SYMMETRIZATION, GREEN'S FUNCTIONS, HARMONIC MEASURES AND DIFFERENCE EQUATIONS

by

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Abstract

Symmetrization methods transform functions or sets into functions or sets having desirable features such as symmetry or some kind of convexity. We consider whether the values of various functionals do or do not increase under the transformation.

First, we answer in the negative a question of W. K. Hayman (1967) on the precise way that Green's functions increase under circular rearrangement.

Next we study symmetrization techniques in discrete cases, obtaining convolution-rearrangement inequalities of the form

$$\sum_{x,y\in G} f(x)K(d(x,y))g(y) \leq \sum_{x,y\in M} f^{\#}(x)K(d(x,y))g^{\#}(y),$$

for several graphs G, where $f^{\#}$ is a carefully chosen rearrangement of a real function f on G. The graphs we work with are primarily the circular graphs $\mathbb{Z}_n$ and the p-regular trees ($p \geq 3$). These inequalities allow us to obtain a full analogue of the classical Faber-Krahn inequality in the case of subsets of a p-regular tree.

We show that the convolution-rearrangement inequalities imply results on the effects of discrete Steiner-type symmetrization on harmonic measures and Green's functions, and we obtain discrete analogues of some of Albert Baernstein II's (1994) results for partial differential equations.

Then, we consider the collection $\mathfrak{B}$ of holomorphic functions f on the unit disc $\mathbb{D}$ with $f(0) = 0$ and $\frac{1}{\pi}\iint_{\mathbb{D}} |f'(x+iy)|^2\,dx\,dy \leq 1$. We study qualitative properties (e.g., symmetry or various kinds of convexity) of the extremals of functionals Λ_Φ on $\mathfrak{B}$ defined by $\Lambda_\Phi(f) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(f(e^{i\theta}))\,d\theta$, for $f \in \mathfrak{B}$ where Φ is a fixed real function on $\mathbb{C}$ satisfying various properties.

S.-Y. A. Chang and D. E. Marshall (1985) proved that Λ_Φ is bounded on $\mathfrak{B}$ if $\Phi(z) = e^{|z|^2}$. While it is not known whether this Λ_Φ attains a maximum over $\mathfrak{B}$, we do show that there is a perturbation of Φ for which the corresponding functional does *not* attain its supremum.

Finally, a study motivated by several conjectures concerning least harmonic majorant functionals and radial rearrangement leads us to obtain, among other things, an improved version of Arne Beurling's (1933) shove theorem for harmonic measures on slit discs.

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Ad maiorem Dei gloriam

Foreword

Most of Section I.7 is taken and/or adapted from the author's paper [89] which will appear in the *Proceedings of the American Mathematical Society*; text and diagrams taken from this paper are copyright © 1994 The American Mathematical Society. The American Mathematical Society copyright agreement permits such use of the text by the author.

Most of Section II.3.3 is taken and/or adapted from the author's paper [88] which will appear in the *Canadian Mathematical Bulletin*; text taken from this paper is copyright © 1994 The Canadian Mathematical Society. Permission to use the text in the thesis has been secured.

Scattered major portions of the text in Sections III.1–III.4.4 (with a noteworthy and particular exception of the above-discussed Section III.3.3) are taken and/or adapted from the author's joint paper with Alec Matheson [75] which will appear in the *Transactions of the American Mathematical Society*; text taken from this paper is copyright © 1995 The American Mathematical Society. The author thanks Professor Alec Matheson for granting permission for the author to use his own judgement in describing the author's contribution to the joint work and for granting permission to use the text in the thesis. The American Mathematical Society copyright agreement permits such use of the text by the authors.

Most of Section IV.9 is taken from the author's paper [87] which will appear in the *Annales de l'Institut Henri Poincaré – Probabilités & Statistiques*. As of the date of submission of this thesis, the copyright in this paper remains vested in the author. Permission of the journal editor to use the text in this thesis has nonetheless been secured.

Introduction

We study symmetrization and certain nonlinear functionals on collections of sets or functions.

Symmetrization theory strives to replace a given object (function or set) by one which is somehow related to it so that (a) the replacement object has some desirable features, such as symmetry or some kind of convexity, and (b) the replacement "improves" the values of various functionals associated with the object. Note that given a symmetrization method which replaces sets with sets, we can often automatically get a method which replaces functions with functions by applying the method to the level sets of the functions and reassembling the symmetrized level sets to form a new function.

A typical kind of desirable feature that fits under (a) is **circular symmetry**: a subset of the complex plane is said to be circularly symmetric if the intersection of this subset with every circle centred about the origin is a connected arc centred about the positive real axis. Thus, circular symmetry includes reflection symmetry about the real axis and a kind of angular convexity condition. Given a measurable set S in the complex plane, we can define the **circular symmetrization** $S^{\circledcirc}$ by requiring that $S^{\circledcirc}$ be circularly symmetric and satisfy the following three conditions for every $r \geq 0$:

(i) the circle of radius r about the origin is contained in $S^{\circledcirc}$ if and only if it is contained in S

(ii) the angular measure of the intersection of the circle of radius r about the origin with $S^{\circledcirc}$ equals the angular measure of the intersection of the same circle with S

(iii) the intersection of $S^{\circledcirc}$ with the circle of radius r about the origin is open in that circle.

See Figure 0.1. Condition (ii) implies that the area of $S^{\circledcirc}$ equals that of S.

We now describe the kinds of functionals that interest us in connection with circular symmetrization. Let D be a domain containing the origin (i.e., a connected open subset of $\mathbb{C}$ with $0 \in D$). Let $\phi\colon [0, \infty) \to [-\infty, \infty)$ be such that $t \mapsto \phi(e^t)$ is convex and increasing (in particular if ϕ is convex and increasing then this condition will hold). Let h be the infimum of all real functions g which are harmonic on D and satisfy $g(z) \geq \phi(|z|)$ for all z in D. Then put $\Gamma_\phi(D) = h(0)$.

An alternate probabilistic description of $\Gamma_\phi(D)$ is as follows. Let B_t be a Brownian motion in the complex plane starting at the origin. Let $\tau = \inf\{t \geq 0 : B_t \notin D\}$ be the first time that the Brownian motion leaves D. Then $\Gamma_\phi(D)$ is the expected value of $\phi(|B_\tau|)$. It is thus a kind of weighted average of $\phi(|z|)$ as z ranges over the boundary of D, with the weight at a point being given by the probability density that a Brownian motion first impacts the boundary at that point.

An important result on the Γ_ϕ functionals is a consequence of a theorem of Baernstein's groundbreaking paper [7] and says that $\Gamma_\phi(D) \leq \Gamma_\phi(D^{\circledcirc})$. Hence, circular symmetrization increases the above-defined Γ_ϕ functionals. Circular symmetrization also increases convex circular means of Green's functions. The Green's function of a sufficiently regular domain D with pole at $w \in D$ is defined to be the (unique) function $g(\cdot, w; D)$ vanishing on the boundary of D, harmonic on $D \backslash \{w\}$ and such that $z \mapsto g(z, w; D) - \log \frac{1}{|z-w|}$ remains harmonic in a neighbourhood of w.

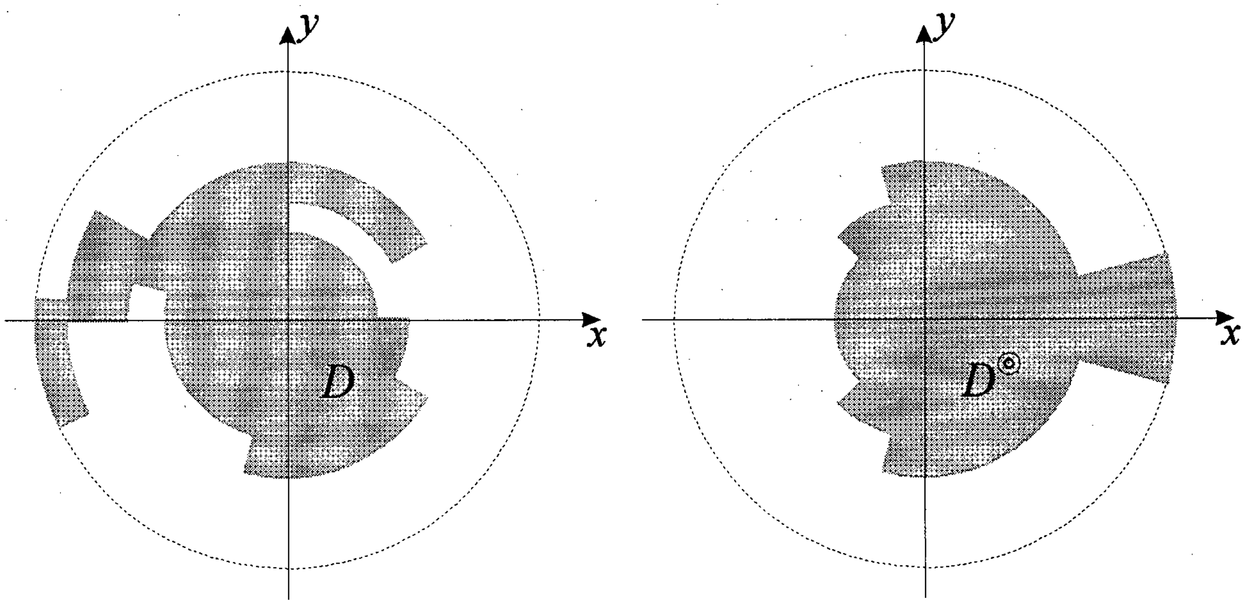

Figure 0.1: An example of circular symmetrization

Baernstein's theorem [7] then states that for every $r \geq 0$ we have

$$\int_0^{2\pi} \psi(g(re^{i\theta}, w; D))\, d\theta \leq \int_0^{2\pi} \psi(g(re^{i\theta}, |w|; D^{\circledcirc}))\, d\theta,$$

for every convex increasing function ψ. This is of particular interest in the case of $w = 0$. The case of $w = 0$ and $\psi(t) = t$ implies the above-mentioned result on the increase of the Γ_ϕ under circular symmetrization. An interesting question is whether we may take ψ to be any increasing function, not necessarily convex. This was asked by Hayman [59]. The answer turns out to be negative, even in the case of $w = 0$, as we shall prove in §I.7.

A second set of interesting functionals are the w_r harmonic measure functionals. Fix $r > 0$. Let D be a domain containing the origin. Let D_r be the intersection of D with the disc of radius r centred at 0. Then, let h be the harmonic function on D_r which equals 1 on the parts of the boundary of D_r which lie on the circle of radius r about the origin (i.e., h equals 1 on $(\partial D_r) \cap \{z \in \mathbb{C} : |z| = r\}$) and which equals 0 on all the other parts of the boundary of D_r. (See Figure 0.2.) Put $w_r(D) = h(0)$. A probabilistic interpretation of this functional is that it is the probability that a Brownian motion starting at the origin reaches the circle of radius r centred at 0 before impacting on any other part of the boundary of D. Hence, intuitively it measures how easy it is for a Brownian particle to reach the circle of radius r while staying in D.

Once again, Baernstein [7] has shown that $w_r(D) \leq w_r(D^{\circledcirc})$ and hence circular symmetrization increases the w_r functionals. The intuitive reason for this in terms of the probabilistic interpretation is that the circular symmetrization straightens out and consolidates the roads leading from 0 to the circle of radius r; see Figure 0.1.

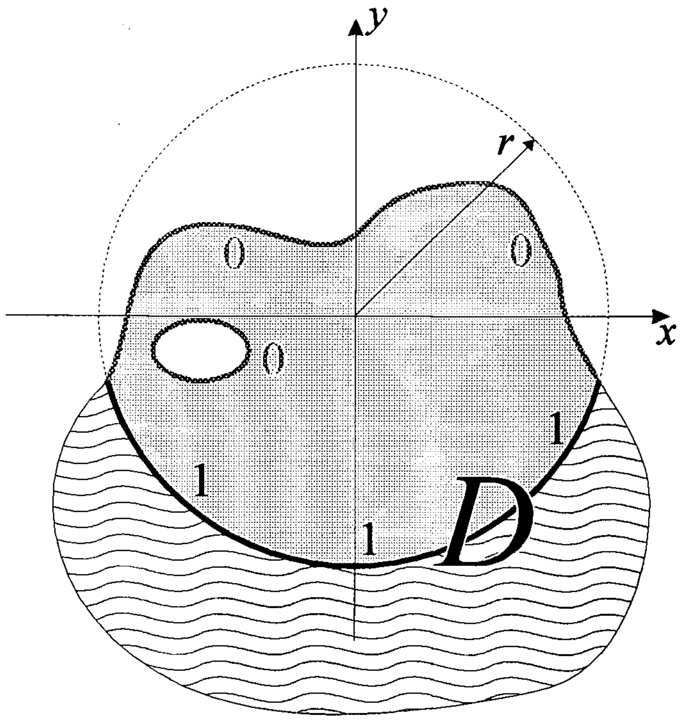

Figure 0.2: Boundary values for the Dirichlet problem associated with the w_r functional on D

In Chapter I we shall set up definitions and give some theorems on general and specific symmetrization methods, as well as cite and summarize some material on the notions needed to define and study our functionals. We shall also give the most important results of Baernstein's famous paper [7], as well as our answers to the question of Hayman [59] mentioned above. Finally, we shall conclude Chapter I by obtaining a lower bound on the size of the set on which the function $g(\cdot, 0; D)$ is radially decreasing; this will be useful to us in Chapter IV.

In Chapter II, we shall consider symmetrization theory in discrete settings. For instance, we shall prove a generalization of a full analogue of Baernstein's above-cited results on the increase in Green's functions and w_r-functionals under circular symmetrization in the setting of subsets of the discrete cylinder $\mathbb{Z} \times \mathbb{Z}_m$. (Note that special cases of our results can be proved by the methods of Quine [90].) Of course the discrete cylinder is a discrete version of the continuous cylinder $\mathbb{R} \times \{z \in \mathbb{C} : |z| = 1\}$, which is conformally equivalent to the punctured plane $\mathbb{C}\backslash\{0\}$ under a natural exponential conformal equivalence, and Baernstein's results can be lifted to the continuous cylinder, which is why we can say that our results on the discrete cylinder are analogous to his theorems.

The method of Chapter II proceeds by proving convolution-symmetrization inequalities of the form

$$\sum_{x,y \in G} f(x)K(d(x,y))g(y) \leq \sum_{x,y \in G} f^{\#}(x)K(d(x,y))g^{\#}(y),$$

where G is a specific connected graph such as $\mathbb{Z}_m$, d is the shortest-distance metric on G, K is a decreasing function and $f^{\#}$ and $g^{\#}$ indicate symmetrizations of arbitrary functions f and g on

G. Symmetrizations inducing such inequalities do not exist for all graphs; indeed, we shall use a computer-based proof to show that they do not exist for the cube $G = \mathbb{Z}_2^3$ or for the ternary plane $G = \mathbb{Z}_3^2$. However, we shall prove the inequalities in the special cases of the circular graph $G = \mathbb{Z}_m$, the p-regular tree $G = T_p$ and the octahedron edge graph $G = H_8$. And, of course, the case of the linear graph $G = \mathbb{Z}$ goes back to Hardy and Littlewood (see [58, Thm. 371]). Our method for proving these kinds of convolution-symmetrization inequalities will be a discrete version of a method of Beckner [18, 19, 20, 21], and Baernstein and Taylor [15], as presented by Baernstein [11]. This method in the discrete setting is in fact completely elementary.

Given appropriate convolution-symmetrization inequalities, we then get several consequences. First, given the case of $G = T_p$ we obtain a full Faber-Krahn inequality for subsets of the p-regular tree. Recall that the classical Faber-Krahn [47, 68] inequality stated that out of all domains D in $\mathbb{R}^n$ of a fixed area, the first non-zero Dirichlet[1] eigenvalue of the negated Laplacian $-\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is minimized precisely for a ball. Our Faber-Krahn inequality will provide a characterization of the subsets D of T_p of a given size which minimize the first non-zero eigenvalue of the negated combinatorial Laplacian with the Dirichlet boundary condition that our functions vanish outside our subset D. Not surprisingly, these subsets look like a discrete tree version of the balls which were optimal in the classical Faber-Krahn case (see Figure II.6.3 on p. 104). Our proofs, are completely elementary, although the proof of the uniqueness of the optimal subsets is quite involved.

Secondly, given a convolution-symmetrization inequality on a constant degree graph G, we can, as already mentioned, obtain analogues of Baernstein's results. Indeed, we shall obtain results on symmetrization for domains which are subsets of $\mathbb{Z} \times G$ (where G is $\mathbb{Z}$, $\mathbb{Z}_m$, T_p or H_8); these will concern the increase in certain generalized harmonic measures and generalized Green's functions. In fact, we shall proceed by two methods. Firstly, we shall prove these results via a probabilistic approach going back to Haliste [56]. Secondly, we shall show even more general results for difference equations on $\mathbb{Z} \times G$ (even for some non-linear ones) by a modified version of a method of Baernstein [11] and Weitsman [99] who proved such results for partial differential equations.

In Chapter III we shall engage in an analysis of various functionals on some collections of holomorphic functions. For instance, let $\mathfrak{B}$ be the collection of all functions f holomorphic on the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ which satisfy $f(0) = 0$ and have

$$\frac{1}{\pi} \iint_{\mathbb{D}} |f'(x+iy)|^2 \, dx \, dy \leq 1.$$

Note that $|f'(x+iy)|^2$ is the Jacobian of the mapping f, and hence the displayed condition can be interpreted as saying that the area of the image of f counting multiplicities does not exceed π. If f is one-to-one then this condition simply says that the area of the image of f does not exceed π. Let Φ be a Borel-measurable function on $\mathbb{C}$. Put

$$\Lambda_\Phi(f) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(f(e^{i\theta})) \, d\theta,$$

[1]The Dirichlet eigenvalues are obtained by having $-\Delta$ act on those functions on our domain D which vanish on the boundary of D.

for $f \in \mathfrak{B}$. (This makes sense since functions in $\mathfrak{B}$ have radial limits almost everywhere on the unit circle.) If ϕ is a Borel-measurable function on $[0, \infty)$, then we abuse notation and write Λ_ϕ for Λ_Φ where $\Phi(z) = \phi(|z|)$.

The Λ_ϕ functionals are closely related to the Γ_ϕ functionals. Indeed, we have the inequality

$$\Lambda_\phi(f) \leq \Gamma_\phi(D) \tag{0.1}$$

whenever $\phi\colon [0, \infty) \to [-\infty, \infty)$ is such that $t \mapsto \phi(e^t)$ is convex and increasing while D is the image of f (assuming f is non-constant). In the displayed inequality, we will actually have equality if f is one-to-one. Moreover, if $f \in \mathfrak{B}$ then $\operatorname{Area} D \leq \pi$. Hence, there are interesting connections between the Λ_ϕ functionals on $\mathfrak{B}$ and the Γ_ϕ functionals on $\mathcal{B}$, where $\mathcal{B}$ is the collection of all domains containing the origin and having area at most π.

The result that had sparked our interest in the Λ_ϕ functionals was the Chang-Marshall inequality [32] (another proof was later given by Marshall [72]) which says that

$$\sup_{f \in \mathfrak{B}} \Lambda_\phi(f) < \infty, \tag{0.2}$$

if $\phi(t) = e^{t^2}$ for $t \in [0, \infty)$. This inequality is rather difficult to prove and very sharp. Indeed, we shall show that if ϕ is a function on $[0, \infty)$ such that $\sup_{f \in \mathfrak{B}} \Lambda_\phi(f) < \infty$ then there exists a $C < \infty$ such that for all $t \in [0, \infty)$ we have $\phi(t) \leq Ce^{t^2}$. The Chang-Marshall inequality was improved by Essén [44] to assert that

$$\sup_{D \in \mathcal{B}} \Gamma_\phi(D) < \infty, \tag{0.3}$$

where $\phi(t)$ again equals e^{t^2}. (That this is actually an improvement follows from (0.1).)

We spend some time in Chapter III examining the Λ_Φ functionals for their own sake. We shall in particular be interested in the **extremals** of these functionals, namely functions $f \in \mathfrak{B}$ such that $\Lambda_\Phi(f) \geq \sup_{g \in \mathfrak{B}} \Lambda_\Phi(g)$. For instance, as a generalization of a result of Matheson [73], we have a theorem that if Φ is upper semicontinuous on $\mathbb{C}$ and $\Phi(z) = o(e^{|z|^2})$ as $|z| \to \infty$ (uniformly in $\arg z$), then Λ_Φ has an extremal on $\mathfrak{B}$. We also obtain some results on the Λ_ϕ functionals in a few other settings such as analogous functionals acting on balls of Hilbert spaces of measurable functions.

The question of existence of an extremal function is a quite interesting one in the case of the Chang-Marshall inequality. Indeed it is still not known whether Λ_ϕ has an extremal if $\phi(t) = e^{t^2}$. We shall prove, however, that even if it does, still there exists an infinitely differentiable convex increasing function $\phi_1(t)$ satisfying $\phi_1(t) \leq e^{t^2}$ for all $t \in [0, \infty)$ such that $\phi_1(t)/e^{t^2} \to 1$ as $t \to \infty$ but Λ_{ϕ_1} has no extremal. On the other hand, we shall also show that there exists an infinitely differentiable convex increasing function $\phi_2(t)$ satisfying $\phi_2(t) \geq e^{t^2}$ for all $t \in [0, \infty)$ and again such that $\phi_2(t)/e^{t^2} \to 1$ as $t \to \infty$, but this time with Λ_{ϕ_2} achieving a maximum. These results will show that the existence or nonexistence of an extremal function in $\mathfrak{B}$ for Λ_ϕ if $\phi(t) = e^{t^2}$ depends on the precise nature of the function e^{t^2} and not just on its asymptotics, and that this existence or nonexistence is not stable under perturbations of ϕ.

Given an extremal for Λ_Φ, we will under some hypotheses on Φ be able to obtain a variational equation that this extremal will have to satisfy. The variational equation is not a differential equation, but involves a pseudo-differential operator, which complicates the analysis. However, the equation is good enough to yield qualitative results. For example, jointly with Alec Matheson [75] it has been shown that if Φ is infinitely differentiable on $\mathbb{C}$ then the extremals, if they exist, must extend to be infinitely differentiable functions on the unit circle (recall that they were assumed to be holomorphic *inside* the unit disc; this did not by itself say much about their regularity on the boundary of the unit disc). We will see that some of the results mentioned in this paragraph continue to work in more general settings such as α-weighted Dirichlet spaces.

Finally, we wrap up Chapter III by connecting the results with symmetrization theory. We shall prove a result on the relation of α-weighted Dirichlet norms ($0 < \alpha < 2$) and symmetric decreasing rearrangement, and use this result to prove that if f is extremal for Λ_Φ where $\Phi(z)$ is of the form $\phi(\operatorname{Re} z)$ for ϕ a convex function on $\mathbb{R}$ strictly convex at 0, then f is one-to-one and has a Steiner symmetric image. We also present an alternative rearrangement method due to Baernstein replacing Steiner symmetrization but keeping some of its desirable properties. We shall examine the relation between this alternative method and Steiner symmetrization.

In Chapter IV we come back to symmetrization theory. This time, our interest is in replacing a domain D by a domain $\tilde{D}$ of not larger area which is simply connected and hopefully star-shaped[2] as well, satisfying $\Gamma_\phi(\tilde{D}) \geq \Gamma_\phi(D)$ for all ϕ such that $t \mapsto \phi(e^t)$ is convex and increasing. We are unable to determine if such a replacement exists. If it does exist, then this shows that Essén's inequality (0.3) is a consequence of the Chang-Marshall inequality (0.2). This is so because of the relation between the Λ_ϕ and Γ_ϕ functionals for one-to-one functions and simply connected domains. Moreover, if such a $\tilde{D}$ exists then we can obtain results on the existence of extremal domains in $\mathcal{B}$ for the Γ_ϕ functionals.

In fact, we make an explicit conjecture on what we think $\tilde{D}$ should be, namely a combination of circular symmetrization (which by itself does not always produce simply connected domains) and Marcus' radial rearrangement [70] (which may sometimes decrease the Γ_ϕ if applied by itself, as we shall show). If our conjecture is correct, then moreover we have $w_r(D) \leq w_r(\tilde{D})$. While this weaker inequality is also still open, we shall use another method of Haliste [56] to prove that $w_r(D) \leq w_r(\tilde{D})$ for our conjectured choice of $\tilde{D}$ if D is simply connected. Of course, the main point in the construction of $\tilde{D}$ was to obtain a simply connected domain so that if D is *a priori* simply connected, we do not gain much. Although we do gain something: our $\tilde{D}$ is star-shaped, whereas D need not be.

We shall also discuss some other partial results giving support to our conjectures. For instance we shall note that it essentially follows from known results that $\Gamma_\phi(D) \leq \Gamma_\phi(\tilde{D})$ if $t \mapsto \frac{d\phi(e^t)}{dt}$ is concave (which is of course a major restriction). We also consider some generalizations of Beurling's shove theorem [23]. These will allow us to, for instance, prove our full conjecture on the increase of the Γ_ϕ for domains D of the form $\mathbb{D}\backslash I$, where $\mathbb{D}$ is the unit disc and I is a finite collection of closed intervals on the negative real axis.

We shall also describe what our conjectures look like when transferred to the continuous cylinder. In particular, our conjectures can be transformed so as to concern Brownian motion on the cylinder. We shall show that the transformed version of the weaker $w_r(D) \leq w_r(\tilde{D})$ conjecture does not hold if the lengthwise component of the Brownian motion on the cylinder is replaced by a uniform motion to the right. This would seem to provide evidence against our conjectures,

[2]A set $D \subseteq \mathbb{C}$ is star-shaped if for all $z \in D$ the line segment joining z with 0 lies in D.

but the counterexample in fact satisfies the original $w_r(D) \leq w_r(\tilde{D})$ conjecture as it comes from a simply connected domain D.

Then, we shall discretize and collapse the cylinder to the set $\mathbb{Z}$. We formulate a simple analogue of the $w_r(D) \leq w_r(\tilde{D})$ conjecture as a conjecture on a random walk on $\{1, 2, \ldots, N+1\}$ with a reflecting boundary condition at $\frac{1}{2}$ and with dangers distributed on the points $1, 2, \ldots, N$. The latter conjecture then concerns the optimal distribution of the dangers so that the random walk has greatest chance of surviving to reach $N + 1$ when started at 1. We shall prove this conjecture (which incidentally is a variation on a problem considered by Essén [43]) via an involved but completely elementary method.

Now, our main conjectures were formulated for functionals Γ_ϕ defined as infima of the values at 0 of the harmonic majorants of $\phi(|z|)$. We can likewise consider similar functionals which are infima of the values at 0 of the harmonic majorants of $\phi(\operatorname{Re} z)$ for convex ϕ. In the final section of our thesis we prove a result in this case, analogous to a weak form of our conjectures. The proof of this result connects in a crucial way with the work of Chapter III by using the variational expression for the extremals of the Λ_Φ. Moreover, the proof uses either Steiner symmetrization discussed in Chapter I or Baernstein's alternative to it discussed in Chapter III.

Note on numbering and organization

Theorems are numbered in the form $X.Y$, where X is the section number and Y the theorem number. A reference to "Theorem $X.Y$" refers to theorem Y of section X of the current chapter. A reference to "Theorem $A.X.Y$" refers to theorem Y of section X of chapter A. Note that $A \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, while X and Y are Arabic numerals. What has just been said about theorems also applies to lemmas, definitions, remarks, etc. Sections are sometimes subdivided into subsections and even sub-subsections. For instance, a reference to §2.1.4 would refer to sub-subsection 4 of subsection 1 of section 2 of the current chapter. A reference to §III.2.3 would refer to subsection 3 of section 2 of chapter III. Note that theorems, lemmas, etc., are not renumbered by subsections or sub-subsections, but their numbering is reset precisely each time one comes to a new section.

Each chapter begins with an overview.

Some possible reading tracks

A reader whose only interest is symmetrization theory can read Chapters I, II and IV as well as sections I.1.2, III.5 (omitting Corollary III.5.2) and III.6 (omitting Corollary III.6.2). The only adverse effect on symmetrization theory that this will have is that some of the motivations in §IV.1.2 for various conjectures will be obscure and some of the proofs in §IV.10 will refer to unread material.

A reader whose only interest lies in *discrete* symmetrization theory can read §I.1, §I.2, all of Chapter II as well as §IV.9. In doing so, no necessary background material will be omitted, though it may be desirable to read sections I.5 and I.6 to see continuous versions of some of the discrete theorems; these continuous versions are from time to time parenthetically mentioned in the discussion of the discrete results.

A reader whose only interest is in the Λ_Φ and Γ_Φ functionals as well as in the Chang-Marshall and Essén inequalities can read §I.3–I.5 together with §III.1–III.4.4. The omission of background material on rearrangements will necessitate the dropping of §III.5, §III.6 and §III.10 which do contain some information on the functionals.

If a reader is not interested in discrete symmetrization theory, Chapter II can be omitted with no loss of continuity. Likewise, so can §IV.9, although if the latter is dropped then the proof of Theorem IV.7.1 will have to remain a mystery.

There are a few results throughout the thesis which are of some interest in and of themselves and which are independent of other work:

(i) A reader interested in an answer to Hayman's problem on circular symmetrization and Green's functions and already familiar with the basic notions of symmetrization need only read §I.7.

(ii) A reader already familiar with basic notions about Green's functions and the Riesz representation theorem for subharmonic functions, and whose interest lies in the radial monotonicity properties of the Green's function need only read §I.8.

(iii) A reader interested only in the Faber-Krahn inequality on regular trees need only read sections 1–3 and 6 in Chapter II.

(iv) A reader interested in the results on the existence/nonexistence of extremal functions for perturbations of the Chang-Marshall or Moser-Trudinger inequality need only read sections 2, 3.1, 3.2 and 3.3 of Chapter III together with the basic definitions in §III.1. A reader already familiar with the Chang-Marshall or Moser-Trudinger inequality and interested only in the existence/nonexistence result for perturbations will only really need a quick glance at the sections other than §III.3.3.

(v) A reader whose only interest lies in the elementary combinatorial rearrangement result for random walks in a dangerous blind alley need only read §IV.9.

Chapter I

Definitions, background material and introductory results

Overview

After having given some basic conventions (§1), we begin the thesis proper by defining the notion of a rearrangement method and stating a few useful and simple results on rearrangements (§2). Of particular importance will be the fundamental Hardy-Littlewood rearrangement inequalities (Theorems 2.3 and 2.4) which, following Kawohl [65], we shall give for all measure-preserving rearrangement methods. A standard construction central to §2 will be the rearrangement of a function based upon the rearrangement of its level sets (equation (2.1)). Then, we shall review basic results on Hardy spaces on the unit disc and on Poisson integrals there (§3). Then, we discuss basic notions concerning subharmonic functions (§4). Then, in §5 we shall review such notions as harmonic measure, least harmonic majorants, the connection between Brownian motion and harmonic measure, the uniformizer, Green's functions and finally the Riesz decomposition theorem for subharmonic functions (Theorem 5.9). These notions and standard results will be used in the subsequent parts of Chapter I and in Chapters III and IV. Having reviewed these notions, we then consider in §6 the circular and Steiner rearrangements as well as the symmetric decreasing rearrangement, give a few very basic facts, and then discuss various results of Baernstein [7], [9], [14], Beckner [20] and Haliste [56].

After these basic results and reviews, we proceed in §7 to give a negative answer to a question of Hayman [59] concerning circular rearrangements and Green's functions. Note that §7 is essentially taken from the author's paper [89]. Then, we discuss the question of how large we can take the set on which the Green's function is radially decreasing away from its pole (§8). In particular, we shall show that this happens on the largest disc centred on the pole and contained in the domain in question (Theorem 8.1). In fact this will usually happen on a larger set, and we obtain another lower bound on this set (Theorem 8.2). While our result here may be of some interest in and of itself, the reason we give it is because we will use it in §IV.8 where we shall obtain an improved version of Beurling's shove theorem [23]. The material of §8 is adapted from the author's paper [84].

1. Some conventions and notations

We have $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{Z}^- = \{-1, -2, \ldots\}$, $\mathbb{Z}_0^+ = \{0\} \cup \mathbb{Z}^+$, $\mathbb{Z}_0^- = \{0\} \cup \mathbb{Z}^-$ and $\mathbb{Z} = \mathbb{Z}_0^+ \cup \mathbb{Z}^-$.

The terms "positive" and "negative" shall be taken to mean "non-negative" and "non-positive", respectively. Likewise, the terms "increasing", "decreasing", "decrease", "increase", "smaller" and "greater" shall be understood in the weaker sense. When we wish to make a stronger statement, we will explicitly add an auxiliary term such as "strictly".

The term "countable" means "at most countable".

We shall use the term **domain** to mean any non-empty connected open subset of $\mathbb{C}$. We call a set in the plane **convex** if whenever z and w are two points in it, then the line segment joining z and w lies in this set. We call a set **star-shaped** if whenever z is a point in the set, then the line segment joining z and 0 lies in this set. We call a set **horizontally convex** if whenever z and w are two points in this set lying on the same horizontal line (i.e., having $\operatorname{Im} z = \operatorname{Im} w$), then the line segment joining z and w lies in the same set.

Measurable functions and semicontinuous functions are assumed to be almost everywhere finite unless otherwise provided for.

We shall use $|\cdot|$ for absolute values of numbers, for cardinalities of discrete sets, and for (sometimes normalized) Lebesgue measures in non-discrete settings. We write

$$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$$

for the support of a function f defined on a topological space. If f is defined on a discrete set, then $\operatorname{supp} f = \{x : f(x) \neq 0\}$.

Let $\mathbb{D}(w; r) = \{z \in \mathbb{C} : |w - z| < r\}$ be the disc of radius r centred on w. Put $\mathbb{D}(r) = \mathbb{D}(0; r)$ and let $\mathbb{D} = \mathbb{D}(1)$ be the unit disc. Put $\mathbb{T}(w; r) = \{z \in \mathbb{C} : |z - w| = r\}$ for $r \geq 0$. Set $\mathbb{T}(r) = \mathbb{T}(0; r)$ and let $\mathbb{T} = \mathbb{T}(1)$ be the unit circle. Note that $\mathbb{T}(0) = \{0\}$.

We write A^c for the complement of a set A and 2^A for its power set, namely $2^A = \{B : B \subseteq A\}$. Put $A \backslash B = \{a \in A : a \notin B\}$. If $\mathcal{A}$ is a collection of sets, write

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A = \{x : \exists A \in \mathcal{A}.\ x \in A\}.$$

If A is a set then write 1_A for the function which is 1 on A and 0 outside A. If P is a proposition, then put $1_{\{P\}} = 1$ if P is true and $1_{\{P\}} = 0$ if P is false.

Write $\delta_{x,y} = 1_{\{x=y\}}$ for the Kronecker δ.

Put $t^+ = \max(t, 0)$ and $t^- = (-t)^+$ so that $t = t^+ - t^-$ for all $t \in \mathbb{R}$. For a real function f write $f^\pm$ for the function $x \mapsto (f(x))^\pm$. Write $\operatorname{sgn} x = x/|x|$ if $x \neq 0$. The choice of $\operatorname{sgn} 0$ will never be relevant. Write $\arg z$ for any choice of the argument of a complex number z so that $z = |z| e^{i \arg z}$.

For a subset S of $\mathbb{C}$ and a complex number λ we write

$$\lambda S = \{\lambda z : z \in S\}.$$

Likewise,

$$\lambda + S = \{\lambda + z : z \in S\}.$$

The terms "analytic" and "holomorphic" are synonymous for us. The term **univalent** means "one-to-one and holomorphic".

A real function f on an open interval (a, b) is said to be **convex** if whenever $a < \alpha \leq \beta < b$ and $t \in [0, 1]$ then

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t)f(\beta).$$

It is said to be **strictly convex at** $x \in (a, b)$ if whenever $a < \alpha < \beta < b$ and $t \in (0, 1)$ are such that $x = t\alpha + (1-t)\beta$ then

$$f(x) < tf(\alpha) + (1-t)f(\beta).$$

Given a function Φ on $\mathbb{R}^n$, we write $\Phi_{,j} = \frac{\partial \Phi}{\partial x_j}$ for the partial derivative of Φ with respect to x_j $(j = 1, \ldots, n)$.

Given a measure space (I, μ), we say that the measurable functions f_n on I **converge in measure** to a measurable function f on I providing

$$\lim_n \mu\{x \in I : |f(x) - f_n(x)| > \varepsilon\} = 0$$

for every fixed $\varepsilon > 0$.

2. General definitions for rearrangement of functions and some basic results

We are interested in ways of rearranging a set so as to make it more symmetric, or at least so as to give it some special property such as star-shapedness. We wish to formulate the definitions in the greatest generality we can.

Definition 2.1. Let $\mathcal{F}$ be a collection of subsets of a set X. We call $\mathcal{F}$ a **σ-pseudotopology** providing:

(i) $\varnothing \in \mathcal{F}$,

(ii) $X \in \mathcal{F}$, and

(iii) whenever $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ are members of $\mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

A σ-pseudotopology is essentially a one-sided monotone class.

Clearly, any topology is a σ-pseudotopology. Likewise, a σ-algebra is a σ-pseudotopology. Now let $\mathcal{F}$ and $\mathcal{G}$ be σ-pseudotopologies on X and Y, respectively.

Definition 2.2. A map $\#\colon \mathcal{F} \to \mathcal{G}$ is a **rearrangement** providing:

(i) we have $A^{\#} \subseteq B^{\#}$ whenever A and B are two sets in $\mathcal{F}$ such that $A \subseteq B$,

(ii) we have
$$\left(\bigcup_{n=1}^{\infty} A_n\right)^{\#} = \bigcup_{n=1}^{\infty} A_n^{\#},$$
whenever $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ are sets in $\mathcal{F}$, and

(iii) $\varnothing^{\#} = \varnothing$ and $X^{\#} = Y$.

Remark 2.1. Note that condition (i) actually follows from condition (ii), but we retain the definition as above for clarity.

Given an extended real-valued function f on X, define the level set at height λ to be

$$f_\lambda \stackrel{\text{def}}{=} \{x \in X : f(x) > \lambda\}.$$

Likewise, for an extended real-valued g on Y, put

$$g_\lambda = \{y \in Y : g(y) > \lambda\}.$$

Remark 2.2. Let f be any extended real function. Then,

$$f^+(x) = \int_0^\infty 1_{f_\lambda}(x)\, d\lambda$$

as can be easily seen. Likewise,

$$f^-(x) = \int_{-\infty}^0 (1 - 1_{f_\lambda}(x))\, d\lambda.$$

Thus, if f is any extended real function then

$$f(x) = \int_0^\infty 1_{f_\lambda}(x)\,d\lambda + \int_{-\infty}^0 (1_{f_\lambda}(x) - 1)\,d\lambda.$$

Note that this last expression always makes sense, because if one of the two terms on the right side is $\pm\infty$ then the other vanishes.

Remark 2.3. Let f and g be any extended real functions on a set X. Then $f \equiv g$ if and only if $f_\lambda = g_\lambda$ for every $\lambda \in \mathbb{R}$. This follows immediately from the preceding remark.

Definition 2.3. Say that f is **$\mathcal{F}$-lower measurable** providing $f_\lambda \in \mathcal{F}$ for every $\lambda \in \mathbb{R}$.

If $\mathcal{F}$ is a topology, then $\mathcal{F}$-lower measurability agrees with $\mathcal{F}$-lower semicontinuity. If $\mathcal{F}$ is a σ-algebra, then f is $\mathcal{F}$-lower measurable if and only if it is $\mathcal{F}$-measurable.

Now, for an $\mathcal{F}$-lower measurable extended real-valued function f on X let

$$f^\#(y) = \sup\{\lambda : y \in (f_\lambda)^\#\}, \tag{2.1}$$

where $\#: \mathcal{F} \to \mathcal{G}$ is a rearrangement. This is a standard and well-known technique for taking a rearrangement of sets and making it into a rearrangement of functions by rearranging the level sets of the function and then reassembling the rearranged level sets into a new function.

Theorem 2.1. *Let f be an extended real $\mathcal{F}$-lower measurable function on X and let $\#: \mathcal{F} \to \mathcal{G}$ be a rearrangement. Then for every $\lambda \in [-\infty, \infty]$ we have*

$$(f^\#)_\lambda = (f_\lambda)^\#. \tag{2.2}$$

In particular, $f^\#$ is $\mathcal{G}$-lower measurable.

Proof. Fix $\lambda \in [-\infty, \infty)$ first. Then from the definition of $f^\#(y)$ we see that $f^\#(y) > \lambda$ if and only if there exists a $\rho > \lambda$ such that $y \in (f_\rho)^\#$. Let $\rho_n = \lambda + \frac{1}{n}$. If there is a $\rho > \lambda$ such that $y \in (f_\rho)^\#$ then choose $n \in \mathbb{Z}^+$ so that $\rho_n \in (\lambda, \rho]$. Then we have $(f_\rho)^\# \subseteq (f_{\rho_n})^\#$ by property (i) of rearrangements since evidently $f_\rho \subseteq f_{\rho_n}$ as $\rho \geq \rho_n$. Hence, $y \in (f_{\rho_n})^\#$. Since all the ρ_n are strictly larger than λ, it follows that $f^{\#}(y) > \lambda$ if and only if there exists $n \in \mathbb{Z}^+$ such that $y \in (f_{\rho_n})^{\#}$. But $f^{\#}(y) > \lambda$ if and only if $y \in (f^{\#})_\lambda$. Thus,

$$\begin{aligned}(f^{\#})_\lambda &= \{y : \exists n \in \mathbb{Z}^+.\ y \in (f_{\rho_n})^{\#}\} \\ &= \bigcup_{n=1}^{\infty} (f_{\rho_n})^{\#} \\ &= \left(\bigcup_{n=1}^{\infty} f_{\rho_n}\right)^{\#},\end{aligned} \tag{2.3}$$

where we have used property (ii) of rearrangements since $f_{\rho_1} \subseteq f_{\rho_2} \subseteq f_{\rho_3} \subseteq \cdots$ as $\rho_1 \geq \rho_2 \geq \rho_3 \geq \cdots$. But, it is easy to see from the definition of f_λ that

$$f_\lambda = \bigcup_{n=1}^{\infty} f_{\rho_n},$$

so that by (2.3) we see that $(f^{\#})_\lambda = (f_\lambda)^{\#}$ for all $\lambda \in [-\infty, \infty)$ as desired.

The remaining case is if $\lambda = \infty$. But then both sides of (2.2) are empty sets and we are done.

The proof is now complete since the lower measurability of $f^{\#}$ follows from the fact that $(f^{\#})_\lambda = (f_\lambda)^{\#} \in \mathcal{G}$ for all real λ. □

Remark 2.4. If Y is a topological space and $\#$ is a rearrangement such that $A^{\#}$ is open whenever $A \in \mathcal{F}$, then $f^{\#}$ is lower semicontinuous whenever f is $\mathcal{F}$-lower measurable. To see this, it suffices to note that by Theorem 2.1 we have $(f^{\#})_\lambda$ open for every real λ which precisely says that $f^{\#}$ is lower semicontinuous.

We say that a function ϕ on $[-\infty, \infty]$ is **left lower semicontinuous** providing

$$\liminf_{u \to t-} \phi(u) \geq \phi(t)$$

for all $t \in (-\infty, \infty]$.

Theorem 2.2. *Let f be an extended real $\mathcal{F}$-lower measurable function on X, and suppose that $\phi\colon [-\infty, \infty] \to [-\infty, \infty]$ is a monotone increasing and right lower semicontinuous function. Then $(\phi \circ f)^{\#} = \phi \circ (f^{\#})$ if $\#\colon \mathcal{F} \to \mathcal{G}$ is a rearrangement.*

Proof. In light of Remark 2.3, it suffices to prove that $(\phi \circ f)^{\#}_{\lambda} = (\phi \circ (f^{\#}))_{\lambda}$ whenever $\lambda \in \mathbb{R}$. So fix $\lambda \in \mathbb{R}$.

Let $t = \inf\{u : \phi(u) > \lambda\}$. Then if $u < t$, we have $\phi(u) \le \lambda$. By left lower semicontinuity we have $\phi(t) \le \lambda$. Of course, for $u > t$ we have $\phi(u) > \lambda$. Then,

$$(\phi \circ f)_{\lambda} = \{x : f(x) > t\} = f_t \tag{2.4}$$

and

$$(\phi \circ f^{\#})_{\lambda} = \{y : f^{\#}(x) > t\} = (f^{\#})_t = (f_t)^{\#},$$

by Theorem 2.1. But by Theorem 2.1 and (2.4) we have

$$(\phi \circ f)^{\#}_{\lambda} = ((\phi \circ f)_{\lambda})^{\#} = (f_t)^{\#},$$

so that $((\phi \circ f)^{\#})_{\lambda} = (\phi \circ (f^{\#}))_{\lambda}$ as desired. □

Definition 2.4. Suppose that we have a measure μ on X such that all elements of $\mathcal{F}$ are μ-measurable and a measure ν on Y such that all elements of $\mathcal{G}$ are ν-measurable. We say that a rearrangement $\#: \mathcal{F} \to \mathcal{G}$ is **measure-preserving** providing $\mu(A) = \nu(A^{\#})$ for all $A \in \mathcal{F}$.

*Example 2.1 (**Decreasing rearrangement to** $\mathbb{R}_0^+$).* Let Y be the interval $[0, \infty)$ equipped with Lebesgue measure. Let $(X, \mathcal{F}, \mu)$ be any measure space. Given $A \in \mathcal{F}$, let

$$A^* = [0, \mu(A)).$$

Then $*$ is easily seen to be a rearrangement, and it is also clearly measure-preserving. Since A^* is always open, by Remark 2.4 it follows that f^* is lower semicontinuous on $[0, \infty)$ whenever f is $\mathcal{F}$-measurable.

*Example 2.2 (**Decreasing rearrangement on** $\mathbb{Z}_0^+$).* Consider the discrete space $X = Y = \mathbb{Z}_0^+$. Put $\mathcal{F} = \mathcal{G} = 2^{\mathbb{Z}_0^+}$. Define

$$S^* = \{i \in \mathbb{Z}_0^+ : i < |S|\},$$

where $|S|$ is the cardinality of a subset S of X. Then, clearly $|S^*| = |S|$ and it is easy to see that $*$ is a measure preserving rearrangement.

Given an extended real f on $\mathbb{Z}_0^+$, we may describe the f^* in a very intuitive way. Indeed, for any $n \in \mathbb{Z}^+$, the numbers $f^*(0), \ldots, f^*(n-1)$ are a list of the n largest values of f. We call f^* the **decreasing rearrangement** of f.

The reader is advised to often keep in mind the previous two examples which are very typical and are really the most basic types of rearrangement.

*Example 2.3 **(Schwarz symmetrization in** $\mathbb{R}^n$).* Let U be an arbitrary Lebesgue measurable subset of $\mathbb{R}^n$. Let $U^{\circledast}$ be an open ball in $\mathbb{R}^n$ centred on the origin and having the same volume as U. (If U has infinite volume, then put $U^{\circledast} = \mathbb{R}^n$.) It is easy to see that $\circledast$ is a measure preserving rearrangement on the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}^n$, where "measure preserving" is asserted with respect to Lebesgue volume measure.

*Example 2.4 **(Steiner symmetrization on** $\mathbb{C}$).* Let U be a subset of the plane. Define

$$Y(x; U) = |\{y : x + iy \in U\}|,$$

where $|\cdot|$ is Lebesgue measure on $\mathbb{R}$. Set

$$U^{\boxminus} = \{x + iy : x \in \mathbb{R}, |y| < Y(x; U)/2\}.$$

Call $U^{\boxminus}$ the **Steiner symmetrization of U about the real axis** (frequently the words "about the real axis" will be omitted). It is easy to verify that $\boxminus$ is a measure preserving rearrangement on the σ-algebra of all Lebesgue measurable subsets of $\mathbb{C}$, where "measure preserving" is predicated with respect to Lebesgue area measure.

Steiner symmetrization was invented by J. Steiner who proved that it decreased circumferences of sets and used it in 1838 to prove that if there is a domain in the plane which minimizes circumference for fixed area, then this domain is a circular disc [98]. Note that in Brascamp, Lieb and Luttinger [26, Appendix] one may find a modern proof of the fact that given a bounded measurable set in the plane, one can find an infinite sequence of Steiner symmetrizations about different axes which transforms the set into a disc.

See Figure 2.1 for a simple example of Steiner symmetrization at work.

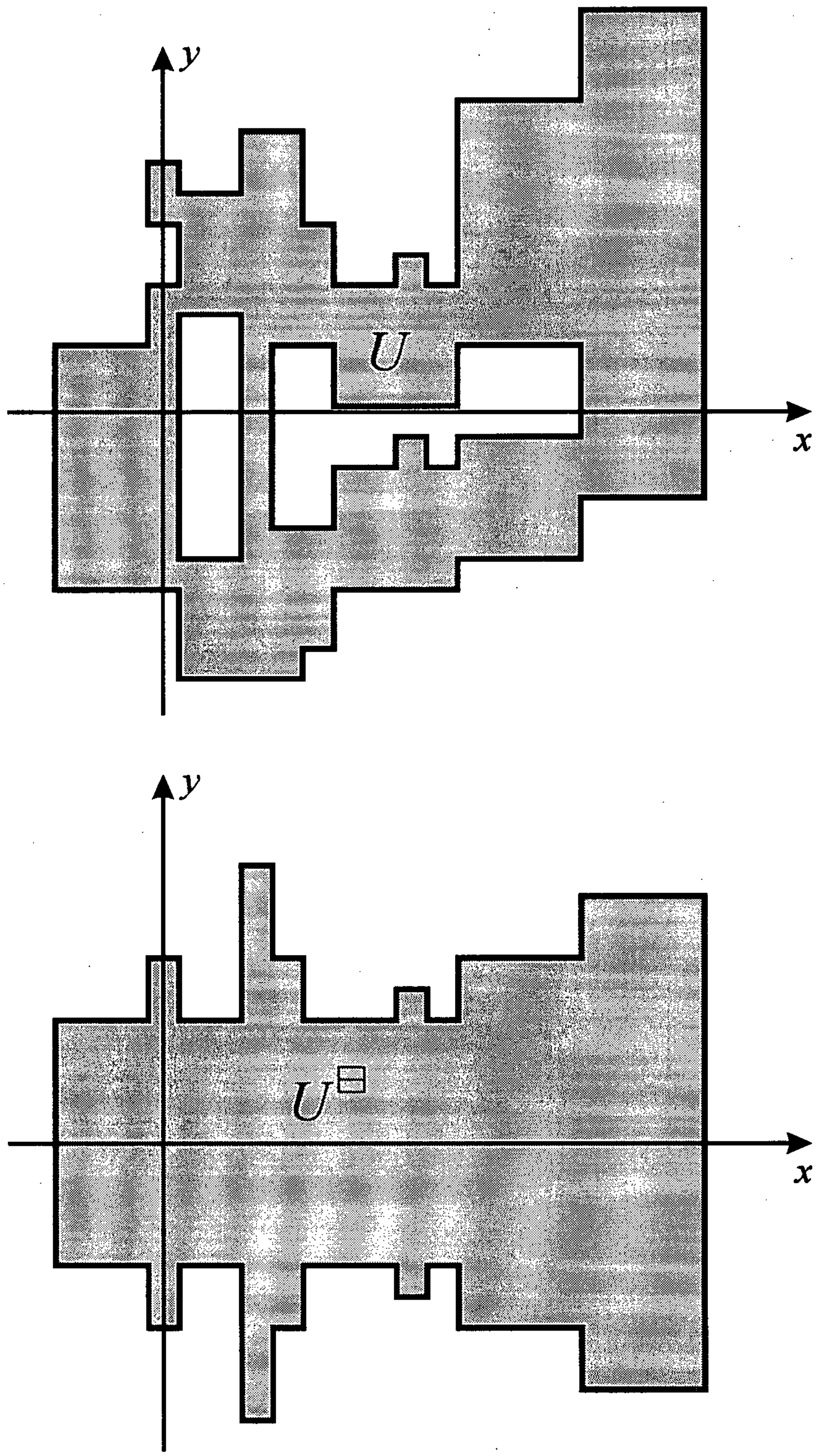

Figure 2.1: Steiner symmetrization about the real axis.

*Example 2.5 (**Circular symmetrization on** $\mathbb{C}$).* Let U be a measurable subset of $\mathbb{C}$. Fix $r \in [0, \infty)$. If $\{|z| = r\} \subseteq U$ then let $\theta(r; U) = \infty$, otherwise let $\theta(r; U) = |\{\theta \in [0, 2\pi) : re^{i\theta} \in U\}|$, where $|\cdot|$ indicates Lebesgue measure on $\mathbb{R}$. Define

$$U^{\circledcirc} = \{re^{i\theta} : |\theta| < \theta(r; U)/2\}$$

to be the **circular symmetrization** of U. Note that $\text{Area}(U^{\circledcirc}) = \text{Area}(U)$ as can be easily seen. An example of circular symmetrization is given in Figure 0.1 on p. xiii of our Introduction. Note that $\circledcirc$ is a measure preserving rearrangement on the collection of all open subsets of $\mathbb{C}$ (see Remark 6.1 in §6.1). Again, "measure preserving" is meant with respect to Lebesgue area measure.

Circular and Steiner symmetrizations are two of the main symmetrizations in which we are interested. Some further properties of them can be found in sections 6.1 and 6.3, respectively.

The following proposition is useful. It is not intended to have optimal conditions, but it suffices for our purposes.

Proposition 2.1. *Let ϕ be any Borel measurable function on $\mathbb{R}$, f any $\mathcal{F}$-lower measurable function, and $\#$ a measure-preserving rearrangement. Then,*

$$\int_X \phi(f)\, d\mu = \int_Y \phi(f^{\#})\, d\nu, \tag{2.5}$$

providing either side makes sense and at least one of the following conditions is satisfied:

(i) $\mu(X) < \infty$,

(ii) $f \geq 0$ *and ϕ is monotone increasing and left lower semicontinuous with* $\phi(0) = 0$.

Proof. Suppose first that (i) holds. Then, if we let $F(\lambda) = \mu(f_\lambda)$, we will have

$$\int_X \phi(f)\, d\mu = \int_{-\infty}^{\infty} \phi(t)\, dF(t),$$

while

$$\int_Y \phi(f^{\#})\, d\nu = \int_{-\infty}^{\infty} \phi(t)\, dG(t),$$

where $G(x) = \nu((f^{\#})_\lambda)$. But by Theorem 2.1 and the assumption that $\#$ is measure-preserving, we have $F = G$ and so (2.5) follows.

Suppose now that (ii) holds. Then, using Theorem 2.2 we may replace f by $\phi \circ f$ and $f^{\#}$ by $\phi \circ f^{\#} = (\phi \circ f)^{\#}$ and thus we may assume that ϕ is the identity function. But, by Remark 2.2 and Fubini's theorem we have,

$$\int_X f\,d\mu = \int_0^\infty \mu(f_\lambda)\,d\lambda$$

and

$$\int_Y f^{\#}\,d\nu = \int_0^\infty \nu((f^{\#})_\lambda)\,d\lambda.$$

Hence,

$$\int_X f\,d\mu = \int_Y f^{\#}\,d\nu,$$

since $\mu(f_\lambda) = \nu((f^{\#})_\lambda)$ by Theorem 2.1 and by the measure-preserving property of our rearrangement. $\square$

The following result is very well known. It is essentially due to Hardy and Littlewood (cf. [58, Thm. 368]) in the case of decreasing rearrangement on $\mathbb{Z}_0^+$, and has been more generally expounded by Kawohl [65, Lem. 2.1] whose proof we adopt. Assume that X and Y are equipped with the measures μ and ν respectively.

Theorem 2.3 (Hardy-Littlewood inequality). *Let $\#: \mathcal{F} \to \mathcal{G}$ be a measure-preserving rearrangement. Let f and g be $\mathcal{F}$-lower measurable extended real functions on X. Then, if f and g are both positive, then we will have*

$$\int_X fg\,d\mu \leq \int_Y f^{\#} g^{\#}\,d\nu. \tag{2.6}$$

Moreover, if f is any $\mathcal{F}$-lower measurable function in $L^p(\mu)$ and g is any $\mathcal{G}$-lower measurable function in $L^q(\mu)$, where $p^{-1} + q^{-1} = 1$ and either $1 < p < \infty$ or μ is σ-finite and $1 \leq p \leq \infty$, then (2.6) continues to hold providing at least one of the following conditions holds:

(i) *one of f and g is positive*

(ii) $\mu(X) < \infty$.

It is not intended that the disjunction of conditions (i) and (ii) should be optimal, but only that it should cover the cases which we need.

Before we prove Theorem 2.3 we give a result which is of independent interest although it is doubtless also well known.

Theorem 2.4. *Let $\#\colon \mathcal{F} \to \mathcal{G}$ be a measure-preserving rearrangement. Let $f_1, f_2, \ldots, f_n$ be any positive $\mathcal{F}$-lower measurable functions on X. Then,*

$$\int_X f_1 f_2 \ldots f_n \, d\mu \leq \int_Y f_1^\# f_2^\# \ldots f_n^\# \, d\nu. \tag{2.7}$$

Proof. We first prove (2.7) for $f_i = 1_{A_i}$ where $A_i \in \mathcal{F}$ for $i = 1, \ldots, n$. It is clear that $(1_{A_i})^\# = 1_{A_i^\#}$. Thus, the desired inequality in this case is equivalent to the relation

$$\mu(A_1 \cap \cdots \cap A_n) \leq \nu(A_1^\# \cap \cdots \cap A_n^\#).$$

Let $A = A_1 \cap \cdots \cap A_n$ and $B = A_1^\# \cap \cdots \cap A_n^\#$. Then, $A \subseteq A_i$ for all i so that $A^\# \subseteq A_i^\#$ for all i since rearrangements preserve inclusions. Thus, $A^\# \subseteq B$. Hence,

$$\mu(A) = \nu(A^\#) \leq \mu(B),$$

where we have used the assumption that our rearrangement was measure preserving.

Now, consider the case of general functions f_i. Let $f_i^\lambda = 1_{(f_i)_\lambda}$. In light of Remark 2.2 and Theorem 2.1 we have

$$f_i = \int_0^\infty f_i^\lambda \, d\lambda$$

and

$$f_i^\# = \int_0^\infty (f_i^\lambda)^\# \, d\lambda.$$

By Fubini's theorem, we then have

$$\int_X f_1 \cdots f_m \, d\mu = \int_0^\infty \cdots \int_0^\infty \int_X f_1^{\lambda_1} \cdots f_n^\lambda \, d\mu \, d\lambda_1 \cdots d\lambda_n, \tag{2.8a}$$

and

$$\int_X f_1^\# \cdots f_m^\# \, d\mu = \int_0^\infty \cdots \int_0^\infty \int_X (f_1^{\lambda_1})^\# \cdots (f_n^\lambda)^\# \, d\mu \, d\lambda_1 \cdots d\lambda_n. \tag{2.8b}$$

But

$$\int_X f_1^{\lambda_1} \cdots f_n^\lambda \, d\mu \le \int_X (f_1^{\lambda_1})^\# \cdots (f_n^\lambda)^\# \, d\mu,$$

since we have already proved our inequality for characteristic functions. Then (2.7) follows from (2.8a) and (2.8b). □

The following result will be of some use in the proof of Theorem 2.3.

Lemma 2.1. *Let f_n be a sequence of extended-real $\mathcal{F}$-lower measurable functions increasing pointwise to a function f. Let $\#: \mathcal{F} \to \mathcal{G}$ be a rearrangement. Then, $f_n^\#$ is a sequence of extended-real functions increasing pointwise to $f^\#$.*

Proof of Lemma. It is clear from the definition of $f_n^\#$ and property (i) of rearrangements that the sequence $f_n^\#$ increases pointwise. Let g be the pointwise limit of the $f_n^\#$.

Fix $\lambda \in \mathbb{R}$. Note that $(f_1)_\lambda \subseteq (f_2)_\lambda \subseteq \cdots$ since we have $f_1 \le f_2 \le \cdots$ pointwise. Moreover, note that

$$f_\lambda = \bigcup_{n=1}^\infty (f_n)_\lambda. \tag{2.9}$$

For, if $x \in f_\lambda$ then $f(x) > \lambda$ and hence for sufficiently large n we must have $f_n(x) > \lambda$ so that $x \in (f_n)_\lambda$, while, conversely, if $x \in (f_n)_\lambda$ then $f(x) \ge f_n(x) > \lambda$ and so $x \in f_\lambda$. In the same way,

$$g_\lambda = \bigcup_{n=1}^\infty (f_n^\#)_\lambda.$$

But, by Theorem 2.1 we have $(f_n^\#)_\lambda = ((f_n)_\lambda)^\#$. Hence,

$$g_\lambda = \bigcup_{n=1}^\infty ((f_n)_\lambda)^\#.$$

But $(f_1)_\lambda \subseteq (f_2)_\lambda \subseteq \cdots$ so that by property (ii) of rearrangements we have

$$g_\lambda = \left(\bigcup_{n=1}^{\infty} (f_n)_\lambda \right)^{\#}.$$

Thus, $g_\lambda = (f_\lambda)^{\#}$ by (2.9). By Theorem 2.1 it follows that $g_\lambda = (f^{\#})_\lambda$ and, since λ was arbitrary it follows that $g = f^{\#}$, which completes the proof. □

Proof of Theorem 2.3. We have already proved the result in the positive case in Theorem 2.4. Suppose now that $f \in L^p(\mu)$ and $g \in L^q(\mu)$ for $1 \le p \le \infty$, and that if $p \in \{1, \infty\}$ then μ is σ-finite.

Let $\phi(t) = t^+$ and $\psi(t) = -(t^-)$. Then, ϕ and ψ are continuous monotone functions, and hence commute with rearrangements by Theorem 2.2. It follows that

$$f^{\#} = \phi \circ (f^{\#}) + \psi \circ (f^{\#}) = (\phi \circ f)^{\#} + (\psi \circ f)^{\#},$$

while of course

$$f = \phi \circ f + \psi \circ f.$$

Suppose we could prove (2.6) in the special case where f and g have constant (but not necessarily the same) sign. Then, the general case would follow by linearity and the above two displayed equations (together with their analogues for g) since ϕ and ψ have constant sign. Hence we need only prove our result in the case of functions f and g with constant sign.

We now reduce the problem further. Let $\phi_n(x) = x$ if $|x| < n$ and let $\phi_n(x) = n \operatorname{sgn} x$ otherwise. This is a continuous monotone function so that

$$(\phi_n \circ f)^{\#} = \phi_n \circ f^{\#}$$

and

$$(\phi_n \circ g)^{\#} = \phi_n \circ g^{\#}.$$

Thus, $\phi_n \circ f^{\#} \to f^{\#}$ and $\phi_n \circ g^{\#} \to g^{\#}$ as $n \to \infty$. In the constant sign case, if we could prove (2.6) for $\phi_n \circ f$ and $\phi_n \circ g$, then we could take a limit as $n \to \infty$ (using the monotone convergence theorem) and thus obtain the result for f and g. But the functions $\phi_n \circ f$ and $\phi_n \circ g$ are bounded.

Hence, we may assume that f and g are bounded and of constant (but not necessarily the same) sign. Suppose first that $f \leq 0$ and $g \geq 0$. If $g \in L^q(\mu)$ for some $q < \infty$ then g is integrable, since it is bounded and $L^1 \subseteq L^q \cap L^\infty$; let $g_n = g$ in that case for every $n \in \mathbb{Z}^+$. If $q = \infty$ then we are in the σ-finite case, so let A_n be an increasing sequence of sets with finite μ-measure whose union is X, and put $g_n = g \cdot 1_{A_n}$, so that g_n is integrable because of the boundedness of g.

Choose M so that $f + M \geq 0$ (we can do this as f is bounded). By (2.6) for positive functions (which is already proved) we have

$$\int_X (f + M) g_n \, d\mu \leq \int_Y (f + M)^\# g_n^\# \, d\nu.$$

But $(f + M)^\# = f^\# + M$, and $\int_X g_n \, d\mu = \int_Y g_n^\# \, d\nu < \infty$ (the last equality follows from Proposition 2.1). Hence,

$$\int_X f g_n \, d\mu \leq \int_Y f^\# (g_n)^\# \, d\nu.$$

Taking the limit as $n \to \infty$ and applying the monotone convergence theorem as well as Lemma 2.1 we obtain the inequality

$$\int_X f g \, d\mu \leq \int_Y f^\# g^\# \, d\nu,$$

as desired.

The case where $f \geq 0$ and $g \leq 0$ is analogous.

Consider now the case where $f \leq 0$ and $g \leq 0$. Under our assumed conditions, this case can only occur if $\mu(X) < \infty$. Let M be again such that $f + M \geq 0$. We then have

$$\int_X (f + M) g \, d\mu \leq \int_Y (f + M)^\# g^\# \, d\mu,$$

since we have already proved (2.6) in the case of functions of which one is positive and the other is negative. But $(f + M)^\# = f^\# + M$ and $\int_X g \, d\mu = \int_Y g^\# \, d\mu < \infty$ since g is bounded and μ is finite, and so (2.6) follows. $\square$

Finally we give the following definition.

Definition 2.5. A rearrangement $\#$ mapping a σ-pseudotopology $\mathcal{F}$ into itself is said to be **a symmetrization** if $(A^{\#})^{\#} = A^{\#}$ for all $A \in \mathcal{F}$.

Example 2.6. Schwarz, Steiner and circular symmetrizations (Examples 2.3, 2.4 and 2.5, respectively) are easily seen to be symmetrizations in the sense of the above definition. Likewise, the decreasing rearrangement on $\mathbb{Z}_0^+$ (Example 2.2) is a symmetrization.

Useful results on and applications of various concrete symmetrizations can be found in [16], [40] and [82].

3. Hardy spaces, Poisson integrals and further background material

3.1. Definitions of Hardy spaces

Useful references on Hardy spaces are Duren [41], Garnett [55], Hoffman [63] and Koosis [67].

A real or complex valued function f is said to be **harmonic** on a domain D if $f \in C^2(D)$ and $\Delta f = 0$ everywhere on D, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian. The following mean value property is very basic. It follows from rescaling the $z = 0$ case of [94, Thm. 11.9].

Theorem 3.1. *Let f be harmonic on D. Let $z \in D$ and choose any $r > 0$ such that $\overline{\mathbb{D}}(z;r) \subseteq D$. Then,*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta.$$

Remark 3.1. Let f be harmonic on D and let $g\colon U \to D$ be holomorphic. Then $f \circ g$ is harmonic on U, i.e., harmonicity is conformally invariant. To see this, note that by linearity it suffices to show it for real f. But any real harmonic function is locally the real part of a holomorphic function (see, e.g., [94, Thm. 11.9]) so that $f = \operatorname{Re} F$ for some holomorphic F. Then, $f \circ g = \operatorname{Re}(F \circ g)$. Now, $F \circ g$ is holomorphic. Moreover, the Cauchy-Riemann equations easily show that the real part of a holomorphic function is harmonic, and so $f \circ g = \operatorname{Re}(F \circ g)$ is harmonic as desired.

Let f be a holomorphic or harmonic function on $\mathbb{D}$. Define

$$m_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}$$

for $0 < p < \infty$ and $0 < r < 1$. Set

$$m_\infty(r,f) = \sup_{\theta\in[0,2\pi]} |f(re^{i\theta})|.$$

Then, the **holomorphic Hardy space** $H^p = H^p(\mathbb{D})$ is defined to be the collection of all functions f holomorphic on $\mathbb{D}$ with

$$||f||_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} m_p(r,f) < \infty.$$

Likewise, the **harmonic Hardy space** $h^p = h^p(\mathbb{D})$ is defined as the collection of all real harmonic functions f on $\mathbb{D}$ with

$$||f||_{h^p} \stackrel{\text{def}}{=} \sup_{0<r<1} m_p(r,f) < \infty.$$

Definition 3.1. Let D be a domain in the plane. Fix $p \in (0,\infty)$. Then, D is said to be an H^p **domain** providing that every holomorphic function f on $\mathbb{D}$ whose image lies in D satisfies $f \in H^p$. In such a case, we write $D \in H^p$.

Theorem 3.2 (Sakai [95, Thm. 8.1]). *If* $\operatorname{Area} D < \infty$ *then* $D \in H^p$ *for every finite* p. *Moreover, if* f *is holomorphic on* $\mathbb{D}$ *with* $f(0) = 0$ *then for* $1 \le p < \infty$ *we have*

$$||f||_{H^p} \le c(p)\sqrt{\operatorname{Area} f[\mathbb{D}]},$$

for some constant $c(p)$ *depending only on* p.

3.2. Nontangential limits

Define the cone

$$C_\alpha = \{1 - re^{i\theta} : r > 0, |\theta| < \alpha\}$$

with opening angle 2α at the point 1.

Definition 3.2. Let $z \in \mathbb{T}$. We say that a sequence $z_n \in \mathbb{D}$ **tends to** z **nontangentially** providing $|z_n - z| \to 0$ and there exists $\alpha \in (0, \pi/2)$ such that $z_n \in C_\alpha$ for all sufficiently large n.

Definition 3.3. Let f be a function on $\mathbb{D}$. Say that f **has a nontangential limit** L **at** $z \in \mathbb{D}$ if $f(z_n) \to L$ whenever z_n tends nontangentially to z. Write $\mathrm{n.t.}\lim f(z) = L$.

Given $f \in L^1(\mathbb{T})$, define

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta$$

to be the nth Fourier coefficient. Let

$$P_r(e^{i\alpha}) = \frac{1 - r^2}{1 - 2r\cos\alpha + r^2}$$

be the **Poisson kernel.** Given a function $f \in L^1(\mathbb{T})$, define

$$(P * f)(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\phi}) P_r(e^{i(\theta - \phi)}) \, d\phi, \tag{3.1}$$

for $re^{i\theta} \in \mathbb{D}$. Then, $P * f$ is necessarily harmonic [55, p. 11]. Moreover, it is well known that

$$P * f = \sum_{n=0}^{\infty} \hat{f}(n) z^n + \sum_{n=-\infty}^{-1} \hat{f}(n) \bar{z}^n.$$

The crucial result on non-tangential limits is as follows. The H^p case for $p = 1$ is essentially due to F. and M. Riesz and is given in Koosis [67, p. 49]. The h^p case is due to Fatou (cf. Garnett [55, Thm. I.5.3]). The last "Moreover" can be found in Garnett [55, p. 59].

Theorem 3.3. *Let f be in $H^p(\mathbb{D})$ for some $p \in [1, \infty]$ or in $h^p(\mathbb{D})$ for some $p \in (1, \infty]$. Then f has a nontangential limit at almost every point of $\mathbb{T}$. Moreover, the function* $\mathrm{n.t.}\lim f \in L^p(\mathbb{T})$ *and* $f = P * \mathrm{n.t.}\lim f$.

For $1 < p \leq \infty$, the map $f \mapsto \mathrm{n.t.}\lim f$ is an isometry of $h^p(\mathbb{D})$ onto $\mathrm{Re}\, L^p(\mathbb{T})$, *so that if $F \in \Re L^p(\mathbb{T})$ and $f = P * F$ then $F = \mathrm{n.t.}\lim f$. Moreover, for $1 \leq p \leq \infty$, the map* $\mathrm{n.t.}\lim$ *is an isometry of $H^p(\mathbb{D})$ onto $H^p(\mathbb{T}) \stackrel{\mathrm{def}}{=} \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \forall n \in \mathbb{Z}^-\}$.*

Frequently, we shall identify n. t. lim f with f. In (3.1), we say that f is the **Poisson extension of F into $\mathbb{D}$.**

3.3. The conjugate function and the M. Riesz theorem

Let u be a real harmonic function on $\mathbb{D}$. It is well known that there exists a unique real harmonic function $\tilde{u}$ on $\mathbb{D}$ such that $\tilde{u}(0) = 0$ and $u + i\tilde{u}$ is holomorphic on $\mathbb{D}$. We call $\tilde{u}$ the **conjugate function** of u. We may likewise define $\tilde{u}$ for a complex valued harmonic function u by

$$\tilde{u} = \tilde{u}_1 + i\tilde{u}_2,$$

where $u_1 = \operatorname{Re} u$ and $u_2 = \operatorname{Im} u$. Then, $u + i\tilde{u} = (u_1 + i\tilde{u}_1) + i(u_2 + i\tilde{u}_2)$ is also holomorphic. Write $\mathcal{P}u = u + i\tilde{u}$ for u harmonic on $\mathbb{D}$. Note that $\mathcal{P}\mathcal{P}u = \mathcal{P}u$ for all harmonic u on D with $u(0) = 0$ since $\mathcal{P}f = f$ if f holomorphic on $\mathbb{D}$ with $f(0) = 0$ by the uniqueness of the conjugate function. We call $\mathcal{P}$ the **Szegö projection.**

Let u be a function in $L^p(\mathbb{T})$ for some $p > 1$. Then, $P * u$ is harmonic on $\mathbb{D}$ (cf. Theorem 3.3) and satisfies n. t. lim $P * u = u$. (Of course Theorem 3.3 only says this for real valued functions, but this extends to the complex case by linearity.) Then, write $\tilde{u} =$ n. t. lim $\widetilde{P * u}$. Likewise, put $\mathcal{P}u =$ n. t. lim $\mathcal{P}(P * u)$.

The following theorem is very important.

Theorem 3.4 (M. Riesz; see, e.g., [55, Thm. III.2.3]). *The map $u \mapsto \tilde{u}$ is a bounded operator from $L^p(\mathbb{T})$ to itself for every $p > 1$, and the map $\mathcal{P}$ is a bounded operator from $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$.*

Of course, as mentioned in §3.2, there is an identification between $H^p(\mathbb{T})$ and $H^p(\mathbb{D})$ via $f \mapsto P * f$.

The following remark is somewhat useful.

Remark 3.2. Let u be a real valued harmonic function on $\mathbb{D}$ satisfying $u(\bar{z}) = u(z)$ for all $z \in \mathbb{D}$. Then, $\tilde{u}(\bar{z}) = -\tilde{u}(z)$. The easiest way to see this is to note that, by replacing u with

$u_r(z) = u(rz)$ for $0 < r < 1$, it suffices to prove the remark for u bounded on $\mathbb{D}$. But then the desired result follows from the expression (see [55, p. 102])

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi} Q_r(\theta - \phi) u(e^{i\phi})\, d\phi,$$

where

$$Q_r(\theta) = \frac{-2r sin\theta}{1 - 2r\cos\theta + r^2}$$

is antisymmetric.

Finally note that $L^2(\mathbb{T})$ is identified naturally with the space $\ell^2(\mathbb{Z})$ via the map $F : f \mapsto \hat{f}$. Under this identification, the operator $\mathcal{P}$ (or, more precisely, $F\mathcal{P}F^{-1}$) acts on a function $\hat{f} \in \ell^2(\mathbb{Z})$ by sending $\hat{f}$ to $1_{\mathbb{Z}_0^+} \cdot \hat{f}$. In other words, $\mathcal{P}$ is the orthogonal projection operator from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$.

3.4. Disc algebra and BMO

We define the **disc algebra** to be the collection of all functions f continuous on $\bar{\mathbb{D}}$ and holomorphic on $\mathbb{D}$.

Now, let f be a locally integrable function on $\mathbb{T}$. Let I be an arc of $\mathbb{T}$. Let

$$f_I = \frac{1}{|I|} \int_I f$$

be the mean of f on I, where $|\cdot|$ is normalized Lebesgue measure on $\mathbb{T}$. Define

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f - f_I|.$$

Write

$$\mathrm{BMO} = \{f : \|f\|_* < \infty\},$$

for the space of functions of **bounded mean oscillation**. If we identify functions which differ almost everywhere by a constant, BMO will become a Banach space. It is known that $L^p(\mathbb{T}) \supset \mathrm{BMO} \supset L^\infty(\mathbb{T})$ for all $p < \infty$, with all inclusions proper.

Define $\mathrm{BMOA} = \{f \in H^1(\mathbb{D}) : \mathrm{n.t.}\lim f \in \mathrm{BMO}\}$.

3.5. The Nevanlinna class N

Let f be a holomorphic function on $\mathbb{D}$. Then, we say that f is in the **Nevanlinna class** N providing

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty,$$

where $\log^+ t = \max(0, \log t)$. We say that a domain D is a **Nevanlinna domain** if for every holomorphic f on $\mathbb{D}$ whose image lies in D we have $f \in N$. By Theorem 3.2, any domain of finite area is a Nevanlinna domain since clearly $H^p \subseteq N$ for any strictly positive p.

Theorem 3.5 ([55, Thm. II.5.3]). *Let $f \in N$. Then f has a nontangential limit almost everywhere on $\mathbb{T}$, and $\log^+ |\,\mathrm{n.t.}\lim f| \in L^1(\mathbb{T})$.*

4. Subharmonic functions

Definition 4.1. A real function $f\colon D \to [-\infty, \infty)$ where $D \subseteq \mathbb{C}$ is a domain is said to be **subharmonic** providing:

(i) f is upper semicontinuous

(ii) f for every $z \in D$ there is an $r_0 > 0$ such that $\mathbb{D}(z; r_0) \subseteq D$ and

$$f(z) \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(z;r)} f(w) \, dA(w), \tag{4.1}$$

for all $r \in (0, r_0]$, where A is Lebesgue area measure on $\mathbb{C}$.

It is clear that subharmonicity is a local criterion. Note that any harmonic function is subharmonic by Theorem 3.1. A positive multiple of a subharmonic function is obviously subharmonic, and the sum of two subharmonic functions is subharmonic.

Definition 4.2. A real function $f\colon D \to (-\infty, \infty]$ where $D \subseteq \mathbb{C}$ is a domain is said to be **superharmonic** if $-f$ is subharmonic.

Theorem 4.1 (Maximum principle). *Let $f\colon D \to [-\infty,\infty)$ be a subharmonic function. Assume that f attains a maximum on D. Then f is constant.*

Proof. Suppose that $z \in D$ is such that $f(z) = \sup_D f$. Then, let $U = \{w \in D : f(w) = f(z)\}$. This is a relatively closed set in D since $\{w \in D : f(w) = f(z)\} = \{w \in D : f(w) \geq f(z)\}$ and f is upper semicontinuous. We shall show that U is likewise open. To see this, fix $w \in U$. Let $r > 0$ be such that $\mathbb{D}(w;r) \subseteq D$ and

$$f(w) \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(w;r)} f(v)\, dA(v).$$

But $f(v) \leq f(w)$ for all $v \in \mathbb{D}(w;r)$ by choice of w. Hence, since $A(\mathbb{D}(w;r)) = \pi r^2$ it follows that $f(v) = f(w)$ for almost every v in $\mathbb{D}(w;r)$. By upper semicontinuity then it follows that $f(v) \geq f(w)$ for every $v \in \mathbb{D}(w;r)$. But $f(v) \leq f(w)$ and so $f(v) = f(w)$ for all $v \in \mathbb{D}(w;r)$. Hence $\mathbb{D}(w;r) \subseteq U$. Hence U is both open and closed. Since it is nonempty and D is connected, it follows that $U = D$ and we are done. $\square$

The following well-known theorem explains the reason for the term "subharmonic".

Theorem 4.2 ([55, Thm. I.6.3]). *Let $f\colon D \to [-\infty,\infty)$ be upper semicontinuous. Then f is subharmonic if and only if whenever u is a harmonic function on a bounded open subset U of D with*

$$\limsup_{z\to\zeta}(f(z) - u(z)) \leq 0$$

for every $\zeta \in \partial U$, where the lim sup *is taken as z tends to ζ from within U, then*

$$f \leq u$$

everywhere on U.

The above condition basically says that if f is smaller than a harmonic function on the boundary of a bounded domain U, then it is smaller than that harmonic function inside U, too.

Remark 4.1. Let $f\colon D \to U$ be univalent and let g be subharmonic on U. Then $f \circ g$ is subharmonic on D. This follows from Theorem 4.2, the conformal invariance of harmonic functions (Remark 3.1) and the fact that the definition of subharmonicity was a local one.

The following well-known result will often be implicitly used.

Theorem 4.3. *Let $\Phi(z) = \phi(\operatorname{Re} z)$ for some function ϕ on $\mathbb{R}$. Then Φ is subharmonic on $\{z : \operatorname{Im} z \in (a,b)\}$ for (a,b) a non-empty open interval in $\mathbb{R}$ if and only if ϕ is convex on $\mathbb{R}$.*

Proof. Suppose first that ϕ is convex and put $\Phi(z) = \phi(\operatorname{Re} z)$. Let $D = \{z : \operatorname{Im} z \in (a,b)\}$. Choose any $z \in D$ and any r sufficiently small that $\mathbb{D}(z;r)$ is contained in D. Then,

$$\begin{aligned}\frac{1}{\pi r^2}\iint_{\mathbb{D}(z;r)} \Phi(z)\, dA(z) &= \frac{1}{\pi r^2}\iint_{\mathbb{D}(z;r)} \phi(\operatorname{Re} w)\, dA(w)\\ &\le \phi\left(\frac{1}{\pi r^2}\iint_{\mathbb{D}(z;r)} \operatorname{Re} w\, dA(w)\right)\\ &= \phi(\operatorname{Re} z),\end{aligned}$$

where we have used Jensen's inequality (see, e.g., [55, Thm. I.6.1]) and the easy fact that the mean of $w \mapsto \operatorname{Re} w$ over a disc centred at z is $\operatorname{Re} z$. Hence, Φ is subharmonic as desired.

Conversely, suppose that $\Phi(z) = \phi(\operatorname{Re} z)$ is subharmonic on D. In fact, by translation invariance of subharmonicity and since subharmonicity is a local condition, it follows that Φ is subharmonic on all of $\mathbb{C}$. Then, to obtain a contradiction, assume that ϕ fails to be convex. We can then find real numbers $x < y$ in (a,b) and $t \in (0,1)$ such that

$$\phi(tx + (1-t)y) > t\phi(x) + (1-t)\phi(y). \tag{4.2}$$

Let $\alpha = \frac{\phi(y)-\phi(x)}{y-x}$. Let $v = tx + (1-t)y$. Put

$$\psi(u) = \phi(u) + \alpha(y-u) - \phi(y).$$

Then $\psi(x) = \psi(y) = 0$. Moreover, it is easy to see that $\psi(v) > 0$ in light of (4.2). Let $\Psi_1(z) = \psi(\operatorname{Re} z)$. Since Ψ_1 differs from Φ only by a linear function and linear functions are harmonic, it follows that Ψ_1 is subharmonic. Let $\Psi_2(z) = \psi(\operatorname{Im} z)$. This too must be subharmonic, since subharmonicity is invariant under rotation. Thus,

$$\Psi \stackrel{\text{def}}{=} \Psi_1 + \Psi_2$$

is subharmonic. Let Q be the square

$$\{z : x < \operatorname{Re} z < y, x < \operatorname{Im} z < y\}.$$

I claim that Ψ attains a maximum on Q and is moreover non-constant on Q; by the maximum principle we will then have a contradiction. To prove the claim, first let $w \in (x, y)$ be such that $\psi(w)$ is maximal. Since $v \in (x, y)$ has $\psi(v) > 0$ it follows that $\psi(w) > 0$. Now, because Ψ is continuous on $\overline{Q}$, it suffices for us to verify that

$$\sup_{z \in \partial Q} \Psi(z) < \Psi(w + iw)$$

since $w + iw \in Q$. Fix $z \in \partial Q$. Then either $\operatorname{Re} z \in \{x, y\}$ or $\operatorname{Im} z \in \{x, y\}$ or both. Consider the case where $\operatorname{Re} z \in \{x, y\}$, as the other case is analogous. Then, $\Psi(z) = \psi(\operatorname{Re} z) + \psi(\operatorname{Im} z) = \psi(\operatorname{Im} z)$ since ψ vanishes on $\{x, y\}$. But $\Psi(w + iw) = 2\psi(w)$ and $\psi(w) \geq \psi(\operatorname{Im} z)$ by the choice of w since $\operatorname{Im} z \in [x, y]$. Since $\psi(w) > 0$ it follows that $\Psi(w + iw) > \Psi(z)$ as desired.

Hence indeed Ψ attains a strictly positive maximum in Q and thus must be constant on Q. But Ψ is not constant since $\Psi(x + ix) = \psi(x) + \psi(x) = 0$ and Ψ is continuous. Hence we have a contradiction as desired. ☐

Theorem 4.4. *Let $\Phi(z) = \phi(|z|)$ for some function $\phi\colon [0, \infty) \to [-\infty, \infty)$. Then Φ is subharmonic if and only if $t \mapsto \phi(e^t)$ is convex and increasing on $\mathbb{R}$ and ϕ is continuous at* 0 *(though possibly equal to $-\infty$ at* 0*).*

Proof. We shall first show that Φ is subharmonic on $\mathbb{C}\backslash(-\infty, 0]$ if and only if $t \mapsto \phi(e^t)$ is convex on $\mathbb{R}$. To see this, let $\psi(t) = \phi(e^t)$ for $t \in \mathbb{R}$. Consider the exponential map exp sending $D \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi\}$ into $\mathbb{C}\backslash(-\infty, 0]$. Then, by conformal invariance of subharmonicity (Remark 4.1), we have Φ subharmonic on $\mathbb{C}\backslash(-\infty, 0]$ if and only if $\Phi \circ \exp$ is subharmonic on D.

But $(\Phi \circ \exp)(z) = \phi(|e^z|) = \psi(\operatorname{Re} z)$. Hence, by Theorem 4.3, Φ is subharmonic on $\mathbb{C}\backslash(-\infty, 0]$ if and only if ψ is convex.

In fact, by rotation invariance, if $\phi(|z|)$ is subharmonic on $\mathbb{C}\backslash(-\infty, 0]$ then it is subharmonic on all of $\mathbb{C}\backslash\{0\}$. Hence, Φ is subharmonic on $\mathbb{C}\backslash\{0\}$ if and only if $t \mapsto \phi(e^t)$ is convex. We now complete our proof. Suppose first that Φ is subharmonic on $\mathbb{C}$. We have only to prove that ϕ is increasing and has $\phi(0) = \lim_{r\to 0+} \phi(r)$. Suppose first that ϕ is not increasing on $(0, \infty)$. Then, since $t \mapsto \phi(e^t)$ is convex, it follows that $\lim_{t\to-\infty} \phi(e^t) = \infty$. But this is impossible, since Φ is upper semicontinuous at 0 and has $\Phi(0) < \infty$. Thus, ϕ indeed is increasing on $(0, \infty)$. Moreover, for sufficiently small r we have

$$\phi(0) \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(r)} \Phi(z)\, dA(z).$$

Since $\Phi(z) = \phi(|z|)$ and $\phi(r)$ is continuous increasing function on $(0, \infty)$, it follows that $\phi(0) \leq \liminf_{r\to 0+} \phi(r)$ so that ϕ is lower semicontinuous at 0. By upper semicontinuity of Φ at 0 we see that we must have ϕ continuous at 0 as desired.

Conversely, suppose that $t \mapsto \phi(e^t)$ is convex and increasing and ϕ is continuous at 0. Then Φ is subharmonic on $\mathbb{C}\backslash\{0\}$. Clearly it is also continuous at 0. Moreover, the continuity of ϕ at 0 and the increasing character of ϕ imply that

$$\phi(0) \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(r)} \phi(|z|)\, dA(z)$$

for every $r \in (0, \infty)$, and so we see that indeed Φ must be subharmonic everywhere. □

Remark 4.2. In particular, if ϕ is convex and increasing on $[0, \infty)$ then $z \mapsto \phi(|z|)$ is subharmonic on $\mathbb{C}$. To see this, note that in such a case $t \mapsto \phi(e^t)$ must also be convex since $t \mapsto e^t$ is convex.

5. Least harmonic majorants, harmonic measures and uniformizers

This section still contains no really new results, but is intended to give a precise meaning to our terminology and to collect some known facts which we will later use.

We work all the time on domains D in the complex plane $\mathbb{C}$. For harmonic measures, Dirichlet problems, Brownian motions, etc., our basic reference is the book of Doob [39].

5.1. Dirichlet problem and harmonic measure

Definition 5.1. Let D be a domain and f a function on the Euclidean boundary ∂D. Then we say that **the Dirichlet problem on D with boundary value f is solvable** if f is resolutive, i.e., if there exists a PWB solution [39, §1.VIII.2] F on D. We then say that F is **the solution to the Dirichlet problem on D.**

The above definition is rather technical, but in practice this shall not concern us. All that is necessary for intuition is to note that F is a harmonic function on D which in some sense (which sense is made precise by the invocation of the PWB method) has the limit f at the boundary of D.

Definition 5.2. A domain D is **Greenian** if there exists a positive nonconstant superharmonic function on D.

Remark 5.1. As Doob [39, §1.II.13] notes, any domain D which is not dense in $\mathbb{C}$ is Greenian. For, if $w \in \mathbb{C}\backslash\overline{D}$, then the function $f(z) = c + \log|z - w|$ is harmonic (hence superharmonic) and nonconstant on D, while for large enough c it will be positive.

Remark 5.2. By Doob [39, §1.V.6], the plane $\mathbb{C}$ is not Greenian.

The domains with which this thesis will be concerned are primarily the domains of finite area. The following proposition implies that all domains of finite area are Greenian.

Proposition 5.1. *Any domain D which is simply connected or whose complement has positive Lebesgue area measure must necessarily be Greenian.*

The proof of the case of the complement having positive Lebesgue area measure will be delayed until §5.4.

Proof in simply connected case. Suppose first that D is simply connected. Let f be a **Riemann map** from D onto $\mathbb{D}$, i.e., a univalent map sending D onto $\mathbb{D}$ whose existence is guaranteed by the Riemann mapping theorem [94, Thm. 14.8]. Then $z \mapsto -\log|f(z)|$ is superharmonic on D (see [55, p. 34]) and clearly positive and nonconstant. $\square$

Definition 5.3. Let A be a subset of ∂D where D is domain. Then, the **harmonic measure of A in D** is defined to be the function $z \mapsto \omega(z, A; D)$ which is the solution of the Dirichlet problem on D with boundary value 1 on A and 0 on $\partial D \setminus A$, if this solution exists.

Harmonic measure exists for any Borel set [39, §§1.VIII.4 and 1.VIII.6].

Remark 5.3. Harmonic measure is monotone with respect to A and D. More precisely, if $A \subseteq A'$ and $D \subseteq D'$ with $A \subseteq \partial D$ and $A' \subseteq \partial D'$ then $\omega(z, A; D) \leq \omega(z, A'; D')$. This follows from the fact that $\omega(\cdot, A; D)$ is easily seen to lie in the lower PWB class for the Dirichlet problem of which $\omega(\cdot, A'; D')$ is the solution. (See [39, §1.VIII.2] for definitions.)

For convenience, we will sometimes write $\omega_z^D = \omega(z, \cdot; D)$. The following result is very important, although rather technical.

Theorem 5.1 (cf. Doob [39, §1.VIII.8]). *Harmonic measure exists for every Borel subset A of ∂D where D is a Greenian domain. The set of subsets A for which harmonic measure exists is a σ-algebra $\mathcal{H}_D$, and for each fixed z, the function $A \mapsto \omega(z, A; D)$ is a measure ω_z^D on $\mathcal{H}_D$.*

Technical remark 5.1. This is given by Doob [39, §1.VIII.8] in the case of ∂D being given by a metric compactification, and not the Euclidean boundary. However, the case of the Euclidean boundary follows from the fact that for an unbounded Greenian domain, the singleton $\{\infty\}$ has null harmonic measure [39, Example 1.VIII.5(a)].

Definition 5.4. Let D be a Greenian domain. We say that $f \in L^1(\omega^D)$ if f is a $\mathcal{H}_D$ measurable function on ∂D such that

$$\int_{\partial D} |f| \, d\omega_z^D < \infty \tag{5.1}$$

for every $z \in D$.

Remark 5.4. In fact, it suffices to verify (5.1) for any single point $z \in D$; see [39, §1.VIII.8].

The following result is quite important.

Theorem 5.2 ([39, §1.VIII.8]). *Let $f \in L^1(\omega^D)$ for a Greenian domain D. Then, the Dirichlet problem with boundary value f on ∂D has a solution F given by*

$$F(z) = \int_{\partial D} f \, d\omega_z^D.$$

Remark 5.5. The harmonic measure $\omega_0^{\mathbb{D}}$ coincides with normalized Lebesgue measure on $\mathbb{T}$ for Borel sets. The easiest way to see this is to simply note that both measures are rotation invariant finite measures, and hence must coincide on the Borel sets by the uniqueness of Haar measure (see, e.g., [93, Thm. 14.19]).

5.2. Regularity for the Dirichlet problem

Definition 5.5. Let D be a Greenian domain. Call a point z of ∂D **regular** if for any $f \in L^1(\omega^D)$ on ∂D which is continuous at $z \in \partial D$ we have

$$F(\zeta) \to f(z),$$

as $\zeta \to z$ from within D, where F is the solution of the Dirichlet problem on D with boundary values f. A domain D is said to be regular if every point of its boundary is regular.

The results given in Doob's book [39, §1.VIII] show that these definitions are equivalent to the standard definitions in the case of Greenian domains.

Note that if $f \in L^1(\omega^D)$ is continuous on ∂D for a regular domain D, then the solution to the Dirichlet problem on D extends to a continuous function on $\overline{D}$ which agrees with f on ∂D.

We shall have occasion to use the following simple criterion for regularity and for the Greenian character of a domain.

Theorem 5.3. *Let D be any domain. Let $z \in \partial D$. Suppose that there is an curve*

$$w(r) = z + re^{i\theta(r)}, \qquad 0 \leq r \leq \varepsilon,$$

lying outside D, for some $\varepsilon > 0$ and a continuous function θ. Then D is Greenian and z is regular.

Only the Greenian character needs to be proved, since, given this Greenian character, the regularity is a consequence of [60, Thm. 2.11]. We shall prove the Greenian character in §5.4.

Definition 5.6. We say that a domain D is **a C^1 domain** if for every $z \in \partial D$ there exists a function $\phi\colon (-1,1) \to \partial D$ which is continuously differentiable with $\phi(0) = z$ and $\phi'(0) \neq 0$.

Corollary 5.1. *Let $D \neq \mathbb{C}$ be a C^1 domain. Then D is Greenian and regular.*

Proof of Corollary. Fix $z \in \partial D$. (Of course $\partial D \neq \varnothing$ since $D \neq \mathbb{C}$ and since domains are by definition non-empty.) Let ϕ be as in the definition of a C^1 boundary. Then $\phi(0) = z$. Choose $\delta_1 \in (0,1)$ such that $\phi(t) \neq z$ for all $t \in (0, \delta_1)$. (Such a δ_1 exists since $\phi'(0) \neq 0$.) Define $\rho(t) = |\phi(t) - z|$ for $t \in [0, \delta_1)$. Using the fact that $\phi'(0) \neq 0$ it is easy to see that ρ is continuously differentiable on $[0, \delta_1)$, with $\rho'(0) > 0$. Thus, there is an $\varepsilon > 0$ such that there is a continuous function $\rho^{-1}\colon [0, \varepsilon] \to [0, \delta)$ with $\rho(\rho^{-1}(r)) = r$ for all $r \in [0, \varepsilon]$, and $\delta \in (0, \delta_1)$.

We have $\rho(t) > 0$ for $t \in [0, \delta]$. For $t \in (0, \delta]$ let

$$\psi(t) = \frac{\phi(t) - z}{|\phi(t) - z|}.$$

Then, ψ is a continuous map of $(0, \delta]$ into $\mathbb{T}$. The (one-sided) differentiability of ρ at 0 and the strict positivity of this derivative implies that $\lim_{t \to 0+} \psi(t)$ exists by L'Hôpital's rule, so that we may extend ψ to a continuous function from $[0, \delta]$ to $\mathbb{T}$. We may then choose a continuous logarithmic function $L\colon [0, \delta] \to \mathbb{C}$ such that $\psi(t) = e^{L(t)}$ for all $t \in [0, \delta]$. In fact L will map $[0, \delta]$ into $i\mathbb{R}$.

Define $\theta(r) = L(\rho^{-1}(r))/i$. Then, for $0 \leq r \leq \varepsilon$ we have

$$z + re^{i\theta(r)} = z + r\frac{\phi(\rho^{-1}(r)) - z}{\rho(\rho^{-1}(r))} = \phi(\rho^{-1}(r)) \in \partial D \subseteq D^c.$$

□

5.3. Least harmonic majorants

Let Φ be a subharmonic function on a domain D. Write

$$\mathrm{LHM}(z, \Phi; D) = \inf\{h(z) : h \text{ is a harmonic function on } D \text{ with } h \geq \Phi\},$$

for the least harmonic majorant of Φ at $z \in D$. Note that $\mathrm{LHM}(\cdot, \Phi; D)$ is either identically $+\infty$ on D or else it is harmonic there [39, §1.III.1]. If it is harmonic, then we say that Φ **has a harmonic majorant on** D.

The following result is quite well-known.

Theorem 5.4 (cf. [46, equation (1.4)] and [39, Example 1.VIII.3(a)]). *Assume that Φ is a subharmonic function on D with an extension to $\overline{D}$ such that the extended function is continuous at every point of $\overline{D}$ while $\Phi|_{\partial D} \in L^1(\omega^D)$. Assume that Φ (as a function on $\overline{D}$) is continuous at every point of ∂D. Then,*

$$\mathrm{LHM}(z, \Phi; D) = \int_{\partial D} \Phi \, d\mu_z^D.$$

Proof. It is clear that Φ is in the lower PWB class for $\phi \stackrel{\text{def}}{=} \Phi|_{\partial D}$ (see [39, §1.VIII.2] for definitions). Let Ψ be the solution of the Dirichlet problem on D with boundary value ϕ on ∂D (this exists by Theorem 5.2). Since Φ is in the lower PWB class, it follows that $\Phi \leq \Psi$ on D. I now claim that $\mathrm{LHM}(\cdot, \Phi; D)$ is in the upper PWB class for ϕ. If this is true then

$$\Phi \leq \Psi \leq \mathrm{LHM}(\cdot, \Phi; D)$$

on D. But since $\mathrm{LHM}(\cdot, \Phi; D)$ is the least harmonic majorant while Ψ is harmonic, it then follows from the above inequalities that $\Psi = \mathrm{LHM}(\cdot, \Phi; D)$. The conclusion of the theorem then follows from Theorem 5.2.

Thus we must prove that $\mathrm{LHM}(\cdot, \Phi; D)$ is in the upper PWB class for ϕ. To do this, let $z_n \to w$ where $z_n \in D$ and $w \in \partial D$. We must prove that

$$\liminf_{n\to\infty} \mathrm{LHM}(z_n, \Phi; D) \geq \phi(w).$$

But $\mathrm{LHM}(z_n, \Phi; D) \geq \Phi(z_n) \to \Phi(w) = \phi(w)$, where we have used the continuity of Φ on ∂D. □

5.4. Brownian motion and harmonic measure

Standard Brownian motion on $\mathbb{R}^n$ is a Markov process $\{B_t\}_{t\in[0,\infty)}$ with almost surely continuous paths, values in $\mathbb{R}^n$ and Gaussian increments; see [39, §2.VII.2] for a rigorous definition (in the case $n = 2$, see also [38]). We often drop the word "standard" from the term "standard Brownian motion". We use $P^z(\cdot)$ and $E^z[\cdot]$ to indicate probabilities and expectations when the Brownian motion is conditioned to start from the point $z \in \mathbb{R}^n$ at time 0.

Given a domain $D \subseteq \mathbb{R}^2$, let

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\}$$

be the **first exit time of Brownian motion from the domain** D. We then have the following very useful connection between Brownian motion, harmonic measure and PWB solutions of Dirichlet problems.

Theorem 5.5 (see, e.g., [39, §2.IX.10 and §2.IX.13]). *Let D be a Greenian domain. Then $P(\tau_D < \infty) = 1$. Let A be an $\mathcal{H}_D$-measurable subset of ∂D. Then,*

$$\omega(z, A; D) = P^z(B_{\tau_D} \in A). \tag{5.2}$$

Moreover if $f \in L^1(\omega^D)$ and F is the solution of the Dirichlet problem on D with boundary value f on ∂D, then

$$F(z) = E^z[f(B_{\tau_D})]. \tag{5.3}$$

Note that (5.2) is equivalent to (5.3) by Theorem 5.2. The above result shows that harmonic measure of A at z in D is the probability that when a Brownian motion started at z hits ∂D, it hits it within the set A. Thus, it measures how large A is compared to the rest of ∂D as seen from the point of view of z.

A very important result in two dimensions is the theorem of Lévy that if f is a non-constant analytic function and B_t a Brownian motion in the plane, then $f(B_t)$ is a Brownian motion moving perhaps at a variable speed. This is known as the conformal invariance of Brownian motion. More precisely, we have the following result.

Theorem 5.6. *If f is a non-constant analytic function on a domain D and z a fixed starting point in D then there exists a strictly increasing continuous function α (depending on z and f) such that the process $\{f(B_{\alpha(t)})\}_{0\leq t<\alpha^{-1}(\tau_D)}$ is a standard Brownian motion started from $f(z)$ if $B_0 = z$.*

A proof can be found in Doob [39, §2.VIII.14] or in McKean's book [76, p. 108].

Taking this together with the connection between harmonic measure and Brownian motion we can, if we like, obtain a result on the conformal invariance of harmonic measure.

Remark 5.6. We may extend the process $f(B_{\alpha(t)})$ to be a Brownian motion for all time if we wish; see Davis [38, Thm. 2.4].

We may now give two proofs which we have hitherto delayed.

Proof of Proposition 5.1. Suppose that $\mathbb{C}\backslash D$ has positive Lebesgue area measure. Fix any $z \in D$. Then, there exists an $r > 0$ such that $\mathbb{T}(z;r)\backslash D$ has positive one dimensional Lebesgue measure. Then, $P^z(\tau_{\mathbb{D}(z;r)} \notin D)$ equals the harmonic measure of $\mathbb{T}(z;r)\backslash D$ at z in $\mathbb{D}(z;r)$. But this harmonic measure is equal to precisely the angular measure of $\mathbb{T}(z;r)\backslash D$ in light of Remark 5.5, and this angular measure is strictly positive. Hence, $P^z(\tau_{\mathbb{D}(z;r)} \notin D) > 0$. Hence, $P^z(\exists t.\ B_t \notin D) > 0$. By [39, Thm. 2.IX.10], we see that D is Greenian. $\square$

Proof of Greenian character in Theorem 5.3. The condition we have assumed implies that there exist two points $z \neq w$ in D^c and a continuous path γ lying in D^c joining z to w. We shall use a very visual proof. Fix a large positive integer N to be specified later. Let $w_1, w_2, \ldots, w_N$ be the vertices of a regular polygon with N sides of length $|z - w|$ centred on 0. Put $w_0 = w_N$. For $n = 1, \ldots, N$, let γ_i be a translated and rotated copy of γ so that γ_i starts at w_{n-1} and ends at w_n. As N tends to infinity, the distance of the w_i from the origin also tends to ∞. Moreover, γ is a bounded set. Hence, we may choose N sufficiently large that the unit disc $\mathbb{D}$ is contained in the complement of the union of the convex hulls of the sets γ_i. Let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_N$. Then, Γ is a bounded curve winding its way precisely once around the origin. Hence, any continuous path from the origin to infinity must intersect Γ. Since, with probability 1, Brownian motion is an unbounded process, we conclude that with probability 1 a Brownian motion starting at the origin must intersect Γ and thus it must intersect one of the γ_i. By rotation invariance of Brownian motion, the probability of intersecting any particular γ_i is the same, it follows that with probability at least $1/N$, the Brownian motion must intersect γ_1. By [39, Thm. 2.IX.10] it follows that any domain contained in the complement of γ_1 is Greenian. Rotating and translating this statement, we see that any domain contained in the complement of γ is Greenian, and hence D is Greenian. □

5.5. The uniformizer and harmonic measure

Let D be a domain in the plane. Let $\check{D}$ be the universal covering surface of D and let $\pi : \check{D} \to D$ be the universal covering map. See Beardon [17] for details. The Riemann surface $\check{D}$ will be conformally equivalent to one of $\mathbb{C} \cup \{\infty\}$, $\mathbb{C}$ and $\mathbb{D}$ [17, §9.1].

The following result is very standard.

Proposition 5.2. *Let D be a Greenian domain. Then $\check{D}$ is conformally equivalent to $\mathbb{D}$.*

Proof. If $\check{D}$ is conformally equivalent to $\mathbb{C} \cup \{\infty\}$ then it is compact, and hence $\pi[\check{D}] = D$ is compact, a contradiction. Suppose now that $\check{D}$ is conformally equivalent to $\mathbb{C}$. By the Greenian property, there exists a positive nonconstant superharmonic function f on D. Then, $f \circ \pi$ is a positive nonconstant superharmonic function on $\check{D}$. But $\mathbb{C}$ is not Greenian (Remark 5.2), and hence we have a contradiction. Thus the only remaining case is that of $\check{D}$ conformally equivalent to $\mathbb{C}$. $\square$

If D is a Greenian domain, then let $\rho\colon \mathbb{D} \to \check{D}$ be a conformal isomorphism, and put $f = \pi \circ \rho\colon \mathbb{D} \to D$. Then f is a surjective map. We call f **a uniformizer of** D. Note that if D is simply connected then f is just a Riemann mapping. The following result gives us another way to compute harmonic measures, and was kindly pointed out to the author by Professor Alec Matheson. Recall that the notion of a Nevanlinna domain was defined in §3.5.

Theorem 5.7 (cf. Fisher [50, §2.4]). *Let D be a Greenian domain, and suppose that A is a Borel subset of ∂D. Let $f\colon \mathbb{D} \to D$ be a uniformizer. Assume that f lies in the Nevanlinna class N. Then,*

$$\omega(f(0), A; D) = \frac{1}{2\pi} \int_0^{2\pi} 1_A(f(e^{i\theta}))\, d\theta. \tag{5.4}$$

If ϕ is a Borel measurable function on ∂D with $f \in L^1(\omega^D)$ and Φ is the PWB solution to the Dirichlet problem on D with boundary value ϕ, then

$$\Phi(f(0)) = \frac{1}{2\pi} \int_0^{2\pi} \phi(f(e^{i\theta}))\, d\theta. \tag{5.5}$$

Remark 5.7. The assumption that $f \in N$ will be necessarily satisfied if D is a Nevanlinna domain, and in particular if D is an H^p domain.

We give a Brownian motion proof of Theorem 5.7.

Proof. Equation (5.5) follows from (5.4) and Theorem 5.2. Thus we need only prove (5.4). Without loss of generality, $f(0) = 0 \in D$. Identifying $\mathbb{D}$ and $\check{D}$, we may moreover assume that $\mathbb{D} = \check{D}$ and that $f = \pi$.

Let B_t be a Brownian motion in the plane starting at $0 = B_0$. Let $X_t = f(B_{\alpha(t)})$ be a standard Brownian motion, where α is as in Theorem 5.6. Let $T = \alpha^{-1}(\tau_{\mathbb{D}})$. Note that $P(T < \infty) = 1$.

For, X_t is a Brownian motion in the Greenian domain D until time T, while with probability 1 it exits D at some finite time by Theorem 5.5. Moreover, since X_t extends to a Brownian motion for all time (Remark 5.6), the expression X_T makes sense and satisfies $X_T = \lim_{t\to T-} X_t$ with probability 1. I now claim that under this extension

$$T = \inf\{t \geq 0 : X_t \notin D\} \qquad \text{with probability 1.} \tag{5.6}$$

It is clear that $T \geq \inf\{t \geq 0 : X_t \notin D\}$ since $X_t \in D$ whenever $t < T$. Thus, to prove our claim it will suffice to show that

$$\lim_{t\to T-} X_t \notin D \qquad \text{with probability 1.}$$

To do this, let S be the set of points of our underlying probability space such that on S we have:

(i) $T < \infty$ and $\tau_{\mathbb{D}} < \infty$

(ii) $t \mapsto X_t$ is continuous on $[0, T]$

(iii) $t \mapsto B_t$ is continuous on $[0, \tau_{\mathbb{D}}]$

(iv) $X_T = f(B_{\tau_{\mathbb{D}}})$.

We have already seen that (i)–(iii) happen with probability 1. We now remark that so does (iv). To see this, note that $B_T \in \mathbb{T}$ since B_t is continuous on $[0, \tau_{\mathbb{D}}]$, and remark that the "f" in "$f(B_T)$" is short for the non-tangential limit n. t. lim f. (Of course, the non-tangential limit of f exists almost everywhere by Theorem 3.5.) But the standard connection between non-tangential limits of harmonic functions and their limits along Brownian paths (see Brelot and Doob [27], Constantinescu and Cornea [37], as well as Burkholder and Gundy [29]) then shows that $f(B_{\tau_{\mathbb{D}}}) = \lim_{t\to\tau_{\mathbb{D}}-} f(B_t)$ almost surely, from which the desired result follows upon replacing t with $\alpha(t)$.

We shall prove that everywhere on S we have $\lim_{t\to T-} X_t \notin D$. For suppose that we are working at a point $\omega \in S$ and that all our random variables are sampled at precisely ω, and finally assume that $\lim_{t \to T-} X_t \in D$. We shall obtain a contradiction. To do this, let $\Gamma(t) = B_t$ and $\gamma(t) = f(B_t)$ for $t \in [0, \tau_{\mathbb{D}})$. Set $\gamma(\tau_{\mathbb{D}}) = X_T$. Since

$$\lim_{t \to T-} X_t = \lim_{t \to \tau_{\mathbb{D}}-} \gamma(t),$$

it follows that γ is a continuous function on $[0, \tau_{\mathbb{D}}]$. Because $\pi \colon \check{D} = \mathbb{D} \to D$ is a universal covering map, it follows that there is a continuous lifted path $\check{\gamma} \colon [0, \tau_{\mathbb{D}}] \to \mathbb{D}$ such that $\pi \circ \check{\gamma} = \gamma$ on $[0, \tau_{\mathbb{D}}]$ and $\check{\gamma}(0) = 0$ (see, e.g., [17, Chapter 7]). By uniqueness of lifts [17, Thm. 7.4.3], since $\check{\gamma}(0) = \Gamma(0)$ and $\pi \circ \check{\gamma} = \gamma = \pi \circ \Gamma$ on $[0, \tau_{\mathbb{D}})$, it follows that $\check{\gamma} = \Gamma$ on $[0, \tau_{\mathbb{D}})$. But

$$\lim_{t \to \tau_{\mathbb{D}}-} \check{\gamma}(t) = \check{\gamma}(\tau_{\mathbb{D}}) \in \mathbb{D},$$

while

$$\lim_{t \to \tau_{\mathbb{D}}-} \Gamma(t) = B_{\tau_{\mathbb{D}}} \notin \mathbb{D},$$

a contradiction.

Hence, (5.6) is valid. Applying Theorem 5.5 to the standard Brownian motion X_t, we see that

$$\omega(0, A; D) = P(X_T \in A).$$

But, given the set S defined before, which event has probability 1, we have $X_T = f(B_{\tau_D})$. Thus,

$$\omega(0, A; D) = P(f(B_{\tau_D}) \in A).$$

But B_{τ_D} has uniform distribution on $\mathbb{T}$ when $B_0 = 0$. (This can be seen from rotation invariance of Brownian motion and uniqueness of Haar measure on $\mathbb{T}$ [93, Thm. 14.19].) Hence

$$\omega(0, A; D) = P(f(B_{\tau_D}) \in A) = \frac{1}{2\pi} \int_0^{2\pi} 1_A(f(e^{i\theta}))\, d\theta.$$

□

5.6. Green's functions

There is more than one equivalent definition of a Green's function that could be given. The one that we shall give will be a two step definition. First suppose that D has regular boundary and that $\mathbb{C}\backslash\overline{D}$ is non-empty. Then, a Green's function $g(\cdot, w; D)$ for D is defined to be any function such that:

(i) $g(z, w; D) = 0$ whenever $z \notin D$ or $w \notin D$

(ii) $g(\cdot, w; D)$ is continuous on $\mathbb{C}\backslash\{w\}$ for each fixed w

(iii) $g(\cdot, w; D)$ is harmonic on $D\backslash\{w\}$ for each fixed w

(iv) $z \mapsto g(z, w; D) - \log \frac{1}{|z-w|}$ is a harmonic function in a neighbourhood of w if $w \in D$ is fixed.

For uniqueness and existence see [60, Thms. 1.14 and 3.13].

Now suppose that D is any Greenian domain. It is easy to see that we can find a sequence D_n of regular domains with compact closures such that $D_1 \subseteq D_2 \subseteq \cdots$ and $D = \bigcup_{n=1}^{\infty} D_n$. (It is easy to construct such domains, using Theorem 5.3 for this purpose. As Hayman and Kennedy [60, p. 253] remark, it is easy to even make sure that each D_n is a union of finitely many discs.) We then define the Green's function of D as

$$g(z, w; D) = \lim_{n\to\infty} g(z, w; D_n).$$

This limit exists and is independent of the choice of the D_n [60, Lem. 5.6].

It is worth noting that $g(z, w; D) = g(w, z; D)$ for all z and w [60, Thm. 5.26].

Remark 5.8. Note that

$$g(z, 0; \mathbb{D}) = \log \frac{1}{|z|}, \tag{5.7}$$

for $z \in \mathbb{D}$. This is easily verified since $\log \frac{1}{|z|}$ is locally the real part of an analytic function away from the origin, and hence harmonic there, since condition (iv) is trivial and since $\log \frac{1}{|z|}$ vanishes for $z \in \partial\mathbb{D}$. More generally, for z and w in $\mathbb{D}$ we have

$$g(z, w; \mathbb{D}) = \log \left| \frac{1 - z\overline{w}}{z - w} \right|. \tag{5.8}$$

To see this, note that the function vanishes as desired for z and w in $\partial\mathbb{D}$ and that it is locally the real part of an analytic function of z on $\mathbb{D}\backslash\{w\}$. The only thing left to verify is condition (iv). But,

$$g(z,w;\mathbb{D}) - \log\frac{1}{|z-w|} = \log|1-z\overline{w}| = \operatorname{Re}\log(1-z\overline{w}),$$

and hence is locally the real part of an analytic function for z near w.

The following very well known result is worth noting.

Theorem 5.8. *Let $U \neq \mathbb{C}$ be a simply connected domain. Let $w \in U$ and suppose that $f\colon \mathbb{D} \to U$ is a Riemann map from $\mathbb{D}$ onto U with $f(0) = w$. Then U is Greenian and*

$$g(z,w;U) = \log\frac{1}{|f^{-1}(z)|}$$

for all $z \in U$.

5.7. Riesz' theorem and representation of least harmonic majorants

Theorem 5.9 (Riesz; see [60, Thm. 3.9]). *Let Φ be subharmonic in a domain D, with $\Phi \not\equiv -\infty$. Then there exists a unique positive Borel measure μ in D such that for any compact subset E of D the function*

$$z \mapsto \Phi(z) - \int_E \log|z-w|\,d\mu(w)$$

is harmonic on the interior of E.

We shall write $\mu = \mu_\Phi$ and call it the **Riesz measure of Φ**.

Remark 5.9. The uniqueness of μ_Φ implies that μ_Φ depends only locally on Φ. More precisely, given Φ on D and given a subdomain $U \subseteq D$ we have $\mu_{\Phi|_U} = \mu_\Phi$ on U.

If $\Phi \in C^2(D)$ then $\mu_\Phi = \frac{1}{2\pi}\Delta\Phi$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. (See [39, §I.8].) This together with the local dependence implies that if $U \subseteq D$ is a subdomain in which Φ is harmonic, then μ has support in $D\backslash U$ since $\Delta\Phi = 0$ on U.

The following result, although very trivial, will in fact be useful.

Theorem 5.10. *Let $\Phi(z) = \phi(|z|)$ be subharmonic on a disc $\mathbb{D}(R)$. Then there is a positive measure ν_ϕ on $[0, R)$ such that for every Borel set $A \subseteq \mathbb{D}(R)$ we have*

$$\mu_\Phi(A) = \int_0^R \int_0^{2\pi} 1_A(re^{i\theta})\, d\theta\, d\nu_\phi(r). \tag{5.9}$$

Proof. The uniqueness of $\mu = \mu_\Phi$ implies that it is rotation invariant (since Φ is rotation invariant and so are the conditions of Theorem 5.9). Now, define the measure ν on $[0, R)$ as follows:

$$\nu(S) = (2\pi)^{-1}\mu(\{re^{i\theta} : r \in S, \theta \in [0, 2\pi)\}).$$

We must prove the validity of (5.9). In fact, it suffices to show that (5.9) holds for all sets A of the form $\{re^{i\theta} : r \in S, \theta \in T\}$ for S a Borel subset of $[0, R)$ with $\overline{S} \subset [0, R)$ and for T a Borel subset of $[0, 2\pi)$. The general case then follows from the fact that the collection of these kinds of sets generates all Borel sets in $\mathbb{D}(R)$. Now then, suppose we have a set such as we mentioned.

First note that $\nu(S) < \infty$ since $\overline{A} \subseteq \mathbb{D}(R)$ and if $\nu(S) = \infty$ then $\{re^{i\theta} : r \in \overline{S}, \theta \in [0, 2\pi)\}$ has infinite μ-measure, and since this set is a compact subset of $\mathbb{D}(R)$, this contradicts Riesz's theorem. Consider the measure α on $\mathbb{T}$ defined by

$$\alpha(U) = \mu(\{re^{i\theta} : r \in S, e^{i\theta} \in U\}).$$

This is then a finite rotation invariant measure on the Borel sets. Hence, it must be a multiple of the Haar measure on $\mathbb{T}$ (see, e.g., [93, Thm. 14.19]) so that

$$\alpha(U) = \frac{\alpha(\mathbb{T})}{2\pi} \int_0^{2\pi} 1_U(e^{i\theta})\, d\theta.$$

But $\alpha(\mathbb{T}) = 2\pi\nu(S)$. Hence,

$$\mu(\{re^{i\theta} : r \in S, e^{i\theta} \in U\}) = \nu(S) \int_0^{2\pi} 1_U(e^{i\theta})\, d\theta.$$

Putting $U = \{e^{i\theta} : \theta \in T\}$, we easily obtain (5.9) in the case of our set $A = \{re^{i\theta} : r \in S, \theta \in T\}$. □

Remark 5.10. We would like to say something about the Riesz measure of the function $\Phi(z) = \phi_t(\operatorname{Re} z)$ for $z \in \mathbb{C}$ where $\phi_t(x) = \max(0, x-t)$. I claim that in this case

$$\mu_\Phi = c\delta_t \times m,$$

where $c \in (0, \infty)$ is some constant independent of t, δ_t is the point mass measure in $\mathbb{R}$ concentrated at t and m is Lebesgue measure on $\mathbb{R}$. This claim will be of some use to us in §III.6. Without loss of generality $t = 0$ (the general case follows by translation). Note that Φ is infinitely differentiable away from imaginary axis $i\mathbb{R}$ with $\Delta\Phi = 0$ there, and hence μ_Φ has support on $i\mathbb{R}$. Thus,

$$\mu_\Phi = \delta_0 \times \lambda$$

for some measure λ on $\mathbb{R}$. Now, λ is a translation invariant measure on $\mathbb{R}$ as Φ is invariant with respect to translation in the direction of the imaginary axis so that μ_Φ is likewise invariant with respect to such translation. Moreover, by Riesz's theorem, λ must be a finite measure on the compact subsets of $\mathbb{R}$. Hence, λ is a translation invariant measure on the Borel subsets of $\mathbb{R}$ and finite on compacta. Thus, $\lambda = c \cdot m$ for some $c \in [0, \infty)$ (see, e.g., [93, Thm. 14.21] Of course, $c \neq 0$ since otherwise μ_Φ vanishes which by Riesz's theorem implies that Φ is harmonic, and this is evidently not true on a neighbourhood of $i\mathbb{R}$.

Finally, the following result known as the "Riesz decomposition of subharmonic functions" is quite important.

Theorem 5.11 ([60, Thm. 5.25]). *Suppose that Φ is subharmonic in a Greenian domain D and has a harmonic majorant. Then,*

$$\mathrm{LHM}(z, \Phi; D) = \Phi(z) + \int_D g(z, w; D)\, d\mu_\Phi(w).$$

6. Some known results in symmetrization theory

6.1. Circular symmetrization

Let U be a measurable subset of $\mathbb{C}$ and define $U^{\circledcirc}$ as in Example 2.5. Call a set U **circularly symmetric** if $U = U^{\circledcirc}$. Note that for any set U we have $(U^{\circledcirc})^{\circledcirc} = U^{\circledcirc}$, which shows that $\circledcirc$ is indeed a symmetrization on the collection of all open sets in the plane.

Remark 6.1. It is not quite true that $\circledcirc$ is a rearrangement from the σ-algebra of all Lebesgue measurable sets in $\mathbb{C}$ to that same σ-algebra in the sense of Definition 2.2. The difficulty is that Definition 2.2(ii) fails to be satisfied.[1] However, it *is* a rearrangement on the collection (topology) of all open subsets of $\mathbb{C}$. In fact, to see this we need only prove that for all $r \in [0, \infty)$ we have

$$\mathbb{T}(r) \cap A^{\circledcirc} = \bigcup_{n=1}^{\infty} \mathbb{T}(r) \cap A_n^{\circledcirc}, \tag{6.1}$$

where $A = \bigcup_{n=1}^{\infty} A_n$. Now, if $\mathbb{T}(r) \cap A \neq \mathbb{T}(r)$ then this result easily follows from the fact that Lebesgue measures behave nicely with respect to increasing unions. Suppose now that $A \cap \mathbb{T}(r) = \mathbb{T}(r)$. Let $F_n = \mathbb{T}(r) \backslash A_n$. This is a decreasing sequence of compact sets. Since $A \cap \mathbb{T}(r) = \mathbb{T}(r)$ and A is the union of the A_n it follows that $\bigcap_{n=1}^{\infty} F_n = \varnothing$. Hence, by compactness it follows that for sufficiently large n we have $F_n = \varnothing$ so that $\mathbb{T}(r) \subseteq A_n$ for large n and (6.1) follows.

Hence, when we apply Theorem 2.1, we must be careful to ensure we only apply it to functions f which are lower semicontinuous since lower measurability in our case coincides with lower semicontinuity. But the careful reader will notice that this will always be the case.

The following proposition will henceforth be often implicitly used.

Proposition 6.1. *Let D be an open subset of $\mathbb{C}$. Then, the function $r \mapsto \theta(r; D)$ is lower semicontinuous on $\{r \in [0, \infty) : \theta(r; D) < \infty\}$ and $D^{\circledcirc}$ is open.*

Proof. Write $\theta(r) = \theta(r; D)$ for short. First we prove the lower semicontinuity of θ on $R \stackrel{\text{def}}{=} \{r \in [0, \infty) : \theta(r) < \infty\}$. Define $\tilde{\theta}(r) = |\{\theta \in [0, 2\pi) : re^{i\theta} \in D\}|$. Then, $\tilde{\theta} = \theta$ on R. I claim that $\tilde{\theta}$ is lower semicontinuous on $[0, \infty)$. Let $r_n \to r \in [0, \infty)$ be a sequence of positive numbers. We must prove that $\tilde{\theta}(r) \leq \liminf \tilde{\theta}(r_n)$. Fix $\varepsilon > 0$. Assume $\varepsilon < \tilde{\theta}(r)$. Let $S = D \cap \mathbb{T}(r)$. Then,

[1] Let $A_n = \{e^{i\theta} : |\theta| < \pi - \frac{1}{n}\} \cup \{-1\}$. Then $A_n^{\circledcirc} = \{e^{i\theta} : |\theta| < \pi - \frac{1}{n}\}$ and $\bigcup_{n=1}^{\infty}(A_n^{\circledcirc}) = \mathbb{T}\backslash\{-1\}$ while $\bigcup_{n=1}^{\infty} A_n = \mathbb{T}$ so that $(\bigcup_{n=1}^{\infty} A_n)^{\circledcirc} = \mathbb{T} \neq \mathbb{T}\backslash\{-1\} = \bigcup_{n=1}^{\infty}(A_n^{\circledcirc})$ and Definition 2.2(ii) indeed fails.

there exists a compact set $K \subseteq S$ such that $|\{\theta : re^{i\theta} \in K\}| \geq \tilde{\theta}(r) - \varepsilon$. Since D is open and $K \subset D$ is compact, we have

$$\delta \stackrel{\text{def}}{=} \inf_{z \in \mathbb{C} \setminus D, w \in K} |z - w| > 0.$$

Then, if n is sufficiently large that $|r_n - r| < \delta$, we must then have $(r_n/r)K \subseteq D$. Thus, for such n,

$$\tilde{\theta}(r_n) \geq |\{\theta : r_n e^{i\theta} \in (r_n/r)K\}| \geq \tilde{\theta}(r) - \varepsilon.$$

Hence $\liminf \tilde{\theta}(r_n) \geq \tilde{\theta}(r) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we see that $\tilde{\theta}(r) \leq \liminf \tilde{\theta}(r_n)$.

We must now prove that $D^{\circledcirc}$ is open. Let

$$D_1 = \{re^{i\theta} : |\theta| < \tilde{\theta}(r)\}.$$

Because $\tilde{\theta}$ is lower semicontinuous, it follows that $D_1 \backslash \{0\}$ is open. Put

$$D_2 = \bigcup_{r \in [0,\infty) \backslash R} \mathbb{T}(r).$$

It is easy to see that $[0, \infty)\backslash R$ is open in $[0, \infty)$, since $[0, \infty)\backslash R = \{r \in [0, \infty) : \mathbb{T}(r) \subseteq D\}$ and D is open. Hence, it follows that D_2 is open. But it is clear that

$$D^{\circledcirc} = D_1 \cup D_2. \tag{6.2}$$

Moreover,

$$D^{\circledcirc} = (D_1 \backslash \{0\}) \cup D_2.$$

To see this, note that by (6.2) it suffices to show that if $0 \in D^{\circledcirc}$ then $0 \in D_2$. But this is clear since $\mathbb{T}(0) = \{0\}$. Hence $D^{\circledcirc}$ is open since $D_1 \backslash \{0\}$ and D_2 are both open. $\square$

Given a measurable function f on $\mathbb{C}$, we may define $f^{\circledcirc}$ via (2.1), even if f is not lower semicontinuous. Call a function f on $\mathbb{C}$ **circularly symmetric** if $f = f^{\circledcirc}$. Call it **symmetric decreasing** if for every $r \in (0, \infty)$ we have $f(re^{i\theta}) = f(re^{-i\theta})$ for each $\theta \in [0, \pi]$ while $\theta \mapsto f(re^{i\theta})$ is decreasing on $[0, \pi]$.

Remark 6.2. Clearly a circularly symmetric function is symmetric decreasing. Conversely, a lower semicontinuous symmetric decreasing function is circularly symmetric. For, to see this it is only necessary to note that if f is lower semicontinuous and symmetric decreasing, then each level set f_λ is necessarily a circularly symmetric set since it is open and of the form $\{re^{i\theta} : |\theta| < \theta(r)/2\}$.

We now define the Baernstein $*$-function of a measurable function f on $\mathbb{C}$. First put

$$Jg(re^{i\theta}) = \int_{-|\theta|}^{|\theta|} g(re^{i\varphi})\, d\varphi,$$

for a measurable g on $\mathbb{C}$ and $\theta \in [-\pi, \pi]$. Then, the **(circular) Baernstein $*$-function** $f^{\circlearrowleft}$ of a function f is defined to be

$$f^{\circlearrowleft} = J(f^{\circledcirc}).$$

Remark 6.3. We always have $Jf \leq f^{\circlearrowleft}$ everywhere. To see this, fix $r \in [0, \infty)$. Consider $\circledcirc$ restricted to measurable subsets of $\mathbb{T}(r)\backslash\{-1\}$. This is a rearrangement, and it is measure-preserving with regard to the one-dimensional Lebesgue measure λ on $\mathbb{T}(r)$. Let $u(re^{\varphi}) = 1_{(-\theta,\theta)}(\varphi)$ on $\mathbb{T}(r)$. Then, $u^{\circledcirc} = u$ so that by the Hardy-Littlewood inequality (Theorem 2.3) we have

$$\int_{\mathbb{T}(r)} u \cdot f \leq \int_{\mathbb{T}} u \cdot f^{\circledcirc}.$$

But it is clear that $\int_{\mathbb{T}(r)} u \cdot g = Jg(re^{i\theta})$ for any g, and so we are done.

Remark 6.4. For $\theta \in [-\pi, \pi]$ and $r \geq 0$ we have

$$f^{\circlearrowleft}(re^{i\theta}) = \sup_A \int_A f(re^{i\phi})\, d\phi,$$

where the supremum is taken over all measurable subsets A of $[-\pi, \pi]$ with measure $2|\theta|$. Baernstein uses the above identity as a definition of $f^{\circlearrowleft}(re^{i\theta})$ and proves that this agrees with our definition of $f^{\circlearrowleft}$ (see [7, Prop. 2]).

We now give Baernstein's result on circular symmetrization and Green's functions.

Theorem 6.1 (Baernstein [7]). *Let D be a Greenian domain in the plane, and let $u(z) = g(w, z; D)$ for some fixed $w \in \mathbb{C}$. Let $v(z) = g(|w|, z; D^{\circledcirc})$. Then, v is symmetric decreasing and $u^{\circlearrowleft} \leq Jv$ everywhere on $\mathbb{C}$.*

The following Corollary is well known.

Corollary 6.1. *Let D be a Greenian domain in the plane, and let $\Phi(z) = \phi(|z|)$ be a subharmonic function in the plane. Then,*

$$\mathrm{LHM}(w, \Phi; D) \leq \mathrm{LHM}(|w|, \Phi; D^{\circledcirc}).$$

This corollary is a direct consequence of the following result together with Theorem 6.1 and the fact that if w is fixed and $u(z) = g(w, z; D)$ then $u^{\circlearrowleft}(re^{i\pi}) = \int_0^{2\pi} g(w, re^{i\theta}; D)\, d\theta$.

Proposition 6.2. *Let D and D_1 be Greenian domains in the plane. Fix $w \in D$ and $w_1 \in D_1$ with $|w| = |w_1|$. Assume that*

$$\int_0^{2\pi} g(w, re^{i\theta}; D)\, d\theta \leq \int_0^{2\pi} g(w_1, re^{i\theta}; D_1)\, d\theta$$

for every $r \in [0, \infty)$. Then, for a subharmonic Φ of the form $\Phi(z) = \phi(|z|)$ we have

$$\mathrm{LHM}(w, \Phi; D) \leq \mathrm{LHM}(w_1, \Phi; D_1).$$

The above proposition is quite well-known. The author would like to thank Professor Albert Baernstein II for having pointed it out to him.

Proof of Proposition. Let ν_ϕ be the measure on $[0, \infty)$ given by Theorem 5.10. Fix $w \in \mathbb{C}$. Then, by Theorem 5.11 we have

$$\mathrm{LHM}(w, \Phi; D) = \Phi(w) + \int_0^\infty \int_0^{2\pi} g(w, re^{i\theta}; D)\, d\theta\, d\nu_\phi(r)$$

and

$$\mathrm{LHM}(w_1, \Phi; D_1) = \Phi(w_1) + \int_0^\infty \int_0^{2\pi} g(w_1, re^{i\theta}; D_1)\, d\theta\, d\nu_\phi(r).$$

(Actually, the integrals should really be restricted a little so as to be taken over the sets where $re^{i\theta} \in D$ and $re^{i\theta} \in D_1$, respectively, but this will make no difference since the Green's functions vanish outside their respective domains.) The desired result follows immediately from Fubini's theorem and the positivity of ν_ϕ. $\square$

Finally, Baernstein [7] gives the following symmetrization theorem for harmonic measures. In the language of Brownian motion, it says that the probability that a Brownian motion starting at $w \in D$ hits the circle of radius r about the origin before hitting any other part of ∂D is increased if we replace w by $|w|$ and D by $D^{\circledcirc}$. Intuitively, the obstacles for the Brownian motion to move outward become less prominent. (See Figure 0.1 on p. xiii of the Introduction.)

Theorem 6.2 (Baernstein [7]). *Let D be a domain containing the point w and contained in the disc $\mathbb{D}(r)$. Then,*

$$\omega(w, \mathbb{D}(r) \cap \partial D; D) \leq \omega(|w|, \mathbb{D}(r) \cap \partial D^{\circledcirc}; D^{\circledcirc}).$$

Remark 6.5. Actually, Baernstein [7] proves that if $u(w) = \omega(w, \mathbb{D}(r) \cap \partial D; D)$ and $v(w) = \omega(w, \mathbb{D}(r) \cap \partial D^{\circledcirc}; D^{\circledcirc})$ (with u and v set identically to zero in $\mathbb{D}(r) \backslash D$ and $\mathbb{D}(r) \backslash D^{\circledcirc}$, respectively, and both set identically to one in $\mathbb{C} \backslash \mathbb{D}(r)$) then $u^{\circlearrowleft} \leq Jv$.

However, the inequality $u(w) \leq v(|w|)$ follows. To see this, fix $\varepsilon > 0$. If $w \notin D$ then $u(w) = 0$ and the inequality is trivial as $v \geq 0$. Hence assume that $w \in D$. Then $|w| \in D^{\circledcirc}$ as is easily seen. Write $w = re^{i\theta}$. There exists $\delta > 0$ such that if $|\theta' - \theta| < \delta$ then $u(re^{i\theta'}) > u(w) - \varepsilon$ since u is continuous at w. It follows that $u^{\circledcirc}(re^{i\theta'}) > u(w) - \varepsilon$ for $\theta' \in (-\delta/2, \delta/2)$ by definition of $u^{\circledcirc}$. Hence, for $0 < \theta' < \delta/2$ we have

$$u^{\circlearrowleft}(re^{i\theta'}) > 2\theta' \cdot (u(w) - \varepsilon).$$

On the other hand, v is continuous at $|w|$ from which it follows that

$$v(|w|) = \lim_{\theta' \downarrow 0} (2\theta')^{-1} v^{\circlearrowleft}(re^{i\theta'}) \geq \limsup_{\theta' \downarrow 0} (2\theta')^{-1} u^{\circlearrowleft}(re^{i\theta'}) \geq u(w) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we are done.

6.2. Symmetric decreasing rearrangement

Let S be a subset of $\mathbb{T}$. Let ω be Haar measure on $\mathbb{T}$, i.e., one-dimensional Lebesgue measure normalized so that $\omega(\mathbb{T}) = 1$. If $S = \mathbb{T}$ then let $S^{\circledcirc} = \mathbb{T}$. Otherwise, let

$$S^{\circledcirc} = \{e^{i\theta} : |\theta| < \pi\omega(S)\}.$$

Then $\circledcirc$ is almost a rearrangement on the σ-algebra of all Lebesgue measurable subsets of T (the same difficulty as in Remark 6.1 occurs), and in fact *is* a rearrangement on the collection of all open subsets of $\mathbb{T}$. We shall, however, feel free to continue to define $f^{\circledcirc}$ by (2.1) even if f is not lower semicontinuous; however, in such a case, Theorem 2.1 will be unavailable.

As in the previous section, call a function f on $\mathbb{T}$ **symmetric decreasing** if $f(e^{i\theta}) = f(e^{-i\theta})$ for all θ and if $f(e^{i\theta})$ is decreasing in θ for $\theta \in [0, \pi]$. For a function f on $\mathbb{T}$, we shall call $f^{\circledcirc}$ its **symmetric decreasing rearrangement.**

Remark 6.6. Let f be symmetric decreasing on $\mathbb{T}$. Then $f = f^{\circledcirc}$ at all but countably many points. To see this, let E be the set of all points of $\mathbb{T}$ at which f is discontinuous. This is a countable set by the monotonicity properties of f. Let

$$g(z) = \liminf_{\zeta \to z} f(\zeta)$$

for $z \in \mathbb{T}$. Then, g is lower semicontinuous and agrees with f outside E. Since E has measure zero, it easily follows from the definition of our symmetric decreasing rearrangement that $g^{\circledcirc} = f$ everywhere on $\mathbb{T}$ except possibly at -1. But $g^{\circledcirc} = g$ by Remark 6.2 specialized from circular rearrangement on all of $\mathbb{C}$ to circular rearrangement on $\mathbb{T}$. Hence $f = f^{\circledcirc}$ outside $E \cup \{-1\}$.

Remark 6.7. Say that functions f and g are **equimeasurable** if for all $\lambda \in \mathbb{R}$ the measures of f_λ and g_λ are equal. I claim that for any measurable function f on $\mathbb{T}$ we have f and $f^{\circledcirc}$ equimeasurable. If f is lower semicontinuous then f is lower measurable with respect to the standard topology on $\mathbb{T}$, and the desired result follows from Theorem I.2.1 and the measure-preserving character of $\circledcirc$. To handle the general case, we proceed by defining a new rearrangement $\overline{\circledcirc}$ on the σ-algebra of all Lebesgue measurable subsets of $\mathbb{T}$. Simply let

$$S^{\overline{\circledcirc}} = \{e^{i\theta} : |\theta| < \pi\omega(S)\},$$

where $\theta(r; S)$ is as in §6.1. It is clear that $\overline{\circledcirc}$ is a rearrangement, and that $S^{\overline{\circledcirc}}$ and $S^{\circledcirc}$ can differ by at most the point -1. Hence, $f^{\overline{\circledcirc}}$ and f are equimeasurable by Theorem 2.1 and the measure-preserving character of $\overline{\circledcirc}$, while $f^{\overline{\circledcirc}}$ and $f^{\circledcirc}$ are equal almost everywhere (in fact, equal everywhere except possibly at -1).

Theorem 6.3 (Baernstein [9, 14]). *Let f, g and h be measurable functions on $\mathbb{T}$, with $f \in L^p(\mathbb{T})$, $g \in L^q(\mathbb{T})$ and $h \in L^r(\mathbb{T})$ for $1 \le p, q, r \le \infty$ and $1 = p^{-1} + q^{-1} + r^{-1}$. Then:*

$$\int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i(\theta-\phi)}) h(e^{i\phi})\, d\theta\, d\phi \le \int_0^{2\pi} \int_0^{2\pi} f^{\circledcirc}(e^{i\theta}) g^{\circledcirc}(e^{i(\theta-\phi)}) h^{\circledcirc}(e^{i\phi})\, d\theta\, d\phi.$$

This is a circular version of the well known Riesz-Sobolev rearrangement inequality (see [92]; in [26] an improved version can be found).

Given two functions f and g on $\mathbb{T}$, write

$$(f * g)(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i(\theta-\phi)}) g(e^{i\phi})\, d\phi.$$

Baernstein's inequality then says that $(f * g * h)(1) \le (f^{\circledcirc} * g^{\circledcirc} * h^{\circledcirc})(1)$. This is of particular interest when $g = g^{\circledcirc}$, and it is in that case that we shall use it.

Definition 6.1. A function f on $\mathbb{T}$ is said to be **strictly symmetric decreasing** if f is symmetric decreasing and $\theta \mapsto f(e^{i\theta})$ is one-to-one on $[0, \pi]$.

In the case where g is strictly symmetric decreasing, the following result is an improvement on Theorem 6.3.

Theorem 6.4 (Beckner [20, Lemma on p. 225]). *Let f and h be positive functions on $\mathbb{T}$, and let g be symmetric decreasing. Assume moreover that f, g and h satisfy the conditions listed in Theorem 6.3. Then,*

$$(f * g * h)(1) \le (f^{\circledcirc} * g * h^{\circledcirc})(1). \tag{6.3}$$

Suppose moreover that g is strictly symmetric decreasing and neither f nor h is almost everywhere equal to a constant function on $\mathbb{T}$. Then equality holds in (6.3) if and only if there exists a $w \in \mathbb{T}$ such that for almost all $z \in \mathbb{T}$ we have $f^{\circledcirc}(z) = f(zw)$ and $g^{\circledcirc}(z) = g(zw)$.

Technical remark 6.1. The statement of the result in Beckner [20, Lemma on p. 225] erroneously omits the hypothesis that neither f nor h is almost everywhere constant. If, say, h is almost everywhere constant then $g * h$ is constant and coincides everywhere with $g * h^{\circledcirc}$, so that

$(f*g*h)(1) = (f^{\circledcirc}*g*h^{\circledcirc})$ since f and $f^{\circledcirc}$ are equimeasurable. The error in Beckner's proof is on the top of his p. 227 where he asserts under some conditions the existence of a certain pair of sets A and B, which existence cannot be guaranteed if either of f and h is constant.

Corollary 6.2. *Let g and h be symmetric decreasing functions on $\mathbb{T}$ with $g \in L^p(\mathbb{T})$ and $h \in L^q(\mathbb{T})$ where $1 \le p, q \le \infty$ and $p^{-1} + q^{-1} = 1$. Then $g*h$ is symmetric decreasing and continuous. Moreover, if g is strictly symmetric decreasing and h fails to be almost everywhere constant, then $g*h$ is strictly symmetric decreasing.*

Proof. Put $F = g*h$. The continuity assertion is a standard fact (see, e.g., Hewitt and Ross [61, Thm. 20.16]) We shall prove that for every set $A \subseteq [-\pi, \pi]$ of normalized measure 2α, we have

$$\int_A F(e^{i\theta})\, d\theta \le \int_{-\alpha}^{\alpha} F(e^{i\theta})\, d\theta. \tag{6.4}$$

Suppose that this is so. By Remark 6.4 (specialized to the case of circular symmetrization on $\mathbb{T}$ and not on all of $\mathbb{C}$) we have

$$\int_{-\alpha}^{\alpha} F^{\circledcirc}(e^{i\theta})\, d\theta = \sup_A \int_A F(e^{i\theta})\, d\theta,$$

where the supremum is taken over all sets A of measure $2|\alpha|$. Since $[-\alpha, \alpha]$ is such a set, it follows from the above identity and (6.4) that

$$\int_{-\alpha}^{\alpha} F^{\circledcirc}(e^{i\theta})\, d\theta = \int_{-\alpha}^{\alpha} F(e^{i\theta})\, d\theta.$$

Hence $F = F^{\circledcirc}$ almost everywhere on $\mathbb{T}$, and since F is continuous it follows that F is symmetric decreasing on $\mathbb{T}$.

We now need to prove (6.4). But (6.4) follows from Theorem 6.3 with $f = 1_{\{e^{i\theta}:\theta\in A\}}$ since $g^{\circledcirc} = g$ and $h^{\circledcirc} = h$ almost everywhere (Remark 6.6).

Finally, suppose that g is strictly symmetrically decreasing and h is not almost everywhere constant. We have F symmetric decreasing. Suppose that F is not strictly symmetric decreasing. Then, there exist $0 < \theta_1 < \theta_2 < \pi$ such that $F(e^{i\theta})$ is constant for $\theta \in [\theta_1, \theta_2]$.

Let $\delta = \theta_2 - \theta_1$. Put $A = (-\theta_1, \theta_1) \cup (\theta_1 + \delta/2, \theta_1 + \delta)$. Let $f = 1_{\{e^{i\theta}:\theta \in A\}}$. Clearly $f^{\circledcirc} = 1_{(-\theta_1 - \delta/4, \theta_1 + \delta/4)}$ and so f is not a rotation of $f^{\circledcirc}$. Thus, by Theorem 6.4 we have $(f * F)(1) = (f * g * h)(1) < (f^{\circledcirc} * g * h^{\circledcirc})(1) = (f^{\circledcirc} * F)(1)$. But, the choice of θ_1 and θ_2 implies that $(f * F)(1) = (f^{\circledcirc} * F)(1)$ and we have a contradiction, as desired. $\square$

Corollary 6.3. *Let u be a function in $h^1(\mathbb{D})$. Suppose that the boundary values $u|_{\mathbb{T}}$ are symmetric decreasing. Then, $u_r|_{\mathbb{T}}$ is symmetric decreasing for every $r \in [0,1]$, where $u_r(z) = u(rz)$. Also, $u(x)$ is monotone increasing for $x \in (-1,1)$. Moreover, if u is non-constant then $u_r|_{\mathbb{T}}$ is strictly symmetric decreasing.*

Proof. Without loss of generality $r \in (0,1)$. Let

$$P_r(e^{i\theta}) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Then, the Poisson extension formula (Theorem 3.3) shows that $u_r|_{\mathbb{T}} = P_r * (u|_{\mathbb{T}})$, and since P_r is evidently symmetric decreasing, we thus have $u_r|_{\mathbb{T}}$ symmetric decreasing by Corollary 6.2.

We prove that $u(x)$ is monotone increasing for $x \in [0,1)$. The result on $(-1,0]$ follows by applying the result on $[0,1)$ to the function $z \mapsto -u(-z)$ which is evidently also symmetric decreasing on $\mathbb{T}$. Now, fix $0 \le x_1 < x_2 < 1$. By the maximum principle we have

$$u(x_1) \le \sup_\theta u(x_2 e^{i\theta}).$$

But $u_{x_2}|_{\mathbb{T}}$ is symmetric decreasing so that $\sup_\theta u(x_2 e^{i\theta}) = u(x_2)$ and the proof of the monotone character on $[0,1)$ is complete.

Clearly P_r is strictly symmetric decreasing, and so the "moreover" also follows from Corollary 6.2. $\square$

Finally we give the following result which we will have an occasion to use in Chapter IV.

Theorem 6.5. *Let $\Phi(z) = \phi(|z|)$ be upper semicontinuous on $\mathbb{C}$. Let D be a circularly symmetric Greenian domain containing the origin. Assume that $(-L, 0] \subseteq D$. Let*

$$h(z) = \mathrm{LHM}(z, \Phi; D).$$

Then h is monotone increasing on $(-L, 0]$.

Proof. By circular symmetry we have $\mathbb{D}(L) \subseteq D$. Replacing $\phi(t)$ by $\max(\phi(t), \phi(L))$, which does not change h (this can be seen from Theorem 5.4 and the fact that the change does not affect ϕ on ∂D) and which preserves subharmonicity as can be easily seen, we may assume that Φ is constant on $\mathbb{D}(L)$. By Theorems 5.10 and 5.11 we have

$$h(z) = \Phi(z) + \int_0^\infty \int_0^{2\pi} g(z, re^{i\theta}; D)\, d\theta\, d\nu_\phi(r)$$

for a positive measure ν_ϕ. Since Φ is constant on $\mathbb{D}(L)$, the support of ν_ϕ lies in $[L, \infty)$. It thus suffices to show that for any fixed $r \in [L, \infty)$ we have g_r monotone increasing on $(-L, 0]$, where

$$g_r(z) = \frac{1}{2\pi} \int_0^{2\pi} g(z, re^{i\theta}; D).$$

I claim that g_r is circularly symmetric on $\mathbb{D}(L)$ if $r \geq L$. To see this note that for $z \in \mathbb{D}(L)$ we have

$$\begin{aligned} g_r(z) = &= \frac{1}{2\pi} \int_0^{2\pi} \left(g(z, re^{i\theta}; D) - \log \frac{1}{|z - re^{i\theta}|} \right) d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|z - re^{i\theta}|}\, d\theta \\ &= g(z, 0; D) - \log \frac{1}{|0 - re^{i\theta}|} + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|z - re^{i\theta}|}\, d\theta, \end{aligned}$$

since $g(z, \cdot; D) - \log \frac{1}{|z - \cdot|}$ is harmonic on $\mathbb{D}(L)$. Now, on the right hand side of the above displayed equation, the only term which has any dependence on $\arg z$ is $g(z, 0; D)$. But $z \mapsto g(z, 0; D)$ is circularly symmetric (Theorem 6.1) so that indeed g_r is circularly symmetric on $\mathbb{D}(L)$. Moreover, it is harmonic on $\mathbb{D}(L)$. The desired monotonicity of g_r follows by scaling and an application of Corollary 6.3. □

6.3. Steiner symmetrization

Recall the definition of Steiner symmetrization as given in Example 2.4. We call a set U with $U = U^{\boxminus}$ **Steiner symmetric about the real axis.** We shall sometimes omit the words "about the real axis". Note that $U^{\boxminus}$ is always Steiner symmetric. Note also that Steiner symmetrization is a rearrangement in the sense of Definition 2.2 on the σ-algebra of all Lebesgue measurable subsets of $\mathbb{C}$.

Note that a set is Steiner symmetric about the real axis if and only if every vertical line meets this set in an open interval symmetric about the real axis.

As in Proposition 6.1, we easily see that if D is open then $Y(\cdot; D)$ is lower semicontinuous on $\mathbb{R}$ and $D^{\boxminus}$ is open. (Actually, the proofs are now easier because we do not need to distinguish any exceptional set such as the previously exceptional set of those r where $\theta(r; D) = \infty$.)

Analogues of Theorems 6.1 and Theorem 6.2 can also be proved in this setting by the methods of Baernstein [7]. We shall not give the proofs, but we do state the following two results.

Theorem 6.6. *Let $\Phi(z) = \phi(\operatorname{Re} z)$ be subharmonic and let D be Greenian. Then,*

$$\operatorname{LHM}(z, \Phi; D) \leq \operatorname{LHM}(\operatorname{Re} z, \Phi; D^{\boxminus}),$$

where $D^{\boxminus}$ denotes Steiner symmetrization.

The proof of this result follows via the Steiner analogue of Theorem 6.1 analogously to Corollary 6.1. We only give this result because in §IV.10.3 we shall parenthetically mention that it could be used in a slightly modified proof of Theorem IV.10.1.

The following result is an analogue of Theorem 6.2. It could also be proved by the same methods of Baernstein [7], although it was first proved by Haliste [56] by Brownian motion methods similar to the methods we shall employ in §II.9. Haliste actually only gave the result for certain sufficiently regular domains. However, an approximation such as the one in the proof of [7, Thm. 7] easily yields the general case.

Theorem 6.7 (Haliste [56]). *Let D be a Greenian domain. Assume that $D \subsetneq \{z \in \mathbb{C} : \operatorname{Re} z < M\}$. Then, for any $z \in D$ we have*

$$\omega(z, \{\operatorname{Re} z = M\} \cap \overline{D}; D) \leq \omega(\operatorname{Re} z, \{\operatorname{Re} z = M\} \cap \overline{D^{\boxminus}}; D^{\boxminus}).$$

Remark 6.8. A Steiner symmetric domain is necessarily simply connected. (This does not hold for a circularly symmetric domain. For instance, $\mathbb{D} \backslash \mathbb{D}(-\frac{1}{2}; \frac{1}{4})$ is circularly symmetric but evidently not simply connected.) This observation will be of great importance in §IV.10. To see the validity of this observation, let γ be an arbitrary curve in a Steiner symmetric domain D. Define $F_t(z) = \operatorname{Re} z + i(1-t)\operatorname{Im} z$ for $t \in [0,1]$ and $z \in \mathbb{C}$. Then, F_t is a homotopy with $F_t \circ \gamma \in D$ for all $t \in [0,1]$, $F_0 \circ \gamma = \gamma$, and $F_1 \circ \gamma \subseteq \mathbb{R}$. Hence, γ is homotopic in D with a curve lying on the real axis. But clearly any such curve is homotopic with the trivial curve as can be seen by applying the homotopy $G_t(z) = (1-t)\operatorname{Re} z + i\Im z$ for $t \in [0,1]$ and $z \in \mathbb{C}$.

7. Counterexamples to a question of Hayman

7.1. Hayman's problem

The material of the present section is basically taken from the author's paper [89].

Let U be a Greenian domain with $w \in U$.

$$U_{w,\lambda} = \{z \in U : g(z,w;U) > \lambda\}.$$

Then, $U_{w,\lambda}$ is circularly symmetric for every $\lambda \geq 0$ if and only if U is circularly symmetric and $w \geq 0$ (one implication follows from the fact that $U_0 = U$; the other is due to Baernstein [7, Corollary on p. 154].) Hayman [59, Question 5.17] had asked whether we necessarily have $(U_{w,\lambda})^{\circledcirc} \subseteq (U^{\circledcirc})_{|w|,\lambda}$. As Baernstein [7] notes, this is the same as asking whether we always have

$$g(re^{i\theta}, |w|; U^{\circledcirc}) \geq \tilde{g}(re^{i\theta}, w; U), \tag{7.1}$$

where $\tilde{g}(re^{i\theta}, w; U)$ is $u^{\circledcirc}(re^{i\theta})$ for $u(re^{i\theta}) = g(re^{i\theta}, w; U)$. The equivalence of the two questions then follows from the fact that $\{z \in U : \tilde{g}(z,w;U) > \lambda\} = (U_{w,\lambda})^{\circledcirc}$ (see Theorem 2.1 as $g(\cdot, w; U)$ is lower semicontinuous and $\circledcirc$ is a rearrangement when confined to open sets).

Recall that Baernstein [7] had proved the weaker inequality that

$$\int_0^{\theta_1} g(re^{i\theta}, |w|; U^{\circledcirc})\, d\theta \geq \int_0^{\theta_1} \tilde{g}(re^{i\theta}, w; U)\, d\theta,$$

for $0 \leq \theta_1 \leq \pi$ (Theorem 6.1, above). However, we will show that in general the stronger inequality (7.1) is *not* valid, and the answer to Hayman's question is negative, even when

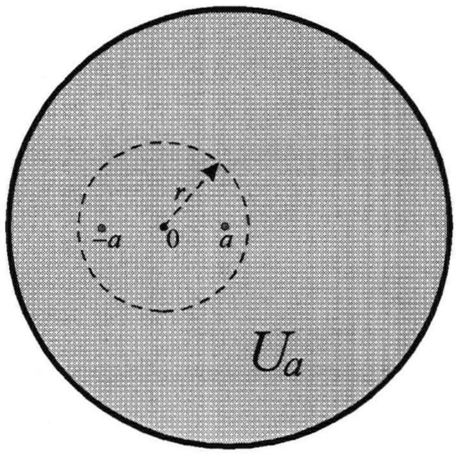

Figure 7.1: The circularly symmetric domain U_a. The pole will be at a or at $-a$.

restricted to U being simply connected and $w = 0$. In one of our examples, (7.1) will be false even though U is circularly symmetric (but of course w cannot lie on the non-negative real axis then).

7.2. The three counterexamples

We give three counterexamples. The first is the easiest, and this is the one with U circularly symmetric. Fix any $0 < a < \frac{1}{2}$. Let U_a be a disc of unit radius centred on the point a. Clearly, U_a is circularly symmetric and $U_a^{\circledcirc} = U_a$. (See Figure 7.1.)

Theorem 7.1. *There exists $r_1 \in (a, 1-a)$ such that for any $r \in (a, r_1]$ we have*

$$\min_\theta g(re^{i\theta}, a; U_a) = g(-r, a; U_a) < \min_\theta g(re^{i\theta}, -a; U_a).$$

The completely elementary proof will be given later. This gives a counterexample to (7.1) since $U_a = U_a^{\circledcirc}$ and since $\min_\theta g(re^{i\theta}, -a; U_a) = \tilde{g}(-r, -a; U_a)$ by definition of $\tilde{g}$.

We now restrict the pole to lie at zero. This will make things a little more difficult.

Theorem 7.2. *There exists a domain U in the plane and strictly positive numbers r and ε such that*

$$g(re^{i\theta}, 0; U^{\circledcirc}) < \tilde{g}(re^{i\theta}, 0; U),$$

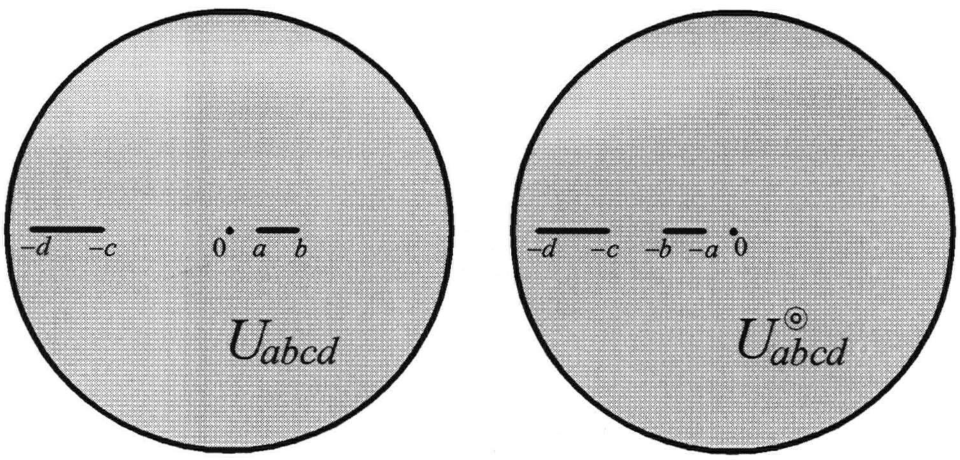

Figure 7.2: The unsymmetrized domain U_{abcd} and the symmetrized domain $U^{\circledcirc}_{abcd}$. The poles will be at the origin.

whenever $0 < |\pi - \theta| < \varepsilon$. *Moreover, U may be taken to be simply connected.*

This is of course also a counterexample to (7.1). We shall present two such counterexamples, one simply connected and one not, because the two examples have rather interesting and different proofs.

The multiply connected example is constructed as follows. Fix $0 < a < b \leq c < d \leq 1$. Let $U_{abcd} = \mathbb{D}\backslash([-d, -c] \cup [a, b])$. Clearly $U^{\circledcirc}_{abcd} = \mathbb{D}\backslash([-d, -c] \cup [-b, -a])$. (See Figure 7.2.)

Lemma 7.1. *There exists $r \in (a, b) \cup (c, d)$ and $\varepsilon > 0$ such that*

$$g(re^{i\theta}, 0; U^{\circledcirc}_{abcd}) < \tilde{g}(re^{i\theta}, 0; U_{abcd}),$$

whenever $0 < |\pi - \theta| < \varepsilon$.

The simply connected example is constructed as follows. Fix $0 < a < b < 1$. Let $V_b = \{z : |z| < 1, \operatorname{Re} z > -b\}$ be a disc with a piece sliced off, and let U_{ab} be V_b slit along the positive real axis starting at a, namely

$$U_{ab} = V_b\backslash[a, 1).$$

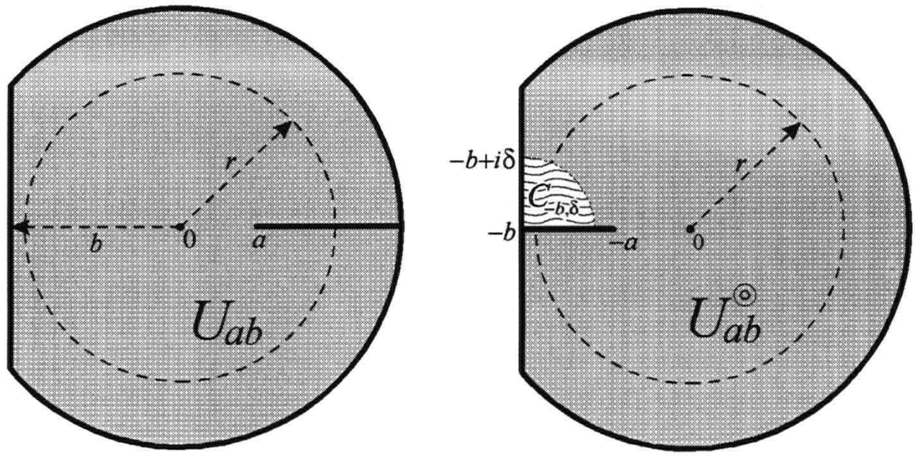

Figure 7.3: The unsymmetrized domain U_{ab} and the symmetrized domain $U_{ab}^{\circledcirc}$, together with the cone $C_{-b,\delta} \subset U_{ab}^{\circledcirc}$ used in the proof. The poles will be at the origin.

Then clearly U_{ab} is simply connected and $U_{ab}^{\circledcirc} = V_b \backslash (-b, -a]$. (See Figure 7.3, ignoring the cone $C_{-b,\delta} \subset U_{ab}^{\circledcirc}$ for now.)

Lemma 7.2. *There exists $a' \in (a, b)$ with the property that for every $r \in [a', b)$ there is an $\varepsilon > 0$ such that*

$$g(re^{i\theta}, 0; U_{ab}^{\circledcirc}) < \tilde{g}(re^{i\theta}, 0; U_{ab}),$$

whenever $0 < |\pi - \theta| < \varepsilon$.

7.3. Proofs that the counterexamples truly contradict Hayman's conjecture

Proof of Theorem 7.1. Let $\mathfrak{g}(\cdot, w)$ be Green's function for the unit disc with a pole at w, so that by (5.8) we have

$$\mathfrak{g}(z, w) = \log \left| \frac{1 - z\overline{w}}{z - w} \right|. \tag{7.2}$$

Since $1 - a > \frac{1}{2} > a$, we need only work with $0 \leq r < 1 - a$. Write $U = U_a = U_a^{\circledcirc}$. Then, $g(re^{i\theta}, -a; U) = \mathfrak{g}(re^{i\theta} - a, -2a)$ and $g(re^{i\theta}, a; U) = \mathfrak{g}(re^{i\theta} - a, 0)$. We can see directly that

$$\min_\theta g(re^{i\theta}, a; U) = \mathfrak{g}(-r - a, 0) = \log \frac{1}{|r + a|}. \tag{7.3}$$

Note that in fact in general if V is a circularly symmetric domain and v is on the positive real axis then $g(re^{i\theta}, v; V)$ is symmetrically decreasing with respect to $\theta \in [-\pi, \pi]$ by Theorem 6.1.

On the other hand,

$$g(re^{i\theta}, -a; U) = \mathfrak{g}(re^{i\theta} - a, -2a) = \log\left|\frac{1 + 2a(re^{i\theta} - a)}{re^{i\theta} + a}\right|. \tag{7.4}$$

Using the `Maple` computer algebra package (one could presumably also do this by hand), we find that

$$\frac{dg(re^{i\theta}, -a; U)}{d\theta} = \frac{(-1 + 6a^2 + 2r^2 - 8a^4 - 8a^2r^2)\, ar\sin\theta}{\Delta},$$

where

$$\begin{aligned}\Delta =& 4a^2r^2 - 8a^2r^2c^2 - a^2 - 2arc + 4a^3rc + 4a^4 \\ &- 4ar^3c - 4a^6 + 16a^4r^2c^2 - 8a^4r^2 - 4a^2r^4 - r^2,\end{aligned}$$

and $c = \cos\theta$. Since $g(re^{i\theta}, -a; U)$ is certainly not constant in θ, it follows that $dg(re^{i\theta}, -a; U)/d\theta$ can only vanish at $\theta = 0$ and at $\theta = \pi$, so that

$$\min_{\theta} g(re^{i\theta}, -a; U) = \min(g(r, -a; U), g(-r, -a; U)). \tag{7.5}$$

But a completely elementary analysis of the explicit formulae (7.3) and (7.4) shows that for $0 < a < \frac{1}{2}$ and r sufficiently close to a we have $g(-r, -a; U) > g(-r, a; U)$, while the inequality $a < r < 1 - a$ implies that $g(r, -a; U) > g(-r, a; U)$. By (7.5), the proof is complete. By a more precise but still elementary analysis it should in principle be possible to determine the exact range of values of r for which the result holds. □

Proof of the Lemmas. We first give the common part of the two proofs. Let U be U_{abcd} or U_{ab}, depending on which of the two examples we wish to work with. By symmetry in both cases we need only consider $\pi < \theta < \pi + \varepsilon$. Define $G(z) = g(z, 0; U)$ and $G^*(z) = g(z, 0; U^{\circledcirc})$, and for $x \in (-1, 1)$ let

$$H(x) = \lim_{y \to 0+} \frac{G(x + iy) - G(x)}{y} \quad \text{and} \quad H^*(x) = \lim_{y \to 0+} \frac{G^*(x + iy) - G^*(x)}{y}.$$

We shall later prove that if $U = U_{abcd}$ then

$$H(x) > H^*(-|x|) \text{ for some } x \in (a,b) \cup (-d,c). \tag{7.6}$$

Assume this for now. In this case we let $r = |x|$. We may assume that $x \in (a,b)$ since the reader will easily see that the proof in the case of $x \in (-d,-c)$ would be quite analogous. On the other hand, in the case of $U = U_{ab}$ we shall show that

$$H^*(x) \to 0 \text{ as } x \downarrow -b. \tag{7.7}$$

Again, assume this for now. But, it is easy to see that $\liminf_{r \uparrow b} H(r) > 0$, so that we may choose $a' \in (a,b)$ such that for every $r \in [a',b)$ we have $H^*(-r) < H(r)$. Fix such an r, then.

Thus, in either case we are working with an r such that $H^*(-r) < H(r)$. Assuming this, the proof from now on will actually be the same in both of the cases. For $t \in [0,\pi]$, let $f(t) = G(re^{it})$ and $g(t) = G^*(re^{i(t+\pi)})$. Then, from the inequality $H(r) > H^*(-r)$ together with the standard fact that in both of the cases under consideration we have $f'(t) \to rH(r)$ and $g'(t) \to rH^*(-r)$ as $t \to 0+$, it follows that there must then be an $\varepsilon_1 > 0$ such that $f'(t) > g'(t)$ whenever $0 < t < \varepsilon_1$. Since the Green's function of a domain vanishes on the boundary, in both of our cases we easily see that $f(0) = g(0) = 0$ and it follows that $f(t) > g(t)$ for $0 < t < \varepsilon_1$.

Let F be the increasing rearrangement[2] of f on $[0,\pi]$, i.e., an increasing function equimeasurable with f on $[0,\pi]$. The positivity of f' near zero, the vanishing of f at zero, together with the easy fact that in both of our cases f is bounded away from zero on every interval $[\delta,\pi]$ where $\delta > 0$, all imply that we may find an $\varepsilon_2 > 0$ such that $f(t) < f(t')$ whenever $0 \le t < \varepsilon_2$ and $t < t' \le \pi$. Then, it follows that $f(t) = F(t)$ whenever $0 \le t < \varepsilon_2$. Hence, $F(t) = f(t) > g(t)$ for $0 < t < \varepsilon$, where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. By the symmetry of U we have $F(t) = \tilde{G}(re^{i(t+\pi)})$, and the desired conclusions of the lemmas follow.

It remains to prove (7.6) for $U = U_{abcd}$ and (7.7) for $U = U_{ab}$. First let $U = U_{abcd}$. Let $\mathfrak{g}$ be Green's function for the unit disc as in (2). By Green's formula, much as in [23, pp. 46–48] or

[2] We can define $F = f^{\curlywedge}$ via (2.1), where $\curlywedge$ is the symmetrization on $[0,\pi)$ defined by $S^{\curlywedge} = (\pi - |S|, \pi)$. This will give us F on $(0,\pi)$. To get F on $[0,\pi]$, impose continuity at the endpoints.

[79, p. 105, eqn. (5.4)], we can see that for $z \in D\backslash\{0\}$ we have:

$$G(z) = \log \frac{1}{|z|} - 2\int_a^b \mathfrak{g}(z, x)H(x)\, dx - 2\int_c^d \mathfrak{g}(z, -x)H(-x)\, dx \tag{7.8}$$

and

$$G^*(z) = \log \frac{1}{|z|} - 2\int_a^b \mathfrak{g}(z, -x)H^*(-x)\, dx - 2\int_c^d \mathfrak{g}(z, -x)H^*(-x)\, dx. \tag{7.9}$$

Now in order to obtain a contradiction, suppose that we had $H(x) \leq H^*(-|x|)$ for every $x \in (-d, -c) \cup (a, b)$. By (7.9), we would have

$$G^*(z) \leq \log \frac{1}{|z|} - 2\int_a^b \mathfrak{g}(z, -x)H(x)\, dx - 2\int_c^d \mathfrak{g}(z, -x)H(-x)\, dx. \tag{7.10}$$

Now, fix $z \in (-d, -c)$. From the explicit formula for $\mathfrak{g}(z, w)$ one can verify that $\mathfrak{g}(z, -x) > \mathfrak{g}(z, x)$ whenever $x > 0$ and $z < 0$. Since H is known to be strictly positive on (a, b), it would follow from (10) that

$$G^*(z) < \log \frac{1}{|z|} - 2\int_a^b \mathfrak{g}(z, x)H(x)\, dx - 2\int_c^d \mathfrak{g}(z, -x)H(-x)\, dx.$$

But by (7.8), the right hand side is precisely $G(z)$ so that we would have $G^*(z) < G(z)$. On the other hand, since both G and G^* vanish on $(-d, -c)$, we have $G(z) = G^*(z) = 0$. We thus obtain a contradiction, and so (7.6) holds.

Now let $U = U_{ab}$. Note that G^* is harmonic and bounded on the cone $C_{-b,\delta} = \{-b + re^{i\theta} : 0 < r < \delta,\, 0 < \theta < \pi/2\}$, where δ is chosen sufficiently small so that the cone fits inside $U^{\circledcirc}$ and $\delta < b - a$ so that G^* vanishes on the two edges $[-b, -b + \delta]$ and $[-b, -b + i\delta]$ of the cone. (See Figure (7.3).)

Thus, (7.7) will clearly follow as soon as can we prove that whenever h is a harmonic function on the translated and dilated cone $C = C_{0,1} = \{re^{i\theta} : 0 < r < 1,\, 0 < \theta < \pi/2\}$, with h bounded on C and vanishing on the two edges $[0, 1]$ and $[0, i]$, then the normal derivative of h at $x \in (0, 1)$ tends to zero as $x \downarrow 0$. To prove this claim, note that

$$R(z) = \left(\frac{1 + z^2}{1 - z^2}\right)^2$$

is a univalent map of C onto the upper half plane, with $R(0) = 1$. Now, define $h_1 = h \circ R^{-1}$ on the upper half plane. This will be a bounded harmonic function, vanishing on the interval $(0, \infty)$ since this interval is the image under R of the edges $[0, 1)$ and $[0, i)$ of C. It is easy then to see that the normal derivative of h_1 will have to be bounded by some finite constant K on the interval $[\frac{1}{2}, \frac{3}{2}]$. Then, since R is analytic in a neighbourhood of x if $x \in (0, 1)$, it follows that the normal derivative of $h = h_1 \circ R$ for x sufficiently close to 0 is bounded by $K|R'(x)|$. But as $x \downarrow 0$ then we see directly that $R'(x) \to 0$, and so the normal derivative tends to zero as desired. This proves (7.7). □

8. Radial monotonicity of Green's functions

The results of the present section are taken from the author's paper [84]. In this section we study the radial monotonicity properties of the Green's function.

Let D be a Greenian domain in the plane with $0 \in D$. Let G_D be Green's function for D with pole at 0, i.e., put $G_D(z) = g(z, 0; D)$. Because of the character of the logarithmic pole of G_D at 0, it is easy to see that G_D is radially decreasing on some neighbourhood U of 0, i.e., if z_1 and z_2 are in U, with $\arg z_1 = \arg z_2$ and $0 < |z_1| \leq |z_2|$, then $G_D(z_1) \geq G_D(z_2)$. It is natural to ask how large we may take U. In answer to this, one may well conjecture the following result which we shall prove later in this section.

Theorem 8.1. *Let D be a domain in the plane containing the origin. Let U be the largest open disc centred about the origin and contained in D. Then G_D is strictly radially decreasing on U.*

In fact, we can do even better than this. Given a point $z \in \mathbb{C}\backslash\{0\}$, let L_z be the line through z perpendicular to the ray from the origin to z. Let H_z be the component of $\mathbb{C}\backslash L_z$ which contains the origin. The domain H_z is a half-plane with z lying on its boundary. Explicitly,

$$H_z = \{w \in \mathbb{C} : \langle w, z \rangle \leq |z|^2\},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{C} = \mathbb{R}^2$. See Figure 8.1 for an example of $\bar{H}_z$ where $z \in \partial D$.

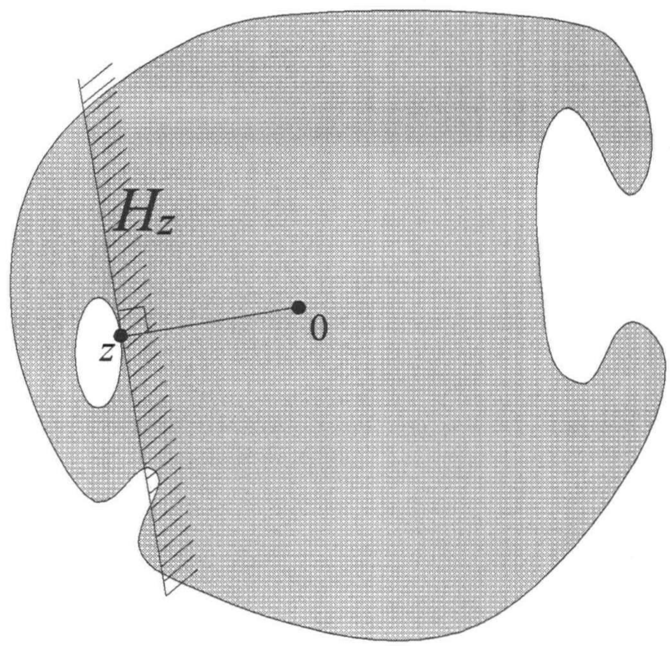

Figure 8.1: The construction of the set H_z for z in the complement of D.

Define

$$D' = \bigcap_{z \in \mathbb{C} \setminus D} \bar{H}_z. \tag{8.1}$$

Note that D' is convex, being an intersection of half-planes.

Theorem 8.2. *Let D be a domain in the plane containing the origin. Then G_D is radially decreasing on $D' \cap D$ and radially strictly decreasing in the interior of D'.*

Corollary 8.1. *Let D be a simply connected domain in the plane with $0 \in D$. Let f be the Riemann map from D onto the unit disc, with $f(0) = 0$. Then f is starlike on the interior of D', i.e., $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$ there.*

Proof of Corollary. One may use Theorem 8.2 and the fact that

$$G_D(z) = \log \frac{1}{|f(z)|}$$

(Theorem 5.8) to prove that $|f|$ is radially increasing on $\operatorname{Int} D'$. It follows that $\operatorname{Re} \frac{zf'(z)}{f(z)} \geq 0$ there. An easy application of the maximum principle shows that, since f is non-constant and $\frac{zf'(z)}{f(z)}$ is holomorphic near 0, we must in fact have strict inequality, as desired. □

In §IV.8 we shall apply Theorem 8.2 to produce an improvement of Beurling's shove theorem [23, pp. 58–62].

Clearly Theorem 8.1 follows from Theorem 8.2 since if a disc centred about the origin is contained in D then it is also contained in D'.

With regard to Theorem 8.2, we may ask how good is the idea of choosing H_z to be a half-plane. To answer this, fix $z \in \mathbb{C}$ and let $D_\varepsilon = \mathbb{C}\backslash\mathbb{D}(z;\varepsilon)$. It is not difficult to see that if we let T_ε be the set of points where the radial derivative of G_{D_ε} is non-positive, then T_ε tends to $\bar{H}_z$ as $\varepsilon \to 0$. (To prove this, one can either compute the G_{D_ε} explicitly, or else one should be able to extract this information from the proof of Theorem 8.2 given below.)

It now remains for us to give a proof of Theorem 8.2. To do it, we need a certain auxiliary result on harmonic measures.

Lemma 8.1. *Let F be a compact set in the plane with C^1 boundary. Fix a disc U outside F with the property that if $w \in F$ and $z \in \bar{U}$, then $z \notin \bar{\mathbb{D}}(w/2;|w|/2)$. Assume that $0 \notin \bar{U}$. Let R be sufficiently large that $\mathbb{D}(R) \supset F \cup \bar{U}$. Define $\phi_R(z) = \omega(z, \mathbb{T}(R); \mathbb{D}(R)\backslash F)$. Then there exists $R_0 < \infty$ such that for all $R \geq R_0$ and every $z \in U$, the radial derivative of ϕ_R is positive at z.*

Assume the Lemma for now.

Proof of Theorem 8.2. Consider the function

$$E_D(z) \stackrel{\text{def}}{=} \left.\frac{d}{d\lambda}\right|_{\lambda=1} G_D(\lambda z),$$

which is harmonic[3] in $D\backslash\{0\}$, and strictly negative near the origin because of the character of the pole of G_D at 0. If we could show that G_D is radially decreasing on $\operatorname{Int} D'$, then it would

[3]The easiest way to see the harmonicity of E_D on $D\backslash\{0\}$ is to note that G_D is locally the real part of a

follow that $E_D \leq 0$ there, and by the maximum principle we would in fact have $E_D < 0$ on $\operatorname{Int} D'$, so that G_D would be strictly decreasing there. And, of course, it would then also follow that G_D is radially decreasing on $D' \cap D$ by continuity of G_D on $D\backslash\{0\}$.

We now prove that G_D is radially decreasing on $\operatorname{Int} D'$. By approximation we may assume that D is bounded and has a C^1 boundary. Fix $w_0 \in \operatorname{Int} D'\backslash\{0\}$. Choose a disc W containing w_0 such that $\bar{W} \subset \operatorname{Int} D'\backslash\{0\}$. Fix w_1 in W with $\arg w_1 = \arg w_0$ and $|w_1| \geq |w_0|$. If we can show that $G_D(w_0) \geq G_D(w_1)$ then we will be done. So fix $\varepsilon > 0$. Because of the character of the pole of G_D at 0, it follows that we may choose a small $\eta = \eta(\varepsilon) > 0$ with the properties that $\mathbb{D}(\eta) \subset D\backslash\bar{W}$ and that for every $\delta \in (0, \eta]$ and every point $\zeta \in \mathbb{T}(\delta)$, we have $|G_D(\zeta) - G_D(\delta)| \leq \varepsilon$. Fix $\delta \in (0, \eta]$. Since G_D is harmonic in $D\backslash\{0\}$ and vanishes on ∂D, it follows that for any $w \in D\backslash\mathbb{D}(\delta)$ we have

$$G_D(w) = \int_{\mathbb{T}(\delta)} G_D(\zeta)\, d\omega_w^{D\backslash\mathbb{D}(\delta)}(\zeta),$$

where the measure $\omega_w^{D\backslash\mathbb{D}(\delta)}$ is the harmonic measure

$$\omega_w^{D\backslash\mathbb{D}(\delta)}(A) = \omega(w, A; D\backslash\mathbb{D}(\delta))$$

for $A \subseteq D\backslash\mathbb{D}(\delta)$. But because every value of G_D on $\mathbb{T}(\delta)$ is within ε of $G_D(\delta)$, it follows that

$$|G_D(w) - G_D(\delta) \cdot \omega(w, \mathbb{T}(\delta); D\backslash\mathbb{D}(\delta))| \leq \varepsilon. \tag{8.2}$$

This estimate holds uniformly for all $w \in D\backslash\mathbb{D}(\delta)$. Now let

$$\Phi_\delta(w) = \omega(w, \mathbb{T}(\delta); D\backslash\mathbb{D}(\delta)).$$

We shall prove that for $\delta > 0$ sufficiently small we have

$$\Phi_\delta(w_1) \leq \Phi_\delta(w_0). \tag{8.3}$$

It will then follow by (8.2) that $G_D(w_1) - G_D(w_0) \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary it will follow that $G_D(w_1) \leq G_D(w_0)$ as desired.

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holomorphic function there. But if F is holomorphic, then $z \mapsto \frac{d}{d\lambda}F(\lambda z)$ is holomorphic, too, and the real part of a holomorphic function is harmonic.

Now, to prove (8.3), we use the conformal invariance of harmonic measure under the conformal map $\tau(w) = 1/w$ on $\mathbb{C}\backslash\mathbb{D}(\delta)$ as follows. Let $F = \{1/w : w \in \mathbb{C}\backslash D\}$. Since D was bounded and contained the origin, it follows that F is compact. Let $R = 1/\delta$. Let $U = \{1/w : w \in W\}$. This will be a disc. Moreover, by conformal invariance we have

$$\Phi_\delta(w) = \phi_R(1/w), \tag{8.4}$$

where ϕ_R is defined as in Lemma 8.1. Since τ maps the closed half-planes $\bar{H}_w$ onto the closed sets $\mathbb{C}\backslash\mathbb{D}(z/2; |z|/2)$, where $z = 1/w$, and since $\bar{W}$ lies in the interior of D', it is easy to see that the conditions of the Lemma are satisfied. It follows that $\phi_R(1/w_1) \leq \phi_R(1/w_0)$ for all sufficiently large R, which in light of (8.4) completes the proof. □

Proof of Lemma 8.1. By dilation invariance we may assume that $\bar{U}$ and F are both contained in $\mathbb{D}$. Let $\mathfrak{g}(\cdot, w)$ be Green's function for the unit disc with pole at w. Then,

$$\mathfrak{g}(z, w) = \log\left|\frac{1 - z\bar{w}}{z - \bar{w}}\right|. \tag{8.5}$$

Extend ϕ_R to all of $\mathbb{D}(R)$ by setting $\phi_R(z) = 0$ for $z \in F$. Then ϕ_R is a subharmonic function on $\mathbb{D}(R)$. Since it is harmonic outside F, we only need to verify subharmonicity on F. But on F it equals 0 and it is positive everywhere else, so that the only part of subharmonicity to be verified is the upper semicontinuity. But, $\mathbb{D}\backslash F$ is a regular domain (Corollary 5.1) so that in fact ϕ_R is continuous.

Now ϕ_R is identically 1 on $\mathbb{T}(R)$, and hence its least harmonic majorant on $\mathbb{D}(R)$ is the function which is identically 1. From Theorem 5.11 we immediately conclude that

$$\phi_R(z) = 1 - \int_F \mathfrak{g}(z/R, w/R)\, d\mu_R(w),$$

where μ_R is short for the positive Riesz measure μ_{ϕ_R}, since

$$\mathfrak{g}(z/R, w/R) = g(z, w; \mathbb{D}(R))$$

by scaling. I claim that $z \mapsto \mathfrak{g}(z/R, w/R)$ is radially decreasing on U for any fixed $w \in F$ and for R sufficiently large, with the size of R depending on U and F but not on the particular choice of $w \in F$. The desired result follows immediately from this claim and the above representation of $\phi_R(z)$.

Given any two points z and w of $\mathbb{C}$, let $\theta(z,w) \in [0,\pi]$ be the size of the angle subtended by the rays from the origin to the points z and w, providing z and w are both non-zero. If at least one of z and w vanishes, then arbitrarily define $\theta(z,w) = 0$. Now, if $z \in \bar{U}$ and $w \in F$, then by the condition $z \notin \bar{\mathbb{D}}(w/2; |w|/2)$, it follows that $|w|\cos\theta(z,w) - |z| < 0$. By a compactness argument it follows that there in fact exists $\varepsilon > 0$ (depending only on the choice of the sets U and F) such that $|w|\cos\theta(z,w) - |z| < -\varepsilon$ whenever $z \in \bar{U}$ and $w \in F$.

We now prove our claim about the radial monotonicity of $z \mapsto \mathfrak{g}(z/R, w/R)$ for large R. Fix any $z \in U$ and $w \in F$. Write $z = Ze^{i\alpha}$ and $w = We^{i\beta}$ for real α and β, and non-negative Z and W. For conciseness let $\theta = \theta(z,w)$. Then,

$$\mathfrak{g}(z,w) = \tfrac{1}{2}\log\frac{1 - 2ZW\cos\theta + Z^2W^2}{Z^2 - 2ZW\cos\theta + W^2},$$

so that

$$\frac{\partial}{\partial Z}\mathfrak{g}(Ze^{i\alpha}, w) = \frac{(1-W^2)(W\cos\theta + WZ^2\cos\theta - ZW^2 - Z)}{(1 - 2ZW\cos\theta + Z^2W^2)(Z^2 + W^2 - 2ZW\cos\theta)}.$$

Providing $z \neq w$, the denominator is strictly positive (and, in fact, we must have $z \neq w$, since U and F are disjoint). Thus, for $R \geq 1$ we may write

$$\frac{\partial}{\partial Z}\mathfrak{g}\left(\left(\frac{Z}{R}\right)e^{i\alpha}, \frac{w}{R}\right) = h(z,w,R)\left(W\cos\theta - Z + \frac{WZ(Z\cos\theta - W)}{R^2}\right), \tag{8.6}$$

for some strictly positive function $h(z,w,R)$. But Z and W do not exceed 1 since we have assumed that $U \cup F \subset \mathbb{D}$. Moreover, since $z \in \bar{U}$ and $w \in F$, we must have $W\cos\theta - Z < -\varepsilon$. It is easy to see then that there exists an $R_0 \geq 1$ depending only on ε (and on the fact that $F \cup \bar{U} \subset \mathbb{D}$) such that for all $R \geq R_0$ the expression (8.6) is strictly negative (indeed any $R_0 \geq \max(1, \sqrt{2/\varepsilon})$ will work here). This proves the claim. $\square$

Chapter II
Discrete symmetrization

Overview

In this chapter we shall examine various discrete symmetrization results. Our first concern (sections 1 through 6.1) is with proving convolution-rearrangement inequalities of the form

$$\sum_{x,y\in G} f(x)K(d(x,y))g(y) \leq \sum_{x,y\in G} f^{\#}(x)K(d(x,y))g^{\#}(y),$$

on some graphs G, for decreasing functions K. Our graphs will be equipped with an ordering order-isomorphic to a subset of $\mathbb{Z}^+$. Our approach will be the a discrete version of a method of Beckner [18, 19, 20, 21], and Baernstein and Taylor [15], as presented by Baernstein [11]. This method will in fact give us some even more general inequalities, known as the "master inequalities" (inequality (2.1)). The method is based on a partial reordering induced by an involution of G, and it is valid only in the presence of a set of involutions satisfying certain stringent requirements.

In §1 we shall define the kind of rearrangement induced by our ordering that we are interested in. In §2 we shall describe how under some rather difficult to satisfy assumptions the master inequality holds on a discrete metric space, and how the master inequality implies a convolution-rearrangement inequality such as the one mentioned above. (Note that it is not thought that the assumptions are *necessary* for the master inequality, though a counterexample is not at present available.) In §3 we shall define some graph-theoretic notions and will give a proposition simplifying, in the case of a graph, the verification of the assumptions under which in §2 we have proved the master inequality. In §4 we shall apply this to verify that the master inequality holds for the edge graph H_8 of an octahedron. Of course, this is not a major result, since after all the octahedron is only a 6-vertex graph, but we give it in order to warm up to proving the master inequality in more interesting cases. In §5 we shall prove the master inequality for the case of the circular graphs $\mathbb{Z}_n$. This result is of some interest. Note that the analogous convolution-rearrangement inequality for $\mathbb{Z}$ has been proved by Hardy and Littlewood (see [58, Thm. 371]). The case of the circular graphs will allow us in §9 to generalize some results of Quine [90] on discrete circular rearrangement and harmonic measures. Finally, after handling the circular graphs, we proceed to the most difficult case, namely that of the regular tree T_p (§6). In that case, after having reviewed some basic notions, we also prove that the assumptions for the master inequality hold (§6.1).

Next, we proceed to apply the master inequality on trees to obtain an analogue of the classical Faber-Krahn inequality for the first nonzero Dirichlet eigenvalue of the negated Laplacian (§6.2). The most difficult part of our proof, and probably of the whole thesis, will be the proof of the uniqueness of the extremal domains (§6.2.4). The reader may wish to skip that section on a first reading.

In §7 we give some open problems connected with the above-mentioned material. Moreover, we use a computer-based proof to show that no convolution-rearrangement inequality of the type we are interested in can be found on the cube $\mathbb{Z}_2^3$ or the ternary plane $\mathbb{Z}_3^2$.

Then, in §8 we examine two types of discrete rearrangements, which we shall call "Schwarz" and "Steiner", respectively. All the rearrangements considered in the previous sections of this chapter were Schwarz type symmetrizations. However, for some of the following material it is good to generalize to what we call a "Steiner type" rearrangement, namely a rearrangement which is essentially a disjoint combination of Schwarz rearrangements. We state and prove a number of basic results on Schwarz and Steiner rearrangements in §8.1. Of particular usefulness will be the rather trivial Proposition 8.4 which characterizes those functions which arise out Steiner rearrangement. Also very useful will be Proposition 8.6 which in a sense allows us to "undo" rearrangements in an appropriate way.

In §8.2, we shall give a construction of a Steiner type rearrangement on the subsets of a product set $Z \times X$ from a Schwarz (or even Steiner) type rearrangement on the subsets of X. In §8, we shall show how a convolution-symmetrization inequality in general implies the preservation of symmetry under certain convolutions; this fact is similar to Corollary I.6.2.

The results of sections 8–8.3 will be used in the remaining three sections of this chapter.

The remaining three sections are concerned with discrete analogues of rearrangement results for harmonic measures, Green's functions and partial differential equations. In §9 we shall work by using a probabilistic method of Haliste [56]. We will obtain discrete rearrangement results for generalized harmonic measures (Theorem 9.1), Green's functions (Theorem 9.2) and exit times (Theorem 9.4). We shall be working with a random walk on the product set $\mathbb{Z} \times X$, where there is assumed to be an appropriate convolution-rearrangement inequality for a Steiner type symmetrization on X. In §9.1.1 we shall state our basic assumptions on the kernel of our random walk. In §9.1.2 we shall show that our basic assumptions are satisfied if X is one of our previously considered graphs $\mathbb{Z}$, $\mathbb{Z}_n$, T_p and H_8. In §9.1.3 we shall define our random walk on $\mathbb{Z} \times X$ given the kernel discussed in §9.1.1.

In §9.1.4 we will construct a generalized harmonic measure based on our random walk encountering dangers of various probabilities at various points. More classical discrete harmonic measures are given by the special case where these probabilities can only have the values 0 and 1. In the same section we shall state our main result on rearrangement and generalized harmonic measure (Theorem 9.1), which result generalizes work of Quine [90]. Then, in §9.1.5 we shall define an analogous generalized Green's function as the expected number of times that our random walk visits a given point. We shall also give our main result on rearrangement and this generalized Green's function (Theorem 9.2). Then, we proceed to the proofs. In §9.2 we make a preliminary reduction. Finally, in §9.3 we shall give our result on exit times (Theorem 9.4) and the probabilistic proofs of our theorems. These proofs will be based on an iteration of our assumed convolution-rearrangement inequality (see Lemma 9.1). Much of the material from §9 is adopted and extended from the author's paper [83].

In §10 we shall prove a discrete version of Beurling's shove theorem [23, pp. 58–62] by using the results of §9. We shall return to the shove theorem and consider the continuous case in §IV.8. Much of the material of §10 is also taken from the author's paper [83].

Finally, in §11 we develop a very general method for proving results such as Theorems 9.1 and 9.2. We shall prove a quite general version of these results (Theorem 11.1) by using a modification of the methods of Baernstein [11] and Weitsman [99]. The technical difference is that while Baernstein had used a specially-constructed elliptic differential operator and a standard maximum principle, we instead use a customized maximum principle and no specially-constructed differential operator since the construction of such an operator would likely run into problems in our discrete setting. Our methods allow us to take a convolution-symmetrization inequality and from it obtain a rearrangement theorem for discrete analogues of the nonlinear partial differential equation $-\Delta u(x) = -c(x)u(x) + \phi(u(x)) + \lambda(x)$, where $c \geq 0$ and ϕ is increasing and convex. First in §11 we define our difference operators and give the assumptions on our kernel functions, effectively assuming that we have a convolution-rearrangement inequality. In §11.1 we give a precise statement of the difference equations (more generally, difference inequalities) that we are working with. In §11.2 we give our main rearrangement theorem and prove it. Then, in §11.3.1 we prove that this rearrangement theorem even gives a meaningful result for the identity rearrangement—it yields a monotonicity result for our system of difference equations. In §11.3.2 we demonstrate how Theorem 11.1 implies Theorem 9.1 and mention that it also similarly implies Theorem 9.2. Finally, in §11.3.3 we show that Theorem 9.1 implies the exit times result of Theorem 9.4. This shows that Theorem 11.1 can also be used to prove results about exit times.

On first reading, the reader may wish to omit sections 6.2.2, 6.2.4, 7.1, 9.2, 9.3, 10 and 11.3.3.

It is worth noting that results in a spirit somewhat resembling some of our work on discrete rearrangement can be found in the book of Marshall and Olkin [71].

1. Definitions and basic results

For the convenience of the reader, while developing the general theory we shall use the concrete metric space $\mathbb{Z}$ equipped with the standard metric $d(m,n) = |m-n|$ as an example.

Let M be a countable set. In this chapter, we say that two real functions f and g on M are **equimeasurable** if they are equimeasurable with respect to counting measure on M, i.e., if

$$|f_\lambda| = |g_\lambda|,$$

for every positive λ, where $|\cdot|$ indicates the cardinality of a set and f_λ and g_λ are defined as in §I.2.

We assume that we are given a specific fixed well-ordering $\prec$ of M with the property that every element of M has at most finitely many predecessors, i.e., an ordering under which is M is order isomorphic to a subset of $(\mathbb{Z}_0^+, <)$. (If $M = \mathbb{Z}$ then we will use the ordering $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec \cdots$.) Note that in our notation $x \prec y$ implies that $x \neq y$. Then, the (discrete) **symmetric decreasing rearrangement** of a real function f on M is defined to be the unique $\prec$-decreasing[1] function $f^\#$ which is equimeasurable with f. We call a $\prec$-decreasing function **symmetrically decreasing**. Given a subset S of M, if $|S| = \infty$ then define $S^\# = M$, and otherwise let $S^\# = \{e_1, e_2, \ldots, e_{|S|}\}$, where $e_1 \prec e_2 \prec e_3 \prec \cdots$ is an enumeration of M. We call S **symmetric** if $S = S^\#$.

Note that if we let $\mathcal{F} = \mathcal{G}$ be all the subsets of M then it is easy to see that $\#\colon \mathcal{F} \to \mathcal{G}$ is a symmetrization in the technical sense of §I.2. Moreover, $f^\#$ as defined above is easily seen to agree with the definition in (I.2.1) since it is easy to see that in our current definition we have $(f^\#)_\lambda = (f_\lambda)^\#$, which agrees with Theorem I.2.1.

Definition 1.1. Two real functions f and g on a set X are said to be **similarly ordered** providing that for every x and y in X we have

$$f(x) \leq f(y) \text{ if and only if } g(x) \leq g(y).$$

[1] A function F is $\prec$-decreasing if $x \prec y$ always implies $F(x) \geq F(y)$.

The following result is very well known [58, Thm. 368][2] and has slightly less restrictive hypotheses than Theorem I.2.3.

Theorem 1.1 (Hardy-Littlewood). *For a pair of real functions f and g we have*

$$\sum_{x\in M} f(x)g(x) \leq \sum_{x\in M} f^{\#}(x)g^{\#}(x), \tag{1.1}$$

assuming that both sides make sense. If moreover the left hand side is finite, then equality holds if and only if f and g are similarly ordered.

2. A general framework for proving discrete master inequalities

Let (M,d) be a metric space such that M is at most countable.

Given two real functions f and g on M, an increasing positive convex function Φ on $[0,\infty)$ and a decreasing positive function K on $[0,\infty)$, define

$$Q(f,g;\Phi,K) = \sum_{x,y\in M} \Phi(|f(x)-f(y)|)K(d(x,y)).$$

Then, we say that **the master inequality holds for M with the ordering $\prec$** providing that for all real f and g, and all Φ and K as above, we have

$$Q(f,g;\Phi,K) \geq Q(f^{\#},g^{\#};\Phi,K). \tag{2.1}$$

Such inequalities were considered in continuous cases by Beckner [18, 19, 20, 21] who generalized the work of Baernstein and Taylor [15]; see also [11]. We now outline a general approach to proving master inequalities in our discrete setting via an adaptation of the Baernstein-Taylor-Beckner approach; our proofs are based on the description of the approach as given by Baernstein [11].

Because we cannot hope to get (2.1) for a general M and a general $\prec$, we must make a number of assumptions. (Indeed, not every M has a well-ordering $\prec$ under which the master inequality holds—see Theorem 7.1 in §7, below, for two very simple and natural counterexamples.)

[2]To see that [58, Thm. 368] implies our result, note that $(M,\prec)$ is order isomorphic to $([0,N]\cap\mathbb{Z}_0^+,<)$ for some $N\in\mathbb{Z}_0^+\cup\{\infty\}$.

Basically, we shall make assumptions which will ensure that the method of proof given for the continuous case in [11] works. Then, to prove the master inequality in any concrete case, we will only need to prove the assumptions in that case.

Recall that an isometry ρ of M onto itself is said to be an **involution** if $\rho \circ \rho$ is the identity function. Whenever we say "involution" we shall mean "isometric involution"; the term "isometric" will sometimes be explicitly given and sometimes dropped.

Given an involution ρ of M, we set

$$\mathfrak{H}_\rho = \{x \in M : x \prec \rho x\},$$

and

$$\operatorname{Fix} \rho = \{x \in M : \rho x = x\}.$$

Then, $\mathfrak{H}_\rho$, $\operatorname{Fix} \rho$ and $\rho \mathfrak{H}_\rho$ are three disjoint sets whose union is M.

Assumption A. There is a transitive set $\mathfrak{I}$ of isometric involutions of M such that whenever ρ is in $\mathfrak{I}$, and the points x and y are both in $\mathfrak{H}_\rho$, then the inequality

$$d(x, y) \leq d(x, \rho y) \tag{2.2}$$

holds.

The assumption of transitivity just says that for all x and y in M there exists a $\rho \in \mathfrak{I}$ such that $\rho x = y$.

In the case $M = \mathbb{Z}$, we may take $\mathfrak{I}$ to be the collection of all isometric involutions of $\mathbb{Z}$. It is easy to see that any element of $\mathfrak{I}$ is defined by $\rho x = x_0 - x$ for some $x_0 = x_0(\rho) \in \frac{1}{2}\mathbb{Z} \stackrel{\text{def}}{=} \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots\}$. It is also easy to see that for $x_0 \leq 0$, we have $\mathfrak{H}_\rho = \{x \in \mathbb{Z} : x > x_0\}$, while for $x_0 > 0$ we have $\mathfrak{H}_\rho = \{x \in \mathbb{Z} : x < x_0\}$. Finally, it is likewise easy to verify that Assumption A holds.

The principal result of the present section is as follows and as mentioned before is essentially due to Baernstein, Taylor and Beckner. A proof will be given later in this section.

Theorem 2.1 (Master Inequality). *If Assumption* A *holds, then the master inequality for* M *with ordering* $\prec$ *must also hold. I.e., if Assumption* A *holds, then for any real* f *and* g*, and for any functions* Φ *and* K *from* $[0,\infty)$ *to* $[0,\infty)$ *such that* Φ *is increasing and convex and* K *is decreasing, we must have*

$$Q(f,g;\Phi,K) \geq Q(f^{\#},g^{\#};\Phi,K). \tag{2.3}$$

Now, for $x \in M$ and $r \in [0,\infty)$, let

$$V(x;r) = |\{y : d(x,y) < r\}|.$$

In order to get an appropriate convolution inequality, we make the following assumption.

Assumption B. For every fixed $r \in [0,\infty)$ the number $V(x;r)$ is finite and independent of $x \in M$.

An easy immediate consequence of Assumption B is that if K is any positive real function then the function

$$x \mapsto \sum_{y \in M} K(d(x,y))$$

is actually constant.

Remark 2.1. Note that if $V(x;r)$ is always finite and there exists a transitive set of isometric automorphisms of M, then Assumption B holds automatically. In particular, if $V(x;r)$ is always finite and Assumption A holds, then Assumption B holds likewise.

Our convolution-rearrangement inequality is then as follows.

Theorem 2.2. *Assume that the master inequality holds for* M *with the ordering* $\prec$*, and that Assumption* B *also holds. Let* K *be a decreasing positive function on* $[0,\infty)$*. Then, for any positive functions* f *and* g *we have*

$$\sum_{x,y \in M} f(x)K(d(x,y))g(y) \leq \sum_{x,y \in M} f^{\#}(x)K(d(x,y))g^{\#}(y). \tag{2.4}$$

Moreover, this also holds in the case where f *and* g *are not necessarily positive, providing that they have finite support.*

Proof. By approximation (monotone convergence theorem) it suffices to consider the case where f and g have finite support. In that case we may also assume that K has bounded support; indeed, if we set $K(t) = 0$ for t sufficiently large that $t \geq d(x, y)$ for all $x \in \operatorname{supp} f \cup \operatorname{supp} f^\#$ and $y \in \operatorname{supp} g \cup \operatorname{supp} g^\#$, then neither the left nor the right side of (2.4) will change. Write

$$\alpha = \sum_{y \in M} K(d(x, y)).$$

By Assumption B, as noted above, α does not depend on x and is finite (the latter fact uses the boundedness of the support of K). Let $\Phi(t) = t^2$. Then we have

$$Q(f, g; \Phi, K) = -2 \sum_{x,y \in M} f(x) K(d(x, y)) g(y) + \alpha \sum_{x \in M} f^2(x) + \alpha \sum_{y \in M} f^2(y), \tag{2.5}$$

and the analogous identity for $Q(f^\#, g^\#; \Phi, K)$ also holds. But,

$$\sum_{x \in M} f^2(x) = \sum_{x \in M} (f^\#)^2(x), \tag{2.6}$$

since f and $f^\#$ are equimeasurable. The identity (2.6) also holds with g in place of f. Combining (2.6) and its analogue for g with the master inequality $Q(f, g; \Phi, K) \geq Q(f^\#, g^\#; \Phi, K)$, and with (2.5) and its analogue for $f^\#$ and $g^\#$, we obtain (2.4) as desired. □

Note that we have seen that if $M = \mathbb{Z}$ then Assumption A holds. Assumption B is trivial in this case, so that (2.4) must also hold. In fact, (2.4) in this case precisely coincides with [58, Thm. 371].

Theorem 2.3 (cf. [58, Thm. 375]). *Suppose that Assumptions* A *and* B *are both satisfied. Given a symmetrically decreasing real valued function g on M which either has finite support or is positive, and given a decreasing function K on $[0, \infty)$, the function $K * g$ defined by*

$$(K * g)(x) = \sum_{y \in M} K(d(x, y)) g(y)$$

is symmetric decreasing on M.

Proof. Our proof is adapted from that of [58, Thm. 375]. Fix any positive function f on M with finite support. Then, by Theorem 2.2 we have

$$\sum_{x\in M} f(x)(K*g)(x) \leq \sum_{x,y\in M} f^{\#}(x)K(d(x,y))g^{\#}(y) = \sum_{x\in M} f^{\#}(x)(K*g)(x),$$

since $g = g^{\#}$. Therefore, by [58, Thm. 369] it follows that $f^{\#}$ and $K*g$ are similarly ordered. Since this holds for any non-negative f, it follows that $K*g$ must be symmetric decreasing. □

We now proceed to prove Theorem 2.1. We need a lemma first, which we state in a slightly stronger form than is needed for Theorem 2.1; we will later need the stronger form in the proof of the condition for equality in Theorem 6.2. Recall that a function Φ is said to be strictly convex if we have the strict inequality

$$\Phi(tx+(1-t)y) < t\Phi(x)+(1-t)\Phi(y),$$

whenever $x \neq y$ and $t \in (0,1)$.

Lemma 2.1 (cf. [11, Thm. 1]). *The master inequality holds for any two point metric space $M = \{\xi, \upsilon\}$ where $\xi \neq \upsilon$. Moreover, if Φ is strictly convex while $K(0) > K(\delta)$ where $\delta = d(\xi,\upsilon)$ is the distance between the two points of M, then $Q(f,g;\Phi,K) = Q(f^{\#},g^{\#};\Phi,K)$ if and only if f and g are similarly ordered.*

The straightforward proof by consideration of the various cases is left to the reader.

Now, for $\rho \in \mathfrak{I}$, define

$$f_\rho(x) = \begin{cases} \max(f(x), f(\rho x)), & \text{if } x \in \mathfrak{H}_\rho \\ \min(f(x), f(\rho x)), & \text{if } x \in \rho\mathfrak{H}_\rho \\ f(x) & \text{if } x \in \operatorname{Fix}\rho. \end{cases}$$

It is easy to see that f and f_ρ are equimeasurable. Note that intuitively the rearrangement $f \mapsto f_\rho$ will often bring f closer to $f^{\#}$. Indeed, our general strategy for the proof of Theorem 2.1 will be to choose a sequence of isometric involutions ρ_n such that if $f_0 = f$ and $g_0 = g$ while

$f_n = (f_{n-1})_{\rho_n}$ and $g_n = (g_{n-1})_{\rho_n}$, then $f_n \to f^\#$ and $g_n \to g^\#$. Given such a sequence, we will basically derive the master inequality from the following lemma which is a precise analogue of [11, Thm. 2].

Lemma 2.2. *Suppose that $\rho \in \mathfrak{J}$ and that Assumption* A *holds. Then,*

$$Q(f, g; \Phi, K) \geq Q(f_\rho, g_\rho; \Phi, K), \tag{2.7}$$

whenever f, g, Φ and K are as in the definition of the master inequality. Moreover if the right hand side of (2.7) *is finite, while Φ is strictly convex and there exist x and y in $\mathfrak{H}_\rho$ such that both of the following two conditions hold:*

(i) $K(d(x, y)) > K(d(x, \rho y))$

(ii) $(f(x) - f(\rho x))(g(y) - g(\rho y)) < 0$,

then strict inequality holds in (2.7).

Proof. Assume that $Q(f_\rho, g_\rho; \Phi, K) < \infty$ (otherwise use an approximation argument). We now have

$$Q(f, g; \Phi, K) = \sum_{(x,y) \in M^2} a(x, y),$$

where

$$a(x, y) = \Phi(|f(x) - g(y)|) K(d(x, y)),$$

and

$$Q(f_\rho, g_\rho; \Phi, K) = \sum_{(x,y) \in M^2} b(x, y),$$

where

$$b(x, y) = \Phi(|f_\rho(x) - g_\rho(y)|) K(d(x, y)).$$

Now, I claim that $\sum_{(x,y) \in M'} a(x, y) = \sum_{(x,y) \in M'} b(x, y)$, where

$$M' = \{(x, y) \in M \times M : x \in \operatorname{Fix} \rho \text{ or } y \in \operatorname{Fix} \rho\}.$$

For, write $M' = M_1 \cup M_2 \cup M_3$, where

$$M_1 = \operatorname{Fix}\rho \times \operatorname{Fix}\rho,$$

$$M_2 = \operatorname{Fix}\rho \times (\mathfrak{H}_\rho \cup \rho\mathfrak{H}_\rho),$$

and

$$M_3 = (\mathfrak{H}_\rho \cup \rho\mathfrak{H}_\rho) \times \operatorname{Fix}\rho.$$

Note that the M_i are pairwise disjoint. To prove the claim it suffices to prove that

$$\sum_{(x,y)\in M_i} a(x,y) = \sum_{(x,y)\in M_i} b(x,y)$$

for $i = 1,2,3$. In fact, it suffices to show it for $i = 1,2$ since the case $i = 3$ is analogous to $i = 2$. Now, if $(x,y) \in M_1$ then $f_\rho(x) = f(x)$ and $g_\rho(y) = g(y)$ so that $a(x,y) = b(x,y)$, and so the proof in the case where $i = 1$ is complete. Suppose $i = 2$. Note that

$$\sum_{(x,y)\in M_2} a(x,y) = \sum_{\substack{x\in\operatorname{Fix}\rho \\ y\in\mathfrak{H}_\rho}} (a(x,y) + a(x,\rho y)), \tag{2.8}$$

and the analogous expression holds with b in place of a. But if $x \in \operatorname{Fix}\rho$ and $y \in \mathfrak{H}_\rho$ then $d(x,y) = d(\rho x, \rho y) = d(x, \rho y)$ since ρ is an isometry fixing x. Then,

$$a(x,y) + a(x,\rho y) = K(d(x,y))[\Phi(|f(x) - g(y)|) + \Phi(|f(x) - g(\rho y)|)],$$

and

$$b(x,y) + b(x,\rho y) = K(d(x,y))[\Phi(|f_\rho(x) - g_\rho(y)|) + \Phi(|f_\rho(x) - g_\rho(\rho y)|)].$$

But it is easy to see that the sets

$$\{f(x) - g(y), f(x) - g(\rho y)\}$$

and

$$\{f_\rho(x) - g_\rho(y), f_\rho(x) - g_\rho(\rho y)\}$$

are equal since $f_\rho(x) = f(x)$ and since the sets

$$\{g(y), g(\rho y)\}$$

and

$$\{g_\rho(y), g_\rho(\rho y)\}$$

are equal by definition of g_ρ. It follows that

$$a(x,y) + a(x,\rho y) = b(x,y) + b(x,\rho y),$$

so that by (2.8) the proof of the case $i = 2$ is completed, and so the claim is proved.

Because of the claim, we have

$$Q(f,g;\Phi,K) - Q(f_\rho,g_\rho;\Phi,K) = \sum_{(x,y)\in M^2\setminus M'} (a(x,y) - b(x,y)). \tag{2.9}$$

But,

$$\sum_{(x,y)\in M^2\setminus M'} a(x,y) = \sum_{(x,y)\in \mathfrak{H}_\rho^2} (a(x,y) + a(x,\rho y) + a(\rho x,y) + a(\rho x,\rho y)), \tag{2.10}$$

and of course the same expression holds with b in place of a. Let $A(x,y) = a(x,y) + a(x,\rho y) + a(\rho x,y) + a(\rho x,\rho y)$ and $B(x,y) = b(x,y) + b(x,\rho y) + b(\rho x,y) + b(\rho x,\rho y)$, for x and y in $\mathfrak{H}_\rho$. Then by (2.9) and (2.10) we have

$$Q(f,g;\Phi,K) - Q(f_\rho,g_\rho;\Phi,K) = \sum_{(x,y)\in \mathfrak{H}_\rho^2} (A(x,y) - B(x,y)). \tag{2.11}$$

Fix x and y in $\mathfrak{H}_\rho$. Consider the two point metric space $X = \{1,2\}$ with a metric D defined by $D(1,2) = 1$, and with the well-ordering $1 \prec 2$. Define $F(1) = f(x)$, $F(2) = f(\rho x)$, $G(1) = g(y)$ and $G(2) = g(\rho y)$. Then, $F^\#(1) = f_\rho(x)$, $F^\#(2) = f_\rho(\rho x)$, $G^\#(1) = g_\rho(y)$ and $G^\#(2) = g_\rho(\rho y)$, where the symmetric decreasing rearrangement on X is defined in terms of the ordering $\prec$. Define $K'(0) = K(d(x,y))$ and $K'(1) = K(d(x,\rho y))$. Note that $d(x,y) = d(\rho x,\rho y)$ and $d(x,\rho y) = d(\rho x,y)$ since ρ is an isometric involution. It is easy to see then that $A(x,y) = Q(F,G;\Phi,K')$ and $B(x,y) = Q(F^\#,G^\#;\Phi,K')$. Thus, $A(x,y) \geq B(x,y)$ by Lemma 2.1. The desired inequality then follows by (2.11).

Moreover, if Φ is strictly convex while x and y are in $\mathfrak{H}_\rho$ and satisfy conditions (i) and (ii), then F and G will fail to be similarly ordered (because of (ii)) and we will have $K'(1) < K'(0)$

(because of (i)). Thus, in that case we will have $A(x,y) > B(x,y)$ by Lemma 2.1, and since we have already seen that $A(x',y') \geq B(x',y')$ for all other pairs of x' and y' in $\mathfrak{H}_\rho$, we conclude by (2.11) that in that case $Q(f,g;\Phi,K) > Q(f_\rho,g_\rho;\Phi,K)$. $\square$

It only remains to prove Theorem 2.1 itself.

Proof of Theorem 2.1. For $m \in \mathbb{Z}^+$, let $[m]$ be the set of the first m elements of M (with respect to $\prec$). Let $[0]$ be the empty set. For conciseness, write $Q(\cdot,\cdot)$ instead of $Q(\cdot,\cdot;\Phi,K)$. Define a real function ψ on pairs of non-negative integers by $\psi(m,n) = 2\min(m,n)$ if $m \neq n$ and $\psi(m,m) = 2m-1$. Given a real function F on M, let

$$\mu(F) = \sup\{m : F|_{[m]} = F^{\#}\Big|_{[m]}\},$$

where $F|_{[m]}$ is the restriction of F to the set $[m]$. Clearly $F = F^{\#}$ if and only if $\mu(F) = |M|$.

We shall inductively define a sequence $\{(f_n,g_n)\}_{n=0}^{\infty}$ of pairs of functions with the following properties:

(a) $f_0 = f$ and $g_0 = g$

(b) f_n and g_n are equimeasurable with f and g respectively

(c) $Q(f_{n-1},g_{n-1}) \geq Q(f_n,g_n)$

(d) $\psi(\mu(f_n),\mu(g_n)) \geq \min(n-1, 2|M|-1)$.

Assume this has been done. By (a) and an inductive application of (c) we find that

$$Q(f,g) \geq Q(f_n,g_n) \tag{2.12}$$

for all n. Conditions (b) and (d) imply that $\Phi(|f_n(x)-g_n(y)|)$ tends to $\Phi(|f^{\#}(x)-g^{\#}(y)|)$ as $n \to \infty$ for any fixed x and y in M; indeed for n sufficiently large (the size of n depending on x and y) we will have $f_n(x) = f^{\#}(x)$ and $g_n(y) = g^{\#}(y)$. Also observe that if M is finite, then in fact f_n and g_n eventually stop changing with n, since if $\psi(\mu(f_n),\mu(g_n)) \geq 2|M|-1$ then we must have $\mu(f_n) = \mu(g_n) = |M|$ and so $f_n = f^\#$ and $g_n = g^\#$. Now in any case by Fatou's lemma (the application of which is not necessary if $|M| < \infty$),

$$\begin{aligned}\liminf_{n\to\infty} Q(f_n, g_n) &\geq \sum_{x,y\in M} \liminf_{n\to\infty} \Phi(|f_n(x) - g_n(y)|)K(d(x,y)) \\ &= \sum_{x,y\in M} \Phi(|f^\#(x) - g^\#(y)|)K(d(x,y)) = Q(f^\#, g^\#).\end{aligned}$$

From this and (2.12) we obtain (2.3) as desired.

Define $f_0 = f$ and $g_0 = g$. We must construct f_n and g_n for $n \geq 1$. Before we proceed to do this, I first claim that if we have any real function F on M and $\rho \in \mathfrak{J}$, then

$$\mu(F_\rho) \geq \mu(F). \tag{2.13}$$

For, let $m = \mu(F)$, so that $F|_{[m]} = F^\#|_{[m]}$. It suffices for us to show that $F_\rho|_{[m]} = F|_{[m]}$. But first fix $x_1 \in [m]$. Choose any $x_2 \in M$ such that $x_1 \prec x_2$. I claim that we then have $F(x_1) \geq F(x_2)$. For, if $x_2 \in [m]$ then $F(x_1) = F^\#(x_1) \geq F^\#(x_2) = F(x_2)$. On the other hand, if $x_2 \notin [m]$ then we can use the equimeasurability of F and the definition of $F^\#$ together with the fact that $F|_{[m]} = F^\#|_{[m]}$ to prove the inequality $F(x_1) \geq F(x_2)$.

We can now show that $F_\rho(x) = F(x)$ if $x \in [m]$. For, x must lie in $\mathfrak{H}_\rho$, in $\operatorname{Fix}\rho$ or in $\rho\mathfrak{H}_\rho$. Suppose first that $x \in \mathfrak{H}_\rho$. Then $x \prec \rho x$. Also, $F_\rho(x) = \max(F(x), F(\rho x))$, and so $F_\rho(x) = F(x)$ as $F(x) \geq F(\rho x)$ by the work of the previous paragraph (just let $x_1 = x$ and $x_2 = \rho x$). Now, if $x \in \operatorname{Fix}\rho$ then $F_\rho(x) = F(x)$ automatically. Finally, consider the case where $x \in \rho\mathfrak{H}_\rho$. Then, $\rho x \prec x$ and so $F_\rho(x) = \min(F(x), F(\rho x))$. Since $x \in [m]$, we must likewise have $\rho x \in [m]$ and so $F(\rho x) \geq F(x)$, upon letting $x_1 = \rho x$ and $x_2 = x$ in the work of the previous paragraph. Thus, $F_\rho(x) = \min(F(x), F(\rho x)) = F(x)$ as desired, and so (2.13) has been proved in all cases.

Now, suppose f_{n-1} and g_{n-1} have been defined and that

$$n - 2 \leq \psi(\mu(f_{n-1}), \mu(g_{n-1})) < 2|M| - 1. \tag{2.14}$$

(If $\psi(\mu(f_{n-1}), \mu(g_{n-1})) = 2|M| - 1$, then just let $f_n = f_{n-1}$ and $g_n = g_{n-1}$.) We either have $\mu(f_{n-1}) \leq \mu(g_{n-1})$ or $\mu(g_{n-1}) < \mu(f_{n-1})$. We only consider the first case, since the second can be handled by just using the first and interchanging f and g in the construction. Let $m = \mu(f_{n-1})$. If we can construct f_n and g_n such that

$$\mu(f_n) > m \qquad \text{and} \qquad \mu(g_n) \geq \mu(g_{n-1}), \tag{2.15}$$

then we will have $\psi(\mu(f_n), \mu(g_n)) > \psi(\mu(f_{n-1}), \mu(g_{n-1}))$, and (d) will be verified. To do this construction, let x be the $(m+1)$st element of M, counting with respect to $\prec$ (such an element exists since (2.14) implies that $m < |M|$). Since $m = \mu(f_{n-1})$, it follows that $f_{n-1}(x) \neq f^{\#}(x)$. The equimeasurability of f and $f^{\#}$ together with the fact that $f_{n-1}|_{[m]} = f^{\#}|_{[m]}$ implies that there exists a y with $x \prec y$ and $f_{n-1}(y) = f^{\#}(x)$; moreover, we will have $f_{n-1}(y) > f_{n-1}(x)$. Let $\rho \in \mathfrak{I}$ be an isometric involution swapping x and y. Then, since $f_{n-1}(y) > f_{n-1}(x)$ while $x \in \mathfrak{H}_\rho$ as $x \prec y = \rho x$, it follows that $(f_{n-1})_\rho(x) = f_{n-1}(y) = f^{\#}(x)$. On the other hand, $\mu((f_{n-1})_\rho) \geq m$ by (2.13) so that $(f_{n-1})_\rho|_{[m]} = f^{\#}|_{[m]}$. Since $[m+1] = [m] \cup \{x\}$ it follows that $(f_{n-1})_\rho|_{[m+1]} = f^{\#}|_{[m+1]}$ and so $\mu((f_{n-1})_\rho) \geq m+1$. Furthermore, again by (2.13) we also have $\mu((g_{n-1})_\rho) \geq \mu(g_{n-1})$. Letting $f_n = (f_{n-1})_\rho$ and $g_n = (g_{n-1})_\rho$, we obtain (2.15). Condition (b) follows from the definition of $(\cdot)_\rho$, while condition (c) follows from Lemma 2.2. $\square$

3. The general case of graphs

The purpose of this section is to show that in the case of a graph, the full Assumption A from the previous section is implied by an apparently weaker variant.

We first give a definition of a graph in the sense in which we will be using this term.

Definition 3.1. A **graph** G consists of a countable set $\operatorname{Vert} G$ of vertices and a set $\operatorname{Edge} G$ of edges, where each edge is a two point set $\{v, w\}$ with $v \neq w$ in $\operatorname{Vert} G$, and where we assume that each vertex lies on at most finitely many edges.

Thus, our graphs are undirected, at most countable, with all vertices of finite degree, with no self-loops, and with no multiple edges between a pair of vertices. We will often identify a graph G with the collection $\operatorname{Vert} G$ of its vertices, using the same notation G to denote both.

Definition 3.2. A sequence of vertices $v_1, v_2, \ldots, v_n$ of a graph is said to be a **path of length** $n-1$ **joining** v_1 **with** v_n if $\{v_i, v_{i+1}\}$ is an edge for all $i \in \{1, \ldots, n-1\}$. The points v_1 and v_n are referred to as the **endpoints** of the path. For convenience, we consider a vertex v standing by itself to be a path of length 1 joining v with v.

Definition 3.3. A graph is said to be **connected** if for every pair $v \neq w$ of distinct vertices there is a path joining v with w.

Suppose that G is a connected graph. Let d be **geodesic distance** on G, i.e., for a pair of vertices v and w we let $d(v,w)$ be the length of the shortest path from v to w if $v \neq w$. We put $d(v,v) = 0$. Moreover, the triangle inequality can be easily verified, so that (G,d) is indeed a metric space. A **geodesic** is defined to be any path joining v and w whose length is precisely $d(v,w)$. In general, there may be more than one geodesic joining v and w.

We say that the vertices v and w of a graph G are **adjacent** if $\{v,w\}$ is an edge of G. Note that no vertex is adjacent to itself.

We now give a few more definitions which will be useful later.

Definition 3.4. A **full subgraph** H of a graph G is a graph such that $\operatorname{Vert} H \subseteq \operatorname{Vert} G$ and such that whenever v and w are vertices of H then $\{v,w\}$ is an edge of H if and only if it is an edge of G. We will often identify a full subgraph with its collection of vertices.

Definition 3.5. The degree $\delta(v) = \delta_G(v)$ of a vertex v in a graph G is the number of edges of G containing v.

Definition 3.6. A graph is said to be **of constant degree** if every one of its vertices has the same degree.

Definition 3.7. If G and G' are two graphs, then a **graph isomorphism** ϕ of G onto G' is a bijective map from $\operatorname{Vert} G$ to $\operatorname{Vert} G'$ such that whenever v and w are vertices of G, then $\{v,w\}$

is an edge of G if and only if $\{\phi(v), \phi(w)\}$ is an edge of G'. In such a case the graphs G and G' are said to be isomorphic. A **graph automorphism** ϕ of G is a graph isomorphism of G onto itself.

If G and G' are connected graph and equipped with the metric d described above, then the notion of a graph isomorphism of G onto G' precisely corresponds to the notion of an isometry of $\operatorname{Vert} G$ onto $\operatorname{Vert} G'$.

Suppose now that the connected graph G is equipped with a well-ordering $\prec$ such that each element has at most finitely many predecessors.

The following result shows that when verifying Assumption A for a concrete graph, we only need to check (2.2) for x, y and ρ such that $d(x, \rho y) = 1$.

Proposition 3.1. *Let G be a connected graph with geodesic distance d and an ordering $\prec$. Fix a graph involution ρ of G. Suppose that for every x and y in $\mathfrak{H}_\rho$ if x and ρy are adjacent then x and y are either adjacent or equal. Then, for any x and y in $\mathfrak{H}_\rho$ we have*

$$d(x, y) \leq d(x, \rho y).$$

Proof. Proceed by induction on $n = d(x, \rho y)$. First, if $n = 0$ then $x = \rho y$. Since ρ is an involution, we likewise have $\rho x = y$. Now, $x \prec \rho x$ as $x \in \mathfrak{H}_\rho$, hence $x \prec y$. On the other hand, $y \prec \rho y = x$, likewise, and we have a contradiction. Secondly, if $n = 1$, then we are done by our hypotheses, since $d(x, \rho y) = 1$ means precisely that x and ρy are adjacent. Thus, assume that $n > 1$ and that the proposition has already been proved whenever $d(x, \rho y) < n$.

Let $x = x_0, x_1, \ldots, x_{n-1}, x_n = \rho y$ be a geodesic joining x with ρy, where x_i is adjacent to x_{i-1} whenever $1 \leq i \leq n$. Note that

$$d(x_1, \rho y) = n - 1. \tag{3.1}$$

We split the rest of the proof into two cases depending on whether x_1 lies in $\mathfrak{H}_\rho \cup \operatorname{Fix} \rho$ or in $\rho \mathfrak{H}_\rho$. Suppose first that $x_1 \in \mathfrak{H}_\rho \cup \operatorname{Fix} \rho$. I claim that then $d(x_1, y) \leq n - 1$. Assuming this claim, we have $d(x,y) \le d(x,x_1) + d(x_1,y) \le 1 + (n-1) = n$, as desired, and the proof is complete.

To prove the claim, there are two subcases to consider, depending on whether x_1 lies in $\operatorname{Fix}\rho$ or in $\mathfrak{H}_\rho$. Consider first the subcase of $x_1 \in \operatorname{Fix}\rho$. Then, $d(x_1,y) = d(\rho x_1, \rho y) = d(x_1, \rho y) = n-1$ by (3.1), where we have used the fact that ρ is a graph automorphism fixing x_1, and so the proof of the claim is complete in this case. Now, suppose that $x_1 \in \mathfrak{H}_\rho$. By (3.1) we have $d(x_1, \rho y) = n-1 < n$ and thus by the induction hypothesis we conclude that $d(x_1,y) \le n-1$, as desired.

The remaining case is that of $x_1 \in \rho\mathfrak{H}_\rho$. But in that case we have $z \stackrel{\text{def}}{=} \rho x_1 \in \mathfrak{H}_\rho$. Now, x and $\rho z = x_1$ are adjacent, so that by assumption we must have x and z adjacent or equal, so that $d(x,z) \le 1$. But,

$$d(x,y) \le d(x,z) + d(z,y) \le 1 + d(z,y) = 1 + d(\rho x_1, y) = 1 + d(x_1, \rho y) = n,$$

where in the second last equality we have used the fact that ρ is a graph involution, while the last equality was a consequence of (3.1). The proof is thus complete. □

4. The octahedron edge graph

We now proceed to prove that Assumptions A and B of §2 are satisfied for a few concrete graphs. Our first and simplest example is the octahedron edge graph, a graph with only 6 vertices so that the value of the resulting inequalities is probably only didactic.

Let H_8 be the edge graph of an octahedron (see Figure 4.1). We shall use the ordering $\prec$ induced by the labels given in Figure 4.1.

Theorem 4.1. *The octahedron H_8 with the ordering shown in Figure 4.1 satisfies Assumptions A and B of §2.*

Proof. Assumption B is trivial. Now, let $\mathfrak{I}$ be the collection of all involutions of H_8. The easy verification of the transitivity of $\mathfrak{I}$ is left to the reader.

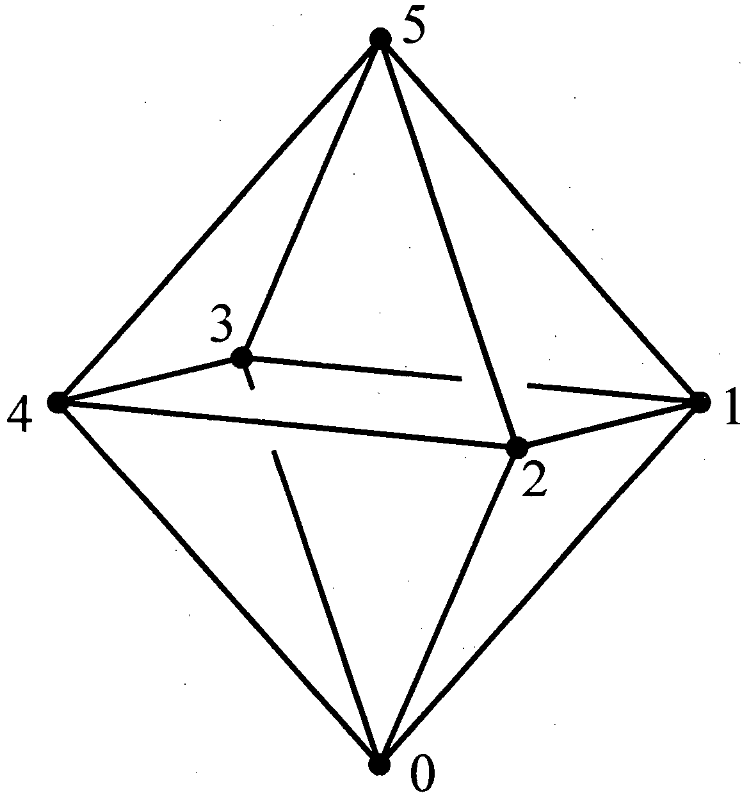


Figure 4.1: The edge graph H_8 of the octahedron

In light of Proposition 3.1, it suffices to prove that if ρ is in $\mathfrak{I}$ while x and y are in $\mathfrak{H}_\rho$ with x and ρy adjacent, then x and y are either adjacent or equal. To obtain a contradiction, suppose that we are given x and y in $\mathfrak{H}_\rho$ with x and ρy adjacent, but $d(x, y) \geq 2$. Without loss of generality we may assume that $x \prec y$ (otherwise, simply exchange x and y and note that the involutive character of the automorphism ρ implies that we have $d(x, \rho y) = d(\rho x, y)$). Now, then, we have four vertices x, y, ρx and ρy in the octahedron. These vertices are distinct, and moreover $d(x, y) \geq 2$ and $d(\rho x, \rho y) \geq 2$. Since x and y are in $\mathfrak{H}_\rho$, we must have $x \prec \rho x$ and $y \prec \rho y$.

Given any vertex $X \in H_8$, there exists a unique vertex $\overline{X} \in H_8$ such that $d(X, \overline{X}) \geq 2$. Note that $d(X, \overline{X}) = 2$ and $\overline{\overline{X}} = X$. Thus, since $d(x, y) \geq 2$ and $d(\rho x, \rho y) \geq 2$, we must have $y = \overline{x}$ and $\rho y = \overline{\rho x}$. Recall that we have assumed that $x \prec y$. Looking at Figure 4.1 we see that $\overline{0} = 5$, $\overline{1} = 4$ and $\overline{2} = 3$ (where we have identified the vertices with their labels). Thus, our assumption that $x \prec y = \overline{x}$ implies that $x \in \{0, 1, 2\}$. Now, if $x = 0$ then $y = \overline{0} = 5$. But $y \prec \rho y$, and this is a contradiction since there is no vertex bigger than 5. Suppose now that $x = 1$. Then $y = \overline{1} = 4$. Since $y \prec \rho y$, we must have $\rho y = 5$. But $d(\rho x, \rho y) \geq 2$, so $\rho x = \overline{\rho y} = \overline{5} = 0$. But then $\rho x \prec x$, contradicting the fact that $x \prec \rho x$. The remaining case is when $x = 2$. Then $y = \overline{2} = 3$. We have $x \prec \rho x$ and $\rho x \neq y$; also, $y \prec \rho y$. Thus, ρx and ρy both lie in $\{4, 5\}$. But $\rho x = \overline{\rho y}$ and so we have a contradiction as $\overline{4} \neq 5$. □

The author is grateful to Professor Greg Kuperberg for drawing the author's attention to the octahedron by pointing out that there is a set of reflections in planes about the origin which is transitive on the octahedron's vertices when the octahedron is inscribed in the sphere. (Note that this fact could be used to give another proof of Theorem 4.1, essentially by restriction of the spherical case of Baernstein and Taylor [15].)

5. The circle graphs $\mathbb{Z}_n$

Before proceeding to the case of the regular tree, we give the easier case of the cyclic group $\mathbb{Z}_n$. Define $d(m, m')$ to be $\frac{n}{2\pi}$ times the length of the shortest arc of the unit circle in the complex plane joining the points $e^{2\pi i m/n}$ with $e^{2\pi i m'/n}$. Note that d agrees with the graph distance if we consider $\mathbb{Z}_n$ to be the graph composed of n points joined together in a circle (see Figure 5.1). Order $\mathbb{Z}_n$ by $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec \cdots$, where the "$\cdots$" here indicates that we keep on going until we exhaust all the elements of $\mathbb{Z}_n$.

See Figure 5.2 for an example of how symmetrization works on $\mathbb{Z}_n$.

Now, let L be a line through the origin of the complex plane at an angle which is an integer multiple of π/n. Define $\rho_L m$ to be m' where m' is chosen so that $e^{2\pi i m'/n}$ is the reflection of $e^{2\pi i m/n}$ in the line L. Let $\mathfrak{I}$ be the collection of all such ρ_L. It is easy to see that $\mathfrak{I}$ is a transitive collection of isometric involutions of $\mathbb{Z}_m$. Now, given L as above, if $L \neq \mathbb{R}$ then let H be the component of $\mathbb{C} \backslash L$ which contains the point $1 \in \mathbb{C}$, and if $L = \mathbb{R}$ then let H be the open upper half plane. It is easy to verify by drawing diagrams that $m \in \mathfrak{H}_{\rho_L}$ if and only if $e^{2\pi i m/n} \in H$. But from this and the definition of the metric on $\mathbb{Z}_m$ it is easy to see that we

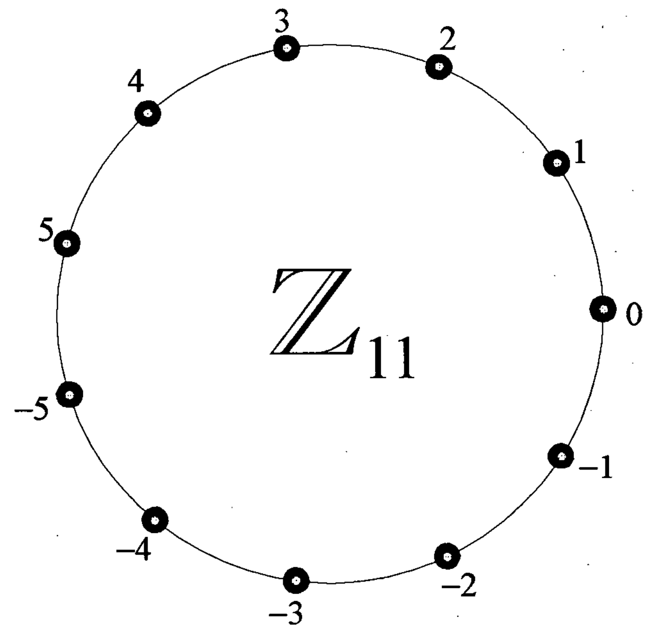


Figure 5.1: The graph $\mathbb{Z}_{11}$.

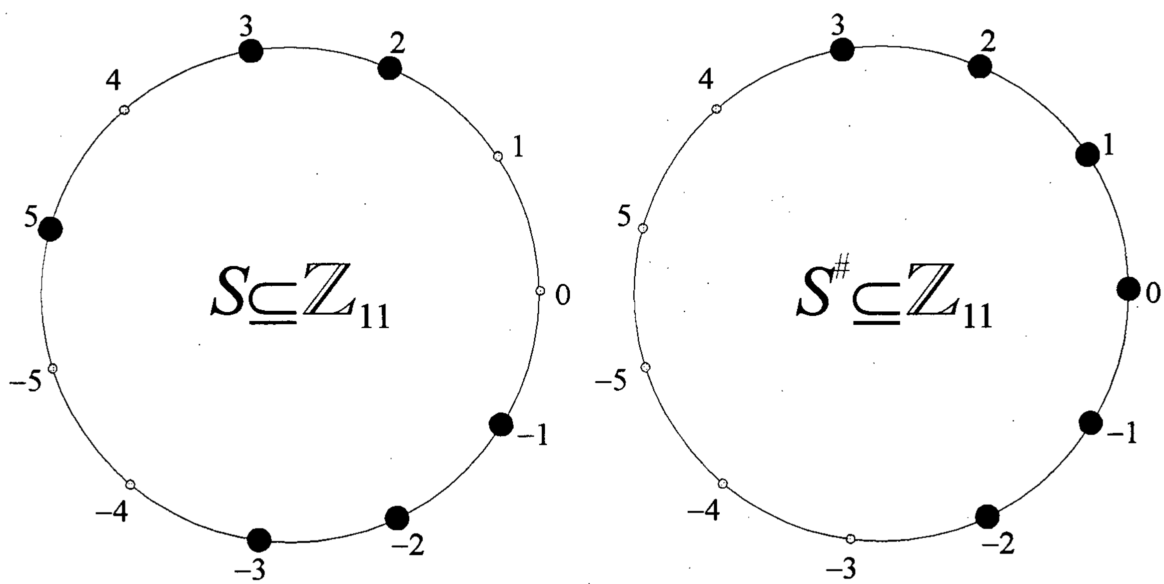


Figure 5.2: Symmetrization of subsets of $\mathbb{Z}_{11}$.

have Assumption A satisfied.

On the other hand, Assumption B is trivial. Hence, we see that (2.3) and (2.4) both hold for $\mathbb{Z}_n$ with the ordering as above. This is a discrete analogue of Theorem I.6.4, minus the discussion of the case of equality.

We summarize the findings of this section as follows.

Theorem 5.1. *The group $\mathbb{Z}_n$ equipped with the ordering $\prec$ satisfies Assumptions* A *and* B *of* §2.

6. Regular trees

Definition 6.1. A path $v_1, v_2, \ldots, v_n$ in a graph is said to contain a **backtracking** if there is an $i \in \{3, \ldots, n\}$ such that $v_i = v_{i-2}$.

Definition 6.2. A **tree** is a graph such that between every pair of vertices there is at most one path without any backtrackings.

It follows that in a connected tree, geodesics are unique.

Definition 6.3. The p**-regular tree** T_p, where $p \in \{2, 3, \ldots\}$, is any connected tree such that every vertex has degree exactly p.

It is easy to see that a p-regular tree T_p exists, and that any two such trees are isomorphic as graphs.

Note that T_2 can be naturally identified with $\mathbb{Z}$. See Figure 6.1 for a subset of T_3.

The unique path without backtracking joining vertices v and w will necessarily be a geodesic. Its length (i.e., the number of edges in it) shall be denoted by $d(v, w)$ as before; we will write $[v, w]$ for the set of vertices (including the endpoints) lying on this path. We write $[v, v] = \{v\}$.

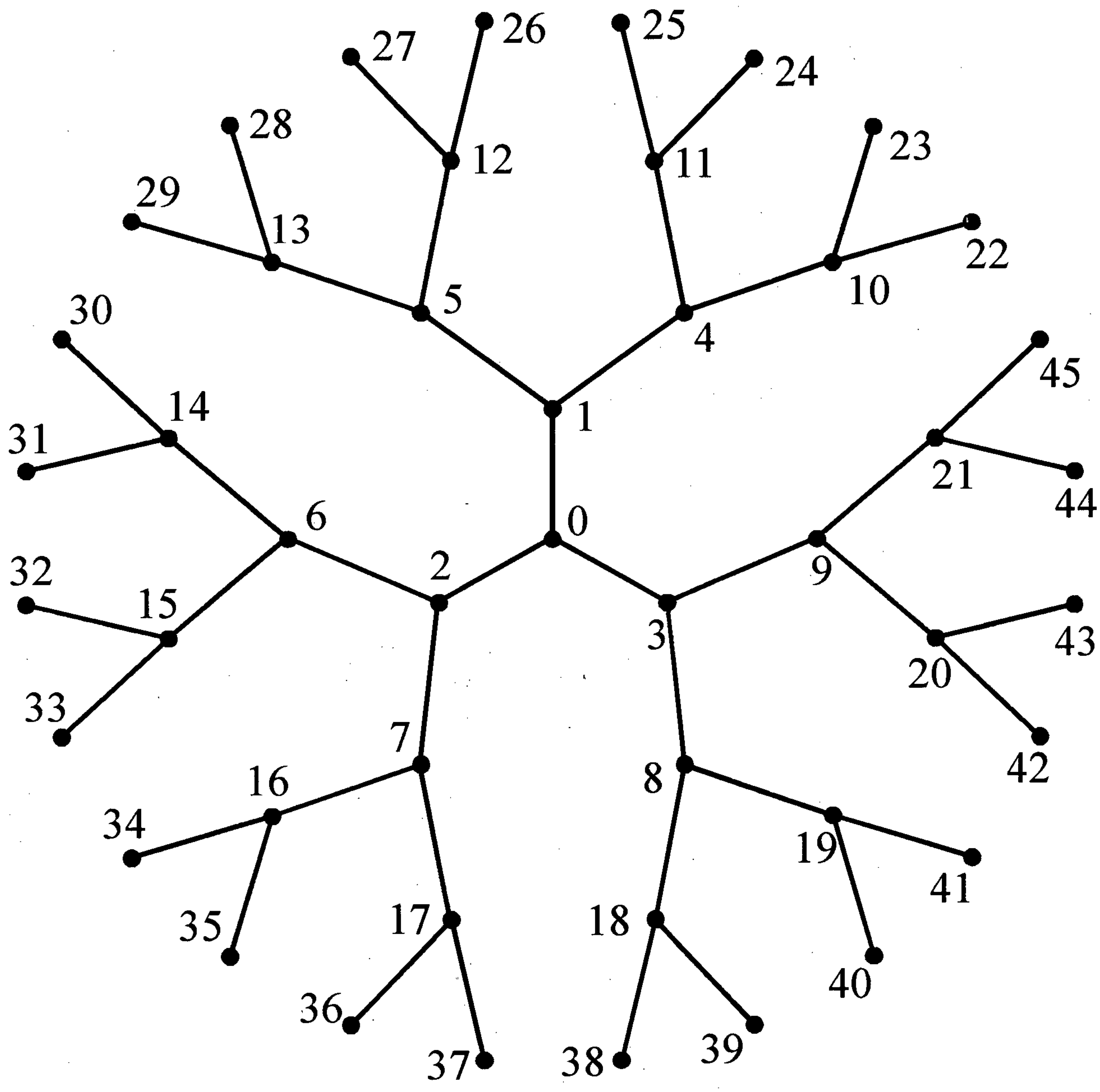


Figure 6.1: The ordering on the tree T_3. The portion $T_{3,4}$ of the tree is shown. Note that $h(0) = 0$, $h(1) = h(2) = h(3) = 1$, $h(4) = \cdots = h(9) = 2$, $h(10) = \cdots = h(21) = 3$ and $h(22) = \cdots = h(45) = 4$.

In Figure 6.1, for instance, we have $[5, 24] = \{5, 1, 4, 11, 24\}$ and $d(5, 24) = 4$. In Figures 6.5 and 6.6, below, two more examples of geodesics are shown.

We distinguish one vertex O which we shall call the **root** of T_p. Given a vertex v of T_p, we write $h(v) \stackrel{\text{def}}{=} d(v, O)$ for the **height** of this vertex (see Figure 6.1, where we have let $O = 0$.) For $k \geq 1$ there are precisely $p(p-1)^{k-1}$ vertices which have height exactly k. We write $T_{p,k}$ for the full subtree of T_p defined by all vertices whose height does not exceed k. Note that the subtree pictured in Figure 6.1 is $T_{3,4}$. The subtree $T_{p,k}$ can be called a "geodesic ball" in the regular tree.

Given a vertex v, we say that a vertex w is a **descendant** of v and that v is an **ancestor** of w providing v is contained in the geodesic from O to w. We say that w is a **child** of v or that v is the **parent** of w providing that w is a descendant of v which is adjacent to v. Note that in general each vertex other than O has a unique parent. Every vertex other than O has precisely $p-1$ children; the vertex O has p children. Write $\operatorname{Desc} v$ for the set of all descendants of a vertex v; note that we always have $v \in \operatorname{Desc} v$. Write $\operatorname{Children} v$ for the set of all children of a vertex v; note that $\operatorname{Children} v \subset (\operatorname{Desc} v) \backslash \{v\}$. Note also that if the vertices v and w are adjacent then either v is the parent of w or w is the parent of v.

To illustrate these definitions, note for instance that in Figure 6.1 we have

$$T_{3,4} \cap \operatorname{Desc} 4 = \{4, 10, 11, 22, 23, 24, 25\}$$

and

$$\operatorname{Children} 4 = \{10, 11\},$$

while the parent of 4 is 1.

Definition 6.4. We say that a well-ordering $\prec$ of T_p is **spiral-like** providing the following conditions hold for all vertices v, w, v_1, v_2, w_1 and w_2:

(a) if $h(v) < h(w)$ then $v \prec w$

(b) if $h(v_1) = h(v_2)$ while $v_1 \prec v_2$ and w_i is a descendant of v_i for $i = 1, 2$, then $w_1 \prec w_2$.

Such an ordering certainly exists, and may be chosen by induction (see Figure 6.1, where the spiral-like ordering $\prec$ is induced by the standard ordering $<$ on the integers forming the labels). A spiral-like ordering is unique up to isomorphism. (I.e., if $\prec_1$ and $\prec_2$ are spiral-like well-orderings of T_p then there is a unique graph automorphism α of T_p such that $v \prec_1 w$ if and only if $\alpha v \prec_2 \alpha w$. The inductive construction of α is quite easy and left to the reader.)

Given such an ordering, we can form the symmetric decreasing rearrangement $\#$ with respect to it as in §1.

Note that if $G \subset T_p$ is finite then there is a unique $k \in \mathbb{Z}_0^+$ such that $T_{p,k} \subseteq G^{\#} \subset T_{p,k+1}$. Thus $G^{\#}$ is always in some sense close to being a geodesic ball. If for some k we have $|G| = |T_{p,k}|$ then in fact $G^{\#}$ is precisely equal to the geodesic ball $T_{p,k}$.

6.1. The master inequality on regular trees

Our main result about regular trees is that they, like the octahedron and the circle graphs, satisfy Assumptions A and B of §2 and thus also satisfy a master inequality.

Theorem 6.1. *The tree T_p equipped with any spiral-like well-ordering satisfies Assumptions* A *and* B *of* §2.

Before we prove this, we need a crucial result about spiral-like well-orderings and geodesics.

Lemma 6.1. *Let T_p be equipped with a spiral-like well-ordering $\prec$. Then for each geodesic $v_1, v_2, \ldots, v_{n-1}, v_n$ joining v_1 with v_n such that $n \geq 4$ and $v_1 \prec v_n$, we have $v_2 \prec v_{n-1}$.*

Note that in the setting of the Lemma we have $n = 1 + d(v_1, v_n)$.

Proof of Lemma. Let a geodesic $v_1, v_2, \ldots, v_{n-1}, v_n$ be given with $v_1 \prec v_n$. We must have $h(v_2) = h(v_1) \pm 1$ for some choice of $\pm$. Suppose first that $h(v_2) = h(v_1) + 1$. Then since we have a geodesic it follows that $h(v_3) = h(v_2) + 1$, and $h(v_4) = h(v_3) + 1$ and so on, so that it follows that $h(v_{n-1}) > h(v_2)$ and so $v_2 \prec v_{n-1}$ by property (a) of spiral-like orderings. Now suppose that $h(v_2) = h(v_1) - 1$. If $h(v_n) > h(v_1)$ then, since $h(v_{n-1}) = h(v_n) \pm 1$ for some choice of $\pm$, it follows that $h(v_{n-1}) > h(v_1) - 1 = h(v_2)$, and so $v_2 \prec v_{n-1}$, again by property (a) of spiral-like orderings. Thus, the remaining case is when $h(v_n) \leq h(v_1)$. Again by property (a) we must in fact have $h(v_n) = h(v_1)$. The non-backtracking property of geodesics then guarantees that $h(v_{n-1}) = h(v_n) - 1$ and that v_1 and v_n are children of v_2 and v_{n-1}, respectively. Moreover, $h(v_2) = h(v_{n-1})$. Since $n \geq 4$ we have $v_2 \neq v_{n-1}$. If we had $v_{n-1} \prec v_2$ then by property (b) of spiral-like orderings we would have $v_n \prec v_1$, a contradiction. Thus, we must have $v_2 \prec v_{n-1}$, as desired. $\square$

The rest of this section will be occupied with the proof of Theorem 6.1.

Proof of Theorem 6.1. It is clear that Assumption B holds. Thus it suffices to verify Assumption A. Let $\mathfrak{J}$ be the collection of all involutive graph automorphisms of T_p. It is easy to verify that $\mathfrak{J}$ is transitive. By Proposition 3.1 the only other thing that we must verify is that for $\rho \in \mathfrak{J}$ and v and w in $\mathfrak{H}_\rho$, if v and ρw are adjacent, then v and w are either equal or adjacent. We shall in fact prove that they are always equal. For, assume that $v \neq w$. Suppose v and ρw are adjacent. Likewise, then, ρv and w are adjacent since ρ is an involutive graph automorphism. Note that the four points v, ρw, ρv and w are all distinct because of the various assumptions above and as ρ is an involutive automorphism.

In general it is easy to see that given four points a_1, a_2, a_3 and a_4 on a tree, with a_1 and a_2 adjacent and with a_3 and a_4 also adjacent, it follows that there exists an $i \in \{1, 2\}$ and a $j \in \{3, 4\}$ such that all four points a_1, a_2, a_3 and a_4 lie on the geodesic $[a_i, a_j]$. Applying this to our situation, we see that our points v, ρw, ρv and w must all lie on one of the geodesics $[v, \rho v]$, $[\rho w, \rho v]$, $[v, w]$ and $[\rho w, w]$. In fact, we may reduce the cases even further. Let $P = \{v, \rho w, \rho v, w\}$. If $P \subseteq [\rho w, \rho v]$, then likewise we must have $P = \rho P \subseteq \rho[\rho w, \rho v] = [w, v]$, since ρ preserves geodesics. Conversely, if $P \subseteq [\rho w, \rho v]$ then $P \subseteq [w, v]$. Now, if two points x and y are contained in some geodesic $[x', y']$ then $[x, y] \subseteq [x', y']$. It follows from the above, then, that if $P \subseteq [\rho w, \rho v]$ then $P \subseteq [w, v] \subseteq [\rho w, \rho v]$, and if $P \subseteq [w, v]$ then $P \subseteq [\rho w, \rho v] \subseteq [w, v]$.

Hence, in either case we have $[v, w] = [\rho v, \rho w]$, which implies that the sets $\{v, w\}$ and $\{\rho v, \rho w\}$ are equal, whereas we know that all four points in P are distinct, a contradiction.

Moreover, we need not concern ourselves with the case $P \subseteq [v, \rho v]$, since upon exchanging w and v the result in this case will follow from the result in the case $P \subseteq [w, \rho w] = [\rho w, w]$, as our assumption that v and ρw be adjacent is symmetric in v and w as ρ is an involution.

Hence we need only consider the case where $P \subseteq [w, \rho w]$. Write $[w, \rho w] = \{w_1, w_2, \ldots, w_n\}$, where $\{w_i, w_{i+1}\}$ is an edge of T_p whenever $1 \leq i \leq n-1$, and where $w_1 = w$ and $w_n = \rho w$. Since v and ρw are adjacent and $v \in [w, \rho w]$, we must have $w_{n-1} = v$. Now, ρ is a graph isomorphism and it swaps the endpoints of $[w, \rho w]$, so that in fact, since $[w, \rho w]$ is a geodesic while geodesics are unique on a tree, it must map $[w, \rho w]$ onto itself, with $\rho w_i = w_{n+1-i}$. Hence, $\rho v = \rho w_{n-1} = w_2$. Now, $w_1 \prec w_n$ by definition of $\mathfrak{H}_\rho$ as $w_1 = w \in \mathfrak{H}_\rho$ and $w_n = \rho w_1$. The geodesic $[w_1, w_n]$ contains at least four points since P contains four distinct points. Thus, by Lemma 6.1 it follows that $\rho v = w_1 \prec w_{n-1} = v$, contradicting the fact that $v \in \mathfrak{H}_\rho$. $\square$

6.2. The Faber-Krahn inequality for subsets of regular trees

6.2.1. Statement of the Faber-Krahn inequality

If D is an open set in $\mathbb{R}^n$ and $\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ is the ordinary continuous Laplacian, then let $\nu_1(D)$ be the smallest strictly positive eigenvalue of the operator $-\Delta$ acting on functions f on D with the Dirichlet boundary condition that they vanish on ∂D. Then, in the case $n = 2$, Lord Rayleigh [91, §210] conjectured that

$$\nu_1(D^{\circledast}) \leq \nu_1(D),$$

where $D^{\circledast}$ is a Euclidean ball of the same area as D. This is known as the Faber-Krahn inequality [47, 68]. It is in fact true in all dimensions. We shall prove an analogous inequality where D is a subset of the p-regular tree, while $D^{\#}$ is defined in terms of a spiral-like ordering and takes the place of $D^{\circledast}$.

Definition 6.5. Given any graph G, define the **discrete Laplacian** $\Delta = \Delta_G$ on G by

$$\Delta f(v) = -f(v) + \frac{1}{\delta(v)} \sum_{w \in N(v)} f(w),$$

for a function f on G, where $N(v)$ is the set of all vertices of G adjacent to v.

Thus on T_p the discrete Laplacian is given by

$$\Delta f(v) = -f(v) + \frac{1}{p} \sum_{w \in N(v)} f(w).$$

Of course in this case $|N(v)| = p$ for all v.

Given a finite non-empty subset G of T_p, let $\nu_1(G)$ be the smallest strictly positive eigenvalue of $-\Delta$ on G with the Dirichlet boundary condition that our functions vanish outside G. Standard eigenvalue methods (see Theorem 6.3, in §6.2.2, below) let us also compute $\nu_1(G)$ via the expression

$$\nu_1(G) = \inf_{f \in \mathfrak{D}(G)} \mathcal{R}(f), \tag{6.1}$$

where $\mathfrak{D}(G)$ is the set of all real functions $f \not\equiv 0$ which vanish everywhere outside G, while

$$\mathcal{R}(f) = \frac{\sum_{v \in G} -f(v)\Delta f(v)}{\sum_{v \in G} f^2(v)}$$

is the Rayleigh quotient for G. (We shall verify (6.1) formally in the next section.) Note that $\mathcal{R}(f) \geq \mathcal{R}(|f|)$ since

$$\sum_{v \in G} -f(v)\Delta g(v) = \frac{1}{p} \sum_{\{v,w\} \in E} (f(v) - f(w))(g(v) - g(w)),$$

where $E = \operatorname{Edge} T_p$. (To verify this last identity, by linearity it suffices to prove it if the supports of f and g have one point each, in which case the result is easy.)

The main result of this section is as follows. The result was inspired by Friedman [54, Conjecture 4.3].

Theorem 6.2 (Faber-Krahn inequality for subsets of regular trees). *Let G be a finite non-empty subset of T_p. Then,*

$$\nu_1(G) \geq \nu_1(G^{\#}), \tag{6.2}$$

where $G^{\#}$ is defined with respect to any spiral-like well-ordering on T_p. Equality holds if and only if there is an automorphism of T_p mapping G onto $G^{\#}$.

See Figure 6.2 for the extremal subsets $G^{\#}$ of cardinalities from 1 to 12. In Figures 6.3 and 6.4 on pages 104 and 105, one may find two other extremal subsets together with the corresponding eigenfunctions.

Recall that Theorem 6.1 guarantees the validity of (2.4) on T_p. In §6.2.3 we shall use this fact to prove (6.2). Only after that, in section §6.2.4, will we prove the condition for equality.

The proof in §6.2.3 will show that (6.2) is valid for subsets of any constant degree graph on which we have the convolution-rearrangement inequality (2.4). However, it is not known, even given an appropriate convolution-rearrangement inequality, whether for a more general graph we can make a classification of the case of equality similar to the one given in Theorem 6.2 for the case of the regular tree. See Problem 7.3 in §7.

6.2.2. Some useful well-known results

Let G be a non-empty finite subset of a constant degree graph H. Let $\mathfrak{D}$ be the collection of all real functions on H which are zero outside G but do not vanish identically. Let ν_1 be the first strictly positive eigenvalue of the operator $-\Delta$ acting on $\mathfrak{D}$. Write

$$\mathcal{R}(f) = \frac{\sum_{v \in G} -f(v)\Delta f(v)}{\sum_{v \in G} f^2(v)}$$

for $f \in \mathfrak{D}$.

The following result is very well known, but we give a proof for completeness.

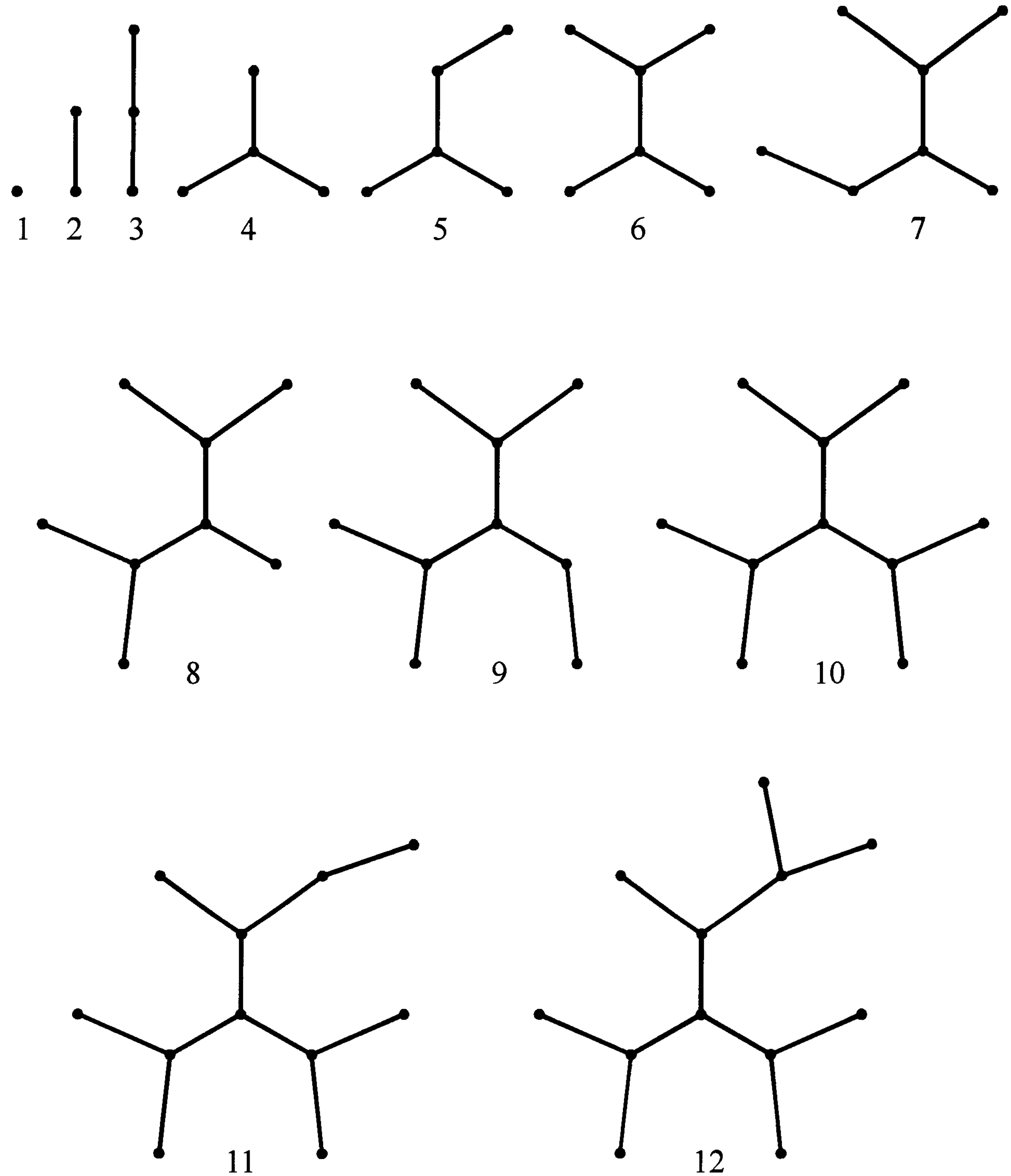


Figure 6.2: The extremal subtrees $G^{\#}$ of T_3 with cardinalities from 1 to 12.

Theorem 6.3. *Assume that G is a non-empty finite subset of a constant degree graph H. Assume that given any vertex v in G there is a vertex w in $H \backslash G$ and a path from v to w in H. Then we have*

$$\nu_1 = \min_{f \in \mathfrak{D}} \mathcal{R}(f).$$

Moreover, the minimum of this functional is achieved at f if and only if f is an eigenfunction of $-\Delta$ with eigenvalue ν_1. If f is an eigenfunction with eigenvalue ν_1, then so is $|f|$.

We have $\nu_1 \in (0, 1]$. Finally, if there exist vertices v_1 and v_2 of G which are adjacent, then $\nu_1(G) \leq 1 - (1/p) < 1$, where p is the degree of each of the vertices of the constant degree graph H.

Proof. Let

$$\lambda = \inf_{f \in \mathfrak{D}} \mathcal{R}(f).$$

For f a function on G, write

$$\|f\|_2 = \left(\sum_{v \in G} f^2(v) \right)^{1/2}.$$

Let S be the collection of all functions $f \in \mathfrak{D}$ such that $\|f\|_2 = 1$. Since $\mathcal{R}(cf) = \mathcal{R}(f)$ for all $c \neq 0$, it is easy to see that then

$$\lambda = \inf_{f \in S} \mathcal{R}(f) = \sum_{v \in G} -f(v) \Delta f(v),$$

and that $f \in \mathfrak{D}$ minimizes $\mathcal{R}$ over $\mathfrak{D}$ if and only if $f/\|f\|_2$ minimizes it over S. Now, S is also easily seen to be homeomorphic in a natural way to the $(|G| - 1)$-dimensional sphere, and the function $\mathcal{R}$ is continuous in the induced topology, so that by compactness the infimum is attained. Hence there exist minimizers for $\mathcal{R}$ over S, and thus also over $\mathfrak{D}$.

Suppose that $f \in \mathfrak{D}$ minimizes λ. We shall prove that f is an eigenfunction of $-\Delta$ on G with eigenvalue λ. For, fix $w \in G$. Without loss of generality we may assume that $\|f\|_2 = 1$ (else replace f by $f/\|f\|_2$.) Let e be the function which is 1 at w and 0 elsewhere. It is clear that for h a sufficiently small real number we have $f + he \in \mathfrak{D}$, and that

$$\left. \frac{d}{dh} \right|_{h=0} \mathcal{R}(f + he) \tag{6.3}$$

is well-defined. By minimality, this derivative must vanish. But, by first-year calculus, this derivative is also equal to

$$-\frac{(\langle f, \Delta e\rangle + \langle e, \Delta f\rangle)\langle f, f\rangle - \langle f, \Delta f\rangle (2\langle f, e\rangle)}{\langle f, f\rangle},$$

where $\langle \alpha, \beta\rangle \stackrel{\text{def}}{=} \sum_{v\in G} \alpha(v)\beta(v)$. Hence, setting (6.3) to zero and using the assumption that $||f||_2 = 1$ as well as the definition of e, we find that

$$\sum_{v\in G} f(v)\Delta e(v) + \Delta f(w) = 2f(w)\sum_{v\in G} f(v)\Delta f(v). \tag{6.4}$$

But if $||f||_2 = 1$ then $\sum_{v\in G} f(v)\Delta f(v) = -\mathcal{R}(f)$ and in our case $\mathcal{R}(f) = \lambda$ by minimality. On the other hand, $\Delta e(w) = -1$, while clearly $\Delta e(v) = 1/p$ for $v \in N(w)$ and $\Delta e(v) = 0$ for $v \notin \{w\} \cup N(w)$. Thus,

$$\sum_{v\in G} f(v)\Delta e(v) = -f(w) + \frac{1}{p}\sum_{w\in N(v)} f(v) = \Delta f(w).$$

Thus, (6.4) becomes

$$2\Delta f(w) = -2\lambda f(w).$$

Since $w \in G$ was arbitrary, it follows that f is indeed an eigenfunction of $-\Delta$ on G with eigenvalue λ.

I now claim that $\lambda > 0$. For, we have

$$\sum_{v\in G} -f(v)\Delta g(v) = \frac{1}{p}\sum_{\{v,w\}\in \operatorname{Edge} G} (f(v) - f(w))(g(v) - g(w)), \tag{6.5}$$

for any f and g, so that $\mathcal{R}(f)$ is always non-negative. (We have already noted the displayed identity in the case of the tree, and said that it is best proved by first verifying it for f and g whose supports have one point each, and then using linearity for the general case. This works just as well for any constant degree graph.)

In fact, $\mathcal{R}(f)$ must be strictly positive, for the above identity shows that if it is equal to zero then $f(v) = f(w)$ whenever $v \in G$ and w is adjacent to it. I claim that this implies that if $\mathcal{R}(f) = 0$ then $f \equiv 0$ if f vanishes outside G. For, fix $v \in G$. Let $v = v_1, v_2, \ldots, v_n$ be a path in H such that $v_n \in H \backslash G$. Such a path exists by our assumptions. Shortening the path if necessary, we may assume that $v_1, \dots, v_{n-1} \in G$. Then, iterating an observation made at the beginning of the paragraph, we see that $f(v) = f(v_1) = f(v_2) = \cdots = f(v_n)$. But $v_n \notin G$ so that $f(v_n) = 0$, and so $f(v) = 0$ as desired. Hence $\mathcal{R}(f) = 0$ and $f \in \mathfrak{D}$ are incompatible assumptions. Thus, indeed, we conclude that $\lambda > 0$.

Now, let f be an eigenfunction of $-\Delta$ with eigenvalue λ'. We have

$$-\Delta f(v) = \lambda' f(v).$$

Thus,

$$\mathcal{R}(f) = \frac{\sum_{v \in G} \nu_1 f^2(v)}{\sum_{v \in G} f^2(v)} = \lambda'.$$

From this and the fact that if $\mathcal{R}$ achieves its minimum at $f \in \mathfrak{D}$ then f is an eigenfunction with eigenvalue λ, we conclude that indeed $\lambda = \nu_1$ and that eigenfunctions corresponding to ν_1 coincide with the minimizers of $\mathcal{R}$ over $\mathfrak{D}$.

The statement that $|f|$ is also an eigenfunction if f is an eigenfunction follows from the observation that $\mathcal{R}(|f|) \le \mathcal{R}(f)$ (this observation is clear from (6.5) and the triangle inequality) which implies that if f is a minimizer of $\mathcal{R}$ then so is $|f|$.

To show that $\nu_1 \le 1$, choose any $w \in G$. Let e be the indicator function of $\{w\}$, as before. Clearly $f \in \mathfrak{D}$. Then $\Delta e(w) = -1$ so that $\nu_1 \le \mathcal{R}(e) = 1$.

Now suppose that v_1 and v_2 are vertices of G which are adjacent. Let f be the indicator function of $\{v_1, v_2\}$. Clearly $f \in \mathfrak{D}$. Then,

$$\Delta f(v_i) = -1 + \frac{1}{p},$$

for $i \in \{1, 2\}$. Let $\rho = 1 - (1/p)$. Note that $\|f\|_2^2 = 2$. Thus,

$$\nu_1 \le \mathcal{R}(f) = \frac{-\Delta(v_1) - \Delta(v_2)}{\|f\|_2^2} = \frac{2\rho}{2} = \rho,$$

as desired. □

Definition 6.6. Let G be a graph. Define the relation $\sim$ on the vertices of G by writing $v \sim w$ whenever v is connected by a path in G to w. This is clearly an equivalence relation. Define a **connected component** of G to be an equivalence class under $\sim$.

Definition 6.7. Let f be a function defined on a graph H, and let G be a subset of H. Then f is said to be **superharmonic** on G if

$$\Delta f(v) \leq 0$$

for all $v \in G$.

Definition 6.8. Let $C \subseteq \operatorname{Vert} H$. Define $\overline{C} = C \cup \partial C$, where ∂C is the set of vertices v of H such that there exists a $v' \in C$ with v and v' adjacent.

The following well-known result is known as the minimum principle for superharmonic functions. It does not require constancy of degree.

Theorem 6.4. *Let G be a finite subset of a graph H. Let f be non-negative on H and superharmonic on G. Assume that there exists a vertex $w \in G$ at which f vanishes. Let C be the connected component of G containing w. Then f vanishes everywhere on $\overline{C}$.*

Proof. We shall prove that if $f(v) = 0$ for $v \in G$ then $f(v') = 0$ for every v' adjacent to v. This will suffice to prove the result in light of the definition of $\overline{C}$.

Now, the condition $\Delta f(v) \leq 0$ implies that

$$0 = f(v) \geq \frac{1}{\delta(v)} \sum_{v' \in N(v)} f(v').$$

But $f(v') \geq 0$ for $v' \in N(v)$. This immediately implies that $f(v') = 0$ for all $v' \in N(v)$, as desired. □

Finally we give the following also well-known result.

Corollary 6.1. *Let G be a connected non-empty finite subset of a constant degree graph H. Then the eigenvalue ν_1 has multiplicity 1, and the nontrivial eigenfunctions corresponding to it do not vanish anywhere on G and have constant sign on G.*

Proof. Let f be an eigenfunction corresponding to ν_1. First we prove that f has constant sign. For, assume that $f(w) \geq 0$ for some $w \in G$. I claim that $f \geq 0$ everywhere on G. For, otherwise the function $g = |f| - f$ is not identically zero. But, by Theorem 6.3, if f is an eigenfunction corresponding to ν_1, then so is $|f|$. Hence, so is g, since the difference of two eigenfunctions with the same eigenvalue is also an eigenfunction with the same eigenvalue. We then have

$$-\Delta g = \nu_1 g$$

on G. But $g \geq 0$ since $|f| \geq f$, so that it follows that g is superharmonic and positive. Moreover, $f(w) \geq 0$ so that $g(w) = 0$, which is an immediate contradiction to Theorem 6.4 if g does not vanish almost everywhere. (Here we have used the connectedness of G).

Thus, indeed f has constant sign. Moreover, just as we argued with g, we can also use the minimum principle (applied to f if f is positive and to $-f$ otherwise) to see that f vanishes nowhere.

Now, suppose that f_1 and f_2 are nontrivial eigenfunctions corresponding to ν_1. We must prove that $f_1 = cf_2$ for some constant c. Since f_1 and f_2 have constant signs, we may assume that they are both positive. Fix any $v \in G$. We then have $f_1(v) > 0$ and $f_2(v) > 0$. Choose c so that $f_1(v) = cf_2(v)$. Let $g = f_1 - cf_2$. If g vanishes everywhere, then we are done. Hence, suppose g does not vanish identically. Then g is a nontrivial eigenfunction corresponding to ν_1, and by our work above, g cannot vanish anywhere. But $g(v) = 0$, and so we have a contradiction. □

6.2.3. Proof of not necessarily strict inequality in Theorem 6.2

Define the operator A on the set of real functions on T_p via

$$Af(v) = f(v) + \sum_{w \in N(v)} f(w).$$

Then, $-(1+p^{-1})f + p^{-1}Af = \Delta f$, so that we can write

$$\mathcal{R}(f) = 1 + \frac{1}{p} - \left(\frac{1}{p}\right) \frac{\sum_{v\in G} f(v)Af(v)}{\sum_{v\in G} f^2(v)}. \tag{6.6}$$

But, clearly,

$$Af(v) = \sum_{w\in T_p} K(d(v,w))f(w), \tag{6.7}$$

where K is the function defined by $K(t) = 1$ for $t \leq 1$ and $K(t) = 0$ for $t > 1$. By (2.4) we thus have

$$\sum_{v\in G} f(v)Af(v) \leq \sum_{v\in G} f^{\#}(v)Af^{\#}(v)$$

for positive f. On the other hand, by the equimeasurability of f and $f^{\#}$ we have

$$\sum_{v\in G} f^2(v) = \sum_{v\in G} (f^{\#}(v))^2.$$

Thus, in general,

$$\mathcal{R}(f) \geq \mathcal{R}(|f|) \geq \mathcal{R}(|f|^{\#}). \tag{6.8}$$

But if $f \in \mathfrak{D}(G)$ then clearly we must likewise have $|f|^{\#} \in \mathfrak{D}(G^{\#})$. Hence (6.2) follows from (6.1).

6.2.4. Proof of condition for strict inequality in Theorem 6.2

The reader is warned that this section is perhaps the most difficult and involved in the whole thesis.

In order to prove the condition for equality in Theorem 6.2, we first examine the properties of the functions f extremal for the Rayleigh quotient on the domain $G^{\#}$. (We say that f is **extremal** for $\mathcal{R}$ if the minimum of $\mathcal{R}$ is achieved at f.) By Theorem 6.3, an extremal f does exist, and may be taken to be positive.

From now on we assume that f is a positive eigenfunction corresponding to $\nu = \nu_1(G^{\#})$. Then, as in the work of the previous section, we see that

$$\mathcal{R}(f^{\#}) \leq \mathcal{R}(f).$$

But by extremality of f and Theorem 6.3 we conclude that $f^\#$ is also an eigenfunction corresponding to $\nu_1(G^\#)$, and so by Corollary 6.1 we see that f is a multiple of $f^\#$. Since f and $f^\#$ are equimeasurable, it follows that $f = f^\#$.

Hence f is symmetrically decreasing with respect to our spiral-like ordering. See Figures 6.3 and 6.4 for examples of two extremal subtrees $G^\#$ and the corresponding eigenfunctions f.

We need to improve the symmetric decrease condition to some sort of *strict* decrease condition. In the case where $G^\# = T_{n,k}$ for some k, this condition will in effect say that if $h(v) < h(w) \leq k$ then $f(v) > f(w) > 0$. In general, however, we cannot hope for this statement, since it fails, e.g., if G has precisely two points, or in the case of the tree in Figures 6.4.

Henceforth, suppose that $G^\#$ contains at least two points (otherwise Theorem 6.2 is trivial) and that f is positive. In light of Theorem 6.3, we have $0 < \nu < 1$. We may rewrite the equation $-\Delta f = \nu f$ on $G^\#$, then, as

$$f(v) = \frac{\kappa}{p} \sum_{w \in N(v)} f(w), \qquad \forall v \in G^\#, \tag{6.9}$$

where $\kappa = (1-\nu)^{-1} > 1$. It is clear that a particular consequence of (6.9) holding on $G^\#$ for a positive f and $\kappa \geq 1$ is that f is superharmonic on $G^\#$, since in that case we have $-\Delta f = \nu' f$ where $\nu' = 1 - \kappa^{-1}$. Hence, if a positive function $f \in \mathfrak{D}(G^\#)$ solves (6.9) for some $\kappa > 1$, then we necessarily have f everywhere strictly positive on $G^\#$.

We now give our characterization of the strictness of the decrease of our eigenfunction f. This characterization will in fact work for any positive solution $f \in \mathfrak{D}(G^\#)$ of (6.9), for any $\kappa > 1$, not just for $\kappa = (1-\nu)^{-1}$.

Proposition 6.1. *Suppose that $f \in \mathfrak{D}(G^\#)$ is positive and symmetrically decreasing and that there is some $\kappa > 1$ such that (6.9) holds. Suppose that v and w are points of T_p such that $v \prec w$, $v \in G^\#$ and $f(v) = f(w)$. Then either $v = O$ and $h(w) = 1$, or else w is not a descendant of v. In either case, there is an involution ρ of T_p interchanging w with v such that $f \circ \rho = f$, and, moreover, f is constant on* (Children v) $\cup$ (Children w).

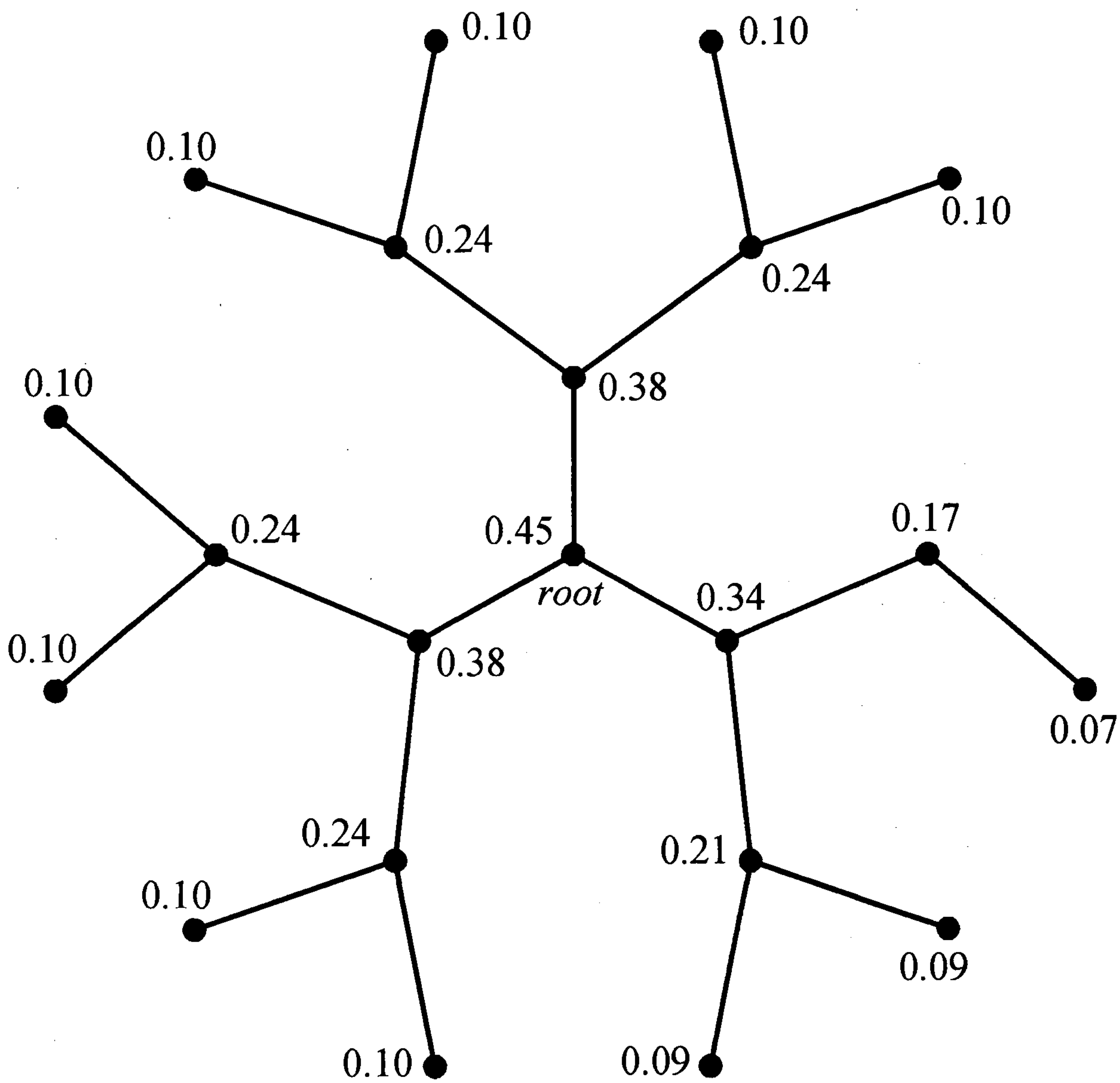


Figure 6.3: An extremal subtree $G^{\#} \subsetneq T_3$ with cardinality 21 and the eigenfunction f corresponding to the first non-zero eigenvalue of $-\Delta$. **Note:** The numerical values of f were computed with `Maple` and have been rounded off to two decimal places. However, it can be proved that for all the pairs of x and y that can be seen in the two displayed figures where $f(x)$ and $f(y)$ agree to two decimal places, we in fact have exactly $f(x) = f(y)$.

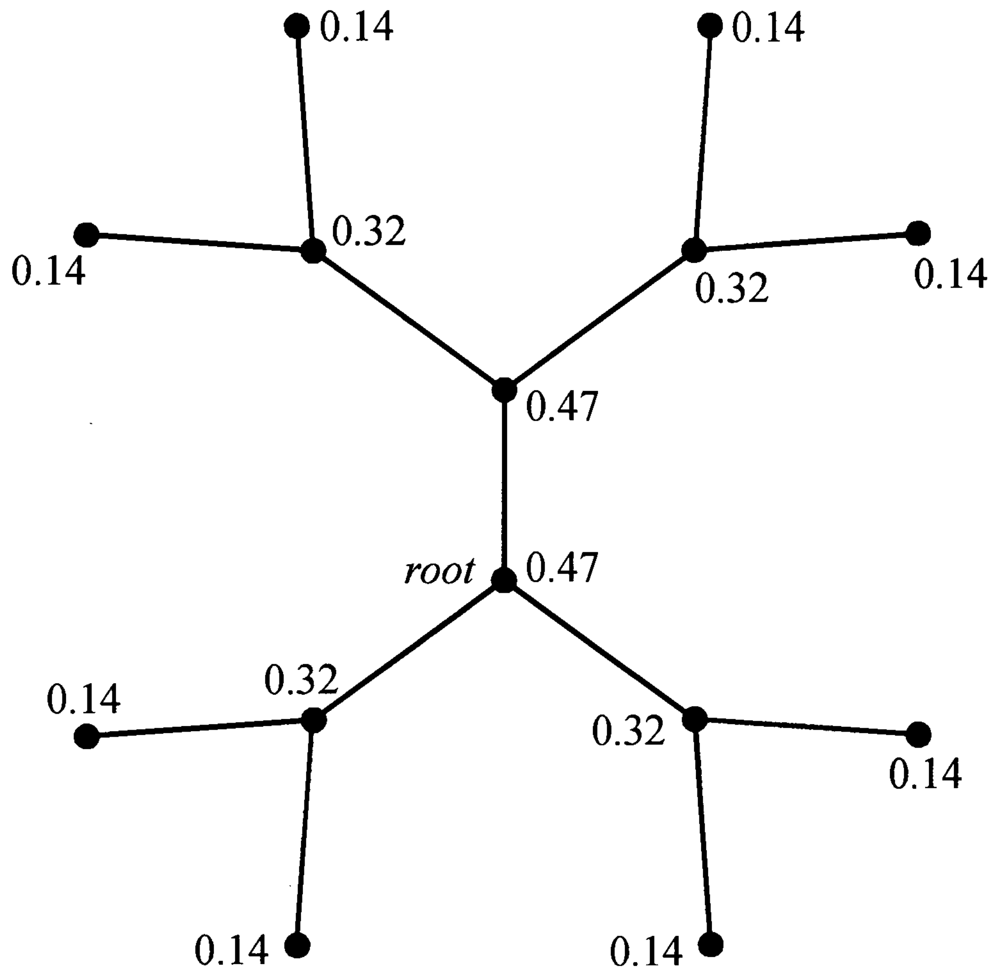


Figure 6.4: An extremal subtree $G^{\#} \subseteq T_3$ with cardinality 14 and once again with the eigenfunction f corresponding to the first non-zero eigenvalue of $-\Delta$. See the note attached to the previous figure.

The reader is encouraged to examine what this result asserts about the functions shown in Figures 6.3 and 6.4.

Corollary 6.2. *Suppose that* $f \in \mathfrak{D}(G^{\#})$ *is positive and symmetrically decreasing and that there is some* $\kappa > 1$ *such that we have* (6.9). *Then the maximum of* f *is attained on a set of cardinality at most* 2.

Proof of Corollary. The maximum of f is clearly attained at O. Suppose $f(O) = f(w)$ with $O \neq w$. Set $v = O$, and let ρ be as in Proposition 6.1. Now every point w' with $f(w') = f(O)$ must have $h(w') = 1$ by Proposition 6.1, and thus must be adjacent to O. Hence, if there exist at least three distinct points at which the maximum of f is attained, there must be at least two distinct points adjacent to O at which the maximum of f is attained. But $f \circ \rho = f$, and ρ preserves adjacency and satisfies $\rho O = w$, so that it follows that there are at least two distinct points adjacent to w at which the maximum of f is attained. But then at least one of these two points must be exactly a distance 2 from the root of T_p, which contradicts the fact that each point w' with $f(w') = f(O)$ has $h(w') = 1$. $\square$

We now give the proof of Proposition 6.1, broken up into four claims. First we note that by Corollary 6.1, f cannot vanish anywhere on $G^{\#}$, since $G^{\#}$ is clearly connected. Moreover if $w \notin G^{\#}$ then $f(w) = f(v) = 0$, and since $v \in G^{\#}$ it follows that $f \equiv 0$, which we assumed was not the case. Thus we may assume that $w \in G^{\#}$.

Claim A1. *Suppose that* w *is a descendant of* v. *Then* $v = O$ *and* $h(w) = 1$.

Proof of Claim A1. Let v' be the parent of w. We then have $v \preceq v' \preceq w$ so that $f(v) \geq f(v') \geq f(w)$. Since $f(v) = f(w)$ we must also have $f(v') = f(w)$. Replacing v by v' if necessary, then, we may assume that v is the parent of w. Let $w_1, \ldots, w_{p-1}$ be the children of w. If we can prove that $v = O$ then we will likewise have $h(w) = 1$ since w is a child of v. Hence, to obtain a contradiction assume that $v \neq O$. Let $v_1, \ldots, v_{p-1}$ be the children of v; note that

$w \in \{v_1, \ldots, v_{p-1}\}$. Then, by (6.9),

$$f(v) = \kappa p^{-1}\left(f(x) + \sum_{n=1}^{p-1} f(v_n)\right), \tag{6.10}$$

where x is the parent of v. Also,

$$f(w) = \kappa p^{-1}\left(f(v) + \sum_{n=1}^{p-1} f(w_n)\right). \tag{6.11}$$

But the symmetrically decreasing character of f ensures that

$$f(v) \leq f(x) \tag{6.12}$$

and

$$\max_n f(w_n) \leq \min_m f(v_m), \tag{6.13}$$

since $h(v) > h(x)$ and $h(w_n) = h(w) + 1 > h(w) = h(v_m)$, for all n and m. Yet, since $f(v) = f(w)$, it follows from (6.10)–(6.13) that we must in fact have

$$f(w) = f(v) = f(x)$$

and

$$f(w_n) = f(v_m), \qquad \forall n, m \in \{1, \ldots, p-1\}.$$

Moreover, $w \in \{v_1, \ldots, v_{p-1}\}$, so we have

$$\alpha \stackrel{\text{def}}{=} f(v) = f(x) = f(w) = f(w_n) = f(v_m)$$

for all n and m. Then, by (6.10),

$$\alpha = f(v) = \kappa p^{-1} p\alpha = \kappa\alpha,$$

and so $\alpha = 0$ as $\kappa > 1$. But we have already seen that f cannot vanish anywhere on $G^{\#}$ and so we have a contradiction as desired. $\square$

It now suffices to show that in general if $v \prec w$ and $f(v) = f(w)$ then there is an involution ρ with the desired properties, and that f is constant on (Children v) $\cup$ (Children w).

Claim A2. *We either have $h(w) = 1 + h(v)$ or $h(w) = h(v)$.*

Proof of Claim A2. If $h(w) > 1 + h(v)$, then let w' be the parent of w. Then $h(v) < h(w') < h(w)$ so that $v \prec w' \prec w$. We then have $f(v) \geq f(w') \geq f(w)$ as f is symmetric decreasing. But $f(v) = f(w)$, so it follows that $f(w') = f(w)$. By Claim A1, we see that $w' = O$. But this is impossible as $v \prec w'$. Hence, $h(w) \leq 1 + h(v)$. On the other hand $h(v) \leq h(w)$ as $v \prec w$, so that $h(w)$ must equal either $h(v)$ or $1 + h(v)$. □

We now make a few definitions. First, let A be the point of $[v, w]$ minimizing $h(A)$.

If $h(w) = h(v)$ then let $B = A$, and let A' and B' be those unique vertices of $[A, v]$ and $[A, w]$, respectively, which are also children of A. See Figure 6.5.

If $h(w) = 1 + h(v)$ then let B be the unique vertex of $[A, w]$ which is a child of A, and set $A' = A$ and $B' = B$. See Figure 6.6.

Note that in any case we have $d(A', v) = d(B', w)$.

Claim A3. *Let $x \in \operatorname{Desc} A'$ and $y \in \operatorname{Desc} B'$, with $d(x, A') = d(y, B')$. Then $f(x) = f(y)$.*

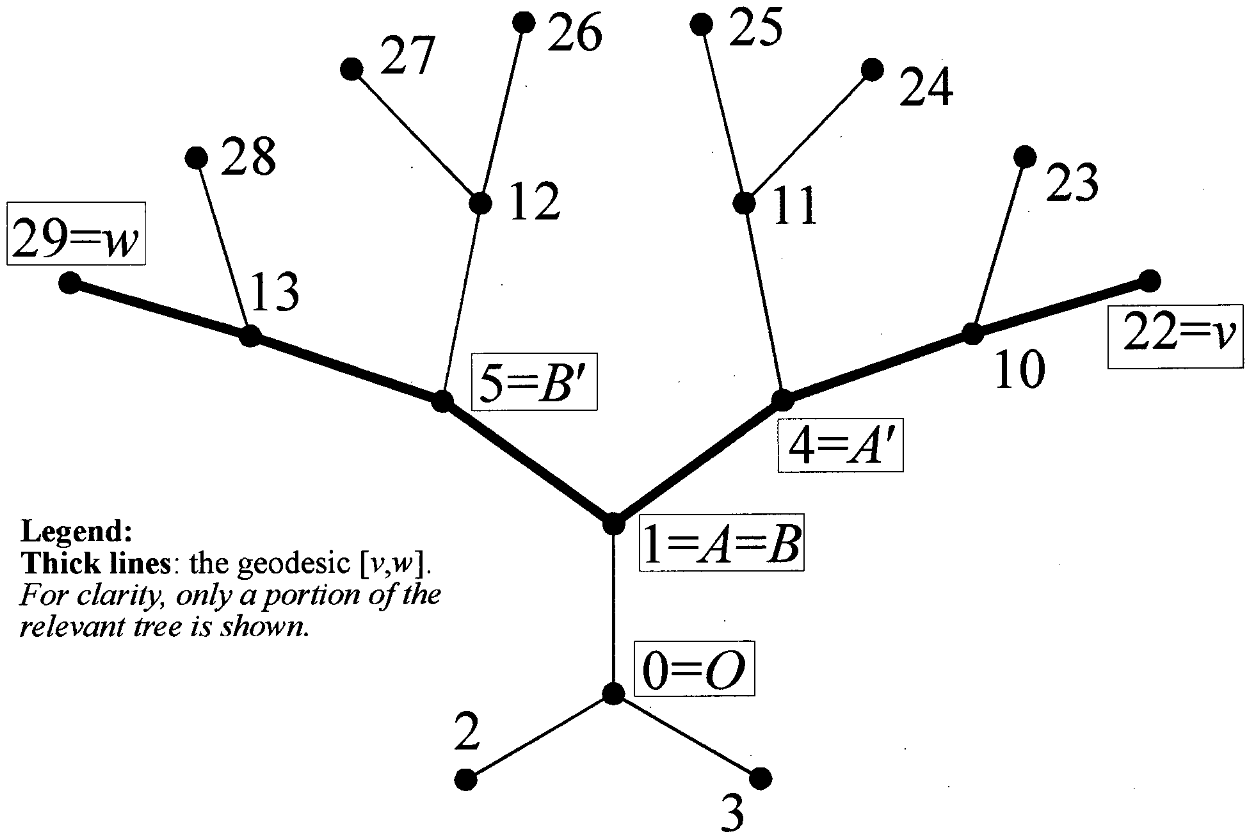
Completion of proof of Proposition 6.1 assuming Claim A3. Assume that the claim is just. The reader is advised to try to follow the proof along by looking at Figure 6.5 if $h(w) = h(v)$ and at Figure 6.6 if $h(w) = 1 + h(v)$.

Note now that if $h(w) = 1 + h(v)$ then B is a descendant of A, $A \in \operatorname{Desc} A'$, $B \in \operatorname{Desc} B'$ and $d(A, A') = d(B, B') = 0$, so that by the claim we have $f(A) = f(B)$. By Claim A1 it follows that $A = O$.

In any case, let

$$H = (\operatorname{Desc} A') \cup (\operatorname{Desc} B') \cup \{A\}.$$

It is easy to see (considering the cases $h(w) = h(v)$ and $h(w) = 1 + h(v)$ separately) that there is an involution ρ of H which interchanges v and w. Then, our involution ρ interchanges A'



Legend:
Thick lines: the geodesic [v,w].
For clarity, only a portion of the relevant tree is shown.

Figure 6.5: Definition of A, B, A' and B' in the case where $h(w) = h(v)$.

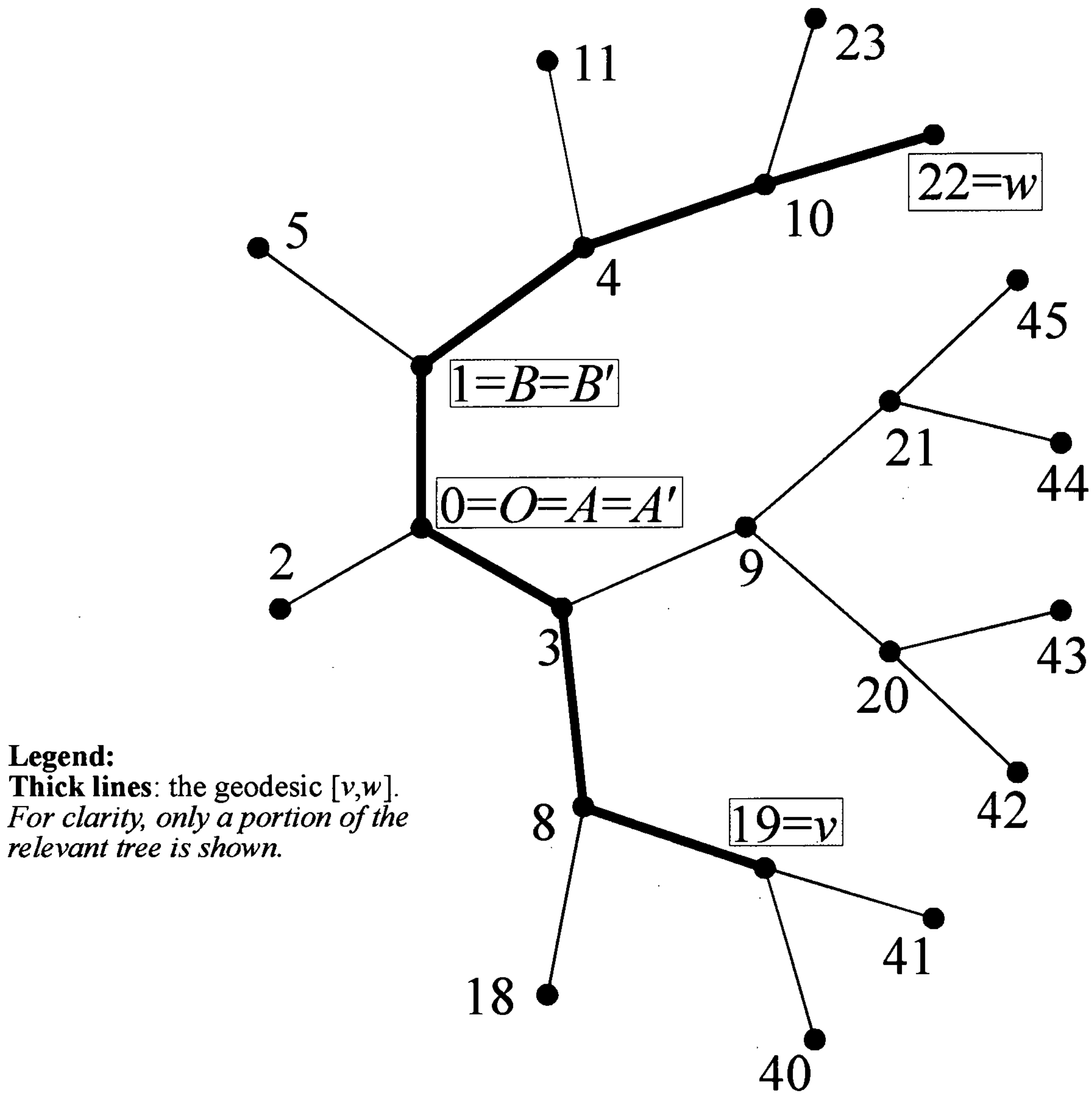


Figure 6.6: Definition of A, B, A' and B' in the case where $h(w) = 1 + h(v)$

with B' and $\operatorname{Desc} A'$ with $\operatorname{Desc} B'$. Let $x \in \operatorname{Desc} A'$. Let $y = \rho x$. Then, $d(x, A') = d(y, B')$ since ρ is an isomorphism and $\rho A' = B'$. Thus, by Claim A3 we have $f(x) = f(\rho x)$. On the other hand if $x \in \operatorname{Desc} B'$ then $\rho x \in \operatorname{Desc} A'$ so that by the above we have $f(\rho x) = f(\rho^2 x) = f(x)$ since ρ is an involution. Hence $f \circ \rho = f$ on $\operatorname{Desc} A' \cup \operatorname{Desc} B'$.

If $A \in \operatorname{Desc} A' \cup \operatorname{Desc} B'$ then we conclude immediately that $f \circ \rho = f$ on H. Otherwise, we must have $h(v) = h(w)$ and it is easy to see that ρ must then fix A, so that $f(\rho A) = f(A)$, and thus we also have $f \circ \rho = f$ on all of H.

Now suppose that $h(w) = 1 + h(v)$. Then as noted above, we have $A = O$ and so $H = T_p$ since $A' = A$ in this case while $\operatorname{Desc} O = T_p$. Thus, $f \circ \rho = f$ everywhere on T_p.

On the other hand, suppose that $h(w) = h(v)$. Then, $A = B$ and it is easy to see that $A \in \operatorname{Fix} \rho$ as ρ swaps A' and B' which are both adjacent to A. Extend ρ to all of T_p by setting $\rho x = x$ for $x \notin H$. It is not difficult to see that ρ is still an involutive graph isomorphism and satisfies $f \circ \rho = f$ everywhere on T_p.

Moreover, in any case, if x_1 and y_1 are children of x and y respectively then $d(x_1, A') = d(x, A') + 1 = d(y, A') + 1 = d(y_1, B')$, and so $f(x_1) = f(y_1)$ by Claim A3, as desired. $\square$

In order to prove Claim A3 we first formulate yet another claim.

Claim A4. *Suppose $O \prec V \prec W$ and $f(V) = f(W)$ for some $V \in G^\#$. Then $f(X) = f(Y)$ where X and Y are the parents of V and W respectively. Moreover, f is constant on* $(\text{Children } V) \cup (\text{Children } W)$.

Proof of Claim A4. If $W \notin G^\#$ then $f(W) = 0$ and so $f(V) = 0$, which is impossible as f does not vanish in $G^\#$. Hence, $W \in G^\#$.

We now employ an argument similar to the one given in the proof of Claim A1. Note that $X \prec Y$ so that $f(X) \geq f(Y)$. Let $V_1, \ldots, V_{p-1}$ and $W_1, \ldots, W_{p-1}$ be the children of V and W

respectively. Then for all m and n we have $V_m \prec W_n$ and so $f(V_m) \geq f(W_n)$. But, by (6.9) we see that $f(V)$ is κ times the average of

$$f(X), f(V_1), \ldots, f(V_{p-1}),$$

while $f(W)$ is κ times the average of

$$f(Y), f(W_1), \ldots, f(W_{p-1}).$$

Now $f(V) = f(W)$, and since $f(X) \geq f(Y)$ and $f(V_m) \geq f(W_n)$ for all m and n, it follows that we must have equality in all these inequalities so that $f(X) = f(Y)$ and $f(V_m) = f(W_n)$ for all m and n as desired. □

Finally we can now prove Claim A3 and thus finish our proof of Proposition 6.1.

Proof of Claim A3. Applying Claim A4 we see that $f(v') = f(w')$ where v' and w' are the parents of v and w respectively. If $v' \neq O$ and $v' \neq w'$ then we may apply Claim A4 again to see that $f(v'') = f(w'')$ where v'' and w'' are the parents of v' and w' respectively. Iterating this procedure, we will eventually conclude that $f(A') = f(B')$ since $d(A', v) = d(B', w)$ and $A' \in [0, v]$ while $B' \in [0, w]$.

To complete the proof, we proceed by induction. Suppose that it has been shown that $x \prec y$ with $x \in \operatorname{Desc} A'$ and $y \in \operatorname{Desc} B'$ and $d(x, A') = d(y, B') = n$ implies $f(x) = f(y)$. Indeed, this has been shown if $n = 0$. We shall show that then the desired relation holds if $d(x, A') = d(y, B') = n + 1 > 0$, and by induction we will have completed the proof of the claim and thus of the Proposition. But, if $d(x, A') = d(y, B') = n + 1 > 0$ and $x \prec y$, then let X be the parent of x and Y the parent of y. We will then have $X \prec Y$, and by the induction hypothesis $f(X) = f(Y)$ as $d(X, A') = d(x, A') - 1 = d(y, B') - 1 = d(Y, B')$. If $X \in G^{\#}$ then it follows by Claim A4 that f is be constant on $(\operatorname{Children} X) \cup (\operatorname{Children} Y)$, and since x and y fall into this set, we are done. But if $X \notin G^{\#}$ then $Y \notin G^{\#}$, and the symmetry of $G^{\#}$ then implies that x and y also fail to be in $G^{\#}$ so that $f(x) = 0 = f(y)$ and we are done. □

Given Proposition 6.1, we will now proceed to prove the condition for equality in Theorem 6.2. Again, we shall do this by breaking the proof up into several claims.

If G is isomorphic to $G^{\#}$ then it is clear that we have equality as desired. Now, assume that there is no automorphism of T_p which maps G onto $G^{\#}$. We must prove that $\nu_1(G) > \nu_1(G^{\#})$. Let $g \in \mathfrak{D}(G)$ be a minimizer of the Rayleigh quotient for G. Since $\mathcal{R}(g) \geq \mathcal{R}(|g|)$ we may assume that g has constant sign, replacing g by $-g$ we may assume that $g \geq 0$. Let $f = g^{\#}$. If f did not minimize the Rayleigh quotient for $G^{\#}$ then

$$\nu_1(G^{\#}) < \mathcal{R}(f) \leq \mathcal{R}(g) = \nu_1(G),$$

where the second inequality used (6.8), and so we would be done as we would have a strict inequality as desired. Hence assume that f does minimize the Rayleigh quotient for $G^{\#}$ and thus satisfies (6.9).

Then, since the supports of f and g are $G^{\#}$ and G, respectively, by our assumption there cannot be an automorphism ϕ of T_p such that $f = g \circ \phi$. Let S be a subset of $G^{\#}$ of maximal cardinality with the property that every ancestor of every element of S is also in S and there exists an automorphism ϕ of T_p with $f|_S = (g \circ \phi)|_S$. It is easy to see that S is non-empty, and in fact $O \in S$. Moreover, $S \neq G^{\#}$ since we have seen that there is no automorphism ψ of T_p with $f = g \circ \psi$.

Replacing G by $\phi[G]$ and g by $g \circ \phi$ (where ϕ is an automorphism of T_p such that $f|_S = (g \circ \phi)|_S$), we may assume that $f|_S = g|_S$ while for no set $S' \subseteq G^{\#}$ containing ancestors of all of its points and of strictly larger cardinality than S is there an automorphism ψ such that $f|_{S'} = (g \circ \psi)|_{S'}$.

Claim B1. *There exist points v, V, w and W of T_p and an involution $\rho \in \mathfrak{I}$ such that the following conditions are satisfied:*

(a) *V is the parent of v*

(b) *if w is a descendant of v then W is a child of w; if w is not a descendant of v then W is the parent of w*

(c) $v \prec w$ *and* $V \prec W$

(d) $\rho v = w$ *and* $\rho V = W$

(e) $g(v) < g(w)$ *and* $g(V) > g(W)$.

The proof of this shall be given later.

Claim B2. *Assume that* ρ, v, V, w *and* W *are as in Claim* B1. *Let* $\Phi(t) = t^2$. *Define* $K(t) = 1$ *for* $t \leq 1$ *and* $K(t) = 0$ *for* $t > 1$. *Then,*

$$Q(g, g; \Phi, K) > Q(g_\rho, g_\rho; \Phi, K).$$

Proof of Claim B2. Lemma 2.2 guarantees that

$$Q(g, g; \Phi, K) \geq Q(g_\rho, g_\rho; \Phi, K).$$

Let $x = v$ and $y = V$. We shall show that conditions (i) and (ii) of Lemma 2.2 are satisfied. Then the desired strict inequality will follow since $\Phi(t) = t^2$ is strictly convex.

The facts that $g(v) < g(w) = g(\rho v)$ and $g(V) > g(W) = g(\rho V)$ imply that (ii) holds.

Now $K(d(v, V)) = K(1) = 1$. If we could show that $d(v, \rho V) > 1$ then it would immediately follow that $K(d(v, \rho V)) = 0$ and (i) would necessarily hold, so that the proof of the claim would be complete.

Hence, to obtain a contradiction, suppose that $d(v, \rho V) \leq 1$. Now, $\rho V = W$. By condition (b) we have $v \neq W$. Thus, the only way we can have $d(v, W) \leq 1$ is if $d(v, W) = 1$, i.e., if W is either the parent of v or a child of v. Suppose first that W is a child of v. Then, since $v \neq w$ and W is adjacent to w, it follows that w is a descendant of v. But were w to be a descendant of v then W would have been a child of w, which would have made it impossible for W to be adjacent to v, since $v \neq w$. Suppose now that W is the parent of v. But then $W = V$, which is impossible since $V \prec W$. Thus in both cases we have a contradiction and the claim is proved. $\square$

Continue to assume Claim B1.

Proof of the condition for equality in Theorem 6.2. Following the proof of Theorem 2.2, we see that our strict inequality

$$Q(g, g; \Phi, K) > Q(g_\rho, g_\rho; \Phi, K)$$

implies the strict inequality

$$\sum_{x,y \in T_p} g(x) K(d(x,y)) g(y) < \sum_{x,y \in T_p} g_\rho(x) K(d(x,y)) g_\rho(y).$$

But, on the other hand, Theorem 2.2 says that

$$\sum_{x,y \in T_p} g_\rho(x) K(d(x,y)) g_\rho(y) \le \sum_{x,y \in T_p} (g_\rho)^\#(x) K(d(x,y)) (g_\rho)^\#(y),$$

and the equimeasurability of g_ρ with g implies that $(g_\rho)^\# = g^\# = f$. Hence,

$$\sum_{x,y \in T_p} g(x) K(d(x,y)) g(y) < \sum_{x,y \in T_p} f(x) K(d(x,y)) f(y).$$

But by (6.6) and (6.7) it then follows that

$$\nu_1(G) = \mathcal{R}(g) > \mathcal{R}(f) = \nu_1(G^\#),$$

as desired. □

All that remains to be proved is Claim B1. Let v be the smallest (with respect to $\prec$) element of $G^\# \backslash S$. Let w be an element of $G^\# \backslash S$ such that $f(v) = g(w)$ (such a w exists because $f = g^\#$ are equimeasurable while $f|_S = g|_S$). We have $f(v) \neq g(v)$ since if $f(v) = g(v)$ then we could set $S' = S \cup \{v\}$ and we would have $f|_{S'} = g|_{S'}$. The minimality and choice of v would then ensure that $|S'| > |S|$ and that all ancestors of elements of S' are in S, thereby yielding a contradiction.

Then, we must have $v \prec w$ (we cannot have $v = w$ since $f(v) \neq g(v)$ and we cannot have $w \prec v$ because of the minimality of v).

Claim B3. *We have $f(v) > f(w)$.*

Proof. To obtain a contradiction, suppose instead that $f(v) \leq f(w)$. But $v \prec w$ so that $f(v) \geq f(w)$ and so $f(v) = f(w)$. Then, by Proposition 6.1 there exists an involution ρ swapping v and w, and satisfying $f \circ \rho = f$. Then, $f(x) = f(\rho x) = g(\rho x)$ for every $x \in \rho S$ since $g|_S = f|_S$ and $\rho = \rho^{-1}$. Moreover, $g(\rho v) = g(w) = f(v)$. Then, $(g \circ \rho)|_{S'} = f|_{S'}$, where $S' = \{v\} \cup \rho S$. We have $|S'| > |S|$ since $v \notin \rho S$ as $\rho v = w \notin S$.

If we can prove that the ancestor of every element of S' lies in S' then we will have obtained a contradiction to the maximal cardinality of S. (It is clear that we must have $\rho S \subseteq G^{\#}$ since f does not vanish on S, hence $f = f \circ \rho$ does not vanish on ρS, while the support of f is precisely $G^{\#}$.)

We now prove the above statement about ancestors of elements of S'. Let $x \in S'$. First consider the case where $x \in \rho S$ so that $\rho x \in S$. We must show that if $x \neq O$ and X is the parent of x, then $X \in \rho S$. But, if $X \notin \rho S$ then $\rho X \notin S$. Since X is adjacent to x, we have ρX adjacent to ρx. As the parent of ρx must lie in S, it follows that ρX is not the parent of ρx, but must instead be a child of it. Thus, $f(\rho X) \leq f(\rho x)$. But $f = f \circ \rho$ so that $f(\rho X) = f(X)$ and $f(\rho x) = f(x)$, while, since X is the parent of x, we have $f(X) \geq f(x)$. Thus, $f(\rho X) = f(\rho x) = f(X) = f(x)$. From Proposition 6.1 it follows that $X = O$ and $h(x) = 1$, and likewise that $\rho x = O$ and $h(\rho X) = 1$. Since f evidently thus attains its maximum at x and at ρX and also at O, while $O \notin \{x, \rho X\}$, it follows from Corollary 6.2 that $x = \rho X$. Thus, $\rho O = \rho^2 x = \rho X \notin S$. Since, $f(\rho O) = f(O)$ is the maximum of f while $\rho O \notin S$, it follows from the minimality of v that $v \preceq \rho O$ so that $f(v) = f(\rho O)$ likewise. But since the maximum of f is attained on a set of cardinality at most 2 and $v \neq O$, it follows that in fact $\rho O = v$. Hence, $O = \rho v = w$, which contradicts the choice of $w \in G^{\#} \backslash S$ since $O \in S$.

It only now remains to show that the parent of x lies in S' if $x = v$. Let V be the parent of v. If $V \notin \rho S$ then $\rho V \notin S$. The minimality of v shows that we have $v \preceq \rho V$. Thus, $f(v) \geq f(\rho V)$. But $f(\rho V) = f(V)$ as $f = f \circ \rho$. Thus, $f(v) \geq f(V)$. Since V is the parent of v it follows that $f(v) \leq f(V)$ so that $f(v) = f(V)$, and then Proposition 6.1 implies that $V = O$. But we have assumed that $f(v) = f(w)$ and there are at most two points at which f attains its maximum, while it evidently attains it at $V = O$, v and w, so that $w = v$ or $w = O$, and either option immediately yields a contradiction. $\square$

Completion of proof of Claim B1. Let V be the parent of v. Note that $V \in S$ by minimality of v. If w is a descendant of v then let W be any child of w; otherwise, let W be the parent of w. Thus conditions (a) and (b) are satisfied.

I claim that $V \neq W$. If w is a descendant of v then this is easy. Otherwise, suppose that V is the parent of both v and w. Then it is easy to see that there exists an involution ρ which fixes S (use here the fact that ancestors of elements of S lie in S) but swaps v and w. Then, we have $f|_{S\cup\{v\}} = (g \circ \rho)|_{S\cup\{v\}}$. But the parent of v lies in S by minimality of v, and so we have a contradiction to the maximal cardinality of S. Hence indeed $V \neq W$. It is clear that from the construction of V and W we obtain the fact that $V \preceq W$ as $v \prec w$. Thus, $V \prec W$ and so condition (c) is satisfied.

I further claim now that we must then have $f(V) > f(W)$. To prove this, note that since $V \prec W$, we must have $f(V) \geq f(W)$. If W is the parent of w and if equality holds here, then by Proposition 6.1 we likewise have $f(v) = f(w)$, since v and w are children of V and W, respectively. But we have seen that $f(v) > f(w)$ and so $f(V)$ cannot equal $f(W)$ in this case. On the other hand if W is a child of w then $h(W) > h(w) \geq h(v) > h(V)$ and so if $f(W) = f(V)$ then $f(W) = f(v)$ since $f(W) \leq f(v) \leq f(V)$. Since in our case W is a descendant of v, it follows from Proposition 6.1 that $v = O$, a contradiction. Hence indeed $f(V) > f(W)$.

Now since V is the parent of v, the minimality of $v \in G^{\#}\backslash S$ implies that $V \in S$ so that $g(V) = f(V)$. I now claim that $g(V) > g(W)$. If $W \in S$ then $g(V) = f(V) > f(W) = g(W)$ and we are done. Suppose thus that $W \notin S$. Since $f|_S = g|_S$ and $f = g^{\#}$, there must be a $W' \notin S$ such that $f(W') = g(W)$. By minimality of v and the choice of V we will have $V \prec v \preceq W'$. If $g(W) \geq g(V)$ then since $g(V) = f(V) \geq f(v) \geq f(W') = g(W)$, we must in fact have $f(V) = f(v) = f(W')$. But v is a descendant of V and so by Proposition 6.1, then, we must in fact have $V = O$. Then, $f(O) = f(v) = f(W')$, and so by Corollary 6.2 we have

$W' = v$. Thus $g(W) = f(W') = g(w) = g(O)$, since $g(w) = f(v)$ and $g(O) = f(O)$. But the points O, w and W are distinct. Hence g attains the value $f(O)$ at three or more vertices of T_p. By equimeasurability, f also attains the value $f(O)$ at three or more vertices, which contradicts Corollary 6.2. Hence, indeed, $g(V) > g(W)$.

On the other hand it is easy to see that $g(v) < f(v)$ by the choice of v and the fact that $f = g^{\#}$. But $f(v) = g(w)$ so that we have $g(v) < g(w)$. We have thus verified condition (e).

If w is a descendant of v then let ρ be an involution of T_p which interchanges W and V; considering ρ restricted to $[V, W]$ we see that it must also interchange v with w. If w is not a descendant of v then let ρ be an involution of T_p which interchanges w and v. It can be seen in this case that ρ must interchange W with V since in this case we must have both W and V contained in $[v, w]$. Thus in any case condition (d) is satisfied. □

7. Some open problems and two counterexamples

We may define a number of classes of graphs depending on which, if any, of the inequalities and conditions considered in §2 hold. Throughout this section, when we speak of an "ordering" we shall mean a "well-ordering such that every element has at most finitely many predecessors." Let G be a graph with an ordering $\prec$. Let $K_1(t) = 1$ if $t \leq 1$ and let $K_1(t) = 0$ otherwise. Consider the following properties of G under the ordering $\prec$:

(A) Condition A holds

(B) Condition B holds

(C) For all positive f and g we have (2.4) for all decreasing positive K

(D') For all positive f and g which are similarly ordered we have (2.4) for all decreasing positive K

(D) For all positive f and $g = f$ we have (2.4) for all decreasing positive K

(M) The master inequality holds

(c) For all positive f and g we have (2.4) for $K = K_1$

(d') For all positive f and g which are similarly ordered we have (2.4) for $K = K_1$

(d) For all positive f and $g = f$ we have (2.4) for $K = K_1$

(m) Inequality (2.1) holds for all real f and g, and all convex increasing Φ, providing $K = K_1$

Let $\mathfrak{G}_P$ be the collection of all graphs for which there exists an ordering $\prec$ under which condition P holds. In light of the results of §2 and some trivial implications, we have:

$$\mathfrak{G}_A \subseteq \mathfrak{G}_M, \tag{7.1}$$

$$\mathfrak{G}_M \cap \mathfrak{G}_B \subseteq \mathfrak{G}_C \subseteq \mathfrak{G}_{D'} \subseteq \mathfrak{G}_D, \tag{7.2}$$

$$\mathfrak{G}_m \cap \mathfrak{G}_B \subseteq \mathfrak{G}_c \subseteq \mathfrak{G}_{d'} \subseteq \mathfrak{G}_d \tag{7.3}$$

and

$$\mathfrak{G}_Q \subseteq \mathfrak{G}_q, \tag{7.4}$$

where Q is C, D', D or M, respectively, while q is c, d', d or m, respectively.

(The first inclusion in (7.3) does not follow directly from Theorem 2.2 but rather from its proof.) It is not known which, if any, of the inclusions in (7.1)–(7.4) can be reversed. In the few examples known to the author, either the graph has all of the properties (A)–(m) or it has none, but the author suspects that this will not be true in general.

Open Problem 7.1. Which, if any, of the inclusions in (7.1)–(7.4) are strict, and which are not?

Open Problem 7.2. Classify the graphs (or at least all finite graphs) lying in the various classes $\mathfrak{G}_P$, where P ranges over the properties (A)–(m). In particular, determine what graphs lie in $\mathfrak{G}_A \cap \mathfrak{G}_B$; also, determine what graphs lie in $\mathfrak{G}_d$.

Condition (d) is of some interest in that it implies the existence of a Faber-Krahn type inequality of the form of inequality (6.2) of §6.2.1. However, we do not know whether the condition for equality given in Theorem 6.2 has universal validity whenever (d) holds.

Open Problem 7.3. Given a graph H with an ordering $\prec$ such that (d) holds, and given a subset G such that $\nu_1(G) = \nu_1(G^{\#})$, must we then have G and $G^{\#}$ isomorphic? What if we additionally require that some of the more stringent conditions from among (A)–(m) hold?

One may also ask the above questions restricted to the class of constant degree graphs, or even to the class of regular graphs (a graph is said to be **regular** if its automorphism group is transitive on vertices).

Note that one can find examples of very nice regular graphs which do not have any ordering under which property (d) holds.

Theorem 7.1. *Let G be either the cube $\mathbb{Z}_2^3$ or the ternary plane $\mathbb{Z}_3^2$. Then there exists no ordering of the graph G under which G has property (d).*

The cube graph $\mathbb{Z}_2^3$ is defined to have as vertices all triples (a, b, c) with a, b and c in $\mathbb{Z}_2$, where two vertices are defined to be adjacent if and only if they differ in precisely one coordinate. The ternary plane graph $\mathbb{Z}_3^2$ is defined to have as vertices all pairs (a, b) with a and b in $\mathbb{Z}_3$, where two distinct pairs (a, b) and (a', b') are said to be adjacent providing $a = a'$ or $b = b'$. (See Figure 7.1.)

Remark 7.1. Let G be one of the two graphs in Theorem 7.1. Then clearly $G \in \mathfrak{G}_B$. By (7.1)–(7.4) it follows that there is no ordering under which G has any of the properties (A)–(m) other than (B).

One could presumably produce a formal paper-and-pencil proof of Theorem 7.1. Instead, however, the author programmed a computer to generate basically all possible orderings on the given graph, and then for each ordering the computer produced enough randomly generated functions f to give a counterexample to property (d). More details will be given below. It

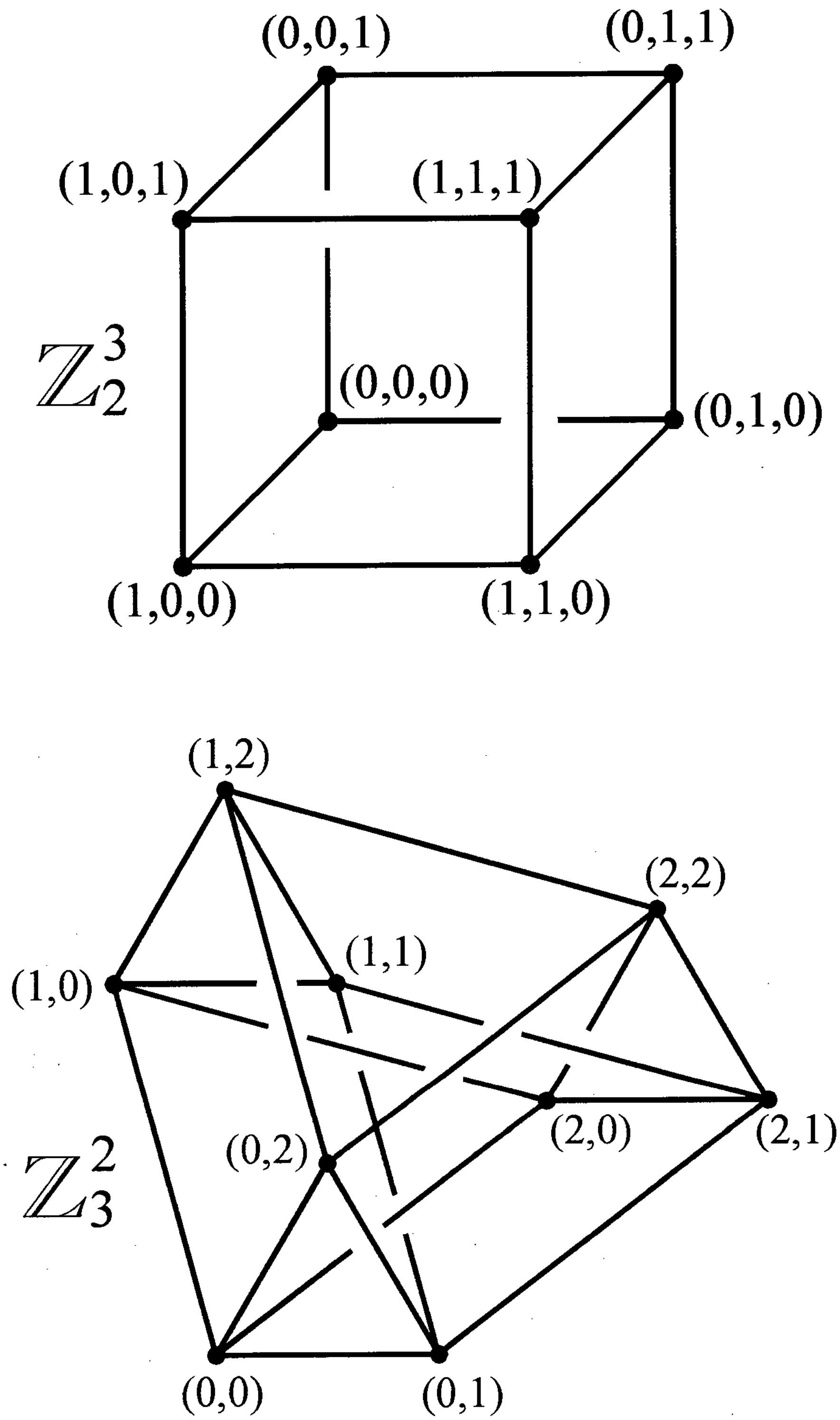


Figure 7.1: The cube $\mathbb{Z}_2^3$ and the ternary plane $\mathbb{Z}_3^2$.

is to be emphasized that, assuming the correctness of the author's simple computer program `cubetern.c` given in Appendix A, the results are exact, and in principle a human being could check that each counterexample is indeed a counterexample. In the next section we will outline how this program works.

Open Problem 7.4. Let P be one of the properties (A)–(m) other than (B). Consider the problem of determining whether a given finite graph G is in $\mathfrak{G}_P$. Is this problem NP-complete?

7.1. How the computer proved Theorem 7.1

Let G be one of the two graphs from Theorem 7.1. In order to prove Theorem 7.1, we must show that for any ordering $\prec$ on G there exists a positive function f on G (depending on the ordering) such that

$$\sum_{v\in G} f(v)Nf(v) > \sum_{v\in G} f^{\#}(v)Nf^{\#}(v), \tag{7.5}$$

where

$$Ng(v) \stackrel{\text{def}}{=} \sum_{w\in N(v)} f(w),$$

and where $\#$ is the rearrangement induced by the ordering $\prec$ while $N(v)$ is the collection of all vertices adjacent to v. To see that this suffices, it is only necessary to note that in such a case if $K_1(t)$ is 1 for $t \leq 1$ and 0 otherwise, then

$$\sum_{v,w\in G} f(v)K_1(d(v,w))f(w) = \sum_{v,w\in G} f(v)Nf(v) + \sum_{v\in G} f^2(v).$$

Since the second summation on the right hand side of this expression is invariant under replacement of f with $f^{\#}$ by equimeasurability, it will follow from (7.5) that

$$\sum_{v,w\in G} f(v)K_1(d(v,w))f(w) > \sum_{v\in G} f^{\#}(v)K_1(d(v,w))f^{\#}(w),$$

which says precisely that (2.4) fails for $g = f$ and $K = K_1$, as desired.

If $G = \mathbb{Z}_2^3$ then let $O = (0,0,0)$ and if $G = \mathbb{Z}_3^2$ then let $O = (0,0)$. What one must prove is that for every ordering $\prec$ there exists a function f such that (7.5) holds. However, because both of our graphs G have the property that all vertices are equivalent (i.e., both graphs are regular so that their automorphism groups are transitive on vertices), it follows by this symmetry property that we need only examine orderings $\prec$ such that O is the $\prec$-initial element. There will be $(|G|-1)!$ such orderings, which number equals 5 040 in the case of the cube graph and 40 320 in the case of the ternary plane.

The computer program `cubetern.c` (see Appendix A) proceeds by looping through all of the $(|G|-1)!$ orderings mentioned above. For each ordering, the program generates pseudorandom functions f (via the built-in Borland `Turbo C++ 3.0` random number generator `rand()`, seeded at the beginning of the program with the arbitrary value 317) with values in $\{0, 1, \ldots, 19\}$. Then, the program checks whether (7.5) holds. (This is an exact computation since the functions are integer valued.) If it does hold, then we have the requisite counterexample for the current ordering. If it does not hold, then we simply keep on generating more pseudorandom functions f as above, until one is found for which (7.5) holds. Of course, in principle one might never find such a function and in such a case we neither have a proof of Theorem 7.1 nor of its negation. However, as it turned out, the program did find such a counterexample for every ordering (for both choices of G) and this shows that Theorem 7.1 is just (assuming correct functioning of the software and hardware).

In the case of the cube, the largest number of tries to find an f satisfying (7.5) happened to be 799. In the case of the ternary plane, this happened to be 777. On a `486sx/20` system under `MS-DOS 5.0`, compiling the code in the `tiny` model under Borland International's `Turbo C++ 3.0` with all speed optimizations enabled gave a run time of 8 seconds for the case of the cube and 60 seconds for the case of the ternary plane.

8. Discrete Schwarz and Steiner type rearrangements

8.1. Basic definitions and results

Let X and Y be countable sets equipped with counting measures and let $\#$ be a rearrangement from the power set 2^X to the power set 2^Y. The term "measure preserving" shall refer to counting measure. We wish to study some general types of discrete rearrangements.

Definition 8.1. The rearrangement $\#$ is said to be of **(discrete) Schwarz type** if it is measure preserving and $A^{\#} = B^{\#}$ whenever $|A| = |B|$ and $A, B \subseteq X$.

Example 8.1. A prototypical example of a discrete Schwarz type rearrangement is the decreasing rearrangement $*$ on $\mathbb{Z}_0^+$ defined in Example I.2.2.

Remark 8.1. Let f and g be equimeasurable functions on X. Then, $f^{\#} = g^{\#}$ if $\#$ is of Schwarz type. For,

$$(f^{\#})_\lambda = (f_\lambda)^{\#} = (g_\lambda)^{\#} = (g^{\#})_\lambda.$$

(The first and last equalities follow from Theorem I.2.1. The middle equality follows from the fact that f_λ and g_λ have the same cardinality because of equimeasurability, and thus have the same rearrangement because of the Schwarz property of $\#$.)

Proposition 8.1. *A Schwarz type rearrangement $\#$ from 2^X to 2^Y induces a unique well ordering $\prec$ on Y with the properties that every element of Y has at most finitely many predecessors and if $y_0 \prec y_1 \prec y_2 \prec \cdots$ is an ordered enumeration of Y then*

$$S^{\#} = \{y_i : i < |S|\}$$

for every $S \subseteq X$.

Proof. Suppose that we have two such orderings $\prec^1$ and $\prec^2$, which induce the enumerations $y_0^j \prec y_1^j \prec y_2^j \prec \cdots$ of Y for $j = 1, 2$, respectively. To obtain the identity between $\prec^1$ and $\prec^2$ it suffices to show that

$$\{y_i^1 : i < n\} = \{y_i^2 : i < n\}$$

for all $n \leq |Y| = |X^{\#}| = |X|$. But to do this for some fixed n, choose a set $S \subseteq X$ with cardinality n, and then the two above quantities will necessarily be equal, since they will be both equal to $S^{\#}$.

Hence we now need only show the existence of such an ordering. We define the ordering as follows. Fix a non-negative integer $i < |X|$. Let S be a subset of X with cardinality $i + 1$

and let $x \in S$. Let $S' = S \backslash \{x\}$. Then $S' \subset S$ so that $(S')^{\#} \subset S^{\#}$, and the latter inclusion is proper since the former is proper and since S and S' are finite while $\#$ is measure preserving. Moreover, $|(S')^{\#}| = |S^{\#}| - 1$. Thus there is a unique element $y_i \in S^{\#} \backslash (S')^{\#}$. Because $S^{\#}$ and $(S')^{\#}$ depend only on the cardinality of S and S', it follows that whenever S and S' are arbitrary sets of cardinalities $i+1$ and i respectively, then $S^{\#} \backslash (S')^{\#} = \{y_i\}$.

I now claim that if S is a finite subset of X then

$$S^{\#} = \{y_i : i < |S|\}. \tag{8.1}$$

This is trivial if $|S| = 0$. Hence assume that $|S| \geq 1$. Write $S = \cup_{i=0}^{|S|} S_i$, where the set S_i has cardinality i for $i = 0, \ldots, |S|$ and where $S_0 \subset S_1 \subset \cdots \subset S_{|S|}$. Then $(S_{i+1})^{\#} \backslash (S_i)^{\#} = \{y_i\}$ because of the remarks in the previous paragraph and because of the cardinalities of S_{i+1} and S_i. Since $\#$ preserves inclusions, it follows that $(S_0)^{\#} \subset (S_1)^{\#} \subset \cdots \subset (S_{|S|})^{\#}$. Thus, since $(S_0)^{\#} = \varnothing$, we must have

$$S^{\#} = (S_{|S|})^{\#} = \bigcup_{i=0}^{|S|} [(S_{i+1})^{\#} \backslash (S_i)^{\#}] = \{y_0, y_1, \ldots, y_{|S|-1}\}$$

as desired. We now verify that (8.1) also holds if S is an infinite subset of X. For, then, let $S_1 \subset S_2 \subset S_3 \subset \cdots$ be a collection of subsets of S whose union equals S and which satisfy $|S_n| = n$ for all $n \in \mathbb{Z}^+$. By the definition of a rearrangement we have

$$S^{\#} = \bigcup_{n=1}^{\infty} (S_n)^{\#}.$$

But

$$(S_n)^{\#} = \{y_0, \ldots, y_{n-1}\}.$$

Thus,

$$S^{\#} = \{y_0, y_1, y_2, \ldots\}$$

as desired and (8.1) holds.

Applying (8.1) with $S = X$, we see that $y_0, y_1, \ldots$ is an enumeration of Y. Defining the well-ordering $\prec$ by $y_0 \prec y_1 \prec y_2 \prec \cdots$ we are done. $\square$

Definition 8.2. The rearrangement $\#$ is said to be of **(discrete) Steiner type** if it is measure preserving and there exists a countable collection $\mathcal{A}$ of disjoint nonempty subsets A of X such that:

(i) $\bigcup \mathcal{A} = X$

(ii) for any $A \in \mathcal{A}$ the rearrangement $\#$ restricted to 2^A is of Schwarz type

(iii) if A and B are two distinct elements of $\mathcal{A}$ then $A^\#$ and $B^\#$ are disjoint.

We call $\mathcal{A}$ the **fibres** of $\#$, and we call any set $A \in \mathcal{A}$ a **fibre**.

The following proposition describes how Steiner rearrangement decomposes into Schwarz rearrangements.

Proposition 8.2. *Let $\#$ be of Steiner type and let $\mathcal{A}$ be its fibres. Then for every $S \subseteq X$ we have*

$$S = \bigcup_{A \in \mathcal{A}} (S \cap A)^\#. \tag{8.2}$$

In particular,

$$Y = \bigcup_{A \in \mathcal{A}} A^\#.$$

Proof. Let $T = \bigcup_{A \in \mathcal{A}} (S \cap A)^\#$. First we note that $T \subseteq S^\#$ since $(S \cap A)^\# \subseteq S^\#$ as $S \cap A \subseteq S$ and $\#$ preserves inclusions.

We now verify the opposite inclusion in the case of a finite set S. If S is finite, by condition (i) in the definition of Steiner type we have a finite collection $A_1, \ldots, A_n$ of distinct (hence, disjoint) elements of $\mathcal{A}$ such that

$$S = (S \cap A_1) \cup \cdots \cup (S \cap A_n).$$

Let

$$T' = (S \cap A_1)^\# \cup \cdots \cup (S \cap A_n)^\#.$$

The sets $(S \cap A_1)^{\#}, \ldots, (S \cap A_n)^{\#}$ are disjoint since $A_1^{\#}, \ldots, A_n^{\#}$ are disjoint. Moreover, the cardinality of $S \cap A_i$ equals the cardinality of $(S \cap A_i)^{\#}$. Hence, the cardinality of S equals the cardinality of T', while of course the cardinality of $S^{\#}$ equals that of S. But $T' \subseteq T \subseteq S^{\#}$. Since $|T'| = |S^{\#}| < \infty$ it follows that $T' = T = S^{\#}$, as desired.

Now consider an infinite set S. Write $S = \bigcup_{n=1}^{\infty} S_n$, where each S_n is finite and where $S_1 \subseteq S_2 \subseteq \cdots$. We then have

$$S^{\#} = \bigcup_{n=1}^{\infty} (S_n)^{\#},$$

since by definition rearrangements preserve countable increasing unions. But

$$(S_n)^{\#} = \bigcup_{A \in \mathcal{A}} (S_n \cap A)^{\#}.$$

Thus,

$$S^{\#} = \bigcup_{A \in \mathcal{A}} \bigcup_{n=1}^{\infty} (S_n \cap A)^{\#}.$$

But $S_1 \cap A \subseteq S_2 \cap A \subseteq \cdots$ and the union of these sets is $S \cap A$, so that

$$\bigcup_{n=1}^{\infty} (S_n \cap A)^{\#} = (S \cap A)^{\#},$$

by the increasing union preservation property of rearrangements. Thus,

$$S^{\#} = \bigcup_{A \in \mathcal{A}} (S \cap A)^{\#},$$

as desired.

The last statement of the Proposition follows from applying the above with $S = X$ and recalling that $X^{\#} = Y$ by Definition I.2.2. □

Definition 8.3. A subset S of Y is said to be **symmetric** if there is a subset T of X such that $T^{\#} = S$. A function f on Y is said to be **symmetric** if there exists a function g on X with $f = g^{\#}$.

Corollary 8.1. *Let $\#$ be of Steiner type, and let $\mathcal{A}$ be its fibres. Then a set $S \subseteq Y$ is symmetric if and only if for every $A \in \mathcal{A}$ the set $S \cap A^{\#}$ is symmetric.*

Proof. Suppose S is symmetric. Then, $S = T^{\#}$ for some $T \subseteq X$. Then, by Proposition 8.2, since the $A^{\#}$ are disjoint, we have

$$S \cap A^{\#} = T^{\#} \cap A^{\#} = (T \cap A)^{\#},$$

and so $S \cap A^{\#}$ s symmetric. Conversely, suppose that for each $A \in \mathcal{A}$ there exists $T_A \subseteq X$ such that

$$T_A^{\#} = S \cap A^{\#}.$$

We must necessarily have $T_A \subseteq A$ because of the disjointness of the $A^{\#}$. Let $T = \bigcup_{A \in \mathcal{A}} T_A$. Then, by Proposition 8.2 we have

$$T^{\#} = \bigcup_{A \in \mathcal{A}} (T \cap A)^{\#} = \bigcup_{A \in \mathcal{A}} T_A^{\#} = \bigcup_{A \in \mathcal{A}} (S \cap A^{\#}) = S,$$

so that S is symmetric as desired. $\square$

The following result shows that the fibres of a Steiner type rearrangement are uniquely determined.

Proposition 8.3. *Let $\#$ be of Steiner type. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of disjoint nonempty subsets of X such that conditions* (i)–(iii) *in the definition of a Steiner type rearrangement are satisfied both for $\mathcal{A}$ and for $\mathcal{B}$. Then $\mathcal{A} = \mathcal{B}$.*

Proof. It clearly suffices to show that $\mathcal{A} \subseteq \mathcal{B}$, since the opposite inclusion then follows upon interchange of $\mathcal{A}$ and $\mathcal{B}$. Let $A \in \mathcal{A}$. Suppose that $A \notin \mathcal{B}$. Now, $A \subseteq X = \bigcup \mathcal{B}$. Thus, we can cover A with sets from $\mathcal{B}$. Since $A \notin \mathcal{B}$, it follows that one of the following two possibilities must hold:

(a) there is a set $B \in \mathcal{B}$ such that $A \cap B \neq \varnothing$ and $B \nsubseteq A$

(b) there is a set $B \in \mathcal{B}$ which is a strict subset of $\mathcal{A}$.

Suppose first that (a) holds, and let B be a set as in (a). Let $x \in A \cap B$ and let $y \in B \backslash A$. Consider the set $S = \{x, y\}$. Since $y \notin A$ and $\mathcal{A}$ covers X, it follows that there exists $A' \in \mathcal{A}$

such that $y \in A'$. In light of Proposition 8.2 applied with the collection $\mathcal{A}$, we see that

$$S^{\#} = (S \cap A)^{\#} \cup (S \cap A')^{\#} = \{x\}^{\#} \cup \{y\}^{\#}$$

(since $\varnothing^{\#} = \varnothing$, all the terms in the union (8.2) drop out except for $S \cap A$ and $S \cap A'$).

Moreover, since S has cardinality 2 and our rearrangement was assumed to be measure preserving, it follows that $S^{\#}$ has cardinality 2. On the other hand, $\{x\}$ and $\{y\}$ are both subsets of $B \in \mathcal{B}$, and by condition (ii) in the definition of a Steiner type rearrangement we must have $\{x\}^{\#} = \{y\}^{\#}$ since a Schwarz rearrangement of a set depends only on the cardinality of a set, while $|\{x\}| = 1 = |\{y\}|$. Hence, $\{x\}^{\#} \cup \{y\}^{\#}$ has cardinality 1, contradicting the previously noted fact that $|S^{\#}| = 2$.

Now assume that (b) holds. Then since $\mathcal{B}$ covers X and hence also A, it follows that there is a set $B' \in \mathcal{B}$ distinct from B and also containing some element of A. Let $x \in A \cap B$ and $y \in A \cap B'$. Applying condition Proposition 8.2 to the collection $\mathcal{B}$, this time we see that if $S = \{x, y\}$ then

$$S^{\#} = (S \cap B)^{\#} \cup (S \cap B')^{\#} = \{x\}^{\#} \cup \{y\}^{\#}.$$

As before, the cardinality of the left hand side is 2. On the other hand, $\{x\}$ and $\{y\}$ are both cardinality 1 subsets of A, and since $\#$ restricted to 2^A is of Schwarz type it follows that $\{x\}^{\#} = \{y\}^{\#}$ so that $|\{x\}^{\#} \cup \{y\}^{\#}| = 1$, a contradiction.

Hence, both possibility (a) and possibility (b) leads to a contradiction, and our proposition has thus been proved. □

Definition 8.4. The **Steiner order $\prec$ on Y** for a Steiner type rearrangement $\#\colon 2^X \to 2^Y$ is given by defining $x \prec y$ for $x, y \in Y$ if and only if there is a fibre A such that $x, y \in A$ and $x \prec y$ according to the well-ordering induced by Proposition 8.1 applied to the Schwarz rearrangement $\#$ on 2^A.

Definition 8.5. A function f on a set Y is said to be **decreasing with respect to a partial ordering $\prec$** providing $f(x_1) \geq f(x_2)$ whenever $x_1 \prec x_2$.

Definition 8.6. A subset S of Y is said to be **symmetric** if there is a subset T of X such that $T^{\#} = S$. A function f on Y is said to be **symmetric** if there exists a function g on X with $f = g^{\#}$.

By Theorem I.2.1, a symmetric function has symmetric level sets.

Proposition 8.4. *The following are equivalent for a Steiner type rearrangement* $\#\colon 2^X \to 2^Y$ *and an extended real function* f *on* Y*:*

(i) f *is symmetric*

(ii) *the level set* f_λ *is symmetric for every real* λ

(iii) f *is decreasing with respect to the Steiner order on* Y*.*

Proof. Because of Proposition 8.2, we may consider the problem separately for each restriction $f|_{A^{\#}}$ of f to a set $A^{\#}$ such that $A \in \mathcal{A}$, and it is easy to see that f is symmetric if and only if each of these restrictions is symmetric with respect to the Schwarz type rearrangement given by the restriction of $\#$ to 2^A.

The above reasoning shows that, replacing X by A if necessary, we may assume that $\#$ is a Schwarz type rearrangement.

First we show that (ii) implies (iii). For, fix x and y in Y with $x \prec y$. Then $x = y_i$ and $y = y_j$ with $i < j$, in the notation of Proposition 8.1. Let $\lambda = f(y_i)$. Of course we have $y_i \notin f_\lambda$. Now, if the set f_λ is symmetric and if it contains y_j then, since $j > i$, it follows from Proposition 8.1 that it contains y_i as well. Hence, since $y_i \notin f_\lambda$, we likewise have $y_j \notin f_\lambda$ so that $f(y) = f(y_j) \leq \lambda = f(y_i) = f(x)$, as desired.

We now prove that (iii) implies (i). To do this, let g be any function on X equimeasurable with f (such a function exists since $|X| = |X^{\#}| = |Y|$). I claim that $g^{\#} = f$. For, fix $\lambda \in \mathbb{R}$. We must prove that $(g^{\#})_\lambda = f_\lambda$. But $(g^{\#})_\lambda = (g_\lambda)^{\#}$ by Theorem I.2.1. Let N be the cardinality of g_λ. If $N = \infty$ then g takes on values strictly bigger than λ infinitely often, and so does

f. Since f is $\prec$-decreasing, it follows that $f > \lambda$ everywhere, so that $f_\lambda = Y$. On the other hand, by Proposition 8.1 it follows that if $N = \infty$ then $(g_\lambda)^\# = \{y_0, y_1, \ldots\} = Y$, too. Suppose now that $N < \infty$. Then g takes on values strictly bigger than λ precisely N times. Since f is equimeasurable to g, the same is true of it. Since f is $\prec$-decreasing, it follows that

$$f_\lambda = \{y_0, y_1, \ldots, y_{N-1}\}.$$

But if the cardinality of g_λ equals N, it follows then from Proposition 8.1 that $(g_\lambda)^\# = \{y_0, y_1, \ldots, y_{N-1}\} = f_\lambda$. Hence, we have proved that $g^\# = f$ so that f is symmetric.

Finally we show that (i) implies (ii). For, fix $\lambda \in R$. Since f is symmetric, there exists g such that $g^\# = f$. Then $f_\lambda = (g_\lambda)^\#$ by Theorem I.2.1 so that it follows that f_λ is symmetric, as noted before. □

Via condition (iii), we immediately obtain as a corollary the following result.

Proposition 8.5. *Let f and g be two symmetric functions on Y with respect to a Steiner type rearrangement $\#: 2^X \to 2^Y$. Then $f + g$ is symmetric. Moreover, if f and g are positive, then fg is symmetric.*

The following result is a very useful property of (discrete) Steiner type rearrangements. In a sense, it lets us "reverse" a Steiner type rearrangement.

Proposition 8.6. *Assume that $\#: 2^X \to 2^Y$ is a discrete Steiner type rearrangement with fibres $\mathcal{A}$. Let f be any positive function on X such that $A \cap f_\lambda$ is finite for every $\lambda > 0$ and $A \in \mathcal{A}$. Then there exists a one-to-one map $\phi: Y \to X$ with the following properties:*

(i) $f^\# = f \circ \phi$

(ii) *if g is any positive symmetric function on Y and if $\check{g}$ is the function on X such that*

$$\check{g}(x) = \begin{cases} g(\phi^{-1}(x)), & \text{if } x \in \phi[Y], \\ 0, & \text{if } x \notin \phi[Y], \end{cases}$$

then $(\check{g})^\# = g$, $(\check{g} \cdot f)^\# = g \cdot f^\#$ *and*

$$\sum_{x \in X} \check{g}(x) f(x) = \sum_{y \in Y} g(y) f^\#(y) \tag{8.3}$$

(iii) *ϕ maps $A^\#$ onto A whenever $A \in \mathcal{A}$*

(iv) *if either $A \cap \operatorname{supp} f$ is finite or f is strictly positive on A for a fibre $A \in \mathcal{A}$, then ϕ maps $A^\#$ into A.*

Moreover, if $S \subseteq A$ is a fixed subset of some fixed fibre A and if we have

$$\inf_S f \geq \sup_{A \setminus S} f,$$

then we may choose ϕ with the additional property that whenever g is a positive function with $\operatorname{supp} g \subseteq S^\#$ *then* $\operatorname{supp} \check{g} \subseteq S$.

Proof. Because of Proposition 8.2, we may restrict all our functions on X to $A \in \mathcal{A}$ and all our functions on Y to $A^\#$, and then construct ϕ: $A^\# \to A$, and prove that all the above results hold. Then, we may piece these things together via Proposition 8.2 and via the fact that the $\{A^\#\}_{A \in \mathcal{A}}$ are disjoint as are the $\{A\}_{A \in \mathcal{A}}$. But $\#$ restricted to subsets of $A \in \mathcal{A}$ is of Schwarz type. Hence, it suffices to prove Proposition 8.6 for rearrangements of Schwarz type.

Thus, assume that $\#$ is of Schwarz type. Then $\mathcal{A} = \{X\}$. Without loss of generality, the support of f is non-empty (otherwise the result is trivial, since we can let ϕ be *any* bijection then).

Write $Y = \{y_0, y_1, \ldots\}$, where the y_i are as in Proposition 8.1. Using the fact that the f_λ are finite for all $\lambda > 0$ we may construct a sequence $x_0, x_1, x_2, \ldots$ with the property that $f(x_0) = \max_X f$ and

$$f(x_i) = \max\{f(x) : x \in X \setminus \{x_0, x_1, \ldots, x_{i-1}\}\}$$

for $i > 0$. Moreover, if $\operatorname{supp} f$ is finite, we may easily ensure that $x_0, x_1, x_2, \ldots$ is an enumeration of X (since after having chosen those x_i which form the support of f, we then choose all the other x_i arbitrarily in such a way that we enumerate all of X.) Let $X' = \{x_0, x_1, x_2, \dots\}$. It is clear that because of the finiteness of f_λ for all λ, the set X' contains at least all the points x of X such that $f(x) > 0$. Hence, if either $f > 0$ everywhere or $\operatorname{supp} f$ is finite, we have $X' = X$.

Now define

$$\phi(y_i) = x_i,$$

for $i \in \mathbb{Z}_0^+$. It is clear that ϕ is a bijection from Y onto X'. Since $\mathcal{A} = \{X\}$, condition (iii) is trivial. Since $\phi[Y] = X'$, we have already verified (iv). Condition (i) is also easy to verify. To do this, fix $\lambda \in \mathbb{R}$. We must prove that $(f^\#)_\lambda = (f \circ \phi)_\lambda$. But $(f^\#)_\lambda = (f_\lambda)^\#$ by Theorem I.2.1. Suppose first that $\lambda < 0$. Then, $f_\lambda = X$ since f is positive, so that $(f_\lambda)^\# = Y$. Likewise $(f \circ \phi)_\lambda = Y$ then, since $f \circ \phi$ is also positive.

Suppose now that $\lambda \geq 0$. Suppose that the cardinality of the set f_λ is i. Consider first the case where $i = \infty$; necessarily we have $\lambda = 0$. Then, f takes on infinitely many strictly positive values. The choice of X' implies that f is strictly positive on X', hence $(f \circ \phi)_\lambda = Y$. But $(f_\lambda)^\# = Y$ as well, since we have a Schwarz rearrangement, and the Schwarz rearrangement of an infinite subset S of X must equal Y since $X^\# = Y$ and $|X| = |S|$. Hence, in that case we have verified the desired result.

Suppose now that $i < \infty$. Then,

$$(f_\lambda)^\# = \{y_0, y_1, \dots, y_{i-1}\}$$

by Proposition 8.1. But f_λ consists of points at which f takes on its i largest values. By the choice of the x_i we have

$$f_\lambda = \{x_0, x_1, \dots, x_{i-1}\}.$$

Thus,

$$(f \circ \phi)_\lambda = \phi^{-1}[f_\lambda] = \{y_0, y_1, \dots, y_{i-1}\} = (f_\lambda)^\#,$$

as desired, where the second last equality follows from the definition of ϕ.

We have thus verified (i). We must now verify (ii). I claim that $\breve{g}$ is equimeasurable with g. For, again fix $\lambda \in \mathbb{R}$. If $\lambda < 0$ then $|\breve{g}_\lambda| = |g_\lambda| = \infty$ as desired. So now suppose that $\lambda \geq 0$.

Then, $\check{g}_\lambda = \phi[g_\lambda]$ since $\check{g}_\lambda \subseteq X'$ for $\lambda \geq 0$ as $\check{g}$ vanishes outside X'. Since ϕ is one-to-one it follows that $|\check{g}_\lambda| = |g_\lambda|$. Hence our claim is proved. Now, g is symmetric so that $g = h^\#$ for some h. Moreover, g and h are equimeasurable, hence $\check{g}$ and h are equimeasurable so that $\check{g}^\# = h^\#$ since we have a Schwarz rearrangement (Remark 8.1).

Now, $\check{g}(x_i)f(x_i)$ is decreasing with respect to i (since $g(y_i)$ is decreasing with respect to i by symmetry), just as f was, and vanishes outside X', just as f did. Applying the same method which we used to show that $f^\# = f \circ \phi$ we find that $(\check{g} \cdot f)^\# = (\check{g} \cdot f) \circ \phi = g \cdot f^\#$, as desired. Since $\#$ is measure preserving it follows immediately that (8.3) holds (Proposition I.2.1).

To verify the "moreover", suppose that we are given such a set S. The condition

$$\inf_S f \geq \sup_{A \setminus S} f,$$

shows that we may choose the x_i satisfying the further constraint that if $x_i \notin S$ for some i then $S \subseteq \{x_0, \ldots, x_{i-1}\}$. (I.e., we may preferentially at each stage choose an element of S over an element of $A \setminus S$.) If we do this, then it is clear that if g is a symmetric function whose support lies in $S^\#$, hence in $\{y_i : i < |S|\}$, then the support of $\check{g}$ will lie in $\{x_i : i < |S|\}$. $\square$

8.2. Rearrangement on a product set

We are still working in the discrete setting of countable sets with counting measure.

Definition 8.7. Let $\#$ be a rearrangement from 2^X to 2^Y. Let Z be any countable set. Then the **Z-product rearrangement $\#$ from $2^{Z \times X}$ to $2^{Z \times Y}$** is defined by:

$$S^\# \stackrel{\text{def}}{=} \bigcup_{z \in Z} [\{z\} \times (\{x : (z, x) \in S\}^\#)],$$

for any $S \subseteq Z \times X$, where the "$\#$" on the right hand side of the above displayed equation is the $\#$-rearrangement for subsets of X.

Remark 8.2. It is clear that the Z-product rearrangement is indeed a rearrangement in the sense of Definition I.2.2 if $\#: 2^X \to 2^Y$ is. Moreover, the Z-product rearrangement is measure preserving if $\#: 2^X \to 2^Y$ is.

Remark 8.3. Suppose that $\#\colon 2^X \to 2^Y$ is a Steiner type rearrangement with $\mathcal{A}$ being its fibres. Then the Z-product rearrangement is also of Steiner type, and its collection of fibres is given by

$$\mathcal{A}_Z \stackrel{\text{def}}{=} \{\{z\} \times A : z \in Z, A \in \mathcal{A}\}.$$

The verification of this fact is almost immediate. In particular, the product construction lets us start with a Schwarz type rearrangement for subsets of X and obtain a Steiner type rearrangement for subsets of $Z \times X$. This will let us construct a number of interesting examples of Steiner type rearrangements.

8.3. Symmetrization and preservation of symmetry

We recall the following definition.

Definition 8.8. A rearrangement $\#$ mapping a σ-pseudotopology $\mathcal{F}$ into itself is said to be **a symmetrization** if $(A^\#)^\# = A^\#$ for all $A \in \mathcal{F}$.

Let K be a positive function on $X \times X$. Given a positive function g on X, define

$$Kg(x) = \sum_{y \in X} K(x, y) g(y),$$

and

$$K^* g(y) = \sum_{x \in X} g(x) K(x, y).$$

The following result is very similar to Theorem 2.3.

Theorem 8.1. *Suppose that $\#\colon 2^X \to 2^X$ is a discrete Steiner type symmetrization. Assume that*

$$\sum_{x,y \in X} f(x) K(x, y) g(y) \leq \sum_{x,y \in X} f^\#(x) K(x, y) g^\#(y)$$

for all positive functions f and g on X. Then Kg and $K^* g$ are both symmetric whenever g is a positive symmetric function.

The above result should remind us of Corollary I.6.2.

Proof. It suffices to prove that Kg is symmetric when g is symmetric, since the other assertion follows upon replacing $K(x,y)$ by $L(x,y) \stackrel{\text{def}}{=} K(y,x)$. For any positive f and symmetric g we have

$$\sum_{x,y \in X} f(x)K(x,y)g(y) \leq \sum_{x,y \in X} f^{\#}(x)K(x,y)g(y),$$

since $g^{\#} = g$ as g is symmetric and $\#$ is a symmetrization. Rewriting the above displayed equation, we see that for any positive f we have

$$\sum_{x \in X} f(x)Kg(x) \leq \sum_{x \in X} f^{\#}(x)Kg(x).$$

The desired result then follows from Proposition 8.7, below. $\square$

Proposition 8.7. *Let $\#$ be a discrete Steiner type symmetrization. Then, an extended real function h on X is symmetric if and only if*

$$\sum_{x \in X} f(x)h(x) \leq \sum_{x \in X} f^{\#}(x)h(x) \tag{8.4}$$

for every positive f on X for which both sums make sense.

Before we outline the proof of this, we need the following result.

Lemma 8.1. *Let $\#$ be a discrete Steiner type symmetrization. Then each fibre $A \in \mathcal{A}$ is symmetric.*

Proof. Suppose that $A, B \in \mathcal{A}$. Then,

$$(A^{\#} \cap B)^{\#} = (A^{\#})^{\#} \cap B^{\#} = A^{\#} \cap B^{\#} \tag{8.5}$$

by Proposition 8.2 and by the fact that $\#$ is a symmetrization.

Suppose that $B \in \mathcal{A}$ is different from A. Then $A^{\#} \cap B^{\#} = \varnothing$ by Definition 8.2(iii), so that (8.5) implies that $A^{\#} \cap B = \varnothing$. Since $\mathcal{A}$ covers X, it follows that $A^{\#} \subseteq A$ for every $A \in \mathcal{A}$.

Now, to obtain a contradiction suppose that $A^{\#}$ is a proper subset of $A \in \mathcal{A}$. Since the $\{B^{\#} : B \in \mathcal{A}\}$ disjointly cover X (Definition 8.2(iii) and Proposition 8.2), it follows that there is a $B^{\#}$ distinct from $A^{\#}$ such that $B \in \mathcal{A}$ and $B^{\#} \cap A \neq \varnothing$. But by what we have already proved, we have $B^{\#} \subseteq B$. Hence, $B \cap A \neq \varnothing$. Hence $B = A$, which contradicts the assumption that $B^{\#}$ should be distinct from $A^{\#}$. Hence, indeed, $A^{\#} = A$. □

Proof of Proposition 8.7. Suppose first that h is symmetric. Then $h^{\#} = h$ and so (8.4) follows from Theorem I.2.3. Suppose now that (8.4) holds for all positive f. To obtain a contradiction, suppose that f is not symmetric. Because of Proposition 8.2, we may decompose X into the fibres $\mathcal{A}$ of Definition 8.2, and consider (8.4) on each of them separately, since the rearrangements can be computed on each separately.

By this argument, we may assume that our symmetrization in fact is of Schwarz type, since it is of Schwarz type on each $A \in \mathcal{A}$ and since $A = A^{\#}$ for all $A \in \mathcal{A}$. However, it is easy to see that any Schwarz type symmetrization of functions on a countable set is equivalent to decreasing rearrangement of functions on $\mathbb{Z}_N \stackrel{\text{def}}{=} \{n \in \mathbb{Z}_0^+ : n < N\}$ for N equal to the cardinality of our countable set; to see this, use the method of proof of Proposition 8.4 and the mapping $y_i \mapsto i$ sending X onto $\mathbb{Z}_N$, where the y_i are as in Proposition 8.1. But a result equivalent to Proposition 8.7 on $\mathbb{Z}_N$ is contained in [58, Thm. 369]. □

Finally, we state a useful generalization of Proposition 8.7.

Proposition 8.8. *Let $\#$ be a discrete Steiner type symmetrization. Then, an extended real function h on X is symmetric if and only if*

$$\sum_{x\in S} h(x) \leq \sum_{x \in S^{\#}} h(x) \tag{8.6}$$

for every finite $S \subseteq X$.

Proof. In light of Proposition 8.7, it suffices to prove that if (8.6) holds for every finite S then (8.4) holds for every positive g. But, given a positive g with finite support we may write

$$\sum_{x\in X} g(x)h(x) = \int_0^\infty \sum_{x\in g_\lambda} h(x)\, d\lambda,$$

in light of Remark I.2.2. Applying (8.6) we see that (8.4) holds for our g. A limiting argument then shows that it likewise holds for any g for which both sides of (8.4) holds. $\square$

Remark 8.4. It is easy to see that in fact it suffices to prove (8.6) for all S which lie completely within a single fibre $A \in \mathcal{A}$, because of Proposition 8.2 and Lemma 8.1.

9. Haliste's method for exit times, discrete harmonic measures and discrete Green's functions

Throughout this section assume that we are given a discrete set X and a Steiner type symmetrization $\#\colon 2^X \to 2^X$.

9.1. Definitions and statement of results for generalized harmonic measures and Green's functions

9.1.1. The kernel and the assumptions on it

Let $V = \mathbb{Z} \times X$. Let $K\colon V \times V \to [0,1]$ be a function such that for all $z \in V$ we have

$$\sum_{w \in V} K(z,w) = 1. \tag{9.1}$$

We call such a function K a **kernel**.

We now consider the following assumptions on a function $L\colon V \times V \to [0,\infty)$.

Assumption 9.1. The following inequality is valid whenever f and g are positive functions on X while m and n are in $\mathbb{Z}$:

$$\sum_{x,y \in X} f(x) L\big((m,x),(n,y)\big) g(y) \le \sum_{x,y \in X} f^{\#}(x) L\big((m,x),(n,y)\big) g^{\#}(y). \tag{9.2}$$

Write $Lg(v) = \sum_{w \in V} L(v,w) g(w)$. Put $L^*(v,w) = L(w,v)$ so that $L^* f(w) = \sum_{v \in V} f(v) L(v,w)$.

From Theorem 8.1 we obtain the following result.

Proposition 9.1. *Assume Assumption* 9.1. *Both* $(Lg)(n,\cdot)$ *and* $(L^*g)(n,\cdot)$ *are symmetric functions from* X *to* $[0,\infty)$ *for each* $n \in \mathbb{Z}$ *providing* $g(m,\cdot)$ *is a symmetric positive function on* X *for every* $m \in \mathbb{Z}$.

Assumption 9.2. For every fixed m and n in $\mathbb{Z}$, the quantity

$$\sum_{y \in X} L((m,x),(n,y))$$

is independent of the choice of $x \in X$.

Finally, we make the following assumption on our kernel K.

Assumption 9.3. There exists $\lambda \in [0,\infty)$ such that $L \stackrel{\text{def}}{=} K + \lambda\delta$ satisfies Assumptions 9.1 and 9.2, where $\delta(z,w) = 1_{\{z=w\}}$.

For $D \subseteq V$, let

$$D^{\#} = \bigcup_{i \in \mathbb{Z}} \Big[\{i\} \times (\{x : (i,x) \in D\}^{\#}) \Big] .$$

This is of course the $\mathbb{Z}$-product rearrangement construction (Definition 8.7) based on the rearrangement $\#$ for subsets of X. As noted in 8.2, $D \mapsto D^{\#}$ is a measure preserving rearrangement. We shall call $D^{\#}$ the **(discrete generalized) Steiner symmetrization** of D. If $D = D^{\#}$ then we say that D is **Steiner symmetric.** See Figure 9.1 for an example of Steiner symmetrization in the case of the discrete cylinder where $X = \mathbb{Z}_{11}$, and Figure 9.2 for an example where $X = \mathbb{Z}$.

9.1.2. The kernel in our main examples

Our primary example will have X a constant degree graph satisfying the master inequality with respect to some well-ordering with each element having finitely many predecessors. The rearrangement $\#$ on X will be defined with respect to this well-ordering as in §1. Such examples which we have proved in this chapter to satisfy the master inequality are the linear graph $\mathbb{Z}$ (with edges $\{j, j+1\}$ as j ranges over $\mathbb{Z}$), the circular graph $\mathbb{Z}_n$ (with edges $\{j, j+1\}$ as j ranges over $\mathbb{Z}_n$), the octahedron edge graph H_8 and the p-regular tree T_p.

Given X a constant degree graph satisfying the master inequality with each vertex of degree p,

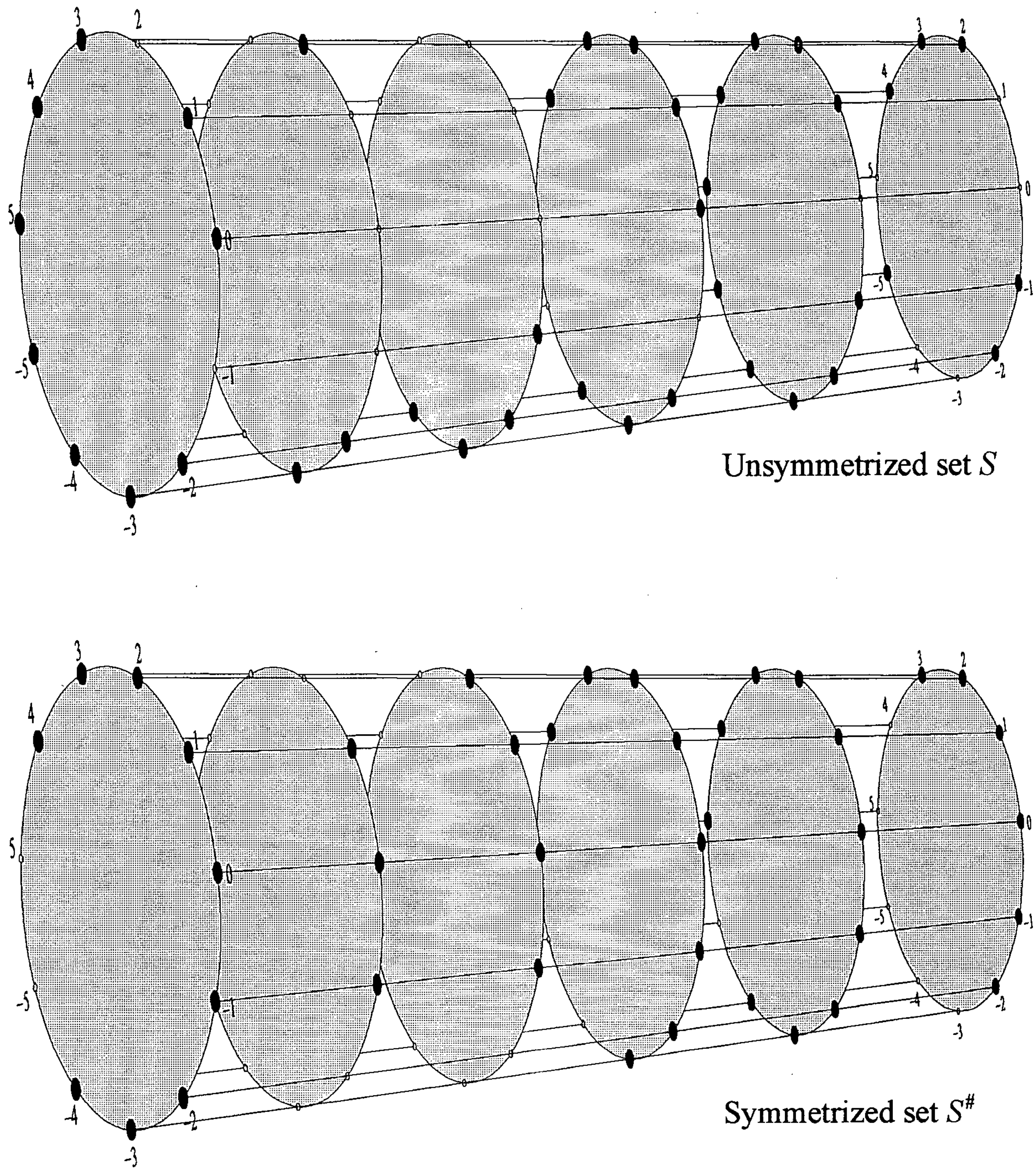


Figure 9.1: Symmetrization on $\mathbb{Z} \times \mathbb{Z}_{11}$. The symmetrized and unsymmetrized sets in question are indicated by black ellipses.

let

$$K^X(x,y) = \begin{cases} p^{-1}, & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\sum_{y\in X} K(x,y) = 1$ for all $x \in X$. Then, let $K^{\mathbb{Z}}$ be any kernel on $\mathbb{Z}$, i.e., any function $K^{\mathbb{Z}}\colon \mathbb{Z}^2 \to [0,1]$ such that $\sum_{y\in\mathbb{Z}} K^{\mathbb{Z}}(x,y) = 1$ for all $x \in \mathbb{Z}$. Two interesting cases are the simple random walk kernel $K^{\mathbb{Z}}_{\mathrm{S}}(x,y) = \frac{1}{2}$ whenever $|x-y| = 1$ and $K^{\mathbb{Z}}_{\mathrm{S}}(x,y) = 0$ otherwise, and the trivial kernel $K^{\mathbb{Z}}_{\delta}(x,y) = \delta_{x,y}$.

Then, given a kernel $K^{\mathbb{Z}}$ on $\mathbb{Z}$ and the kernel K^X on X, we may define a kernel on V. For $r \in [0,1]$, let

$$(K^{\mathbb{Z}} \overset{r}{\otimes} K^X)((m,x),(n,y)) = rK^{\mathbb{Z}}(m,n)\delta_{x,y} + (1-r)K^X(x,y)\delta m,n.$$

It is not difficult to verify that $K^{\mathbb{Z}} \overset{r}{\otimes} K^X$ is a kernel.

Proposition 9.2. *Let X be as above satisfy the master inequality. Then Assumption 9.3 is satisfied by $K^{\mathbb{Z}} \overset{r}{\otimes} K^X$ for any $r \in [0,1]$.*

Proof. Let $\lambda = 1$ and set $L = (K^{\mathbb{Z}} \overset{r}{\otimes} K^X) + \delta$. First we verify Assumption 9.1. To do this, fix positive f and g on X and $m, n \in \mathbb{Z}$. Suppose first that $m \neq n$. Then, for x and y in X we have

$$L((m,x),(n,y)) = \delta_{x,y} rK^{\mathbb{Z}}(m,n),$$

since $\delta((m,x),(n,y)) = 0$ and by the definition of $K^{\mathbb{Z}} \overset{r}{\otimes} K^X$. Hence,

$$\begin{aligned} \sum_{x,y\in X} f(x)L((m,x),(n,y))g(y) &= rK^{\mathbb{Z}}(m,n)\sum_{x\in X} f(x)g(x) \\ &\le rK^{\mathbb{Z}}(m,n)\sum_{x\in X} f^{\#}(x)g^{\#}(x) \\ &= \sum_{x,y\in X} f(x)L((m,x),(n,y))g(y), \end{aligned}$$

as desired, where we have used the Hardy-Littlewood inequality (Theorem 1.1 of this chapter or Theorem I.2.3). Suppose now that $m = n$. Then, for x and y in X we have

$$L((m,x),(n,y)) = (1-r)K^X(x,y) + (1 + rK^{\mathbb{Z}}(m,m))\delta_{x,y}.$$

If we can prove that

$$(1-r)K^X(x,y) + (1+rK^{\mathbb{Z}}(m,m))\delta_{x,y} = K(d(x,y))$$

for some decreasing function K, then by Theorem 2.2 we will obtain (9.2). But if $d(x,y) = 1$ then x and y are adjacent so that

$$(1-r)K^X(x,y) + (1+rK^{\mathbb{Z}}(m,m))\delta_{x,y} = (1-r)p^{-1} \le 1.$$

If $x = y$ then

$$(1-r)K^X(x,y) + (1+rK^{\mathbb{Z}}(m,m))\delta_{x,y} = 1 + rK^{\mathbb{Z}}(m,m).$$

If $d(x,y) > 1$ then

$$(1-r)K^X(x,y) + (1+rK^{\mathbb{Z}}(m,m))\delta_{x,y} = 0,$$

as x and y are then neither equal nor adjacent. Hence, if we let $K(x) = 1 + rK^{\mathbb{Z}}(m,m)$ for $x < 1$, $K(x) = (1-r)p^{-1}$ for $1 \le x < 2$ and $K(x) = 0$ for $x \ge 2$ then we will have $(1-r)K^X(x,y) + (1+rK^{\mathbb{Z}}(m,m))\delta_{x,y} = K(d(x,y))$ as desired. Thus Assumption 9.1 holds.

Only Assumption 9.2 remains. Suppose first that $m \ne n$. Then,

$$\sum_{y \in X} K((m,x),(n,y)) = rK^{\mathbb{Z}}(m,n)$$

which is clearly independent of x. On the other hand, suppose that $m = n$. Then,

$$\sum_{y \in X} K((m,x),(n,y)) = 1 + rK^{\mathbb{Z}}(m,m) + (1-r) \sum_{y \in N(x)} p^{-1} = 1 + rK^{\mathbb{Z}}(m,m) + 1 - r,$$

where $N(x)$ denotes the set of all vertices of X adjacent to x so that $|N(x)| = p$, and hence again we have independence of x. Hence Assumption 9.2 is valid. $\square$

9.1.3. The random walk on V

We now consider the Markov process $\{R_n\}$ on V for $n \in \mathbb{Z}_0^+$ with the transition probabilities

$$P(R_{n+1} = w \mid R_n = z) = K(w,z).$$

Because in our examples this process will always be some kind of random walk, we shall refer to it as our "random walk". We call K the **kernel** of the walk $\{R_n\}$. We shall write $P^z(\cdot)$ and $E^z[\cdot]$ for probabilities and expectations, respectively, where the random walk is conditioned to have the starting value $R_0 = z$.

Remark 9.1. Consider the kernel $K = K^{\mathbb{Z}} \overset{r}{\otimes} K^X$ mentioned in §9.1.2. Then, R_n can be characterized at follows. At each time step, while sitting at the point $(m, x) \in V$, a weighted coin is flipped. With probability r a move is taken with x fixed, but with m changed according to the transition probability $K^{\mathbb{Z}}$ (in the case where $K^{\mathbb{Z}} = K^{\mathbb{Z}}_{\mathrm{S}}$ we move to $(m \pm 1, x)$ with equal probability, and in the case $K^{\mathbb{Z}} = K^{\mathbb{Z}}_{\delta}$ we remain put at (m, x).) With probability $1 - r$, the first coordinate m is left fixed, but x is changed according to the transition probability K^X, i.e., a random point adjacent to x is chosen and the walk moves to it.

Suppose now that $X = \mathbb{Z}_n$ for $n \geq 3$ or $X = \mathbb{Z}$, that $r = \frac{1}{2}$ and that $K^{\mathbb{Z}} = K^{\mathbb{Z}}_{\mathrm{S}}$. Then V is a discrete cylinder $\mathbb{Z} \times \mathbb{Z}_n$ and the walk R_n at each times step can be easily seen to have an equal probability $\frac{1}{4}$ of moving to any of the neighbouring points in $\mathbb{Z} \times \mathbb{Z}_n$ or $\mathbb{Z} \times \mathbb{Z}$. (I.e., with equal probability $\frac{1}{4}$, we can move from (x, y) to any of the four points $(x + 1, y)$, $(x - 1, y)$, $(x, y + 1)$ and $(x, y - 1)$.)

9.1.4. Generalized harmonic measure

Instead of simply considering harmonic measure for $\{R_n\}$, we shall consider a slightly more complicated situation for which the proofs, however, are no more difficult. Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of independent and identically distributed random variables uniformly distributed over $(0, 1]$. For $s \colon V \to [0, 1]$, we define

$$\tau_s = \inf\{n \geq 0 : X_n > s(R_n)\}.$$

This is a stopping time with respect to an appropriate filtration. We may interpret the situation as having $1 - s(z)$ indicate the probability of the random walk being killed while standing at z, so that $s(z)$ is a survival probability; then, τ_s indicates the amount of time for which the random walk survives. We may call the situation described above "**a random walk with dangers.**"

The usual case of harmonic measure will be recovered if we let s be the indicator function 1_D of a set $D \subseteq X$, since then τ_s coincides with the exit time $\tau_D \overset{\text{def}}{=} \inf\{n \geq 0 : R_n \notin D\}$.

For $S \subseteq X$ and $z \in V$, we write

$$\omega(z, S; s) = P^z(\tau_s < \infty \text{ and } R_{\tau_s} \in S).$$

Technical remark 9.1. In some cases the condition $\tau_s < \infty$ is redundant, since if the walk $\{R_n\}$ is recurrent and visits every point, then we have $\tau_s < \infty$ with probability one providing there exists a $z \in V$ with $s(z) < 1$.

Remark 9.2. Let $X = \mathbb{Z}_n$ or $X = \mathbb{Z}$. Consider the case of $K = K_{\mathrm{S}}^{\mathbb{Z}} \overset{1/2}{\otimes} K^X$. Then, R_n is a simple random walk on $\mathbb{Z} \times X$. If $s = 1_D$ and S is a subset of the boundary of D (the boundary being defined as the set of points of $V \backslash D$ which are one simple random walk step away from D), then $\omega(z, S; s)$ is just the ordinary discrete harmonic measure at z in D of X; it is simply the probability that if the random walk started at z ever exits D, its first exit from D lands it at a point of S. Thus, in light of the connection between Brownian motion and harmonic measure (Theorem I.5.5), $\omega(z, S; s)$ is quite analogous to the classical harmonic measure $\omega(z, S; D)$ for $S \subseteq \partial D$ and $D \subseteq \mathbb{C}$.

Note that $\mathbb{Z} \times \mathbb{Z}_n$ is a tube, and thus is the discrete analogue of the tube $\mathbb{R} \times \mathbb{T}$. On the other hand, the latter tube is conformally equivalent to the punctured plane $\mathbb{C}\backslash\{0\}$ via an exponential map.

The effect of a general choice of s is that of making the domain D somewhat "fuzzy." Note that if $s = 1_D$ then $s^{\#} = 1_{D^{\#}}$ so that symmetrization of s corresponds to symmetrization of a domain.

Finally, we give a certain partial ordering on the set of positive functions on V. For $f, g \colon V \to [0, \infty)$, we write $f \preceq g$ providing that for every $i \in \mathbb{Z}$ and every convex increasing function Φ on $[0, \infty)$ we have

$$\sum_{j \in X} \Phi(f(i, j)) \leq \sum_{j \in X} \Phi(g(i, j)).$$

By [58, Thm. 108], this is equivalent to the condition

$$\sup_{|I|=k} \sum_{j\in I} f(i,j) \leq \sup_{|I|=k} \sum_{j\in I} g(i,j) \tag{9.3}$$

being valid for every $k \in \mathbb{Z}_0^+$ and each $i \in \mathbb{Z}$, where the suprema are taken over all k element subsets I of X. In particular if $f \preceq g$ then $\sup_{j\in X} f(i,j) \leq \sup_{j\in X} g(i,j)$. If, moreover, g is Steiner symmetric and $X = \mathbb{Z}$ then the right hand side of (9.3) becomes

$$\sum_{j=-l}^{l} g(i,j),$$

if $k = 2l+1$, and

$$\sum_{j=-l+1}^{l} g(i,j),$$

if $k = 2l$, while on the other hand $\sup_{j\in X} g(i,j) = g(i,0)$.

Definition 9.1. Let X be a non-empty set equipped with a Schwarz type symmetrization $\#$. Then the **initial element of** X is defined to be the unique element O of $\{x\}^\#$ where $x \in X$. (This definition is unique because of the Schwarz type of $\#$.)

If g is Steiner symmetric and $\#$ is of Schwarz type on X with initial element O, then

$$\sup_{j\in X} g(i,j) = g(i,O)$$

for all i.

Our result on the effect of symmetrization on harmonic measure is as follows.

Theorem 9.1. *Let $\{R_n\}$ be a random walk associated to a kernel K which satisfies Assumption 9.3. Fix $k \in \mathbb{Z}$. Let $s\colon V \to [0,1]$ be a function vanishing on $\{k\} \times X$. Let $S \subseteq \{k\} \times X$. Then,*

$$\omega(\cdot, S; s) \preceq \omega(\cdot, S^\#; s^\#).$$

Moreover, $\omega(\cdot, S^\#; s^\#)$ is Steiner symmetric.

In particular, if $\#$ is a Schwarz type symmetrization on X with initial element O, then for every $i \in \mathbb{Z}$ we have

$$\sup_{j \in X} \omega((i,j), S; s) \leq \omega((i,O), S^{\#}; s^{\#}).$$

Remark 9.3. In the case of $X = \mathbb{Z}_n$, $K = K_{\mathrm{S}}^{\mathbb{Z}} \overset{1/2}{\otimes} K^X$, and $s = 1_D$, Theorem 9.1 is an analogue of (a trivial generalization of) a result of Baernstein [7, Thm. 7] (stated in the present thesis in Theorem I.6.2). Of course, Baernstein's result is concerned with circular symmetrization. However, our space $\mathbb{Z} \times \mathbb{Z}_n$ is a discrete analogue of $\mathbb{R} \times \mathbb{T}$, and the latter is conformally equivalent to $\mathbb{C}\backslash\{0\}$ under an exponential map. Using this exponential map, we can define a circular rearrangement on $\mathbb{R} \times \mathbb{T}$ in a natural way by pulling back the rearrangement on $\mathbb{C}$. Our rearrangement $\#$ on $\mathbb{Z} \times \mathbb{Z}_n$ is a discrete analogue of that circular rearrangement on $\mathbb{R} \times \mathbb{T}$. Indeed, it might even be possible to obtain Baernstein's result on $\mathbb{R} \times \mathbb{T}$ as a limiting case of results on $\mathbb{Z} \times \mathbb{Z}_n$ (we would have to take n to infinity and use some sequence of sets $D_n \subseteq \mathbb{Z} \times \mathbb{Z}_n$ which in some way approximates a domain $D \subseteq \mathbb{R} \times \mathbb{T}$).

Strictly speaking, our Theorem improves on a discrete result of Quine [90], but in the special case of the simple random walk, still under the assumption that $s = 1_D$, it is actually a *consequence* of Quine's result on the subharmonicity of the discrete $*$-function [90].

An example of an application of our Theorem with $X = \mathbb{Z}$ can be seen in Figure 9.2. The idea is that we have a random walk starting from the point S which continues until it either bumps into one of the black squares (which represent dangers, such as a dragon or a cliff) or it gets to one of the squares on the right hand edge where it safely exists. Then, our Theorem asserts that the probability of reaching the right hand edge in the original situation (top figure) is less than or equal to the probability of reaching it in the symmetrized situation (bottom figure). (Note that in the figure, the black squares correspond to the complement of the set we are allowed to walk on.)

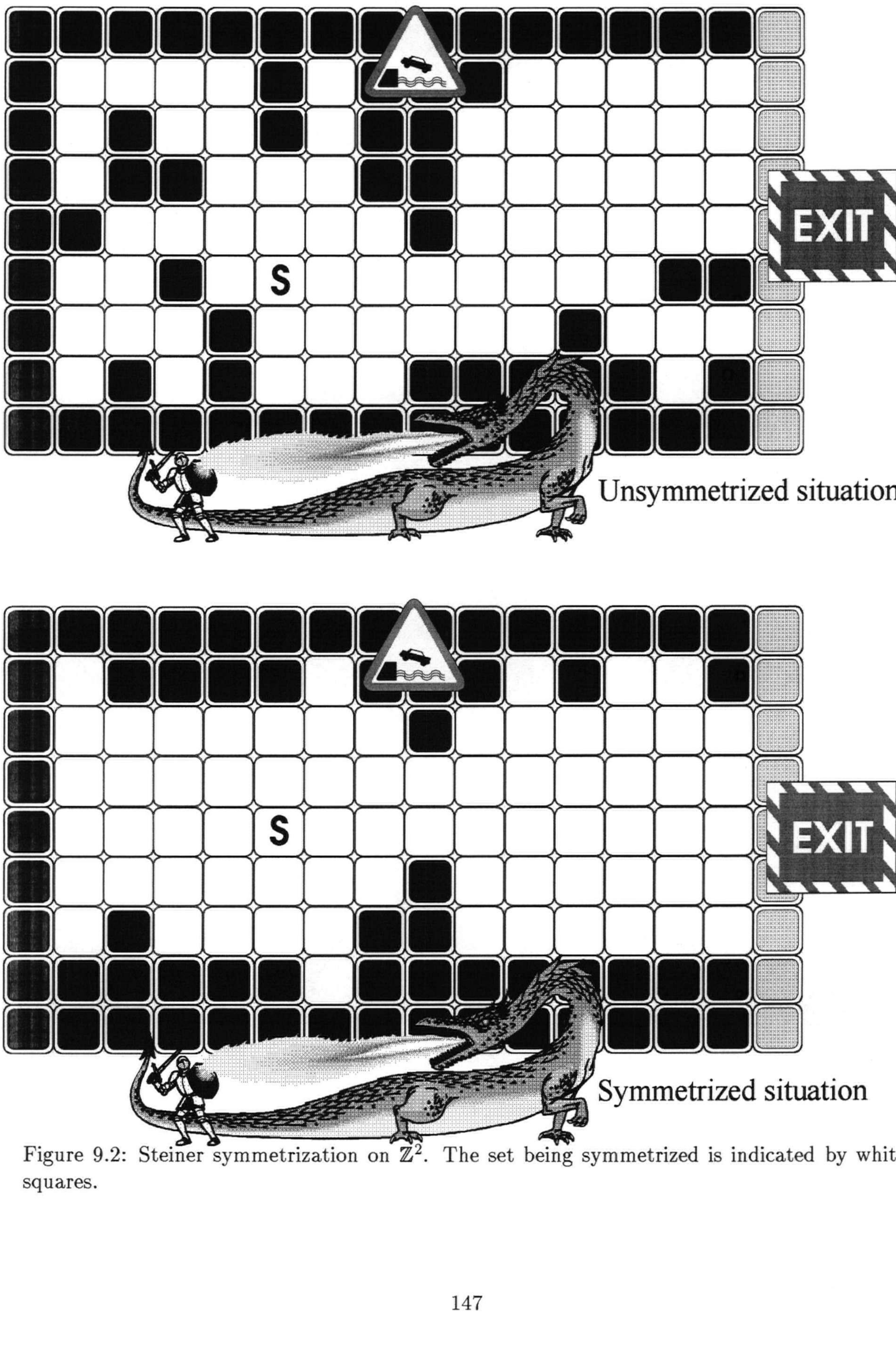


Figure 9.2: Steiner symmetrization on $\mathbb{Z}^2$. The set being symmetrized is indicated by white squares.

9.1.5. Generalized Green's functions

We may now define a **Green's function** for our "fuzzy" domain defined by the survival function s. Simply let

$$g_s(x,y) = E^x\left[\sum_{n=0}^{\tau_s-1} 1_{\{R_n=y\}}\right].$$

This represents the expected number of times that the random walk starting at x visits y before being killed by one of the dangers. Note that, depending on questions of transience, it is quite possible for g_s to be infinite at some pairs of points. If $s = 1_D$ then g_s corresponds to the usual definition of the discrete Green's function g_D for D.

We say that an extended real function G on V^2 is **Steiner bi-symmetric** if for every pair of integers i and j we have

$$\sum_{\alpha\in K,\beta\in L} G((i,\alpha),(j,\beta)) \leq \sum_{\alpha\in K^\#,\beta\in L^\#} G((i,\alpha),(j,\beta)),$$

for all subsets K and L of X. Now, given two positive extended-real functions F and G on V^2, write $F \preceq G$ providing

$$\sup_{|K|=k,|L|=l} \sum_{\alpha\in K,\beta\in L} F((i,\alpha),(j,\beta)) \leq \sup_{|K|=k,|L|=l} \sum_{\alpha\in K,\beta\in L} G((i,\alpha),(j,\beta)),$$

for all integers i and j and all positive integers k and l, where the suprema are taken over all subsets K and L of X containing precisely k and l elements, respectively. If G is in addition Steiner bi-symmetric, then this condition is equivalent to saying that for every pair of integers i and j and every $K \subseteq X$ we have

$$\sum_{\beta\in X} \Phi\left(\sum_{\alpha\in K} F((i,\alpha),(j,\beta))\right) \leq \sum_{\beta\in X} \Phi\left(\sum_{\alpha\in K^\#} G((i,\alpha),(j,\beta))\right),$$

for every convex increasing Φ on $[0,\infty]$. In particular if K is singleton and $\#$ is of Schwarz type on X with initial element O, then it follows that for every fixed $\alpha \in X$ we have

$$\sum_{\beta\in X} \Phi[F((i,\alpha),(j,\beta))] \leq \sum_{\beta\in X} \Phi[F((i,O),(j,\beta))]$$

for any convex non-decreasing Φ and for every pair of integers i and j.

Theorem 9.2. *Consider a random walk $\{R_n\}$ with a kernel K satisfying Assumption 9.3. For any $m \in \mathbb{Z}^+ \cup \{\infty\}$ and any $s\colon V \to [0,1]$ we have*

$$g_s \preceq g_{s\#},$$

and, moreover, $g_{s\#}$ is Steiner bi-symmetric.

In particular, if $\#$ is of Schwarz type on X with initial element O, then for any integers i and j, any $y \in X$ and any convex increasing function Φ we have

$$\sum_{\beta\in X} \Phi(g_s((i,y),(j,\beta))) \leq \sum_{\beta\in X} \Phi(g_{s\#}((i,O),(j,\beta))).$$

Remark 9.4. This is a discrete version of a generalization of a result of Baernstein [7, Thm. 5] (see Theorem I.6.1 of the present thesis). In the special case of the simple random walk on $\mathbb{Z}\times\mathbb{Z}_m$ and $s = 1_D$, this can probably be proved by using Quine's result [90] on the subharmonicity of the discrete $*$-function.

Our proofs, like those of Borell [25], are based on the probabilistic method of Haliste [56, proof of Thm. 8.1].

9.2. Reducing to the case $\lambda = 0$ in Assumption 9.3

We shall show that if we can prove Theorems 9.1 and 9.2 for K satisfying Assumption 9.3 with $\lambda = 0$, then we can prove them for K satisfying Assumption 9.3 with general $\lambda \geq 0$.

For, suppose that K satisfies Assumption 9.3 with some $\lambda > 0$. Define $L = c(K + \lambda\delta)$, where $c > 0$ is chosen so that (9.1) holds with L in place of K, and where $\delta(v,w)$ is 1 if $v = w$ and 0 otherwise. It is easy to see that $c = (1+\lambda)^{-1}$. It is clear that L satisfies Assumptions 9.1 and 9.2.

Use superscripts K and L to distinguish quantities defined in terms of the random walk $\{R_n^K\}$ and the random walk $\{R_n^L\}$ defined by the transition kernels K and L respectively. Professor Gregory Lawler would call R_n^L "the walk R_n^K with geometric waiting times".

Assume now that Theorems 9.1 and 9.2 have been proved for the kernel L. We shall prove them for the kernel K. To do this, proceed as follows. We may describe the random walk $\{R_n^L\}$ slightly differently from before. A step of this random walk consists of first flipping a coin and with probability $p = \lambda/(1+\lambda)$ staying put, while with probability $1-p = 1/(1+\lambda)$ taking a step with transition probabilities defined by the kernel K. Let S_n be the event that the flip of the coin was such that we took the step with transition probabilities defined by K.

Now, note that the walk R_n^L will eventually take a step defined by the kernel K, and that the distribution of R_{n+1}^L conditioned on the event S_n is the same as the unconditioned distribution of R_{n+1}^K. Moreover, the probability that the random walk R_n^L will survive until one of the events S_n happens is equal to

$$(1-p)s(z) + (1-p)p(s(z))^2 + (1-p)p^2(s(z))^3 + \cdots = \frac{(1-p)s(z)}{1-ps(z)} = \phi(s(z)),$$

where $\phi(t) = \frac{(1-p)t}{1-pt}$. Note that ϕ is a strictly increasing function mapping $[0,1]$ onto itself. The above shows that R_n^L with survival probabilities s behaves much as R_n^K with survival probabilities $\phi \circ s$. This observation shows that

$$\omega^K(z, X; \phi \circ s) = \omega^L(z, X; s),$$

or, equivalently,

$$\omega^K(z, X; s) = \omega^L(z, X; \phi^{-1} \circ s),$$

both for any choices of $z \in V$, $S \subseteq V$ and $s\colon V \to [0,1]$. Since ϕ is strictly increasing it follows that $(\phi^{-1} \circ s)^{\#} = \phi^{-1} \circ s^{\#}$ (Theorem I.2.2). These observations show that Theorem 9.1 for the walk $\{R_n^L\}$ with kernel L implies an analogue for the walk $\{R_n^K\}$.

Now, if the random walk $\{R_n^L\}$ is at a point w then the probability that it survives at least k contiguous steps before one of the events S_n happening (i.e., before taking a step in accordance with the kernel K) is

$$s(w)^k p^{k-1}.$$

The expected number of contiguous steps that it survives without taking a step according to

K is then equal to

$$\sum_{k=1}^{\infty} s(w)^k p^{k-1} = \frac{s(w)}{1-s(w)p} = (1-p)^{-1}\phi(s(w)).$$

On the other hand, the expected number of contiguous steps that the random walk $\{R_n^K\}$ equipped with survival probabilities $\phi \circ s$ will survive and stay at w before taking a step according to K is $\phi(s(w))$, if we condition on it starting at w. Combined with the results of the previous paragraph, we can easily convince ourselves that the identity

$$g^K_{\phi \circ s}(z,w) = (1-p)^{-1} g^L_s(z,w),$$

must hold. Using the monotonicity of ϕ and Theorem 9.2 applied with the kernel L, we can see that Theorem 9.2 must also hold for the kernel K.

9.3. Exit times and proofs

Our proofs are based on the methods of Haliste [56, Proof of Thm. 8.1].

Assume that $\{R_n\}$ is a random walk associated with a kernel K. Let $\pi_1 \colon V \to \mathbb{Z}$ be defined by $\pi_1(i,j) = i$ and let $\pi_2 \colon V \to X$ be defined by $\pi_2(i,j) = j$.

Let $\mathcal{G}$ be the σ-field generated by $\{\pi_1 R_n\}_{n=1}^{\infty}$ on our underlying probability space.

Theorem 9.3. *Let $I, J \subseteq X$. Fix $i \in \mathbb{Z}$. Let the kernel K satisfy Assumption 9.3 with $\lambda = 0$. Then*

$$\begin{aligned} &\sum_{j\in I} P^{(i,j)}(\tau_s > N \text{ and } \pi_2(R_N) \in J \mid \mathcal{G}) \\ &\qquad \le \sum_{j \in I^\#} P^{(i,j)}(\tau_s > N \text{ and } \pi_2(R_N) \in J^\# \mid \mathcal{G}), \end{aligned} \tag{9.4}$$

almost surely for every non-negative integer valued random variable N which is measurable with respect to $\mathcal{G}$.

Setting $J = X$ and taking an (unconditional) expectation we will obtain the following theorem.

Theorem 9.4. *Let $I \subseteq X$. Fix an arbitrary $i \in \mathbb{Z}$. Let the kernel K satisfy Assumption 9.3 with $\lambda = 0$. Then,*

$$\sum_{j\in I} P^{(i,j)}(\tau_s \geq N) \leq \sum_{j\in I^{\#}} P^{(i,j)}(\tau_{s^{\#}} \geq N), \tag{9.5}$$

for any non-negative N.

Remark 9.5. Note that if we apply (9.5) with $s^{\#}$ in place of s, then we easily obtain the fact that $z \mapsto P^z(\tau_{s^{\#}} \geq N)$ is Steiner symmetric for every N via Proposition 8.8. Moreover, (9.5) implies that in general

$$(z \mapsto P^z(\tau_s \geq N)) \preceq (z \mapsto P^z(\tau_{s^{\#}} \geq N)).$$

Proof of Theorem 9.1. Assume λ can be taken to be zero in Assumption 9.3. (The case of $\lambda > 0$ then follows from the work of §9.2.) We shall prove that

$$\sum_{j\in I} P^{(i,j)}(\tau_s < \infty \text{ and } R_{\tau_s} \in S) \leq \sum_{j\in I^{\#}} P^{(i,j)}(\tau_{s^{\#}} < \infty \text{ and } R_{\tau_{s^{\#}}} \in S^{\#}), \tag{9.6}$$

for $S \subseteq \{k\} \times X$. This will imply Theorem 9.1 (the requisite Steiner symmetry will follow by applying the result with $s = s^{\#}$ and using Proposition 8.8). We shall now prove that

$$\begin{aligned} &\sum_{j\in I} P^{(i,j)}(\tau_s < \infty \text{ and } R_{\tau_s} \in X \mid \mathcal{G}) \\ &\qquad \leq \sum_{j\in I^{\#}} P^{(i,j)}(\tau_{s^{\#}} < \infty \text{ and } R_{\tau_{s^{\#}}} \in X \mid \mathcal{G}). \end{aligned} \tag{9.7}$$

Inequality (9.6) will then follow by taking an unconditional expectation. Now, to prove (9.7), first suppose $\{\pi_1(R_n)\}$ never visits k. Then both sides of (9.7) vanish and we are done. Hence, suppose that $\{\pi_1(R_n)\}$ visits k and let $N = \inf\{n \geq 0 : \pi_1(R_n) = k\}$. In this case we clearly have $P(\tau_s < \infty \mid \mathcal{G}) = P(\tau_{s^{\#}} < \infty \mid \mathcal{G}) = 1$ since s vanishes on $\{k\} \times X$. For convenience, set $\tilde{s}(z) = s(z)$ for $z \notin \{k\} \times X$ and let $\tilde{s}(z) = 1$ for $z \in \{k\} \times X$. Then, (9.7) is equivalent to the assertion that

$$\sum_{j\in I} P^{(i,j)}(\tau_{\tilde{s}} > N \text{ and } R_N \in X \mid \mathcal{G}) \leq \sum_{j\in I^{\#}} P^{(i,j)}(\tau_{\tilde{s}^{\#}} > N \text{ and } R_N \in X^{\#} \mid \mathcal{G}).$$

But this inequality holds by Theorem 9.3 since N is $\mathcal{G}$-measurable. ☐

We now prove Theorem 9.2.

Proof of Theorem 9.2. It suffices to prove that whenever K and L are finite subsets of X and i and j are integers, then we have

$$\sum_{\alpha\in K,\beta\in L} g_s((i,\alpha),(j,\beta)) \le \sum_{\alpha\in K^\#,\beta\in L^\#} g_{s^\#}((i,\alpha),(j,\beta)).$$

We proceed once more by conditioning on $\mathcal{G}$ and then taking an unconditional expectation. In this way, using the definition of the Green's function, we see that it suffices for us to prove that

$$\begin{aligned}\sum_{\alpha\in K,\beta\in L} & E^{(i,\alpha)}\Big[\sum_{n=0}^{\tau_s-1} 1_{\{R_n=(j,\beta)\}} \mid \mathcal{G}\Big] \\ &\le \sum_{\alpha\in K^\#,\beta\in L^\#} E^{(i,\alpha)}\Big[\sum_{n=0}^{\tau_{s^\#}-1} 1_{\{R_n=(j,\beta)\}} \mid \mathcal{G}\Big].\end{aligned}$$

By Fubini's theorem this is equivalent to the assertion that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{\alpha\in K,\beta\in L} & E^{(i,\alpha)}[1_{\{R_n=(j,\beta)\text{ and }\tau_s>n\}} \mid \mathcal{G}] \\ &\le \sum_{n=0}^{\infty}\sum_{\alpha\in K^\#,\beta\in L^\#} E^{(i,\alpha)}[1_{\{R_n=(j,\beta)\text{ and }\tau_{s^\#}>n\}} \mid \mathcal{G}].\end{aligned} \tag{9.8}$$

Let T be the random set $\{n\in\mathbb{Z}_0^+ : \pi_1(R_n)=j\}$. Then, (9.8) is equivalent to the assertion that

$$\begin{aligned}\sum_{n\in T}\sum_{\alpha\in K} & P^{(i,\alpha)}(\pi_2(R_n)\in L\text{ and }\tau_s>n \mid \mathcal{G}) \\ &\le \sum_{n\in T}\sum_{\alpha\in K^\#} P^{(i,\alpha)}(\pi_2(R_n)\in L^\#\text{ and }\tau_{s^\#}>n \mid \mathcal{G}).\end{aligned}$$

But this follows immediately from Theorem 9.3. $\square$

We now proceed to prove Theorem 9.3 itself. In order to do this, we first state a lemma giving the inequality that lies at its heart. This inequality is a generalization of the idea of the proof of Haliste's [56, Lemma 8.1] and can be called an "iterated convolution-rearrangement inequality".

We now need an assumption on a function $L\colon X\times X\to[0,\infty)$.

Assumption 9.4. The following inequality is valid whenever f and g are positive functions on X:

$$\sum_{x,y\in X} f(x)L(x,y)g(y) \le \sum_{x,y\in X} f^{\#}(x)L(x,y)g^{\#}(y).$$

Write $Lg(v) = \sum_{w\in V} L(v,w)g(w)$ and $L^*f(w) = \sum_{v\in V} f(v)L(v,w)$. By Theorem 8.1 we obtain the following result.

Proposition 9.3. *Assume Assumption* 9.4. *Then the functions $Lg, L^*g\colon X \to [0,\infty)$ are symmetric whenever $g\colon X \to [0,\infty)$ is symmetric.*

Lemma 9.1. *Let $L^0, L^1, \dots, L^{k-1}$ be a sequence of functions on $X \times X$ all satisfying Assumption 9.4. Let $S^0, S^1, \dots, S^k$ also be a collection of positive functions on X. Let*

$$\Psi_k(S,L) = \sum_{\nu_0} \cdots \sum_{\nu_k} S^0(\nu_0)L^0(\nu_0,\nu_1)\cdots S^{k-1}(\nu_{k-1})L^{k-1}(\nu_{k-1},\nu_k)S^k(\nu_k),$$

for $k > 0$, where all sums are taken over X, and let $\Psi_0(S,L) = \sum_{\nu_0} S^0(\nu_0)$. Then for each $k \in \mathbb{Z}_0^+$ we have

$$\Psi_k(S,L) \le \Psi_k(S^{\#},L),$$

where $(S^{\#})^i(j) \overset{\text{def}}{=} (S^i)^{\#}(j)$.

Assuming this lemma, we now give a proof of Theorem 9.3.

Proof of Theorem 9.3. Let $f(n) = \pi_1(R_n)$ so that $\mathcal{G} = \sigma(\{f(n)\}_{n=0}^{\infty})$. Let $\mathcal{F}$ be the σ-field generated by $\{R_n\}_{n=1}^{\infty}$.

Let N be a $\mathcal{G}$-measurable random variable with values in $\mathbb{Z}_0^+$. Let

$$p_s(i,j,N) = P^{(i,j)}(\tau_s > N \text{ and } \pi_2(R_N) \in J \mid \mathcal{G}).$$

We shall now give a formula for p_s. To do this rigorously, let $g(n) = \pi_2(R_n)$. Then,

$$\begin{aligned} E_s(N) &\overset{\text{def}}{=} \{\tau_s > N \text{ and } \pi_2(R_N) \in J\} \\ &= \{g(N) \in J\} \cap \left(\bigcap_{n=0}^{N} \{X_n \le s(f(n), g(n))\} \right). \end{aligned} \tag{9.9}$$

Thus,

$$P(E_s(N) \mid \mathcal{F}) = 1_{\{g(N)\in J\}} \cdot s(f(0), g(0))s(f(1), g(1)) \cdots s(f(N), g(N)). \tag{9.10}$$

Let

$$L^n(x, y) = P(g(n) = y \mid g(n-1) = x, \mathcal{G}),$$

for $1 \leq n \leq N$. Let

$$L^{N+1}(x, y) = \delta_{x,y}.$$

Put $S^{N+2} = 1_J$. Then, taking the conditional expectation of (9.10) given $\mathcal{G}$, we find that we have to basically average $\mathcal{G}$ over all functions g with transition probabilities L^n, and we obtain

$$\begin{aligned} p_s(i, j, N) &= P^{(i,j)}(E_s(N) \mid \mathcal{F}) \\ &= \sum_{\nu_1 \in X} \sum_{\nu_2 \in X} \cdots \sum_{\nu_{N+1} \in X} \\ &\qquad s(f(0), j)L^1(j, \nu_1)s(f(1), \nu_1)L^2(\nu_1, \nu_2) \cdots \\ &\qquad s(f(N), \nu_N)L^{N+1}(\nu_N, \nu_{N+1})S^{N+2}(\nu_{N+1}). \end{aligned} \tag{9.11}$$

Now, let $L^0(x, y) = \delta_{x,y}$. Set $S^0 = 1_I$. Let $S^n(j) = s(f(n-1), j)$ for $1 \leq n \leq N+1$. Then, it follows from (9.11) that

$$\sum_{j \in I} p_s(i, j, N) = \Psi_{N+2}(S, L), \tag{9.12}$$

where Ψ_{N+2} is defined as in Lemma 9.1. Thus the left hand side of (9.4) equals $\Psi_{N+2}(S, L)$.

Henceforth, implicitly condition all our reasoning on $\mathcal{G}$.

Now, we have $(s(i, \cdot))^{\#} = (s^{\#})(i, \cdot)$, the first quantity being defined in terms of the rearrangement on X and the second in terms of the Steiner rearrangement on $V = \mathbb{Z} \times X$ induced by the first. Tracing through the above work we see that the right hand side of (9.4) thus equals $\Psi_{N+2}(S^{\#}, L)$. Hence, the conclusion of our theorem will follow from Lemma 9.1 as soon as we prove that all our kernels L^n satisfy Assumption 9.4.

First of all, L^0 and L^{N+1} both satisfy Assumption 9.4 in light of the Hardy-Littlewood inequality (Theorem 1.1). Thus it suffices to verify that L^n satisfies Assumption 9.4 for $1 \leq n \leq N+1$.

More precisely, we shall prove that there is a set $A \in \mathcal{G}$ such that $P(A) = 1$ and such that (given $\mathcal{G}$) we have Assumption 9.4 holding on A. Then via Lemma 9.1 we will obtain the inequality $\Psi_{N+2}(S, L) \le \Psi_{N+2}(S^{\#}, L)$ on A, and hence almost surely (given $\mathcal{G}$), as desired.

Define

$$k^i(m,n) = \sum_{j \in X} K((m,i),(n,j)),$$

for $i \in X$ and m and n in $\mathbb{Z}$. Note that in fact $k^i(m,n)$ does not depend on i in light of Assumption 9.2; we shall sometimes omit the superscript i and at other times we shall retain it for emphasis. Let

$$B = \bigcup_{n=1}^{N} \{k(f(n-1), f(n)) = 0\}.$$

It is clear that B has probability zero by definition of f and the random walk K. Let A be the complement of B. This is then a $\mathcal{G}$ measurable event which happens almost surely. Henceforth, we assume that A happens. Then,

$$\begin{aligned}
L^n(x,y) &= P(g(n) = y \mid g(n-1) = x, \mathcal{G}) \\
&= P(R_n = (f(n), y) \mid R_{n-1} = (f(n-1), x), \mathcal{G}) \\
&= K((f(n-1),x),(f(n),y))/k^x(f(n-1), f(n)) \\
&= K((f(n-1),x),(f(n),y))/k(f(n-1), f(n)).
\end{aligned}$$

But now on A the quantity $k(f(n-1), f(n))$ is nonzero, and it is independent of x as noted before. Hence, Assumption 9.4 follows from Assumption 9.1 and Proposition 9.1, as desired. $\square$

Our proof of Lemma 9.1 is a generalization of the proof of Haliste's [56, Lemma 8.1].

Proof of Lemma 9.1. Proceed by induction on k. If $k = 1$ then Lemma 9.1 is equivalent to Assumption 9.4. Hence assume that $k > 1$ and that Lemma 9.1 has been proved for all smaller values of k.

To reduce clutter, let $T = S^{\#}$. Put

$$c(\nu) = \sum_{\nu_1} \cdots \sum_{\nu_k} L^1(\nu, \nu_1) \cdots T^{k-1}(\nu_{k-1}) L^{k-1}(\nu_{k-1}, \nu_k) T^k(\nu_k),$$

and

$$B(\nu) = \sum_{\nu_0} S^0(\nu_0)L^0(\nu_0,\nu)S^1(\nu).$$

Since Lemma 9.1 holds for $k-1$ it follows that

$$\sum_{\nu_1}\sum_{\nu_2}\cdots\sum_{\nu_k} B(\nu_1)L^1(\nu_1,\nu_2)\cdots S^{k-1}(\nu_{k-1})L^{k-1}(\nu_{k-1},\nu_k)T^k(\nu_k)$$
$$\leq \sum_{\nu_1} B^{\#}(\nu_1)c(\nu_1).$$

The left hand side of this inequality equals $\Psi_k(S,L)$. Hence,

$$\Psi_k(S,L) \leq \sum_{\nu_1} B^{\#}(\nu_1)c(\nu_1).$$

I claim that

$$\sum_{\nu_1} B^{\#}(\nu_1)c(\nu_1) \leq \Psi_k(T,L). \tag{9.13}$$

If this claim is just then we are obviously done. Now, we may assume that S^1 has finite support (the general case follows by approximation via the monotone convergence theorem and Lemma I.2.1.)

Apply Proposition 8.6 to the function $f = B$ which has finite support if S^1 has finite support. Then, $B^{\#} = B \circ \phi$. Define $\check{c}(\nu) = c(\phi^{-1}(\nu))$ and $d(\nu) = S^1(\nu)\check{c}(\nu)$. Then,

$$\begin{aligned}\sum_{\nu_1} B^{\#}(\nu_1)c(\nu_1) &= \sum_{\nu_1} B(\nu_1)\check{c}(\nu_1) \\ &= \sum_{\nu_0,\nu_1} S^0(\nu_0)L^0(\nu_0,\nu_1)d(\nu_1) \\ &\leq \sum_{\nu_0,\nu_1} T^0(\nu_0)L^0(\nu_0,\nu_1)d^{\#}(\nu_1),\end{aligned} \tag{9.14}$$

by Assumption 9.4. Let

$$E(\nu) = \sum_{\nu_0} T^0(\nu_0)L^0(\nu_0,\nu).$$

By Proposition 9.3, the function E is symmetric. I claim that

$$\sum_{\nu_1} E(\nu_1)d^{\#}(\nu_1) \leq \sum_{\nu_1} E(\nu_1)T^1(\nu_1)c(\nu_1). \tag{9.15}$$

If this claim is true then we are done since the right hand side of this inequality is precisely equal to $\Psi_k(T, L)$, so that (9.13) would immediately follow from (9.14).

Now, since S^1 has finite support, it follows that d has finite support. Applying Proposition 8.6 with $f = d$, let ψ be the function "ϕ" given by that Proposition. Then, $d^{\#}(\nu) = d(\psi(\nu)) = S^1(\psi(\nu))\check{c}\psi(\nu))$. Then, the left hand side of (9.15) equals

$$\sum_{\nu_1} \check{E}(\nu_1)S^1(\nu)\check{c}(\nu),$$

where $\check{E}(\nu) = E(\psi^{-1}(\nu))$. Applying Theorem I.2.4, we see that

$$\sum_{\nu_1} \check{E}(\nu_1)S^1(\nu)\check{c}(\nu) \leq \sum_{\nu_1} E(\nu_1)T^1(\nu)c(\nu),$$

as desired, where we have used the fact that $T^1 = (S^1)^{\#}$ and the facts $\check{E}^{\#} = E$ and $\check{c}^{\#} = c$ which came from our two applications of Proposition 8.6, since E and c are symmetric. The symmetry of E was already noted. The symmetry of c, on the other hand, is also not hard to prove. To prove it, one only needs to inductively apply Proposition 9.3 as well as Corollary 8.5. □

10. A discrete Beurling shove theorem

In this section we are working with the simple random walk R_n^{S} defined by the kernel $K_{\mathrm{S}}^{\mathbb{Z}} \overset{1/2}{\otimes} K^{\mathbb{Z}_m}$ on $\mathbb{Z} \times \mathbb{Z}_m$ for $m \geq 3$ (see §9.1.2), and all quantities (Green's functions, etc.) should be interpreted with respect to it. Given $D \subseteq \mathbb{Z} \times \mathbb{Z}_m$, we write $\omega(\cdot, \cdot; D)$ for $\omega(\cdot, \cdot; 1_D)$ and g_D for g_{1_D}.

Let $D = \mathbb{Z}^- \times \mathbb{Z}_m$, where $\mathbb{Z}^- = \{-1, -2, \ldots\}$. Let $T = \{0\} \times \mathbb{Z}_m$. Then the following result is a discrete analogue of Beurling's shove theorem [23, pp. 58–62] (an account of this theorem and some generalizations in the continuous case will be given in §IV.8).

Theorem 10.1. *Let H be a finite non-empty subset of $\mathbb{Z}^- \times \{0\}$, and set $U = D \backslash H$. Let $U^\diamond = D \backslash H'$, where $H' = \{-|H|, -|H| + 1, \ldots, -1\} \times \{0\}$. Then,*

$$\omega((t, 0), T; U) \leq \omega((t, 0), T; U^\diamond), \tag{10.1}$$

whenever $t < \inf\{t' : (t', 0) \in H\}$.

Remark 10.1. One may conjecture a number of generalizations of this. One such would be to consider a survival function s instead of U, such that s and $s^\diamond$ vanish identically on $V \backslash D$ and are identically 1 everywhere on D except possibly on $\mathbb{Z}^- \times \{0\}$, while $s^\diamond(\cdot, 0)$ is the decreasing rearrangement of $s(\cdot, 0)$. Then the conjecture of course is that the analogous inequality continuous to hold. This conjecture appears to be nontrivial even in the case of $m = 1$ (with appropriate definitions which collapse the random walk to a one-dimensional walk), although in that case it *is* true (Theorem IV.9.1).

The proof of Theorem 10.1 will be done almost exactly as in the continuous case, as soon as we establish two lemmas and discuss the notion of a discrete harmonic function.

Let $U \subseteq V = \mathbb{Z} \times \mathbb{Z}_m$. Write ∂U for the collection of points of $V \backslash U$ which lie precisely one simple random walk step away from U, i.e.,

$$\partial U = \{(x, y) \in V \backslash U : \\ (x-1, y) \in U \text{ or } (x+1, y) \in U \text{ or } (x, y-1) \in U \text{ or } (x, y+1) \in U\}.$$

Write $\bar{U} = U \cup \partial U$. Let f be any function on $\bar{U}$. Define

$$\Delta f(x, y) = \tfrac{1}{4}[f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)] - f(x, y),$$

for $(x, y) \in U$. Then Δ is a discrete Laplacian. Moreover, for any $i \in \mathbb{Z}_0^+$ we have

$$\Delta f(z) = E[f(R_{i+1}) \mid R_i = z] - f(z), \tag{10.2}$$

where $\{R_n\}$ is our simple random walk on V. We say that a function f is a **(discrete) harmonic function on** U if it is defined on $\bar{U}$ and satisfies $\Delta f(z) = 0$ for all $z \in U$.

Example 10.1. The function $z \mapsto g_D(z, w)$ is a discrete harmonic function on $D \backslash \{w\}$ for any fixed $w \in V$. This is easiest seen directly from the definition of $g_D = g_{1_D}$ and from equation (10.2).

Example 10.2. The function $z \mapsto \omega(z, A; D)$ is a harmonic function on D for $A \subseteq \partial D$. This is also easy to see from (10.2) and the definition of $\omega(z, A; D) = \omega(z, A; 1_D)$.

Clearly sums and scalar multiples of harmonic functions are harmonic. A central result about discrete harmonic functions is the following very well-known maximum principle.

Theorem 10.2. *Let f be a harmonic function on $U \subset V$ which is bounded above and let C be a real constant such that $f(z) \leq C$ for all $z \in \partial U$. Assume that $U \neq V$. Suppose that there exists $w \in U$ such that $f(w) \geq C$. Then f is constant on $\bar{U}$.*

We now state our two lemmas which provide the keys to the proof of Theorem 10.1.

Lemma 10.1. *Let $h(t_1, t_2) = g_D((t_1, 0), (t_2, 0))$. Then for fixed $t_2 \in \mathbb{Z}^-$ the function $h(\cdot, t_2)$ is increasing on $(-\infty, t_2] \cap \mathbb{Z}^-$, and decreasing on $[t_2, -1] \cap \mathbb{Z}^-$. Similarly, for fixed $t_1 \in \mathbb{Z}^-$ the function $h(t_1, \cdot)$ is increasing on $(-\infty, t_1] \cap \mathbb{Z}^-$, and decreasing on $[t_2, -1] \cap \mathbb{Z}^-$.*

Proof. Fix $t_2 \in \mathbb{Z}^-$. First suppose $t_1 > t_2$. We shall show that $h(t_1 - 1, t_2) \geq h(t_1, t_2)$. Let $D_1 = \{t_1 - 1, t_1, \ldots, -1\} \times \mathbb{Z}_m$ Then, it is easy to see that

$$h(t_1, t_2) = \sum_{\alpha \in \mathbb{Z}_m} \omega((t_1, 0), \{(t_1 - 1, \alpha)\}; D_1) g_D((t_1 - 1, \alpha), (t_2, 0)).$$

But g_D is bi-symmetric because of Theorem 9.2 so that $g_D((t_1 - 1, \alpha), (t_2, 0)) \leq g_D((t_1 - 1, 0), (t_2, 0))$. Thus,

$$\begin{aligned} h(t_1, t_2) &\leq g_D((t_1 - 1, 0), (t_2, 0)) \sum_{\alpha \in \mathbb{Z}_m} \omega((t_1, 0), \{(t_1 - 1, \alpha)\}; D_1) \\ &= \omega((t_1, 0), \{t_1 - 1\} \times \mathbb{Z}_m\}; D_1) h(t_1 - 1, t_2) \\ &\leq h(t_1 - 1, t_2). \end{aligned}$$

The inequality $h(t_1 + 1, t_2) \geq h(t_1, t_2)$ in the case $t_1 < t_2$ is proved very similarly. The case of t_1 fixed can be handled just as above (or else it can be noted that it follows from the fact that $h(t_1, t_2) = h(t_2, t_1)$ for the simple random walk.) ☐

Now, for a subset S of $\mathbb{Z} \times \mathbb{Z}_m$, let

$$\tau_S = \inf\{n \geq 0 : R_n \notin S\}$$

and

$$\tilde{\tau}_S = \inf\{n > 0 : R_n \notin S\}.$$

The following lemma then is valid for any random walk, not just the simple random walk. It extends in an easy way to a number of situations.

Lemma 10.2. *Let $\phi(z) = 1 - \omega(z, T; U)$. Then*

$$\phi(z) = \sum_{w \in H} g_D(z, w)\psi(w),$$

for a positive function ψ on H. More precisely, we may take

$$\psi(w) = P^w(\tilde{\tau}_U = \tilde{\tau}_D).$$

Proof. For $\psi(w) = P^w(\tilde{\tau}_U = \tilde{\tau}_D)$, by the definition of the Green's function and by Fubini's theorem we have

$$\begin{aligned}
&\sum_{w \in H} g_D(z, w)\psi(w) \\
&\quad = \sum_{w \in H} E^z \left[\sum_{n=0}^{\tau_D - 1} 1_{\{R_n = w\}} \right] P^w(\tilde{\tau}_U = \tilde{\tau}_D) \\
&\quad = \sum_{n=0}^{\infty} \sum_{w \in H} P^z(R_n = w \text{ and } \tau_D > n) P^w(\tilde{\tau}_U = \tilde{\tau}_D) \\
&\quad = \sum_{n=0}^{\infty} P^z(R_n \in H,\, \tau_D > n \text{ and } (R_k \in U, \forall k \in \{n+1, \dots, \tau_D - 1\})).
\end{aligned}$$

But it is easy to see that the events within the $P^z(\cdot)$ are disjoint for distinct values of n since $H \subseteq U^c$. Moreover, it is easy to see that the union over $n \in \mathbb{Z}_0^+$ of these events is the event $\{\exists n \in \mathbb{Z}_0^+ . (R_n \in H \text{ and } n < \tau_D)\}$. But clearly the probability of this event if the random walk starts at z is precisely $\omega(z, H; D) = \phi(z)$. □

Proof of Theorem 10.1. For conciseness, given a subset L of $\mathbb{Z}_0^- \times \{0\}$, write

$$\inf L \overset{\text{def}}{=} \inf\{l : (l, 0) \in L\}.$$

Let $t_0 = \inf H$. We proceed by induction on $N = |t_0| - |H|$. First, if $N = 0$ then $H = H'$ and we are done. Suppose now that the result has been proved whenever $|t_0| - |H| < N$ and that

$$H = \boxed{\begin{array}{cccccccccc} & \boxed{A_1} & \boxed{A_1} & & & \boxed{A_2} & \boxed{A_2} & \boxed{A_2} & & \boxed{A_2} \\ (-10,0) & (-9,0) & (-8,0) & (-7,0) & (-6,0) & (-5,0) & (-4,0) & (-3,0) & (-2,0) & (-1,0) \\ \square & \blacksquare & \blacksquare & \square & \square & \blacksquare & \blacksquare & \blacksquare & \square & \blacksquare \\ & \uparrow & & \uparrow & & & & & & \\ & t_0 & & t_1 & & & & & & \end{array}}$$

$$H_1 = \boxed{\begin{array}{cccccccccc} (-10,0) & (-9,0) & (-8,0) & (-7,0) & (-6,0) & (-5,0) & (-4,0) & (-3,0) & (-2,0) & (-1,0) \\ \square & \square & \blacksquare & \blacksquare & \square & \blacksquare & \blacksquare & \blacksquare & \square & \blacksquare \end{array}}$$

Figure 10.1: An example of the sets H and H_1 (indicated with $\blacksquare$'s) in a case where $t_0 = -9$, $t_1 = -7$, $|H| = 6$ and $N = 3$. The symbol $\boxed{A_i}$ indicates a point contained in the set A_i.

$N \geq 1$. Let $t_1 = \inf\{t \in \mathbb{Z}^- : t \geq t_0, t \notin H\}$. Since $N \geq 1$, we have $t_1 \in \{t_0 + 1, \ldots, 1\}$, and moreover $\{t_0, \ldots, t_1 - 1\} \subseteq H$. (See Figure 10.1.)

Define

$$H_1 = (H \cap [t_1, 1]) \cup \{t_0 + 1, \ldots, t_1 + 1\}.$$

(See Figure 10.1.) It is easy to see that $|H_1| = |H|$ and that H_1 is in fact just H with the hole at t_1 deleted. Moreover, $|\inf H_1| = t_0 + 1$ so that $|\inf H_1| - |H| < N$ as $t_0 < 0$. Thus, if we form $(H_1)'$ from H_1 in the same way that H' is formed from H, by our induction hypothesis we will have

$$\omega((t,0), T; D\backslash H_1) \leq \omega((0,t), T; D\backslash (H_1)'),$$

whenever $t < \inf H_1$, and in particular whenever $t < \inf H$. But $|H_1| = |H|$ so that $(H_1)' = H'$. Thus, the desired inequality (10.1) will follow as soon as we establish the fact that

$$\omega((t,0), T; D\backslash H) \leq \omega((t,0), T; D\backslash H_1) \tag{10.3}$$

whenever $t < \inf H$. Write $H = A_1 \cup A_2$ where $A_1 = \{t_0, \ldots, t_1 - 1\}$ and $A_2 = \{t_1 + 1, \ldots 1\} \cap H$. Let $\phi(z) = \omega(z, T; D\backslash H)$. Then, by Lemma 10.2 we have

$$\phi(z) = \phi_1(z) + \phi_2(z),$$

where

$$\phi_i(z) = \sum_{w \in A_i} g_D(z,w)\psi(w),$$

for $i = 1,2$. For $(x,y) \in \mathbb{Z}_0^- \times \mathbb{Z}_m$, let $\bar{\phi}(x,y) = \phi_1(x-1,y) + \phi_2(x,y)$. I claim that

$$\bar{\phi}(z) \geq 1 - \omega(z,T;D \backslash H_1) \tag{10.4}$$

for all $z \in \mathbb{Z}_0^- \times \mathbb{Z}_m$. Suppose for now that this claim is just. Then, for $t < \inf H$ we have

$$\omega((t,0),T;D\backslash H_1) \geq 1 - \phi_1(t-1,0) - \phi_2(t,0).$$

But $\phi_1(t-1,0) \leq \phi_1(t,0)$ since $g_D((t,0),(u,0))$ is increasing in t for $t < u$ (Lemma 10.1) and since ψ is positive while $A_1 \subseteq [-t_0,-1] \times \{0\}$. Thus,

$$\omega((t,0),T;D\backslash H_1) \geq 1 - \phi_1(t,0) - \phi_2(t,0) = 1 - \phi(t,0) = \omega((t,0),T;D\backslash H),$$

which is precisely what we were supposed to prove.

Thus, we need only verify (10.4). But $\bar{\phi}$ is a bounded[3] discrete harmonic function on $D\backslash H_1$ since ϕ_1 and ϕ_2 are harmonic on $D\backslash A_1$ and $D\backslash A_2$, respectively, (since the ϕ_i are sums of Green's functions to which we can apply Example 10.1), and $\omega(\cdot,T;D\backslash H_1)$ is harmonic in $D\backslash H_1$ (Example 10.2), while

$$H_1 = ((1,0) + A_1) \cup A_2.$$

Thus, the maximum principle (Theorem 10.2) implies that to show (10.4) it suffices to verify that (10.4) holds on $\partial(D\backslash H_1) = H_1 \cup H_2$ and T.

But on T, the inequality (10.4) holds trivially as its right hand side vanishes while the left is positive. Suppose now that $z \in H_1 = ((1,0) + A_1) \cup A_2$. Then the right hand side of (10.4) equals 1. There are two cases to consider. First suppose that $z \in ((1,0) + A_1)$. Then, $z = (x+1,0)$, where $(x,0) \in A_1$. We have

$$\bar{\phi}(z) = \phi_1(x+1-1,0) + \phi_2(x+1,0) = \phi_1(x,0) + \phi_2(x+1,0).$$

[3]The function $\bar{\phi}$ is bounded since ϕ is bounded which implies that ϕ_1 and ϕ_2 are both bounded.

But whenever $w \in A_2$ and $(x,0) \in A_1$, we have $g_D((x+1,0),w) \geq g_D((x,0),w)$ by Lemma 10.1 so that $\phi_2(x+1,0) \geq \phi_2(x,0)$, and so

$$\bar{\phi}(z) \geq \phi_1(x,0) + \phi_2(x,0) = \phi(x,0).$$

Now, $(x,0) \in H$ so that $\phi(x,0) = 1 - \omega((x,0),T;D\backslash H) = 1$, and so (10.4) is verified. Suppose now that $z = (x,0) \in A_2$. Then,

$$\bar{\phi}(z) = \phi_1(x-1,0) + \phi_2(x,0).$$

But whenever $w \in A_1$ and $(x,0) \in A_2$ we have $g_D((x-1,0),w) \geq g_D((x,0),w)$ by Lemma 10.1, so that $\phi_1(x-1,0) \geq \phi_1(x,0)$, and thus

$$\bar{\phi}(z) \geq \phi_1(x,0) + \phi_2(x,0) = \phi(x,0) = 1,$$

since $(x,0) \in H$ as before, so that again (10.4) is verified. □

Remark 10.2. The methods of this section apply equally well to a random walk on $\mathbb{Z}_0^- \times G$ where G is one of the constant degree graphs for which a master inequality holds under some well-ordering $\prec$ with each element having at most finitely many predecessors, and where $0 \in \mathbb{Z}_m$ is replaced by the initial element O with respect to $\prec$. For instance, in the above work we may replace the graph $\mathbb{Z}_m$ by the regular tree T_p.

11. A general rearrangement method for difference equations

In this section we shall present a quite general rearrangement method for difference equations. Our method is a modification of the method of Baernstein [11].

Let X and Y be countable sets equipped with counting measures and let $\#$ be a Steiner type rearrangement from the power set 2^X to the power set 2^Y. Let $\mathcal{A}$ be the fibres of this rearrangement.

Let K be a positive function on $X \times X$. For a function f on X, define

$$Kf(x) = \sum_{y \in X} K(x,y) f(y)$$

and

$$K^*f(y) = \sum_{x \in X} f(x)K(x,y).$$

Let L be a positive function on $Y \times Y$, and define Lf and L^*f analogously to Kf and K^*f. We make the following assumptions on K and L.

Assumption 11.1. For every $x \in X$ we have

$$\sum_{y \in Y} L(x,y) \leq 1.$$

Assumption 11.2. For every pair of positive functions f and g on X we have

$$\sum_{x,y \in X} f(x)K(x,y)g(y) \leq \sum_{x,y \in Y} f^{\#}(x)L(x,y)g^{\#}(y). \tag{11.1}$$

Assumption 11.3. If f is a positive symmetric function on Y then so is L^*f.

Remark 11.1. If $X = Y$, $K = L$ and $\#$ is a symmetrization, and if (11.1) holds for *all* positive f and g, then Assumption 11.3 follows from Theorem 8.1. The setting of $X = Y$, $K = L$ and $\#$ a symmetrization is very close to the setting of the previous section.

Example 11.1 ***(Trivial example).*** Let $X = Y$, $K = L$ and let $S^{\#} = S$ for all S. This is a Steiner type symmetrization (just let the fibres be $\mathcal{A} = \{\{x\} : x \in X\}$), such that all functions are symmetric and all our assumptions are trivial.

We define the operators

$$Df = (K-1)f = (Kf) - f$$

and

$$Ef = (L-1)f = (Lf) - f.$$

In many interesting cases these will essentially be discrete Laplacians.

If f is a real function on X while g is a real function on Y then we write $f \lhd g$ if

$$\sum_S f^{\#} \leq \sum_S g$$

for all symmetric sets S. Since level sets of symmetric functions are symmetric, it easily follows that $f \lhd g$ if and only if for every positive symmetric h we have

$$\sum_Y h \cdot f^{\#} \leq \sum_Y h \cdot g.$$

Let $\mathcal{M}$ be the collection of all positive functions symmetric f on Y such that

$$\sum_Y f \leq 1.$$

Given a function g on Y, define the functional

$$\sigma_g(f) = \sum_Y f \cdot g$$

for $f \in M$. It is clear that $f \lhd g$ if and only if $\sigma_{f^{\#}-g}(h) \leq 0$ for all $h \in M$.

Given a countable set X, let $w(X)$ be the collection of all real functions f such that

$$|S_n|^{-1} \sum_{x \in S_n} f(x) \to 0$$

whenever $\{S_n\}_{n=1}^{\infty}$ is a collection of subsets of X with $|S_n| \to \infty$.

Lemma 11.1. *Suppose that $X = Y$ and that $\#$ is a symmetrization. Let v be a function on X such that $v \lhd v$. Then v is symmetric.*

Proof. Suppose that v and $v^{\#}$ do not coincide. Then there exists $x \in X$ such that $v(x) \neq v^{\#}(x)$. Choose the fibre A such that $x \in A$. By Lemma 8.1 we have $A = A^{\#}$. Let $\prec$ be the ordering on $A = A^{\#}$ induced by Proposition 8.1, and let $x_0 \prec x_1 \prec \cdots$ be an enumeration of A. Choose the smallest $n \in \mathbb{Z}_0^+$ such that $v(x_n) \neq v^{\#}(x_n)$. Then, $v^{\#}(x_i) = v(x_i)$ for $i \in \{0, \ldots, n-1\}$ and $v^{\#}(x_n) > v(x_n)$ since $v^{\#}$ restricted to A is the $\prec$-decreasing rearrangement of v (this follows from Proposition 8.1).

Letting $S = \{x_0, \ldots, x_n\}$, which is symmetric, we have

$$\sum_S v^{\#} = \sum_S v = v^{\#}(x_n) - v(x_n) > 0,$$

contradicting the assumption that $v \lhd v$. $\square$

Proposition 11.1. *Suppose that $u \lhd v$. Then for every convex increasing function Φ and fibre $A \in \mathcal{A}$, we have*

$$\sum_{x\in A}\Phi(u(x)) \leq \sum_{y\in A^{\#}}\Phi(v(y)).$$

Proof. Considering this question only on the fibre A, we see that we may assume that we are working with a Schwarz type rearrangement. Since X and Y have the same cardinality, we may with no loss of assume that $X = Y$. But then we actually have a Schwarz type symmetrization, as can be easily seen. Now, then, it follows that X (equipped with the Schwarz order of Proposition 8.1) is order-isomorphic to a subset of $\mathbb{Z}_0^+$ and then our result follows from [58, Thm. 108]. □

11.1. Our assumptions

Assume that # is a Steiner type rearrangement from 2^X to 2^Y. Now assume Assumptions 11.1, 11.2 and 11.3. Let Ω be a subset of X. Suppose that u and v are positive functions on X and Y, respectively, satisfying

$$-Du \leq \phi(u) - c\cdot u + \lambda, \qquad \text{on } \Omega \tag{11.2a}$$

and

$$-Ev \geq \phi(v) - d\cdot v + \mu, \qquad \text{on } \Omega^{\#}. \tag{11.2b}$$

Assume that ϕ is a real function on $[0,\infty)$ such that:

$$\phi \text{ is convex and increasing on } [0,\infty). \tag{11.3}$$

Assume that c and d are functions on Ω and $\Omega^{\#}$ respectively such that

$$c \geq 0 \tag{11.4a}$$

$$d \geq 0 \tag{11.4b}$$

$$-c \lhd -d. \tag{11.4c}$$

(In interpreting (11.4c), since c and d are only defined on Ω and $\Omega^{\#}$ respectively, we extend c and d to all of X and Y respectively, by setting them to $+\infty$ outside their domains of definition.) Assume that $-d$ (after the extension) is symmetric.

Assume that λ and μ are any real functions on Ω and $\Omega^{\#}$, respectively, such that

$$\lambda \lhd \mu. \tag{11.5}$$

(Again, to interpret this, extend λ and μ to all of X and Y, respectively, by setting them equal to $-\infty$ outside their respective domains of definition.)

Finally, we need two boundary value conditions on u and v. First, assume that

$$\inf_{A\cap\Omega} u \geq \sup_{A\setminus\Omega} u, \qquad \forall A \in \mathcal{A}, \tag{11.6a}$$

where $\mathcal{A}$ is the set of the fibres of our Steiner type rearrangement. Moreover, assume that

$$\sum_{U\setminus\Omega} u \leq \sum_{U^{\#}\setminus\Omega^{\#}} v, \qquad \forall U \subseteq X. \tag{11.6b}$$

Remark 11.2. In particular, (11.6a) and (11.6b) will be satisfied if u and v vanish outside Ω and $\Omega^{\#}$, respectively.

Finally, we make the following assumption on the operator L and the domain $\Omega^{\#}$.

Assumption 11.4. Let f be any positive function with $\operatorname{supp} f$ nonempty. Define $f_0 = f$. Inductively, given f_n, let $f_{n+1} = f_n + L^* f_n$. Then there exists $n \in \mathbb{Z}^+$ such that $\operatorname{supp} f \not\subseteq \Omega^{\#}$.

This assumption essentially says that a random walk defined by L has a positive probability of eventually leaving $\Omega^{\#}$.

11.2. A discrete rearrangement theorem for difference equations or difference inequalities

Our main discrete rearrangement result is encapsulated in the following theorem.

Theorem 11.1. *Suppose that the assumptions give in §11.1 are made. Assume moreover that u is in $w(X)$. Then $u \lhd v$.*

The proof of Theorem 11.1 depends essentially on three main observations, which we shall label as Lemma 11.2, Proposition 11.2 and Proposition 11.3, respectively.

Lemma 11.2. *Let $F \in \mathcal{M}$. Then, there is a sequence of positive numbers α_i and symmetric finite non-empty sets S_i such that for each i there exists $A = A(i) \in \mathcal{A}$ with $S_i \subseteq A^{\#}$, with the property that $\sum_{i=1}^{N} \alpha_i \leq 1$ while*

$$F = \sum_{i=1}^{N} \alpha_i f_{S_i},$$

where

$$f_{S_i} = |S_i|^{-1} 1_{S_i},$$

and $N \in \mathbb{Z}^+ \cup \{\infty\}$.

Proof. First we do this for functions $f_S \in \mathcal{M}$ of the form $f_S = |S|^{-1} 1_S$ for S a nonempty finite symmetric subset of Y. Let $\mathcal{A}' = \{A \in \mathcal{A} : A \cap S \neq \varnothing\}$. This is a countable set. Let $S_A = S \cap A^{\#}$ and $\alpha_A = |S|^{-1}|S \cap A|$ for $A \in \mathcal{A}'$. I claim that then

$$f_S = \sum_{A \in \mathcal{A}'} \alpha_A f_{S_A}.$$

The proof of this is essentially trivial since the $\{A^{\#}\}$ are a countable disjoint cover of Y. Moreover,

$$\sum_{A \in \mathcal{A}'} \alpha_A = |S|^{-1}|S| = 1.$$

Hence, we have proved our Proposition for functions of the form f_S.

Now, suppose that we have a general function $F \in \mathcal{M}$. Let $T = \{f(x) : x \in Y, f(x) \neq 0\}$. Then, for each $\lambda > 0$ the set $T \cap [\lambda, \infty)$ is finite since otherwise F could not be summable while we had assumed that $F \in \mathcal{M}$. It follows that we may find a strictly decreasing enumeration $t_0 > t_1 > t_2 > \cdots$ of T. Let $M = |T|$. Let $\delta_i = t_i - t_{i+1}$ for $0 \leq i < M$. If $M < \infty$, then let $\delta_M = t_M$.

Let

$$T_i = \{x : F(x) \geq t_i\}.$$

The T_i are clearly symmetric if f is symmetric (this need not be the case for a general rearrangement, but is easy enough to check for a discrete Steiner rearrangement by intersecting each T_i with a fibre and checking that the intersection is Schwarz symmetric much as in Proposition 8.4). Then, it is very easy to see that

$$F(x) = \sum_{i=0}^{M} \delta_i 1_{T_i}.$$

Let

$$\beta_i = |S_i|\delta_i.$$

Then,

$$F(x) = \sum_{i=0}^{M} \beta_i f_{T_i},$$

where $f_{T_i} = |T_i|^{-1} 1_{T_i}$. Moreover, $\sum_Y f_{T_i} = 1$ so that

$$\begin{aligned}\sum_{i=0}^{M} \beta_i &= \sum_{i=0}^{M} \beta_i \sum_{x \in Y} f_{T_i}(x) \\ &= \sum_{x \in Y} \sum_{i=0}^{M} \beta_i f_{T_i}(x) \\ &= \sum_{x \in Y} F(x) \leq 1,\end{aligned}$$

by Fubini's theorem and since $F \in \mathcal{M}$. Now, $f_{T_i} \in \mathcal{M}$ is the kind of function for which we have already proved the Lemma so that

$$f_{T_i} = \sum_{A \in \mathcal{A}'_i} \alpha^i_A f_{S^i_A},$$

for some countable set A'_i and for positive α^i_A with $\sum_{A \in \mathcal{A}'_i} \alpha^i_A \leq 1$, and S^i_A a symmetric set contained within a single fibre $A \in \mathcal{A}$. Then,

$$F = \sum_{i=0}^{M} \sum_{A \in \mathcal{A}'_i} \beta_i \alpha^i_A f_{S^i_A},$$

by Fubini's theorem. Moreover, also by Fubini's theorem we have

$$\sum_{i=0}^{M} \sum_{A \in \mathcal{A}'_i} \beta_i \alpha^i_A \leq 1.$$

Renumbering the index set, the proof is complete. □

Proposition 11.2. *Assume that* (11.6a) *and* (11.6b) *hold for some pair of real functions u and v on X and Y respectively, with $u \in w(X)$. Assume that $\sigma_{u^\# - v}$ attains a strictly positive maximum on $\mathcal{M}$ at a function $F \in \mathcal{M}$. Then* $\operatorname{supp} F \subseteq \Omega^\#$.

Proof. Suppose that $\operatorname{supp} F \not\subseteq \Omega^\#$. Write $\sigma = \sigma_{u^\# - v}$ to reduce clutter. Let $G = 1_{\Omega^\#} \cdot F$. The function G is symmetric by Corollary 8.5. I claim that

$$\sigma(G) \geq \sigma(F). \tag{11.7}$$

Suppose that this claim is just. Then, because $\operatorname{supp} F \not\subseteq \Omega^\#$ we have $\sum_Y G < \sum_Y F$. Thus, there exists $\lambda > 1$ such that $\sum_Y \lambda G \leq \sum_Y F \leq 1$ so that $\lambda G \in \mathcal{M}$. But,

$$\sigma(\lambda G) = \lambda \sigma(G) \geq \lambda \sigma(F) > \sigma(F),$$

since $\sigma(F) > 0$, and thus we have a contradiction to the assumption that σ attains a maximum at F.

Hence, it suffices to prove (11.7). Now, by Remark I.2.2 and Fubini's theorem we will have

$$\begin{aligned} \sigma(F) - \sigma(G) &= \sum_{y \in Y \setminus \Omega^\#} (u^\#(y) - v(y)) F(y) \\ &= \int_0^\infty \sum_{y \in Y \setminus \Omega^\#} (u^\#(y) - v(y)) 1_{F_\lambda}(y)\, d\lambda \\ &= \int_0^\infty \sum_{y \in F_\lambda \setminus \Omega^\#} (u^\#(y) - v(y))\, d\lambda. \end{aligned} \tag{11.8}$$

Since F is symmetric, the set F_λ is symmetric. Thus, it suffices to prove that for all finite symmetric sets V we have

$$\sum_{y \in V \setminus \Omega^\#} u^\#(y) \leq \sum_{y \in V \setminus \Omega^\#} v(y), \tag{11.9}$$

since then (11.7) will follow immediately from (11.8). Since the $A^{\#}$ for $A \in \mathcal{A}$ form a disjoint cover of Y and $V \cap A^{\#}$ is symmetric for each symmetric V (Proposition 8.1), it suffices to prove (11.9) for V a subset of $A^{\#}$ for some fibre $A \in \mathcal{A}$. Use the "moreover" in the Proposition with $S = A \cap \Omega$, which is acceptable by (11.6a). Let $g_1 = 1_V$ and $g_2 = 1_{V \cap \Omega^{\#}}$. Then, $\check{g}_1$ is the indicator function of some set $\check{V}$, and $\check{g}_2$ is the indicator function of some set $\check{W}$. By the "moreover", we have $\check{W} \subseteq S$. Clearly, also, $\check{W} \subseteq \check{V}$. Since,

$$\sum_{x \in \check{V}} u(x) = \sum_{y \in Y} u^{\#}(y) g_1(y)$$

and

$$\sum_{x \in \check{W}} u(x) = \sum_{y \in Y} u^{\#}(y) g_2(y),$$

it follows that

$$\sum_{x \in \check{V} \backslash \check{W}} u(x) = \sum_{y \in Y} u^{\#}(y)(g_1(y) - g_2(y)) = \sum_{y \in V \backslash \Omega^{\#}} u^{\#}(y). \tag{11.10}$$

If we could prove that $\check{V} \backslash \check{W} = \check{V} \backslash \Omega$, then (11.9) would follow from (11.10) together with (11.6a) and the fact that $\check{V}^{\#} = V$ since $\check{g}_1^{\#} = g_1$.

Hence, we must show that $\check{V} \backslash \check{W} = \check{V} \backslash \Omega$. Now, $\check{W} \subseteq \Omega$ and $\check{V} \subseteq A$ by Proposition 8.6(iii). We shall show that either $\check{V} = \check{W}$ or $\check{W} = A \cap \Omega$. In either case, it follows that $\check{V} \backslash \check{W} = \check{V} \backslash \Omega$. Suppose first that $\check{V} \neq \check{W}$. Then, g_1 and g_2 do not coincide, and hence V is not a subset of $\Omega^{\#}$. But V is symmetric and contained within the symmetrization a single fibre $A^{\#}$, and $\#$ is of Schwarz type on the fibre A, so that it follows that in fact $\Omega^{\#} \cap A^{\#}$ is a proper subset of V, since of two Schwarz symmetric sets, one must always be contained in the other (this follows from Proposition 8.1). Thus, $g_2 = 1_{\Omega^{\#} \cap A^{\#}}$. Moreover, since V is finite, so is $\Omega^{\#} \cap A$. Then, $|\check{W}| = |\Omega^{\#} \cap A^{\#}|$ and $\check{W}$ is contained in $S = \Omega \cap A$. Since $|S| = |\Omega^{\#} \cap A^{\#}|$ as $\Omega^{\#} \cap A^{\#} = (\Omega \cap A)^{\#}$ (Proposition 8.2), it follows that in fact $\check{W} = \Omega \cap A$, as desired. □

Proposition 11.3. *Make all the assumptions of* §11.1, *except possibly for* (11.6a) *and* (11.6b). *Assume that $\sigma_{u^{\#}-v}$ attains a strictly positive maximum over $\mathcal{M}$ at a function F with* $\operatorname{supp} F \subseteq \Omega^{\#}$. *Then it also attains its maximum over $\mathcal{M}$ at L^*F.*

Assume this proposition for now.

Proof of Theorem 11.1. Write $\sigma = \sigma_{u^\# - v}$ to reduce clutter. The relation $u \lhd v$ is equivalent to the relation $\sigma(F) \leq 0$ for all $F \in \mathcal{M}$. To obtain a contradiction, suppose that $\sigma(F) > 0$ for some $F \in \mathcal{M}$.

For S a finite non-empty subset of Y, let $f_S = |S|^{-1} 1_S$. Put

$$\mathcal{N} \stackrel{\text{def}}{=} \{f_S : \exists A \in \mathcal{A}.\ S \subseteq A^\#, S \text{ symmetric}, 0 < |S| < \infty\}.$$

I claim that $\sup_{F\in\mathcal{M}} \sigma = \sup_{F\in\mathcal{N}} \sigma$. To see this, fix $F \in \mathcal{M}$. By Lemma 11.2, we then have

$$F = \sum_i \alpha_i f_{S_i},$$

with the α_i positive constants whose sum does not exceed 1 and with $f_{S_i} \in \mathcal{N}$. Then,

$$\sigma(F) = \sum_i \alpha_i \sigma(f_{S_i})$$

by Fubini's theorem. It is clear that if F is such that $\sigma(F) > 0$ then there exists i so that $\sigma(f_{S_i}) \geq \sigma(F)$. Hence $\sup_{F\in\mathcal{M}} \sigma \leq \sup_{F\in\mathcal{N}} \sigma$ since we have assumed that $\sup_{F\in\mathcal{M}} \sigma > 0$. But $\mathcal{N} \subseteq \mathcal{M}$ so that σ has the same supremum over $\mathcal{N}$ as it does over $\mathcal{M}$. I claim that it follows that σ attains a maximum over $\mathcal{N}$ and hence over $\mathcal{M}$.

For, let S_n be a sequence of sets such that $|S_n|^{-1} 1_{S_n} \in \mathcal{N}$ and $|S_n|^{-1}\sigma(1_{S_n}) \to \sup_{\mathcal{M}} \sigma$.

If the collection $\{A \in \mathcal{A} : \exists n.\ S_n \subseteq A^\#\}$ is finite then by passing to a subsequence we may assume that there exists $A \in \mathcal{A}$ such that $S_n \subseteq A$ for all n. In that case, since S_n is a symmetric subset of the rearrangement of a single fibre, by the Schwarz property of the rearrangement restricted to subsets of fibres, it follows that $S_n = S_m$ if and only if $|S_n| = |S_m|$. If $|S_n|$ takes on only finitely many values, by passing to a further subsequence we may then assume that all the $|S_n|$, and hence all the S_n, are equal, and it follows immediately that σ attains its maximum over $\mathcal{M}$ (in fact at $|S_1|^{-1} 1_{S_1}$). Hence assume that $|S_n|$ takes on infinitely many values. Then, passing to a subsequence if necessary, we may assume that $|S_n|$ is a strictly increasing sequence tending to ∞. Since the S_n are nested (they are symmetric subsets of a rearrangement of a fibre), it follows from the assumption that $u \in w(X)$ (which implies that $u^\# \in w(Y)$ by equimeasurability) that we must have $\limsup_{n\to\infty} |S_n|^{-1}\sigma_{u^\#-v}(1_{S_n}) \le 0$, so that $\sup_{\mathcal{M}} \sigma \le 0$, which we had assumed not to be the case, and so we have a contradiction.

Suppose then that the collection $\{A \in \mathcal{A} : \exists n.\ S_n \subseteq A^\#\}$ is infinite. Then, passing to a subsequence, since the $A^\#$ are disjoint, it follows that we may assume that the S_n are disjoint. Since $\sup_{\mathcal{M}} \sigma > 0$, there exists $\varepsilon > 0$ and $N \in \mathbb{Z}^+$ such that for $n \ge N$ we will have

$$|S_n|^{-1}\sigma(1_{S_n}) \ge \varepsilon.$$

Define $U_n = \bigcup_{i=N}^{N+n} S_n$. We then have

$$\sigma(1_{U_n}) = \sum_{i=N}^{N+n} \sigma(1_{S_n}) \ge \sum_{i=N}^{N+n} \varepsilon|S_i| = |U_n|\varepsilon. \tag{11.11}$$

It is clear that the U_n are nested and satisfy $|U_n| \to \infty$. Hence

$$\limsup |U_n|^{-1}\sigma(1_{U_n}) \le 0,$$

which immediately contradicts (11.11).

We have thus proved that indeed the maximum of σ is attained over $\mathcal{M}$. By our assumptions, this is a strictly positive maximum. Suppose it is attained at $F_0 \in \mathcal{M}$. By Proposition 11.2, the support of F_0 lies inside $\Omega^\#$. Define $F_n = \frac{1}{2}(F_{n-1} + L^*F_{n-1})$. Suppose that $F_{n-1} \in \mathcal{M}$, that the support of F_{n-1} is contained in $\Omega^\#$ and that σ attains its maximum over $\mathcal{M}$ at F_{n-1}. By Assumption 11.3 the function L^*F_{n-1} is symmetric and by Assumption 11.1 it follows easily that $\sum_Y L^*F_{n-1} \le \sum_Y F_{n-1} \le 1$ since $F_{n-1} \in \mathcal{M}$. Now $\frac{1}{2}(F_{n-1} + L^*F_{n-1})$ is symmetric by Corollary 8.5. It is then clear that $F_n \in \mathcal{M}$. We then have $\sigma(F_n) = \frac{1}{2}(\sigma(F_{n-1}) + \sigma(L^*F_{n-1}))$. By Proposition 11.3 we have $\sigma(L^*F_{n-1}) = \sigma(F_{n-1})$ so that $\sigma(F_n) = \sigma(F_{n-1})$. Since σ attains its maximum at F_{n-1} it follows that it must also attain this maximum at F_n.

We have thus inductively proved that the F_n are a sequence of positive functions whose supports lie in $\Omega^\#$. Since these supports are non-empty (else our maximum of σ would be negative), in light of the choice of F_n we have contradicted Assumption 11.4. $\square$

Our proof of Proposition 11.3 is based on the methods of Baernstein [11]. The methods use a trick of Weitsman [99] to handle the $\phi(u) - c \cdot u$ terms.

Proof of Proposition 11.3. Write $\sigma = \sigma_{u^{\#}-v}$. I claim that it suffices to prove Proposition 11.3 for the special case of a function F of the form

$$F = f_S = |S|^{-1} 1_S,$$

where S is a finite nonempty subset of $A^{\#}$ for some fibre A. For suppose we have done this, and that F is a general function. Apply Lemma 11.2 to write

$$F = \sum_i \alpha_i f_{S_i}$$

with the α_i positive, $\sum_i \alpha_i \leq 1$ and S_i a finite symmetric subset of $A_i^{\#}$ for A_i a fibre. Then, by Fubini's theorem,

$$\sigma(F) = \sum_i \alpha_i \sigma(f_{S_i}).$$

Since $\sigma(F) > 0$ and $\sum_i \alpha_i \leq 1$, and, finally, $\sigma(f_{S_i}) \leq \sigma(F)$ (since σ attains a maximum at F), it follows that $\sigma(f_{S_i}) = \sigma(f)$ for all F and that $\sum_i \alpha_i = 1$. Applying the result for the special case of the f_{S_i}, we will see that

$$\sigma(L^* f_{S_i}) = \sigma(f_{S_i}) \tag{11.12}$$

whenever $\alpha_i \neq 0$. (Of course $\operatorname{supp} f_{S_i} \subseteq \operatorname{supp} F \subseteq \Omega^{\#}$ in such a case.) Then,

$$L^* F = \sum_i \alpha_i L^* f_{S_i}$$

by Fubini's theorem and it follows from (11.12) by another application of Fubini's theorem that

$$\sigma(L^* F) = \sigma(F)$$

as desired.

Hence, it suffices to prove our result in the special case of $F = f_S$, for a finite nonempty symmetric $S \subseteq A^{\#}$, where $A \in \mathcal{A}$ is a fibre. Let $y_0, y_1, \ldots$ be the enumeration of $A^{\#}$ induced by Proposition 8.1, since # is a Schwarz type rearrangement from 2^A to $2^{A^\#}$. Then, $S = \{y_i : i < N\}$, where $N = |S| < \infty$.

Since $u \in w(X)$ it follows that u_λ is finite for all $\lambda > 0$. Use Proposition 8.6 to construct the function $\check{F}$ from the function F such that $\check{F}^\# = F$, $(\check{F} \cdot u)^\# = F \cdot u^\#$, $u \circ \psi = u^\#$,

$$\sum_{x \in X} \check{F}(x)u(x) = \sum_{y \in Y} F(y)u^\#(y) \tag{11.13}$$

and $\operatorname{supp} \check{F} \subseteq (\Omega \cap A)$, where A is the fibre such that $\operatorname{supp} F \subseteq A^\#$. (Here we use the fact that $\operatorname{supp} F \subseteq \Omega^\#$ together with (11.6a). We write "ψ" where Proposition 8.6 has "ϕ" and put $S = \Omega \cap A$ in the "moreover".)

I claim that we then have

$$\begin{aligned}
&-\sum_{y \in Y} F(y)E(u^\# - v)(y) \\
&\qquad \le -\sum_{x \in X} \check{F}(x)Du(x) + \sum_{y \in Y} h(y)Ev(y) && (11.14a) \\
&\qquad \le \sum_{x \in X} \check{F}(x)(\phi(u(x)) - c(x)u(x) + \lambda(x)) && (11.14b) \\
&\qquad\qquad - \sum_{y \in Y} F(y)(\phi(v(x)) - d(y)v(y) + \mu(y)) \\
&\qquad \le \sum_{y \in Y} F(y)[\phi(u^\#(y)) + (-d)(y)u^\#(y) && (11.14c) \\
&\qquad\qquad\qquad + \mu(y) - \phi(v(x)) + d(y)v(y) - \mu(y)] \\
&\qquad \le \sum_{y \in Y} F(y)[\phi(u^\#(y)) - d(y)u^\#(y) && (11.14d) \\
&\qquad\qquad\qquad - [\phi(v(y)) - d(y)v(y)]].
\end{aligned}$$

We now justify this chain of inequalities. First, note that

$$\begin{aligned}\sum_{x\in X}\check{F}(x)Du(x) &= \sum_{x\in X}\check{F}(x)Ku(x) - \sum_{x\in X}\check{F}(x)u(x)\\ &\le \sum_{y\in Y}\check{F}^{\#}(y)Lu^{\#}(y) - \sum_{x\in X}\check{F}(x)u(x)\\ &= \sum_{y\in Y}F(y)Lu^{\#}(y) - \sum_{y\in Y}F(y)u^{\#}(y)\\ &= \sum_{y\in Y}F(y)Eu^{\#}(y).\end{aligned}$$

The first equality followed by definition of D; the subsequent inequality followed from Assumption 11.2; the subsequent equality was given by (11.13); the final equality followed from the definition of E. Hence, we see that the inequality in line (11.14a) is valid.

Line (11.14b) follows from (11.2a) and (11.2b) since $\operatorname{supp}\check{F} \subseteq \Omega$ and $\operatorname{supp} F \subseteq \Omega^{\#}$.

We now justify (11.14c) in three steps. First note that

$$\begin{aligned}\sum_{x\in X}\check{F}(x)\lambda(x) &\le \sum_{y\in Y}\check{F}^{\#}(y)\cdot\lambda^{\#}(x)\\ &= \sum_{y\in Y}F(y)\lambda^{\#}(y)\\ &\le \sum_{y\in Y}F(y)\mu(y),\end{aligned}\tag{11.15}$$

where the first inequality followed from the Hardy-Littlewood inequality (Theorem I.2.3), the middle equality followed from the fact that $\check{F}^{\#} = F$, and the final inequality followed from the symmetry and positivity of F together with condition (11.5).

Now note that

$$\begin{aligned}\sum_{x\in X}\check{F}(x)u(x)(-c(x)) &\le \sum_{y\in Y}(\check{F}\cdot u)^{\#}(y)(-c)^{\#}(y)\\ &\le \sum_{y\in Y}(\check{F}\cdot u)^{\#}(y)(-d)(y)\\ &= \sum_{y\in Y}F(y)u^{\#}(y)(-d)(y).\end{aligned}\tag{11.16}$$

Here, the first inequality followed from the Hardy-Littlewood inequality (Theorem I.2.3). The second came from (11.4c) and the symmetry of $(\check{F}\cdot u)^{\#}$. The final equality came from the relation $(\check{F}\cdot u)^{\#} = F\cdot u^{\#}$.

Finally, note that by the definition of $\check{F}$ and by the relation $u^{\#} = u\circ\psi$, we have

$$\begin{aligned}\sum_{x\in X}\check{F}(x)\phi(u(x)) &= \sum_{y\in Y}F(y)\phi(u(\psi(y)) \\ &= \sum_{y\in Y}F(y)\phi(u^{\#}(y)).\end{aligned}\tag{11.17}$$

Putting (11.15) and (11.16) together, we obtain (11.14c). Trivial manipulation then yields (11.14d).

Let R be the right hand side of (11.14d). We shall show that $R\le 0$. For, $F = N^{-1}1_{\{y_0,\dots,y_{N-1}\}}$ so that

$$\begin{aligned}N\cdot R &= \sum_{i=0}^{N-1}[-d(y_i)(u^{\#}(y_i)-v(y_i))+\phi(u^{\#}(y_i))-\phi(v(y_i))] \\ &\le \sum_{i=0}^{N-1}[-d(y_i)+\phi'(u^{\#}(y_i))]\cdot[u^{\#}(y_i)-v(y_i)],\end{aligned}$$

where ϕ' is a one-sided derivative of our convex function ϕ (the choice of side is irrelevant). Let $\psi(i) = -d(y_i)+\phi'(u^{\#}(y_i))$. Since $-d$ and $u^{\#}$ are symmetric, $-d(y_i)$ and $u^{\#}(y_i)$ must be decreasing as i increases (Proposition 8.4). Now ϕ is convex so ϕ' is increasing, and it follows that Ψ is decreasing in i. For conciseness, let $\alpha(i) = u^{\#}(y_i)-v(y_i)$ and put

$$W(n) = \sum_{i=n}^{N-1}\alpha(i),$$

with $W(N)=0$. By summation by parts, we then have

$$\begin{aligned}N\cdot R &\le \sum_{i=0}^{N-1}\Psi(i)\alpha(i) \\ &= \sum_{i=0}^{N-1}\Psi(i)(W(i)-W(i+1)) \\ &= \Psi(0)W(0)-\Psi(N-1)W(N)+\sum_{i=1}^{N-1}(\Psi(i)-\Psi(i-1))W(i) \\ &= \Psi(0)W(0)+\sum_{i=1}^{N-1}(\Psi(i)-\Psi(i-1))W(i).\end{aligned}$$

Now, $\Psi(i)-\Psi(i-1) \le 0$ since Ψ is decreasing. Moreover, $\Psi(0) \le 0$ since ϕ is increasing. We will then be able to conclude that $R \le 0$ as soon as we prove that $W(i) \ge 0$ for $i \in \{0, 1, \ldots, N-1\}$. Of course $W(0) = \sum_{i=0}^{N-1}(u^\#(y_i) - v(y_i)) = N\sigma(F) > 0$. Let $F_n = n^{-1}1_{\{y_i : i<n\}}$. Then, $F_n \in M$. Thus, for $n = 1, \ldots, N-1$ we have

$$\sigma(F_n) \le \sigma(F).$$

Note that $F_N = f$. But $\sigma(F_n) = n\sum_{i=0}^{n-1}\alpha(i)$. Thus, for $n \le N$ we have

$$n^{-1}\sum_{i=0}^{n-1}\alpha(i) \le N^{-1}\sum_{i=0}^{n-1}\alpha(i) \le n^{-1}\sum_{i=0}^{N-1}\alpha(i),$$

since $\sum_{i=0}^{N-1}\alpha(i) = N\sigma(F) > 0$. It follows from the above displayed equation that

$$W(i) = \sum_{i=n}^{N-1}\alpha(i) \ge 0$$

for $n = 1, \ldots, N-1$, as desired. Hence we have indeed proved that $R \le 0$.

Thus, by (11.14a)–(11.14d) we have

$$-\sum_{y \in Y} F(y)E(u^\# - v)(y) \le 0.$$

Now, $E = L - 1$, so that

$$\sum_{y \in Y} F(y)(u^\# - v)(y) \le \sum_{y \in Y} F(y)L(u^\# - v)(y) = \sum_{y \in Y} L^*F(y)(u^\# - v)(y).$$

Hence $\sigma(F) \le \sigma(L^*F)$, as desired. □

11.3. Applications

11.3.1. Monotonicity of the system

Let $X = Y$ and $K = L$. Let $S^\# = S$ for every subset S of X. This a Steiner type symmetrization with fibres $\{\{x\} : x \in X\}$. Every set and every function is symmetric. The condition $f \lhd g$ means simply that $f \le g$ everywhere. Assumptions 11.2 and 11.3 are trivial in this case.

The content of Theorem 11.1 in this case is as follows.

Corollary 11.1. *Let Ω be a subset of X. Suppose that u and v are positive functions on X satisfying*

$$-Du \leq \phi(u) - c \cdot u + \lambda, \qquad \text{on } \Omega$$

and

$$-Dv \geq \phi(v) - c \cdot v + \lambda, \qquad \text{on } \Omega,$$

where ϕ is a real function on $[0, \infty)$ such that

$$\phi \text{ is convex and increasing on } [0, \infty),$$

c is any positive function on Ω and λ is any real function on Ω. Moreover, assume that

$$\inf_{\Omega} u \geq \sup_{X \setminus \Omega} u$$

and that

$$\sup_{X \setminus \Omega} u \leq \inf_{X \setminus \Omega} v.$$

Finally assume that Ω satisfies Assumption 11.4 (note that of course $\Omega^{\#} = \Omega$ for any $\Omega \subseteq X$). Assume Assumption 11.1. Suppose that $u \in w(X)$. Then $u \leq v$ everywhere on X.

11.3.2. Generalized harmonic measures

We show that Theorem 9.1 can also be seen as a consequence of Theorem 11.1. To see this, note that by the work of §9.2, we may assume that K is a kernel satisfying Assumption 9.3 with $\lambda = 0$. Let $L = K$. Then Assumptions 11.1–11.3 are satisfied.

We first perform a simple approximation argument that allows us to assume that s has finite support. Let $\mathcal{F}$ be the σ-field generated by $\{R_n\}_{n=0}^{\infty}$. Adopt the notation of §9.1. Consider the event

$$A_n(s) = \bigcup_{m=0}^{n} \{X_m \leq s(R_n)\}.$$

Write

$$\rho = \inf\{n \geq 0 : R_n \in \{k\} \times X\}.$$

Then,

$$\omega(z, S; s) = E^z[P(\rho < \infty \text{ and } A_\rho \mid \mathcal{F} \text{ and } R_\rho \in S)].$$

Now, conditioned on $\mathcal{F}$ and $\rho < \infty$ we have A_ρ involving only finitely many of the numbers $s(v)$. Now, let V_N be an increasing sequence of finite subsets of V whose union is all of V. Define $s_N(x) = s(x) \cdot 1_{V_N}(x)$ and $S_N = S \cap V_N$. Then, we have

$$P(\rho < \infty \text{ and } A_\rho(s_N) \text{ and } R_\rho \in S_N \mid \mathcal{F}) \uparrow P(\rho < \infty \text{ and } A_\rho(s) \text{ and } R_\rho \in S \mid \mathcal{F})$$

with probability 1 as $N \to \infty$, since almost surely

$$P(\rho < \infty \text{ and } A_\rho(s) \text{ and } R_N \mid \mathcal{F})$$

depends on at most finitely many of the $s(v)$. Hence,

$$\omega(z, S; s_N) \uparrow \omega(z, S; s).$$

Let

$$u_N(z) = \omega(z, S; s_N).$$

Clearly u_N has at most finite support, since if $s_N(z) = 0$ then $u_N(z) = 0$ while s_N has finite support. Define

$$T = \{m \in \mathbb{Z} : \exists x \in X.\ P^{(m,x)}(\rho < \infty) = 0\}.$$

Note that in fact

$$T = \{m \in \mathbb{Z} : \forall x \in X.\ P^{(m,x)}(\rho < \infty) = 0\},$$

by Assumption 9.2. Note that if $m \in T$ and $x \in X$ then $u_N(m, x) = 0$. Let

$$\Omega_N = (\operatorname{supp} s_N) \backslash [(\{k\} \cup T) \times X].$$

Let

$$v(z) = \omega(z, S^{\#}; s^{\#}).$$

Write $\Delta = D = E = K - 1$. Then, for any function f on V we have

$$\Delta f(z) = E^z[f(R_1)] - f(z),$$

since K precisely gives the transition probabilities of the random walk $\{R_n\}$. Fix $z \in \Omega$. Then,

$$u_N(z) = s_N(z)E^z[u_N(R_1)].$$

For, $u_N(z)$ (if $z \notin \{k\} \times X$) is the probability that we will survive while at point z and that from the next step we will survive until arrival at S; this latter probability is precisely $E^z[u_N(R_1)]$. Hence,

$$u_N(z) = s_N(z)E^z[u_N(R_1)].$$

Thus,

$$\Delta u_N(z) = E^z[u_N(R_1)] - u_N(z) = (s_N(z))^{-1}u_N(z) - u_N(z) = [(s_N(z))^{-1} - 1]u_N(z).$$

Let $c_N(z) = (s_N(z))^{-1} - 1$. (This makes sense since if $z \in \Omega$ then $s_N(z) > 0$.) Then,

$$-\Delta u_N = -c_N u_N \qquad \text{on } \Omega.$$

Set $c_N = +\infty$ outside Ω_N. Now, if s_N is strictly positive on Ω, then $s_N^\#$ is strictly positive on $\Omega_N^\#$ (this follows from the fact that rearrangements preserve inclusions, while $(\operatorname{supp} f)^\# = \operatorname{supp}(f^\#)$ for any positive function f since for a positive function f we have $\operatorname{supp} f = f_0$ and can apply Theorem I.2.1). Then,

$$-\Delta v = -d \cdot v,$$

on Ω_N, where $d(z) = (s^\#(z))^{-1} - 1$. Set $d(z) = +\infty$ for every z such that $s^\#(z) = 0$.

I now claim that $-c_N \lhd -d$. To see this, write $c(z) = (s(z))^{-1} - 1$ for every z such that $s(z) > 0$ and $c(z) = \infty$ if $s(z) = 0$. Then, $-c \geq -c_N$ everywhere on V since $s \geq s_N$ everywhere. Hence, it suffices to show that $-c \lhd -d$. But in fact we have $(-c)^\# = -d$ by Theorem I.2.2 since $-c = \phi \circ s$ and $-d = \phi \circ s^\#$, where ϕ is the monotone function defined by

$$\phi(t) = \begin{cases} -\infty, & \text{if } t \leq 0, \\ 1 - t^{-1}, & \text{if } t > 0. \end{cases}$$

Hence, the relation $-c \lhd -d$ follows trivially.

Now, u is a positive function. Let A be a fibre of V. Then, there is some $m \in \mathbb{Z}$ such that $A \subseteq \{m\} \times X$. (This is because each fibre of V is of the form of a cartesian product of a singleton $\{m\}$ with a fibre of X.) If $m = k$ or $m \in T$ then A does not meet Ω_N. Otherwise, $A \cap \Omega_N = A \cap \operatorname{supp} s_N$, and (11.6a) (with u_N in place of u and Ω_N in place of Ω) follows by positivity of u_N and the fact that u_N vanishes wherever s_N vanishes.

To verify (11.6b) (with u_N in place of u and Ω_N in place of Ω), it suffices for us to verify it in the case of U being a subset of some fibre A of V. There are two cases. Either A and $(T \cup \{k\}) \times X$ are disjoint, or A is a subset of $(T \cup \{k\}) \times X$. Consider first the former case. Then, $U \backslash \Omega_N$ lies outside the support of s_N and hence u_N vanishes there and (11.6b) is trivial as v is positive. Consider now the case when A is a subset of $(T \cup \{k\}) \times X$. Then, since $k \notin T$ it follows that A is either a subset of $T \times X$ or of $\{k\} \times X$. In the former case, u_N and v both vanish identically on T and we are done. Suppose then that $A \subseteq \{k\} \times X$. Now, on $\{k\} \times X$, we have u_N equal to the indicator function of S_N and v equal to the indicator function of $S^\#$. Now, then, on A, the function u_N is smaller than the indicator function of S. Hence, (11.6b) is a consequence of the Hardy-Littlewood inequality (Theorem I.2.3) applied with $f = 1_S$ and $g = u$, restricted to the fibre A.

Note that u has finite support (since s does), so that $u \in w(V)$. Of course Assumption 11.4 follows from the choice of T and the finiteness of Ω_N. Hence, the assumptions of Theorem 11.1 are satisfied. Hence, $u_N \lhd v$. Letting $N \to \infty$, we conclude that $u \lhd v$ (use Lemma I.2.1). This easily implies the conclusion of Theorem 9.1. For we can use condition (9.3) to prove that $u \preceq v$. Then, all that remains to be proved is the Steiner symmetry of v. But this follows by noting that $v \lhd v$ by a second application of the above work, and then using Lemma 11.1.

The crucial fact in the above work was the formula

$$-\Delta\omega(z, S; s) = -[(s(z))^{-1} - 1]\omega(z, S; s),$$

valid outside on $\{z : s(z) > 0\}$ and it is essentially this formula which together with Theorem 11.1 implies Theorem 9.1.

Remark 11.3. We can use a similar technique for proving Theorem 9.2. There, the necessary formula is

$$\Delta g_U^s(z) = 1_U - [(s(z))^{-1} - 1]g_U^s(z),$$

on $\{z : s(z) > 0\}$, where $g_U^s(z) = \sum_{w \in U} g(z, w; s)$ for $U \subseteq V$. The above equation is also of a type that can be handled by Theorem 11.1. Since we have already given one rigorous proof of Theorem 9.2 and have shown how to use Theorem 11.1 to prove Theorem 9.2, we leave it to the interested reader to work out the details of a proof of Theorem 9.2 via Theorem 11.1.

11.3.3. Exit times

By Haliste's method, we were also able to prove a result about exit times, namely Theorem 9.4. It is natural to ask whether that result can also be proved to be a consequence of Theorem 11.1. The answer is positive. Our method here is actually very simple: since we have already shown that Theorem 11.1 implies Theorem 9.1, we need only prove that Theorem 9.1 implies Theorem 9.4. To do this, proceed as follows. Assume that the hypotheses of Theorem 9.4 are verified. Let

$$\mathfrak{V} = \mathbb{Z} \times V.$$

Let $\#$ be the $\mathbb{Z}$-product symmetrization on $\mathfrak{V}$ obtained from the Steiner symmetrization $\#$ on V. Given the kernel K on V, let

$$\mathfrak{K}((m, b), (m', v') = \delta_{m+1,m'} K(v, v').$$

Then, the hypotheses of Theorem 9.1 are satisfied if we put $\mathfrak{K}$ and $\mathfrak{V}$ in the place of K and V, respectively. Let $\mathfrak{S} = \{N\} \times V$. For $m \in \mathbb{Z}$ and $v \in V$ let $\mathfrak{s}(m, v) = s(v)$, where s was our survival function on V. Then, it is not hard to see that

$$\omega^{\mathfrak{K}}((0, z), \mathfrak{S}; \mathfrak{s}) = P^z(\tau_s \geq N), \tag{11.18}$$

for $z \in V$, where $\omega^{\mathfrak{K}}$ is generalized harmonic measure on $\mathfrak{V}$ defined with respect to the kernel $\mathfrak{K}$. Let $\mathfrak{I} = \{i\} \times I$. Then, by Theorem 9.1 we see that

$$\sum_{\mathfrak{j} \in \mathfrak{I}} \omega^{\mathfrak{K}}((0, \mathfrak{j}), \mathfrak{S}; \mathfrak{s}) \leq \sum_{\mathfrak{j} \in \mathfrak{I}^\#} \omega^{\mathfrak{K}}((0, \mathfrak{j}), \mathfrak{S}^\#; \mathfrak{s}^\#).$$

Now, $\mathfrak{S}^{\#} = \mathfrak{S}$ by definition of the $\mathbb{Z}$-product rearrangement on $\mathfrak{V}$, since $V^{\#} = V$. Moreover, $\mathfrak{s}^{\#}(m, v) = s^{\#}(v)$ as is easily seen, and $\mathfrak{I}^{\#} = \{i\} \times I^{\#}$. Hence, by (11.18) (and its analogue for #-rearranged sets and functions) we see that

$$\sum_{\mathfrak{j} \in \{i\} \times I} P^{\mathfrak{j}}(\tau_s \geq N) \leq \sum_{\mathfrak{j} \in \{i\} \times I^{\#}} P^{\mathfrak{j}}(\tau_s \geq N).$$

But this is precisely the conclusion of Theorem 9.4, and so we see that Theorem 9.1 indeed implies Theorem 9.4.

Chapter III

Chang-Marshall inequality, harmonic majorant functionals, and some nonlinear functionals on Dirichlet spaces

Overview

The purpose of the present chapter is to study the Λ_Φ and Γ_Φ functionals. We shall begin by defining the non-linear functionals Λ_Φ acting on functions on an arbitrary finite measure space (§1.1). Then, in §1.2, we shall define various Dirichlet spaces on which our functionals are to act; we shall also review some very basic results about these Dirichlet spaces. In §1.3, we shall define the Γ_Φ functionals and describe the connection between them acting on domains of area at most π and the Λ_Φ acting on functions from the unit ball of the holomorphic Dirichlet space.

In §2 we give the important Chang-Marshall [32] inequality which in fact started our whole investigation. We also give Essén's improvement [44] of the Chang-Marshall inequality, and state the Moser-Trudinger [78] inequality which motivated the Chang-Marshall inequality. Then, we prove that in a strong sense the Moser-Trudinger and Chang-Marshall (and *a fortiori* Essén) inequalities are unimprovable (Theorem 2.1).

Section 3 will be primarily devoted to the study of the Λ_Φ functionals on functions on an arbitrary finite measure space. In §3.1 we shall give some results on the existence of extremal functions for Λ_Φ functionals on balls of Dirichlet spaces by giving an important upper semicontinuity result (Theorem 3.2) from the author's joint paper with Matheson [75]. This result improves on a theorem of Matheson [73]. In §3.2 we examine the Λ_Φ functionals acting on the unit balls of Hilbert spaces of measurable functions. We will be able to obtain some general results which will allow us to give a proof of Cima and Matheson's theorem [35] on the weak continuity of the Chang-Marshall functional on the punctured unit ball of the Dirichlet space. Our results will also be useful for proving weak continuity results in the case of the Moser-Trudinger functional.

Then, in section 3.3 we consider the general notion of a "critically sharp inequality". Much of the material in §3.3 comes from the author's paper [88]. Particular examples of critically sharp inequalities will be the Chang-Marshall and Moser-Trudinger inequalities. We shall prove that given a critically sharp inequality for Λ_Φ we may perturb Φ in such wise as to, depending on our wishes, either gain (Theorem 3.9) or destroy (Theorem 3.7) the existence of a function at which the maximal value in the inequality is attained. Our results are new even in the cases of the Chang-Marshall inequality and the Moser-Trudinger inequality, although in the former case the result was strongly suspected by Cima and Matheson (personal communication) and in the latter it was conjectured by McLeod and Peletier [77]. We also obtain a partial converse (Theorem 3.8) to the upper semicontinuity result of Theorem 3.2.

In §4 we come back to the specific case of the Λ_Φ functionals on balls of Dirichlet spaces. In §4.1 we shall give a variational equation for the extremals of our functionals. This is due to the author, refining a partial result of Andreev and Matheson [5], and is taken from a joint paper with Matheson [75]. In §4.2 we give a joint result of the author with Matheson [75] which shows that under appropriate conditions on Φ the extremals of the Λ_Φ functionals automatically satisfy some regularity conditions. In §4.3 we give some important assumptions on the functions Φ to which our methods are applicable. Then, in §4.4 we give some useful extensions of the results of sections 4.1 and 4.2. We shall use a variational equation from §4.4 later in §IV.10.

We have noted that some of the results of the present chapter come from the paper of Matheson and Pruss [75], and the proofs are sometimes taken from that paper. When this is done, the proof and the result can be assumed to have been due to the author, unless otherwise noted (as in the case of Theorem 4.2).

In §5 we return to symmetrization theory, and study the symmetric decreasing rearrangement acting on boundary values of real parts of functions from the Dirichlet spaces. We prove a strict symmetrization theorem (Theorem 5.1) and use it to prove that if $\Phi(z) = \phi(\operatorname{Re} z)$ for an appropriate function ϕ, then the extremal functions of Λ_Φ are univalent with Steiner symmetric image.

Finally, in §6 we discuss a rearrangement method due Baernstein [12]. Our main result is that the symmetrized domain the method produces is always a subset of the Steiner symmetrization of the original domain (Theorem 6.1). Hence, the method does not increase areas of domains, and this fact allows us to prove the existence of extremals for Γ_Φ where $\Phi(z) = \phi(\operatorname{Re} z)$ for appropriate ϕ by using the rearrangement to reduce the problem to the Λ_Φ functional, and then using the results of §3.1.

On first reading, the reader may wish to omit the proofs of the results in §3.2 and the material of §3.3.4. The proof of Theorem 6.1 can also be omitted.

1. The Λ_Φ and Γ_Φ functionals and Dirichlet spaces

1.1. The Λ_Φ functionals on a finite measure space

Let (I,μ) be a finite measure space. Let B be a set of complex-valued or real-valued functions on I and let Φ be a Borel measurable function from $\mathbb{C}$ or $\mathbb{R}$, respectively, to $\mathbb{R}$. Given $f \in B$, let

$$\Lambda_\Phi(f) = \int_I \Phi(f)\, d\mu.$$

In reference to the Λ_Φ functionals, "bounded" shall mean "bounded above".

Definition 1.1. The functional Λ_Φ on B is said to be **bounded** providing

$$\sup_{f\in B} \Lambda_\Phi(f) < \infty.$$

Definition 1.2. A function f is said to be extremal for Λ_Φ on B if $f \in B$ and

$$\Lambda_\Phi(f) \geq \Lambda_\Phi(g),$$

for all $g \in B$.

By an abuse of notation, if ϕ is defined only the half-line $[0, \infty)$ then we define

$$\Lambda_\phi(f) = \Lambda_\Phi(f),$$

where $\Phi(z) = \phi(|z|)$.

1.2. Dirichlet spaces

For a complex function f on the unit circle $\mathbb{T}$, write $\hat{f}(n)$ for its nth Fourier coefficient. For a real f on $\mathbb{T}$, write $\mathfrak{c}_n(f)$ and $\mathfrak{s}_n(f)$ for its nth Fourier cosine and sine coefficients, respectively. We shall often identify a function f on the (open) unit disc $\mathbb{D}$ with its nontangential boundary values on $\mathbb{T}$.

Definition 1.3. For $0 \le \alpha < \infty$, let the **α-weighted holomorphic Dirichlet space $\mathfrak{D}_\alpha$** be the Hilbert space of all functions f holomorphic on the unit disc $\mathbb{D}$ with $f(0) = 0$ and

$$\|f\|^2_{\mathfrak{D}_\alpha} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} n^\alpha |\hat{f}(n)|^2 < \infty.$$

Definition 1.4. Let the **α-weighted real harmonic Dirichlet space $\mathfrak{d}_\alpha$** be the real Hilbert space of all real functions f harmonic on the unit disc $\mathbb{D}$ with $f(0) = 0$ and

$$\|f\|^2_{\mathfrak{d}_\alpha} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} n^\alpha (\mathfrak{c}_n^2(f) + \mathfrak{s}_n^2(f)) < \infty.$$

Remark 1.1. It is clear that $\mathfrak{D}_\alpha$ has the inner product

$$(f, g) = \sum_{n=1}^{\infty} n^\alpha (\hat{f}(n), \overline{\hat{g}(n)}),$$

whereas $\mathfrak{d}_\alpha$ has the inner product

$$(f, g) = \sum_{n=1}^{\infty} n^\alpha (\mathfrak{c}_n(f)\mathfrak{c}_n(g) + \mathfrak{s}_n(f)\mathfrak{s}_n(g)).$$

Remark 1.2. The map $f \mapsto \operatorname{Re} f$ is an isometry sending $\mathfrak{D}_\alpha$ onto $\mathfrak{d}_\alpha$. To see this, it suffices to note that $\mathfrak{c}_n(\operatorname{Re} f) = \operatorname{Re} \hat{f}(n)$ and $\mathfrak{s}_n(\operatorname{Re} f) = -\operatorname{Im} \hat{f}(n)$.

Definition 1.5. The spaces $\mathfrak{D}_1$ and $\mathfrak{d}_1$ are known as the **holomorphic** and **real harmonic Dirichlet spaces**, respectively. We will write $\mathfrak{D}\stackrel{\text{def}}{=}\mathfrak{D}_1$ and $\mathfrak{d}\stackrel{\text{def}}{=}\mathfrak{d}_1$ for short.

It is easy to verify by expanding f in a series that

$$\|f\|_{\mathfrak{D}}^2 = \frac{1}{\pi}\iint_{\mathbb{D}} |f'(x+iy)|^2\,dx\,dy,$$

and

$$\|f\|_{\mathfrak{d}}^2 = \frac{1}{\pi}\iint_{\mathbb{D}} |\nabla f(x,y)|^2\,dx\,dy.$$

Since for a holomorphic function f the quantity $|f'(x+iy)|^2$ is the Jacobian of the mapping f, it follows that $\pi\|f\|_{\mathfrak{D}}^2$ is the area of the image of $\mathbb{D}$ under f *counting multiplicities*, and if f is univalent (i.e., one-to-one) then $\pi\|f\|_{\mathfrak{D}}^2$ is precisely equal to the area of the image of f.

It is in the space $\mathfrak{D}$ that our greatest interest lies.

Technical remark 1.1. To place $\mathfrak{D}$ in a larger picture, it should be noted that $\mathfrak{D}$ contains unbounded functions (any univalent map to an unbounded region of finite area will be in $\mathfrak{D}$), and that not every function from the disc algebra lies in $\mathfrak{D}$. Indeed, let

$$f(z) = \sum_{m=0}^{\infty} 2^{-m/2} z^{2^m}.$$

Evidently, this series converges uniformly on $\mathbb{T}$ and thus f lies in the disc algebra. However,

$$\sum_{n=1}^{\infty} n|\hat{f}(n)|^2 = \sum_{m=0}^{\infty} 2^m 2^{-m} = \infty,$$

and so $f \notin \mathfrak{D}$. On the other hand, $\mathfrak{D} \subset$ BMOA. One of the easier ways to prove this is to note that from the definition of the norm on Dirichlet space it easily follows that if $f \in \mathfrak{D}$ then

$$\sup_{m\geq 1} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m-1} |\hat{f}(mn+j)| \right)^2 < \infty,$$

which is C. Fefferman's sufficient (and, if we have $\hat{f}(k) \geq 0$ for all $k \geq 0$, then also necessary) condition for a holomorphic function f on $\mathbb{D}$ to be a member of BMOA [24, 97].

Definition 1.6. Let $\mathfrak{B}$, $\mathfrak{b}$, $\mathfrak{B}_\alpha$ and $\mathfrak{b}_\alpha$ be the unit balls of $\mathfrak{D}$, $\mathfrak{d}$, $\mathfrak{D}_\alpha$ and $\mathfrak{d}_\alpha$, respectively.

These unit balls will be the ranges of the Λ_Φ functionals in which we will be interested.

The following result will also be useful. The most useful case is where $\alpha = 0$ and $\beta = 1$ and that case was already proved by Andreev and Matheson [5].

Theorem 1.1. *The inclusions $\mathfrak{D}_\beta \subset \mathfrak{D}_\alpha$ and $\mathfrak{d}_\beta \subset \mathfrak{d}_\alpha$ are compact for $0 \leq \alpha < \beta < \infty$. In particular, if f_n is a weakly null sequence in $\mathfrak{D}_\alpha$ or $\mathfrak{d}_\alpha$ for $\alpha \in (0, \infty)$, then $f_n \to 0$ in $L^2(\mathbb{T})$ and hence f_n tends to 0 in measure on $\mathbb{T}$.*

$||f||_{\mathfrak{D}_0} = ||f||_{L^2(\mathbb{T})}$ for a function f with mean 0 on $\mathbb{T}$,

The following lemma is doubtless well-known but is easier to prove than to attribute.

Lemma 1.1. *Fix $1 \leq p < \infty$. Consider the space $\ell^p(w)$, where w is a non-negative weight with $w_n \to \infty$ as $n \to \infty$. Let a_k be a norm-bounded sequence in $\ell^p(w)$, converging pointwise to 0. Then $a_k \to 0$ in ℓ^p-norm.*

Assume this for now.

Proof of Theorem 1.1. It suffices to prove that if $f_k \to 0$ weakly in $\mathfrak{D}_\beta$ then $f_k \to 0$ in $\mathfrak{D}_\alpha$ norm. The assertion for $f_k \to f$ weakly where $f \not\equiv 0$ then follows by considering instead the sequence $f_n - f$, while the assertion for the harmonic Dirichlet spaces follows from Remark 1.2. Now, let $p = 2$, and put $(a_k)_n = n^{\alpha/2} \widehat{f}_k(n)$. Then,

$$||(a_k)||_{\ell^2} = ||f_k||_{\mathfrak{D}_\alpha}. \tag{1.1}$$

Let $w(n) = n^{\beta/\alpha}$. Note that $w(n) \to \infty$ as $n \to \infty$. Moreover,

$$||(a_k)||_{\ell^2(w)} = ||f_k||_{\mathfrak{D}_\beta}.$$

Hence, a_k satisfies the conditions of Lemma 1.1, and thus $a_k \to 0$ in ℓ^2 norm. In light of (1.1), we conclude that $f_k \to 0$ in $\mathfrak{D}_\alpha$ norm, as desired. The assertion that $||f||_{L^2(\mathbb{T})} \to 0$ follows since $||f||_{\mathfrak{D}_0} = ||f||_{L^2(\mathbb{T})}$ for a function f with mean 0 on $\mathbb{T}$. $\square$

Proof of Lemma 1.1. Without loss of generality assume that $\|a_k\|_{\ell^p(w)} \leq 1$ for all k. Fix $\varepsilon > 0$. Choose N sufficiently large that $w_n \geq 1/\varepsilon$ for $n \geq N$. Then,

$$\sum_{n=N}^{\infty} |(a_k)_n|^p \leq \varepsilon \sum_{n=N}^{\infty} w_n |(a_k)_n|^p \leq (\varepsilon/2)\|a_k\|^p_{\ell^p(w)} \leq \varepsilon.$$

On the other hand, for fixed N,

$$\sum_{n=1}^{N-1} |(a_k)_n|^p \to 0,$$

as $k \to \infty$. It follows that $\limsup_{k\to\infty} \|a_k\|_{\ell^p} \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

1.3. The Γ_Φ functionals acting on domains and the Λ_Φ acting on holomorphic and harmonic functions

We now establish the convention that when Λ_Φ acts on a holomorphic or harmonic function f on $\mathbb{D}$, then $\Lambda_\Phi(f)$ is defined in terms of the normalized Lebesgue measure on $\mathbb{T}$, i.e.,

$$\Lambda_\Phi(f) = \int_{\mathbb{T}} \Phi(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, d\theta,$$

where $f(e^{i\theta})$ as usual is short for $\text{n.t.}\lim f(e^{i\theta})$.

Definition 1.7. Let $\mathcal{B}$ denote the collection of all domains in the plane which contain the origin and whose area does not exceed π.

Given a domain $D \in \mathcal{B}$ and a Borel measurable function Φ on $\mathbb{C}$, let $h(\cdot;\Phi,D)$ be the solution of the Dirichlet problem on D with boundary value $\Phi|_{\partial D}$ on ∂D.

Definition 1.8. Given a Borel measurable function Φ on $\mathbb{C}$ and $D \in \mathcal{B}$, let

$$\Gamma_\Phi(D) = h(0;\Phi,D),$$

providing the right hand side is defined. If $\Phi^+|_{\partial D} \notin L^1(\omega^D)$ and $\Phi^-|_{\partial D} \in L^1(\omega^D)$ then write $\Gamma_\Phi(D) = \infty$; if $\Phi^+|_{\partial D} \in L^1(\omega^D)$ and $\Phi^-|_{\partial D} \notin L^1(\omega^D)$ then write $\Gamma_\Phi(D) = -\infty$; if $\Phi^+|_{\partial D} \notin L^1(\omega^D)$ and $\Phi^-|_{\partial D} \notin L^1(\omega^D)$ then say that $\Gamma_\Phi(D)$ is undefined. Say that the functional Γ_Φ is **bounded** providing it is defined (i.e., the Dirichlet problem in question has a solution) and

$$\sup_{D\in\mathcal{B}} \Gamma_\Phi(D) < \infty.$$

Say that a domain D is **extremal** for Γ_Φ providing $D \in \mathcal{B}$ and

$$\Gamma_\Phi(D) \geq \Gamma_\Phi(D')$$

for all $D' \in \mathcal{B}$. By abuse of notation, if ϕ is defined on $[0, \infty)$ then we write Γ_ϕ for Γ_Φ where $\Phi(z) = \phi(|z|)$.

The case that will interest us the most is when Φ is a continuous subharmonic function, in which case $\Gamma_\Phi(D) = \mathrm{LHM}(0, \Phi; D)$ (Theorem I.5.4).

The Γ_Φ functionals are actually closely related to the Λ_Φ functionals on $\mathfrak{B}$. First note that the map $f \mapsto f[\mathbb{D}]$ sending a function to its image maps $\mathfrak{B}\backslash\{0\}$ into $\mathcal{B}$, since if $f \in \mathfrak{B}$ then $f[\mathbb{D}]$ has area at most π (since the area of the image *counting multiplicities* is at most π).

The following result is essentially well known (the first part is implicit in, e.g., [44]).

Theorem 1.2. *Let f be any Nevanlinna class function whose image is contained in a domain D. Let Φ be subharmonic on $\mathbb{C}$ and continuous at every point of ∂D. We then have*

$$\Lambda_\Phi(f) \leq \Gamma_\Phi(D).$$

If f is univalent and $D = f[\mathbb{D}]$, then equality holds.

Proof. Put $h(z) = \mathrm{LHM}(z, \Phi; D)$. As in [44], note that

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi(f(re^{i\theta}))\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} h(f(re^{i\theta}))\, d\theta = h(f(0)) = h(0),$$

for every $0 \leq r < 1$, since $h \geq \Phi$ on $\mathbb{D}$. Taking the limit as $r \uparrow 1$ we conclude that $\Lambda_\Phi(f) \leq h(0)$ (use Fatou's lemma together with the existence of radial limits of f due to Theorem I.3.5 and the assumption that $f \in N$).

If f is univalent then f is a uniformizer and we may apply Theorem I.5.7 to obtain the desired equality. (Of course, the domain in this case is Greenian since all simply connected domains are Greenian, while all images of univalent functions are simply connected.) □

Corollary 1.1. *Fix $p \in [1, \infty)$. Let D be a Greenian domain containing the origin and let $f: \mathbb{D} \to D$ be a uniformizer for D. Then $D \in H^p$ if and only if $f \in H^p(\mathbb{D})$.*

Proof. It is clear that if $D \in H^p$ then $f \in H^p$. Conversely, suppose that $f \in H^p$. Let g be any holomorphic function on $\mathbb{D}$ with image in D. Let $w = g(0)$. Let $U = -w + D$. Of course, f is surjective so that there exists $w' \in \mathbb{D}$ such that $f(w') = w$. Let ϕ be a Möbius transformation of $\mathbb{D}$ onto itself such that $\phi(0) = w'$. Then, $f \circ \phi$ is easily seen to lie in H^p if and only if f lies in H^p. Let $F(z) = f(\phi(z)) - w'$. Then, F is a uniformizer for U (since ϕ is a conformal automorphism of $\mathbb{D}$) with $F(0) = 0$. We have $F \in H^p$. Let $G(z) = g(z) - w$. Then the image of G lies in U and $G(0) = 0$. Let $\Phi(z) = |z|^p$. We have

$$\Lambda_\Phi(G) \leq \Gamma_\Phi(U).$$

But by Theorem I.5.7, we have

$$\Gamma_\Phi(U) = \Lambda_\Phi(F).$$

But $\Lambda_\Phi(F) < \infty$, so that $\Lambda_\Phi(G) < \infty$ and hence $G \in H^p(\mathbb{D})$ so that $g \in H^p(\mathbb{D})$ as desired. □

2. The Chang-Marshall, Essén and Moser-Trudinger inequalities

The Chang-Marshall [32], Essén [44] and Moser-Trudinger [78] inequalities are all closely related. We now state them, using the notation of the previous sections of this chapter.

Essén Inequality ([44]). *Let $\Phi(t) = e^{t^2}$ for $t \in [0, \infty)$. Then Γ_Φ is bounded on the set $\mathcal{B}$ of all domains in the plane containing 0 and having area at most π.*

Chang-Marshall Inequality ([32]). *Let $\Phi(t) = e^{t^2}$ for $t \in [0, \infty)$. Then Λ_Φ is bounded on the closed unit ball $\mathfrak{B}$ of the Dirichlet space.*

Moser-Trudinger Inequality ([78]). *Let $\Phi(t) = e^{t^2}$. Then Λ_Φ is bounded on the set $\mathcal{F}$ of absolutely continuous real-valued functions f on $[0, \infty)$ satisfying $f(0) = 0$ and $\int_0^\infty (f'(x))^2\, dx \leq 1$, where the integral in the definition of Λ_Φ is taken with respect to the measure $d\mu(x) = e^{-x}\, dx$ on $[0, \infty)$.*

In other words, the Chang-Marshall inequality states that

$$\sup_{f \in \mathfrak{B}} \int_0^{2\pi} e^{|f(e^{i\theta})|^2} < \infty,$$

while the Moser inequality states that

$$\sup_{f \in B} \int_0^\infty e^{f^2(x) - x}\, dx < \infty.$$

Note that in light of Theorem 1.2 and its preceding remarks, the Chang-Marshall inequality is actually a consequence of the Essén inequality while the Essén inequality restricted to $\mathcal{U}$ instead of on all of $\mathcal{B}$ is a consequence of the Chang-Marshall inequality. The Chang-Marshall and Essén inequalities are also closely connected with the Moser-Trudinger inequality since Marshall [72] has found a fairly easy proof of the Chang-Marshall inequality using the Moser-Trudinger inequality[1], while Essén's proof of his inequality [44] also uses the Moser-Trudinger inequality in an essential way.

For more work related to the Moser inequality, see, e.g., [1, 2, 6, 18, 19, 20, 21, 30, 31, 33, 34, 49, 52, 53, 69, 77, 80, 81, 86]. For more work related to the Chang-Marshall inequality, see [5, 35].

All three inequalities (Essén, Chang-Marshall and Moser-Trudinger) are unimprovable in that the function $\Phi(t) = e^{t^2}$ cannot be replaced by any faster growing function. In the case of the Chang-Marshall inequality, the following result is an improvement of a result of Matheson and Pruss [75, Thm. 1].

Theorem 2.1. *Let Ψ be any nonnegative Borel measurable function on $[0, \infty)$ such that atleast one of the following three conditions is fulfilled:*

[1] Of course Marshall's proof uses some rather specialized complex variable techniques, and thus may be considered more difficult if one is more comfortable with real variable methods.

(a) $t \mapsto \Psi(e^t)$ *is convex, increasing and* Ψ *is continuous at* 0 *while* Γ_Ψ *is bounded on* $\mathcal{B}$

(b) Λ_Ψ *is bounded on* $\mathfrak{B}$

(c) Λ_Ψ *is bounded on* $\mathcal{F}$, *in the setting of the Moser-Trudinger inequality.*

Then there exists a finite constant C *such that* $\Psi(t) \leq Ce^{t^2}$ *for all* $t \in [0,\infty)$.

Proof in the case of (c). Suppose that $\sup_{f\in\mathcal{F}} \Lambda_\Psi(f) = C < \infty$. Fix $t \in [0,\infty)$. As in [78], let $f(x) = t\min(x/t^2, 1)$. It is easy to see that $f \in \mathcal{F}$. Define the measure $d\mu(x) = e^{-x}\,dx$. Then, for $x \geq t^2$ we have $f(x) = t$. Thus,

$$C \geq \Lambda_\Psi(f) = \int_0^\infty \Psi(f(x))e^{-x}\,dx \geq \int_{t^2}^\infty \Psi(t)e^{-x}\,dx = \Psi(t)e^{-t^2}.$$

Hence, $\Psi(t) \leq Ce^{t^2}$. Since t was arbitrary, we are done. $\square$

Case (a) of Theorem 2.1 actually follows from case (b), in the same way as we have indicated that Essén inequality implies the Chang-Marshall inequality. (The stated assumptions guarantee that $\Psi(|z|)$ is subharmonic by Theorem I.4.4.)

Before we prove case (b), we need to define the Beurling functions, which also will be useful later, as well as the cut-off Beurling functions. Set

$$B_a(z) = \frac{\log\frac{1}{1-\bar{a}z}}{\sqrt{\log\frac{1}{1-|a|^2}}},$$

for $a \in \mathbb{D}\backslash\{0\}$ and $z \in \mathbb{D}$, where the branch of the logarithm is chosen so that $B_a(a)$ is positive. The denominator was chosen so that $\|B_a\|_{\mathfrak{D}} = 1$. The B_a will be called the **Beurling functions**. They are normalized versions of the reproducing kernels for the space $\mathfrak{D}$.

Note that the Beurling functions are univalent and in fact star-like (i.e., has star-shaped images). To see this fact, note that as is well known a function f holomorphic on $\mathbb{D}$ is star-like if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0$$

on $\mathbb{D}$. To verify this identity is a simple elementary exercise.

Now, set $M_a = \sqrt{\log \frac{1}{1-|a|^2}}$. Given $M > 0$, let $a(M) \in (0,1)$ be such that $M_{a(M)} = M$. Define the domain

$$D_M = B_{a(M)}[\mathbb{D}] \cap \{|z| < M\}.$$

Since $B_{a(M)}$ is starlike, it follows that D_M is star-shaped and hence simply connected. Define the **cut-off Beurling function** $\tilde{B}_M$ as the Riemann map from $\mathbb{D}$ onto D_M sending 0 to 0 with $\tilde{B}'_M(0) > 0$. For convenience, define $\tilde{B}_0 \equiv 0$. The following Lemma is a variation on a result of Beurling [23, pp. 39–41].

Lemma 2.1. *There exists an absolute constant $c > 0$ such that*

$$|\{z \in \mathbb{T} : |\tilde{B}_M(z)| = M\}| \geq ce^{-M^2}$$

for every positive M, where $|\cdot|$ indicates normalized Lebesgue measure on $\mathbb{T}$.

Assuming this, we can finish the proof of case (b) of Theorem 2.1.

Proof of case (b) *of Theorem 2.1.* Suppose that $\sup_{f \in \mathfrak{B}} \Lambda_\Psi(f) = K < \infty$. Fix $M \geq 0$. We then have

$$\begin{aligned} K \geq \Lambda_\Psi(B_M) &= \int_{\mathbb{T}} \Psi(|B_M|) \\ &\geq \Psi(M)|\{z \in \mathbb{T} : |B_M(z)| = M\}| \\ &\geq \Psi(M)ce^{-M^2}, \end{aligned}$$

so that $\Psi(M) \leq Kc^{-1}e^{M^2}$, which completes the proof in light of the arbitrariness of M. $\square$

Proof of Lemma 2.1. Clearly, by possibly adjusting the value of c, we need only consider the case of M *strictly* positive. et $a = a(M)$. Set $U = B_a[\mathbb{D}]$. Put

$$w = |\{z \in \mathbb{T} : \operatorname{Re} B_a \geq M\}|.$$

Then, as in [23], an elementary computation shows that

$$w \geq ce^{-M^2}, \tag{2.1}$$

for some absolute constant $c > 0$. But w is the harmonic measure at 0 of $\{z \in \mathbb{T} : \operatorname{Re} B_a \geq M\}$ in $\mathbb{D}$. By conformal invariance of harmonic measure under the univalent map B_a, the number w is also equal to the harmonic measure at 0 of $\{z \in \partial U : \operatorname{Re} z \geq M\}$ in the domain U. Let

$$w_1(z) = \omega(z, \{z \in \partial U : \operatorname{Re} z \geq M\}; U),$$

and

$$w_2(z) = \omega(z, \{z \in \partial D_a : |z| = M\}; D_a).$$

We shall apply the maximum principle to conclude that $w_2 \geq w_1$ on D_a. For, in order to do this it suffices to verify this inequality on ∂D_a, since both functions are harmonic on $D_a \subset U$. Fix $z \in \partial D_a$. Suppose first that $|z| = M$. In that case $w_2(z) = 1$ and since w_1 nowhere exceeds 1, we are done. Suppose now that $|z| \neq M$, so that $w_2(z) = 0$. Then necessarily $|z| < M$ and $z \in \partial U$ by definition of D_a. Then, $\operatorname{Re} z < M$. Hence, $z \in \partial U \backslash \{z \in \partial U : \operatorname{Re} z \geq M\}$. Thus, $w_1(z) = 0 \leq 0 = w_2(z)$. We have thus verified that $w_2 \geq w_1$. Hence,

$$w_2(0) \geq w_1(0) \geq ce^{-M^2}. \tag{2.2}$$

Applying conformal invariance of harmonic measure under the map $\tilde{B}_a^{-1}$ together with Remark I.5.5, we see as before that

$$w_2(0) = |\{z \in \mathbb{T} : |\tilde{B}_a(z)| = M\}|,$$

which clearly completes the proof in light of (2.2). □

3. General results on the Λ_Φ functionals on measure spaces

In this section we discuss results on the Λ_Φ functionals acting on a collection of functions on a finite measure space. From time to time we shall give applications to the Chang-Marshall or Moser-Trudinger inequality.

In all of this section, (I, μ) will be a finite measure space with $\mu(I) > 0$ and the functionals Λ_Φ will be defined by

$$\Lambda_\Phi(f) = \int_I \Phi(f)\, d\mu,$$

for Φ on $\mathbb{C}$ and by

$$\Lambda_\Phi(f) = \int_I \Phi(|f|)\, d\mu,$$

for Φ on $[0,\infty)$.

3.1. Existence of extremals

We are interested in general criteria for existence of an extremal function. Our method for deriving this existence will be to use a compactness argument.

First we remark that the unit balls $\mathfrak{B}_\alpha$ and $\mathfrak{b}_\alpha$ of the Dirichlet spaces are compact with respect to sequential convergence in measure on $\mathbb{T}$ providing $\alpha > 0$. In the case of $\mathfrak{B} = \mathfrak{B}_1$, this was noted by Andreev and Matheson [5]. The easiest way to see this fact is to note that weak convergence in one of these balls implies $L^2(\mathbb{T})$ convergence by Theorem 1.1. However, these balls are all weakly compact (Banach-Alaoglu), so that they must be compact with respect to sequential convergence in measure on $\mathbb{T}$.

Now, if these balls are sequentially compact with respect to convergence in measure, then extremals for Λ_Φ exist provided that Λ_Φ is upper semicontinuous with respect to the topology of sequential convergence in measure, i.e., providing that

$$\limsup_{n\to\infty} \Lambda_\Phi(f_n) \le \Lambda_\Phi(f)$$

whenever $f_n \to f$ in measure on $\mathbb{T}$ with f and the f_n lying in the appropriate ball.

The following result is a direct consequence of Fatou's lemma.

Lemma 3.1. *Let $\Phi\colon \mathbb{C} \to [0,\infty)$ be lower semicontinuous. Then, given a sequence $\{f_n\}$ of measurable functions on I converging in measure to f, we must have*

$$\liminf_{n\to\infty} \Lambda_\Phi(f_n) \ge \Lambda_\Phi(f).$$

The following simple result is then due to Matheson [73]. We say that Λ_Φ is upper semicontinuous along a sequence $f_n \to f$ converging in measure providing

$$\limsup_{n\to\infty} \Lambda_\Phi(f_n) \le \Lambda_\Phi(f).$$

Theorem 3.1 (cf. Matheson [73]). *Let $f_n \to f$ in measure on I, with f almost everywhere finite. Suppose that $\Phi\colon \mathbb{C} \to \mathbb{R}$ is upper semicontinuous and $\Psi\colon \mathbb{C} \to \mathbb{R}$ is lower semicontinuous with $\Phi \leq \Psi$ everywhere. Then if Λ_Ψ is upper semicontinuous along a sequence $f_n \to f$ converging in measure, so is Λ_Φ.*

Proof. The function $\Psi - \Phi$ is lower semicontinuous and nonnegative, so that by Lemma 3.1 we have $\liminf_{n\to\infty} \Lambda_{\Psi-\Phi}(f_n) \geq \Lambda_{\Psi-\Phi}(f)$, and thus $\Lambda_{\Psi-\Phi}$ is lower semicontinuous along our sequence. But $\Lambda_\Phi = \Lambda_\Psi - \Lambda_{\Psi-\Phi}$, and Λ_Ψ is upper semicontinuous along this sequence, so Λ_Φ must be upper semicontinuous along it as well. □

The above result shows that under some continuity assumptions on the functions Φ, the upper semicontinuity property of the Λ_Φ is preserved under pointwise majorization of the functions Φ.

Corollary 3.1. *The functionals E_M defined for $M \geq 0$ by*

$$E_M(f) = |\{z \in \mathbb{T} : |f(\omega)| \geq M\}|,$$

where $|\cdot|$ is normalized Lebesgue measure on $\mathbb{T}$, attain their maxima on $\mathfrak{B}_\alpha$ for every $\alpha > 0$.

Proof. Let $\Phi(z) = 1_{\{|z|\geq M\}}$ and $\Psi(z) \equiv 1$ so that $\Lambda_\Phi = E_M$ and $\Lambda_\Psi \equiv 1$. Then the conditions of Theorem 3.1 are satisfied along any sequence converging in measure on $\mathbb{T}$, and since $\mathfrak{B}_\alpha$ is sequentially compact with respect to convergence in measure, it follows that if f_n is a maximizing sequence for E_M on $\mathfrak{B}_\alpha$ (i.e., $\lim_{n\to\infty} E_M(f_n) = \sup_{f\in\mathfrak{B}_\alpha} E_M(f)$) then we may, by passing to an appropriate subsequence, assume that it converges in measure on $\mathbb{T}$ and then we will have

$$\limsup_{n\to\infty} E_M(f_n) \leq E_M(\lim f_n),$$

so that the maximum of E_M will be attained at $\lim f_n$. □

It is not known what the extremal functions look like for the E_M on $\mathfrak{B}_\alpha$. It would be of interest to determine this even if only for $\alpha = 1$.

We now state a result which we will use a number of times in the future. Our result is the author's generalization of a result of Matheson [73] and uses a different method of proof from that of Matheson. Recall that $\Phi^+(z) = \max(\Phi(z), 0)$ for all z.

Theorem 3.2 ([75]). *Let $f_n \to f$ in measure on I, with f almost everywhere finite. Let Φ be a finite-valued upper semicontinuous function on $\mathbb{C}$, and let Ψ be a non-negative Borel-measurable function on $\mathbb{C}$ such that Λ_Ψ is uniformly bounded on $\{f_n\} \cup \{f\}$. Suppose that*

$$\Phi^+(z) = o(\Psi(z)),$$

uniformly in $\arg z$, *as* $|z| \to \infty$. *Then* $\limsup_{n\to\infty} \Lambda_\Phi(f_n) \leq \Lambda_\Phi(f)$.

We shall give a (quite elementary) proof of this result at the end of this section. However, first we wish to go back to Dirichlet spaces and discuss some consequences of this result.

Before we do this, we would like to state the following theorem.

Theorem 3.3. *For $f \in \mathfrak{B}_\alpha$ where $\alpha > 1$ we have*

$$\|f\|_{L^\infty(\mathbb{T})} \leq \sqrt{\zeta(\alpha)},$$

where $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$ is the Riemann zeta function. For $\alpha \in [0,1)$ we have Λ_Φ bounded on $\mathfrak{B}_\alpha$, where $\Phi(t) = t^{2/(1-\alpha)}$.

Proof. To prove the first assertion, let $a_n = \hat{f}(n)$ and use the Cauchy-Schwarz inequality to note that

$$\begin{aligned}
|f(z)| &\leq \sum_{n=1}^{\infty} |a_n| \\
&= \sum_{n=1}^{\infty} n^{\alpha/2}|a_n| \cdot n^{-\alpha/2} \\
&\leq \left(\sum_{n=1}^{\infty} n^{\alpha}|a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{-\alpha}\right)^{1/2} \|f\|_{\mathfrak{D}_\alpha} \\
&= \sqrt{\zeta(\alpha)}\|f\|_{\mathfrak{D}_\alpha}.
\end{aligned}$$

To prove the second assertion, note that $\mathfrak{D}_1$ embeds in BMO (see Technical Remark 1.1) while $\mathfrak{D}_0$ is obviously a (closed) subspace of $L^2(\mathbb{T})$. It follows by interpolation that $\mathfrak{D}_\alpha$ embeds in $L^{2/(1-\alpha)}(\mathbb{T})$ whenever $\alpha \in (0,1)$. This interpolation can be done in the complex method, using the Stein-Weiss interpolation theorem [22, Chapter 5] to interpolate between $\mathfrak{D}_0$ and $\mathfrak{D}_1$ (since these are essentially weighted ℓ^2 spaces) and a theorem of Fefferman and Stein [48, equation (5.1), p. 156][2] to interpolate between L^2 and BMO. $\square$

Thus, we obtain the following result. The case where $\alpha = 1$ and $\Phi(z) = \phi(|z|)$ for ϕ monotone increasing is due to Matheson [73].

Theorem 3.4. *Fix $\alpha \in (0,\infty)$. Let Φ be an upper semicontinuous function on $\mathbb{C}$. If $\alpha \le 1$ then make the following additional assumptions:*

(i) *If $\alpha \in (0,1)$ then $\Phi^+(z) = o(|z|^{2/(1-\alpha)})$, uniformly in $\arg z$, as $|z| \to \infty$.*

(ii) *If $\alpha = 1$ then $\Phi^+(z) = o(e^{|z|^2})$, uniformly in $\arg z$, as $|z| \to \infty$.*

Then Λ_Φ attains its maximum on $\mathfrak{B}_\alpha$.

The proof of this follows by sequential compactness of $\mathfrak{B}_\alpha$ with respect to convergence in measure, combined with an application of Theorem 3.2, where we choose Ψ as follows:

(i) If $\alpha \in (0,1)$ then $\Psi(z) = |z|^{2/(1-\alpha)}$.

(ii) If $\alpha = 1$ then $\Psi(z) = e^{|z|^2}$.

(iii) If $\alpha > 1$ then Ψ is any finite-valued upper semicontinuous function such that $\Phi^+(z) = o(\Psi(z))$ as $t \to \infty$. (Existence of such a function is easy; e.g., take $\Psi(z) = (|z| + \Phi^+(z))^2$.)

[2] Fefferman and Stein [48] work with spaces on $\mathbb{R}^n$ and not on $\mathbb{T}$ as we do. But, as is well known, the proofs in the case of spaces on $\mathbb{T}$ work in exactly the same way. A typographical error on p. 156 of [48] should be noted. The inequality "$1 < p \le \infty$" preceding equation (5.1) should instead read "$1 < p < \infty$". The case $p = \infty$ is likewise true, but was not proved in [48]; it was only proved later by Jones [64]. In any case we do not use the case $p = \infty$ but only the case $p = 2$.

In cases (i) and (iii) the boundedness of Λ_Ψ on $\mathfrak{B}_\alpha$ follows from Theorem 3.3. In case (ii), the boundedness is precisely the Chang-Marshall inequality.

Theorem 3.4 gives us a large supply of cases in which extremal functions exist.

Corollary 3.2 (Andreev and Matheson [5]). *If $\Phi(t) = t^p$ for $p \in (0, \infty)$ or $\Phi(t) = e^{\alpha t^2}$ for $\alpha \in (0, 1)$, then Λ_Φ attains its maximum on $\mathfrak{B}$.*

However, the functions at which these maxima are attained are generally not known. It has been proved by Alec Matheson (unpublished) that if $\Phi(t) = t^p$ for $p \in (0, 4]$ and $t \in [0, \infty)$ then Λ_Φ attains its maximum at the identity function. It is also known that this is not the case if $p > 4$; this is essentially due to Sakai [95] (see [75, discussion after Open Problem 4] for details.)

Open Problem 3.1. Does Λ_Φ attain its supremum over $\mathfrak{B}$ when $\Phi(t) = e^{t^2}$?

The above problem was raised by Andreev and Matheson [5]. Cima and Matheson [35] have shown that an affirmative answer can be given if one restricts the problem to the subset of $\mathfrak{B}$ given by the closed convex of the Beurling functions B_a. However, in general the problem remains open. We shall see in §3.3.3 that to answer this problem it does not suffice to use general arguments based on the asymptotic behaviour of e^{t^2} but one must use more quantitative arguments which also take into account the nonasymptotic behaviour of this function.

It has been conjectured by Andreev and Matheson [5] that in fact Λ_Φ attains its maximum at the identity function z. Since $\Lambda_\Phi(z) = e$, this conjecture is equivalent to the conjecture that

$$\Lambda_\Phi(f) \leq e$$

for all $f \in \mathfrak{B}$. This conjecture has been numerically verified by the author for 40 million pseudorandomly chosen degree 6 polynomials f with positive coefficients and $\|f\|_{\mathfrak{D}} = 1$ (see source code in Appendix B). (Note that it does in fact suffice to consider polynomials with positive coefficients; see Chang and Marshall [32, p. 1022] who attribute this observation to J. Clunie.)

We may now give the proof of Theorem 3.2.

Proof of Theorem 3.2. Dropping to a subsequence, we may assume that we have almost everywhere convergence. For, every subsequence of $\{f_n\}$ has an almost everywhere convergent subsubsequence, and it suffices to show that for every subsequence f_{n_k} along which $\Lambda_\Phi(f_{n_k})$ converges the limit is no greater than $\Lambda_\Phi(f)$.

Now fix $\varepsilon > 0$. Let M be a uniform upper bound for Λ_Ψ on $\{f_n\} \cup \{f\}$. Since $\Phi^+(z) = o(\Psi(z))$ and Ψ is nonnegative, we may choose a number T such that $\Phi(z) \leq (\varepsilon/4M)\Psi(z)$ whenever $|z| \geq T$. Furthermore, we may ensure that our choice of T is such that

$$\mu\{x : |f(x)| = T\} = 0 \tag{3.1}$$

and

$$|\Phi(0)|\mu\{x : |f(x)| > T - 1\} \leq \varepsilon/4, \tag{3.2}$$

since f is almost everywhere finite.

Let $U(z) = z$ if $|z| < T$ and $U(z) = 0$ if $|z| \geq T$. Then $U \circ f_n \to U \circ f$ almost everywhere outside the set $\{x : |f(x)| = T\}$. By (3.1) this latter set has measure zero, so we have convergence almost everywhere. Since Φ is upper semicontinuous, we also have

$$\limsup_{n\to\infty} \Phi \circ U \circ f_n \leq \Phi \circ U \circ f, \tag{3.3}$$

almost everywhere. Now the $|U \circ f_n|$ are uniformly bounded by T, so that the $\Phi \circ U \circ f_n$ are uniformly bounded by some constant K independent of n since Φ is bounded on the compact set $\bar{\mathbb{D}}(T)$, being upper semicontinuous and finite. Then using the fact that we have a finite measure space, and applying Fatou's lemma to the non-negative functions $K - \Phi \circ U \circ f_n$ we find that

$$\limsup_{n\to\infty} \Lambda_\Phi(U \circ f_n) \leq \int_I \limsup_{n\to\infty} \Phi \circ U \circ f_n \leq \Lambda_\Phi(U \circ f), \tag{3.4}$$

where the last inequality follows from (3.3).

But by the definition of U we have

$$\begin{aligned}\Lambda_\Phi(f_n) - \Lambda_\Phi(U \circ f_n) &= \int_{\{x:|f_n(x)|\geq T\}} (\Phi(f_n) - \Phi(0)) \\ &\leq \left(\int_{\{x:|f_n(x)|\geq T\}} (\varepsilon/4M)\Psi(f_n) \right) - \Phi(0)\mu\{x : |f_n(x)| \geq T\} \\ &\leq (\varepsilon/4M)M - \Phi(0)\mu\{x : |f_n(x)| \geq T\},\end{aligned}$$

where in the last two inequalities we have used the choices of T and M, respectively. Furthermore, by (3.2) and the fact that f_n converges in measure to f, it follows that

$$\limsup_{n\to\infty} |\Phi(0)|\mu\{x : |f_n(x)| \geq T\} \leq \varepsilon/4.$$

Therefore,

$$\limsup_{n\to\infty} \Lambda_\Phi(f_n) - \Lambda_\Phi(U \circ f_n) \leq \varepsilon/2. \tag{3.5}$$

In much the same way we find that

$$|\Lambda_\Phi(f) - \Lambda_\Phi(U \circ f)| \leq \varepsilon/2. \tag{3.6}$$

Then, putting (3.4), (3.5) and (3.6) together, we find that

$$\begin{aligned}\limsup_{n\to\infty} \Lambda_\Phi(f_n) - \Lambda_\Phi(f) = \limsup_{n\to\infty} \big(&\Lambda_\Phi(f_n) - \Lambda_\Phi(U \circ f_n) \\ &+ \Lambda_\Phi(U \circ f_n) - \Lambda_\Phi(U \circ f) \\ &+ \Lambda_\Phi(U \circ f) - \Lambda_\Phi(f)\big) \\ \leq \varepsilon.&\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

3.2. The Λ_Φ functionals on balls of Hilbert spaces

In this section, assume that H is a (real or complex) separable Hilbert space of measurable functions f on a finite measure space (I, μ). Assume that $H \cap L^\infty(\mu)$ is dense in H. Write $\|\cdot\|$

for the norm in H and $(\cdot,\cdot)$ for the inner product. Define:

$$\begin{aligned} B(r) &= \{f \in H : \|f\| \leq r\} \\ B &= B(1) \\ \partial B &= \{f \in H : \|f\| = 1\} \\ S(r,R) &= \{f \in H : r < \|f\| \leq R\} \\ S(r) &= S(r,1) \end{aligned}$$

Note that $S(r)$ is not weakly closed, its weak closure being all of B.

Given a function Φ on $[0,\infty)$ or on $\mathbb{C}$ and a positive number α, define $\Phi_\alpha(t) = \Phi(\alpha t)$. The following result is based on an insight attributed by Chang and Marshall [32] to Garnett in the case of the Chang-Marshall inequality.

Theorem 3.5. *Suppose that Φ is a finite increasing function on $[0,\infty)$ such that there exists an $r > 0$ with the property that $\Lambda_\Phi(f) < \infty$ for every $f \in B(r)$. Then $\Lambda_\Phi(f) < \infty$ for every $f \in H$.*

Proof. Fix $f \in H$. Then, since we have assumed that $L^\infty \cap H$ is H, we may choose $h \in L^\infty(\mu) \cap H$ such that $s \stackrel{\text{def}}{=} \|f-h\| < r$. Then, choose M so that $|h| \leq M < \infty$ almost everywhere on I, and choose N sufficiently large that $rN/s \geq N+M$. Then,

$$\begin{aligned} \Lambda_\Phi(f) &= \int_I \Phi(|f|)\,d\mu \\ &\leq \int_I \Phi(|f-h|+M)\,d\mu \\ &= \int_{\{|f-h|\geq N\}} \Phi(|f-h|+M)\,d\mu + \int_{\{|f-h|<N\}} \Phi(|f-h|+M)\,d\mu \\ &\leq \int_{\{|f-h|\geq N\}} \Phi(r|f-h|/s)\,d\mu + \Phi(N+M)\mu\{|f-h|<N\} \\ &\leq \int_I \Phi(r|f-h|/s)\,d\mu + \Phi^+(N+M)\mu(I) \\ &= \Lambda_\Phi(r(f-h)/s) + \Phi^+(N+M)\mu(I). \end{aligned}$$

The second term is clearly finite. As to the first, it too is finite because of our assumption of the finiteness of Λ_Φ on $B(r)$ and because of the fact that since $\|r(f-h)/s\| = r$ we must have $r(f-h)/s \in B(r)$. $\square$

The following result uses a similar method of proof. As a special case we will get the result of Cima and Matheson [35] on the weak continuity of Λ_Φ on $\mathfrak{B}\backslash\{0\}$ when $\Phi(t) = e^{t^2}$ for $t \in [0, \infty)$.

Theorem 3.6. *Fix $0 \le r < 1$. Suppose that $\Phi \colon \mathbb{C} \to \mathbb{R}$ is upper semicontinuous. Suppose moreover that there exists an increasing $\Psi \colon [0, \infty) \to [0, \infty)$ such that for every $\alpha > 1$ we have $\Phi^+(z) = o(\Psi(\alpha|z|))$ as $|z| \to \infty$ (uniformly in $\arg z$). Finally assume that Λ_Ψ is bounded on $B(\alpha)$ for every $\alpha < \sqrt{1 - r^2}$. Then Λ_Φ is weak upper semicontinuous on $S(r)$.*

Remark 3.1. We may scale this result so as to replace the assumption $0 \le r < 1$ by the assumption $0 \le r < R$, and to assume that Λ_Ψ is bounded on $B(\alpha)$ for every $\alpha < \sqrt{R^2 - r^2}$.

Proof of Theorem 3.6. Let $f_n \to f$ weakly with $f_n, f \in S(r)$. We must show that

$$\limsup_{n\to\infty} \Lambda_\Phi(f_n) \le \Lambda_\Phi(f).$$

Note that weak convergence implies that

$$\liminf_{n\to\infty} \|f_n\| \ge \|f\| > r. \tag{3.7}$$

(The easiest way to see this is to note that $\|f_n\| \ge |(f_n, f)|/\|f\|$, while $(f_n, f) \to \|f\|^2$.)

Now,

$$r < \|f\| = \sup_{g\in\partial B} (g, f).$$

Using our density assumption, there exists a function $g \in L^\infty(\mu) \cap \partial B$ such that $(g, f) > r$.

Put $g_n = f_n - (f_n, g)g$. Then, $(f_n, g)g$ and g_n are orthogonal, so that

$$\|f_n\|^2 = \|f_n - (f_n, g)g\|^2 + \|(f_n, g)g\|^2 = \|g_n\|^2 + |(f_n, g)|^2.$$

Then,

$$\begin{aligned}\limsup_{n\to\infty} \|g_n\|^2 &= \limsup_{n\to\infty} (\|f_n\|^2 - |(f_n, g)|^2) \\ &\le \limsup_{n\to\infty} (1 - |(f_n, g)|^2) \\ &= 1 - \liminf_{n\to\infty} |(f_n, g)|^2 \\ &= 1 - |(f, g)|^2 < 1 - r^2,\end{aligned} \tag{3.8}$$

since $(f_n, g) \to (f, g)$ by of weak convergence.

Weak convergence implies also that the numerical sequence $\{(f_n, g)\}$ be bounded. Since g is almost everywhere bounded, there exists a finite number M such that

$$|(f_n, g)g(x)| \leq M$$

for every $n \in \mathbb{Z}^+$ and almost every $x \in I$.

Now choose γ such that

$$\limsup_{n\to\infty} \|g_n\|^2 < \gamma^2 < 1 - r^2.$$

For n sufficiently large we then have $g_n \in B(\gamma)$.

Choose $\alpha > 1$ such that $\alpha\gamma < \sqrt{1-r^2}$. I claim that Λ_{Ψ_α} is bounded on the f_n corresponding to those g_n which lie in $B(\gamma)$. Assume for now that this is proved. Then, since Λ_{Ψ_α} is finite on H by Theorem 3.5, and since $g_n \in B(\gamma)$ for sufficiently large n, and finally since $\alpha > 1$ so that $\Phi^+(z) = o(\Psi_\alpha(|z|))$, it follows from Theorem 3.2 that $\limsup_{n\to\infty} \Lambda_\Phi(f_n) \leq \Lambda_\Phi(f)$.

To prove the claim, note that we have

$$|f_n| \leq |g_n| + M$$

almost everywhere, by choice of M.

Since $\alpha\gamma < \sqrt{1-r^2}$, we may choose $\beta > \alpha$ such that we still have $\beta\gamma < \sqrt{1-r^2}$. Choose T sufficiently large that $\beta T \geq \alpha(T + M)$. Let $A_n = \{x \in I : |g_n(x)| \geq T\}$ and put $B_n = I \backslash A_n$. Then,

$$\begin{aligned}
\Lambda_{\Psi_\alpha}(f_n) &= \int_I \Psi(\alpha|f_n|)\,d\mu \\
&\leq \int_I \Psi(\alpha(|g_n| + M))\,d\mu \\
&= \int_{A_n} \Psi(\alpha(|g_n| + M))\,d\mu + \int_{B_n} \Psi(\alpha(|g_n| + M))\,d\mu \\
&\leq \int_{A_n} \Psi(\beta|g_n|)\,d\mu + \mu(B_n)\Psi(\alpha(T + M))\,d\mu \\
&\leq \Lambda_\Psi(\beta g_n) + \mu(I)\Psi(\alpha(T + M)).
\end{aligned}$$

Now, the second term on the right hand side is finite and independent of n. The first term, on the other hand, is uniformly bounded on the set of those n for which $g_n \in B(\gamma)$ as the inequality $\gamma\beta < \sqrt{1-r^2}$ implies that Λ_Ψ is bounded on $B(\gamma\beta)$. $\square$

Corollary 3.3. *Suppose that $\Psi\colon [0,\infty) \to [0,\infty)$ is an increasing function such that Λ_Ψ is bounded on $B(r)$ for some $r > 0$. Let $\Phi^+(z) = o(\Psi(|z|))$ as $t \to \infty$ (uniformly in $\arg z$). Then, Λ_Φ is norm upper semicontinuous on all of H if Φ is upper semicontinuous.*

Remark 3.2. Of course, norm upper semicontinuity does not by itself imply any boundedness results since unit balls of infinite dimensional Hilbert spaces are not norm compact.

Proof of Corollary. Without loss of generality $r = 1$ so that Λ_Ψ is bounded on B.

Let $f_n \to f$ in norm. There are two cases to be considered. Suppose first that f is the zero element of H. For n sufficiently large we will have $f_n \in B$. Then, since Λ_Ψ is bounded by assumption on those f_n for which this happens and since it is everywhere finite on H by Theorem 3.5, it follows that Λ_Ψ is bounded on the $\{f_n\}$ and so $\limsup_{n\to\infty} \Lambda_\Phi(f_n) \leq \Lambda_\Phi(f)$ by Theorem 3.2. (We have, of course, used the fact that if $f_n \to f$ in norm, then it also converges weakly.)

On the other hand if $\|f\| > 0$, then let $M = \|f\|^2$. We have $\|f_n\|^2 \to M$. Fix $\varepsilon > 0$ to be chosen later, supposing for now only that $\varepsilon < M$. For n large enough we will have $M - \varepsilon < \|f\|^2 \leq M + \varepsilon$. Let $r = M - \varepsilon$ and $R = M + \varepsilon$. Then for sufficiently large n we have $f_n \in S(r, R)$. Note that $R^2 - r^2 \to 0$ as $\varepsilon \to 0$ so we may choose ε sufficiently small that $R^2 - r^2 < 1$. We then indeed have Λ_Ψ bounded on $B(\alpha)$ for every $\alpha < \sqrt{R^2 - r^2}$ since Λ_Ψ is bounded on $B(1)$. Applying Theorem 3.6 and Remark 3.1 we see that Λ_Φ is weak upper semicontinuous on $S(r, R)$ so that $\limsup_{n\to\infty} \Lambda_\Phi(f_n) \leq \Lambda_\Phi(f)$ since $f_n \in S(r, R)$ for sufficiently large n and certainly $f \in S(r, R)$ by choice of r and R. $\square$

We now apply these results to the Chang-Marshall inequality. The weak continuity result in the following corollary is due to Cima and Matheson [35].

Corollary 3.4. *Let $\Phi(t) = e^{t^2}$ for $t \in [0,\infty)$ and consider the functional Λ_Φ on the Dirichlet space $\mathfrak{D}$. Then, Λ_Φ is norm continuous on all of $\mathfrak{D}$, and weak continuous on $\mathfrak{B}\backslash\{0\}$.*

Proof. Let $H = \mathfrak{D}$ and $B = \mathfrak{B}$. Applying the Chang-Marshall inequality we see that the assumptions of Theorem 3.6 are satisfied with $r = 0$ and $\Phi = \Psi$. Since $S(0) = B\backslash\{0\}$, this proves weak upper semicontinuity away from 0. As for the norm upper semicontinuity, this follows from Corollary 3.3 whose assumptions are verified with $r = 1$ because of the Chang-Marshall inequality.

As for weak and norm lower semicontinuity, this is a trivial consequence of Lemma 3.1. $\square$

Remark 3.3. Note that Λ_Φ is not weak continuous at $0 \in \mathfrak{B}$. We shall see a proof of this fact in §3.3.3. Note also that the proof of Corollary 3.4 only really needs a weaker form of the Chang-Marshall inequality—the boundedness of Λ_Φ on $B(r)$ for every fixed $r < 1$. This weaker form is much easier to prove and its knowledge preceded that of that Chang-Marshall inequality, since in fact the estimate central to it was already contained in Beurling's thesis [23].

In particular, by Corollary 3.4 we see that Λ_Φ is finite everywhere on $\mathfrak{D}$ when $\Phi(t) = e^{\alpha t^2}$ for $t \in [0,\infty)$ and α is arbitrary.

The following result is attributed by Chang and Marshall [32] to Garnett. It is an immediate consequence of Corollary 3.4 and the fact that $p(|z|)e^{\alpha|z|^2} = O(e^{\beta|z|^2})$ as $|z| \to \infty$ whenever $\beta > \alpha$ and p is a polynomial.

Corollary 3.5 (Garnett). *Let $\Phi(t) = p(t)e^{\alpha t^2}$ for p a polynomial, α any real number and $t \in [0,\infty)$. Then Λ_Φ is finite everywhere on $\mathfrak{D}$.*

3.3. Critically sharp inequalities and nonexistence of extremals

We now discuss a the Chang-Marshall and Moser-Trudinger inequalities in an abstract setting. Much of the material under this heading will be taken from the author's paper [88]. The methods here are entirely real variable based.

3.3.1. The general results

Let $\mathcal{F}$ be a collection of measurable functions on (I, μ). We shall throughout assume that $0 \in \mathcal{F}$ and that $\mathcal{F}$ is sequentially compact with respect to convergence in measure. Throughout §3.3 when we refer to concepts such as compactness, semicontinuity or continuity with respect to convergence in measure we shall mean sequential compactness, sequential semicontinuity or sequential continuity, respectively, all with respect to convergence in measure.

Definition 3.1. We say that an upper semicontinuous $\Phi\colon [0, \infty) \to \mathbb{R}$ is **critical** for $\mathcal{F}$, providing:

(i) Λ_Φ is upper semicontinuous on $\mathcal{F} \backslash \{0\}$ with respect to convergence in measure, and

(ii) Λ_Φ is not upper semicontinuous with respect to convergence in measure at $0 \in \mathcal{F}$.

Condition (ii) says that there is a sequence of $f_k \in \mathcal{F}$ converging to zero in measure, but with $\Lambda_\Phi(f_k)$ converging to some number (possibly $+\infty$), which strictly greater than $\Phi(0) = \Lambda_\Phi(0)$.

The following result then is of the same type as the work of Flores [51]. We write $\Gamma \circ \Phi$ for the composition of the functions Γ and Φ.

Theorem 3.7 (Pruss [88]). *Let Φ be continuous and positive on $[0, \infty)$. Assume that Φ is critical for $\mathcal{F}$. Then there exists a positive, convex, increasing and infinitely differentiable function Γ on $[0, \infty)$ with $\Gamma(y) \le y$ for every $y \in [0, \infty)$, $\lim_{y\to\infty} \frac{\Gamma(y)}{y} = 1$ and support bounded away from zero, such that $\Lambda_{\Gamma\circ\Phi}$ does not attain its supremum on $\mathcal{F}$.*

Moreover, if $\Phi(0) = 0$ then we may require that there be a sequence of $f_k \in \mathcal{F}$ converging to zero in measure such that $\limsup_k \Lambda_\Phi(f_k) = \sup_{f\in\mathcal{F}} \Lambda_{\Gamma\circ\Phi}(f_k)$.

A proof will be given in §3.3.4. It is not known whether the assumption of continuity of Φ can be weakened to upper semicontinuity. It is not hard to see that the continuity of a positive Φ immediately implies the lower semicontinuity of Λ_Φ on all of $\mathcal{F}$ by Fatou's lemma (see Lemma 3.1).

We say that Λ_Φ is bounded on $\mathcal{F}$ if $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$. If Φ is in addition critical for $\mathcal{F}$ then we say that the inequality $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$ is **critically sharp**. Theorem 3.7 then roughly says that if Φ is continuous then even if a critically sharp inequality $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$ attains its maximum, still we may perturb Φ in an asymptotically negligible way and lose the attainment of a maximum.

Now recall that if Φ is any positive measurable function on $[0,\infty)$ with Λ_Φ bounded on $\mathcal{F}$, then, for every upper semicontinuous Ψ with $\Psi(t) = o(\Phi(t))$ as $t \to \infty$, we have Λ_Ψ upper semicontinuous with respect to convergence in measure on $\mathcal{F}$, and in particular attaining its maximum there (Theorem 3.2). Theorem 3.7 shows that $o(\Phi(t))$ cannot be replaced by $O(\Phi(t))$ in that result, even under the assumption that Φ attains its maximum on $\mathcal{F}$.

Now, Theorem 3.2 says that a critically sharp inequality $\sup_{f\in\mathcal{F}} \Lambda_\Phi(f) < \infty$ cannot be improved by replacing Φ by some Ψ with $\Phi(t) = o(\Psi(t))$ as $t \to \infty$ since $\sup_{f\in\mathcal{F}} \Lambda_\Psi(f)$ will then fail to be finite.

We have the following partial converse to Theorem 3.2. As before, $\mathcal{F}$ is a collection of measurable functions on a finite measure space (I,μ), with $0 \in \mathcal{F}$ and $\mathcal{F}$ compact with respect to convergence in measure.

Theorem 3.8 (Pruss [88]). *Let Φ be continuous and positive, and suppose that Λ_Φ is continuous on $\mathcal{F}$ with respect to convergence in measure. Then there exists a positive, convex and increasing $\Gamma \in C^\infty[0,\infty)$ with $\frac{\Gamma(y)}{y} \to \infty$ as $y \to \infty$ and $\Lambda_{\Gamma\circ\Phi}$ bounded on $\mathcal{F}$. Moreover, we may require that $\frac{d\Gamma(y)}{dy} \to \infty$ as $y \to \infty$.*

A proof will be given in §3.3.4. As in the case of Theorem 3.7, it is not known whether the assumption of continuity can be weakened to upper semicontinuity.

Corollary 3.6 (Pruss [88]). *Let Φ be continuous and positive with $\Phi(t) \to \infty$ as $t \to \infty$ and suppose that Λ_Φ is continuous on $\mathcal{F}$ with respect to convergence in measure. Then there is a continuous and positive Ψ with $\Phi(t) = o(\Psi(t))$ as $t \to \infty$ and with Λ_Ψ continuous on $\mathcal{F}$ with respect to convergence in measure, and, in particular, bounded there. Moreover, if Φ is* *convex (respectively, increasing, or convex increasing), then Ψ can be taken to also be convex (respectively, increasing, or convex increasing).*

Proof. First assume that we do not need Ψ to be increasing or convex. Let

$$\Psi(t) = \sqrt{\Gamma(\Phi(t)) \cdot \Phi(t)},$$

where Γ is as in Theorem 3.8. Then it follows that $\Psi(t) = o(\Gamma(\Phi(t)))$ as $t \to \infty$, so that by Theorem 3.2, it follows from the boundedness of $\Lambda_{\Gamma\circ\Phi}$ that Λ_Ψ is continuous on $\mathcal{F}$ with respect to convergence in measure. On the other hand, we also have $\Phi(t) = o(\Psi(t))$ as $t \to \infty$.

Now, if we do want Ψ to be increasing and/or convex, then choose Γ as in the "moreover" of Theorem 3.8. Let

$$\tilde{\Gamma}(y) = \int_0^y \sqrt{1 + \frac{d\Gamma(t)}{dt}}\, dt.$$

Since $\frac{d\Gamma(t)}{dt} \to \infty$ as $t \to \infty$, it follows from L'Hôpital's Rule that $\tilde{\Gamma}(y) = o(\Gamma(y))$ and that $y = o(\Gamma(y))$, both as $y \to \infty$. Furthermore, if Γ is infinitely differentiable, then so is $\tilde{\Gamma}$, and if Γ is convex then so is $\tilde{\Gamma}$. Then the desired result follows upon setting $\Psi = \tilde{\Gamma} \circ \Phi$, and applying Theorem 3.2 as before in order to obtain the continuity of Λ_Ψ on $\mathcal{F}$ with respect to convergence in measure. (Of course we also need to use the general fact that if F is increasing and convex while G is convex, then $F \circ G$ is convex.) □

We also have the following result which is complementary to Theorem 3.7 but which will turn out to be much easier to prove. The proof is again given in §3.3.4. The result is a modified version of a result of Pruss [88].

Theorem 3.9. *Let $\Phi \in C^\infty[0,\infty)$ be positive, convex and increasing with $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Suppose that Λ_Φ is upper semicontinuous on $\mathcal{F}\backslash\{0\}$ with respect to convergence in measure and that Λ_Φ is bounded on $\mathcal{F}$. Then there exists a positive, convex, increasing and infinitely differentiable function Ψ on $[0,\infty)$ with $\Psi(t) \geq \Phi(t)$ for every $t \in [0,\infty)$ and with $\Phi(t) = \Psi(t)$ for all sufficiently large t, such that Λ_Ψ does attain its supremum on $\mathcal{F}$.*

3.3.2. Application to the Moser-Trudinger inequality

In the Moser-Trudinger inequality, $\mathcal{F}$ is the collection of real-valued absolutely continuous functions f on $[0, \infty)$ with $f(0) = 0$ and

$$\int_0^\infty (f'(x))^2 \, dx \leq 1.$$

If $I = [0, \infty)$ and $d\mu(x) = e^{-x} \, dx$, then the inequality

$$\sup_{f \in \mathcal{F}} \Lambda_\Phi(f) < \infty$$

is the Moser-Trudinger inequality, as mentioned before. Carleson and Chang [30] then showed that the supremum is actually achieved at some $f \in \mathcal{F}$, though it is not known what exactly this extremal f nor what the exact value of the supremum is.

As implicitly noted by Carleson and Chang [30, p. 117], $\mathcal{F}$ is compact with respect to uniform convergence on compact subsets of $[0, \infty)$, and in particular with respect to convergence in measure. (Note also that $\mathcal{F}$ is the unit ball of the Hilbert space $\mathcal{H}$ of real valued absolutely continuous functions f on $[0, \infty)$ with $f(0) = 0$ and

$$\|f\|_{\mathcal{H}}^2 \stackrel{\text{def}}{=} \int_0^\infty (f'(x))^2 \, dx < \infty.$$

Note that the map $f \mapsto f(x)$ is a bounded linear functional on $\mathcal{H}$, and compactness of $\mathcal{F}$ with respect to pointwise convergence, and in particular convergence in measure, follows.)

Furthermore, Φ is critical for $\mathcal{F}$. Condition (i) was implicitly shown by Carleson and Chang [30, pp. 117–118], while condition (ii) is implicit in the work of Moser [78].

We may see the validity of condition (i) also from Theorem 3.6 as follows. For it is clear that $\mathcal{H}$ is a Hilbert space of functions on $I = [0, \infty)$. Let $\Psi(t) = \Phi(t) = e^{t^2}$. The Moser inequality implies that Λ_{Φ_α} is bounded on $\mathcal{F}$ for all $\alpha < 1$ (actually, this last fact is considerably simpler than the Moser inequality). Thus, the weak upper semicontinuity of Λ_Φ on $\mathcal{F} \backslash \{0\}$ follows from Theorem 3.6 with $r = 0$. From weak upper semicontinuity we easily obtain upper semicontinuity with respect to convergence in measure by using the weak compactness of $\mathcal{F}$.

On the other hand, Λ_Φ fails to be upper semicontinuous at $0 \in \mathcal{F}$. To prove this we look at Moser's broken line functions, proceeding much like in [78] and the proof of case (c) of our Theorem 2.1. Let $\beta(x) = \min(x, 1)$ and put $f_n(x) = \sqrt{n}\beta(x/n)$. Then clearly $f_n \in \mathcal{F}$ and

$$\int_0^\infty e^{f_n^2(x)-x}\,dt = \int_0^n e^{x^2/n-x}\,dt + \int_n^\infty e^{n-x}\,dt \geq \int_0^n e^{-x}\,dt + 1.$$

Now, the right hand side converges to 2 as $n \to \infty$. On the other hand, it is easy to verify that $f_n \to 0$ in measure and $\Lambda_\Phi(0) = 1$ so that Λ_Φ indeed fails to be upper semicontinuous at $0 \in \mathcal{F}$.

Hence $\Phi(t) = e^{t^2}$ is critical for $\mathcal{F}$. The following result which was conjectured by McLeod and Peletier [77] then follows immediately from Theorem 3.7.

Theorem 3.10. *There exists a convex, increasing and smooth function Γ with $0 \leq \Gamma(y) \leq y$ for every $y \in [0, \infty)$ and with $\lim_{y\to\infty} \frac{\Gamma(y)}{y} = 1$, such that the supremum*

$$\sup_{f\in\mathcal{F}} \int_0^\infty \Gamma(e^{f^2(t)})e^{-t}dt \tag{3.9}$$

is not achieved over $\mathcal{F}$.

Of course it should be noted that (3.9) is finite. Theorem 3.10 shows that the existence of the extremal for Moser's inequality is in some way accidental, relying on non-asymptotic properties of the function e^{t^2}.

3.3.3. Application to the Chang-Marshall inequality

Now consider the setting of the Chang-Marshall inequality, with $I = \mathbb{T}$ and μ being normalized Lebesgue measure. Again let $\Phi(t) = e^{t^2}$.

We have already noted in Theorem 1.1 that weak convergence in $\mathfrak{B}$ implies convergence in measure on $\mathbb{T}$ (and we have remarked that this was already known to Andreev and Matheson [5]), so that $\mathfrak{B}$ is indeed compact with respect to convergence in measure on $\mathbb{T}$.

Cima and Matheson [35] have shown that Λ_Φ is weakly continuous on $\mathfrak{B}\backslash\{0\}$. This is also a direct consequence of the Chang-Marshall inequality and 3.6 with $r = 0$. Thus condition (i) of criticality is satisfied.

On the other hand, Cima and Matheson [35] have shown that Λ_Φ fails to be weakly upper semicontinuous at $0 \in \mathfrak{B}$. We give a concise proof of this fact. For, if Λ_Φ were upper semicontinuous on all of $\mathfrak{B}$, then by Theorem 3.8 there would be a $\Psi = \Gamma \circ \Phi$ which grows strictly faster than Φ and such that Λ_Ψ is bounded. But this contradicts Theorem 2.1. Thus condition (ii) of criticality is satisfied. (The same concise argument could have been used in the previous section to prove condition (ii) in the case of the Moser-Trudinger inequality, but for the sake of variety we preferred to give a more elementary argument there and a less elementary argument here.)

Then, even though we do not know whether Λ_Φ achieves its maximum over $\mathfrak{B}$, we do have the following result which follows from Theorems 3.7 and 3.9.

Theorem 3.11. *There exist two $C^\infty[0,\infty)$ functions Ψ_1 and Ψ_2 such that for every $t \in [0,\infty)$ we have $0 \le \Psi_1(t) \le e^{t^2} \le \Psi_2(t)$ and Λ_{Ψ_i} is bounded on $\mathfrak{B}$ for $i = 1, 2$, but Λ_{Ψ_1} does not achieve its supremum over $\mathfrak{B}$ while Λ_{Ψ_2} does achieve* its *maximum over $\mathfrak{B}$. One may take the Ψ_i to be convex and increasing with $\lim_{t\to\infty} e^{-t^2}\Psi_1(t) = 1$ and $\Psi_2(t) = e^{t^2}$ for all large t.*

Alec Matheson has kindly communicated to the author that he and Joseph Cima had strongly suspected the truth of this result. Theorem 3.11 implies that one cannot use any asymptotic arguments to prove or disprove the existence of an extremal in the Chang-Marshall inequality without taking into account the exact behaviour of the function e^{t^2}, not just asymptotically, but also closer to the origin. But this does not exclude the possibility of some abstract argument based on some *global* properties of e^{t^2} (e.g., analyticity).

3.3.4. Proofs of the results on critically sharp inequalities

If f and ϕ are measurable on $[0,\infty)$, then write

$$\|g\|_{L^1(\phi)} \stackrel{\text{def}}{=} \int_0^\infty |f\phi| = \int_0^\infty |f(x)\phi(x)|\,dx.$$

Then, the main step in the construction of Ψ for Theorem 1 is encapsulated in the following result.

Lemma 3.2. *Let $\mathcal{G}$ be a subset of $L^1[0,\infty)$ containing the zero function, such that for each finite number T we have $\sup_{g\in\mathcal{G}} \|g \cdot 1_{[0,T]}\|_{L^\infty} < \infty$. Assume that for every sequence g_n of elements of $\mathcal{G}$, there exists a subsequence g_{n_k} which converges in measure to some $g \in L^1[0,\infty)$ such that either g is almost everywhere null or else has $\|g\|_{L^1} \geq \limsup_k \|g_{n_k}\|_{L^1}$. Suppose further that $\|\cdot\|_{L^1}$ fails to be upper semicontinuous at $0 \in \mathcal{G}$ with respect to convergence in measure.*

Then, there exists a positive increasing function $\phi \leq 1$ on $[0,\infty)$ such that $\lim_{x\to\infty}\phi(x) = 1$, with $\|\cdot\|_{L^1(\phi)}$ not attaining its maximum on $\mathcal{G}$. Furthermore, ϕ may be taken to be in $C^\infty[0,\infty)$, with support bounded away from 0. Moreover if $\sup_{g\in\mathcal{G}}\|g\|_{L^1} < \infty$ then we may also require that there be a sequence $g_k \in \mathcal{G}$ such that $g_k \to 0$ almost everywhere and $\limsup_k \|g_k\|_{L^1} = \sup_{g\in\mathcal{G}} \|g\|_{L^1(\phi)}$.

Assuming the lemma for now, we may proceed to prove Theorem 3.7.

Proof of Theorem 3.7. Without loss of generality assume that $\Phi(0) = 0$. For $f \in \mathcal{F}$ and $t \in [0,\infty)$, let $m_f(t) = \mu\{x : \Phi(|f(x)|) > t\}$. Let $\mathcal{G} = \{m_f : f \in \mathcal{F}\}$. We shall apply Lemma 3.2 to $\mathcal{G}$. Let us verify its conditions. Clearly, every element of $\mathcal{G}$ is pointwise bounded by $\mu(I) < \infty$. Furthermore, for $m_f \in \mathcal{G}$ we have

$$\|m_f\|_{L^1} = \int_0^\infty m_f(t)\,dt = \Lambda_\Phi(f).$$

Then, using the lack of upper semicontinuity of Λ_Φ at zero, we may choose a sequence $f_k \in \mathcal{F}$ such that $f_k \to 0$ in measure and $\limsup_k \Lambda_\Phi(f_k) > \Lambda_\Phi(0) = 0$. Passing to a subsequence if necessary, we may assume that $f_k \to 0$ almost everywhere. Then, $\limsup_k \Phi(|f_k|) = \Phi(0) = 0$ almost everywhere, by the continuity of Φ. Hence, $\Phi(|f_k|) \to 0$ almost everywhere, too, and hence also in measure. Thus, for every $t > 0$ we have $m_{f_k}(t) \to 0$, and in particular $m_{f_k} \to 0$ in measure while $\limsup_k \|m_{f_k}(t)\|_{L^1} = \limsup_k \Lambda_\Phi(f_k) > 0$, so that $\|\cdot\|_{L^1}$ fails to be upper semicontinuous at zero in $\mathcal{G}$.

Now, given any sequence m_{f_n} of elements of $\mathcal{G}$, we may choose a subsequence $m_{f_{n_k}}$ such that

f_{n_k} converges in measure, using the compactness with respect to convergence in measure of $\mathcal{F}$. Choosing a further subsequence if necessary, we may assume f_{n_k} converges almost everywhere. If the limit is almost everywhere zero then we are done. On the other hand, if $f_{n_k} \to f$ where f does not vanish almost everywhere then we first of all have $\lim_k \Phi(|f_{n_k}|) = \Phi(|f|)$ by continuity of Φ, and secondly, by the upper semicontinuity of Λ_Φ with respect to convergence in measure away from zero, we have $\limsup_k \Lambda_\Phi(|f_{n_k}|) \leq \Lambda_\Phi(|f|)$. Then, since $\Phi(|f_{n_k}|) \to \Phi(|f|)$ in measure, it follows that $m_{f_{n_k}} \to m_f$ almost everywhere (in fact at all points of $[0, \infty)$ other than the at most countably many discontinuities of m_f), as can be easily verified. Also, $\|m_f\|_{L^1} \geq \limsup_k \|m_{f_{n_k}}\|_{L^1}$. Hence, the conditions for the Lemma are satisfied.

Choose ϕ as in Lemma 3.2. Let
$$\Gamma(y) = \int_0^y \phi(x)\, dx.$$
It is easy to verify that $\|m_f\|_{L^1(\phi)} = \Lambda_{\Gamma \circ \Phi}(f)$. Then the Theorem follows from the conclusions of Lemma 3.2. For example, the convexity of Γ follows from the fact that ϕ is monotone increasing. $\square$

Lemma 3.3. *Let $\mathcal{G}$ be a subset of $L^1[0, \infty)$ containing the zero function, such that for each finite number T we have $\sup_{g \in \mathcal{G}} \|g \cdot 1_{[0,T]}\|_{L^\infty} < \infty$. Assume that for every sequence g_n of elements of $\mathcal{G}$, there exists a subsequence g_{n_k} which converges in measure to some $g \in L^1[0, \infty)$ such that either g is almost everywhere null or else has $\|g\|_{L^1} \geq \limsup_k \|g_{n_k}\|_{L^1}$. Suppose further that $\|\cdot\|_{L^1}$ is uniformly bounded on all of $\mathcal{G}$ and upper semicontinuous with respect to convergence in measure at $0 \in \mathcal{G}$.*

Then, there exists a positive and increasing function $\phi \geq 1$ on $[0, \infty)$ with $\lim_{x \to \infty} \phi(x) = \infty$ and $\|\cdot\|_{L^1(\phi)}$ bounded on $\mathcal{G}$.

Theorem 3.8 then follows from Lemma 3.3 in the same way as Theorem 3.7 had followed from Lemma 3.2. We now proceed to prove our two lemmata.

Proof of Lemma 3.2. First suppose $\mathcal{G}$ is not uniformly bounded in L^1 norm. Let ϕ be a positive increasing $C^\infty[0,\infty)$ function whose support is bounded away from zero and which has $\phi(x)=1$ for all $x \geq 1$. Then let g_k be a sequence of elements of $\mathcal{G}$ with $||g_k||_{L^1} \to \infty$. Passing to a subsequence we can assume that for some $g \in L^1[0,\infty)$ we have $g_k \to g$ in measure. If g is almost everywhere null, then it is easy to see that the proof is complete since by the bounded convergence theorem (which is applicable because the $\{g_k\}$ are almost everywhere uniformly bounded on $[0,1]$ by the hypotheses of the Lemma) we have $\int_0^1 |g_k| \to 0$ so that $||g_k||_{L^1(\phi)} \geq \int_1^\infty |g_k| = ||g_k||_{L^1} - \int_0^1 |g_k|$ and the right hand side tends to ∞, so that $||g_k||_{L^1(\phi)} \to \infty$ as desired. Choosing a subsequence if necessary, then, we may assume that $g_k \to 0$ almost everywhere and the Lemma follows. On the other hand, if g is not almost everywhere null then $||g||_{L^1} \geq \limsup_k ||g_k||_{L^1}$ by our hypotheses. But, the right hand side is infinite, and this contradicts the fact that $g \in L^1$.

Now, assume that

$$M \stackrel{\text{def}}{=} \sup_{g\in\mathcal{G}} ||g||_{L^1} < \infty.$$

Let

$$\lambda = \sup_{\substack{\{g_k\}\subseteq\mathcal{G} \\ g_k\to 0}} \limsup_k ||f_k||_{L^1}, \tag{3.10}$$

where the supremum is to be understood as taken over all sequences $\{g_k\}$ in $\mathcal{G}$ tending to zero in measure. Since $||\cdot||_{L^1}$ fails to be upper semicontinuous at $0 \in \mathcal{G}$ with respect to convergence in measure, we have $\lambda > 0$. Obviously, $\lambda \leq M$.

Replacing $\mathcal{G}$ by $\{|g| : g \in \mathcal{G}\}$ if necessary, we may assume all functions in $\mathcal{G}$ are positive. Choose $0 < \alpha < 1$ such that $\alpha M < \lambda$.

For $g \in \mathcal{G}$, let

$$\tau_g = \inf\left\{\tau \geq 0 : \int_\tau^\infty g \leq \alpha \int_0^\infty g\right\},$$

so that

$$\int_{\tau_g}^\infty g = \alpha \int_0^\infty g \tag{3.11}$$

and

$$\int_0^{\tau_g} g = \int_0^\infty g - \int_{\tau_g}^\infty g = (1-\alpha)\int_0^\infty g. \tag{3.12}$$

Now define

$$\mathcal{G}_x = \{g : \tau_g \geq x\}.$$

Let

$$M_x = \sup_{g\in\mathcal{G}_x} \int_0^\infty g.$$

I claim that

$$\limsup_{x\to\infty} M_x \leq \lambda. \tag{3.13}$$

To show this, it suffices to prove that for any sequences $x_k \to \infty$ and $g_k \in \mathcal{G}_{x_k}$ such that $\int_0^\infty g_k$ converges, we have $\lim_k \int_0^\infty g_k \leq \lambda$. Fix such sequences x_k and g_k. Passing to subsequences, if necessary, by our hypotheses we may assume that g_k either converges to 0 in measure, or else it converges in measure to some nonzero $g \in L^1[0,\infty)$ with $\|g\|_{L^1} \geq \limsup_k \|g_k\|_{L^1}$. If it converges to 0 in measure then $\limsup_k \int_0^\infty g_k \leq \lambda$ by definition of λ. Otherwise note that since $g_k \in \mathcal{G}_{x_k}$, we have $\int_{x_k}^\infty g_k \geq \alpha\|g_k\|_{L^1}$. Let $h_k(x) = g_k(x)\cdot 1_{\{x\geq x_k\}}$. We have $h_k \to 0$ pointwise since $x_k \to \infty$. By Fatou's lemma then,

$$\liminf_k \int_0^\infty (g_k - h_k) \geq \int_0^\infty g.$$

But

$$\int_0^\infty (g_k - h_k) = \int_0^{x_k} g_k \leq \int_0^{\tau_{g_k}} g_k = (1-\alpha)\|g_k\|_{L^1},$$

where we have used the fact that $g_k \in \mathcal{G}_{x_k}$ together with (3.12). Thus,

$$\liminf_k (1-\alpha)\|g_k\|_{L^1} \geq \|g\|_{L^1}.$$

But since $\alpha < 1$, this contradicts the facts that $\|g\|_{L^1} \geq \limsup_k \|g_k\|_{L^1}$ and that g does not almost everywhere vanish. Hence, the case where g_k does not converge to zero in measure is impossible, and the claim is proved.

Now define

$$\psi(x) = \left(\frac{x}{1+x}\right)\left(1 \wedge \inf_{g \in \mathcal{G}_x \setminus \{0\}} \frac{\lambda - \int_{\tau_g}^{\infty} g}{\int_0^{\tau_g} g}\right).$$

I claim that $\psi(x) \to 1$ as $x \to \infty$. To prove this, consider the function

$$h(t) = \frac{\lambda - \alpha t}{(1-\alpha)t},$$

which is easily seen to be decreasing for $t \in [0, M]$ since $\alpha M < \lambda$, and which satisfies $h(\lambda) = 1$. By (3.11) and (3.12) we then have

$$\psi(x) = \left(\frac{x}{1+x}\right)\left(1 \wedge \inf_{g \in \mathcal{G}_x \setminus \{0\}} h\left(\int_0^{\infty} g\right)\right).$$

But for $g \in \mathcal{G}_x$ we have $\int_0^\infty g \le M_x$ so that by the monotonicity of h on $[0, M]$ we have

$$\psi(x) \ge \frac{x}{1+x}(1 \wedge h(M_x)).$$

Now by (3.13) we have $\liminf_{x\to\infty} h(M_x) \ge h(\lambda) = 1$ and hence $\lim_{x\to\infty} \psi(x) = 1$ as desired.

Note that ψ is measurable as it is increasing. It is easy to verify that $\|g\|_{L^1(\psi)} < \lambda$ for every $g \in \mathcal{G}$. For, given $g \in \mathcal{G}$ with $\|g\|_{L^1} \ne 0$, we have

$$\int_0^\infty g\psi < \int_0^{\tau_g} g\psi + \int_{\tau_g}^\infty g \le \psi(\tau_g)\int_0^{\tau_g} g + \int_{\tau_g}^\infty g \le \lambda - \int_{\tau_g}^\infty g + \int_{\tau_g}^\infty g = \lambda,$$

where we have used the monotonicity and choice of ψ. The first inequality came from the facts that $\psi < 1$ everywhere and that $\int_{\tau_g}^\infty g > 0$ if $\|g\|_{L^1} \ne 0$.

If we do not need ϕ to be $C^\infty[0,\infty)$ or to have support bounded away from zero, then just let $\phi = \psi$. Otherwise, since ψ is a increasing function $[0,\infty)$ with limit 1, we may easily choose a increasing $C^\infty[0,\infty)$ function ϕ with support bounded away from 0 and with the properties that $0 \le \phi \le \psi$ everywhere and that $\phi(x) \to 1$ as $x \to \infty$. We will then necessarily still have $\|g\|_{L^1(\phi)} < \lambda$ for each $g \in \mathcal{G}$.

We shall now show that

$$\sup_{g \in \mathcal{F}} \|g\|_{L^1(\phi)} = \lambda. \tag{3.14}$$

To do this, fix $\varepsilon > 0$. By definition of λ, let g_k be a sequence in $\mathcal{G}$ with $g_k \to 0$ in measure and $||g_k||_{L^1} \to \lambda$. Choose T sufficiently large that $\phi(x) \geq 1 - \varepsilon$ for $x \geq T$. Since $g_k \to 0$ in measure and, by our hypotheses, the $g_k \cdot 1_{[0,T]}$ are uniformly bounded in L^∞, the bounded convergence theorem tells us that $\int_0^T g_k \to 0$ as $k \to \infty$. Since $||g_k||_{L^1} \to \lambda$, we may choose K sufficiently large that $\int_T^\infty g_k \geq \lambda - \varepsilon$ for $k \geq K$. Then, for $k \geq K$ we have

$$||g_k||_{L^1(\phi)} \geq \int_T^\infty g_k \phi \geq (1-\varepsilon) \int_T^\infty g_k \geq (1-\varepsilon)(\lambda - \varepsilon).$$

Hence $\liminf_k ||g_k||_{L^1(\phi)} \geq (1-\varepsilon)(\lambda-\varepsilon)$ for every $\varepsilon > 0$, and so we see that indeed

$$\liminf_k ||g_k||_{L^1(\phi)} \geq \lambda$$

and (3.14) follows from this and the already proved inequality $||g||_{L^1(\phi)} < \lambda$ valid for every $g \in \mathcal{G}$.

The last sentence of the statement of the Lemma now follows upon taking a subsequence of the above g_k which converges almost everywhere to zero. Thus, the Lemma is proved. □

Proof of Lemma 3.3. As in the proof of Lemma 3.2, we may assume without loss of generality that all functions of $\mathcal{G}$ are positive. Let

$$M = \sup_{g \in \mathcal{G}} \int_0^\infty g.$$

By assumption this will be finite. Let

$$U_x = \sup_{g \in \mathcal{G}} \int_x^\infty g.$$

I claim that $U_x \to 0$ as $x \to \infty$. For, fix any $0 < \alpha < 1$, and define

$$\tau_g^\alpha = \inf \left\{ \tau \geq 0 : \int_\tau^\infty g \leq \alpha \int_0^\infty g \right\}.$$

Let $\mathcal{G}_x^\alpha = \{g \in \mathcal{G} : \tau_g^\alpha \geq x\}$. Exactly as in the proof of Lemma 3.2, we may show that if

$$M_x^\alpha = \sup_{g \in \mathcal{G}_x^\alpha} \int_0^\infty g,$$

then $M_x^\alpha \to 0$ as $x \to \infty$. (For, in the present case λ as defined by (3.10) will be zero, by the assumption of upper semicontinuity with respect to convergence in measure to zero.) Now, fix $\varepsilon > 0$ and choose $0 < \alpha < 1$ such that $\alpha M \leq \varepsilon$. Assume that x is sufficiently large that $M_x^\alpha \leq \varepsilon$. Then, for such x and $g \in \mathcal{G}$, we have $\int_x^\infty g \leq \int_0^\infty g \leq M_x^\alpha \leq \varepsilon$ providing $g \in G_x^\alpha$. On the other hand, if $g \notin G_x^\alpha$ then $\tau_g^\alpha < x$, so that $\int_x^\infty g \leq \int_{\tau_g^\alpha}^\infty g = \alpha \int_0^\infty g \leq \alpha M \leq \varepsilon$. Hence, in either case $\int_x^\infty g \leq \varepsilon$, and so $U_x \to 0$ as desired.

Choose a sequence of finite positive numbers $x_k \to \infty$ with the property that $U_{x_k} \leq 2^{-k}$. Then set

$$\psi(x) = \mathrm{Card}\{k \in \mathbb{Z}^+ : x_k \leq x\}.$$

This is a increasing positive function on $[0, \infty)$, and we have $\psi(x) \to \infty$ as $x \to \infty$. Furthermore, if $g \in \mathcal{G}$ then it is easy to verify that

$$\|g\|_{L^1(\psi)} = \sum_{k=1}^{\infty} \int_{x_k}^{\infty} g \leq \sum_{k=1}^{\infty} U_{x_k} \leq 1,$$

by choice of x_k. In fact, we also have $\|\cdot\|_{L^1(1+\psi)}$ bounded on $\mathcal{G}$ since we had assumed that $\|\cdot\|_{L^1}$ is bounded on $\mathcal{G}$. Now, we may easily choose a increasing function $\phi \in C^\infty[0, \infty)$ with the property that $1 \leq \phi(x) \leq 1 + \psi(x)$ and $\phi(x) \to \infty$ as $x \to \infty$. The Lemma then follows. □

Proof of Theorem 3.9. If necessary replacing Φ by $\Phi - \Phi(0)$, we may assume that $\Phi(0) = 0$. If all functions f in $\mathcal{F}$ satisfy $\Lambda_\Phi(f) = 0$ then we are done. Otherwise, fix $f_0 \in \mathcal{F}$ such that $\Lambda_\Phi(f_0) > 0$. Let

$$M = \sup_{f \in \mathcal{F}} \Lambda_\Phi(f).$$

Choose any $\varepsilon \in (0, \mu(I))$. Choose $\lambda \in (0, \infty)$ such that

$$\mu\{x : \lambda^{-1} \leq \Phi(|f_0(x)|) \leq \lambda\} \geq \varepsilon.$$

(Such a λ exists since Φ is everywhere finite while $\Lambda_\Phi(f_0) > 0$.)

Now, choose a function Ψ on $[0, \infty)$ and a positive number K which satisfy the following properties:

(i) Ψ is convex, positive, increasing and C^∞

(ii) $\Psi(t) = Kt$ for $t \in [0, \lambda]$

(iii) $\Psi(t) = t$ for all sufficiently large t

(iv) $\Psi(t) \geq \Phi(t)$ for all t

(v) $K \geq \lambda\varepsilon^{-1}(1 + M)$.

To do this, first choose T sufficiently large that $T^{-1}\Phi(T) > \lambda\varepsilon^{-1}(1 + M)$ and $T > \lambda$. (This can be done since $t^{-1}\Phi(t) \to \infty$ as $t \to \infty$.) Let $K = T^{-1}\Phi(T)$. It easily follows that we may splice together a convex, positive, increasing C^∞ function $\Psi(t)$ for $t \in [0, \infty)$ which equals Kt for $t \in [0, \lambda]$, coincides with Ψ on $[T + 1, \infty)$ and is at least as large as Φ on (λ, ∞). (It is automatically at least as large as Φ on $[0, \lambda]$ by choice of K and the convexity of Φ.)

I claim that Λ_Ψ attains its maximum on $\mathcal{F}$. For, Λ_Ψ is upper semicontinuous on $\mathcal{F}\backslash\{0\}$ with respect to convergence in measure. To see this, note that

$$\Lambda_\Psi = \Lambda_\Phi + \Lambda_{\Psi-\Phi}.$$

But $\Psi - \Phi$ is a bounded function by condition (iii) of the choice of Ψ, so that $\Lambda_{\Psi-\Phi}$ is continuous with respect to convergence in measure on all of $\mathcal{F}$ by the bounded convergence theorem and the continuity of $\Psi - \Phi$. On the other hand, we had assumed the upper semicontinuity of Λ_Φ on $\mathcal{F}\backslash\{0\}$.

Let f_n be a sequence such that

$$\lim_{n\to\infty} \Lambda_\Psi(f_n) = \sup_{g\in\mathcal{F}} \Lambda_\Psi(g). \tag{3.15}$$

Passing to a subsequence, by compactness with respect to convergence in measure, we may assume that $f_n \to f$ in measure for some $f \in \mathcal{F}$. If f is not almost everywhere null, then $\lim_{n\to\infty} \Lambda_\Psi(f_n) \leq \Lambda_\Psi(f)$ by upper semicontinuity of Λ_Ψ on $\mathcal{F}\backslash\{0\}$, and hence Λ_Ψ attains its maximum at f. Assume now that f vanishes almost everywhere. Then,

$$\lim_{n\to\infty} \Lambda_\Psi(f_n) = \lim_{n\to\infty} (\Lambda_\Phi(f_n) + \Lambda_{\Psi-\Phi}(f_n)).$$

Now, $\Psi - \Phi$ is a bounded continuous function and $f_n \to 0$ in measure so that $\Lambda_{\Psi-\Phi}(f_n) \to \Psi(0) - \Phi(0)$ as $n \to \infty$ by the bounded convergence theorem. Since $\Phi(0) - \Psi(0) = 0$, we see that $\Lambda_{\Psi-\Phi}(f_n) \to 0$. On the other hand, trivially

$$\limsup_{n\to\infty} \Lambda_\Phi(f_n) \leq M.$$

In light of (3.15) we see that

$$\sup_{g\in\mathcal{F}} \Lambda_\Phi(g) \leq M. \tag{3.16}$$

But now note that by conditions (ii) and (v) in the choice of Ψ and by the choice of ε we see that

$$\Lambda_\Phi(f_0) \geq \varepsilon^{-1}(1+M)\lambda\lambda^{-1}\mu\{x : \lambda^{-1} \leq \Phi(|f_0|) \leq \lambda\} \geq 1+M,$$

contradicting (3.16). Thus we see that the limit of the f_n cannot be 0 and so Λ_Ψ does attain its maximum on $\mathcal{F}$. $\square$

4. Properties of extremals of the Λ_Φ on Dirichlet spaces

In this section we shall discuss properties of Λ_Φ functionals acting on the unit balls of Dirichlet spaces.

4.1. A variational equation

Recall that if Φ was a function from $[0,\infty)$ to $\mathbb{R}$ then Λ_Φ was an abbreviation for $\Lambda_{\Phi(|\cdot|)}$. The following result due to the author is taken from the author's joint paper with Alec Matheson [75].

Theorem 4.1. *Let $\Phi\colon [0,\infty) \to \mathbb{R}$. Write $\Psi(t) = \Phi(\sqrt{t})$. Assume that Ψ is differentiable on $(0,\infty)$, with $|\Psi'(t)| \leq ce^{Ct}$ for some finite constants c and C, and every $t > 0$. Suppose $f \in \mathfrak{B}$ is an extremal function for Λ_Φ. Write*

$$S_\Psi(f) = \int_{\mathbb{T}} |f|^2 \Psi'(|f|^2).$$

Then if $S_\Psi(f)$ vanishes, it follows that $f\Psi'(|f|^2)$ vanishes almost everywhere on $\mathbb{T}$. Assuming that $S_\Psi(f)$ does not vanish, we have $f' \in H^2$ and

$$S_\Psi(f) z f' = \mathcal{P}_0\left(f\Psi'(|f|^2)\right) \qquad on\ \mathbb{T}, \tag{4.1}$$

where z stands for the identity function on $\mathbb{T}$, and $\mathcal{P}_0$ is the orthogonal projection from $L^2(\mathbb{T})$ to $H_0^2(\mathbb{T}) \stackrel{\text{def}}{=} \{f \in H^2(\mathbb{T}) : \hat{f}(0) = 0\}$.

Moreover, if we also have $\|f\|_{\mathfrak{D}} < 1$, then $S_\Psi(f)$ must vanish.

Remark 4.1. The functional Λ_Φ makes sense on $\mathfrak{B}$ since the hypotheses guarantee that $|\Psi(t)| \leq cC^{-1}(e^{Ct} - 1) + |\Psi(0)|$, which, together with Corollary 3.5, guarantees the finiteness of $\Lambda_\Phi(f)$ for every $f \in \mathfrak{B}$ (indeed, for every $f \in \mathfrak{D}$). Similarly, the right hand side of (4.1) and the integral defining $S_\Psi(f)$ make sense because of Corollary 3.5 and our assumption on the size of $|\Psi'|$.

Also, we note that the identity function on $\mathbb{D}$ always satisfies (4.1) even though it may not be extremal. (For example, it will not be extremal if $\Phi(t) = t^{2n}$ and n is sufficiently large.)

The interested reader may, of course, translate the conditions on Ψ into conditions on Φ, but they are perhaps more naturally stated for Ψ'. This will be even more true in the next theorem.

If Ψ' (or, equivalently, Φ') does not vanish on $(0, \infty)$, and $S_\Psi(f) = 0$, then the vanishing of $f\Psi'(|f|^2)$ on $\mathbb{T}$ implies that $f \equiv 0$, and (4.1) trivially continues to hold.

The criterion (4.1) was shown by Andreev and Matheson (see [5, Cor. 3 and remarks following it]) in the special cases of the Chang-Marshall functions $\Phi_\alpha(t) = e^{\alpha t^2}$ and of the functions $\Phi(t) = t^{2n}$, under the auxiliary assumption that $f' \in H^1$. Our result above shows that this assumption will automatically be satisfied whenever f is extremal and Φ' does not vanish on $(0, \infty)$.

Note that for *any* Φ the functions f of the form $f(z) = az^n$ where $n|a|^2 = 1$ (note that in particular this includes the identity function) provide solutions to (4.1). However, in general they may not be the only solutions. For instance, if $\Phi(t) = t^{2n}$ where $2n > 4$, then an extremal exists as noted before, and clearly the hypotheses of Theorem 4.1 are satisfied. But we have seen that the extremal in that case is *not* the identity function. Nor can it be of the form $f(z) = az^n$ with $n|a|^2 = 1$ since for any such function we have $\Lambda_\Phi(f) \le \Lambda_\Phi(z)$, where z indicates the identity function.

If one could prove that for all $\alpha < 1$ sufficiently close to 1 all solutions f of (4.1) are of the form $f(z) = az^n$ with $n|a|^2 = 1$, then it would follow that $\sup_{g\in\mathfrak{B}} \Lambda_{\Phi_\alpha}(g) \le e^\alpha$ for $\alpha < 1$ sufficiently close to 1, and so by a limiting argument we would conclude that $\Lambda_{\Phi_1}(g) \le e = \Lambda_{\Phi_1}(z)$, which would give an affirmative answer to Problem 3.1.

Numerical solutions to (4.1) might lead to a better understanding of the extremal functions for Λ_Φ.

Open Problem 4.1. Investigate numerical algorithms for solving pseudodifferential equations on $\mathbb{T}$ of the form

$$zf' = \mathcal{P}_0\left(f\psi(|f|^2)\right),$$

where f is the boundary value of an analytic function and ψ a sufficiently nice real-valued function.

The following question is easily seen to be related to a conjecture that we shall give later (Conjecture IV.1.2).

Open Problem 4.2. If $\Phi(e^t)$ is convex in t and $\Phi'(x) > 0$ for every $x > 0$, then does it follow from (4.1) that f is automatically univalent? If so, then must it also be star-shaped?

We now proceed to the proof of Theorem 4.1.

We shall use the following trivial and very classical lemma to analyze the case where $S_\Psi(f) = 0$. We write $\bar{H}^p(\mathbb{T}) = \{\bar{f} : f \in H^p(\mathbb{T})\}$ for the antianalytic Hardy spaces.

Lemma 4.1 (see Koosis [67, p. 87]). *Let $V \in H^p(\mathbb{T})$ or $V \in \bar{H}^p(\mathbb{T})$ for some p, $1 \le p \le \infty$, be real valued on $\mathbb{T}$. Then V is almost everywhere constant.*

Proof of Lemma. Without loss of generality consider the case $V \in H^p(\mathbb{T})$. By [67, p. 87] we can continue V analytically to all of $\mathbb{C}$ by setting $V(z) = \overline{f(1/\bar{z})}$ for $|z| > 1$. Then V will be a bounded entire function, hence constant. $\square$

The careful reader will note that at several points in the following proof we would be able to assert that various functions lie in L^p for every $p < \infty$, but we only write down that they lie in L^2 or L^1. We do this in order to make it clearer how to generalize the proof to yield more general results.

Proof of Theorem 4.1. We proceed by a simple variational argument. Let f be extremal. Write $f(z) = \sum_{n=1}^{\infty} a_n z^n$.

Suppose first that $\|f\|_{\mathfrak{D}} = 1$. Write $F(\lambda) = \|f + \lambda z^n\|_{\mathfrak{D}}^2$, where we use z^n as a short form for the function $z \mapsto z^n$. Since $\|f\|_{\mathfrak{D}} = 1$ while $\|z^n\|_{\mathfrak{D}} = \sqrt{n}$, it follows that, for $|\lambda| < 1/\sqrt{n}$, we have $F(\lambda) > 0$ and $(f + \lambda z^n)/\sqrt{F(\lambda)}$ has unit Dirichlet norm. Suppose now that λ is real. If f is extremal, then we must have

$$\left.\frac{d\Lambda_\Phi\left((f+\lambda z^n)/\sqrt{F(\lambda)}\right)}{d\lambda}\right|_{\lambda=0} = 0, \tag{4.2}$$

as long as this derivative exists. We shall prove this derivative exists, and compute its form.

First note that

$$\begin{aligned} F(\lambda) &= \left(\sum_{\substack{k=1\\k\neq n}}^{\infty} k|a_k|^2\right) + n|a_n+\lambda|^2 = \left(\sum_{k=1}^{\infty} k|a_k|^2\right) + n[(a_n+\bar{a}_n)\lambda + \lambda^2] \\ &= \|f\|_{\mathfrak{D}}^2 + n(2\lambda \operatorname{Re} a_n + \lambda^2) = 1 + n(2\lambda \operatorname{Re} a_n + \lambda^2). \end{aligned} \tag{4.3}$$

Now we have $f(z)$ finite for almost every $z \in \mathbb{T}$. Furthermore, we may assume that $|f(z)| \neq 0$ for almost every z. For if f vanishes on a set of positive measure then, being a function in $\mathfrak{D} \subset H^1$, it would have to vanish identically and the Theorem would follow trivially. Moreover, if we restrict λ to $[-1/(2\sqrt{n}), 1/(2\sqrt{n})]$ then $F(\lambda)$ will not vanish. Fixing z such that $|f(z)| < \infty$, we then have $\Psi(|f(z)+\lambda z^n|^2/F(\lambda))$ absolutely continuous with respect to λ on $[-1/(2\sqrt{n}), 1/(2\sqrt{n})]$. For, Ψ is differentiable except possibly at zero, the derivative is bounded in a neighbourhood of zero (since $|\Psi'(t)| \leq ce^{Ct}$) and, if z is fixed, then $|f(z)+\lambda z^n|^2$ can vanish for at most finitely many (in fact, for at most two) values of λ. Thus if $-1/(2\sqrt{n}) \leq \lambda \leq 1/(2\sqrt{n})$, we will have

$$\begin{aligned}
\Delta_\lambda(z) &\stackrel{\text{def}}{=} \frac{1}{\lambda}\left(\Psi\left(\frac{|f(z)+\lambda z^n|^2}{F(\lambda)}\right) - \Psi(|f(z)|)\right) \\
&= \frac{1}{\lambda}\int_0^\lambda \left(\frac{d}{dt}\Psi\left(\frac{|f(z)+tz^n|^2}{F(t)}\right)\right) dt \\
&= \frac{1}{\lambda}\int_0^\lambda \Psi'\left(\frac{|f+tz^n|^2}{F(t)}\right) \\
&\qquad \times \frac{F(t)(z^n\bar{f}+\bar{z}^n f + 2t|z|^{2n}) - F'(t)|f+tz^n|^2}{(F(t))^2}\, dt \\
&= \frac{1}{\lambda}\int_0^\lambda \Psi'\left(\frac{|f+tz^n|^2}{F(t)}\right) \\
&\qquad \times \frac{F(t)(2\operatorname{Re}(\bar{z}^n f) + 2t) - n(2\operatorname{Re} a_n + 2t)|f+tz^n|^2}{(F(t))^2}\, dt,
\end{aligned} \tag{4.4}$$

by (4.3) and the fact that $|z| = 1$ on $\mathbb{T}$. Of course, as long as $0 < |f(z)| < \infty$ (which happens for almost every $z \in \mathbb{T}$), if we let $\lambda \to 0$ then this will converge to

$$\delta(z) \stackrel{\text{def}}{=} \left.\frac{d}{d\lambda}\right|_{\lambda=0} \Psi\left(\frac{|f(z)+\lambda z^n|^2}{F(\lambda)}\right) = 2(\operatorname{Re}(\bar{z}^n f) - n\operatorname{Re} a_n|f|^2)\Psi'(|f(z)|^2), \tag{4.5}$$

where we have used the fact that $F(0) = \|f\|_{\mathfrak{D}}^2 = 1$. We wish to conclude that

$$\lim_{\lambda\to 0}\int_{\mathbb{T}} \Delta_\lambda = \int_{\mathbb{T}} \delta, \tag{4.6}$$

and to this end we will dominate Δ_λ by an $L^1(\mathbb{T})$ function not depending on λ.

Now if $-1/(2\sqrt{n}) \leq \lambda \leq 1/(2\sqrt{n})$, then $1/4 \leq F(\lambda) \leq 9/4$, as can be easily verified. Thus, making simple estimates in (4.4) and using the assumption that $|\Psi'(y)| \leq ce^{Cy}$, we obtain, for $-1/(2\sqrt{n}) \leq \lambda \leq 1/(2\sqrt{n})$ and almost every $z \in \mathbb{T}$,

$$|\Delta_\lambda| \leq Ke^{K|f(z)|^2}(|f| + 1 + (n+1)(|f|+1)^2),$$

for some finite constant K depending only on c and C. But by Corollary 3.5, the right hand side of the last expression is integrable on $\mathbb{T}$. Applying Lebesgue's dominated convergence theorem we find that (4.6) holds, so that (4.2) makes sense and we have

$$\begin{aligned} 0 = \left.\frac{d\Lambda_\Phi\left((f+\lambda z^n)/\sqrt{F(\lambda)}\right)}{d\lambda}\right|_{\lambda=0} &= \lim_{\lambda\to 0}\int_{\mathbb{T}}\Delta_\lambda = \int_{\mathbb{T}}\delta \\ = \int_{\mathbb{T}} 2(\operatorname{Re}(\bar z^n f) - n\operatorname{Re} a_n|f|^2)\Psi'(|f|^2), \end{aligned} \tag{4.7}$$

by (4.5). But if f is extremal, then so is $-if$, since $|f| = |-if|$ and the Dirichlet norms are also the same. Applying (4.7) to $-if$ then yields the same expression but with $\operatorname{Im}(\bar z^n f)$ and $\operatorname{Im} a_n$ in place of $\operatorname{Re}(\bar z^n f)$ and $\operatorname{Re} a_n$, respectively. Multiplying the new expression by i, adding to (4.7), and dividing everything by two, we find that

$$\int_{\mathbb{T}}(\bar z^n \cdot f\Psi'(|f|^2)) = na_n\int_{\mathbb{T}}|f|^2\Psi'(|f|^2) = na_nS_\Psi(f), \tag{4.8}$$

for every positive integer n.

Assume first that $S_\Psi(f)$ does not vanish. Then (4.8) would immediately yield (4.1) if we knew that zf' had boundary values whose positive Fourier coefficients were $\{na_n\}$. This would follow if we knew that $f' \in H^1$, but unfortunately, we do not *a priori* know this, and so we must proceed more carefully.

Let $G(z) = f(z)\Psi'(|f(z)|^2)$ on $\mathbb{T}$. By Corollary 3.5 and the hypothesis that $|\Psi'(y)| \le ce^{Cy}$, we have $G \in L^2(\mathbb{T})$. Hence $\mathcal{P}_0(G) \in H^2(\mathbb{T})$. But we see from (4.8) that the nth positive Fourier coefficient of G is $S_\Psi(f)na_n$, which is also the nth Taylor coefficient of $S_\Psi(f)zf'$. Now $\mathcal{P}_0(G)$ extends to a holomorphic function with Taylor coefficients equal to its positive Fourier coefficients, and the positive Fourier coefficients of $\mathcal{P}_0(G)$ must of course match those of G and these match the Taylor coefficients of $S_\Psi(f)zf'$. It follows that $S_\Psi(f)zf' = \mathcal{P}_0(G)$, which is precisely the equation (4.1). Furthermore, $f' \in H^2(\mathbb{T})$ since $\mathcal{P}_0(G) \in H^2(\mathbb{T})$.

Now if $S_\Psi(f)$ vanishes, then by (4.8), the positive Fourier coefficients of the function $f\Psi'(|f|^2) \in L^2(\mathbb{T})$ must vanish. But this implies that $f\Psi'(|f|^2)$ is the boundary value of an antianalytic function, call it $\bar h$. By Corollary 3.5 and the inequality $\Psi'(y) \le ce^{Cy}$, $\bar h$ lies in $\bar H^2(\mathbb{T})$. Therefore, $|f|^2\Psi'(|f|^2) = \bar h \cdot \bar f$ is also the boundary value of an antianalytic function from $\bar H^1(\mathbb{T})$, since $\bar f$ is antianalytic and $f \in \mathfrak{D} \subset H^2$. Applying Lemma 4.1, we see that $|f|^2\Psi'(|f|^2)$ is almost everywhere constant on $\mathbb{T}$. But since $0 = S_\Psi(f)$ equals the integral of this constant about $\mathbb{T}$, it follows that in fact $|f|^2\Psi'(|f|^2) = 0$ almost everywhere on $\mathbb{T}$, so that $f\Psi'(|f|^2) = 0$ almost everywhere on $\mathbb{T}$ as desired.

The only remaining thing to do is to consider the case where $\|f\|_{\mathfrak{D}} < 1$. In this case, for λ sufficiently small we still have $f + \lambda z^n \in \mathfrak{B}$, so that we may use a simpler variation. Much as before, but with the calculations being somewhat simpler, we find that the condition

$$\left.\frac{d\Lambda_\Phi(f + \lambda z^n)}{d\lambda}\right|_{\lambda=0} = 0$$

makes sense, and applied to f and $-if$ leads to the condition

$$\int_{\mathbb{T}} (\bar{z}^n \cdot \Psi'(|f|^2)f) = 0, \tag{4.9}$$

for every positive integer n. As before, this implies that $f\Psi'(|f|^2)$ is the boundary value of an antianalytic function, say $\bar{h}$, which as before will have to be in $\bar{H}^2$. Therefore, $|f|^2\Psi'(|f|^2)$ is the boundary value of $\bar{h} \cdot \bar{f} \in \bar{H}^1$, so that its integral about $\mathbb{T}$ must equal $\bar{h}(0)\bar{f}(0) = 0$. But its integral about $\mathbb{T}$ is just $S_\Psi(f)$. Thus, $S_\Psi(f)$ vanishes and we may complete our argument in the same way as was done in the case of $\|f\|_{\mathfrak{D}} = 1$, above, but using (4.9) in place of (4.8). □

4.2. Regularity of extremals

Finally, as corollary to Theorem 4.1, we obtain a result on the regularity of extremal functions. This result is due to joint work of the author and Alec Matheson [75].

Theorem 4.2 (Matheson and Pruss [75]). *Write $\Psi(t) = \Phi(\sqrt{t})$. Fix an integer $n \geq 0$. Suppose that Ψ is n times differentiable on $(0, \infty)$. If $n > 0$ then assume further that $\Psi^{(n)}$ is Lipschitz on bounded subintervals of $[0, \infty)$, and, if $n = 0$, then assume that Ψ is differentiable on $(0, \infty)$. Also assume that $0 < |\Psi'(t)| \leq ce^{Ct}$ for some finite constants c and C, and for every $t > 0$. Finally assume that $f \in \mathfrak{B}$ is an extremal function for Λ_Φ. Then f is n times continuously differentiable on $\mathbb{T}$. Furthermore, $f^{(n)}$ is absolutely continuous on $\mathbb{T}$, and in fact $f^{(n+1)} \in$ BMOA so that $f^{(n)} \in \Lambda^*$.*

Remark 4.2. The Zygmund class Λ^* is the set of functions g on $\mathbb{T}$ such that for every $\theta \in [0, 2\pi)$ we have $F(e^{i(\theta+h)}) + F(e^{i(\theta-h)}) - 2F(e^{i\theta}) = O(h)$.

Of course if Λ_Φ has no extremal functions, then the content of the Theorem as stated is null. Note, however, that as the proof will show, the conclusion of Theorem 4.2 holds not only for extremal functions f, but indeed for any functions f satisfying the conclusion of Theorem 4.1 with $S_\Psi(f) \neq 0$.

The following result follows immediately from Theorem 4.2.

Corollary 4.1 (Matheson and Pruss [75]). *Suppose $\Phi : [0, \infty) \to \mathbb{R}$ is such that $t \mapsto \Phi(\sqrt{t})$ is infinitely differentiable on $(0, \infty)$ and each of its derivatives is bounded near zero. Also assume that $\Phi'(t) \neq 0$ on $(0, \infty)$, and that there is a finite constant C such that $|\Phi'(t)| = O(e^{Ct^2})$ as $t \to \infty$. Then any extremal function for Λ_Φ must lie in $C^\infty(\overline{D})$, i.e., for every non-negative k we must have $f^{(k)}$ in the disc algebra.*

For the proof of Theorem 4.2, we now recall the following well-known result which is crucial to the proof of our regularity theorem, though in fact we only use the easy case $p = 2$ in our work.

Lemma 4.2 (Privalov; see [41, Thm. 3.11]). *Let $f' \in H^p$ for a holomorphic function f on D and $p \geq 1$. Then f is absolutely continuous on $\mathbb{T}$, continuous on $\bar{D}$, and f' has the boundary values $f'(e^{i\theta}) = -ie^{-i\theta}\frac{df(e^{i\theta})}{d\theta}$.*

We also need the following simple result, which we list here for ease of reference and to establish a convention which we will use in our proof of Theorem 4.2.

Lemma 4.3. *Let F be a Lipschitz function and G absolutely continuous. Then, $F \circ G$ is absolutely continuous and $(F \circ G)' = (F' \circ G)G'$ almost everywhere, with the convention that $(F'(G(x))G'(x) = 0$ whenever $G'(x) = 0$, whether F is differentiable at $G(x)$ or not.*

Given this, we proceed to the proof of our regularity theorem. The main tool in the proof will be a qualitative analysis of condition (4.1). The careful reader will note that at some points in the proof we take some care to conclude that certain quantities are in L^2, and only use the fact that other quantities are in L^2, whereas it may be slightly simpler to replace all the L^2 conditions by the condition $\bigcap_{p<\infty} L^p$, and to use the full M. Riesz theorem. This would allow us not to bother with keeping track of the degrees of the arguments of the polynomial p_{k+1}. However, we choose to argue in L^2 because the M. Riesz theorem is trivial for the L^2 case and generalizes to L^2_M for arbitrary $M \subseteq \mathbb{Z}$, so that it will be possible to adapt our proof to later obtain the more general Theorem 4.4.

The following proof is due to the joint work of Matheson and Pruss [75].

Proof of Theorem 4.2. First note that for every $k \leq n$ the function $\Psi^{(k)}$ is bounded on the intervals $[0, A]$ for every finite A. This follows by $(n-k)$ integrations of the function $\Psi^{(n)}$ which is Lipschitz there.

We may also assume that $S_\Psi(f) \neq 0$. For if it does equal zero, then $f\Psi'(|f|^2)$ vanishes on $\mathbb{T}$. But Ψ' can only vanish at zero, so we see that $f \equiv 0$ in this case, and the Theorem is trivial then.

We note that (4.1) can be written in the form

$$Szf' = \mathcal{P}_0\left[p(f, \overline{f}; \Psi'(|f|^2))\right], \tag{4.10}$$

where p is a polynomial in $2+1$ variables and $S = S_\Psi(f)$. The basic idea is to differentiate this expression repeatedly. Write

$$\Delta g(e^{i\theta}) = -i\frac{dg(e^{i\theta})}{d\theta},$$

for a function g on $\mathbb{T}$. Note that $\Delta z^n = nz^n$. If g is holomorphic on D and $g' \in H^1$, then $\Delta g = zg'$ by Lemma 4.2.

Thus we can rewrite (4.10) as

$$S\Delta f = \mathcal{P}_0\left[p\left(f, \overline{f}; \Psi'(|f|^2)\right)\right]. \tag{4.11}$$

The main thing to do now is to prove that our hypotheses in effect allow us to commute Δ with $\mathcal{P}_0$ the right number of times.

We shall proceed iteratively. For the convenience of the reader, however, we first outline the idea of the proof in a less formal way in the case where $\Phi \in C^\infty[0,\infty)$. In that case we must show that $f \in C^\infty(\mathbb{T})$. By Theorem 4.1 and Lemma 4.2 we have $\Delta f = zf' \in H^2$ and f is absolutely continuous on $\mathbb{T}$. Moreover,

$$S\Delta f = F_1,$$

where $F_1 = p(f, \overline{f}; \Psi'(|f|^2))$. Now,

$$\Delta F_1 = p_2(z, \overline{z}, f, \overline{f}, f', \overline{f'}; \Psi'(|f|^2), \Psi''(|f|^2)), \tag{4.12}$$

for some polynomial p_2. Moreover, the right hand side of (4.12) is only linear in f' and $\overline{f'}$ (i.e., f' and $\overline{f'}$ in it are never multiplied together nor are they raised to a power bigger than 1). The absolutely continuity of f implies that $F_1 \stackrel{\text{def}}{=} p(f, \overline{f}; \Psi'(|f|^2))$ is absolutely continuous on $\mathbb{T}$ since p is a polynomial. Together with an integration by parts and the fact that $S\Delta f = \mathcal{P}_0 F_1$, this shows that the positive Fourier coefficients of ΔF_1 coincide with the positive Fourier coefficients of $S\Delta^2 f$ considered as a distribution on $\mathbb{T}$. But by (4.12) since $f' \in H^2$ it follows that $\Delta F_1 \in L^2(\mathbb{T})$ (here we have used the fact that the right side of (4.12) is only linear in f' and $\overline{f'}$). Thus, $S\Delta^2 f \in L^2(\mathbb{T})$ and

$$S\Delta^2 f = \mathcal{P}_0 F_2,$$

where F_2 denotes the right hand side of (4.12). From the fact that $\Delta^2 f \in L^2(\mathbb{T})$, it follows (using Lemma 4.2) that f' is absolutely continuous on $\mathbb{T}$ and that $f'' \in H^2$. Thus F_2 is absolutely continuous on $\mathbb{T}$. Applying the above method one more time we conclude that ΔF_2 is a polynomial in

$$z, \overline{z}, f, \overline{f}, f', \overline{f'}, f'', \overline{f''}, \Psi'(|f|^2), \Psi''(|f|^2), \Psi'''(|f|^2),$$

and that

$$S\Delta^3 f = \mathcal{P}_0 \Delta F_2 \in L^2(\mathbb{T}),$$

so that $f''' \in H^2(\mathbb{T})$. Iterating we conclude that $f^{(k)} \in H^2(\mathbb{T})$ for all natural k, which implies that $f \in C^\infty(\mathbb{T})$ as desired.

We now return to the general case and proceed more rigorously. For the sake of our iteration, suppose that on $\mathbb{T}$ we have

$$\begin{aligned} S\Delta^k f = \mathcal{P}_0[p_k(z, \overline{z}, f, \overline{f}, f', \overline{f'}, ..., f^{(k-1)}, \overline{f^{(k-1)}}; \\ \Psi'(|f|^2), \Psi''(|f|^2), ..., \Psi^{(k)}(|f|^2))], \end{aligned} \tag{4.13}$$

with $k \leq n$, with p_k a polynomial in $2(k+1)+k$ variables, and with $\Delta^j f \in H^2$ for every $0 \leq j \leq k$. By Theorem 4.1 and (4.11), we have this for $k = 1$.

Using the fact that $\Delta g = zg'$, we easily see that for any natural j we can write (in the sense of power series)

$$z^j f^{(j)} = \sum_{l=0}^{j} a_{j,l} \Delta^l f$$

for some set of universal constants $a_{j,l}$. Since $\Delta^j f \in H^2$ for every $0 \leq j \leq k$ it follows that likewise $f^{(j)} \in H^2$ whenever $0 \leq j \leq k$. Thus by Lemma 4.2 we have that $f^{(j)}$ is absolutely continuous on $\mathbb{T}$ for $0 \leq j \leq k-1$. Since sums and products of absolutely continuous functions are absolutely continuous, and since $\Psi^{(j)}$ for $0 \leq j \leq k$ is Lipschitz on bounded subintervals of $[0, \infty)$ while f is absolutely continuous on $\mathbb{T}$ so that $\Psi^{(j)}(|f|^2)$ must be absolutely continuous on $\mathbb{T}$ by Lemma 4.3, it follows that

$$\begin{aligned} F_k \stackrel{\text{def}}{=} p_k(z, \overline{z}, f, \overline{f}, f', \overline{f'}, ..., f^{(k-1)}, \overline{f^{(k-1)}}; \\ \Psi'(|f|^2), \Psi''(|f|^2), ..., \Psi^{(k)}(|f|^2)) \end{aligned} \tag{4.14}$$

is absolutely continuous on $\mathbb{T}$. But, in the presence of absolute continuity, we may use integration by parts to see that for every positive integer m,

$$\begin{aligned} \int_0^{2\pi} e^{-im\theta} \Delta F_k(e^{i\theta})\, d\theta &= m \int_0^{2\pi} e^{-im\theta} F_k(e^{i\theta})\, d\theta \\ &= Sm \int_0^{2\pi} e^{-im\theta} \Delta^k f(e^{i\theta})\, d\theta, \end{aligned} \tag{4.15}$$

where we have applied (4.13) to obtain the last equality.

Now it is easy to see from (4.14) and the definition of Δ that

$$\Delta F_k = p_{k+1}(z, \overline{z}, f, \overline{f}, f', \overline{f'}, ..., f^{(k)}, \overline{f^{(k)}}; \Psi'(|f|^2), \Psi''(|f|^2), ..., \Psi^{(k+1)}(|f|^2)) \tag{4.16}$$

for some polynomial p_{k+1} in $2(k+1)+(k+1)$ variables, such that $f^{(k)}$ and $\overline{f^{(k)}}$ are not raised to any power greater than one nor are they multiplied together on the right hand side of (4.16) (i.e., fixing all arguments other than the $(2k+1)$st and $(2k+2)$nd in p_{k+1} we obtain something of degree one), and where we have used the convention of Lemma 4.3 at points z of $\mathbb{T}$ where $\Psi^{(k+1)}(|f(z)|^2)$ is undefined. This convention may be necessary if $k = n$ and $n > 0$, since then we only know that $\Psi^{(k+1)}$ exists almost everywhere, and the set of points z of $\mathbb{T}$ at which $\Psi^{(k+1)}(|f|^2(z))$ is undefined may well have positive measure, since after all $|f|$ is free to be constant (as long as this constant is not zero) on a subset of $\mathbb{T}$ with positive measure, even if f itself is not everywhere constant in D.

Since we know that f is absolutely continuous on $\mathbb{T}$, it must be bounded there. As noted at the beginning of the proof, the hypotheses of the Theorem imply that $\Psi^{(j)}$ is bounded everywhere on $[0, ||f||^2_\infty]$ for $j \leq n$. This is also obviously true for $j = n+1$ if $n = 0$ by our condition on the size of $|\Psi'|$. In the case where $j = k = n+1$ and $n > 0$, we note that the derivative of the function $\Psi^{(n)}$ which is Lipschitz on $[0, ||f||^2_\infty]$ is bounded by the Lipschitz constant wherever it exists. Moreover, for almost every $z \in \mathbb{T}$ such that $|f(z)|^2$ falls into the exceptional set where $\Psi^{(n)}$ is not defined, we have, following the convention of Lemma 4.3, some factor equal to zero multiplying the $\Psi^{(n)}(|f(z)|)$ in the right hand side of (4.16). Finally, since $f^{(k)}$ is in H^2, from Lemma 4.2 we conclude that all arguments of p_{k+1} in (4.16) are bounded, except possibly for $f^{(k)}$ and $\overline{f^{(k)}}$. But these latter two arguments are never multiplied together, nor are they ever raised to any power, so we see that since they lie in $L^2(\mathbb{T})$, it follows that in fact the right hand side of (4.16) must lie in $L^2(\mathbb{T})$. Thus, $\mathcal{P}_0(\Delta F_k) \in H^2(\mathbb{T})$, as well, by the L^2 case of the theorem of M. Riesz (Theorem I.3.4).

Now write $g(z) = S\Delta^k f(z) = \sum_{m=1}^{\infty} a_m z^m$ for $z \in D$. We have $zg'(z) = \sum_{m=1}^{\infty} m a_m z^m$ in D. In this notation, (4.15) tells us that for m positive, the mth Fourier coefficient of ΔF_k on $\mathbb{T}$ is

ma_m, and since $\mathcal{P}_0(\Delta F_k) \in H^2(\mathbb{T})$, it follows as in the proof of Theorem 4.1 that we must have $zg' = \mathcal{P}_0(\Delta F_k)$. But $zg' = S\Delta^{k+1}f$, so that we obtain

$$\begin{aligned} S\Delta^{k+1}f = \mathcal{P}_0[p_{k+1}(z,\overline{z},f,\overline{f},f',\overline{f'},...,f^{(k)},\overline{f^{(k)}};\\ \Psi'(|f|^2),\Psi''(|f|^2),...,\Psi^{(k+1)}(|f|^2))], \end{aligned} \tag{4.17}$$

with $\Delta^{k+1}f = S^{-1}zg' \in H^2$ since $S \neq 0$. But this was precisely what was needed for the iteration to continue.

Thus iterating, we see that $\Delta^k f \in H^2$ for each $k \leq n+1$. As argued before, it follows that $f^{(n+1)} \in H^2$, and so by Lemma 4.2 we have $f^{(n)}$ absolutely continuous on $\mathbb{T}$, and of course then f must be n times differentiable, as desired. Finally, since $f^{(n)}$ must be bounded, being absolutely continuous, and on the bounded interval $[0, \|f\|_\infty^2]$ we have $\Psi^{(n+1)}$ bounded wherever defined, it follows from (4.16) that ΔF_n must lie in $L^\infty(\mathbb{T})$ (where as before we use the convention of Lemma 4.3) so that by (4.17) it follows that $\Delta^{n+1}f \in$ BMOA, and so $f^{(n+1)} \in$ BMOA, by the argument which we have used twice before in order to pass from estimates on the $\Delta^j f$ to ones on the $f^{(j)}$. But, BMOA is contained in the Bloch space, and a derivative g' of a holomorphic function g is in the Bloch space if and only if $g \in \Lambda^*$ (see, e.g., [100, vol. I, p. 163]) so that $f^{(n)} \in \Lambda^*$. □

4.3. The strict analytic radial increase property (SARIP)

Definition 4.1. We say that $\Phi\colon \mathbb{C} \to \mathbb{R}$ has the **strict analytic radial increase property (SARIP)** if the following conditions are satisfied:

(i) Φ is Borel measurable

(ii) Φ is lower semicontinuous at 0

(iii) whenever $f \in \mathfrak{B}$ has $f(0) = 0$ and $f \not\equiv 0$ then we have

$$\int_{\mathbb{T}} \Phi(\lambda f) > \int_{\mathbb{T}} \Phi(f)$$

for every $\lambda > 1$.

Definition 4.2. We say that $\Phi\colon \mathbb{C} \to \mathbb{R}$ has the **smooth strict analytic radial increase property (SSARIP)** if it has the SARIP and the following additional conditions are satisfied:

(i) $\Phi \in C^1(\mathbb{R}^2)$

(ii) if $f \in \mathfrak{B}$ has $f(0) = 0$ and $f \not\equiv 0$ then we have

$$\int_{\mathbb{T}} \langle f, (\nabla\Phi)(f)\rangle > 0, \tag{4.18}$$

providing $|\langle f, (\nabla\Phi)(f)\rangle| \in L^1(\mathbb{T})$.

On the left hand side of (4.18), we use $\langle\cdot,\cdot\rangle$ for the Euclidean inner product on $\mathbb{R}^2$ and we make the identification $\mathbb{C} = \mathbb{R}^2$.

Note that non-trivial positive linear combinations of functions having SARIP (respectively, SSARIP) also have SARIP (respectively, SSARIP).

The properties SARIP and SSARIP will make useful assumptions later in the study of the extremals of the Λ_Φ.

Remark 4.3. Let Φ have the SARIP (respectively, SSARIP). Let L be a linear functional from $\mathbb{R}^2$ to $\mathbb{R}$. Then $\Phi + L$ has the SARIP (respectively, SSARIP). To see this, note that

$$\int_{\mathbb{T}} (\Phi + L)(f) = \int_{\mathbb{T}} \Phi(f) + \int_{\mathbb{T}} L(f) = \int_{\mathbb{T}} \Phi(f) + 0,$$

for all $f \in \mathfrak{B}$ with $f(0) = 0$ since $L \circ f$ is harmonic for f holomorphic so that $\int_{\mathbb{T}} L(f) = L(f(0)) = L(0) = 0$.

The following two examples are easy to verify.

Example 4.1. If $\lambda \mapsto \Phi(\lambda z)$ is strictly increasing on $[0,\infty)$ for every $z \in \mathbb{C}\backslash\{0\}$ then Φ has the SARIP. In particular, if $\Phi(z) = \phi(|z|)$ for ϕ strictly increasing on $[0,\infty)$, then Φ has SARIP.

Example 4.2. Let ϕ be a function on $\mathbb{R}$ which is strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty, 0]$ then $\Phi(z) = \phi(\operatorname{Re} z)$ has SARIP.

Finally we have the following example.

Example 4.3. Let ϕ be a convex function on $\mathbb{R}$ which is strictly convex at 0. Then $\Phi(z) = \phi(\operatorname{Re} z)$ has SARIP. To see this, by strict convexity at 0 choose a $k \in \mathbb{R}$ such that $\phi(0) + kx < \phi(x)$ for all $x \neq 0$. Let $\psi(x) = \phi(x) - kx - \phi(0)$. This is a convex function which is everywhere strictly positive except at 0 where it vanishes. I claim that ψ is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$. It suffices to prove the strict increase on $[0, \infty)$ since the other strict monotonicity property follows by applying the same argument to $\phi(-(\cdot))$ and $\psi(-(\cdot))$. To obtain a contradiction, suppose that there exist $0 \leq x_1 < x_2 < \infty$ such that $\psi(x_1) \geq \psi(x_2)$. By convexity, the line joining $(x_1, \psi(x_1))$ with $(x_2, f(x_2))$ will be below the graph of f except over the interval $[x_1, x_2]$. In particular, this line will be below $(0, \psi(0))$. But since $\psi(x_1) \geq \psi(x_2)$, it follows that this line is above the point $(0, \psi(x_2))$. Hence, $\psi(x_2) \leq \psi(0) = 0$, which is a contradiction to the strict positivity of ψ away from 0. Hence, $\Psi(z) = \psi(\operatorname{Re} z)$ has SARIP by Example 4.2. But $\Phi - \Phi(0)$ differs from Ψ only by a linear functional so that by Remark 4.3 it follows that $\Phi - \Phi(0)$ has SARIP. It follows that Φ has SARIP.

We may modify the preceding example as follows.

Example 4.4. Let $\phi \in C^1(\mathbb{R})$ be convex and strictly convex at 0. Then $\Phi(z) = \phi(\operatorname{Re} z)$ has SSARIP. To see this, note first that by the previous example it has SARIP. Then, by adding a linear function if necessary, we may assume that $\phi'(0) = 0$. Let $f \in \mathfrak{B}$ be such that $f \not\equiv 0$. Then,

$$\int_{\mathbb{T}} \langle f, (\nabla\Phi)(f) \rangle = \int_{\mathbb{T}} (\operatorname{Re} f)\phi'(\operatorname{Re} f). \tag{4.19}$$

Now, because $\phi'(0) = 0$ and ϕ is convex, it follows that $\phi'(x) \geq 0$ for $x > 0$ and $\phi'(x) \leq 0$ for $x < 0$. Hence, $(\operatorname{Re} f)\phi'(\operatorname{Re} f)$ is everywhere non-negative on $\mathbb{T}$ and vanishes only where $\phi'(\operatorname{Re} f)$ vanishes. Thus, (4.19) is non-negative. To obtain a contradiction, assume that it vanishes. Then, $\phi'(\operatorname{Re} f)$ vanishes almost everywhere on $\mathbb{T}$. Now, strict convexity of ϕ at 0 implies that either $\phi'(x) > 0$ for all $x > 0$ or $\phi'(x) < 0$ for all $x < 0$ or both. Moreover, f does not almost everywhere vanish on $\mathbb{T}$ and f has mean zero over $\mathbb{T}$ so that neither f^+ nor f^- can almost everywhere vanish on $\mathbb{T}$. Hence, $\phi'(\operatorname{Re} f)$ cannot almost everywhere vanish on

$\mathbb{T}$ because ϕ' is non-zero on at least one of the two half lines $(-\infty, 0)$ and $(0, \infty)$.

Part of the usefulness of SARIP lies in the following proposition.

Proposition 4.1. *Fix $\alpha \in [0, \infty)$. Let Φ be any real-valued SARIP function on $\mathbb{C}$ such that Λ_Φ achieves a maximum over $\mathfrak{B}_\alpha$ at $f \in \mathfrak{B}_\alpha$. Then $||f||_{\mathfrak{D}_\alpha} = 1$.*

Proof. By SARIP the function $\lambda \mapsto \Lambda_\Phi(\lambda \cdot \mathrm{Id})$ is strictly increasing on $(0, 1]$, where Id is the identity function on $\mathbb{D}$. Taking the limit as $\lambda \to 0$ and using the lower semicontinuity of Φ at 0, we see that in fact $\lambda \mapsto \Lambda_\Phi(\lambda \cdot \mathrm{Id})$ is strictly increasing on $[0, 1]$. Hence, $\Lambda_\Phi(0) < \Lambda_\Phi(\mathrm{Id})$. Hence, our extremal f cannot be the zero function.

If $0 < ||f||_{\mathfrak{D}_\alpha} < 1$ then let $\lambda = ||f||_{\mathfrak{D}_\alpha}^{-1} > 1$. We have $\lambda f \in \mathfrak{B}_\alpha$. By SARIP we have $\Lambda_\Phi(\lambda f) > \Lambda_\Phi(f)$, contradicting the assumption that Λ_Φ achieves a maximum at f. □

4.4. Some extensions

In this section we consider the more general Λ_Φ functionals for Φ a real function on $\mathbb{C}$. The results of §4.1 and §4.2 will essentially be special cases of the work of the present section.

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$. When convenient, we shall make the identification $\mathbb{C} = \mathbb{R}^2$.

In the foregoing, $(\nabla\Phi)(f)$ will indicate $\nabla\Phi$ evaluated at f, where $\nabla\Phi$ is the gradient. We denote the euclidean inner product of $x, y \in \mathbb{R}^2$ by $\langle x, y \rangle$. Let $\mathcal{P}_0$ and z be as in Theorem 4.1.

The following result generalizes Theorem 4.1.

Theorem 4.3. *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$. Assume that Λ_Φ is defined on $\mathfrak{B}$, and that Φ has first partials $\Phi_{,j}$ everywhere on $\mathbb{R}^2$, satisfying $|\Phi_{,j}(z)| \le ce^{C|z|^2}$, for every $z \in \mathbb{R}^2$, where $j = 1, 2$, and c and C are any finite constants. Suppose $f \in \mathfrak{B}$ is an extremal function for Λ_Φ. Write*

$$Q_\Phi(f) = \int_{\mathbb{T}} \langle f, (\nabla\Phi)(f) \rangle .$$

Assuming that $Q_\Phi(f)$ does not vanish, we have $f' \in H^2$ and

$$Q_\Phi(f) z f' = \mathcal{P}_0 \left[(\nabla\Phi)(f) \right], \qquad \text{on } \mathbb{T}. \tag{4.20}$$

Moreover, if we also have $\|f\|_{\mathfrak{D}} < 1$ then $Q_\Phi(f)$ must vanish.

Remark 4.4. The proof of the last sentence of the Theorem is an easy to justify differentiation of the function $r \mapsto \Lambda_\Phi(rf)$ at $r = 1$. Note also that if Φ satisfies SSARIP then $Q_\Phi(f)$ cannot be equal to zero. For, by SSARIP if $Q_\Phi(f) = 0$ then $f \equiv 0$, while 0 cannot be extremal by SARIP and Proposition 4.1.

One proves the rest of the Theorem by using exactly the same variational methods as in the proof of Theorem 4.1. It should also be noted that at the point at which we had previously asserted that if f is extremal then so is $-if$, one must modify the remark to note that if f is extremal for $\Lambda_{\Phi(\cdot)}$ then $-if$ is extremal for $\Lambda_{\Phi(i(\cdot))}$.

The careful reader will note that Theorem 4.3 is not a complete generalization of Theorem 4.1. First of all, Theorem 4.1 had a discussion of what happens when $S_\Psi(f)$ vanishes. Secondly, Theorem 4.1, when translated into the language of Theorem 4.3, did not require the first partials of Φ to exist at zero.

We may explain this as follows. The proof of Theorem 4.1 which was adapted to yield Theorem 4.3 still goes through if we only assume that first of all, $\Phi_{,j}$ exists on $\mathbb{R}^2 \backslash E$ for some set E such that $|\{e^{i\theta} : f(e^{i\theta}) \in E\}| = 0$, and that secondly, Φ is Lipschitz on compact subsets of some ε-neighbourhood of the image of f. In Theorem 4.3 we in effect took $E = \{0\}$ and noted that if $|f| \in E$ on a set of positive measure then $f \equiv 0$ and the results are trivial then. Hence, if we wish we could loosen the condition of the differentiability of Φ at zero.

The following result is a generalization of Theorem 4.2, which, as we had noted, was due to joint work of Matheson and Pruss [75].

Theorem 4.4 ([75]). *Fix $n \geq 0$. Let $\Phi : \mathbb{C} \to \mathbb{R}$. Assume that Λ_Φ is defined on $\mathfrak{B}$, and that Φ has first partials $\Phi_{,j}$ everywhere on $\mathbb{R}^2$, satisfying $|\Phi_{,j}(z)| \leq c e^{C|z|^2}$ for every $z \in \mathbb{R}^2$, where*

$1 \le j \le 2$ and c and C are any finite constants. Assume that Φ satisfies SSARIP. Suppose that all the (pure and mixed) partials of Φ of order $\le n$ exist. If $n > 0$ then assume further that all the (pure and mixed) nth order partials are Lipschitz on compact subsets of $\mathbb{R}^2$. Suppose $f \in \mathfrak{B}$ is an extremal function for Λ_Φ. Then f is n times continuously differentiable on $\mathbb{T}$. Furthermore, the nth derivative of f on $\mathbb{T}$ is absolutely continuous and lies in $\Lambda^*(\mathbb{T})$.

Remark 4.5. It is in fact possible to obtain analogues of Theorems 4.3 and 4.4 for $\mathfrak{B}_\alpha$ in place of $\mathfrak{B}$, for $\alpha \ge 1$. The methods are essentially the same, except that the variational equation (4.20) has

$$\sum_{n=1}^{\infty} n^\alpha \hat{f}(n) z^n$$

in place of zf'. Indeed, this much will even work for $\alpha \in (0,1)$, and so we can find an analogue of Theorem 4.3 for all $\alpha > 0$. Of course, for $\alpha < 1$, the square exponential growth condition on $\Phi_{,j}(z)$ will have to be replaced by a polynomial condition, with the degree of the polynomial depending on α and determined by use of Theorem 3.3 and the requirements of the proof. On the other hand, for $\alpha > 1$ the square exponential growth condition can be dropped, and replaced by boundedness on compacta.

The proof of the modified Theorem 4.4 for $\mathfrak{B}_\alpha$, $\alpha \ge 1$, is much the same as that of the ordinary Theorems 4.2 and 4.4. The only thing to note is that on the left hand sides of various equations instead of having

$$\Delta^k f$$

we will have

$$\Delta^{k-1} \sum_{n=1}^{\infty} n^\alpha \hat{f}(n) z^n.$$

However, this does not affect the conclusions very much, since if this quantity lies in $L^2(\mathbb{T})$ then *a fortiori* so does $\Delta^k f$. Actually, we can do better: we can conclude that $f^{(n+\lfloor\alpha\rfloor)} \in L^2(\mathbb{T})$, where $\lfloor\alpha\rfloor$ is the smallest integer greater than or equal to α. We can also conclude that f is $n + \lfloor\alpha\rfloor - 1$ times continuously differentiable. Since these various extensions are not of great interest to us at present, we leave the details to the reader.

5. Symmetric decreasing rearrangement and Dirichlet norms

Let $f \in \mathfrak{d}_\alpha$ for $\alpha \in [0, \infty)$. Then f has finite nontangential boundary values almost everywhere on $\mathbb{T}$ since $\mathfrak{d}_\alpha \subseteq \mathfrak{d}_0 \subset h^2$. Let $f^{\circledcirc}$ be the function on $\mathbb{T}$ given by the symmetric decreasing rearrangement of n. t. $\lim f$ (see §I.6.2). Since $f \in L^2(\mathbb{T})$, we must likewise have $f^{\circledcirc} \in L^2(\mathbb{T})$. Moreover, $f^{\circledcirc}$ has mean zero on $\mathbb{T}$ since f does and since $f^{\circledcirc}$ and f are equimeasurable. (Given equimeasurability, use Remark I.2.2. Equimeasurability follows from Remark I.6.7.) Let $f^{\circledcirc} : \mathbb{D} \to \mathbb{R}$ be the Poisson extension of $f^{\circledcirc} : \mathbb{T} \to \mathbb{R}$ via the Poisson integral (Theorem I.3.3); we will then have $f^{\circledcirc} \in h^2(\mathbb{D})$.

Theorem 5.1. *Let $f \in \mathfrak{d}_\alpha$ for $0 \le \alpha \le 2$. Let $f^{\circledcirc}$ be as above. Then,*

$$\|f^{\circledcirc}\|_{\mathfrak{d}_\alpha} \le \|f\|_{\mathfrak{d}_\alpha}. \tag{5.1}$$

Suppose moreover that $0 < \alpha < 2$ and that equality holds in (5.1). Then f is a rotation of $f^{\circledcirc}$, i.e., there exists $w \in \mathbb{T}$ such that $f(z) = f^{\circledcirc}(zw)$ for all $z \in \mathbb{D}$.

The proof will be given later.

Remark 5.1. For $\alpha = 0$, we always have equality in (5.1), since both sides of (5.1) are then equal to the $L^2(\mathbb{T})$ norm, and symmetric decreasing rearrangement preserves L^2 norms.

Remark 5.2. If $\alpha = 2$ then it is well known that there exists $f \in \mathfrak{d}_\alpha$ such that f is not a rotation of $f^{\circledcirc}$ but $\|f^{\circledcirc}\|_{\mathfrak{d}_\alpha} = \|f\|_{\mathfrak{d}_\alpha}$. To see this, note that for any function $g \in C^1(\bar{\mathbb{D}}) \cap \mathfrak{d}_2$ we have

$$\|g\|_{\mathfrak{d}_2} = \frac{1}{2\pi} \int_0^{2\pi} |g'(e^{i\theta})|^2 \, d\theta, \tag{5.2}$$

where $g'(e^{i\theta}) = \frac{d}{d\theta} g(e^{i\theta})$. (This is easy to see by writing the right hand side of (5.2) in terms of the Fourier series expansion of g'.) Now, let f_1 and f_2 be functions in $C^1(\mathbb{T})$ with the following properties:

(i) f_1 and f_2 are symmetric decreasing

(ii) f_1 is constant on $A_1 \stackrel{\text{def}}{=} \{e^{i\theta} : |\theta| < \pi/2\}$, while f_2 is constant on $A_2 \stackrel{\text{def}}{=} \{e^{i\theta} : |\theta - \pi| < 3\pi/4\}$

(iii) $f_1 + f_2$ has mean zero.

Now, define $f(e^{i\theta}) = f_1(e^{i\theta}) + f_2(e^{i(\theta+(\pi/4))})$. Then, $f^{\circledcirc} = f_1 + f_2$. This is easiest seen in light of Figure 5.1 by comparing sizes of level sets of f and of $f_1 + f_2$. It is moreover clear that f is not a rotation of $f^{\circledcirc}$. Let $g_2(e^{i\theta}) = f_2(e^{i(\theta+(\pi/4))})$. Then g_2 is constant on $e^{-i\pi/4}A_2$, while f_1 is constant on A_1, so that the supports of g_2' and f_1' are disjoint. Likewise, f_2 is constant on A_2, so that the supports of f_2' and f_1' are disjoint.

Hence,

$$\begin{aligned}\int_0^{2\pi} |f'(e^{i\theta})|^2\,d\theta &= \int_0^{2\pi} |f_1'(e^{i\theta})|^2\,d\theta + \int_0^{2\pi} |g_2'(e^{i\theta})|^2\,d\theta \\ &= \int_0^{2\pi} |f_1'(e^{i\theta})|^2\,d\theta + \int_0^{2\pi} |f_2'(e^{i\theta})|^2\,d\theta \\ &= \int_0^{2\pi} |(f^{\circledcirc})'(e^{i\theta})|^2\,d\theta.\end{aligned}$$

From (5.2) we conclude that $\|f^{\circledcirc}\|_{\mathfrak{d}_2} = \|f\|_{\mathfrak{d}_2}$ as desired.

Now, given an analytic function $F \in \mathfrak{D}_\alpha$ for some finite $\alpha \in [0,2]$, let $f = \operatorname{Re} F$. Then, $f \in \mathfrak{d}_\alpha$, and $\|f\|_{\mathfrak{d}_\alpha} = \|F\|_{\mathfrak{D}_\alpha}$ (Remark 1.2). Let $f^{\circledcirc}$ be the symmetric decreasing rearrangement of f as before; then $f^{\circledcirc} \in \mathfrak{d}_\alpha$ and there exists a unique function $G \in \mathfrak{D}_\alpha$ such that $\operatorname{Re} G = f^{\circledcirc}$; we will have $\|G\|_{\mathfrak{D}_\alpha} = \|f^{\circledcirc}\|_{\mathfrak{d}_\alpha}$. (We put $G = \mathcal{P} f^{\circledcirc}$ in the notation of §I.3.3.)

We then define $F^{\circledcirc} = G$ and can reformulate Theorem 5.1 as follows.

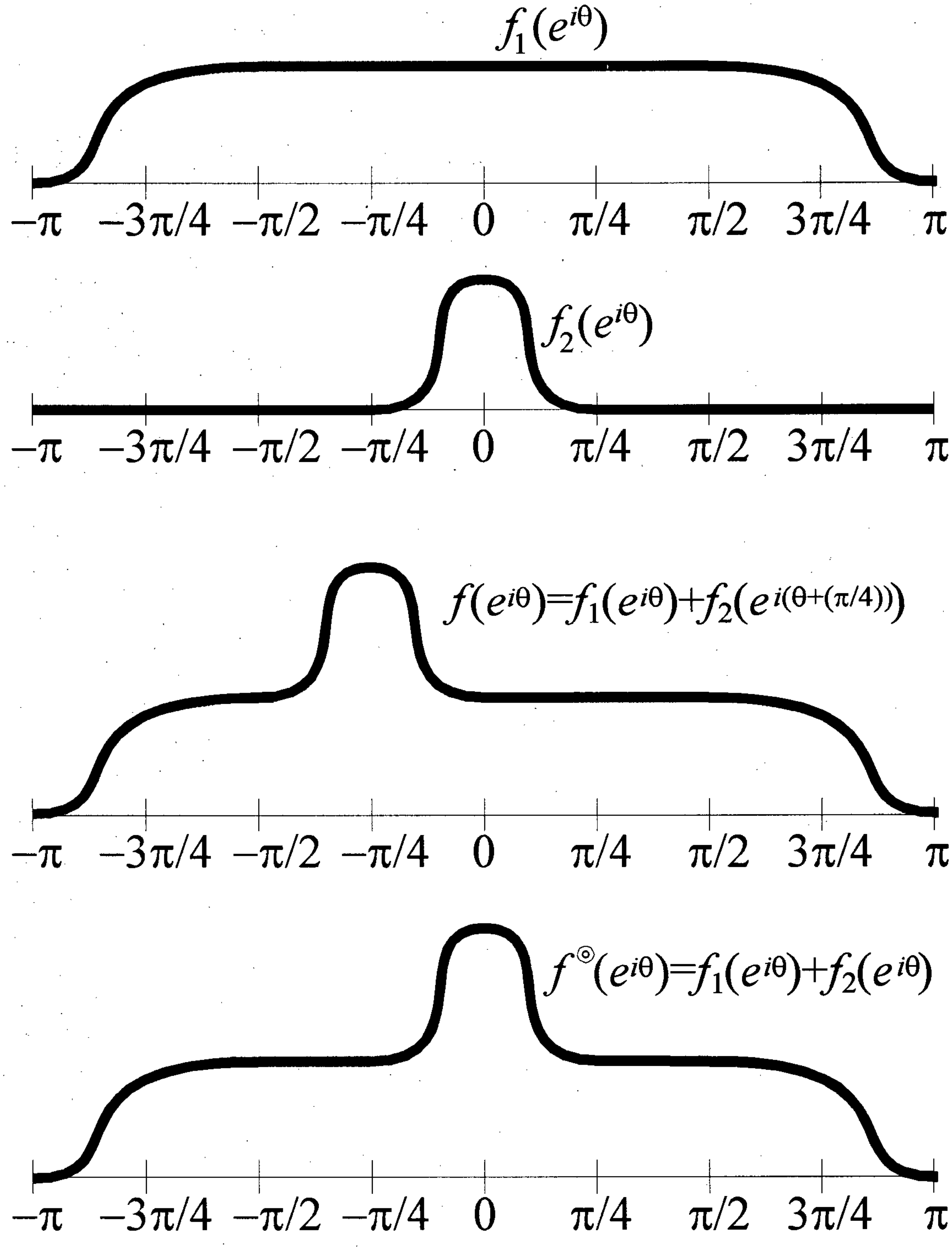
Corollary 5.1. *Let $F \in \mathfrak{D}_\alpha$ for $0 \le \alpha \le 2$. Let $F^{\circledcirc}$ be as above. Then,*

$$\|F^{\circledcirc}\|_{\mathfrak{d}_\alpha} \le \|F\|_{\mathfrak{d}_\alpha}.$$

Suppose moreover that $0 < \alpha < 2$ and that equality holds in (5.1). *Then F is a rotation of $F^{\circledcirc}$, i.e., there exists $w \in \mathbb{T}$ such that $F(z) = F^{\circledcirc}(zw)$ for all $z \in \mathbb{D}$.*

The function $F^{\circledcirc}$ is in fact univalent on $\mathbb{D}$ and the image $F^{\circledcirc}[\mathbb{D}]$ is Steiner symmetric. This follows from the following Proposition.

Proposition 5.1. *Let $f \in H^1(\mathbb{D})$ be such that* $\operatorname{Re} f$ *is symmetric decreasing on $\mathbb{T}$. Assume f is not constant. Then f is univalent and the image $f[\mathbb{D}]$ is Steiner symmetric about the real axis.*



Note: appropriate constants should be added to the functions to ensure that f has mean zero.

Figure 5.1: The functions f_1, f_2, f and $f^{\circledcirc}$

The proof will be given later.

Corollary 5.2. *Let ϕ be a function on $\mathbb{R}$ such that $\Phi(z) = \phi(\operatorname{Re} z)$ has SARIP. Fix $0 < \alpha < 2$. Suppose that Λ_Φ attains its maximum over $\mathfrak{B}_\alpha$ at $f \in \mathfrak{B}_\alpha$. Then, there exists a $w \in \mathbb{T}$ such that $z \mapsto \operatorname{Re} f(zw)$ is symmetric decreasing on $\mathbb{T}$ (i.e., $\operatorname{Re} f$ is the rotation of a symmetric decreasing function). Moreover, the image $f[\mathbb{D}]$ is Steiner symmetric and f is univalent.*

Proof of Corollary 5.2. Suppose that $\operatorname{Re} f$ is not the rotation of a symmetric decreasing function in the sense specified in the statement of the Corollary. Then, $\|f^{\circledcirc}\|_{\mathfrak{D}_\alpha} < \|f\|_{\mathfrak{D}_\alpha} \leq 1$ in light of Corollary 5.1. But $\Lambda_\Phi(f^{\circledcirc}) = \Lambda_\Phi(f)$ by Proposition I.2.1 since $\mathbb{T}$ has finite measure. Hence, Λ_Φ attains a maximum at $f^{\circledcirc}$. But by Proposition 4.1 we then obtain a contradiction since $\|f\|_{\mathfrak{D}_\alpha} < 1$.

Hence $\operatorname{Re} f$ is the rotation of a symmetric decreasing function, i.e., $g(z) = \operatorname{Re} f(zw)$ is symmetric decreasing. Moreover, by Proposition 4.1 we have $f \not\equiv 0$ so that the rest of the conclusions of the theorem follow by applying Proposition 5.1 to g. □

Proof of Proposition 5.1. First, fix $r \in (0,1)$. Let $f_r(z) = f(zr)$. Put $u = \operatorname{Re} f_r$ and $v = \operatorname{Im} g_r$. By Corollary I.6.3, the function u is symmetrically strictly decreasing on $\mathbb{T}$.

Now, let $D_r = f_r[\mathbb{D}]$. We shall prove that D_r is Steiner symmetric about the real axis and that f_r is univalent on $\mathbb{D}$. It will then follow that f is univalent on $\mathbb{D}(r)$. Taking $r \to 1-$ it will follow that f is univalent on $\mathbb{D}$. Moreover, $f[\mathbb{D}] = \bigcup_{0<r<1} D_r$, and the union of Steiner symmetric domains is Steiner symmetric, so that we will thus be done. Note that by another limiting argument it suffices to consider only those values of r for which f has no zeroes on $\mathbb{T}(r)$.

The symmetry of u implies that $u(\bar{z}) = u(z)$ for all $z \in \overline{\mathbb{D}}$. It follows from Remark I.3.2 that $v(\bar{z}) = -v(z)$ since v is the conjugate function of u. In particular, it follows that $v(1) = v(-1) = 0$ and that D_r is symmetric under reflection in the real axis. I first claim that it is false that for every $\theta \in (0, \pi)$ we have $v(e^{i\theta}) \leq 0$. For, suppose that indeed $v(e^{i\theta}) \leq 0$ for each $\theta \in (0, \pi)$.

We have assumed that f_r has no zeroes on $\mathbb{T}$. Hence, since u is symmetric strictly decreasing while $v(e^{i\theta}) \leq 0$ for $\theta \in (0, \pi)$ so that $v(e^{i\theta}) \geq 0$ for $\theta \in (-\pi, 0)$ since $v(\bar{z}) = -v(z)$, it follows that $f_r(e^{i\theta})$ winds once around 0 in the negative direction as θ goes from 0 to 2π. But this is impossible if f_r is to be analytic. Hence we see that $v(e^{i\theta})$ cannot be non-positive for every $\theta \in (0, \pi)$.

I now claim that $v(e^{i\theta}) > 0$ everywhere on $(0, \pi)$. For suppose that on the contrary there exists $\theta_0 \in (0, \pi)$ such that $v(e^{i\theta_0}) < 0$. Since we have already seen that $v(e^{i\theta})$ cannot be non-positive for every θ, it follows that it must be strictly positive at some $\theta = \theta_1 \in (0, \pi)$, and by continuity it follows then that there must be a $\theta \in (0, \pi)$ for which $v(e^{i\theta}) = 0$. We shall prove that this leads to a contradiction. For, let $x = u(e^{i\theta})$.

We have $x \in \overline{D_r}$. Let $y = \sup\{y \in \mathbb{R} : x + iy \in D_r\}$. If $y = -\infty$ then the vertical line at abscissa x never meets D_r and so x must either be the maximum or the minimum of $u(e^{i\varphi})$ so that $\theta \in \{0, \pi\}$ by the strictness of the symmetric decreasing character of u, and so we have a contradiction as $\theta \in (0, \pi)$. Since D_r is an open bounded domain symmetric under reflection in the real axis, it follows that $0 < y < \infty$. Moreover, $x + iy \in \overline{D}_r = \overline{f_r[\mathbb{D}]}$ so that there is a sequence $z_n \in \mathbb{D}$ such that $f_r(z_n) \to x + iy$. Passing to a subsequence if necessary, we may assume that z_n converges to some point z of $\overline{\mathbb{D}}$. By continuity of f_r on $\overline{\mathbb{D}}$ we have $f_r(z) = x + iy$. First suppose that $z \in \mathbb{D}$. Then, since f_r is non-constant it follows that $f_r(z) \notin \partial D_r$ since f_r is an open mapping, and we have a contradiction since $x + iy \in \partial D_r$. Suppose now that $z \in \partial\mathbb{D}$. Then $x + iy = u(z) + iv(z)$. Write $z = e^{i\varphi}$ for $\theta \in (-\pi, \pi]$. We have $x = u(e^{i\varphi})$. But $x = u(e^{i\theta})$. Since u is symmetrically strictly decreasing it follows that $\varphi = \pm\theta$. But $v(e^{i\theta}) = 0$ and $v(e^{-i\theta}) = 0$ likewise since $v(\bar{Z}) = -v(Z)$. Hence, $v(e^{i\varphi}) = 0$. Hence $y = 0$. But this is an immediate contradiction since we have already seen that $y > 0$.

Thus, indeed $v(e^{i\theta}) > 0$ on $(0, \pi)$, and $v(e^{i\theta}) < 0$ on $(-\pi, 0)$. Since u is symmetrically strictly decreasing it follows that $u + iv$ is one-to-one on $\mathbb{T}$. By Darboux's theorem it follows that $f_r = u + iv$ is univalent in $\mathbb{D}$. We must now show that D_r is Steiner symmetric about the real axis. We have already noted that D_r is reflection symmetric about the real axis. Fix $x + iy \in D_r$

with $y > 0$ and $x \in \mathbb{R}$. We must show that the line segment joining $x+iy$ with $x-iy$ lies in D_r. Let $y_1 = \sup\{y' : x+iy' \in D_r\}$. As before, we have $x+iy_1 \in \partial D_r$ and $x+iy_1 = f_r(z_1)$ for some $z_1 \in \mathbb{T}$. Moreover $y \in (0, y_1)$. To obtain a contradiction, suppose that there exists $y' \in (-y, y)$ such that $x+iy' \notin D_r$. Let $y_2 = \sup\{y' < y : x+iy' \notin D_r\}$. We then have $y_2 \in (-y, y)$ and $x+iy_2 \in \partial D_r$. As before, it follows that $x+iy_2 = f_r(z_2)$ for some $z_2 \in \mathbb{T}$. Then, it follows that $u(z_2) = u(z_1) = x$. Hence, $z_1 = z_2$ or $z_1 = \overline{z}_2$. In either case it follows that $|v(z_1)| = |v(z_2)|$ because of the reflection antisymmetry of v. Thus, $|y_1| = |y_2|$. But this contradicts the facts that $y \in (0, y_1)$ and $y_2 \in (-y, y)$.

Hence we see that D_r is indeed Steiner symmetric. $\square$

We now proceed to the proof of Theorem 5.1.

Proof of Theorem 5.1. Assume that $0 < \alpha < 2$, since the case $\alpha = 0$ is trivial, and the case $\alpha = 2$ can be obtained from the case $\alpha \in (0, 2)$ by taking the limit as $\alpha \to 2-$ and noting that then $||f||_{\mathfrak{d}_\alpha} \to ||f||_{\mathfrak{d}_2}$ for every fixed $f \in \mathfrak{d}_2$.

We use ideas from a proof of the Stein-Weiss interpolation theorem [22, Chapter 5]. For $f \in \mathfrak{d}_0$, define

$$\mathfrak{K}_2^2(f,t) \overset{\text{def}}{=} \inf_{f_0+f_2=f} ||f_0||_{\mathfrak{d}_0}^2 + t||f_2||_{\mathfrak{d}_2}^2,$$

where the infimum is taken over all decompositions $f = f_0 + f_2$ where $f_i \in \mathfrak{d}_i$ for $i \in \{0, 2\}$.

We compute an explicit formula for $\mathfrak{K}_2^2(f,t)$. To do this, note that elementary considerations show that for any fixed $w \geq 0$, $t > 0$ and $a \in \mathbb{R}$ the quantity

$$(a-x)^2 + twx^2$$

attains its minimum over $x \in \mathbb{R}$ precisely at

$$x = \frac{a}{1+tw}.$$

Let

$$\mathfrak{K} = \{\mathfrak{c}_n : n \in \mathbb{Z}^+\} \cup \{\mathfrak{s}_n : n \in \mathbb{Z}^+\}.$$

For $\mathfrak{k} \in \mathfrak{K}$, let $n(\mathfrak{k}) = m$ if $\mathfrak{k} = \mathfrak{c}_m$ or $\mathfrak{k} = \mathfrak{s}_m$. Then,

$$\|f\|_{\mathfrak{d}_\alpha}^2 = \sum_{\mathfrak{k} \in \mathfrak{K}} n^\alpha(\mathfrak{k})\mathfrak{k}^2(f).$$

Letting $w = n^2(\mathfrak{k})$, $a = \mathfrak{k}(f)$ and $x = \mathfrak{k}(f_2)$, it follows that the infimum in the definition of $\mathfrak{K}_2^2(f,t)$ is actually attained and

$$\mathfrak{K}_2^2(f,t) = \|f - f_2\|_{\mathfrak{d}_0}^2 + t\|f_2\|_{\mathfrak{d}_2}^2,$$

where for every $\mathfrak{k} \in \mathfrak{K}$ we have

$$\mathfrak{k}(f_2) = \frac{\mathfrak{k}(f)}{1 + tn^2(\mathfrak{k})},$$

so that

$$\mathfrak{k}(f_0) = \mathfrak{k}(f - f_2) = \frac{tn^2(\mathfrak{k})}{1 + tn^2(\mathfrak{k})}\mathfrak{k}(f).$$

(We use the short forms $n^p(\mathfrak{k}) = (n(\mathfrak{k}))^p$ and $\mathfrak{k}^p(f) = (\mathfrak{k}(f))^p$.)

Thus,

$$\begin{aligned} \mathfrak{K}_2^2(f,t) &= \sum_{\mathfrak{k} \in \mathfrak{K}} \frac{t^2n^4(\mathfrak{k}) + tn^2(\mathfrak{k})}{(1 + tn^2(\mathfrak{k}))^2}\mathfrak{k}^2(f) \\ &= \sum_{\mathfrak{k} \in \mathfrak{K}} \frac{tn^2(\mathfrak{k})\mathfrak{k}^2(f)}{1 + tn^2(\mathfrak{k})}. \end{aligned} \tag{5.3}$$

Suppose now that ϕ is a smooth non-negative function on $(0, \infty)$. Using Fubini's theorem and then making the change of variable $1 + tn^2(\mathfrak{k}) = u$ we obtain

$$\int_0^\infty \mathfrak{K}(f,t)\phi(t)\,dt = \sum_{\mathfrak{k} \in \mathfrak{K}} \mathfrak{k}^2(f) \int_1^\infty (u-1)u^{-1}\phi\left(\frac{u-1}{n^2(\mathfrak{k})}\right)n^{-2}(\mathfrak{k})\,du. \tag{5.4}$$

Now, let $\phi(t) = c_\theta t^{-1-\theta}$ for a positive constant c_θ. Then, it is easy to see that the right hand side of (5.4) will be of the form

$$\sum_{\mathfrak{k} \in \mathfrak{K}} C_\theta n^{2\theta}(\mathfrak{k})\mathfrak{k}^2(f) = C_\theta\|f\|_{\mathfrak{d}_{2\theta}}^2,$$

where

$$C_\theta = c_\theta \int_1^\infty (u-1)u^{-1}(u-1)^{-1-\theta}\,du.$$

It is easy to see that for $0 < \theta < 1$ this integral is finite. Then, choose c_θ so that $C_\theta = 1$, and so we will have

$$\int_0^\infty \mathfrak{K}_2^2(f,t)\phi(t)\,dt = \|f\|_{\mathfrak{D}_\alpha}^2, \tag{5.5}$$

where $\alpha = 2\theta$.

I now claim that for $0 < t < \infty$ we have $\mathfrak{K}_2^2(f^\circledcirc, t) \le \mathfrak{K}_2^2(f,t)$ with equality if and only if f is a rotation of $f^\circledcirc$. Once we prove this claim, we will be done by (5.5).

Rewrite (5.3) as

$$\begin{aligned}\mathfrak{K}_2^2(f,t) &= \sum_{\mathfrak{k}\in\mathfrak{K}} \left(\mathfrak{k}^2(f) - \frac{\mathfrak{k}^2 f}{1+tn^2(\mathfrak{k})}\right)\\ &= 2\|f\|_{L^2(\mathbb{T})}^2 - \sum_{\mathfrak{k}\in\mathfrak{K}} \frac{\mathfrak{k}^2 f}{1+tn^2(\mathfrak{k})}.\end{aligned}$$

Of course $\|f\|_{L^2(\mathbb{T})} = \|f^\circledcirc\|_{L^2(\mathbb{T})}$. Let

$$\gamma_t(f) = \sum_{\mathfrak{k}\in\mathfrak{K}} \frac{\mathfrak{k}^2 f}{1+tn^2(\mathfrak{k})},$$

for $0 < t < \infty$. To prove the claim we now need only show that $\gamma_t(f) \le \gamma_t(f^\circledcirc)$ with equality if and only if f is a rotation of $f^\circledcirc$. Define

$$K_t(e^{i\theta}) = \sum_{n=1}^\infty \frac{1}{1+tn^2}\cos n\theta.$$

Then,

$$\gamma_t(f) = 2\langle f, K_t * f\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{T})$. But it is not difficult to verify that

$$K_t(e^{i\theta}) = \frac{a\cosh a(\pi-\theta)}{4\sinh \pi a} - \frac{1}{2},$$

for $0 \le \theta \le 2\pi$, where $a = t^{-1/2}$. Thus, K_t is symmetrically strictly decreasing. Thus, the inequality $\gamma_0(f) < \gamma_0(f^\circledcirc)$ follows from Beckner's rearrangement theorem (Theorem I.6.4). $\square$

6. Baernstein's sub-Steiner rearrangement

We now define Baernstein's sub-Steiner rearrangement [12]. The terminology is slightly misleading in that it is actually not known whether this is a rearrangement in the sense of Definition I.2.2.

Let D be an H^p domain containing the origin for some $p > 1$. Let $f\colon \mathbb{D} \to D$ be a uniformizer with $f(0) = 0$ (see §I.5.5). Then $f \in H^p(\mathbb{D})$. Let $f^{\circledcirc}$ be the unique H^p function such that $f^{\circledcirc}(0) = 0$ and $\operatorname{Re} f^{\circledcirc}$ is the symmetric decreasing rearrangement of $\operatorname{Re} f$. Here we use the M. Riesz theorem (Theorem I.3.4) which asserts that the conjugate function of an $L^p(\mathbb{T})$ function lies in $L^p(\mathbb{T})$, for $p > 1$, while of course $\operatorname{Re} f^{\circledcirc} \in L^p(\mathbb{T})$ since $f \in L^p(\mathbb{T})$. In light of Proposition 5.1, the function $f^{\circledcirc}$ is univalent and its image is Steiner symmetric about the real axis. Let $D^{\mathrm{B}} = f^{\circledcirc}[\mathbb{D}]$. We call D^{B} the **Baernstein sub-Steiner rearrangement of** D.

Theorem 6.1. *Let D be a Greenian H^p domain containing the origin for some $p > 1$. Let ϕ be any Borel measurable function on D such that $\Gamma_\Phi(D)$ is defined for $\Phi(z) = \phi(\operatorname{Re} z)$. Then, $\Gamma_\Phi(D^{\mathrm{B}}) = \Gamma_\Phi(D)$.*

Proof. We have $f^{\circledcirc}(0) = 0$. Moreover, $f^{\circledcirc}$ is a uniformizer (in fact, it is a Riemann map since D^{B} is simply connected) and has Steiner symmetric image. Hence, by Corollary 1.1, the domain D^{B} is an H^p domain since $f^{\circledcirc} \in H^p$. In particular, it is a Nevanlinna domain. The image of $f^{\circledcirc}$ cannot be all of $\mathbb{C}$ (otherwise the inverse function $(f^{\circledcirc})^{-1}$ is a bounded non-constant entire function), and it is easy to see if that $z \in \partial D^{\mathrm{B}}$ then the Steiner symmetry guarantees that Theorem I.5.3 is applicable (just take $\theta(r) \equiv (\pi/2)\operatorname{sgn}\operatorname{Im} z$ if $\operatorname{Im} z \neq 0$, and $\theta(r) = \pi/2 \pm (\pi/2)\operatorname{sgn}\operatorname{Re} z$ if $\operatorname{Im} z = 0$; of course $z \neq 0$ since $0 \in D^{\mathrm{B}}$), and so D^{B} is Greenian. Hence, Theorem I.5.7 is applicable and

$$\Gamma_\Phi(D^{\mathrm{B}}) = \int_{\mathbb{T}} \Phi(f^{\circledcirc}) = \int_{\mathbb{T}} \phi(\operatorname{Re} f^{\circledcirc}) = \int_{\mathbb{T}} \phi(\operatorname{Re} f) = \int_{\mathbb{T}} \Phi(f) = \Gamma_\Phi(D),$$

providing that $\Gamma_\Phi(D)$ and $\Gamma_\Phi(D^{\mathrm{B}})$ are finite. (The first and last equality followed from Theorem I.5.7. The third equality followed from the definition of $f^{\circledcirc}$ and Proposition I.2.1.) In case one of $\Gamma_\Phi(D)$ and $\Gamma_\Phi(D^{\mathrm{B}})$ is infinite, then they both must be infinite and of the same sign, because we can approximate our Φ by bounded functions. $\square$

Let $\boxminus$ now indicate Steiner symmetrization about the real axis. Then, we know that $\Gamma_\Phi(D) \leq \Gamma_\Phi(D^\boxminus)$ whenever $\Phi(z) = \phi(\operatorname{Re} z)$ for a convex function ϕ (Theorem I.6.6). Since $\Gamma_\Phi(D) = \Gamma_\Phi(D^{\mathrm{B}})$ and D^{B} is Steiner symmetric, it would seem that D^{B} is some "smaller" version of the Steiner symmetrization. In fact, we have the following result, which provides an answer to a question of Baernstein [12] and justifies the name **sub-Steiner** as applied to the $(\cdot)^{\mathrm{B}}$ operation. The main result of the present section is as follows.

Theorem 6.2. *Let D be an H^p domain containing the origin for some $p > 1$. Then*

$$D^{\mathrm{B}} \subseteq D^\boxminus. \tag{6.1}$$

More generally, if f is any holomorphic function whose image is contained in D, then

$$f^{\circledast}[\mathbb{D}] \subseteq D^\boxminus.$$

Since $\operatorname{Area}(D^\boxminus) = \operatorname{Area}(D)$, we obtain the following Corollary.

Corollary 6.1. *Let D be an H^p domain for some $p > 1$ with $0 \in D$. Then,*

$$\operatorname{Area}(D^{\mathrm{B}}) \leq \operatorname{Area}(D).$$

Recall that $\mathcal{B}$ is the collection of all domains of area at most π which contain the origin.

Corollary 6.2. *Let ϕ be a measurable function on $\mathbb{R}$ such that Λ_Φ attains a maximum over $\mathfrak{B}$ where $\Phi(z) = \phi(\operatorname{Re} z)$. Then either $\Gamma_\Phi(D) \leq \Phi(0)$ for all $D \in \mathcal{B}$ or there is a univalent function $f \in \mathfrak{B}$ with Steiner symmetric image such that Γ_Φ attains a maximum over $\mathcal{B}$ at $f[\mathbb{D}]$.*

Note that unfortunately do not have any uniqueness result, even under some assumption such as SARIP. It would be nice, for instance, to be able to say that all domains at which Γ_Φ attains its maximum are of the form $f[\mathbb{D}]$ where f is as in the Corollary.

It would also be nice if we knew that if Λ_Φ attains a maximum over $\mathfrak{B}$ at $f \in \mathfrak{B}$ then Γ_Φ attains its maximum at $f[\mathbb{D}]$.

Note that it is easy to see that the condition $\Gamma_\Phi(D) \leq \Phi(0)$ for all $D \in \mathcal{B}$ cannot happen if, for instance, ϕ has a minimum at 0 and is not constant. (Just choose D a domain of area π which has the property that it reaches far enough so that a part of its boundary with strictly positive harmonic measure lies in $\{z : \Phi(z) > \Phi(0)\}$ so that $\Gamma_\Phi(D)$, being a weighted average of Φ with the weight given by harmonic measure, must be strictly larger than $\Phi(0)$.)

Proof of Corollary 6.2. Let $f_1 \in \mathfrak{B}$ be such that Λ_Φ attains its maximum at f_1. Assume first that $f_1 \not\equiv 0$. Let $f = f_1^{\circledcirc}$. Then f is univalent with Steiner symmetric image. Moreover, $f \in \mathfrak{B}$ (Corollary 5.1) so that $f[\mathbb{D}] \in \mathcal{B}$. I claim that Γ_Φ attains a maximum over $\mathcal{B}$ at $f[\mathbb{D}]$. To see this, let $D \in \mathcal{B}$. Then,

$$\Gamma_\Phi(D^{\mathrm{B}}) = \Gamma_\Phi(D).$$

Let h be the Riemann map from $\mathbb{D}$ onto D^{B} with $h(0) = 0$. By Theorem 1.2 we have

$$\Gamma_\Phi(D^{\mathrm{B}}) = \Lambda_\Phi(h).$$

But $\operatorname{Area} D^{\mathrm{B}} \leq \operatorname{Area} D \leq \pi$ so that $h \in \mathfrak{B}$ as h is univalent and has image area at most π. Thus,

$$\Lambda_\Phi(h) \leq \Lambda_\Phi(f).$$

But f is univalent also, so that

$$\Lambda_\Phi(f) \leq \Gamma_\Phi(f[\mathbb{D}]).$$

Putting all the preceding displayed inequalities together we see that

$$\Gamma_\Phi(D) \leq \Gamma_\Phi(f[\mathbb{D}]),$$

and we conclude that indeed Γ_Φ attains a maximum at $f[\mathbb{D}]$.

Suppose now that $f_1 \equiv 0$. Then, Λ_Φ is bounded above by $\Phi(0)$. The same argument as above then shows that Γ_Φ is bounded above by $\Phi(0)$. $\square$

To prove Theorem 6.2 we now define the Nevanlinna counting function. Given an analytic function F on $\mathbb{D}$ and $r \in (0,1)$, let $n(r, w; F)$ be the number of solutions $z \in \overline{\mathbb{D}}(r)$ of the equation $w = F(z)$, *counting multiplicities.* The **Nevanlinna counting function** then is:

$$N_F(w) = \int_0^1 n(r, w; F)\, r^{-1} dr.$$

Remark 6.1. The function N_F vanishes at $w \in \mathbb{C}$ if and only if $w \notin F[\mathbb{D}]$.

We recall the following very useful theorem of Baernstein. Use $\boxminus$ to denote Steiner symmetrization about the real axis. Write $N_F^{\boxminus}$ for $(N_F)^{\boxminus}$. The function $f^{\circledcirc}$ is defined as before.

Theorem 6.3 (Baernstein [8]). *Let $F \in H^1(\mathbb{D})$. Then,*

$$\int_{-Y}^{Y} N_F^{\boxminus}(x+iy)\, dy \le \int_{-Y}^{Y} N_{F^{\circledcirc}}(x+iy)\, dy, \tag{6.2}$$

for every $Y \in (0, \infty]$, with equality for $Y = \infty$.

Assume this Theorem. The following Lemma is also useful in conjunction with it.

Lemma 6.1. *Let $F \in H^p(\mathbb{D})$ for some $p > 1$. Then,*

$$\int_{-\infty}^{\infty} N_F(x+iy)\, dy < \infty,$$

for every $x \in \mathbb{R}$.

Proof of Lemma. Note that

$$\int_{-\infty}^{\infty} N_F(x+iy)\, dy = \int_{-\infty}^{\infty} N_F^{\boxminus}(x+iy)\, dy.$$

This follows by the fact that (for fixed x) $\boxminus$ is a measure preserving rearrangement when restricted to $x + i\mathbb{R}$ (with respect to one-dimensional Lebesgue measure) and we can apply Proposition I.2.1(ii). Hence since Theorem 6.3 guarantees equality for $Y = \infty$ in (6.2), we need only prove the lemma for $F^{\circledcirc}$ in place of F. Let $G = F^{\circledcirc}$. Then $G \in H^p$ since $F \in H^p$. As in the proof of Theorem 6.1, we have G a univalent function onto a Steiner symmetric Greenian domain. Define

$$\Phi(z) = \max(0, (\operatorname{Re} z) - x).$$

This is a subharmonic function (Theorem I.4.3) and we may write

$$\mathrm{LHM}(z, \Phi; D) = \Phi(z) + \int_D g(z, w; D)\, d\mu_\Phi(w).$$

Now, by Theorem I.5.11 we then have

$$\mathrm{LHM}(z, \Phi; D) = \Phi(z) + c \int_{-\infty}^{\infty} g(z, x + iy; D)\, dy, \tag{6.3}$$

for some strictly positive constant c (we used here the fact that $g(z, \cdot; D)$ vanishes outside D to extend the range of integration from $\{y : x + iy \in D\}$ to all of $\mathbb{R}$). By Theorems III.1.2 and I.5.4 we have

$$\mathrm{LHM}(0, \Phi; D) = \Gamma_\Phi(D) = \Lambda_\Phi(G) = \int_{\mathbb{T}} \max(0, (\mathrm{Re}\, G) - x).$$

The right hand side is finite since $G \in H^p(\mathbb{D})$ for some $p > 1$ so that $G \in L^p(\mathbb{T})$ (all we need is $G \in H^1$ here in fact). Since $\Phi(0)$ is finite, it follows from (6.3) that

$$\int_{-\infty}^{\infty} g(z, x + iy; D)\, dy < \infty. \tag{6.4}$$

But now G is univalent. Thus,

$$n(r, w; G) = 1_{\{r > |G^{-1}(w)|\}},$$

and so

$$N_G(w) = \log \frac{1}{|G^{-1}(w)|}.$$

Hence $N_G(w) = g(w, 0; D)$ by Theorem I.5.8. Since $g(w, 0; D) = g(0, w; D)$, we are done by (6.4). □

Proof of Theorem 6.2. We need only prove the "Moreover", since the rest follows from the special case where f is a uniformizer of $\mathbb{D}$ sending 0 to 0. Fix $x_0 \in \mathbb{R}$. Let $L_0 = \{x_0 + iy : y \in \mathbb{R}\}$. Put $L = L_0 \cap D^{\mathsf{B}}$ and $M = L_0 \cap D^{\boxminus}$. Write λ_1 for one-dimensional Lebesgue measure. We must prove that $L \subseteq M$. Because both D^{B} and $D^{\boxminus}$ are Steiner symmetric, it suffices to prove that $\lambda_1(L) \leq \lambda_1(M)$.

If $\lambda_1(M)$ is infinite, then we are done. Hence, suppose that $Y_1 \stackrel{\text{def}}{=} \frac{1}{2}\lambda_1(M)$ is finite. Since N_f vanishes outside D it follows that N_f can be nonzero on a set of measure at most $2Y_1$, and hence $N_f^{\boxminus}$ vanishes on $L\backslash\{x_0 + iy : |y| > Y_1\}$ by definition of the Steiner rearrangement. But we have equality in (6.2) for $Y = \infty$ so that it follows that

$$\int_{-Y_1}^{Y_1} N_f^{\boxminus}(x+iy)\,dy = \int_{-\infty}^{\infty} N_{f^{\circledcirc}}(x+iy)\,dy.$$

On the other hand,

$$\int_{-Y_1}^{Y_1} N_f^{\boxminus}(x+iy)\,dy \leq \int_{-Y_1}^{Y_1} N_{f^{\circledcirc}}(x+iy)\,dy,$$

by (6.2) with $Y = Y_1$. Since $N_{f^{\circledcirc}} \geq 0$ and by Lemma 6.1 all our integrals are finite, it follows that

$$\int_{|y|>Y_1} N_{f^{\circledcirc}}(x+iy)\,dy = 0.$$

Now, L is precisely the set of points of L_0 at which $N_{f^{\circledcirc}}$ is non-zero. It follows that $\lambda_1(L) \leq 2Y_1 = \lambda_1(M)$ as desired. $\square$

Chapter IV
Radial rearrangement

Overview

In this chapter, "domain" shall mean "Greenian domain".

Our interest in this chapter is a conjecture of Matheson and Pruss [75] to the effect that there exists a way of replacing a domain D by a simply connected and star-shaped domain $\tilde{D}$ of not bigger area such that the Γ_Φ functionals are increased by the replacement, where Φ is a continuous function on $[0, \infty)$ such that $t \mapsto \Phi(e^t)$ is increasing and convex. A positive answer to this conjecture would allow for a tighter connection between the Λ_Φ and Γ_Φ functionals. Unfortunately, we do not manage to obtain an answer to the conjecture, although we do get some partial results and some interesting evidence.

We state the relevant conjecture and discuss it in §1.1. In §1.2 we discuss some consequences of our conjectures. Then, in §1.3, we state a conjecture (Conjecture 1.4) as to how we think $\tilde{D}$ should be defined via circular symmetrization and Marcus' radial rearrangement [70], and we discuss a few cases in which the conjecture holds. We state a weaker conjecture concerning harmonic measures (Conjecture 1.6). We also give a few counterexamples contradicting some possible extensions of our various conjectures.

It is unfortunate that all our main conjectures remain open. However, in §2 we do give some partial results. For instance, we show that the harmonic measure functionals w_r are increased by our conjectured choice of $\tilde{D}$ providing the original D is simply connected (Theorem 2.3). The proof uses simple connectivity in an essential way.

In §3 we discuss our problems as transferred to the cylinder. This makes the constructions and conjectures a little more intuitive. We also discuss two-sided lengthwise Steiner symmetrization, and obtain a partial result (Theorem 3.1) analogous to our Conjecture 1.6. Finally, we discuss a "cutting" operation which one might conjecture to increase the w_r functionals for circularly symmetric domains.

In §4 we discuss a formulation of our Conjecture 1.4 in terms of Green's functions, and discuss why our Conjecture 1.6 on harmonic measures is weaker than Conjecture 1.4, and how Marcus' result [70] on the increase of the inner radius under radial rearrangement is connected with our Conjecture 1.4. Then, in §5 we generalize Marcus' above-mentioned result, proving our Conjecture 1.4 in the special case of $t \mapsto \frac{\partial}{\partial t}\Phi(e^t)$ being concave.

In §6 we discuss a one-sided Steiner rearrangement due to Haliste [56] and state her result on the effect of this rearrangement on harmonic measures (Theorem 6.1).

Then, in §7, we discuss a Brownian motion formulation of our conjectures. Having done so, we state a rather natural one-dimensional discrete version of Conjecture 1.6 which we shall prove in §9. We note that while this one-dimensional discrete version holds not only for a simple random walk, but also for a random walk with probability p of going to the right and $1-p$ of going to the left, this is not the case with Conjecture 1.6. Indeed, in §7.1 we prove that the analogue of the cylindrical Brownian motion version of Conjecture 1.6 fails when the lengthwise component of the Brownian motion is replaced by a uniform motion to the right. This seems to provide some evidence against Conjecture 1.6, although the particular counterexample domain produced here does have Conjecture 1.6 satisfied because of simple connectivity and Theorem 2.3. Finally, in §7.2 we prove that a somewhat natural conjecture about Brownian motion exit times, analogous to Conjecture 1.4, is false. However, this too does not really provide evidence against Conjecture 1.4, but is mainly a consequence of the fact that radial rearrangement does not preserve areas.

Most of the material in sections 1–7 is taken from the author's paper [85].

In §8 we prove extensions of the Beurling shove theorem on the harmonic measure of slit discs [23, pp. 58–62]. In particular, we shall prove Conjecture 1.6 in the case of a circularly symmetric and starshaped domain slit along the negative real axis. Our extensions of the Beurling shove theorem in the 2-dimensional case go further than the extensions of Essén and Haliste [45]; however, their methods are also valid in higher dimensions, while confine ourselves to domains in $\mathbb{R}^2$. Parts of §8 are taken from the author's paper [84].

Then, in §9 we give the discrete one-dimensional version of Conjecture 1.6 and prove it. We also give a few other results. The discrete one-dimensional result is actually completely elementary, as are its proofs (broken up between sections 9.2 and 9.3). The proofs are based on an explicit formula (Theorem 9.6) for the probability of a random walk surviving various dangers on its route so as to exit a blind alley.

Our results in §9 are similar to a discrete result of Essén [43], and in §9.4 we discuss the connection with a continuous result of Essén [42]. All the material in §9, excepting Theorem 9.2, is taken from the author's paper [87].

Finally, in §10 we prove a partial result concerning a horizontal-convexity analogue of a weaker version of the Matheson-Pruss Conjecture 1.1. This result provides some evidence for that conjecture. Our proof uses the variational equation for extremals of Λ_Φ given in §III.4.4. It also uses Baernstein's sub-Steiner rearrangement from §III.5, though the proof could also be done with standard Steiner rearrangement and Steiner analogues of the results of Baernstein [7].

On first reading, the reader may wish to omit sections 5, 6 and 9.3, the details of the constructions in sections 7.1 and 7.2, as well as some of the details in the proof of Lemma 10.1 of §10.1.

1. Conjectures and counterexamples

1.1. The primary conjectures

Throughout this chapter, let $\mathcal{F}$ be the collection of all functions Φ on $[0, \infty)$ such that $t \mapsto \Phi(e^t)$ is convex increasing and Φ is continuous at 0. By Theorem I.4.4, $\mathcal{F}$ is precisely the set of functions Φ such that $z \mapsto \Phi(|z|)$ is subharmonic on $\mathbb{C}$.

Consider the following conjecture.

Conjecture 1.1 (Matheson and Pruss [75]). For any domain U containing the origin and of finite area there exists a star-shaped domain $\tilde{U}$ with $\operatorname{Area}\tilde{U} \leq \operatorname{Area} U$ such that $\Gamma_\Phi(\tilde{U}) \geq \Gamma_\Phi(U)$ for every $\Phi \in \mathcal{F}$.

A weaker conjecture but still very much of interest would be as follows.

Conjecture 1.2. For any domain U containing the origin and of finite area and any fixed $\Phi \in \mathcal{F}$ there exists a star-shaped domain $\tilde{U}$ with $\operatorname{Area}\tilde{U} \leq \operatorname{Area} U$ and $\Gamma_\Phi(\tilde{U}) \geq \Gamma_\Phi(U)$.

The difference between Conjectures 1.1 and 1.2 is that the latter allows the domain U to depend on the choice of the particular function $\Phi \in \mathcal{F}$.

Recall that if $\tilde{U}$ is star-shaped then it is simply connected (this is easy to verify directly since $\tilde{U}$ is then contractible).

Unfortunately, Conjecture 1.2 and *a fortiori* Conjecture 1.1 are still open.

1.2. Consequences of a positive answer to Conjecture 1.2

Let us formulate an even weaker version of Conjecture 1.2. It will be weaker because star-shaped domains are automatically simply connected.

Conjecture 1.3. For any domain U containing the origin and of finite area and any $\Phi \in \mathcal{F}$ there exists a simply connected $\tilde{U}$ with $\operatorname{Area}\tilde{U} \leq \operatorname{Area} U$ and $\Gamma_\Phi(\tilde{U}) \geq \Gamma_\Phi(U)$.

This conjecture is still open, too.

Let $\mathcal{B}$, as before, be the set of all domains of area at most π which contain 0.

Proposition 1.1. *Suppose that Conjecture 1.3 is valid for some $\Phi \in \mathcal{F}$ such that $\Phi(t) = o(e^{t^2})$ as $t \to \infty$. Then Γ_Φ attains its supremum over $\mathcal{B}$. Moreover, there exists a simply connected extremal domain in $\mathcal{B}$ at which Γ_Φ is maximized.*

Proof. By Theorem 3.4, there exists $f \in \mathfrak{B}$ such that $\Lambda_\Phi(f) \geq \Lambda_\Phi(g)$ for all $g \in \mathfrak{B}$. Let $D = f[\mathbb{D}]$. Let U be an arbitrary domain in $\mathcal{B}$. Then by Theorem III.1.2 and Conjecture 1.3 we have

$$\Gamma_\Phi(U) \leq \Gamma_\Phi(\tilde{U}) = \Lambda_\Phi(g),$$

where g is a Riemann map from $\mathbb{D}$ onto U with $g(0) = 0$. But $\operatorname{Area} U \leq \pi$ so $g \in \mathfrak{B}$ and hence $\Lambda_\Phi(g) \leq \Lambda_\Phi(f)$. By Theorem III.1.2 we then have $\Lambda_\Phi(f) \leq \Gamma_\Phi(D)$, and so $\Gamma_\Phi(U) \leq \Gamma_\Phi(D)$ for all $U \in \mathcal{B}$. Hence Γ_Φ attains its maximum over $\mathcal{B}$. Moreover, if it attains this maximum at D, it likewise attains it at $\tilde{D}$ and hence there exists an extremal simply connected domain. □

Sakai [95] had conjectured that Γ_{Φ_p} attains a maximum over $\mathcal{B}$ where $\Phi_p(t) = t^p$ for $0 < p < \infty$. Hence, an affirmative answer to Conjecture 1.3 implies an answer to Sakai's conjecture.

Proposition 1.2. *Let $\Phi_p(t) = t^p$. Then $\Gamma_{\Phi_p}(D) \leq \Gamma_\Phi(\mathbb{D})$ for every $D \in \mathcal{B}$ providing $p \in [0, 2]$. If Conjecture 1.3 holds for Φ_p then this is also true for $p \in (2, 4]$.*

Proof. The case of $p \in [0, 2]$ is the well-known Alexander-Taylor-Ullman inequality [3] (see Kobayashi [66] for another proof). (More precisely, the Alexander-Taylor-Ullman inequality is the case $p = 2$, and, as Sakai [95] notes, the case $p < 2$ follows from Hölder's inequality.)

The case $p \in (2, 4]$ follows from Conjecture 1.3 and the fact that the inequality is valid for simply connected domains D. To see the validity for simply connected domains, it suffices to use Theorem III.1.2 and the fact that $\Lambda_{\Phi_p}(f) \leq \Lambda_{\Phi_p}(\mathrm{Id})$ for $p \in [0, 4]$, where Id is the identity function and f is any function in $\mathfrak{B}$. This latter inequality has been proved by Matheson [74]. Professor

Sakai has kindly informed the author that the desired inequality in the simply connected case was also obtained by Professors N. Suita and S. Kobayashi. □

The inequality in the above proposition was conjectured for $p \in (2,4]$ by Sakai [95]. Hence, an affirmative answer to Conjecture 1.3 would imply an affirmative answer to yet another conjecture of Sakai. As can be seen, it would also simplify the proof of the Alexander-Taylor-Ullman inequality, since the inequality $\Lambda_{\Phi_p}(f) \le \Lambda_\Phi(\mathrm{Id})$ for $f \in \mathfrak{B}$ is quite easy to prove for $p \in [0,2]$. Indeed, it suffices to prove the latter inequality for $p = 2$ since the general case then follows by Hölder's inequality. But for $p = 2$ the inequality is essentially trivial since then $\Lambda_{\Phi_2}(f) = \sum_{n=1}^\infty |\hat{f}(n)|^2$ while $\|f\|_{\mathfrak{D}}^2 = \sum_{n=1}^\infty n|\hat{f}(n)|^2 \le 1$, which easily shows that Λ_{Φ_2} attains its maximum over $\mathfrak{B}$ precisely at the functions of the form $c \cdot \mathrm{Id}$ where $|c| = 1$.

Finally, we note that if Conjecture 1.2 holds, then the Essén inequality

$$\sup_{D \in \mathcal{B}} \Gamma_\Phi(D) < \infty$$

for $\Phi(t) = e^{t^2}$ follows from the Chang-Marshall inequality

$$\sup_{f \in \mathfrak{B}} \Lambda_\Phi(f) < \infty.$$

Indeed, it follows from the Chang-Marshall inequality for univalent functions. Since the proof of the Chang-Marshall inequality given in [72] is much simpler (and it also simplifies considerably in the univalent case) than the proof of Essén's inequality [44], we see that an affirmative answer to Conjecture 1.2 would imply a simpler proof of Essén's inequality.

1.3. Radial rearrangement

A tool which one would think is very natural for attacking Conjecture 1.1 is Marcus' radial rearrangement [70]. The author is grateful to Professor Albert Baernstein II for having suggested the use of Marcus' radial rearrangement for this purpose. Let U be a set in the plane containing a neighbourhood of the origin and choose $\varepsilon > 0$ such that $\mathbb{D}(\varepsilon) \subseteq U$, where $\mathbb{D}(r)$ indicates an open disc of radius r about 0. Define

$$R_\varepsilon(\theta; U) = \int_\varepsilon^\infty 1_{\{\rho e^{i\theta} \in U\}} \rho^{-1} d\rho.$$

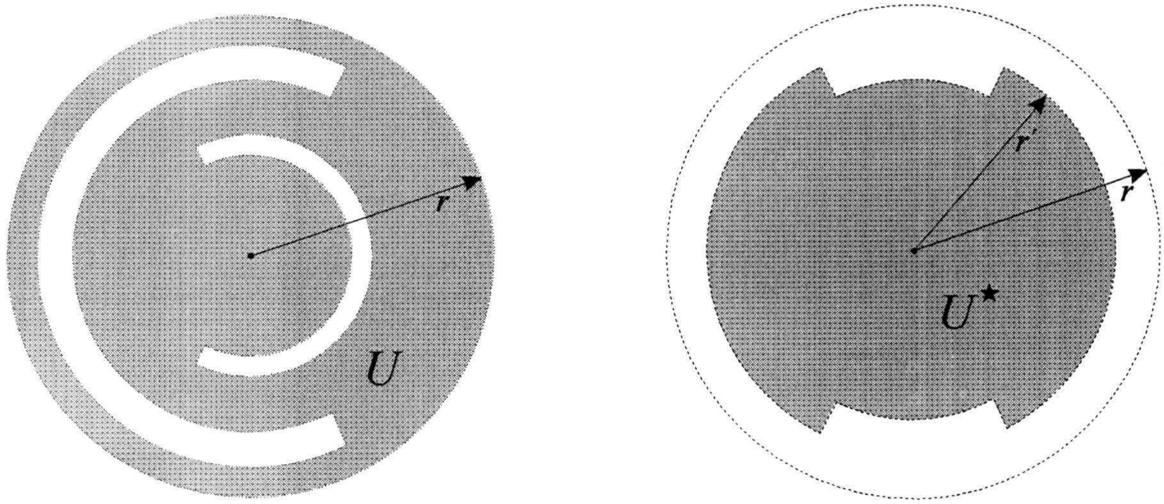


Figure 1.1: A multiply connected domain for Example 1.1, together with its radial rearrangement.

Then the set

$$U^{\bigstar} = \left\{ re^{i\theta} : \int_{\varepsilon}^{r} \rho^{-1} d\rho < R_{\varepsilon}(\theta; U) \right\}$$

is called the (Marcus) **radial rearrangement** of U. Observe that we are using a logarithmic metric in the definitions. Note that $\mathbb{D}(\varepsilon) \subseteq U^{\bigstar}$ and that if U is open then $\theta \mapsto R_{\varepsilon}(\theta; U)$ is lower semicontinuous (cf. Proposition I.6.1) and hence $U^{\bigstar}$ is open. It is easy to verify that $U^{\bigstar}$ is independent of the choice of ε and that $\text{Area}(U^{\bigstar}) \leq \text{Area}(U)$ with equality if and only if $U^{\bigstar}$ and U coincide almost everywhere with respect to Lebesgue area measure. Note that $U^{\bigstar}$ is always star-shaped and that $U^{\bigstar} = U$ if and only if U is star-shaped.

Finally, one may observe that $\bigstar$ is a rearrangement in the sense of Definition I.2.2 on the σ-pseudotopology of all measurable subsets of $\mathbb{C}$ containing a neighbourhood of the origin.

Marcus [70] has shown that $\bigstar$ decreases neither inner radii (see inequality (4.2), below) nor certain capacities.

Example 1.1. One might naturally conjecture that $U^{\bigstar}$ has the desired property that $\Gamma_{\Phi}(U^{\bigstar}) \geq \Gamma_{\Phi}(U)$. However, this is not the case. Let U be any domain with the following properties for some positive r and δ, where ε is as before:

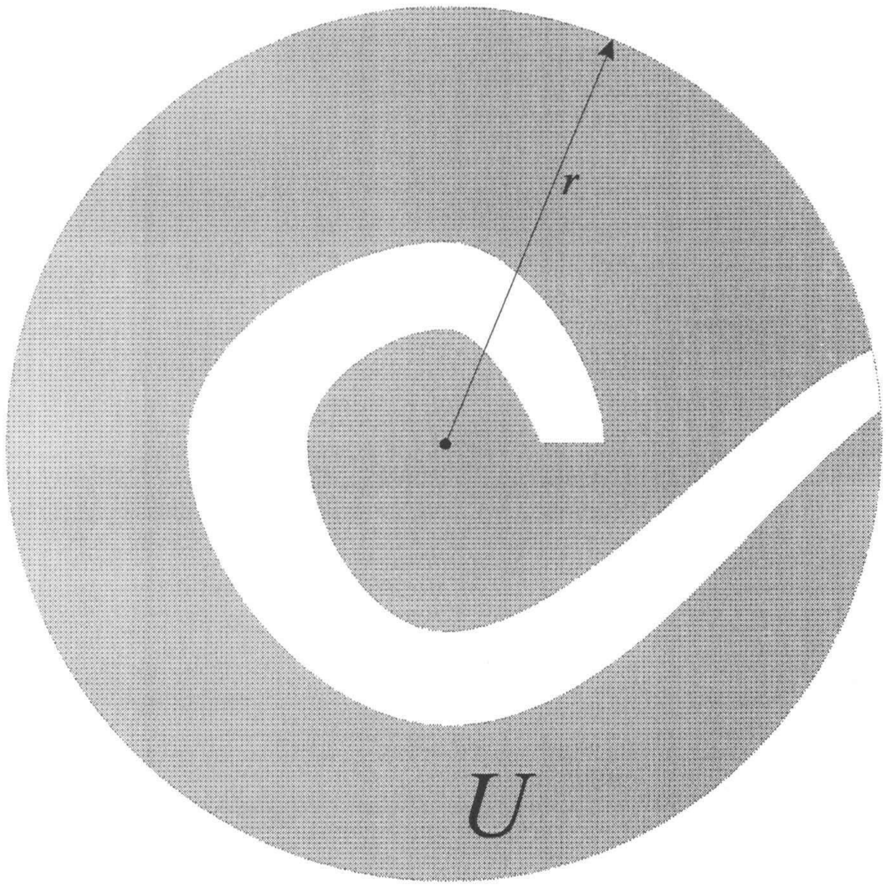


Figure 1.2: A simply connected domain for Example 1.1. The radial rearrangement of this domain will be contained in some disc $\mathbb{D}(r')$ for $0 < r' < r$.

(i) The harmonic measure of $\{|z| = r\} \cap \partial(\mathbb{D}(r) \cap \bar{U})$ at zero in U does not vanish.

(ii) $R_\varepsilon(\theta; U) \leq \int_\varepsilon^{r-\delta} \rho^{-1} d\rho$ for every θ.

Such a domain can easily be exhibited; see Figure 1.1 (left) for a multiply-connected example, and Figure 1.2 for a simply-connected example.

Given such a domain, (ii) implies that we will have $U^\star \subseteq \mathbb{D}(r')$ where $r' = r - \delta$ (see Figure 1.1, right). Choose $x \in (r', r)$. Define $\Phi(t) = \max(0, t - x)$. Then Φ is convex, hence in $\mathcal{F}$, and it is easy to verify that $\Gamma_\Phi(U) > 0$ by (i). On the other hand, as $U^\star \subseteq D(r')$ we must have $\Phi(|z|)$ vanishing on the closure of $U^\star$ so that $\Gamma_\Phi(U^\star) = 0$, and thus radial rearrangement by itself does not give a solution to Conjecture 1.1.

Now, given a domain U, let $U^{\circledcirc}$ be its circular symmetrization (see §I.6.1). We have $\mathrm{Area}(U^{\circledcirc}) = \mathrm{Area}(U)$. And $\Gamma_\Phi(U^{\circledcirc}) \geq \Gamma_\Phi(U)$ for every $\Phi \in \mathcal{F}$ by Corollary I.6.1.

Note that if U is circularly symmetric then so is $U^\star$ though the converse does not in general hold. It is easy to see that no domain satisfying (i) and (ii) of Example 1.1 can be circularly symmetric.

Conjecture 1.4. Confined to the class of circularly symmetric domains, radial rearrangement does not decrease any of the functionals Γ_Φ for $\Phi \in \mathcal{F}$.

An affirmative answer to this question would give an affirmative answer to Conjecture 1.1 since we could then let $\tilde{U} = (U^{\circledcirc})^\star$. While Conjecture 1.4 is in general open, we can prove it for a very simple class of domains.

Example 1.2. Consider the closed polar rectangle

$$H = \{re^{i\theta} : r_1 \leq r \leq r_2, |\theta - \pi| \leq \theta_0\},$$

where $0 < r_1 < r_2 \leq 1$ and $\theta_0 < \pi$. Let

$$U = \mathbb{D} \setminus H.$$

Then U is circularly symmetric. It is easy to verify that

$$U^\star = \mathbb{D} \backslash H',$$

where

$$H' = \left\{ re^{i\theta} : \frac{r_1}{r_2} \leq r \leq 1, |\theta - \pi| \leq \theta_0 \right\}.$$

Letting $\lambda = r_2^{-1}$, we see that we may also write

$$U^\star = \mathbb{D} \backslash \lambda H,$$

with $\lambda \geq 1$, and where $\lambda H = \{\lambda z : z \in H\}$.

Fix $r \leq 1$. We then have

$$\begin{aligned} \omega(0, (\partial U) \cap \{|z| < r\}; U) &= \omega(0, (\partial H) \cap \{|z| < r\}; \mathbb{D} \backslash H) \\ &\geq \omega(0, (\partial H) \cap \{|z| < r\}; \mathbb{D}(\lambda^{-1}) \backslash H) \\ &\geq \omega(0, (\partial H) \cap \{|z| < \lambda^{-1} r\}; \mathbb{D}(\lambda^{-1}) \backslash H) \\ &= \omega(0, (\partial \lambda H) \cap \{|z| < r\}; \mathbb{D} \backslash \lambda H), \end{aligned} \tag{1.1}$$

where we have used the monotonicity of $\omega(z, A; V)$ with respect to A and V (Remark I.5.3), together with invariance under scaling. Now, let

$$v(r; V) = \omega(z, \{|z| \geq r\}; V). \tag{1.2}$$

It is not difficult to see that if $\Phi \in \mathcal{F}$ and h is the least harmonic majorant of $z \mapsto \Phi(|z|)$ in a domain V, then

$$h(0) = \Phi(0) + \int_0^\infty \Phi'(r) v(r; V) \, dr.$$

From (1.1) it follows that $v(r; U) \leq v(r; U^\star)$ and thus it follows that $\Gamma_\Phi(U) \leq \Gamma_\Phi(U^\star)$ as desired.

In the example above, we had a stronger conclusion than simply increase of the Γ_Φ for $\Phi \in \mathcal{F}$. This is contained in the following conjecture.

Conjecture 1.5. (Known to be false!) If U is circularly symmetric then

$$v(r;U^\star) \geq v(r;U)$$

for every $r \geq 0$, where $v(\cdot;\cdot)$ is defined by (1.2).

To see that this is false, let

$$U = \mathbb{D}\setminus\left((-1,-\tfrac{1}{3}]\cap\partial\mathbb{D}(\tfrac{1}{2})\right).$$

Evidently U is circularly symmetric and $U^\star = \mathbb{D}\setminus(-1,-\frac{1}{3}]$. Set $r=\frac{1}{2}$. Clearly,

$$v(\tfrac{1}{2};U) = \omega\left(0,\partial\mathbb{D}(\tfrac{1}{2});\mathbb{D}(\tfrac{1}{2})\setminus(-\tfrac{1}{2},-\tfrac{1}{3}]\right).$$

Define

$$g(z) = \omega\left(z,\partial\mathbb{D}(\tfrac{1}{2});\mathbb{D}(\tfrac{1}{2})\setminus(-\tfrac{1}{2},-\tfrac{1}{3}]\right),$$

and

$$h(z) = \omega\left(z,\{|z|\geq\tfrac{1}{2}\}\cap\partial U^\star\right).$$

Then, it is easy to see that $h(z)<1$ for every $z\in\mathbb{T}(\frac{1}{2})\setminus\{-\frac{1}{2}\}$ and that $h(z)=0$ for every $z\in(-\frac{1}{2},-\frac{1}{3}]$. On the other hand $g(z)=1$ for each $z\in\partial D(\frac{1}{2})$ and again $g(z)=0$ for $z\in(-\frac{1}{2},-\frac{1}{3}]$. Hence the maximum principle applied in the domain $\mathbb{D}(\frac{1}{2})\setminus(-\frac{1}{2},-\frac{1}{3}]$ implies that $h(z)<g(z)$ for every z in this domain. In particular $v(\frac{1}{2};U)=g(0)>h(0)=v(\frac{1}{2};U^\star)$. Of course, one might say that this is not really a counterexample to Conjecture 1.5 since U is not connected and domains are usually taken to be connected. But we can make U connected! Fix a small $\varepsilon>0$. Let

$$U_\varepsilon = \mathbb{D}\setminus\left((-1,-\tfrac{1}{3}]\cup\{\tfrac{1}{2}e^{i\theta}:\varepsilon\leq\theta\leq 2\pi-\varepsilon\}\right).$$

The U_ε are connected and circularly symmetric for every positive ε. But as $\varepsilon\to 0$, we have $v(\frac{1}{2};U_\varepsilon)\to v(\frac{1}{2};U)$, and certainly $U_\varepsilon^\star = \mathbb{D}\setminus(-1,-\frac{1}{3}] = U^\star$, so that for sufficiently small ε we have $v(\frac{1}{2};U_\varepsilon) > v(\frac{1}{2};U^\star) = v(\frac{1}{2};U_\varepsilon^\star)$ and we truly have a counterexample.

We now present the following *bona fide* Conjecture.

Conjecture 1.6. Let U be a circularly symmetric domain, and let $U_r=\mathbb{D}(r)\cap U$. Let $w_r(U) = \omega(0,\partial\mathbb{D}(r)\cap\partial U_r;U_r)$. Then $w_r(U)\leq w_r(U^\star)$.

Note that by Theorem I.6.2 (which is due to Baernstein [7]), an analogue of this for circular symmetrization holds, i.e., $w_r(U) \le w_r(U^{\circledast})$ for any domain U containing the origin.

Since $w_r(U) = v_r(U_r)$, Conjecture 1.6 is weaker than the false Conjecture 1.5. It is also weaker than Conjecture 1.4, as will be seen in §4. However, Conjecture 1.6 has the one advantage that, as we shall see below in Theorem 2.3, it is actually known to be true when the domain U is simply connected. It is also known to be true in the case where U is of the form $\mathbb{D}\backslash I$ where I is a finite union of closed intervals on the negative real axis. This fact is known as Beurling's shove theorem and is proved in his thesis [23, pp. 58–62]; an account of this may also be found in [79, §IV.5.4], and a more general result also valid in higher dimensions was obtained by Essén and Haliste [45]. We note here that in fact Beurling's method of proof readily also gives the following result which is a full answer to Conjecture 1.4 in the case of the slit disc mentioned above. A proof of a generalization will be given in §8.

Theorem 1.1. *Let I be a finite union of closed intervals in $[-1, 0)$. Then*

$$\Gamma_\Phi(\mathbb{D}\backslash I) \le \Gamma_\Phi((\mathbb{D}\backslash I)^\star)$$

for any $\Phi \in \mathcal{F}$.

One possible approach to proving Conjecture 1.6 given the solution in the simply connected case would be to attempt to reduce a general circularly symmetric domain to a simply connected one by using a shoving argument like Beurling's, this time moving holes which are circularly symmetric about the negative real axis instead of intervals. We shall make this idea precise in Problem 3.1 of §3. However, Beurling's proof does not seem to go through directly for this case; the reader cognizant of Beurling's proof will recognize that the difficulty is that the various monotonicity properties of the Green's functions used in the proof are no long applicable when the holes are not contained in the negative real axis.

Finally we remark that if the full Conjecture 1.6 were known to be true, then the proof of Essén's inequality [44] would be greatly simplified.

2. Some positive results

The following result, which will be proved in Remark 8.5 of §8, generalizes Theorem 1.1, above.

Theorem 2.1. *Let D be a star-shaped and circularly symmetric domain. Let I be a finite union of closed intervals in $(-\infty, 0)$. Let $U = D \backslash I$. Then $\Gamma_\Phi(U) \leq \Gamma_\Phi(U^\bigstar)$ for all $\Phi \in \mathcal{F}$.*

The following result is a special case of Conjecture 1.4.

Theorem 2.2. *Let $\Phi \in \mathcal{F}$ be such that $t \mapsto \frac{d}{dt}\Phi(e^t)$ is concave on $(-\infty, \log R)$. Then*

$$\Gamma_\Phi(U) \leq \Gamma_\Phi(U^\bigstar)$$

for any domain $U \subseteq \mathbb{D}(R)$.

A proof of this as a consequence of a variant of a special case of a result of Alvino, Lions and Trombetti [4] will be outlined in §5, below. However, Theorem 2.2 does not assume circular symmetry of U, and as seen in Example 1.1, a method that does not use circular symmetry (or something like it) cannot in general be expected to yield a full positive answer to Conjecture 1.4.

We now state the positive result for simply connected and circularly symmetric domains which constitutes the main part of the evidence for Conjecture 1.6.

Theorem 2.3. *Let U be a simply connected circularly symmetric domain. Then $w_r(U) \leq w_r(U^\bigstar)$, with definitions as in Conjecture 1.6.*

This shall be seen to follow from the following more general result, the proof of which takes only a little more effort. Given $z \in \mathbb{C}$, write $[0, z) = \{\lambda z : \lambda \in [0, 1)\}$.

Theorem 2.4. *Let U be a simply connected domain which is symmetric with respect to reflection about the real axis and satisfies $[-\rho, r) \subset U \subseteq \mathbb{D}(r)$ for some $\rho \geq 0$. Let C be a symmetric arc in $\mathbb{T}(r) \cap \partial U$, centred on the point r. Let*

$$C' = \{z \in C : [0, z) \subset U\}.$$

Then

$$\omega(-\rho, C; U) \le \omega(-\rho, C'; U^{\bigstar}).$$

It is easy to see that $C' \subseteq \partial U^{\bigstar} \cap C$. Theorem 2.3 then follows by applying Theorem 2.4 to the U_r with $C = \partial D(r) \cap \partial U_r$ and $\rho = 0$. It is easy to see that Theorem 2.4 need no longer hold if $\rho < 0$. To see this, consider the circularly symmetric domains

$$U_\varepsilon = \mathbb{D} \backslash \{re^{i\theta} : \tfrac{1}{4} \le r \le \tfrac{1}{2}, \varepsilon \le \theta \le 2\pi - \varepsilon\}$$

for $\varepsilon > 0$, note that

$$U_\varepsilon^{\bigstar} = \mathbb{D} \backslash \{re^{i\theta} : \tfrac{1}{2} \le r < 1, \varepsilon \le \theta \le 2\pi - \varepsilon\}$$

and fix $\rho \in (-1, -\frac{1}{2}]$. Then, it is easy to see that $\omega(-\rho, \partial\mathbb{D} \cap \partial U_\varepsilon; U_\varepsilon)$ is bounded away from zero as $\varepsilon \to 0$ while $\omega(-\rho, \partial\mathbb{D} \cap \partial U_\varepsilon^{\bigstar}; U_\varepsilon^{\bigstar}) \to 0$ as $\varepsilon \to 0$ so that Theorem 2.4 will not hold if ε is sufficiently small.

Finally, note that C' will be connected in Theorem 2.4 if U is simply connected.

Now we proceed to the proof of Theorem 2.4. First however we need some background results so that we can use a special case of a quite general result of Baernstein [11]. The reader interested in the many different kinds of rearrangements which all yield analogous results is referred to [11]. Let I be the interval $(-\pi, \pi]$, and let F be a positive Lipschitz function on $I \times \mathbb{R}$. Recall that the Steiner rearrangement about the real axis $S^{\boxminus}$ of a set $S \subseteq I \times \mathbb{R}$ was defined by

$$S^{\boxminus} = \bigcup_{x \in I} \left[\{x\} \times \{y : |y| < \tfrac{1}{2}|\{t : (x,t) \in S\}|\}\right],$$

where $|\{t : (x,t) \in S\}|$ indicates the Lebesgue measure of $\{x : (x,t) \in S\}$. Given a function F on $I \times \mathbb{R}$ we may define $F^{\boxminus}$ as in §I.2.

Then we have the following result, where both integrals are taken with respect to Lebesgue area measure. For $\Phi(t) = t^2$, these integrals are known as **Dirichlet integrals.**

Theorem 2.5 (Baernstein [11, Cor. 3]). *Let F be Lipschitz and positive on $I \times \mathbb{R}$ and assume that for every y we have $F(x,y) \to 0$ as $y \to \pm\infty$. Then $F^{\boxminus}$ is also Lipschitz and for any* *convex increasing function* Φ *we have*

$$\int_{I\times\mathbb{R}} \Phi(|\nabla F^{\boxminus}|) \le \int_{I\times\mathbb{R}} \Phi(|\nabla F|).$$

As a corollary, we obtain the following modified version of a result of Marcus [70, Thm. 1]. For a positive function f which is Lipschitz on $\bar{D}$ and satisfies $f(0) = 0$, and for $(x, y) \in I \times \mathbb{R}$, we define $F(x, y) = f(e^{-|y|+ix})$. It is easy to verify that this is Lipschitz on $I \times \mathbb{R}$. Then, identifying $\mathbb{R}^2$ with $\mathbb{C}$, let $\log z$ be the branch of the logarithm with $\operatorname{Im} \log z \in I$, and set $f_\bigstar(z) = F^{\boxminus}(-i\log(z))$ for $z \in \mathbb{D}$. Then, $f_\bigstar$ will be radially increasing on D. Note that $f_\bigstar = -(-f)^\bigstar$ as can be easily verified.

Theorem 2.6. *Let* f *be Lipschitz and positive on* $\bar{\mathbb{D}}$ *and assume that* $f(0) = 0$. *Then* $f_\bigstar$ *is Lipschitz on compact subsets of* $\bar{\mathbb{D}}\backslash\{0\}$, *satisfies* $f_\bigstar(0) = O(z)$ *as* $z \to 0$, *and has*

$$\iint_{\mathbb{D}} |\nabla f_\bigstar|^2 \le \iint_{\mathbb{D}} |\nabla f|^2.$$

This follows immediately from Theorem 2.5 with $\Phi(t) = t^2$ and from the well-known conformal invariance of Dirichlet integrals, where we use the conformal map $-i\log z$ from $\mathbb{D}\backslash(-1, 0]$ onto the upper half of $I \times \mathbb{R}$, and then note that the Dirichlet integral for $F^{\boxminus}$ over the lower half of $I \times \mathbb{R}$ is the same as that over the upper half of it. The only subtlety is with proving the Lipschitz character of $f_\bigstar$. From Theorem 2.5 we find that $f_\bigstar$ is Lipschitz on compact subsets of $\bar{\mathbb{D}}\backslash[-1, 0]$. Rotational symmetry in the definition of $f_\bigstar$ (i.e., applying the above to $f_\varphi(z) \stackrel{\text{def}}{=} f(e^{i\varphi}z)$ and noting that $(f_\varphi)_\bigstar(z) = f_\bigstar(e^{i\varphi}z)$) shows that in fact it must be Lipschitz on compact subsets of all of $\bar{\mathbb{D}}\backslash\{0\}$. Now $f(0) = 0$ and so I claim that for every $r > 0$ we have

$$\sup_{z\in\mathbb{D}(r)} f(z) \ge \sup_{z\in\mathbb{D}(r)} f_\bigstar(z).$$

Thus $f_\bigstar(z) = O(z)$ as $z \to 0$ since $f(z) = O(z)$ as $z \to 0$. To prove the claim, note that

$$\{z \in \mathbb{D} : f(z) < \lambda\}^\bigstar = \{z \in \mathbb{D} : f_\bigstar(z) < \lambda\}, \tag{2.1}$$

for any $\lambda > 0$. To prove this it suffices to show that equality holds when we intersect both sides with a ray starting from the origin, and to do this one needs to note that the logarithmic metric defining radial rearrangement precisely corresponds to the composition with $-i \log z$ in the definition of $f_{\bigstar}$.

We now proceed to the proof of our Theorem 2.4.

Proof of Theorem 2.4. Without loss of generality set $r = 1$. As usual, by an internal exhaustion like the one in [7, Proof of Thm. 7] we may assume that all our domains have smooth boundaries. We shall assume for now that $\rho = 0$ and at the end of the proof we discuss the minor modifications necessary to take care of the case $\rho > 0$. We now use the method of Haliste [56]. Let V be any subdomain of $\mathbb{D}$ such that $[0, 1) \subset V$ and V is symmetric under reflection in the real axis. Let E be a symmetric arc about 1 in $\partial\mathbb{D} \cap \partial V$. Let f be the (unique) holomorphic map of V onto the disc $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$. Then $f(E)$ is a symmetric arc of $\partial\mathbb{D}$ centred about 1, and its harmonic measure at 0 in $\mathbb{D}$ equals its normalized Lebesgue measure. By conformal invariance, this normalized harmonic measure also equals $\omega(0, E; V)$. Now, as in [56] (but for convenience with reversed boundary values so that Theorem 2.6 would work), let $u = u_{E,V}$ be the solution of the following mixed Dirichlet-Neumann problem on $\mathbb{D}\backslash[0, 1)$:

$$u(z) = 0, \qquad z \in [0, 1), \tag{2.2a}$$

$$u(z) = 1, \qquad z \in \partial\mathbb{D}\backslash f(E), \tag{2.2b}$$

$$\frac{d}{dn} u(z) = 0, \qquad z \in f(E), \tag{2.2c}$$

$$\Delta u(z) = 0, \qquad z \in \mathbb{D}\backslash[0, 1), \tag{2.2d}$$

where $\frac{d}{dn}$ denotes a normal derivative. Let

$$\phi(E, V) = \iint_{\mathbb{D}\backslash[0,1)} |\nabla u_{E,V}|^2,$$

where all such integrals are understood to be taken with respect to Lebesgue area measure. Then, gluing two copies of $\mathbb{D}$ together along the arc $f(E)$ to form a Riemann surface, and applying the Dirichlet and maximum principles on it, we easily see that $\phi(E, V)$ must be strictly decreasing with respect to the length of the arc $f(E)$. But since the length of this arc is proportional to $\omega(0, E; V)$, there must be a strictly decreasing function $\psi : [0, \infty) \to [0, \infty)$

such that

$$\psi(\omega(0,E;V)) = \phi(E,V). \tag{2.3}$$

Haliste [56, equation (3.6)] gives an explicit expression for ψ in terms of elliptic integrals. Now, by conformal invariance and the fact that f sends 0 to 0, E onto $f(E)$ and $[0,1)$ onto $[0,1)$ (this last fact holding due to the reflection symmetry of V), it follows that we may instead consider the function $s_{E,V} \stackrel{\text{def}}{=} f^{-1} \circ u_{E,V}$ and we will have

$$\phi(E,V) = \iint_{V\setminus[0,1)} |\nabla s_{E,V}|^2, \tag{2.4}$$

and more moreover the function $s = s_{E,V}$ will be the solution to the following mixed Dirichlet-Neumann problem on $V\setminus[0,1)$:

$$s(z) = 0, \qquad z \in [0,1), \tag{2.5a}$$

$$s(z) = 1, \qquad z \in \partial V\setminus E, \tag{2.5b}$$

$$\frac{d}{dn}s(z) = 0, \qquad z \in E, \tag{2.5c}$$

$$\Delta s(z) = 0, \qquad z \in V\setminus[0,1). \tag{2.5d}$$

The proof of Theorem 2.4 is now quite simple. Take $s = s_{C,U}$ with the above definition and set $s(z) = 1$ for $z \in \bar{\mathbb{D}}\setminus U$. If U has a nice boundary then s has no problem with satisfying the conditions of Theorem 2.6. Hence,

$$\iint_{\mathbb{D}} |\nabla s_\bigstar|^2 \le \iint_{\mathbb{D}} |\nabla s|^2. \tag{2.6}$$

Now, it is easy to verify that s and $s_\bigstar$ are identically 1 in $\overline{D}\setminus\overline{U}$ and $\overline{D}\setminus\overline{U^\bigstar}$, respectively; for $s_\bigstar$ this follows from (2.1) with $\lambda = 1$, together with the fact that $\{z \in D : s(z) < 1\} = U$. Thus the integrands in (2.6) are supported on $\overline{U}$ and $\overline{U^\bigstar}$ respectively, so that

$$\iint_{U^\star} |\nabla (s_{C,U})_\bigstar|^2 \le \iint_{U} |\nabla s_{C,U}|^2, \tag{2.7}$$

since we have a nice boundary which thus has Lebesgue measure zero. Furthermore, if U is symmetric about the real axis and simply connected then so is $U^\bigstar$, and $s_\bigstar$ is identically 1 on $\partial U^\bigstar\setminus C'$ since for $z \in C'$ we have $s_\bigstar(z) = \max_{[0,z)} s = 1$. Clearly, too, $s_\bigstar$ is identically 0 on $[0,1)$. Hence $s_\bigstar = (s_{C,U})_\bigstar$ satisfies the two Dirichlet boundary conditions which would be imposed on $s_{C',(U^\bigstar)}$ by (2.5a) and (2.5b), though in general it will fail to satisfy the Neumann condition (2.5c) and the harmonicity condition (2.5d). Then, it follows by the Dirichlet principle with free boundary values that

$$\iint_{U^\star} |\nabla s_{C',(U^\star)}|^2 \leq \iint_U |\nabla (s_{C,U})_\bigstar|^2,$$

which combined with (2.4) and (2.7) yields

$$\phi(C', U^\bigstar) \leq \phi(C, U).$$

By (2.3) it follows that

$$\omega(0, C'; U^\bigstar) \geq \omega(0, C; U),$$

since ψ is strictly decreasing. This completes the proof in the case of $\rho = 0$.

If $\rho > 0$ then we proceed just as above, the main difference being that the map f, instead of taking 0 to 0, is now required to take $-\rho$ to 0; note that the condition $f'(0) > 0$ is equivalent to the condition $f'(-\rho) > 0$ since f is to be univalent. Then, instead of considering the solution $s = s_{E,V}$ to (2.5a)–(2.5d) we now consider the solution $s = s_{-\rho,E,V}$ to the mixed Dirichlet-Neumann problem (2.5a′)–(2.5d′) obtained from (2.5a)–(2.5d) by replacing $[0,1)$ in (2.5a) and (2.5d) by $[-\rho, 1)$. The rest of the proof goes through. For, we still have

$$\psi(\omega(-\rho, E; V)) = \iint_{V\setminus[0,1)} |\nabla s_{-\rho,E,V}|^2,$$

with exactly the same function ψ as before. Moreover, if $s = s_{-\rho,E,U}$ then $s_\bigstar$ satisfies the two Dirichlet boundary conditions that would be imposed on $s_{-\rho,E,(U^\star)}$ by (2.5a′) and (2.5b), where (2.5a′) is (2.5a) with $[0,1)$ replaced by $[-\rho, 1)$. The reader waiting to see where the assumption $\rho \geq 0$ is used may be pleased to note that it is used precisely in the assertion that $s_\bigstar$ satisfies (2.5a′) for the rearranged domain $V = U^\bigstar$. $\square$

3. Transferring the problems to the cylinder and the question of two-sided lengthwise Steiner symmetrization

With no loss of generality we may assume that our domain U is contained in the unit disc (use scaling and approximation to handle the general case). Let $f\colon \mathbb{D}\backslash\{0\} \to \mathbb{V}^- \stackrel{\text{def}}{=} (-\infty, 0) \times \mathbb{T}$ be the conformal isomorphism

$$f(z) = (\log|z|, e^{i \arg z}). \tag{3.1}$$

Note that $\mathbb{V}^-$ is a semi-infinite cylinder. We shall sometimes have a certain tendency to implicitly consider the set $(-\infty, 0] \times \mathbb{T}$ as the closure $\overline{\mathbb{V}^-}$ of $\mathbb{V}^-$, and to extend things defined on $\mathbb{V}^-$ automatically to $\overline{\mathbb{V}^-}$. Now, for $W \subset \mathbb{V}^-$ and $u \in \mathbb{T}$, let

$$X(u; W) = |\{x \leq 0 : (x, u) \notin W\}|$$

and set

$$W^{\blacktriangleleft} = \{(x, u) \in \mathbb{V}^- : x < -X(u; W)\}.$$

It is easy to verify that $\text{Area}(\mathbb{V}^-\backslash W) = \text{Area}(\mathbb{V}^-\backslash W^{\blacktriangleleft})$ and

$$f[U\backslash\{0\}]^{\blacktriangleleft} = f[U^{\bigstar}\backslash\{0\}].$$

We say that a domain $W \subseteq \mathbb{V}^-$ is **circularly symmetric** if $\{x\} \times \mathbb{T}$ is the cartesian product of $\{x\}$ with a circular interval centred about $1 \in \mathbb{T}$, for each $x \in (-\infty, 0]$. Clearly W is a circularly symmetric subset of $\mathbb{V}^-$ if and only if $f^{-1}[W]$ is a circularly symmetric subset of the plane. Let $\mathcal{G}$ be the collection of convex increasing functions Ψ on $(-\infty, 0]$. Then, it is easy to see that we may reformulate Conjecture 1.4 and Conjecture 1.6 to be conjectures concerning least harmonic majorants of $(x, t) \mapsto \Psi(t)$ for $\Psi(t)$, harmonic measures and circularly symmetric domains in $\mathbb{V}^-$. This is done explicitly in the language of Brownian motion in §7.

Now, we present a conjecture which is more general than Conjecture 1.6. An analogous conjecture more general that Conjecture 1.4 could perhaps also be formulated. Let $\mathbb{V} = \mathbb{R} \times \mathbb{T}$ be a doubly infinite cylinder. We may of course easily define what it means for a domain on $\mathbb{V}$

to be circularly symmetric. We now say that a domain $W \subseteq \mathbb{V}$ is **flip-symmetric** if for every $(x,t) \in \mathbb{V}$ we have $(x,t) \in W$ if and only if $(-x,t) \in W$. Now, given $t \in \mathbb{T}$, let

$$L(t;W) = |\{x : (x,t) \in W\}|,$$

where $|\cdot|$ indicates Lebesgue measure on $\mathbb{R}$. Define the (two-sided) **lengthwise Steiner symmetrization** of $W \subseteq \mathbb{V}$ to be

$$W^{\triangleq} \stackrel{\text{def}}{=} \{(x,t) : |x| < L(t,W)/2\}.$$

Clearly $\text{Area}(W^{\triangleq}) = \text{Area}(W)$. We may analogously define flip symmetry and lengthwise Steiner symmetrization for domains which are subsets of $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, except that in the above definitions we replace $t \in \mathbb{T}$ by $y \in \mathbb{R}$. Given W a subset of $\mathbb{V}$ or $\mathbb{C}$, let Y be $\mathbb{T}$ or $\mathbb{R}$, respectively. Given $M \geq 0$, we define[1]

$$\Omega_M(W) = \omega\left((0,0), (\{-M,M\} \times Y) \cap \overline{W}; ((-M,M) \times Y) \cap W\right),$$

where of course $(-M,M)$ is an open interval with boundary $\{-M,M\}$. It would certainly be hopeless to suppose that $\Omega_M(W) \leq \Omega_M(W^{\triangleq})$ in general since Example 1.1 can be adapted to this situation. Hence, if $W \subseteq \mathbb{V}$ then we will impose circular symmetry on W while if $W \subseteq \mathbb{C}$ then we will usually impose Steiner symmetry about the real axis. From Theorem I.6.7 it follows that $\Omega_M(U) \leq \Omega_M(U^{\boxminus})$ since

$$\Omega_M(U) = \Omega'(U') + \Omega'(-U'),$$

where $U' = U \cap ((-M,M) \times \mathbb{R})$ and $\Omega'(D) = \omega(0, \{\operatorname{Re} z = M\} \cap \overline{D}; D)$.

But Steiner symmetry is still insufficient, since we may easily come up with examples where $\Omega_M(W) > 0$ while $\{-M,M\} \times Y$ is disjoint from $\overline{W^{\triangleq}}$ so that $\Omega_M(W^{\triangleq}) = 0$. For example, let $W = (-M/3, 4M/3) \times Y$ and note that $W^{\triangleq} = (-5M/6, 5M/6) \times Y$. Hence we also require flip-symmetry, which in the case of $Y = \mathbb{R}$, is by definition the same as reflection symmetry about the real axis.

[1] Of course harmonic measures can be easily defined on the manifold $\mathbb{V}$.

Conjecture 3.1. Let $W \subseteq \mathbb{V}$ be both flip-symmetric and circularly symmetric. Then

$$\Omega_M(W^{\hat{=}}) \geq \Omega_M(W)$$

for every $M > 0$.

To see that this is more general than Conjecture 1.6, note that given a domain $W \subseteq V$ with $W = f[U \backslash \{0\}]$ for $U \subseteq \mathbb{D}$ with $0 \in U$ (where f is our conformal map from $\mathbb{D}\backslash\{0\}$ to $\mathbb{V}^-$), there is an $R > 0$ such that $(-\infty, -R] \times \mathbb{T} \subseteq W$ since U is open at 0. Then, let

$$W_r = ((-r, r) \times \mathbb{T}) \cup \{(\pm(r + x), t) : x \in [0, R), (x - R, t) \in W\},$$

where the $\pm$ means that for each admissible x and t, we throw *both* $((r+x), t)$ and $(-(r+x), t)$ into W_r. Then, it is easy to see that $\Omega_{R+r}(W_r) \to w_1(U)$, and all the rearrangements are nicely preserved by the correspondences so that Conjecture 3.1 indeed implies Conjecture 1.6.

Theorem 3.1. *Let $W \subseteq \mathbb{C}$ be a simply connected domain which is symmetric under reflection in the real and imaginary axes. Moreover, assume that the interval $(-M, M)$ lies in W. Let C_1 be an interval of $(\{M\} \times \mathbb{R}) \cap \overline{W}$ centred on M, and let $C = C_1 \cup -C_1$. Let*

$$C' = \{(\pm M, y) : (-M, M) \times \{y\} \subset W\}.$$

Assume that C' is an interval. Then

$$\omega(0, C; ((-M, M) \times \mathbb{R}) \cap W) \leq \omega(0, C'; ((-M, M) \times \mathbb{R}) \cap W^{\hat{=}}).$$

If W is Steiner symmetric about the real axis then C' will automatically be an interval. Moreover, we see that $\Omega_M(W) \leq \Omega_M(W^{\hat{=}})$ under the conditions of the Theorem with $C = (\{M\} \times \mathbb{R}) \cap W$. In particular we have the following result.

Corollary 3.1. *If W is both flip-symmetric (about the imaginary axis) and Steiner symmetric about the real axis then*

$$\Omega_M(W) \leq \Omega_M(W^{\hat{=}})$$

for every $M > 0$.

This is a special case of Conjecture 3.1. For, given a domain W, by approximation we may assume that W is bounded, and then by scaling we may assume that it is a subset of $\mathbb{R} \times (-\pi, \pi)$. But such a subset can be mapped conformally into $\mathbb{V}^-$ via $(x, y) \mapsto (x, e^{iy})$, and this map preserves all the relevant symmetries and symmetrizations so that Conjecture 3.1 would indeed imply the inequality $\Omega_M(W) \leq \Omega_M(W^{\hat{=}})$.

Outline of proof of Theorem 3.1. The proof is done in a manner very similar to the proof of Theorem 2.4. We assume W is a subset of $(-M, M) \times \mathbb{R}$. We map W onto D by a holomorphic map g in such a way that 0 goes to 0 while the interval $(-M, M)$ goes to the interval $(-1, 1)$. Then, $g[C]$ consists of two circular intervals of equal length, centred on 1 and -1 respectively. The length of each of these intervals then is proportional to the harmonic measure $\omega(0, C; W)$. Moreover, this length is strictly inversely monotonely related to the Dirichlet integral of the solution $s = s_{C,W}$ of the boundary value problem

$$\begin{aligned} s(z) &= 1, & z &\in (-1, 1), \\ s(z) &= 0, & z &\in \partial\mathbb{D} \backslash g[C], \\ \frac{d}{dn} s(z) &= 0, & z &\in g[C], \\ \Delta s(z) &= 0, & z &\in \mathbb{D} \backslash (-1, 1), \end{aligned}$$

in $\mathbb{D}$. And as in the proof of Theorem 2.4, this problem can be pulled back by g^{-1} to a problem on W. We may now complete the proof as in Theorem 2.4, except that we can no longer use Theorem 2.5 or 2.6. Let $u = u_{C,W} = s_{C,W} \circ g^{-1}$. By rescaling, we may assume that $M = \pi$. Define $u = 0$ on $((-\pi, \pi) \times \mathbb{R}) \backslash W$. Thus extended, u on $(-\pi, \pi) \times \mathbb{R}$ will satisfy any regularity conditions that we may later need, providing W is nice enough. We could just apply a theorem on Steiner symmetrization (about the imaginary axis) on $(-\pi, \pi) \times \mathbb{R}$ to u, but at least in [11] we do not seem to find an explicit statement of the kind of theorem we need. Instead, [11] gives a $(1, 2)$-cap symmetrization theorem which we can use as follows. The set $(-\pi, \pi) \times \mathbb{R}$ can of course be mapped conformally via $(t, y) \mapsto e^{y+it}$ onto $\mathbb{T} \times \mathbb{R}$. This lets us transfer u to a function on $\mathbb{T} \times \mathbb{R}$. We then apply $(1, 2)$-cap symmetrization (see [11]) to the transferred function and we pull back to get a function on $(-\pi,\pi)\times\mathbb{R}$. This process will not increase Dirichlet integrals [11, Cor. 3]. The function we obtain from u by this process will turn out to satisfy the Dirichlet boundary conditions (but perhaps not the Neumann or harmonicity conditions) that would be imposed on the function $u_{C',(W^{\doteq})}$. The proof is then completed just as in the case of Theorem 2.4 by the Dirichlet principle with free boundary values. □

Corollary 3.1 was also independently obtained by Baernstein [13] via much the same adaptation of the method of proof of Theorem 2.4.

Conjecture 3.2. Let Φ be an even convex function on $\mathbb{R}$. Let U be a flip-symmetric subset of, respectively, $\mathbb{V}$ or $\mathbb{C}$, and assume it to be, respectively, circularly symmetric or Steiner symmetric about the real axis. Then if Γ_Φ indicates, respectively, the value at $(0,1)\in\mathbb{V}$ or the value at $0\in\mathbb{C}$ of the least harmonic majorant of $(x,t)\mapsto\Phi(x)$ on U, then $\Gamma_\Phi(U^{\doteq})\geq\Gamma_\Phi(U)$.

Actually, while the Conjecture 3.2 is still open, in the special case of $U\subseteq\mathbb{C}$ in §10 we will obtain a certain partial replacement.

Now, given a domain $W\subseteq\mathbb{V}^-$, and given a half-open interval $(a,b]\subset(-\infty,0]$, we may define

$$W_{(a,b]}=\operatorname{Int}(\{(x+b-a,t):x\leq a,(x,t)\in W\}\cup\{(x,t):b<x,(x,t)\in W\}),$$

where $\operatorname{Int}X$ denotes the union of all open subsets of X. The domain $W_{(a,b]}$ may be considered to be W with the ring $(a,b]\times\mathbb{T}$ cut out. Intuitively, then, one would expect that the probability that a Brownian motion starting at a point at infinity (to the left of $\{0\}\times\mathbb{T}$) arrives at $\{0\}\times\mathbb{T}$ without touching the complement W would increase if we replace W with $W_{(a,b]}$, since $W_{(a,b]}$ is in some way "shorter." Of course, in reality, one cannot talk about a finite shortening of the length of a set which has infinite length.

In the most interesting cases we will have $(a,b]\times\mathbb{T}\subset W$. We may pull the transformation $(\cdot)_{(a,b]}$ to the punctured disc to get

$$U^{(r,R]}=\operatorname{Int}\left(\left\{\frac{R}{r}se^{i\theta}:0\leq s\leq r,se^{i\theta}\in U\right\}\cup\left\{se^{i\theta}:r<s,se^{i\theta}\in U\right\}\right),$$

for $U \subseteq \mathbb{D}$ and $0 < r < R \leq 1$. We will have $f[U^{(r,R]}\backslash\{0\}] = f[U\backslash\{0\}]_{(\log r, \log R]}$, where f is our conformal map of $\overline{\mathbb{D}}\backslash\{0\}$ onto $\mathbb{V}^-$ defined by (3.1). Note also that $\operatorname{Area} U^{(r,R]} \leq \operatorname{Area} U$ if $r < R$.

The above-mentioned intuition concerning Brownian motion then suggests the following problem.

Open Problem 3.1. Do we have $w_1(U) \leq w_1(U^{(r,R]})$? More generally, do we have $\Gamma_\Phi(U) \leq \Gamma_\Phi(U^{(r,R]})$ for every $\Phi \in \mathcal{F}$?

As it stands, the answer is negative, and a counterexample is provided by the domain on the left side of Figure 1.1 (cf. Example 1.1). However, the Problem is still open under the additional assumption that U be circularly symmetric, and this is the case we consider in the discussion below. We should also remark here that the fact that the second part of the Problem is more general than the first is a consequence of Theorem 4.1 in §4, below.

Note that $(U^{(r,R]})^\bigstar \subseteq U^\bigstar$ with equality if and only if the annulus $\{se^{i\theta} : s \in (r,R]\}$ is contained in U possibly modulo a set whose intersection with any ray through the origin has null Lebesgue length measure. Note also that if a domain U is finitely connected and circularly symmetric then a finite number of applications of $(\cdot)^{(r,R]}$ with $(r,R]$ a maximal interval chosen so that the annulus $\{se^{i\theta} : s \in (r,R]\}$ is contained in U will reduce U to a simply connected and circularly symmetric domain U' with $(U')^\bigstar = U^\bigstar$. A limiting argument could then be employed in the case of an infinitely connected domain U to reduce it, too, by an infinite number of applications of $(\cdot)^{(r,R]}$ to a simply connected domain U' with $(U')^\bigstar = U^\bigstar$. Then, an affirmative answer to the first part of the Problem would imply that $w_1(U') \geq w_1(U)$. But by Theorem 2.3 we would then have $w_1(U^\bigstar) = w_1((U')^\bigstar) \geq w_1(U')$ since U' is simply connected. Hence, an affirmative answer to the first part of the Problem would imply an affirmative answer to Conjecture 1.6.

It may also be worth noting Problem 3.1 can be considered as concerned about a generalization of the method of proof of Beurling's shove theorem.

As further motivation, note that since an affirmative answer to the second part of the Problem would give us a way to increase the functionals Γ_Φ by replacing a domain with a simply connected one without increasing the area of the domain, it would yield an affirmative answer to Conjecture 1.3.

4. Formulation in terms of Green's functions

Now we would like to note a well-known result which was alluded to before. The following result is an immediate consequence of Proposition I.6.2.

Theorem 4.1. *For any pair of domains U and V the following are equivalent:*

(a) *For every $\Phi \in \mathcal{F}$ we have $\Gamma_\Phi(V) \geq \Gamma_\Phi(U)$.*

(b) *For every $r > 0$ we have*

$$\int_0^{2\pi} g(re^{i\theta}, 0; V)\, d\theta \geq \int_0^{2\pi} g(re^{i\theta}, 0; U)\, d\theta.$$

It might well eventually turn out that to prove Conjecture 1.4 it would easier to prove condition (b) of Theorem 4.1 with $V = U^\bigstar$.

We now note that Conjecture 1.4 implies Conjecture 1.6 because harmonic measures on the boundary of a domain correspond to normal derivatives of Green's functions, so that (at least for a sufficiently nice domain)

$$\frac{2\pi}{r} w_r(U) = -\int_0^{2\pi} \frac{\partial g(re^{i\theta}, 0; U_r)}{\partial r}\, d\theta = \lim_{\rho \to r-} (r-\rho)^{-1} \int_0^{2\pi} g(re^{i\theta}, 0; U_r)\, d\theta, \qquad (4.1)$$

as noted in [10]. Now if Conjecture 1.6 holds then by Theorem 4.1 we have an inequality between the right hand side of (4.1) and the same right hand side but with U_r replaced by $(U_r)^\bigstar$, so that the desired inequality $w_r(U) \leq w_r(U^\bigstar)$ then follows from the easy observation that $(U_r)^\bigstar \subseteq (U^\bigstar)_r$ and $w_r((U_r)^\bigstar) \leq w_r(U^\bigstar)$. (Of course, if our domains are not sufficiently nice then we have to use an approximation argument, but this is easy.)

Assuming $U \subseteq \mathbb{D}(r)$, Conjecture 1.6 is an inequality between $g(\cdot, 0; U)$ and $g(\cdot, 0; U^\bigstar)$ near $\mathbb{T}(r)$. Let us consider such inequalities near 0. Given a domain U containing the origin, its

Green's function can be written in the form

$$g(0, z; U) = \log \frac{1}{|z|} + \log \rho + o(1)$$

as $z \to 0$, where $\rho = \rho(U)$ is a constant known as the inner radius of U about 0. Then, inequalities between $g(\cdot, 0; U)$ and $g(\cdot, 0; U^\star)$ near zero correspond to inequalities between $\rho(U)$ and $\rho(U^\star)$. Marcus [70, Thm. 3] had shown that for any domain U (not necessarily simply connected or circularly symmetric) we have

$$\rho(U) \leq \rho(U^\star). \tag{4.2}$$

(See the following section for a reference to another proof of this fact.) Hence, in the simply connected circularly symmetric case we know that we have the correct inequality between $g(\cdot, 0; U)$ and $g(\cdot, 0; U^\star)$ near $\partial\mathbb{D}(r)$ and near 0; for general domains we only know that we have it near 0.

5. The case where $t \mapsto \frac{d\Phi(e^t)}{dt}$ is concave

Now, return to the cylinder $\mathbb{V}$ considered in §3. Given a real-valued function f on $\mathbb{V}$, consider the function $f^\leftrightarrow$ defined on $\mathbb{V}$ by

$$f^\leftrightarrow(x, t) = \sup_E \int_E f(w, t)\, dw,$$

where the supremum is taken over all measurable sets $E \subseteq \mathbb{R}$ with Lebesgue measure $2|x|$. The function $f^\leftrightarrow$ may be called the (lengthwise) Baernstein $*$-function of f. See Baernstein [11] where many similar objects are considered.

We recall the following theorem which is a modified version of a very special case of a theorem of Alvino, Lions and Trombetti [4] (see also [11, Thm. 7] for another way to prove such results). Say that a function f on $\mathbb{R}$ is **symmetric decreasing** if $f(x) = f(-x)$ for every $x \in \mathbb{R}$ and f is decreasing on $[0, \infty)$.

Theorem 5.1. *Let $V \subseteq \mathbb{V}$ and let $\lambda(\cdot, t)$ be a symmetric decreasing function on $\mathbb{R}$ for every $t \in \mathbb{T}$. Let u be the solution of*

$$-\Delta u = \lambda$$

in V, with boundary value 0 on ∂V. Let v be the solution of the analogous problem on $V^{\triangleq}$. Extend u and v to vanish identically outside V and $V^{\triangleq}$, respectively. Then

$$u^{\leftrightarrow}(x,t) \leq \int_{-|x|}^{|x|} v(s,t)\,ds \leq v^{\leftrightarrow}(x,t) \tag{5.1}$$

everywhere on $\mathbb{V}$.

Note that the second inequality in (5.1) is a trivial consequence of the definition of $(\cdot)^{\leftrightarrow}$. Now, suppose Ψ is a convex increasing function in $C^2[0,R)$ with Ψ' concave and $\Psi''(0)$ finite. Assume that $V \subseteq (-R,R) \times \mathbb{T}$ is sufficiently regular. Let h be the least harmonic majorant of $(x,t) \mapsto \Psi(x)$ in V. Then,

$$-\Delta(h-\Psi)(x,t) = \Psi''(x).$$

And of course $h - \Psi$ vanishes on ∂V. Let H be the least harmonic majorant of $(x,t) \mapsto \Psi(x)$ on $V^{\triangleq}$. Then by Theorem 5.1 we have

$$u^{\leftrightarrow}(x,t) \leq \int_{-|x|}^{|x|} v(s,t)\,ds,$$

where $u = h - \Psi$ and $v = H - \Psi$. It easily follows from this and the definition of $u^{\leftrightarrow}$ that for every $t \in T$,

$$\sup_x u(x,t) \leq v(0,t).$$

Thus, if we apply an approximation argument, we will obtain the following result.

Corollary 5.1. *Fix $t \in \mathbb{T}$. Let Ψ be a convex increasing function on $[0,R)$ such that Ψ' is concave and $\Psi''(0)$ is finite. Suppose $V \subseteq \mathbb{V}$. Let h be the least harmonic majorant of $(x,t) \mapsto \Psi(|x|)$ on V, and let H be the least harmonic majorant of $(x,t) \mapsto \Psi(|x|)$ on $V^{\triangleq}$. Then*

$$h(0,t) \leq H(0,t)$$

for every $t \in \mathbb{T}$.

At this point we note that to prove Theorem 2.2, one may use a limiting argument together with Corollary 5.1, following the method that was used in §3 to show that Conjecture 3.1

implies Conjecture 1.6. Note that Theorem 2.2 generalizes the inequality (4.2), above, which was due to Marcus [70]. To see this, assume without loss of generality that $U \subseteq \mathbb{D}$ be sufficiently regular and let $\Phi(t) = \log t$ on $[0,1]$, so that the least harmonic majorant of Φ on U equals $g(\cdot, 0; U) - \log \frac{1}{|\cdot|}$. Also note that Theorem 2.2 does not need any circular symmetry condition on U, so it is difficult to imagine using this method to prove results like Conjecture 1.4 in light of the fact that Conjecture 1.4 need not hold if U fails to be circularly symmetric.

6. Haliste's one-sided lengthwise Steiner rearrangement

We take the occasion to discuss a situation analogous to Theorems 2.4 and 3.1. As before let U be a domain in the plane. Now, for $M > 0$, define

$$W_M(U) = \omega(0, \{\operatorname{Re} z < M\} \cap U, \{\operatorname{Re} z = M\} \cap \bar{U}).$$

Given a domain U, we define a domain $U^{(M)}$, which we shall term the **one-sided lengthwise Steiner rearrangement at abscissa** M, by the equality

$$U^{(M)} = \{(x,y) : M - |\{t < M : (t,y) \in U\}| < x < M\}.$$

This definition also makes sense for $M \leq 0$, and was first studied by Haliste [56]. As in the case of radial and circular rearrangement, for any real M the set $U^{(M)}$ is open if U is open. Furthermore, $\operatorname{Area}(U^{(M)}) = \operatorname{Area}(U \cap \{\operatorname{Re} z < M\})$.

Now, it would be hopeless to desire that in general $W_M(U^{(M)}) \geq W_M(U)$ since $U^{(M)}$ need not contain 0 even if U does. However, we can assert that $W_M(U^{(M)}) \geq W_M(U)$ provided U is Steiner symmetric.

Theorem 6.1 (Haliste [56, Thm. 4.2]). *Let U be a simply connected domain in the plane which is reflection symmetric about the real axis and contains the interval $[0, M)$. Then*

$$W_M(U) \leq W_M(U^{(M)}). \tag{6.1}$$

In particular, (6.1) will hold whenever U is Steiner symmetric. Moreover, by Theorem I.6.7, we have $W_M(U) \leq W_M(U^{\boxminus})$, so that we also have the following inequality.

Corollary 6.1. *Let U be any domain containing the origin. Then for any positive M we have*

$$W_M(U) \leq W_M((U^{\boxminus})^{(M)}).$$

Haliste's proof [56] of Theorem 6.1 basically follows along much the same lines as that of Theorem 2.4 or Theorem 3.1.

7. Brownian motion, simple discrete analogues and exit times

We may now use Brownian motion to transfer Conjectures 1.4 and 1.6 to the cylinder $\mathbb{V}^-$ considered in §3. Let B_t be a Brownian motion on $\mathbb{V}^-$. Let

$$\tau_W = \inf\{t \geq 0 : B_t \notin W\}.$$

We use $P^z(\cdot)$ and $E^z[\cdot]$ to indicate probabilities and expectations, respectively, when B_0 is conditioned to be z. Moreover, we use $P^{-\infty}(\cdot)$ and $E^{-\infty}[\cdot]$ to indicate the limit of $P^z(\cdot)$ and $E^z[\cdot]$, respectively, as $z \to (-\infty, u)$ (in all cases which we consider, this limit will not depend on u). Throughout, W will indicate an open subset of $\mathbb{V}^-$ which contains some semi-infinite cylinder $(-\infty, -R] \times \mathbb{T}$ (for some $R > 0$), so that $f^{-1}[W] \cup \{0\}$ will be an open subset of $\mathbb{D}$, where f is given by (3.1).

Write $\pi : \mathbb{V}^- \to (-\infty, 0)$ for the projection onto the first coordinate. Recalling the well-known connection between Brownian motion, harmonic measure and harmonic majorants, the following two conjectures are equivalent to Conjecture 1.4 and Conjecture 1.6, respectively. Let $\mathcal{G}$ be the collection of all increasing convex functions on $[-\infty, 0]$.

Conjecture 1.4′. Let W be circularly symmetric and let $\Phi \in \mathcal{G}$. Then

$$E^{-\infty}[\Phi(\pi(B_{\tau_W}))] \leq E^{-\infty}[\Phi(\pi(B_{\tau_{W^{\blacktriangleleft}}}))].$$

Conjecture 1.6′. Let W be circularly symmetric. Then

$$P^{-\infty}(\pi(B_{\tau_W}) = 0) \leq P^{-\infty}(\pi(B_{\tau_{W^{\blacktriangleleft}}}) = 0).$$

By Theorem 2.3, Conjecture 1.6′ will be true if $\mathbb{V}^-\backslash W$ is connected.

We now give a simple one-dimensional discrete model for these problems. Let R_i, $i \in \mathbb{Z}_0^+$, be a simple random walk on $Z_0^- = \{0, -1, -2, \ldots\}$ with $P(R_{i+1} = R_i \pm 1 \mid R_i) = \frac{1}{2}$ for all i. Let $\{s_n\}_{n=-\infty}^0$ be a sequence of numbers in $[0, 1]$ with $s_0 = 0$. Assume that $s_n = 1$ for n sufficiently close to $-\infty$. Consider the following process. Let X_i, $i \in \mathbb{Z}_0^+$, be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. We will kill the random walk at time

$$\tau_s = \inf\{i \geq 0 : X_i > s_{R_i}\}.$$

Thus, the random walk R_i at step i has a probability of surviving equal to s_{R_i}. This τ_s is quite analogous to our previous τ_W, since the likelihood of our Brownian motion terminating in $\{x\} \times \mathbb{T}$ is heuristically expected to depend inversely on the size of the set $(\{x\} \times \mathbb{T}) \cap W$, so that the size of this set is analogous to s_x. What is the discrete analogue of $(\cdot)^{\blacktriangleleft}$ then? Well, if W is circularly symmetric, then $W^{\blacktriangleleft}$ is W with the slices $(\{x\} \times \mathbb{T}) \cap W$ reordered so their sizes decrease as x increases. Hence, let $\{s_n^{\blacktriangleleft}\}_{n=-\infty}^0$ be the decreasing rearrangement of the s_n. We will have $0 = s_0^{\blacktriangleleft} \leq s_{-1}^{\blacktriangleleft} \leq s_{-2}^{\blacktriangleleft} \leq \cdots$. Let $P^{-\infty}(\cdot)$ and $E^{-\infty}[\cdot]$ indicate probabilities and expectations, respectively, after having taken the limit of the starting point of the random walk tending to $-\infty$. The obvious analogue of Conjecture 1.6 is as follows.

Theorem 7.1.

$$P^{-\infty}(R_{\tau_s} = 0) \leq P^{-\infty}(R_{\tau_{s^{\blacktriangleleft}}} = 0).$$

This will be a consequence of Theorem 9.1 in §9, below. To get this out of Theorem 9.1, we note that $E^n[\Phi(R_{\tau_s})]$ and $P^n(R_{\tau_s} = 0)$, as well as their analogues with $s^{\blacktriangleleft}$ in place of s, will both necessarily be constant for $n \leq N$ where $N < 0$ is chosen so that $s_n = 1$ whenever $n \leq N$, so that nothing will change if we make things finite by introducing a reflecting boundary condition at n, or even the somewhat different boundary condition used in §9.

The natural analogue of Conjecture 1.4 is still open.

7.1. Uniform motion to the right: a counterexample

Now, in §9, Theorem 7.1 was in fact proved not just for random walks with equal probabilities of going forward and backward, but also for walks which have a probability p of going forward and $1-p$ of going backward. (For $p < \frac{1}{2}$ we have to change the statement of Theorem 7.1 somewhat, though, since otherwise both of the limits in it are zero. However, if instead of taking the limit we simply start the random walk at some point x such that $s_{x'} = 1$ for every $x' \leq x$ then Theorem 7.1 will continue to hold and will make more sense for $p < \frac{1}{2}$.) One might hope that such a generalization to non-equal forward/reverse movement probabilities would also be possible in the two dimensional case of Conjecture 1.6′. But, this is not true in the natural adaptation of Conjecture 1.6′ to the case $p = 1$.

We now show this impossibility of extending Conjecture 1.6′ to the case $n = 1$. Of course, $p = 1$ means that the random walk constantly moves to the right. Hence, in our analogue we shall replace the process B_t in Conjecture 1.6′ by a process whose first coordinate moves uniformly to the right while whose second coordinate is a Brownian motion S_t on the circle $\mathbb{T}$. For convenience, we index our process with time on $(-\infty, 0]$ so that the process is $t \mapsto (t, S_t)$ on $\overline{\mathbb{V}^-}$. Let $W \subseteq \mathbb{V}^-$ be circularly symmetric and such that $(-\infty, -T] \times \mathbb{T} \subseteq W$ for some finite T. Then, the analogue of $p = 1$ in Conjecture 1.6′ is that

$$P((t, S_t) \in W, \forall t \in (-\infty, 0)) \leq P((t, S_t) \in W^{\blacktriangleleft}, \forall t \in (-\infty, 0)).$$

Write

$$F(W) = P((t, S_t) \in W, \forall t \in (-\infty, 0)).$$

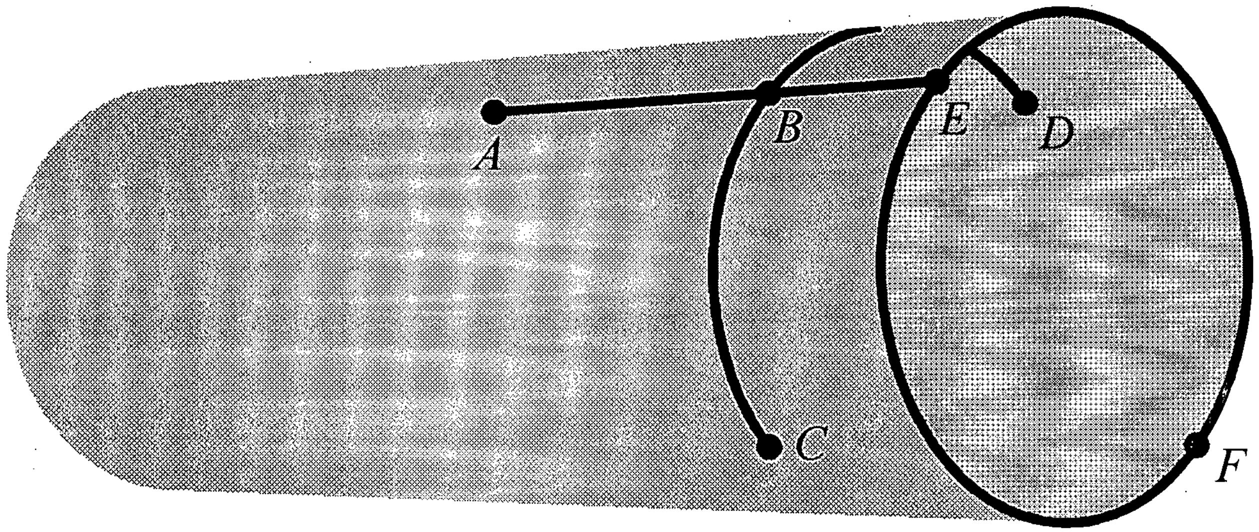
To prove the existence of a counterexample, first let H be the semicircle $\{e^{i\theta} : |\theta| < \pi/2\}$, let $H^c = \mathbb{T} \backslash H$ be its complement, and then define

$$W_{T,\varepsilon} = \mathbb{V}^- \backslash [([-T-\varepsilon, 0) \times \{-1\}) \cup (\{-\varepsilon\} \times H^c)].$$

(See Figure 7.1.) Then,

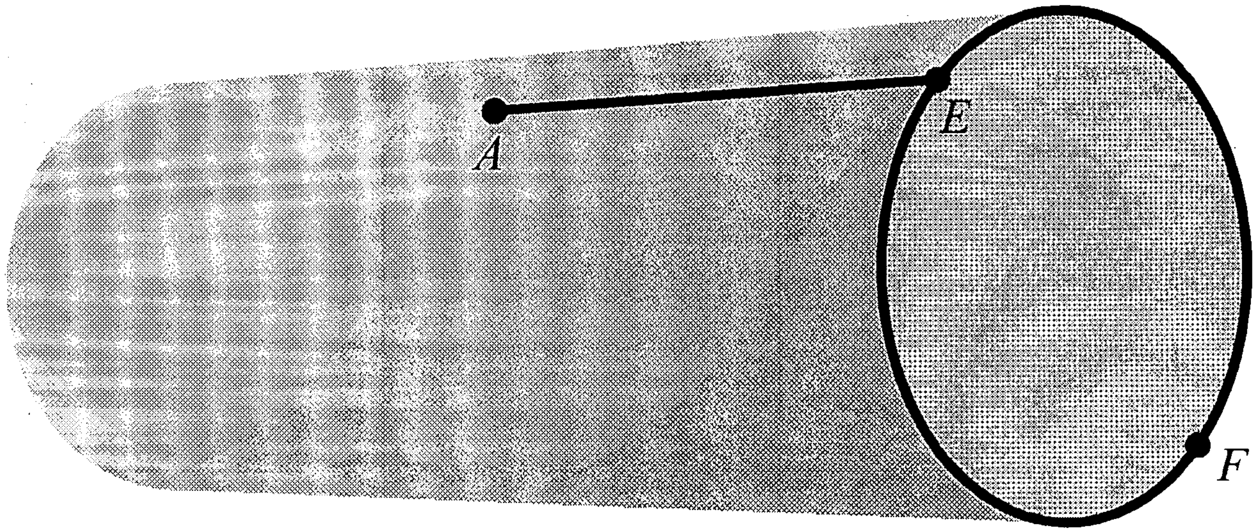
$$W_{T,\varepsilon}^{\blacktriangleleft} = \mathbb{V}^- \backslash ([-T-\varepsilon, 0) \times \{-1\}).$$

(See Figure 7.2.)



Legend:
Black lines indicate the complement of $W_{T,\varepsilon}$.
$A=(-T-\varepsilon,-1)$
$B=(-\varepsilon,-1)$
$C=(-\varepsilon,-i)$
$D=(-\varepsilon,i)$
$E=(-1,0)$
$F=(1,0)$

Figure 7.1: The domain $W_{T,\varepsilon}$.



Legend:
Black lines indicate the complement of $W^{\blacktriangleleft}_{T,\varepsilon}$.
A=($-T$$-\varepsilon$,$-1$)
E=(-1,0)
F=(1,0)

Figure 7.2: The domain $W^{\blacktriangleleft}_{T,\varepsilon}$.

For a domain W, let

$$H(W) = P((t, S_t) \in W, \forall t \in (-\infty, 0) \text{ and } S_0 \in H).$$

I claim that one can choose $\varepsilon > 0$ and $T \in (0, \infty)$ such that

$$H(W^{\blacktriangleleft}_{T,\varepsilon}) < F(W_{T,\varepsilon}). \tag{7.1}$$

Assume this claim for now. Let

$$W_\delta = \mathbb{V}^- \backslash [([-T-\varepsilon, 0) \times \{1\}) \cup ([-\varepsilon, -\varepsilon + \delta] \times H^c)], \tag{7.2}$$

for $0 < \delta < \varepsilon$, and note that

$$W^{\blacktriangleleft}_\delta = \mathbb{V}^- \backslash ([-T-\varepsilon, 0) \times \{1\} \cup [-\delta, 0) \times H^c).$$

It is easy to verify that

$$F(W_\delta) \to F(W_{T,\varepsilon}),$$

as $\delta \to 0$ for fixed $\varepsilon > 0$, while

$$F(W_\delta^{\blacktriangleleft}) \to H(W_{T,\varepsilon}),$$

again as $\delta \to 0$. Choosing δ sufficiently small, may easily ensure that $F(W_\delta^{\blacktriangleleft}) < F(W_\delta)$ because of (7.1).

It remains to prove (7.1). To do this, let

$$\alpha(T, \varepsilon) = F(W_{T,\varepsilon}),$$

and

$$\beta(T) = H(W_{T,0}^{\blacktriangleleft}).$$

Of course, $W_{T,0}^{\blacktriangleleft} = W_{T,0}$. It is easy to see that $\beta(T+\varepsilon) = H(W_{T,\varepsilon}^{\blacktriangleleft})$. Also easy to verify is the fact that as $\varepsilon \downarrow 0$ then $\alpha(T,\varepsilon) \uparrow H(W_{T,0}) = H(W_{T,0}^{\blacktriangleleft}) = \beta(T)$. We need an estimate of the rate of convergence in this limit. To obtain this estimate, note that we may write

$$\alpha(T,\varepsilon) = F(W_{T,\varepsilon}) = H(W_{T,0}) P(S_t \in \mathbb{T}\backslash\{-1\}, \forall t \in (-\varepsilon, 0) \mid S_{-\varepsilon} \in H).$$

For, we may use the Markov property of Brownian motion together with the fact that $H(W_{T,0})$ can be seen to be equal to the probability of our process arriving from $(-\infty, 0)$ to $\{-\varepsilon\} \times H$ without having touched $\partial(W_{T,\varepsilon})$, because of translation invariance and the relation

$$(\partial(W_{T,\varepsilon})) \cap ((-\infty, -\varepsilon) \times \mathbb{T}) = \{(t-\varepsilon, w) : t < 0, (t,w) \in \partial(W_{T,0})\}.$$

Now,

$$\begin{aligned} 1 - P(S_t \in \mathbb{T}\backslash\{-1\}, \forall t \in (-\varepsilon, 0) \mid S_{-\varepsilon} \in H) \\ &= P(S_t = 1 \text{ for some } t \in (-\varepsilon, 0) \mid S_{-\varepsilon} \in H) \\ &\leq c_1 e^{-c_2/\varepsilon}, \end{aligned}$$

for some finite positive constants c_1 and c_2 depending only on the rate of the Brownian motion S_t. To obtain the last inequality, we have noted that there is a positive distance between H and the point -1, and so we can get a bound of the form $c_1 e^{-c_2/\varepsilon}$ for the probability of a Brownian motion crossing this distance within a time ε. Hence,

$$\begin{aligned}\beta(T) &- \alpha(T,\varepsilon)\\ &= H(W_{T_0})(1 - P(S_t \in \mathbb{T}\backslash\{-1\}, \forall t \in (-\varepsilon, 0) \mid S_{-\varepsilon} \in H)) \\ &\leq H(W_{T,0})c_1 e^{-c_2/\varepsilon} \leq c_1 e^{-c_2/\varepsilon},\end{aligned} \tag{7.3}$$

for every $\varepsilon > 0$. To obtain a contradiction, suppose now that (7.1) is always false. Then, for every $\varepsilon > 0$ and each finite positive T we have $\alpha(T,\varepsilon) \leq \beta(T+\varepsilon)$ as $\beta(T+\varepsilon) = H(W^{\blacktriangleleft}_{T,\varepsilon})$. Thus, by (7.3) we must have $\beta(T) - \beta(T+\varepsilon) \leq \beta(T) - \alpha(T,\varepsilon) \leq c_1 e^{-c_2/\varepsilon}$. Since β is monotone decreasing, it follows that

$$|\beta(T) - \beta(T')| \leq c_1 e^{-c_2/|T-T'|}.$$

Dividing both sides by $T - T'$ and taking the limit as $T' \to T$ we see that β is differentiable everywhere on $(0,\infty)$ and that $\beta' \equiv 0$. Hence, β is constant. But, $\lim_{T\to 0+} \beta(T) = |H| = \frac{1}{2}$ while $\lim_{T\to\infty} \beta(T) = 0$, so that β is not constant. Hence we have a contradiction, and so (7.1) must be true for some choice of ε and T.

Note that the counterexample is of the form (7.2). It is easy to see that the complement of a domain defined by (7.2) is connected, so that such a domain corresponds to a simply connected domain on the disc. By Theorem 2.3, then, it follows that Conjecture 1.6′ does hold for our present counterexamples to the $p = 1$ analogue of Conjecture 1.6′. Hence, these particular counterexamples cannot be used to disprove Conjecture 1.6′.

7.2. Exit times of Brownian motion

We now should mention something about exit times of Brownian motion, given their connection with the Γ_Φ as discussed by Burkholder [28]. As before, let $\tau_W = \inf\{t \geq 0 : B_t \notin W\}$. It makes little sense to ask whether we have

$$P^{-\infty}(\tau_W > \lambda) \leq P^{-\infty}(\tau_{W^{\blacktriangleleft}} > \lambda),$$

for each $\lambda \geq 0$, because if $(-\infty, T] \times \mathbb{T} \subseteq W$ for some $T > -\infty$ then both sides of the above inequality will equal 1 for all finite λ. We may, however, obtain such a result in the setting of

§3 and §5 for the symmetrization $\doteq$ where the domains W are subsets of the full cylinder $\mathbb{V}$, and the Brownian motions start somewhere on $\{0\} \times \mathbb{T}$. Indeed, if $z \in \{0\} \times \mathbb{T}$ and W is an arbitrary domain on $\mathbb{V}$ (not necessarily circularly symmetric) then

$$P^z(\tau_W > \lambda) \leq P^z(\tau_{W^{\doteq}} > \lambda),$$

for all $\lambda \geq 0$. The proof is almost identical to that of Haliste [56, Thm. 8.1] (see also [25]).

At this point we demonstrate why the cylindrical setting is more natural in terms of Brownian motion. For, suppose now that B_t is Brownian motion in the plane. Let $\tau_U = \inf\{t \geq 0 : B_t \notin U\}$ as before. We may ask whether it is true that

$$P^0(\tau_U > \lambda) \leq P^0(\tau_{U^\star} > \lambda) \tag{7.4}$$

for all $\lambda \geq 0$ if U is a planar domain. We might wish to restrict this question to circularly symmetric U. But, as it turns out, this inequality is false, even for circularly symmetric U. (The technical reason why a proof along the lines of [56, Proof of Thm. 8.1] cannot be constructed is the lack of an appropriate convolution-rearrangement inequality caused by the fact that the rearrangement $U \mapsto U^\star$ is not measure preserving.) We now prove that (7.4) is not true in general.

Example 7.1. Fix a small positive δ, a large positive R and an even larger positive L. Let

$$U_{\delta,L,R} = \{z : |z| < \delta\} \cup \{x + iy : 0 \leq x \leq L, |y| < \delta\} \cup \{z : |z - L| < R\}.$$

The domain $U_{\delta,R,L}$ can be visualized as a balloon tied with a string to the origin, with the radius of the balloon being R, and the string being of width 2δ and length $L - R$. I claim that for appropriate choices of δ, R, L and for every $\lambda \geq \lambda_0$ for some $\lambda_0 = \lambda_0(\delta, R, L) < \infty$, the inequality (7.4) is false for $U = U_{\delta,L,R}$. To prove this, first let

$$h(\delta, L, R) = \sup\{|\operatorname{Im} z| : z \in U^\star_{\delta,L,R}\}.$$

It is easy to verify that for fixed δ and R we have $h(\delta, L, R) \to \delta$ as $L \to \infty$. From now on, then, for any fixed δ and R we will be implicitly choosing a specific $L = L(\delta, R)$ such that $h(\delta, L, R) \leq 2\delta$.

We may write $B_t = (B_t^{(1)}, B_t^{(2)})$, where the $B_t^{(j)}$ are independent Brownian motions on $\mathbb{R}$. Define

$$\psi(\lambda, H) = P^0 \left(\sup_{0 \le t \le \lambda} |B_t^{(1)}| < H \right),$$

for nonnegative λ and positive H. If we put $B_t^{(2)}$ in place of $B_t^{(1)}$ in the above definition this will of course change nothing.

Note that

$$P^x(\sup_{0 \le t \le \lambda} |B_t^{(1)}| < H) \le \psi(\lambda, H) \tag{7.5}$$

for any $x \in \mathbb{R}$. One way to see this is to note that, as is well known, [56, Proof of Thm. 8.1] implies that for any domain D we have

$$P^z(\tau_D \ge \lambda) \le P^{\operatorname{Re} z}(\tau_{D^\boxminus} \ge \lambda), \tag{7.6}$$

for any real λ. Let $D = \{z \in \mathbb{C} : |\operatorname{Im} z| < H\}$. Then,

$$P^z(\tau_D \ge \lambda) = P^{\operatorname{Im} z}(\sup_{0 \le t \le \lambda} |B_t^{(2)}| < H),$$

and $D^\boxminus = D$. Setting $z = ix$ and noting that $B_t^{(1)}$ and $B_t^{(2)}$ behave in exactly the same way, we obtain (7.5) from (7.6).

Inequality (7.5) combined with the Markov property of Brownian motion implies that

$$\psi(S + T, H) \le \psi(S, H)\psi(T, H).$$

Let $\phi(\lambda) = \psi(\lambda, 1)$. By scaling of Brownian motion we have $\psi(\lambda, H) = \phi(\lambda/\sqrt{H})$. As $\phi(S+T) \le \phi(S)\phi(T)$, the function $\log\phi$ is a negative subadditive function on $(0, \infty)$. By [62, Thm. 6.6.1], the limit $\lim_{\lambda\to\infty} \lambda^{-1} \log \phi(\lambda)$ exists and equals $\inf_{\lambda\in(0,\infty)} \lambda^{-1} \log \phi(\lambda)$. Let γ be this limit. Then, since $\phi(\lambda) < 1$ for $t > 0$, it follows that $\gamma < 0$, and so $\phi(\lambda)$ behaves like $e^{\gamma\lambda}$ for large t. We now estimate $P^0(\tau_{U_{\delta,L,R}} > \lambda)$ and $P^0(\tau_{U^\star_{\delta,L,R}} > \lambda)$. Since $h(\delta, L, R) \le 2\delta$, we have $U_{\delta,L,R} \subseteq \mathbb{R} \times (-2\delta, 2\delta)$. Thus,

$$P^0(\tau_{U^\star_{\delta,L,R}} > \lambda) \le P^0(\tau_{\mathbb{R}\times(-2\delta,2\delta)} > \lambda) = \psi(\lambda, 2\delta) = \phi(\lambda/\sqrt{2\delta}). \tag{7.7}$$

On the other hand, let $\varepsilon = \varepsilon(\delta, L, R)$ be the probability that the random walk B_t starting at 0 hits the interval $[L - R/2, L + R/2]$ before hitting $\partial U_{\delta,L,R}$. Clearly $\varepsilon > 0$. Then we may apply the strong Markov property to conclude that

$$P^0(\tau_{U_{\delta,L,R}} > \lambda) \geq \varepsilon \inf_{x\in[L-R/2,L+R/2]} P^x(\tau_{U_{\delta,L,R}} > \lambda).$$

Let $Q_{R,x}$ be the square $(x - \kappa R, x + \kappa R) \times (-\kappa R, \kappa R)$, where $\kappa = \frac{1}{4}(\sqrt{7} - 1) \approx 0.4114$. It is easy to see that for $x \in [L - R/2, L + R/2]$, the square $Q_{R,x}$ is contained in $U_{\delta,L,R}$. Thus,

$$P^0(\tau_{U_{\delta,L,R}} > \lambda) \geq \varepsilon \inf_{x\in[L-R/2,L+R/2]} P^x(\tau_{Q_{R,x}} > \lambda). \tag{7.8}$$

But, by translation invariance and the splitting of B_t into the independent coordinates $B_t^{(1)}$ and $B_t^{(2)}$, we have

$$\begin{aligned} P^x(\tau_{Q_{R,x}} > \lambda) &= P^0(\tau_{Q_{R,0}} > \lambda) \\ &= P^0\left(\sup_{0\leq t\leq\lambda} |B_t^{(1)}| < \kappa R\right) P^0\left(\sup_{0\leq t\leq\lambda} |B_t^{(2)}| < \kappa R\right) \\ &= (\psi(\lambda, \kappa R))^2 \\ &= (\phi(\lambda/\sqrt{\kappa R}))^2. \end{aligned} \tag{7.9}$$

Now, choose R and δ such that $2/\sqrt{\kappa R} < 1/\sqrt{2\delta}$. Then I claim that

$$\lim_{\lambda\to\infty} \frac{(\phi(\lambda/\sqrt{\kappa R}))^2}{\phi(\lambda/\sqrt{2\delta})} = \infty. \tag{7.10}$$

If this claim is just then we are finished by (7.7), (7.8) and (7.9). But to prove the claim it suffices to take a logarithm of the ratio within the limit on the left hand side of (7.10), and to apply the choice of R and δ as well as the fact that $t^{-1}\log\phi(t) \to \gamma < 0$.

8. The Beurling shove theorem and extensions

Let D be a domain in the plane, containing the interval $(-1, 0]$. Let I be a finite union of closed intervals in $[-1, 0)$. Define

$$W(I; D) = \omega(0, \partial D; D\backslash I).$$

Let $I^{\leftarrow}$ be a single interval of the form $[-1,-\alpha]$ such that the logarithmic length of $I^{\leftarrow}$ equals that of I. More precisely, the logarithmic length of I is

$$\rho(I) \overset{\text{def}}{=} \int_I \frac{dx}{|x|}.$$

Thus, $I^{\leftarrow} = [-1, -e^{-\rho(I)}]$. Then Beurling in his thesis has shown that

$$W(I;\mathbb{D}) \geq W(I^{\leftarrow};\mathbb{D}).$$

This inequality is known as Beurling's shove theorem [23, pp. 58–62][2]; the name (suggested by Professor Baernstein) comes from the fact that $I^{\leftarrow}$ can be visualized as I with all of its intervals "shoved" over to the right. A generalization of Beurling's shove theorem to higher dimensions and to some more general domains D has been formulated by Essén and Haliste [45].

Remark 8.1. If D is a star-shaped domain with $-1 \in \partial D$ and $(-1,0] \subseteq D$, then for any finite union I of closed subintervals of $[-1,0)$ we have

$$(D \backslash I)^{\bigstar} = D \backslash I^{\leftarrow}, \tag{8.1}$$

where $(\cdot)^{\bigstar}$ is Marcus' radial rearrangement. Thus, Beurling's shove theorem says precisely that $w_1(\mathbb{D}\backslash I) \leq w_1((\mathbb{D}\backslash I)^{\bigstar})$.

In fact Beurling's shove theorem implies Conjecture 1.6 for the domain $D = \mathbb{D}\backslash I$ (we already alluded to this in §1.3). To see this, we must prove that

$$w_r(\mathbb{D}\backslash I) \leq w_r(\mathbb{D}\backslash I^{\leftarrow})$$

for all $r > 0$. For $r > 1$ this is trivial, since both sides of the above inequality vanish. The case of $r = 1$ has already been done and is a direct consequence of the shove theorem. It remains to consider the case of $r < 1$. Given a subset J of $[-1,0)$, write $J_r = J \cap [-r, 0)$. I claim that the

[2]Unfortunately the thesis [23] is not widely available, but an account of the theorem and of its proof can also be found in [79, §IV.5.4].

following chain is valid:

$$w_r(\mathbb{D}\backslash I) = \omega(0, \partial\mathbb{D}(r); \mathbb{D}(r)\backslash I) \tag{8.2a}$$

$$= \omega(0, \partial\mathbb{D}(r); \mathbb{D}(r)\backslash I_r) \tag{8.2b}$$

$$= \omega(0, \partial\mathbb{D}; \mathbb{D}\backslash r^{-1} I_r) \tag{8.2c}$$

$$\leq \omega(0, \partial\mathbb{D}; \mathbb{D}\backslash (r^{-1} I_r)^{\leftarrow}) \tag{8.2d}$$

$$= \omega(0, \partial\mathbb{D}(r); \mathbb{D}(r)\backslash r[(r^{-1} I_r)^{\leftarrow}]) \tag{8.2e}$$

$$\leq \omega(0, \partial\mathbb{D}(r); \mathbb{D}(r)\backslash (I^{\leftarrow})_r) \tag{8.2f}$$

$$= \omega(0, \partial\mathbb{D}(r); \mathbb{D}(r)\backslash I^{\leftarrow}) \tag{8.2g}$$

$$= w_r(\mathbb{D}\backslash I^{\leftarrow}). \tag{8.2h}$$

We now justify each step here. First, (8.2a) and (8.2h) follow directly from the definitions of w_r. Equalities (8.2b) and (8.2g) are trivial since $\mathbb{D}(r)\backslash I = \mathbb{D}(r)\backslash I_r$ and $\mathbb{D}(r)\backslash I^{\leftarrow} = \mathbb{D}(r)\backslash (I^{\leftarrow})_r$. Also, (8.2c) and (8.2e) follow from the scaling invariance of harmonic measure. Inequality (8.2d) comes from Beurling's shove theorem. Only inequality (8.2f) remains to be verified. It follows from the monotonicity of harmonic measure with respect to increase of the domain as soon as we prove the inclusion

$$(I^{\leftarrow})_r \subseteq r[(r^{-1} I_r)^{\leftarrow}]. \tag{8.3}$$

Now, to prove (8.3), since both sides of (8.3) are possibly empty intervals of the form $[-r, -\alpha]$, it suffices to prove that

$$\rho((I^{\leftarrow})_r) \leq \rho(r(r^{-1} I_r)^{\leftarrow}). \tag{8.4}$$

Without loss of generality assume that $(I^{\leftarrow})_r \neq \varnothing$. Now, the logarithmic length ρ is invariant under dilation and under the $(\cdot)^{\leftarrow}$ operation so that

$$\begin{aligned}
\rho(r(r^{-1}I_r)^{\leftarrow}) &= \rho((r^{-1}I_r)^{\leftarrow}) = \rho(r^{-1}I_r) = \rho(I_r) \\
&= \int_{-r}^{0} 1_{I_r}(x)\frac{dx}{|x|} = \int_{-1}^{0} 1_{I_r \cup (-1,-r]}(x)\frac{dx}{|x|} + \log r \\
&= \rho(I_r \cup (-1,r]) + \log r = \rho((I_r \cup (-1,r])^{\leftarrow}) + \log r \\
&\geq \rho(I^{\leftarrow}) + \log r \\
&= \int_{-1}^{0} 1_{I^{\leftarrow}}(x)\frac{dx}{|x|} + \log r \\
&= \int_{-r}^{0} 1_{I^{\leftarrow}}(x)\frac{dx}{|x|} = \rho((I^{\leftarrow})_r),
\end{aligned}$$

where we have used the fact that $[-1,-r] \subseteq I^{\leftarrow}$ which follows from the assumption that $(I^{\leftarrow})_r \neq \varnothing$.

The following result is implicit in Beurling's proof [23] and in the work of Essén and Haliste [45].

Proposition 8.1 (Beurling [23]). *Let D be a domain in the plane containing the interval $(-1,0]$. Suppose that D satisfies both of the following two conditions:*

(i) *for any fixed $x \in (-1,0)$, the Green's function $g(\cdot,x;D)$ is increasing on the interval $(-1,x)$ and decreasing on the interval $(x,0)$*

(ii) *the domain D is star-shaped.*

Then, for every finite union I of closed subintervals of $[-1,0)$ we have

$$W(I;D) \geq W(I^{\leftarrow};D). \tag{8.5}$$

A proof of a more general result (Theorem 8.2) will be given later.

Given this, to prove Beurling's shove theorem for $D = \mathbb{D}$ it suffices to verify (i), which can easily be done since $g(\cdot,\cdot;\mathbb{D}) = \mathfrak{g}$ has an explicit expression given by (I.8.5), so that (i) can be verified by elementary methods. We now use Proposition 8.1 to show that the inequality (8.5) also holds for two quite different classes of star-shaped domains D. Under assumption (a), the following result generalizes the one-dimensional case of Essén-Haliste's [45, Thm. 1]. Under assumption (b), we shall use Theorem I.8.2 in our proof.

Theorem 8.1. *Let D be a domain in the plane with $(-1,0] \subset D$. Assume that D satisfies at least one of the following auxiliary conditions:*

(a) *D is simply connected and reflection symmetric about the real axis*

(b) *D contains the disc $\mathbb{D}(-\frac{1}{2};\frac{1}{2})$.*

Then condition (i) *of Proposition* 8.1 *holds. Hence if D is also star-shaped, then for any finite union I of closed subintervals of* $[-1,0)$ *we have $W(I;D) \geq W(I^{\leftarrow};D)$.*

A proof will be given after the remarks below. The above theorem was given in Pruss [84].

Remark 8.2. Note that if D is star-shaped then it is automatically simply connected so that (a) in that case only requires reflection symmetry about the real axis.

It is not hard to trace through the proofs to conclude that if D is star-shaped and satisfies at least one of the conditions (a) and (b) of Theorem 8.1, then equality holds in (8.5) if and only if $I = I^{\leftarrow}$.

A discrete analogue of the Beurling shove theorem for $D = \mathbb{D}$ was given in §II.10. It is not known whether an analogue of Theorem 8.1 under either of its auxiliary conditions can also be proved in the discrete setting of that section. A generalization of the shove theorem to a case where $D = \mathbb{D}$ but I is allowed to range over all closed subsets of $[-1,1]\setminus\{0\}$ has been conjectured by Segawa [96, Remark on p. 183].

It can be seen without undue difficulty that the star-shapedness of D does not by itself imply condition (i) of Proposition 8.1. (A counterexample may be constructed by letting $D_\varepsilon = \mathbb{C}\setminus\{-re^{i\varepsilon} : r \geq 1\}$, fixing $0 < \alpha < 1 < \beta < \gamma < \infty$ and noting that if $\varepsilon > 0$ is sufficiently small then $g(\beta,\gamma;D_\varepsilon) < g(\alpha,\gamma;D_\varepsilon)$.) However, the author does not know of any counterexample to the inequality $W(I;D) \geq W(I^{\leftarrow};D)$ for D star-shaped, and so the question of whether the star-shapedness of D is by itself sufficient to guarantee (8.5) appears to be open.

Proof of Theorem 8.1. First assume that condition (a) holds. Let $L = D \cap (-\infty, 0] \supseteq (-1, 0]$. By simple connectivity and reflection symmetry, the set L is an interval. Let f be the Riemann map of D onto $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$. By reflection symmetry it is easy to see that f is real on L and in fact that it is a strictly increasing bijection of L onto the interval $(-1, 0]$. But, $g(r, R; D) = g(f(r), f(R); \mathbb{D})$, and so by the monotonicity of f and by the fact that $\mathbb{D}$ satisfies (i) of Proposition 8.1, it follows that D must also satisfy it.

Now assume instead that condition (b) holds. Given any domain D in the plane and a point $z \in D$, define $D'_z = z + (-z + D)'$, where $(-z + D)'$ is defined as in §I.8 via equation (I.8.1). Then, using condition (b), it is easy to verify on geometric grounds that for any $R \in (-1, 0)$ we have $(-1, 0) \subset D'_R$. But by Theorem 8.2 we have $g(\cdot, R; D)$ radially decreasing away from R on D'_R, hence *a fortiori* also on $(-1, 0)$, so that condition (i) of Proposition 8.1 follows immediately. $\square$

We now state a very general proposition which together with Theorem 8.1 gives a number of results of the type of the shove theorem.

Theorem 8.2. *Let D be a domain in the plane containing the interval $(-1, 0]$. Suppose that D satisfies both of the following two conditions:*

(i) *for any fixed $x \in (-1, 0)$, the Green's function $g(\cdot, x; D)$ is increasing on the interval $(-1, x)$ and decreasing on the interval $(x, 0)$*

(ii) *the domain D is star-shaped.*

Let Φ be a real function on $\overline{D}$. For $\zeta \in \partial D$ let

$$\Psi(\zeta) = \limsup_{z \to \zeta} \Phi(z)$$

as z tends to ζ from within D. Make the following assumptions:

(a) *Φ is subharmonic on D and continuous on $(-1, 0)$; moreover, $\Phi \geq \Psi$ on ∂D*

(b) *the solution h of the Dirichlet problem on D with boundary value $\Phi|_{\partial D}$ on ∂D has $h(x)$ monotone increasing on $(-1, 0)$*

(c) Φ *is monotone decreasing and continuous on* $(-1,0)$.

Then,

$$\Gamma_\Phi(D\backslash I) \leq \Gamma_\Phi(D\backslash I^{\leftarrow}),$$

where $\Gamma_\Phi(U)$ *is the value at* 0 *of the solution of the Dirichlet problem on* U *with boundary value* $\Phi|_{\partial U}$ *on* ∂U.

Remark 8.3. Theorem 8.1 gives us sufficient conditions for (i) and (ii) to hold, so that in combination with it we get a generalized shove theorem.

Remark 8.4. Proposition 8.1 follows from Theorem 8.2. To see this, let Φ be the function which is identically 1 on ∂D and identically 0 in D. Clearly Φ satisfies (a)–(c) (in (b) we will of course have $h \equiv 1$). Moreover, it is clear that in this case $\Gamma_\Phi(D\backslash I) = W(I;D)$ and $\Gamma_\Phi(D\backslash I^{\leftarrow}) = W(I^{\leftarrow};D)$.

Remark 8.5. Theorem 8.2 implies Theorem 2.1. To see this, note that D satisfies conditions (i) and (ii) of Theorem 8.2 by Theorem 8.1, since any circularly symmetric set is reflection symmetric with respect to the real axis. Since $D \neq \mathbb{C}$ (as D is Greenian since we had assumed that all domains are Greenian in this chapter) and as D is circularly symmetric, we will have $L \stackrel{\text{def}}{=} \sup\{L' < 0 : L' \notin D\}$ not equal to $-\infty$. Rescaling the problem if necessary we may assume that $L = -1$, so that $-1 \in \partial D$. Replacing I by $I \cap [-1,0)$, we may assume that I is a subset of $[-1,0)$. Let $\Phi \in \mathcal{F}$. This Φ satisfies (a)–(c). Indeed, (a) and (c) follow from the assumption that $\Phi \in \mathcal{F}$. Now, to verify condition (b) note that $h(z) = \mathrm{LHM}(z,\Phi;D)$ by Theorem I.5.4 and so condition (b) follows by Theorem I.6.5. Then the conclusion of Theorem 2.1 follows from Theorems 8.1 and 8.2 together with equality (8.1), since D is star-shaped and $-1 \in \partial D$.

Theorem 8.2 and Theorem 8.1 together yield the following generalization of Beurling's shove theorem which is interesting even in the case $D = \mathbb{D}$. We call a set A **a smooth arc** if it is the homeomorphic image of either an interval or the circle.

Theorem 8.3. *Let* D *be a star-shaped* C^1 *domain which is reflection symmetric about the real axis and contains the interval* $(-1,0]$. *Let* $A \subseteq D$ *be a smooth arc which is also reflection* *symmetric about the real axis and which intersects the positive real axis. Then,*

$$\omega(0, A; D\backslash I) \leq \omega(0, A; D\backslash I^{\leftarrow}).$$

Proof. Conditions (i) and (ii) of Theorem 8.2 hold by Theorem 8.1(i). Define Φ to vanish on D and be equal to the indicator function 1_A of A on ∂D. Conditions (a) and (c) of Theorem 8.2 then are trivial. It suffices to verify condition (b). We have $\Psi \equiv 0$ so that

$$h(z) = \omega(z, A; D).$$

Let f be a Riemann map of D onto the left half plane $H = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ with $f(0) = -1$ and $f'(0) > 0$. Then, f maps A into interval of the imaginary axis which is symmetric about the positive real axis. Moreover, f is monotone increasing on $(-\infty, 0] \cap D$ and maps $(-\infty, 0] \cap D$ onto $(-\infty, 0]$. (The preceding two sentences both use the reflection symmetry of D.) By conformal invariance of harmonic measure we have

$$h(z) = \omega(f(z), f(A); H).$$

To verify condition (b) of Theorem 8.2, because of the monotonicity of f on $(-\infty, 0] \cap D$ it suffices to prove that

$$\omega(w, B; H)$$

is increasing for $w \in (-\infty, 0)$ and B an interval of $i\mathbb{R}$ symmetric about the point 0. We now do this. Let $-\infty < w' \leq w < 0$. Let $\lambda = w'/w \geq 1$. Then since both the domain H and harmonic measure are invariant under dilations we have

$$\omega(w, B; H) = \omega(\lambda w, \lambda B; H) = \omega(w', \lambda B; H) \geq \omega(w', B; H),$$

as desired. The last inequality followed from the fact that $\lambda B \supseteq B$ and from the monotonicity of harmonic measure with respect to the set being measured. $\square$

It remains to prove Theorem 8.2. The careful reader will notice that the proof bears some resemblance to that of Theorem II.10.1. Indeed, both proofs are based on the ideas of Beurling's original proof of his shove theorem (see [23, pp. 58–62] or [79, §IV.5.4]).

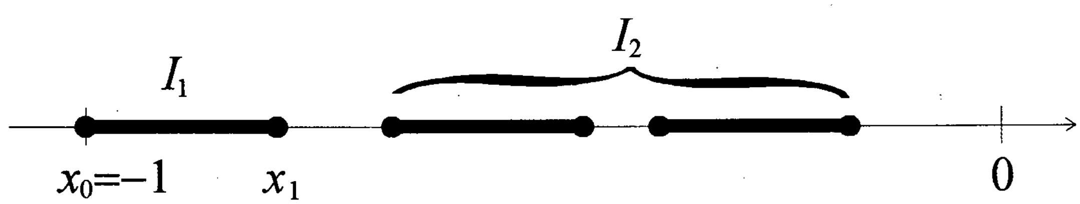


Figure 8.1: The decomposition of I into I_1 and I_2 in a case where $n = 3$.

Proof of Theorem 8.2. Let n be the number of disjoint intervals in the set $(-1,0)\backslash I$. If $n = 1$ then I must be of the form $[-1, -\alpha]$ and $I^{\twoheadleftarrow} = I$ so that we are done. Proceed by induction. Suppose that $n > 1$ and that our Proposition has already been proved for $n - 1$. Let

$$x_0 = \inf[-1,0)\backslash I.$$

Then x_0 either equals -1 or is the right endpoint of one of the disjoint intervals making up I. Let

$$x_1 = \inf\{x > x_0 : x \notin I\}.$$

Then x_1 is the left endpoint of an interval of I, $x_1 > x_0$ and (x_0, x_1) is one of the disjoint intervals making up the set $(-1,0)\backslash I$. Let $\lambda = x_0/x_1 > 0$. Put

$$I_1 = I \cap [-1, x_0]$$

and

$$I_2 = I \cap [x_1, 0).$$

(See Figure 8.1.) Then $I = I_1 \cup I_2$.

Put

$$J_1 = I_1$$

and

$$J_2 = \lambda I_2.$$

Finally, set $J = J_1 \cup J_2$. Then, because x_1 is the left endpoint of an interval of I_2 and x_0 is either -1 or the right endpoint of an interval of I_1 it follows that $(-1,0)\backslash J$ consists of precisely

$n-1$ disjoint intervals, since dilation of I_2 by the factor λ will merge the left-most interval of I_2 with the right-most interval of I_1. The induction hypothesis then implies that

$$\Gamma_\Phi(D\backslash J) \leq \Gamma_\Phi(D\backslash J^{\leftarrow}).$$

Moreover,

$$\rho(I) = \rho(I_1) + \rho(I_2) = \rho(J_1) + \rho(J_2) = \rho(J)$$

because of the dilation invariance of the logarithmic measure ρ. Hence, $I^{\leftarrow} = J^{\leftarrow}$ so that

$$\Gamma_\Phi(D\backslash J) \leq \Gamma_\Phi(D\backslash I^{\leftarrow}).$$

We will thus be done as soon as we can prove that

$$\Gamma_\Phi(D\backslash I) \leq \Gamma_\Phi(D\backslash J). \tag{8.6}$$

(Note the analogy between inequality (8.6) and inequality (II.10.3) which played a similar role in the proof of Theorem II.10.1.)

Let f be the solution of the Dirichlet problem in $D\backslash I$ with boundary value Φ on $\partial(D\backslash I)$. I claim that f is subharmonic on D if we define $f(x) = \Phi(x)$ for $x \in I$. To see this, let Ψ be as in the statement of our proposition and additionally let $\Psi(z) = \Phi(z)$ for $z \in D\backslash I$. Let g be the solution of the Dirichlet problem on $D\backslash I$ with boundary value Ψ on $\partial(D\backslash I)$. Then, by the subharmonic function inequality [39, inequality (1.VIII.8.4)] which is essentially a refined version of Theorem I.4.2 we have $g \geq \Phi$ everywhere on D.

Now to show that f is subharmonic on D, it suffices to verify the subharmonicity at every point x of $I \cap D$ since f is harmonic everywhere else. Such a point x is regular (with respect to the domain $D\backslash I$) by Theorem I.5.3 since I is a collection of intervals and lies in the complement of D. Hence, f is continuous at x. Moreover, we have $f \geq g$ on D since $\Phi \geq \Psi$ everywhere. But $g(x) = f(x) = \Phi(x)$ by regularity of x and continuity of Φ and Ψ at x. We have already seen that $g \geq \Phi$ on D. Hence $f \geq \Phi$ on D. Then, if $r > 0$ is sufficiently small, we have

$$f(x) = \Phi(x) \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(x;r)} \Phi \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(x;r)} f,$$

where we have used the subharmonicity of Φ on D. Hence f is indeed subharmonic at x, and thus everywhere on D.

Note that D is regular by Theorem I.5.3 since D is star-shaped so that for each point of its boundary the ray pointed directly away from the origin must lie in the complement of D.

I claim that h is the least harmonic majorant of f on D, where h is as in condition (b) of our Proposition. To see this, note first that h indeed is harmonic on D. We must prove that $h \geq f$ there. Let $\phi = h - f$. Then, ϕ is the solution of the Dirichlet problem with boundary value 0 on ∂D and $h - f$ on $I \cap D$. But on $I \cap D$, f agrees with Φ. But h is larger than Φ on ∂D, and so by the maximum principle must everywhere be larger than Φ and hence must be larger than f on $I \cap D$ so that ϕ indeed is positive on $\partial(D\backslash I)$ and hence is positive on all of D by the maximum principle. Hence h is a harmonic majorant of f. Moreover, since ϕ vanishes on ∂D it follows that we must indeed have h equal to the least harmonic majorant of f.

Then, by the Riesz decomposition (Theorem I.5.11) write

$$f(z) = h(z) - \int_D g(z, w; D)\, d\mu_f.$$

But f is harmonic on $D\backslash I$ so that the support of μ_f must lie within I (Remark I.5.9). Hence,

$$f(z) = h(z) - \int_{I_1} g(z, w; D)\, d\mu_f - \int_{I_2} g(z, w; D)\, d\mu_f.$$

Let

$$f_k(z) = \int_{I_k} g(z, w; D)\, d\mu_f,$$

for $k = 1, 2$. Then f_k is a harmonic function on $D\backslash I_k$ (this follows from the harmonicity of $g(\cdot, w; D)$ away from w). Define

$$F(z) = h(z) - f_1(z) - f_2(\lambda^{-1}z)$$

on D. Then, F is harmonic on $D\backslash J$ as can be easily seen (use star-shapedness to note that if $z \in D$ then $\lambda^{-1}z \in D$ as well). Let $\tilde{f}$ be the solution of the Dirichlet problem on $D\backslash J$ with boundary value Φ on $\partial(D\backslash J)$. I claim that $F \leq \tilde{f}$. Suppose that this is true. Then, we will have

$$\begin{aligned}\Gamma_\Phi(D\backslash I) &= f(0) = h(0) - f_1(0) - f_2(0)\\ &= h(0) - f_1(0) - f_2(\lambda^{-1}0) = F(0) \le \tilde{f}(0) = \Gamma_\Phi(D\backslash J),\end{aligned}$$

and the proof of (8.6) will be complete.

Hence we must verify that $F \le \tilde{f}$. By the maximum principle since both F and $\tilde{f}$ are harmonic in $D\backslash J$ it suffices to show that if $z_n \to \zeta \in \partial(D\backslash I)$ with $z_n \in D\backslash I$ then

$$\limsup_n (F(z_n) - \tilde{f}(z_n)) \le 0.$$

Note that $F(z_n) - \tilde{f}(z_n) = f_2(z_n) - f_2(\lambda^{-1}z_n)$. Suppose first that $z \in \partial D$. Since we may neglect subsets of the boundary with null harmonic measure, we may assume that $z \notin \overline{D \cap J}$ since $\partial D \cap \overline{D \cap J}$ can contain at most one point by the star-shapedness of D. Then, recall that D was a regular domain and that $f - h$ was the solution of a Dirichlet problem in $D\backslash I$ with boundary value 0 on ∂D. The boundary value function for this Dirichlet problem is continuous on $\partial D\backslash\overline{D \cap I}$ (indeed, it is identically zero there) so that it follows that $(f-h)(z_n) \to 0$. Hence, $f_1(z_n)+f_2(z_n) \to 0$. Since both f_1 and f_2 are positive, it follows that $\limsup f_2(z_n) = 0$. Hence,

$$\limsup_n (F(z_n) - \tilde{f}(z_n)) = \limsup_n (f_2(z_n) - f_2(\lambda^{-1}z_n)) \le 0,$$

as desired.

Now consider $z \in J$. Set both $\tilde{f}$ equal to Φ on J and f equal to Φ on I. Regularity of $D\backslash I$ and $D\backslash J$ (easy to check via Theorem I.5.3) shows that F, f and $\tilde{f}$ are continuous in D since Φ is continuous on $I \cap D$. Hence, what we must verify is that $F(z) \le \tilde{f}(z)$. Now, $\tilde{f}(z) = \Phi(z)$. Thus, we must prove that

$$F(z) \le \Phi(z).$$

Consider first the case where $z \in J_1 = I_1$. Then,

$$\begin{aligned}F(z) &= h(z) - f_1(z) - f_2(\lambda^{-1}z)\\ &\le h(z) - f_1(z) - f_2(z)\\ &= \Phi(z),\end{aligned}$$

where the inequality followed from the fact that $g(\lambda^{-1}z, w; D) \geq g(z, w; D)$ for $w \in I_2$ and $z \in I_1$ since then $z \leq \lambda^{-1}z \leq w$ and we may apply assumption (i). Consider now the case where $z \in J_2$. Then, $\lambda^{-1}z \in I_2$ and

$$\begin{aligned} F(z) &= h(z) - f_1(z) - f_2(\lambda^{-1}z) \\ &\leq h(\lambda^{-1}z) - f_1(z) - f_2(\lambda^{-1}z) \\ &\leq h(\lambda^{-1}z) - f_1(\lambda^{-1}z) - f_2(\lambda^{-1}z) \\ &= f(\lambda^{-1}z) \\ &= \Phi(\lambda^{-1}z) \\ &\leq \Phi(z), \end{aligned}$$

as desired. To justify the above inequalities, note that the first inequality followed from assumption (b) on the monotonicity of h. The second inequality followed from the inequality $g(z, w; D) \geq g(\lambda^{-1}z, w; D)$ valid for $z \in J_2$ and $w \in I_1$ by assumption (i) as then $w \leq z \leq \lambda^{-1}z$. The final inequality followed from assumption (c).

Hence we have seen that indeed $F \leq \Phi$ on J and our proof is complete. □

9. A discrete one-dimensional analogue

9.1. Statement of results

Fix $p \in [0, 1]$. Let $\{r_i^p : i \in \mathbb{Z}_0^+\}$ be a random walk on $\{1, 2, \ldots, N+1\}$, with $r_0^p = 1$,

$$P(r_{i+1}^p = r_i^p + 1 \mid r_i^p) = p,$$

$$P(r_{i+1}^p = n - 1 \mid r_i^p = n) = 1 - p, \qquad \text{if } n > 1,$$

and

$$P(r_{i+1}^p = 1 \mid r_i^p = 1) = 1 - p.$$

Thus, we have a simple random walk on a "blind alley," with the boundary condition that at the "wall" (i.e., at 1) when we try to go to the left then we stay put. The open end of the blind alley is at $N + 1$.

Let $s_1, s_2, \ldots, s_N \in [0,1]$ be given. Every time the random walk r_i^p is at a point $n \in \{1, 2, \ldots, N\}$, let there be a new danger (independent of anything that had happened until that time, and in particular independent of the outcomes of any previous visits to the point n) and let the probability of surviving it be s_n. Let $P_N^p(s_1, \ldots, s_N)$ be the probability that the random walk has survived all the time up to its arrival at the point $N+1$. More precisely, let $X_0, X_1, \ldots$ be random variables which are independent and identically uniformly distributed on $[0,1]$. Let

$$T_N = \inf\{i \geq 0 : r_i^p = N+1\}.$$

Of course $P(T_N < \infty) = 1$ if $p > 0$. Then we have

$$P_N^p(s_1, \ldots, s_N) = P\left(\bigcap_{i=0}^{T_N - 1} \{X_i \leq s_{r_i^p}\}\right).$$

Note that

$$P_N^1(s_1, \ldots, s_N) = s_1 s_2 \ldots s_N, \tag{9.1}$$

$$P_N^0(s_1, \ldots, s_N) \equiv 0$$

and

$$P_N^p(1, \ldots, 1) = 1,$$

for every $p > 0$. Since we have a random walk in a blind alley, and there are dangers, we may term the model a **random walk in a dangerous blind alley**.

Theorem 9.1. *Let $s_1, \ldots, s_N \in [0,1]$, and let $s_1^\curlyvee, \ldots, s_N^\curlyvee$ be $s_1, \ldots, s_N$ rewritten in decreasing order. Then for $p \in [0,1]$ we have*

$$P_N^p(s_1, \ldots, s_N) \leq P_N^p(s_1^\curlyvee, \ldots, s_N^\curlyvee),$$

with equality if and only if at least one of the following conditions holds:

(a) $(s_1, \ldots, s_N) = (s_1^\curlyvee, \ldots, s_N^\curlyvee)$

(b) $s_k = 0$ *for some* $k \in \{1, \ldots, N\}$

(c) $p = 1$

(d) $p = 0$.

This result is analogous to an inequality of Essén [43, Thm. 2] concerning rearrangement in a certain second order difference equation. His difference equation is very similar to that which must be solved to compute $P_N^{1/2}(s_1, \ldots, s_N)$, but there are still some essential differences. We will say more regarding the work of Essén in §9.3, below, where we shall state the actual difference equations whose solution gives $P_N^{1/2}$, and in §9.4 where shall discuss the connection with Essén's analogous continuous case [42, Thm. 5.2].

It is quite possible that Essén's methods [43] could be adapted to prove Theorem 9.1, at least in the case $p = \frac{1}{2}$, even though his results do not appear to apply directly. However, we prefer to use different tactics (keeping the same overall strategy) which, in an elementary way, exploit the linearity properties of a function appearing in the explicit formula for P_N^p. Our proof will be given in §9.2. Finally, it should be noted that it does not seem that the methods of Baernstein [11] can be used to prove results like Theorem 9.1.

The heuristics behind Theorem 9.1 say that if we consider the random walk only until such time as it hits the point $N+1$, then it will spend more time further away from this point than it does nearer to it, so we will improve safety if we reorder the dangers so the more dangerous ones are near $N+1$ where the random walk spends less time. The author has not found a way of making this intuition into a rigorous proof. One might hope to find a probabilistic proof along these lines, but no such proof appears to be available right now, and it does not appear at all easy to produce such a proof.

If $p = \frac{1}{2}$ then Theorem 9.1 may be thought of as a discrete one-dimensional analogue of Conjecture 1.6. The author is grateful to Professor Albert Baernstein II for suggesting that the author also consider the case $p \neq \frac{1}{2}$.

Theorem 9.1 says that $P_N^p(s_1, \ldots, s_N)$ increases if we rewrite the s_n in decreasing order. One might hope that on the other hand if we rewrite them in increasing order, then $P_N^p(s_1, \ldots, s_N)$ decreases. However, the hope would be in vain, as can be seen from the following theorem.

Theorem 9.2. *Fix $p \in (0,1)$. Then, for all sufficiently small $\varepsilon > 0$ we have*

$$P_4^p(\varepsilon, 1, \varepsilon, 1) < P_4^p(\varepsilon, \varepsilon, 1, 1).$$

We now give a certain simple result which is not *completely* trivial.

Theorem 9.3. *Let $s_1, \ldots, s_N \in [0,1]$ be given. Fix $j \in \{1, \ldots, N\}$. Then for $p \in [0,1]$ we have*

$$P_N^p(s_1, \ldots, s_N) \le P_{N-1}^p(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N), \tag{9.2}$$

with the obvious conventions if j is 1 or N. Moreover, equality holds if and only if at least one of the following conditions holds:

(a) *there is a $k \in \{1, \ldots, j-1, j+1, \ldots, N\}$ with $s_k = 0$*

(b) *$s_k = 1$ for every $k \in \{1, \ldots, j\}$*

(c) *$p = 1$ and $s_j = 1$*

(d) *$p = 0$.*

Intuitively Theorem 9.3 says that if we make a dangerous road shorter by removing a segment then the road becomes safer for a random walk. We will give a proof of Theorem 9.3 in §9.2 as a by-product of our proof of Theorem 9.1. This result is an analogue to the first part of Problem 3.1, in that the passage from $s_1, \ldots, s_N$ to $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N$ is analogous to the cutting operation in that Problem.

Now, let $s_1, s_2, \cdots \in [0,1]$ be an infinite sequence. Define the random walk r_i^p on $\mathbb{Z}^+$ with the same transition probabilities as the previous walk on $\{1, \ldots, N+1\}$. Let L_s be the first time that the random walk fails to survive a step. More precisely, we define

$$L_s = \inf\{i \ge 0 : X_i > s_{r_i}\}.$$

The proof of the following result is given in [87] where it is done via the methods of §II.9.3. We shall not give the proof in this thesis.

Theorem 9.4. *Let $s_1, s_2, \cdots \in [0,1]$ and let $s_1^\curlyvee, s_2^\curlyvee, \ldots$ be the decreasing rearrangement of the s_i. Let $p \in [0, \frac{1}{2}]$. Then*

$$P(L_s > n) \leq P(L_{s^\curlyvee} > n),$$

for every $n \geq 0$.

It is not known whether the condition $p \in [0, \frac{1}{2}]$ can be relaxed to $p \in [0,1]$, although it is easy to see that Theorem 9.4 does hold for $p = 1$.

Open Problem 9.1. Does Theorem 9.4 also hold for $p \in (\frac{1}{2}, 1)$?

Now, fix $p \in [0,1]$. Let Φ be a real valued function on $\mathbb{Z}^+$ satisfying the "convexity" (one might also use the term "subharmonicity") condition

$$\Phi(n) \leq (1-p)\Phi(n-1) + p\Phi(n+1), \tag{9.3}$$

for $n \in \mathbb{Z}^+$, where $\Phi(0) \stackrel{\text{def}}{=} \Phi(1)$. The condition (9.3) is equivalent to positing that $\Phi(r_i)$ is a submartingale. It is easy to inductively see (starting with the fact that $\Phi(1) = \Phi(0)$ so that $\Phi(1) \geq \Phi(0)$) that (9.3) implies that Φ is increasing.

Open Problem 9.2. Does it follow from (9.3) that

$$E[\Phi(r_{L_s})] \leq E[\Phi(r_{L_{s^\curlyvee}})]?$$

If $p = \frac{1}{2}$ then this is a one-dimensional discrete analogue of Conjecture 1.4. Now we just wish to note that some sort of convexity condition like (9.3) on Φ in addition to the increasing character of Φ is necessary if $p \in (0,1)$. For, if we do not have this condition, then we may adapt our counterexample to Conjecture 1.5. Set $s_1 = \frac{1}{2}$, $s_2 = 0$, $s_3 = \frac{1}{2}$ and $s_4 = s_5 = \cdots = 0$, and let $\Phi(n)$ be 0 for $n \leq 1$ and 1 otherwise; a simple computation then shows that then the answer to Problem 9.2 would be negative. Note also that if we let $s_{N+1} = s_{N+2} = \cdots = 0$ and set $\Phi(n) = \max(n - N, 0)$ then $E[\Phi(r_{L_s})] = P_N^p(s_1, \ldots, s_N)$ so that Theorem 9.1 is a special case of Problem 9.2.

Finally, the following result should surprise no one, but we state it for completeness. If we increase the probability of going towards our goal then certainly the probability of arriving at it should increase.

Theorem 9.5. *Let* $0 \le p < r \le 1$ *and let* $s_1, \ldots, s_N \in [0,1]$. *Then*

$$P_N^p(s_1, \ldots, s_N) \le P_N^r(s_1, \ldots, s_N),$$

with equality if and only if one of the following conditions holds:

(a) $s_k = 0$ *for some* $k \in \{1, \ldots, N\}$

(b) $s_1 = \cdots = s_N = 1$ *and* $p > 0$.

We now outline a proof of Theorem 9.5, leaving the details as an exercise to the reader. Consider a more general case of a random walk defined as above, but instead of having a constant probability p of going to the right and $1-p$ of going to the left, allow this probability to vary with position, so that the probability of moving to the right from $n \in \{1, \ldots, N\}$ is $t_n \in [0,1]$ and the probability of moving to the left is $1-t_n$. As before, moving to the left from 1 results in standing still. Just as before, we can define the probability of the random walk getting from 1 to $N+1$ without having fallen into any of the dangers. I claim that this probability will increase if any one of the t_n is increased; clearly this would a more general result than Theorem 9.5 (though of course we would have to ensure that appropriate conditions of equality hold, the verification of which we leave as an exercise for the reader).

To prove the claim, fix n. Assume $n > 1$; the case $n = 1$ is handled similarly. We want to see the dependence on t_n. So, let A be the probability that a random walk (with movement probabilities defined by the t_j) starting from $n-1$ will eventually arrive at n without having fallen into any of the dangers. Let B be the probability that such a random walk starting from $n+1$ eventually arrives at n without having fallen into any of the dangers and without having first arrived at $N+1$. Let C be the probability that such a random walk random when started from $n+1$ eventually arrives at $N+1$ without having fallen into any of the dangers and without having first arrived at n. Finally, let P be the probability that a random walk starting at n eventually arrives at $N+1$ without having fallen into any of the dangers. The probability of a random walk from 1 arriving safely at $N+1$ is proportional to P, so we need only compute how P depends on t_n. Also, A, B and C are independent of t_n and satisfy the equation

$$P = s_n(1-t_n)AP + s_n t_n(BP + C).$$

From this point on it is an elementary exercise to verify that P increases with t_n, and to determine the conditions under which the increase fails to be strict.

9.2. Various useful identities, formulae and some proofs

In this section we shall prove Theorems 9.1, 9.2 and 9.3, assuming an explicit formula (Theorem 9.6, below) for $P_N^p(s_1, \ldots, s_N)$. The proof of this formula will be given in §9.3.

First we note a simple probabilistic identity which will later prove to be of use. Suppose $p \in (0,1)$, $N \geq 2$ and $s_1 = 1$. Then it does not matter how long the random walk spends at the point 1, since it will survive to eventually leave 1 and go to 2. Whenever it subsequently goes left from 2, it will survive until its eventual return to 2. Hence, we may form a certain correspondence between random walks on $\{1, 2, \ldots, N\}$ and those on $\{2, \ldots, N\}$ in such a way as to prove that

$$P_N^p(1, s_2, \ldots, s_N) = P_{N-1}^p(s_2, \ldots, s_N). \tag{9.4}$$

It is trivial to also verify that this continues to hold if $p \in \{0, 1\}$.

Now, for positive n, let $\psi_{N,n}(a_1, \ldots, a_N)$ be the sum of all terms of the form

$$a_{i_1} a_{i_1+1} a_{i_2} a_{i_2+1} \cdots a_{i_n} a_{i_n+1} \tag{9.5}$$

with

$$1 \leq i_1 < i_1 + 1 < i_2 < i_2 + 1 < \cdots < i_n < i_n + 1 \leq N.$$

Explicitly we have

$$\psi_{N,n}(a_1,\ldots,a_N) = \sum_{i_1=1}^{N-2n+1} \sum_{i_2=i_1+2}^{N-2n+3} \cdots \sum_{i_n=i_{n-1}+2}^{N-1} a_{i_1}a_{i_1+1}a_{i_2}a_{i_2+1}\ldots a_{i_n}a_{i_n+1},$$

with the convention that empty sums are equal to zero. Clearly $\psi_{N,n}$ is a function of N variables, is linear in each variable if the others are fixed, and vanishes identically for $2n > N$. Let

$$\Psi_N(a_1,\ldots,a_N) = 1 + \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^n \psi_{N,n}(a_1,\ldots,a_N),$$

for $N \in \mathbb{Z}_0^+$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Note that $\Psi_N \equiv 1$ for $N \in \{0,1\}$.

Now, I claim that

$$\Psi_{N+1}(a_1,\ldots,a_{N+1}) = \Psi_N(a_2,a_3,\ldots,a_{N+1}) - a_1a_2\Psi_{N-1}(a_3,\ldots,a_{N+1}), \tag{9.6}$$

for $N \geq 1$. This identity is central to our work.

The proof of the identity is not very difficult. For, take one of the terms in $\Psi_{N+1}(a_1,\ldots,a_{N+1})$. It will be either of the form

$$(-1)^n a_{i_1}a_{i_1+1}a_{i_2}a_{i_2+1}\ldots a_{i_n}a_{i_n+1}$$

with $1 \leq n \leq \lfloor \frac{N+1}{2} \rfloor$ and

$$1 \leq i_1 < i_1+1 < i_2 < i_2+1 < \cdots < i_n < i_n+1 \leq N+1,$$

or else it will be identically 1. If a_1 occurs in this term then $i_1 = 1$ so that a_2 must also occur in it. It is easy to see by the definitions that it must then also be a term of $-a_1a_2\Psi_{N-1}(a_3,\ldots,a_{N+1})$. On the other hand, if a_1 fails to occur in the term, then this term must be a term of $\Psi_N(a_2,\ldots,a_{N+1})$. Conversely, it is easy to verify that any term of the right hand side of (9.6) is also a term of the left hand side, and the proof of the claim is complete.

As a corollary of (9.6), we can see that

$$\Psi_{N+1}(0,a_1,\ldots,a_N) = \Psi_N(a_1,\ldots,a_N), \tag{9.7}$$

for $N \geq 1$. For $N = 0$ this also holds trivially, and hence (9.7) is valid for all $N \geq 0$. Also, by (9.6) and (9.7) we obtain

$$\Psi_{N+1}(a_1, 0, a_3, \ldots, a_{N+1}) = \Psi_N(0, a_3, \ldots, a_{N+1}) = \Psi_{N-1}(a_3, \ldots, a_{N+1}), \tag{9.8}$$

for $N \geq 1$.

Note that $\Psi_N(a_1, \ldots, a_N) = \Psi_N(a_N, \ldots, a_1)$, so that

$$\Psi_N(a_1, \ldots, a_N) = \Psi_{N-1}(a_1, \ldots, a_{N-1}) - a_N a_{N-1} \Psi_{N-2}(a_1, \ldots, a_{N-2}), \tag{9.9}$$

whenever $N \geq 2$, by (9.6).

Now, define

$$\phi_n(p) = \begin{cases} p, & \text{if } n \text{ is even,} \\ 1-p, & \text{if } n \text{ is odd.} \end{cases}$$

Note that $\phi_{n+1}(p) = \phi_n(1-p) = 1 - \phi_n(p)$ for every n and p. Because the expressions that will be involved would be unmanageable otherwise, it will be useful to have two more abbreviations. Let

$$\overline{\Psi}_N^p(a_1, \ldots, a_N) = \Psi_{N+1}(1, \phi_1(p)a_1, \ldots, \phi_N(p)a_N)$$

and

$$\Psi_N^p(a_1, \ldots, a_N) = \Psi_N(\phi_1(p)a_1, \ldots, \phi_N(p)a_N).$$

At times the reader will be implicitly expected to be able to use the definitions to mentally rewrite the $\overline{\Psi}_N^p$ and Ψ_N^p in terms of the Ψ_N.

The following result then gives a formula for the probability of traversal; a proof will be given in §9.3.

Theorem 9.6. *For $p \in (0, 1]$ and $s_1, \ldots, s_N \in [0, 1]$ we have*

$$P_N^p(s_1, \ldots, s_N) = \frac{p^N s_1 s_2 \ldots s_N}{\overline{\Psi}_N^p(s_1, s_2, \ldots, s_N)}. \tag{9.10}$$

Moreover, the denominator is always strictly positive under the above conditions.

Assuming Theorem 9.6, I claim that

$$\overline{\Psi}^p_N(1, a_2, \dots, a_N) = p\overline{\Psi}^p_{N-1}(a_2, \dots, a_N) \tag{9.11}$$

For, if p is fixed then both sides are linear in any one variable when the others are fixed, so that it is enough to verify (9.11) for $a_2, \dots, a_N \in (0, 1]$. Moreover, both sides of (9.11) are continuous in p and hence it suffices to consider $p \in (0, 1]$. But under such circumstances (9.11) follows from (9.4) and Theorem 9.6. Note that if $p = \frac{1}{2}$ then (9.11) takes the particularly simple form

$$\Psi_{N+1}(1, \tfrac{1}{2}, x_2, \dots, x_N) = \tfrac{1}{2}\Psi_N(1, x_2, \dots, x_N).$$

Lemma 9.1. *Let $N \geq 1$ and fix $a_1, \dots, a_N \in [0, 1]$. Suppose $p \in (0, 1]$. Then*

$$\Psi_{N+1}(x, \phi_1(p)a_1, \dots, \phi_N(p)a_N)$$

is strictly positive for every $x \in [0, 1]$.

Proof. Fix $a_1, \dots, a_N$. Now,

$$x \mapsto \Psi_{N+1}(x, \phi_1(p)a_1, \dots, \phi_N(p)a_N)$$

is a linear function and hence it suffices to verify its strict positivity for $x \in \{0, 1\}$. If $x = 1$, then the strict positivity immediately follows from the "moreover" in Theorem 9.6. Now, for $x = 0$, by (9.8) we may write

$$\begin{aligned}
&\Psi_{N+1}(0, \phi_1(p)a_1, \dots, \phi_N(p)a_N) \\
&\qquad = \Psi_{N+2}(1, 0, \phi_1(p)a_1, \dots, \phi_N(p)a_N) \\
&\qquad = \Psi_{N+2}(1, \phi_1(1-p) \cdot 0, \phi_2(1-p)a_1, \dots \phi_{N+1}(1-p)a_N).
\end{aligned}$$

The strict positivity of this again immediately follows from the "moreover" of Theorem 9.6. $\square$

We also note that

$$\Psi_{M+N+1}(a_1, \dots, a_M, 0, b_1, \dots, b_N) = \Psi_M(a_1, \dots, a_M)\Psi_N(b_1, \dots, b_N). \tag{9.12}$$

The easiest way to prove this is to note that every term of the right hand side is a term of the left hand side and vice versa, much as in the proof of (9.6).

Finally, it is easy to use the fact that $\phi_n(p)\phi_{n+1}(p) = p(1-p) = \phi_n(1-p)\phi_{n+1}(1-p)$ for every n together with the way that Ψ_M is defined to show that

$$\Psi_M(\phi_1(p)a_1, \ldots, \phi_M(p)a_M) = \Psi_M(\phi_1(1-p)a_1, \ldots, \phi_M(1-p)a_M). \tag{9.13}$$

We can write this concisely as $\Psi_M^p = \Psi_M^{1-p}$. Now, recalling that $1 - \phi_n(p)$ is either p or $1-p$ for any n, and applying (9.12), followed by (9.13) if necessary, we see that

$$\begin{aligned} \overline{\Psi}^p_{M+N+1}(a_1, \ldots, a_M, 0, b_1, \ldots, b_N) &= \overline{\Psi}^p_M(a_1, \ldots, a_M)\Psi^r_N(b_1, \ldots, b_N) \\ &= \overline{\Psi}^p_M(a_1, \ldots, a_M)\Psi^p_N(b_1, \ldots, b_N), \end{aligned} \tag{9.14}$$

where $r = 1 - \phi_{M+2}(p)$.

Lemma 9.2. *Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be in $[0,1]$. Let $p \in (0,1)$. Suppose that*

$$\min(a_1, \ldots, a_m) \geq \max(b_1, \ldots, b_n). \tag{9.15}$$

Then

$$\overline{\Psi}^p_{m-1}(a_1, \ldots, a_{m-1})\Psi^p_n(b_1, \ldots, b_n) \geq \overline{\Psi}^p_m(a_1, \ldots, a_m)\Psi^p_{n-1}(b_2, \ldots, b_n). \tag{9.16}$$

Moreover if equality holds then at least one of the a_j vanishes.

Proof. We proceed by induction on $\max(m,n)$. If $\max(m,n) = 1$ then (9.16) becomes

$$1 \geq \overline{\Psi}^p_1(a_1) = 1 - \phi_1(p)a_1.$$

This is clearly true, and strict inequality holds unless $a_1 = 0$.

Now suppose that Lemma 9.2 has been proved when $\max(m,n) = N-1$ and also assume that we have $\max(m,n) = N > 1$. By (9.6) and (9.9), we see that (9.16) is equivalent to the inequality

$$\begin{aligned}
&\overline{\Psi}^p_{m-1}(a_1,\ldots,a_{m-1})\Psi^p_{n-1}(b_2,\ldots,b_n)\\
&\quad-\overline{\Psi}^p_{m-1}(a_1,\ldots,a_{m-1})\phi_1(p)\phi_2(p)b_1b_2\Psi^p_{n-2}(b_3,\ldots,b_n)\\
&\geq\overline{\Psi}^p_{m-1}(a_1,\ldots,a_{m-1})\Psi^p_{n-1}(b_2,\ldots,b_n)\\
&\quad-\phi_{m-1}(p)\phi_m(p)a_{m-1}a_m\overline{\Psi}^p_{m-2}(a_1,\ldots,a_{m-2})\Psi^p_{n-1}(b_2,\ldots,b_n).
\end{aligned}$$

Note that we have implicitly used (9.13) after applying (9.6). Clearly our last inequality is equivalent to

$$\begin{aligned}
&p(1-p)a_{m-1}a_m\overline{\Psi}^p_{m-2}(a_1,\ldots,a_{m-2})\Psi^p_{n-1}(b_2,\ldots,b_n)\\
&\qquad\geq p(1-p)b_1b_2\overline{\Psi}^p_{m-1}(a_1,\ldots,a_{m-1})\Psi^p_{n-2}(b_3,\ldots,b_n).
\end{aligned}$$

But this is true by the induction hypothesis since $\max(m-1,n-1)=N-1$ and since $a_{m-1}a_m\geq b_1b_2$ because of (9.15). Moreover, if equality holds then either $a_{m-1}a_m$ vanishes or, again by the induction hypothesis, one of $a_1,\ldots,a_{m-1}$ vanishes. □

The following lemma is an exact equivalent of Essén's [43, Lemma 1], and indeed our strategy for the proof of Theorem 9.1 is quite similar to that of Essén. Of course we use the convention that the infimum of an empty set is equal to $+\infty$.

Lemma 9.3. *Fix $p\in(0,1)$. Suppose that $a_1,\ldots,a_N\in[0,1]$ and assume that $i\in\{1,\ldots,N-1\}$ has the property that*

$$\inf\{a_1,\ldots,a_{i-1}\}\geq\max(a_i,\ldots,a_N)\tag{9.17}$$

(this condition on i is trivially satisfied if $i=1$). Finally suppose that

$$a_i<\max(a_i,\ldots,a_N)\tag{9.18}$$

and that $j\in\{i+1,\ldots,N\}$ is such that $a_j=\max(a_i,\ldots,a_N)$. Then

$$\overline{\Psi}^p_N(a_1,\ldots,a_N)>\overline{\Psi}^p_N(a_1,\ldots,a_{i-1},a_j,a_i,a_{i+1}\ldots,a_{j-1},a_{j+1},\ldots,a_N).\tag{9.19}$$

For the rest of this section, in interpreting (9.19) and similar expressions we use the convention that a sequence of the form $a_k,\ldots,a_n$ is empty and omitted if $n<k$. We shall assume

Lemma 9.3 for now and show how it implies Theorems 9.1 and 9.3. A more elementary method of proof of Theorem 9.3 was kindly communicated to the author by Mr. Ravi Vakil. His approach in effect reduces the question to consideration of the movement of the system between the points $j-1$, j, $j+1$, N and ∞, where ∞ indicates that the random walk has been terminated by having fallen into one of the dangers. This new system is sufficiently small that explicit computation can be used to prove the desired result (cf. the outline of proof of Theorem 9.5, above). However, since we have Lemma 9.3 available (and we will definitely need it for Theorem 9.1), we proceed as follows.

Proof of Theorem 9.3. Assume that $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N \in (0,1]$. (If one of these vanishes then the result is trivial.) The result is easy if $p \in \{0,1\}$ so assume $0 < p < 1$. It is clear on probabilistic grounds that we may assume that $s_j = 1$ since changing $s_j = 1$ to $s_j < 1$ would strictly decrease the left side of (9.2) and leave the right side unchanged. By Theorem 9.6, we need only show that

$$p^{-1}\overline{\Psi}^p_N(s_1, \ldots, s_N) \geq \overline{\Psi}^p_{N-1}(s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, a_N) \tag{9.20}$$

and that equality holds if and only if $s_1 = \cdots = s_j = 1$. We shall prove this by induction on N. If $N = 1$ then the result follows immediately from the definition of the Ψ_N. Suppose that $N > 1$ and the desired result has been proved for $N - 1$. Assume first that $s_1 = 1$. If $j = 1$ then by (9.11) we do have equality in (9.20) as desired. If $j > 1$, on the other hand, then we may apply (9.11) to both sides of (9.20) and the desired result will then follow by the induction hypothesis. Hence we may assume that $s_1 < 1$. Then, the hypotheses of Lemma 9.3 are satisfied with $i = 1$ and j as above so that

$$\overline{\Psi}^p_N(s_1, \ldots, s_N) > \overline{\Psi}^p_N(s_j, s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N).$$

Now since $s_j = 1$, an application (9.11) to the right hand side of the above inequality proves that strict inequality holds in (9.20) as desired. □

Proof of Theorem 9.1. Again, we may assume that $0 < p < 1$ and that the s_k are all strictly positive. Then, assuming Lemma 9.3 and given $s_1, \ldots, s_N \in (0,1]$, I claim that

$$\overline{\Psi}_N^p(s_1, \ldots, s_N) \geq \overline{\Psi}_N^p(s_1^\curlyvee, \ldots, s_N^\curlyvee)$$

with equality if and only if $s_1 \geq s_2 \geq \cdots \geq s_N$. For, if it is not true that $s_1 \geq s_2 \geq \cdots \geq s_N$, then we may let i be the maximum of the numbers $i_1 \in \{1, \ldots, N\}$ which have the properties that $s_1, \ldots, s_{i_1-1}$ are in decreasing order and that whenever $1 \leq k < i_1$ then $s_k \geq \max(s_{i_1}, \ldots, s_N)$ (note that the conditions on i_1 are automatically satisfied for $i_1 = 1$). Because $s_1, \ldots, s_N$ are *not* all in decreasing order, it follows that $i < N$ and the maximality of i implies that $s_i < \max(s_i, \ldots, s_N)$. We may then apply Lemma 9.3, and let

$$(s_1', \ldots, s_N') = (s_1, \ldots, s_{i-1}, s_j, s_i, s_{i+1}, \ldots, s_{j-1}, s_{j+1}, \ldots, s_N).$$

Note that $s_1', \ldots, s_N'$ are a permutation of $s_1, \ldots, s_N$. Hence, if $s_1', \ldots, s_N'$ are in decreasing order then we are done. Otherwise, proceed just as before and define i' in terms of the s_k' just as i was defined in terms of the s_k. Then it is easy to see that $i' > i$. We may iterate this procedure at most $N - 1$ times until we have sorted the s_k into decreasing order, and so the claim is proved. Theorem 9.1 then follows from Theorem 9.6 and this claim. □

We now prove Lemma 9.3 by exploiting the linearity properties of the Ψ_N, using a reduction reminiscent of Hardy and Littlewood's [57] reduction of a certain rearrangement inequality to the case where all the variables were in $\{0, 1\}$.

Proof of Lemma 9.3. Throughout $p \in (0,1)$ shall be fixed. Let j be as in the statement of the Lemma and set $\lambda = a_j$. Note that by (9.18) we have $\lambda > 0$. Fix a_j as well as $a_1, \ldots, a_{i-1}$. What we must prove is that

$$\overline{\Psi}_N^p(a_1, \ldots, a_N) - \overline{\Psi}_N^p(a_1, \ldots, a_{i-1}, a_j, a_i, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N) > 0 \qquad (9.21)$$

whenever $a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N \in [0, \lambda]$ and $0 \leq a_i < \lambda$. We first consider the case when $j = i + 1$. In that case, the two N-tuples serving as arguments to the $\overline{\Psi}_N^p$ in (9.21) will only differ by an exchange of their ith and jth elements. Moreover, if all variables other than a_i are fixed, then the left hand side of (9.21) will be a linear function of a_i. If we had $a_i = \lambda$ then the left side of (9.21) would vanish since $a_j = \lambda$ too. On the other hand if we had $a_i = 0$ then this left hand side would become

$$\overline{\Psi}^p_N(a_1, \ldots, a_{i-1}, 0, a_j, \ldots, a_N) - \overline{\Psi}^p_{N+1}(a_1, \ldots, a_{i-1}, a_j, 0, a_{j+1}, \ldots, a_N).$$

But applying (9.14) to both terms and then using Lemma 9.2, we see that this is strictly positive. Note that Lemma 9.2 is applicable since by choice of j and by (9.17), we have

$$\min(a_1, \ldots, a_{i-1}, a_j) = \lambda \geq \lambda = \max(a_j, \ldots, a_N),$$

and moreover $\lambda > 0$ so that strict inequality must hold. Hence, the left side of (9.21) is strictly positive if $a_i = 0$, vanishes if $a_i = \lambda$ and hence by linearity is strictly positive if $a_i \in [0, \lambda)$. This completes the proof if $j = i + 1$.

Now suppose $j > i + 1$. By linearity considerations we need only verify (9.21) when

$$a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N \in \{0, \lambda\}$$

and the conclusion for them lying in the full interval $[0, \lambda]$ will immediately follow. Of course we always have $a_j = \lambda$. Thus from now on we assume that $a_{i+1}, \ldots, a_N \in \{0, \lambda\}$. Now, take the least integer $j_1 \in \{i+1, \ldots, j\}$ with the property that $\lambda = a_{j_1} = a_{j_1+1} = \cdots = a_j$. Then, the N-tuple

$$(a_1, \ldots, a_{i-1}, a_j, a_i, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N) \tag{9.22}$$

will not at all change if we replace j by j_1 throughout its definition, since when we are moving one of the λ's from the string $a_{j_1}, \ldots, a_j$, then it clearly does not matter which one we move (see Figure 9.1). Thus, we may replace j by j_1 and by minimality of j_1 assume that either $j = i + 1$ or that $a_{j-1} \neq \lambda$ (or both). We have already handled the case $j = i + 1$.

Hence, we have $a_{j-1} \neq \lambda$ and $j > i + 1$. Moreover $a_{j-1} \in \{0, \lambda\}$ so that $a_{j-1} = 0$. Now keep $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N$ fixed. We shall show that in our present case (9.21) holds whenever $a_i \in [0, \lambda]$. By linearity it suffices to consider $a_i \in \{0, \lambda\}$. We first note that we can reduce that

$$\begin{array}{cccccccccccccccccccc}
 & & & i & & & i_1 & & & & & & & & & j_1 & & j & & \\
 & & & \downarrow & & & \downarrow & & & & & & & & & \downarrow & & \downarrow & & \\
a_1 & a_2 & a_3 & \lambda & \lambda & \lambda & 0 & \lambda & 0 & \lambda & \lambda & 0 & \lambda & \lambda & 0 & \lambda & \lambda & \lambda & a_{19} & a_{20}
\end{array}$$

Figure 9.1: An example of the original N-tuple $a_1, \ldots, a_N$ for $N = 20$, $i = 4$ and $j = 18$. The new N-tuple (9.22) will be formed from this N-tuple by cutting out the jth element and pasting it to the left of the ith. Clearly the result of this operation will be the same whether it is the element in position j or the element in position j_1 that we cut out. The result will also be the same whether it is to the left of position i or to the left of position i_1 that we paste this element.

case $a_i = \lambda$ to the case $a_i = 0$ as follows. Suppose $a_i = \lambda$. Then, let i_1 be the greatest integer $i_1 \in \{i, \ldots, N\}$ with the property that $a_i = a_{i+1} = \cdots = a_{i_1} = \lambda$. Since $a_{j-1} = 0$, we have $i_1 < j - 1$. Just as in our work with j_1 we can see that the N-tuple (9.22) will not change if i is replaced by $i_1 + 1$ (this is so because $a_i, \ldots, a_{i_1}$ is a string of λ's and it does not matter on which side of this string we insert $a_j = \lambda$; see Figure 9.1). But the maximality of i_1 implies that $a_{i_1+1} \neq \lambda$, hence $a_{i_1+1} = 0$. Hence, indeed, replacing i by $i_1 + 1$ if necessary, we may assume that $a_i = 0$.

We now thus need only consider the case where $a_i = a_{j-1} = 0$ and $a_j = \lambda$. The case $j = i + 1$ was already handled, so we may still assume that $j > i + 1$. Then, we may rewrite the left hand side of (9.21) as

$$\begin{aligned}
&\overline{\Psi}^p_{N+1}(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{j-2}, 0, a_j, \ldots, a_N) \\
&\qquad - \overline{\Psi}^p_{N+1}(a_1, \ldots, a_{i-1}, a_j, 0, a_{i+1}, \ldots, a_{j-2}, 0, a_{j+1}, \ldots, a_N).
\end{aligned}$$

Applying (9.14) twice in each of the two terms, we see that this equals

$$\begin{aligned}
&\overline{\Psi}^p_{i-1}(a_1, \ldots, a_{i-1}) \Psi^p_{j-2-i}(a_{i+1}, \ldots, a_{j-2}) \Psi^p_{N-j+1}(a_j, \ldots, a_N) \\
&\qquad - \overline{\Psi}^p_i(a_1, \ldots, a_{i-1}, a_j) \Psi^p_{j-2-i}(a_{i+1}, \ldots, a_{j-2}) \Psi^p_{N-j}(a_{j+1}, \ldots, a_N).
\end{aligned} \tag{9.23}$$

But the middle factor in both terms is the same, and by Lemma 9.1 it is strictly positive. Moreover,

$$\min(a_1, \ldots, a_{i-1}, a_j) = \lambda \geq \lambda = \max(a_j, \ldots, a_N)$$

and $\lambda > 0$ so that the left hand side of (9.23) is strictly positive by Lemma 9.2. □

Finally we give a proof of Theorem 9.2.

Proof of Theorem 9.2. Fix $p \in (0,1)$. To obtain a contradiction, suppose that $\varepsilon_n \downarrow 0$ as $n \to \infty$, with $\varepsilon_n \in (0,1]$ and

$$P_4^p(\varepsilon_n, 1, \varepsilon_n, 1) \geq P_4^p(\varepsilon_n, \varepsilon_n, 1, 1).$$

Then, by Theorem 9.6

$$\frac{p^{-4}\varepsilon_n^2}{\overline{\Psi}_4^p(\varepsilon_n, 1, \varepsilon_n, 1)} \geq \frac{p^{-4}\varepsilon_n^2}{\overline{\Psi}_4^p(\varepsilon_n, \varepsilon_n, 1, 1)}.$$

Hence,

$$\overline{\Psi}_4^p(\varepsilon_n, \varepsilon_n, 1, 1) \geq \overline{\Psi}_4^p(\varepsilon_n, 1, \varepsilon_n, 1).$$

Taking the limit as $n \to \infty$, we see that

$$\overline{\Psi}_4^p(0,0,1,1) \geq \overline{\Psi}_4^p(0,1,0,1). \tag{9.24}$$

But by (9.14) and the definitions of the $\overline{\Psi}_k^p$ and the Ψ_k^p, we have

$$\begin{aligned} \overline{\Psi}_4^p(0,0,1,1) &= \overline{\Psi}_1^p(0)\Psi_2^p(1,1) \\ &= \Psi(1,0)\Psi_2(1-p,p) \\ &= 1 \cdot (1-(1-p)p) \\ &= 1 - p(1-p) < 1. \end{aligned} \tag{9.25a}$$

On the other hand, by the definition of $\overline{\Psi}_4^p$ and by two applications of (9.12),

$$\begin{aligned} \overline{\Psi}_4^p(0,1,0,1) &= \Psi_5(1,0,p,0,p) \\ &= \Psi_1(1)\Psi_1(p)\Psi_1(p) = 1. \end{aligned} \tag{9.25b}$$

Clearly (9.25a) and (9.25b) contradict (9.24), and so the proof is complete. □

9.3. Proof of the formula for the probability of safe traversal

Instead of giving a probabilistic proof, we give one coming from a solution of an associated system of simultaneous equations.

Proof of Theorem 9.6. If $p = 1$ then $\Psi_N^1 \equiv 1$ for all $N \geq 1$ by a repeated application of (9.8), so that the content of the Theorem for $p = 1$ follows from (9.1). From now on we assume that $p \in (0,1)$. Let $q = 1 - p$.

Consider a random walk with the same transition probabilities as $\{r_i^p\}$, with the same boundary condition at 1, but not necessarily starting at the point 1. Let p_n be the probability that when started at n, it arrives at N without having fallen into any of the dangers along the route. Then,

$$p_1 = P_N^p(s_1, \ldots, s_N).$$

The following equations are easy to verify:

$$\begin{aligned} p_1 &= s_1(qp_1 + pp_2) \\ p_2 &= s_2(qp_1 + pp_3) \\ p_3 &= s_3(qp_2 + pp_4) \\ &\cdots \\ p_{N-1} &= s_{N-1}(qp_{N-2} + pp_N) \\ p_N &= s_N(qp_{N-1} + p). \end{aligned}$$

This is a tridiagonal system of N equations in the N unknowns $p_1, \ldots, p_N$. If $p = q = \frac{1}{2}$ then all but the first and last equations can be rewritten as

$$D^2 p_j - \delta_j p_j = 0$$

where $2 \leq j \leq N-1$, $D^2 p_j = \frac{1}{2}(p_{j-1} + p_{j+1}) - p_j$ and $\delta_j = s_j^{-1} - 1$. This shows the similarity with the work of Essén [43] who considers a similar question but with different boundary conditions and with D^2 replaced by Δ^2, where $\Delta^2 p_j = 2D^2 p_{j-1}$ so that $\Delta^2 p_j = \Delta(\Delta p_j)$ where $\Delta p_j = p_j - p_{j-1}$.

In fact, for general $p \in (0,1)$, our system can be solved by a simple and standard elimination scheme. First we transform it into the upper triangular system of equations

$$\begin{pmatrix} A_1 & ps_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & A_2 & ps_2 & \dots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & A_{N-1} & ps_{N-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & A_N \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N-1} \\ p_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -ps_N \end{pmatrix},$$

where the A_i are inductively defined by

$$A_1 = qs_1 - 1$$

and by

$$A_{n+1} = -1 - \frac{pqs_n s_{n+1}}{A_n},$$

for $n = 2, \dots, N$. It is easy to inductively verify that we will have

$$A_n \leq -\min(p,q) < 0$$

for $n = 1, \dots, N$ so that everything is well defined.

Then, a further reduction which transforms the system starting from the bottom up into a diagonal system easily shows that

$$p_1 = (-1)^N \frac{(ps_1)(ps_2)\dots(ps_N)}{A_1 A_2 \dots A_N}.$$

Comparing this with (9.10), we see that we will be done as soon as we show that

$$(-A_1)(-A_2)\dots(-A_N) = \Psi_{N+1}(1, \phi_1(p)s_1, \dots, \phi_N(p)s_N). \tag{9.26}$$

Since we have seen that $A_n < 0$ for $n = 1, \dots, N$, the positivity of the denominator in (9.10) will also follow from (9.26).

Let

$$B_n = -A_{N-n+1},$$

for $n = 1, \ldots, N$ and set

$$t_n = pqs_{N-n}s_{N-n+1},$$

for $n = 1, \ldots, N-1$. Define $t_N = qs_1$. Then from the inductive definition of the A_n we find that

$$B_N = 1 - t_N$$

while

$$B_n = 1 - t_n B_{n+1}^{-1}, \tag{9.27}$$

for $n = 1, \ldots, N-1$. Let

$$B_{N+1} = 1.$$

Then (9.27) also holds for $n = N$. We then have

$$B_n B_{n+1} = B_{n+1} - t_n, \tag{9.28}$$

for $n = 1, \ldots, N$. Let $\Gamma_n = B_1 B_2 \ldots B_n$. Then since $B_{N+1} = 1$, and since $\Psi_{N+1}(a_1, \ldots, a_{N+1}) = \Psi_{N+1}(a_{N+1}, \ldots, a_1)$, we see that (9.26) is equivalent to the assertion that

$$\Gamma_{N+1} = \Psi_{N+1}(a_1, a_2, \ldots, a_{N+1}), \tag{9.29}$$

where $a_n = \phi_{N-n+1}(p)s_{N-n+1}$ for $n = 1, \ldots, N$ and $a_{N+1} = 1$. Recall that

$$\phi_n(p)\phi_{n+1}(p) = pq$$

for every n and that $\phi_1(p) = q$ so that $t_n = a_n a_{n+1}$ for $n = 1, \ldots, N$. We shall now work exclusively in terms of the a_n, t_n, B_n and Γ_n.

To compute Γ_n, note that

$$\Gamma_1 = B_1.$$

Suppose that

$$\Gamma_n = \alpha_n B_n + \beta_n.$$

Then

$$\begin{aligned}\Gamma_{n+1} &= \Gamma_n B_{n+1} \\ &= \alpha_n B_n B_{n+1} + \beta_n B_{n+1} \\ &= \alpha_n (B_{n+1} - t_n) + \beta_n B_{n+1} \\ &= (\alpha_n + \beta_n) B_{n+1} - t_n \alpha_n,\end{aligned}$$

where we have used (9.28) to obtain the second-last equality. Thus, if we define α_n and β_n inductively by

$$\alpha_1 = 1$$
$$\beta_1 = 0$$

and

$$\alpha_{n+1} = \alpha_n + \beta_n \tag{9.30a}$$

$$\beta_{n+1} = -t_n \alpha_n, \tag{9.30b}$$

for $n = 1, \ldots, N$, then we will always have

$$\Gamma_n = \alpha_n B_n + \beta_n.$$

Since $B_{N+1} = 1$, it follows that

$$\Gamma_{N+1} = \alpha_{N+1} + \beta_{N+1}. \tag{9.31}$$

I claim that

$$\alpha_n = \Psi_{n-1}(a_1, \ldots, a_{n-1}) \tag{9.32a}$$

$$\beta_n = \Psi_n(a_1, \ldots, a_n) - \Psi_{n-1}(a_1, \ldots, a_{n-1}), \tag{9.32b}$$

for $n = 1, \ldots, N+1$. If this were true then (9.29) would immediately follow from (9.31). We prove (9.32a) and (9.32b) by induction. For $n = 1$ they are true since Ψ_1 and Ψ_0 are both identically 1. Suppose that they hold for n. Then by applying (9.30a) to (9.32a) and (9.32b), we see that

$$\alpha_{n+1} = \Psi_n(a_1, \ldots, a_n)$$

as desired. Now applying (9.30b) to (9.32a) we find that

$$\beta_{n+1} = -t_n \Psi_{n-1}(a_1, \ldots, a_{n-1}).$$

Thus, to obtain (9.32b) for $n+1$ we must show that

$$\Psi_{n+1}(a_1, \ldots, a_{n+1}) - \Psi_n(a_1, \ldots, a_n) = -t_n \Psi_{n-1}(a_1, \ldots, a_{n-1}). \tag{9.33}$$

But $t_n = a_n a_{n+1}$ so that (9.33) follows from (9.9). □

9.4. The one-dimensional continuous case

We now suppose that $p = \frac{1}{2}$. For a sequence p_j, let $D^2 p_j = \frac{1}{2}(p_{j-1} + p_{j+1}) - p_j$. Then, it is not difficult to verify (cf. the proof of Theorem 9.6, above) that to find $P_N^{1/2}(s_1, \ldots, s_N)$ one needs to solve

$$D^2 p_j - \delta_j p_j,$$

for $j \in \{-N+1, \ldots, N\}$, subject to the conditions

$$p_{N+1} = p_{-N} = 1$$

and

$$p_0 = p_1,$$

where $\delta_j = \delta_{1-j} = s_j^{-1} - 1$ if $j \in \{1, \ldots, N\}$. Then one will have

$$p_0 = P_N^{1/2}(s_1, \ldots, s_N).$$

The symmetry of the problem easily shows that the solution will have the property that if $j \in \{1, \ldots, N\}$ then $p_j = p_{1-j}$, and this symmetry easily shows why this system is equivalent to the one exhibited at the beginning of the proof of Theorem 9.6. (Note that we are in effect now considering a random walk on $\{-N+1, \ldots, N\}$ in place of our reflecting walk on $\{1, \ldots, N\}$.)

The reason for writing the system as above is that it suggests as a continuous analogue the differential equation

$$p''(x) - \delta(x) p(x) = 0 \tag{9.34a}$$

on $[-L, L]$, where δ is even, and where p is subject to the conditions that

$$p(L) = p(-L) = 1$$

and

$$p'(0) = 0. \tag{9.34b}$$

To solve this, by symmetry we need only solve (9.34a) on $[0, L]$ subject to (9.34b) and to the condition that

$$p(L) = 1. \tag{9.34c}$$

We now define the function $\delta^{\curlywedge}$ on $[0, L]$ to be the equimeasurable increasing rearrangement[3] of the restriction of δ to $[0, L]$. (Note that we are rearranging in the opposite order from the way we rearranged the p_j because $\delta(x)$ corresponds to $p_j^{-1} - 1$.)

The following result is then an exact continuous equivalent of the $p = \frac{1}{2}$ case of Theorem 9.1.

Theorem 9.7 (special case of Essén [42, Thm. 5.2]). *Let δ be a nonnegative lower semicontinuous piecewise constant function on $[0, L]$, and let p be the solution of (9.34a), (9.34b) and (9.34c). Let P be the solution of (9.34a), (9.34b) and (9.34c) after replacing δ with $\delta^{\curlywedge}$. Then $P(0) \geq p(0)$.*

It is not unlikely that the above theorem can be given some probabilistic interpretation in terms of Brownian motion, but such an interpretation is not as interesting as the probabilistic interpretation of our discrete results.

10. Horizontal convexity of extremals for some least harmonic majorant functionals

Conjecture 1.2 can be rephrased as follows.

[3]For $S \subseteq (0, L)$, put $S^{\curlywedge} = (L - |S|, L)$ and use (I.2.1) to define $f^{\curlywedge}$ on $(0, L)$. Extend $f^{\curlywedge}$ to $[0, L]$ by requiring continuity at the endpoints.

Conjecture 1.2′. Let $\Phi(z)$ be a subharmonic function on $\mathbb{C}$ which depends only on $|z|$. Then given any domain U which contains the origin and has finite area, there exists a domain $\tilde{U}$ with the following properties:

(i) $0 \in \tilde{U}$

(ii) $\operatorname{Area} \tilde{U} \leq \operatorname{Area} U$

(iii) $\Gamma_\Phi(\tilde{U}) \geq \Gamma_\Phi(U)$

(iv) $\tilde{U}$ is "radially convex", i.e., the intersection of $\tilde{U}$ with any ray starting at the origin is a convex set.

What is worth noting is the connection between (iv) and the assumption that $\Phi(z)$ depends only on $|z|$. This connection can be summarized in effect as saying that $\tilde{U}$ has a convex intersection with all lines which orthogonally intersect the level sets of Φ.

A natural analogue of Conjecture 1.2′ is then to consider a subharmonic function $\Phi(z)$ which depends only on $\operatorname{Re} z$ and to look for a domain $\tilde{U}$ satisfying (i)–(iii) and (iv′), where (iv′) is the requirement that $\tilde{U}$ be horizontally convex, i.e., that its intersection with any horizontal line be convex. Since the level sets of this Φ are vertical lines, it is the horizontal lines that intersect the level sets of Φ orthogonally. Under an auxiliary growth condition on Φ, this modified conjecture is actually true as the following theorem states, and by analogy this truth provides evidence for the likelihood of the truth of Conjecture 1.2′.

Theorem 10.1. *Let $\Phi(z)$ be a subharmonic function on $\mathbb{C}$ depending only on $\operatorname{Re} z$. Assume that*

$$\Phi(\operatorname{Re} z) = o(e^{(\operatorname{Re} z)^2}) \qquad \text{as} \qquad |\operatorname{Re} z| \to \infty. \tag{10.1}$$

Then for any domain U of finite area containing the origin, there exists a horizontally convex domain $\tilde{U}$ also containing the origin and satisfying $\operatorname{Area} \tilde{U} \leq \operatorname{Area} U$ and $\Gamma_\Phi(\tilde{U}) \geq \Gamma_\Phi(U)$. Moreover, the domain $\tilde{U}$ can be taken to be Steiner symmetric.

There is really nothing new in the "Moreover"; it follows from Steiner analogues of known results of Baernstein [7].

Conjecture 10.1. Theorem 10.1 remains true even if the assumption (10.1) is dropped.

The idea of the proof of Theorem 10.1 is roughly as follows:

(I) Use either Steiner symmetrization or Baernstein's sub-Steiner rearrangement (see §II.6) to reduce the problem to considering Steiner symmetric domains which are necessarily simply connected. This will allow us to consider Λ_Φ functionals instead of Γ_Φ functionals.

(II) Use an approximation argument to allows us to assume that Λ_Φ has an extremal over $\mathfrak{B}$ and Φ is in nice enough that we can analyze that extremal via the variational equation in §4.4.

(III) Prove from the variational equation that this extremal in fact has a horizontally convex image.

First we recall that $\phi(\operatorname{Re} z)$ is subharmonic on $\mathbb{C}$ if and only if ϕ is convex on $\mathbb{R}$ (Theorem I.4.3).

We shall actually go through the three steps of the proof in reverse order.

10.1. Step III of the proof of Theorem 10.1

Step III is encapsulated in the following Theorem which may of some independent interest.

Theorem 10.2. *Let $\Phi(z) = \phi(\operatorname{Re} z)$ for a convex function ϕ which is differentiable on $\mathbb{R}$ and whose derivative satisfies $|\phi'(t)| \le Ce^{Ct^2}$ for all $t \in \mathbb{R}$ and some finite C. Assume also that ϕ is strictly convex at 0 and that Λ_Φ attains a maximum over $\mathfrak{B}$ at $f \in \mathfrak{B}$. Then:*

(i) f is absolutely continuous on $\mathbb{D}$; in fact, $f \in \Lambda^*(\mathbb{T})$

(ii) f is univalent and its image $f[\mathbb{D}]$ is Steiner symmetric about the real axis and horizontally convex.

Moreover, there exists $w \in \mathbb{T}$ such that if $g_r(z) = f(rwz)$ then:

(iii) for every fixed $r \in (0,1)$, the function $z \mapsto \operatorname{Re} g_r(z)$ is symmetric decreasing on $\mathbb{T}$

(iv) for every fixed $r \in (0,1)$ the function $z \mapsto \operatorname{Re}(zg_r'(z))$ is symmetric decreasing on $\mathbb{T}$

The proof will require the following Lemma which will also be useful in Step II, below.

Lemma 10.1. *Let g be a holomorphic function on $\mathbb{D}$ satisfying* (iii) *and* (iv) *of Theorem* 10.2, *where $g_r(z) \stackrel{\text{def}}{=} g(rz)$. Assume that $g(0) = 0$ but that g does not vanish identically. Then, g satisfies the same conditions that* (ii) *imposes on f.*

Open Problem 10.1. Must a holomorphic function g satisfying the conditions of the Lemma necessarily be convex?

Proof of Lemma. Write $g_r(z) = g(rz)$. Fix $r \in (0,1)$. It suffices to prove that g_r satisfies (ii), since if g_r is univalent for all $r \in (0,1)$ then likewise g is univalent, while the union of an increasing sequence of horizontally convex Steiner symmetric domains is evidently horizontally convex and Steiner symmetric.

Put $f = g_r$. The Steiner symmetry of the image of f and the univalence of f follow from Proposition III.5.1. Write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Then for $z = e^{i\theta}$ we have

$$\operatorname{Re} zf'(z) = \operatorname{Re} -i\frac{df(e^{i\theta})}{d\theta} = \frac{dv(e^{i\theta})}{d\theta}.$$

Now, $\operatorname{Re} zf'$ is symmetric decreasing on $\mathbb{T}$. Hence, $\frac{dv}{d\theta}$ is decreasing for $\theta \in [0,\pi]$ and increasing for $\theta \in [\pi, 2\pi]$. In other words, $v(e^{i\theta})$ is a concave function of θ for $\theta \in [0,\pi]$ and is convex for $\theta \in [\pi, 2\pi]$.

Moreover, for all $z \in \overline{\mathbb{D}}$ we have $v(z) = -v(\bar{z})$, $v(1) = v(-1) = 0$ and $v(e^{i\theta})$ is positive on $[0,\pi]$ and negative on $[\pi, 2\pi]$ (see the proof of Proposition III.5.1). This, together with our concavity/convexity conditions on $v(e^{i\theta})$ implies that there is an $\alpha \in (0,\pi)$ such that $v(e^{i\theta})$ is increasing for $\theta \in [0,\alpha]$ and decreasing for $\theta \in [\alpha, \pi]$. By the reflection symmetry condition $v(z) = -v(\bar{z})$ we have $v(e^{i\theta})$ decreasing for $\theta \in [\pi, 2\pi - \alpha]$ and increasing for $\theta \in [2\pi - \alpha, 2\pi]$.

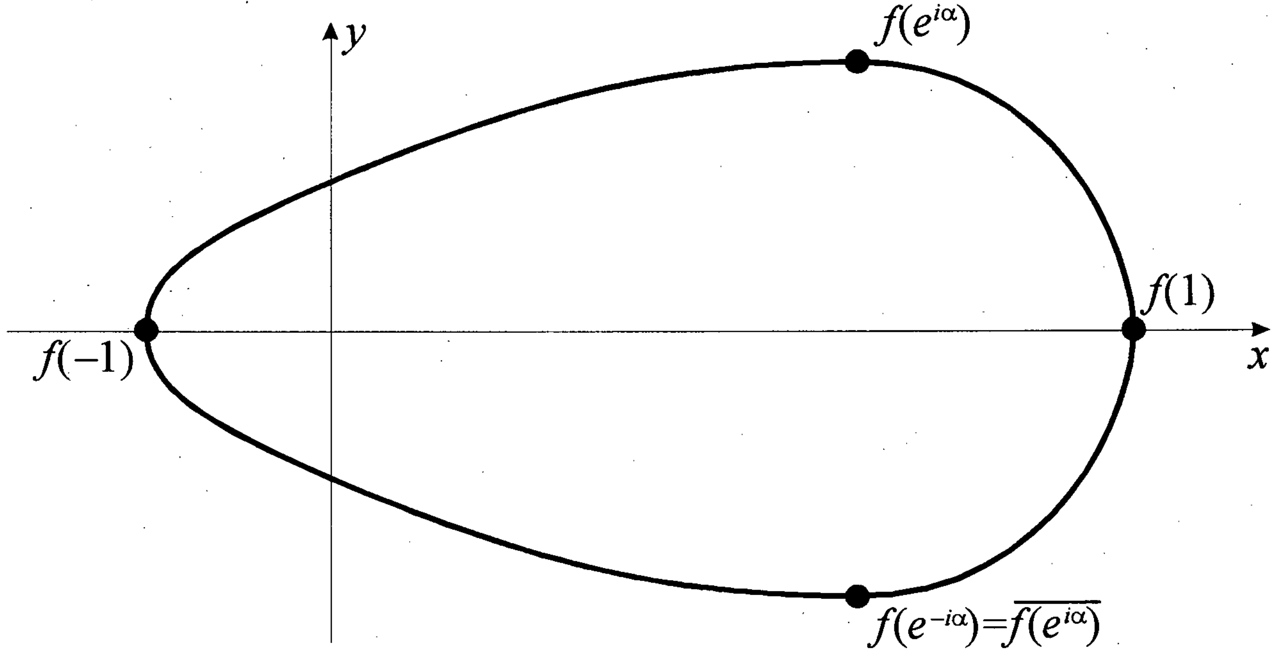


Figure 10.1: The graph of $f(e^{i\theta})$.

Together with the symmetric decreasing character, this lets us draw a rough sketch of the graph of $f(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta})$; see Figure 10.1. The graph makes it intuitively clear that the image of f must be horizontally convex, but we shall prove it rigorously.

Fix $x \in [u(-1), u(1)]$. Define

$$g(x) = \inf\{v(e^{i\theta}) : \theta \in [0, \pi], u(e^{i\theta}) = x\}.$$

Note that $u(e^{i\theta})$ is continuous and decreasing for $\theta \in [0, \pi]$. The fact that $v(e^{i\theta})$ is increasing for $\theta \in [0, \alpha]$ and decreasing for $\theta \in [\alpha, \pi]$ then implies that $g(x)$ is increasing for $x \in [u(-1), u(e^{i\alpha})]$ and decreasing for $x \in [u(e^{i\alpha}), u(1)]$. Define

$$U = \{x + iy : x \in (u(-1), u(1)), |y| < g(x)\}.$$

The increasing character of g on $[u(-1), u(\alpha)]$ and the decreasing character of g on $[u(\alpha), u(1)]$ implies that U is horizontally convex.

I claim that $U = f[\mathbb{D}]$. To see this, because both U and $f[\mathbb{D}]$ are reflection symmetric, it suffices to verify that U^+ and $f[\mathbb{D}]^+$ are equal, where for a set $D \subseteq \mathbb{C}$ we write $D^+ = \{z \in D : \operatorname{Im} z \geq 0\}$. Fix $z \in U^+$. We shall show that $z \in f[\mathbb{D}]$. Write $z = x + iy$ with $y \geq 0$. Now, because $u(-1) = f(-1) < x < f(1) = u(1)$ and we have Steiner symmetry, it follows that $x \in f[\mathbb{D}]$. We may thus dismiss the case $y = 0$ and assume $y > 0$. Let

$$y_1 = \sup\{y' : x + iy' \in f[\mathbb{D}]\}.$$

We have $y_1 > 0$ since $x \in f[\mathbb{D}]$. I claim that $y_1 > y$. If so, then we are done, since it follows immediately by the Steiner symmetry and openness of $f[\mathbb{D}]$ that $y \in f[\mathbb{D}]$. We now prove the claim. Hence, assume that $y_1 \leq y$. But $x + iy_1 \in \partial f[\mathbb{D}]$ as is easy to see, and hence $x + iy_1 = f(e^{i\theta})$ for some $\theta \in [0, 2\pi]$ as can be seen by the methods that were so often used in the proof of Proposition III.5.1. Since $y_1 > 0$ it follows in fact that $\theta \in (0, \pi)$. We have $u(e^{i\theta}) = x$. Thus, $v(e^{i\theta}) = y_1 \geq g(x) > |y|$ and so the claim is proved.

Hence, $U^+ \subseteq f[\mathbb{D}]^+$. We now prove the opposite inclusion. Suppose that $x + iy \in f[\mathbb{D}]^+$ for $y \geq 0$. Then $u(-1) < x < u(1)$ as is easily seen, since u takes on its minimum and maximum at -1 and 1, respectively. By continuity of v, there exists $\theta \in [0, \pi]$ such that $u(e^{i\theta}) = x$ and $v(e^{i\theta}) = g(x)$. Now, $f(e^{i\theta}) \in \partial f[\mathbb{D}]$ by the univalence of f and the continuity of f on $\overline{\mathbb{D}}$. Thus, $x + ig(x) \notin f[\mathbb{D}]$. By Steiner symmetry of $f[\mathbb{D}]$, since $g(x) \geq 0$ it follows that since $x + iy \in f[\mathbb{D}]$, we must have $y < g(x)$. But if $y < g(x)$ then $x + iy \in U^+$ as desired, and the desired inclusion is proved.

Thus, $U^+ = f[\mathbb{D}]^+$ which combined with previous remarks completes the proof. □

Proof of Theorem 10.2. We have already proved (i) in Theorem III.4.4, since the strict convexity of ϕ at 0 implies SSARIP of Φ by Example III.4.4.

Moreover, the univalence of f and Steiner symmetry of its image follows from Corollary III.5.2. That same Corollary implies that we may choose w so that $\operatorname{Re} g$ is symmetric decreasing on $\mathbb{T}$ if $g(z) = f(zw)$. Without loss of generality we may assume that in fact $w = 1$, since otherwise we need only replace g by f.

Write $f = u + iv$. Theorem III.4.3 then says that

$$Qzf' = \mathcal{P}_0[(\nabla\Phi)(f)] \qquad \text{on } \mathbb{T},$$

for a real constant Q which is in fact strictly positive by SSARIP. Now, $\nabla\Phi(f) = \phi'(u)$ is real. Thus, $\mathcal{P}_0\nabla\Phi(f) = \phi'(u) + C + iw$ on $\mathbb{T}$ for some real function w on $\mathbb{T}$ with mean zero and a constant C chosen so that $\int_{\mathbb{T}}(\phi'(u) + C) = 0$. Thus,

$$Q\,\mathrm{Re}\,zf' = \phi'(u) + C \qquad \text{on } \mathbb{T}.$$

Now u is symmetric decreasing on $\mathbb{T}$ and ϕ' is a monotone increasing function as ϕ is convex. Hence, $z \mapsto \phi'(u(z)) + C$ is symmetric decreasing on $\mathbb{T}$ and hence so is $\mathrm{Re}\,zf'$.

But $W(z) \stackrel{\text{def}}{=} \mathrm{Re}\,zf'(z)$ is a harmonic function on D. Hence if it is symmetric decreasing on $\mathbb{T}$, likewise we must have $W_r(z) = W(rz)$ symmetric decreasing on $\mathbb{T}$ for all $r < 1$ because of Corollary I.6.3. But the symmetric decrease of w_r is precisely equivalent to statement (iv). In the same way since u is symmetric decreasing, so is $u_r(z) = u(rz)$ and this proves statement (iii). Finally, statement (ii) follows from Lemma 10.1. □

10.2. Step II of the proof of Theorem 10.1

Step II is encapsulated in the proof of the following Proposition. This proof will in an essential way make use of Theorem 10.2.

Proposition 10.1. *Let $\Phi(z) = \phi(\mathrm{Re}\,z)$ for a convex function ϕ on $\mathbb{R}$ with $\phi(t) = o(e^{t^2})$ as $|t| \to \infty$. Then, there exists an extremal $f \in \mathfrak{B}$ for Λ_Φ on $\mathfrak{B}$ such that f is univalent while its image $f[\mathbb{D}]$ is Steiner symmetric about the real axis and horizontally convex.*

The proof of this will need the following elementary lemma whose proof we leave to the reader.

Lemma 10.2. *Let ϕ be a convex non-linear function on $\mathbb{R}$. Then, there is a sequence of continuously differentiable convex functions such that:*

(a) $\phi_1 \le \phi_2 \le \cdots$ on $\mathbb{R}$

(b) $\phi_n(x) \to \phi(x)$ *for every* $x \in \mathbb{R}$ *as* $n \to \infty$

(c) ϕ *is differentiable on* $\mathbb{R}$ *and for every* n *there is a finite constant* C_n *such that* $|\phi_n'(x)| \le C_n(1+|x|)^{C_n}$ *for all* $x \in \mathbb{R}$

(d) ϕ_n *is strictly convex at* 0.

Proof of Proposition 10.1. If ϕ is linear then we are done since then

$$\Lambda_\Phi(f) = \phi(0)$$

for every f and so we may set $f(z) = z$ and we will have the maximum achieved at f. Hence suppose that ϕ is non-linear. Then, choose a sequence of functions ϕ_n as in Lemma 10.2. Let $\Phi_n(z) = \phi_n(\operatorname{Re} z)$. Extremals for Λ_{Φ_n} over $\mathfrak{B}$ exist (Theorem III.3.4). Let $f_n \in \mathfrak{B}$ be an extremal for Λ_{Φ_n}. Moreover, Φ_n satisfies the hypotheses of Theorem 10.2. Replacing $f_n(z)$ by $f_n(zw)$ if necessary (for some $w \in \mathbb{T}$) we may assume that w can be taken to be 1 in the conclusions of that Theorem.

Now, $f_n \in \mathfrak{B}$ and $\mathfrak{B}$ is weakly compact. Passing to a subsequence if necessary, we may assume that f_n converges weakly to some $f \in \mathfrak{B}$. Then, f_n converges to f uniformly on compact subsets of $\mathbb{D}$ as is well known (see Lemma 10.3, below). It follows that because f_n satisfied conclusions (iii) and (iv) of Theorem 10.2, likewise f must satisfy them, since these conclusions are clearly preserved under uniform convergence on compacta. Hence, by Lemma 10.1 it follows that f is univalent and has a Steiner symmetric horizontally convex image providing f does not vanish identically. We shall take care of this last proviso later.

We now show that Λ_Φ actually attains a maximum over $\mathfrak{B}$ at f. To see this, suppose that on the contrary there is an $h \in \mathfrak{B}$ with $\Lambda_\Phi(h) > \Lambda_\Phi(f)$. Now, by monotone convergence we have $\Lambda_{\Phi_n}(h) \to \Lambda_\Phi(h)$. Hence there exists $\varepsilon > 0$ such that for sufficiently large n we have $\Lambda_{\Phi_n}(h) > \varepsilon + \Lambda_\Phi(f)$. But,

$$\Lambda_\Phi(f) \ge \limsup_n \Lambda_\Phi(f_n)$$

by Theorem III.3.2 applied with $\Psi(t) = e^{(\mathrm{Re}\, t)^2}$ (of course Λ_Ψ is bounded on $\mathfrak{B}$ by the Chang-Marshall inequality). Hence, for sufficiently large n we have $\Lambda_\Phi(f_n) < \Lambda_\Phi(f) + \varepsilon/2$. Thus, for sufficiently large n we have

$$\Lambda_{\Phi_n}(h) > \varepsilon/2 + \Lambda_\Phi(f_n).$$

But $\Lambda_\Phi(f_n) \geq \Lambda_{\Phi_n}(f_n)$. Hence, $\Lambda_{\Phi_n}(h) > \Lambda_{\Phi_n}(f_n)$ for all large n, which contradicts the fact that f_n maximizes Λ_{Φ_n}.

Thus, indeed Λ_Φ achieves a maximum over $\mathfrak{B}$ at f. If $f \not\equiv 0$ then we are done by previous remarks. Suppose now that $f \equiv 0$. Then, $\Lambda_\Phi(f) = \Phi(0)$. Set $F(z) = z$. Then, $\Lambda_\Phi(F) \geq \Phi(0)$ by the subharmonicity of Φ. Hence, we are also done, since in that case the maximum of Λ_Φ must be achieved at F and F is univalent and has a Steiner symmetric and horizontally convex image. □

The following lemma is well-known and is a consequence of a more general proposition of Cima and Matheson [36]. We shall give a very simple proof of the lemma for the reader's convenience.

Lemma 10.3. *Let f_n be a sequence of functions in $\mathfrak{D}$ converging weakly to a function $f \in \mathfrak{D}$. Then $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.*

Proof. We have $f_n \to f$ in $L^1(\mathbb{T})$ by Andreev and Matheson's [5] result on the weak continuity of $L^p(\mathbb{T})$ norms (finite p) on $\mathfrak{D}$. By Theorem I.3.3, since $\mathfrak{D} \subseteq H^1$, we have

$$f_n(z) = (P * f_n)(z),$$

and

$$f(z) = (P * f)(z),$$

for all $z \in \mathbb{D}$. It is clear from the form of the Poisson kernel that $L^1(\mathbb{T})$ convergence implies convergence on compacta within $\mathbb{D}$. □

10.3. Step I and the rest of the proof of Theorem 10.1

Proof of Theorem 10.1. Without loss of generality, by scaling assume that Area $U = \pi$. Let $V = U^{\mathrm{B}}$, as in §III.6. (One could also use Steiner symmetrization here and Theorem I.6.6.) Then, U^{B} has area not exceeding π by Corollary III.6.1. Moreover, U^{B} is Steiner symmetric and hence simply connected. Let f_1 be the Riemann map sending $\mathbb{D}$ onto U^{B} and satisfying $f_1(0) = 0$. Then $f_1 \in \mathfrak{B}$ and $\Lambda_\Phi(f_1) = \Gamma_\Phi(U^{\mathrm{B}}) = \Gamma_\Phi(U)$ by Theorems III.1.2 and III.6.1. Apply Proposition 10.1 to obtain a univalent function $f \in \mathfrak{B}$ with Steiner symmetric and horizontally convex image such that Λ_Φ attains its maximum over $\mathfrak{B}$ at f. In particular $\Lambda_\Phi(f) \geq \Lambda_\Phi(f_1) = \Gamma_\Phi(U)$. Let $\tilde{U} = f[\mathbb{D}]$. Then, by Theorem III.1.2 we have $\Gamma_\Phi(\tilde{U}) = \Lambda_\Phi(\tilde{U}) \geq \Gamma_\Phi(U)$ as desired. $\square$

List of notations and symbols

1. Rearrangement-type operators

Notation	First use	Description
$\#$	p. 5	a generic rearrangement
$\circledast$	p. 9	Schwarz symmetrization
$\boxminus$	p. 9	Steiner symmetrization about the real axis
$\circledcirc$	p. 11	circular symmetrization
$\circlearrowleft$	p. 44	the circular $*$-function; $f^{\circlearrowleft} = J(f^{\circledcirc})$
$F^{\circledcirc}$	p. 244	a symmetrized version of a holomorphic F
$\bigstar$	p. 263	radial rearrangement
$\blacktriangleleft$	p. 275	a cylindrical version of radial rearrangement
$\leftrightarrow$	p. 282	a lengthwise Steiner-type $*$-function
$\simeq$	p. 276	two-sided lengthwise Steiner rearrangement
$\twoheadleftarrow$	p. 295	$I^{\twoheadleftarrow}$ is a single interval with same logarithmic length as I
$\curlyvee$	p. 307	decreasing rearrangement
$\curlywedge$	p. 328	increasing rearrangement
$U^{(M)}$	p. 284	one-sided Steiner rearrangement of U at abscissa M
$U^{(r,R]}$	p. 279	a certain cutting operation applied to U
U^{B}	p. 251	Baernstein's sub-Steiner rearrangement of U
D'	p. 61	a certain set containing D
f_ρ	p. 75	partial rearrangement of f with respect to involution ρ
$\breve{g}$	p. 131	a "de-rearrangement" of g

2. Some other operators and relations for sets and functions

Notation	First use	Description
A^c	p. 3	complement of A
$\|A\|$	p. 3	measure or cardinality of A
2^A	p. 3	power set of A
$\bigcup \mathcal{A}$	p. 3	union of all elements of $\mathcal{A}$
$A \setminus B$	p. 3	set of elements of A not lying in B
1_A	p. 3	indicator function of a set A
f_λ	p. 5	level set of function f at height λ
$\preceq$	p. 144	a domination between functions (only in §II.9)
$\lhd$	p. 165	another domination between functions
L^*	p. 138	$L^*(x,y) \stackrel{\text{def}}{=} L(y,x)$
$f * g$	p. 48	convolution of two functions f and g on $\mathbb{T}$
$K * g$	p. 74	the function $x \mapsto \sum_y K(d(x,y))g(y)$
$\check{D}$	p. 34	the universal covering surface of a domain D

3. Numerical operators

Notation	First use	Description
t^+	p. 3	positive part of a number or function t
t^-	p. 3	positive part of a number or function t
$\|t\|$	p. 3	absolute value of a number or function t
$\Phi_{,j}$	p. 4	partial derivative of Φ with respect to x_j

4. Miscellaneous

Notation	First use	Description
$[v, w]$	p. 88	geodesic joining v with w
$1_{\{P\}}$	p. 3	indicator function of a proposition P
$K^{\mathbb{Z}} \overset{r}{\otimes} K^X$	p. 141	product kernel on $\mathbb{Z} \times X$

5. Greek alphabetical index

Notation	First use	Description
Γ_Φ	p. 192	a certain functional on domains
Δ	p. 39	continuous Laplacian operator $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$
Δ, Δ_G	p. 94	discrete Laplacian operator on a graph G
$\delta_{x,y}$	p. 3	Kronecker delta function
$\delta(v), \delta_G(v)$	p. 82	degree of vertex v in graph G
$\theta(r; U)$	p. 11	angular measure of $U \cap \mathbb{T}(r)$
Λ^*	p. 231	Zygmund class
Λ_Φ	p. 188	a functional on measurable functions
μ_Φ	p. 39	Riesz measure of Φ
$\nu_1(G)$	p. 94	first nonzero Dirichlet eigenvalue of $-\Delta$ on $G \subseteq T_p$
π_1	p. 151	projection of $V = \mathbb{Z} \times X$ onto $\mathbb{Z}$
π_2	p. 151	projection of $V = \mathbb{Z} \times X$ onto X
$\rho(I)$	p. 295	logarithmic length of I
$\sigma_g(f)$	p. 166	$\sum_Y g \cdot f$
τ_D	p. 32	first exit time of Brownian motion from domain D
τ_s	p. 143	first time that the random walk does not survive a time step
$\phi_n(p)$	p. 314	p for even n and $1-p$ for odd n
Ψ_N	p. 313	a useful auxiliary function
Ψ_N^p	p. 314	another useful abbreviation
$\overline{\Psi}_N^p$	p. 314	yet another useful abbreviation
$\psi_{N,n}$	p. 312	an auxiliary function
ω	p. 46	Haar measure on $\mathbb{T}$
$\omega(z, A; D)$	p. 28	harmonic measure of A at z in D

$\omega(z, A; s)$	p. 144	generalized harmonic measure
ω_z^D	p. 28	$\omega(z, \cdot; D)$

6. Latin alphabetical index

Notation	First use	Description
arg	p. 3	argument of a complex number
$\mathcal{A}$	p. 126	fibres of our Steiner type rearrangement #
B	p. 206	unit ball of Hilbert space (in §III.3.2)
$B(r)$	p. 206	ball of radius r in Hilbert space (in §III.3.2)
B_a	p. 196	Beurling function
B_t	p. 32	Brownian motion process
$\tilde{B}_M$	p. 197	cut-off Beurling function
BMO	p. 21	the space BMO on $\mathbb{T}$
BMOA	p. 21	the analytic functions from BMO
$\mathcal{B}$	p. 192	set of all domains of area $\leq \pi$ containing 0
$\mathfrak{B}$	p. 190	unit ball of $\mathfrak{D}$
$\mathfrak{B}_\alpha$	p. 190	unit ball of $\mathfrak{D}_\alpha$
$\mathfrak{b}$	p. 190	unit ball of $\mathfrak{d}$
$\mathfrak{b}_\alpha$	p. 190	unit ball of $\mathfrak{d}_\alpha$
Children v	p. 90	the set of children of a vertex v of T_p
$\mathfrak{c}_n f$	p. 189	the nth Fourier cosine coefficient of f
D	p. 165	the difference operator $K - 1$ (only in §II.11)
D_M	p. 197	image of cut-off Beurling function
Desc v	p. 90	the set of descendants of a vertex v of T_p
$\mathfrak{D}(G)$	p. 94	set of all functions vanishing outside G but not identically zero
$\mathfrak{D}$	p. 190	holomorphic Dirichlet space $\mathfrak{D}_1$
$\mathfrak{D}_\alpha$	p. 189	α-weighted holomorphic Dirichlet space
$\mathfrak{d}$	p. 190	real harmonic Dirichlet space $\mathfrak{d}_1$
$\mathfrak{d}_\alpha$	p. 189	α-weighted real harmonic Dirichlet space
$\mathbb{D}$	p. 3	open unit disc
$\mathbb{D}(r)$	p. 3	open disc of radius r about the origin
$\mathbb{D}(z; r)$	p. 3	open disc of radius r about z
E	p. 165	the difference operator $L - 1$ (only in §II.11)
$E_M(f)$	p. 200	the measure of the level set f_M
$E^{-\infty}[\cdot]$	p. 285	a limiting case of $E^z[\cdot]$
$E^z[\cdot]$	p. 32	expectation conditioned on the process starting at z
Edge G	p. 81	set of edges of a graph G
Fix ρ	p. 72	fixed point set of involution ρ
$\mathcal{F}$	p. 211	a set of measurable functions on I (in §III.3.3)
$\mathcal{F}$	p. 260	set of Φ with $\Phi(\|z\|)$ subharmonic (in Chapter IV)
G_D	p. 60	Green's function of D with pole at 0
$\mathcal{G}$	p. 217	a subset of $L^1[0, \infty)$ (in §III.3.3)
$\mathfrak{G}_P$	p. 119	collection of graphs on which there is an ordering for which P holds

$g(z, w; D)$	p. 38	Green's function of D evaluated at z and with pole at w
$\tilde{g}(z, w; D)$	p. 53	circular rearrangement of $g(z, w; D)$ with respect to z
$\mathfrak{g}(z, w)$	p. 56	$g(z, w; \mathbb{D})$
H	p. 205	Hilbert space (in §III.3.2)
H_8	p. 84	edge graph of the octahedron
H_0^2	p. 226	$\{f \in H^2(\mathbb{D}) : f(0) = 0\}$
H^p	p. 18	holomorphic Hardy space
h^p	p. 18	harmonic Hardy space
$h(v)$	p. 90	height of a vertex v in T_p
$\mathcal{H}_D$	p. 28	σ-algebra of sets for which harmonic measure exists
$\mathfrak{H}_\rho$	p. 72	the set of points mapped by ρ to something strictly $\prec$-larger
$\mathfrak{J}$	p. 72	a fixed set of involutions
$Jg(re^{i\theta})$	p. 44	$\int_{-\lvert\theta\rvert}^{\lvert\theta\rvert} g(re^{i\varphi})\, d\varphi$
K	p. 138	kernel function (in §II.9)
K^X	p. 141	simple random walk kernel on the graph X
$K^{\mathbb{Z}}$	p. 141	any kernel on $\mathbb{Z}$
$K^{\mathbb{Z}}_\delta$	p. 141	trivial kernel on $\mathbb{Z}$
$K^{\mathbb{Z}}_S$	p. 141	simple random walk kernel on $\mathbb{Z}$
$K^{\mathbb{Z}} \overset{r}{\otimes} K^X$	p. 141	product kernel on $\mathbb{Z} \times X$
$L(t; W)$	p. 276	a measure of W along a line
$L^1(\omega^D)$	p. 28	functions integrable with respect to harmonic measure
L_s	p. 309	first time random walk fails to survive a step
$\mathrm{LHM}(z, \Phi; D)$	p. 31	value at z of the least harmonic majorant of Φ on D
$\mathcal{M}$	p. 166	a certain collection of positive symmetric functions
$N(v)$	p. 94	set of vertices adjacent to v
N_F	p. 254	Nevanlinna counting function of F
n. t. lim	p. 19	nontangential limit operator
O	p. 90	root of T_p
P_N^p	p. 307	a certain survival probability
P_r	p. 19	Poisson kernel
$P^{-\infty}(\cdot)$	p. 285	a limiting case of $P^z(\cdot)$
$P^z(\cdot)$	p. 32	probability conditioned on the process starting at z
p	p. 88	degree of our tree T_p
$\mathcal{P}_0$	p. 226	projection operator from $L^2(\mathbb{T})$ onto $H_0^2(\mathbb{T})$
$Q(f, g; \Phi, K)$	p. 71	$\sum_{x,y} \Phi(\lvert f(x) - f(y)\rvert) K(d(x, y))$
$R_\varepsilon(\theta; U)$	p. 262	logarithmic measure of a ray intersected with U
R_n	p. 142	the random walk on V
r_i^p	p. 306	a reflecting random walk on $\mathbb{Z}^+$
$\mathcal{R}(f)$	p. 94	Rayleigh quotient for f
$S(r)$	p. 206	$S(r; 1)$ (in §III.3.2)
$S(r; R)$	p. 206	$\{f : r < \lVert f\rVert \le R\}$ (in §III.3.2)
s	p. 143	survival probabilities on V (in §II.9)
$\operatorname{sgn} z$	p. 3	$\frac{z}{\lvert z\rvert}$
$\operatorname{supp} f$	p. 3	support of a function f
$\mathfrak{s}_n f$	p. 189	the nth Fourier sine coefficient of f

T_N	p. 307	first time random walk reaches $N+1$
T_p	p. 88	p-regular tree
$T_{p,k}$	p. 90	the height k subtree of T_p
$\mathbb{T}$	p. 3	unit circle
$\mathbb{T}(r)$	p. 3	circle of radius r about the origin
$\mathbb{T}(z;r)$	p. 3	circle of radius r about z
U_a	p. 54	$\mathbb{D}(a;1)$
U_{ab}	p. 55	disc with a slit and with a piece sliced off
U_{abcd}	p. 55	$\mathbb{D}\backslash([-d,-c]\cup[a,b])$
V	p. 138	$\mathbb{Z}\times X$ (in §II.9)
$\operatorname{Vert} G$	p. 81	set of vertices of a graph G
$v(r;V)$	p. 266	a certain harmonic measure functional
$\mathbb{V}^-$	p. 275	the semi-infinite cylinder $(-\infty,0)\times\mathbb{T}$
$W(I;D)$	p. 294	$\omega(0,\partial D;D\backslash I)$
W_M	p. 284	a certain harmonic measure functional
$w(X)$	p. 166	set of functions on X satisfying a certain size property
w_r	p. 267	a certain harmonic measure functional
X	p. 138	a discrete set on which a Steiner type symmetrization is given (in §II.9)
$X(u;W)$	p. 275	a measure of W along a line
X_n	p. 143	sequence of i.i.d. random variable uniformly distributed on $(0,1]$
$Y(x;U)$	p. 9	linear measure of $\{z:\operatorname{Re} z=x\}\cap U$
$\mathbb{Z}$	p. 2	the set of integers
$\mathbb{Z}^-$	p. 2	$\{-1,-2,\ldots\}$
$\mathbb{Z}_0^-$	p. 2	$\{0,-1,-2,\ldots\}$
$\mathbb{Z}^+$	p. 2	$\{1,2,\ldots\}$
$\mathbb{Z}_0^+$	p. 2	$\{0,1,2,\ldots\}$
$\mathbb{Z}_2^3$	p. 120	edge graph of cube
$\mathbb{Z}_3^2$	p. 120	ternary plane graph
$\mathbb{Z}_n$	p. 86	the circular graph on n vertices

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Appendix A

Source code for cubetern.c

```
/* cubetern.c by Alexander R. Pruss <pruss@math.ubc.ca>, 1995. */
/* Turbo C++ (C mode) code */
/*
   Check whether there exists a measure preserving rearrangement
   # under which we have
       sum_{v\in G} f(v) Nf(v) <= sum_{v \in G} f^#(v) Nf^#(v),
   where Nf=sum_{w \in N(v)} f(w).  We do this for the cube graph
   G=Z_2^3 and for the ternary plane graph G=Z_3^2.
 */

#define CUBE_GRAPH /* comment this out to do the
                      ternary plane instead */

#include <stdio.h>
#include <stdlib.h>
#include <time.h>

#ifdef CUBE_GRAPH    /* case of the cube graph */
# define NAME "cube"
# define ALL_VERTICES_EQUIVALENT
# define DEGREE 3       /* graph has degree 3 */
# define N_VERT 8       /* graph has 8 vertices */
  enum vertices
  { v000=0, v001=1, v010=2, v011=3, v100=4, v101=5, v110=6, v111=7 };
                        /* name the vertices by their coordinates */
  int N[N_VERT][DEGREE]  /* neighbour list */
   = {
       { v001, v010, v101 },    /* neighbours of v000 */
       { v000, v011, v101 },    /* neighbours of v001 */
       { v011, v000, v110 },    /* neighbours of v010 */
       { v010, v001, v100 },    /* neighbours of v011 */
       { v101, v110, v000 },    /* neighbours of v100 */
       { v100, v111, v001 },    /* neighbours of v101 */
       { v111, v100, v010 },    /* neighbours of v110 */
```

```
        { v110, v101, v011 }     /* neighbours of v111 */
     };
#else                   /* case of the ternary plane graph */
# define NAME "ternary plane"
# define ALL_VERTICES_EQUIVALENT
# define DEGREE 4       /* graph has degree 4 */
# define N_VERT 9       /* graph has 9 vertices */
  enum vertices
  { v00=0, v01=1, v02=2, v10=3, v11=4, v12=5, v20=6, v21=7, v22=8 };
                        /* name the vertices by their coordinates */
  int N[N_VERT][DEGREE]  /* neighbour list */
   = {
        { v01, v02, v10, v20 },     /* neighbours of v00 */
        { v02, v00, v11, v21 },     /* neighbours of v01 */
        { v00, v01, v12, v22 },     /* neighbours of v02 */
        { v11, v12, v00, v20 },     /* neighbours of v10 */
        { v12, v10, v21, v01 },     /* neighbours of v11 */
        { v10, v11, v22, v02 },     /* neighbours of v12 */
        { v21, v22, v00, v10 },     /* neighbours of v20 */
        { v22, v20, v01, v11 },     /* neighbours of v21 */
        { v20, v21, v02, v12 }      /* neighbours of v22 */
     };
#endif         /* the rest of the code is the same for both graphs */

#define LARGEST_VAL 20 /* try values of functions from 0 to 20-1 */
#define NUM_TRIES   20000 /* don't bother trying more than 20000
                             functions per order */
unsigned f[N_VERT];   /* the input function f */
int order[N_VERT];    /* the order we are sorting in:  this defines
                         our measure preserving rearrangement # */
unsigned sorted_f[N_VERT]; /* the function f sorted according to
                              order[] */

unsigned sum_fNf(unsigned *F) /* compute sum_{v\in G} F(v)NF^#(v) */
{
  /* With LARGEST_VAL=20, the biggest sum_fNf can be is
     19*19*DEGREE*N_VERT, which, in the case of the ternary plane is
     12996, and in the case of the cube is 8664.  Both values will
     nicely fit in an unsigned variable. */
  unsigned sigma=0;
  int v,w;
  for(v=0; v<N_VERT; v++)
  {
    int NF=0;
    for(w=0; w<DEGREE; w++)
      NF+=F[N[v][w]];
```

```
    sigma+=F[v]*NF;
  }
  return sigma;
}

void do_sort(unsigned *F)    /* Sort the input in descending
                                order according to order[].  Use
                                the bubble sort. */
{
  int i,j;
  for(i=0; i<N_VERT; i++)  /* at the beginning of the i-th loop,
                              the first i elements are sorted
                              correctly */
   for(j=i+1; j<N_VERT; j++) /* put the (i+1)st largest
                                element in the right place */
     if(F[order[j]] > F[order[i]])
     {
       unsigned swap;
       swap=F[order[j]];
       F[order[j]]=F[order[i]];
       F[order[i]]=swap;
     }
}

unsigned max_num_tries=0;
int proved=1;

void test_order(void)
     /* check if the current order[] induces the right inequality */
     /* If this function returns without printing any messages, then
        it is guaranteed that order[] does not induce the right
rearrangement inequality. If this guarantee cannot be given,
        then proved is set to 0. */
{
  unsigned tries;
  int i;
  for(tries=0; tries<NUM_TRIES; tries++)
  {
    for(i=0; i<N_VERT; i++)
      sorted_f[i]=f[i]=rand()%LARGEST_VAL;
    do_sort(sorted_f);
    if(sum_fNf(sorted_f) < sum_fNf(f))
    {
      /* this order doesn't give the desired inequality */
      if(tries+1>max_num_tries) max_num_tries=tries+1;
      return;
```

```
    }
  }
  /* we haven't found a counterexample! */
  printf("We haven't found a counterexample for order:\n");
  for(i=0; i<N_VERT; i++)
    printf("%d ",order[i]);
  putchar('\n');
  proved=0;
}

void loop_orders(int depth)
        /* recursively generate permutations on
           ( 0,...,N_VERT-1 ) and run test_order
           on them */
{
  int i,j;
  if(depth==N_VERT)
  {
    test_order();
    return;
  }
  for(i=0; i<N_VERT; i++)
  {
     for(j=0; j<depth; j++)
       if(order[j]==i) goto TRY_NEXT_i; /* this value already chosen
   earlier in order[] */
     order[depth]=i;
     loop_orders(depth+1);
TRY_NEXT_i: ;
# ifdef ALL_VERTICES_EQUIVALENT
     if(depth==0)
       break;    /* all vertices are equivalent, so only need to try
                    one value for order[0] */
# endif
  }
  if(depth==0) fprintf(stderr,"[done!]\n");
}

int main()
{
  time_t t_start;
  srand(317);
  printf("Computing for the %s...\n",NAME);
  t_start=time(NULL);
  loop_orders(0);
  printf("Run time: %ld seconds.\n",time(NULL)-t_start);
```

```
  if(proved)
  {
    printf("We have proved that there is no order giving the "
           "inequality.\n");
    printf("The maximum number of random functions per order "
           "was %u.\n", max_num_tries);
  }
  return 0;
}
```

Appendix B

Source code for cm.f

```
C
C FORTRAN 77 source code "cm.f"
C
C Check whether the Chang-Marshall functional is really less
C than e as it is conjectured to be.
C Copyright (c) 1994-1995 Alexander Pruss.
C
      program chmrsh
      real p(1:100)
      real L

      integer iy,niter,nterms

      open(unit=80,file='cmdcfg')
      read(80,*) iy
      read(80,*) niter
      read(80,*) nterms
      close(unit=80)
      print*, 'iy=',iy
      print*, 'niter=',niter
      print*, 'nterms=',nterms

      istop=1
      nstops=0
      do 1 i=1,niter
      do 2 j=1,nterms
2       p(j)=urand(iy)
      call cmfunc(nterms,p,L)
      if(log(L) .ge. 1) print*, L,(p(k),k=1,nterms)
      if(i.eq.istop) then
        nstops=nstops+1
        istop=nstops/20.*niter
        print*, '[',real(i)/niter*100., '% finished ]'
      endif
```

```
1     continue
      print *, 'Finished'
      stop
      end

      complex function crpol(n,p,z)
C  evaluate the real-coefficient polynomial
C  p(0)+p(1)z+...+p(n)z^(n) at complex z
      integer n
      complex z

      real p(0:*)

      integer j

      crpol=p(n)

      do 1 j=n-1,0,-1
        crpol=p(j)+crpol*z
1     continue

      return
      end

      subroutine dirnor(n,p)
C  normalize (wrt the Dirichlet norm) the real-coefficient polynomial p
C  (p(0)=0)
      integer n
      real p(1:*)

      real dnor

      integer j

      dnor=0

      do 1 j=1,n
        dnor=dnor + j * p(j)**2
1     continue

      if (dnor.eq.0) return

      dnor=sqrt(dnor)

      do 2 j=1,n
        p(j)=p(j)/dnor
```

```
2     continue
      return
      end

      subroutine cmfunc(n,p,L)
      integer j
      integer n
      real p(1:*)
      real L

      real simpsn,Lambda
      external Lambda

      real d0,d1,d2,d3,d4,m4

      real epsilon
      integer niter

      real pi,two pi,e
      parameter (two pi=2*3.141592653589793)
      parameter (pi=3.14159 26535 89793)
      parameter (e=2.71828 18284 59045 23536)

C  normalize our polynomial
      call dirnor(n,p)
C  load all polynomials
      call loadp(n,p)

C  compute derivatives of p(z)/z at z=1 of orders 0,1,2,3 and 4
      d0 = 0
      d1 = 0
      d2 = 0
      d3 = 0
      d4 = 0
      do 10 j=1,n
        d0 = d0+p(j)
        if(j.ge.1) d1 = d1+p(j)*(j-1)
        if(j.ge.2) d2 = d2+p(j)*(j-1)*(j-2)
        if(j.ge.3) d3 = d3+p(j)*(j-1)*(j-2)*(j-3)
        if(j.ge.4) d4 = d4+p(j)*(j-1)*(j-2)*(j-3)*(j-4)
   10 continue

C  Maple generated formula
      m4 = 2*exp(d0**2)
     #*(3*d2**2+36*d1**2*d0*d2+4*d1*d3+6*d1**4+24*d1**4
     #*d0**2+d0*d4+8*d0**2*d3*d1+6*d0**2*d2**2+24*d0**3
```

```
     #*d2*d1**2+8*d0**4*d1**4)/180.
C  End Maple generated formula

      epsilon=.01
      do 1 j=1,16
        if(j.ne.1) then
          if(abs(L-e) .ge. 1e-5) then
            epsilon=min(epsilon,abs(L-e))/2
          else
            epsilon=epsilon/8.
          endif
        endif

C  Simpson's rule guarantees that niter will be a sufficient
C  number of time steps to obtain precision epsilon.

        niter=2*int(1.+ pi * (m4/epsilon)**.25)
        if(niter .gt. 130) niter=132
        L=simpsn(Lambda,0.,two pi,niter)/two pi
C
C  The 1.1 in the next line is a safety factor to take care of
C  possible roundoff error.
C
        if(abs(L-e) .ge. 1.1*epsilon .or. m4 .eq. 0 .or. niter
+           .eq. 132) goto 2
1     continue

C  Return an upper bound for L
C  The 1.1 in the next line is a safety factor to take care of
C  possible roundoff error.

2     L=L+1.1*epsilon
      return
      end

      subroutine loadp(n,p)
      real p(1:100)
      integer n,nn
      real pp(1:100)
      common/mainp/pp,nn
      integer j
      nn=n
      Dw=0
      do 1 j=1,n
        pp(j)=p(j)
1     continue
```

```
      return
      end

      real function Lambda(t)
      real t,abs2
      complex z,crpol
      real p(1:100)
      integer n
      common/mainp/p,n

      abs2(z)=real(z)**2+imag(z)**2
C  our integrand
      Lambda=exp(abs2( crpol( n-1,p,exp(cmplx(0.,t)) ) ))
      return
      end

      real function simpsn(f,a,b,niter)
C
C  Find $\int_a^b f(x) dx$ via Simpson's rule with niter intervals.
C
      real f,a,b
      integer niter

      real h
      integer j

C  number of intervals must be even
      if(mod(niter,2).eq.1) niter=niter+1

      h=(b-a) / niter

C  here we have f(x_0) + 4f(x_1) + f(x_n)
      simpsn=f(a)+4*f(a+h)+f(b)

      do 1 j=2,niter-2,2
        simpsn=simpsn+2*f(j*h)+4*f((j+1)*h)
1     continue

      simpsn=simpsn*(b-a)/(3*niter)

      return
      end

C
C The following function was obtained via the Internet
C from NETLIB
```

```
C
      REAL FUNCTION URAND(IY)
      INTEGER  IY
C
C URAND IS A UNIFORM RANDOM NUMBER GENERATOR  BASED
C ON  THEORY  AND  SUGGESTIONS GIVEN IN D.E. KNUTH
C (1969),  VOL 2.   THE  INTEGER  IY SHOULD  BE
C INITIALIZED  TO AN ARBITRARY INTEGER PRIOR TO THE
C FIRST CALL TO URAND. THE CALLING PROGRAM  SHOULD
C NOT  ALTER  THE  VALUE  OF  IY BETWEEN SUBSEQUENT
C CALLS TO URAND.  VALUES OF URAND WILL BE RETURNED
C IN THE INTERVAL (0,1).
C
```

[the source code for this standard NETLIB function is omitted]

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