)(1+i\omega)}{4\lambda^2 P\omega(\omega - i\delta)}\left\{-\frac{\sigma R_0}{P(1+i\omega)}\left[\frac{\pi^2}{2P^2(1+i\omega)(\varpi+2i\omega)}\right.\right. \\
&\quad \left. - \frac{\pi d_{11}}{4\lambda P(1+i\omega)} + \pi d_{21}/2\right] - \frac{\pi^2\sigma\zeta Q\lambda(d_{31} + 2d'_{41})}{P(\zeta+i\omega)} \\
&\quad \left. + \frac{\sigma\zeta Q d'_{41}\lambda^2}{\zeta+i\omega} + \frac{\sigma\zeta Q\lambda\pi d_{11}}{4P(\zeta+i\omega)^2} - \frac{\pi d_{11}(3\lambda^2-1)}{4\lambda}\right\}, \qquad (4.77)
\end{aligned}$$

where and λ, ω, P, R_0 satisfy equations (2.58), (2.59), (2.66) and (2.67) respectively, $\varpi = 4\pi^2/P$, and

$$\begin{aligned}
c_{21} &= \frac{\pi}{P^2(1+i\omega)(\varpi+2i\omega)}, \qquad (4.78)\\
c'_{41} &= \frac{2\pi^2\lambda^2}{P^2(\zeta+i\omega)(\varpi\zeta + 2i\omega\lambda^2)}, \qquad (4.79)\\
d_{11} &= -\frac{\lambda i\omega Q\pi}{P(\zeta^2+\omega^2)(4\pi^2+Q)}, \qquad (4.80)\\
d_{22} &= \frac{1}{4\pi P(1+\omega^2)}, \qquad (4.81)
\end{aligned}$$

$$d_{33} = -\frac{2\lambda i\omega\pi^2}{\zeta P(4\pi^2+Q)(\zeta^2+\omega^2)}, \tag{4.82}$$

$$d'_{41} = \frac{\lambda^2}{2P(\zeta^2+\omega^2)}. \tag{4.83}$$

4.2.2 Case II

For (4.2) and (4.3) we obtain (for any $k > 0$)

$$\lim_{m\to\infty} a_1 = \lim_{m\to\infty} a_2 = \check{a}, \tag{4.84}$$

where

$$\check{a} = \frac{(\sigma+1)\sigma\pi^2\check{\omega} - i\sigma\pi^2 + \omega^2}{4\check{\lambda}^2\check{P}^2\check{\omega}(\check{\omega}^2+(\sigma+1)^2)}, \tag{4.85}$$

and $\check{\lambda}, \check{P}, \check{\omega}$ are satisfying (2.72), (2.73) and (2.74). Using (4.70) We also obtain

$$\check{b}_{1R} = -\frac{4\sigma(\sigma+1)\left(\pi^2\check{\omega}^2+\check{P}\right)}{\check{\lambda}^3\check{P}(\omega^2+(\sigma+1)^2)}\epsilon^2 + O(\epsilon^{2+(k/2)}), \tag{4.86}$$

and

$$\check{b}_{2R} = \frac{4\sigma(\sigma+1)\left(\pi^2\check{\omega}^2+\check{P}\right)}{\check{\lambda}^3\check{P}(\check{\omega}^2+(\sigma+1)^2)}\epsilon^2 + O(\epsilon^{2+(k/2)}). \tag{4.87}$$

$$\check{b}_{1I} = \left[\frac{4\pi^2\sigma\check{\omega}}{\check{\lambda}^3} + \frac{2\pi^2\sigma\left(\check{\lambda}^2-2\right)(1+\check{\omega}^2)\check{\omega}}{((\sigma+1)^2+\check{\omega}^2)\check{\lambda}^3}\right]\epsilon + O(\epsilon^{1+(k/2)}), \tag{4.88}$$

$$\check{b}_{2I} = \left[\frac{4\pi^2\sigma\check{\omega}}{\check{\lambda}^3} + \frac{2\pi^2\sigma\left(\check{\lambda}^2-2\right)(1+\check{\omega}^2)\check{\omega}}{\left((\sigma+1)^2+\check{\omega}^2\right)\check{\lambda}^3}\right]\epsilon + O(\epsilon^{1+(k/2)}), \tag{4.89}$$

as $m \to \infty$. However, since by (2.77)–(2.80), $P_1 - P_2$ and $\omega_1 - \omega_2$ are $O(\epsilon)$, it is easy to show that $b_{1I} - b_{2I} = O(\epsilon^2)$. This will be important in our analysis in Chapter 6.

In Appendix B, we show that if $k > 2$, then C_2, C_3, C_4 and C_6 become unbounded as $m \to \infty$, and if $k = 2$, then limits exist but

$$\lim_{L_m\to\infty} C_2 \neq \lim_{L_m\to\infty} C_4, \quad \text{and} \quad \lim_{L_m\to\infty} C_3 \neq \lim_{L_m\to\infty} C_6.$$

In order to keep the limiting behavior of the normal form coefficients similar to that for Case I, we restrict $k < 2$. Then we find

$$\begin{aligned}
\lim_{m\to\infty} C_1 &= \lim_{m\to\infty} C_5 = \check{A} + \check{B}, \\
\lim_{m\to\infty} C_2 &= \lim_{m\to\infty} C_4 = \check{A}, \\
\lim_{m\to\infty} C_3 &= \lim_{m\to\infty} C_6 = \check{C},
\end{aligned}$$

where

$$\begin{aligned}
\check{A} + \check{B} &= \frac{i\pi^2\sigma\check{R}_0}{4\check{\lambda}^2\check{P}^2(\check{\omega} - i(\sigma+1))}\left[\frac{1}{4\check{P}(1+\check{\omega}^2)} + \frac{\pi^2}{2\check{P}^2(1+i\check{\omega})(\check{\varpi}+2i\check{\omega})}\right] \\
&\quad + \frac{\sigma\zeta Q\pi^4(1+i\check{\omega})}{8\check{\omega}^3\check{P}^3(\check{\omega} - i(\sigma+1))}, && (4.90) \\
\check{A} &= \frac{\pi^2\sigma\check{R}_0 i}{4\check{\lambda}^2\check{P}^2(\check{\omega} - i(\sigma+1))}\left[\frac{\pi(\check{c}_{21} + \check{d}_{21})}{2} + \frac{\pi\check{d}_{11}}{4\check{\lambda}\check{P}(1+i\check{\omega})} + \frac{1}{4\check{P}(1+\check{\omega}^2)}\right] \\
&\quad - \frac{\pi^2(1+i\check{\omega})i}{4\check{P}\check{\lambda}^2(\check{\omega} - i(\sigma+1)}\left[\frac{\pi^2\sigma\zeta Q(2\check{c}'_{41} - 2\check{d}'_{41} + \check{\lambda}\check{d}_{31})}{i\check{P}\check{\omega}} + \frac{\pi\sigma\zeta Q\check{d}_{11}}{4\check{P}\check{\omega}^2}\right. \\
&\quad \left. + \frac{\pi\check{d}_{11}(3\check{\lambda}^2 - 1)}{4\check{\lambda}} + \frac{\check{\lambda}^2\sigma\zeta Q\check{d}'_{41}}{i\check{\omega}}\right], && (4.91) \\
\check{C} &= \frac{i\pi^2\sigma R_0}{4\check{\lambda}^2\check{P}^2[\check{\omega} - i(\sigma+1)]}\left[\frac{\pi^2}{2\check{P}^2(1+i\check{\omega})(\check{\varpi}+2i\check{\omega})} - \frac{\pi\check{d}_{11}}{4\check{\lambda}\check{P}(1+i\check{\omega})} + \frac{\pi\check{d}_{21}}{2}\right] \\
&\quad - \frac{i\pi^2(1+i\check{\omega})}{4\check{P}\check{\lambda}^2(\check{\omega} - i(\sigma+1))}\left[-\frac{\pi^2\sigma\zeta Q\check{\lambda}(\check{d}_{31} + 2\check{d}'_{41})}{i\check{P}\check{\omega}} - \frac{\pi\check{d}_{11}(3\check{\lambda}^2 - 1)}{4\check{\lambda}}\right. \\
&\quad \left. + \frac{\sigma\zeta Q\check{d}'_{41}\check{\lambda}^2}{i\check{\omega}} - \frac{\sigma\zeta Q\lambda\pi\check{d}_{11}}{i4\check{P}\check{\omega}^2}\right], && (4.92)
\end{aligned}$$

and $\check{\varpi} = \dfrac{4\pi^2}{\check{P}}$,

$$\begin{aligned}
\check{c}_{21} &= \frac{\pi}{\check{P}^2(1+i\check{\omega})(4\pi^2/\check{P} + 2i\check{\omega})}, && (4.93) \\
\check{c}'_{41} &= -\frac{\pi^2}{\check{\omega}^2\check{P}^2}, && (4.94) \\
\check{d}_{11} &= -\pi\check{\lambda}i/\check{P}\check{\omega}, && (4.95)
\end{aligned}$$

$$\check{d}_{21} = \frac{1}{4\pi\check{P}(1+\check{\omega}^2)}, \tag{4.96}$$

$$\check{d}_{31} = -\frac{2\pi i\check{\lambda}}{\zeta Q\check{P}\check{\omega}}, \tag{4.97}$$

$$\check{d}'_{41} = \frac{\check{\lambda}^2}{2\check{P}\check{\omega}}. \tag{4.98}$$

4.3 Numerical results

Because of the complexity of the formulas for the normal coefficients $C_1, ..., C_6$, we were not able to find the signs of the real and imaginary parts of these coefficients analytically. The signs will be important in our bifurcation and stability analysis in Chapters 5 and 6. Therefore we have evaluated the coefficients numerically, using Maple to carry out the numerical computations. The results are summarized in Appendix B, and a representative selection of them is presented in this section. The symbols ∞ in the tables correspond to the limiting values of the coefficients as $m \to \infty$, as calculated in §4.2. We note that numerical results appear to verify our analytic asymptotic results on the normal form coefficients.

The values of L_m were calculated by using (2.35). We have also calculated the values of C_1 and C_5 for a relatively large set of parameter values σ, ζ and Q (see Appendix B). We found that C_{1R} and C_{5R} are negative for all values of σ, ζ, Q and m we used. The numerical computations of C_2, C_3, C_4 and C_6 took much longer and more care was required to avoid memory overflow and round-off error. For $\zeta = .1, \sigma = 1$ and all the different values of Q and m that we used, we found that C_{2R}, C_{4R} are negative, and

$$C_{1R}C_{5R} - C_{2R}C_{4R} < 0. \tag{4.99}$$

We have also checked this inequality for a wider set of parameter values of σ, ζ and Q in the limit as $m \to \infty$. This will be important for our results in Chapter 5.

In our calculations we observed that the signs of the real and imaginary parts of C_1, C_2, C_4 and C_5 for all finite m and in the limit as $m \to \infty$ are the same, but the signs of C_{3R}, C_{6R}, C_{3I} and C_{6I} changed as m was increased.

We have also computed the values of normal form coefficients in Case II with $\sigma = 1, \hat{\zeta} = .1, \hat{Q} = 100\pi^2, k = 1$ and for increasing values of m (Tables 4.3-4.6). The essential features are preserved, and our analytic asymptotic results seem verified numerically. However, convergence appears to be slower: C_1 and C_5, C_2 and C_4, C_3 and C_6 approach their limits much slower than their differences approach zero as $m \to \infty$. For example, it is easy to verify that both $C_1 = \check{A} + \check{B} + O(\epsilon^{k/2})$ and $C_5 = \check{A} + \check{B} + O(\epsilon^{k/2})$, but $C_1 - C_5 = O(\epsilon)$.

Table 4.1: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 100\pi^2$.

m	$2L_m/m$	R_m	$100a_1$	$100a_2$
1	1.411	2399.	$.1762 - .1372i$	$.2296 - .1146i$
11	1.022	2018	$.2270 - .1446i$	$.2342 - .1405i$
101	.9835	2013	$.2307 - .1431i$	$.2315 - .1426i$
1001	.9791	2013	$.2310 - .1428$	$.2311 - .1428i$
10001	.9787	2013	$.2311 - .1428i$	$.2311 - .1428i$
∞	.9787	2013	$.2311 - .1428i$	$.2311 - .1428i$

Table 4.2: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 100\pi^2$ (continued).

m	$L_m^2 b_{1R}$	$L_m^2 b_{2R}$	$L_m b_{1I}$	$L_m b_{2I}$
1	-5.305	8.642	17.18	24.04
11	-6.621	7.037	19.85	20.83
101	-6.803	6.850	20.28	20.39
1001	-6.824	6.829	20.33	20.34
10001	-6.827	6.827	20.33	20.33
∞	-6.827	6.827	20.33	20.33

Table 4.3: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 100\pi^2$ (continued)

m	C_1	C_2	C_4	C_5
1	$-.0460 + .1125i$	$-.0743 - .0352i$	$-.0686 - .0026i$	$-.0595 + .0931i$
11	$-.0523 + .1045i$	$-.1360 + .0849i$	$-.1360 + .0849i$	$-.0541 + .1020i$
101	$-.0531 + .1034i$	$-.1000 + .1300i$	$-.0859 + .1363i$	$-.0533 + .1031i$
1001	$-.0532 + .1033i$	$-.0936 + .1333i$	$-.0921 + .1339i$	$-.0532 + .1032i$
10001	$-.0532 + .1033i$	$-.0928 + .1336i$	$-.0927 + .1336i$	$-.0532 + .1033i$
100001	$-.0532 + .1033i$	$-.0929 + .1336i$	$-.0928 + .1336i$	$-.0532 + .1033i$
∞	$-.0532 + .1033i$	$-.0928 + .1336i$	$-.0928 + .1336i$	$-.0532 + .1033i$

Table 4.4: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .1, \hat{Q} = 100\pi^2$, $k = 1$.

m	$2L_m/m$	R_m	$100a_1$	$100a_2$
1	1.411	2399.	$.1762 - .1372i$	$.2296 - .1146i$
11	1.011	1832	$.2362 - .1379i$	$.2442 - .1315i$
101	.9692	1775	$.2432 - .1333i$	$.2441 - .1325i$
1001	.9638	1757	$.24470 - .1322i$	$.2447 - .1321i$
10001	.9630	1752	$.2450 - .1320i$	$.2450 - .1320i$
100001	.9629	1750	$.2451 - .1319i$	$.2451 - .1319i$
1000001	.9628	1749	$.2452 - .1319i$	$.2452 - .1319i$
∞	.9628	1749	$.2452 - .1319i$	$.2452 - .1319i$

Table 4.5: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .1, \hat{Q} = 100\pi^2$, $k = 1$ (continued).

m	$L_m^2 b_{1R}$	$L_m^2 b_{2R}$	$L_m b_{1I}$	$L_m b_{2I}$
1	-5.305	8.642	17.18	24.04
11	-6.181	6.583	17.96	18.947
101	-6.229	6.274	17.92	18.03
1001	-6.206	6.210	17.82	17.83
10001	-6.193	6.195	17.78	17.78
100001	-6.190	6.190	17.77	17.77
1000001	-6.188	6.188	17.76	17.76
∞	-6.188	6.188	17.76	17.76

Table 4.6: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .1$, $\hat{Q} = 100\pi^2$, $k = 1$ (continued).

m	C_1	C_2	C_4	C_5
1	$-.0460 + .1125i$	$-.0743 - .0352i$	$-.0686 - .0026i$	$-.0595 + .0931i$
11	$-.04321 + .0968i$	$-.1061 + .0066i$	$.0119 + .0512i$	$-.0444 + .0946i$
101	$-.0412 + .0938i$	$-.1205 + .0747i$	$-.0109 + .1131i$	$-.0413 + .0935i$
1001	$-.0404 + .0930i$	$-.0947 + .1116i$	$-.0517 + .1264i$	$-.0404 + .0930i$
10001	$-.0402 + .0928i$	$-.0810 + .1196i$	$-.0668 + .1244i$	$-.0402 + .0928i$
100001	$-.0401 + .0927i$	$-.0762 + .1215i$	$-.0717 + .1230i$	$-.0401 + .0927i$
1000001	$-.0401 + .0927i$	$-.0746 + .1220i$	$-.0732 + .1225i$	$-.0401 + .0927i$
$10^7 + 1$	$-.0401 + .0927i$	$-.0785 + .1722i$	$-.0784 + .1723i$	$-.0401 + .0927i$
$10^8 + 1$	$-.0401 + .0927i$	$-.0739 + .1223i$	$-.0738 + .1223i$	$-.0401 + .0927i$
$10^9 + 1$	$-.0401 + .0927i$	$-.0739 + .1223i$	$-.0739 + .1223i$	$-.0401 + .0927i$
∞	$-.0401 + .0927i$	$-.0739 + .1223i$	$-.0739 + .1223i$	$-.0401 + .0927i$

Table 4.7: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .1$, $\hat{Q} = 100\pi^2$, $k = 1$ (continued).

m	C_3	C_6
1	$.1376 - .0938i$	$.0309 - .0866i$
11	$.0890 - .0031i$	$-.0405 - .0381i$
101	$.0539 + .0673i$	$-.0569 + .0305i$
1001	$.0112 + .0820i$	$-.0319 + .0674i$
10001	$-.0042 + .0803i$	$-.0184 + .0754i$
100001	$-.0092 + .0789i$	$-.0137 + .0774i$
1000001	$-.0107 + .0784i$	$-.0121 + .0780i$
$10^7 + 1$	$-.0107 + .0784i$	$-.0121 + .0780i$
$10^8 + 1$	$-.0113 + .0782i$	$-.0115 + .0782i$
$10^9 + 1$	$-.0114 + .0782i$	$-.0114 + .0782i$
∞	$-.0114 + .0782i$	$-.0114 + .0782i$

Chapter 5

Existence of invariant tori

In Chapter 3 we have shown that the dynamics of the magnetoconvection equation in a neighborhood of the origin in the Hilbert space X, for L sufficiently close to L_m and R sufficiently close to R_m (for all finite L_m, and in both Case I and II limits as $m \to \infty$), will be determined by the dynamics of the four-dimensional ordinary differential equation (3.74), which we rewrite here as

$$\begin{aligned} \dot{Z}_1 &= Z_1 \left[iP_1\omega_1 + a_1\mu + b_1\nu + C_1|Z_1|^2 + C_2|Z_2|^2 \right] + \text{ h.o.t. }, \\ \dot{Z}_2 &= Z_2 \left[iP_2\omega_2 + a_2\mu + b_2\nu + C_5|Z_2|^2 + C_4|Z_1|^2 \right] + \text{ h.o.t.}, \end{aligned} \tag{5.1}$$

where

$$\text{h.o.t.} = O(|\mu,\nu|^2|Z|, |\mu,\nu||Z|^3 + |Z|^5), \quad \mu = R - R_m, \quad \nu = L - L_m.$$

and $Z = (Z_1, Z_2) \in \mathbf{C}^2$. The normal form coefficients $C_j, j = 1, 2, 4, 5$, and $a_i, b_i, i = 1, 2$ were given and evaluated in Chapter 4. Recall that system (5.1) possesses the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry (3.59).

In this chapter, we determine the dynamics of (5.1), for Z, μ, ν near zero. First, in §5.1 we study the "truncated" normal form where we ignore the higher-order terms (h.o.t.). In this case, the equations decouple and the system is reduced to a planer one which is straightforward to analyze. We must then determine whether certain structures "persist" when the higher-order terms are restored, and in the rest of this chapter we prove results for (5.1) that are valid in the generic case, i.e. when the higher-order terms are not necessarily identically zero. In §5.2 we prove the existence of primary Hopf bifurcations of symmetric "standing wave" periodic orbits SW_0 and SW_π from the trivial solution, along two curves Γ_1, Γ_2 in the (μ, ν) parameter plane. In §5.3 we prove the occurrence of bifurcations of invariant tori from the periodic orbits SW_0 and SW_π, along additional curves Λ_1 and Λ_2 in the parameter plane, which implies the existence of these tori for parameter values in a region adjacent to the two curves Λ_1 and Λ_2. Finally, in §5.4 we prove the existence of normally hyperbolic invariant tori for parameter values in the interior of a wedge in the (μ, ν) plane bounded by Λ_1 and Λ_2, but away from the boundaries. Then we combine this result with the result of §5.3 to prove the existence of normally hyperbolic invariant tori for parameter values throughout the wedge bounded by Λ_1 and Λ_2.

From our analytic results in §4.1, we know that a_{1R}, a_{2R} and b_{2R} are positive, while b_{1R} is negative for all odd m. Also, based on our numerical results in §4.3, throughout this chapter we assume

Hypothesis 5.1 *$C_{1R}, C_{2R}, C_{4R}, C_{5R}$ are all negative.*

5.1 The truncated normal form

We write the normal form (5.1) in polar coordinates $Z_1 = r_1 e^{i\theta_1}, Z_2 = r_2 e^{i\theta_2}$ and obtain

$$
\begin{aligned}
\dot{r}_1 &= r_1 \left(a_{1R}\mu + b_{1R}\nu + C_{1R}r_1^2 + C_{2R}r_2^2\right) + O(r^5), \\
\dot{r}_2 &= r_2 \left(a_{2R}\mu + b_{2R}\nu + C_{4R}r_1^2 + C_{5R}r_2^2\right) + O(r^5), \\
\dot{\theta}_1 &= P_1\omega_1 + a_{1I}\mu + b_{1I}\nu + C_{1I}r_1^2 + C_{2I}r_2^2 + O(r^4), \\
\dot{\theta}_2 &= P_2\omega_2 + a_{2I}\mu + b_{2I}\nu + C_{4I}r_1^2 + C_{5I}r_2^2 + O(r^4).
\end{aligned}
\tag{5.2}
$$

We observe that the first two equations of (5.2) decouple from the last two, up to terms of lower order, but generically the higher-order terms of $O(r^5)$ and $O(r^4)$ depend nontrivially on θ_1 and θ_2. As an approximation, we ignore temporarily the higher-order terms and consider the truncated normal form

$$\begin{aligned}
\dot{r}_1 &= r_1\left(a_{1R}\mu + b_{1R}\nu + C_{1R}r_1^2 + C_{2R}r_2^2\right), \\
\dot{r}_2 &= r_2\left(a_{2R}\mu + b_{2R}\nu + C_{4R}r_1^2 + C_{5R}r_2^2\right), \\
\dot{\theta}_1 &= P_1\omega_1 + a_{1I}\mu + b_{1I}\nu + C_{1I}r_1^2 + C_{2I}r_2^2, \\
\dot{\theta}_2 &= P_2\omega_2 + a_{2I}\mu + b_{2I\nu} + C_{4I}r_1^2 + C_{5I}r_2^2.
\end{aligned} \tag{5.3}$$

Later, we will restore the higher order terms and prove results for the original system (5.1), based on the analysis of truncated system (5.3). Since the last two equations of (5.3) are decoupled from the first two, we need only consider the two dimensional system

$$\begin{aligned}
\dot{r}_1 &= r_1\left(a_{1R}\mu + b_{1R}\nu + C_{1R}r_1^2 + C_{2R}r_2^2\right), \\
\dot{r}_2 &= r_2\left(a_{2R}\mu + b_{2R}\nu + C_{4R}r_1^2 + C_{5R}r_2^2\right).
\end{aligned} \tag{5.4}$$

This system has been discussed by Guckenheimer and Holmes [16, §7.5]. To make use of their analysis, we simplify (5.4) by using the following scaling:

$$\bar{r}_1 = r_1\sqrt{|C_{1R}|}, \quad \bar{r}_2 = r_2\sqrt{|C_{5R}|}, \quad \bar{t} = sgn(C_{1R})t.$$

Then (5.4) becomes

$$\begin{aligned}
\frac{d\bar{r}_1}{d\bar{t}} &= \bar{r}_1\left[sgn(C_{1R})(a_{1R}\mu + b_{1R}\nu) + \bar{r}_1^2 + \frac{sgn(C_{1R})C_{2R}}{|C_{5R}|}\bar{r}_2^2\right], \\
\frac{d\bar{r}_2}{d\bar{t}} &= \bar{r}_2\left[sgn(C_{1R})(a_{2R}\mu + b_{2R}\nu) + \frac{C_{4R}}{C_{1R}}\bar{r}_1^2 + sgn(C_{5R})sgn(C_{1R})\bar{r}_2^2\right].
\end{aligned} \tag{5.5}$$

By Hypothesis 5.1, $sgn(C_{1R}) = sgn(C_{5R}) = -1$, and we are reduced to

$$\begin{aligned}
\dot{\bar{r}}_1 &= \bar{r}_1(-\mu_{1R} + \bar{r}_1^2 + B\bar{r}_2^2), \\
\dot{\bar{r}}_2 &= \bar{r}_2(-\mu_{2R} + C\bar{r}_1^2 + \bar{r}_2^2).
\end{aligned} \tag{5.6}$$

where

$$\mu_{iR} = a_{iR}\mu + b_{iR}\nu, \quad i = 1, 2, \quad B = C_{2R}/C_{5R}, \quad C = C_{4R}/C_{1R}.$$

According to the sign of B, C and $1-BC$, there are six different cases. The coefficients B and C are positive in our application, so our study of (5.6) falls into cases I_a (if $1 - BC > 0$) and I_b (if $1 - BC < 0$) of [16, p.399]. In our application for the parameter values that we have checked, only the case I_b is possible (see §4.3), but most of our analysis will be valid for both I_a and I_b cases.

For (5.6), the origin $(\bar{r}_1, \bar{r}_2) = (0, 0)$ is always an equilibrium point, and there are up to three other equilibrium points in the first quadrant:

$$\begin{aligned}
(\bar{r}_1, \bar{r}_2) &= (\sqrt{\mu_{1R}}, 0), \ \text{ if } \mu_{1R} > 0; \\
(\bar{r}_1, \bar{r}_2) &= (0, \sqrt{\mu_{2R}}), \ \text{ if } \mu_{2R} > 0; \\
(\bar{r}_1, \bar{r}_2) &= ({r^*}_1, {r^*}_2) = \left(\sqrt{\frac{\mu_{1R} - B\mu_{2R}}{1 - BC}}, \sqrt{\frac{\mu_{2R} - C\mu_{1R}}{1 - BC}}\right),
\end{aligned}$$

if

$$\frac{\mu_{1R} - B\mu_{2R}}{1 - BC}, \frac{\mu_{2R} - C\mu_{1R}}{1 - BC} > 0.$$

These fixed points correspond to the following equilibrium points of (5.4):

$$\begin{aligned}
SW_0^{(0)} : (r_1, r_2) &= \left(\sqrt{-\frac{a_{1R}\mu + b_{1R}\nu}{C_{1R}}}, 0\right), \ \text{ if } \mu > -\frac{b_{1R}}{a_{1R}}\nu; \\
SW_\pi^{(0)} : (r_1, r_2) &= \left(0, \sqrt{-\frac{a_{2R}\mu + b_{2R}\nu}{C_{5R}}}\right), \ \text{ if } \mu > -\frac{b_{2R}}{a_{2R}}\nu; \\
T^{(0)} : (r_1, r_2) &= (r_1^*(\mu, \nu), r_2^*(\mu, \nu)),
\end{aligned}$$

where

$$\begin{aligned}
r_1^*(\mu, \nu) &= \sqrt{\frac{\mu(a_{2R}C_{2R} - a_{1R}C_{5R}) + \nu(b_{2R}C_{2R} - b_{1R}C_{5R})}{C_{1R}C_{5R} - C_{2R}C_{4R}}}, \\
r_2^*(\mu, \nu) &= \sqrt{\frac{\mu(a_{1R}C_{4R} - a_{2R}C_{1R}) + \nu(b_{1R}C_{4R} - b_{2R}C_{1R})}{C_{1R}C_{5R} - C_{2R}C_{4R}}},
\end{aligned}$$

if (μ, ν) belongs to a wedge $W^{(0)}$ in the parameter plane defined by

$$\frac{\mu(a_{2R}C_{2R} - a_{1R}C_{5R}) + \nu(b_{2R}C_{2R} - b_{1R}C_{5R})}{C_{1R}C_{5R} - C_{2R}C_{4R}} > 0,$$
$$\frac{\mu(a_{1R}C_{4R} - a_{2R}C_{1R}) + \nu(b_{1R}C_{4R} - b_{2R}C_{1R})}{C_{1R}C_{5R} - C_{2R}C_{4R}} > 0.$$

Thus, the boundaries of $W^{(0)}$ lie along the lines

$$\Lambda_1^{(0)} : \mu = \lambda_0^* \nu, \quad sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R}),$$
$$\Lambda_2^{(0)} : \mu = \lambda_\pi^* \nu, \quad sgn(\nu) = sgn(\lambda_\pi^* + b_{2R}/a_{2R}),$$

where

$$\lambda_0^* = \frac{C_{4R}b_{1R} - C_{1R}b_{2R}}{C_{1R}a_{2R} - C_{4R}a_{1R}}, \tag{5.7}$$
$$\lambda_\pi^* = \frac{C_{2R}b_{2R} - C_{5R}b_{1R}}{C_{5R}a_{1R} - C_{2R}a_{2R}}. \tag{5.8}$$

See Figure 5.1.

The family of equilibrium points $SW_0^{(0)}$ of (5.4) correspond to a family of periodic orbits of (5.3) which bifurcate from the trivial solution at the origin as the parameters cross the line $a_{1R}\mu + b_{1R}\nu = 0$. A similar correspondence holds for the family $SW_\pi^{(0)}$ of (5.4). In §4.2 we will prove the existence of these two families of periodic orbits for the non-truncated system (5.1). The family of equilibrium points $T^{(0)}$ of (5.4) corresponds to a family of invariant tori for (5.3), for parameter values in the wedge $W^{(0)}$. In §5.3 and §5.4 we show that a family of invariant tori for the non-truncated system (5.1) exists for parameter values inside a wedge W in the (μ, ν) plane, that is approximated by $W^{(0)}$.

To study the stability of the equilibrium points $T^{(0)}$, we linearize the vector field of (5.3) about the fixed point $(\bar{r}_1^*, \bar{r}_2^*)$, obtaining the matrix

$$E(\mu_{1R}, \mu_{2R}) = \begin{bmatrix} 2(\bar{r}_1^*)^2 & 2B\bar{r}_1^*\bar{r}_2^* \\ 2C\bar{r}_1^*\bar{r}_2^* & 2(\bar{r}_2^*)^2 \end{bmatrix}. \tag{5.9}$$

Since $BC \neq 1$, E has eigenvalues

$$\lambda_{1,2} = \left(Tr(E) \pm \sqrt{Tr(E)^2 - 4Det(E)}\right)/2,$$

where

$$Tr(E) = 2((\bar{r}_1^*)^2) + (\bar{r}_2^*)^2), \quad Det(E) = 4(\bar{r}_1^*\bar{r}_2^*)^2(1 - BC).$$

Depending on the sign of $Det(E)$, $(\bar{r}_1^*, \bar{r}_2^*)$ is either a sink or saddle point for (5.6).

The corresponding linearized vector field of (5.4) about (r_1^*, r_2^*) is given by

$$\hat{E}(\mu, \nu) = \begin{bmatrix} 2C_{1R}(r_1^*)^2 & 2C_{2R}r_1^*r_2^* \\ 2C_{4R}r_1^*r_2^* & 2C_{5R}(r_2^*)^2 \end{bmatrix}, \tag{5.10}$$

which has eigenvalues

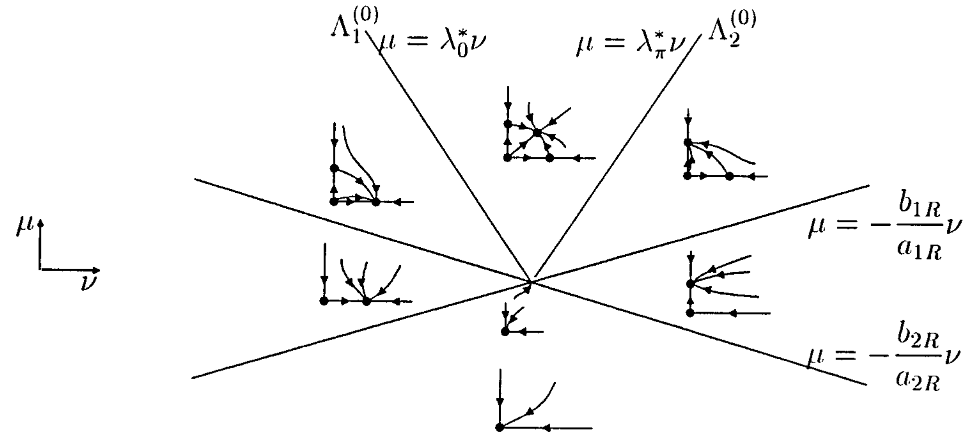
$$\hat{\lambda}_{1,2} = \left(-Tr(\hat{E}) \pm \sqrt{Tr(\hat{E})^2 - 4Det(\hat{E})}\right)/2.$$

Depending on the sign of $C_{1R}C_{5R} - C_{2R}C_{4R}$, (r_1^*, r_2^*) is either a sink or saddle for (5.4). These fixed points correspond to either normally hyperbolic attracting invariant tori, or normally hyperbolic invariant tori of saddle type for (5.3). Bifurcation sets for (5.4), corresponding to the two different cases depending on the sign of $C_{1R}C_{5R} - C_{2R}C_{4R}$, are given in Figure 5.1.

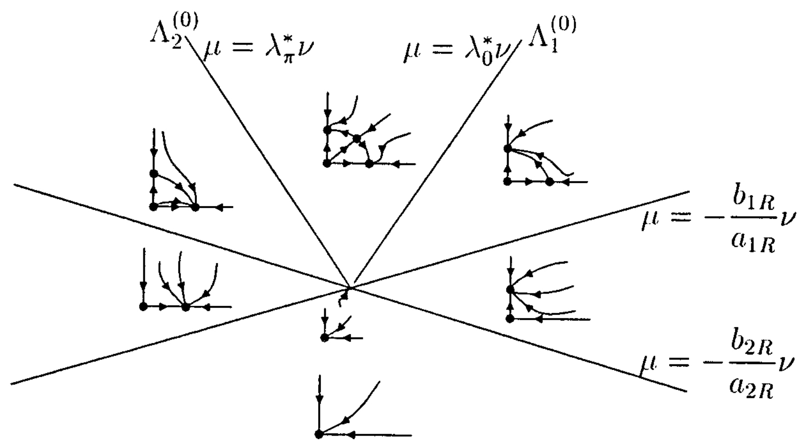
5.2 Bifurcating periodic orbits

In this section we return to consider the reduced four dimensional system of ordinary differential equations (5.1), and prove the existence of bifurcating periodic solutions. These solutions correspond to nonlinear standing waves in the magnetoconvection problem, so we will denote them by SW solutions. First, we note that the subspaces

$$V_0 = \{(Z_1, 0) : Z_1 \in \mathbf{C}\}, \quad V_\pi = \{(0, Z_2) : Z_2 \in \mathbf{C}\},$$



(a)



(b)

Figure 5.1: Bifurcation set for (5.4): (a) $C_{1R}C_{5R} - C_{2R}C_{4R} > 0$; (b) $C_{1R}C_{5R} - C_{2R}C_{4R} < 0$.

are invariant manifolds for (5.1) due to the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry (3.59). So we may study some of the dynamics of (5.1) by restricting it to these subspaces.

The system on V_0 becomes:

$$\dot{Z}_1 = Z_1(iP_1\omega_1 + \mu_1 + C_1|Z_1|^2) + O(|Z_1|^5 + |\mu,\nu||Z_1|^3 + |\mu,\nu|^2|Z_1|), \tag{5.11}$$

where $\mu_1 = a_1\mu + b_1\nu$. Now let $Z_1 = r_1 e^{i\theta_1}$, so in polar coordinates (5.11) becomes

$$\begin{aligned} \dot{r}_1 &= r_1(\mu_{1R} + C_{1R}r_1^2) + O(r_1^5 + |\mu,\nu|r_1^3 + |\mu,\nu|^2 r_1), \\ \dot{\theta}_1 &= P_1\omega_1 + \mu_{1I} + C_{1I}r_1^2 + O(r_1^4 + |\mu,\nu|r_1^2 + |\mu,\nu|^2). \end{aligned} \tag{5.12}$$

Since $C_{1R} < 0$, by [16, Theorem 3.4.2], (5.12) undergoes a supercritical Hopf bifurcation as we increase μ_{1R} through some value near zero, and there exists a family of periodic orbits $Z_0(t) = Z_0(\mu,\nu)(t)$, which we denote by the branch of SW_0 solutions. The SW_0 solutions bifurcate from the trivial solution for parameter (μ,ν) along the curve Γ_1 defined by $\mu = \mu_0(\nu)$, where

$$\mu_0(\nu) = -\nu b_{1R}/a_{1R} + O(|\nu|^2),$$

and the SW_0 solutions satisfy

$$|Z_1|^2 = -\frac{a_{1R}\mu + b_{1R}\nu}{C_{1R}} + O(|\mu,\nu|^2), \quad Z_2 \equiv 0. \tag{5.13}$$

By a similar argument, we restrict (5.1) to the the subspace V_π, and prove the Hopf bifurcation (supercritical, since $C_{5R} < 0$) of a family of periodic orbits $Z_\pi(t) = Z_\pi(\mu,\nu)(t)$ from the trivial solution for parameters along the curve Γ_2 defined by $\mu = \mu_\pi(\nu)$, where

$$\mu_\pi(\nu) = -\nu b_{2R}/a_{2R} + O(|\nu|^2).$$

We call these solutions, satisfying

$$Z_1 \equiv 0, \quad |Z_2|^2 = -\frac{a_{2R}\mu + b_{2R}\nu}{C_{5R}} + O(|\mu,\nu|^2), \tag{5.14}$$

the branch of SW_π solutions. The SW solutions Z_0, Z_π are periodic solutions of (5.1) with periods $\tau_0 = \tau_0(\mu,\nu), \tau_\pi = \tau_\pi(\mu,\nu)$ respectively, with τ_0 near $2\pi/P_1\omega_1$, and τ_π near $2\pi/P_2\omega_2$. The solutions $Z_0(t)$ and $Z_\pi(t)$ satisfy

$$\begin{aligned} JZ_0(t) &= Z_0(t) \ \ \forall\, t \in \mathbf{R}, \\ \beta Z_\pi(t) &= Z_\pi(t) \ \ \forall\, t \in \mathbf{R}. \end{aligned} \tag{5.15}$$

To consider the stabilities of the SW solutions, and secondary bifurcations, we transform coordinates near each periodic orbit. For $\nu \neq 0$, we rescale variables by putting

$$Z_1 = |\nu|^{1/2}u_1, \ \ Z_2 = |\nu|^{1/2}u_2, \ \ \mu = |\nu|\lambda,$$

where $u_1, u_2 \in \mathbf{C}, \ \lambda \in \mathbf{R}$. therefore we have

$$\begin{aligned} \dot{u}_1 &= |\nu|u_1\left[iP_1\omega_1 + a_1\lambda + b_1 sgn(\nu) + C_1|u_1|^2 + C_2|u_2|^2\right] + O(|\nu|^2) \\ \dot{u}_2 &= |\nu|u_2\left[iP_2\omega_2 + a_2\lambda + b_2 sgn(\nu) + C_5|u_2|^2 + C_4|u_1|^2\right] + O(|\nu|^2). \end{aligned} \tag{5.16}$$

In order to study stability and bifurcation of the SW_0 solutions ($u_1 \neq 0, u_2 \equiv 0$) we let $u_1 = re^{i\theta}$ and $u_2 = u$, and obtain

$$\begin{aligned} \dot{u} &= iP_2\omega_2 u + |\nu|u\left[a_2\lambda + b_2 sgn(\nu) + C_5|u|^2 + C_4 r^2\right] + O(|\nu|^2), \\ \dot{r} &= |\nu|r\left[a_{1R}\lambda + sgn(\nu)b_{1R} + C_{1R}r^2 + C_{2R}|u|^2\right] + O(|\nu|^2), \\ \dot{\theta} &= P_1\omega_1 + |\nu|\left[a_{1I}\lambda + sgn(\nu)b_{1I} + C_{1I}r^2 + C_{2I}|u|^2\right] + O(|\nu|^2). \end{aligned} \tag{5.17}$$

Since $\dot{\theta} \neq 0$ for all sufficiently small $|\nu|$, we can parameterize the flow away from $r = 0$ by θ, obtaining

$$\begin{aligned} du/d\theta &= i\Delta u + |\nu|\mathcal{U}(u,r,\lambda) + |\nu|^2\tilde{\mathcal{U}}(u,r,\theta,\lambda,\nu), \\ dr/d\theta &= |\nu|\mathcal{R}(u,r,\lambda) + |\nu|^2\tilde{\mathcal{R}}(u,r,\theta,\lambda,\nu), \end{aligned} \tag{5.18}$$

where $\Delta = P_2\omega_2/P_1\omega_1$, $\tilde{\mathcal{U}}(u,r,\theta,\lambda,\nu)$ and $\tilde{\mathcal{R}}(u,r,\theta,\lambda,\nu)$ are periodic in θ with period 2π, and

$$\begin{aligned}
\mathcal{U}(u,r,\lambda) &= (u/P_1\omega_1)\left[\lambda(a_2 - i\Delta a_{1I}) + sgn(\nu)(b_2 - i\Delta b_{1I})\right.\\
&\quad \left. +(C_4 - i\Delta C_{1I})r^2 + (C_5 - i\Delta C_{2I})|u|^2\right],\\
\mathcal{R}(u,r,\lambda) &= (r/P_1\omega_1)[a_{1R}\lambda + sgn(\nu)b_{1R} + C_{1R}r^2 + C_{2R}|u|^2].
\end{aligned}$$

The SW_0 solution (5.13) now corresponds to the branch of 2π-periodic solutions

$$u = u_{SW_0}(\theta,\lambda,\nu) \equiv 0, \quad r = r_{SW_0}(\theta,\lambda,\nu) = r_0(\lambda) + O(\nu),$$

of the rescaled, reparameterized nonautonomous system (5.18), where

$$r_0^2(\lambda) = -\frac{a_{1R}\lambda + sgn(\nu)b_{1R}}{C_{1R}}.$$

Now we define a moving coordinate system by

$$u = v, \quad r = r_{SW_0}(\theta,\lambda,\nu) + x,$$

where $v \in \mathbf{C}, x \in \mathbf{R}$. Then (5.18) becomes

$$\begin{aligned}
dv/d\theta &= i\Delta v + |\nu|\mathcal{V}(v,\bar{v},x,\lambda) + |\nu|^2\tilde{\mathcal{V}}(v,\bar{v},x,\theta,\lambda,\nu),\\
dx/d\theta &= |\nu|\mathcal{X}(v,\bar{v},x,\lambda) + |\nu|^2\tilde{\mathcal{X}}(v,\bar{v},x,\theta,\lambda,\nu),
\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}
\mathcal{V}(v,\bar{v},x,\lambda) &= (v/P_1\omega_1)\left\{\lambda(a_2 - \Delta a_{1I}) + sgn(\nu)(b_2 - i\Delta b_{1I})\right.\\
&\quad \left. + 2(C_4 - i\Delta C_{1I})r_0(\lambda)x + (C_4 - i\Delta C_{1I})r_0^2(\lambda) + (C_5 - i\Delta C_{2I})|v|^2\right\}\\
&\quad + O(|x|^2|v|),\\
\mathcal{X}(v,\bar{v},x,\lambda) &= (1/P_1\omega_1)[2C_{1R}r_0^2(\lambda)x + C_{2R}r_0(\lambda)|v|^2] + O(|x|^2 + |x||v|^2),
\end{aligned}$$

and $\tilde{\mathcal{V}}(v,\bar{v},x,\theta,\lambda,\nu), \tilde{\mathcal{X}}(v,\bar{v},x,\theta,\lambda,\nu)$ are both periodic in θ with period 2π. (Note that $\bar{v}$ satisfies the complex conjugate of the equation for v.) By Floquet theory, there is a 2π-periodic coordinate transformation, linear in the spatial variables, of the form

$$(v,\bar{v},x)^T = [I + \nu P(\theta,\lambda,\nu)](w,\bar{w},y)^T, \quad w \in \mathbf{C}, y \in \mathbf{R}, \tag{5.20}$$

which removes the θ-dependence of the linear terms in $w,\bar{w},y$ to all orders in $|\nu|$. Equations (5.19) are then replaced by

$$\begin{aligned} dw/d\theta &= i\Delta w + |\nu|\mathcal{W}(w,\bar{w},y,\lambda,\nu) + |\nu|^2\tilde{\mathcal{W}}(w,\bar{w},y,\theta,\lambda,\nu), \\ dy/d\theta &= |\nu|\mathcal{Y}(w,\bar{w},y,\lambda,\nu) + |\nu|^2\tilde{\mathcal{Y}}(w,\bar{w},y,\theta,\lambda,\nu), \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} \mathcal{W}(w,\bar{w},y,\lambda,\nu) &= \mathcal{V}(w,\bar{w},y,\lambda) + O(|\nu||w,y|), \\ \mathcal{Y}(w,\bar{w},y,\lambda,\nu) &= \mathcal{X}(w,\bar{w},y,\lambda) + O(|\nu||w,,y|), \\ \tilde{\mathcal{W}}(w,\bar{w},y,\theta,\delta,\nu) &= O(|y|^2 + |y||w| + |w|^2), \\ \tilde{\mathcal{Y}}(w,\bar{w},y,\theta,\delta,\nu) &= O(|y|^2 + |y||w| + |w|^2), \end{aligned}$$

and $\tilde{\mathcal{W}}, \tilde{\mathcal{Y}}$ are both periodic in θ with period 2π, and $\bar{w}$ satisfies the complex conjugate of the equation for w.

The trivial solution $w \equiv 0, y \equiv 0$ of (5.21) now corresponds to the SW_0 solutions of (5.1). The linearization about $w = 0, \bar{w} = 0, y = 0$, is given by the matrix

$$A = \begin{bmatrix} \tilde{\Omega}_0(\nu,\lambda) & O(|\nu|^2) & 0 \\ O(|\nu|^2) & \bar{\tilde{\Omega}}_0(\nu,\lambda) & 0 \\ 0 & 0 & \dfrac{2|\nu|C_{1R}r_0^2(\lambda)}{P_1\omega_1} + O(|\nu|^2) \end{bmatrix}, \tag{5.22}$$

where $\tilde{\Omega}_0(\nu,\lambda) = i\Delta + |\nu|\Omega_0(\lambda) + O(|\nu|^2)$, and

$$\Omega_0(\lambda) = (1/P_1\omega_1)\left[\lambda(a_2 - i\Delta a_{1I}) + sgn(\nu)(b_2 - i\Delta b_{1I}) + (C_4 - i\Delta C_{1I})r_0^2(\lambda)\right].$$

Due to the Floquet change of coordinates (5.20), the entries of the matrix A are independent of θ. Moreover, its zero entries are due to the reflection symmetry. The eigenvalues of A are Floquet exponents for the SW_0 solution. Since the entries of the Jacobian matrix A are analytic in ν and its eigenvalues are simple for small $|\nu|$, by [23] the Floquet exponents of SW_0 solutions depend analytically in ν, for small $|\nu|$ and have the form

$$2\pi\tilde{\kappa}_0(\lambda,\nu), \;\; 2\pi\bar{\tilde{\kappa}}_0(\lambda,\nu), \;\; 2\pi\tilde{\gamma}_0(\lambda,\nu),$$

where

$$\begin{aligned} \tilde{\kappa}_0(\lambda,\nu) &= \tilde{\Omega}_0(\nu,\lambda) + O(|\nu|^2), \\ \tilde{\gamma}_0(\lambda,\nu) &= \frac{2|\nu|C_{1R}r_0^2(\lambda)}{P_1\omega_1} + O(|\nu|^2), \\ \lambda &> -sgn(\nu)b_{1R}/a_{1R}. \end{aligned} \tag{5.23}$$

Similarly, Floquet exponents for the SW_π solutions have the form

$$2\pi\tilde{\kappa}_\pi(\lambda,\nu), \;\; 2\pi\bar{\tilde{\kappa}}_\pi(\lambda,\nu) \;\; 2\pi\tilde{\gamma}_\pi(\lambda,\nu),$$

where

$$\begin{aligned} \tilde{\kappa}_\pi(\lambda,\nu) &= i\Delta^{-1} + |\nu|\Omega_\pi(\lambda) + O(|\nu|^2), \\ \tilde{\gamma}_\pi(\lambda,\nu) &= \frac{2|\nu|C_{5R}r_\pi^2(\lambda)}{P_2\omega_2} + O(|\nu|^2), \\ \Omega_\pi(\lambda) &= (1/P_2\omega_2)\left[\lambda(a_1 - i\Delta a_{2I}) + sgn(\nu)(b_1 - i\Delta b_{2I}) + (C_2 - i\Delta C_{5I})r_\pi^2(\lambda)\right], \\ r_\pi^2(\lambda) &= -\frac{a_{2R}\lambda + sgn(\nu)b_{2R}}{C_{5R}}, \\ \lambda &> -sgn(\nu)b_{2R}/a_{2R}. \end{aligned} \tag{5.24}$$

If ν is fixed so that

$$\left(\frac{a_{2R}b_{1R} - a_{1R}b_{2R}}{a_{1R}a_{2R}}\right)\nu + O(\nu^2) > 0,$$

which corresponds to $\nu < 0$, then the SW_0 solutions bifurcate before the SW_π solutions as μ increases. Near the bifurcation of the SW_0 solutions, $r_0(\lambda)$ is small and therefore, $\Re(\tilde{\kappa}_0(\lambda,\nu) < 0$ for small $|\nu|$, and all three nontrivial Floquet exponents of the SW_0 solutions have negative real parts, implying that the solutions are stable. Near the bifurcation of the SW_π solutions where $r_\pi(\lambda)$ is small, we have $\Re(\tilde{\kappa}_\pi(\nu) > 0$ for small ν, implying that the SW_π solutions are unstable. On the other hand, if $\nu > 0$, the roles of the SW_π and SW_0 solutions are interchanged. We summarize the results of this section in the following lemma:

Lemma 5.1 *Assume a_{iR}, $i = 1, 2$, and C_{1R}, C_{5R} in (5.1) are not equal to zero. Then there are primary Hopf bifurcations of symmetric periodic solutions of (5.1) from the trivial solution, for parameter values (μ,ν) belonging to the curves*

$$\begin{aligned} \Gamma_1: \ \mu &= \mu_0(\nu) = -\nu b_{1R}/a_{1R} + O(|\nu|^2), \\ \Gamma_2: \ \mu &= \mu_\pi(\nu) = -\nu b_{2R}/a_{2R} + O(|\nu|^2), \end{aligned} \tag{5.25}$$

near the origin in $\mathbf{R}^2$. We denote the periodic solutions in the two branches by SW_0 and SW_π respectively. If $a_{iR} > 0$, $i = 1, 2$, $C_{1R} < 0$ and $C_{5R} < 0$ (in our application these conditions are satisfied) then for fixed sufficiently small ν, both solutions bifurcate supercritically as μ increases and the solutions that bifurcate at the lower value of μ are stable, while the other solutions are unstable.

5.3 Bifurcation of invariant tori

In this section, we show that for parameter values (μ,ν) belonging to one of two curves Λ_1 or Λ_2, one of the SW solutions for (5.1) has pure imaginary Floquet exponents. We then prove that parameter values along these curves correspond to secondary torus bifurcations from one of the SW solutions, which implies the existence of invariant tori for (5.10), for (μ,ν) near Λ_1 or Λ_2.

We first consider secondary bifurcations from the SW_0 solutions. We write

$$\tilde{\kappa}_0(\lambda, \nu) = i\Delta + |\nu|\hat{\kappa}_0(\lambda, \nu),$$

and notice that $\Re[\hat{\kappa}_0(sgn(\nu)\lambda_0^*, 0)] = 0$, where λ_0^* is given by (5.7). The implicit function theorem now implies the existence of a unique smooth curve

$$\hat{\Lambda}_1 : \quad \lambda = \tilde{\lambda}_0^*(\nu), \quad sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R}), \tag{5.26}$$

such that $\Re[\hat{\kappa}_0(\tilde{\lambda}_0^*(\nu), \nu)] \equiv 0$, with $\tilde{\lambda}_0^*(\nu) = sgn(\nu)\lambda_0^* + O(|\nu|)$, ν sufficiently close to 0 . Thus for parameters λ, ν along $\hat{\Lambda}_1$, the SW_0 solutions have conjugate pairs of pure imaginary Floquet exponents. In terms of original parameters of (5.1), $\hat{\Lambda}_1$ corresponds to the smooth curve

$$\Lambda_1 : \quad \mu = \tilde{\mu}_0^*(\nu) = |\nu|\tilde{\lambda}_0^*(\nu) = \nu\lambda_0^* + O(|\nu|^2), \quad sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R}), \tag{5.27}$$

such that $\Re[\tilde{\kappa}_0(\mu_0^*(\nu), \nu)] \equiv 0$

Similarly, for the SW_π solutions we write

$$\tilde{\kappa}_\pi(\lambda, \nu) = i\Delta^{-1} + |\nu|\hat{\kappa}_\pi(\lambda, \nu),$$

and note that $\Re[\hat{\kappa}_\pi(sgn(\nu)\lambda_\pi^*, 0)] = 0$, where λ_π^* is given by (5.8), and the implicit function theorem implies the existence of a unique smooth curve

$$\hat{\Lambda}_2 : \quad \lambda = \tilde{\lambda}_\pi^*(\nu), \quad sgn(\nu) = sgn(\lambda_\pi^* + b_{2R}/a_{2R}), \tag{5.28}$$

such that $\Re[\hat{\kappa}_\pi(\tilde{\lambda}_\pi^*(\nu), \nu)] \equiv 0$, with $\tilde{\lambda}_\pi^*(\nu) = sgn(\nu)\lambda_\pi^* + O(|\nu|)$. In terms of the parameters of (5.1), $\hat{\Lambda}_2$ corresponds to the smooth curve

$$\Lambda_2 : \quad \mu = \tilde{\mu}_\pi^*(\nu) = |\nu|\tilde{\lambda}_\pi^*(\nu) = \nu\lambda_\pi^* + O(|\nu|^2), \quad sgn(\nu) = sgn(\lambda_\pi^* + b_{2R}/a_{2R}), \tag{5.29}$$

such that $\Re[\tilde{\kappa}_\pi(\mu_\pi^*(\nu), \nu)] \equiv 0$. We summarize the above arguments in the following lemma:

Lemma 5.2 *There exist smooth curves Λ_1, Λ_2 in the (μ, ν) parameter plane, along which the SW_0 and SW_π solutions of (5.1) have conjugate pairs of pure imaginary Floquet exponents.*

To consider the behavior of the system near the SW_0 solutions for parameter values near Λ_1 we let

$$\lambda = \tilde{\lambda}_0^*(\nu) + \delta, \quad sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R})$$

in (5.21) to obtain

$$\begin{aligned} \frac{dw}{d\theta} &= \kappa(\delta,\nu)w + |\nu|\mathcal{W}(w,\bar{w},y,\delta) + |\nu|^2\tilde{\mathcal{W}}(w,\bar{w},y,\theta,\delta,\nu), \\ \frac{dy}{d\theta} &= |\nu|\gamma(\delta,\nu)y + |\nu|\mathcal{Y}(w,\bar{w},y,\delta) + |\nu|^2\tilde{\mathcal{Y}}(w,\bar{w},y,\theta,\delta,\nu), \end{aligned} \tag{5.30}$$

where

$$\kappa(\delta,\nu) = |\nu|\alpha(\delta,\nu) + i[\Delta + |\nu|\beta(\delta,\nu)], \tag{5.31}$$

$$\alpha(0,\nu) \equiv 0, \tag{5.32}$$

$$\partial\alpha/\partial\delta(0,0) = \frac{C_{1R}a_{2R} - C_{4R}a_{1R}}{C_{1R}P_1\omega_1}, \tag{5.33}$$

$$\beta(0,0) = (1/P_1\omega_1)\left[\Omega_0 + \lambda_0^*(a_{2I} - \Delta a_{1I})\right], \tag{5.34}$$

$$\gamma(0,0) = \frac{2C_{1R}(r_0^*)^2}{P_1\omega_1}, \tag{5.35}$$

$$r_0^* = r_0(\lambda_0^*), \tag{5.36}$$

$$\Omega_0 = -(b_{2I} - \Delta b_{1I}) + (C_{4I} - \Delta C_{1I})(r_0^*)^2, \tag{5.37}$$

$$\mathcal{W}(w,\bar{w},y,\delta) = \left(\frac{2r_0^* y w}{P_1\omega_1}\right)[C_4 - i\Delta C_{1I}] + \frac{w|w|^2}{P_1\omega_1}[C_5 - i\Delta C_{2I}], \tag{5.38}$$

$$+ O(|y|^2 w + |\delta||w|^3 + |\delta||y||w|), \tag{5.39}$$

$$\mathcal{Y}(w,\bar{w},y,\delta) = \frac{C_{2R}r_0^*|w|^2}{P_1\omega_1} + O(|y|^2 + |y||w| + |\delta||w|^2), \tag{5.40}$$

$$\tilde{\mathcal{W}}(w,\bar{w},y,\theta,\delta,\nu) = O(|y|^2 + |y||w| + |w|^2), \tag{5.41}$$

$$\tilde{\mathcal{Y}}(w,\bar{w},y,\theta,\delta,\nu) = O(|y|^2 + |y||w| + |w|^2), \tag{5.42}$$

and $\tilde{\mathcal{W}}, \tilde{\mathcal{Y}}$ are both periodic in θ with period 2π.

Since $\alpha(0,\nu) = 0$ and $\gamma(0,\nu) < 0$ for $|\nu|$ sufficiently small, we can we apply an invariant manifold theorem for periodically forced systems [18, Theorem VII.2.1] to obtain an attracting center manifold, which can be represented as a smooth function

$$y = h(w, \bar{w}, \theta, \delta, \nu),$$

defined in a neighborhood of $w = 0, \delta = 0$, with

$$h(0,0,\theta,0,\nu) = 0, \ \ \frac{\partial h}{\partial w}(0,0,\theta,0,\nu) = \frac{\partial h}{\partial \bar{w}}(0,0,\theta,0,\nu) = \frac{\partial h}{\partial \delta}(0,0,\theta,0,\nu) = 0,$$

for all $\theta \in S^1$ and all sufficiently small $|\nu|$. To find h, we use the fact that the center manifold is invariant, which implies that $h = h(w, \bar{w}, \theta, \delta, \nu)$ satisfies

$$\begin{aligned}
& \frac{\partial h}{\partial w}\left\{|\nu|\kappa(\delta,\nu)w + |\nu|\mathcal{W}(w,\bar{w},h,\delta) + |\nu|^2\tilde{\mathcal{W}}(w,\bar{w},h,\theta,\delta,\nu)\right\} \\
+ \ & \frac{\partial h}{\partial \bar{w}}\left\{|\nu|\bar{\kappa}(\delta,\nu)\bar{w} + |\nu|\overline{\mathcal{W}}(w,\bar{w},h,\delta) + |\nu|^2\overline{\tilde{\mathcal{W}}}(w,\bar{w},h,\theta,\delta,\nu)\right\} \\
= \ & |\nu|\gamma(\delta,\nu)h + |\nu|\mathcal{Y}(w,\bar{w},h,\delta) + |\nu|^2\tilde{\mathcal{Y}}(w,\bar{w},h,\theta,\delta,\nu),
\end{aligned} \tag{5.43}$$

Substituting the Taylor series expansion

$$y = h(w,\bar{w},\theta,\delta,\nu) = a_{11}(\theta,\nu)w^2 + a_{12}(\theta,\nu)|w|^2 + a_{13}(\theta,\nu)\bar{w}^2 + O(|w|^3 + |\delta||w|)$$

into (5.43), we identify the coefficients of $w^2, \bar{w}^2$ and $|w|^2$ in both sides of equation (5.43) and obtain

$$a_{11}(\theta,\nu) = O(|\nu|), \ \ a_{13} = O(|\nu|), \ \ a_{12}(\theta,\nu) = -C_{2R}/(2C_{1R}r_0^*) + O(|\nu|), \tag{5.44}$$

We then obtain the following periodically forced equation in the complex plane which represents the flow restricted to the (attracting) invariant center manifold:

$$\frac{dw}{d\theta} = \kappa(\delta,\nu)w + |\nu|\mathcal{W}_1(w,\bar{w},\delta) + |\nu|^2\tilde{\mathcal{W}}_1(w,\bar{w},\theta,\delta,\nu), \tag{5.45}$$

where

$$\begin{aligned}
\mathcal{W}_1(w,\bar{w},\delta) &= \frac{w|w|^2}{P_1\omega_1}\left[[C_5 - i\Delta C_2 I] - \frac{C_{2R}}{C_{1R}}[C_4 - i\Delta C_{1I}]\right] + O(|w|^5 + |\delta||w|^3) \\
\tilde{\mathcal{W}}_1(w,\bar{w},\theta,\delta,\nu) &= O(|w|^2),
\end{aligned}$$

and $\tilde{\mathcal{W}}_1$ is periodic in θ with period 2π. We write (5.45) in polar coordinates $w = \rho e^{i\psi}$, and then apply the theory of normal forms for periodically forced systems [1, section 26]: if $q|\nu|\beta(0,\nu)$ is not an integer for $q = 1, 2, 3, 4$ or 5 (this is satisfied for all sufficiently small $|\nu|$) then we may change coordinates to put (5.45) into the normal form

$$\begin{aligned}
\frac{d\rho}{d\theta} &= |\nu|\alpha_1(\nu)\delta\rho + |\nu|\mathcal{A}_1(\nu)\rho^3 + |\nu|\tilde{\mathcal{A}}(\rho,\psi,\theta,\delta,\nu), \qquad (5.46)\\
\frac{d\psi}{d\theta} &= \Delta + |\nu|\beta_0(\nu) + |\nu|\beta_1(\nu)\delta + |\nu|\mathcal{B}_1(\nu)\rho^2 + |\nu|\tilde{\mathcal{B}}(\rho,\psi,\theta,\delta,\nu),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(\nu) &= \frac{\partial\alpha}{\partial\delta}(0,\nu) = \frac{C_{1R}a_{2R} - C_{4R}a_{1R}}{C_{1R}P_1\omega_1}, \\
\beta_0(\nu) &= \beta(0,\nu), \quad \beta_1(\nu) = \frac{\partial\beta}{\partial\delta}(0,\nu), \\
\mathcal{A}_1(\nu) &= \frac{C_{5R}C_{1R} - C_{2R}C_{4R}}{P_1\omega_1 C_{1R}} + O(|\nu|), \\
\mathcal{B}_1(\nu) &= \frac{C_{1R}C_{5I} - C_{2R}C_{4I}}{C_{1R}} - \frac{P_2\omega_2(C_{1R}C_{2I} - C_{2R}C_{1I})}{P_1\omega_1 C_{1R}} + O(|\nu|), \\
\tilde{\mathcal{A}}(\rho,\psi,\delta,\nu) &= O(\rho^5 + |\delta|\rho^3 + |\delta|^2\rho), \\
\tilde{\mathcal{B}}(\rho,\psi,\delta,\nu) &= O(\rho^4 + |\delta|\rho^2 + |\delta|^2),
\end{aligned}$$

and $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$ are periodic in both θ and ψ, with periods 2π. In our application $\alpha_1(\nu) < 0$ and $\mathcal{A}_1(\nu) > 0$ for all sufficiently small ν. We now state and prove a result on the existence of bifurcating tori for parameter values near the curve Λ_1.

Lemma 5.3 *Suppose $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$ in (5.46) are C^1 functions with $\mathcal{A}_1(\nu) \neq 0$ and $\alpha_1(\nu) \neq 0$. Then there exist $\nu_0 > 0$, $\delta_0 > 0$ such that for all*

$$0 < |\nu| \leq \nu_0, \quad sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R})$$

and all

$$0 < |\delta| \leq \delta_0, \quad sgn(\delta) = -sgn(\alpha_1(0)/\mathcal{A}_1(0)),$$

the system (5.46) possesses a C^1 invariant torus $\rho = \rho^*(\psi, \theta, \delta, \nu)$. Moreover, the torus is attracting for (5.46) if $\alpha_1(0)\mathcal{A}_1(0) > 0$ and is of saddle type if $\alpha_1(0)\mathcal{A}_1(0) < 0$.

Proof: The proof of existence of ρ^* follows from [5, Lemma 12.6.1], all the conditions of the lemma being obviously satisfied for sufficiently small $|\nu|$ and $|\delta|$. Now we follow [5, Lemma 12.5.2], adopting its notation, to find an estimate on δ_0. Since (5.46) is not in a form that we can directly apply this lemma, we first use the standard rescaling

$$\rho = \left(-\frac{\alpha_1(\nu)\delta}{\mathcal{A}_1(\nu)}\right)^{1/2} \left(1 + |\delta|^{1/2}\hat{\rho}\right). \tag{5.47}$$

Then (5.46) becomes

$$\begin{aligned} \dot{\hat{\rho}} &= -2\alpha_1(\nu)|\nu|\delta\hat{\rho} + \delta^{3/2}|\nu|\mathcal{R}(\psi, \theta, \hat{\rho}, \nu, \delta), \\ \dot{\psi} &= \Delta + |\nu|\beta_0(\nu) + \delta|\nu|\Psi(\psi, \theta, \hat{\rho}, \nu, \delta), \\ \dot{\theta} &= 1, \end{aligned} \tag{5.48}$$

where $\mathcal{R}, \Psi$ are periodic in ψ and θ with period 2π and are bounded, together with their derivatives, as $|\delta| \to 0$ and $|\nu| \to 0$. We note that Ψ has the form

$$\Psi(\psi, \theta, \hat{\rho}, \nu, \delta) = \left[\beta_1(\nu) - \frac{\mathcal{B}_1(\nu)\alpha_1(\nu)}{\mathcal{A}_1(\nu)}\right] + O(|\delta|^{1/2}).$$

Now we let

$$\hat{\theta} = (\psi, \theta)^T, \quad \hat{\lambda} = (\nu, \delta)^T,$$

and write (5.48) in the form

$$\begin{aligned} \dot{\hat{\rho}} &= A(\hat{\lambda})\hat{\rho} + R(\hat{\theta}, \hat{\rho}, \hat{\lambda}), \\ \dot{\hat{\theta}} &= \omega(\hat{\lambda}) + \Theta(\hat{\theta}, \hat{\rho}, \hat{\lambda}), \end{aligned} \tag{5.49}$$

where $A(\hat{\lambda}) = 2\nu\alpha_1(\nu)\delta$ and

$$\omega(\hat{\lambda}) = \begin{pmatrix} \Delta + |\nu|\beta_0(\nu) \\ 1 \end{pmatrix}, \qquad \Theta(\psi, \theta, \hat{\rho}, \nu, \delta) = \begin{pmatrix} \delta|\nu|\Psi(\psi, \theta, \hat{\rho}, \nu, \delta) \\ 0 \end{pmatrix}.$$

Although [5, Lemma 12.5.2] does not actually apply to (5.49), mainly due to the dependence of $A(\hat{\lambda})$ on $\hat{\lambda}$, we can use the same proof with minor changes. We are interested in integral manifolds in the form $\hat{\rho} = f(\hat{\theta}, \hat{\lambda})$, with f being periodic in each component of $\hat{\theta}$. Following [5], for each integer $p \geq 1$ we put

$$F_p = \{f \in C^p(\mathbf{R}^2, \mathbf{R}) : f(\hat{\theta}) \text{ is } 2\pi\text{-periodic in each components of } \hat{\theta}\},$$

With the C^p-topology, F_p is a Banach space. For any $f \in F_p$, we now consider

$$\begin{aligned} \dot{\hat{\rho}} &= A(\hat{\lambda})\hat{\rho} + Q(\hat{\theta}, \hat{\lambda}, f), \\ \dot{\hat{\theta}} &= \omega(\hat{\lambda}) + P(\hat{\theta}, \hat{\lambda}, f). \end{aligned} \tag{5.50}$$

where

$$Q(\hat{\theta}, \hat{\lambda}, f) = R(\hat{\theta}, f(\hat{\theta}), \hat{\lambda}), \text{ and } P(\hat{\theta}, \hat{\lambda}, f) = \Theta(\hat{\theta}, f(\hat{\theta}), \hat{\lambda}).$$

For a fixed $\hat{\phi} \in \mathbf{R}^2$, let $\hat{\theta}^*(t, \hat{\phi}, \hat{\lambda}, f)$ denote the solution of

$$\dot{\hat{\theta}} = \omega(\hat{\lambda}) + P(\hat{\theta}, \hat{\lambda}, f)$$

with initial condition $\hat{\theta}^*(0, \hat{\phi}, \hat{\lambda}, f) = \hat{\phi}$. Now assuming that $A(\hat{\lambda}) > 0$, we define

$$\begin{aligned} K(u, \hat{\lambda}) &= 0 \qquad \text{for } u < 0, \\ &= -e^{-A(\hat{\lambda})u} \quad \text{for } u \geq 0, \end{aligned}$$

and form the mapping

$$T(\hat{\lambda}, f)(\hat{\phi}) = -\int_{-\infty}^{\infty} K(u, \hat{\lambda}) Q(\hat{\theta}^*(u, \hat{\phi}, \hat{\lambda}, f), \hat{\lambda}, f) du. \tag{5.51}$$

We note that

$$|K(u,\hat{\lambda})| \leq e^{-\alpha(\hat{\lambda})|u|} \quad \forall\, u \in \mathbf{R},$$

where

$$\alpha(\nu) = |2\alpha_1(\nu)\nu\delta| = O(|\nu||\delta|),$$

and

$$\gamma(\hat{\lambda}, f) = \sup_{\hat{\theta}} \left|\frac{\partial P}{\partial \hat{\theta}}(\hat{\theta}, \hat{\lambda}, f)\right| = O(|\nu||\delta|^{3/2}) \cdot (|\delta|^{1/2} + |f|_1).$$

Thus, if we choose a bounded neighborhood V of 0 in F_1 and choose $|\nu|, |\delta|$ sufficiently small so that $\gamma(\hat{\lambda}, f) < \alpha(\hat{\lambda})$, then by proof of of [5, Lemma 12.5.2] we have

$$T(\hat{\lambda}, \cdot) : V \to F_1.$$

In particular we may take

$$V = \{f \in F_1 : |f|_1 \leq 1\}.$$

Also by of [5, Lemma 12.5.2], (5.48) has an integral manifold of the form $\hat{\rho} = \hat{f}(\hat{\theta}, \hat{\lambda})$ if and only if T has a fixed point in F_1. We prove the existence of such a fixed point, by proving that T is a contraction mapping. By the mean value theorem we have

$$|T(\hat{\lambda}, \tilde{f})(\hat{\phi}) - T(\hat{\lambda}, \bar{f})(\hat{\phi})| \leq \sup |D_f T(\hat{\lambda}, f)(\hat{\phi})||\tilde{f} - \bar{f}|_1,$$

where the supremum is taken over all $\hat{\phi} \in \mathbf{R}^2,\;\; f \in F_1, |f|_1 \leq 1$. But

$$D_f T(\hat{\lambda}, f)(\hat{\phi}) = -\int_{\infty}^{\infty} K(u, \hat{\lambda}) \left[\frac{\partial Q(\hat{\theta}^*(u), \hat{\lambda}, f)}{\partial f} + \frac{\partial Q(\hat{\theta}^*(u), \hat{\lambda}, f)}{\partial \hat{\theta}} \cdot \frac{\partial \hat{\theta}^*(u)}{\partial f}\right] du$$

where $\hat{\theta}^*(u) = \hat{\theta}^*(u, \hat{\phi}, \hat{\lambda}, f)$. In our problem

$$\frac{\partial Q(\hat{\theta}^*(u), \hat{\lambda}, f)}{\partial f} = O(|\delta|^{3/2}|\nu|), \quad \frac{\partial Q(\hat{\theta}^*(u), \hat{\lambda}, f)}{\partial \hat{\theta}} = O(|\delta|^{3/2}|\nu|)$$

and

$$\left|\frac{\partial \theta^*(u)}{\partial f}\right| \leq C e^{\hat{\gamma}(\hat{\lambda}, f)u}, \text{ where } \hat{\gamma}(\hat{\lambda}, f) = O(|\nu||\delta|^{3/2}),$$

and therefore

$$|D_f(T(\hat{\lambda}, f)(\bar{\phi})| \le |\nu|\delta^{3/2}\hat{K}\int_0^\infty e^{-[\alpha(\hat{\lambda})-\hat{\gamma}(\hat{\lambda},f)]u}du,$$

for some real constant $\hat{K}$. Since $\alpha(\hat{\lambda}) = O(|\nu||\delta|)$, we have $|D_f(T(\hat{\lambda}, f)(\hat{\phi})| = O(|\delta|^{1/2})$ uniformly for all $\hat{\phi} \in \mathbf{R}^2$, $f \in F_1$ with $|f|_1 \le 1$. Similarly, $|D_{\hat{\phi}}(T(\hat{\lambda}, f)(\hat{\phi})| = O(|\delta|^{1/2})$, hence there is a positive constant C such that

$$|T(\hat{\lambda}, \tilde{f}) - T(\hat{\lambda}, \bar{f})| \le C\delta^{1/2}|\tilde{f} - \bar{f}|_1$$

for all sufficiently small $|\nu|$ and $|\delta|$, and for all $|\tilde{f}|_1, |\bar{f}|_1 \le 1$. Also we observe that

$$|T(\hat{\lambda}, f)(\hat{\phi})| = O(|\delta|^{1/2}), \quad |D_{\hat{\phi}}(T(\hat{\lambda}, f)(\hat{\phi})| = O(|\delta|^{1/2}),$$

and therefore

$$|T(\hat{\lambda}, f)|_1 \le C|\delta|^{1/2}$$

for all sufficiently small $|\nu|$ and $|\delta|$. Thus for all sufficiently small $|\nu|$ and $|\delta|$, $T(\hat{\lambda}, \cdot)$ is a contraction mapping on the closed unit ball in F_1. Q.E.D.

Remark 5.1 a. *With more lengthy estimates, one can prove that $T(\hat{\lambda}, \cdot)$ maps the unit ball in F_p into itself for $p \ge 2$ by showing that $|T(\hat{\lambda}, f)|_p = O(|\delta|^{1/2})$ and therefore showing that the bifurcating tori is C^p. However, the region of parameter values for which C^p tori exist may shrink as p increases (see [5, p.492]).*

b: *In the above proof, based on our numerical results in §4.3, we have assumed that $A(\hat{\lambda}) > 0$. However the proof easily can be modified if $A(\hat{\lambda}) < 0$. In that case we simply redefine $K(u, \hat{\lambda})$ in the obvious way, and the rest of the proof will be the same.*

We observe that ν_0, δ_0 in Lemma 5.3 can be chosen independently of each other. In terms of the system (5.1), this implies that the bifurcating invariant tori exist for parameters (μ, ν) in a thin wedge-shaped region of width $O(|\nu|)$ adjacent to the curve Λ_1. In a similar manner, we can show that invariant tori will bifurcate from the SW_π solutions along the curve Λ_2. See Figure 5.2.

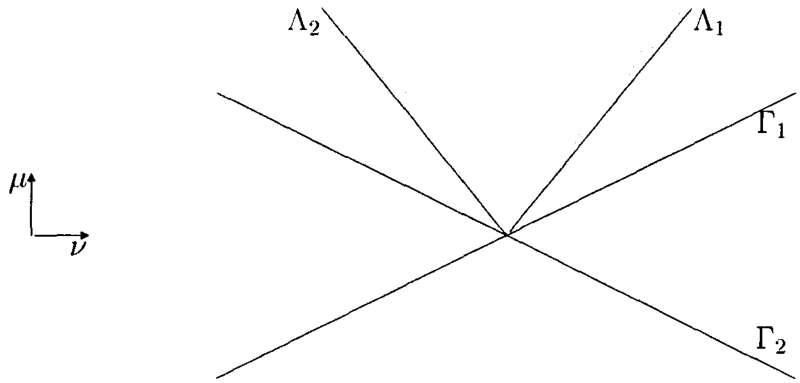


Figure 5.2: Parameter values (shaded regions) for which bifurcating invariant tori exist for (5.1), when $C_{1R}C_{5R} - C_{2R}C_{4R} < 0$.

5.4 Persistence of invariant tori

In this section we prove the existence of invariant tori, when the higher-order terms are restored in the full system (5.2) or (5.1), for parameters (μ, ν) in the wedge-shaped region bounded by the curves Λ_1 and Λ_2, but sufficiently far from the boundaries. Combining this with the results of the previous section, on the bifurcation of tori along the curves Λ_1 and Λ_2, we thus prove the existence of invariant tori for parameters (μ, ν) throughout the interior of the wedge bounded by the above curves.

It is convenient to use the parameters $\mu_{iR} = a_{iR}\mu + b_{iR}\nu$, $i = 1, 2$ introduced in (5.6). In terms of these parameters, the curves Λ_1, Λ_2 (see Figure 5.3) become

$$\begin{aligned}
\Lambda_1^* : \mu_{2R} &= \mu_0^*(\mu_{1R}) = C\mu_{1R} + O(\mu_{1R}|^2), \\
\Lambda_2^* : \mu_{2R} &= \mu_\pi^*(\mu_{1R}) = \mu_{1R}/B + O(|\mu_{1R}|^2),
\end{aligned}$$

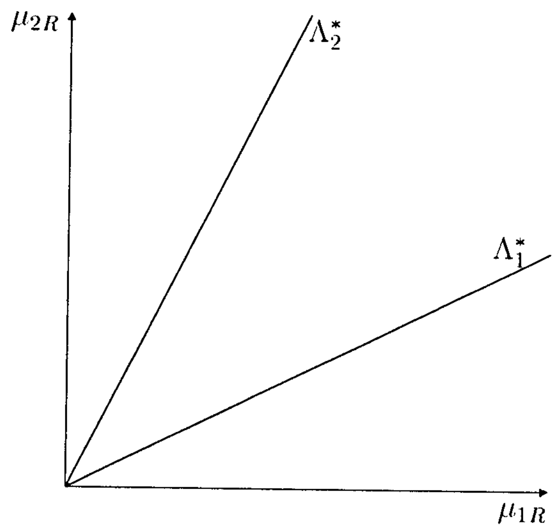


Figure 5.3: Curves in the (μ_{1R}, μ_{2R}) parameter plane corresponding to Λ_1 and Λ_2.

where $\mu_{1R} > 0$ and

$$C = \frac{C_{4R}}{C_{1R}}, \quad B = \frac{C_{2R}}{C_{5R}}.$$

Based on our numerical results of Chapter 4, we assume that $C_{1R}C_{5R} - C_{2R}C_{4R} < 0$, and thus $0 < 1/B < C$. We introduce the scaling

$$\mu_{1R} = \epsilon, \quad \mu_{2R} = \epsilon\lambda, \quad r_i = \epsilon^{1/2}\hat{r}_i, \ i = 1, 2.$$

for $\epsilon > 0$, then after dropping the hats, (5.2) becomes

$$\begin{aligned}
\dot{r}_1 &= \epsilon r_1 \left(1 + C_{1R}r_1^2 + C_{2R}r_2^2\right) + O(\epsilon^2), \qquad (5.52)\\
\dot{r}_2 &= \epsilon r_2 \left(\lambda + C_{4R}r_1^2 + C_{5R}r_2^2\right) + O(\epsilon^2),\\
\dot{\theta}_1 &= P_1\omega_1 + \epsilon\left[\frac{a_{1I}(b_{2R} - b_{1R}\lambda)}{a_{1R}b_{2R} - a_{2R}b1R} + \frac{b_{1I}(a_{2R} - a_{1R}\lambda)}{a_{2R}b_{1R} - a_{1R}b_{2R}} + C_{1I}r_1^2 + C_{2I}r_2^2\right] + O(\epsilon^2),\\
\dot{\theta}_2 &= P_2\omega_2 + \epsilon\left[\frac{a_{2I}(b_{2R} - b_{1R}\lambda)}{a_{1R}b_{2R} - a_{2R}b1R} + \frac{b_{2I}(a_{2R} - a_{1R}\lambda)}{a_{2R}b_{1R} - a_{1R}b_{2R}} + C_{4I}r_1^2 + C_{5I}r_2^2\right] + O(\epsilon^2).
\end{aligned}$$

Now we use the coordinate transformation defined by

$$r_1 = r_1^*(\lambda) + \epsilon^{1/2}\rho_1, \quad r_2 = r_2^*(\lambda) + \epsilon^{1/2}\rho_2,$$

where

$$\begin{aligned} r_1^*(\lambda) &= \sqrt{\frac{1 - B\lambda}{C_{1R}(BC-1)}} = \sqrt{\frac{C_{5R} - \lambda C_{2R}}{C_{2R}C_{4R} - C_{1R}C_{5R}}}, \\ r_2^*(\lambda) &= \sqrt{\frac{\lambda - C}{C_{5R}(BC-1)}} = \sqrt{\frac{\lambda C_{1R} - C_{4R}}{C_{2R}C_{4R} - C_{1R}C_{5R}}}. \end{aligned}$$

and λ satisfies

$$C_{5R} - \lambda C_{2R}, \quad \lambda C_{1R} - C_{4R} > 0.$$

Note that this is equivalent to $1/B < \lambda < C$. We then find that $\rho = (\rho_1, \rho_2)$, $\hat{\theta} = (\theta_1, \theta_2)$ satisfy an equation of the form

$$\begin{aligned} \dot{\rho} &= \epsilon E(\lambda)\rho + \epsilon^{3/2}\mathcal{F}(\hat{\theta}, \rho, \lambda, \epsilon), \\ \dot{\hat{\theta}} &= \omega(\lambda, \epsilon) + \epsilon^{3/2}\Theta_1(\rho, \lambda, \epsilon) + \epsilon^2\Theta_2(\hat{\theta}, \rho, \lambda, \epsilon), \end{aligned} \tag{5.53}$$

where

$$E(\lambda) = \begin{bmatrix} 2C_{1R}(r_1^*(\lambda))^2 & 2C_{2R}r_1^*(\lambda)r_2^*(\lambda) \\ 2C_{4R}r_1^*(\lambda)r_2^*(\lambda) & 2C_{5R}(r_2^*(\lambda))^2 \end{bmatrix}, \tag{5.54}$$

$$\omega(\lambda, \epsilon) = \begin{pmatrix} P_1\omega_1 + O(\epsilon) \\ P_2\omega_2 + O(\epsilon) \end{pmatrix}, \tag{5.55}$$

and $\mathcal{F}, \Theta_2$ are 2π-periodic in both components of $\hat{\theta}$. Furthermore, for all sufficiently small $\sigma > 0, \epsilon_0 > 0$ and for fixed q, $0 < q < 1/2$, the functions $\mathcal{F}, \Theta_1$ and Θ_2 are continuously differentiable on

$$(\theta_1, \theta_2, \rho_1, \rho_2, \lambda, \epsilon) \in \mathbf{R}^2 \times \Omega(\sigma, \epsilon_0),$$

where

$$\Omega(\sigma, \epsilon_0) = \left\{(\rho_1, \rho_2, \lambda, \epsilon) : |(\rho_1, \rho_2)| < \sigma, \ 1/B + \epsilon^q < \lambda < C - \epsilon^q, \ 0 < \epsilon \leq \epsilon_0\right\}.$$

Now we use a linear coordinate transformation of the form $\rho = S(\lambda)\hat{\rho}$, where $S(\lambda)$ is a nonsingular 2×2 matrix, that diagonalizes $\hat{E}(\lambda)$, and we get

$$\begin{aligned}\dot{\hat{\rho}} &= A(\hat{\lambda})\rho + R(\hat{\theta}, \hat{\rho}, \hat{\lambda}), \\ \dot{\hat{\theta}} &= \omega(\hat{\lambda}) + \Theta(\hat{\theta}, \hat{\rho}, \hat{\lambda}),\end{aligned} \tag{5.56}$$

where $\hat{\lambda} = (\lambda, \epsilon)$,

$$A(\hat{\lambda}) = \epsilon \begin{pmatrix} \alpha_1(\lambda) & 0 \\ 0 & \alpha_2(\lambda) \end{pmatrix}, \quad \alpha_1(\lambda) < 0 < \alpha_2(\lambda),$$

and

$$\begin{aligned}R(\hat{\theta}, \hat{\rho}, \hat{\lambda}) &= \epsilon^{3/2} S(\lambda)^{-1} \mathcal{F}(\hat{\theta}, S(\lambda)\hat{\rho}, \lambda, \epsilon), \\ \Theta(\hat{\theta}, \hat{\rho}, \hat{\lambda}) &= \epsilon^{3/2}\Theta_1(S(\lambda)\hat{\rho}, \lambda, \epsilon) + \epsilon^2 \Theta_2(S(\lambda)\hat{\rho}, \lambda, \epsilon).\end{aligned}$$

We observe that (5.56) has the same form as (5.49) (the dimension of $\hat{\rho}$ is different), and we can use the same method used in Lemma 5.3 to prove the existence of invariant tori. We construct the 2×2 matrix

$$\begin{aligned}K(u) &= diag(e^{-\epsilon\alpha_1(\lambda)u}, 0) \quad \text{for } u < 0 \\ &= diag(0, e^{-\epsilon\alpha_2(\lambda)u}) \quad \text{for } u \geq 0,\end{aligned}$$

then form the mapping

$$T(\hat{\lambda}, f)(\hat{\phi}) = \int_{-\infty}^{\infty} K(u, \hat{\lambda}) Q(\theta^*(u, \hat{\phi}, \hat{\lambda}, f), \hat{\lambda}, f) du. \tag{5.57}$$

for $f \in F_1, \hat{\phi} \in \mathbf{R}^2$, where Q and θ^* have the same meaning as in the proof of Lemma 5.3. In the present case, we have

$$\gamma(\hat{\lambda}, f) = \sup_{\hat{\theta}} \left| \frac{\partial P}{\partial \hat{\theta}}(\hat{\theta}, \hat{\lambda}, f) \right| = O(\epsilon^{3/2}) \cdot (\epsilon^{1/2} + |f|_1),$$

where P has the same meaning as in the proof of Lemma 5.3. To obtain useful estimates for T we estimate the eigenvalues of $\alpha_1(\lambda), \alpha_2(\lambda)$ of $E(\lambda)$. Near a boundary of the wedge, one of the eigenvalues approaches zero, so first we put $\lambda = C - \epsilon^q$, and after some simplification get

$$\begin{aligned} \alpha_1(\lambda) &= \epsilon^q/2 + O(|\epsilon|^{2q}), \\ \alpha_2(\lambda) &= -2 + O(|\epsilon|^{2q}), \end{aligned} \tag{5.58}$$

If we let $\lambda = 1/B + \epsilon^q$, we get the similar result, and thus for all λ, $C + \epsilon^q \leq \lambda \leq 1/B - \epsilon^q$, there are positive constants β, k such that

$$|K(u, \hat{\lambda})| \leq \beta e^{-k\epsilon^{1+q}|u|}, \quad u \in (-\infty, \infty),$$

for all sufficiently small ϵ. Since $0 < q < 1/2$, we choose a bounded neighborhood V of 0 in F_1 and choose $|\lambda|, |\epsilon|$ sufficiently small so that

$$\gamma(\hat{\lambda}, f) < k\epsilon^{1+q},$$

then by proof of Lemma 12.5.2 of [5] we again have

$$T(\hat{\lambda}, \cdot) : V \to F_1,$$

We prove the existence of invariant tori, by proving that T is a contraction mapping on the close unit ball in F_1. In a way similar to how we obtained the analogous estimates in the proof of Lemma 5.3, we obtain

$$|D_f(T(\hat{\lambda}, f)|_1 = O(\epsilon^{1/2-q}), \quad |T(\hat{\lambda}, f)|_1, = O(\epsilon^{1/2-q})$$

for all $f \in F_1$ with $|f|_1 \leq 1$, and for all λ with $C + \epsilon^q \leq \lambda \leq 1/B - \epsilon^q$. Thus for all sufficiently small ϵ, T is a contraction mapping on the closed unit ball in F_1, and we have

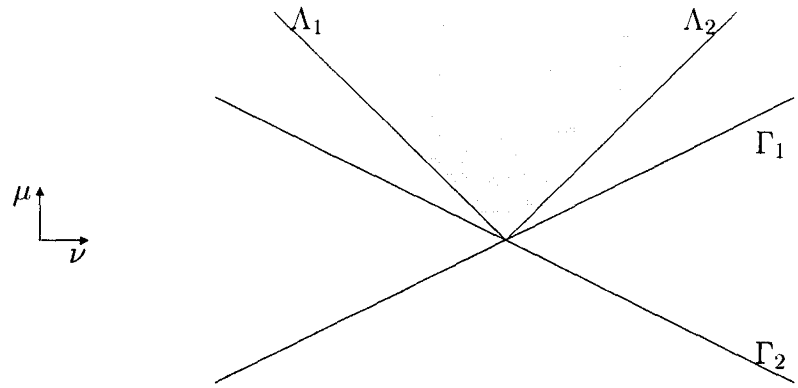


Figure 5.4: Parameter values (shaded region) for which normally hyperbolic invariant tori exist for (5.1), with $C_{1R}C_{5R} - C_{2R}C_{4R} < 0$.

Lemma 5.4 *For any fixed q, $0 < q < 1/2$, system (5.4) possesses C^1 invariant tori for*

$$1/B + \epsilon^q \leq \lambda \leq C - \epsilon^q, \quad 0 < q < 1/2,$$

for all ϵ sufficiently small. The tori are normally hyperbolic, and have the same stability type as the invariant tori $T^{(0)}$ of the truncated system (5.3).

In terms of the parameters of (5.1), Lemma 5.4 implies that the invariant tori for (5.1) exist in a region bounded by curves that are tangent to Λ_1 and Λ_2 at the origin (see Figure 5.4).

Since Lemma 5.3 already implies that tori exist in thin wedge-shaped regions of width $O(|\nu|)$ adjacent to Λ_1 and Λ_2 (see Figure 5.2), the parameter regions corresponding to the two lemmas overlap in a neighborhood of the origin:

Theorem 5.1 *System (5.1) possesses invariant tori (denoted T_1) for the parameter values (μ, ν) throughout the interior of the wedge of the parameter plane bounded by the curves Λ_1 and Λ_2, and the tori have the same stability type as the invariant tori $T^{(0)}$ of the truncated system (5.3).*

We summarize the results of this chapter in Figure 5.5, showing the schematic bifurcation set for (5.1). Corresponding to one-parameter paths with fixed ν and increasing μ, are diagrams shown in Figure 5.6. We note for μ sufficiently large there exists the phenomenon of bistability: both SW_0 and SW_π solutions are asymptotically stable, and the behavior of a typical solution (transient) as $t \to \infty$ is determined by its initial condition. The boundary between the basins of attraction for the two SW solutions contains the invariant torus T_1 and its stable manifold.

We briefly mention some implications of the result of this chapter for the original magnetoconvection problem. For a fixed σ, ζ, Q and odd integer m, for $L < L_m$ and sufficiently close to L_m ($\nu < 0$ and sufficiently close to 0: see Figure 5.6(a)), as we increase the Rayleigh number R through $R_m(L)$, stable standing wave solutions SW_0 (corresponding to odd mode solutions) bifurcate from the motionless solution and an odd number of time-periodic rolls will be observed in the fluid. As we increase the Rayleigh number further through $R_{m+1}(L)$, a branch of unstable SW_π solutions (corresponding to even mode solutions) will bifurcate from the motionless solution. Increasing the Rayleigh numbers still further, the competition between odd and even modes produces a branch of invariant tori T_1 (typically corresponding to quasiperiodic or weakly resonant solutions) which bifurcate from the branch of even mode SW_π standing wave solutions, and coexists with the branches of stable SW_0 and SW_π solutions. For $L = L_m$ ($\nu = 0$) and $L > L_m$ ($\nu > 0$), we have similar interpretations of the bifurcation structure. See Figure 5.7.

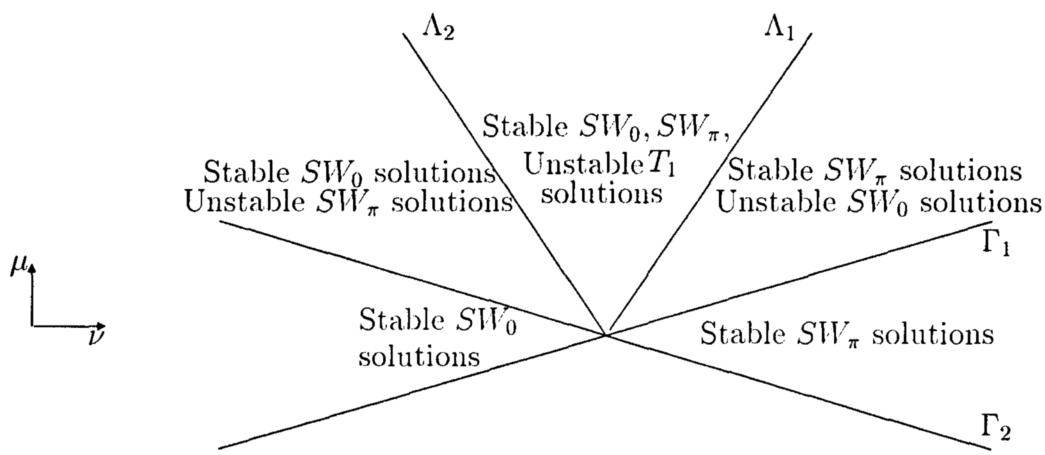


Figure 5.5: Schematic bifurcation set for (5.1), with $C_{1R}C_{5R} - C_{2R}C_{4R} < 0$. (Compare with Figure 5.1(b)).

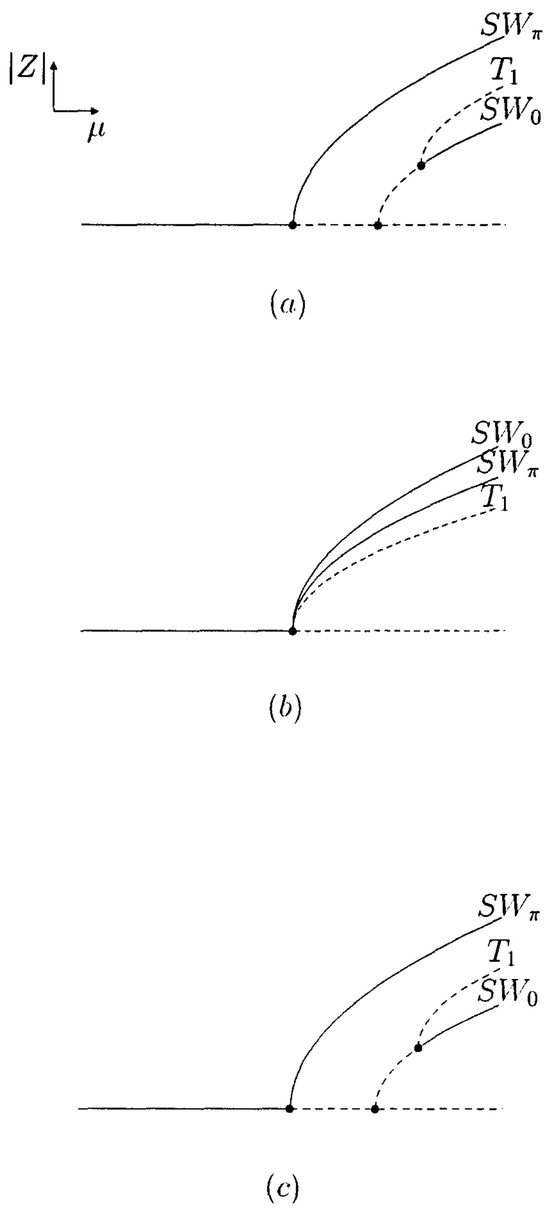


Figure 5.6: Bifurcation diagrams for (5.1), for fixed ν and increasing μ: (a) $\nu < 0$; (b) $\nu = 0$; (c) $\nu > 0$.

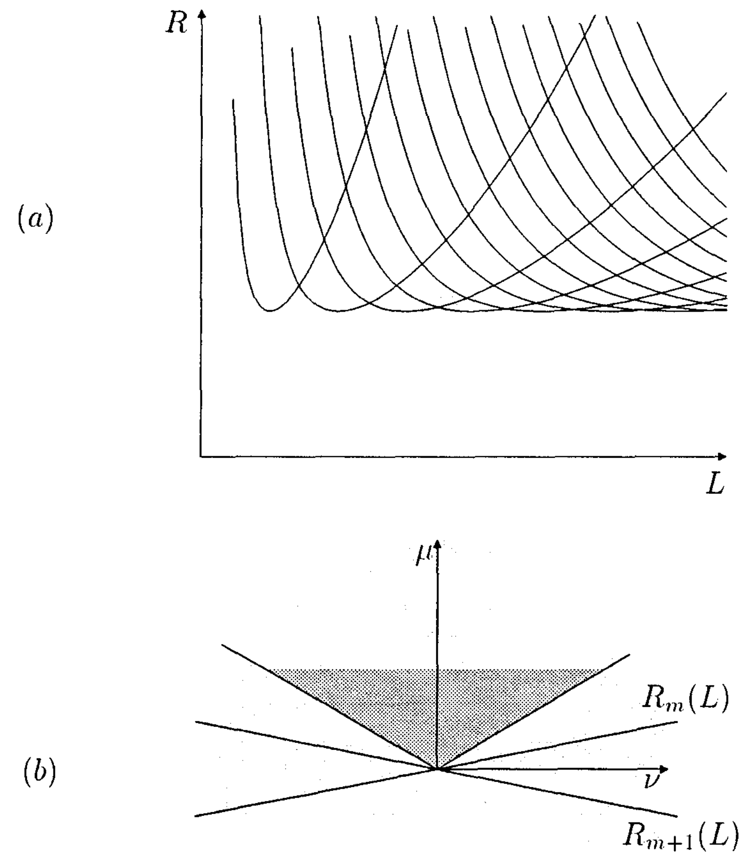


Figure 5.7: (*a*) Typical region (inside shaded circle) in (R, L) parameter plane for which results of this chapter apply to the magnetoconvection equations; (*b*) Magnification of the region in (*a*), showing parameter values (darker shaded region) corresponding to invariant tori and bistability of standing wave solutions.

Chapter 6

Secondary Bogdanov-Takens bifurcations

The results of the previous chapter give rigorous results on the dynamics of small-amplitude oscillatory magnetoconvection, when L is sufficiently close to L_m and R is sufficiently close to R_m, so that $|a_{jR}\mu + b_{jR}\nu| \ll |P_1\omega_1 - P_2\omega_2|$, $j = 1, 2$. However, as we discussed in §3.5, for small ζ, large Q and large L, the quantity $|P_1\omega_1 - P_2\omega_2|$ is extremely small, so the rigorous results can be expected to be valid only for a small range of parameter values. To get a more complete picture of the dynamics of our problem for a wider range of parameters values in the Case II limit, in this chapter we will analyze the alternate normal form (3.79), which we rewrite here as:

$$\begin{aligned} \dot{Z}_1 &= Z_1[iP_1\omega_1 + a_1\mu + b_1\nu + C_1|Z_1|^2 + C_2|Z_2|^2] + C_3\bar{Z}_1 Z_2^2 + \text{h.o.t.} \qquad (6.1) \\ \dot{Z}_2 &= Z_2[iP_2\omega_2 + a_2\mu + b_2\nu + C_4|Z_1|^2 + C_5|Z_2|^2] + C_6\bar{Z}_2 Z_1^2 + \text{h.o.t.}, \end{aligned}$$

where h.o.t. are $O(|\mu,\nu|^2|Z| + |\mu,\nu||Z|^3 + |Z|^5)$ and $a_1, b_1, a_2, b_2, C_1, ..., C_6$ are the same complex numbers as in (3.73).

In §6.1 we consider some coordinate transformations of (3.79) which make the analysis of the normal form easier. Then in §6.2 we consider a three dimensional reduced system obtained from the third order truncation of the normal form (6.1) and find bifurcation parameters μ, ν for which the linearized vector field about its nontrivial fixed points have double zero eigenvalues. These nontrivial fixed points for the reduced system correspond to the periodic orbits SW_0 and SW_π of the full system. In §6.3 we consider the Bogdanov-Takens singularities which correspond to the double zero eigenvalue, and their unfoldings for the reduced system. In this way we predict the existence of secondary and tertiary pitchfork, Hopf and global bifurcations from the SW solutions. In §6.4 we consider the reduced system as a small perturbation of a similar reduced system coming from a Hopf bifurcation with D_4 symmetry that was considered before by Swift [41].

6.1 Preliminary coordinate transformations

To study the bifurcation of solutions in (6.1) we would like to decouple the average phase from the radial direction. The most straightforward way to accomplish this (up to finite order) is to write the complex amplitudes in terms of polar coordinates $Z_1 = r_1 e^{i\theta_1}$, $Z_2 = r_2 e^{i\theta_2}$ as we did in Chapter 5. Then the average phase $(\theta_1 + \theta_2)/2$ does not appear in the equations for $\dot{r}_1, \dot{r}_2$, or $(\dot{\theta}_1 - \dot{\theta}_2)$ up to cubic order. But coupling of the phase difference $(\theta_1 - \theta_2)$ with r_1 and r_2 in the system makes calculations somewhat awkward. We will instead use a different coordinate transformation which is easier to work with. Like the polar coordinates, the phase angle of the S^1 symmetry in the normal form will be decoupled from the other variables up to finite order in this coordinate system.

We use the coordinate transformation [41]

$$U + iV = 2Z_1\bar{Z}_2, \quad W = \mid Z_1 \mid^2 - \mid Z_2 \mid^2, \quad e^{i\Psi} = \frac{Z_1^2 - Z_2^2}{|Z_1^2 - Z_2^2|}. \tag{6.2}$$

Although this coordinate transformation is singular at origin, we will use it away from the origin. System (6.1) in this new coordinate system becomes:

$$\begin{aligned} \dot{U} &= U[\check{a}_R\mu + \check{b}_R\nu + r(B_{4R} + B_{5R} + B_{6R})/2] - V[\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu] \\ &\quad -Vr[B_{1I} - B_{2I} - B_{3I}]/2 - VW[B_{1I} + B_{2I} + B_{6I}]/2 \\ &\quad + UW[B_{4R} - B_{5R} - B_{3R}]/2 + \mathcal{U}(U, V, W, \Psi, \mu, \nu), \end{aligned} \tag{6.3}$$

$$
\begin{aligned}
\dot{V} &= U[\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu + r(B_{1I} - B_{2I} + B_{3I})/2] + V[\check{a}_R\mu + \check{b}_R\nu] \\
&\quad + Vr(B_{4R} + B_{5R} - B_{6R}]/2 + UW[B_{1I} + B_{2I} - B_{6I}]/2 \\
&\quad + VW[B_{4R} - B_{5R} + B_{3R}]/2 + \mathcal{V}(U, V, W, \Psi, \mu, \nu), \\
\dot{W} &= W[\check{a}_R\mu + \check{b}_R\nu + r(C_{1R} + C_{5R})] + r[\hat{a}_R\mu + \hat{b}_R\nu] + UVB_{6I} \\
&\quad + U^2(B_{1R} - B_{2R} + B_{3R})/2 + V^2(B_{1R} - B_{2R} - B_{3R})/2 \\
&\quad + W^2(C_{1R} - C_{5R}) + \mathcal{W}(U, V, W, \Psi, \mu, \nu), \\
\dot{\Psi} &= P_1\omega_1 + P_2\omega_2 + O(r),
\end{aligned}
$$

where $r = \sqrt{U^2 + V^2 + W^2}$, and

$$
\mathcal{U}, \mathcal{V}, \mathcal{W} = O(r^3 + |\ \mu, \nu\ |r^2 + |\ \mu, \nu\ |^2 r)
$$

are 2π-periodic in Ψ, and

$$
\begin{aligned}
B_1 &= C_1 - C_4, \quad B_2 = C_5 - C_2, \quad B_3 = C_3 - C_6, \\
B_4 &= C_1 + C_4, \quad B_5 = C_5 + C_2, \quad B_6 = C_3 + C_6, \\
\hat{a} &= a_1 - a_2, \qquad \hat{b} = b_1 - b_2, \qquad \hat{\omega} = P_1\omega_1 - P_2\omega_2, \\
\check{a} &= a_1 + a_2, \qquad \check{b} = b_1 + b_2.
\end{aligned}
$$

Due to the symmetry (3.55) of system (3.79), (6.3) is equivariant under $(U, V, W, \Psi) \to (-U, -V, W, \Psi)$, therefore

$$
\begin{aligned}
\mathcal{U}(-U, -V, W, \Psi, \mu, \nu) &= -\mathcal{U}(U, V, W, \Psi, \mu, \nu), \\
\mathcal{V}(-U, -V, W, \Psi, \mu, \nu) &= -\mathcal{V}(U, V, W, \Psi, \mu, \nu), \\
\mathcal{W}(-U, -V, W, \Psi, \mu, \nu) &= \mathcal{W}(U, V, W, \Psi, \mu, \nu),
\end{aligned}
\tag{6.4}
$$

and this implies that the set $U = V = 0$ is an invariant subset for (6.3). This invariant subset corresponds to the invariant subspaces V_0 (if $W > 0$) and V_π (if $W < 0$) in the (Z_1, Z_2) coordinates. We can find some information on the dynamics of (6.3) by restricting to invariant subset $U = V = 0$. Then

$$\begin{aligned} \dot{W} &= W[\check{a}_R\mu + \check{b}_R\nu + |W|(C_{1R} + C_{5R})] + |W|[\hat{a}_R\mu + \hat{b}_R\nu] \\ &\quad + W^2(C_{1R} - C_{5R}) + \mathcal{W}(0, 0, W, \Psi), \\ \dot{\Psi} &= P_1\omega_1 + P_2\omega_2 + O(|W|). \end{aligned} \tag{6.5}$$

Since $\dot{\Psi} > 0$ for $|W|$ sufficiently small, we can reparametrize (6.5) by Ψ and obtain

$$\begin{aligned} \frac{dW}{d\Psi} &= [P_1\omega_1 + P_2\omega_2]^{-1}\left\{W[\check{a}_R\mu + \check{b}_R\nu + W(C_{1R} - C_{5R}) + |W|(C_{1R} + C_{5R})]\right. \\ &\quad \left. + |W|[\hat{a}_R\mu + \hat{b}_R\nu]\right\} + \mathcal{W}_1(W, \Psi), \end{aligned} \tag{6.6}$$

where $\mathcal{W}_1(W, \Psi, \mu, \nu)$ is periodic in Ψ with period 2π and is $O(|W|^3 + |\mu, \nu||W|^2 + |\mu, \nu|^2|W|)$. Now the Hopf bifurcation theorem implies that (6.6) will have two different periodic orbits

$$\tilde{W}_0(\Psi, \mu, \nu) = W_0(\mu, \nu) + O(|\mu, \nu|^2), \tag{6.7}$$

$$\tilde{W}_\pi(\Psi, \mu, \nu) = W_\pi(\mu, \nu) + O(|\mu, \nu|^2), \tag{6.8}$$

where

$$\begin{aligned} W_0(\mu, \nu) &= -\frac{a_{1R}\mu + b_{1R}\nu}{C_{1R}}, \quad a_{1R}\mu + b_{1R}\nu > 0, \\ W_\pi(\mu, \nu) &= \frac{a_{2R}\mu + b_{2R}\nu}{C_{5R}}, \quad a_{2R}\mu + b_{2R}\nu > 0. \end{aligned} \tag{6.9}$$

Solutions $\tilde{W}_0(\Psi, \mu, \nu)$ and $\tilde{W}_\pi(\Psi, \mu, \nu)$ are the periodic orbits SW_0 and SW_π of Chapter 5, which bifurcate from the trivial solution along the curves

$$\begin{aligned} \Gamma_1 : \mu &= -(b_{1R}/a_{1R})\nu + O(|\nu|^2), \\ \Gamma_2 : \mu &= -(b_{2R}/a_{2R})\nu + O(|\nu|^2), \end{aligned} \tag{6.10}$$

respectively.

To study secondary bifurcations from the SW_0 solutions we use a moving coordinate transformation

$$\hat{U} = U, \quad \hat{V} = V, \quad \hat{W} = W - \tilde{W}_0(\Psi, \mu, \nu).$$

After dropping the hats, (6.3) becomes

$$\begin{aligned}
\dot{U} &= U[a_{2R}\mu + b_{2R}\nu + (C_{4R} + C_{6R})W_0] - V[\hat{\omega} + a_I\mu + b_I\nu] \\
&\quad - VW_0[B_{1I} + C_{6I}] - VW[B_{1I} + C_{6I}] + UW[B_{4R} + C_{6R}] + \tilde{\mathcal{U}}(U, V, W, \Psi, \mu, \nu) \\
\dot{V} &= U[\hat{\omega} + a_I\mu + b_I\nu + W_0(B_{1I} - C_{6I})] + V[a_{2R}\mu + b_{2R}\nu + W_0(C_{4R} - C_{6R})] \\
&\quad + UW[B_{1I} - C_{6I}] + VW[B_{4R} - C_{6R}] + \tilde{\mathcal{V}}(U, V, W, \Psi, \mu, \nu) \\
\dot{W} &= -2W[a_{1R}\mu + b_{1R}\nu] + UV[B_{6I}] + U^2(B_{1R} - B_{2R} + B_{3R})/2 \\
&\quad + V^2(B_{1R} - B_{2R} - B_{3R})/2 + 2W^2C_{1R} + \tilde{\mathcal{W}}(U, V, W, \Psi, \mu, \nu), \\
\dot{\Psi} &= P_1\omega_1 + P_2\omega_2 + O(r),
\end{aligned} \tag{6.11}$$

where the higher-order terms $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}, \tilde{\mathcal{W}}$ satisfy the symmetry condition (6.2), and are 2π-periodic in Ψ.

6.2 The reduced system

If the higher-order terms in (6.11) are ignored then the first three equations decouple from the fourth equation, although in general Ψ will appear in the periodic coefficients of higher-order terms. As an approximation we consider the following reduced system, obtained from a truncation of (6.11) that ignores the higher-order terms $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$:

$$\begin{aligned}
\dot{U} &= U[a_{2R}\mu + b_{2R}\nu + (C_{4R} + C_{6R})W_0] - V[\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu] \\
&\quad - VW_0[B_{1I} + C_{6I}] - VW[B_{1I} + C_{6I}] + UW[B_{4R} + C_{6R}], \\
\dot{V} &= U[\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu + W_0(B_{1I} - C_{6I})] + V[a_{2R}\mu + b_{2R}\nu + W_0(C_{4R} - C_{6R})]
\end{aligned} \tag{6.12}$$

$$
\begin{aligned}
& +UW[B_{1I} - C_{6I}] + VW[B_{4R} - C_{6R}], \\
\dot{W} \;=\; & -2W[a_{1R}\mu + b_{1R}\nu] + UV[B_{6I}] + U^2(B_{1R} - B_{2R} + B_{3R})/2 \\
& +V^2(B_{1R} - B_{2R} - B_{3R})/2 + 2W^2 C_{1R}.
\end{aligned}
$$

The origin in (6.12) now corresponds to the SW_0 solution.

Remark 6.1 *By using a similar moving coordinate transformation*

$$\hat{U} = U, \quad \hat{V} = V, \quad \hat{W} = W - \tilde{W}_\pi(\Psi, \mu, \nu),$$

we could consider the dynamics of (6.3) near the SW_π solution. Due to the symmetry (6.4) of (6.11), the analysis about the SW_π solution is quite similar to that about the SW_0 solution, therefore we only present our analysis of (6.12).

The eigenvalues of the linearization of vector field to (6.12) about the origin (i.e., about the SW_0 solution) are

$$
\begin{aligned}
\lambda^0_{1,2}(\mu,\nu) &= \frac{Tr_{0,\pi}(\mu,\nu) \pm \sqrt{Tr^2_{0,\pi}(\mu,\nu) - 4Det_{0,\pi}(\mu,\nu)}}{2}, \qquad (6.13)\\
\lambda^0_3(\mu,\nu) &= -2(a_{1R}\mu + b_{1R}\nu) < 0,
\end{aligned}
$$

where

$$
\begin{aligned}
Tr_{0,\pi}(\mu,\nu) &= 2[a_{2R}\mu + b_{2R}\nu + W_0 C_{4R}], \qquad (6.14)\\
Det_{0,\pi}(\mu,\nu) &= -|C_6|^2 W_0^2 + [\hat{a}_R\mu + \hat{b}_R\nu + B_{1R}W_0]^2 \qquad (6.15)\\
&\quad + [\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu + B_{1I}W_0]^2,
\end{aligned}
$$

and $a_{1R}\mu + b_{1R}\nu > 0$. We notice that one of the eigenvalues of the linearized vector field, which corresponds to the W direction, is always negative .

Remark 6.2 *By considering the truncation of (6.3) about the SW_π solutions we find the corresponding expressions for eigenvalues*

$$\begin{aligned} \lambda_{1,2}^{\pi}(\mu,\nu) &= \frac{Tr_{\pi,0}(\mu,\nu) \pm \sqrt{Tr_{\pi,0}^2(\mu,\nu) - 4Det_{\pi,0}(\mu,\nu)}}{2}, \\ \lambda_3^{\pi}(\mu,\nu) &= -2(a_{2R}\mu + b_{2R}\nu) < 0, \end{aligned} \tag{6.16}$$

where

$$Tr_{\pi,0}(\mu,\nu) = 2[a_{1R}\mu + b_{1R}\nu - W_\pi C_{2R}], \tag{6.17}$$

$$\begin{aligned} Det_{\pi,0}(\mu,\nu) &= -|C_3|^2 W_\pi^2 + [\hat{a}_R\mu + \hat{b}_R\nu + B_{2R}W_\pi]^2 \\ &\quad + [\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu + B_{2I}W_\pi]^2, \end{aligned} \tag{6.18}$$

and $a_{2R}\mu + b_{2R}\nu > 0$.

We expect pitchfork bifurcations of fixed points of the three-dimensional system (6.12) along one-parameter paths transversal to the curve

$$\Gamma_3 : Det_{0,\pi}(\mu,\nu) = 0. \tag{6.19}$$

Such bifurcations correspond to secondary pitchfork bifurcations of periodic solutions from the periodic SW_0 solutions in the four-dimensional normal form. Along one-parameter paths transversal to

$$\Gamma_4 : Tr_{0,\pi}(\mu,\nu) = 0, \quad Det_{0,\pi}(\mu,\nu) > 0, \tag{6.20}$$

we expect Hopf bifurcations of periodic orbits of (6.12) from the origin. These correspond to secondary bifurcations of invariant tori from the SW_0 solutions.

To find multiple bifurcation points where Γ_3 and Γ_4 intersect, we look at the sign of $Det_{0,\pi}$ when $Tr_{0,\pi} = 0$. The line $\mu = \nu\lambda_0^*$, where λ_0^* satisfies (5.7), corresponds to $Tr_{0,\pi} = 0$. If we put $W_0 = |\nu| W_0^*$, we have

$$W_0^* = -sgn(\nu)\frac{b_{1R}a_{2R} - a_{1R}b_{2R}}{C_{1R}a_{2R} - C_{4R}a_{1R}}, \quad sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R}). \tag{6.21}$$

Remark 6.3 *Based on our numerical results in Chapter 4, in our application* $sgn(\nu) = sgn(\lambda_0^* + b_{1R}/a_{1R}) = 1$ *when* m *is odd. To simplify our notation, we continue our analysis assuming that* $sgn(\nu) = 1$*; however if* $sgn(\nu) = -1$ *we can treat the problem similarly.*

Therefore for $sgn(\nu) = 1$ we have

$$W_0^* = -\frac{b_{1R}a_{2R} - a_{1R}b_{2R}}{C_{1R}a_{2R} - C_{4R}a_{1R}}, \tag{6.22}$$

and then

$$Det_{0,\pi}(\mu_0^*(\nu), \nu) = -|C_6|^2 W_0^{*2}\nu^2 + \{\hat{\omega} + [\hat{a}_I\lambda_0^* + \hat{b}_I + B_{1I}W_0^*]\nu\}^2. \tag{6.23}$$

Then $Det_{0,\pi}(\mu_0^*(\nu), \nu) = 0$ if

$$\nu = \nu_j^* = \frac{\hat{\omega}}{[\pm|C_6| - B_{1I}]W_0^* - \hat{b}_I - \hat{a}_I\lambda_0^*} > 0, \quad j = 1, 2. \tag{6.24}$$

In the above equation, and in what will follow throughout this chapter, j can be either 1 or 2, with the "plus" sign in front of $|C_6|$ corresponding to $j = 1$, while the "minus" sign corresponds to $j = 2$. Using (6.22) and (5.7) and after some simplification we write (6.24) in terms of normal form coefficients:

$$\nu = \nu_j^* = \frac{\hat{\omega}(C_{1R}a_{2R} - a_{1R}C_{4R})}{\pm|C_6|[a_{1R}b_{2R} - b_{1R}a_{2R}] + a_{1R}\Im(B_1\bar{\hat{b}}) - b_{1R}\Im(B_1\bar{\hat{a}}) - C_{1R}\Im(\hat{a}\bar{\hat{b}})}. \tag{6.25}$$

Let us denote

$$\mu_j^* = \nu_j^*\lambda_0^*, \quad j = 1, 2, \tag{6.26}$$

and note that

$$\frac{\partial Det_{0,\pi}(\mu_j^*, \nu_j^*)}{\partial \nu} = -2\hat{\omega}\overline{W}_0^*(\pm|C_6|) \neq 0, \tag{6.27}$$

which implies that any point of intersection of the curves $Tr_{0,\pi}(\mu, \nu) = 0$ and $Det_{0,\pi}(\mu, \nu) = 0$ is transversal. To find a condition on the number of solutions for ν in (6.25), we consider

$$M = -\left[a_{1R}\Im(B_1\bar{\hat{b}}) - b_{1R}\Im(B_1\bar{\hat{a}}) - C_{1R}\Im(\hat{a}\bar{\hat{b}})\right]^2 \tag{6.28}$$

$$
\begin{aligned}
&+|C_6|^2[a_{1R}\hat{b}_R-\hat{a}_R b_{1R}]^2, \\
=\ & (a_{1R})^2\left[|C_6|^2\hat{b}_R^2-[\Im(B_1\bar{\hat{b}})]^2\right]+(b_{1R})^2\left[|C_6|^2\hat{a}_R^2-[\Im(B_1\bar{\hat{a}})]^2\right] \\
&-2a_{1R}b_{1R}\left[|C_6|^2\hat{a}_R\hat{b}_R-\Im(B_1\bar{\hat{b}})\Im(B_1\bar{\hat{a}})\right]-(C_{1R})^2[\Im(\hat{a}\bar{\hat{b}})]^2 \\
&-2C_{1R}\Im(\hat{a}\bar{\hat{b}})[a_{1R}\Im(B_1\bar{\hat{b}})-b_{1R}\Im(B_1\bar{\hat{a}})].
\end{aligned}
\tag{6.29}
$$

There will be one solution ν_1^* if and only if $M > 0$. On the other hand $Det_{0,\pi}(\mu_0^*(\nu),\nu) = 0$ will have two solutions $\nu = \nu_j^*$ if

$$
-[|C_6|+B_{1I}]W_0^*-\hat{b}_I-\hat{a}_I\lambda_0^*>0, \tag{6.30}
$$

or equivalently, if

$$
|C_6|<\frac{a_{1R}\Im(B_1\bar{\hat{b}})-b_{1R}\Im(B_1\bar{\hat{a}})-C_{1R}\Im(\hat{a}\bar{\hat{b}})}{a_{1R}\hat{b}_R-b_{1R}\hat{a}_R}. \tag{6.31}
$$

Now we consider the curve $Det_{0,\pi}=0$ in general, after substituting for W_0 from (6.9) into (6.15). We have

$$
Det_{0,\pi}(\mu,\nu)=D\mu^2+F\nu^2+2E\mu\nu+G\mu+H\nu+\hat{\omega}^2, \tag{6.32}
$$

where

$$
\begin{aligned}
D &= (a_{1R}/C_{1R})^2\left[|B_1|^2-|C_6|^2\right]+|\hat{a}|^2-2(a_{1R}/C_{1R})\Re(B_1\bar{\hat{a}}) \qquad (6.33)\\
E &= \left(\frac{a_{1R}b_{1R}}{C_{1R}^2}\right)\left[|B_1|^2-|C_6|^2\right]+\Re(\hat{a}\bar{\hat{b}}) \\
&\quad -(b_{1R}/C_{1R})\Re(B_1\bar{\hat{a}}))-(a_{1R}/C_{1R})\Re(B_1\bar{\hat{b}})), \\
F &= (b_{1R}/C_{1R})^2\left[|B_1|^2-|C_6|^2\right]+|\hat{b}|^2-2(b_{1R}/C_{1R})\Re(B_1\bar{\hat{b}}), \\
G &= -2\hat{\omega}[(a_{1R}/C_{1R})B_{1I}-\hat{a}_I], \\
H &= -2\hat{\omega}[(b_{1R}/C_{1R})B_{1I}-\hat{b}_I].
\end{aligned}
$$

The graph of $Det_{0,\pi}=0$ is a conic section in μ and ν, the type of this conic section being determined by the sign of E^2-DF; it will be an hyperbola, parabola or ellipse if

$E^2 - DF$ is positive, zero or negative respectively. After some simplification we get

$$\begin{aligned}(E^2 - DF)(C_{1R})^2 &= |C_6|^2\left[a_{1R}^2|\hat{b}|^2 + b_{1R}^2|\hat{a}|^2 - 2a_{1R}b_{1R}|R(\hat{a}\bar{\hat{b}})\right] \\ &\quad -\left[a_{1R}\Im(B_1\bar{\hat{b}}) - b_{1R}\Im(B_1\bar{\hat{a}}) - C_{1R}\Im(\hat{a}\bar{\hat{b}})\right]^2, \\ &> -\left[a_{1R}\Im(B_1\bar{\hat{b}}) - b_{1R}\Im(B_1\bar{\hat{a}}) - C_{1R}\Im(\hat{a}\bar{\hat{b}})\right]^2 \\ &\quad + |C_6|^2\left[a_{1R}|\hat{b}| - b_{1R}|\hat{a}|\right]^2,\end{aligned} \tag{6.34}$$

or in another form

$$(C_{1R})^2(E^2 - DF) = M + [|C_6|(\hat{b}_I a_{1R} - \hat{a}_I b_{1R})]^2. \tag{6.35}$$

This implies that in the region in (μ, ν) plane for which there is only one positive root ν_1^* for (6.23), the graph of $Det_{0,\pi} = 0$ will be a hyperbola.

Since we are interested in the behavior of the system for large m, let us denote $\epsilon = m^{-1}$ as in Chapters 2 and 4. We recall some of our asymptotic results on the normal form coefficients. For fixed σ, ζ and Q (i.e., Case I) we have, as $m \to \infty$:

$$\hat{\omega} = P_1\omega_1 - P_2\omega_2 = \frac{\zeta^2\pi^2}{\lambda\omega}\epsilon + O(\epsilon^2), \tag{6.36}$$

$$\begin{aligned} C_1 &= A + B + O(\epsilon), & C_5 &= A + B + O(\epsilon), \\ C_2 &= A + O(\epsilon), & C_4 &= A + O(\epsilon), \\ C_3 &= C + O(\epsilon), & C_6 &= C + O(\epsilon), \\ \hat{b} &= \epsilon^2 b + O(\epsilon^3), & \hat{a} &= O(\epsilon), \\ b_{1R} &= \epsilon^2 b_R/2 + O(\epsilon^3), & b_{1R} &= -\epsilon^2 b_R/2 + O(\epsilon^3), \\ B_1 &= B + O(\epsilon), \\ B_2 &= B + O(\epsilon), \end{aligned} \tag{6.37}$$

where P_j, ω_j, $j = 1, 2$ and P, ω are as in Chapter 2, and C_j $(j = 1, ..., 6)$, A, B, C, a_1, a_2, b_1, b_2 are as in Chapter 4, and $\hat{a}, \hat{b}, \hat{\omega}$ are as in equation (6.3). Now we consider the

ϵ-dependence of the terms involved in $Det_{0,\pi}$ and $Tr_{0,\pi}$. From (6.36) we observe that the slopes of the curves Γ_1, Γ_2 and Γ_4 near origin are $O(\epsilon^2)$. This implies that in Case I for a fixed σ, ζ and Q and for sufficiently large m, $Det_{0,\pi}(\mu_0^*(\nu), \nu)$ cannot be zero for small μ and ν. For example, for $m = 10^7, Q = 100\pi^2, \sigma = 1$, and $\zeta = .1$, there is one intersection point, but with the value of $\nu \simeq 9519$. In fact we have

$$\begin{aligned} \nu_j^* &= O(m), \\ Det_{0,\pi} &= \hat{\omega}^2 + O(\epsilon^3). \end{aligned} \tag{6.38}$$

However, if we consider a decreasing sequence in ζ and an increasing sequence in Q as in our Case II for large aspect ratios that was discussed in Chapters 3 and 4, we will get different results. Recall that for fixed $\hat{\zeta}$ and $\hat{Q}$, we put

$$\zeta = \epsilon^{k/2}\hat{\zeta}, \quad Q = \epsilon^{-k/2}\hat{Q}, \tag{6.39}$$

where $0 < k < 2$. Under the scaling (6.39) we have checked numerically the conditions on the number of intersection points of the curves $Tr_{0,\pi} = 0$ and $Det_{0,\pi} = 0$, for different parameters of the original magnetoconvection problem. We found that it is possible to have both one or two intersection points. When the graph of Γ_3 is an hyperbola there is a very small region in parameter values for which there are two intersection points, however the value of ν at the second intersection point is large. For fixed σ and $\hat{\zeta}$, as the value of $\hat{Q}$ decreases, the value of ν_1^* seems to increase without bound, but as the value of $\hat{Q}$ increases, the values of ν_1^* and ν_2^* become smaller and the intersection points appear to approach the origin. We did not find parameter values for which there are no intersection points (see Table 6.1).

We have sketched the scaled graphs of $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ for $\sigma = 1, \hat{\zeta} = .1$, and for $\hat{Q} = 100\pi^2$ and $300\pi^2$, which correspond to cases with one or two intersection points, respectively, in Figures 6.1 and 6.2. The directions of the axes in all the figures in this chapter are as in Figure 6.1.

Table 6.1: Values of $\nu_j^*, \lambda_0^*, -b_{1R}/a_{1R}$ for $\sigma = 1, \hat{\zeta} = .1, m = 10^7$, and different values of $\hat{Q}$.

$\hat{Q}$	ν_1^*	ν_2^*	$10^{-4}m^2\lambda_0^*$	$10^{-4}m^2(-b_{1R}/a_{1R})$
$10^{-4}\pi^2$	4.75	–	.247965	.247964
$.01\pi^2$	.4814	–	.2481	.2480
π^2	.0546	–	.2718	.2580
$100\pi^2$	$.804 \times 10^{-3}$	–	3.9458	1.0887
$200\pi^2$	$.417 \times 10^{-3}$	–	6.330	1.808
$220.04793\pi^2$	$.3901 \times 10^{-3}$	–	6.733	1.945
$220.04794\pi^2$	$.3901 \times 10^{-3}$	76667	6.733	1.945
$300\pi^2$	$.319 \times 10^{-3}$	$.407 \times 10^{-2}$	8.238	2.477
$400\pi^2$	$.259 \times 10^{-3}$	$.143 \times 10^{-2}$	9.997	3.113
$500\pi^2$	$.241 \times 10^{-3}$	$.127 \times 10^{-2}$	11.682	3.727
$1000\pi^2$	$.173 \times 10^{-3}$	$.551 \times 10^{-3}$	19.616	6.577
$10^5\pi^2$	$.248 \times 10^{-4}$	$.312 \times 10^{-4}$	$.111 \times 10^4$	$.312 \times 10^3$
$10^8\pi^2$	$.101 \times 10^{-5}$	$.104 \times 10^{-5}$	$.475 \times 10^6$	$.100 \times 10^6$

To consider the asymptotic behaviour of the location of the intersection points, we put

$$\begin{aligned} \lambda_0^* &= \epsilon^2\bar{\lambda}_0^* + O(\epsilon^3) = \frac{b_R(2A_R + B_R)}{a_R B_R}\epsilon^2 + O(\epsilon^3), \\ W_0^* &= \epsilon^2\bar{W}_0^* + O(\epsilon^3) = \frac{b_R}{B_R}\epsilon^2 + O(\epsilon^3), \end{aligned} \tag{6.40}$$

where λ_0^*, W_0^* are as in (5.7) and (6.21). Recall that we have shown that in Case II, we have

$$\hat{\omega} = \frac{\zeta^2\pi^2}{\lambda\omega}\epsilon^{k+1} + O(\epsilon^{k+2}) = \epsilon^{k+1}\omega_0 + O(\epsilon^{k+2}). \tag{6.41}$$

For $1 < k < 2$ we have

$$Det_{0,\pi}(\nu,\epsilon) = \epsilon^4\nu^2\{-|C|^2W_0^{*2} + [-B_I W_0^* + b_I]^2\} + O(\epsilon^{2k+2} + \epsilon^5), \tag{6.42}$$

and if $k = 1$ we have

$$Det_{0,\pi}(\nu,\epsilon) = \left\{\hat{\omega} + \epsilon^2[-B_I W_0^* + b_I]\nu\right\}^2$$

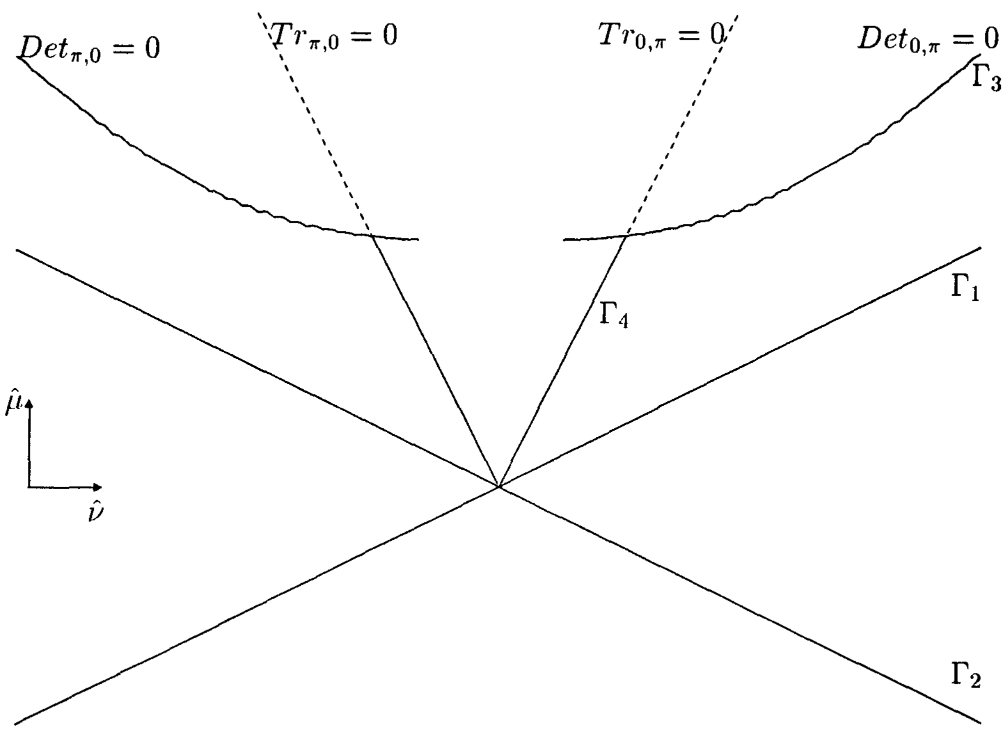


Figure 6.1: The curves $\Gamma_1, ..., \Gamma_4$ for $\sigma = 1, \hat{\zeta} = .1, \hat{Q} = 100\pi^2, k = 1, m = 10^7$, using the scaling $\mu = 200\epsilon^2\hat{\mu}, \nu = .01\hat{\nu}$. Note that $\nu_1^* = .000804$. Dotted lines show the parts of curves $Tr_{0,\pi} = 0$, $Tr_{\pi,0} = 0$ for which their corresponding values for $Det_{0,\pi}$ and $Det_{\pi,0}$ are negative.

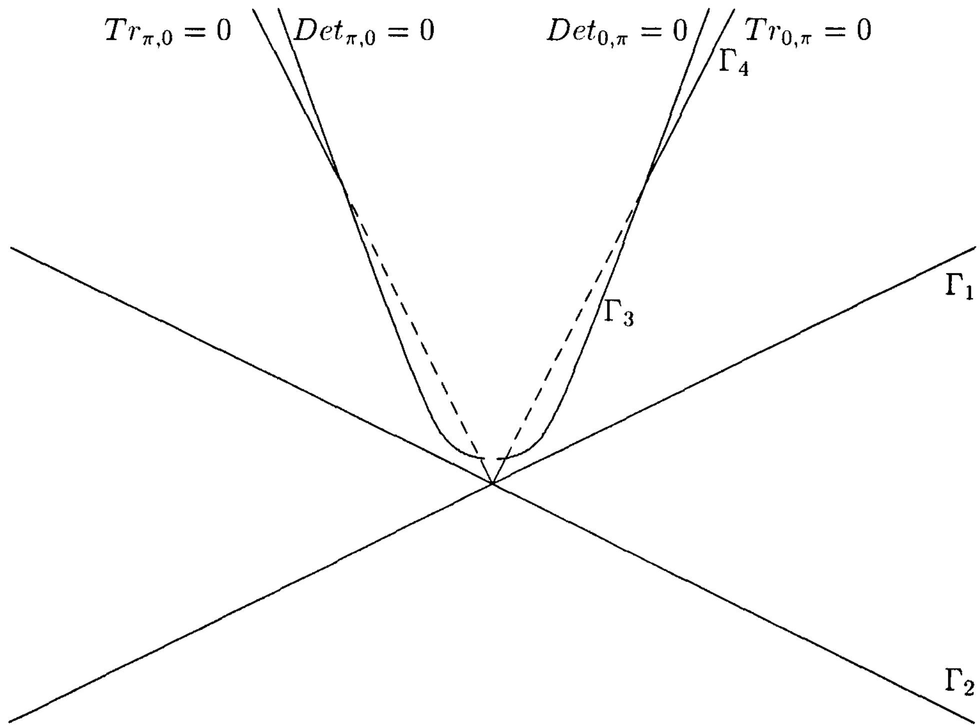


Figure 6.2: The curves $\Gamma_1, ..., \Gamma_4$ for $\sigma = 1, \hat{\zeta} = .1, \hat{Q} = 300\pi^2, k = 1, m = 10^7$, using the scaling $\mu = 400\epsilon^2\hat{\mu}, \nu = .01\hat{\nu}$. Note that $\nu_1^* = .000319$, $\nu_2^* = .00407$. Dashed lines show the parts of graph $Tr_{0,\pi} = 0$, $Tr_{\pi,0} = 0$ for which their corresponding value for $Det_{0,\pi}$ and $Det_{\pi,0}$ are negative.

$$-\epsilon^4|C|^2(\nu W_0^*)^2 + O(\epsilon^5). \tag{6.43}$$

The implicit function theorem implies that $Det_{0,\pi}(\mu,\nu,\epsilon)$ along the line $Tr_{0,\pi}(\mu,\nu,\epsilon) = 0$ is zero if

$$\nu = \nu_j^*(\epsilon) \quad = \quad \bar{\nu}_j^* + O(\epsilon), \tag{6.44}$$

where

$$\begin{aligned} \bar{\nu}_j^* &= \frac{\omega_0}{(\pm|C| - B_I)\bar{W}_0^* - b_I} \\ &= \frac{-B_R\omega_0}{-\Im(B\bar{b}) \pm b_R|C|} + O(\epsilon). \end{aligned} \tag{6.45}$$

6.3 Bogdanov-Takens bifurcations in the truncated system

In the previous section we showed that the linearization of vector field of (6.12) about $(U,V,W) = (0,0,0)$ (the SW_0 solutions) has double zero eigenvalues when $(\mu,\nu) = (\mu_j^*,\nu_j^*)$, i.e., where the curves Γ_3 ($Det_{0,\pi} = 0$) and Γ_4 ($Tr_{0,\pi} = 0$) intersect. Similar results hold for the SW_π solution. The parameter values (μ_j^*,ν_j^*) correspond to Bogdanov-Takens singularities in (6.12). In this section we unfold the singularities, and analyze the non-linear dynamics of (6.16) for (μ,ν) near (μ_j^*,ν_j^*).

We first use the coordinate transformation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \Delta_{2j} & \Delta_{1j} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}, \tag{6.46}$$

where

$$\Delta_{1j} = (a_{2R}\mu + b_{2R}\nu) + \overline{W}_0[C_{4R} - C_{6R}], \tag{6.47}$$

$$\Delta_{2j} = \hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu + \overline{W}_0[B_{1I} - C_{6I}]. \tag{6.48}$$

Under this transformation (6.12) becomes

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -Det_{0,\pi} & Tr_{0,\pi} & 0 \\ 0 & 0 & -2\mu_{1R} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} \mathcal{X}^0(X,Y,W) \\ \mathcal{Y}^0(X,Y,W) \\ \mathcal{Z}^0(X,Y,W,) \end{pmatrix}, \tag{6.49}$$

where

$$\begin{aligned} \mathcal{X}^0(X,Y,Z) &= Z\left\{\frac{[-\Delta_{1j}X+Y][B_{1I}-C_{6I}]}{\Delta_{2j}} + X[B_{4R}-C_{6R}]\right\}, \\ \mathcal{Y}^0(X,Y,Z,) &= Z\left\{\frac{(-\Delta_{1j}X+Y)[B_{4R}+C_{6R}) + \Delta_{1j}(B_{1I}-C_{6I}]}{\Delta_{2j}}\right. \\ &\quad \left. -X[\Delta_{2j}(C_{6I}+B_{1I}) + \Delta_{1j}(C_{6R}-B_{4R})]\right\}, \\ \mathcal{Z}^0(X,Y,Z) &= X\left(\frac{(-\Delta_{1j}X+Y)B_{6I}}{\Delta_{2j}}\right) + \left(\frac{-\Delta_{1j}X+Y}{\Delta_{2j}}\right)^2 [B_{1R}-B_{2R}+B_{3R}]/2 \\ &\quad + X^2[B_{1R}-B_{2R}-B_{3R}]/2 + 2C_{1R}Z^2. \end{aligned}$$

When $Tr_{0,\pi} = 0, Det_{0,\pi} = 0$, the linear part of the vector field for (6.49) has double zero eigenvalue, while the third eigenvalue $-2\lambda_j^*$, where

$$\lambda_j^* = a_{1R}\mu_j^* + b_{1R}\nu_j^*, \tag{6.50}$$

is negative. By the center manifold theorem, there exists an attracting center manifold represented by a smooth surface $Z = h(X,Y,\mu,\nu)$ for X,Y sufficiently small and μ,ν close to μ_j^*,ν_j^*. Moreover, the reflection symmetry (6.4) implies that h can be chosen so that

$$h(-X,-Y,\mu,\nu) = h(X,Y,\mu,\nu).$$

This center manifold can be represented by its Taylor series to any finite order, and to the lowest order it will be in the form

$$h(X,Y,\mu_j^*,\nu_j^*) = e_{1j}X^2 + e_{2j}XY + e_{3j}Y^2 + O(|X,Y|^4). \tag{6.51}$$

To calculate e_{1j}, e_{2j}, e_{3j} we substitute $h = h(X, Y, \mu_j^*, \nu_j^*)$ given by (6.51) into

$$(2e_{1j}X + e_{2j}Y)(Y + \mathcal{X}^0(X,Y,h)) + (e_{2j}X + 2e_{3j}Y)\mathcal{Y}^0(X,Y,h) =$$
$$-2\lambda_j^*(e_{1j}X^2 + e_{2j}XY + e_{3j}Y^2) + \mathcal{Z}^0(X,Y,h,\mu_j^*,\nu_j^*), \tag{6.52}$$

Then by equating coefficients of powers of X and Y in both sides of (6.52), we get

$$\begin{aligned} e_{1j} &= \frac{-(\Delta_{1j}/\Delta_{2j})\,[B_{6I}] + (\Delta_{1j}/\Delta_{2j})^2[B_{1R} - B_{2R} + B_{3R}]/2}{2\lambda_j^*} \\ &\quad + \frac{[B_{1R} - B_{2R} - B_{3R}]/2}{2\lambda_j^*}, \end{aligned} \tag{6.53}$$

$$e_{2j} = -\frac{e_{1j}}{\lambda_j^*} + \frac{B_{6I}}{2\Delta_{2j}\lambda_j^*} - \frac{\Delta_{1j}(B_{1R} - B_{2R} + B_{3R})}{2\Delta_{2j}^2\lambda_j^*}, \tag{6.54}$$

$$e_{3j} = -\frac{e_{2j}}{2\lambda_{3j}^*} + \frac{B_{1R} - B_{2R} + B_{3R}}{4\Delta_{2j}^2\lambda_j^*}. \tag{6.55}$$

Then the dynamics of (6.49) restricted to the attracting center manifold when $\mu = \mu_j^*, \nu = \nu_j^*$, is represented by

$$\begin{aligned} \dot{X} &= Y + g_{1j}X^3 + g_{2j}X^2Y + g_{3j}XY^2 + g_{4j}Y^3, \\ \dot{Y} &= g_{5j}X^3 + g_{6j}X^2Y + g_{7j}XY^2 + g_{8j}Y^3, \end{aligned} \tag{6.56}$$

where we have ignored the higher order terms. Using normal form theory, we can remove six of the eight nonlinear terms at cubic order. After transformation, (6.56) in normal form can be taken as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \alpha_j x^3 + \beta_j x^2 y, \end{aligned} \tag{6.57}$$

where

$$\begin{aligned} \alpha_j &= g_{5j}, \\ \beta_j &= g_{6j} + 3g_{1j}. \end{aligned} \tag{6.58}$$

The dynamics of this normal form has been discussed in [16]. Allowing μ, ν to vary near μ_j^* and ν_j^*, we obtain the unfolding

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \gamma_1 x + \gamma_2 y + \alpha_j x^3 + \beta_j x^2 y, \end{aligned} \tag{6.59}$$

where $\gamma_1 = -Det_{0,\pi}$ $\gamma_2 = Tr_{0,\pi}$. When $(\mu, \nu) = (\mu_j^*, \nu_j^*)$, the system (6.59) has a Bogdanov-Takens singularity at the origin and undergoes a codimension two bifurcation in a small neighbourhood of the origin, for μ, ν close to μ_j^*, ν_j^* i.e., (γ_1, γ_2) close to $(0, 0)$. See Figures 6.3 and 6.4. The shaded circular and square regions in these figures correspond to two different case of bifurcations.

System (6.59) can further be simplified. Using the scalings

$$\begin{aligned} x &= -\left(\sqrt{|\alpha_j|}/\beta_j\right)\bar{x}, \quad y = \left(|\alpha_j|^{3/2}/\beta_j^2\right)\bar{y}, \quad t = -\left(\beta_j/|\alpha_j|\right)\bar{t}, \\ \gamma_1 &= \left(\alpha_j/\beta_j\right)^2 \bar{\gamma}_1, \quad \gamma_2 = -\left(|\alpha_j|/\beta_j\right)\bar{\gamma}_2, \end{aligned} \tag{6.60}$$

then dropping the "bars", (6.59) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \gamma_1 x + \gamma_2 y + sgn(\alpha_j) x^3 - x^2 y, \end{aligned} \tag{6.61}$$

Depending on the sign of

$$\alpha_j = -a_1 \left\{ 2\Delta_{1,j}[C_{6R}] + \Delta_{2,j}[B_{1I} + C_{6I}] + (\Delta_{1,j}^2/\Delta_{2,j})[B_{1I} - C_{6I}] \right\}, \tag{6.62}$$

for each j we will have two distinct cases (up to time reversal). In the above expression for α_j we assumed the nondegeneracy condition

$$\beta_j = 3g_{1j} + g_{6j} \neq 0, \tag{6.63}$$

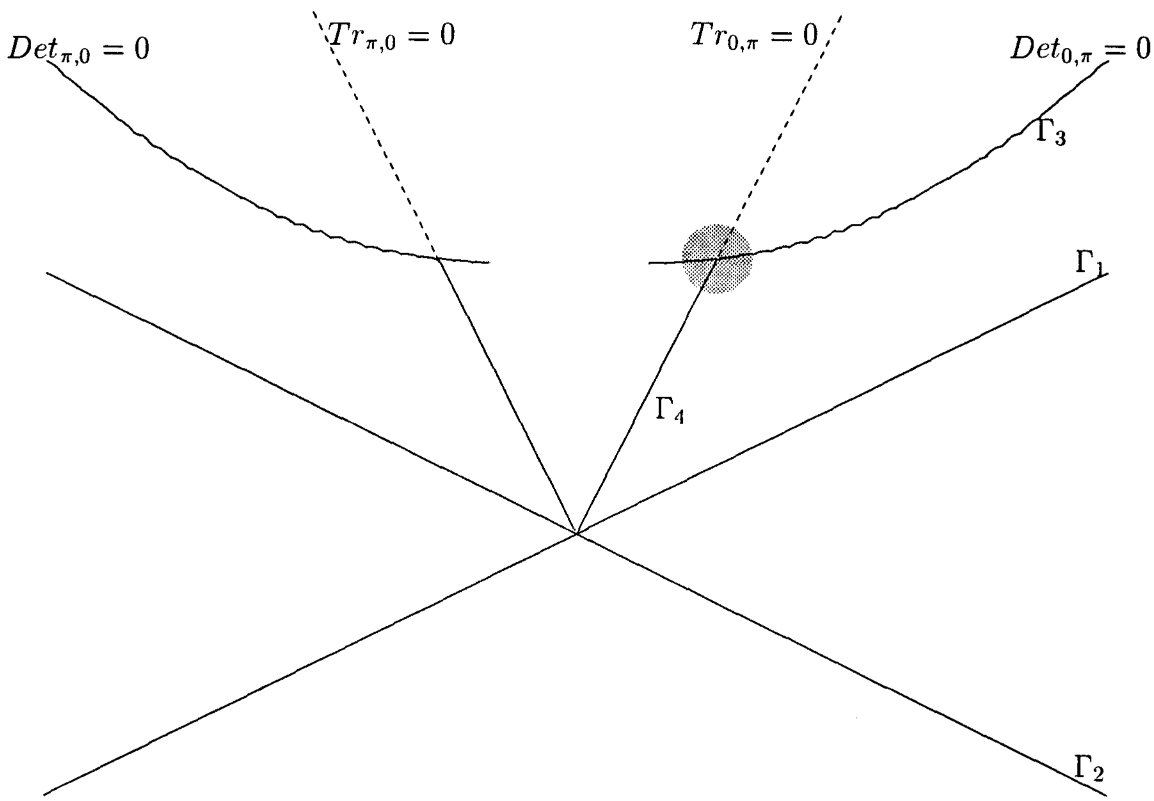


Figure 6.3: Parameters as in Figure 6.1. The circular shaded region shows the parameter values that correspond to our bifurcation analysis.

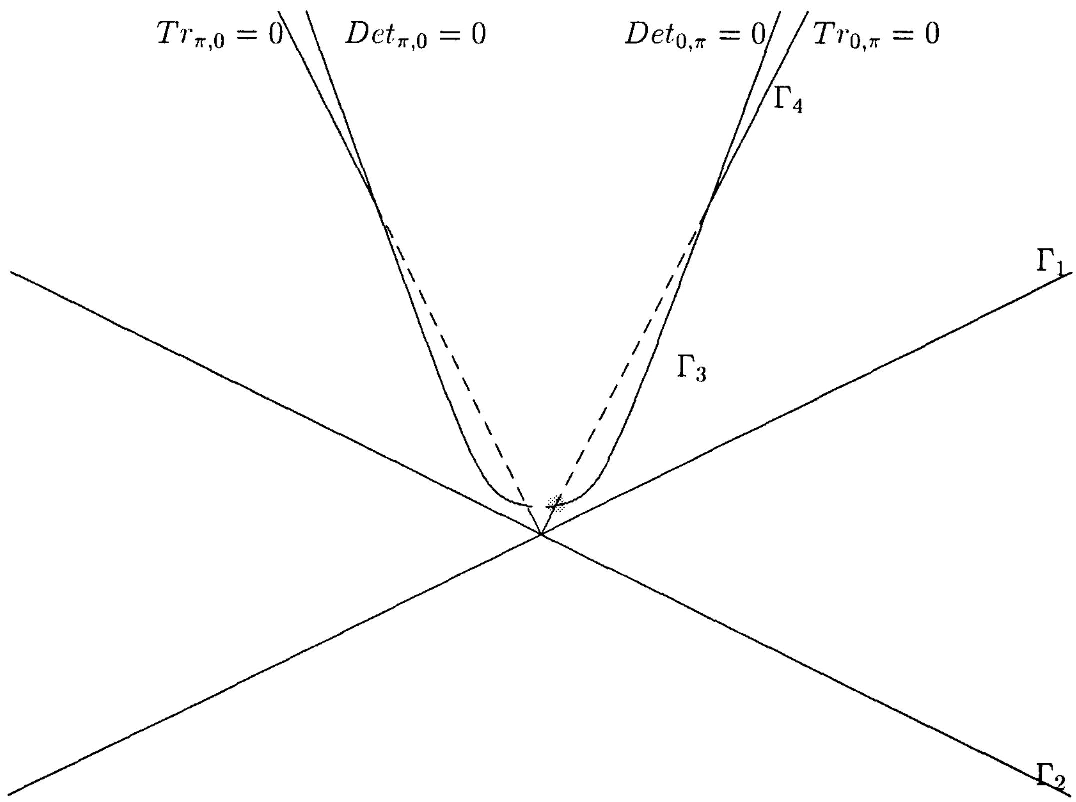


Figure 6.4: Parameters as in Figure 6.2. The circular and square shaded regions show the parameter values that correspond to our bifurcation analysis.

where

$$
\begin{aligned}
g_{1j} &= e_{1,j}\left\{-(\Delta_{1,j}/\Delta_{2,j})[B_{1I} - C_{6I}] + [B_{4R} - C_{6R}]\right\}, && (6.64)\\
g_{6j} &= e_{1j}\left[B_{4R} + C_{6R} + (\Delta_{1j}/\Delta_{2j})[B_{1I} - C_{6I}] + (e_{2j}/e_{1j})\alpha_j\right. && (6.65)
\end{aligned}
$$

From (6.60) we observe that the sign of β_j determines the orientation of t, and hence affects stability types. If $\beta_j > 0$, there will be time reversal and the sign of γ_2 will be changed, however if $\beta_j < 0$, then system (6.59) and (6.61) have the same dynamics. Also, from (6.62) and (6.63) it is clear that α_j and β_j depend only on e_{1j} and e_{2j}, so an explicit calculation of e_{3j} is not necessary. The calculation of e_{2j} is necessary for the nondegeneracy condition and stability type of the solutions, but the calculation of e_{1j} is crucial to determine the dynamics. Because of the complicated form of β_j we could not find simple expression for the nondegeneracy condition in terms of the original magnetoconvection problem. However, for given set of magnetoconvection parameter it is very simple to check the condition numerically (see Table 6.2).

To calculate e_{1j} we note that when $Tr_{0,\pi} = Det_{0,\pi} = 0$ we have

$$
\begin{aligned}
\Delta_{1j} &= -W_0 C_{6R}, && (6.66)\\
\Delta_{2j} &= W_0[\pm|C_6| - C_{6I}]. && (6.67)
\end{aligned}
$$

Now let

$$\Delta_{3j} = -\left[\hat{\omega} + \hat{a}_I\mu + \hat{b}_I\nu + W_0[B_{1I} - C_{6I}\right]. \quad (6.68)$$

After simplication at $(\mu,\nu) = (\mu_j^*, \nu_j^*)$, we have

$$\Delta_{3j} = -W_0[\pm|C_6| + C_{6I}]. \quad (6.69)$$

At this parameter value $\Delta_{1j}^2 = -\Delta_{3j}\Delta_{2j}$, therefore after simplification

$$\alpha_j = 2e_{1j}W_0[\pm|C_6|]\left[\pm|C_6| - (B_{1I})\right], \quad (6.70)$$

Table 6.2: Values of α_j, β_j for $\sigma = 1, \hat{\zeta} = .1, m = 10^7$, and different values of $\hat{Q}$. (Since for $\hat{Q} = 100\pi^2$ there is only one bifurcation point for $\nu > 0$, the values for α_2 and β_2 are left blank.)

$\hat{Q}$	α_1	α_2	$m^{-2}\beta_1$	$m^{-2}\beta_2$
$100\pi^2$	-5820	$-$	51604	$-$
$300\pi^2$	-7496	14.89	105551	-16.45
$500\pi^2$	-8233	46.73	129794	-139.8

where

$$e_{1j} = \frac{\Im(C_3 C_6) \pm |C_6|[B_{1R} - B_{2R}]}{2\lambda_j^*[\pm|C_6| - C_{6I}]}. \tag{6.71}$$

For m sufficiently large, $B_{1R} - B_{2R} = O(\epsilon)$ and therefore

$$\begin{aligned} sgn(\alpha_1) &= sgn(\Im(C^2)(|C| - B_I)), \\ sgn(\alpha_2) &= -sgn(\Im(C^2)(|C| + B_I)). \end{aligned} \tag{6.72}$$

For both $j = 1$ and $j = 2$, under the non-degeneracy condition $\beta_j \neq 0$,, we will have two different cases, according to the sign of α_j, which have been discussed in [16, §7.3]. Now we give a summary of their results and its implication for our problem. Along a 1-parameter path transversal to the line $\gamma_1 = 0$ there will be a subcritical (if $\alpha_j > 0$) or supercritical (if $\alpha_j < 0$) pitchfork bifurcation as two other fixed points will bifurcate from the trivial solution. Also, by the Bendixson criterion there is no periodic orbit when $\gamma_2 < 0$. Along a 1-parameter path transversal to Γ_4 ($\gamma_2 = 0$, $\gamma_1 > 0$) there is a Hopf bifurcation of periodic orbits from the trivial solution.

Case a ($\alpha_j > 0$): The periodic orbit created by the Hopf bifurcarion is destroyed in a heteroclinic (saddle connection) bifurcation along any 1-parameter path transversal to

$$\Gamma_5 = \{(\gamma_1, \gamma_2) : \gamma_2 = -\gamma_1/5 + O(\gamma_1^2),\ \ \gamma_1 < 0\}. \tag{6.73}$$

Case b ($\alpha_j < 0$): In addition to the periodic orbit created by Hopf bifurcation along Γ_4, two other periodic orbits bifurcate from nontrivial fixed points $(\pm\sqrt{\gamma_1}, 0)$ in Hopf bifurcations along the curve

$$\Gamma_6 = \{(\gamma_1, \gamma_2) : \gamma_1 = \gamma_2, \ \ \gamma_1 > 0\}. \tag{6.74}$$

These two periodic orbits are destroyed while another periodic orbit will be created in a global homoclinic bifurcation, along any path transversal to

$$\Gamma_7 : \{(\gamma_1, \gamma_2) : \gamma_2 = (4/5)\gamma_1 + O(\gamma_1^2), \ \ \gamma_1 > 0\}. \tag{6.75}$$

The two hyperbolic periodic orbits coalesce into a non-hyperbolic periodic orbit and disappear in a saddle-node bifurcation of periodic orbits, along paths transversal to

$$\Gamma_8 = \{(\gamma_1, \gamma_2) : \gamma_2 = c\gamma_1 + O(\gamma_1^2), \ \ c \approx 0.752, \ \ \gamma_1 > 0\}. \tag{6.76}$$

To apply the above analysis to our problem we observe that fixed points in (6.61) correspond to periodic orbits in (6.1). In particular the origin in (6.61) corresponds to SW_0 periodic orbits, and non–trivial fixed points $(\pm\sqrt{\gamma_1}, 0)$ correspond to two further periodic orbits, which we denote by Ω_i, $i = 1, 2$. Periodic orbits in (6.61) correspond to invariant tori in (6.11). In fact, the periodic orbits that bifurcate from Γ_4 correspond to the invariant tori for which we have established existence in Chapter 5. In our case, since both α_j and β_j could be either positive and negative, several possibilities exist. In Figures 6.5–6.10 we have considered two of these cases. The other cases can be analyzed in the obvious way.

In parameter region I in Figure 6.7, the only periodic solution in a neighborhood of the SW_0 solution is SW_0 itself (the SW_π solution exists, but it is not close), and there are no invariant tori. Along the curve Γ_3 there will be a subcritical pitchfork bifurcation of periodic orbits Ω_i, $i = 1, 2$ from the SW_0 solution and they exist in regions II, III

and IV. Along the curve Γ_4 invariant tori bifurcate from the SW_0 solutions. These are unstable (of saddle type) and exist in region III. (These are the same tori whose existence was proved in Chapter 5). Along Γ_5 there is a heteroclinic manifold between the periodic orbits Ω_i and as we cross Γ_5 the invariant tori are destroyed in a global heteroclinic bifurcation. See Figure 6.5, 6.6 and 6.7.

In Figure 6.8 the only periodic orbits near the SW_0 solutions are SW_0 solutions themselves, for parameters in region I. Invariant tori denoted by T_1 are created along the curve Γ_4 in Hopf bifurcations. These invariant tori correspond to those for which we established existence in Chapter 5, and they exist in regions II, III, IV and V. Periodic orbits Ω_i bifurcate from SW_0 solutions in pitchfork bifurcations, for parameters on Γ_3. These periodic orbits exist in every parameter region except regions I and II. Two invariant tori denoted by T_2 and T_3 bifurcate from Ω_i, $i = 1, 2$ respectively, in Hopf bifurcations for parameters on Γ_6. These invariant tori exist for parameters in region IV, and are destroyed, while another family of invariant tori, denoted by T_4 are created in a global bifurcation along the curve Γ_7. Hyperbolic invariant tori T_1 and T_4 persist until for parameters in region V they coalesce into non-hyperbolic invariant tori for parameters along Γ_8 and then disappear in a saddle-node bifurcation of tori. See Figures 6.9-6.13.

We expect that if we restore the higher order terms to our truncated equations (6.12) most of the above behaviour persists. Since Hopf and pitchfork bifurcations of periodic orbits are structurally stable, using similar methods as we did in Chapter 5, we could prove that for a curve $\tilde{\Gamma}_3$ close to Γ_3 (see [16, theorem 4.3.1]) periodic orbits $\tilde{\Omega}_i$, $i = 1, 2$ for (6.11) bifurcate from SW_0 solutions in a pitchfork bifurcations of periodic orbits, since the higher order terms respect the required symmetry. Also, there are Hopf bifurcations of periodic orbits about curves $\tilde{\Gamma}_4$ and $\tilde{\Gamma}_6$ close to Γ_4 and Γ_6, and invariant hyperbolic tori will bifurcate from $SW_0, \tilde{\Omega}_1, \tilde{\Omega}_2$. These invariant tori will have the same stability types as their corresponding periodic orbits of the truncated system (6.12).

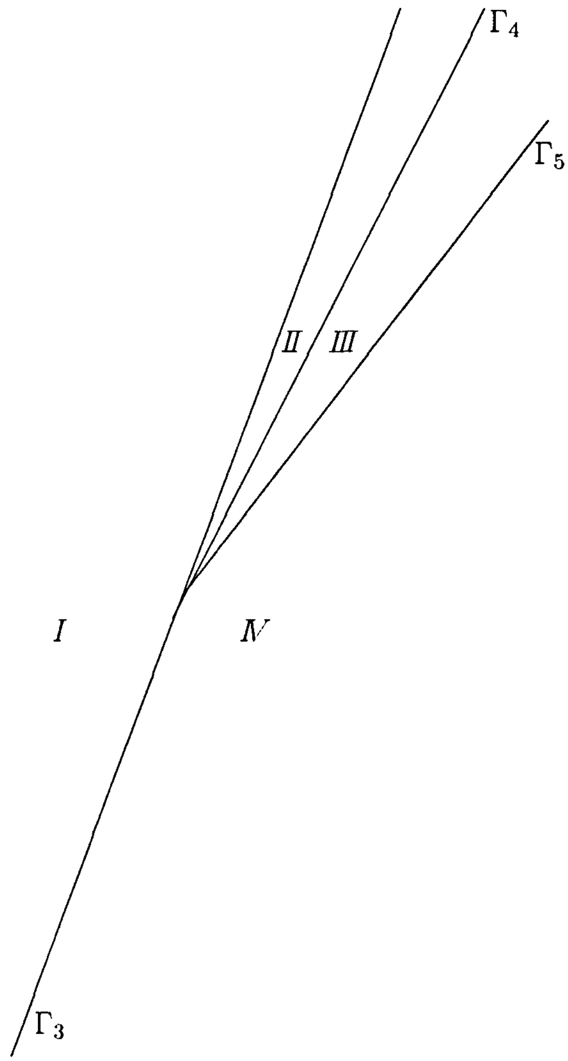


Figure 6.5: Magnification of the square region in Figure 6.4 with $\alpha_2 > 0, \beta_2 < 0$ (Case a).

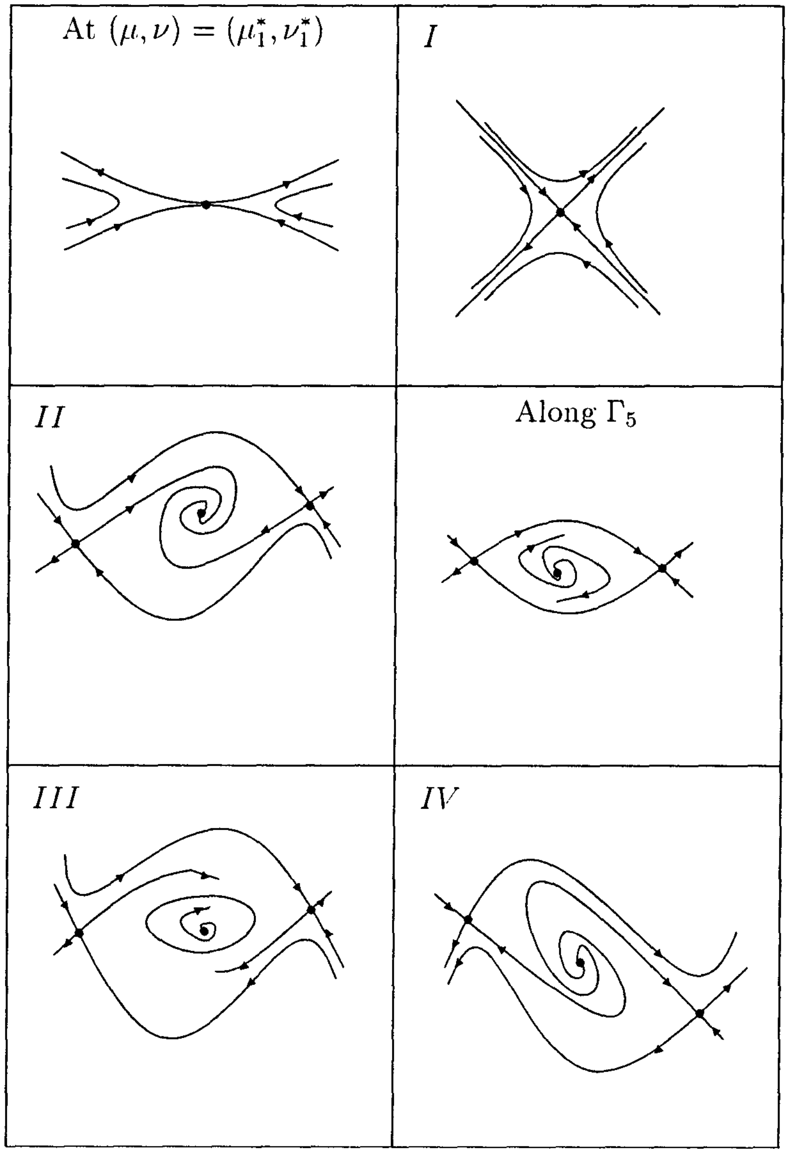
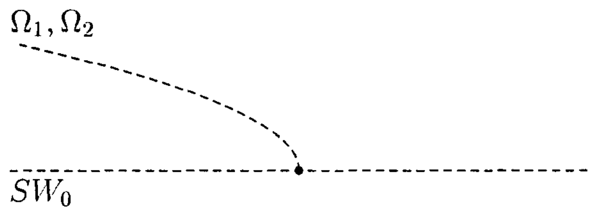
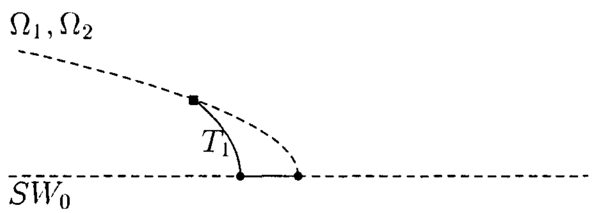


Figure 6.6: Phase portraits for regions I-IV and along Γ_5 of Figure 6.5 and at $(\mu, \nu) = (\mu_1^*, \nu_1^*)$ of equation (6.60) with $\alpha_2 > 0$ and $\beta_2 < 0$.



$(a)\ \nu < \nu_2^*$



$(b)\ \nu > \nu_2^*$

Figure 6.7: Bifurcation diagrams for Figure 6.5, corresponding to one-parameter paths obtained by increasing μ, for fixed ν. Dots represent local bifurcations and the rectangle represents a global (heteroclinic) bifurcation. Solid lines represent stable solutions and dashed lines represent unstable solutions.

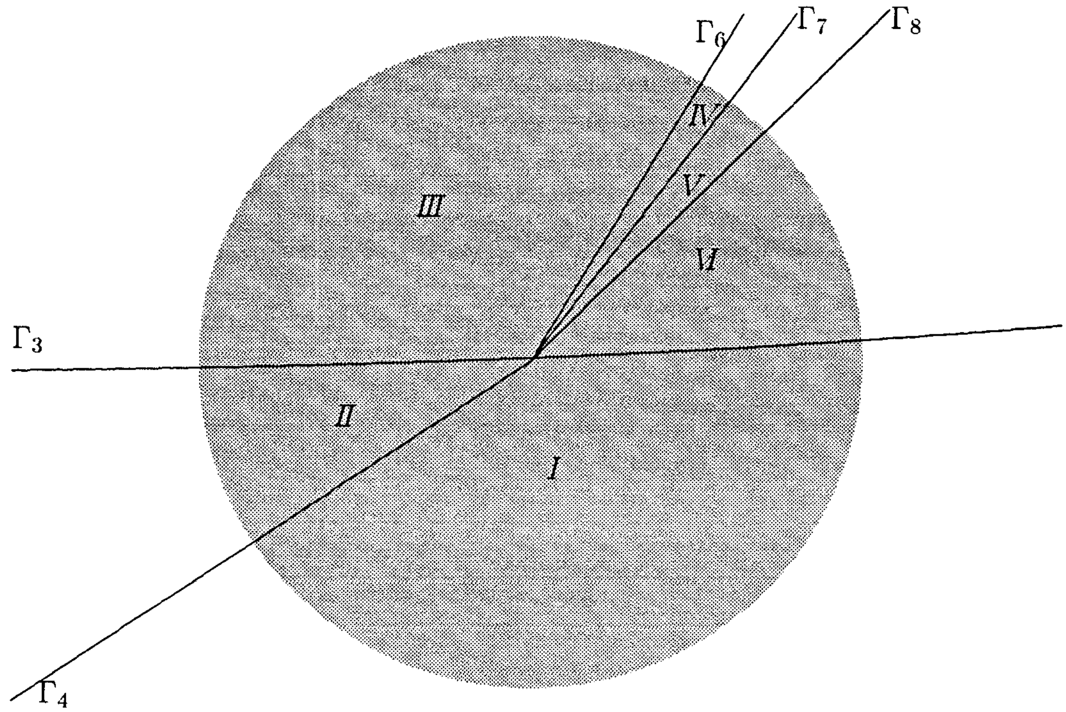


Figure 6.8: Magnification of the circular regions in Figures 6.3 and 6.4 with $\alpha_1 < 0, \beta_1 > 0$ (Case b).

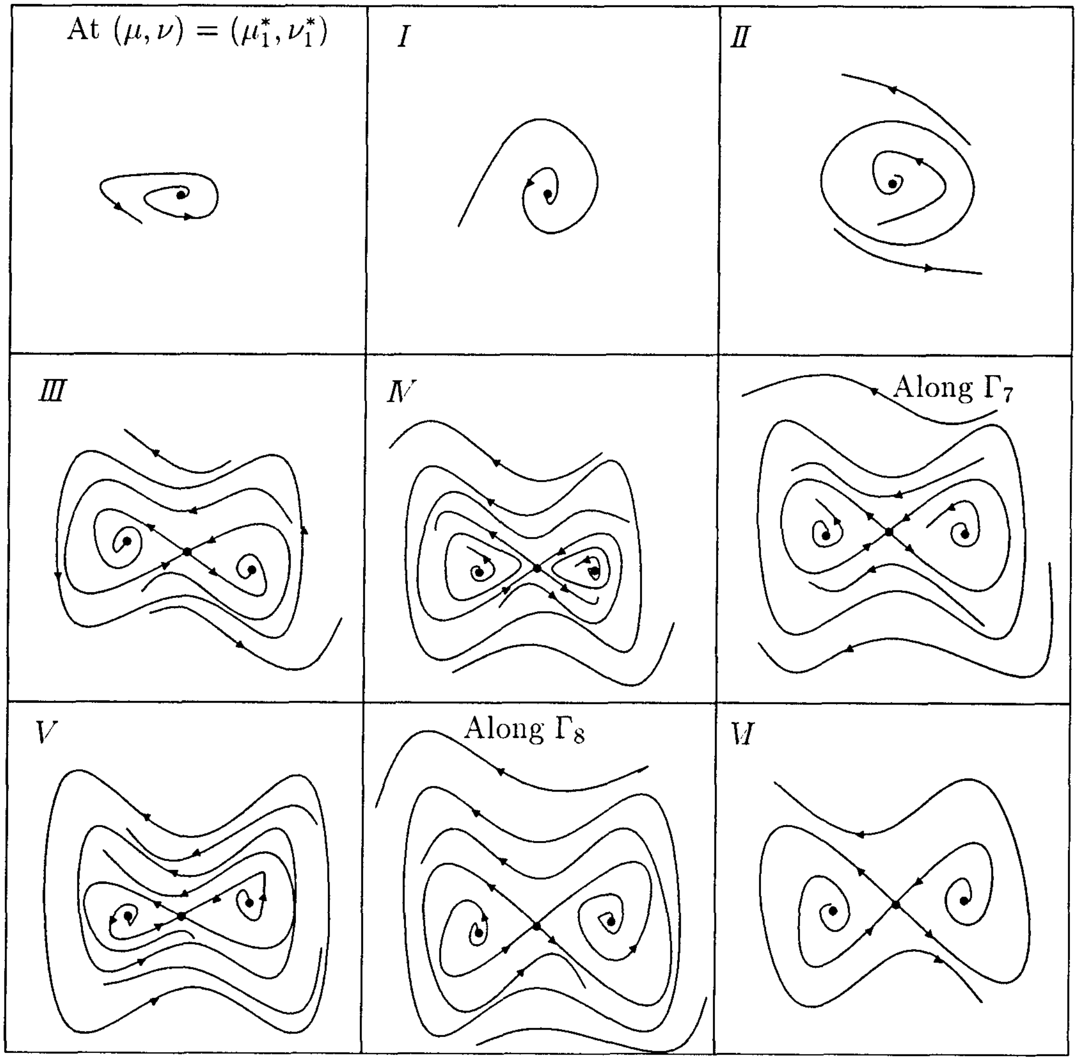
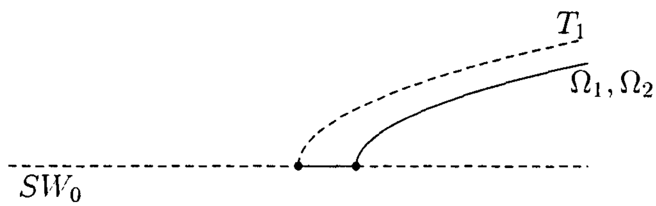
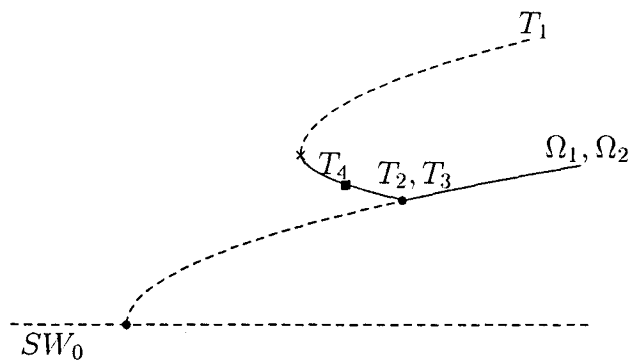


Figure 6.9: Phase portraits for regions I-VI and along Γ_7, Γ_8 of Figure 6.8, and at $(\mu, \nu) = (\mu_1^*, \nu_1^*)$ of equation (6.60) with $\alpha_1 < 0$ and $\beta_1 > 0$.



$(a)\ \nu < \nu_1^*$



$(b)\ \nu > \nu_1^*$

Figure 6.10: Bifurcation diagrams for Figure 6.8 corresponding to one-dimensional paths obtained by increasing μ, for fixed ν. Dots represent local bifurcations, the rectangle represents a global homoclinic bifurcation and the cross represents the saddle-node bifurcation of periodic orbits. Solid lines represent stable solutions and dashed lines represent unstable solutions.

We now indicate some differences that would be expected due to the Ψ-dependence of higher-order terms, which were neglected in the truncated normal form. When the Ψ-dependence is restored, the phase portraits of Figures 6.6 and 6.9, represent approximate Poincaré maps for (6.11), restricted to two-dimensional, invariant manifolds. The curves Γ_5 in Figure 6.6 and Γ_7 in Figure 6.9 correspond to heteroclinic and homoclinic manifolds, but for maps such behavior is nongeneric. It is known that transverse heteroclinic and homoclinic points (and consequently chaos) will exist generically in exponentially thin wedges in the (μ, ν) parameter plane, near the curves Γ_5 and Γ_7, i.e., for generic higher-order terms these two curves will be replaced by exponentially thin parameter regions $\tilde{\Gamma}_5$ and $\tilde{\Gamma}_7$ corresponding to the existence of transverse heteroclinic and homoclinic orbits, and away from these regions no such orbits exist [21]. Another situation where the Ψ-dependence of higher-order terms would be expected to affect dynamics is for parameters near the curve Γ_8, which correspond to saddle-node bifurcation of tori. For generic higher order terms depending on Ψ, the curve Γ_8 will be replaced by a Cantor set $\tilde{\Gamma}_8$ that corresponds to quasi-periodic saddle-node bifurcations of invariant tori [31, Theorem 1.1]. Moreover, it can be expected that near $\tilde{\Gamma}_8$ there exist open sets of parameter values (called "bubbles" in [4]) that correspond to resonant and chaotic behavior.

6.4 Approximate D_4 symmetry

Using the asymptotic results of Chapter 4 on the normal form coefficients as $m \to \infty$ in Case II, we observe that (6.3) approaches a small perturbation of a system with D_4 symmetry,

$$\begin{aligned} \dot{U}/2 &= U[a_R\mu + r(2A_R + B_R + C_R)/2] - VW(BI + C_I)/2 \\ &\quad + O(r^3 + |\mu|r^2 + |\mu|^2 r) \qquad (6.77) \\ \dot{V}/2 &= V[a_R\mu + r(2A_R + B_R - C_R)/2] + UW(BI - C_I)/2 \end{aligned}$$

$$
\begin{aligned}
& \quad + O(r^3 + |\mu| r^2 + |\mu|^2 r) \\
\dot{W}/2 &= W[a_R \mu + r(A_R + B_R)] + UVC_I + O(r^3 + |\mu| r^2 + |\mu|^2 r) \\
\dot{\Psi}/2 &= P\omega + O(r).
\end{aligned}
$$

Ignoring higher-order terms, system (6.77) has an additional symmetry generated by

$$(U, V, W, \Psi) \xrightarrow{\gamma} (U, -V, -W, \Psi), \tag{6.78}$$

which corresponds to

$$(Z_1, \bar{Z}_1, Z_2, \bar{Z}_2) \xrightarrow{\gamma} (Z_2, \bar{Z}_2, Z_1, \bar{Z}_1) \tag{6.79}$$

in (6.1). Using (3.45) this corresponds to the symmetry

$$(\Phi_1, \bar{\Phi}_1, \Phi_2, \bar{\Phi}_2) \xrightarrow{\gamma} (\Phi_2, \bar{\Phi}_2, \Phi_1, \bar{\Phi}_1) \tag{6.80}$$

in the original magnetoconvection equation. Using (2.21) and (2.22), this in turn corresponds to a fixed translational symmetry in spatial variable

$$\gamma\Phi(x, y, t) = \Phi(x - \lambda/2, y, t), \tag{6.81}$$

where $\lambda = \lim_{m\to\infty}(2L_m/m)$ satisfies (2.27). Therefore for large m in Case II we can consider our magnetoconvection problem with sidewalls as corresponding to a small perturbation of magnetoconvection in an infinite layer with the D_4 symmetry generated by the actions of γ, β and J, if we identify $\Phi(x, y, t)$ with $\Phi(x + 2\lambda, y, t)$.

System (6.77) has been studied by Swift [41]. In system (6.77), there are at least three invariant subspaces $V = W = 0$, $U = W = 0$ and $U = V = 0$. In each of these subspaces there are two pairs of periodic orbits that are denoted by "U", "V" and "W" solutions respectively. The periodic solutions $\tilde{W}_0(\psi, \mu, \nu)$ and $\tilde{W}_\pi(\psi, \mu, \nu)$ of our perturbed system correspond to the pair of "W" solutions of the unperturbed D_4-symmetric system, and agree with them at the lowest order in ϵ. Using the implicit function theorem, we can prove the existence of periodic solutions corresponding to "U" and "V" solutions for sufficiently small ϵ under the following nondegeneracy conditions. The perturbed "U" solutions exist if

$$|C|^2 - \Re(B\bar{C}) \neq 0, \tag{6.82}$$

and they have the form

$$(\tilde{U}_i(\psi,\mu,\nu,\epsilon), \tilde{V}_i(\psi,\mu,\nu,\epsilon), \tilde{W}_i(\psi,\mu,\nu,\epsilon)) = (\pm U_0 + O(\epsilon), O(\epsilon), O(\epsilon)),\ i = 1,2$$

where

$$|U_0| = \frac{-2a_R\mu}{2A_R + B_R + C_R},\ sgn(\mu) = -sgn(2A_R + B_R + C_R).$$

Similarly, the perturbed "V" solutions exist if

$$|C|^2 + \Re(B\bar{C}) \neq 0, \tag{6.83}$$

and they have the form

$$(\hat{U}_i(\psi,\mu,\nu,\epsilon), \hat{V}_i(\psi,\mu,\nu,\epsilon), \hat{W}_i(\psi,\mu,\nu,\epsilon)) = (O(\epsilon), \pm V_0 + O(\epsilon), O(\epsilon)),\ i = 1,2,$$

where

$$|V_0| = \frac{-2a_R\mu}{2A_R + B_R - C_R},\ sgn(\mu) = -sgn(2A_R + B_R - C_R).$$

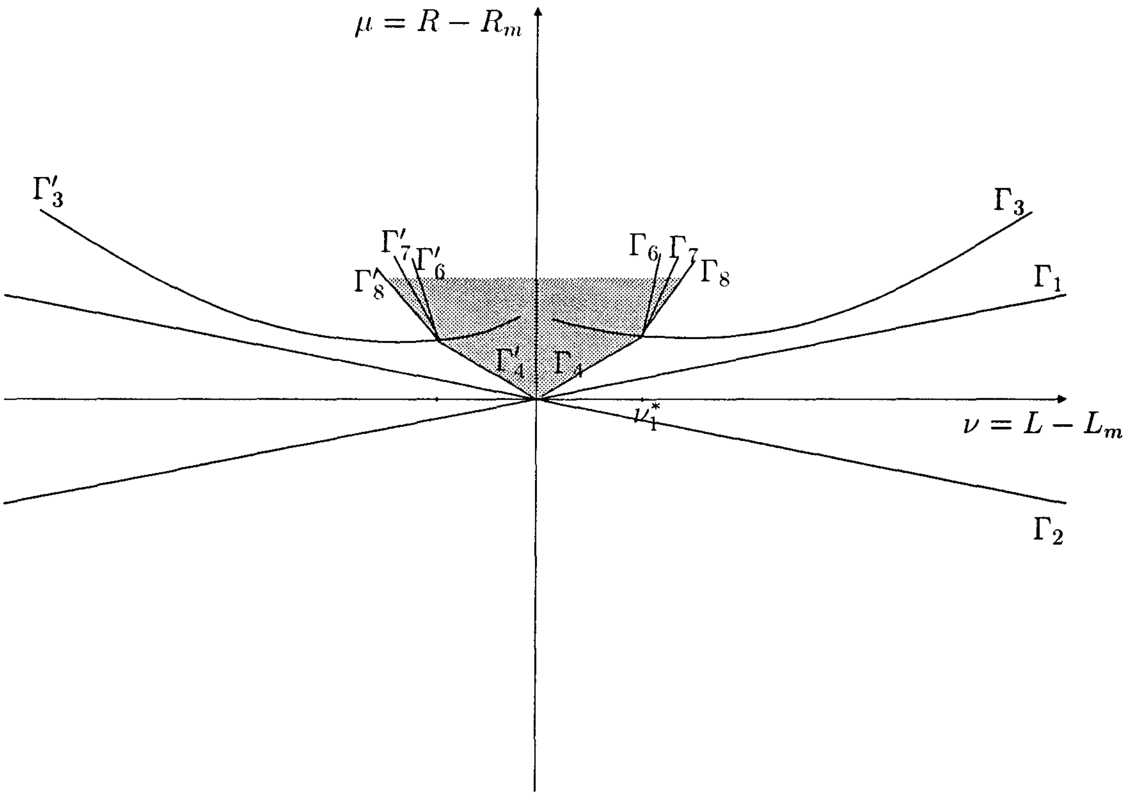
These solutions have the same stability type as their corresponding unperturbed solutions. If the nondegeneracy conditions (6.82) and (6.83) are satisfied, no nonsymmetric solutions bifurcate from "U" and "V" solutions (6.77). In the unperturbed case two non-symmetric solutions bifurcate from the "W" solutions in a pitchfork bifurcation as we crossed the parametric surface $|C|^2 - |B|^2 = 0$.

Chapter 7

Conclusion

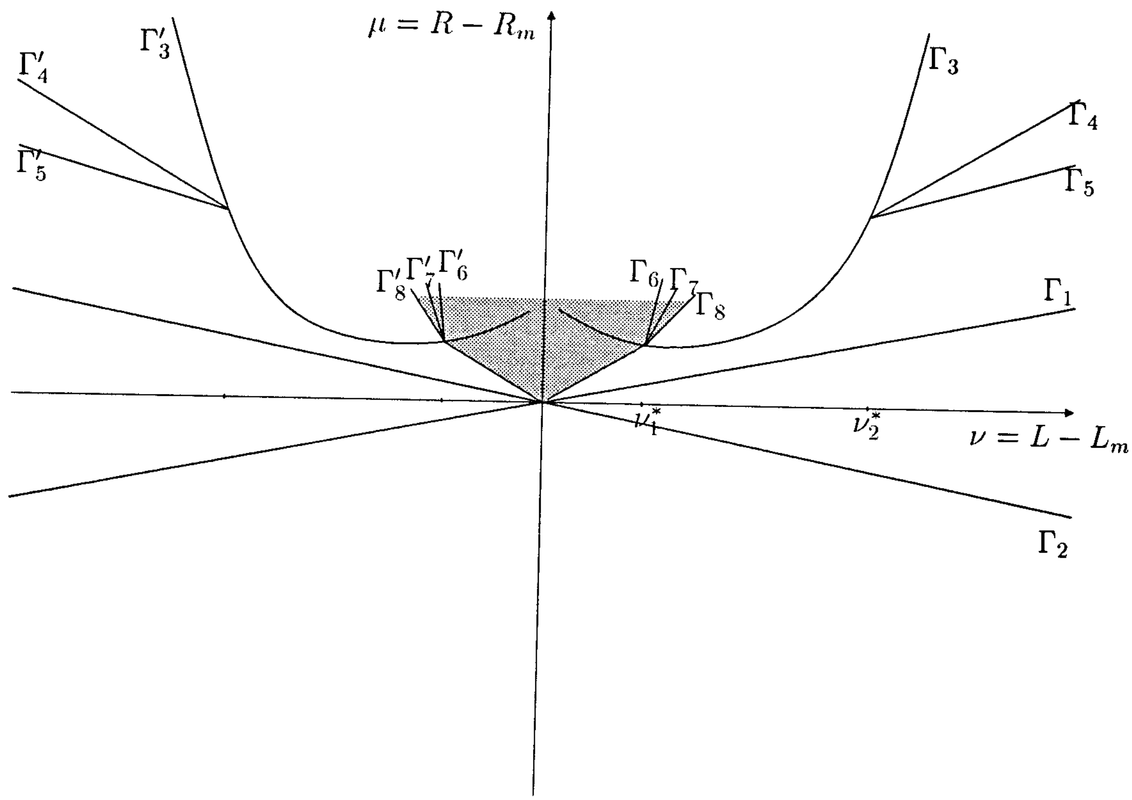
In this chapter we summarize the results of Chapters 5 and 6, and also make some remarks on our magnetoconvection problem. For fixed magnetoconvection parameters σ, ζ and Q, we have proved that the motionless conduction state loses its stability as we increase the Rayleigh number R. For L close to one of the $L_m, m = 1, 2, \ldots$, two standing wave solutions, which we denote by SW_0 and SW_π solutions, bifurcate in primary Hopf bifurcations of periodic orbits, and the SW solutions that bifurcate at the lower value of R are asymptotically stable, and the other SW solutions are unstable. As we increase R further there occurs a secondary Hopf bifurcation of invariant tori (e.g. unstable quasiperiodic solutions), which we denote by T_1 solutions, from the branch of unstable SW solutions. After the T_1 solutions bifurcate, both SW solutions are stable. We have also proved that the tori T_1 persist in wedges in the parametric plane.

By considering a decreasing sequence in ζ and an increasing sequence in Q, for fixed σ, we were able to extend the regions of validity of our bifurcation analysis by considering an alternate normal form. We showed the existence of Bogdanov-Takens singularities, and these codimension two singularities lead to more complicated dynamics such as further secondary and tertiary bifurcations of invariant tori and generically some open parameter regions corresponding to chaos. Figure 7.1 shows schematically the regions in the parametric plane for which our bifurcation analysis of Case II is valid. The curves $\Gamma_1, \ldots, \Gamma_8$ are the same as Chapter 6, and by reflection symmetry there are curves $\Gamma_3', \ldots, \Gamma_8'$ corresponding to bifurcations about the SW_π solutions. Note that the curves Γ_4 and Γ_4'



(a)

Figure 7.1: Schematic bifurcation sets for magnetoconvection equations in Case II, for L near L_m and R near R_m: (a) when Γ_3 and Γ_4 only intersect once ($\alpha_1 < 0, \beta_1 > 0$); (b) when Γ_3 and Γ_4 intersect twice ($\alpha_1 < 0, \beta_1 > 0$ and $\alpha_2 > 0$ and $\beta_2 < 0$). The shaded regions show parameter values for which the invariant tori T_1 occur.



(b)

Figure 7.1 (continued).

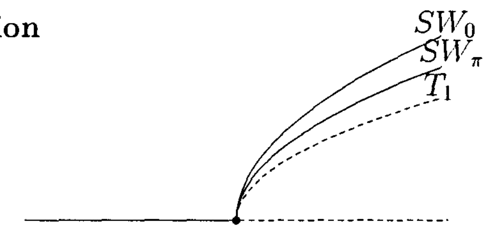
near the origin correspond to the curves Λ_1 and Λ_2 in Chapter 5.

In Figure 7.2 we give bifurcation diagrams for fixed $\nu = L - L_m \geq 0$, as we increase $\mu = R - R_m$. The bifurcation diagrams for $\nu < 0$ are the same as those for $\nu > 0$ if we interchange the roles of SW_0 and SW_π solutions. If we compare the bifurcation diagrams in Figure 7.2(a, b) with Figure 5.6(b, c), we observe that they are the same, since for ν sufficiently small the analysis of Chapter 5 is valid. However, for ν near ν_1^* we have more bifurcations. Also, we observe that the bifurcation diagrams in Figure 7.2(b) and (c), and also Figure 7.2(d) and (e), are rather different. This implies the existence of more global or local bifurcations for ν between 0 and ν_1^* and between ν_1^* and ν_2^*. It seems very hard to locate these bifurcations analytically, but it should be possible to get some results using numerical methods and computer software packages like AUTO [11] to locate some of these bifurcations. AUTO is able to follow branches of periodic solutions and analyze changes in their Floquet multipliers, as we change parameters. Also, we note that as long as the point (μ, ν) in parameter plane lies below the curve of Γ_4, for $\nu > 0$, the SW_0 solution is unstable, which is in agreement with our primary bifurcation results. For large m, the slopes of the curves $\Gamma_1, ..., \Gamma_8$ are $O(m^{-2})$. Therefore all of these curves lie very close to the line $\mu = 0$, and regions where our bifurcation analysis applies become small as $m \to \infty$.

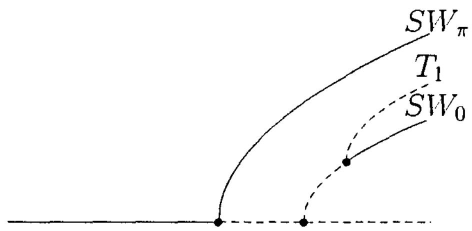
We have also shown that in the Case II limit as $m \to \infty$ our system becomes a small perturbation of a system with D_4 symmetry. Using this fact we found other periodic solutions.

This project by no means complete and it can be continued in number of ways. First, the results of Chapter 6 can be continued by proving that the results we have persist after restoring the higher-order terms (and therefore Ψ-dependence) in the equations. There are some technical difficulties in achieving this, but in principle it could be done. Also, one could attempt to complete the bifurcation sets in Figure 7.1 as we mentioned above.

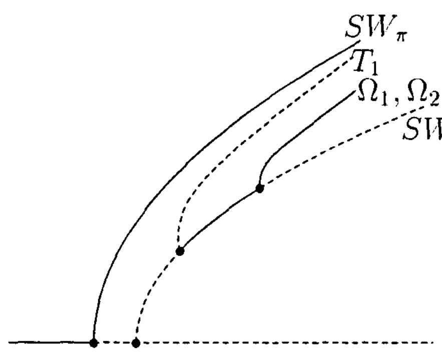
- **local bifurcation**



$(a)\ \nu = 0$

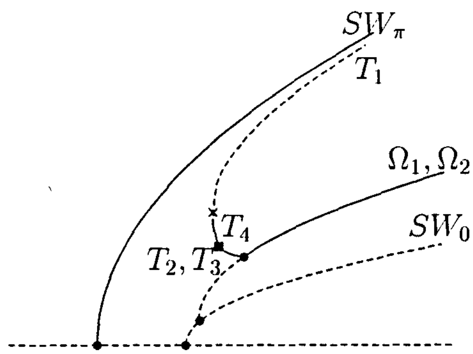


$(b)\ \nu > 0, \nu$ near 0

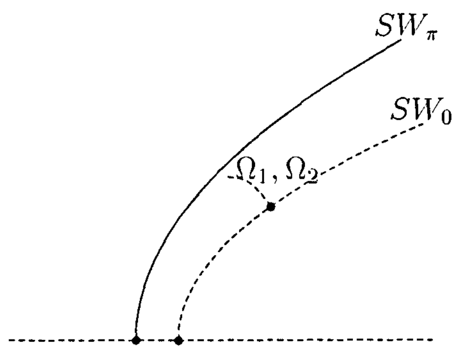


$(c)\ 0 < \nu < \nu_1^*,\ \nu$ near ν_1^*

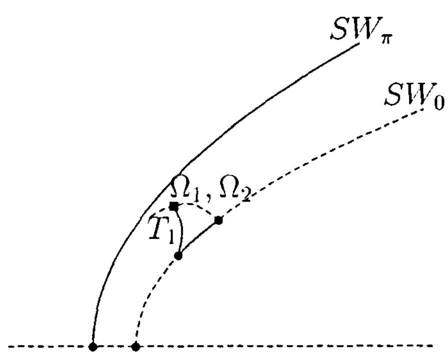
Figure 7.2: Bifurcation diagrams for magnetoconvection equations in Case II.



(d) $\nu > \nu_1^*$, ν near ν_1^*



(e) $\nu_1^* < \nu < \nu_2^*$, ν near ν_2^*



(f) $\nu > \nu_2^*$, ν near ν_2^*

Figure 7.2 (continued).

One possibility is to try to exploit further the small perturbations from D_4 symmetry.

In our bifurcation analysis, we restricted to those magnetoconvection parameter values for which the Hopf bifurcation was preferred. However, as it was mentioned in Chapter 2, there are parameter values for which steady state bifurcation is preferred, and also critical parameter values for which the linearization of the convection problem has double zero eigenvalues. The bifurcation analysis at these higher codimension singularities would be very interesting but would need another thesis for a complete analysis.

We simplified our original magnetoconvection problem in number of ways. First we restricted ourselves to two-dimensional flows by assuming the velocity, temperature and magnetic field remain constant in third z direction. In three dimensions, depending on the shape of the container, there could be more than two different spatial Hopf modes for some parameter values. This would increase the dimension of reduced ordinary differential equation and therefore a more complicated analysis would be needed. The calculation of center manifold coefficients would also be much longer.

We adopted boundary conditions that made our eigenfunction calculations possible by hand. If we adopted different boundary conditions, the computation of the eigenfunctions for the linearized magnetoconvection equations would take long hours of computer programming and numerical calculations. The center manifold coefficients would also need to be computed numerically. Also, we chose our boundary conditions so that the system has $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ symmetry. This symmetry can be easily broken by perturbations that does not respect the symmetry and might correspond to more physically more realistic situations. For example, it can be arranged that (2.5) satisfy boundary conditions of the form

$$\begin{aligned}\tilde{T} &= T_0[1+\tilde{\theta}(x)] \quad \text{on } y=0,\\ \tilde{T} &= T_1 \quad \text{on } y=h.\end{aligned}$$

where $\tilde{\theta}$ is not an even function of x, so that there is no longer a reflection symmetry under $x \to -x$.

Bibliography

[1] Arnold, V.I. [1983]. *Geometrical Methods in the Theory of Differential Equations.* Springer-Verlag, New York.

[2] Bedford, T., Swift, J., eds. [1988]. *New Directions in Dynamical Systems.* Cambridge University Press.

[3] Chandrasekhar, S. [1961]. *Hydrodynamic and Hydromagnetic Stability.* Oxford University Press, Oxford.

[4] Chenciner, A. [1988]. Bifurcation de points fixes elliptiques, III. Orbites périodiques de 'petites' périodes et élimination résonnante des couples de courbes invariantes. *Publ. Math. IHES*, **66**, 5-91.

[5] Chow, S.N., Hale, J.K. [1982]. *Methods of Bifurcation Theory.* Springer-Verlag, New York.

[6] Crawford, J.D., Knobloch,E. [1988]. Degenerate Hopf bifurcation with broken $O(2)$ symmetry. *Nonlinearity,* **1**, 617–652.

[7] Dangelmayr, D., Knobloch, E. [1991]. Hopf bifurcation with broken circular symmetry. *Nonlinearity*, **2**, 485–523.

[8] Daniels, P.G. [1978]. The effect of distant sidewalls effect on the transition to finite amplitude Bénard convection. *Mathematika*, **25**, 216–235.

[9] Daniels, P.G. [1977]. Asymptotic sidewall effects in rotating Bénard convection. *J. Appl. Math. Phys. (ZAMP)*, **28**, 575–584.

[10] Devaney, R.L. [1989] *An Intoduction to Chaotic Dynamical Systems.* Addison-Wesley, New York.

[11] Doedel, E.J., Kernevez, J.P. [1986]. *AUTO: Software for Continuation and Bifurcation Problems in Ordinary Differential Equations.* Applied Mathematics Report, California Institute of Technology, Pasadena.

[12] Gibson, R.D. [1966]. Overstability in the magnetohydrodynamic Bénard problem at large Hartmann numbers. *Proc. Camb. Phil. Soc.*, **62**, 287-299.

[13] Golubitsky, M., Schaeffer, D.G. [1985]. *Singularities and Groups in Bifurcation Theory, Vol. I.* Springer-Verlag, New York.

[14] Golubitsky, M., Stewart, I. [1986]. Hopf bifurcation with dihedral coupled non-linear oscillators. *Contemporary Mathematics*, **56**, 131–173.

[15] Golubitsky, M., Stewart, I., Schaeffer, D.G. [1987]. *Singularities and Groups in Bifurcation Theory, Vol. II.* Springer-Verlag, New York.

[16] Guckenheimer, J., Holmes, P. [1983]. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, New York.

[17] Guillemin, E.A. [1949]. *The Mathematics of Circuit Analysis.* Wiley, New York.

[18] Hale, J. [1969]. *Ordinary Differential Equations.* Wiley, New York.

[19] Hassard, B.D., Kazarinoff, N.D., Wan,Y.H. [1980]. *Theory and Applications of the Hopf Bifurcation.* Cambridge University Press, Cambridge.

[20] Henry, D. [1981]. *Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Mathematics, **840**. Springer-Verlag, Berlin-Heidelberg..

[21] Holmes, P., Marsden, J., Scheurle, J. [1988]. Exponentially small splittings of separatrices with applications to K.A.M.theory and degenerate bifurcations. *Contemporary Mathematics*, **81**, 213–244.

[22] Iooss, G., Lozi, R. [1977]. Convection entre deux plaques planes en rotation, et effet dynamo résultant d'une bifurcation secondaire. *Journal de Mécanique,* **16**, 675–703.

[23] Kato, T. [1981]. *Perturbation Theory for Linear Operators.* Springer-Verlag, New York.

[24] Kidachi, H. [1982]. Sidewall effect on pattern formation of the Rayleigh-Bénard convection. *Progress of Theoretical Physics*, **68**, 49–63.

[25] Knobloch, E., Weiss, N.O., Da Costa, L.N. [1981]. Oscillatory and steady convection in a magnetic field. *J. Fluid Mech.*, **113**, 153–186.

[26] Ladyzhenskaya, O.A. [1985]. *The Boundary Value Problems of Mathematical Physics.* Springer-Verlag, New York.

[27] Lions, J.L., Magenes, E. [1972]. *Nonhomogeneous Boundary Value Problems and Applications*, Springer-Verlag, New York.

[28] Lorenz, E.N. [1963]. Deterministic non-periodic flow. *J. Atmos. Sci.*, **20**, 130–141.

[29] Mallet-Paret, J., van Gils, S.A. [1986]. Hopf bifurcation and symmetry: Travelling and standing waves on the circle. *Proc. Roy. Soc. Edinburgh*, **104A**, 279–307.

[30] Nagata, W. [1993]. Non-hyperbolic invariant tori in a double Hopf bifurcation with reflection symmetry. *Nonlinearity*, **6**, 259–275.

[31] Nagata, W. [1991]. Quasiperiodic saddle-node bifurcations near a differentiable singularity for forced oscillations. *J. Diff. Equations*, to appear.

[32] Nagata, W. [1991]. A perturbed Hopf bifurcation with reflection symmetry. *Proc. Roy. Soc. Edinburgh*, **117A**, 1–20.

[33] Nagata, W. [1990]. Convection in a layer with side walls: bifurcation with reflection symmetry. *J. Appl. Math. Phys. (ZAMP)*, **41**, 812–828.

[34] Nagata, W. [1988]. Dynamics near a symmetric Hopf bifurcation. *Proc. Roy. Soc. Edinburgh*, **108A**, 249–267.

[35] Nagata, W. [1987] in *Oscillations, Bifurcations and Chaos*, Langford, W.F, Atkinson, F.V. (eds.). Canadian Mathematical Society Conference Proceedings, **8**, 545

[36] Nagata, W., Thomas, J.W. [1986]. Bifurcation in doubly-diffusive systems I. Equilibrium solutions. *SIAM J. Math. Anal.*, **17**, 91–113.

[37] Nagata,W., Thomas, J.W. [1986]. Bifurcation in doubly-diffusive systems II. Time periodic solutions. *SIAM J. Math. Anal.*, **17**, 114–127.

[38] Nagata, W., Thomas, J.W. [1986]. Bifurcation in doubly-diffusive systems III. Interaction of equilibrium and the periodic solutions. *SIAM J. Math. Anal*, **17**, 289–311.

[39] Nakagawa, Y. [1955]. An experiment on the inhibition of thermal convection by a magnetic field. *Nature*, **175**, 417-419.

[40] Nakagawa, Y. [1957]. Experiments on the inhibition of thermal convection by a magnetic field. *Proc. Roy. Soc. London* , **240A**, 108-113.

[41] Swift, J.W. [1988]. Hopf bifurcation with the symmetry of the square. *Nonlinearity*, **1**, 333–377.

[42] Smale, S. [1963]. Diffeomorphisms with many periodic points. In *Differential and Combinatorial Topology*, Cairns, S.S.(ed). 63–80. Princeton University Press, Princeton.

[43] Smale, S. [1967]. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, **73**, 747–817.

[44] Takens, F. [1974]. Forced oscillation and bifurcations. *Comm. Math. Inst. Rijksuniversiteit Utrecht*, **3**, 1–59.

[45] Takens, F. [1973]. Unfoldings of certain singularities of vector fields. *J. Diff. Equations*, **14**, 476–493.

[46] Temam, R. [1988]. *Infinite Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York.

[47] Vanderbauwhede, A., Iooss, G. [1992]. Center manifold theory in infinite dimensions. *Dynamics Reported, New Series*, **1**, 125–163.

[48] Weiss, N.O. [1981]. Convection in an imposed magnetic field. Part 1. The devolepment of nonlinear convection. *J. Fluid Mech.*, **108**, 247–272.

Appendix A

Normal form coefficients

A.1 Computation of normal form coefficients

In this section, we give more details of the calculations outlined in §4.1, to obtain the normal form coefficients $C_1, \ldots, C_6$. To find explicit formulae we use (4.33)–(4.38); first let us give $M_0(\Phi, \Phi') = M(\Phi, \Phi', L_m)$ in vector form explicitly from (3.44), as

$$M_0(\Phi, \Phi') = \begin{pmatrix} \sigma\zeta Q b'_y \left[\dfrac{\partial b_x}{\partial y} - L_m^{-2}\dfrac{\partial b_y}{\partial x}\right] - \left[u\dfrac{\partial u'}{\partial x} + v\dfrac{\partial u'}{\partial y}\right] + L_m^{-2}\dfrac{\partial \chi}{\partial x} \\ L_m^2 \sigma\zeta Q b'_x \left[\dfrac{\partial b_x}{\partial y} - L_m^{-2}\dfrac{\partial b_y}{\partial x}\right] - \left[u\dfrac{\partial v'}{\partial x} + v\dfrac{\partial v'}{\partial y}\right] + \dfrac{\partial \chi}{\partial y} \\ -\left[u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial v}\right] \\ \dfrac{\partial}{\partial y}(u b'_y - v b'_x) \\ -\dfrac{\partial}{\partial x}(u b'_y - v b'_x) \end{pmatrix}, \tag{A.1}$$

where χ is chosen so that the divergence of the first two components vanishes. Using (A.1) and (3.39)-(3.40) we observe that $\tilde{P} M_0(\Phi_j, \Phi_k) = 0$ for $j, k = 1, 2$, due to elementary trigonometric identities, and we calculate

$$-(I - \tilde{P}) M_0(\Phi_1, \Phi_1) \quad = \quad \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi \sin(2\pi y)}{2P_1(1 + i\omega_1)} \\ 0 \\ \dfrac{\pi^2 \cos(m\pi x)}{P_1(\zeta + i\omega_1)} \end{pmatrix}, \tag{A.2}$$

$$-(I-\tilde{P})M_0(\Phi_2,\Phi_2) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi\sin(2\pi y)}{2P_2(1+i\omega_2)} \\ 0 \\ -\dfrac{\pi^2\cos((m+1)\pi x)}{P_2(\zeta+i\omega_2)} \end{pmatrix}. \tag{A.3}$$

Since

$$M_0(\bar{\Phi}_j,\Phi_j) = M_0(\Phi_j,\Phi_j), \quad j=1,2,$$

and

$$M_0(\Phi_j,\bar{\Phi}_j) = \overline{M_0(\Phi_j,\Phi_j)}, \quad j=1,2,$$

we have

$$-(I-\tilde{P})\left(M_0(\Phi_1,\bar{\Phi}_1)+M_0(\bar{\Phi}_1,\Phi_1)\right) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi\sin(2\pi y)}{P_1(1+\omega_1^2)} \\ 0 \\ \dfrac{2\pi^2\zeta\cos(m\pi x)}{P_1(\zeta^2+\omega_1^2)} \end{pmatrix}, \tag{A.4}$$

$$-(I-\tilde{P})\left(M_0(\Phi_2,\bar{\Phi}_2)+M_0(\bar{\Phi}_2,\Phi_2)\right) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi\sin(2\pi y)}{P_2(1+\omega_2^2)} \\ 0 \\ -\dfrac{2\pi^2\zeta\cos((m+1)\pi x)}{P_2(\zeta^2+\omega_2^2)} \end{pmatrix}. \tag{A.5}$$

We also have

$$-(I-\tilde{P})\left(M_0(\Phi_1,\Phi_2)+M_0(\Phi_2,\Phi_1)\right)=\begin{pmatrix} \dfrac{\pi^2 A_1}{A_0}\cos(\pi x/2)\cos(2\pi y) \\ \dfrac{\pi^2 A_1}{4A_0}\sin(\pi x/2)\sin(2\pi y) \\ A_2\sin(\pi x/2)\sin(2\pi y) \\ A_3\cos(\pi x/2)\sin(2\pi y) \\ A_4\sin(\pi x/2)-\dfrac{A_3}{4}\sin(\pi x/2)\cos(2\pi y) \end{pmatrix}$$

$$+\begin{pmatrix} \dfrac{\pi^2 B_1}{B_0}\cos((2m+1)\pi x/2)\cos(2\pi y) \\ \dfrac{(2m+1)\pi^2 B_1}{4B_0}\sin((2m+1)\pi x/2)\sin(2\pi y) \\ B_2\sin((2m+1)\pi x/2)\sin(2\pi y) \\ B_3\cos((2m+1)\pi x/2)\sin(2\pi y) \\ B_4\sin((2m+1)\pi x/2)-\dfrac{B_3(2m+1)}{4}\sin((2m+1)\pi x/2)\cos(2\pi y) \end{pmatrix}, \tag{A.6}$$

where

$$A_0 = \pi^2(1/4L_m^2+4), \tag{A.7}$$

$$B_0 = \pi^2((2m+1/2L_m)^2+4), \tag{A.8}$$

$$B_1 = -\frac{\sigma\zeta Q\pi(P_2-P_1)}{m(m+1)P_1P_2(\zeta+i\omega_1)(\zeta+i\omega_2)}-\frac{(2m+1)\pi}{4L_m^2 m(m+1)}, \tag{A.9}$$

$$A_1 = (2m+1)B_1, \tag{A.10}$$

$$A_2 = -\frac{\pi(2m+1)\left[mP_2(1+i\omega_2)+(m+1)P_1(1+i\omega_1)\right]}{4m(m+1)P_1P_2(1+i\omega_1)(1+i\omega_2)}, \tag{A.11}$$

$$B_2 = -\frac{\pi\left[-mP_2(1+i\omega_2)+(m+1)P_1(1+i\omega_1)\right]}{4m(m+1)P_1P_2(1+i\omega_1)(1+i\omega_2)}, \tag{A.12}$$

$$B_3 = \frac{\pi^2\left[P_1(\zeta+i\omega_1)-P_2(\zeta+i\omega_2)\right]}{m(m+1)P_1P_2(\zeta+i\omega_1)(\zeta+i\omega_2)}, \tag{A.13}$$

$$A_3 = (2m+1)B_3, \tag{A.14}$$

$$A_4 = -\frac{\pi^2\left[P_1(\zeta+i\omega_1)+P_2(\zeta+i\omega_2)\right]}{4m(m+1)P_1P_2(\zeta+i\omega_1)(\zeta+i\omega_2)}, \tag{A.15}$$

$$B_4 = (2m+1)^2 A_4. \tag{A.16}$$

We now solve (4.23), using (A.2):

$$(K_0 - 2iP_1\omega_1)\Psi_{2000} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi\sin(2\pi y)}{2P_1(1+i\omega_1)} \\ 0 \\ \dfrac{\pi^2\cos(m\pi x)}{P_1(\zeta+i\omega_1)} \end{pmatrix}, \tag{A.17}$$

i.e.,

$$\begin{aligned} \sigma\left[(\Delta^1 - 2iP_1\omega_1/\sigma)\mathbf{u} - \nabla^1\chi + R_m\theta\mathbf{e}_y + \zeta Q(\nabla^1\times\mathbf{b})\times\mathbf{e}_y\right] &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ (\Delta^1 - 2iP_1\Omega_1)\theta + \mathbf{u}\cdot\mathbf{e}_y &= \frac{\pi\sin(2\pi y)}{2P_1(1+i\omega_1)}, \\ (\zeta\Delta^1 - 2iP_1\Omega_1)\mathbf{b} + \nabla\times(\mathbf{u}\times\mathbf{e}_y) &= \begin{pmatrix} 0 \\ \dfrac{\pi^2\cos(m\pi x)}{P_1(\zeta+i\omega_1)} \end{pmatrix}, \end{aligned} \tag{A.18}$$

where $\Psi_{2000} = (\mathbf{u}, \theta, \mathbf{b})^T$ has the general form

$$\begin{pmatrix} c_1\sin(m\pi x)\cos(2\pi y) \\ -(mc_1/2)\cos(m\pi x)\sin(2\pi y) \\ c_2\cos(m\pi x)\sin(2\pi y) + c_3\sin(2\pi y) \\ c_4\sin(m\pi x)\sin(2\pi y) \\ mc_4/2\cos(m\pi x)\cos(2\pi y) + c_5 cos(m\pi x) \end{pmatrix}, \tag{A.19}$$

and $\chi = c_6\cos(m\pi x) + c_7\cos(2\pi y) + c_8\cos(m\pi x)\cos(2\pi y)$ is chosen so that the divergence of the first two components of Ψ_{2000} become zero. Substituting (A.19) into (A.17), we solve for the coefficients $c_1, ..., c_5$ in (A.19) to get

$$\Psi_{2000} = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\pi \sin(2\pi y)}{2P_1^2(1+i\omega_1)(\varpi_1 + 2i\omega_1)} \\ 0 \\ -\dfrac{4\pi^2 L_m^2 \cos(m\pi x)}{P_1^2(\zeta + i\omega_1)(m^2\varpi_1\zeta + 8i\omega_1 L_m^2)} \end{pmatrix}, \tag{A.20}$$

where $\varpi_1 = 4\pi^2/P_1$. Similarly, we solve

$$(K_0 - 2iP_2\omega_2)\Psi_{0020} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi}{2P_2(1+i\omega_2)} \sin(2\pi y) \\ 0 \\ -\dfrac{\pi^2}{P_2(\zeta + i\omega_2)} \cos((m+1)\pi x) \end{pmatrix}, \tag{A.21}$$

to obtain

$$\Psi_{0020} = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\pi}{2P_2^2(1+i\omega_2)(\varpi_2 + 2i\omega_2)} \sin(2\pi y) \\ 0 \\ \dfrac{4\pi^2 L_m^2}{P_2^2(\zeta + i\omega_2)\left((m+1)^2\varpi_2\zeta + 8i\omega_2 L_m^2\right)} \cos((m+1)\pi x) \end{pmatrix}, \tag{A.22}$$

where $\varpi_2 = 4\pi^2/P_2$. We also solve

$$K_0\Psi_{1100} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi}{P_1(1+\omega_1^2)}\sin(2\pi y) \\ 0 \\ \dfrac{2\pi^2\zeta}{P_1(\zeta^2+\omega_1^2)}\cos(m\pi x) \end{pmatrix}, \tag{A.23}$$

to obtain

$$\Psi_{1100} = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{1}{4\pi P_1(1+\omega_1^2)}\sin(2\pi y) \\ 0 \\ -\dfrac{2L_m^2}{m^2 P_1(\zeta^2+\omega_1^2)}\cos(m\pi x) \end{pmatrix}, \tag{A.24}$$

and similarly,

$$K_0\Psi_{0011} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\pi}{P_2(1+\omega_2^2)}\sin(2\pi y) \\ 0 \\ -\dfrac{2\pi^2\zeta}{P_2(\zeta^2+\omega_2^2)}\cos((m+1)\pi x) \end{pmatrix}, \tag{A.25}$$

has solution

$$\Psi_{0011} = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{1}{4\pi P_2(1+\omega_2^2)}\sin(2\pi y) \\ 0 \\ \dfrac{2L_m^2}{(m+1)^2 P_2(\zeta^2+\omega_2^2)}\cos((m+1)\pi x) \end{pmatrix}. \tag{A.26}$$

By (4.29) we have

$$(K_0 - iP_1\omega_1 - iP_2\omega_2)\Psi_{1010} = -(I-\tilde{P})\left(M_0(\Phi_1,\Phi_2) + M_0(\Phi_2,\Phi_1)\right), \tag{A.27}$$

where

$$-(I-\tilde{P})\left(M_0(\Phi_1,\Phi_2) + M_0(\Phi_2,\Phi_1)\right)$$

is given by (A.6), therefore Ψ_{1010} has the form

$$\Psi_{1010} = \begin{pmatrix} c_1\cos(\pi x/2)\cos(2\pi y) \\ (c_1/4)\sin(\pi x/2)\sin(2\pi y) \\ c_2\sin(\pi x/2)\sin(2\pi y) \\ c_3\cos(\pi x/2)\sin(2\pi y) \\ -(c_3/4)\sin(\pi x/2)\cos(2\pi y) + c_4\sin(\pi x/2) \end{pmatrix} + \tag{A.28}$$

$$\begin{pmatrix} c_1'\cos((2m+1)\pi x/2)\cos(2\pi y) \\ \dfrac{(2m+1)c_1'}{4}\sin((2m+1)\pi x/2)\sin(2\pi y) \\ c_2'\sin((2m+1)\pi x/2)\sin(2\pi y) \\ c_3'\cos((2m+1)\pi x/2)\sin(2\pi y) \\ -\dfrac{(2m+1)c_3'}{4}\sin((2m+1)\pi x/2)\cos(2\pi y) + c_4'\sin((2m+1)\pi x/2) \end{pmatrix},$$

and

$$\chi = [c_5\sin(\pi x/2) + c_5'\sin((2m+1)\pi x/2)]\cos(2\pi y) + c_6\sin(\pi x/2) + c_6'\sin((2m+1)\pi x/2).$$

To simplify our notation, let

$$\eta_1 = A_0 + iP_1\omega_1 + iP_2\omega_2, \qquad \Psi_1 = B_0 + iP_1\omega_1 + iP_2\omega_2, \tag{A.29}$$

$$\eta_2 = \sigma A_0 + iP_1\omega_1 + iP_2\omega_2, \qquad \Psi_2 = \sigma B_0 + iP_1\omega_1 + iP_2\omega_2, \tag{A.30}$$

$$\eta_3 = \zeta A_0 + iP_1\omega_1 + iP_2\omega_2, \qquad \Psi_3 = \zeta B_0 + iP_1\omega_1 + iP_2\omega_2, \tag{A.31}$$

$$\eta_4 = \zeta(\frac{\pi}{2L_m})^2 + iP_1\omega_1 + iP_2\omega_2, \quad \Psi_4 = \zeta(\frac{(2m+1)\pi}{2L_m})^2 + iP_1\omega_1 + iP_2\omega_2. \tag{A.32}$$

Now by using (A.6), (A.27) and (A.28) we get

$$\zeta Q c_4 + c_6 = 0, \tag{A.33}$$

$$-\eta_2 c_1 + \frac{\sigma\zeta Q A_0 c_3}{2\pi} - \frac{\sigma\pi c_5}{2L_m^2} = \frac{\pi^2 A_1}{A_0}, \tag{A.34}$$

$$-\frac{\eta_2 c_1}{4}) + \sigma R_m c_2 + 2\pi\sigma c_5 = \frac{\pi^2 A_1}{4A_0}, \tag{A.35}$$

$$-\eta_1 c_2 + c_1/4 = A_2, \tag{A.36}$$

$$-\eta_3 c_3 - 2\pi c_1 = A_3, \tag{A.37}$$

$$-\eta_4 c_4 = A_4, \tag{A.38}$$

$$\zeta Q c_4' + c_6' = 0, \tag{A.39}$$

$$-\Psi_2 c_1' + \frac{\sigma\zeta Q B_0 c_3'}{2\pi} - \frac{(2m+1)\sigma\pi c_5'}{2L_m^2} = \pi^2 B_1/B_0, \tag{A.40}$$

$$-\frac{(2m+1)\Psi_2 c_1'}{4} + \sigma R_m c_2' + 2\pi\sigma c_5' = \frac{(2m+1)\pi^2 B_1}{4B_0}, \tag{A.41}$$

$$-\Psi_1 c_2 + c_1'(2m+1)/4 = B_2, \tag{A.42}$$

$$-\Psi_3 c_3' - 2\pi c_1' = B_3, \tag{A.43}$$

$$-\Psi_4 c_4' = B_4. \tag{A.44}$$

Explicit calculation of pressure term is not neccessary, and we solve the system (A.33)–(A.44) of algebraic equations to find $c_1, ..., c_4,\ c_1', ..., c_4'$, to get

$$c_1 = -\frac{A_1/4 + \dfrac{\sigma\zeta Q A_0 A_3}{2\pi\eta_3} + \dfrac{\sigma R_m A_2}{4L_m^2\eta_1}}{\dfrac{\eta_2 A_0}{4\pi^2} + \dfrac{\sigma\zeta Q A_0}{\eta_3} - \dfrac{\sigma R_m}{16L_m^2\eta_1}}, \tag{A.45}$$

$$c_2 = \frac{c_1 - 4A_2}{4\eta_1}, \tag{A.46}$$

$$c_3 = -\frac{2\pi c_1 + A_3}{\eta_3}, \tag{A.47}$$

$$c_4 = -A_4/\eta_4, \tag{A.48}$$

$$c'_1 = -\frac{B_1/4 + \dfrac{\sigma\zeta Q B_0 B_3}{2\pi\Psi_3} + \dfrac{(2m+1)\sigma R_m B_2}{4L_m^2\Psi_1}}{\dfrac{\Psi_2 B_0}{4\pi^2} + \dfrac{\sigma\zeta Q B_0}{\Psi_3} - \dfrac{(2m+1)^2\sigma R_m}{16L_m^2\Psi_1}}, \tag{A.49}$$

$$c'_2 = \frac{(2m+1)c'_1 - 4B_2}{4\Psi_1}, \tag{A.50}$$

$$c'_3 = -\frac{2\pi c'_1 + B_3}{\Psi_3}, \tag{A.51}$$

$$c'_4 = -B_4/\Psi_4, \tag{A.52}$$

We observe that Ψ_{1001} can be obtained from Ψ_{1010} by changing $\omega_2 \to -\omega_2$ and therefore

$$\Psi_{1001} = \begin{pmatrix} d_1\cos(\pi x/2)\cos(2\pi y) \\ (d_1/4)\sin(\pi x/2)\sin(2\pi y) \\ d_2\sin(\pi x/2)\sin(2\pi y) \\ d_3\cos(\pi x/2)\sin(2\pi y) \\ -(d_3/4)\sin(\pi x/2)\cos(2\pi y) + d_4\sin(\pi x/2) \end{pmatrix} + \tag{A.53}$$

$$\begin{pmatrix} d'_1\cos((2m+1)\pi x/2)\cos(2\pi y) \\ \dfrac{(2m+1)c'_1}{4}\sin((2m+1)\pi x/2)\sin(2\pi y) \\ d'_2\sin((2m+1)\pi x/2)\sin(2\pi y) \\ d'_3\cos((2m+1)\pi x/2)\sin(2\pi y) \\ -\dfrac{(2m+1)d'_3}{4}\sin((2m+1)\pi x/2)\cos(2\pi y) + c'_4\sin((2m+1)\pi x/2) \end{pmatrix},$$

and

$$\chi = [d_5\sin(\pi x/2) + d'_5\sin((2m+1)\pi x/2)]\cos(2\pi y) + d_6\sin(\pi x/2) + d'_6\sin((2m+1)\pi x/2).$$

where $d_i, d'_i,\ i = 1,...,4$ are $c_i, c'_i,\ i = 1,...,4$ under the change $\omega_2 \to -\omega_2$. Also, by (4.30) and (4.31) we have $\Psi_{0110} = \bar{\Psi}_{1001}$.

We now compute the terms $M_0(\Psi_{ijkl}, \Phi_q)$, etc. that appear in (4.17)–(4.22). To simplify our notation, from now on we give only $\tilde{M}_0(\cdot,\cdot)$, which is the same as $M_0(\cdot,\cdot)$

except for terms that are eliminated in the inner products with $\Phi_j^*, j = 1, 2$, i.e.,

$$\left\langle M_0(\cdot,\cdot), \Phi_j^* \right\rangle = \left\langle \tilde{M}_0(\cdot,\cdot), \Phi_j^* \right\rangle . \tag{A.54}$$

We first calculate

$$\tilde{M}_0(\Phi_2, \Psi_{0011}) = \begin{pmatrix} \frac{\sigma\zeta Q(2L_m/(m+1))^3}{4L_m P_2(\zeta^2+\omega_2^2)(\zeta+i\omega_2)} \sin((m+1)\pi x/2)\cos(\pi y) + L_m^{-2}\partial\chi/\partial x \\ \partial\chi/\partial y \\ -\frac{1}{4P_2(1+\omega_2^2)} \cos((m+1)\pi x/2)\sin(\pi y) \\ -\frac{\pi(2L_m/(m+1))^3}{4L_m P_2(\zeta^2+\omega_2^2)} \sin((m+1)\pi x/2)\sin(\pi y) \\ -\frac{\pi(2L_m/(m+1))^2}{4P_2(\zeta^2+\omega_2^2)} \cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.55}$$

where $\chi = c\cos((m+1)\pi x/2)\cos(\pi y)$ is chosen so that the divergence of the first two components of (A.55) is zero. If we denote the first two components of $\tilde{M}_0(\Phi_2, \Psi_{0011})$ by M_1, M_2, then we should have

$$div(M_1, M_2) = \left(\frac{\sigma\zeta Q(2L_m/(m+1))^2\pi}{4P_1(\zeta^2+\omega_1^2)(\zeta+i\omega_1)} - P_1 c \right) \cos((m+1)\pi x/2)\cos(\pi y) = 0,$$

therefore

$$\begin{aligned} M_1 &= \frac{\pi^2\sigma\zeta Q(2L_m/(m+1))^3}{4L_m P_1^2(\zeta^2+\omega_1^2)(\zeta+i\omega_1)} \sin((m+1)\pi x/2)\cos(\pi y), \\ M_2 &= -\frac{\pi^2\sigma\zeta Q(2L_m/(m+1))^2}{4P_1^2(\zeta^2+\omega_1^2)(\zeta+i\omega_1)} \cos((m+1)\pi x/2)\sin(\pi y). \end{aligned}$$

Similarly,

$$\tilde{M}_0(\bar{\Phi}_1, \Psi_{1100}) = \begin{pmatrix} -\frac{\pi^2\sigma\zeta Q(2L_m/m)^3}{4L_m P_1^2(\zeta^2+\omega_1^2)(\zeta+i\omega_1)} \cos(m\pi x/2)\cos(\pi y) \\ \frac{\pi^2\sigma\zeta Q(2L_m/m)^2}{4P_1^2(\zeta^2+\omega_1^2)(\zeta+i\omega_1)} \sin(m\pi x/2)\sin(\pi y) \\ -\frac{1}{4P_1(1+\omega_1^2)} \sin(m\pi x/2)\sin(\pi y) \\ -\frac{\pi(2L_m/m)^3}{4L_m P_1(\zeta^2+\omega_1^2)} \cos(m\pi x/2)\sin(\pi y) \\ \frac{\pi(2L_m/m)^3}{4P_1(\zeta^2+\omega_1^2)} \sin(m\pi x/2)\cos(\pi y) \end{pmatrix}. \tag{A.56}$$

We also have

$$\tilde{M}_0(\Psi_{1100}, \Phi_1) = \tilde{M}_0(\Psi_{0011}, \Phi_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.57}$$

$$\tilde{M}_0(\Psi_{2000}, \bar{\Phi}_1) = \tilde{M}_0(\Psi_{0020}, \bar{\Phi}_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.58}$$

$$\tilde{M}_0(\Phi_2, \Psi_{2000}) = \begin{pmatrix} \frac{\pi^4 \sigma \zeta Q (2L_m/(m+1))^3}{2P_2^3 L_m (\zeta^2 + \omega_2^2)(\varpi_2 \zeta + 2i\omega_2 (2L_m/(m+1))^2)} \sin((m+1)\pi x/2) \cos(\pi y) \\ -\frac{\pi^4 \sigma \zeta Q (2L_m/(m+1))^2}{2P_2^3 (\zeta^2 + \omega_1^2)(\varpi_2 \zeta + 2i\omega_2 (2L_m/(m+1))^2)} \cos((m+1)\pi x/2) \sin(\pi y) \\ -\frac{\pi^2}{2P_2^2 (1 + i\omega_2)(\varpi_2 + 2i\omega_2)} \cos((m+1)\pi x/2) \sin(\pi y) \\ -\frac{\pi^3 (2L_m/(m+1))^3}{2P_2^2 L_m (\zeta + i\omega_2)(\varpi_2 \zeta + 2i\omega_2 (2L_m/(m+1))^2)} \sin((m+1)\pi x/2) \sin(\pi y) \\ -\frac{\pi^3 (2L_m/(m+1))^2}{2P_2^2 (\zeta + i\omega_2)(\varpi_2 \zeta + 2i\omega_2 (2L_m/(m+1))^2)} \cos((m+1)\pi x/2) \cos(\pi y) \end{pmatrix}, \tag{A.59}$$

and

$$\tilde{M}_0(\bar{\Phi}_1, \Psi_{2000}) = \begin{pmatrix} -\dfrac{\pi^4\sigma\zeta Q(\pi^2 2L_m/m)^3}{2P_1^3 L_m(\zeta^2+\omega_1^2)(\varpi_1\zeta+2i\omega_1(2L_m/m)^2)}\cos(m\pi x/2)\cos(\pi y), \\ -\dfrac{\pi^4\sigma\zeta Q(2L_m/m)^2}{4L_m P_1^3(\zeta^2+\omega_2^2)(\varpi_2\zeta+2i\omega_2(2L_m/m)^2)}\sin(m\pi x/2)\sin(\pi y) \\ -\dfrac{\pi^2}{2P_1^2(1+i\omega_1)(\varpi_1+2i\omega_1})\sin(m\pi x/2\sin(\pi y) \\ \dfrac{\pi^3(2L_m/m)^3}{2P_1^2 L_m(\zeta+i\omega_1)(\varpi_1\zeta+2i\omega_1(2L_m/m)^2)}\cos(m\pi x/2)\sin(\pi y) \\ -\dfrac{\pi^3(2L_m/m)^2}{2P_1^2(\zeta+i\omega_1)(\varpi_1\zeta+2i\omega_1(2L_m/m)^2)}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.60}$$

Now using (4.33) and (4.37) we have

$$\begin{aligned} C_1 &= \bar{C}_m/2\left\{\frac{\sigma R_m}{P_1(1+i\omega_1)}\left[-\frac{1}{4P_1(1+\omega_1^2)}-\frac{\pi^2}{2P_1^2(1+i\omega_1)(\varpi_1+2i\omega_1)}\right]\right. \\ &\quad\left.-\frac{i\sigma\zeta Q(2L_m/m)^4\pi^2\omega_2}{P_1^2(\zeta+i\omega_1)(\zeta^2+\omega_1^2)(\varpi_1\zeta+2i\omega_1(2L_m/m)^2)}\right\}, \qquad\qquad (A.61) \\ C_5 &= \bar{C}_{m+1}/2\left\{\frac{\sigma R_m}{P_2(1+i\omega_2)}\left[-\frac{1}{4P_2(1+\omega_2^2)}-\frac{\pi^2}{2P_2^2(1+i\omega_2)(\varpi_2+2i\omega_2)}\right]\right. \\ &\quad\left.-\frac{i\sigma\zeta Q(2L_m/(m+1))^4\pi^2\omega_2}{P_2^2(\zeta+i\omega_2)(\zeta^2+\omega_2^2)(\varpi_2\zeta+2i\omega_2(2L_m/(m+1))^2)}\right\}. \qquad\qquad (A.62) \end{aligned}$$

where $\bar{C}_m, \bar{C}_{m+1}$ are given by (2.55).

To compute C_2, C_3, C_4 and C_6, we also need

$$\tilde{M}_0(\Psi_{0011}, \Phi_1) = \tilde{M}_0(\Psi_{1100}, \Phi_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.63}$$

and

$$\tilde{M_0}(\Phi_2, \Psi_{1100}) = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{1}{4L_m P_1(1+\omega_1^2)}\cos((m+1)\pi x/2)\sin(\pi y) \\ 0 \\ 0 \end{pmatrix}. \tag{A.64}$$

In the following equations, $c_i, d_i, c_i', d_i', i = 1, ..., 4$ are as in equations (A.45)-(A.52). We have:

$$\tilde{M_0}(\bar{\Phi}_1, \Psi_{1010}) = \begin{pmatrix} D_1 \sin((m+1)\pi x/2)\cos(\pi y) \\ -\dfrac{(m+1)D_1}{2}\cos((m+1)\pi x/2)\sin(\pi y) \\ \dfrac{\pi\left[(2m+1)c_2 + c_2'\right]}{4m}\cos((m+1)\pi x/2)\sin(\pi y) \\ E_1 \sin((m+1)\pi x/2)\sin(\pi y) \\ (m+1)E_1 2\cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.65}$$

where

$$D_1 = \frac{\pi^2}{P_2}\left\{-\frac{\sigma\zeta Q\left((2m+1)c_3 + c_3' + 8(c_4 + c_4')\right)}{8m(\zeta - i\omega_1)}\right. \tag{A.66}$$

$$\left. + \frac{\pi\left((2m+1)c_1 + c_1'\right)}{4m} + \frac{\pi(m+1)(2m+1)(c_1 + c_1')}{32mL_m^2}\right\},$$

$$E_1 = \frac{\pi\left((2m+1)c_3 + c_3'\right) - 8\pi(c_4 + c_4')}{8m}; \tag{A.67}$$

$$\tilde{M_0}(\Phi_1, \Psi_{1001}) = \begin{pmatrix} \tilde{D}_1 \sin((m+1)\pi x/2)\cos(\pi y) \\ -\dfrac{(m+1)\tilde{D}_1}{2}\cos((m+1)\pi x/2)\sin(\pi y) \\ -\dfrac{\pi\left((2m+1)d_2 + d_2'\right)}{4m}\cos((m+1)\pi x/2)\sin(\pi y) \\ \tilde{E}_1 \sin((m+1)\pi x/2)\sin(\pi y) \\ (m+1)\tilde{E}_1 2\cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.68}$$

where

$$\tilde{D}_1 = \frac{\pi^2}{P_2}\left\{-\frac{\sigma\zeta Q\left[(2m+1)d_3 + d_3' + 8(d_4+d_4')\right]}{8m(\zeta+i\omega_1)} + \frac{\pi\left((2m+1)d_1 + d_1'\right)}{4m} + \frac{\pi(m+1)(2m+1)(d_1+d_1')}{32mL_m^2}\right\}, \tag{A.69}$$

$$\tilde{E}_1 = \frac{\pi\left[(2m+1)d_3 + d_3' - 8(d_4+d_4')\right]}{8m}; \tag{A.70}$$

$$\tilde{M}_0(\Psi_{1010},\bar{\Phi}_1) = \begin{pmatrix} D_2\sin((m+1)\pi x/2)\cos(\pi y) \\ -\frac{(m+1)D_2}{2}\cos((m+1)\pi x/2)\sin(\pi y) \\ \frac{\pi\left[(2m+1)c_1 - c_1'\right]}{16P_1(1-i\omega_1)}\cos((m+1)\pi x/2)\sin(\pi y) \\ E_2\sin((m+1)\pi x/2)\sin(\pi y) \\ \frac{(m+1)E_2}{2}\cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.71}$$

where

$$D_2 = \frac{\pi^2}{P_2}\left\{\frac{\pi^2\sigma\zeta Q\left[(2m+1)c_3 + c_3'\right]}{2mP_1(\zeta-i\omega_1)} + \frac{\pi^2\sigma\zeta Q(2m+1)c_3}{32mL_m^2P_1(\zeta-i\omega_1)} + \frac{\pi^2\sigma\zeta Q\left[(2m+1)^3c_3' + 8\left(c_4 + (2m+1)^2c_4'\right)\right]}{32mL_m^2P_1(\zeta-i\omega_1)} + \frac{\pi\left((2m+1)c_1 + c_1'\right)}{8m} - \frac{\pi(m+1)\left[(2m+1)c_1 - c_1'\right]}{32L_m^2}\right\}, \tag{A.72}$$

$$E_2 = -\frac{\pi^2\left[(2m+1)c_1 + c_1'\right]}{8mP_1(\zeta-i\omega_1)}; \tag{A.73}$$

$$\tilde{M}_0(\Psi_{1001},\Phi_1) = \begin{pmatrix} \tilde{D}_2\sin((m+1)\pi x/2)\cos(\pi y) \\ -\frac{(m+1)\tilde{D}_2}{2}\cos((m+1)\pi x/2)\sin(\pi y) \\ \frac{\pi\left((2m+1)d_1 - d_1'\right)}{16P_2(1+i\omega_1)}\cos((m+1)\pi x/2)\sin(\pi y) \\ \tilde{E}_2\sin((m+1)\pi x/2)\sin(\pi y) \\ \frac{(m+1)\tilde{E}_2}{2}\cos((m+1)\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.74}$$

where

$$\tilde{D}_2 = \frac{\pi^2}{P_2}\left\{\frac{\pi^2\sigma\zeta Q\left[(2m+1)d_3+d_3'\right]}{2mP_1(\zeta+i\omega_1)}+\frac{\pi^2\sigma\zeta Q(2m+1)d_3}{32mL_m^2P_1(\zeta+i\omega_1)}\right.$$
$$+\frac{\pi^2\sigma\zeta Q\left[(2m+1)^3d_3'+8\left(d_4+(2m+1)^2d_4'\right)\right]}{32mL_m^2P_1(\zeta+i\omega_1)}$$
$$\left.+\frac{\pi\left((2m+1)d_1+d_1'\right)}{8m}-\frac{\pi(m+1)\left[(2m+1)d_1-d_1'\right]}{32L_m^2}\right\}, \quad \text{(A.75)}$$

$$\tilde{E}_2 = -\frac{\pi^2\left[(2m+1)d_1+d_1'\right]}{8mP_1(\zeta+i\omega_1)}; \quad \text{(A.76)}$$

$$\tilde{M}_0(\bar{\Phi}_2,\Psi_{1010}) = \begin{pmatrix} D_3\cos(m\pi x/2)\cos(\pi y) \\ \frac{mD_3}{2}\sin(m\pi x/2)\sin(\pi y) \\ \frac{-\pi\left((2m+1)c_2+c_2'\right)}{4(m+1)}\sin(m\pi x/2)\sin(\pi y) \\ E_3\cos(m\pi x/2)\sin(\pi y) \\ -\frac{mE_3}{2}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \quad \text{(A.77)}$$

where

$$D_3 = \frac{\pi^2}{P_1}\left\{\frac{-\sigma\zeta Q\left[(2m+1)c_3+c_3'-8(c_4+c_4')\right]}{8(m+1)(\zeta-i\omega_2)}\right.$$
$$\left.+\frac{\pi\left[(2m+1)c_1+c_1'\right]}{4(m+1)}-\frac{\pi m(2m+1)(c_1-c_1')}{32L_m^2(m+1)}\right\}, \quad \text{(A.78)}$$

$$E_3 = \frac{\pi\left[(2m+1)c_3+c_3'+8(c_4+c_4')\right]}{8(m+1)}; \quad \text{(A.79)}$$

$$\tilde{M}_0(\Phi_2,\Psi_{0110}) = \begin{pmatrix} \tilde{D}_3\cos(m\pi x/2)\cos(\pi y) \\ \frac{m\tilde{D}_3}{2}\sin(m\pi x/2)\sin(\pi y) \\ \frac{-\pi\left((2m+1)\bar{d}_2+\bar{d}_2'\right)}{4(m+1)}\sin(m\pi x/2)\sin(\pi y) \\ \tilde{E}_3\cos(m\pi x/2)\sin(\pi y) \\ -\frac{m\tilde{E}_3}{2}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \quad \text{(A.80)}$$

where

$$\tilde{D}_3 = \frac{\pi^2}{P_1}\left[-\frac{\sigma\zeta Q\left[(2m+1)\bar{d}_3 + \bar{d}_3' - 8(\bar{d}_4 + \bar{d}_4')\right]}{8(m+1)(\zeta + i\omega_2)} + \frac{\pi\left[(2m+1)\bar{d}_1 + \bar{d}_1'\right]}{4(m+1)} - \frac{m(2m+1)\pi(\bar{d}_1 - \bar{d}_1')}{32L_m^2(m+1)}\right], \tag{A.81}$$

$$\tilde{E}_3 = \frac{\pi\left[(2m+1)\bar{d}_3 + \bar{d}_3' + 8(\bar{d}_4 + \bar{d}_4')\right]}{8(m+1)}; \tag{A.82}$$

$$\tilde{M}_0(\Psi_{1010}, \bar{\Phi}_2) = \begin{pmatrix} D_4 \cos(m\pi x/2)\cos(\pi y) \\ \frac{mD_4}{2}\sin(m\pi x/2)\sin(\pi y) \\ -\frac{\pi\left((2m+1)c_1 - c_1'\right)}{16P_2(1 - i\omega_2)}\sin(m\pi x/2)\sin(\pi y) \\ E_4\cos(m\pi x/2)\sin(\pi y) \\ -\frac{mE_4}{2}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.83}$$

where

$$D_4 = \frac{\pi^2}{P_1}\left\{\frac{\pi^2\sigma\zeta Q\left[(2m+1)c_3 + c_3'\right]}{2(m+1)P_2(\zeta - i\omega_2)} + \frac{\pi^2\sigma\zeta Q(2m+1)c_3}{32(m+1)L_m^2 P_2(\zeta - i\omega_2)} + \frac{\pi^2\sigma\zeta Q\left[(2m+1)^3 c_3' - 8\left(c_4 + (2m+1)^2 c_4'\right)\right]}{32(m+1)L_m^2 P_2(\zeta - i\omega_2)} + \frac{\pi\left[(2m+1)c_1 + c_1'\right]}{8(m+1)} - \frac{m\pi\left[(2m+1)c_1 - c_1'\right]}{32L_m^2}\right\}, \tag{A.84}$$

$$E_4 = -\frac{\pi^2\left[(2m+1)c_1 + c_1'\right]}{8P_2(m+1)(\zeta - i\omega_2)}; \tag{A.85}$$

and

$$\tilde{M}_0(\Psi_{0110}, \Phi_2) = \begin{pmatrix} \tilde{D}_4 \cos(m\pi x/2)\cos(\pi y) \\ \frac{m\tilde{D}_4}{2}\sin(m\pi x/2)\sin(\pi y) \\ -\frac{\pi\left((2m+1)\bar{d}_1 - \bar{d}_1'\right)}{16P_2(1 + i\omega_2)}\sin(m\pi x/2)\sin(\pi y) \\ \tilde{E}_4\cos(m\pi x/2)\sin(\pi y) \\ -\frac{m\tilde{E}_4}{2}\sin(m\pi x/2)\cos(\pi y) \end{pmatrix}, \tag{A.86}$$

where

$$\tilde{D}_4 = \frac{\pi^2}{P_1}\left\{\frac{\pi^2\sigma\zeta Q\left[(2m+1)\bar{d}_3+\bar{d}_3'\right]}{2(m+1)P_2(\zeta+i\omega_2)}+\frac{\pi^2\sigma\zeta Q(2m+1)\bar{d}_3}{32(m+1)L_m^2P_2(\zeta+i\omega_2)}\right. \tag{A.87}$$
$$+\frac{\pi^2\sigma\zeta Q\left[(2m+1)^3\bar{d}_3'-8\left(\bar{d}_4+(2m+1)^2\bar{d}_4'\right)\right]}{32(m+1)L_m^2P_2(\zeta+i\omega_2)}$$
$$\left.+\frac{\pi\left[(2m+1)\bar{d}_1+\bar{d}_1'\right]}{8(m+1)}-\frac{m\pi\left[(2m+1)\bar{d}_1-\bar{d}_1'\right]}{32L_m^2}\right\},$$

$$\check{E}_4 = -\frac{\pi^2\left[(2m+1)\bar{d}_1+\bar{d}_1'\right]}{8P_2(m+1)(\zeta+i\omega_2)}. \tag{A.88}$$

We observe that the inner products

$$\langle M_0(\Phi_2,\Psi_{1001}),\Phi_1^*\rangle \text{ and } \langle M_0(\Psi_{1001},\Phi_2),\Phi_1^*\rangle$$

are obtained from

$$\langle M_0(\bar{\Phi}_2,\Psi_{1010}),\Phi_1^*\rangle \text{ and } \langle M_0(\bar{\Phi}_2,\Psi_{1010}),\Phi_1^*\rangle,$$

respectively, by changing $\omega_2 \to -\omega_2$. Similarly,

$$\langle M_0(\Phi_1,\Psi_{0110}),\Phi_2^*\rangle \text{ and } \langle M_0(\Psi_{0110},\Phi_1),\Phi_2^*\rangle$$

are obtained from

$$\langle M_0(\bar{\Phi}_1,\Psi_{1010}),\Phi_2^*\rangle \text{ and } \langle M_0(\Psi_{1010},\bar{\Phi}_1),\Phi_2^*\rangle$$

by changing $\omega_1 \to -\omega_1$. Therefore we also have

$$\langle M_0(\Psi_{2000},\bar{\Phi}_2),\Phi_2^*\rangle = \left\langle M_0(\Psi_{0020},\bar{\Phi}_1),\Phi_1^*\right\rangle = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, \tag{A.89}$$

also

$$\tilde{M}_0(\bar{\Phi}_1, \Psi_{0020}) = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\pi^2}{2P_1^2(1+i\omega_2)(\varpi_2+2i\omega_2)}\sin(\pi y) \\ 0 \\ 0 \end{pmatrix}, \tag{A.90}$$

and

$$\tilde{M}_0(\bar{\Phi}_2, \Psi_{2000}) = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\pi^2}{2P_1^2(1+i\omega_1)(\varpi_1+2i\omega_1)}\sin(\pi y) \\ 0 \\ 0 \end{pmatrix}. \tag{A.91}$$

Finally, we observe that

$$\langle M_0(\Psi_{0110}, \Phi_2), \Phi_1^*\rangle \text{ and } \langle M_0(\Phi_2, \Psi_{0110}), \Phi_1^*\rangle$$

are obtained from

$$\langle M_0(\Psi_{1010}, \bar{\Phi}_2), \Phi_1^*\rangle \text{ and } \langle M_0(\bar{\Phi}_2, \Psi_{1010}), \Phi_1^*\rangle,$$

respectively by changing $\omega_2 \to -\omega_2$ and $c_i, c_i' \to \bar{d}_i, \bar{d'}_i, i = 1, ..., 4$. Therefore by (4.34)-(4.36) and (4.38) we can now compute

$$\begin{aligned} C_2 = \bar{C}_m/2 \Bigg\{ &-\frac{\sigma R_m}{P_1(1+i\omega_1)}\left[\frac{\pi\left[(2m+1)(c_2+d_2)+c_2'+d'_2\right]}{4(m+1)} + \right. \\ &\left.\frac{\pi\left((2m+1)c_1+c_1'\right)}{16P_2(1-i\omega_2)} + \frac{\pi\left((2m+1)d_1+d'_1\right)}{16P_2(1+i\omega_2)} + \frac{1}{4P_2(1+\omega_2^2)}\right] \\ &+ \frac{2L_m^2P_1}{\pi^2 m}\left[D_3+D_4+\hat{D}_3+\hat{D}_4+\frac{\sigma\zeta Q\pi}{(\zeta+i\omega_1)P_1}(E_3+E_4+\hat{E}_3+\hat{E}_4)\right]\Bigg\}, \end{aligned} \tag{A.92}$$

$$C_4 = \bar{C}_{m+1}/2\left\{\frac{\sigma R_m}{P_2(1+i\omega_2)}\left[-\frac{\pi\left((2m+1)(c_2+\bar{d}_2)+c_2'+\bar{d'}_2\right)}{4m}+\right.\right. \tag{A.93}$$
$$\left.\frac{\pi\left((2m+1)c_1+c_1'\right)}{16P_1(1-i\omega_1)}+\frac{\pi\left((2m+1)\bar{d}_1+\bar{d'}_1\right)}{16P_1(1+i\omega_1)}-\frac{1}{4P_1(1+\omega_1^2)}\right]$$
$$\left.-\frac{2L_m^2P_2}{(m+1)\pi^2}\left[D_1+D_2+\overline{\hat{D}_1}+\overline{\hat{D}_2}+\frac{\sigma\zeta Q\pi}{P_2(\zeta+i\omega_2)}(E_1+E_2+\overline{\hat{E}_1}+\overline{\hat{E}_2})\right]\right\},$$

$$C_3 = \bar{C}_m/2\left\{\frac{2L_m^2P_1}{\pi^2 m}\left[\tilde{D}_3+\tilde{D}_4+\frac{\sigma\zeta Q\pi}{(\zeta+i\omega_1)P_1}(\tilde{E}_3+\tilde{E}_4)\right]\right. \tag{A.94}$$
$$-\frac{\sigma R_m}{P_1(1+i\omega_1)}\left[\frac{\pi^2}{2P_2^2(1+i\omega_2)(\varpi_2+2i\omega_2)}+\frac{\pi\left((2m+1)\bar{d}_1-\bar{d'}_1\right)}{16P_2(1+i\omega_2)}\right.$$
$$\left.\left.+\frac{\pi\left((2m+1)\bar{d}_2+\bar{d'}_2\right)}{4(m+1)}\right]\right\},$$

$$C_6 = \bar{C}_{m+1}/2\left\{-\frac{2L_m^2P_2}{(m+1)\pi^2}\left[\tilde{D}_1+\tilde{D}_2+\frac{\sigma\zeta Q\pi}{P_2(\zeta+i\omega_2)}(\tilde{E}_1+\tilde{E}_2)\right]\right. \tag{A.95}$$
$$-\frac{\sigma R_m}{P_2(1+i\omega_2)}\left[\frac{\pi^2}{2P_1^2(1+i\omega_1)(\varpi_1+2i\omega_1)}-\frac{\pi\left((2m+1)d_1-d_1'\right)}{16P_1(1+i\omega_1)}\right.$$
$$\left.\left.+\frac{\pi\left((2m+1)d_2+d'_2\right)}{4m}\right]\right\},$$

where $\hat{D}_i, \hat{E}_i, i = 1, ..., 4$ are $D_i, E_i, i = 1, ..., 4$ under the change $\omega_2 \to -\omega_2$, and overbar denotes the complex conjugation.

A.2 Limiting values of normal form coefficients

In this section we calculate limiting values the normal form coeficients $C_1, ..., C_6$ as $m \to \infty$, for both Cases I and II described in §2.5.

A.2.1 Case I

We first need certain auxilliary limits. Using the equations (4.61)–(4.68) in Chapter 4 , we find that as $m \to \infty$ in Case I we have

$$c_{11} = \lim_{m\to\infty} L_m c_1 = 0, \tag{A.96}$$

$$c_{21} = \lim_{m\to\infty} c_2 = \frac{\pi}{P^2(1+i\omega)(\varpi+2i\omega)} \quad \text{where } \varpi = 4\pi^2/P, \tag{A.97}$$

$$c_{31} = \lim_{m\to\infty} L_m c_3 = 0, \tag{A.98}$$

$$c_{41} = \lim_{m\to\infty} c_4 = 0, \tag{A.99}$$

$$c'_{11} = \lim_{m\to\infty} L_m c'_1 = 0, \tag{A.100}$$

$$c'_{21} = \lim_{m\to\infty} c'_2 = 0, \tag{A.101}$$

$$c'_{31} = \lim_{m\to\infty} L_m c'_3 = 0, \tag{A.102}$$

$$c'_{41} = \lim_{m\to\infty} c'_4 = \frac{2\pi^2\lambda^2}{P^2(\zeta+i\omega)(\varpi\zeta+2i\omega\lambda^2)}, \tag{A.103}$$

$$d_{11} = \lim_{m\to\infty} L_m d_1 = -\frac{\lambda i\omega Q\pi}{P(\zeta^2+\omega^2)(4\pi^2+Q)}, \tag{A.104}$$

$$d_{21} = \lim_{m\to\infty} d_2 = \frac{1}{4\pi P(1+\omega^2)}, \tag{A.105}$$

$$d_{31} = \lim_{m\to\infty} L_m d_3 = -\frac{2\lambda i\omega\pi^2}{\zeta P(4\pi^2+Q)(\zeta^2+\omega^2)}, \tag{A.106}$$

$$d_{41} = \lim_{m\to\infty} d_4 = 0, \tag{A.107}$$

$$d'_{11} = \lim_{m\to\infty} L_m d'_1 = 0, \tag{A.108}$$

$$d'_{21} = \lim_{m\to\infty} d'_2 = 0, \tag{A.109}$$

$$d'_{31} = \lim_{m\to\infty} L_m d'_3 = 0, \tag{A.110}$$

$$d'_{41} = \lim_{m\to\infty} d'_4 = \frac{\lambda^2}{2P(\zeta^2+\omega^2)}, \tag{A.111}$$

where P, ω and λ are given by (2.59), (2.60) and (2.27). Then (A.96)-(A.110) imply that

$$\lim_{m\to\infty} L_m D_1 = -\lim_{m\to\infty} L_m D_3 = -\frac{\pi^2\sigma\zeta Q\lambda c'_{41}}{2P(\zeta-i\omega)}, \tag{A.112}$$

$$\begin{aligned}
\lim_{m\to\infty} L_m E_1 &= -\lim_{m\to\infty} L_m E_3 = -\pi\lambda c'_{41}/2, && \text{(A.113)}\\
\lim_{m\to\infty} L_m D_2 &= -\lim_{m\to\infty} L_m D_4 = \frac{\pi^4\sigma\zeta Q\lambda c'_{41}}{2P^2(\zeta - i\omega)}, && \text{(A.114)}\\
\lim_{m\to\infty} L_m E_2 &= \lim_{m\to\infty} L_m E_4 = 0, && \text{(A.115)}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{m\to\infty} L_m \bar{\hat{D}}_1 = -\lim_{m\to\infty} L_m \hat{D}_3 &= (\pi^2/P)\left\{-\frac{\sigma\zeta Q(-d_{31} + 2\lambda d'_{41})}{4(\zeta + i\omega)}\right.\\
&\qquad \left. -\frac{\pi d_{11}}{2}\right\}, && \text{(A.116)}\\
\lim_{m\to\infty} L_m \bar{\hat{E}}_1 = -\lim_{m\to\infty} L_m \hat{E}_3 &= -\pi\lambda d'_{41}/2 - \pi d_{31}/4, && \text{(A.117)}\\
\lim_{m\to\infty} L_m \bar{\hat{D}}_2 = -\lim_{m\to\infty} L_m \hat{D}_4 &= (\pi^2/P)\left\{\frac{\pi^2\sigma\zeta Q(-d_{31} + (2/\lambda)d'_{41})}{P(\zeta + i\omega)}\right.\\
&\qquad \left. -\frac{\pi d_{11}(\lambda^2 - 1)}{4\lambda^2}\right\}, && \text{(A.118)}\\
\lim_{m\to\infty} L_m \bar{\hat{E}}_2 = -\lim_{m\to\infty} L_m \hat{E}_4 &= \frac{\pi^2 d_{11}}{4P(\zeta + i\omega)}. && \text{(A.119)}
\end{aligned}$$

We also have

$$\begin{aligned}
\lim_{m\to\infty} L_m \tilde{D}_1 = -\lim_{m\to\infty} L_m \tilde{D}_3 &= -(\pi^2/P)\left\{\frac{\sigma\zeta Q(d_{31} + 2\lambda d'_{41})}{4(\zeta + i\omega)} - \pi d_{11}/2\right\}, && \text{(A.120)}\\
\lim_{m\to\infty} L_m \tilde{E}_1 = -\lim_{m\to\infty} L_m \tilde{E}_3 &= -\pi\lambda d'_{41}/2 + \pi d_{31}/4, && \text{(A.121)}\\
\lim_{m\to\infty} L_m \tilde{D}_2 = -\lim_{m\to\infty} L_m \tilde{D}_4 &= (\pi^2/P)\left\{\frac{\pi^2\sigma\zeta Q(d_3 + (2/\lambda)d'_{41})}{P(\zeta + i\omega)}\right. .\\
&\qquad \left. +\frac{\pi d_{11}(\lambda^2 - 1)}{(4\lambda^2}\right\}, && \text{(A.122)}\\
\lim_{m\to\infty} L_m \tilde{E}_2 = -\lim_{m\to\infty} L_m \tilde{E}_4 &= -\frac{\pi^2 d_{11}}{8(\zeta + i\omega)}, && \text{(A.123)}
\end{aligned}$$

From (2.49) we get

$$\lim_{m\to\infty} \bar{C}_m = \lim_{m\to\infty} \bar{C}_{m+1} = -\frac{\pi^2(1 + i\omega)(\zeta + i\omega)}{2\lambda^2 P\omega(\omega - i\delta)}, \tag{A.124}$$

where $\delta = 1 + \sigma + \zeta$. Using (4.39)–(4.44), (A.96)–(A.110) and (A.112)-(A.123) we get

$$\lim_{L_m\to\infty} C_1 = \lim_{L_m\to\infty} C_5 = A+B, \tag{A.125}$$

$$\lim_{L_m\to\infty} C_2 = \lim_{L_m\to\infty} C_4 = A, \tag{A.126}$$

$$\lim_{L_m\to\infty} C_3 = \lim_{L_m\to\infty} C_6 = C, \tag{A.127}$$

where

$$\begin{aligned} A+B &= -\frac{\pi^2\sigma R_0(\zeta+i\omega)}{4\lambda^2P^2\omega(\omega-i\delta)}\left[-\frac{1}{4P(1+\omega^2)}-\frac{\pi^2}{2P^2(1+i\omega)(\varpi+2i\omega)}\right] \\ &+\frac{i\sigma\zeta Q\lambda^2\pi^4(1+i\omega)}{4P^3(\zeta^2+\omega^2)(\varpi\zeta+2i\omega\lambda^2)(\omega-i\delta)}, \end{aligned} \tag{A.128}$$

$$\begin{aligned} A &= \frac{-\pi^2(\zeta+i\omega)(1+i\omega)}{4\lambda^2P\omega(\omega-i\delta)}\left\{-\frac{\sigma R_0}{P(1+i\omega)}\left[\frac{\pi(c_{21}+d_{21})}{2}+\frac{\pi d_{11}}{4\lambda P(1+i\omega)}\right.\right. \\ &\left.+\frac{1}{4P(1+\omega^2)}\right]+\frac{\sigma Q\zeta^2\lambda^2 c_{41}'}{(\zeta^2+\omega^2)}-\frac{2\pi^2\sigma\zeta Q c_{41}'}{P(\zeta-i\omega)}+\frac{\sigma\zeta Q d_{41}'\lambda^2}{\zeta+i\omega} \\ &\left.+\frac{\pi d_{11}(3\lambda^2-1)}{4\lambda}+\frac{\pi^2\sigma\zeta Q(\lambda d_{31}-2d_{41}')}{P(\zeta+i\omega)}-\frac{\pi\sigma\zeta Q d_{11}}{4P(\zeta+i\omega)^2}\right\}, \end{aligned} \tag{A.129}$$

$$\begin{aligned} C &= \frac{-\pi^2(\zeta+i\omega)(1+i\omega)}{4\lambda^2P\omega(\omega-i\delta)}\left\{-\frac{\sigma R_0}{P(1+i\omega)}\left[\frac{\pi^2}{2P^2(1+i\omega)(\varpi+2i\omega)}\right.\right. \\ &\left.-\frac{\pi d_{11}}{4\lambda P(1+i\omega)}+\pi d_{21}/2\right]-\frac{\pi^2\sigma\zeta Q\lambda(d_{31}+2d_{41}')}{P(\zeta+i\omega)} \\ &\left.+\frac{\sigma\zeta Q d_{41}'\lambda^2}{\zeta+i\omega}+\frac{\sigma\zeta Q\lambda\pi d_{11}}{4P(\zeta+i\omega)^2}-\frac{\pi d_{11}(3\lambda^2-1)}{4\lambda}\right\}. \end{aligned} \tag{A.130}$$

A.2.2 Case II

In Case II, when

$$\zeta = m^{-k/2}\check{\zeta},\ Q = m^{k/2}\check{Q}, \tag{A.131}$$

for fixed $\sigma, \check{\zeta}, \check{Q}$ and $0<k<2$, the limits in equations (A.96)-(A.111) are replaced by

$$\check{c}_{11} = \lim_{m\to\infty} L_m^m c_1 = 0, \tag{A.132}$$

$$\check{c}_{21} = \lim_{m\to\infty} c_2 = \frac{\pi}{\check{P}^2(1+i\check{\omega})(\check{\varpi}+2i\check{\omega})} \quad \text{where } \check{\varpi} = 4\pi^2/\check{P}, \tag{A.133}$$

$$\check{c}_{31} = \lim_{m\to\infty} L_m^m c_3 = 0, \tag{A.134}$$

$$\check{c}_{41} = \lim_{m\to\infty} c_4 = 0, \tag{A.135}$$

$$\check{c}'_{11} = \lim_{m\to\infty} L_m^m c'_1 = 0, \tag{A.136}$$

$$\check{c}'_{21} = \lim_{m\to\infty} c'_2 = 0, \tag{A.137}$$

$$\check{c}'_{31} = \lim_{m\to\infty} L_m^m c'_3 = 0, \tag{A.138}$$

$$\check{c}'_{41} = \lim_{m\to\infty} c'_4 = -\frac{\pi^2}{\check{\omega}^2\check{P}^2}, \tag{A.139}$$

$$\check{d}_{11} = \lim_{m\to\infty} L_m^m d_1 = -\pi\check{\lambda}i/\check{P}\check{\omega}, \tag{A.140}$$

$$\check{d}_{21} = \lim_{m\to\infty} d_2 = \frac{1}{4\pi\check{P}(1+\check{\omega}^2)}, \tag{A.141}$$

$$\check{d}_{31} = \lim_{m\to\infty} L_m^m d_3 = -\frac{2\pi^2 i\check{\lambda}}{\zeta Q\check{P}\check{\omega}}, \tag{A.142}$$

$$\check{d}_{41} = \lim_{m\to\infty} d_4 = 0, \tag{A.143}$$

$$\check{d}'_{11} = \lim_{m\to\infty} L_m^m d'_1 = 0, \tag{A.144}$$

$$\check{d}'_{21} = \lim_{m\to\infty} d'_2 = 0, \tag{A.145}$$

$$\check{d}'_{31} = \lim_{m\to\infty} L_m^m d'_3 = 0, \tag{A.146}$$

$$\check{d}'_{41} = \lim_{m\to\infty} d'_4 = \frac{\check{\lambda}^2}{2\check{P}\check{\omega}^2}, \tag{A.147}$$

where $\check{P}, \check{\omega}$ and $\check{\lambda}$ satisfy (2.72)-(2.74). If $k = 2$ in (A.131), all the limits given by (A.132)-(A.147) are the same, except for $\check{d}_{41}$. In this case (A.146) is replaced by

$$\check{d}_{41} = \lim_{m\to\infty} d_4 = \frac{\check{\lambda}}{8\check{P}(\check{\omega}+2i\check{\zeta})}. \tag{A.148}$$

If $k > 2$, $\check{d}_{41}$ becomes unbounded as $m \to \infty$, and this in turn will imply that C_2, C_4, C_3 and C_6 become unbounded. Therefore we only consider the Case II with $k \leq 2$. Then (A.135)-(A.115) imply that

$$\lim_{m\to\infty} L_m^m D_1 = -\lim_{m\to\infty} L_m^m D_3 = \frac{\pi^2\sigma\zeta Q\check{\lambda}\check{c}'_{41}}{2\check{P}i\check{\omega}}, \tag{A.149}$$

$$\lim_{m\to\infty} L_m^m E_1 = -\lim_{m\to\infty} L_m^m E_3 = -\pi\check{\lambda}\check{c}'_{41}/2, \tag{A.150}$$

$$\lim_{m\to\infty} L_m^m D_2 = -\lim_{m\to\infty} L_m^m D_4 = -\frac{\pi^4\sigma\zeta Q\check{\lambda}\check{c}'_{41}}{2\check{P}^2 i\check{\omega}}, \tag{A.151}$$

$$\lim_{m\to\infty} L_m^m E_2 = \lim_{m\to\infty} L_m^m E_4 = 0, \tag{A.152}$$

and

$$\lim_{m\to\infty} L_m^m \bar{\hat{D}}_1 = -\lim_{m\to\infty} L_m^m \hat{D}_3 = (\pi^2/\check{P})\left\{\frac{\sigma\zeta Q(-\check{d}_{31} + 2\check{\lambda}\check{d}'_{41})}{4\check{\omega}} - \pi\check{d}_{11}/2\right\}, \tag{A.153}$$

$$\lim_{m\to\infty} L_m^m \bar{\hat{E}}_1 = -\lim_{m\to\infty} L_m^m \hat{E}_3 = -\pi\check{\lambda}\check{d}'_{41}/2 - \pi\check{d}_3/4, \tag{A.154}$$

$$\lim_{m\to\infty} L_m^m \bar{\hat{D}}_2 = -\lim_{m\to\infty} L_m^m \hat{D}_4 = (\pi^2/\check{P})\left\{-\frac{\pi^2\sigma\zeta Q(-\check{d}_{31} + (2/\check{\lambda})\check{d}'_{41})}{iP\check{\omega}} - \frac{\pi\check{d}_{11}(\check{\lambda}^2-1)}{4\check{\lambda}^2}\right\}, \tag{A.155}$$

$$\lim_{m\to\infty} L_m^m \bar{\hat{E}}_2 = -\lim_{m\to\infty} L_m^m \hat{E}_4 = -\frac{\pi^2\check{d}_{11}}{8i\check{\omega}}. \tag{A.156}$$

However, for $k = 2$ we have

$$\lim_{m\to\infty} L_m^m \bar{\hat{D}}_1 = (\pi^2/\check{P})\left\{\frac{\sigma\zeta Q(-\check{d}_{31} + 4\check{\lambda}(\check{d}'_{41} + \bar{\check{d}}_{41}))}{4\check{\omega}} - \pi\check{d}_{11}/2\right\}, \tag{A.157}$$

$$-\lim_{m\to\infty} L_m^m \hat{D}_3 = (\pi^2/\check{P})\left\{\frac{\sigma\zeta Q(-\check{d}_{31} + 4\check{\lambda}(\check{d}'_{41} + \check{d}_{41}))}{4\check{\omega}} - \pi\check{d}_{11}/2\right\}, \tag{A.158}$$

$$\lim_{m\to\infty} L_m^m \bar{\hat{E}}_1 = -\pi\check{\lambda}(\check{d}'_{41} + \bar{\check{d}}_{41}) - \pi\check{d}_{31}/4, \tag{A.159}$$

$$-\lim_{L_m^m\to\infty} L_m^m \hat{E}_3 = -\pi\check{\lambda}(\check{d}'_{41} + \check{d}_{41}) - \pi\check{d}_{31}/4, \tag{A.160}$$

$$\lim_{m\to\infty} L_m^m \bar{\hat{D}}_2 = (\pi^2/\check{P})\left\{-\frac{\pi^2\sigma\zeta Q(-\check{d}_{31} + (2/\check{\lambda})(\check{d}'_{41} + \bar{\check{d}}_{41})}{i\check{P}\check{\omega}} - \frac{\pi\check{d}_{11}(\check{\lambda}^2-1)}{4\check{\lambda}^2}\right\}, \tag{A.161}$$

$$-\lim_{m\to\infty} L_m^m \hat{D}_4 = (\pi^2/\check{P})\left\{-\frac{\pi^2\sigma\zeta Q(-\check{d}_{31} + (2/\check{\lambda})(\check{d}'_{41}) + \check{d}_{41})}{i\check{P}\check{\omega}} - \frac{\pi\check{d}_{11}(\check{\lambda}^2-1)}{4\check{\lambda}^2}\right\}, \tag{A.162}$$

and this implies that the limiting behaviours of the normal form coefficients are qualitatively different from those in Case I, and in Case II with $0 < k < 2$, since $\hat{d}_{41}$ is not real for $k = 2$. In fact it is easy to show that when $k = 2$ (A.157)–(A.162) imply that

$$\lim_{m\to\infty} C_2 \neq \lim_{m\to\infty} C_4, \quad \text{and} \quad \lim_{m\to\infty} C_3 \neq \lim_{m\to\infty} C_6. \tag{A.163}$$

unlike in Case I, and in Case II with $0 < k < 2$. For this reason, from now on we restrict ourselves in Case II to only $0 < k < 2$.

We also have

$$\begin{aligned}
\lim_{m\to\infty} L_m^m \tilde{D}_1 = -\lim_{m\to\infty} L_m^m \tilde{D}_3 &= -(\pi^2/\check{P})\left\{\frac{\sigma\zeta Q(\check{d}_{31} + 2\check{\lambda}\check{d}_{41})}{4i\check{\omega}} - \pi\check{d}_{11}/2\right\}, && \text{(A.164)}\\
\lim_{m\to\infty} L_m^m \tilde{E}_1 = -\lim_{m\to\infty} L_m^m \tilde{E}_3 &= -\pi\check{\lambda}\check{d}'_{41}/2 + \pi\check{d}_{31}/4, && \text{(A.165)}\\
\lim_{m\to\infty} L_m^m \tilde{D}_2 = -\lim_{m\to\infty} L_m^m \tilde{D}_4 &= (\pi^2/\check{P})\left\{\frac{\pi^2\sigma\zeta Q(\check{d}_{31} + (2/\check{\lambda})\check{d}_{41})}{i\check{P}\check{\omega}}\right. \\
&\qquad \left. + \frac{\pi\check{d}_{11}(\check{\lambda}^2 - 1)}{4\check{\lambda}^2}\right\}, && \text{(A.166)}\\
\lim_{m\to\infty} L_m^m \tilde{E}_2 = -\lim_{m\to\infty} L_m^m \tilde{E}_4 &= -\frac{\pi^2\check{d}_{11}}{4i\check{\omega}}, && \text{(A.167)}
\end{aligned}$$

From (2.55) we get

$$\lim_{m\to\infty} \bar{C}_m = \lim_{m\to\infty} \bar{C}_{m+1} = -\frac{\pi^2(1 + i\check{\omega})i}{2\check{\lambda}^2\check{P}(\check{\omega} - i(\sigma + 1))}, \tag{A.168}$$

Now using (A.132)–(A.147) and (A.149)–(A.150) we finally get

$$\begin{aligned}
\lim_{L_m^m\to\infty} C_1 &= \lim_{L_m^m\to\infty} C_5 = \check{A} + \check{B}, && \text{(A.169)}\\
\lim_{L_m^m\to\infty} C_2 &= \lim_{L_m^m\to\infty} C_4 = \check{A}, && \text{(A.170)}\\
\lim_{L_m^m\to\infty} C_3 &= \lim_{L_m^m\to\infty} C_6 = \check{C}, && \text{(A.171)}
\end{aligned}$$

where

$$\begin{aligned}
\check{A} + \check{B} &= \frac{i\pi^2\sigma\check{R}_0}{4\check{\lambda}^2\check{P}^2(\check{\omega} - i(\sigma + 1))}\left[\frac{1}{4\check{P}(1 + \check{\omega}^2)} + \frac{\pi^2}{2\check{P}^2(1 + i\check{\omega})(\check{\varpi} + 2i\check{\omega})}\right] \\
&\quad + \frac{\sigma\zeta Q\pi^4(1 + i\check{\omega})}{8\check{\omega}^3\check{P}^3(\check{\omega} - i(\sigma + 1))}, && \text{(A.172)}
\end{aligned}$$

$$\begin{aligned}
\check{A} &= \frac{i\pi^2\sigma\check{R}_0}{4\check{\lambda}^2\check{P}^2(\check{\omega}-i(\sigma+1))}\left[\frac{\pi(\check{c}_{21}+\check{d}_{21})}{2}+\frac{\pi\check{d}_{11}}{4\check{\lambda}\check{P}(1+i\check{\omega})}+\frac{1}{4\check{P}(1+\check{\omega}^2)}\right] \\
&\quad -\frac{\pi^2(1+i\check{\omega})i}{4\check{P}\check{\lambda}^2(\check{\omega}-i(\sigma+1)}\left[\frac{\pi^2\sigma\zeta Q(2\check{c}'_{41}-2\check{d}'_{41}+\check{\lambda}\check{d}_{31})}{i\check{P}\check{\omega}}+\frac{\pi\sigma\zeta Q\check{d}_{11}}{4\check{P}\check{\omega}^2}\right. \\
&\quad \left.+\frac{\pi\check{d}_{11}(3\check{\lambda}^2-1)}{4\check{\lambda}}+\frac{\check{\lambda}^2\sigma\zeta Q\check{d}'_{41}}{i\check{\omega}}\right], \qquad \text{(A.173)} \\
\check{C} &= \frac{i\pi^2\sigma R_0}{4\check{\lambda}^2\check{P}^2[\check{\omega}-i(\sigma+1)]}\left[\frac{\pi^2}{2\check{P}^2(1+i\check{\omega})(\check{\varpi}+2i\check{\omega})}-\frac{\pi\check{d}_{11}}{4\check{\lambda}\check{P}(1+i\check{\omega})}+\frac{\pi\check{d}_{21}}{2}\right] \\
&\quad -\frac{i\pi^2(1+i\check{\omega})}{4\check{P}\check{\lambda}^2(\check{\omega}-i(\sigma+1))}\left[-\frac{\pi^2\sigma\zeta Q\check{\lambda}(\check{d}_{31}+2\check{d}'_{41})}{i\check{P}\check{\omega}}-\frac{\pi\check{d}_{11}(3\check{\lambda}^2-1)}{4\check{\lambda}}\right. \\
&\quad \left.+\frac{\sigma\zeta Q\check{d}'_{41}\check{\lambda}^2}{i\check{\omega}}-\frac{\sigma\check{\zeta}Q\lambda\pi\check{d}_{11}}{i4\check{P}\check{\omega}^2}\right]. \qquad \text{(A.174)}
\end{aligned}$$

Appendix B

Numerical values of normal form coefficients

In this section we give more numerical results. In Tables B.1–B.6, numerical values in Case I of normal form coefficients for $\sigma = 1, \zeta = .1$ and for $Q = \pi^2, 10^4\pi^2$ and $10^6\pi^2$ and for increasing values of m, are given. As in Chapter 4, the symbols ∞ in the tables show the corresponding values of coefficients as $m \to \infty$. In our numerical calculations we have also calculated the values of C_1, C_5 for a wider set of parameter values (i) $\sigma = 1, \zeta = .01$, (ii) $\sigma = 1, \zeta = .5$, (iii) $\sigma = 10^{-6}, \zeta = .01$ and increasing values of Q and m. We have also considered, for fixed ζ, a decreasing sequence of values of σ and an increasing sequence of values of Q such that $P_j\omega_j,\ j = 1, 2$ remains fixed. Because of lack of space we give only a sample of these numerical calculations. In all of these calculations we find that C_{1R} and C_{5R} are negative. Because the numerical calculations of C_2 and C_4 were very time consuming, we have looked at the values of $C_{1R} - C_{2R}$ for large m for several different set of values of σ, ζ and Q. In all of these calculations we found that $C_{2R} < 0$ and $C_{1R} - C_{2R} > 0$. These values are important in stability results of Chapter 5. We also give another sample of our numerical calculation of normal form coefficients in Case II.

Table B.1: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = \pi^2$.

m	$2L_m/m$	R_m	$100a_1$	$100a_2$
1	2.007	950.3	$.1822 - .1517i$	$.2871 - .8320i$
101	1.407	811.5	$.2555 - .2828i$	$.5720 - .2881i$
10001	1.400	811.5	$.2563 - .2854i$	$.2563 - .2854i$
100001	1.400	811.5	$.2563 - .2854i$	$.2563 - .2854i$
∞	1.400	811.5	$.2563 - .2854i$	$.2563 - .2854i$

Table B.2: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = \pi^2$ (continued).

m	$L_m^2 b_{2R}$	$L_m b_{1I}$	$L_m b_{2I}$	
1	−2.882	5.294	3.942	−2.8277
101	−3.893	3.927	3.379	3.337
10001	−3.909	3.910	3.358	3.358
100001	−3.910	3.910	3.358	3.358
∞	−3.910	3.910	3.358	3.358

Table B.3: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = \pi^2$ (continued).

m	C_1	C_2	C_4	C_5
1	$-.3061 + .7467i$	$-1.868 - .4878i$	$-1.714 - .3182i$	$-.6399 + .3554i$
101	$-.4912 + .6478i$	$-1.673 - .3706i$	$-1.665 - .3708i$	$-.4973 + .6422i$
10001	$-.4942 + .6451i$	$-1.668 - .3706i$	$-1.667 - .3706i$	$-.4943 + .6450i$
100001	$-.4943 + .6450i$	$-1.668 - .3706i$	$-1.668 - .3706i$	$-.4943 + .6450i$
∞	$-.4943 + .6450i$	$-1.668 - .3706i$	$-1.668 - .3706i$	$-.4943 + .6450i$

Table B.4: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 10^4\pi^2$.

m	$2L_m/m$	$.1R_m$	$1000a_1$	$1000a_2$
1	.6145	7596	$.2637 - .7716i$	$.5153 - .5004i$
101	.4229	6919	$.4417 - .7250i$	$.4460 - .7203i$
10001	.4208	6919	$.4438 - .7227i$	$.4439 - .7226i$
100001	.4208	6919	$.4438 - .7227i$	$.4439 - .7227i$
∞	.4208	6919	$.4439 - .7227i$	$.4439 - .7227i$

Table B.5: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 10^4\pi^2$ (continued).

m	$L_m^2 b_{1R}$	$L_m^2 b_{2R}$	$L_m b_{1I}$	$L_m b_{2I}$
1	−5.734	15.91	95.51	204.7
101	−9.839	9.989	124.6	126.4
10001	−9.913	9.915	125.5	125.5
100001	−9.914	9.914	125.5	125.5
∞	−9.914	9.914	125.5	125.5

Table B.6: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 10^4\pi^2$ (continued).

m	C_1	C_2	C_4	C_5
1	$-.0045 + .0211i$	$-.0132 + .0528i$	$-.0209 + .0349i$	$-.0213 + .0274i$
101	$-.0115 + .0259i$	$-.0260 + .0625i$	$-.0222 + .0632i$	$-.0117 + .0261$
10001	$-.0116 + .0260i$	$-.0242 + .0632i$	$-.0241 + .0632i$	$-.0116 + .0260i$
100001	$-.0116 + .0260i$	$-.0242 + .0632i$	$-.0242 + .0632i$	$-.0116 + .0260i$
∞	$-.0116 + .0260i$	$-.0242 + .0632i$	$-.0242 + .0632i$	$-.0116 + .0260i$

Table B.7: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 10^6\pi^2$.

m	$2L_m/m$	$10^{-7}R_m$	$10^4 a_1$	$10^4 a_2$
1	.2778	.5795	$.1430 - 1.0887i$	$.4582 - .9322i$
101	.1904	.5651	$.2873 - 1.071i$	$.2922 - 1.068i$
10001	.1895	.5651	$4.2897 - 1.070i$	$.2898 - 1.070i$
100001	.1895	.5651	$.2897 - 1.070i$	$.2898 - 1.070i$
∞	.1895	.5651	$.2898 - 1.070i$	$.2898 - 1.070i$

Table B.8: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 10^6\pi^2$ (continued).

m	$L_m^2 b_{1R}$	$L_m^2 b_{2R}$	$L_m b_{1I}$	$L_m b_{2I}$
1	−2.938	13.943	245.8	825.4
101	−6.163	6.308	351.3	359.3
10001	−6.234	6.236	355.2	355.3
100001	−6.235	6.235	355.3	355.3
∞	−6.235	6.235	355.3	355.3

Table B.9: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .1$, $Q = 10^6\pi^2$ (continued).

m	$100C_1$	$100C_2$	$100C_4$	$100C_5$
1	$-.06848 + .5881i$	$-.2550 + 2.072i$	$-.3500 + .8118i$	$-.7566 + 1.603i$
101	$-.2637 + 1.049i$	$-.4901 + 1.936i$	$-.4136 + 1.928i$	$-.2730 + 1.064i$
10001	$-.2683 + 1.056i$	$-.4545 + 1.948i$	$-.4536 + 1.948i$	$-.2684 + 1.056i$
100001	$-.2685 + 1.057i$	$-.4540 + 1.948i$	$-.4540 + 1.948i$	$-.2685 + 1.057i$
∞	$-.2685 + 1.057i$	$-.4540 + 1.948i$	$-.4540 + 1.948i$	$-.2685 + 1.057i$

Table B.10: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .01$, $Q = \pi^2$.

m	C_1	C_5
1	$-.2247 + 1.814i$	$-.5729 + 1.724i$
11	$-.3288 + 1.801i$	$-.3711 + 1.792i$
101	$-.3468 + 1.797i$	$-.3516 + 1.796i$
1001	$-.3489 + 1.797i$	$-.3494 + 1.797i$
10001	$-.3491 + 1.797i$	$-.3492 + 1.797i$
∞	$-.3491 + 1.797i$	$-.3492 + 1.797i$

Table B.11: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .01$. $Q = 100\pi^2$.

m	C_1	C_5
1	$-.0837 + .2139i$	$-.0755 + .1997i$
11	$-.0802 + .2078i$	$-.0791 + .2059i$
101	$-.0798 + 2069i$	$-.0796 + .2067i$
1001	$-.0797 + .2068i$	$-.0797 + .2068i$
∞	$-.0797 + .2068i$	$-.0797 + .2068i$

Table B.12: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .01$, $Q = 10^4\pi^2$.

m	$10C_1$	$10C_5$
1	$-.1006 + .4180i$	$-.2597 + .3921i$
11	$-.1705 + .4252i$	$-.1926 + .4218i$
101	$-.1803 + .4240i$	$-.1828 + .4236i$
1001	$-.1814 + .4238i$	$-.1817 + .4238i$
10001	$-.1816 + .4238i$	$-.1816 + .4238i$
∞	$-.1816 + .4238i$	$-.1816 + .4238i$

Table B.13: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .5$, $Q = 100\pi^2$.

m	C_1	C_5
1	$-.06479 + .1288i$	$-.08207 + .1111i$
11	$-.07989 + .1154i$	$-.08240 + .1119i$
101	$-.0811 + .1138i$	$-.0814 + .1134i$
1001	$-.0812 + .1136i$	$-.0812 + .1135i$
10001	$-.0812 + .1136i$	$-.0812 + .1136i$
∞	$-.0812 + .1136i$	$-.0812 + .1136i$

Table B.14: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .5$, $Q = 10^4\pi^2$.

m	$10C_1$	$10C_5$
1	$-.0904 + .3310i$	$-.4095 + .5270i$
11	$-.2110 + 4417i$	$-.2559 + .4681i$
101	$-.2306 + .4540i$	$-.2357 + .4570i$
1001	$-.2329 + .4553i$	$-.2334 + .4556i$
10001	$-.2331 + .4555i$	$-.2331 + .4555i$
∞	$.2331 + .4555i$	$-.2331 + .4555i$

Table B.15: Normal form coefficients (Case I) for $\sigma = 1, \zeta = .5$, $Q = 10^6\pi^2$.

m	$10C_1$	$10C_5$
1	$-.01411 + .1185i$	$-.1538 + .3537i$
11	$-.0476 + .2103i$	$-.0476 + .2103i$
101	$-.0546 + .2242i$	$-.0565 + .2278i$
1001	$-.0555 + .2258i$	$-.0557 + .2262i$
10001	$-.0557 + .2262i$	$-.0557 + .2262i$
∞	$-.0556 + .2261i$	$-.0556 + .2261i$

Table B.16: Normal form coefficients (Case I) for $\sigma = 10^{-6}, \zeta = .01$, $Q = 40000\pi^2$.

m	C_1	C_5
1	$-2.298 + 7.327i$	$-6.313 + .8211i$
11	$-4.286 + 6.332i$	$-4.938 + 5.676i$
101	$-4.578 + 6.069i$	$-4.652 + 5.994i$
1001	$-4.611 + 6.036i$	$-4.618 + 6.028i$
10001	$-4.614 + 6.032i$	$-4.615 + 6.031i$
100001	$-4.615 + 6.032i$	$-4.615 + 6.032i$
∞	$-4.615 + 6.032i$	$-4.615 + 6.032i$

Table B.17: Normal form coefficients (Case I) for $\sigma = 10^{-6}, \zeta = .01$, $Q = 10^6\pi^2$.

m	C_1	C_5
1	$-.0644 + 1.263i$	$-.2487 + 1.232i$
11	$-.1289 + 1.258i$	$-.1314 + 1.257i$
101	$-.1300 + 1.258i$	$-.1303 + 1.258i$
1001	$-.1301 + 1.258i$	$-.1301 + 1.258i$
∞	$-.1301 + 1.258i$	$-.1301 + 1.258i$

Table B.18: Normal form coefficients (Case I) for $\sigma = 10^{-6}, \zeta = .01$, $Q = 10^8\pi^2$.

m	$100C_1$	$100C_5$
1	$-.1788 + 12.66i$	$-.4366 + 12.62i$
11	$-.2577 + 12.65i$	$-.2889 + 12.64i$
101	$-.2710 + 12.65i$	$-.2746 + 12.65i$
1001	$-.2726 + 12.65i$	$-.2730 + 12.65i$
10001	$-.2728 + 12.65$	$-.2728 + 12.65i$
∞	$-.2728 + 12.65$	$-.2728 + 12.65i$

Table B.19: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .01$, $\hat{Q} = \pi^2$, $k = 1$.

m	$2L_m/m$	R_m	$10^3 a_1$	$10^3 a_2$
1	2.024	786.5	$1.975.4116i$	$3.133 - .9750i$
101	1.419	660.3	$2.802 - .1252i$	$2.821 - .1260i$
10001	1.412	659.1	$2.814 - .0722i$	$2.814 - .0722i$
100001	1.412	659.0	$2.814 - .0681i$	$2.814 - .0681i$
100000001	1.412	658.9	$2.814 - .0663i$	$2.814 - .0663i$
∞	1.412	658.9	$2.814 - .0663i$	$2.814 - .0663i$

Table B.20: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .01$, $\hat{Q} = \pi^2$, $k = 1$ (continued).

m	$L_m^2 b_{1R}$	$L_m^2 b_{2R}$	$L_m b_{1I}$	$L_m b_{2I}$
1	-2.597	4.794	.8240	$-.6923$
101	-3.484	3.515	.4655	.4673
10001	-3.496	3.497	.4657	.4657
100001	-3.496	3.496	.4657	.4657
100000001	-3.496	3.496	.4656	.4656
∞	-3.496	3.496	.4656	.4656

Table B.21: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .01, \hat{Q} = \pi^2$, $k = 1$ (continued).

m	C_1	C_2	C_4	C_5
1	$-.2247 + 1.814i$	$-6.969 - 2.731i$	$-12.79 - 4.230i$	$-.5729 + 1.724i$
101	$-.1133 + 1.773i$	$-24.93 - 1.101i$	$-23.02 - 1.050i$	$-.1135 + 1.773i$
10001	$-.0909 + 1.771i$	$-32.18 - 1.290i$	$-31.94 - 1.284i$	$-.0908 + 1.771i$
100001	$-.0886 + 1.771i$	$-33.21 - 1.352i$	$-33.19 - 1.351i$	$-.0886 + 1.771i$
$10^8 + 1$	$-.0883 + 1.771i$	$-33.33 - 1.359i$	$-33.33 - 1.359i$	$-.0883 + 1.771i$
∞	$-.0883 + 1.771i$	$-33.33 - 1.359i$	$-33.33 - 1.359i$	$-.0883 + 1.771i$

Table B.22: Normal form coefficients (Case II) for $\sigma = 1, \hat{\zeta} = .01, \hat{Q} = \pi^2$, $k = 1$ (continued).

m	C_{3R}	C_{3I}	C_{6R}	C_{6I}
1	9.188	−4.566	11.40	−10.87
101	24.91	−.6175	22.91	−.6644
10001	32.11	.1286	31.86	.1230
100001	33.14	.1968	33.12	.1963
$10^8 + 1$	33.26	.2044	33.26	.2044
∞	33.26	.2044	33.26	.2044