HYPER-FINITE METHODS FOR MULTI-DIMENSIONAL STOCHASTIC PROCESSES

By

MARK ALLAN REIMERS

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Department of Mathematics

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date Dec 11 1986

ABSTRACT

In this thesis we introduce Non-Standard Methods, in particular the use of hyperfinite difference equations, to the study of space-time random processes. We obtain a new existence theorem in the spirit of Keisler (1984) for the one dimensional heat equation forced non-linearly by white noise. We obtain several new results on the sample path properties of the Critical Branching Measure Diffusion, and show that in one dimension it has a density which satisfies a non-linearly forced heat equation. We also obtain results on the dimension of the support of the Fleming-Viot Process.

Edwin Perkins

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TABLE 1

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Symbol	Meaning
Δx	Infinitesimal grid spacing in space
Δt	Infinitesimal grid spacing in time
α	$\Delta t/\Delta x^2$
$\mathcal{X}$	*-countable grid representing R^d
$\mathcal{T}$	*-finite grid representing $[0, t_f]$
$\xi_{\underline{t}\underline{x}}$	Internal I.I.D. $S\text{-}L^2$ random variables on $\mathcal{T} \times \mathcal{X}$
$\mathcal{B}(X)$	Borel subsets of X
$Q^{\underline{t}}_{\underline{x}}$	Coefficients for a discrete *-finite Green's formula; also density of an infinitesimal random walk
$U_{\underline{t}\underline{x}}$	Internal solution to stochastic *-finite difference equations
$\delta_{\underline{t}}$	Internal *-finite analogue of $\frac{\partial}{\partial t}$
$\underline{\Delta}$	Internal *-finite analogue of Δ
$\hat{U}_{\underline{t}\underline{x}}$	Internal solution to unforced *-finite analogue of the heat equation
λ	Lebesgue measure on R^d
$\lambda_{\underline{x}}$	Internal measure on $\mathcal{X}$, which assigns mass Δx^d at each grid point
$M_F(R^d)$	Space of positive finite measures on R^d
$M_1(R^d)$	Space of probability measures on R^d
$C_c^\infty(R^d)$	Space of infinitely differentiable functions with compact support in R^d

$C_b^2(R^d)$	Space of C^2 functions bounded on R^d
$C_{b,2}^2(R^d)$	Space of C_b^2 functions whose second derivatives are bounded on R^d
$\langle x \rangle_t$	Predictable square (increasing) process associated with a process x_t

CHAPTER ONE

Introduction

1.1 Why this Thesis

The aim of this work is to introduce nonstandard methods, and in particular the use of hyperfinite difference equations, to the theory of multi-dimensional stochastic processes. Non-standard analysis is a particularly appropriate tool when investigating random processes over a region in space, which evolve in time, especially when the support of these processes is confined to a region of infinitesimal volume, as is frequently the case. The evolution through time of all the processes discussed in this work may be described by the heat operator $\frac{\partial}{\partial t} - \Delta$, but in the opinion of the author, nonstandard techniques may be equally fruitful in the analysis of processes whose development through time is described by different operators as well.

1.2 SPDEs

The approach to the theory of SPDEs which we will follow has its home in the theory of multiparameter processes, and in particular in the theory of multiparameter stochastic integration that has been developed in recent years. Walsh (1986) contains a systematic treatment of this theory. This approach emphasizes sample path properties. An alternative approach considers SPDEs as stochastic evolutions on a space of functions, and emphasizes analytic properties. See Dawson (1975) and (1985) and the references there for further information on this approach. We will not follow it closely here.

The type of SPDE we will be considering most often in this work, is

$$\frac{\partial u}{\partial t} = \Delta u + f(u)\,\dot{W}_{tx}\,, \tag{1-1}$$

where $t \in R^+$ and $x \in R^d$. $\dot{W}_{tx}$ is "white noise" on $R^+ \times R^d$; that is, the derivative, in the sense of distributions (or generalized functions) of a random process W_A, indexed by sets $A \subset R^+ \times R^d$, such that

i) $E(W_A) = 0$ ii) $E(W_A^2) = \lambda(A)$ where λ is Lebesgue measure

iii) if $A \cap B = \emptyset$, then W_A is independent of W_B. For further information on white noise see Walsh (1986) Chapter 1. $f(u)$ is a real-valued function of the point values of u.

Equation (1-1) cannot possibly hold in the classical sense of an equation between the values of functions at every point in the domain. The $\dot{W}_{tx}$ term is far too rough. Rather we usually interpret (1-1) in the weak sense. That is, if we multiply (1-1) by a $C_c^\infty(R^d)$ (smooth, with compact support) function $\phi(x)$ and integrate over a rectangle $[0,T] \times A$, where A contains supp ϕ, then

$$(1\text{-}2) \qquad \int_A u_{tx}\phi(x)dx - \int_A u_{0x}\phi(x)dx = \int_0^T \left(\int_A u_{sx}\Delta\phi(x)dx\right)ds$$

$$+ \int_0^T \int_A f(u_{sx})\phi(x)dW_{sx}$$

where the last integral on the right is the multiparameter stochastic integral in the sense of Itô discussed in Walsh (1986) Chapter 1. Equation (1-2) may be derived from (1-1) taken in the classical sense, by integration of the middle term by parts.

An existence theory for (1-1) has been developed in the case $d = 1$, when $f : R \to R$, is a Lipshitz - continuous function which grows at most linearly at infinity. Dawson (1972) established existence and uniqueness under these conditions, using a Hilbert space approach.

Funaki (1983), established the same result, with joint continuity of sample paths in t and x. Walsh (1981) established a modulus of continuity in t and in x for solutions of an equation similar to (1-1) and investigated finer sample path properties, under the same conditions on f.

In Chapter Three of this thesis an existence result is established for (1-1) assuming only continuity and linear growth of f for $d = 1$.

The situation for $d \geq 2$ is entirely different. The term $\dot{W}_{tx}$ in (1-1) may be regarded as a derivative of order $d + 1$ of a continuous function of unbounded variation (the Brownian Sheet) on $R^+ \times R^d$. When $d \geq 2$ there is no hope of finding a continuous function u to satisfy the equation, even in the weak sense of (1-2). The most we can hope for is to find a continuous process v, such that u may be regarded as a derivative, in the sense of distributions, of order $d - 1$, of v. In this case u will not, in general, have point values, and it is difficult to see what sense can be made of the term $f(u_{sx})$ occurring in the stochastic integral on the r.h.s. of (1-2). In order for our theory of stochastic integration to make sense of (1-2) we would need $f(u_{sx})$ to be an adapted continuous process. This is out of the question if f is supposed to be a real function of the (non-existent) point values of u.

The functions f for which (1-2) can be reasonably expected to make sense are the constant functions. Walsh (1984) has shown existence and uniqueness of solutions to (1-2) in this case. We were not able to extend his results (see Appendix B).

1.3 The Dawson Critical Measure Valued Diffusion

Measure-Valued Branching Processes (MB Processes) were first obtained by Jirina (1958) as a limit of a branching diffusion of a large number of particles. These processes were studied extensively by Watanabe (1968) and co-workers, and lately many details of the fine structure have been obtained by Dawson and Hochberg (1979). The name of Dawson is particularly associated with the case we shall study here, hence we refer to it often as the "Dawson Process". We will however most often make use of a martingale characterization of this process described in Roelly-Coppoletta (1986).

A simple construction, for the case of a initial Lebesgue measure is as follows. Let particles be distributed initially on R^d or a portion thereof, according to a Poisson point process with intensity λ. Suppose that thereafter each particle independently executes a Brownian motion on R^d. Also suppose that each particle independently undergoes critical branching with rate μ, i.e. at fixed or exponentially distributed times, whose number in a unit time interval has expectation μ, the particle dies or splits into two particles, each outcome being equally likely. If the particle splits, both daughter particles begin independent careers from the point of bifurcation. We now assign a mass of $\frac{1}{\lambda}$ to each particle, and obtain a random measure on R^d at each time.

Now suppose that both the initial density of particles λ, (which is the reciprocal of the weight assigned to each particle) and the branching rate μ, are allowed to go to infinity in such a way that $\frac{\mu}{\lambda}$ is constant. Then there is a limiting process taking values in the space of positive measures on R^d.

Dawson (1972) indicated a connection in $d = 1$ between this process obtained as a limit of a particle system, and the solution to the SPDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sqrt{u}\ \dot{W}_{tx} \quad , \tag{1-3}$$

but this connection has not yet been made rigorous, since the function $f : u \longrightarrow \sqrt{u}$ is not Lipshitz, and hence (1-3) does not fall under the purview of the existence theory discussed in 1.2. With the existence theorem in chapter three, we may now close the gap in Chapter **Five**, and establish the identity of the solution of (1-3) with the MB process in one dimension.

As mentioned in 1.2, it is difficult to make sense of an equation like (1-3) in dimensions $d \geq 2$. However a non-standard analogue of (1-3) may be constructed, and its solution may be shown to coincide with the Dawson process. This construction is used in Chapter Four to establish several previously unknown results on the sample paths of the Dawson Process. Specifically we show that i) any given Lebesgue null set is a.s. never charged, ii) the mass on any given Lebesgue set is a.s. a continuous function of time, iii) convergence of a sequence of bounded functions in measure, implies the a.s. convergence of the integrals of those functions with respect to the random measure, uniformly on finite time intervals.

1.4 The Fleming-Viot Process

In Fleming and Viot (1979) a measure-valued process was introduced as a limit, under suitable scalings of time and space, of the Ohta-Kimura stepwise mutation model. Further results on the structure of the sample paths have been obtained by Dawson and Hochberg (1982). Among these results is the fact that for fixed times the support of the random measure has Hausdorff dimension not greater than 2, almost surely. In Chapter Six we establish this result for all times simultaneously using Non-standard methods to amplify some of the ideas of Dawson and Hochberg (1982). For further information on the Fleming-Viot process, the reader is referred to section 6.1.

CHAPTER TWO

Non-Standard Analysis and Probability

2.1 Some Definitions and Notations from Non-Standard Analysis

For a real introduction the ideas of non-standard analysis, with a minimum of technical apparatus, we refer the reader to Cutland (1983).

An internal object in the non-standard universe is one which may be referred to in the non-standard language. One of the consequences of the transfer principle is that internal objects described in the non-standard language inherit all the qualities of standard objects that are described in analogous standard language. This is useful when dealing with hyper-finite collections, which may be treated as finite sets, though they are generally infinite.

We will usually denote non-standard objects by capital letters or underlined letters. Unless otherwise noted, lower case roman letters will stand for standard objects. The embedding of a standard object into the non-standard universe will be denoted by an asterisk (*) to the left.

We say that $\underline{x} \in {}^*R$ is infinitesimal if $|\underline{x}| < \frac{1}{n}$ for every $n \in N$; we denote this $\underline{x} \approx 0$; $\underline{x} \approx \underline{y}$ means $\underline{x} - \underline{y} \approx 0$. We say $\underline{x} \in {}^*R$ is infinite, if $|\underline{x}| > n$, for every $n \in N$. If $\underline{x} \in {}^*R$ is finite (not infinite) then there is a unique $x \in R$ such that $\underline{x} \approx {}^*x$. We say that $\underline{x}$ is near standard and call x the standard part of $\underline{x}$, denoted ${}^\circ\underline{x}$ or $st(\underline{x})$. These concepts may be extended, in the obvious fashion, to any space. In particular, if $f \in C(R^k; R^n)$ and $F \in {}^*C(R^k; R^n)$, then $st(F) = f \iff {}^\circ F(\underline{x}) = f({}^\circ\underline{x})$ for all nearstandard $\underline{x} \in {}^*R^k$.

We shall require the following axiom of saturation (see Cutland 1983, 1.9).

If $\{A_n\}_{n\in N}$ is a decreasing family of non-empty internal sets, then $\bigcap_{n\in N} A_n$ is not empty. Two consequences of this are

a) Denumerable Comprehension. For every internal set A, and every function $f : N \to A$, there is an internal function $F : {}^*N \to A$ which extends f.

b) Infinitesimal Underflow. Let S be internal, $S \subseteq {}^*R$, and suppose for some $a > 0$, $\underline{x} \in S$ whenever $0 < {}^\circ\underline{x} \leq a$. Then for some $\varepsilon \approx 0$, $\underline{x} \in S$ whenever $\varepsilon \leq \underline{x} \leq {}^*a$.

A useful notion is S-continuity (S for standard).

Definition 2.1.1. An internal function $F : E \subset {}^*R^d \to {}^*R^n$ is called S-continuous iff $\underline{x} \approx \underline{y} \Rightarrow F(\underline{x}) \approx F(\underline{y}) \in ns({}^*R^n)$. That this is the appropriate notion of continuity to link standard and non-standard, is shown by

Theorem 2.1.2. (Cutland (1983), Theorem 1.6).

Let $F : E \subset {}^*R \to {}^*R$ be internal. Then ${}^\circ F$ exists and is continuous iff F is S-continuous on E.

Nonstandard Probability Theory really came into its own after the development of Loeb Measure.

Theorem 2.1.3. (see Cutland (1983), Theorem 3.1).

Every internal *finitely additive measure space $(\Gamma, F, \underline{\mu})$ gives rise to a classical σ-additive measure space $(\Gamma, L(F), L(\underline{\mu}))$, such that $F \subset L(F)$, and if $A \in F$, then $L(\underline{\mu})(A) = {}^\circ\underline{\mu}(A)$. If $A \in L(F)$, then $L(\underline{\mu})(A) = \inf\{{}^\circ\underline{\mu}(B) \mid A \subseteq B, A \in F\}$. Further, if $L(\underline{\mu})(A) < \infty$, then there is $B \in F$, with $L(\underline{\mu})(B \,\Delta\, A) = 0$.

The (complete) measure space $(\Gamma, L(F), L(\underline{\mu}))$ is called the Loeb extension of $(\Gamma, F, \underline{\mu})$. In the case of probability measures, we will denote the Loeb extension of $(\Omega, F, \underline{P})$ by $(\Omega, \mathcal{F}, P)$, contrary to our usual convention.

Theorem 2.1.4. (see Cutland (1983) Theorems 3.1, 3.5).

If E is an internal field of subsets of Γ, and $F : \Gamma \to {}^*R^d$ is an internal E-measurable function then the projection $st \circ F : \Gamma \to R^d \cup \{\infty\}$ is $L(E)$ measurable. Conversely if $f : \Gamma \to R^d$ is $L(E)$ measurable, and Γ is σ-finite with respect to $L(\underline{\mu})$ (as is always the case here), then there is an internal E-measureable function $F : \Gamma \to {}^*R^d$ such that $st \circ F(x) = f(x)$ $L(\mu)$ - a.e. Such an internal F is called a lifting of f. If f has finite support, i.e. $L(\underline{\mu})(\{\underline{x} \mid f(\underline{x}) \neq 0\}) < \infty$, then f admits a lifting F with the same property. If $\Gamma \subseteq {}^*R^d$ if $f : \Gamma \to R$ is continuous, then we may obtain a lifting F which is S-continuous and for which $st \circ F(x) = f({}^\circ\underline{x})$, $\forall\, \underline{x} \in \Gamma \cap st^{-1}(R^d)$ (the near-standard points). Such an F is called a uniform lifting of f.

Definition 2.1.5. An internal function F on $(\Gamma, F, \underline{\mu})$ is called S-integrable, iff $\int |F| d\underline{\mu}$ is finite, and

$$\int_{\{x:\ |F(\underline{x})| < \delta\}} |F(\underline{x})| d\underline{\mu} + \int_{\{\underline{x}:\ |F(\underline{x})| > H\}} |F(\underline{x})| d\underline{\mu} \approx 0 \text{ , for all}$$

infinitesimal δ and infinite H. F is said to be S-L^q with respect to $\underline{\mu}$, if $|F|^q$ is S- integrable.

Theorem 2.1.6. (see Cutland (1983) Theorem 3.9).

If F is S-integrable on $(\Gamma, F, \underline{\mu})$, then for all $A \in F$,

$$^\circ\int_A F\, d\underline{\mu} = \int_A {}^\circ F\, d\, L(\underline{\mu}) \ .$$

Definition 2.1.7. An internal measure space $(\Gamma, F, \underline{\mu})$ is called a hyperfinite representation of a topological measure space $(E, \mathcal{F}, \mu)$ iff

(i) Γ is a hyperfinite internal subset of ${}^*E$

(ii) F is the internal power set of Γ

(iii) A set $\mathcal{B} \subset E$ is μ-measureable, iff $st^{-1}(\mathcal{B}) \cap \Gamma$ is $L(\underline{\mu})$-measureable.

In that case $\mu(\mathcal{B}) = L(\underline{\mu})(st^{-1}(\mathcal{B}) \cap \Gamma)$.

The canonical example of a hyperfinite representation is the discrete representation of Lebesgue measure on R^d , which we will use frequently.

Theorem 2.1.8. (see Cutland (1983) Theorem 4.1).

Let Δx_i be any infinitesimal, i = 1,...,d. Let $M_i \in {}^*N \setminus N$ be infinite, i = 1, 2,...,d such that ${}^\circ M_i \Delta x_i \neq 0$. Then let $X = \{k_1 \Delta x_i, \ldots, k_j \Delta x_j) | k_i \in {}^*Z, |k_i| \leq M_i\}$. Define a measure $\underline{\lambda}$ on X by setting the value of $\underline{\lambda}$ on each point $\underline{x} \in X$ to be $\prod_{i=1}^{d} \Delta x_i$. Then $(X, P(X), (\underline{\lambda}))$ is a hyperfinite representation of the rectangle $\{\underline{x} | |x_i| \leq {}^\circ M_i \Delta x_i\} \subseteq R^d$, equipped with Lebesgue measure.

In this thesis we will prove results about S-continuity of functions from hyperfinite grids X to ${}^*R$. All the preceding definitions and theorems apply as if these functions were step functions on a rectangle in ${}^*R^d$.

Theorems about S-continuity on a hyperfinite grid X , can be translated into theorems about the weak convergence of a sequence of processes on a sequence of finite grids to a continuous limit. We will not explicitly make such a translation.

2.2 Non Standard White Noise

Anderson (1976) introduced a hyperfinite representation of Brownian Motion, namely an infinitesimal random walk with spatial excursions of size $\sqrt{\Delta t}$ in a time step Δt. Implicit in this construction was a non-standard representation of white noise on the line, as a sum of IID random variables $\xi_{\underline{t}}$, each of mean 0 and variance Δt, on a hyperfinite time-line of spacing Δt.

Recently Andreas Stoll (1985) has generalized Anderson's construction to arbitrary σ-finite Radon spaces. For our purposes we only need representations of white noise on rectangles in R^d, or on all of R^d.

Let X be a hyperfinite lattice as described above, and let Ω be an internal space on which are defined a family $\{\xi_{\underline{x}}\}_{\underline{x}\in X}$ of I.I.D. $S\text{-}L^2$ internal random variables, such that $\underline{E}(\xi_{\underline{x}}) = 0$ and $\mathrm{var}(\xi_{\underline{x}}) = 1$. For most of our applications we will need finiteness of all the higher moments of $\xi_{\underline{x}}$ as well.

The existence of such a space Ω may be shown by example. Let $\Omega = \{-1, 1\}^X$. Let F be the family of internal subsets of Ω, and define $\underline{P}(A) = \dfrac{|A|}{|\Omega|}$ for $A \in F$. Then $(\Omega, F, \underline{P})$ is an internal probability space, and we may define $\xi_{\underline{x}}$ as the coordinate maps.

We will most often be using a *countable lattice $X = \{(k_1\Delta x,\dots,k_j\Delta x) \mid k_1,\dots,k_d \in {}^*Z\}$. An exemplary space Ω is then $\{-1, 1\}^X$. We may take F to be the ${}^*\sigma$-field generated by the *finite subsets of Ω (closed under ${}^*\sigma$-unions and ${}^*\sigma$-intersections). The internal probability measure $\underline{P}$ may be defined as in the standard analogue: $\underline{P}$ is the unique ${}^*\sigma$-additive measure on $\underline{F}$ such that, $\{\omega_{\underline{x}} : \underline{x} \in \mathcal{X}\}$ are independent and for any $\underline{x} \in \mathcal{X}$,

$$\underline{P}(\{\omega_{\underline{x}} = 1\}) = \underline{P}(\{\omega_{\underline{x}} = -1\}) = \frac{1}{2} .$$

The transfer principle guarantees that the Kolmogorov Extension Theorem carries over to the non-standard setting, and therefore that such a measure $\underline{P}$ exists. In this case again we will take the probability space $(\Omega, \mathcal{F}, P)$ to be the Loeb extension of $(\Omega, F, \underline{P})$.

Given such a space Ω, and random variables $\xi_{\underline{x}}$, define for internal sets $A \subseteq \mathcal{X}$, $W(A) = \sum_{\underline{x} \in A} \xi_{\underline{x}} \prod_{i=1}^{d} \Delta x_i$; the map $W : A \to W(A)$ is called <u>d-dimensional S-white noise</u> on $\mathcal{X}$. Stoll (1986) shows

<u>Lemma 2.2.2</u>. If $^{\circ}\underline{\lambda}(A) < \infty$ and $\underline{\lambda}(A \Delta B) = 0$, then $^{\circ}W(A) = {}^{\circ}W(B)$ P-a.s.

Thus we may make

Definition 2.2.3. For each Loeb measureable set $\mathcal{A} \subset \mathcal{X}$, with $L(\underline{\lambda})(\mathcal{A}) < \infty$, a standard random variable $w(\mathcal{A})$ is well defined (up to a null set) by

$$w(\mathcal{A}) = {}^{\circ}W(A) \quad P\text{-a.s.}$$

whenever A is internal and $L(\underline{\lambda})(\mathcal{A} \Delta A) = 0$.

<u>Theorem 2.2.4</u>. (Stoll (1986) Theorem 2.5).

The family $\{W(\mathcal{A}) \mid L(\underline{\lambda})(\mathcal{A}) < \infty\}$, is a white noise on the Loeb extension of $(X, \underline{\lambda})$ with respect to the Loeb probability space $(\Omega, \mathcal{F}, P)$.

2.3 Adapted Stochastic Integrals

Let Δt and Δx be infinitesimals, and let T be a lattice of spacing Δt representing a line segment in R^+, and let X be a lattice of spacing Δx in each direction representing a rectangle in R^d. Let Ω be an internal space supporting a collection $\xi_{\underline{tx}}$, $(\underline{t},\underline{x}) \in T \times X$, of I.I.D. S-$L^2$ random variables, as described above.

In this section we will use the notation $dW_{\underline{tx}}$ to represent $\xi_{\underline{tx}}\sqrt{\Delta t\ \Delta x^d}$. We define an internal filtration $F_{\underline{t}}$, $\underline{t} \in T$ on Ω to be the algebra of internal sets generated by $\{W_{\underline{sx}} \mid 0 \leq \underline{s} < \underline{t},\ \underline{x} \in X\}$. On the Loeb space (Ω, F, P) we define a filtration F_t, $t \in st(T)$, by $F_t = \bigcap_{{}^\circ\underline{t}>t} \sigma(F_{\underline{t}}) \vee N$, where N is the collection of P-null sets. For properties of this filtration see Hoover and Perkins (1983) §3.

We say that an internal process $U_{\underline{tx}}(\omega)$ is a <u>lifting</u> of a process $u_{tx}(\omega)$ on $R^+ \times R^d \times \Omega$, if for $H \in {}^*\mathbf{N}\setminus N$ and $n \in N$, $\int_T\int_X U^2_{\underline{tx}} I(\{|\underline{x}| \geq H\ ;\ \underline{t} \leq n\})\, d\lambda_{\underline{x}}\, d\lambda_{\underline{t}} \approx 0$, and

$$L(\lambda_{\underline{t}} \times \lambda_{\underline{x}} \times \underline{P})(\{(\underline{t}, \underline{x}, \omega) \mid {}^\circ U_{\underline{tx}}(\omega) \neq u_{{}^\circ\underline{t},\, {}^\circ\underline{x}}(\omega)\}) = 0 .$$

We say that an internal process $U_{\underline{tx}}(\omega)$ on $T \times X$ is $F_{\underline{t}}$-adapted, if $U_{\underline{sy}}$, $0 \leq \underline{s} \leq \underline{t}$, $\underline{y} \in X$, is $F_{\underline{t}}$ measureable.

We shall call a process u_{tx} on $R^+ \times R^d$, F_t-adapted, if, for each $t \in R^+$, $u_{sx}(\omega)$, $(s \in [0, t],\ x \in R^d,\ \omega \in \Omega)$, is $B([0,t]) \times B(R^d) \times F_t$ measureable. Since our filtration F_t is continuous and we do not wish to integrate discontinuous integrands, we shall not make distinctions between adapted, progressively measureable, optional, and predictable processes.

We shall usually be working with $\underline{F}_t$-adapted liftings of $\mathcal{F}_t$-adapted processes.

Theorem 2.3.1. Suppose $u_{tx}(\omega)$ is an L^2 $\mathcal{F}_t$-adapted process on $R^+ \times R^d \times \Omega$, and that an internal $\underline{F}_{\underline{t}}$ adapted process $U_{\underline{t}\underline{x}}(\omega)$ lifts u, and is $S\text{-}L^2(\mathcal{T} \times \mathcal{X} \times \Omega;\ L(\lambda_{\underline{t}} \times \lambda_{\underline{x}} \times \underline{P}))$. Then for any $t \in R^+$ and $\underline{t} \approx t$

$$\int_0^t \int_{R^d} u_{sx}\, dW_{sx} \overset{\text{a.s.}}{=} {}^{\circ}\sum_{0\le \underline{s}<\underline{t}} \ \sum_{\underline{x}\in \mathcal{X}} U_{\underline{s}\underline{x}}\, dW_{\underline{s}\underline{x}} \tag{2-1}$$

where the integral on the left is in the sense of Itô.

Proof: We first establish an isometry property:

$$\underline{E}\left| \sum_{0\le \underline{s}<\underline{t}} \ \sum_{\underline{x}\in \mathcal{X}} U_{\underline{s}\underline{x}}\, dW_{\underline{s}\underline{x}} \right|^2$$

$$= \underline{E}\left(\sum_{0\le \underline{s}<\underline{t}} \left(\sum_{\underline{x}\in \mathcal{X}} U_{\underline{s}\underline{x}}\, dW_{\underline{s}\underline{x}} \right)^2 + 2 \times \sum_{0\le \underline{s}<\underline{t}} \ \sum_{\underline{s}<\underline{s}'<\underline{t}} \left(\sum_{\underline{x}\in \mathcal{X}} U_{\underline{s}\underline{x}}\, dW_{\underline{s}\underline{x}} \right) \underline{E}\left(\sum_{\underline{x}\in \mathcal{X}} U_{\underline{s}'\underline{x}}\, dW_{\underline{s}'\underline{x}} \,\middle|\, \underline{F}_{\underline{s}} \right) \right)$$

(since $dW_{\underline{s}'\underline{x}}$ is (conditionally) independent of $\underline{F}_{\underline{s}}$)

$$= \underline{E}\left(\sum_{0\le \underline{s}<\underline{t}} \ \sum_{\underline{x}\in \mathcal{X}} \Big[U^2_{\underline{s}\underline{x}}\, dW^2_{\underline{s}\underline{x}} + \sum_{\substack{\underline{x}'\neq \underline{x} \\ \underline{x}'\in \mathcal{X}}} U_{\underline{s}\underline{x}} U_{\underline{s}\underline{x}'}\, dW_{\underline{s}\underline{x}}\, dW_{\underline{s}\underline{x}'} \Big] + 2 \times 0 \right)$$

$$= \sum_{0\le \underline{s}<\underline{t}} \ \sum_{\underline{x}\in \mathcal{X}} \underline{E}(U^2_{\underline{s}\underline{x}})\, \Delta x^d \Delta t + 0 \quad \text{(using } E(dW^2_{\underline{s}\underline{x}}) = \Delta x^d \Delta t\text{)} .$$

$$= ||U||^2_{L^2(\mathcal{T}\times\mathcal{X}\times\Omega)}$$

Hence the mapping $U \longrightarrow \sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} U_{\underline{sx}} dW_{\underline{sx}}$ acts isometrically from the space of $F_{\underline{t}}$-adapted $S\text{-}L^2(T \times X \times \Omega)$ processes to ${}^*L^2(\Omega)$. Hence, if U and U' are any two $S\text{-}L^2$ $F_{\underline{t}}$-adapted liftings of u ,

$$\underline{E} \left| \sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} U_{\underline{sx}} dW_{\underline{sx}} - \sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} U'_{\underline{sx}} dW_{\underline{sx}} \right|^2$$

$$= \sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} \underline{E} \left| U_{\underline{tx}} - U'_{\underline{tx}} \right|^2 \Delta x^d \Delta t$$

$$\approx 0$$

Hence $${}^\circ\sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} U_{\underline{sx}} dW_{\underline{sx}} \overset{a.s.}{=} {}^\circ\sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} U'_{\underline{sx}} dW_{\underline{sx}}$$

A similar argument shows that if $\underline{t}' \lessapprox \underline{t}$, then

$${}^\circ\sum_{\underline{t}' \le \underline{s} < \underline{t}} \sum_{\underline{x} \in X} U_{\underline{sx}} dW_{\underline{sx}} \overset{a.s.}{=} 0$$, so that the r.h.s. of (2-1) is well defined up to a null set.

We must show that the r.h.s. of (2-1) coincides with the Itô integral. Consider a process u of the form

$$u_{tx}(\omega) = I_{[t_1, t_2)}(t) \cdot I_A(x) \cdot I_R(\omega) ,$$

where R is an F_{t_1} measureable set in Ω , and A is Lebesgue measureable in R^d. Pick $\underline{t}_2 \approx {}^*t_2$, and pick A internal in $\mathcal{X}$ such that $L(\lambda_{\underline{x}})(A \,\Delta\, st^{-1}(\mathcal{A})) = 0$. By Theorem 3.2 in Hoover and Perkins (1983), we may find $\underline{t}_1 \approx {}^*t_1$ and an internal subset R of Ω, such that $R \in F_{\underline{t}_1}$ and $P(\mathcal{R} \,\Delta\, R) = 0$.

Let $U_{\underline{tx}}(\omega) = I_{[\underline{t}_1, \underline{t}_2]}(\underline{t})\, I_A(\underline{x})\, I_R(\omega)$

Then $$\int_0^t \int_{R^d} u_{sx}\, dw_{sx} = I_{\mathcal{R}}(\omega) \cdot \int_{[t_1, t\wedge t_2)\times \mathcal{A}} dw_{sx}$$

$$\overset{a.s.}{=} I_R(\omega) \cdot \sum_{\substack{\underline{t}_1 \le \underline{s} \le \underline{t}\wedge\underline{t}_2 \\ \underline{x}\in A}} dW_{\underline{sx}}, \text{ (by definition of the white noise } dw_{tx})$$

$$= \sum_{0 \le \underline{s} < \underline{t}} \sum_{\underline{x}\in \mathcal{X}} U_{\underline{sx}}\, dW_{\underline{sx}}$$

Thus the left hand side and rhs of Theorem 2.3.1 coincide for simple functions u of the form indicated, and hence for linear combinations thereof. Such linear combinations are clearly dense in the Hilbert space of L^2 $\mathcal{F}_t$-adapted processes on $[0, t]$, as in the case of the ordinary stochastic integral. Hence the two sides must coincide for all such u and U by the isometry property of the stochastic integral, and of the hyperfinite sum.

CHAPTER THREE

The Heat Equation with Non-Linear Stochastic Forcing

3.1 Scope

In this chapter I will use hyperfinite methods to prove a weak existence theorem for solutions of equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)\,\dot{W}_{tx}\,, \tag{3-1}$$

on $R^+ \times R$. Here $\dot{W}_{tx}$ is "white noise" on $R^+ \times R$. We suppose the function f to be continuous, and to satisfy a growth condition: there is a real K , such that

$$f^2(u) \le K(1+u^2) \quad \text{for all } u\,. \tag{3-2}$$

We solve (3-1) subject to an initial condition

$$u(0,x) = u_0(x)\,, \tag{3-3}$$

where $u_0(x)$ is a bounded continuous function.

With minor changes in notation f could be made to depend on t and x as well as u , and $u_0(x)$ could be taken as a random function. With more significant changes, the same construction can be made to work if (3-1) is modified by the introduction of an additional forcing term $+\,g(t,\, x,\, u)$, or by the introduction of a bounded, non-zero, continuous non-linear function of u multiplying the term $\frac{\partial^2 u}{\partial x^2}$. With appropriate inequalities analogous to those in Appendix A, a treatment very similar to the remainder of this chapter can be done for equation (3-1) on a strip $R^+ \times [a,\, b]$ with

Dirichlet or Neumann type boundary conditions specified at a and b. A random initial condition u_0 independent of the white noise $\dot{W}_{tx}$ may be handled by enlarging the probability space Ω.

As discussed in section 1.2, (3-1) is only solvable in a 'weak' sense, that is, if $\phi \in C_c^\infty(R)$,

$$(3\text{-}4)\quad \int_{-\infty}^{\infty} u_{tx}\phi(x)\,dx - \int_{-\infty}^{\infty} u_{0x}\phi(x)\,dx = \int_0^t \int_{-\infty}^{\infty} u_{sy}\Delta\phi(y)\,dy\,ds$$

$$+ \int_0^t \int_{-\infty}^{\infty} f(u_{sy})\phi(y)\,dW_{sy}$$

This is called the "weak" form of (3-1). The main theorem (3.9.2) of this chapter asserts that there exists a space Ω, such that, for any f, a stochastic process u_{tx}, jointly continuous in t and x, may be defined on Ω, for which (3-4) holds for any $\phi \in C_c^\infty(R)$.

Section 3.2 introduces the probability space Ω and discusses the construction of a white noise on Ω.

Section 3.3 exhibits a hyperfinite analogue of (3-1) and shows how it may be solved internally, for a solution $U_{\underline{\underline{tx}}}$.

Section 3.4 contains several inequalities which are used subsequently. Proofs are delayed until Appendix A.

Section 3.5 contains estimates on the moments of the internal solution $U_{\underline{\underline{tx}}}$, which are necessary for 3.6 and 3.7.

Section 3.6, obtains bounds on the moments of spatial differences $U_{\underline{\underline{tx}}} - U_{\underline{\underline{ty}}}$. In section 3.7 we obtain bounds on the moments of temporal differences $U_{\underline{tx}} - U_{\underline{rx}}$. We use the results of 3.6 and 3.7 in 3.8 to show that $U_{\underline{tx}}$ is, with probability 1, a lifting of a jointly continuous process u_{tx} .

In 3.9 we verify that the process u_{tx} actually satisfies (3-4), hence is a weak solution of (3-1).

3.2 White Noise on the space Ω

Let a positive infinitesimal Δx be given. Let X be the set $\{k\Delta x | k \in {}^*Z\}$. Now pick α such that $0 < {}^\circ\alpha < \frac{1}{3}$, and let $\Delta t = \alpha\, \Delta x^2$. Now suppose $t_f > 0$ is given and let $\underline{t}_f$ be any number of the form $\{k\Delta t,\ k \in {}^*N\}$ such that $\underline{t}_f \approx t_f$. Let T be $\{k\Delta t | k \in {}^*N,\ k \leq \underline{t}_f / \Delta t\}$.

We have mentioned in section 2.2 that if Ω is a *countable space (such as $\{-1, 1\}^{T\times X}$) on which are defined a family of I.I.D. random variables $\{\xi_{\underline{tx}} | \underline{t} \in T,\ \underline{x} \in X\}$ such that $\underline{E}(\xi_{\underline{tx}}) = 0$ & $\underline{E}(\xi^2_{\underline{tx}}) = 1\ \forall(\underline{t},\underline{x})$, and such that $\xi_{\underline{tx}}$ possesses finite higher moments of all orders, then the random (internal) set function $A \longrightarrow \sum_{(\underline{t},\underline{x})\in A} \xi_{\underline{tx}} \sqrt{\Delta t\Delta x}$ induces a "white noise" $\{w_A(\omega) | A \in \mathcal{B}([0, t_f] \times R)\}$. Further, this white noise is adapted to the filtration F_t derived as per section 2.3 from the internal filtration $F_{\underline{t}}$ generated by the values $\{\xi_{\underline{sx}} | \underline{s} \leq \underline{t}\}$. It is with respect to this white noise that we shall solve (3-1). The method we shall use is similar in spirit to that employed in Keisler (1983) for stochastic O.D.E.'s.

3.3 Hyper-Finite Difference Equations

Let $F(\underline{u})$ be a uniform lifting of $f(u)$, subject also to the growth condition (3-2). Let U_0 be a uniform lifting of u_0 which is also uniformly bounded on X. Then consider the hyperfinite analogue of (3-1),

$$\text{(3-5)} \qquad \frac{U_{\underline{t}+\Delta t,\underline{x}} + U_{\underline{t},\underline{x}}}{\Delta t} = \frac{U_{\underline{t},\underline{x}+\Delta x} - 2\,U_{\underline{t},\underline{x}} + U_{\underline{t},\underline{x}-\Delta x}}{\Delta x^2} + \frac{F(U_{\underline{tx}})\,\xi_{\underline{tx}}}{\sqrt{\Delta t\,\Delta x}}$$

or equivalently,

$$U_{\underline{t}+\Delta t,\underline{x}} = U_{\underline{t},\underline{x}} + \Delta t\,\Delta x^{-2}[U_{\underline{t},\underline{x}+\Delta x} - 2\,U_{\underline{t},\underline{x}} + U_{\underline{t},\underline{x}-\Delta x}] + F(U_{\underline{tx}})\,\xi_{\underline{tx}}\,\frac{\Delta t}{\Delta x}\,.$$

We may solve (3-5) in principle inductively. The specification of $U_{\underline{0},\underline{x}}$ and $\xi_{\underline{0},\underline{x}}$ for $\underline{x} \in X$ gives us enough information to find $U_{\Delta t,\cdot}$. Knowing $U_{\Delta t,\cdot}$ and $\xi_{\Delta t,\cdot}$, we may solve for $U_{2\Delta t,\cdot}$, and so on.

Continuing in this manner, we define an internal process $U_{\underline{tx}}(\omega)$. We may find a closed form expression for this inductive definition as follows. From (3-5) it is clear that the value $F(U_{\underline{tx}})\,\xi_{\underline{tx}}$ enters into the definition of $U_{\underline{t}+\Delta t,\underline{x}}$, then $U_{\underline{t}+2\Delta t,\underline{x}-\Delta x}$, $U_{\underline{t}+2\Delta t,\underline{x}}$, $U_{\underline{t}+2\Delta t,\underline{x}+\Delta x}$, and then five values of $U_{t+3\Delta t,\cdot}$, and so on. The coefficient $Q^{n\Delta t}_{\underline{x}-\underline{y}}$ with which $F(U_{\underline{tx}})\,\xi_{\underline{tx}}$ enters into the definition of $U_{\underline{t}+n\Delta t,\underline{y}}$, may be found as the solution of a difference equation: $Q^{\Delta t}_0 = 1$; $Q^{\Delta t}_{\underline{x}} = 0\ \forall \underline{x} \neq 0$;

$$\text{(3-6)} \qquad Q^{(n+1)\Delta t}_{\underline{x}} = \alpha\,Q^{n\Delta t}_{\underline{x}-\Delta x} + (1-2\alpha)Q^{n\Delta t}_{\underline{x}} + \alpha\,Q^{n\Delta t}_{\underline{x}+\Delta x}\,.$$

But we may recognize (3-6) as the difference equation governing the density of a Markov Process, in fact, a random walk. Thus we have

Lemma 3.3.1. The coefficients $Q_{\underline{x}}^{\underline{t}}$ are the "density" of an infinitesimal random walk on the lattice X, starting at $\underline{x} = \underline{0}$, at time Δt, and taking one step to the right (or left) in each time period Δt with probability α.

With this notation, we may write

$$(3\text{-}7) \qquad U_{\underline{tx}} = \sum_{0 \leq \underline{s} < \underline{t}} \sum_{\underline{y} \in X} Q_{\underline{x}-\underline{y}}^{\underline{t}-\underline{s}} \; F(U_{\underline{sy}}) \; \xi_{\underline{sy}} \sqrt{\frac{\Delta t}{\Delta x}}$$

$$+ \sum_{\underline{y} \in X} Q_{\underline{x}-\underline{y}}^{t+\Delta t} \; U_{0\underline{y}}$$

The second term on the r.h.s. of (3-7) is the deterministic solution to the hyperfinite heat equation (3-5), which we will designate hereafter $\hat{U}_{\underline{tx}}$.

3.4 Some Useful Inequalities

We will find the following inequalities helpful in the next three sections. Proofs are deferred to Appendix A, since they do not illuminate the particular subject matter of this chapter.

Lemma 3.4.1. There is a finite constant K_α, depending on α, such that for

$$\underline{t}/\Delta t \in {}^*N \, , \quad \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{t}})^2 \le K_\alpha \sqrt{\frac{\Delta t}{\underline{t}}} \; .$$

Lemma 3.4.2. There is a finite constant K_α, depending on α, such that for all $\underline{t}$, $\displaystyle\sum_{0<\underline{s}<\underline{t}} \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{s}})^2 \le K_\alpha \sqrt{\underline{t}/\Delta t}$.

Lemma 3.4.3. There is a constant K_α, depending on α, such that $\forall\, \underline{z} \in X$

$$\sum_{0<\underline{s}<t} \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{s}} - Q_{\underline{x}+\underline{z}}^{\underline{s}})^2 \le K_\alpha \, |\underline{z}| \, / \, \Delta x \; .$$

Lemma 3.4.4. There is a constant K_α, depending on α, such that, for all $\underline{t} \in T$, and $\underline{r} \le \underline{t}$

$$\sum_{0\le\underline{s}<\underline{r}} \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{t}-\underline{s}} - Q_{\underline{x}}^{\underline{r}-\underline{s}})^2 \le K_\alpha \sqrt{(\underline{t}-\underline{r})/\Delta t} \; .$$

We shall also require the following theorem of Burkholder (1973, Theorem 2.1.1., specialized slightly).

Theorem 3.4.5. Let M_n, $n \in {}^*N$ be a hypermartingale and let $\langle M\rangle_n$ $n \in {}^*N$ be the associated predictable square function, and let $p > 1$ be finite. Then there is a finite constant K_p depending only on p, such that

$$\underline{E}(M_n^p) \le K_p \, \underline{E}(\langle M\rangle_n^{p/2}) + K_p \, \underline{E}(\max_{0\le k<n} |M_{k+1} - M_k|^p) \; .$$

3.5 Bounds on Moments of $U_{\underline{tx}}$

Pick $q > 1$, and let $R_q(\underline{t},\underline{x}) = \underline{E}|U_{\underline{tx}}|^{2q}$. Let $H_q(\underline{t}) = \sup_{\underline{x}\in X} R_q(\underline{t},\underline{x})$.

Consider any fixed $(\underline{t},\underline{x})$, and let μ be the measure on $[\underline{0}, \underline{t}] \times X$ defined by $\mu(\{(\underline{s},\underline{y})\}) = (Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}})^2 \frac{\Delta t}{\Delta x}$. Let $|\mu|$ denote $\mu([0, \underline{t}] \times X)$.

Let c denote, in what follows, a finite constant, depending only on q and t_f, which may change its exact value from line to line.

Now

$$R_q(\underline{t},\underline{x}) \leq c\, \underline{E} \left| \sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} F(U_{\underline{sy}})\, Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}}\, \xi_{\underline{sy}} \left(\frac{\Delta t}{\Delta x}\right)^{1/2} \right|^{2q} + c\, |\hat{U}_{\underline{tx}}|^{2q}$$

In the summation above, the term corresponding to $(\underline{s}, \underline{y})$ may be regarded as a martingale increment with respect to the filtration $F_{\underline{s}}$. Hence we may apply Burkholder's inequality (3.4.5) to obtain

$$R_q(\underline{t},\underline{x}) \leq c\, \underline{E}[\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} F^2(U_{\underline{sy}})\, \mu(\underline{s},\underline{y})]^q$$

(3-8)

$$+ c\, \underline{E}| \sup_{\substack{0\leq\underline{s}<\underline{t} \\ \underline{y}\in X}} | F(U_{\underline{sy}})\, |\xi_{\underline{sy}}\, \sqrt{\mu(\underline{s},\underline{y})}|^{2q} + c\, |\hat{U}_{\underline{tx}}|^{2q} .$$

The second term on the r.hs. of (3-8) is clearly equal to

$c\, \underline{E}[\max_{0\leq\underline{s}<\underline{t}} F^2(U_{\underline{sy}})\, \xi^2_{\underline{sy}}\, \mu(s,\underline{y})]^q$ which is bounded by

$c\, \underline{E}[\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} F^2(U_{\underline{sy}})\, \xi^2_{\underline{sy}}\, \mu(\underline{s},\underline{y})]^q$. Thus

$$R_q(\underline{t},\underline{x}) \leq c\, \underline{E}| \sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} F^2(U_{\underline{sy}})(1 + \xi^2_{\underline{sy}})\, \frac{\mu(\underline{s},\underline{y})}{|\mu|}|^q |\mu|^q + c|\hat{U}_{\underline{tx}}|^{2q} .$$

Applying Jensen's Inequality to the probability measure $\mu(\cdot)/|\mu|$, we find

$$R_q(\underline{t},\underline{x}) \leq c\,\underline{E}[\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} |F(U_{\underline{sy}})|^{2q}(1+|\xi_{\underline{sy}}^{2q}|)\frac{\mu(\underline{s},\underline{y})}{|\mu|}]|\mu|^q + c|\hat{U}_{\underline{tx}}|^{2q}.$$

Now we use the facts that $|\mu|^q/|\mu| = |\mu|^{q-1}$, that $|\mu|$ is bounded uniformly for $°\underline{t} \leq t_f$ by Lemma 3.4.2, that the moments of $\xi_{\underline{sy}}$ are finite, that $\xi_{\underline{sy}}$ is independent of $U_{\underline{sy}}$, and that $\hat{U}_{\underline{tx}}$ is bounded uniformly in $(\underline{t},\underline{x})$, since U_0 is bounded, to obtain

$$R_q(\underline{t},\underline{x}) \leq c(1+\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} \underline{E}|F(U_{\underline{sy}})|^{2q}(1+E|\xi_{\underline{sy}}|^{2q})(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}})^2\frac{\Delta t}{\Delta x})$$

$$\leq c(1+\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} K^{2q}(1+\underline{E}|U_{\underline{sy}}|^{2q})(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}})^2\frac{\Delta t}{\Delta x})$$

$$\leq c(1+\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} R_q(\underline{s},\underline{y})(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}})^2\frac{\Delta t}{\Delta x})$$

$$\leq c(1+\sum_{0\leq\underline{s}<\underline{t}} H_q(\underline{s})(\sum_{\underline{y}\in X}\frac{(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}})^2}{\Delta x}\Delta t)\ .$$

Now using $\Delta x = \sqrt{\Delta t/\alpha}$, and lemma 3.4.1 we find

Lemma 3.5.1.

(3-9) $H_q(\underline{t}) = \sup_{\underline{y}} R_q(\underline{t},\underline{y}) \leq c(1+\sum_{0\leq\underline{s}<\underline{t}} H(\underline{s})(\underline{t}-\underline{s})^{-1/2}\Delta t)$, for all t in $[0, \underline{t}_f]$, where c is a constant depending on α, q and t_f.

Now, iterating (3-9) and integrating by parts,

$$H_q(\underline{t}) \leq c \sum_{0\leq \underline{s}<\underline{t}} (1 + c \sum_{0<\underline{u}<\underline{s}} (1 + H_q(\underline{u}))(\underline{s}-\underline{u})^{-1/2}\Delta t)(\underline{t}-\underline{s})^{-1/2}\Delta t$$

$$\leq c(\sqrt{\underline{t}_f} + \sum_{0\leq \underline{u}<\underline{t}} (1 + H_q(\underline{u})) \left(\sum_{\underline{u}<\underline{s}<\underline{t}} (\underline{s}-\underline{u})^{-1/2}(\underline{t}-\underline{s})^{-1/2}\Delta t \right) \Delta t$$

(3.10)

$$\leq c(1 + \sum_{0\leq \underline{u}<\underline{t}} (1 + H_q(\underline{u})) \left(\sum_{0\leq \underline{s}<\underline{t}_f} \underline{s}^{-1/2}(\underline{t}_f-\underline{s})^{-1/2} \Delta t \right) \Delta t$$

$$\leq c(1 + \sum_{0\leq \underline{u}<\underline{t}} H_q(\underline{u}) \Delta t)$$

where c is another constant depending only on α, q, and t_f .

We now require a type of Gronwall's lemma.

<u>Lemma 3.5.2</u>. There is a constant c depending only on q, $\max_{\underline{Y}}|U_{0\underline{y}}|^{2q}$, α , and t_f , such that $\underline{E}|U_{\underline{tx}}|^{2q} \leq c \exp(c\underline{t})$, for $\underline{t} \leq \underline{t}_f$, and for any $\underline{x} \in X$.

<u>Proof</u>. We may take c to be the maximum of $H_q(0) (= \max_{\underline{Y}}|U_{\underline{0y}}|)^{2q}$ and the c of equation (3-10). We proceed by induction, on $n \in {}^*Z^+$. For $n = 0$, the lemma holds for $\underline{t} = n\Delta t$.

Suppose now that for $n \in {}^*N$

$$H(k\Delta t) \leq c(1 + c\,\Delta t)^k \text{ , for } k = 0, 1, \ldots, n-1 \text{ .}$$

Then $H(n\Delta t) \leq c(1 + c \sum_{0\leq k<n} (1 + c\,\Delta t)^k\, \Delta t)$ (by (3-10))

$$= c(1 + c\,\frac{(1+c\Delta t)^n - 1}{c\Delta t}\,\Delta t)$$

(summing a geometric series)

$$= c(1 + c\,\Delta t)^n$$

This is the induction step.

By the transfer principle, we may conclude that this internal argument verifies $H(n\Delta t) \leq c(1+c\Delta t)^n$, for all $n \in {}^*N$ such that $n\Delta t \leq t_f$. Then we notice that $(1+c\Delta t)^n \leq \exp(cn\Delta t)$, (in fact they are infinitesimally close). □

3.6 Bounds on Moments of Spatial Differences

Let $U_{\underline{tx}} = \hat{U}_{\underline{tx}} + V_{\underline{tx}}$, where V represents the contribution from the random forcing. $\hat{U}$ is S-continuous (in fact S-smooth), and we are interested in the continuity of V. In this section we obtain estimates on the moments of the differences $V_{\underline{tx}} - V_{\underline{ty}}$.

Lemma 3.6.1. There is a constant c, depending only on $q > 1$, $\max_{\underline{y}} |U^{2q}_{0\underline{y}}|$, α, and t_f, such that, for all $\underline{x}, \underline{y}$ in X, and for $0 \le \underline{t} \le \underline{t}_f$,

$$\underline{E}|V_{\underline{tx}} - V_{\underline{ty}}|^{2q} \le c|\underline{x} - \underline{y}|^q .$$

Proof: We may write $V_{\underline{tx}} - V_{\underline{ty}}$ as a sum of martingale increments with respect to the internal filtration $F_{\underline{s}}$; by (3-7),

$$V_{\underline{tx}} - V_{\underline{ty}} = \sum_{0<\underline{s}<\underline{t}} \sum_{\underline{z}\in X} (Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{z}} - Q^{\underline{t}-\underline{s}}_{\underline{y}-\underline{z}}) F(U_{\underline{sz}})\, \xi_{\underline{sz}} \left(\frac{\Delta t}{\Delta x}\right)^{1/2} .$$

We will estimate the 2q-th moment of this using Burkholder's Inequality (Theorem 3.4.5).

We will designate by $\mu(\underline{s},\underline{z})$ the measure on $[0, \underline{t}] \times X$ which assigns to each point $(\underline{s},\underline{z})$ the weight $(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{z}} - Q^{\underline{t}-\underline{s}}_{\underline{y}-\underline{z}})^2 \frac{\Delta t}{\Delta x}$.

$$\underline{E}|V_{\underline{tx}} - V_{\underline{ty}}|^{2q} \le c\, \underline{E}\Big(\int_{[0,\underline{t})\times X} F^2(U_{\underline{sz}})\, d\mu(\underline{s},\underline{z})\Big)^q$$

$$+ c\, \underline{E}\Big(\max_{0\le \underline{s}<\underline{t}} F^2(U_{\underline{sz}})\, \xi^2_{\underline{sz}}\, \mu(\underline{s},\underline{z})\Big)^q$$

$$\leq c\ \underline{E}\Big(\int_{[0,\underline{t}]\times X} F^2(U_{\underline{s}\underline{z}}) \frac{d\mu(\underline{s},\underline{z})}{|\mu|} \Big)^q |\mu|^q$$

$$+ c\ \underline{E}\Big(\int F^2(U_{\underline{s}\underline{z}})\ \xi^2_{\underline{s}\underline{z}} \cdot \frac{d\mu(\underline{s},\underline{z})}{|\mu|} \Big)^q |\mu|^q$$

$$\leq c\ \underline{E} \int [F^2(U_{\underline{s}\underline{z}})]^q [1 + |\xi|^{2q}_{\underline{s}\underline{z}}] \frac{d\mu(\underline{s},\underline{z})}{|\mu|} \cdot |\mu|^q$$

(Jensen's Inequality applied to $\frac{\mu}{|\mu|}$)

$$\leq c \int K^{2q}\ H_q(\underline{s}) \frac{d\mu(\underline{s},\underline{z})}{|\mu|} \cdot |\mu|^q$$

(using independence of $U_{\underline{s}\underline{z}}$ and $\xi_{\underline{s}\underline{z}}$, and also finite moments of $\xi_{\underline{s}\underline{z}}$)

$$\leq c\ |\underline{x} - \underline{y}|^q$$

(using lemmas 3.5.2 and 3.4.3, noting that $\frac{\Delta t}{\Delta x} = \alpha \Delta x$) . □

3.7 Bounds on Moments of Temporal Differences

In this section we obtain estimates on the moments of the differences $V_{\underline{tx}} - V_{\underline{rx}}$.

Lemma 3.7.1. There is a constant c, depending only on $q > 1$, $\max_{\underline{y}} |U_{0\underline{y}}|^{2q}$, α, and t_f, such that, for all $\underline{t}, \underline{r} \leq \underline{t}_f$, and all $\underline{x} \in X$,

$$\underline{E}|V_{\underline{tx}} - V_{\underline{rx}}|^{2q} \leq c\,|\underline{t} - \underline{r}|^{q/2} .$$

Proof: We may suppose w.l.o.g. that $\underline{r} < \underline{t}$.

We may write $V_{\underline{tx}} - V_{\underline{rx}}$ as a sum of martingale differences with respect to the internal filtration $F_{\underline{s}}$.

$$V_{\underline{tx}} - V_{\underline{rx}} = \sum_{0 \leq \underline{s} < \underline{r}} \sum_{\underline{z} \in X} (Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{z}} - Q^{\underline{r}-\underline{s}}_{\underline{x}-\underline{z}}) F(U_{\underline{sz}})\, \xi_{\underline{sz}} \left(\frac{\Delta t}{\Delta x}\right)^{1/2}$$

$$+ \sum_{\underline{r} \leq \underline{s} < \underline{t}} \sum_{\underline{z} \in X} Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{z}} F(U_{\underline{sz}})\, \xi_{\underline{sz}} \left(\frac{\Delta t}{\Delta x}\right)^{1/2}$$

We will designate by $\mu(\underline{s},\underline{z})$ the measure on $[0, \underline{t}) \times X$ which assigns to each point $(\underline{s},\underline{z})$ the weight $(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{z}} - Q^{\underline{r}-\underline{s}}_{\underline{x}-\underline{z}})^2 \frac{\Delta t}{\Delta x}$, if $\underline{s} < \underline{r}$, and the weight $(Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{z}})^2 \frac{\Delta t}{\Delta x}$, if $\underline{r} \leq \underline{s} < \underline{t}$. Then using Burkholder's Inequality (Theorem 3.4.5)

$$\underline{E}|V_{\underline{tx}} - V_{\underline{rx}}|^{2q} \leq c\, \underline{E}\Big(\int_{[0,\underline{t})\times X} F^2(U_{\underline{sz}})\, d\mu(\underline{s},\underline{z})\Big)^q$$

$$+ c\, \underline{E}\Big(\max_{\substack{0 \leq \underline{s} < \underline{t} \\ \underline{z} \in X}} F^2(U_{\underline{sz}})\, \xi^2_{\underline{sz}}\, \mu(\underline{s},\underline{z})\Big)^q$$

$$\leq c\ \underline{E} \int (F^2(U_{\underline{s}\underline{z}}))^q (1 + |\xi_{\underline{s}\underline{z}}|^{2q}) \frac{d\mu(\underline{s},\underline{z})}{|\mu|} \cdot |\mu|^q$$

(using Jensen's Inequality)

$$\leq c\ \underline{E} \int K^{2q}\ H_q(\underline{s}) \frac{d\mu}{\mu} \cdot |\mu|^q$$

(using independence of $U_{\underline{s}\underline{z}}$ and $\xi_{\underline{s}\underline{z}}$, and the finiteness of $\underline{E}|\xi_{\underline{s}\underline{z}}|^{2q}$)

$$\leq c(\underline{t}-\underline{r})^{q/2}$$

(using lemma 3.5.2, and lemmas 3.4.2 and 3.4.4, with $\frac{\Delta t}{\Delta x} = \sqrt{\alpha \Delta t}$)

3.8 S-Continuity and the Standard Part

The main result in this section is that $U_{\underline{tx}}$, the solution to the hyperfinite difference equations (3-5) is a.s. S-continuous. We shall obtain this by applying a non-standard version of Kolmogorov's Continuity Criterion:

Theorem 3.8.1. Let $U_{\underline{x}} : \Omega \times \Gamma \to {}^*R$ be an internal process on a hyperfinite lattice Γ which represents a finite rectangle in R^d. If there exist positive real numbers $\beta_1,\ldots,\beta_d$, $\gamma_1,\ldots,\gamma_d$, K such that for $k = 1,\ldots,d$ $\underline{E}|U_{\underline{x}} - U_{\underline{y}}|^{\beta_k} \leq K|\underline{x} - \underline{y}|^{d+\gamma_k}$, whenever $\underline{x}, \underline{y} \in \Gamma$ are such that $\underline{x} - \underline{y}$ lies along the k^{th} coordinate axis, then if $\delta_k < \hat{\gamma}_k/\beta_k$, $k = 1,\ldots,d$ there is a set $\Omega' \subset \Omega$ of Loeb Probability 1, a function $\delta(\omega)$, ${}^\circ\delta(\omega) > 0$ on Ω', and a constant c, such that for $k = 1,\ldots,d$ $|U_{\underline{x}} - U_{\underline{y}}| \leq c|\underline{x} - \underline{y}|^{\delta_k}$ whenever $\underline{x}, \underline{y} \in \Gamma$, $|\underline{x} - \underline{y}| < \delta(\omega)$ and $\underline{x} - \underline{y}$ lies along the k^{th} coordinate axis. In particular U is a.s. S-continuous on Γ.

Proof: See Stoll (1984), Lemma 3.2. The result he states is not as detailed as 3.8.1, but his proof is sufficient.

Theorem 3.8.2. The hyperfinite process $U_{\underline{tx}}(\omega)$ constructed by the solution of (3-5), is a.s. S-continuous on near standard points in $T \times X$. Moreover, if $\beta_1 < \frac{1}{4}$ and $\beta_2 < \frac{1}{2}$ and $A \subset X$ is a rectangle whose sides have finite length, there is a set $\Omega' \subset \Omega$ of probability 1, a positive real function $\delta(\omega)$ on Ω', and a constant c, depending on β_1, β_2 $\max_{\underline{y}\in X} U_{0\underline{y}}$, α, and t_f, such that $\forall\, \omega \in \Omega'$, $\underline{x}, \underline{y} \in A$, $\underline{t}, \underline{r} \leq \underline{t}_f$

$$|U_{\underline{tx}} - U_{\underline{ry}}| \leq c(|\underline{t} - \underline{r}|^{\beta_1} + |\underline{x} - \underline{y}|^{\beta_2}) \ , \text{ if } \ |t-r| + |x-y| < \delta \ .$$

<u>Proof</u>: Pick $q \in R^+$ such that $\beta_1 < \frac{q/2 - 2}{2q}$ and $\beta_2 < \frac{q-2}{2q}$. By lemmas 3.6.1 and 3.7.1, there are constants c such that

$$\underline{E}|V_{\underline{tx}} - V_{\underline{rx}}|^{2q} \leq c \ |\underline{t} - \underline{r}|^{2+(q/2 - 2)}$$

$$\underline{E}|V_{\underline{tx}} - V_{\underline{ty}}|^{2q} \leq c \ |\underline{x} - \underline{y}|^{2+(q-2)}$$

Hence by Theorem 3.8.1, the statement of the lemma is true with V in place of U on any set $T \times A$, where A is an internal finite rectangle in X; but the near standard part of X is a σ-union of such A. Now $U = \hat{U} + V$ and $\hat{U}$ is a lifting of a smooth function, which is a solution to the heat equation. An examination of the explicit form (3-7) for $\hat{U}$ yields quickly that

$$|\hat{U}_{tx} - \hat{U}_{ry}| \leq c(|\underline{t} - \underline{r}| + |\underline{x} - \underline{y}|)$$

Hence the theorem is true for $U = \hat{U} + V$. □

In general the exponents $\frac{1}{4}$ and $\frac{1}{2}$ are best possible (see Walsh (1986), Corollary 3.4).

We may allow slightly more general initial conditions, if we are prepared to relax the conclusion slightly. The arguments in lemmas 3.6.1 and 3.7.1 depend only on the boundedness of $\max_{\underline{y} \in X} |U_{0\underline{y}}|$. Thus the boundedness of U_0 is enough to ensure the S-continuity of $U - \hat{U}$. However, in this case $\hat{U}$ itself will not be S-continuous in the monad of zero. If U_0 is bounded but discontinuous the conclusion of the theorem will have to be restricted to

$°\underline{t}, °\underline{r} > 0$.

Returning to the case when U_0 is continuous, we find that another way of phrasing the conclusion of Theorem 3.8.2 is that U is nearstandard almost surely in $^*C([0, t_f] \times R : R)$. Hence we may define a process $u(\omega)$ as the standard part of $U(\omega)$, or equivalently $u_{°\underline{t}, °\underline{x}} = °U_{\underline{t}\underline{x}}$ for all $\underline{t}, \underline{x}$, a.s. It is clear that that set $\Omega' = \{\omega : \forall n \in N, \exists k \in N, \forall \underline{x}, \underline{y} \in X$, $\underline{t}, \underline{r} \leq \underline{t}_f$, $|\underline{x} - \underline{y}| < \frac{1}{k}$, $|\underline{t} - \underline{r}| \leq \frac{1}{k} \Rightarrow |U_{\underline{t}\underline{x}} - U_{\underline{r}\underline{y}}| \leq \frac{1}{n}\}$ is in the σ-algebra generated by the internal sets, hence is Loeb-measureable.

Thus the process

$$u_{tx}(\omega) = \begin{cases} °U_{\underline{t}\underline{x}}(\omega) \text{ for any } (\underline{t},\underline{x}) \approx (\underline{t},\underline{x}) \text{ , if } \omega \in \Omega' \\ 0 \text{ , if } \omega \notin \Omega' \end{cases}$$

has sample paths in $C([0, t_f] \times R : R)$.

We are of course really interested in solutions for all t . Thus take the hyperfinite time line T up to some infinite number L , and take Ω larger enough to support a white noise on $T \times X$. Construct the solution $U_{\underline{t}\underline{x}}$ as before. All theorems proved previously hold true up until any finite time $\underline{t}_f$. Thus we have

<u>Corollary 3.8.3</u>. The solution $U_{\underline{t}\underline{x}}$ on $T \times X$, where T now is an infinitesimal grid representing R^+ , constructed from the difference equations (3-5) is S-continuous in $\{°|\underline{x}| < \infty\} \cap \{°\underline{t} < \infty\}$, a.s.

We note also that $u_{t\cdot}$ is $\sigma(F_{\underline{s}})$ measureable for any $\underline{s}$ with $°\underline{s} > t$ hence by definition (in section 2.3) u is F_t-adapted.

3.9 Solution of the SPDE

We now show that the process u_{tx} of section 3.8 is in fact a solution of (3-1) with respect to the white noise defined on Ω in sections 2.2 and 3.2. We must check condition (3-4).

We need first a new definition.

Definition 3.9.1. An internal function Φ on an infinitesimal lattice $X \subseteq {}^*R^d$, is called a <u>lifting to order k</u> of a C^k function $\phi : R^d \to R$, if ${}^\circ\delta_{x_{i_1}}(\ldots(\delta_{x_{i_k}}\Phi)\ldots)(\underline{x}) = \frac{\partial^k}{\partial x_{i_1}\ldots\partial x_{i_k}}\phi({}^\circ\underline{x})$ for all $\underline{x} \in X$. Here δ_{x_i} is the finite difference operator in the direction x_i: $(\delta_{x_i}(\Phi)(\underline{x}) = [\Phi(\underline{x}_1,\ldots,\underline{x}_i+\Delta x,\ldots,\underline{x}_d) - \Phi(\underline{x}_1,\ldots,\underline{x}_d)]/\Delta x$. A consequence of this definition is that for nearstandard $\underline{x}$, ${}^\circ[\Phi(\underline{x}+\Delta x) - 2\Phi(\underline{x}) + \Phi(\underline{x}-\Delta x)]/\Delta x^2 = \frac{\partial}{\partial x^2}\phi({}^\circ\underline{x})$ if Φ is a lifting to order 2 of ϕ.

We note that every C^k function ϕ has a canonical lifting to order k, namely ${}^*\phi$ restricted to the lattice.

Now fix any $\phi \in C_c^\infty(R)$, and let Φ be a lifting to order 2 which is exactly 0 on values of $\underline{x}$ whose standard parts ${}^\circ\underline{x}$ lie outside the closed support of ϕ, (to avoid (unnecessary) concern over the convergence of *-countable sums). Then $U_{\underline{\underline{t}}\underline{\underline{x}}}\cdot\Phi(\underline{x})$ is a (uniform) lifting of $u_{tx}\phi(x)$ a.s. for any $\underline{t} \approx t$. Thus

$$\int_R u_{tx}\phi(x)\,dx - \int_R u_{0x}\phi(x)\,dx \overset{a.s.}{=} {}^\circ\sum_{\underline{x}\in X}(U_{\underline{\underline{t}}\underline{\underline{x}}} - U_{0\underline{x}})\,\Phi(\underline{x})\,\Delta x$$

$$= {}^\circ\sum_{\underline{x}\in X}\ \sum_{0\le\underline{s}<\underline{t}} \left[\frac{U_{\underline{s}+\Delta t,\underline{x}} - U_{\underline{s},\underline{x}}}{\Delta t}\right]\cdot\Delta t\cdot\Phi(\underline{x})\cdot\Delta x$$

$$= {}^\circ\sum_{0\le\underline{s}\le\underline{t}}\ \sum_{\underline{x}\in X}\left[\frac{U_{\underline{s},\underline{x}+\Delta x} - 2U_{\underline{sx}} + U_{\underline{s},\underline{x}-\Delta x}}{\Delta x^2}\Phi(\underline{x}) + \frac{F(U_{\underline{sx}})}{\sqrt{\Delta t\Delta x}}\xi_{\underline{sx}}\Phi(\underline{x})\right]\Delta x\,\Delta t$$

$$(3\text{-}11)\quad = {}^\circ\sum_{0\le\underline{s}<\underline{t}}\ \sum_{\underline{x}\in X} U_{\underline{sx}}\frac{\Phi(\underline{x}+\Delta x) - 2\Phi(\underline{x}) + \Phi(\underline{x}-\Delta x)}{\Delta x^2}\Delta x\,\Delta t$$

$$+ {}^\circ\sum_{0\le\underline{s}<\underline{t}}\ \sum_{\underline{x}\in X} F(U_{\underline{sx}})\ \Phi(x)\ \sqrt{\Delta t\Delta x}\ \xi_{\underline{sx}}\ .$$

Now $U_{\underline{sx}}[\Phi(\underline{x}+\Delta x) - 2\Phi(\underline{x}) + \Phi(\underline{x}-\Delta x)]/\Delta x^2$ is a uniform lifting a.s. of $u_{sx}\phi''(x)$. Hence the first term on the r.h.s. of (3-11) is a.s. $\int_0^t\int_R u_{sx}\phi''(x)dx$.

Now $F(U_{\underline{sx}})\Phi(\underline{x})$ is a (uniform) $F_{\underline{s}}$-adapted lifting of $f(u_{sx})\phi(x)$.

Hence by Theorem 2.3.2, the second term on the r.h.s. of (3-11) is a.s. equal to $\int_0^t\int_R f(u_{sx})\phi(x)dw_{sx}$. Thus we have, $\phi\in C_c^\infty(R)$ $t\in R^+$

$$(3\text{-}12)\quad \int_R u_{tx}\ \phi(x)dx - \int_R u_{0x}\ \phi(x)dx \overset{a.s.}{=} \int_0^t\int_R u_{sx}\ \phi(x) + \int_0^t\int_R f(u_{s,x})\phi(x)dw_{sx}\ .$$

<u>Theorem 3.9.2</u>. There is a Loeb space Ω , on which any equation of the form (3-1) has a solution jointly continuous in t and x with respect to the canonical white noise on Ω .

NOTE: We believe it is possible to extend Keisler's internal transformation principle to this Loeb space Ω (see Keisler (1984)). In this case, the equation (3-1) has a solution with respect to any white noise on $R^+ \times R^d$ supported by Ω .

Walsh (1986) has established uniqueness for the case when f is Lipshitz. Presumably, this is false in general but we do not know a convenient counterexample.

CHAPTER FOUR

The Dawson Critical Branching Diffusion

4.1 Introduction

The use of hyperfinite difference equations, which was successful in Chapter 3 for one spatial dimension may be attempted for higher dimensions as well. This approach does not succeed to the same extent as it does in one dimension, as is spelled out in Appendix B. However it does succeed with the Dawson Critical Branching Diffusion, as will be explained here.

It is possible to represent the Dawson process non-standardly, by placing, on a hyperfinite grid X, representing a portion of R^d, $d \geq 1$, an infinite number of particles, each executing an infinitesimal random walk, and undergoing branching. If we let $U_{\underline{t}\underline{x}}$ stand for the density of particles at any grid point $\underline{x} \in X$, at time $\underline{t}$, then, if the initial density is taken large enough, it is possible to show that U satisfies a hyperfinite difference equation of the form

$$\delta_{\underline{t}} U = \underline{\Delta}\, U + \dot{Z}$$

where $\dot{Z}$ is an internal noise with $E(\dot{Z}_{\underline{t}\underline{x}}) = 0$, $E(\dot{Z}^2_{\underline{t}\underline{x}} | U_{\underline{t}\underline{x}}) = \dfrac{U_{\underline{t}\underline{x}}}{\Delta t \Delta x^d}$.

Hence $\dot{Z}$ may be written as $\sqrt{U}\,\dot{W}$ where $\dot{W}$ is like an S-white noise in many respects. This noise $\dot{W}$ is however, a little awkward to work with, and therefore we adopt the simpler scheme of difference equations set out in 4.2.

In 4.4 we examine the total mass of the process constructed in 4.2, and use this in 4.5 to establish some continuity results, which yield easily that the standard part is well-defined. In 4.6 we verify that this standard part does indeed coincide with the Dawson process. In 4.7 we obtain several new results about the pathwise regularity of the Dawson process, using our nonstandard construction.

4.2 A Hyperfinite Difference Equation

Let Δx be any infinitesimal, and let X be $\{\underline{x} \mid \underline{x} = (k_1\Delta x, \ldots, k_d\Delta x)\ ,\ k_i \in {}^*Z\ ,\ d \geq 1\}$ so that X represents R^d. We will treat here the construction of the Dawson Process only on the whole of R^d. A very similar treatment is possible if reflecting boundary conditions are imposed on several hyper-planes in R^d, or along the edges of a rectangle in R^d, but the inequalities are messier, and indeed depend on those for the unbounded domain.

Let Δt be an infinitesimal, such that $\Delta t/\Delta x^d \approx 0$. This makes some parts of the treatment much easier. In case $d = 1$ we require $\Delta t \leq \frac{1}{3}\Delta x^2$, as in Chapter 3. Let T be a hyperfinite time line of spacing Δt:

$$T = \{\underline{t} : \underline{t} = k\Delta t\ ,\ k \in {}^*Z^+\ ,\ k \leq M\}\ .$$

We will suppose $0 < {}^\circ(M\Delta t) < \infty$. Let $\underline{t}_f = M\Delta t$.

It is easier, and it suffices for our purposes, to take $\Omega = \{-1, 1\}^{T \times X}$, and to let $\xi_{\underline{tx}}(\omega)$ be the coordinate map, as outlined in section 2.2. Thus $\underline{P}(\xi_{\underline{tx}} = 1) = \underline{P}\{\xi_{\underline{tx}} = -1\} = \frac{1}{2}$.

The analogue of equation (3-5) in higher dimensions is:

$$(\delta_{\underline{t}} U)_{\underline{tx}} = (\underline{\Delta} U)_{\underline{tx}} + \left(\frac{U_{\underline{tx}}}{\sqrt{\Delta t \Delta x^d}} \wedge \frac{U_{\underline{tx}}}{2\Delta t} \right) \xi_{\underline{tx}}\ , \tag{4-1}$$

or equivalently,

$$U_{t+\Delta t,\underline{x}} = U_{\underline{tx}} + \Delta t(\underline{\Delta} U)_{\underline{tx}} + \left(\sqrt{\frac{U_{\underline{tx}}\Delta t}{\Delta x^d}} \wedge \frac{U_{\underline{tx}}}{2} \right) \xi_{\underline{tx}}\ ,$$

where $\delta_{\underline{t}}$ is a finite difference analogue of $\frac{\partial}{\partial t}$:

$(\delta_{\underline{t}}U)_{\underline{tx}} = [U_{\underline{t}+\Delta t,\underline{x}} - U_{\underline{t},\underline{x}}] / \Delta t$, and $\underline{\Delta}$ is the finite difference analogue of the Laplacian in R^d :

$$(4\text{-}2)\quad (\underline{\Delta}U)_{\underline{t}(\underline{x}_1,\ldots,\underline{x}_d)} = \frac{1}{\Delta x^2}\left[\sum_{i=1}^{d} (U_{\underline{t}(\underline{x}_1,\ldots,\underline{x}_i+\Delta x,\ldots,\underline{x}_d)} + U_{\underline{t}(\underline{x}_1,\ldots,\underline{x}_i-\Delta x,\ldots,\underline{x}_d)} - 2d\, U_{\underline{t}(\underline{x}_1,\ldots,\underline{x}_d)}\right]$$

The term $\sqrt{\frac{U_{\underline{tx}}\Delta t}{\Delta x^d}} \wedge \frac{U_{\underline{tx}}}{2}$ is substituted for simply $\sqrt{\frac{U_{\underline{tx}}\Delta t}{\Delta x^d}}$ in (4-1), in order to ensure, that, a non-negative value at $(\underline{t},\underline{x})$, all of whose nearest neighbours are non-negative, will not become negative at the next time step. The values of U for which the linear term is taken are $0 < U < \frac{4\Delta t}{\Delta x^d}$.

For an initial condition for (4-1) we may use any non-negative internal function $U_{0\underline{x}}$ on X , requiring only that $M_0 \overset{\text{def}}{=} \sum_{\underline{x}\in X} U_{0\underline{x}}\Delta x^d$, the total initial mass, be finite and that the mass on points of X which are not near-standard be infinitesimal in sum. We may represent any finite positive Borel measure on R^d (we will denote the space of all such measures $M_F(R^d)$) by such an internal function U_0 (see Cutland (1983) Theorem 4.7 and preceeding remarks).

4.3 The Coefficients $\underline{Q}$

From (4-1) and (4-2) we observe that each value of $\sqrt{\frac{U_{\underline{sy}}\Delta t}{\Delta x^d}} \wedge \frac{U_{\underline{sy}}}{2} \quad \xi_{\underline{sy}}$ enters into the definition of subsequent $U_{\underline{tx}}$'s . We denote the coefficient of the former term in the definition of the latter by $Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}}$, observing that these coefficients are homogeneous in space and time. We may then write the analogue of a Green's function formula:

$$(4\text{-}3) \qquad U_{\underline{tx}} = \sum_{\underline{0}\leq\underline{s}<\underline{t}} \ \sum_{\underline{y}\in X} Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}} \left(\sqrt{\frac{U_{\underline{sy}}\Delta t}{\Delta x^d}} \wedge \frac{U_{\underline{sy}}}{2} \right) \xi_{\underline{sy}} + \sum_{\underline{y}\in X} Q^{\underline{t}+\Delta t}_{\underline{x}-\underline{y}} \ U_{\underline{0y}} \ .$$

<u>Lemma 4.3.1</u>. The coefficients $Q^{\underline{t}+\Delta t}_{\underline{x}}$ are the internal density for an internal infinitesimal random walk $B_{\underline{t}}$ on X . The standard part of $B_{\underline{t}}$ is a.s. d-dimensional Brownian motion of rate 2.

<u>Proof</u>: As in Lemma 3.3.1 we may construct difference equations for the coefficients $Q^{\underline{t}}_{\underline{x}}$ from (4-1) and (4-2). We observe that these difference equations correspond to a Markov process, $B_{\underline{t}}$, on X , with parameter $\underline{t} \in T$ where $B_{\underline{t}}$ starts at $\underline{0} \in X$, and at each time step $B_{\underline{t}}$ takes a step to <u>one</u> of the 2d nearest neighbours of its current position, with infinitesimal probability $\alpha = \Delta t/\Delta x^2$ for each of the 2d possibilities. The process stays put with probability $1 - 2d\alpha \approx 1$. $Q^{\underline{t}+\Delta t}_{\underline{x}} = P(B_{\underline{t}} = \underline{x})$.

Let $B^i_{\underline{t}}$ denote the i^{th} coordinate of $B_{\underline{t}}$. Since steps in opposite directions have equal probability, B^i is an internal martingale. Also $E((B^i_{\underline{t}+\Delta t} - B^i_{\underline{t}})^2 \mid B_{\underline{s}} ; 0 \leq \underline{s} \leq \underline{t}) = 2\alpha\Delta x^2 = 2\Delta t$. Hence the (internal) predictable quadratic variation $\langle B^i\rangle_{\underline{t}} = 2\underline{t}$. Using Burkholder's Inequality on the higher moments of $B^i_{\underline{t}}$, we may conclude that $(B^i_{\underline{t}})^2$ is S-integrable $\underline{t}$. Now a step in one direction excludes a step in any other direction during the same time step. Hence $E[(B^j_{\underline{t}+\Delta t} - B^j_{\underline{t}})(B^i_{\underline{t}+\Delta t} - B^i_{\underline{t}})] = 0$ if $i \neq j$. Successive steps are independent, so that the internal process $\langle B^i, B^j\rangle_{\underline{t}} = 2\delta_{ij}\underline{t}$. Now we invoke Hoover and Perkins (1983) Theorem 8.5 to assert that B has a.s. a standard part b, and we observe that b satisfies

i) b^i_t is a martingale $i = 1,\ldots,d$

ii) $\langle b^i, b^j\rangle_t = 2\delta_{ij}t \quad i, j = 1,\ldots,d$

(again with reference to Hoover and Perkins (1983)). Now i) and ii) above characterize d-dimensional Brownian Motion. □

We need a little more information about the Q's.

<u>Lemma 4.3.2</u>: If $^\circ\underline{t} > 0$, $\displaystyle {}^\circ\frac{Q^{\underline{t}}_{\underline{x}}}{\Delta x^d} = P_{^\circ\underline{t}}(^\circ\underline{x})$, where $P_t(x) = \dfrac{1}{\sqrt{4\pi t}}\exp\{\dfrac{||x||^2}{4t}\}$ is the density for Brownian motion of rate 2. Hence $\dfrac{Q^{\underline{t}}_{\underline{x}}}{\Delta x^d}$ is S-continuous in $\underline{t}$ for $^\circ\underline{t} > 0$.

<u>Proof</u>: We know from 4.3.1 that the distribution for $^\circ B_{\underline{t}}$ is the same as that for $b_{^\circ\underline{t}}$, which has density $P_{^\circ\underline{t}}$, for any $^\circ\underline{t} > 0$. The statement of the lemma will follow then, if $Q^{\underline{t}}_{\underline{x}}/\Delta x^d$ is S-continuous for $^\circ\underline{t} > 0$.

We observe that the following equation holds:

$$(\delta_{\underline{t}} Q^{\cdot}_{\underline{x}})_{\underline{t}} = (\underline{\Delta}\, Q^{\underline{t}}_{\cdot})_{\underline{x}} \tag{4-4}$$

Now it is clear from the definition of the random walk $B_{\underline{t}}$ that the coefficients $Q_{\underline{x}}^{\underline{t}}$ are symmetric with respect to change of sign on any of the d indices $\underline{x}_k$.

<u>Claim</u>: For each $\underline{t}$, $\underline{x}$, $Q_{\underline{x}}^{\underline{t}} \geq Q_{\underline{x}+\Delta x\vec{e}_k}^{\underline{t}}$ for $k = 1,\ldots,d$, where $\vec{e}_k$ represents a unit vector in the x_k direction, which points <u>away</u> from 0 .

We prove this by *-finite induction on $\underline{t}$. It is clearly true for $\underline{t} = \Delta t$: $Q_{\underline{x}}^{\Delta t} = P(B_0 = \underline{x}) = \delta_{\underline{0}}^{\underline{x}}$. Suppose the claim holds for some $\underline{t} \in T$. Then by (4-4) and (4-2),

$$(4\text{-}5) \qquad Q_{\underline{x}}^{t+\Delta t} = (1-2d\alpha)Q_{\underline{x}}^{\underline{t}} + \alpha \left(\sum_{j=1}^{d} Q_{\underline{x} \pm \Delta x e_j}^{\underline{t}} \right) \text{, and}$$

$$(4\text{-}6) \qquad Q_{\underline{x}+\vec{e}_k\Delta x}^{t+\Delta t} = (1-2d\alpha)Q_{\underline{x}+\vec{e}_k\Delta x}^{\underline{t}} + \alpha \left(\sum_{j=1}^{d} Q_{\underline{x}+\Delta x\vec{e}_k \pm \Delta x e_j}^{\underline{t}} \right)$$

Now if $\underline{x}_k \neq 0$ then all terms appearing in (4-5) and (4-6) above lie in the same half-plane; each term in (4-6) is shifted by $\vec{e}_k\Delta x$ relative to, and hence, by the induction assumption, is not greater than, the corresponding term in (4-5). Hence $Q_{\underline{x}}^{\underline{t+\Delta t}} \geq Q_{\underline{x}+\Delta x\vec{e}_k}^{\underline{t+\Delta t}}$.

Now suppose (w.l.o.g. $k = 1$) $\underline{x}_1 = 0$. We pick one of the two possibilities for $\vec{e}_1$ and stick with it. Let $\underline{y}$ be a $(k-1)$ tuple.

By our assumptions on Δt , in any dimension d , $(2d+1)\alpha \leq 1$. Hence

$$(4\text{-}7) \quad (1-(2d+1)\alpha)Q_{0\underline{y}}^{\underline{t}} \geq (1-(2d+1)\alpha)Q_{\Delta x,\underline{y}}^{\underline{t}} \geq (1-(2d+2)\alpha)Q_{\Delta x,\underline{y}}^{\underline{t}} + \alpha\, Q_{2\Delta x,\underline{y}}^{\underline{t}}$$

Now adding $\alpha\, Q_{\underline{y}}^{\underline{t}} + \alpha\, Q_{\Delta x,\underline{y}}^{\underline{t}} + \alpha\, Q_{-\Delta x,\underline{y}}^{\underline{t}}$ to both sides of (4-7) and using $Q_{-\Delta x,\underline{y}}^{\underline{t}} = Q_{\Delta x,\underline{y}}^{\underline{t}}$, we obtain

$$(4\text{-}8)\quad (1-2d\alpha)\, Q^{\underline{t}}_{0\underline{y}} + \alpha(Q^{\underline{t}}_{\Delta x,\underline{y}} + Q^{\underline{t}}_{-\Delta x,\underline{y}}) \geq (1-2d\alpha) Q^{\underline{t}}_{\Delta x,\underline{y}} + \alpha\,(Q^{\underline{t}}_{0\underline{y}} + Q^{\underline{t}}_{2\Delta x,\underline{y}})$$

Now by the induction assumption

$$(4\text{-}9)\qquad \alpha \sum_{j=2}^{d} Q^{\underline{t}}_{0\underline{y}\pm\Delta x\underline{e}_j} \geq \alpha \sum_{j=2}^{d} Q^{\underline{t}}_{\Delta x,\underline{y}\pm\Delta x\underline{e}_j} \ .$$

Adding (4-8) and (4-9) we obtain

$$Q^{\underline{t}+\Delta t}_{0\underline{y}} \geq Q^{\underline{t}+\Delta t}_{\Delta x,\underline{y}} \ .$$

This establishes the claim for $\{Q^{\underline{t}+\Delta t}_{\underline{x}} \mid \underline{x} \in \mathcal{X}\}$. Now we invoke the principle of induction under the transfer principle to establish the claim for all $\underline{t} \in \mathcal{T}$.

Now by Lemma 4.3.1, for any internal rectangle A

$$(4\text{-}10)\quad {}^{\circ}\sum_{x\in A} \frac{Q^{\underline{t}}_{\underline{x}}}{\Delta x^d}\,\Delta x^d = \int_{{}^{\circ}A} P_{\underline{t}}(x)\,dx \ , \text{ for } {}^{\circ}\underline{t} > 0 \ .$$

In light of the monotonicity claim just established, this means

$${}^{\circ}(Q^{\underline{t}}_{\underline{x}}/\Delta x^d) = P_{{}^{\circ}\underline{t}}({}^{\circ}\underline{x}) \quad \text{if} \quad {}^{\circ}\underline{t} > 0 \ , \text{ and } \ \underline{x} \in ns(\mathcal{X}) \ .$$

Therefore also $\dfrac{Q^{\underline{t}}_{\underline{x}}}{\Delta x^d}$ is S-continuous in $\underline{t}$ for ${}^{\circ}\underline{t} > 0$, since $P_t(x)$ is continuous in t , in $t > 0$. □

We prove the following inequality in order to obtain moment bounds in section 4.5. It has no independent interest. We introduce the notation

$$Q^{\underline{t}}_{A:\underline{y}} = \sum_{\underline{x}\in A} Q^{\underline{t}}_{\underline{x}-\underline{y}} \ .$$

<u>Lemma 4.3.3</u>. There is a constant $K < \infty$ and a positive infinitesimal $\Delta t'$

such that for all internal sets A, and all $\underline{y} \in X$, whenever $\underline{t}, \underline{r}, \underline{s} \in T$ and $\underline{r}-\underline{s} \geq \underline{t}-\underline{r} \geq \Delta t'$,

$$\left| Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:\underline{y}} \right| \leq K\frac{(\underline{t}-\underline{r})}{(\underline{r}-\underline{s})}$$

<u>Proof</u>: Suppose first that ${}^{\circ}(\underline{t}-\underline{r}) > 0$.

Now clearly for fixed $\underline{t}$, $\underline{r}$ and $\underline{s}$,

$$\max_{A,\underline{y}} |Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:y}| = Q^{\underline{r}-\underline{s}}_{A_{\underline{y},\underline{t},\underline{r},\underline{s}}:\underline{y}} - Q^{\underline{t}-\underline{s}}_{A_{\underline{y},\underline{t},\underline{r},\underline{s}}:\underline{y}}$$

$$\text{where } A_{\underline{y},\underline{t},\underline{r},\underline{s}} = \{\underline{x} \in X | Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}} < Q^{\underline{r}-\underline{s}}_{\underline{x}-\underline{y}}\}$$

The standard part of this set is easy to identify, using Lemma 4.3.2. It is the ball

$$A_{y,t,r,s} = \{x \in R^d | \; ||x-y||_2^2 \leq 2d\,\frac{(t-s)(r-s)}{(t-r)}\,\log[1 + (\frac{t-r}{r-s})]\},$$

where $y = {}^{\circ}\underline{y}$, $t = {}^{\circ}\underline{t}$, $r = {}^{\circ}\underline{r}$ and $s = {}^{\circ}\underline{s}$. The radius $\delta_{t,r,s}$ of this ball is computed by solving $P_{r-s}(x) = P_{t-s}(x)$

$$\text{i.e.} \quad \frac{1}{[4\pi(r-s)]^{d/2}}\exp[-\frac{\delta^2}{4(r-s)}] = \frac{1}{[4\pi(t-s)]^{d/2}}\exp[-\frac{\delta^2}{4(t-s)}]$$

$$\Leftrightarrow \quad (\frac{t-s}{r-s})^{d/2} = \exp[\frac{\delta^2}{4}\,\frac{(t-s)-(r-s)}{(r-s)(t-s)}]$$

$$\Leftrightarrow \quad \delta^2 = 2d\,\frac{(t-s)(r-s)}{t-s}\,\log[\frac{t-s}{r-s}].$$

Now $\displaystyle\int_{A_{y,t,r,s}} (P_{r-s}(x-y) - P_{t-s}(x-y))dx$

$$= \frac{|S_{d-1}|}{(4\pi)^{d/2}} \int_0^{\delta_{t,r,s}} \frac{1}{(r-s)^{d/2}} \exp(-\rho^2/4(r-s))\rho^{d-1} d\rho \ ,$$

$$- \frac{|S_{d-1}|}{(4\pi)^{d/2}} \int_0^{\delta_{t,r,s}} \frac{1}{(t-s)^{d/2}} \exp(-\rho^2/4(t-s))\rho^{d-1} d\rho$$

where $|S_{d-1}|$ is the area of the surface, S_{d-1}, of the unit ball in R^d. Now let $\rho' = \rho/2\sqrt{t-s}$ in the first integral, and $\rho' = \rho/2\sqrt{r-s}$ in the second, to obtain

$$\frac{|S_{d-1}|}{(4\pi)^{d/2}} \int_{\delta_{t,r,s}/2\sqrt{t-s}}^{\delta_{t,r,s}/2\sqrt{r-s}} \rho^{d-1} e^{-\rho^2} d\rho \ .$$

Since the function $f(\rho) = \rho^{d-1} e^{-\rho^2}$ is bounded, we may find K to bound this integral by

$$K\ \delta_{t,r,s}\left(\frac{1}{\sqrt{r-s}} - \frac{1}{\sqrt{t-s}}\right)$$

$$= K\sqrt{\log\left(1+\frac{t-r}{r-s}\right)}\ \left(\frac{\sqrt{t-s} - \sqrt{r-s}}{\sqrt{t-r}}\right)$$

Now if $r-s \geq t-r$ we may bound this further, by

$$K\sqrt{\frac{t-r}{r-s}}\ \left(\frac{(t-s)-(r-s)}{\sqrt{t-r}(\sqrt{t-s} + \sqrt{r-s})}\right)$$

$$\leq K\,\frac{t-r}{r-s}\ .$$

Thus, if $°(t-r) > 0$

$$\text{(4-11)} \quad \underline{r} - \underline{s} \geq \underline{t} - \underline{r} \Rightarrow \sup_{A\subseteq X} \left|Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:\underline{y}}\right| \leq K\,\frac{(\underline{t}-\underline{r})}{(\underline{r}-\underline{s})}, \quad \forall \underline{y} \in X\ .$$

Since the internal statement (4-11) is true for $\underline{t} - \underline{r} > \varepsilon$ for all real $\varepsilon > 0$, then by the principle of infinitesimal overflow it must hold for $\underline{t} - \underline{r} > \Delta t' \approx 0$. □

We believe that this Lemma is true for $\underline{t} - \underline{r}$ down to Δt but an internal proof of this is not easy.

4.4 The Total Mass Process M

Now let $M_t = \sum_{x \in X} U_{tx} \Delta x^d$ be the total mass of the internal measure whose density U is obtained by solving (4-1) inductively. We have by definition,

$$M_t = M_0 + \sum_{0 \le s < t} \sum_{y \in X} \Big(\sum_{x \in X} Q^{t-s}_{x-y} \Big) \left(\sqrt{\frac{U_{sy} \Delta t}{\Delta x^d}} \wedge \frac{U_{sy}}{2} \right) \xi_{sy} \Delta x^d .$$

In what follows c denotes a flexible constant whose values depend only on q and t_f, and which values may change from line to line.

Lemma 4.4.1. There is a constant c depending on $q > 1$, M_0, and t_f, such that $\forall t \le t_f$,

$$E(M_t^{2q}) \le c\, e^{ct}$$

Proof: Using Theorem 3.4.5 (Burkholder's Inequality), the fact that $\sum_{x \in X} Q^{t-s}_{x-y} = 1$ and the fact that for any a, b, $a \wedge b \le a$

$$E(M_t^{2q}) \le c\, E\Big(\Big| \sum_{0 \le s < t} \sum_{y \in X} 1 \cdot U_{sy} \Delta t \Delta x^d \Big|^q \Big)$$

$$+\, c\, E\Big(\max_{\substack{0 \le s < t \\ y \in X}} \big| U_{sy} \Delta t \Delta x^d\, \xi^2_{sy} \big|^q \Big) \tag{4-12}$$

$$+\, c\, M_0^{2q}$$

Now the second term in (4-12) above may be bounded by

$c\, E(\max_{0 \le s < t} | M_s \Delta t|^q) \le c\, E \big| \sum_{0 \le s < t} M_s \Delta t \big|^q$. Hence (4-12) becomes

$$E(M_t^{2q}) \leq c\, E\Big|\sum_{0\leq s<t} M_s\, \Delta t\Big|^q + c\, M_0^{2q}$$

$$\leq c\, E\Big|\sum_{0\leq s<t} M_s\, \Delta t\Big|^{2q} + c(M_0^{2q}+1)$$

$$\leq c \sum_{0\leq s<t} E(M_s^{2q})\Delta t + c(M_0^{2q}+1) \tag{4-13}$$

for all $t \leq t_f$.

We complete the proof with an appeal to the appropriate version of Gronwall's Lemma (Lemma 3.5.2). □

We may now estimate the differences $M_t - M_r$ as follows, in order to obtain continuity of the mass process. Let $q > 2$, and w.l.o.g. take $0 \leq r < t \leq t_f$. Then

$$E|M_t - M_r|^{2q} \leq c\, E\, \Big|\sum_{r\leq s<t}\ \sum_{y\in X} U_{sy}\, \Delta t \Delta x^d\Big|^q \tag{4-14}$$

(using Burkholder's Inequality and applying the same reasoning as in going from (4-12) to (4-13))

$$\leq c\, E\, \Big|\sum_{r\leq s<t} M_s \frac{\Delta t}{(t-r)}\Big|^q\, (t-r)^q$$

$$\leq c\, E\, \Big(\sum_{r\leq s<t} M_s^q \frac{\Delta t}{t-r}\Big)(t-r)^q$$

(applying Jensen's Inequality to the probability measure $\frac{\Delta t}{t-r}$) on the interval $r \leq s < t$)

$$\leq c \sum_{\underline{r} \leq \underline{s} < \underline{t}} \underline{E}(M^q_{\underline{s}}) \frac{\Delta t}{(\underline{t}-\underline{r})} (\underline{t}-\underline{r})^q .$$

(4-15) $$\leq c \exp(c\ \underline{t}_f)(\underline{t}-\underline{r})^q \quad ,$$

by Lemma 4.4.1.

Thus we have

Lemma 4.4.2. The total mass process $M_{\underline{t}}$, defined above, is a.s. S-continuous on $0 \leq \underline{t} \leq \underline{t}_f$.

Proof: Apply the Kolmogorov Continuity Criterion (3.8.1) to (4-15). □

In fact $M_{\underline{t}}$ is S-Hölder continuous of order $\frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$.

4.5 S-Continuity of the Process

For internal sets A, and internal functions F let the notations $X_{\underline{t}}^{A}$ and $X_{\underline{t}}^{F}$ stand for $\sum\limits_{\underline{x}\in A} U_{\underline{t}\underline{x}}\,\Delta x^d$ and $\sum\limits_{\underline{x}\in X} F(\underline{x})\,U_{\underline{t}\underline{x}}\,\Delta x^d$. We introduce the class of (standard) functions $C^2_{b,2} = \{f \in C^2(R^d) \mid \exists K < \infty$ such that $|f(x)| + |\Delta f(\underline{x})| \le K\ \forall x \in R^d\}$. Recall from section 3.9 that if an internal function F <u>lifts to order 2</u> a function $f \in C^2_{b,2}$, then $°(\underline{\Delta}F)(\underline{x}) = \Delta f(°\underline{x})$, $\forall \underline{x} \in ns(X)$.

<u>Lemma 4.5.1.</u> If F is an internal function such that $|F(\underline{x})| + |\underline{\Delta}F(\underline{x})| \le K$, $\forall\ \underline{x} \in X$, then the process $X_{\underline{t}}^{F}$ is S-continuous on T a.s.

<u>Proof</u>: For $\underline{r} < \underline{t} \in T$,

$$X_{\underline{t}}^{F} - X_{\underline{r}}^{F} = \sum_{\underline{x}\in X} F(\underline{x})\,(U_{\underline{t}\underline{x}} - U_{\underline{r}\underline{x}})\,\Delta x^d$$

$$= \sum_{\underline{r}\le \underline{s}<\underline{t}}\ \sum_{\underline{x}} F(\underline{x}) \left[\frac{U_{\underline{s}+\Delta t,\underline{x}} - U_{\underline{s}\underline{x}}}{\Delta t}\right] \Delta t\Delta x^d$$

$$= \sum_{\underline{r}\le \underline{s}<\underline{t}}\ \sum_{\underline{x}} F(\underline{x}) \left[(\underline{\Delta}U_{\underline{s}\cdot})_{\underline{x}} + \left(\sqrt{\frac{U_{\underline{s}\underline{x}}}{\Delta t\Delta x^d}} \wedge \frac{U_{\underline{s}\underline{x}}}{2\Delta t}\right)\xi_{\underline{s}\underline{x}}\right] \Delta t\Delta x^d$$

$$\text{(4-16)}\quad = \sum_{\underline{r}\le \underline{s}<\underline{t}}\ \sum_{\underline{x}} (\underline{\Delta}F)(\underline{x})\,U_{\underline{s}\underline{x}}\,\Delta x^d\,\Delta t$$

$$+ \sum_{\underline{r}\le \underline{s}<\underline{t}}\ \sum_{\underline{x}} F(\underline{x})\left(\sqrt{U_{\underline{s}\underline{x}}} \wedge \frac{U_{\underline{s}\underline{x}}}{2}\ \frac{\Delta x^d}{\Delta t}\right)\xi_{\underline{s}\underline{x}}\sqrt{\Delta t\Delta x^d}$$

The first term in (4-16) above we recognize as $\sum_{\underline{r}\le \underline{s}<\underline{t}} X_{\underline{s}}^{\Delta F} \Delta t$.

Now $X_{\underline{s}}^{\Delta F} \le K\, M_{\underline{s}}$. Hence if $q \ge 1$

$$\underline{E}|X_{\underline{t}}^{F} - X_{\underline{r}}^{F}|^{2q} \le c\, \underline{E}\Big(\sum_{\underline{r}\le \underline{s}<\underline{t}} M_{\underline{s}} \Delta t\Big)^{2q}$$

$$+ c\, \underline{E}\Big(\sum_{\underline{r}\le \underline{s}<\underline{t}} \sum_{\underline{x}\in X} F^2(\underline{x})\, U_{\underline{sx}}\, \Delta x^d \Delta t\Big)^{q}$$

(using Burkholder's Inequality for the second term on the right and incorporating the term involving "max" as we did in (4-12))

$$\le c \sum_{\underline{r}\le \underline{s}<\underline{t}} \underline{E}(M_{\underline{s}}^{2q}) \frac{\Delta t}{\underline{t}-\underline{r}} (\underline{t}-\underline{r})^{2q}$$

$$+ c \sum_{\underline{r}\le \underline{s}<\underline{t}} \underline{E}\Big(\sum_{\underline{x}\in X} F^2(\underline{x})\, U_{\underline{sx}}\, \Delta x^d\Big)^{q} \Delta t$$

(we now recognize the second terms as $\sum_{\underline{r}\le \underline{s}<\underline{t}} \underline{E}|X_{\underline{s}}^{F^2}|^{q}\, \Delta t$ and $X_{\underline{s}}^{F^2} \le K^2 M_{\underline{s}}$)

$$\le c\, (\underline{t}-\underline{r})^{2q} + c\, (\underline{t}-\underline{r})^{q} , \tag{4-17}$$

if $\underline{t} \le \underline{t}_f$, using Jensen's Inequality a second time.

Now we apply the N.S. version of the Kolmogorov Continuity Criterion (Theorem 3.8.1). □

<u>Lemma 4.5.2.</u> $\lim_{n\to\infty} \sup_{\underline{t}\in[0,t_f]} {}^\circ X_{\underline{t}}^{A_n} = 0$ a.s. where

$A_n = \{\underline{x} \in \mathcal{X} \mid |\underline{x}| \geq n\}$; that is, $X_{\underline{t}}$ is nearstandard in ${}^*M_F(R^d)$ for all $\underline{t}$, a.s.

<u>Proof</u>: Let H be infinite. Let F be an internal function such that $0 \leq F \leq 1$, $F = 1$ on A_H, $F = 0$ on A_{H-1}^c and $\underline{\Delta}F$ is bounded. Then

$$\underline{E}(X_{\underline{t}}^F) \leq \underline{E}(X_{\underline{t}}^{A_{H-1}}) = \sum_{\underline{y}\in\mathcal{X}} \Big(\sum_{\underline{x}\in A_{H-1}} Q_{\underline{x}-\underline{y}}^{\underline{t}+\Delta t} \Big) U_{0\underline{y}} \Delta x^d .$$

Now by Lemma 4.3.1, for each $\underline{y} \in ns(\mathcal{X})$

$$\sum_{\underline{x}\in A_{H-1}^c} Q_{\underline{x}-\underline{y}}^{\underline{t}-\underline{s}} \approx 1 \text{ , so that } \varepsilon_{\underline{y}} = \sum_{\underline{x}\in A_{H-1}} Q_{\underline{x}-\underline{y}}^{\underline{t}+\Delta t} \approx 0 .$$

By assumption U_0 is nearstandardly concentrated, so that

$$\lim_{n\to\infty} {}^\circ\Big(\sum_{\underline{y}\in A_n} U_{0\underline{y}} \Delta x^d \Big) = 0 .$$

Hence,

$$E(X_t^F) \leq \sum_{\underline{y}\in\mathcal{X}} \varepsilon_{\underline{y}} U_{0\underline{y}} \approx 0 \implies X_{\underline{t}}^F \approx 0 \text{ a.s.} \tag{4-18}$$

Now by Lemma 4.5.1 $X_{\underline{t}}^F$ is a.s. S-continuous.

There is a countable S-dense subset of $\mathcal{T}$ for which (4-18) holds. Hence

$${}^\circ X_{\underline{t}}^F = 0 \ \forall \underline{t} \in \mathcal{T} \text{ , a.s.} \implies \sup_{\underline{t}\in[0,\underline{t}_f]} {}^\circ X_{\underline{t}}^{A_H} = 0 \text{ a.s.}$$

Therefore $\sup_{\underline{t}\in[0,\underline{t}_f]} {}^\circ X_{\underline{t}}^{A_n} \xrightarrow{P} 0$ as $n \to \infty$,

but since $X^{A_n} \geq X^{A_{n+1}}$ we must have convergence a.s. $\square$

Theorem 4.5.3. $X : \mathcal{T} \to {}^*M_F(\mathbb{R}^d)$ is a.s. S-continuous. Thus X is nearstandard in ${}^*C(\mathbb{R}; M_F(\mathbb{R}^d))$, and $({}^\circ X)_{{}^\circ \underline{t}} = L(X_{\underline{t}}) \circ st^{-1}$ for all $\underline{t} \in [0, \underline{t}_f]$ a.s.

Proof: Recall the weak topology on $M_F(\mathbb{R}^d)$: $x_n \to x \iff$ for all bounded continuous f, $x_n(f) \to x(f)$. Let $\{\phi_k\}$ be a countable collection of $C^2_{b,2}(\mathbb{R}^d)$ functions which constitute a convergence determining class for $M_F(\mathbb{R}^d)$. Let $\{\Phi_k\}$ be liftings to order 2 of the $\{\phi_k\}$. Then by Theorem 4.5.1 each $X_{\underline{t}}^{\Phi_k}$ is a.s. S-continuous. Now by the Loeb construction (see section 2.2)

$$\text{(4-19)} \qquad \int \phi_k \, d\, L(X_{\underline{t}}) \circ st^{-1} = {}^\circ X_{\underline{t}}^{\Phi_k} .$$

Therefore the l.h.s. of (4-19) is continuous a.s. for each ϕ_k in the convergence determining class. Hence, almost surely $L(X_{\underline{t}}) \circ st^{-1} = L(X_{\underline{s}}) \circ st^{-1}$, for all $\underline{t} \approx \underline{s} \in \mathcal{T}$.

From Lemmas 4.4.1 and 4.5.2, $X_{\underline{t}}$ is in fact nearstandard in ${}^*M_F(\mathbb{R}^d)$ for all $\underline{t}$, a.s., so that ${}^\circ(X_{\underline{t}})$ exists and equals $L(X_{\underline{t}}) \circ st^{-1} \; \forall \, \underline{t}$ a.s. □

Now we show a stronger form of continuity.

Theorem 4.5.4. Let X_0 be nearstandard in ${}^*M_F(\mathbb{R}^d)$ (as discussed at the end of section 4.2). Let A be internal. Then $X_{\underline{t}}^A$ is S-continuous for ${}^\circ\underline{t} > 0$ on a coarser grid $\mathcal{T}' \subset \mathcal{T}$ of infinitesimal spacing $\Delta t'$ independent of A.

Proof: We take $\Delta t'$ from Lemma 4.3.3.

Let $Y_{\underline{t}}^1 = \sum_{\underline{y} \in X} Q_{A:\underline{y}}^{\underline{t}+\Delta t} U_{0\underline{y}} \Delta x^d$ and let

$$Y^2_{\underline{t}} = \sum_{0\le \underline{s}<\underline{t}} \sum_{\underline{y}\in X} Q^{\underline{t}-\underline{s}}_{A:\underline{y}} \left(\sqrt{U_{\underline{sy}}\Delta t \Delta x^d} \wedge \frac{U_{\underline{sy}}}{2} \Delta x^d \right) \xi_{\underline{sy}}$$

Then $X^A_{\underline{t}} = Y^1_{\underline{t}} + Y^2_{\underline{t}}$.

From Lemma 4.3.3, there is a K such that, if

$\underline{r} < \underline{t} < 2\underline{r}$ in T' , $|Q^{\underline{t}}_{A:\underline{y}} - Q^{\underline{r}}_{A:\underline{y}}| \le K(\underline{t}-\underline{r})/r$. Hence if $\underline{t} \approx \underline{r}$ and $^\circ\underline{r} > 0$

$$\sum_{\underline{y}\in X} |Q^{\underline{t}}_{A:\underline{y}} - Q^{\underline{r}}_{A:\underline{y}}| \, U_{0\underline{y}} \Delta x^d$$

$$\le \frac{K}{\underline{r}} (\underline{t}-\underline{r}) \, M_0$$

$$\approx 0 \, .$$

Thus $Y^1_{\underline{t}}$ is S-continuous in $T' \cap st^{-1}(t>0)$.

Now suppose $\underline{r} < \underline{t} \in T'$, $\underline{t} - \underline{r} < 1$, $0 < \gamma < 1$, and $q \ge 2$. Then using Burkholder's Inequality and absorbing the terms involving a maximum into the summations, as we did in section 4.4, ((4-12) and (4-13)),

$$\underline{E}|Y^2_{t} - Y^2_{\underline{r}}|^{2q} \le c\,\underline{E}\left| \sum_{0\le\underline{s}<\underline{r} - (\underline{t}-\underline{r})^\gamma} \sum_{\underline{y}\in X} (Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:y})^2 \, U_{\underline{sy}} \Delta t \Delta x^d \right|^q$$

$$+ c\,\underline{E}\left| \sum_{r-(t-r)^\gamma \le \underline{s}<\underline{r}} \sum_{\underline{y}\in X} (Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:y})^2 \, U_{sy} \Delta t \Delta x^d \right|^q$$

$$+ c\,\underline{E}\left| \sum_{\underline{r}\le\underline{s}<\underline{t}} \sum_{\underline{y}\in X} (Q^{\underline{t}-\underline{s}}_{A:y})^2 \, U_{\underline{sy}} \Delta t \Delta x^d \right|^q$$

$$\le c\,\underline{E}\left| \sum_{0\le\underline{s}<\underline{r}-(\underline{t}-\underline{r})^\gamma} (\underline{t}-\underline{r})^{2-2\gamma} \, M_{\underline{s}} \, \Delta t \right|^q$$

(we invoke Lemma 4.3.3 and note $(\underline{r}-\underline{s}) \ge (\underline{t}-\underline{r})^\gamma$.)

$$+ c\,\underline{E}\left|\sum_{\underline{r}-(\underline{t}-\underline{r})^{\gamma}\le\underline{s}<\underline{r}} 1\cdot M_{\underline{s}}\,\Delta t\right|^{q}$$

(since $|Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:\underline{y}}| \le 1$)

$$+ c\,\underline{E}\left|\sum_{\underline{r}\le\underline{s}<\underline{t}} M_{\underline{s}}\,\Delta t\right|^{q} .$$

(4-20) $$\le c((\underline{t}-\underline{r})^{(2-2\gamma)q} + (\underline{t}-\underline{r})^{\gamma q} + (\underline{t}-\underline{r})^{q})$$

(following the usual Jensen's inequality train of development, by now so familiar).

Our faithful servant the constant c has now acquired dependence on M_0 , α , $\underline{t}_f$, and q , but not on $\underline{t}$ and $\underline{r}$.

The best choice of γ for (4-20) seems to be $\frac{2}{3}$. Using the Kolmogorov criterion (Theorem 3.8.1) we get S-continuity with a modulus $\frac{1}{3} - \varepsilon$ for any $\varepsilon > 0$. □

Remark: Of course if X_0 puts all its mass on a null set A , we get failure of S-continuity in the monad $\{\underline{t} \approx 0\}$.

<u>Corollary 4.5.5</u>. Let F be any bounded internal function on $\mathcal{X}$. Then the process $X^F_{\underline{t}}$ is S-continuous on $\mathcal{T}' \cap st^{-1}(t>0)$, a.s.

<u>Proof</u>: Let K be a bound for $|F|$. Let $\hat{X}^F_{\underline{t}} = \sum_{\underline{y}\in\mathcal{X}}\sum_{\underline{x}\in\mathcal{X}} F(\underline{x})\, Q^{\underline{t}+\Delta t}_{\underline{x}-\underline{y}}\, U_{0\underline{y}}\, \Delta x^d$, and let $$\tilde{X}^F_t = \sum_{0\le\underline{s}<\underline{t}}\sum_{\underline{y}\in\mathcal{X}}\Big(\sum_{\underline{x}\in\mathcal{X}} F(\underline{x})\, Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}}\Big)\Big(\sqrt{U_{\underline{sy}}\Delta t\Delta x^d} \wedge \frac{U_{\underline{sy}}}{2}\Delta x^d\Big)\xi_{\underline{sy}} .$$

Then $X^F_{\underline{t}} = \hat{X}^F_{\underline{t}} + \tilde{X}^F_{\underline{t}}$. Now if $\underline{t} > \underline{r} \in \mathcal{T}'$

$$|\hat{X}^F_{\underline{t}} - \hat{X}^F_{\underline{r}}| \leq K|\hat{X}^{\{F>0\}}_{\underline{t}} - \hat{X}^{\{F>0\}}_{\underline{r}}| + K|\hat{X}^{\{F<0\}}_{\underline{t}} - \hat{X}^{\{F<0\}}_{\underline{r}}| \ .$$

Following the first part of Lemma 4.5.4, $\hat{X}^F$ is S-continuous in $\mathcal{T}' \cap \{{}^\circ\underline{t} > 0\}$.

Now if $q > 1$,

$$\underline{E}|\tilde{X}^F_{\underline{t}} - \tilde{X}^F_{\underline{r}}|^{2q} \leq c\ \underline{E}\left| \sum_{0\leq \underline{s}<\underline{r}} \sum_{\underline{y}\in X} \left(\sum_{\underline{x}\in X} F(\underline{x}) (Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}} - Q^{\underline{r}-\underline{s}}_{\underline{x}-\underline{y}}) \right)^2 U_{\underline{s}\underline{y}} \Delta t \Delta x^d \right|^{2q}$$

$$+ c\ \underline{E}\left| \sum_{\underline{r}\leq\underline{s}<\underline{t}} \sum_{\underline{y}\in X} K^2\ U_{\underline{s}\underline{y}} \Delta t \Delta x^d \right|^{2q} .$$

Now

$$\left(\sum_{\underline{x}\in X} F(\underline{x}) (Q^{\underline{t}-\underline{s}}_{\underline{x}-\underline{y}} - Q^{\underline{r}-\underline{s}}_{\underline{x}-\underline{y}}) \right)^2 \leq 2\ K^2 \sup_{\underline{A}\subset X} (Q^{\underline{t}-\underline{s}}_{A:\underline{y}} - Q^{\underline{r}-\underline{s}}_{A:\underline{y}})^2 \leq c\ \frac{\underline{t}-\underline{r}}{\underline{r}-\underline{s}} ,$$

by Lemma 4.3.3. Hence following the second part of the argument in 4.5.4 we obtain the similar result

$$\underline{E}|\tilde{X}^F_{\underline{t}} - \tilde{X}^F_{r}|^{2q} \leq c(\underline{t}-\underline{r})^{2q/3} ,$$

and thus $\tilde{X}^F$ is S-continuous on $\mathcal{T}'$ a.s. □

Remark: We believe that $\tilde{X}^F$ is Hölder S-continuous of index $\frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$.

Remark: If we took $\underline{t}_f$ to be hyperfinite, ${}^\circ\underline{t}_f = \infty$, then 4.5.3, 4.5.4 and 4.5.5 will hold a.s. until all finite times, and hence a.s. on $ns(\mathcal{T})$. We will use this fact without further ado in section 4.7.

4.6 Characterization by a Martingale Problem

We show now that the measure-valued process x_t, which we constructed in the last section as $L(X_{\underline{t}}) \circ st^{-1}$ is in fact the measure diffusion studied by Dawson and others. We will use one of the martingale characterizations given in Roelly Coppoletta (1986), which are refinements of the larger class of martingale problems investigated by Holley and Stroock (1978), Dawson and Kurtz (1982) and others.

The characterization of the critical branching measure diffusion x_t given in Roelly-Coppoletta (1986; Theorem 1.3, condition iii)) is as follows:

Theorem 4.6.1. x_t is the (unique) $M_F(\mathbb{R}^d)$ valued process with weakly continuous paths, and the given initial condition such that for all $\phi \in D(\Delta)$ (the domain of the Laplacian Δ)

a) $m_t^\phi \overset{\text{def}}{=} x_t^\phi - x_0^\phi - \int_0^t x_s \Delta\phi \, ds$ is a continuous martingale.

b) $\langle m^\phi \rangle_t = \int_0^t x_s \phi^2 \, ds$.

We know already (Lemma 4.5.3) that our process is weakly continuous in $M_F(\mathbb{R}^d)$. We will verify a) and b) above for functions $\phi \in C^2_{b,2}(\mathbb{R}^d)$ only. The extension to $D(\Delta)$ (which is the closure under Δ of $C^2_{b,2}(\mathbb{R}^d)$ in $C_b(\mathbb{R}^d)$) may be done by a limiting argument. It is in fact unnecessary to Ms. Roelly-Coppoletta's argument in her Theorem 1.3 to go beyond $C^2_{b,2}(\mathbb{R}^d)$.

For such a ϕ let Φ be a lifting to order 2, and suppose $|\Phi| \leq K$. Then

$$x^\phi_{{}^\circ\underline{t}} = \int_{ns(X)} \phi({}^\circ\underline{x}) d\, L(X_{\underline{t}}) \ , \ \underline{t} \in T' \ , \text{ a.s.}$$

(recall from 4.5.2 that the mass on $X \setminus ns(X)$ is always infinitesimal a.s.)

$$= {}^\circ \sum_{\underline{x} \in X} \Phi(\underline{x}) \, U_{\underline{t}x} \Delta x^d \ , \text{ by Loeb Lifting Theorem (Theorem 2.1.6)}$$

$$= {}^\circ X^\Phi_{\underline{t}}$$

Hence,

$$m_t^\phi = x_t^\phi - x_0^\phi - \int_0^t x_s^{\Delta\phi}\, ds$$

is indistinguishable from the standard part of

$$M_{\underline{t}}^\Phi \overset{\text{def}}{=} X_{\underline{t}}^\Phi - X_0^\Phi - \sum_{0\leq \underline{s}<\underline{t}} X_{\underline{s}}^{\Delta\Phi}\,\Delta t$$

$$= \sum_{\underline{x}\in X} \Phi(\underline{x})(U_{\underline{tx}} - U_{0\underline{x}})\Delta x^d - \sum_{0\leq \underline{s}<\underline{t}} \sum_{\underline{x}\in X} U_{\underline{sx}}(\Delta\Phi)(\underline{x})\ \Delta x^d \Delta t$$

$$= \sum_{0\leq \underline{s}<\underline{t}} \sum_{\underline{x}\in X} \Phi(\underline{x}) \frac{(U_{\underline{s}+\Delta t\underline{x}} - U_{\underline{sx}})}{\Delta t} \Delta t \Delta x^d$$

$$- \sum_{0\leq \underline{s}<\underline{t}} \sum_{\underline{x}\in X} \Phi(x)(\underline{\Delta}\ U_{\underline{t}\cdot})_{\underline{x}}\ \Delta t\Delta x^d$$

(using the infinitesimal analogue of integration-by-parts)

$$(4\text{-}21) \qquad = \sum_{0\leq \underline{s}<\underline{t}} \sum_{\underline{x}\in X} \Phi(\underline{x})[\sqrt{U_{sx}\ \Delta t\Delta x^d} \wedge \frac{U_{\underline{sx}}}{2}\ \Delta x^d]\xi_{sy}\ .$$

Now $M_{\underline{t}}^\Phi$ is a martingale with respect to the internal filtration $F_{\underline{t}}$. Hence $m_{\circ\underline{t}}^\phi = {}^\circ M_{\underline{t}}^\Phi\ \forall \underline{t} \in T'$, a.s. implies that m_t^ϕ is an F_t-martingale, by Theorem 5.2 of Hoover and Perkins (1983). This verifies a) of Theorem 4.6.1.

Further, since m_t^ϕ is a.s. continuous in t , the predictable square function $\langle m^\phi\rangle_t$ coincides a.s. with the square function $[m^\phi]_t$. Now by another result of Hoover and Perkins (1983) (Theorem 6.7)

$$[st(M_{\underline{t}}^\Phi)] = st[M_{\underline{t}}^\Phi]\ ,$$

the standard part of the internal square function, and by Theorem 8.5 of the same paper,

$$[M^{\Phi}]_{\underline{t}} \approx \langle M^{\Phi}\rangle_{\underline{t}} \ \forall \underline{t}\ , \text{ a.s.,}$$

provided $(M^{\Phi}_{\underline{t}})^2$ is S-integrable, where $\langle M^{\Phi}\rangle$ is the internal predictable (with respect to $F_{\underline{t}}$) square process.

Now by Burkholder's Inequality, if $p > 2$

$$\underline{E}\big(\sup_{\underline{t}\leq\underline{t}_f} (M^{\Phi}_{\underline{t}})^p\big) \leq c_p \, \underline{E}\left|\sum_{0<\underline{s}<\underline{t}} \ \sum_{\underline{x}\in X} \Phi^2(\underline{x})\, U_{\underline{sx}}\, \Delta x^d \Delta t\right|^{p/2}$$

$$\leq c_p K^p \, \underline{E}\left|\sum_{0\leq\underline{s}<\underline{t}} M_{\underline{s}}\, \Delta t\right|^{p/2} .$$

(where $M_{\underline{s}}$ is the total mass process)

which is finite by Lemma 4.4.1.

Now apply Chebychev's Inequality to conclude that $(M^{\Phi}_{\underline{t}})^2$ is S-integrable as required.

Now $\langle M^{\Phi}\rangle_{\underline{t}}$

$$= \sum_{0\leq\underline{s}<\underline{t}} \ \sum_{\underline{x}\in X} \Phi^2(\underline{x}) \left[U_{\underline{sx}} \wedge \frac{U^2}{4}\frac{\Delta x^d}{\Delta t}\right] \Delta t \Delta x^d$$

$$(4\text{-}22) \quad = \sum_{0\leq\underline{s}<\underline{t}} \ \sum_{\underline{x}\in X} \Phi^2(\underline{x})\, U_{\underline{sx}}\, \Delta t \Delta x^d$$

$$+ \sum_{0\leq\underline{s}<\underline{t}} \ \sum_{\underline{x}\in X} \Phi^2(\underline{x})\, I_{\{U_{\underline{sx}} < \frac{4\Delta t}{\Delta x^d}\}} \left(\frac{U^2_{\underline{sx}}}{4}\,\frac{\Delta x^d}{\Delta t} - U_{\underline{sx}}\right)\Delta t \Delta x^d$$

Now the second term above may be bounded by

$$||\Phi||^2_{\infty} \sum_{0\leq\underline{s}<\underline{t}} \ \sum_{\substack{\underline{x}\in X \\ \{|\underline{x}| \leq L\}}} U_{\underline{sx}}\, I_{\{U_{\underline{sx}} < \frac{4\Delta t}{\Delta x^d}\}}\, \Delta t \Delta x^d$$

$$+ \quad ||\Phi||_\infty^2 \sum_{0 \leq \underline{s} < \underline{t}} \sum_{\substack{\underline{x} \in X \\ \{|\underline{x}| > L\}}} U_{\underline{s}\underline{x}} \, 1_{\{U_{\underline{s}\underline{x}} < \frac{4\Delta t}{\Delta x^d}\}} \Delta x^d \Delta t \, .$$

$$\leq K^2 \cdot \underline{t} \cdot (2L)^d \cdot \frac{4\Delta t}{\Delta x^d}$$

$$(4\text{-}23) \qquad + K^2 \sum_{0 \leq \underline{s} < \underline{t}} X_{\underline{s}}^{\{|\underline{x}| \geq L\}} \Delta t$$

By Lemma 4.5.2, the standard part of the second term in (4-23) goes to 0 a.s. as $L \to \infty$. The first term is infinitesimal for all L finite.

Hence, for each $\underline{t}$

$$(4\text{-}24) \qquad \langle m^\phi \rangle_{{}^\circ \underline{t}} \overset{a.s.}{=} {}^\circ \langle M^\Phi \rangle_{\underline{t}} \overset{a.s.}{=} {}^\circ \sum_{0 \leq \underline{s} < \underline{t}} X_{\underline{s}}^{\Phi^2} \Delta t \overset{a.s.}{=} \int_0^{{}^\circ \underline{t}} x_s^{\phi^2} ds \, ,$$

as required by condition b) of Theorem 4.6.1. Hence

<u>Theorem 4.6.2</u>. The standard part x_t of the hyperfinite process $X_{\underline{t}}$ constructed by solving the difference equations (4-1) for an internal density U, is the critical branching measure diffusion considered by Dawson and others.

4.7 New Results on the Dawson Measure Valued Diffusion

This section represents joint work with Ed Perkins, my thesis supervisor.

Now that we know that the measure valued process x_t which we constructed via solution of the hyperfinite difference equations (4-1) for an internal "density" $U_{\underline{t}\underline{x}}$, is in fact a construction of the Dawson Process, we may use this construction to ascertain some regularity properties of the process. First we need

Lemma 4.7.1. Let $A \subseteq X$ be a Loeb-measureable set of $L(\mu_{\underline{x}})$ measure zero. Then on a set Ω_A of probability 1 $L(X_{\underline{t}})(A) = 0$, $\forall\, \underline{t} \in T' \cap st^{-1}(t>0)$.

Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a nested sequence of internal sets such that $A \subseteq A_n$ for each $n \in \mathbb{N}$, and $^\circ(\mu_{\underline{x}}(A_n)) \to 0$. Extend $\{A_n\}$ to $^*\mathbb{N}$ in such a way that the A_n's are still nested (that this may be done, follows from ω_1-saturation).

Let H be infinite in $^*\mathbb{N}$. Then $\mu_{\underline{x}}(A_H) \approx 0$. Suppose $\underline{t} \in T'$, with $^\circ\underline{t} > 0$. Now by Lemma 4.3.2, if K is a bound for $p_{^\circ\underline{t}}(x)$,

$$Q^{\underline{t}+\Delta t}_{A_H:\underline{y}} = \sum_{\underline{x}\in A_H} \frac{Q^{\underline{t}+\Delta t}_{\underline{x}-\underline{y}}}{\Delta x^d} \Delta x^d \leq K\mu_{\underline{x}}(A_H) \quad . \qquad \text{Hence}$$

$$E(X_t^{A_H}) = \underline{E}\Big(\sum_{\underline{x}\in A_H} U_{\underline{t}\underline{x}}\, \Delta x^d\Big) = \sum_{\underline{y}\in X} Q^{\underline{t}+\Delta t}_{A_H:\underline{y}}\, U_{0\underline{y}}\, \Delta x^d \leq K\mu_{\underline{x}}(A_H) M_0 \approx 0 \; .$$

Thus for each $\underline{t}$ there is a set $\Omega^{\underline{t}}$ of probability 1 on which $X_{\underline{t}}^{A_H} \approx 0$. Let $\{\underline{t}_k\}_{k\in N}$ be a countable S-dense set in T'. Let Ω_H' be the set of probability one on which $X_{\underline{t}}^{A_H}$ is S-continuous for $\underline{t} \in T'$. Let $\Omega_H = \bigcap_{k=1}^{\infty} \Omega_{\underline{t}_k} \cap \Omega_H'$. Then on Ω_H, $^\circ(X_{\underline{t}}^{A_H}) = \lim_{^\circ\underline{t}_{k_i} \to {}^\circ\underline{t}} {}^\circ(X_{\underline{t}_{k_i}}^{A_H}) = 0$ for some sequence $\{\underline{t}_{k_i}\}$. Thus $X_{\underline{t}}^{A_H} \approx 0 \; \forall \underline{t} \in T'$, hence by infinitesimal underflow for each $\omega \in \Omega_H$, there is an infinitesimal $\varepsilon(\omega)$ such that

$$X_{\underline{t}}^{A_H} \leq \varepsilon \; \forall \, \underline{t} \in T' .$$

For $n \in {}^*\mathbb{N}$, let $Y_n = \sup_{\underline{t} \in T'} X_{\underline{t}}^{A_n} \wedge 1$. Then Y_n is S-L^1 for each n, and the sequence $\{Y_n\}_{n \in {}^*\mathbb{N}}$ is internal. Now $\underline{E}(Y_H) \approx 0$, for any infinite $H \in {}^*\mathbb{N}$. Hence $°(\underline{E}(Y_n)) \downarrow 0$ as $n \to \infty$ through $\mathbb{N}$, and thus $°Y_n \downarrow 0$ on a set Ω_A of probability one. Now $L(X_{\underline{t}})(A) \leq L(X_{\underline{t}})(A_n) = °X_{\underline{t}}^{A_n} \leq °Y_n$

$$\forall \underline{t} \in T' , \; \forall n \in \mathbb{N} . \text{ Hence } L(X_{\underline{t}})(A) = 0 , \; \forall \underline{t} \in T' \text{ on } \Omega_A .$$ □

Hence, we may draw

<u>Corollary 4.7.2</u>. If $A \subseteq \mathbb{R}^d$ is a Lebesgue null set $(\lambda(A) = 0)$, then $x_t(A) = 0 \; \forall t > 0$, a.s.

<u>Proof</u>: $st^{-1}(A)$ is a Loeb null set. □

We introduce the notations $x_t^f = \int f \, dx_t$ and $x_t^A = x_t(A)$.

<u>Theorem 4.7.3</u>. If $f : \mathbb{R}^d \to \mathbb{R}$ is a bounded measureable function, then x_t^f is a.s. continuous in $t > 0$. In particular x_t^A is a.s. continuous in $t > 0$ for any Lebesgue set A.

<u>Proof</u>: Let $F : X \to {}^*\mathbb{R}$ be a lifting with respect to $L(\mu_{\underline{x}})$ of f. Then let A be $\{\underline{x} : \underline{x} \in X \; °F(\underline{x}) \neq f(°\underline{x})\}$ which is a Loeb null set. By Lemma 4.7.1, $L(X_{\underline{t}})(A) = 0 \; \forall \underline{t} \in T'$ a.s. Now F is S-L^1 of $(X, X_{\underline{t}}) \; \forall \underline{t}$ a.s. (since $M_{\underline{t}}$ is finite $\forall \underline{t}$ a.s.). Hence $x_{°\underline{t}}^f = \int °F(\underline{x}) dL(X_{\underline{t}}) = °X_{\underline{t}}^F \; \forall \underline{t} \in T'$ a.s. By Lemma 4.5.5 $X_{\underline{t}}^F$ is S-continuous a.s. □

Theorem 4.7.4. Let ϕ be a bounded measureable function on R^d, and suppose a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ of uniformly bounded measureable functions converges to ϕ in Lebesgue measure. Then for any $\varepsilon > 0$

$$\sup_{[\varepsilon,t_f]} |x_t^{\phi_k} - x_t^{\phi}| \xrightarrow{P} 0 ,$$

Proof: Let Φ and Φ_k $k \in \mathbb{N}$, be liftings of ϕ and ϕ_k, $k \in \mathbb{N}$ respectively. Extend $\{\Phi_k\}_{k\in\mathbb{N}}$ to an internal sequence $\{\Phi_k\}$ $k \in {}^*\mathbb{N}$.

Then if $°\underline{t} > 0$

$$°\underline{E}(X_{\underline{t}}^{\Phi-\Phi_k}) = °\sum_{\underline{x}\in X} \hat{U}_{\underline{t}\underline{x}} (\Phi(\underline{x}) - \Phi_k(\underline{x}))\Delta x^d$$

$$= \int_{R^d} \hat{u}_{°\underline{t}x} (\phi(x) - \phi_k(x))dx \quad \text{where} \quad \hat{u} = st(\hat{U}) .$$

$\to 0$ as $k \to \infty$, since $\hat{u}_{tx}$ is bounded and $L^1(R)$ for $t > 0$.

Hence if $H \in {}^*\mathbb{N}$ is infinite,

$$°\underline{E}(X_{\underline{t}}^{\Phi-\Phi_H}) = 0 \Rightarrow X_{\underline{t}}^{\Phi-\Phi_H} \approx 0 \text{ a.s. } \forall \underline{t} \in T . \tag{4-25}$$

Now $X_{\underline{t}}^{\Phi}$ and $X_{\underline{t}}^{\Phi_H}$ are both S-continuous on $T' \cap st^{-1}(t>0)$ a.s. by Lemma 5.5.2. Therefore since (4-25) holds for a countable S-dense subset of T', hence $X_{\underline{t}}^{\Phi-\Phi_H} \approx 0$ $\forall \underline{t} \in T'$ a.s., and

$$Y_H \overset{def}{=} \max_{\underline{t}\in T'} X_{\underline{t}}^{\Phi-\Phi_H} \approx 0 \text{ a.s.}$$

Therefore $°Y_k \xrightarrow{P} 0$ as $k \to \infty$ through $\mathbb{N}$.

Now we have seen (Lemma 4.7.3) that for $k \in \mathbb{N}$ $x_t^{\phi_k}$ (resp. x_t^{ϕ}) and $st(X_{\underline{t}}^{\Phi_k})$ (resp. $st(X_{\underline{t}}^{\Phi})$) are indistinguishable processes. Hence

$$\sup_{t\in[\varepsilon,t_f]} |x_t^{\phi_k} - x_t^{\phi}| \overset{a.s.}{=} {}^{\circ}\max_{\underline{t}\in T'} |X_{\underline{t}}^{\Phi-\Phi_k}| = {}^{\circ}Y_k \xrightarrow{P} 0 \qquad \square$$

CHAPTER 5

The Critical Branching Diffusion in One Dimension

5.1 Introduction

In Chapter Three we obtained an existence theorem for SPDEs of the form

$$\frac{\partial u}{\partial t} = \Delta u + f(u)\ \dot{w}_{tx}$$

where f grows at most linearly at infinity, without the necessity of imposing a Lipshitz condition on f. As discussed in Chapter One, the one dimensional Dawson critical branching diffusion has been believed to satisfy the SPDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sqrt{u}\ \dot{w}_{tx}\ . \tag{5-1}$$

However the theory of such an equation has not been well-known since the function $u \to \sqrt{u}$ is not Lipshitz. In this section we use the results of Chapter Three to show that the critical branching diffusion of Dawson does indeed satisfy (5-1) in one spatial dimension, and thus it has a.s. a jointly continuous density.

5.2 The SPDE and the Measure Diffusion

Let $d = 1$, and let Ω be the space $\{-1, 1\}^{T\times X}$ as described in section 4.2. Let $U_{\underline{\underline{tx}}}$ be the solution to the hyperfinite difference equations (4-1):

$$(5\text{-}2) \qquad \delta_{\underline{t}}\, U_{\underline{\underline{tx}}} = \underline{\Delta}\, U_{\underline{\underline{tx}}} + \left(\sqrt{U_{\underline{\underline{tx}}}} \wedge \frac{U_{\underline{\underline{tx}}}}{2}\,\frac{\Delta x}{\Delta t}\right) \frac{\xi_{\underline{\underline{tx}}}}{\sqrt{\Delta t \Delta x}} ,$$

and let X be the measure-valued process, whose internal density is U.

In section 4.6 we verified that the process $x_t = {}^\circ(X_{\underline{t}})$ was in fact the critical branching diffusion. In the case $d = 1$ however, we have the additional information that if U_0 is S-continuous and S-L^1, then $U_{\underline{\underline{tx}}}$ is S-continuous on $T \times X$ a.s. by Corollary 3.8.3. Furthermore $u = {}^\circ U$ satisfies (3-12), the weak form of an SPDE; to complete the identification of u as the solution to (5-1), we need only note that, since $\frac{\Delta t}{\Delta x} \approx 0$, then for $u \in \mathrm{ns}({}^*R)$

$${}^\circ\left(\sqrt{\underline{u}} \wedge \frac{\underline{u}}{2}\,\frac{\Delta x}{\Delta t}\right) = \sqrt{{}^\circ\underline{u}} .$$

On the other hand, starting from the SPDE (5-1) we note that the martingale problem is easily satisfied since for $\phi \in C_c^2(R)$

$$\int_0^t \int_R \sqrt{u_{sx}}\ \phi(x)\,dw_{sx}$$

is a martingale, whose increasing process is

$$\int_0^t \int_R u_{sx}\ \phi^2(x)\,ds\,dx$$

If we also assume that u_0 is L^1, then

$$\int_0^t \int u_{sx} \, dsdx < \infty \quad \text{a.s.}$$

and a dominated convergence argument ensures that the martingale problem is satisfied for $\phi \in C^2_{b,2}(\mathbb{R})$ (which coincides with $D(\Delta)$ in this case). Thus we have

Theorem 5.2.1: Let $u_0(x)$ be continuous and $L^1(\mathbb{R})$. Let w_{tx} be the white noise constructed in section 2.2 from the Loeb space of $\{-1, 1\}^{T\times X}$ onto $\mathbb{R}^+ \times \mathbb{R}$. There is a jointly continuous non-negative process u_{tx} such that

$$u_{0x} = u_0(x) ,$$

and for all $\phi \in C^2_{b,2}(\mathbb{R})$, and all $t \in \mathbb{R}^+$

$$\int_{\mathbb{R}} u_{tx}\phi(x)\,dx = \int_{\mathbb{R}} u_{0x}\,\phi(x)\,dx + \int_0^t \int_{\mathbb{R}} u_{sx} \frac{\partial^2\phi}{\partial x^2}(x)\,dxds \tag{5-3}$$

$$+ \int_0^t \int_{\mathbb{R}} \sqrt{u_{sx}}\,\phi(x)\,dw_{sx} .$$

Moreover any solution to (5-3) is the jointly continuous density of the unique measure-valued solution to the martingale problem described in Theorem 4.6.1.

Remark: The existence of a jointly continuous density holds for all realizations of the one-dimensional Dawson branching diffusion, not only the one constructed here. This follows from the fact that the existence of a jointly continuous density is a measureable property of the sample paths of this process (see Cutler (1985)).

CHAPTER SIX

The Support of the Fleming-Viot Process

6.1 Introduction and Construction

As mentioned in Chapter One, the Fleming-Viot process is the limiting case of a model used in theoretical genetics: the Ohta-Kimura model for d quantitative characters. Briefly, in this model, the total number of individuals is conserved, and the dynamics involve two processes, genetic "drift", and mutation. Mutation is modelled by a random walk on Z^d. "Drift" is modelled by replacing individuals at random by new individuals whose genetic type matches another individual chosen at random from the rest of the population. If we denote the types by points $k \in Z^d$, then we may denote by $p(t,k)$, the number of individuals of type k alive at time $t \in R^+$, divided by N, the total number of individuals (conserved), then $p(t,\cdot)$ forms a continuous-time countable state space Markov jump process with generator:

$$(6\text{-}1) \qquad Lf(p) = \sum_{i \neq j \in Z^d} [\gamma\, p(i)p(j) + D\, p(i)\, \theta_{ij}](f(p^{ij}) - f(p))$$

$$\text{where } p^{ij}(k) = \begin{cases} p(k) + \frac{1}{N}, & \text{if } k = j \\ p(k) - \frac{1}{N}, & \text{if } k = i \\ p(k), & \text{otherwise, and} \end{cases}$$

$$\theta_{ij} = \begin{cases} 1, & \text{if } |i-j| = 1 \\ -2d, & \text{if } i = j \\ 0, & \text{otherwise, and} \end{cases}$$

γ and D are positive constants describing the rates of "drift" and mutation respectively.

Each of the N individuals takes one step ("mutates") in one of the $2d$ possible directions at Poisson times whose rate is D/N. Similarily each individual dies and is replaced by another whose type coincides with that of a given other of the $N-1$ individuals (i.e. an "offspring" of that individual) according to a Poisson process with rate γ/N^2.

To construct the Fleming-Viot process, we re-scale time and space by $1/N^2$ and $1/N^{1/2}$ respectively: for $A \in \mathcal{B}(R^d)$ let

$$(6\text{-}2) \qquad X_t^A = \sum_{\frac{k}{\sqrt{N}} \in A} p(N^2 t, k)$$

$X_t^{\cdot}$ takes values in $M_1(R^d)$, the space of probability measures on R^d.

The result of Fleming and Viot (1979) is that in the limit as $N \to \infty$, the measure-valued process defined by (6-2) converges in the space of $M_1(R^d)$-valued processes provided the initial measure converges in $M_1(R^d)$. We refer to Fleming and Viot (1979) and Dawson and Hochberg (1982) for more details on this.

However, a non-standard construction of the Fleming-Viot process is immediate from (6-2). We simply take N infinite and let the state space be ${}^*Z^d$. Stated in non-standard language the result of Fleming and Viot is

<u>Theorem 6.1.1.</u> For N infinite, and $t \in n.s.({}^*R^+)$ (6-2) defines a.s. a process with S-continuous paths in $ns({}^*M_1(R^d))$, if initially all but an infinitesimal fraction of the N particles are on nearstandard points in ${}^*Z^d/\sqrt{N}$.

Note that this result asserts that although the parameter space in (6-2) is technically ${}^*R^+$, that $X_t^{\cdot} \approx X_r^{\cdot}$ in the weak topology whenever $t \approx r \in st^{-1}(R^+)$, a.s. Hence X has a standard part, x, which is a measure-valued Strong Markov process called the Fleming-Viot Process.

6.2 The Dimension of a Putative Support Set

We use here the terminology of Dawson and Hochberg (1982) and some of their results. What we aim to show in this chapter is that a non-standard construction makes much of their work more natural (and a good deal easier!) as well as extending their results. The main result (Theorem 6.4.4) of this chapter asserts that the dimension of the support of the Fleming-Viot process is at most 2 for all times simultaneously, a.s. (Dawson and Hochberg were able to show this only for fixed times). In this section we derive the dimension of a set, that in section 6.4 will be shown to be the supporting set.

Consider a particle (or individual) alive at some time t. As the process evolves and the particle wanders, at some time $r > t$ this particle may disappear (to be replaced by another particle somewhere else), or else it may serve as the 'type-model' for the replacement of some other particle which disappears at time r. In this latter case we say that both particles at time r are 'descendants' of the original particle at time t. For ease of terminology we will say that the particle at any time $s > t$, up until the time of disappearance of that particle, is the descendant of that particle at time t. Note that ancestry is a transitive relation. Furthermore every particle at time t has a unique ancestor at any time $s < t$; if we follow the paths of particles backward in time, they may converge, but they will never split. Two particles at a time $r > t$ are said to have a common ancestor at time t, if they are both descendants of a given particle at time t.

We will construct a supporting set for the mass of N particles at any time, by looking for a small (finite) set of ancestors, at an earlier time whose descendants comprise all of the N particles at the later time.

Consider any finite time interval $[0, T]$. Let $\varepsilon > 0$, and let $\{a_n\}$ and $\{\Delta_n\}$ be strictly decreasing sequences of positive numbers such that $\Delta_n e^{a_n^2/\Delta_n} \to \infty$ and

$a_n^{2+\varepsilon}/\Delta_n \to 0$ as $n \to \infty$. (A convenient choice would be $a_n = \Delta_n^{(4+\varepsilon)/(4+2\varepsilon)}$.)

We suppose w.l.o.g. that $\{T/\Delta_n\}$ are all integers so that for each n, the (finite) sets $\{t_{n,k} | t_{n,k} = k\,\Delta_n,\ 0 \le k \le T/\Delta_n\}$ form a partition of $[0, T]$. We let $N_{n,k}$ be the number of ancestors at time $t_{n,k-1}$ of the N particles alive at time $t_{n,k}$. Let $A_{n,k}$ be the union of balls of radius a_n, centered at each of the $N_{n,k}$ ancestors at time $t_{n,k-1}$ of the system of N particles at time $t_{n,k}$. Let $\hat{A}_{n,k}$ be the union of smaller balls of radius $\frac{a_n}{2}$, centered at the same points. (Technically all of the above are non-standard (internal) objects, but since they are all near-standard I will make the distinction between standard and internal only when necessary.)

Let $k_n(t) = [t/\Delta_n]$ identify in which interval $[t_{n,k}, t_{n,k+1})$, t lies. Then let

$$\Lambda_t = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \operatorname{st}(A_{n,k_n(t)}) \tag{6-3}$$

which is a (standard) Borel set for each standard t in $[0, T]$. This will turn out to be our supporting set. The dimension of this set depends on the number of ancestors at time $t_{n,k}$, of the N particles at the times $t_{n,k}$ (and hence of all the particles alive between times $t_{n,k}$ and $t_{n,k}$).

Dawson and Hochberg treated this problem for fixed times by constructing an infinite particle system to describe the Fleming-Viot process at that particular time. Using our hyperfinite model, we can use the same system of N particles at all times. Dawson and Hochberg (1982; (6.23)) showed that the distribution of the time T^N_m taken (in reverse time) to reduce the number of ancestors of N particles to m only, had Laplace transform

$$\text{(6-4)} \qquad \underline{E}(e^{-sT^N_m}) = \Pi^N_{k=m+1} \left(1 + \frac{s}{\gamma k(k-1)}\right)^{-1} .$$

(Their argument applies *verbatim* to our hyperfinite scheme.) We may use (6-4) to estimate $P(N_{n,k} > \frac{c}{\Delta_n}) = P({}^\circ T^N_{c/\Delta_n} > {}^\circ\Delta_n)$ for integral $\frac{c}{\Delta_n}$. From (6-3)

$$\text{(6-5)} \qquad \underline{E}(T^N_{c/\Delta_n}) = \sum_{k=\frac{c}{\Delta_n}+1} \frac{1}{\gamma k(k+1)} = \frac{1}{\frac{c}{\Delta_n}} - \frac{1}{\gamma N} \approx \frac{\Delta_n}{\gamma c} \text{, and}$$

$$\mathrm{Var}(T^N_{c/\Delta_n}) = \sum^N_{k=\frac{c}{\Delta_n}+1} \frac{1}{[\gamma k(k-1)]^2}$$

$$\leq \sum_{k=\frac{c}{\Delta_n}+1} \frac{1}{3\gamma^2}\left[\frac{1}{(k-1)^3} - \frac{1}{k^3}\right]$$

$$= \frac{1}{\gamma^2}\left[\frac{1}{(\frac{c}{\Delta_n})^3} - \frac{1}{N^3}\right] \approx \frac{\Delta_n^3}{3\gamma^2 c^3}$$

Now by Chebychev's Inequality, $P\left({}^\circ T^N_{c/\Delta_n} > \frac{{}^\circ\Delta_n}{\gamma c} + h\,\frac{{}^\circ\Delta_n^{3/2}}{\sqrt{3}\gamma c^{3/2}}\right) < \frac{1}{h^2}$,

hence, taking $h = {}^\circ\sqrt{3}c(\gamma c-1)/\sqrt{\Delta_n}$,

$$\text{(6-6)} \qquad P(N_{n,k} > \frac{c}{\Delta_n}) < \frac{\Delta_n}{3c(\gamma c-1)^2} \qquad (\text{for } c > \gamma^{-1}) .$$

Let $\delta > 0$. Let $\mu_\varepsilon(A)$ denote the Hausdorff $x^{2+\varepsilon}$ measure of a set A . Then

$$P(\max_{k\le T/\Delta_n} N_{n,k} > \delta/a_n^{2+\varepsilon})$$

$$\le \frac{T}{\Delta_n} P(N_{n,1} > \delta/a_n^{2+\varepsilon})$$

$$\le \frac{T}{\Delta_n} \frac{\Delta_n}{3 \frac{\delta\Delta_n}{a_n^{2+\varepsilon}} (\gamma \frac{\delta\Delta_n}{a_n^{2+\varepsilon}} - 1)^2} \qquad \text{(from (6-6))}$$

$$(6\text{-}7) \qquad = \frac{T}{3\ \delta \frac{\Delta_n}{a_n^{2+\varepsilon}} (\gamma\delta \frac{\Delta_n}{a_n^{2+\varepsilon}} - 1)^2}$$

$$\to 0 \quad \text{as} \quad n \to \infty \quad \text{since} \quad \frac{\Delta_n}{a_n^{2+\varepsilon}} \to \infty .$$

Hence $P(\max_{t\in[0,T]} a_n^{2+\varepsilon} N_{n,k_n(t)} > \delta) \to 0$, and by taking a subsequence, if necessary, we can ensure

$$(6\text{-}8) \qquad \max_{t\in[0,T]} a_n^{2+\varepsilon} N_{n,k_n(t)} \to 0 \quad \text{a.s.}$$

Now for each m , $A_{m,k_m(t)}$ is a covering of $\bigcap_{n=m}^{\infty} A_{n,k_n(t)}$ by $N_{m,k_m(t)}$ balls of radius a_m . Hence (6-8) ensures that

$$\mu_\varepsilon(\bigcap_m^{\infty} A_{n,k_n(t)}) = 0 \quad \text{uniformly in} \quad t , \text{ a.s.}$$

and hence $\mu_\varepsilon(\bigcup_{m=1}^{\infty} \bigcap_m^{\infty} A_{n,k_n(t)}) = 0$, for all $t \in [0, T]$ a.s. This is true for any $\varepsilon > 0$.

<u>Lemma 6.2.1</u>: The random set-valued frunction Λ_t defined by (6-3) has Hausdorff dimension at most 2 for all time, a.s.

6.3 A Useful Stochastic Differential Equation

We study the numbers of descendants of a specified group of particles. At any time, we stop the Fleming-Viot process, designate n particles out of the N, and restart this (Markov) process. For this section w.l.o.g. suppose the process is re-started at time 0. Let Y_t denote the mass associated with the designated n particles at time 0, and with their descendants at times $t > 0$. $Y_t = \frac{1}{N}$ (# of particles alive at time t which are descended from the original n). Recall that we are including particles in the original n that have remained unchanged until time t, in this count.

Now Y_t changes only when

i) one particle out of the NY_t disappears, and is replaced by a particle of a type-model not included in the NY_t, or

ii) one particle out of the $N-NY_t$ excluded particles disappears and is replaced according to the type of one of the NY_t descendant particles.

Disappearances and replacements happening entirely within the context of the NY_t designated particles or amongst the $N-NY_t$ excluded particles, make no change to Y_t.

Now the replacement of any given particle according to the type model of any other given particle happens according to a Poisson process with rate γ (see (6-1)).

There are $NY_t \cdot (N-NY_t)$ possible ways for an event of type i) to occur, each happening according to a Poisson process with rate γ, and each causing a decrease of size $\frac{1}{N}$ in Y_t. There are $(N-NY_t) \cdot NY_t$ possible ways for an event of type ii) to occur, each with the effect of increasing Y_t by $\frac{1}{N}$, and occurring at the same rate γ as an event of type i). Thus Y_t is an (internal) martingale. The associated predictable increasing process is easy to compute, since the change to Y_t at any time is the sum of $2NY_t(N-NY_t)$ independent Poisson processes, each of rate γ, whose predictable quadratic variations are each $\frac{t}{N^2}$. Thus

$$\langle Y\rangle_t = \int_0^t 2\gamma\, Y_t(1-Y_t)dt$$

(6-9)

$$d\langle Y\rangle_t = 2\gamma\, Y_t(1-Y_t)dt$$

Hence by Hoover and Perkins (1983)[1] we may conclude that the paths of Y are a.s. nearstandard in the space ${}^*C((0,\infty);\, R)$. Let $y = {}^\circ Y$. Then from (6-9) $d\langle y\rangle_t = 2\gamma\, y_t(1-y_t)dt$ and hence $b_t' = \int_0^t \frac{1}{\sqrt{2\gamma\, y_t(1-y_t)}}\, dy_t$ is a standard Brownian motion (up till the time of extinction of y). Thus by enlarging our probability space we may find a Brownian motion b_t such that

$$dy_t = \sqrt{2\gamma\, y_t(1-y_t)}\; db_t\,. \tag{6-10}$$

Furthermore we have the following lemma.

<u>Lemma 6.3.1</u>: Suppose $y_0 = \varepsilon$ where $0 < \varepsilon < \frac{1}{2}$. Then there are finite

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[1]Note. Hoover and Perkins (1983) Theorem 8.5 refers to *-discrete time processes X, such that if Δx is the change in X over an infinitesimal time step Δt, then $\sup {}^\circ|\Delta x| = 0$ a.s. By looking at our process Y only at intervals of $\Delta t = 1/N^2$, we may ensure $\sup {}^\circ|\Delta Y| = 0$ a.s. and bring Y into the framework of the theorem quoted.

constants K_1, K_2 independent of ε, such that

$$P(\exists s \in [0,t] \text{ such that } y_s \le \tfrac{3}{4}\varepsilon) \le K_1 \exp\left[-\frac{\varepsilon}{K_2 t}\right].$$

Proof: Write y_t as the time change of a Brownian motion b_t. As long as y_t stays in the range $[\frac{3}{4}\varepsilon, \frac{5}{4}\varepsilon]$ the derivative of the time change must be at least $\dfrac{1}{\frac{5}{2}\gamma\varepsilon(1-\frac{3}{4}\varepsilon)}$. Now use the estimate

$$P(\exists s \in [0,t] \text{ such that } |b_s| \ge c) \le 2\,P(|b_t| \ge c)$$

$$= 4\,P(b_t \ge c) \le K_1\, e^{-\frac{c^2}{2t}} \qquad \square$$

6.4 Verification of Support

We now check that the random set Λ_t of section 6.2 does indeed support the measure x_t of Theorem 6.1.1 (the Fleming-Viot Process).

Lemma 6.4.1: Consider at time t any particle p. Let $s < t$, and trace the path of the (unique) ancestor of p at each time r, for $s \leq r < t$. Then this path is an infinitesimal random walk whose standard part is a d-dimensional Brownian motion of rate $2D$.

Proof: Between the appearance and ultimate disappearance of any given particle, it takes a step of size $\frac{1}{\sqrt{N}}$ from its current position to any one of the $2d$ neighbouring positions according to a Poisson process with rate DN. Replacement (which is bifurcation of an ancestor particle) occurs according to a Poisson process which is independent of the motion of the particle. If we imagine such a motion continuing indefinitely and call this process B_t, and its coordinates B_t^i, then clearly each B_t^i is an internal martingale, since steps to the right occur at the same rate as steps to the left. Now $\langle B^i\rangle_t = 2DNt(\frac{1}{\sqrt{N}})^2 = 2Dt$, since B^i is the sum of 2 independent Poisson processes, of rate $2DN$, and of amplitude $\frac{1}{\sqrt{N}}$. The motions B^i and B^j happen according to independent Poisson processes, hence $\langle B^i, B^j\rangle_t = 0$ if $i \neq j$. Thus by Hoover and Perkins (1983) (once again!)[2] B_t has a standard part b_t a.s., and this b_t satisfies the characterization of d-dimensional Brownian motion: $E(b_t | b_r \; 0 \leq r \leq s) = b_s$; $\langle b\rangle_t = 2DIt$. □

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[2] Again we must look at the process B_t at discrete intervals (of size $\Delta t = 1/N$) to put it in the framework of Theorem 8.5 of this paper.

Referring to 6.2 for the definitions of the sets $A_{n,k}$ and $\hat{A}_{n,k}$.

<u>Lemma 6.4.2</u>: $P(x_{\circ t_{n,k}}((^\circ\hat{A}_{n,k})^c) > \varepsilon) \leq \frac{K_1}{\varepsilon} \, {}^\circ e^{-a_n^2/K_2\Delta_n}$ for finite constants K_1, K_2.

<u>Proof</u>: Consider the family of descendants of any one of the $N_{n,k}$ particles at time $t_{n,k-1}$ used in defining $\hat{A}_{n,k}$. If we trace back from time $t_{n,k}$ the movements of any one of the particles in this family, and its' progenitors, until we come to the position of the one ancestral particle at time $t_{n,k-1}$, we find a motion of the kind described in Lemma 6.4.1, whose standard part is a Brownian Motion. Hence the displacement of any given particle at time $t_{n,k}$ from its' ancestor at time $t_{n,k-1}$ is distributed $N(0, 2D\Delta_n I)$. For any particle p at time $t_{n,k}$,

$$E(I_{\hat{A}^c_{n,k}}(p)) \leq P(B_{D\Delta_n} \geq \frac{a_n}{2}) \qquad \leq K_1 \, e^{-a_n^2/K_2\Delta_n}$$

Hence

$$E(x_{\circ t_{n,k}}((^\circ\hat{A}_{n,k})^c)) \leq {}^\circ E(X_{t_{n,k}}(\hat{A}^c_{n,k})) = {}^\circ \frac{1}{N} \sum_{i=1}^{N} E I_{\hat{A}^c_{n,k}}(p_i) \leq {}^\circ K_1 \, e^{-a_n^2/K_2\Delta_n}$$

and

$$P(x_{\circ t_{n,k}}((^\circ\hat{A}_{n,k})^c) > \varepsilon) < \frac{K_1}{\varepsilon} \, {}^\circ e^{-a_n^2/K_2\Delta_n}.$$

<u>Lemma 6.4.3</u>: $P(x_{\circ t_{n,k+1}}((^\circ\hat{A}_{n,k})^c) > \varepsilon) < \frac{K_1}{\varepsilon} \, {}^\circ e^{-a_n^2/K_2\Delta_n}$

<u>Proof</u>: As above, with larger K_2, since displacements are distributed $N(0, 4D\,\Delta_n I)$.

Let $\varepsilon > 0$. From Lemmas 6.4.2 and 6.4.3 we may deduce

$$P\big(\max_{k \le \frac{T}{\Delta_n}} \{x_{t_{n,k}}({}^{\circ}\hat{A}^c_{n,k}) \vee x_{t_{n,k+1}}({}^{\circ}\hat{A}^c_{n,k})\} > \tfrac{\varepsilon}{2}\big)$$

$$\le \frac{K_1 T}{\varepsilon \Delta_n}\, e^{-a_n^2/K_2\Delta_n} \to 0 \qquad \text{as} \quad n \to \infty \ .$$

(as per usual, our trusty servants, the constants K_1 and K_2 are changing values when necessary).

Hence *a fortiori*

$$P\big(\max_{k \le \frac{T}{\Delta_n}} \{x_{t_{n,k}}({}^{\circ}A^c_{n,k}) \vee x_{t_{n,k+1}}({}^{\circ}A^c_{n,k})\} > \tfrac{\varepsilon}{2}\big) \to 0 \ .$$

Now consider the possibility that the sets ${}^{\circ}\hat{A}_{n,k}$ contain $1 - \frac{\varepsilon}{2}$ of the mass of the process x_t at times $t_{n,k}$ and $t_{n,k+1}$, but that more than ε of mass lies outside the sets ${}^{\circ}A_{n,k}$ at some time in between $t_{n,k}$ and $t_{n,k+1}$. Let $s_{n,k}$ be the first such time. $s_{n,k}$ is a stopping time for the Markov process x_t and we may consider it restarted at time $s_{n,k}$ from its configuration at $s_{n,k}$. One of the following must occur during the interval $(s_{n,k}, t_{n,k+1})$.

a) at least one fourth of the mass ε that lies outside $A_{n,k}$ initially travels a distance $\frac{a_n}{2}$ (to re-enter $\hat{A}_{n,k}$)

b) the mass ε that lies outside $A_{n,k}$ decreases by at least one fourth (that is the $\frac{\varepsilon}{4}N$ particles have at most $\frac{3}{4}\varepsilon N$ descendants at time $t_{n,k+1}$)

Consider case a).

For each particle p at time $t_{n,k+1}$, the displacement from its ancestor at time $s_{n,k}$ is distributed $N(0, D(t_{n,k+1} - s_{n,k})I)$. Recall from section 6.3 that the number of descendants at a future time of any given subset of particles, forms a martingale. Hence the expected mass at time $t_{n,k+1}$ of descendants of the $\varepsilon_n N$ particles at time $s_{n,k}$, which are outside balls of radius $\frac{a_n}{2}$ centered on their ancestors, is equal to the initial mass ε , times the probability that any one particle is outside such a ball, which product is bounded by $\varepsilon K_1 e^{-a_n^2/K_2\Delta_n}$ (as in Lemma 6.4.2). Hence using Chebychev's Inequality,

$$P\text{ (for a fixed } n \text{ and } k\text{, case a) occurs)} < K_1 e^{-a_n^2/K_2\Delta_n}$$

Hence $P(\text{for some } k < \frac{T}{\Delta_n}, s_{n,k} < t_{n,k+1} \text{ and case a) occurs}) < \frac{K_1 T}{\Delta_n} e^{-a_n^2/K_2\Delta_n}$

which goes to zero as $n \to \infty$.

Now by Lemma 6.3.1

$P(\text{for some } k < \frac{T}{\Delta_n}, s_{n,k} < t_{n,k+1} \text{ and case b) occurs})$

$$\leq \frac{T}{\Delta_n} P(\exists s \leq t_{n,k+1} - s_{n,k} < \Delta_n \text{ s.t. } y_s = \tfrac{3}{4}\varepsilon \mid y_0 = \varepsilon)$$

(where y is the process mentioned in 6.3.1)

$$\leq \frac{T}{\Delta_n} K_1 e^{-\varepsilon/K_2\Delta_n}$$

which also goes to zero.

Hence $P(\exists k < \frac{T}{\Delta_n}$ such that $x_{t_{n,k}}(\mathring{\hat{A}}^c_{n,k}) < \frac{\varepsilon}{2}$ and $x_{t_{n,k+1}}(\mathring{\hat{A}}^c_{n,k}) < \frac{\varepsilon}{2}$

and $\exists s \in (t_{n,k}, t_{n,k+1})$ such that $x_s(\mathring{A}^c_{n,k}) > \varepsilon) \to 0$.

Thus $\sup_{t\in[0,T]} x_t({}^\circ A^c_{n,k_n(t)}) \xrightarrow{P} 0$ as $n \to \infty$.

By taking a subsequence, if necessary, we may ensure

$\sup_{t\in[0,T]} x_t({}^\circ A^c_{n,k_n(t)}) \xrightarrow{a.s.} 0$. Again by taking a subsequence we may assert

$$\sum_{n=1}^{\infty} \sup_{t\in[0,T]} x_t({}^\circ A^c_{n,k_n(t)}) < \infty \text{ a.s.}$$

Then $\sup_{t\in[0,T]} x_t(\Lambda_t^c)$

$$\le \sup_{t\in[0,T]} x_t\Big(\bigcup_{n=m}^{\infty} {}^\circ A^c_{n,k_n(t)}\Big) \quad \text{for any } m \in N,$$

$$\le \sup_{t\in[0,T]} \sum_{n=m}^{\infty} x_t({}^\circ A^c_{n,k_n(t)})$$

$$\le \sum_{n=m}^{\infty} \sup_{t\in[0,T]} x_t({}^\circ A^c_{n,k_n(t)})$$

$$\to 0 \text{ as } m \to \infty .$$

Thus we have

<u>Theorem 6.4.4</u>: The random measures x_t, which are the realizations of the Fleming-Viot process, are supported for all times t on a set Λ_t of dimension at most 2, a.s.

APPENDIX A

Some Inequalities Used in Chapter 3

A.1 Purpose

In Appendix A.3, we derive some inequalities involving the coefficients $Q^{\underline{s}}_{\underline{x}}$ which were presented in Chapter 3.4. In section A.2 we derive some identities which provide a neat route to the inequalities in A.3.

A.2 Some Identities

As we saw in Lemma 3.3.1, $Q^{\underline{s}}_{\underline{x}-\underline{y}} = \underline{P}(B^{\underline{y}}_{\underline{s}-\Delta t} = \underline{x})$ where $B^{\underline{y}}$ is an infinitesimal random walk starting at $\underline{y}$ at time 0, and thereafter taking steps of size Δx to the right or left with probability α, at each time interval Δt, where $\alpha = \Delta t/\Delta x^2 \leq \frac{1}{3}$. The reason for not considering $\frac{1}{3} < \alpha \leq \frac{1}{2}$ will become clear later.

As we saw in Lemma 4.3.1 the above remarks are true also in d-dimensions, where the random walk takes steps in any of 2d directions, each with probability $\alpha = \Delta t/\Delta_1^2$. The identities in A.2 are true in d-dimensions although the d-dimensional versions will not be used

The first identity is trivial.

Lemma A.2.1. $$\sum_{\frac{\underline{x}}{\Delta x} \in {}^*Z} Q^{\underline{s}}_{\underline{x}} = 1 \text{ , if } \frac{\underline{s}}{\Delta t} \in {}^*N$$

Lemma A.2.2. $$\sum_{\frac{\underline{x}}{\Delta x} \in {}^*Z} (Q^{\underline{s}}_{\underline{x}})^2 = Q_0^{2\underline{s}-\Delta t} \text{ , if } \underline{s}/\Delta t \in {}^*N$$

Proof: $\sum_{\underline{x}} (Q^{\underline{s}}_{\underline{x}})^2 = (\sum_{\underline{x}} Q^{\underline{s}}_{\underline{x}})^2 - \sum_{\underline{x}} \sum_{\underline{y}\neq\underline{x}} Q^{\underline{s}}_{\underline{x}} Q^{\underline{s}}_{\underline{y}}$

$$= 1^2 - \sum_{\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x}) \sum_{\underline{y}\neq\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = \underline{y})$$

(by A.1)

$$= 1 - \sum_{\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x}) \cdot \underline{P}(B^0_{s-\Delta t} \neq \underline{x})$$

$$= 1 - \sum_{\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x}) \cdot \underline{P}(B^0_{\underline{s}-\Delta t} \neq -\underline{x})$$

(by symmetry; this step fails if we consider a reflecting random walk.)

$$= 1 - \sum_{\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x}) \ \underline{P}(B^0_{2\underline{s}-2\Delta t} \neq 0 \mid B^0_{s-\Delta t} = \underline{x})$$

$$= 1 - \underline{P}(B^0_{2\underline{s}-2\Delta t} \neq 0)$$

$$= Q_0^{2\underline{s}-\Delta t}$$

(by definition). □

Lemma A.2.3. If $\underline{z}/\Delta x \in {}^*Z$, and $\underline{s}/\Delta t \in {}^*N$

$$\sum_{\underline{x}/\Delta x\in {}^*Z} (Q^{\underline{s}}_{\underline{x}} - Q^{\underline{s}}_{\underline{x}+\underline{z}})^2 = 2Q_0^{2\underline{s}-\Delta t} - 2Q_{\underline{z}}^{2\underline{s}-\Delta t} .$$

Proof: $\sum_{\underline{x}/\Delta x\in {}^*Z} (Q^{\underline{s}}_{\underline{x}} - Q^{\underline{s}}_{\underline{x}+\underline{z}})^2$

$$= \sum_{\underline{x}} (Q^{\underline{s}}_{\underline{x}})^2 - 2 \sum_{\underline{x}} Q^{\underline{s}}_{\underline{x}} Q^{\underline{s}}_{\underline{x}+\underline{z}} + \sum_{\underline{x}} (Q^{\underline{s}}_{\underline{x}+\underline{z}})^2$$

$$= 2\, Q_0^{2\underline{s}-\Delta t} - 2 \sum_{\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x})\underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x} + \underline{z})$$

(by Lemma A.2.2)

$$= 2\, Q_0^{2\underline{s}-\Delta t} - 2 \sum_{\underline{x}} \underline{P}(B^0_{\underline{s}-\Delta t} = -\underline{x})P(B^0_{2\underline{s}-2\Delta t} = \underline{z} \mid B^0_{\underline{s}-\Delta t} = -\underline{x})$$

(by symmetry)

$$= 2\, Q_0^{2\underline{s}-\Delta t} - 2\, \underline{P}(B^0_{2\underline{s}-2\Delta t} = \underline{z})$$

$$= 2(Q_0^{2\underline{s}-\Delta t} - Q_{\underline{z}}^{2\underline{s}-\Delta t}) \quad . \qquad \square$$

Lemma A.2.4. If $\frac{\underline{r}}{\Delta t}, \frac{\underline{s}}{\Delta t} \in {}^*N$,

$$\sum_{\underline{x}/\Delta x \in {}^*Z} (Q_{\underline{x}}^{\underline{r}+\underline{s}} - Q_{\underline{x}}^{\underline{s}})^2 = Q_0^{2\underline{r}+2\underline{s}-\Delta t} + Q_0^{2\underline{s}-\Delta t} - 2\, Q_0^{\underline{r}+2\underline{s}-\Delta t}$$

Proof: $\displaystyle\sum_{\frac{\underline{x}}{\Delta x} \in {}^*Z^d} (Q_{\underline{x}}^{\underline{r}+\underline{s}} - Q_{\underline{x}}^{\underline{s}})^2$

$$= \sum_{\underline{x}} (Q_{\underline{x}}^{\underline{r}+\underline{s}})^2 - 2 \sum_{\underline{x}} \underline{P}(B^0_{\underline{r}+\underline{s}-\Delta t} = \underline{x})\underline{P}(B^0_{\underline{s}-\Delta t} = \underline{x}) + \sum_{\underline{x}} (Q_{\underline{x}}^{\underline{s}})^2$$

$$= Q_0^{2\underline{r}+2\underline{s}-\Delta t} - 2 \sum_{\underline{x}} \underline{P}(B^0_{\underline{r}+\underline{s}-\Delta t} = -\underline{x})\underline{P}(B^0_{\underline{r}+2\underline{s}-2\Delta t} = 0 \mid B^0_{\underline{r}+\underline{s}-\Delta t} = -\underline{x}) + Q_0^{2\underline{s}-\Delta t}$$

(by symmetry and A.2.2)

$$= Q^{2\underline{r}+2\underline{s}-\Delta t} - 2\, \underline{P}(B^0_{\underline{r}+2\underline{s}-2\Delta t} = 0) + Q_0^{2\underline{s}-\Delta t} \quad . \qquad \square$$

A.3 Some Inequalities

In this section I prove the four inequalities 3.4.1 through 3.4.4 of Chapter 3.

Lemma A.3.1. (3.4.1). There is a constant K such that

$$\sum_{\underline{x}\in X} \left(Q^{\underline{t}}_{\underline{x}}\right)^2 \leq K\sqrt{\Delta t/\underline{t}} \quad , \text{ if } \frac{\underline{t}}{\Delta t} \in {}^*N .$$

Proof: Clearly

$$\sum_{\underline{x}\in X} \left(Q^{\underline{t}}_{\underline{x}}\right)^2 = \sum_{\underline{x}/\Delta x\in {}^*Z} \left(Q^{\underline{t}}_{\underline{x}}\right)^2 = Q_0^{2\underline{t}-\Delta t} \quad \text{by A.2.2.}$$

By definition, $Q_0^{(k+1)\Delta t} = P(|S_k| \leq \frac{1}{2})$, where S_k is the sum of k I.I.D. random variables taking the values -1, 0, $+1$ with probabilities α , $1-2\alpha$, α respectively. $\mathrm{Var}(S_k) = 2k\alpha$.

Since $°\alpha > 0$, then by Corollary 2.2.3 of Bhattacharya and Rao (1976),

$$\left| Q_0^{(k+1)\Delta t} - \int_{-\frac{1}{2}\cdot\frac{1}{\sqrt{2k\alpha}}}^{\frac{1}{2}\cdot\frac{1}{\sqrt{2k\alpha}}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \right| \leq \frac{C}{\sqrt{k}} \quad \text{as } k \to \infty$$

Now $\displaystyle\int_{-\frac{1}{2}\cdot\frac{1}{\sqrt{2k\alpha}}}^{\frac{1}{2}\cdot\frac{1}{\sqrt{2k\alpha}}} e^{-z^2/2} dz$ is asymptotically $\dfrac{1}{2\sqrt{2\pi k\alpha}} + o\left(\dfrac{1}{\sqrt{k}}\right)$ as $k \to \infty$.

Thus there is a finite K such that

$$Q_0^{(k+1)\Delta t} \leq \frac{K}{\sqrt{k}} \quad k \in N .$$

By the transfer principle this must hold for all of ${}^*N$. Thus (checking $\underline{t} = \Delta t$ separately)

$$Q_0^{2\underline{t}-\Delta t} \leq K \frac{\Delta t}{2\underline{t}-2\Delta t} \leq K' \frac{\Delta t}{\underline{t}} . \qquad \square$$

<u>Lemma A.3.2 (3.4.2)</u>. There is a constant K such that for $\underline{t}/\Delta t \in {}^*N$

$$\sum_{0<\underline{s}\leq\underline{t}} \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{s}})^2 \leq K\sqrt{\underline{t}/\Delta t} .$$

<u>Proof</u>: Follows from A.3.1 and the fact that $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq C\sqrt{n}$ for all $n \in N$, hence for all $n \in {}^*N$. $\square$

<u>Lemma A.3.3 (3.4.3)</u>. There is a finite constant K such that if $\frac{\underline{z}}{\Delta x} \in {}^*Z$, and $\frac{\underline{t}}{\Delta t} \in {}^*N$, then $\sum_{0\leq\underline{s}<\underline{t}} \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{s}} - Q_{\underline{x}+\underline{z}}^{\underline{s}})^2 \leq K \frac{|\underline{z}|}{\Delta x}$.

<u>Proof</u>: Suppose w.l.o.g. that $\underline{z} \geq 0$. Let $B_{\underline{s}}$ be the infinitesimal random walk whose density is $Q_{\underline{x}}^{\underline{s}+\Delta t}$.

Let $J(\underline{t},\underline{x}) = \sum_{0\leq s<t} I(B_{\underline{s}} > \underline{x})(B_{\underline{s}+\Delta t} - B_{\underline{s}})$.

Let $\hat{L}(\underline{t},\underline{x}) = \sum_{0\leq\underline{s}<\underline{t}} I(B_{\underline{s}} = \underline{x})\Delta x$.

Let $L(\underline{t},\underline{x}) = \sum_{0\leq\underline{s}<\underline{t}} I(B_{\underline{s}} = \underline{x} ; B_{\underline{s}+\Delta t} = \underline{x} + \Delta x)\Delta x$

Then $\hat{L}$ is the true occupation density ("local time"), for our random walk with 'pauses', while L is the discrete analogue of Brownian Local Time.

Now $\underline{E}(L(\underline{t},\underline{x})) = \sum_{0\leq s<t} \underline{P}(B_{\underline{s}} = \underline{x} ; B_{\underline{s}+\Delta t} - B_{\underline{s}} = \Delta x)\Delta x$

$$= \sum_{0 \le s < t} P(B_s = x)\alpha \cdot \Delta x$$

$$= \alpha\, E(\hat{L}(t,x)) \; .$$

In Perkins (1982) Lemma 3.1, Tanaka's formula is established by internal induction on t :

$$(B_t - x)^+ = J(t,x) + L(t,x)$$

Thus

$$L(t,0) - L(t,x) = B_t^+ - (B_t - x)^+ - J(t,0) + J(t,x)$$

Now $J(t,0)$ and $J(t,x)$ are both internal martingales, hence

$$E(L(t,0) - L(t,x)) = E[B_t^+] - E[(B_t - x)^+]$$

(A.3.1)

$$\le |x|$$

Now since $0 < {}^\circ\alpha < \frac{1}{3}$, $Q_0^s = P(B_{s-\Delta t} = 0) \ge P(B_{s-\Delta t} = x) = Q_x^s$, as shown in Lemma 4.3.2. Now by A.2.3

$$\sum_{0<s\le t} \sum_{x \in X} (Q_x^s - Q_{x+z}^s)^2 \le 2 \sum_{0<s\le t} Q_0^{2s-\Delta t} - Q_z^{2s-\Delta t}$$

$$\le 2 \sum_{\Delta t < s \le 2t} (Q_0^{s-\Delta t} - Q_z^{s-\Delta t})$$

$$= 2 \sum_{0 \le s \le 2t-2\Delta t} P(B_s = 0) - P(B_s = z)$$

$$\le 2\, E(\hat{L}(2t-\Delta t,\, 0) - \hat{L}(2t-\Delta t,\, z))$$

$$= \frac{2}{\alpha}\, \underline{E}(L(2\underline{t}-\Delta t,0) - L(2\underline{t}-\Delta t,\underline{z})$$

$$\le \frac{2}{\alpha}\; \frac{\underline{z}}{\Delta x} \;, \text{ by (A.3.1)} \qquad \square$$

Lemma A.3.4 (3.4.4). There is a finite constant K such that if $\underline{r}/\Delta t < \underline{t}/\Delta t$ are in ${}^*N$,

$$\sum_{0 \le \underline{s} < \underline{r}} \; \sum_{\underline{x} \in X} \left(Q_{\underline{x}}^{\underline{t}-\underline{s}} - Q_{\underline{x}}^{\underline{r}-\underline{s}}\right)^2 \le K\sqrt{(\underline{t}-\underline{r})/\Delta t} \;.$$

Proof: The l.h.s. above is equal to

$$\sum_{0 < \underline{s} \le \underline{r}} \; \sum_{\underline{x} \in X} \left(Q_{\underline{x}}^{\underline{s}+(\underline{t}-\underline{r})} - Q_{\underline{x}}^{\underline{s}}\right)^2 \text{, which by A.2.4 is bounded by}$$

(A-2) $$\sum_{0 < \underline{s} \le \underline{r}} \left(Q_0^{2\underline{s}+2(\underline{t}-\underline{r})-\Delta t} + Q_0^{2\underline{s}-\Delta t} - 2\, Q_0^{(\underline{t}-\underline{r})+2\underline{s}-\Delta t}\right)$$

First, suppose $\underline{t}-\underline{r} \le \underline{r}$ and $(\underline{t}-\underline{r})/\Delta t$ is even. Then many of the terms in (A-2) cancel leaving

(A-3) $$\sum_{0 < \underline{s} \le \frac{\underline{t}-\underline{r}}{2}} Q_0^{2\underline{s}-\Delta t} - \sum_{\frac{\underline{t}-\underline{r}}{2} < \underline{s} < \underline{t}-\underline{r}} Q_0^{2\underline{s}-\Delta t}$$

$$\sum_{\underline{r} < \underline{s} < \underline{r}+\frac{\underline{t}-\underline{r}}{2}} Q_0^{2\underline{s}-\Delta t} + \sum_{\underline{r}+\frac{\underline{t}-\underline{r}}{2} < \underline{s} \le \underline{t}} Q_0^{2\underline{s}-\Delta t}$$

Since $\overset{\circ}{\alpha} \le \frac{1}{3}$, the coefficients $Q_0^{\underline{s}}$ are monotone decreasing as $\underline{s}$ increases. Hence the sum of the second, third, and fourth terms in (A-3) above is bounded above by 0 . The first term in (A-3) is bounded by $K\sqrt{(\underline{t}-\underline{r})/\Delta t}$ by A.3.2, as required.

Now if $\underline{t}-\underline{r} \leq \underline{r}$ and $\frac{\underline{t}-\underline{r}}{\Delta t}$ is odd the sum in (A-2) is bounded by $\sum_{0<\underline{s}\leq\underline{r}} [Q_0^{2\underline{s}+2(\underline{t}-\underline{r})-\Delta t} + Q_0^{2\underline{s}-\Delta t} - 2\, Q_0^{(\underline{t}-\underline{r})+2\underline{s}}]$. Applying the same cancellation argument, this is bounded by $\sum_{0<\underline{s}\leq\frac{\underline{t}-\underline{r}}{2}+\Delta t} Q_0^{2\underline{s}-\Delta t} \leq K\sqrt{\underline{t}-\underline{r}/\Delta t}$.

Now if $\underline{t}-\underline{r} > \underline{r}$ (i.e. $\underline{r} < \underline{t}/2$) , then

$$\sum_{0<\underline{s}\leq\underline{r}} \sum_{\underline{x}\in X} (Q_{\underline{x}}^{\underline{s}+(\underline{t}-\underline{r})} - Q_{\underline{x}}^{\underline{s}})^2$$

$$\leq \sum_{0<\underline{s}\leq\underline{r}} \sum_{\underline{x}\in X} [Q_{\underline{x}}^{\underline{s}+(\underline{t}-\underline{r})}]^2 + [Q_{\underline{x}}^{\underline{s}}]^2$$

$$= \sum_{0<\underline{s}\leq\underline{r}} Q_0^{2\underline{s}+2(\underline{t}-\underline{r})-\Delta t} + Q_0^{2\underline{s}-\Delta t}$$

(by A.2.2)

$$\leq 2 \sum_{0<\underline{s}\leq\underline{r}} Q_0^{2\underline{s}-\Delta t}$$

$$\leq K \frac{\underline{r}}{\Delta t} \quad \text{(by A.3.2)}$$

$$\leq K \frac{\underline{t}-\underline{r}}{\Delta t} .$$

□

Remark: If $\alpha = \frac{1}{2}$ then Lemmas A.3.3 and A.3.4 are false. The pattern of non-zero coefficients in the array $\{Q_{\underline{x}}^{\underline{s}}\}$, $\underline{s}/\Delta t \in {}^*N$, $\underline{x}/\Delta x \in {}^*Z$, is a checkerboard pattern. Thus $U_{\underline{t}\underline{x}}$ is independent of $U_{\underline{t}-\Delta t,\underline{x}}$ and of any $U_{\underline{s}\underline{y}}$ for which $\frac{\underline{t}-\underline{s}}{\Delta t} + \frac{\underline{x}-\underline{y}}{\Delta x}$ is an odd hyper-integer. Thus the moment inequalities on spatial and temporal differences will fail. This is the reason for not using the simplest finite difference scheme in Chapter 3.

APPENDIX B

Internal Solutions to SPDEs in Higher Dimensions

The success of the hyperfinite difference equation approach in Chapters 3 and 4 to the existence of solutions to SPDEs in one dimension, and to the Dawson measure diffusion in higher dimensions, leads one to wonder if the use of hyperfinite difference equations might lead to a general theory of SPDEs in higher dimensions.

So far at least, this hope has not borne fruit. The kind of equations that we would be led to consider after the analogy of those in Chapters 3 and 4, would be of the form

$$\text{(B-1)} \qquad \frac{\partial u}{\partial t} = \Delta u + f(u)\,dW_{tx} \ , \ t \in R^{+} \ , \ x \in R^{d}$$

and the corresponding internal equation

$$\text{(B-2)} \qquad (\delta_{\underline{t}} U_{\cdot \underline{x}})_{\underline{t}} = (\underline{\Delta}\, U_{\underline{t}\cdot})_{\underline{x}} + F(U_{\underline{tx}})\ \xi_{\underline{tx}} \,/\, \Delta t \Delta x^{d} \ , \quad \text{for}$$

$\underline{t} \in \mathcal{T}$ and $\underline{x} \in \mathcal{X}$, hyperfinite grids representing R^{+} and R^{d} respectively.

Now it is easy to see that an internal solution to (B-2) exists, by the usual inductive construction. What is not so clear is whether or not this internal solution U has a non-trivial standard part u, presumeably in some space of distributions. It is also not clear what it would mean for such a distribution-valued process u to be a solution of (B-1), in general, since non-linear operations on point values of distributions are undefined or discontinuous at best. It seems possible to make sense of (B-1) if $f(u)$ is an operator valued function of u with values in a class of operators on a space of distributions, but to pursue this possibility would take us too far afield from the ideas of this thesis.

If we restrict ourselves to the case where f is a real valued function of a real variable, and that F is some natural lifting of f, such as $^*f$, then we may still ask whether or not the internal solution to (B-2) has an interesting standard part.

Sadly, in several cases the answer seems to be 'no'.

The first case we considered is when f is a continuous function of compact support, (and F is an S-continuous lifting of compact support). Then since $\Delta t << \Delta x^d$, changes to the internal solution U_{tx} to (B-2) are infinitesimal at each step and an easy induction argument shows that the internal solution $U_{\underline{tx}}$ is bounded by a finite constant. If we seek to estimate the variance of $\sum_{\underline{x}\in A} U_{\underline{tx}} \Delta x^d$ for some finite internal rectangle (or other set) A, then we are naturally led to examine $\underline{E}(F^2(U_{\underline{sy}}))$. We may set up a difference equation for this quantity, and show that it is everywhere a.s. infinitesimal.

Thus the variance of the integral over any finite region of $U_{\underline{sy}}$, is infinitesimal. Thus though the values taken by U lie almost always at either end of the (connected) support of F, these values balance very precisely on each monad. The balance point is infinitesimally close to the value of the deterministic solution to the heat equation with the given initial condition, at the standard point corresponding to the monad in question.

The other cases which we examined were when $F(U) = U^p$, when either $0 < p < \frac{1}{2}$, or $\frac{1}{2} < p \leq 1$. We may try to follow the development of section 4.4 to find the total mass. In the case $0 < p < \frac{1}{2}$, we find that $\underline{E}(M_{\underline{t}}-M_0)^{2q}$ is finite, at least in the case of reflecting boundary conditions on a finite rectangle, simply by bounding U^{2p} by $1 + U$. However, a closer examination of the difference equations (B-2), will show that, the values $U_{\underline{tx}}$ are almost always infinite or infinitesimal. Thus the quantity $\sum_{\underline{x}\in X} U^{2p}_{\underline{tx}}$ is actually infinitesimally smaller than $\sum_{\underline{x}\in X} U_{tx}$, since almost all the mass comes from points $\underline{x}$ where $U_{\underline{tx}}$ is infinite. Thus we end up with, in fact $\underline{E}|M_{\underline{t}} - M_0|^{2q} << \underline{E}(\sum_{0\leq s<\underline{t}} M^q_{\underline{s}})$, and thus the total mass is infinitesimally close to M_0 a.s. If we examine the predictable increasing process associated with $\sum_{\underline{x}\in X}(U_{\underline{tx}} - U_{0\underline{x}})\Phi(\underline{x}) - \sum_{0\leq s<\underline{t}} X^{\Delta\Phi}_{\underline{s}}\Delta t$, as in section 4.5 we find it to be $\sum_{0\leq \underline{s}<\underline{t}}\sum_{\underline{x}\in X} U^{2p}_{\underline{sx}}\Phi^2(\underline{x})\Delta x^d$, which is likewise infinitesimally close to 0. Thus the process we get from solving (B-2) for $F(U) = U^p$, $0 < p < \frac{1}{2}$ turns out to have a deterministic standard part.

If $F(U) = U^p$, $\frac{1}{2} < p \leq 1$, then, again $U_{\underline{tx}}$ must almost always be infinite or infinitesimal. If we obtain an estimate of the variance of the total mass M_t, we find $\underline{E}(M_{\underline{t}} - M_0)^2 = \sum_{0\leq \underline{s}<\underline{t}}\sum_{\underline{x}\in X} U^{2p}_{\underline{sx}}\Delta t\Delta x^d$.

Whenever U is infinite $U << U^{2p}$, and hence $\sum_{\underline{x}\in X} U^{2p}_{\underline{sx}}\Delta x^d >> \sum_{\underline{x}\in X} U_{\underline{sx}}\Delta x^d = M_{\underline{s}}$. The moment bounds on $M_{\underline{t}}$ cannot be obtained. An examination of the predictable increasing process for $X^{\Phi}_{\underline{t}} - X^{\Phi}_0 - \sum_{0\leq \underline{s}<\underline{t}} X^{\Delta\Phi}_{\underline{s}}\Delta t$ as in section 4.5, shows that this is a.s. infinite, and thus the quadratic variation is simply too large for U to be nearstandard as a measure-valued stochastic process.

The only cases where the formulation (B-1) appears to yield anything are if a) F is essentially like $F(\underline{x}) = C$, in which case we recover the linear theory of SPDEs in higher dimensions which was originally developed by Walsh (1986), or b) $F(x) = c\sqrt{\underline{x}}$ essentially, which is treated in Chapter 4.

REFERENCES

R. M. Anderson, "A Non-Standard Representation for Brownian Motion and Itô Integration", Israel J. Math., 25, 15-46 (1976).

K. B. Athreya and P. E. Ney, Branching Processes, Springer Verlag, Heidelberg, 1972.

R. N. Bhattacharya and R. R. Rao, Normal Approximations and Asymptotic Expansions. John Wiley & Sons, New York, 1976.

D. L. Burkholder, "Distribution Function Inequalities for Martingales", Ann. Prob., 1, 19-42 (1973).

C. Cutler, "Some measure-theoretic and topological results for measure-valued and set-valued processes", Technical Report #49, Laboratory for Research in Statistics and Probability, Carleton U., (1985).

N. J. Cutland, "Nonstandard Measure Theory and its Applications", Bull. London Math. Soc., 15, 529-589, (1983).

D. A. Dawson, "Stochastic Evolution Equations". Math. Biosciences 15, 287-316, (1972).

D. A. Dawson, "Stochastic Evolution Equations and Related Measure Processes". J. Multiv. Anal., 3, 1-52, (1975).

D. A. Dawson and K. J. Hochberg, "The Carrying Dimension of a Stochastic Measure Diffusion", Ann. Prob., 7, 693-703, (1979).

D. A. Dawson and K. J. Hochberg, "Wandering Random Measures in the Fleming-Viot model", Ann. Prob., 10, 554-580, (1982).

D. A. Dawson, and T. G. Kurtz. "Application of Duality to Measure-valued Processes", pp. 91-105 in Lecture Notes in Control and In. Sci. 42 Information Sciences, ed. W. Fleming and L. C. Gorostiza, Springer-Verlag (1982).

D. A. Dawson, "Measure-Valued Processes: Construction, Qualitative Behaviour, and Stochastic Geometry", Technical Report #53, Laboratory for Research in Statistics and Probability, Carleton U., (1985).

W. H. Fleming, and M. Viot, "Some Measure-Valued Markov Processes in Population Genetics Theory", Indiana Univ. Math. J., 28, 817-843 (1979).

R. A. Holley, and D. W. Stroock, "Generalized Ornstein-Uhlenbeck Processes and Infinite Particle Branching Brownian Motions", Publ. R.I.M.S., Kyoto Univ., 14, 741-788 (1978).

D. N. Hoover, and E. Perkins, "Non-Standard Construction of the Stochastic Integral and Applications to Stochastic Differential Equations", parts I and II, Trans. Amer. Math. Soc. 275, pp. 1-58, (1983).

H. J. Keisler, "An Infinitesimal Approach to Stochastic Analysis", Memoirs of the A.M.S., 297, (1984).

E. Perkins, "Weak Invariance Principles for Local Time", Z. Wahr. Verw. Geb., 60, 437-451, (1982).

S. Roelly-Coppoletta, "A Criterion of Convergence of Measure-Valued Processes: Application to Measure Branching Processes", Stochastics, 17, 43-65, (1986).

A. Stoll, "A Non-Standard Construction of Levy Brownian Motion", Technical Report, Universität Bochum (1984).

A. Stoll, "A Nonstandard Construction of Levy Brownian Motion", Probab. Th. Rel. Fields, 71, 321-334 (1986).

J. B. Walsh, "A Stochastic Model of Neural Response", Adv. Appl. Prob., 13, 231-281, (1981).

J. B. Walsh, "Regularity Properties of a Stochastic Partial Differential Equation", pp. 257-290 in Seminar on Stochastic Process, Birkhäuser, Boston, (1984).

J. B. Walsh, "Martingales with a multidimensional parameter and stochastic integrals in the plane", in Ecole d'Eté de St. Flour, Lecture Notes in Math. v. 1180, Springer-Verlag (1986).

S. Watanabe, "A Limit Theory of Branching Processes and Continuous State Branching Processes", J. Math. Kyoto University, 8, 141-167, (1968).

M. Jirina, "Stochastic Branching Processes with Continuous State Space", Czechoslovak Math. J., 8, (1958).