CONSTRUCTION OF STRONG MARKOV PROCESSES THROUGH EXCURSIONS,
AND A RELATED MARTIN BOUNDARY

By

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ABSTRACT

For certain Markov processes, K. Ito has defined the Poisson point process of excursions away from a fixed point. The law of this process is determined by a certain measure, called its Characteristic measure. He gives a list of conditions this measure must obey. I add to these conditions, obtaining necessary and sufficient conditions for a measure to arise in this way. The main technique is to use a 'last exit decomposition' related to those of Getoor and Sharpe. The more general problem of excursions away from a fixed set is treated using the Exit system of B. Maisonneuve.

This gives a useful technique for constructing new Markov processes from old ones. For example, we obtain a rigorous construction of the Skew Brownian motion of Ito and McKean, and another proof of results of Pittenger and Knight on excision of excursions.

A related question is that of determining whether an entrance point for a Markov process remains an entrance point for an h-transform of that process. Let E be an open subset of Euclidean space, with a Green function, and let λ be harmonic measure on the Martin boundary Δ of E. I show that, except for a λ\(\cap\)λ null set of \((x,y)\in\Delta^2\), x is an entrance point for Brownian motion conditioned to leave E at y. R.S. Martin gave examples in dimension 3 or higher, for which there exist minimal accessible Martin boundary points x\#y for which this condition fails. I give a similar example in dimension 2. The argument uses recent results of M. Cranston and T. McConnell, together with Schwarz-Christoffel transformations.
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PART 1. EXCURSIONS

1. Introduction

In the theory of Markov chains, it is easily shown that the 'excursions' away from a fixed state are IID random variables, with values in path space. This fact is used extensively in the analysis of recurrence, and similar questions.

In his classic paper, "Poisson Point Processes attached to Markov Processes" \[^{[32]}\], K. Itô obtains the corresponding property for the excursions of a continuous time standard process, away from a recurrent point \( a \). He constructs a point process of excursions, and shows that it is a Poisson point process (PPP). This decomposition, of the original process into its excursions, has proved useful in analysing the original standard process. For example, J.B. Walsh has in \[^{[51]}\] given a simple proof of the Ray-Knight theorems on the Markov property of local time, using the PPP of excursions of Brownian Motion away from 0. A different application lies in the results of P. Greenwood and E. Perkins \[^{[28]}\] who analyse domain of attraction problems for random walk, using yet another type of excursion process.

We will be examining a converse problem; that of constructing a Strong Markov process from its excursions. In \[^{[32]}\], Itô shows that the characteristic measure of the excursion process he constructs, satisfies certain conditions, and then indicates how to reverse his construction provided those conditions hold. That is, given a PPP whose characteristic measure obeys these conditions, it should be possible to construct a process whose excursion process is that PPP. Itô doesn't state this converse result as a formal theorem, and hence doesn't write out a proof.
Our first goal will be to make his argument rigorous. This is perhaps more interesting in that the converse is not true as stated above. In section 4, examples will be given showing that unless we strengthen two of his conditions, the process constructed from the PPP may fail to be strong Markov or right continuous. This strengthening gives us necessary and sufficient conditions for the process obtained to be a right process.

The argument for sufficiency is presented in section 3, and that for necessity in section 5. Section 6 will examine the Ray property and 'hypothèses droites' of the constructed process. Section 2 will give notation and a statement of the results. Section 7 contains a variation on the main lemma used in the proof of sufficiency.

Section 9 will treat a generalization of Ito's result and of our converse, to the situation of excursions away from a set (rather than a point). This is done using the Exit System of B. Maisonneuve (generalizing the excursion measure of Ito's PPP). Section 8 contains some technical facts about the excursion process in this new situation. Section 10 gives applications to skew Brownian motion, and to the results of F. Knight and O. Pittenger on excision of excursions.

2. Notation and Results

E will be a separable metric space, and $E^0$ the σ-field of its Borel subsets. $E$ will be the universal completion of $E^0$. $U$ will be the set of right-continuous $E$-valued paths, and, for $a \in E$, $U^a$ will be the set of all such paths $u$,
satisfying \( u(0) = a \). \((W_t)_{t \geq 0}\) will be the coordinate process on \( U \); 
\( W_t(u) = u(t) \). For \( t \geq 0 \), \( U_t^\circ \) will be the smallest \( \sigma \)-field such that 
for every \( s \in [0, t] \), \( W_s \) is a measurable function from \((U, U_t^\circ)\) to \((E, E^0)\). 
\( U_t \) will be the universal completion of \( U_t^\circ \), and \( U \) that of \( U^\circ \).

\((\Pi, P)\) will be the measurable space of \((U, U)\)-valued point functions. That is, we adjoin a point \( \delta \) to \( U \), and let \( \Pi \) be the set of functions \( p : [0, \omega) \to U \cup \{\delta\} \) such that \( p(t) = \delta \) except for countably many \( t \). \( P \) is the \( \sigma \)-field on \( \Pi \) generated by the functions \( p \mapsto N(A, p) \), where \( N(A, p) \) is the number of times \( t \) such that 
\((t, p(t)) \in A \subseteq [0, \omega) \times U \). Here \( A \) belongs to the product \( B \otimes U \) of the Borel field \( B \) on \([0, \omega)\), and of \( U \). \((y_t)_{t \geq 0}\) will be the coordinate process on \( \Pi \); \( y_t(p) = p(t) \).

For \( A \in B \otimes U \), we define the restriction of \( p \in \Pi \) to \( A \) to be:

\[
\left. p \right|_A (t) = \begin{cases} 
p(t), & \text{if } (t, p(t)) \in A \\
\delta, & \text{otherwise}
\end{cases}
\]

Special cases of this will be the killing operators:

\[
a_s (p) = \left. p \right|_{[0, s] \times U} \\
k_s (p) = \left. p \right|_{(0, s] \times U}
\]

For \( t \geq 0 \), \( P_t \) will be the sub \( \sigma \)-field \( \left. \alpha_t^{-1}(P) \right|_{(0, \omega)} \), of \( P \).

We define the shift operators of \( \Pi \) to be:

\[
\Theta_s (p) = p(t+s), \text{ for } t \geq 0 \\
\Theta_s^0 (p) = \left. \Theta_s (p) \right|_{(0, \omega)}
\]

The same notation will be used for the corresponding shift operators on \( U \).
For \( u \in U \) we define the hitting time and début of \( \{a\} \subset E \) to be
\[
\sigma_a(u) = \inf \{t > 0; u(t) = a\}
\]
\[
\tau_a(u) = \inf \{t \geq 0; u(t) = a\}.
\]
It is well known that both \( \sigma_a \) and \( \tau_a \) are \((\mathbb{U}_t,+)\) stopping times.

A Poisson point process on a probability space \((\Omega, F, \mathbb{P})\), with values in \((U, \mathbb{U})\) is a measurable function \(Y:(\Omega, F) \rightarrow (\mathbb{U}, \mathbb{P})\), together with a filtration \((F_t)_{t \geq 0}\) of \((\Omega, F)\) \((F_t \subset F_s \subset F\) for \(t \leq s\)), such that:

(a) \(\alpha_t(Y) \in F_t\) for \(t \geq 0\).

(b) \(\Theta^t_0(Y)\) is independent of \(F_t\) and has the same law as \(Y|_{(0,\infty)}\) for \(t \geq 0\).

(c) There exist sets \(A_k \in \mathbb{U}\), \(A_k \uparrow U\) such that
\[ N([0,t] \times A_k, Y) < \infty \text{ a.s. for each } k,t. \]
The special role of time \(t = 0\) in (b) will be useful when we consider the point process of excursions away from a point \(a \in E\), of a Markov process taking values in \(E\). Time \(t = 0\) corresponds to the first excursion, which is exceptional in that we will want to allow it to start in any initial distribution. In contrast, the other excursions will start in a manner dictated by the transition probabilities of the given Markov process.

Under conditions (a), (b), (c) above, there exists a \(\sigma\)-finite measure \(\mathfrak{n}\) on \(U\) (the characteristic measure of \(Y\)) such that
\[ E[N((0,t) \times A, Y)] = t \cdot n(A) \text{ for } A \in \mathbb{U}, t \geq 0. \]
This measure determines the law of \(Y|_{(0,\infty)}\). In fact, if \((s_i, t_i) \times A_i\) are disjoint and \(0 \leq s_i \leq t_i\), for \(i = 1 \ldots k\), then the
\(N((s_i, t_i) \times A_i, Y)\) are independent Poisson random variables, of means \((t_i - s_i) \cdot n(A_i)\).

In the theory of right processes, it is customary to equip a right continuous, \(E\)-valued process \(X\) based on \((\Omega, F)\), with laws \(P^b\) on \((\Omega, F)\), one for each \(b \in E\). We will reserve the notation \(P^b\) for laws on the canonical space \((U, \mathcal{U})\). Later on we will work with a right process \(X\), and the \(P^b\) will be the image laws under \(X\) of the \(P^b\).

In general, on a fixed measurable space \((\Omega, F)\), let \((X_t)_{t \geq 0}\) be a right continuous process, based on \(\Omega\), with values in \(E\). We will say that \((X_t, F_t, \mu, P^b)\) has the strong Markov property at \(T\) if the following situation holds; \((F_t)_{t \geq 0}\) is a filtration of \((\Omega, F)\), \(X^{-1}(U_t) \subset F_t\), \(\mu\) is a \(\sigma\)-finite measure on \((\Omega, F)\), \((P^b)_{b \in E}\) is a family of probability measures on \((U, \mathcal{U})\) such that \(b \mapsto P^b(A)\) is \(E\)-measurable for each \(A \in \mathcal{U}\), \(T\) is a stopping time for the filtration \((F_t)\), and

\[
\mu(X_{T+}, \epsilon A, T < \infty, B) = \int \mathbb{P}^{X_T}(A) d\mu \quad \text{for} \quad A \in \mathcal{U}, \ B \in F_{T+}.
\]

Except in Lemma 6 below, we will always take \(\mu\) to be a probability measure.

For \((X_t)\) right continuous with values in \(E\), we will say that \((X_t, F_t, \mu, P^b)\) is strong Markov if \((X_t, F_t, \mu, P^b)\) has the strong Markov property at each \((F_t)\) stopping time. For \(a \in E\), \((L_t)_{t \geq 0}\) is a local time at \(a\), if \(L\) is continuous, nondecreasing, and adapted to \((F_t)\), with set of increase exactly \(\{t; X_t = a\}\), such that for every \((F_{t+})\) stopping time \(T\) with \(X_T = a\), \((X_{T+}, L_{T+} - L_T)\) is independent of \(F_{T+}\) with the same law as \((X_{T+}(x), L_{T+} - a(x))\).
Ito performed the following construction (Actually, he considered only the case of a standard process, but as pointed out to me by J. Pitman, his arguments apply without change. Henceforth similar qualifications will be omitted.); Let \( P \) be a probability measure on \( (\Omega, F) \) under which \( F \) is complete, and suppose the following conditions hold:

1. \((X_t)\) is right continuous with values in \( E \), \((X_t, F_t, P, P^b)\) is strong Markov, and each \( F_t \) contains all the \( P \)-null sets of \( F \).
2. \( X \) is recurrent at a point \( a \in E \). \((P^b(a < \infty) = 1 \text{ for } b \in E)\).
3. If \( P^a(\sigma_a = 0) = 1 \), then there is a local time \((L_t)\) for \( X \) at \( a \), which is canonical in the sense that it is normalized to make
   \[ E[e^{-\sigma_a(X)}] = E[\int_0^\infty e^{-t} \, dL_t]. \]

(Condition (2.3) holds if \( X \) is a right process).

If \( P^a(\sigma_a = 0) = 1 \), let \( S(s) \) be the right continuous inverse local time: \( S(s) = \inf\{t \geq 0; L_t > s\} \), \( S(0-) = 0 \). Let

\[
Y_s(t) = \begin{cases} 
X(S(s-)+t), & \text{if } 0 \leq t < S(s) - S(s-) \\
q & \text{if } t \geq S(s) - S(s-) > 0 \\
0 & \text{if } S(s) - S(s-) = 0 
\end{cases}
\]

Then Ito shows that \( Y \) is a \((U, U)\)-valued PPP with respect to \( P \) and the filtration of \( Y \).

If, on the contrary, \( P^a(\sigma_a = 0) = 0 \), then \( X \) visits \( a \) at a discrete set of times. In this case, let \( S(k) \) be the \( k \)'th hitting time of \( a \);

\( S(0) = 0, S(k+1) = \inf\{t > S(k); X_t = a\} \). Let

\[
Y_k(t) = \begin{cases} 
X(S(k) + t), & \text{if } 0 \leq t < S(k+1) - S(k) \\
q & \text{if } t \geq S(k+1) - S(k). 
\end{cases}
\]
Then Ito shows that under \( P \), the \( Y_k, k \geq 1 \) are IID, \((U,U)\)-valued, \( F \)-measurable random variables.

Let \( n \) be the characteristic measure of \( Y \) in the first case, and the common distribution of the \( Y_k, k \geq 1 \) in the second. We call \( n \) the excursion measure of \( X \) from \( a \).

Ito is concerned with classifying all processes that agree up till the début of a point \( a \in E \). Specifically, suppose that

(2.4) \( (P^b) \) is a family of probability measures on \((U,U)\) such that

for each \( c \in E \), the coordinate process \((W_t, U_t, P^c, P^b)\) is strong Markov. (Note that here, \( c \) is fixed, and \( b \) ranges over \( E \))

(2.5) \( P^b\{u: \tau^a(u) < \infty, \text{ and } u(t) = a \text{ for } t \geq \tau^a(u)\} = 1 \) for each \( b \in E \).

The problem is to classify all families \( (P^b) \) of probability measures on \((U,U)\) for which there exists \((X^T, F^T, P, P^b)\) on some probability space, which is a recurrent extension of \( (P^b) \) in the sense that

(2.1), (2.2), (2.3) hold, and

(2.6) \( P^b\{u; u(\cdot \land \tau^a(u)) \in A\} = P^b_o(A) \) for each \( b \in E, A \in U \).

Ito achieves this classification in terms of the excursion measure \( n \) of \( X \) from \( a \). He shows that the \( P^b_o \) and \( n \) determine the \( P^b \), and then derives the following list of conditions that \( n \) obeys.

**Theorem 1 (Ito)** Let \((X^T, F^T, P, P^b)\) satisfy (2.1), (2.2), (2.3).

Let \( n \) be the excursion measure of \( X \) from \( a \), and define \( P^b_o(A) \) to be

\[ P^b\{u; u(\cdot \land \tau^a(u)) \in A\} \].

Then the following conditions are satisfied:

(i) \( n \) is concentrated on \( \{u; 0 < \sigma^a(u) < \infty, u(t) = a \text{ for } t \geq \sigma^a(u)\} \).
(ii) \( n\{u; u(0) \notin V \} < \infty \) for every open neighborhood \( V \) of \( a \).

(iii) \( \int (1 - e^{-\sigma_a}) \, dn \leq 1 \).

(iv) \( n\{u; \sigma_a(u) > t, u \in \Lambda, \theta_t(u) \in M\} = \int \mathbf{P}_o^{u(t)} (M) n(du) \Lambda_n\{\sigma_a > t\} \)

for \( t > 0, \Lambda \in \mathcal{U}_t, M \in \mathcal{U} \).

(v) \( n\{u; u(0) \in B, u \in M\} = \int \mathbf{P}_o^{u(0)} (M) n'(du) \}

\( \{u; u(0) \in B\} \)

for \( M \in \mathcal{U}, \) and \( B \in \mathcal{E} \) such that \( a \notin B \).

(vi) Either (a) \( n \) is a probability measure concentrated on \( U^a = \{u; u(0) = a\} \) (discrete visiting case);

or (b) \( n \) is finite, \( n(U^a) = 0 \), and \( \int (1 - \exp(-\sigma_a)) \, dn < 1 \) (exponential holding case);

or (c) \( n \) is infinite and \( n(U^a) = 0 \) or \( \infty \) (instantaneous case).

The main result of this paper is that if conditions (ii) and (vi) are strengthened, we obtain conditions that are necessary and sufficient for a \( \sigma \)-finite positive measure \( n \) to arise as the excursion measure of a recurrent extension of a family \( (P_o^b) \) satisfying (2.4) and (2.5).

The conditions are:

(ii') \( n\{u; u \text{ leaves } V \} < \infty \) for every open neighbourhood \( V \) of \( a \).

(vi') Either (a) \( n \) is a probability measure concentrated on \( U^a \). If \( n \geq n' \geq 0 \) and \( n' \) satisfies (iv), then \( n' \) is a multiple of \( n \);

or (b) as in (vi b);

or (c) \( n \) is infinite. If \( n \geq n' \geq 0 \) and \( n' \) satisfies

(iv) \( \), then \( n'(U^a) = 0 \) or \( \infty \).
The statement of necessity is:

**Proposition 1** under the conditions of Theorem 1, conditions (ii') and (vi') also hold.

A strong form of sufficiency is:

**Theorem 2** Assume that \((\Omega,F,P)\) is complete, and that \((p^b_0)\) satisfies (2.4) and (2.5).

a) If \((y_t^*,F_t^*)\) is a PPP with values in \((U,U)\) and with characteristic measure \(n\), such that:

- \((F_t^*)\) is right continuous, and each \(F_t^*\) contains all the \(P\)-null sets of \(F\);
- \(P(y_0 \in M) = \int P y_0(0) (M) \ dP \quad \text{for} \ M \in U\);

(i), (ii'), (iii), (iv), (v) and either (b) or (c) of (vi') hold.

Then there is a right continuous strong Markov process \((X_t,G_t,P,p^b)\) such that:

- \(Y\) is the PPP constructed from \(X\) as above, \(P\)-a.s.;
- \((X_t,G_t,P,p^b)\) is a recurrent extension of \((p^b_0)\);
- \(G_{s(t+)} = F_t^*\);
- \((G_t)\) is right continuous, and each \(G_t^*\) contains all the \(P\)-null sets of \(F\).

b) If \((F^*_k)_{k \geq 0}\) is an increasing family of sub \(g\)-fields of \(F\), each containing all the \(P\)-null sets of \(F\), and for each \(k \geq 0\), \(Y^*_k\) is a measurable function from \((\Omega,F^*_k)\) to \((U,U)\) such that:

For \(k \geq 1\) the \(Y^*_k\) have a common distribution \(n\);
\[ P(Y_0 \in M) = \int_{U \setminus U^a} P^U(o) M P(Y_0 \in du) + n(M) P(Y_0 \in U^a) \]

for \( M \epsilon U \); 

\( \sigma(Y_i ; i > k) \) is independent of \( F^k \) for \( k \geq 0 \); 

(i), (ii'), (iv) and (a) of (vi') hold.

Then there is a strong Markov right continuous process \( (X_t, G_t, P, P^b) \) such that:

The \( Y_k \) are the excursion random variables constructed from \( X \) as above (discrete visiting case), \( P \)-a.s.; 

\( (X_t, G_t, P, P^b) \) is a recurrent extension of \( (P^o) \); 

\( G_{S(k+1)}^c \subseteq F^k \subseteq G_{S(k+1)}^c \), for \( k \geq 0 \); 

\( (G_t) \) is right continuous, and each \( G_t \) contains all the \( P \)-null sets of \( F \).

This result can be used to give a rigorous construction of processes such as skew Brownian motion (see J. Walsh [50]). Similar results in special cases, obtained by different techniques can be found in Blumenthal [4] and S. Watanabe [31], [54].

The key to the proof of Theorem 2 given here is to find an expression for conditioning \( Y_T \) on the strict past \( F_{T^-} \), for \( T \) an \( (F_t) \) stopping time. This is done in Lemma 7, which is related to similar results of M. Weil [53], [54] as follows.

A version of Lemma 7 can be proven for \( (Y_t, F_t) \) now a càdlàg Markov process, and \( T \) an \( (F_t) \) stopping time such that

\[ || T || \cup \bigcup_{q \in Q^+} || q \circ S \circ T^+ \cup || q \],

where \( S \) is a terminal time for the natural filtration of \( Y \) (the smallest right continuous filtration to which \( Y \) is adapted).
\( P(Y_{T^e} \cdot |F_{T^-}) \) is identified in terms of \( P(Y_{S^e} \cdot |F_{S^-}) \), and Weil's results identify the latter in terms of a Levy system for \( Y \) (assuming that \( Y \) is standard.) See section 7.

After completion of this work, it was pointed out to me that parts of it have appeared before; In his Thesis [34], S. Kabbaj obtained a result similar to Theorem 2. He shows that under Ito's conditions, and with \( (\mathcal{G}_t) \) the minimal filtration to which the reconstructed process \( (X_t) \) is adapted, \( (X_t) \) is strong Markov at all \( (\mathcal{G}_t) \) stopping times. His proof uses a weaker form of Lemma 7 below, and relies heavily on the powerful tools of martingale theory. The proof presented here works for \( T \) a \( (\mathcal{G}_t) \) stopping time, is more elementary, and applies in greater generality (we assume no compactness conditions on \( E \)).

Also, when we are given a right process \( (X_t) \), and let \( (X_t) \) be its excursion process, Lemma 7 is still of interest. In this context, it is closely related to Getoor and Sharpe's last exit decompositions, and a very similar result has appeared in Getoor and Sharpe [25]. Some of the methods used in the proof of Lemma 7 appear, both in [25] and in Pitman [45]. Part of the interest of the results presented here, however, lies in that they apply in a more elementary context than the theory of right processes. As pointed out in section 6, the proof of Theorem 2 can be greatly simplified in the case that \( (p_0^b) \) comes from a right process.

I would like to thank Yves LeJan for bringing [34] to my attention.
3. Proof of Theorem 2

Part (a) is shown first, and then the arguments are modified to show part (b). In part (a), conditions (i) and (iii) are used in the construction; the latter so that the normalization of local time agrees with (2.3). Condition (ii') appears in Lemma 3, in the proof of the right continuity of paths. Condition (vi'b) also appears in this lemma, and is used to make the "inverse local time" strictly increasing, so that local time will be continuous. Conditions (iv) and (v) are put in a more convenient form in Lemma 6, which, together with Lemma 7, yields Corollary 2. Lemma 7 is also used with conditions (vi'b) and (vi'c) to give Corollary 1. Note that these two lemmas essentially show the strong Markov property. In part (b), the conditions are put to the same uses, except that we use condition (vi'a) instead of conditions (vi'b) and (vi'c).

We start the proof of part (a), by constructing X as an explicit measurable function of Y.

Put

\[ m = 1 - \int (1 - e^{-a}) \, dn \]

\[ S^-(s,p) = ms + \sum_{r<s} \sigma_a(p(r)), \quad \text{for } s \geq 0, \, p \in \Pi \]

(with the convention that \( \sigma_a(\delta) = 0 \)), and

\[ S^+(s,p) = \lim_{r \uparrow s} S^-(r,p). \]

Then \( S^-(\cdot,p), S^+(\cdot,p) \) are nondecreasing, and respectively left and right continuous, with values in \([0,\infty]\). If \( S^+(s,p) < \infty \),
then $S^-(s,p) = S^-(s,p) + \sigma_a(p(s))$. $S(s,\cdot) \in P_s^-$ since it is left continuous, and $S^-(s,p) = S^- (s, \sigma_s(p))$. Thus also $S^+(s,\cdot) \in P_s^+$. Put

$$\ell^*(p) = \inf \{ s \geq 0; \infty > S^+(s) \geq t \},$$

with the usual convention that $\inf(\emptyset) = +\infty$. Then $\ell^* \in \mathcal{F}_s^+$ is a stopping time, and

$$S^-(\ell^*(\cdot),\cdot) \leq t \leq S^+(\ell^*(\cdot),\cdot)$$

(with the convention that $S^+(\infty, p) = \infty$). Put

$$x_t = \begin{cases} y_{\ell^*} (t - S^- (\ell^*, \cdot)), & \text{if } y_{\ell^*} \neq \delta, \ell^* \\ a, & \text{otherwise} \end{cases}$$

We will show next that $x_t$ is measurable from $P_{\ell^*}$ to $E^0$. Since $(u,r) \mapsto u(r)$ is measurable from $\mathcal{B}$ to $E^0$, and $S^-(\ell^*(\cdot),\cdot) \in P_{\ell^*}$ since $S^-$ is predictable, we need only show that $y_{\ell^*} \in P_{\ell^*}$.

We state this in a more general form, to be useful later, as;

**Lemma 1.** Let $(F_t)$ be a filtration of a measurable space $(\Omega,F)$, and let $Y$ be a function which is measurable from $(\Omega,F_t)$ to $(\mathbb{R},P_t)$, for every $t \geq 0$. Let $R$ be an $(F_t)$ stopping time such that for each $\varepsilon > 0$,

$$\{ R < \infty \} \subset \{ \sigma (Y_t) > \varepsilon \text{ for only finitely many times } t \text{ in any compact time set} \}.$$

Then $Y_R \in F_R$.

**Remark:** In the present situation, we apply the lemma with $Y$ the identity map, and with $F_t = P_{t+}$. 

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proof: Let \( A \in \mathcal{U}, s \geq 0 \). Then

\[
\{Y \in A, R \leq s\} = \{Y \in A, R \leq s, \sigma_{Y_R} = 0\} \cup \\
\bigcup_{m \geq 1} \{Y \in A, R \leq s, \sigma_{Y_R} \geq \frac{1}{m}\}.
\]

Let \( B_n, n = 1, 2, \ldots \) be an open base in the space \([0, s]\). We can write

\[
\{\sigma_{Y_R} = 0, R \leq s\} = \\
n \cup \bigcup_{m \geq 1} \{\{R \in B_n\} \cap \{N(B_n \times \{\sigma_{Y_R} \geq \frac{1}{m}\}, Y) = 0\}\}.
\]

and

\[
\{Y \in A, R \leq s, \sigma_{Y_R} \geq \frac{1}{m}\} = \\
n \cup \bigcup_{n \geq 1} \{\{R \leq s\} \cap \{R \in B_n\} \cup \{N(B_n \times \{\sigma_{Y_R} \geq \frac{1}{m}\}, Y) > 1\}\}.
\]

Thus \( \{Y \in A, R \leq s\} \in F_s \), as required. \( \square \)

Put

\[
M = \{p; S^-(s, p) < \infty \text{ for each } s \geq 0, S^-(s, p) \to \infty \text{ as } s \to \infty, S^-(\cdot, p) \text{ is strictly increasing, and for each open neighbourhood } V \text{ of } a, p(s) \text{ leaves } V \text{ for only finitely many times } s \text{ in any compact set of times}\}.
\]

Thus on \( M, \ell_t < \infty \) for every \( t, t \mapsto \ell_t \) is continuous, and

\[
x_t = \\
\begin{cases} 
  y_s(t - S^-(s, \cdot)) & \text{if } S^-(s, \cdot) \leq t < S^+(s, \cdot) \\
  a & \text{if } t = S^+(s, \cdot)
\end{cases}
\]
Lemma 2 \( t \mapsto x_t \) is right continuous on \( M \).

**proof:** Fix some element \( p \) of \( M \). By definition of \( U \), \( x_t(p) \) is right continuous on each interval \( [S^-(s,p), S^+(s,p)) \). If \( t \) lies in no such interval, then it must equal \( S^+(s,p) \) for some \( s \), and hence \( x_t(p) = a \). Let \( V \) be an open neighborhood of \( a \). Since \( p \in M \), there is an \( s' > s \) such that \( y_x(p) \) remains in \( V \) for any \( r \in (s, s') \). Thus \( x_t(p) \in V \) for any \( q \in (S^+(s, p), S^-(s', p)) \). But by definition of \( M \), \( S^-(s', p) \) is strictly greater than \( S^+(s, p) \), so that as \( V \) was arbitrary, \( x_t(p) \) must be right continuous at \( t \).

(The same argument would show that if we had taken \( U \) to be the space of càdlàg paths, then on \( M \), \( x_t \) would be càdlàg as well.)

For convenience, we separate out the contribution of \( p(o) \); we have shown that there is a measurable function

\[
F: (U \times \Pi, (U \otimes P) \rightarrow (U, \mathcal{U})
\]

such that \( x_t = F(y_o, y_{-}(o, \infty))(t) \) on \( M \).

Put

\[
S^-(s) = S^-(s, Y) \\
S^+(s) = S^+(s, Y) \\
L_t = L_t(Y) \\
x_t = x_t(Y)
\]

Since \( Y \) is, by the hypotheses of Theorem 2, a measurable function from \( (\Omega, \mathcal{F}_S) \) to \( (\Pi, \mathcal{P}_S^+) \), for each \( s \), the above results imply that \( S^- \) and \( S^+ \) are adapted to \( (\mathcal{F}_S) \), and that for each \( t \geq 0 \), \( L_t \) is an \( (\mathcal{F}_S) \) stopping time and \( x_t \) is measurable from \( F_{L_t} \) to \( E \) (as \( F_{L_t} \) is complete).
Lemma 3. $P(Y \in M) = 1$.

Proof: For $f \in B$ with $f(0) = 0$,

$$\sum_{0 < t < s} f(\sigma_a(Y_t)) = \int f(r)N(\sigma_a \in dr \times (o,s), Y) (o,\infty)$$

is $F$-measurable, and has expectation

$$s \int f(r) N(\sigma_a \in dr) = s \int f(\sigma_a) \, dn.$$

In particular,

$$E[\sum_{0 < t < s} \sigma_a(Y_t)] = s \int \sigma_a \, dn_{\sigma_a \leq 1} \{\sigma_a \leq 1\}$$

$$\leq \frac{s}{1-e^{-1}} \int (1-e^{-\sigma_a}) \, dn$$

$$= \frac{s}{1-e^{-1}} (1-m) < \infty.$$ 

Also, there are only finitely many times $t \in [0,s]$ with $\sigma_a(Y_t) > 1$, as

$$n(\sigma_a > 1) \leq \frac{1}{1-e^{-1}} \int (1-e^{-\sigma_a}) \, dn < \infty,$$

so that $S^-(s) < \infty$ a.s., for each $s$.

If $n(U) = 0$ (so that $a$ is a trap), then $m = 1$ so that

$S^-(s) = s$ a.s. If $n(U) \neq 0$, then by (i) there is an $\varepsilon > 0$ such that $n(\sigma_a > \varepsilon) > 0$. Thus, there are a.s. infinitely many times $s$ such that $\sigma_a(Y_s) > \varepsilon$, so that in either case, $S^-(s) \to \infty$ as $s \to \infty$, a.s.

If $m > 0$, it is clear that $S^-$ is strictly increasing. If $m = 0$, 
then by (vib) it follows that $n(U) = \infty$, and hence that \( \{s; \sigma_a(Y_s) > \omega \} \)

is dense in \([0, \infty)\), a.s.

The last condition is obtained from (ii'), letting \( V \) run

through a countable base of open neighborhoods of \( a \). \( \square \)

Appealing to the completeness of \( F \), we will without loss of
generality assume that \( Y \in M \) surely, hence that \( X \) is right continuous

and by Lemma 1, that \( Y_R \in F_R \) for every \( (F_S) \) stopping time \( R \).

If we are given a filtration \( (V_t)_{t \geq 0} \), a \( \sigma \)-field \( V_{o-} \subset V_o \),

and a random variable \( R \) with values in \([0, \infty]\), recall that \( V_{R-} \) is
defined to be the filtration generated by \( V_{o-} \) and by sets of the form

\( A \cap \{ R > s \} \), for \( A \in F \) and \( s \geq 0 \). In our case, let \( F_{o-} \) be generated

by all the \( P \)-null sets of \( F \). For \( R \) a random time, and \( r \geq 0 \), put

\[
H_{o-}^R = F_{R-}
\]

\[
H_{r}^R = F_{R-} \vee \sigma(Y_R(s); 0 \leq s \leq r)
\]

Put

\[
G^o_t = H_{(t-S)^-(L_t)}^-
\]

\[
G_t = G^o_{t+}
\]

\[
G^o_{o-} = G_{o-} = F_{o-}
\]

**Lemma 4**

(a) \( (G_t) \) is right continuous, increasing, and each \( G_t \) contains all

the \( P \)-null sets of \( F \). \( X_t \in G_t \) for each \( t \geq 0 \).
(b) If \( T \) is a \((G_t)\) stopping time then \( L_T \) is an \((F_s)\) stopping time, and
\[
F_{L_T} \subset G_{T-} \subset G_T \subset F_{L_T}
\]

(c) \( S^+(s) \) is a \((G_t)\) stopping time, for \( s \geq 0 \). If \( T \) is any \((G_t)\) stopping time such that \( S^+(L_T) = T \) then \( G_T = F_{L_T} \).

(d) If \( R \) is an \((F_s)\) stopping time, and \( T \) is a \((G_t)\) stopping time such that \( T < S^+(L_T) \) on \( \{T < \infty\} \), put
\[
V = \begin{cases} 
T - S^-(R) & \text{if } L_T = R < \infty \\
\infty & \text{otherwise}
\end{cases}
\]

Then \( V \) is an \((H^R_{t+})\) stopping time.

**proof:** (a) We know that each \( L_t \) is an \((F_s)\) stopping time. Thus, if \( t < t' \) then
\[
G^O_t \cap \{L_t < L_{t'}\} \subset F_{L_t} \cap \{L_t < L_{t'}\}
\]
\[
\subset F_{L_t} \cap \{L_t < L_{t'}\} \quad (\text{as } V_{L_t} \in F_{L_t})
\]
\[
\subset F_{L_{t'}} \subset G^O_{t'}
\]
and \( G^O_t \cap \{L_t = L_{t'}\} \) is, by monotone class arguments, and Prop 18 of [Z], generated by
\[
F_{L_t} \cap \{L_t = L_{t'}\} \subset F_{L_{t'}} \cap \{L_t = L_{t'}\}
\]
\[
\subset G^O_{t'}
\]
and by
Thus, $G_t^c \subseteq G_t^c$.

For $s > t$ we can write $X_t$ as a measurable function of $L_s$,

\[ k_{L_s}(Y), \text{ and } k_{s-s}(L_s) (Y_{L_s}) \] (let

\[ \hat{y}_r = \begin{cases} 
  k_{L_s}(Y)(r), & \text{if } r < L_s \\
  k_{s-s}(L_s) (Y_{L_s}), & \text{if } r = L_s \\
  \delta, & \text{if } r > L_s
\end{cases} \]

Then $X_t = x_t(\hat{y}_r)$.

As in [7], Prop. 25, the first two are $F_{L_s}$ measurable and the latter lies in $H_{(s-s)(L_s)}$.

(b) For $r > t \geq 0$ we have that

\[ F_{L_{t-r}} \subseteq G_t \subseteq G_t^c \subseteq H_{L_r} \subseteq F_{L_r} \].

By definition, $L_r \downarrow L_t$ as $t \downarrow r$, hence

\[ F_{L_{t-r}} = \cap F_{L_r} \]

so that $G_t \subseteq F_{L_t}$.

If now $T$ is a $(G_t)$ stopping time, then by the right continuity of $L$,

\[ \{L_t < \lambda\} = \cup \{T < q\} \cap \{L_q < \lambda\} \].
But \( \{T < q\} \in G_q \subset F_{Lq} \), so that since \( L_q \) is an \((F_s)\) stopping time,

\[
\{L_T < \lambda\} \in F_{\lambda}.
\]

Further, if \( A \in G_T \), then

\[
A \cap \{T < q\} \in G_q \subset F_{Lq}
\]

so that

\[
A \cap \{T < q\} \cap \{L_q < \lambda\} \in F_{\lambda}.
\]

Taking the union over \( q \in \Omega \), we get that \( A \cap \{L_T < \lambda\} \in F_{\lambda} \), hence that \( A \in F_{LT} \).

We will prove (c) before showing the remainder of (b); that

\[
F_{L_T} \subset G_{T^-}.
\]

(c) Since \( S^- \) and \( S^+ \) are strictly increasing, and \( S^-(L_T) \leq t \leq S^+(L_T) \),

we see that

\[
\{s^+(s) < t\} = \{s < L_T\} \in F_{L^-} \subset G_t,
\]

hence \( S^+(s) \) is a \((G_t)\) stopping time.

Similarly, if \( S^+(L_T) = T \), then

\[
\{L_T < L_T\} = \{T < t\},
\]

and hence if \( A \in F_{L_T} \), then by [2] Prop. 16,

\[
A \cap \{T < t\} = A \cap \{L_T < L_T\} \in F_{L^-} \subset G_t.
\]

That is, \( A \in G_T \). Since \( F_{L_T} \supset G_T \) by the part of (b) already shown,

we get that \( G_T = F_{L_T} \).

To finish the proof of (b), let \( A \in F_s = G_{S^+(s)} \). Then
A \cap \{L_T > s\} = A \cap \{T > S^+(s)\} \in G_{T-}

by [7], Prop. 16 again.

(d) 

\{V < h\} = \{T < S^-(R) + h, L_T \leq R < \infty\}\{L_T < R\}

= \{T < (S^-(R) + h) \wedge S^+(R)\}\{L_T < R\}

as \( T < S^+(L_T) \) on \( \{T < \infty\} \). Because \( T \) is a \( (G^O_{t+}) \) stopping time,

\{L_T < R\} \in F_{R-} \subset H^R_h

and

\{T < (S^-(R) + h) \wedge S^+(R)\} \in G^O_{[(S^-(R) + h) \wedge S^+(R)]-}.

We must therefore show that the latter field lies in \( H^R_h \). It is generated by \( G^O_{\infty-} = F_{\infty-} \subset H^R_h \), and by

\( G^O_t \cap \{(S^-(R) + h) \wedge S^+(r) > t\} \), for \( t \geq 0 \).

By monotone class arguments, the latter is generated by

\( H^L_t \cap \{(S^-(R) + h) \wedge S^+(r) > t\} \)

and by

\( H^L_r \cap \{\tau - S^-(L_\tau) > r\} \cap \{(S^-(R) + h) \wedge S^+(r) > t\} \), for \( r \geq 0 \).

\( \sigma_a(Y^R) \) is an \( (H^R_{S^+}) \) stopping time, since \( \sigma_a \) is a \( (U_{S^+}) \) stopping time, each \( \sigma \)-field \( H^R_S \) is complete, and

\( Y^{-1}_R(\cup_{S^+}) \subset H^R_S \). \( (\tau - S^-(R)) \vee 0 \) is also an \( (H^R_{S^+}) \) stopping time, since \( S^-(R) \in F_{R-} \) as \( S^- \) is predictable. Thus
\{(S^{-}(R) + h) \land S^{+}(R) > t\} = \{\sigma_{a}(Y_{R}) \land h > (t - S^{-}(R)) \lor 0\}

\epsilon R^{H}_{(\sigma_{a}(Y_{R}) \land h)} - c H^{R}_{h}.

Also,

\[F_{t} \cap \{L_{t} \leq R\} \subset F_{R}^{-} \cap \{L_{t} \leq R\}\]

by [7] Prop 18. Therefore

\[H^{L}_{R} \cap \{(S^{-}(R) + h) \land S^{+}(R) > t\}

\subset F_{R}^{-} \cap \{(S^{-}(R) + h) \land S^{+}(R) > t\} \subset H^{R}_{h},\]

as \(\{(S^{-}(R) + h) \land S^{+}(R) > t\} \subset \{L_{t} \leq R\}\).

We argue as in part (a) to see that

\[H^{L}_{R} \cap \{t - S^{-}(L_{t}) > r\} \cap \{(S^{-}(R) + h) \land S^{+}(R) > t\} \cap \{L_{t} < R\}\]

\[\subset F_{R}^{-} \cap \{(S^{-}(R) + h) \land S^{+}(R) > t\} \subset H^{R}_{h},\]

and

\[H^{L}_{R} \cap \{t - S^{-}(L_{t}) > r\} \cap \{(S^{-}(R) + h) \land S^{+}(R) > t\} \cap \{L_{t} = R\}\]

\[\subset H^{R}_{R} \cap \{\sigma_{a}(Y_{R}) \land h > t - S^{-}(R) > r\}\]

\[\subset H^{R}_{R} (\sigma_{a}(Y_{R}) \land h) - c H^{R}_{h} \]

Lemma 5. Let \((\Omega_{1}, F_{t}^{1}), (\Omega_{2}, F_{t}^{2})\) be right continuous filtered spaces.

Let \(Z: \Omega_{1} \rightarrow \Omega_{2}\), be such that \(Z^{-1} F_{t}^{2} \supset F_{t}^{1}\) for every \(t\). Then for every \((F_{t}^{1})\) stopping time \(T_{1}\), there is an \((F_{t}^{2})\) stopping time \(T_{2}\) such that \(T_{2} \circ Z = T_{1}\).
proof: For $r \in \mathcal{Q}$, let $B_r = \{T_1 < r\}$, and find $A_r \in \mathcal{F}_r^2$ such that $B_r = Z^{-1} A_r$. Set

$$\hat{A}_t = \bigcup_{r \in \mathcal{Q}} A_r.$$ 

Thus, putting $T_2(\omega) = \inf\{t; \omega \in \hat{A}_t\}$, we have that

$$\{T_2 < t\} = \hat{A}_t \in \mathcal{F}_t^2.$$ 

Also,

$$Z^{-1} \hat{A}_t = \bigcup_{r < t} Z^{-1} A_r = \{T_1 < t\},$$ 

for every $t \geq 0$, so that $T_2 \circ Z = T_1$. 

The following lemma would be much simplified if, instead of the conditions of Theorem 2, we had assumed that the coordinate process $(W_t)$ was a right process under $(\mathcal{P}_b^b)$. (It would follow as usual from the right continuity of the $U^a f(W_t)$ at $R$, $(U^a)$ being the resolvant for $(\mathcal{P}_b^b)$.)

Lemma 6. Let $(\mathcal{P}_o^b)$ satisfy (2.4) and (2.5), and suppose that $\mu$ is a $\sigma$-finite positive measure on $(U, \mathcal{U})$ satisfying (i), (iv) and (v).

Let $R$ be a $(U_{t+}^\mu)$ stopping time such that $\mu(U_{t+}^\mu, R=0) = 0$. Then the coordinate process $(W_t, U_t, n, \mathcal{P}_o^b)$ is strong Markov at $R$.

**Proof:** By replacing $R$ by the $(U_{t+}^\mu)$ stopping times $R_B = \begin{cases} R & \text{on } B \\ \infty & \text{off } B \end{cases}$ for $B \in U_{R+}^\mu$, we see that it suffices to show that for $A \in \mathcal{U}$,

$$\mu(\mathcal{C}_R^{-1} A, R < \infty) = \int_{\{R < \infty\}} p_o^u(R(u)) (A) \mu(du).$$

Let $h > 0$. By (iv),
\[ (3.1) \int f(u,0,u) \, n(du) \]

\[
\left\{ \sigma > h \right\} = \int \left[ \int f(u,v) \, F_0^u(h) \, (dv) \right] n(du),
\]

for \( f \) of the form \( 1_{B \times C} \) where \( B \in \mathcal{U}_h, C \in \mathcal{U} \). Thus, this holds for every \( f \in \mathcal{U}_h \circ \mathcal{U}, f \geq 0 \).

Put

\[ \hat{\mathcal{U}}_t = \mathcal{U}_h \circ \mathcal{U}_t. \]

Since \( n \) is \( \sigma \)-finite, it follows as in Theorem 7.3 of Blumenthal and Getoor [5], that there is a \( (\mathcal{U}_t^0) \) stopping time \( \hat{R} \) such that \( n(\hat{R} \neq \hat{R}) = 0 \). Define \( Z: \mathcal{U} \to \mathcal{U} \times \mathcal{U} \) by \( Z(u) = (u, \theta U, u) \).

Then \( Z^{-1}(\hat{U}_t) = U_{(t+h)} \). (Note that this might fail for the universal completion \( U_{(t+h)} \).) Thus by Lemma 5, there is a \( (\hat{U}_t^+) \) stopping time \( \hat{R} \) with

\[
\hat{R}(u, \theta U, u) = \begin{cases} \hat{R}(u) - h, & \text{if } \hat{R}(u) > h \\ \infty & \text{otherwise.} \end{cases}
\]

Let

\[ f = 1_{\{ (u,v); \hat{R}(u,v) < \infty, \theta R (u,v) (v) \in A \}}. \]

Since \( \hat{R} \) is a \( (\hat{U}_t^+) \) stopping time, it is immediate that \( \hat{R}(u, \cdot) \) is a \( (\hat{U}_t^+) \) stopping time, for each \( u \in \mathcal{U} \). Since also \( (W_t, \mathcal{U}_t^0, \mathcal{P}_0^C, \mathcal{P}_0^B) \) was assumed to be strong Markov for \( c \in E \), we can use (3.1) twice to get that

\[
\left\{ \sigma > h \right\} \circ \mathcal{U}_h < \infty, \theta R (\hat{R}, u) (v, v) \in A\right\}
\]

\[
= \int f(u, \theta U, u) \, n(du) \left\{ \sigma > h \right\}
\]

\[
= \int \left[ \int P_0^v(\hat{R}(u,v)) (A) 1_{\{ \hat{R} < \infty \}}(u,v) \, p^u(h) (dv) \right] n(du).
\]
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\[ = \int_{\{\sigma > h\}} \mathbb{P}_0^h u(R(\theta, \xi)) \text{ (A) } 1_{\{R < \infty\}}(u, \theta, u) \ n(du) \]

\[ = \int_{\{\sigma > h\}} \mathbb{P}_0^u(R(u)) \text{ (A) } 1_{\{h < R < \infty\}}(u) \ n(du). \]

Letting \( h \downarrow 0 \), we get that

\[ n(0 < R < \infty, 0, -1, A) = \int_{\{0 < R < \infty\}} \mathbb{P}_0^u(R(u)) \ n(du). \]

From (v), we see that

\[ (3.2) \quad \int_{U \setminus A} f(u(0)) g(u) n(du) = \int_{U \setminus A} f(u(0)) E_o^u (g) n(du) \]

for nonnegative \( f \in E, g \in U \).

Let the set of branch points be

\[ E_{br} = \{ b \in E; \mathbb{P}_0^b(W_o \neq b) > 0 \} \in E. \]

The strong Markov property of \( (W, U, t, P^c, P^b, P) \) shows that \( \mathbb{P}_0^c(W_o \in E_{br}) = 0 \) for every \( c \in E \). Thus,

\[ n(R = 0, U \setminus A, A) \]

\[ = \int_{U \setminus A} \mathbb{P}_0^u(0) \text{ (R = 0, A) } n(du) \]

\[ = \int_{U \setminus A} \left[ \int_{\{R = 0\}} \mathbb{P}_0^v(0) \text{ (A) } \mathbb{P}_0^u(0) (dv) \right] n(du) \] \quad \text{(SMF)}

\[ = \int_{U \setminus A} \mathbb{P}_0^u(0) \text{ (A) } \mathbb{P}_0^u(0) \text{ (R = 0) } n(du) \]

\[ = \int_{U \setminus A} \mathbb{P}_0^u(0) \text{ (A) } 1_{\{R = 0\}} (u) \ n(du). \] \quad \text{(by 3.2)}

Since \( n\{R=0\} \cap U^a = 0 \) by hypothesis, we can see that (3.1) holds. \( \square \)
The heart of the proof of Theorem 2 lies in the next result, whose proof we defer until later.

**Lemma 7.** Let $T$ be a $(\mathcal{G}_t)$ stopping time such that $L_T > 0$ and $T < S^+(L_T)$ on \{T < $\infty$\}. Let $\mathcal{H}_T$ be the $\sigma$-field $\mathcal{F}_{L_T} \otimes \mathcal{U}_T$ on $\Omega \times U$. Then there is an $(\mathcal{H}_{t+})$ stopping time $R$ such that

\begin{align*}
(3.3) \quad & P(\omega; n(u; R(\omega, u) < \infty) < \infty) = 1 \\
(3.4) \quad & R(\omega, Y(L_T)) = (T-S^-(L_T))(\omega), \text{ if } T(\omega) < \infty \\
(3.5) \quad & R(\omega, u) = \infty \text{ for every } u \in U, \text{ if } T(\omega) = \infty \\
(3.6) \quad & P(Y_{L_T} \in A, T-S^-(L_T) \in B, T < \infty | F_{L_T}^-)(\omega)
\end{align*}

\[
= \frac{n(u; u \in A, R(\omega, u) \in B, R(\omega, u) < \infty)}{n(u; R(\omega, u) < \infty)} \quad \text{for } P\text{-a.e. } \omega,
\]

where $A \in \mathcal{U}$, $B \in \mathcal{B}$, and we take the convention that $0/0 = 0 = \infty$.

**Corollary 1.** Let $T$ be a $(\mathcal{G}_t)$ stopping time such that $X_T = a$. Then $T = S^+(L_T)$ a.s.

**proof:** By the strong Markov property of $(\mathcal{W}_t, \mathcal{U}_t, \mathbb{P}^c_t, \mathbb{P}_c^b)$ for $c \in E$, and the hypothesis that $\mathbb{P}_0^c(\tau_a < \infty, W_t = a$ for every $t \geq \tau_a) = 1$, we get $\tau_a = 1$, we get that $\mathbb{P}_0^a(\sigma_a = 0) = 1$, and hence that $\mathbb{P}_0^c(W_0 = a, \sigma_a > 0) = 0$ for every $t \geq \tau_a = 1$, we get that $\mathbb{P}_0^a(\sigma_a = 0) = 1$, and hence that $\mathbb{P}_0^c(W_0 = a, \sigma_a > 0) = 0$ for every $t \geq \tau_a$.

Since also $\mathbb{P}(Y_{\infty} \in M) = \left[ \mathbb{P}_0 Y_{\infty}(0) (M) \right] d\mathbb{P}$, it follows that $S^+(0) = 0$ a.s. on $\{X_{\infty} = 0\}$, hence that $T = S^+(L_T)$ a.s. on $\{L_T = 0\}$. Thus putting

$$B = \{L_T > 0 \text{ and } T < S^+(L_T)\},$$

we may replace $T$ by $T_B$, to get that $L_T > 0$ and $T < S^+(L_T)$ on $\{T < \infty\}$. We need to show that $T = \infty$ a.s.
Apply Lemma 7 to $T$, to get an $(H^c_{t+})$ stopping time $R$. For $\omega \in \Omega$, let

$$H(\omega) = \{u; u(\omega) = a, R(\omega, u) = 0\}.$$ 

Since $R$ is an $(H^c_{t+})$ stopping time, it is immediate that $R(\omega, \cdot)$ is a $(U^c_{t+})$ stopping time, hence that $H(\omega) \in U^a$ for each $\omega \in \Omega$. Since also $H(\omega) \subseteq U^a$, we have that

$$n \geq n|_{H(\omega)} \geq 0,$$

and that $n|_{H(\omega)}$ satisfies (iv) and (v). If $n(U) < \infty$, then $n(H(\omega)) \leq n(U^a) = 0$ by (vi'b). If $n(U) = \infty$, then $n(H(\omega)) = 0$ or $\infty$ for each $\omega$, by (vi'c) (this is the only place we use this condition!). But $n(H(\omega)) \leq \{u; R(\omega, u) < \infty\}$, which is itself finite for $P$-a.e. $\omega$. Thus in either case, $n(H) = 0$ a.s., so

$$P(T < \infty) = E[n(H) \cdot \mathbb{1}_{R < \infty}] = 0$$

by (3.6) (recall that $0/0 = 0$). 

Using these three results, we will show

**Corollary 2.** Let $T$ be a $(G^c_t)$ stopping time, and $A \in \mathcal{U}$. Then

$$P[Y_{L^c_T} (\cdot + T - S^+(L^c_T)) \in A, T < \infty] = E[P^T_0 (A), T < \infty]$$

**proof:** Replacing $T$ by various $(G^c_t)$ stopping times $T_B$, it suffices to treat several distinct cases, namely that $X_T = a$ on $\{T < \infty\}$, that $X_T \neq a$ and $L_T > 0$ on $\{T < \infty\}$, and that $X_T \neq a$ and $L_T = 0$ on $\{T < \infty\}$.

In the first case, Corollary 1 shows that the conclusion is trivial.

In the second case, also $T < S^+(L^c_T)$ on $\{T < \infty\}$, so that we can apply Lemma 7 to obtain $R$ as in that result. It follows that if $f \in F_{L^c_T - \emptyset \cup U}$ is bounded, then
(3.7) \( E[f(\cdot, Y_{L_T}^{-}(\cdot)), T < \infty] \)

\[
= \int \left[ \frac{1}{n\{u; R(\omega, u) < \infty\}} \int f(\omega, u) n(du) \right] P(d\omega) 
\{R(\omega, \cdot) < \infty\}
\]

Take

\( f = 1\{(\omega, u); R(\omega, u) < \infty, \Theta_{R(\omega, u)} u \in A\} \),

to obtain that

\[
P(Y_{L_T}^{-}(\cdot+T-S^{-}(L_T)) \in A, T < \infty) \]

\[
= n(\Theta_{R}^{-1}(A), R < \infty) 
\]

\[
= E \left[ \frac{n(R < \infty)}{n(R < \infty)} \right].
\]

Since \( R \) is an \((H_{t+}^+)\) stopping time, it is immediate that \( R(\omega, \cdot) \)
is a \((U_{t+}^+)\) stopping time, for each \( \omega \in \Omega \). Also, since \( X_T \neq a \) on
\( \{T < \infty\} \) we get that \( n(\Theta^A, R(\omega, \cdot) = 0) = 0 \) for \( P\)-a.e. \( \omega \). Thus, by
Lemma 6

\[
n(\Theta_{R(\omega, \cdot)}^{-1}(A), R(\omega, \cdot) < \infty) = \int \frac{p^{u}_O(R(\omega, u))}{n(R < \infty)} (A) n(du) 
\{R(\omega, \cdot) < \infty\}
\]

for \( P\)-a.e. \( \omega \). Therefore by (3.7) again,

\[
P(Y_{L_T}^{-}(\cdot+T-S^{-}(L_T)) \in A, T < \infty) 
\]

\[
= E[p^{u}_O(R(\cdot, Y_{L_T}^{-}(\cdot))) \in A, T < \infty] 
\]

\[
= E[p^{u}_O(A), T < \infty].
\]

In the third case, that \( L_T = 0 \) and \( X_T \neq a \) on \( \{T < \infty\} \), then
also \( T < S^+(L_T) \) on \( \{T < \infty\} \). Thus Lemma 4, part (d) applies, to
show that $T$ is an $(\mathcal{H}_T^0)$ stopping time. Since $(\mathcal{H}_T^0)$ is a completion of the natural right continuous filtration of the process $X_{\cdot \wedge \sigma_a^{\cdot}}(x)$, it is easy to use the hypotheses that $P(Y \in A) = \int P^u(0) A P(Y \in du)$ for $A \in \mathcal{U}$, and that for $c \in E$ the coordinate process $(W_t', U_t', P^c_o, P^c_o)$ is strong Markov, to conclude that the process

$$(X_t^{\cdot \wedge \sigma_a^{\cdot}}, \mathcal{H}_T^0, P^b, P^c_o)$$

is strong Markov. This suffices. \(\Box\)

The remainder of Theorem 2 part (a) now follows easily; For $A \in \mathcal{U}$ and $b \in E$ put

$$P^b(A) = \int \lambda_A(F(u,v)) P(Y|_{(0,\infty)}) \in dv) \ P^b_o(du).$$

Since $b \mapsto P^b_o(A)$ is $E$-measurable for $A \in \mathcal{U}$, the same holds for $b \mapsto P^b(A)$. If $T$ is any $(\mathcal{G}_T)$ stopping time, then by the strong Markov property of PPP's (see Ito [32] Theorem 5.1), $\mathcal{G}_T^o (Y)$ is independent of $F_{L_T}$, with the same law as $Y|_{(0,\infty)}$.

By replacing $T$ with the $(\mathcal{G}_T)$ stopping times

$$T_B = \begin{cases} T & \text{on } B \\ \infty & \text{off } B \end{cases},$$

where $B \in \mathcal{G}_T$, it follows from Corollary 2 that for $A \in \mathcal{U}$,

$$P(Y_{L_T}^{\cdot + T - S^-(L_T)}) \in A|_{\mathcal{G}_T} = \mathcal{X}_T^o (A) \text{ on } \{T < \infty\}.$$

Since also $\mathcal{G}_T \subseteq F_{L_T}$, $Y_{L_T}^{\cdot + T - S^-(L_T)} \in F_{L_T}$, and

$$X_{\cdot + T} = F(Y_{L_T}^{\cdot + T - S^-(L_T)}, \mathcal{X}_T^o Y),$$
this yields the strong Markov property of \((X_t, G_t, P, P^b)\) at \(T\).

By definition of \((P^b)\), \((X_t, G_t, P, P^b)\) is a recurrent extension of \((P^b)\). The remaining points have already been dealt with, except for showing that \(Y\) is the PPP constructed from \(X\) as in Ito [32]. Since \(P^a(\sigma_a = 0) = 1\), this will follow provided \((L_t)\) satisfies (2.3).

The set of increase of \((L_t)\) is exactly \(\{t; X_t = a\}\), since \(X_t = a\) only if \(t = S^+(s)\) for some \(s\). The normalization

\[
E[e^{-\sigma_a(x)}] = E \left[ e^{-t} dL_t \right]
\]

follows easily from the PPP nature of \(Y\), and Theorem 4.5 of Ito [32]. Finally, we can write

\[
(X_{*+T}, L_{*+T} - L_T) = (F(Y_{L_T} (\cdot + T - S^- (L_T)), \theta_{L_T} Y), \ell (\theta_{L_T} Y))
\]

so that by Corollary 1, if \(T\) is any \((G_t)\) stopping time with \(X_T = a\), then \((X_{*+T}, L_{*+T} - L_T)\) is independent of \(G_T\) with a law not depending on the choice of \(T\), as required. Thus, except for the proof of Lemma 7, the proof of part (a) is complete.

**Proof of Lemma 7:** Choose \(\delta, \epsilon > 0\) with \(n(\sigma > \delta, > 0)\) for each \(k\). For \(q \in \mathbb{Q}, q \geq \epsilon\), let

\[
S^k_q = \inf \{ s > q; \sigma_a(Y_s) > \delta_k \}.
\]

The \(S^k_q\) are \((F_s)\) stopping times. By completeness of \(F_s\), we may assume without loss of generality that \(N(\{\sigma_a > \delta_k\} \times [0, s], Y(\omega)) < \infty\) for every \(\omega, k\) and \(s\), and that \(N(\{\sigma_a > \delta_k\} \times [0, \infty), Y(\omega)) = \infty\) for every \(\omega\) and \(k\). Thus, each \(S^k_q\) is surely finite, and (writing \([V]\)) for the graph \(\{(t, \omega); 0 \leq t < \infty, t = V(\omega)\}\)
\[ \{ \sigma_a(L_T) > s_k \} \subset \bigcup_{q \in \mathbb{Q}_+} \{ S^k_q \} \]

Also, for any \(k, k', q, q'\) we have that

\[ s^k_q \leq s^{k'}_q, \text{ if } k \geq k' \text{ and } q \leq q', \]

\[ \{ s^k_q = s^{k'}_q \} \in F^k_{s_q} \text{ if } k \leq k'. \]

Put

\[ R^k_q = (T - S^-(L_T)) \{ L_T = S^k_q \} \]

\[ H^k_t = H^s_t. \]

\[ ^{k'}_t \in F^k_{s_q} \theta \bigcup_t \]

By Lemma 4(d), \( R^k_q \) is an \( (H^k_t, q) \) stopping time.

Use Lemma 5 with \( \Omega_1 = \Omega, F^1_t = H^k_t, \Omega_2 = \Omega \times U, F^2_t = H^k_t, Z(\omega) = (\omega, Y^k_s(\omega)), \)

to obtain an \( (H^k_t, q) \) stopping time \( R^k_q \) with

\[ R^k_q(\omega, Y^k_s(\omega)) = R^k_q(\omega) \text{ for each } \omega \in \Omega. \]

Let

\[ \begin{cases} \tilde{R}^k_q(\omega, u), & \text{if } k' \geq k, q' \in \mathbb{Q}_+, s^k_q(\omega) = s^{k'}_{q'}(\omega), \\ \infty, & \text{if for every } k'' \geq k', q'' \in \mathbb{Q}_+ \text{ this fails to hold.} \end{cases} \]

\[ \tilde{R}^k_q(\omega, u) \text{ such that } S^k_{q''}(\omega) = S^{k'}_{q'}(\omega), \text{ also } R^k_q(\omega, u) = R^k_q(\omega, u). \]
Thus, for every $k, k' \in \mathbb{Z}_+$ and $q, q' \in \mathbb{Q}_+$, we have that

$$\hat{R}^k_q = \hat{R}^{k'}_{q'} \text{ on } \{s^k_q = s^{k'}_{q'}\} \times U,$$

and

$$\hat{R}^k_q(\omega, X^k(\omega)) = (T - S^- (L^t))(\omega) \text{ for } \omega \in \{L^t = s^k_q\}.$$ 

We now show that $\hat{R}^k_q$ is an $(\hat{H}^k, q)$ stopping time. Let $t > 0$, and choose an open base $B_1, B_2, \ldots$ of the space $(0, t)$. Then

$$\{\hat{R}^k_q < t\} = \bigcup_{k' \geq k} \bigcap_{k'' > k} \bigcup_{q' \in \mathbb{Q}_+} \bigcup_{q'' \in \mathbb{Q}_+} \left( \bigcup_{i \geq 1, j \geq i} \left( \{\hat{R}^{k''}_q \in B_j\} \cap \{s^{k''}_{q''} = s^k_q\} \times U \right) \cap \bigcup_{i \geq 1} \{R^k_q \in B_j\} \cap \{s^{k'}_{q'} = s^k_q\} \times U \right).$$

Since $\{s^{k'}_{q'} = s^k_q\} \in F_{s^k_q}$ whenever $k' \geq k$, it will thus suffice to show that

$$\hat{H}^{k', q'}_t \cap \{s^{k'}_{q'} = s^k_q\} \times U \subset \hat{H}^k,q_t$$

for every $k' \geq k, q' \in \mathbb{Q}_+$. This holds, since by monotone class arguments,

$$(F_{s^{k'}_{q'} \ominus U_t}) \cap \{s^{k'}_{q'} = s^k_q\} \times U = (F_{s^{k'}_{q'} \ominus \{s^{k'}_{q'} = s^k_q\}} \oplus U_t \cap \{s^{k'}_{q'} = s^k_q\}) \oplus U_t \subset F_{s^k_q \ominus U_t}.$$
Put
\[
R = \bigwedge_{k,q} (\mathbf{R}_q^k) \{ L_T = S_q^k \} \times U.
\]

Then \( R = R_q^k \) on \( \{ L_T = S_q^k \} \times U \). We will show that \( R \) is an \( (H_{t+}) \) stopping time. Let \( t > 0, k \in \mathbb{N}_+ \). Then
\[
\{ R < t \} \cap \{ \sigma (Y_L) > \delta_k \} \times U
= \bigcup_{q \in \mathbb{N}_+} \{ R_q^k < t \} \cap \{ S_q^k = L_T \} \times U.
\]

As before,
\[
\hat{\mu}_{k,q}^{k} \cap \{ S_q^k = L_T \} \times U
= H_t \cap \{ S_q^k = L_T \} \times U.
\]

Since also
\[
\{ S_q^k = L_T \} = \{ q < L_T \leq S_q^k \} \cap \{ \sigma (Y_L) > \delta_k \},
\]
we get that for each \( k \),
\[
\{ R < t \} \cap \{ \sigma (Y_L) > \delta_k \} \times U
= H_{k} \cap \{ \sigma (Y_L) > \delta_k \} \times U,
\]
for some set \( H_{k} \in H_t \). Using that \( \sigma (Y_L) > 0 \) on \( \{ T < \infty \} \), it is easy to see that in fact
\[
\{ R < t \} = \bigcap_{k \geq 1} \bigcup_{\ell \geq k} \{ T = \infty \} \times U \in H_t,
\]
as required.

Properties (3.4) and (3.5) are now immediate. (3.3) will follow
from (3.5), (3.6), and the convention that $\infty/\infty = 0$. Thus all that remains is to show (3.6).

Let $A \in F_r$. By the Markov property of $Y$ at $r$, and Theorem 4.4A of Ito [32],

$$P(Y^k \in B, S^k_q > r, A) = P(A, S^k_q > r)P(Y^k_{(q-r)\sigma^o} \in B)$$

$$= P(A, S^k_q > r) \frac{n(B, \sigma_a > \delta_k)}{n(\sigma_a > \delta_k)},$$

for $B \in U$. That is, for $f = 1_A \times B, A \in F_{S^k_q}$, $B \in U$ we have

$$E[f(\cdot, Y^k_{(\cdot))}, q] = \frac{1}{n(\sigma_a > \delta_k)} \int \int f(\omega, u) n(du) P(d\omega)$$

Therefore this holds for $f \in F_{S^k_q}$, $f \geq 0$. Take

$$f(\omega, u) = 1_B(\omega) l_A(u) l_B(R^k_{q}(\omega, u)) l_{\{R^k_{q} \leq \infty\}}(\omega, u),$$

where $C \in F_{S^k_q}$, $A \in U$, $B \in B$. Then

$$P(C, Y^{L_T} \in A, T - S^-(L_T) \in B, L_T = S^k_q) = E[f(\cdot, Y^k_{(\cdot))}, q]$$

$$= \int \frac{n(A, R^k_{q}(\omega, \cdot) \in B, R^k_{q}(\omega, \cdot) << \infty, \sigma_a > \delta_k)}{n(\sigma_a > \delta_k)} P(d\omega)$$

$$= \int \frac{P(L_T = S^k_q \mid F_{S^k_q}(\omega)) n(A, R^k_{q}(\omega, \cdot) \in B, R^k_{q}(\omega, \cdot) << \infty, \sigma_a > \delta_k)}{n(R^k_{q}(\omega, \cdot) << \infty, \sigma_a > \delta_k)} P(d\omega)$$

$$= \int \frac{n(A, R^k_{q}(\omega, \cdot) \in B, R^k_{q}(\omega, \cdot) << \infty, \sigma_a > \delta_k)}{n(R^k_{q}(\omega, \cdot) << \infty, \sigma_a > \delta_k)} P(d\omega).$$
Enumerate \( \emptyset_+ \) as \( q_1, q_2, \ldots \), and let \( C \in F_{T^{-\infty}} \). Then there are \( \mathcal{C}_k \in F_{S_k} \) such that

\[
C \cap \{ L_T = S_k^q \} = \mathcal{C}_k \cap \{ L_T = S_k^q \}.
\]

Thus

\[
P(C, Y_T < A, T - S^{-}(L_T) < B, \sigma_{A(T)} > \delta_k, T < \infty) = \sum_j \left[ \frac{n(A, R_{q_j}^k(\omega, *)) \in B, R_{q_j}^k(\omega, *) < \infty, \sigma > \delta_k}{n(R_{q_j}^k(\omega, *) < \infty, \sigma > \delta_k)} \cdot P(d\omega) \right],
\]

\[
= \sum_j \left[ \frac{n(A, R(\omega, *)) \in B, R(\omega, *) < \infty, \sigma > \delta_k}{n(R(\omega, *) < \infty, \sigma > \delta_k)} \cdot P(d\omega) \right],
\]

\[
C \cap \{ S_j^k = L_T \} \cap \{ S_j^k \neq S_i^k \text{ for } i < j \}
\]

Thus \( \{ S_j^k \neq S_i^k \} \in F_{S_k} \cdot \)

If \( A \) is chosen so that \( n(A) < \infty \), then the integrand converges boundedly to

\[
\frac{n(A, R(\omega, *)) \in B, R(\omega, *) < \infty}{n(R(\omega, *) < \infty)}.
\]
Thus
\[ P(C, Y_L \in A, T - S^-(L_T) \in B, T < \infty) \]
\[ = \int \frac{n(A, R(\omega, \cdot) \in B, R(\omega, \cdot) < \infty)}{n(R(\omega, \cdot) < \infty)} P(\omega) \]
whenever \( n(A) < \infty \). Since \( n \) is \( \sigma \)-finite, this holds for every \( A \in \mathcal{U} \), which yields (3.6). \( \square \)

Part (a) is proven. The proof of part (b) is similar. For \( u_0, u_1, u_2, \ldots \in U \), put
\[ S_k(u_0, u_1, \ldots) = \sum_{i \leq k} a_i, k \geq 0 \]
\[ F(u_0, u_1, \ldots)(t) = \begin{cases} u_k(t - S_k(u_0, u_1, \ldots)), & \text{if } S_k(u_0, u_1, \ldots) \leq t \\ a, & \text{if } t > S_k(u_0, u_1, \ldots). \end{cases} \]
Put
\[ X_t = F(Y_0, Y_1, \ldots)(t), S(k) = S_k(Y_0, Y_1, \ldots). \]

For \( A \in \mathcal{U} \), put
\[ P^b(A) = \begin{cases} \int \int 1_A(F(u, v_1, v_2, \ldots)) P((Y_1, Y_2, \ldots) \in d(v_1, v_2, \ldots)) P_o(du), & \text{if } b \neq a \\ \int \int 1_A(F(u, v_1, v_2, \ldots)) P((Y_1, Y_2, \ldots) \in d(v_1, v_2, \ldots)) n(du), & \text{if } b = a. \end{cases} \]

We let \( F_{-1} \) be the set of \( P \)-null sets of \( F \). For \( R \) a random variable with values in \( \{-1, 0, 1, 2, \ldots\} \), \( F^R \) is defined to be the \( \sigma \)-field generated by sets in \( F_{-1} \cap \{R = r\} \), for \( -1 \leq r < \infty \). For \( R \) a non negative integer valued random time, we put
For \( r \geq 0 \), let \( L_r = \inf \{ k; S(k+1) > r \} \). We put

\[
\mathcal{G}_t^0 = H_t^{L_t} - (t - S(L_t))^{-}, \quad G_t = \mathcal{G}_t^0, \quad G_0^- = \mathcal{G}_0^0 = \mathcal{F}_1
\]

Since \( n(U \setminus U^\Delta) = 0 \), we have that \( S(k) \to \infty \), a.s., and that for each \( k \geq 1 \), \( S(k) < \infty \), \( S(k-1) < S(k) \), and \( X_{S(k)} = a \), a.s. Delete a null set of \( \Omega \) to ensure that these statements hold surely. The discrete-time analogue to Lemma 4 is;

**Lemma 4'**

(a) \( (\mathcal{G}_t) \) is right continuous, increasing, and each \( \mathcal{G}_t \) contains all the \( \mathbb{P} \)-null sets of \( \mathcal{F} \). \( X_t \in \mathcal{G}_t \) for each \( t \geq 0 \).

(b) If \( T \) is a \( (\mathcal{G}_t) \) stopping time, then \( L_T \) is an \( (\mathcal{F}_k) \) stopping time, and \( F_{L_T-1} \subset \mathcal{G}_T \subset F_{L_T} \). If further \( X_T \neq a \) on \( \{ T < \infty \} \), then

\[
F_{L_T-1} \subset \mathcal{G}_T^-.
\]

(c) \( S(k) \) is a \( (\mathcal{G}_t) \) stopping time, for \( k \geq 0 \). If \( T \) is a \( (\mathcal{G}_t) \) stopping time such that \( X_T = a \) on \( \{ T < \infty \} \), then \( F_{L_T-1} \supset \mathcal{G}_T^- \).

(d) If \( R \) is an \( (\mathcal{F}_k) \) stopping time, and \( T \) is a \( (\mathcal{G}_t) \) stopping time, put

\[
V = \begin{cases} 
T - S(R) & \text{on } \{ L_T = R < \infty \} \\
\infty & \text{otherwise}
\end{cases}
\]

Then \( V \) is an \( (\mathcal{H}_t^R) \) stopping time.

**proof:** The proofs of part (a), and that for \( T \) a \( (\mathcal{G}_t) \) stopping time, \( L_T \) is an \( (\mathcal{F}_k) \) stopping time and \( \mathcal{G}_T \subset F_{L_T} \), are as in Lemma 4. Further,
\[ F_{L_t-1} \cap \{ T < t \} = F_{L_t-1} \cap \{ L_t - 1 \leq L_t - 1 \} \cap \{ T < t \} \]

\[ \subset F_{L_t-1} \cap \{ L_t - 1 \leq L_t - 1 \} \cap \{ T < t \} \]

\[ = F_{L_t-1} \cap \{ T < t \} \subset G_t, \]

so that \( F_{L_t-1} \subset G_t. \)

For \( k \geq 0 \) we have that

\[ \{ S(k) \leq t \} = \{ k \leq L_t \} \subset F_{L_t-1} \subset G_t, \]

so that \( S(k) \) is a \( (G_t) \) stopping time. Applying the previous computation, we see that

\[ F_k \subset G_{S(k + 1)}. \]

Let \( T \) be a \( (G_e) \) stopping time such that \( X_T \neq a \) on \( \{ T < \infty \} \), and let \( A \in F_k. \) Then

\[ A \cap \{ L_T - 1 \geq k \} = A \cap \{ L_T \geq k + 1 \} \]

\[ = A \cap \{ T > S(k + 1) \} \]

\[ \subset G_{S(k + 1)} \cap \{ T > S(k + 1) \} \subset G_{T-}. \]

Now let \( T \) be a \( (G_e) \) stopping time such that \( X_T = a \) on \( \{ T < \infty \} \). Then

\[ \{ T > r \} = \{ L_T > L_T \}. \]

By monotone class arguments, \( G_{T-} \) is generated by

\[ G_0 = F_{L_T-1} \subset F_{L_T-1}, \]

and by

\[ G_r \cap \{ T > r \} \subset F_{L_T} \cap \{ L_T - 1 \geq L_T \} \subset F_{L_T-1}, \]

for \( r \geq 0 \). Thus part (c) is shown. Part (d) follows as in Lemma 4. \( \Box \)
The discrete time version of Lemma 7 is

**Lemma 7'** Let \( T \) be a \((G_t)\) stopping time such that \( X_0 = a \) on \( \{L_T = 0, T < \infty\} \). Let \( H_t = F_{L_T-1} \otimes U_t \). Then there is an \((H^+_t)\) stopping time \( R \) such that

- (3.8) \( R(\omega, Y_{L_T}(\omega)) = (T - S(L_T))(\omega) \) if \( T(\omega) < \infty \)
- (3.9) \( R(\omega, u) = \infty \) for every \( u \in U \) if \( T(\omega) = \infty \)
- (3.10) \( P(Y_{L_T} \in A, T - S(L_T) \in B, T < \infty | F_{L_T-1}(\omega)) = \frac{n(u; u \in A, R(\omega, u) \in B, R(\omega, u) < \infty)}{n(u; R(\omega, u) < \infty)} \) for \( P\)-a.e. \( \omega \),

where \( A \in U, B \in \mathcal{B}, \) and we take the convention that \( 0/0 = 0 \).

**Proof:** For \( k \geq 0 \), let

\[
R^k = (T - S(L_T)) \{L_T = k\}
\]

\[
\hat{H}^k_t = F_{k-1} \otimes U_t
\]

By Lemma 4', (d), \( R^k \) is an \((H^+_t)\) stopping time. Use Lemma 5 with

\[
\Omega_1 = \Omega, F^1_t = \hat{H}^k_t, \Omega_2 = \Omega \times U, F^2_t = \hat{H}^k_t, \text{ and } Z(\omega) = (\omega, Y_k(\omega)), \text{ to obtain an } \hat{R}^k \text{ stopping time } \hat{R}^k \text{ such that for every } \omega \in \Omega,
\]

\[
\hat{R}^k(\omega, Y_k(\omega)) = R^k(\omega).
\]

Let

\[
R = \Lambda(\hat{R}^k)
\]

\[
k \{L_T = k\} \times U.
\]

Thus for \( t \geq 0 \),
\{ R < t \} = \bigcup_k \{ R^k < t \} \cap \{ L_T = k \} \times U.

By monotone class arguments,
\[ \hat{H}_t^k \cap \{ L_T = k \} \times U = \left( F_{k-1} \cap \{ L_T = k \} \right) \otimes U_t \]
where \( c F_{L_T-1} \otimes U_t = H_t^k \),
so that \( R \) is an \((H^+_{t})\) stopping time. Properties (3.8) and (3.9) are immediate. For \( k \geq 1 \), \( Y_k \) is independent of \( F_{k-1} \) with law \( n \), so that
\[
E[f(\cdot, Y_k(\cdot))] = \int (f(\omega, u) n(du) P(\omega))
\]
for \( f = 1_A \times B, A \in F_{k-1}, B \in U \). Thus this holds for \( f \in F_{k-1} \otimes U \), \( f \geq 0 \). As in the proof of Lemma 7 it follows from this that for \( k \geq 1 \), \( C \in F_{k-1}, A \in U, \) and \( B \in B \), we have that
\[
P(C, Y_{L_T} \in A, T - S(L_T) \in B, L_T = k, T < \infty)
= \int \frac{n(A, R^k(\omega, \cdot) \in B, R^k(\omega, \cdot) < \infty) P(\omega)}{C \cap \{ L_T = k \}} n(R^k(\omega, \cdot) < \infty)
\]

We will show this for \( k = 0 \) as well, as then the argument of Lemma 7 will give (3.10). Let
\[ E_{br} = \{ b \in E: P_0^b(W_0 = b) \neq 1 \} . \]
Since \((W_t, U, P_o^c, P_o^b)\) is strong Markov for each \( c \in E \), we get that
\[ P_c^o(W_0 \in E_{br}) = 0, \text{ for every } c \in E. \]
By hypothesis,
\[
P(Y_o \in M) = \int_{U \setminus U^a} P_o^{u(o)}(M) P(Y \in du) + n(M) P(Y_o \in U^a),
\]
so that as \( a \not\in E_{br} \) and \( n(U\setminus U^a) = 0 \), we see that

\[
P(Y_0(o) \in E_{br}) = 0
\]

Thus,

\[
P(Y_o \in M \cap U^a) = \int_{U\setminus U^a} P(Y_0 \in du) + n(M) P(Y_0 \in U^a)
\]

\[
= n(M) P(Y_0 \in U^a).
\]

Since \( F^{-1} \) consists of sets of \( P \)-measure 0, it follows that for \( f \in F^{-1} \) and \( f \geq 0 \),

\[
E[f(\cdot, Y_0(\cdot)), Y_0 \in U^a] = E[\int f(\cdot, u) n(du)] P(Y_0 \in U^a).
\]

Taking \( f \) as before and using the hypothesis that \( X_0 = a \) if \( L_T = 0 \), we see that for \( C \in F^{-1}, A \in U, B \in B \),

\[
P(C, Y_{L_T} \in A, T - S(L_T) \in B, L_T = 0, T < \infty)
\]

\[
= E(C, n(A, R_0^B, R_0^{<\infty})] P(Y_0 \in U^a)
\]

\[
= \frac{E(C, n(A, R_0^B, R_0^{<\infty})]}{E(n(R_0^{<\infty})]} P(L_T = 0)
\]

\[
= \int_{C \cap \{L_T = 0\}} \frac{n(A, R_0^B, R_0^{<\infty})]}{n(R_0^{<\infty})} P(\omega) .
\]

Once the following result is established, the remainder of part (b) follows immediately, as before.

**Corollary 2'** Let \( T \) be a \((G_t)\) stopping time, and \( A \in U \). Then

\[
P(Y_{L_T}(\cdot + T - S(L_T)) \in A, T < \infty)
\]

\[
= E[P_{X_T}(A), X_T \neq a, T < \infty] + n(A) P(X_T = a, T < \infty) .
\]
proof: As before, it suffices to treat three cases; that on \( \{T < \infty \} \), respectively \( X_T = a, X_T \neq a \) and \( L_T > 0 \), or \( X_T \neq a \) and \( L_T = 0 \). The second case is handled as before, using Lemmas 7' and 6. In the third case, we use Lemma 4'(d) to see that \( T \) is an \((H^0_{t+})\) stopping time. Since \((H^0_{t+})\) is a completion of the natural right continuous filtration of the process \( X, \wedge_{a}^{\infty}(X) \), and

\[
P(Y \in M) = \int_{\bigcup U^a} P^U(0)(M) P(Y \in du), \text{ for } M \in U, M \cap U^a = \emptyset
\]

it is simple to use the strong Markov property of the processes \((W, U, P^C, P^b)\), for \( c \in E \), to conclude that the process

\[
(X_{t \wedge_{a}^{\infty}}(X), H^0_t, P, P^b)
\]

is strong Markov. This suffices.

In the third case, we apply Lemma 7' to obtain an \( R \) satisfying the conclusions of that result. Put \( H(\omega) = \{ u; R(\omega, u) = 0 \} \). Then for \( A \in U \),

\[
P(Y_{L_T} \in A, T < \infty) = \int \frac{n(H(\omega) \cap A)}{n(H(\omega))} P(d\omega)
\]

as \( T - S(L_T) = 0 \) on \( \{ T < \infty \} \). Since \( R \) is an \((H^0_{t+})\) stopping time, also \( R(\omega, \cdot) \) is a \((U_{t+})\) stopping time for each \( \omega \in \Omega \), so that \( H(\omega) \in U_{0+} \). Thus \( n|_{H(\omega)}^{H(\omega)} \) satisfies (iv), so that by (vi'a), it must be a multiple of \( n \). Since

\[
n(H(\omega)) = n|_{H(\omega)}^{H(\omega)}(H(\omega))
\]

we must in fact have \( n|_{H(\omega)}^{H(\omega)} = n \) or 0, so that \( n(H(\omega)) = 1 \) or 0 for each \( \omega \). Thus
\[ P(Y_t \in A, T < \infty) = n(A) P(T < \infty, n(H) = 1), \text{ for } A \in \mathcal{U}. \]

Putting \( A = \mathcal{U} \), we have that \( n(H) = 1 \) a.s. on \( \{T < \infty\} \), so that

\[ P(Y_t \in A, T < \infty) = n(A) P(T < \infty), \]

as required. \( \square \)

4. **Insufficiency of conditions (ii) and (vi)**

First, we present some examples to show that the conditions (vi) and (ii) do not suffice.

**Example 1**: (vi a) is not sufficient:

Let \( P_0 \) correspond to uniform motion as indicated, with absorption at \( a \).

There are two excursion measures \( n_1, n_2 \) corresponding to strong Markov processes visiting \( a \) discretely. \( \frac{n_1 + n_2}{2} \) satisfies (vi a), and gives a Markov process which visits \( a \) discretely, but is not strong Markov.

**Example 2**: (vi c) is not sufficient:

Consider a Bessel process on \( (0, \infty) \) (so that \( 0 \) is an entrance, non-exit point), and make the point \( 1 \) absorbing. Wrap \( (0,1] \) around to make a circle \( E \), and let \( P_0 \) correspond to the resulting process on \( E \). Let \( a = \{1\} \). \( P_0 \) corresponds to a continuous process that is absorbed at \( a \), but which approaches \( a \) only from the counter clockwise direction.

We have strong Markov continuous recurrent extensions with a
instantaneous, corresponding to making the Bessel process (slowly) reflecting at 1, with various delay coefficients. Let the excursion measure with delay coefficient \( m \) be \( n_m \), so that
\[
m = 1 - \int (1 - e^{-\sigma a}) \, dn_m.
\]

There are also continuous 'recurrent extensions' which are not strong Markov, corresponding to stopping the original process at \( a \), holding it there an exponential time, and then making it enter \( E \{a\} \) in the counterclockwise direction (this is possible, as 0 is an entrance point for the Bessel process). For \( m \in (0,1) \), this gives an 'excursion measure' \( n'_m \) such that
\[
m = 1 - \int (1 - e^{-\sigma a}) \, dn'_m.
\]

(\( m \) determines the mean of the holding time at \( a \)).

Though condition (vi'c) fails for the \( n'_m \), it will hold for any measure \( n_m + n'_m \) for which \( m + q \geq 1 \) and \( m, q \in (0,1) \). These measures are ruled out by (vi'c).

**Example 3:** (ii) does not suffice:

Let \( E \) be the subset of \( \mathbb{R}^2 \) described in polar coordinates as
\[
\{ (r \cos \theta, r \sin \theta); r = R \cos \theta \geq 0 \}
\]
\( R \in (1,2] \)

Let \( a \) be the origin, and let \( (P^b_o) \) correspond to uniform clockwise motion around the circles \( r = R \cos \theta \), at speed \( (\pi R(1-R))^{-1} \), with absorption at \( a \).
For any $\sigma$-finite measure on $E\setminus\{a\}$, $n = p_{0}^{u}$ satisfies (i), (iv), (v), and (vi'). Let $n_{0}$ be the excursion measure of one dimensional Brownian motion, from 0, and let $M(u) = \max\{|u(t)|; 0 \leq t < \infty\}$, $u \in U$.

Let $f:E \to (0,1] \times (0,2\pi)$ be given by

$$f(r \cos \theta, r \sin \theta) = \left(\frac{r}{\cos \theta} - 1, \frac{\pi}{2} - \theta\right).$$

Then for $\mu(A) = n_{0}((\sigma_{a}, M) \in f(A))$, we get properties (ii) and (iii) for free, but (ii') fails, so that the resulting process is not right continuous.

We can replace (ii') by the more appealing condition (ii), provided we assume some regularity of $(P_{o}^{b})$;

**Proposition 2** (a) Suppose that $(P_{o}^{b})$ satisfies (2.4) and (2.5), and that $n$ satisfies (i), (iii), (iv) and (v). Suppose also that

$$(4.1) \quad \text{for every open neighbourhood } V \text{ of } a, \text{ there is an open neighborhood } V' \text{ of } a, V' \subset V, \text{ such that }$$

$$\sup_{b \in V'} \frac{P_{o}^{b}((W_{t}) \text{ leaves } V)}{P_{o}^{b}(1-e^{-\sigma_{a}})} < \infty$$

(with the convention that $0/0 = 0$). Then (ii) holds if and only if (ii') does.

(b) Conversely, suppose that for every initial measure $\mu$ with

$\mu(a) = 0$ and $E_{o}^{u}[1 - \exp(-\sigma_{a})] < 1$, the measure $P_{o}^{u}$ is the excursion measure of a right continuous process. Then (4.1) holds.

**Proof:** (a) Let $V, V'$ be as above, and assume that (ii) holds.

Then
The first term is finite by (ii), and the second term is

\[ n(W_0 \in V', (W_t) \text{ leaves } V) \]

\[ = \int_{\{W_0 \in V' \setminus \{a\}\}} p^{u(o)}(W_t \text{ leaves } V)n(du) + \lim_{\delta \to 0} \int_{\{W_0 = a, W_t \in V' \text{ for } t \in [0, \delta]\}} p^{u(\delta)}(W_t \text{ leaves } V)n(du) \]

\[ \leq \sup_{b \in V' \setminus \{a\}} \frac{p^b((W_t) \text{ leaves } V)}{b^a} \left[ \int_{\{W_0 \in V' \setminus \{a\}\}} E^{u(o)}(1-e^{-\sigma a})n(du) + \right. \]

\[ + \lim \inf_{\delta \to 0} \left. \int_{\{W_0 = a, W_t \in V' \text{ for } t \in [0, \delta]\}} E^{u(\delta)}(1-e^{-\sigma a})n(du) \right] \]

\[ \leq \sup_{b \in V' \setminus \{a\}} \frac{p^b((W_t) \text{ leaves } V)}{b^a} \left[ \int_{\{W_0 \in V' \setminus \{a\}\}} (1-e^{-\sigma a})dn + \lim \inf_{\delta \to 0} \left. \int_{\{W_0 = a, W_t \in V' \text{ for } t \in [0, \delta]\}} (1-e^{-\sigma a - \delta})dn \right] \]

\[ \leq \sup_{b \in V' \setminus \{a\}} \frac{p^b((W_t) \text{ leaves } V)}{b^a} \left( 1-e^{-\sigma a} \right)dn < \infty. \]

(b) Assume (4.1) fails for some open neighborhood V of a. Then there are \( b_k \in E \setminus \{a\}, b_k \to a \) such that

\[ a_k = \frac{p^{b_k}(W_t \text{ leaves } V)}{E^{b_k}(1-e^{-\sigma a})} \to \infty. \]

By passing to a subsequence, we may assume that \( a_k \geq k \) for each k.
Let
\[ \lambda_k = \left( k^2 E^b \left( 1 - e^{-\sigma a} \right) \right)^{-1} \]
\[ \mu = \sum_{k=2}^{\infty} \lambda_k e^b_k \]
(where \( e_b \) is the point mass concentrated at \( b \)). Then
\[ E^b \left( 1 - e^{-\sigma a} \right) = \sum_{k=2}^{\infty} k^{-2} < 1, \]
while
\[ p^b_0((W_t) \text{ leaves } V) = \sum_{k=2}^{\infty} \lambda_k a^k E^b_0 \left( 1 - e^{-\sigma a} \right) \]
\[ \geq \sum_{k=2}^{\infty} k^{-1} = \infty \]

The corresponding condition on \( (p^b_0) \) under which \((vi')\) may be replaced by \((vi)\) is that the class of positive measures \( \mu \) satisfying (i), (iii), (iv) and \( n(U \setminus U^a) = 0 \) consists of either multiples of a single probability measure, or consists completely of infinite measures. It is perhaps worth mentioning that though this condition fails for example 3, (4.1) is not in general a consequence of this condition. As example we can replace the space \( E \) of example 3 by
\[ E' = E \cap \{(x,y); y \leq \sqrt{2x}\}. \]
In this case there are no measures satisfying (i) and (iv), and concentrated on \( U^a \), since no path \( r = R \cos \theta, R \in (1,2] \) lies entirely within \( E' \), whereas by Proposition 2(b), there do exist \( n \) satisfying (i), (ii), (iii), (iv), (v), (vi') but not (ii').
5. Proof of Proposition 1

Condition (ii') is clearly necessary. Assume that \( a \) is instantaneous \((n(U) = \infty)\), but (vi'c) fails. We will find a set \( H \in U_{\omega^+} \) such that \( 0 < n(H) < \infty \), and \( H \subset U \cap \{\sigma_a > 0\} \). Let \( H^0 \in U_{\omega^+}^0 \) satisfy \( H^0 \subset H \) and \( n(H\setminus H^0) = 0 \). By completeness of each \( F_t \), we obtain easily that for \( h > 0 \), the \((U,H_t^0)\)-valued process \((X_t^0)\) is progressively measurable, for the filtration \((F_t+)_t\). Thus,

\[
T = \inf \{t > 0; X_{t^+} \in H^0\}
\]

is an \((F_{t+h^+})_{t \geq 0}\) stopping time, for every \( h > 0 \), so that it is in fact an \((F_{t^+})\) stopping time. \( T \) is finite almost surely since \( n(H^0) > 0 \), and \( X_{T^+} \in H^0 \) a.s., since \( n(H^0) < \infty \). This contradicts the strong Markov property of \((X_t,F_t,P,P^b)\), as \( a \) is instantaneous, yet \( X_T = a \) and \( \sigma_a(Y_{L_T}) > 0 \).

Similarly, in the case that \( X \) visits \( a \) discretely, but (vi'a) fails, we will obtain a contradiction to the strong Markov property of \( X \) by finding a set \( H \in U_{\omega^+} \) such that \( H \subset U \cap \{\sigma_a > 0\} \) and \( 0 < n(H) < 1 \).

In both cases, the argument would be simplified if we had assumed that \( X \) was a right process, and we had the Ray-Knight compactification at our disposal.

In the first case, let \( D \) be the set of dyadic rational numbers. For \( \varepsilon, \delta > 0 \) let

\[
B_{\varepsilon,\delta} = \{b \in E; \ P^b_0 (\sigma_a \geq \delta) \geq \varepsilon \} \in E
\]

\[
H_{\varepsilon,\delta} = \bigcup_{n>0} \{ W_0 = a, W_t \in B_{\varepsilon,\delta} \ \text{for} \ t \in D \cap (0,\eta), \ \sigma_a > 0 \}.
\]
We will show that for some $\epsilon, \delta > 0$ we have

$$0 < n(H_{\epsilon, \delta}) < \infty,$$

so that $H_{\epsilon, \delta}$ satisfies the above conditions.

For $\epsilon$ and $\delta$ fixed, let

$$\tau(u) = \inf \{ t > o; t \in D \text{ and } u(t) \notin B_{\epsilon, \delta} \}.$$

Then $\tau$ is a $(\mathcal{U}_{t^+}^\tau)$ stopping time. For $\rho \in D \cap (o, \infty)$, we have

$$\{ \tau > \rho \} \in T_{\rho^+},$$

so that by Lemma 6,

$$n(\sigma_a \geq \delta) \geq n(\tau > \rho, \sigma_a \geq \rho + \delta)$$

$$= \int_{\{ \tau > \rho \}} p_{\rho}^{u(\rho)}(\sigma_a \geq \delta) n(du)$$

$$\geq \epsilon n(\tau > \rho).$$

Letting $\rho \downarrow o, \rho \in D$ we obtain that $n(\sigma_a \geq \delta) \geq \epsilon n(\sigma_a \geq \delta)$. Thus

$$n(H_{\epsilon, \delta}) < \infty$$

for every $\epsilon, \delta > 0$.

Conversely, since (vi'c) fails, there is some measure $n'$ satisfying (iv) and (v) such that $n \geq n' \geq o$, and $o < n'(U^a) < \infty$. Fix $\epsilon, \delta > 0$ such that $n(H_{\epsilon, \delta}) = o$. Then also $n'(H_{\epsilon, \delta}) = 0$. Thus, for $\eta > o$,

$$n'(W_o = a, W_t \in B_{\epsilon, \delta} \text{ for every } t \in (o, \eta) \text{ such that } t = j 2^{-k} \text{ for some } j) \downarrow o \text{ as } k \to \infty.$$

That is,

$$\sum_{j=1}^{2^k \eta - 1} n'(W_o = a, W_{j2^{-k}} \notin B_{\epsilon, \delta}, W_{i2^{-k}} \in B_{\epsilon, \delta} \text{ for } 1 \leq i < j)$$

$$\downarrow n'(U^a) \text{ as } k \to \infty.$$
Thus, for \( \eta \in (0, \delta) \),

\[
\begin{align*}
n'(W_0 = a, \sigma_a \geq 2\delta) \\
&= \lim_{k \to \infty} \sum_{j=1}^{2^k-1} n'(W_0 = a, \sigma_a \geq 2\delta, W_{j2^{-k}} \notin \mathcal{B}_\varepsilon, \delta, W_{i2^{-k}} \in \mathcal{B}_\varepsilon, \delta) \\
& \quad \text{for } 1 \leq i < j
\end{align*}
\]

\[
\begin{align*}
&= \lim_{k \to \infty} \sum_{j=1}^{2^k-1} \left\{ \int_{0}^{u(j2^{-k})} (\sigma_a \geq 2\delta - j2^{-k}) \, n(du) \right\} \\
& \quad \{W_0 = a, \sigma_a \geq j2^{-k}, W_{j2^{-k}} \notin \mathcal{B}_\varepsilon, \delta, W_{i2^{-k}} \in \mathcal{B}_\varepsilon, \delta \} \\
& \quad \text{for } 1 \leq i < j
\end{align*}
\]

\[
\leq (1-\varepsilon) \, n'(U^a).
\]

Since \( n'(U^a) \in (0, \alpha) \), this cannot happen for every \( \varepsilon, \delta > 0 \), so that indeed \( n(H, \varepsilon, \delta) > 0 \) for some \( \varepsilon, \delta \).

In the second case, suppose that \( n \) is a probability measure concentrated on \( U^a \), yet (vi'a) fails. Then we may find \( \mu_1, \mu_2 \in (0,1) \) and probability measures \( n_1 \) and \( n_2 \), each concentrated on \( U^a \) and satisfying (i), (iv) and (v), such that \( n_1 \neq n_2 \) and \( n = \mu_1 n_1 + \mu_2 n_2 \).

Since \( n_1 \neq n_2 \), we obtain from Lemma 6 that there is an open neighbourhood \( V \) of \( a \), a set \( A \in \mathcal{E} \) with \( A \subset \mathcal{E} \setminus V \); and numbers \( \lambda_1, \lambda_2 \) such that

\[
n_1(W_t \in A) > \lambda_1 > \lambda_2 > n_2(W_t \in A),
\]

where \( \tau(u) = \inf \{ t > 0; u(t) \notin V \} \). Let \( V' \) be an open neighborhood of \( a \), with \( \overline{V'} \subset V \). Let

\[
B = \{ b \in V'; P^b_0 (W_t \in A) \geq \lambda_1 \}.
\]

Let \( D \) be the dyadic rationals, and put
\[ H = \bigcup_{n>0} \{ W_t = a, \sigma_a > 0, W_t \in B \text{ for every } t \in D \cap (0, n) \} . \]

Then as before we obtain that
\[ \lambda_2 > n_2 (W_t \in A) \geq \lambda_1 n_2 (H) , \]
so that \( n_2 (H) < \frac{\lambda_2}{\lambda_1} < 1 \), and hence
\[ n(H) = \mu_1 n_1 (H) + \mu_2 n_2 (H) < \mu_1 \mu_2 = 1. \]

Conversely, if \( n(H) = 0 \) then also \( n_1 (H) = 0 \), so that as before,
\[ \lambda_1 < n_1 (W_t \in A) \leq \lambda_1 n_1 (U^a) = \lambda_1 , \]
which is impossible. \( \square \)

6. Ray and Right Processes

Let \( (P^b) \) satisfy (2.4), and put
\[ U^a f(b) = E^b [ \int_0^\infty e^{-at} f (W_t) \, dt] , \]
for \( f \in E, \alpha > 0 \). \( (P^b) \) is said to be Ray if \( E \) is compact, and if for each \( \alpha > 0 \), \( U^a f \) is continuous whenever \( f \) is.

**Proposition 3** Let \( (P^b) \) be Ray, and satisfy (2.5). Let \( (P^b) \) satisfy (2.4) and (2.6). Then \( (P^b) \) is Ray.

**proof:**
\[ U^a f(b) = E^b [ \int_0^{T_a} e^{-at} f (W_t) \, dt] + E^b [ e^{-\alpha T_a} ] U^a f(a) \]
\[ U^a f(b) = E^b [ \int_0^{T_a} e^{-at} f (W_t) \, dt] + \frac{1}{\alpha} E^b [ e^{-\alpha T_a} ] f(a) . \]

Thus
\[ U^a f(b) = U^a f(b) + E^b [ e^{-\alpha T_a} (U^a f(a) - \alpha^{-1} f(a)) , \]
and so we need only show that \( E^b [ \exp(-\alpha T_a)] \) is continuous in \( b \).
Using partitions of unity, choose continuous functions \( f_n \) on \( E \) such that \( 0 \leq f_n \leq 1 \) and

\[
1_{E \setminus \{a\}} = \sum_{n=1}^{\infty} f_n.
\]

Each \( U_{\alpha}^{a} \) is continuous, so that by dominated convergence so is \( U_{\alpha_{\alpha}}^{a} \), whenever \( \alpha > 0 \). But

\[
U_{\alpha}^{a} = E_{\alpha}^{b} \left[ \int_{0}^{\tau_{a}} e^{-\alpha t} dt \right] = \frac{1 - E_{\alpha}^{b}[e^{-\alpha a}]}{\alpha}
\]

Proposition 4 Let \( (P_{\alpha}^{b}) \) be Ray, and satisfy (2.5). Suppose also that \( P_{\alpha}^{b}(U_{\alpha}^{a}) < 1 \) for each \( b \neq a \). Let \( n \) be a positive measure on \( (U,U) \) satisfying (i), (iv) and (v), and suppose that \( n(\sigma_{a} \geq \delta) < \infty \) for every \( \delta > 0 \). Then \( n \) satisfies (ii'). If \( n(U_{\alpha}^{a}) > 0 \) then \( n \) also satisfies (vi'c).

Remark: The condition that \( n(\sigma_{a} \geq \delta) < \infty \) for \( \delta > 0 \) will be satisfied provided (iii) holds. In particular, in the above situation, there are no 'discrete-visiting' extensions of \( (P_{\alpha}^{b}) \).

Proof: Suppose \( n(U_{\alpha}^{a}) > 0 \). If \( n \) is finite, or (vi'c) fails, then there is a finite non zero measure \( n' \) concentrated on \( U_{\alpha}^{a} \) which satisfies (iv), so that

\[
\mu_{\tau} (A) = n' (W_{\tau} \in A)
\]

defines a bounded system of entrance laws for \( (P_{\alpha}^{b}) \). Because \( (P_{\alpha}^{b}) \) is Ray, there is a finite measure \( \mu_{\alpha} \) on \( E \) such that \( \mu_{\tau} = P_{\alpha}^{\mu_{\alpha}}(W_{\tau} \in \cdot) \).

Thus \( n' = P_{\alpha}^{\mu_{\alpha}} \), which is impossible, as \( n' \) is concentrated on \( U_{\alpha}^{a} \). Thus (vi'c) holds, and (vi'a) is vacuous.
To show (ii'), observe as in Proposition 3, that \( E_o^b[1-\exp(-\sigma_a)] \)
is continuous in \( b \), hence
\[
V_\varepsilon = \{ b; \ E_o^b[1-e^{-\sigma_a}] < \varepsilon \}, \quad \varepsilon > 0
\]
define a nested family of open neighborhoods of \( a \). Since \( P_o^b(u^a) < 1 \)
for \( b \neq a \), their intersection is \( \{ a \} \). Thus if \( W \) is any other open
neighborhood of \( a \),
\[
\{ W \} \cup \{ E \setminus V_\varepsilon ; \; \varepsilon > 0 \}
\]
forms an open cover of \( E \). Since \( E \) is compact, this shows that the \( V_\varepsilon \)
form a base of open neighborhoods of \( a \). Thus, it suffices to show
that
\[
n(\{ W \}) \text{ leaves } V_\varepsilon ) < \infty
\]
for each \( \varepsilon > 0 \). Fix \( \varepsilon > 0 \), and let
\[
\tau(u) = \inf \{ t > 0; u(t) \notin V_\varepsilon \}.
\]
Then \( W_t \notin V_\varepsilon \) on \( \{ \tau < \infty \} \), since \( V_\varepsilon \) is open. Also, for \( b \in E \setminus V_\varepsilon \)
and \( \delta > 0 \) we have that
\[
\varepsilon \leq E_o^b(1 - e^{-\sigma_a}) \leq 1 - e^{-\delta} + P_o^b(\sigma_a \geq \delta),
\]
so that by lemma 6,
\[
\infty > n(\sigma_a \geq \delta) \geq n(\tau < \infty, \; \sigma_a \geq \delta + \tau)
\]
\[
= \int_{\{ \tau < \infty \}} P_o^u(\tau(u))(\sigma_a \geq \delta) n(du)
\]
\[
\geq (\varepsilon - (1 - e^{-\delta})) n (\tau < \infty).
\]
Choosing \( \delta \) small, we obtain that \( n(\tau < \infty) < \infty \),
We could use this result to verify the conditions of Theorem 2 for the measures $n$ considered in [4].

The condition that $P^b_o(U^a) < 1$ for each $b \neq a$ rules out the following pathology:

**Example 4:** Let $E = [0,1] \times \{0,1\}$, $a = (0,0)$. Make $a$ absorbing, and on $[0,1] \times \{1\}$ let $(P^b_o)$ correspond to Brownian motion, reflecting at $(1,1)$, with $(0,1)$ a branch point to $a$. On $(0,1] \times \{0\}$, let $(P^b_o)$ correspond to a Brownian motion reflected at $(1,0)$, except that there is a jump from $(x,0)$ to $(x,1)$ at rate $g(x)$, where $g(x) \to \infty$ fast enough as $x \to 0$ so that

$$P^b_o(\{W_t \text{ hits } [0,1] \times \{1\}\} = 1$$

for every $b \in E \setminus \{a\}$. Then we can find $\mu$ on $E \setminus \{a\}$ such that $n = P^\mu_o$ satisfies (i), (ii), (iii), (iv), (v), and (vi'), but not (ii'), even though $(P^b_o)$ is Ray.

We now consider Right processes; Let $E$ be a $U$-space, that is, a universally measurable subset of some compact metric space. Let $E$ be the $\sigma$-field of its universally measurable subsets. Let $(\Omega,F)$ be a measurable space, and let $(P^b)_{b \in E}$ be a family of probability measures on $(\Omega,F)$, such that $P^\cdot(B)$ is $E$-measurable for each $B \in F$. Let $(F^t)$ be a filtration of $(\Omega,F)$ which is right continuous, and satisfies

$$F^t = \bigcap_{\mu} F^\mu_t$$

where for each finite positive measure $\mu$ on $(\Omega,F)$, $F^\mu_t$ denotes the $\sigma$-field obtained from $F_t$ by adjoining all the $P^\mu$-null sets of the $\hat{F}^\mu$ completion of $F$. These conditions will be assumed throughout the
remainder of the section. If further
\[
\begin{align*}
& X \text{ is a right continuous process with values in } E, \\
& \text{which is adapted to } (F_t).
\end{align*}
\]
\begin{align}
\label{eq:6.1}
& \hat{\mathbb{P}}^b_{-} (X_0 = b) = 1 \text{ for each } b \in E, \\
& \hat{\mathbb{P}}^b_{-} (X_{t+h} \in B | F_t) = \hat{\mathbb{P}}^X_t (X_h \in B), \hat{\mathbb{P}}^b_{-} \text{-a.s., for each } b \in E, B \in \mathcal{U} \text{ and } t, h \geq 0
\end{align}
\begin{align}
\label{eq:6.2}
f(X_t) \text{ is } \hat{\mathbb{P}}^\mu_{-} \text{ a.s. right continuous, for each } \alpha > 0, \\
& \text{finite positive measure } \mu, \text{ and } \alpha \text{-excessive function } f.
\end{align}

Then \((X_t, F_t, F^C_t, \hat{\mathbb{P}}^b_t)\) is called a right process.

Under condition (6.1), we will write
\[
\hat{\mathbb{P}}^b (B) = \hat{\mathbb{P}}^b_{-} (X \in B), \text{ for } b \in E, B \in \mathcal{U}.
\]

Following Sharpe [46] we call a process \((Z_t)\), adapted to \((F_t)\), nearly optional if it is optional with respect to each filtration \((F^\mu_t)\).

A function \(f \in E\) is nearly Borel if for each \(\mu\) there exist Borel functions \(f_1\) and \(f_2\) on \(E\) such that \(f_1 \leq f \leq f_2\), and
\[
\hat{\mathbb{P}}^\mu (f_1 (X_t) \neq f_2 (X_t) \text{ for some } t \geq 0) = 0.
\]

Recall that in the presence of (6.1), condition (6.2) is equivalent to the conditions
\begin{align}
\label{eq:6.3}
& (X_t, F_t, F^C_t, \hat{\mathbb{P}}^b_t) \text{ is strong Markov for each } c \in E, \\
\label{eq:6.4}
& f(X_t) \text{ is nearly optional, for each } \alpha > 0 \text{ and each } \\
& \text{\alpha-excessive function } f.
\end{align}

A stronger condition than (6.4) is
\begin{align}
\label{eq:6.5}
& \text{For every } \alpha > 0, \text{ each } \alpha \text{-excessive function is nearly Borel.}
\end{align}
This property has the advantage of being invariant under choices of right continuous realizations, hence is a property of the transition laws \((P^b)\). That is, if \((X_t, F_t, F_t^b)\) satisfies (6.1), and the \((P^b)\) are the transition laws of some right process with all \(\alpha\)-excessive functions nearly Borel, for \(\alpha > 0\), then in fact \((X_t, F_t, F_t^b)\) has nearly Borel \(\alpha\)-excessive functions for \(\alpha > 0\) as well, so that it is a right process.

It is unknown (see Sharpe [46]) whether (6.4) (and hence (6.2)) are invariant in the same manner.

Suppose now that \(Y\) is a measurable function \((\Omega, F) \to (\Pi, P)\) such that with respect to each \(P^b\), \((Y_t, F_t)\) is a PPP with characteristic measure \(\nu\) not depending on \(b\). Define \((P^0_b)\) by

\[ P^0_b(B) = P^b(Y_0 \in B),\]

for \(B \in U\), and suppose that \((P^0_b)\) satisfies (2.4), (2.5) and

\[ P^0_b(W = b) = 1,\]

for each \(b \in E\). Suppose finally that \(\nu\) satisfies (i), (ii'), (iii), (iv), (v), and (vi'). In the proof of theorem 2, we used the completeness of our filtration to discard certain subsets of \(\Omega\). Under the present conditions on \((F_t)\), these sets will still lie in \(F_0\), so that the proof of theorem 2 will give a single process \((X_t, G_t)\) and a family of laws \((P^b)\), such that for every \(c \in E\), \((X_t, G_t, P^c, P^b)\) is strong Markov, and

\[ P^c(X \in B) = P^c(B)\]

for each \(B \in U\). We can ask when \((X_t, F_t, F_t^b)\) is a right process.
By the above discussion, the following gives a sufficient condition
for this to hold.

**Proposition 5**  Suppose that for each \( \alpha > 0 \), the \( \alpha \)-excessive functions
for \( (P_0^b) \) are nearly Borel. Then so are those for \( (P^b) \).

**proof:**  Let \( f \) be \( \alpha \)-excessive for \( (P^b) \), for some \( \alpha > 0 \). Then
\( f \) is \( \alpha \)-excessive for \( (P_0^b) \), so that by hypothesis,
it is also nearly Borel for \( (P_0^b) \). Using (iv) and (v) we
\( \exists \) Borel functions \( f_1 \) and \( f_2 \) with \( f_1 \leq f \leq f_2 \) and
\( n(f_1(W_t) \neq f_2(W_t) \text{ for some } t \geq 0) = 0 \).

It follows that \( f \) is nearly Borel for \( (P^b) \), as required. 

Until the question is settled, of whether (6.4) is invariant under
choices of right continuous realizations, it will be impossible to show
that \( (X_t, F, G_t, P^b) \) is in general a right process, even when the \( (P_0^b) \)
arise from one. We can show, however, that this is the only obstruction
to a proof of this result;

Assume that the \( (P_0^b) \) arise from a right process. Then there
is a compact metric space \( \hat{E} \) (the Ray-Knight compactification),
containing \( E \) as a universally measurable subset, such that the resolvant
of \( (P_0^b) \) extends to a Ray resolvant on \( \hat{E} \), separating points thereon.
Further, any right process on \( E \) with transition laws \( (P_0^b) \), is almost
surely right continuous in the topology of \( \hat{E} \).

Replace \( U \) by the set
\( \tilde{U} = \{ u; u \text{ is right continuous: } [0, \infty) \to E, \text{ in the E-topology,} \)
and is right continuous in the \( \hat{E} \) topology at all times \( t \)
such that \( u(t) \neq a \}. \)
\( \tilde{U} \) and \( (\tilde{U}_t) \) will be as before, with \( \tilde{U} \) replacing \( U \). The \( (P^b) \) induce laws \( (\tilde{P}^b) \) on this new \( \tilde{U} \). Suppose we are given \( n \) and \( Y \) satisfying the condition preceding Proposition 5, but for our new objects \( \tilde{U} \) and \( (\tilde{P}^0) \). Then the same argument applies to give a \( \tilde{U} \) valued process \( (X_t) \), a filtration \( (G_t) \), and laws \( (P^b) \) such that for each \( c \in \hat{E} \), \( (X_t, G_t, \hat{P}^c, P^b) \) is strong Markov, and \( \hat{P}^c(X \in B) = P^b(B) \) for \( B \in \tilde{U} \).

Proposition 6 \((X_t, F, G_t, \hat{P}^b)\) is a right process.

proof: As in Sharpe [46] we need only show that \( U^a f(X_t) \) is nearly optional whenever \( a > 0 \) and \( f \) is positive, bounded and \( E \)-measurable. Since \( E \) is universally measurable in \( \hat{E} \), we need only check this for \( f \) the restriction to \( E \) of a positive, bounded, \( \hat{E} \)-Borel function, and hence by monotone class arguments, only for \( f \) the restriction to \( E \) of a positive, \( \hat{E} \)-continuous function. The argument of Proposition 3 shows that for such \( f \), \( U^a f \) is the restriction to \( E \) of an \( \hat{E} \)-continuous function, so that in fact, we need only show that \( X \) is \( \hat{E} \)-right continuous, \( \hat{P}^b \) - a.s., for \( \mu \) finite and positive.

By the proof of Proposition 4, there is a base of \( \hat{E} \)-neighbourhoods \( V_\varepsilon \) of \( a \), such that

\[(\hat{E}^* \setminus V_\varepsilon) \text{ leaves } V_\varepsilon \text{ is finite.}\]

(6.6) \( n((W_t) \leq V_\varepsilon) < \infty \)

for each \( \varepsilon > 0 \). (Note that since the resolvent of \( (\tilde{P}^b) \) on \( \hat{E} \) separates the points of \( \hat{E} \), the condition that \( \tilde{P}^b(U^a) < 1 \) for each \( b \in \hat{E} \) is satisfied). In case (vi'b), this suffices, as in Theorem 2.

Suppose that (vi'c) holds. Then

\[ \{W_0 = a, W_{t_k} \in V_\varepsilon \text{ for some sequence } t_k \to 0 \} \in \tilde{U}_{O+}, \]

so that by (vi'c), it has \( n \)-measure zero. Thus
As in Theorem 2, (6.6) and (6.7) suffice to show the result in case (vi'c).

Suppose finally, that (vi'a) holds. We use entrance laws, as in the proof of Proposition 4 to conclude the analogue of (6.7); that

\begin{equation}
\text{(6.8) There is a point } \mathbf{C} \in \hat{E} \setminus E \text{ such that } \nonumber
\end{equation}

n(W_o = a, W_t \notin c \text{ in } \hat{E} \text{ as } t \to 0) = 0.

Since \( c \neq a \), \( \tilde{X}_t \) will not be \( \hat{E} \)-right continuous, however, the \( \tilde{U}^a f(X_t) \) will be, since \( a \) is an \( \hat{E} \)-branch point to \( c \), so that \( \tilde{U}^a f(a) = \tilde{U}^a f(c) \). \qed

We can reinterpret (vi') in terms of the Ray Knight compactification;

**Proposition 7** Let \((P_o^b), E, \text{ and } (\tilde{P}_o^b)\) be as above. Let \( n \) be a positive measure on \((\hat{U}, \tilde{U})\) satisfying (i), (iv), and (v) (for \( \tilde{E}, \tilde{U}, \text{ and } (\tilde{P}_o^b)\)), and suppose that \( n(\sigma_a > \delta) < \infty \) for every \( \delta > 0 \). Then

(a) \( n(\hat{E} - \lim W_t \text{ does not exist}) = 0 \)

(b) Suppose \( n \) is a probability measure concentrated on \( \tilde{U}^a = \{ u \in \tilde{U}; u(o) = a \} \). Then (vi'a) is equivalent to (6.8).

(c) Suppose \( n \) is an infinite measure, and \( n(\tilde{U}^a) > 0 \). Then (vi'c) is equivalent to (6.7).

**Proof:** (a) Clearly

\[ n(W_o \neq a, \hat{E} - \lim W_t \neq W_o) = 0, \]

so that without loss of generality, we may assume that

\[ n(\tilde{U} \setminus \tilde{U}^a) = 0. \]

As in the proof of Proposition 4, there is a base of open \( \hat{E} \)-neighborhoods \( V_\varepsilon \) of \( a \), such that
\( n(W_t) \text{ leaves } V_\varepsilon < \infty \),

for each \( \varepsilon > 0 \). Fix \( \varepsilon > 0 \), and let

\[
B_\varepsilon = \{ \text{for every } \delta > 0, W_s \not\in V_\varepsilon \text{ for some } s \in (0, \delta) \}.
\]

Then

\[
\mu_t(A) = n(W_t \in A, B_\varepsilon)
\]

defines a bounded system of entrance laws for \((P^b_o)\), so that, as usual, there is a finite positive measure \( \mu_o \) on \( \hat{E} \) with

\[
\mu_t(A) = \hat{P}^{\mu_o}_o(W_t \in A),
\]

for \( t > 0 \). Thus

\[
n|_B = \hat{P}^{\mu_o}_o,
\]

so that

\[
n(B_\varepsilon, \hat{E} - \lim W_t \text{ does not exist}) = 0,
\]

for every \( \varepsilon > 0 \). But

\[
\tilde{U} \setminus \bigcup_{\varepsilon > 0} B_\varepsilon = \{ \hat{E} - \lim W_t = a \}.
\]

so that (a) is proven.

(b) It was shown in the proof of Proposition 6, that (vi'a) implies (6.8). Conversely, if (6.8) holds for \( n \), then whenever \( n \geq n' \geq 0 \), it also holds for \( n' \). Thus if \( n' \) satisfies (iv) as well, also

\[
n' = n' (\tilde{U}) \hat{P}^c_o = n' (u) n.
\]

(c) It was shown in the proof of Proposition 6, that (vi'c) implies (6.7). The converse is an immediate consequence of part (a), and Proposition 4.
Note that the proof of Theorem 2 could be shortened under the conditions of Proposition 6; it is well known that a Markov process with a Ray resolvant, is in fact strong Markov. Thus, by the proof of Proposition 3, if we start with a $(P^b_0)$ which is Ray, then it suffices to check the Markov property of the constructed process $(X_t)$, at deterministic times $t$. This is simpler because no excursion can start at $t$ with positive probability, but as well, because an explicit expression can be given for the times $R^k_q$. If instead, only the conditions of Proposition 6 are verified, we construct a right continuous process $(X_t)$ as in Theorem 2. Proposition 7(a) shows that $(X_t)$ has an $E$-right continuous modification $(\hat{X}_t)$, to which the argument just given applies, showing that $(\hat{X}_t)$ is strong Markov. Then (6.7) and (6.8), which are equivalent to (vi'a) and (vi'c), are seen to be exactly the conditions required to infer the strong Markov property of $(X_t)$, from that of $(\hat{X}_t)$.

7. Variations on Lemma 7

We turn to the variations of Lemma 7 alluded to in section 2. Let $E$ be a topological space and $\mathcal{E}$ its Borel field. Recall that a function $K(x, dy)$ is called a kernel, if for each $x \in E$, $K(x, \cdot)$ is a probability measure on $(E, \mathcal{E})$, and if $K(\cdot, A) \in E$ whenever $A \in \mathcal{E}$.

**Proposition 8** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $(Y_t)$ be a càdlàg process with values in $E$ and suppose it is adapted to a filtration $(\mathcal{F}_t)$ of $(\Omega, \mathcal{F})$. Let $K(x, dy)$ be a kernel on $E$. Let $Q$ be a countable ordered set, and for each $q \in Q$, let $S_q$ be an $(\mathcal{F}_t)$ stopping time. Suppose that the following conditions hold:
If \( q \leq q' \) then \( S_q \leq S_{q'} \), and \( \{ S_q \leq S_{q'} \} \in F_{q'} \).

\[
P \left( Y_S \in A \mid F_{q'} \right) = K \left( Y_S, A \right) \text{ a.s., for } q \in Q, A \in E.
\]

Let \( T \) be an \( (F_t) \) stopping time such that

\[
\{ T = S \} \in F_{q'} \bigvee \sigma \left( Y_S \right), \text{ for } q \in Q.
\]

Then there is an \( F_{T^-} \)-measurable random subset \( H \) of \( E \) (that is,

\[
\{ (\omega, e) ; e \in H(\omega) \} \in F_{T^-} \Theta E \), such that

\[
P \left( Y_T \in A \mid F_{T^-} \right) = \frac{K \left( Y_{T^-}, H \cap A \right)}{K \left( Y_{T^-}, H \right)} \text{ a.s.}
\]

for any \( A \in E \).

**proof:** Since

\[
\{ T \leq S \} \in F_{q'} \bigvee \sigma \left( Y_S \right),
\]

there is a function \( \tilde{f}_q : \Omega \times E \to \{0,1\} \), measurable with respect to \( F_{q'} \Theta E \), with

\[
\{ T = S \} = \{ \omega ; \tilde{f}_q (\omega, Y_S (\omega)) = 1 \}.
\]

Put

\[
f_q = \nu \left[ \tilde{f}_r \wedge 1_{\{ S_q = S_r \}} \right] \times E.
\]

Then \( f_q \in F_{q'} \Theta E \), since
\[ \{ f_r \land 1_{\{ S_q = S_r \}} \times E = 1 \} = \{ f_r = 1 \} \cap \{ S_q = S_r \} \times E \]

\[ \subseteq (F_{S_r} - \emptyset E) \cap \{ \{ S_q = S_r \} \times E \} \]

\[ = (F_{S_r} - \{ S_q = S_r \}) \emptyset E \]

\[ = (F_{S_r} - \{ S_q = S_r \}) \emptyset E \]

\[ \subseteq F_{S_r} \times E, \]

as \( \{ S_q = S_r \} \in F_{S_r} \). Also, \( f_q = f_r \) on \( \{ S_q = S_r \} \times E \), and

\[ \{ T = S \} = \{ \omega ; f_q (\omega , Y_{S_q} (\omega )) = 1 \} . \]

Put

\[ f = \bigvee q [ f_q \land 1_{\{ S_q = T \}} ] \times E ] . \]

Because \( f_q = f_r \) on \( \{ S_q = S_r \} \times E \), it follows that \( f = f_q \) on \( \{ S_q = T \} \times E \). We will show that \( f \in F_{T'} \emptyset E \). This follows as above, once we show that \( \{ S_q = T \} \in F_{T'} \). To show this, observe that since the \( S_q \) are increasing in \( q \in Q \), and

\[ \{ T \} \subseteq \bigcup q [ S_q ] \]

we can write

\[ \{ S_q = T \} = \bigcup q [ \{ S_q \geq T \} \cap \{ S_q = S_r \} \cap \{ S_r < T \} ] . \]

But

\[ \{ S_r \geq T \} \cap \bigcap q [ \{ S_r , T' \} \in F_{T'} \]
and

\[ \{ S_r = s_r \} \cap \{ S_r \leq T \} \in F_{S_r}^- \cap \{ S_r \leq T \} \subset F_{T_r^-} \cap \{ S_r \leq T \} , \]

so that for \( r \leq q \),

\[ \{ S_r \geq T \} \cap \{ S_r = s_r \} \cap \{ S_{r'} < T \} \]

\[ \in (F_{T_r^-} \cap \{ S_r \leq T \}) \cap (\{ S_r \geq T \} \cap \{ S_{r'} < T \}) \]

\[ = F_{T_r^-} \cap (\{ S_r \geq T \} \cap \{ S_{r'} < T \}) \]

\[ \subset F_{T_r^-} , \]

as required.

Let

\[ H_q (\omega) = \{ e; f_q (\omega, e) = 1 \} \]

\[ H(\omega) = e; f(\omega, e) = 1 \} . \]

We have assumed that

\[ E [g(\cdot, X_{S, q} (\cdot))] = \int [ \int g(\omega, x) K (X_{S, q} (\omega), dx) ] P(d\omega) \]

for \( g \) of the form \( 1_{A \times B} \), \( A \in F_{S, q}^- \), \( B \in E \). This therefore extends to general \( g \in F_{S, q}^- \cap \emptyset E, g \geq 0 \). Take

\[ g = 1_{A \times B} \cdot f_q , \]

where \( A \in F_{S, q}^- \), \( B \in E \). Then
\[ P(A, X_T \in B, S_q = T) = E \left( g(\cdot, X_{S_q} (\cdot)) \right) \]

\[ = \int \left\{ \int_{A,B} f_{q} (\omega, x) K(X_{S_q} (\omega), dx) \right\} P(d\omega) \]

\[ = \int_{A} P(S_q = T \mid F_{S_q})(\omega) \frac{K(X_{S_q} (\omega), B \cap H(\omega))}{K(X_{S_q} (\omega), H(\omega))} P(d\omega) \]

\[ = \int \frac{K(X_{S_q} (\omega), B \cap H(\omega))}{K(X_{S_q} (\omega), H(\omega))} dP. \]

Enumerate \( Q \) as \( q_1, q_2, \ldots, \), and let \( A \in F_{S_q} \). Then there are \( q_{a} \in F_{S_q} \) such that

\[ A \cap \{T = S_{q_i}\} = A_{q_i} \cap \{T = S_{q}\}. \]

Thus

\[ P(A, X_T \in B) = \sum_{j} P(A_{q_j}, X_T \in B, S_{q_j} = T, S_{q_i} \neq S_{q_j} \text{ for } i < j) \]

\[ = \sum_{j} \int \frac{K(X_{S_{q_j}} (B \cap H))}{K(X_{S_{q_j}}, H)} dP \]

\[ A_{q_j} \cap \{S_{q_j} = T, S_{q_i} \neq S_{q_j} \text{ for } i < j\} \]

\[ = E \left[ A, \frac{K(X_{T_{q_j}} (H \cap B)}{K(X_{T_{q_j}}, H)} \right] \]

\[ \square \]
A similar result holds in discrete time. For clarity, the proof will be given with a discrete state space, but the general version can easily be obtained by modifying the preceding proof.

**Proposition 9** Let $E$ be a countable set, and $(X_n)_{n \geq 1}$ an $E$-valued Markov chain with transition probabilities $P(x;dy)$. Let $T$ be a stopping time with respect to the natural filtration of $(X_n)$. Then there is an $F_{T-1}$ measurable random subset $H$ of $E$ such that

$$P(X_T \in B | F_{T-1}) = \frac{P(X_{T-1}; B \cap H)}{P(X_{T-1}; H)}, \text{ for } B \subseteq E.$$ 

**proof:** Atoms of $F_{T-1}$ are of the form

$$A = \{T=n, X_1 = a_1, \ldots, X_{n-1} = a_{n-1}\}.$$ 

Since $T$ is a stopping time with respect to the natural filtration, it follows that for $a \in E$,

$$\{T=n, X_1 = a_1, \ldots, X_{n-1} = a_{n-1}\} \cap \{X_{T-1} = a\} = \emptyset \text{ or } \{X_1 = a_1, \ldots, X_{n-1} = a_{n-1}, X_n = a\}.$$ 

Put

$$H(\omega) = \{a; A \cap \{X_T = a\} \neq \emptyset\},$$

where $A$ is the atom of $F_{T-1}$ containing $\omega$. Thus, for every atom $A$ of $F_{T-1}$,

$$P(A, X_T = a) = P(X_1 = a_1, \ldots, X_{n-1} = a_{n-1}) P(X_{T-1}; H \cap \{a\}), \text{ on } A$$

so that

$$\frac{P(A, X_T = a)}{P(A)} = \frac{P(X_{T-1}; H \cap \{a\})}{P(X_{T-1}; H)}, \text{ on } A.$$

□
8. Regulation, Uniqueness and Construction of Point Processes

This section deals with some general results that will be needed later, when we discuss point processes of excursions away from sets (as opposed to points, as in the preceding sections). In general, these processes will not be PPP's, but will instead have property (8.7) below. The results we will discuss will be analogous to those found in Itô [32], in the PPP case.

Our point processes will be connected to a Markov process \( (\hat{X}_t) \) (the extraneous " \( \hat{\cdot} \) " is for consistency with the next section). Throughout this section, we could have assumed that \( (\hat{X}_t) \) was a right process with nearly Borel excessive functions (actually, we only need that the version of \( (\hat{X}_t) \) on the canonical space be a right process). In applications it will in fact be a time change of such a process, and Gzyl [28] has shown that such objects are right processes (the nearly Borel proviso follows from his proof). Being unaware of this until just before completing the final draft, I have not made use of Gzyl's results. The relevant properties of right processes will thus be specified, and proved directly when needed. The net effect is that we show an apparently stronger form of Corollary 6, than is needed for applications, and we work harder than need be in Theorem 3. The two relevant properties are called 'regulation' and 'strong regulation' in the following. The latter plays a role only in uniqueness results, while the former is also used for existenc. See Weidenfeld [55] for an analysis of similar ideas appearing in a general study of time changes.

(A) Hypotheses. First, some general definitions: Given two general measurable spaces \((\Omega^i, F^i)\) \(i=1,2\), an \((\Omega_1, F_1; \Omega_2, F_2)\) kernel is a function
\[
n : \Omega_1 \times F_2 \to [0, \infty]
\]
such that \(n(\omega_1, \cdot)\) is a measure on \((\Omega_2, F_2)\) for fixed \(\omega_1 \in \Omega_1\),
and \( n(\cdot, d\omega) \) is \( F_1 \) measurable for fixed \( d\omega_2 \in F_2 \).

If \((\Omega, F)\) is a measurable space, and \((F_t)\) is a filtration thereof, recall that the 'usual conditions' (with respect to a measure \( \nu \)) state that \( F \) is \( \nu \)-complete; \( F_0^- \) consists of the \( \nu \)-null sets of \( F \) together with their complements; \((F_t)\) is right continuous; and \( F_0 \supset F_0^- \). If now

\[
(\nu_a)_{a \in A}
\]
is a family of measures on \((\Omega, F)\), we let \( F^a \) be the \( \nu_a \)-completion of \( F \). We let \( F_0^a \) consist of all the \( \nu_a \)-null sets of \( F^a \), together with their complements, and then put

\[
F^a_t = F_{t+} \cap F^a_0.
\]

This lets us form the augmented fields (following Sharpe [46]);

\[
\bar{F} = \cap_a F^a, \quad \bar{F}_t = \cap_a F^a_t.
\]

If \( F = \bar{F} \) and \( F_t = \bar{F}_t \), we say that \((F, F_t)\) is augmented (with respect to the \( \nu_a \)). We will in any case use the notation \( F^a, F^a_t, F^a_0 \) later, without further explanation. In this context, we will as usual say that a property holds almost surely (a.s.), if it holds \( \nu_a \)-a.s. for each \( a \). Note that \((F^a, F^a_t, \nu_a)\) satisfies the 'usual condition', for each \( a \).

We will now fix a \( \mathbb{U} \)-space \( F \), together with a metric \( d \) thereon. We define objects

\[
\mathbb{E}^F, \mathbb{E}^F_t, U^F, U^F_t, \omega_t, etc...
\]
as in section 2 (we will later have a second space \( E \), and reserve
the notation \( E^0 \), \( U \), etc... for the corresponding objects, with respect
to \( E \)). Now let \( Q^x(du) \) be an

\[
(F, E^F; U^F, \bar{U}^F)
\]

kernel, such that for every \( c \in F \), \( Q^c \) is a probability measure, and

\[
(W^F_t, U^F_t, Q^c, Q^x)
\]

is strong Markov. Let \( \bar{U} \), \( \bar{U}_t \) be the augmentations of \( U^F \), \( U^F_t \) under
the \( Q^\mu \), \( \mu \) positive and \( \sigma \)-finite. Recall that \((K_t)\) is called a
perfect additive functional (PAF), adapted to \((\bar{U}_t)\), if it is right
continuous and nondecreasing in \( t \geq 0 \), is adapted to \((\bar{U}_t)\),
and satisfies

\[
K_{t+s} = K_t + K_s \circ \theta_t \quad \text{for every } t, s \geq 0.
\]

It is called pure jump if \( K_t \) is the sum of the jumps of \( K \) at
times \( s \leq t \). Regardless of whether this holds, we may always find
\((\bar{U}_t)\) stopping times \( \tau_k = \tau^K_k \), such that

\[
\{ t > 0 ; K_t - K_s \neq \tau^K_k \} = \{ \tau_k ; k \geq 1 \} \cap [0, \infty).
\]

Since \( F \) is a \( U \)-space, it follows from the proof of (13.4) of Getoor
[23], that for every \( \mu \),

\[(8.1) \{ (t,u) ; W^F_{t-}(u) \text{ does not exist} \} \text{ is } Q^\mu \text{-indistinguishable from the union of the graphs of countably many predictable } \bar{U}_t \text{ stopping times.} \]
Thus, for any totally inaccessible $(\mathcal{U}_t^\mu)$ stopping time $\tau$, $W^F_\tau$ exists $Q^\mu$-a.s.

Our main definition can now be stated. We say that a pure jump PAF $(K^0_t)$ regulates $(Q^X)$, if there exists a strictly positive $b \in \ell^2 \oplus \ell^2$ such that the following conditions hold for each $\mu$:

(8.2) $K^0_t < \infty$ a.s., for each $t$;

(8.3) $\tau_k = \tau^K_k$ is $(\mathcal{U}_t^\mu)$-totally inaccessible,

and $W^F_{\tau_k^+} \neq W^F_{\tau_k^+}, K^0_{\tau_k^+} = K^0_{\tau_k^+} + b(W^F_{\tau_k^+}, W^F_{\tau_k^+}) Q^\mu$-a.s. on $\{\tau_k^+ < \infty\}$.

(8.4) If $\tau$ is any $(\mathcal{U}_t^\mu)$-totally inaccessible stopping time, such that $W^F_{\tau^+} \neq W^F_{\tau^+}$ $Q^\mu$-a.s. on $\{\tau < \infty\}$, then $K^0_{\tau^+} \neq K^0_{\tau} Q^\mu$-a.s. on $\{\tau < \infty\}$.

$(Q^X)$ is said to be strongly regulated if there exists a regulator $(K^0_t)$ for it, and the following additional condition holds:

(8.5) If $\tau$ is any $(\mathcal{U}_t^\mu)$ totally inaccessible stopping time, then $W^F_{\tau^+} \neq W^F_{\tau^+} Q^\mu$-a.s. on $\{\tau < \infty\}$.

(so that the jumps of $(K^0_t)$ are exactly the totally inaccessible times with respect to the natural filtration).

Note that if

$$(W^F_t, U_t, \mathcal{U}_t, Q^X)$$

is a right process, then (7.6) of Getoor [23] identifies the totally inaccessible times $\tau$ as times at which $W^F_\tau$ exists, lies in $\mathcal{F}$, and does
not equal \( W^F_t \) (here \( \tilde{W}^F_t \) denotes the left limit at \( t \), of \( W^F_t \) in the topology of its Ray-Knight compactification.) At such times, 
\[ \tilde{W}^F_t = W^F_t, \]
so that (3.3) of Benveniste et Jacod becomes the statement that \( (Q^X) \) is strongly regulated. Thus,

**Corollary 3** If \( (W^F_t, \tilde{U}, \tilde{U}_t, Q^X) \) is a right process then it is strongly regulated.

**Lemma 8** Let \( (K^O_t) \) regulate \( (Q^X) \). Then there is a continuous PAF \( (K_t) \), and an \( (F, E^F; F, E^F) \) kernel \( N(x,dy) \) such that \( N(x,F) \leq 1 \) for each \( x \), and the following two conditions hold for each \( u \):

\( (K^O_t) \) is a version of the \( (\tilde{U}^u_t, Q^u) \)-dual predictable projection of \( (K^O_t) \); and for every positive \( (\tilde{U}^u_t) \) predictable process \( (Z_t) \), and every positive \( f \in E^F \otimes E^F \),

\[
E^u_Q \int_0^\infty Z_s f(W^F_s, W^F_s) dK^O_s
\]

\[ = E^u_Q \int_0^\infty Z_s \left[ F(W^F_s, y) N(W^F_s, dy) dK_s \right]. \]

**proof:** Theorem 2 of Benveniste et Jacod [2] applies to construct \( (K_t) \). It is continuous since the jumps of \( (K^O_t) \) are totally inaccessible. Mokobodski's absolute continuity argument ((2.5) of [2]) works in our case, so that as in Benveniste et Jacod [2], we may construct the \( N(x,g), \ g \in E^F \). We make them into a kernel using the 'classical argument' at the end of Maisonneuve [38].

**(B) Uniqueness.** Turning now to point processes let \( (V, V^0) \) be a
separable measurable space and let \( V \) be its universal completion. In applications, \((V, V^0)\) will be \((U, U^0)\), where \( E \) is a separable metric space, so that no confusion will arise if we call the corresponding space of point functions, etc... by the names \( \mathbb{M}, \mathbb{P} \), etc..., that were used in section 2. Let \( n(x, dv) \) be an \((F, E^F; V, V)\) kernel satisfying the (unnecessarily strong) condition that:

\[
(8.6) \text{ There exist } V_k \in V, V_k \uparrow V \text{ such that } \sup_x n(x, V_k) < \infty \text{ for each } k.
\]

Let \( Q^X(du) \) be an \((F, E^F; U^F, U^F)\) kernel consisting of probability measures, and let \((\Omega, F, P)\) be a complete probability space, with a filtration \((F_t)\) satisfying the 'usual conditions'. Let \((\hat{X}_t)\) be a right continuous process with values in \( F \) and adapted to \((F_t)\), such that

\[
(\hat{X}_t, F_t, P, Q^X)
\]

is strong Markov. We will consider point processes \((Y_t)\) adapted to \((F_t)\) (for a point process, this means that \( Y: \Omega \to \mathbb{M} \) satisfies \( Y^{-1}(P_t) \subset F_t \)) and satisfying the condition:

\[
(8.7) \text{ For every positive } (F_t) \text{ predictable process } (Z_t), \text{ and every positive } f \in V, \quad
E[ \sum_{s>0} Z_s f(Y_s) ] = E[ \int_0^\infty Z_s \int f(v)n(\hat{X}_s, dv) ds ] .
\]

We will determine to what degree \((8.7)\) determines the joint law of
of \((\hat{X}, Y)\), consider the strong Markov property of the pair \((\hat{X}, Y)\), and then show how to construct such processes \((Y_t)\). It turns out that solutions to (8.7) do not have unique conditional laws. The reason this is so is that (8.7) does not tell us how \(Y\) behaves at the totally inaccessible times for the minimal filtration of \((\hat{X}_t)\). We will soon impose conditions on the \((Q^X)\) that give us information about the form of these times. To obtain uniqueness, we also fix a function \(G\), measurable from \((F \times V, \mathcal{F}^F \otimes V)\) to \((F, \mathcal{F}^F)\), and we will impose the further condition that

\[(8.8) \ \hat{X}_t = G(\hat{X}_t, Y_t) \text{ for every } t > 0 \text{ such that } Y_t \neq \delta, \text{ a.s.}\]

(Note that by (8.7), \(Y_T = \delta\) a.s. for any \((F_T)\) predictable time \(T\), so that by (8.1), \(\hat{X}_t\) exists for every time \(t\) such that \(Y_t \neq \delta\), \(P\)-a.s.). Let

\[G_0 = \{(x,v); \ G(x,v) = x\} .\]

Then for positive \(f \in V\), \(g \in E^F\),

\[x \to \int_{C_0} 1_{C_0}(x,v)g(G(x,v))F(v)n(x,dv)\]

is \(E^F\) - measurable, so that by (8.6), V.58 of Dellacherie et Meyer [14], and the seperability of \(V^0\), there exists a positive \(E^F \otimes E^F\) measurable function \(n\):

\[n(\cdot, \cdot; f): F \times F \to [0,1],\]

for each \(f \in V\) with \(\sup_v |f(v)| \leq 1\), such that
\[
\int_{G(x,v)} (G(x,v)f(v)n(x,dv)
= \int_{A \setminus \{x\}} (G(x,v))n(x,G(x,v);f)n(x,dv)
\]
for each \( A \in \mathcal{V} \).

[Note: if \( V \) is itself a \( \mathcal{U} \)-space, and \( V^0 \) is its Borel field, then
the 'classical argument' in the last section of Maisonneuve [38] shows
that we can find \((F \times F, \mathcal{F} \otimes \mathcal{F}; V, V)\) kernels \( n(x,y; dv) \)
with \( n(x,x; V) = 0 \) and \( n(x,y; V) \leq 1 \), such that
\[
n(x,y;f) = \int f(v)n(x,y;dv)
\]
will work in the above. This is the case when we take \((V,V^0) = (U,U^0)\),
provided \( E \) is a \( \mathcal{U} \)-space; see p.217 of Maisonneuve et Meyer [39].]

**Lemma 9** Let \((Y_t)\) be adapted to \((F_t)\), and satisfy (8.7). Then
for every positive \((F_t)\) predictable process \((Z_t)\), and positive
\( h \in \mathcal{E} \otimes \mathcal{E} \) we have that
\[
E[ \sum_{s>0} Z_s h(\hat{X}_{s-}, \hat{X}_s)] = E[ \int_0^\infty Z_s \int_{Y_s \neq \delta} h(\hat{X}_{s'},v)n(\hat{X}_s,dv)ds]
\]
**proof:** Let \( h(x,v) = g(x)f(v) \) with \( g \in E^{F,0} \) positive and bounded
and \( f \in \mathcal{V} \) positive. The argument of (13.4) of Getoor [23] yields
even more than (8.1), namely that
\[
g(\hat{X}_{t-})1_{\{\hat{X}_t \text{ exists}\}}
\]
is \((F_t)\) predictable. Thus by (8.7),
But because $F$ is a metric space and $(\hat{X}_t)$ is right continuous, $(\hat{X}_t)$ can have only countably many discontinuities. Thus, the right hand side equals
\[
E\left[ \int_0^\infty Z_s h(\hat{X}_s, \nu) n(\hat{X}_s, d\nu) \, ds \right].
\]
By monotone class and completion arguments, this extends to general positive $h \in \mathbb{E}^F \otimes \nu$.

The final condition that we will assume, really belongs in a discussion of the problem of constructing $(Y_t)$ as above. It is:

(8.9) For every $x \in F$, $t \geq 0$, and positive $f \in \mathbb{E}^F \otimes \mathbb{E}^F$, we have that
\[
E^X_Q \left[ \sum_{s \leq 0, t} f(W^F_s, W^F_s) 1_{\{K^0_s \neq K^0_s\}} \right] \\
\geq E^X_Q \left[ \int_0^t 1_{\mathcal{C}_c}(W^F_s, \nu) f(W^F_s, G(W^F_s, \nu)) n(W^F_s, d\nu) \, ds \right].
\]

Observe that by (8.8), this would be a special case of Lemma 9, provided we assumed that for each $x \in F$, there exist $\Omega, F, (F_t), P, (\hat{X}_t), (Y_t)$ as above, with $Q^X$ the image law of $P$ under $\hat{X}$.

**Lemma 10** Let $n$ satisfy (8.6, 7, 8, and 9), and suppose that $(Q^X)$ is regulated by $(K^0_t)$. Then
(a) Recall that (8.3) gave us a function \( b \in \mathbb{E}^F \otimes \mathbb{E}^F \). There exists an \( \mathbb{E}^F \otimes \mathbb{E}^F \) measurable function \( J : \mathbb{F} \times \mathbb{F} \to [0,1] \), such that for each \( \mu, t \geq 0 \), and positive \( g \in \mathbb{E}^F \otimes \mathbb{E}^F \), we have that \( Q^\mu \)-a.s.,

\[
\int_0^t \int_{c_0} \left( \mathbb{F}_{s,y}^W, \mathbb{F}_{s,y}^W, \mathbb{F}_{s,y}^W, \mathbb{F}_{s,y}^W \right) g(\mathbb{F}_{s,y}^W, \mathbb{F}_{s,y}^W) n(\mathbb{F}_{s,y}^W, dv) ds \\
= \int_0^t \int_{c_0} \frac{J(\mathbb{F}_{s,y}^W, y) g(\mathbb{F}_{s,y}^W, y)}{b(\mathbb{F}_{s,y}^W, y)} 1_{\{b \neq 0\}} (\mathbb{F}_{s,y}^W, y) N(\mathbb{F}_{s,y}^W, dy) dK_s
\]

(b) If \( \varepsilon > 0 \) and \( r \geq 0 \), put

\[
R = \inf\{t > r; K_t^0(\hat{\mathbb{X}}_t) > K_t^0(\hat{\mathbb{X}}_{t-}) + \varepsilon\}
\]

Then the inf is attained, and if \( J \) is any function as in (a), and \( f \in \mathbb{V} \) is positive and bounded by 1, then

\[
E[f(Y_R) , Y_R \neq \delta \mid \mathbb{F}_{R-\nu}^\sigma(\hat{\mathbb{X}}_t; t \geq 0)] \\
= J(\hat{\mathbb{X}}_{R-}, \hat{\mathbb{X}}_R)n(\hat{\mathbb{X}}_{R-}, \hat{\mathbb{X}}_R; f), \text{ P-a.s. on } \{R < \infty\}
\]

(c) If \( r \geq 0 \) and \( k \geq 1 \), set

\[
T = \inf\{t > r; Y_T \in V_k, \hat{\mathbb{X}}_T = \hat{\mathbb{X}}_T\}
\]

Then the infimum is attained, and for each positive \( f \in \mathbb{V} \),

\[
E[f(Y_T) , T < \infty \mid \mathbb{F}_{T-\nu}^\sigma(\hat{\mathbb{X}}_t; t \geq 0)] \\
\int_{V_k} \int_{c_0} \left( \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T, \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T, \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T, \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T \right) f(v) n(\hat{\mathbb{X}}_T, dv) ds \\
\int_{V_k} \int_{c_0} \left( \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T, \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T, \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T, \mathbb{F}_{T,y}^\mathbb{X}_T^\mathbb{X}_T \right) P-a.s. \text{ on } \{T < \infty\}
\]
(d) If in fact \((Q^X)\) is strongly regulated, then

\[
P(T > t \mid F_r \land \sigma(\hat{X}_s; s \geq 0))
\]

\[
= \exp\left(-\int_r^t \int V_k (\hat{X}_s, v) n(\hat{X}_s, dv) ds\right)
\]

for every \(t \geq r\).

**Proof:** Suppose that \(f \in \mathcal{E}^F\) is positive, and

\[
\mathbb{E}_Q^X [\int_0^\infty f(W_s^F) dK_s] = 0 \quad \text{for each } x.
\]

Then by Lemma 8,

\[
\mathbb{E}_Q^X [\int_0^\infty f(W_s^F) dK_s^0] \leq \mathbb{E}_Q^X [\int_0^\infty f(W_s^F) dK_s] = 0
\]

for each \(x\). Let

\[
G_1 = \{x; \int_{G_0} 1_{c}(x, v) n(x, dv) > 0\}.
\]

Thus by (8.9) we have that

\[
\mathbb{E}_Q^X [\int_0^\infty f(W_s^F) 1_{G_1} (W_s^F, v) n(W_s^F, dv) ds]
\]

\[
\leq \mathbb{E}_Q^X [\sum_{s > 0} f(W_s^F) 1_{\{K_s^0 \neq K_s^0\}}] = 0,
\]

and hence

\[
\mathbb{E}_Q^X [\int_0^\infty f(W_s^F) 1_{G_1} (W_s^F) ds] = 0.
\]
As remarked in the proof of Lemma 8, (2.5) of Benveniste et Jacod [2] applies to our situation, and hence there is a positive \( h \in E^F \) such that for every \( \mu \),

\[
\int_0^t 1_{E} \left( W_s^F \right) ds = \int_0^t h(W_s^F) dK_s, \quad Q^\mu \text{-a.s.}
\]

As before, we use V.58 of Dellacherie et Meyer [14] to obtain measurable densities \( J_1(x,y) \) and \( J_2(x,y) \), for

\[
n_1(x,dy) = 1_{\{x \neq y\}} n(x,\{v; G(x,v) \in dy\}),
\]

and for \( N(x,dy) \), each with respect to \( n_1(x,dy) + N(x,dy) \). Let

\[
J(x,y) = h(x)b(x,y)J_1(x,y)J_2^{-1}(x,y)1_{\{J_2 \neq 0\}}(x,y).
\]

By (8.9),

\[
E^\mu_0 \langle J \rangle \int_0^\infty \left( \int_0^1 \frac{1_{\{J_1(w_s^F) = 0\}}(w_s^F,y)n_1(w_s^F,dy)ds}{n_1(w_s^F,dy)} - \right) = 0
\]

for every \( x \), and hence similarly,

\[
E^\mu_0 \langle J \rangle \int_0^\infty \left( \int_0^1 \frac{1_{\{J_2(w_s^F) = 0\}}(w_s^F,y)g(w_s^F,y)n_1(w_s^F,dy)ds}{b(w_s^F,y)} - \right) = 0
\]

for every \( \mu \), every positive \( g \in E^F \otimes E^F \), and every positive \((\bar{U}^\mu_t)\) predictable process \((Z_t)\).
(b) Since \( Y_0 \in F_0 \), and \( X_{R+} \) is conditionally independent of \( F_R \) given \( X_R \), it will suffice to show that the right hand side of (a) equals
\[
E[f(Y_0), Y_0 \neq \delta, R < \infty | F_{R-} \vee \sigma(\hat{X}_R)].
\]

Let
\[
R_0 = \inf\{t > 0; \kappa_t^0 > \kappa_t^0 + \epsilon\}.
\]

Let \( C \in F_t \), \( t \in \mathbb{R} \), and let \( g \in E^F \) be positive. Then
\[
E[C, t < R < \infty, Y_0 \neq \delta, f(Y_0)g(\hat{X}_R)]
\]
\[
= E[C, \sum_{s \in (t, R]} f(Y_s)g(G(\hat{X}_s, Y_s)) \cdot \mathbb{1}\{(x,v); G(x,v) \neq x, b(x,G(x,v)) > \epsilon\}(\hat{X}_s, Y_s)]
\]
\[
= E[C, \int_{t}^{R} f(v)g(G(\hat{X}_s, v)) \mathbb{1}\{(x,v); G(x,v) \neq x, b(x,G(x,v)) > \epsilon\}(\hat{X}_s, v)n(\hat{X}_s, dv)ds]
\]
\[
= E[C, t < R, E_{Q}^{\hat{X}^*_t}[\int_{0}^{R_0} \mathbb{1}\{(x,z); x \neq z, b(x,z) > \epsilon\}(W^F_s, G(W^F_s, v)) g(G(W^F_s, v))n(W^F_s, G(W^F_s, v); f)n(W^F_s, dv)ds]]
\]
\[
= E[C, t < R, E_{Q}^{\hat{X}^*_t}[\int_{0}^{R_0} J(W^F_s, y)g(y)n(W^F_s, y; f) \mathbb{1}\{(x,z); x \neq z, b(x,z) > \epsilon\}(W^F_s, y)n(W^F_s, dy)dk_s]]
\]
\[
= E[C, t < R, E_{Q}^{\hat{X}^*_t}[\int_{0}^{R_0} J(W^F_s, W^F_s)g(W^F_s)n(W^F_s, W^F_s; f) \mathbb{1}\{(x,z); x \neq z, b(x,z) > \epsilon\}(W^F_s, y)n(W^F_s, dy)dk_s]]
\]
\[ 1 \{ (x,z); x \not= z, b(x,z) > \varepsilon \} \int_{y \in \mathbb{R}^d} \frac{w(y)}{w(x)} \int_{x' \in \mathbb{R}^d} \frac{w(x')}{w(x)} dK_0(x) \]

\[ = E[C,t < R, \mathbb{E} _Q [J(w_{R_0}^F)^F g(w_{R_0}^F) n(w_{R_0}^F, w_{R_0}^F ; f)] ] \]

\[ = E[C,t < R < \infty, J(\hat{x}_{R^-}, \hat{x}_R) g(\hat{x}_R) n(\hat{x}_{R^-}, \hat{x}_R ; f)] , \]

by Lemma 9 part (a), and (8.3), as required.

(c) We argue as above, and use that \( \hat{x}_t = \hat{x}_{t^-} \in F_{T^-} \), to see that it suffices to show that the right hand side of (c) equals

\[ E[f(Y_T), T < \infty \mid F_{T^-}] . \]

Let \( C \in F_t \), \( t \geq r \). Then by Lemma 9, applied twice,

\[ E[C, t < T < \infty, f(Y_T)] \]

\[ = E[C, \sum_{s \in (t, T]} f(Y_s) 1_{V_k} (Y_s) 1_{G_0} (\hat{x}_{s^-}, Y_s)] \]

\[ = E[C, \int_t^T 1_{V_k} (\hat{x}_s, v) f(v) n(\hat{x}_s, dv) ds] \]

\[ = E[C, \sum_{s \in (t, T]} \int_{V_k} 1_{G_0} (\hat{x}_{s^-}, v) f(v) n(\hat{x}_{s^-}, dv)] \]

\[ = E[C, \int_{V_k} 1_{G_0} (\hat{x}_{T^-}, v) f(v) n(\hat{x}_{T^-}, dv)] , \]

\[ as \ required. \]

(d) For \( t \geq r \), let \( T_t \) be a version of
which is right continuous, decreasing, and equals 1 at \( t = r \). Let

\[
P(T > t \mid F_r \vee \sigma(\hat{X}_s; s \geq 0))
\]

which is right continuous, decreasing, and equals 1 at \( t = r \). Let

\[
z = z_1 \cdot z_2(\hat{X}_{r+}),
\]

where \( z_1 \in F_r \) and \( z_2 \in U^F \) is a function of the form

\[
\frac{k}{f_i} \circ W^F_{r_i}.
\]

Write \((^oZ^2_t)\) for a right continuous version of the \((P, F_t)\)-martingale \(E[Z \mid F_t] \). Write \((^oZ^2_t)\) for the right continuous process that is a version of the \(Q^u\)-martingale

\[
E^u_Q[Z^2 \mid U^u_t],
\]

for each \( u \) (It exists by Lemma 1 of Benveniste et Jacod [3] ).

Then a direct computation shows that

\[
^oZ^2_t = z_1 \cdot ^oZ^2_{t-r}(\hat{X}_{r+})
\]

\[
= z_1 \cdot ^oZ^2_t(\hat{X}_r) \text{ for every } t > r, \ P\text{-a.s.}
\]

Because \((Q^X)\) is strongly regulated, it follows that for each \( u \),

\[
\{(t, u); ^oZ^2_{t-}(u) \neq ^oZ^2_t(u), K^0_{t-}(u) = K^0_t(u)\}
\]

is \(Q^u\)-indistinguishable from the union of the graphs of countably many \((U^u_t)\) accessible stopping times. Since \( T \) is totally inaccessible for \((F_t)\), and \( \hat{X}_{T-} = \hat{X}_T \), it follows that

\[
^oZ^2_{T-}(\hat{X}_r) = ^oZ^2_T(\hat{X}_r) \ P\text{-a.s.},
\]
and hence also that \( ^{0}_{\tau}Z_{t-} = ^{0}_{\tau}Z_{t} \) \( \text{P-a.s.} \). But V.10 of Dellacherie [12] shows that

\[
z_{t} = ^{0}_{\tau}Z_{t-}
\]
is the \((P,F_{t})\) predictable projection of \( Z \), so in fact,

\[
Z_{t} = \mathbb{E}[Z | F_{t}]
\]

Thus, for positive bounded \( g \in \mathcal{B} \),

\[
\mathbb{E}[Z \int_{r}^{\infty} g(t) dT_{t}] = -\mathbb{E}^{[1]}[Zg(T), r < T < \infty]
\]

\[
= -\mathbb{E}[Z_{T}g(T), R < T < \infty]
\]

\[
= -\mathbb{E}[\sum_{s \in (r,T]} Z_{s}g(s)1_{V_{k}}(s)1_{C_{0}}(\hat{X}_{s-},Y_{s}) | \mathcal{F}_{s}]_{s \neq \delta}
\]

\[
= -\mathbb{E}[\int_{r}^{T} g(s) \int_{V_{k}} 1_{C_{0}}(\hat{X}_{s},v.)n(\hat{X}_{s},dv)ds]
\]

\[
= \mathbb{E}[\int_{r}^{\infty} g(s) \int_{V_{k}} 1_{C_{0}}(\hat{X}_{s},v)n(\hat{X}_{s},dv)dsdT_{t} -
\]

\[
- T_{r-} \int_{r}^{\infty} g(s) \int_{V_{k}} 1_{C_{0}}(\hat{X}_{s},v)n(\hat{X}_{s},dv)ds]
\]

Taking \( g \) in a countable dense subset of \( C_{0}([r,\infty)) \), we see that

\[
\int_{r}^{\infty} g(t) dT_{t} = \int_{r}^{\infty} g(s) \int_{V_{k}} 1_{C_{0}}(\hat{X}_{s},v)n(\hat{X}_{s},dv)dsdT_{t}
\]
for every bounded positive \( g \in \mathcal{B}, \text{ P-a.s.} \). We put

\[
g(t) = \int_{[r, t_0]} (t) \exp(\int_{rV_k} \int_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv)ds),
\]

for \( t_0 \geq r \), to obtain that

\[
\int_{t_0}^{t_0} \exp(\int_{rV_k} \int_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv)ds)dT_t
\]

\[
= \int_{t_0}^{t_0} (\exp(\int_{rV_k} \int_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv)ds) - 1)dT_t^+
\]

\[
+ (T_{\infty} - T_{t_0}^-) (\exp(\int_{rV_k} \int_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv)ds) - 1) -
\]

\[
- T_{\infty} (\exp(\int_{rV_k} \int_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv)ds) - 1),
\]

and hence that

\[
T_{t_0} = \exp(-\int_{rV_R} \int_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv)ds)
\]

as required. \( \Box \)

We gather together the current hypotheses for the next result:

**Proposition 10** Let \( (Q^X) \) be strongly regulated, and let \( n \) satisfy (8.6) and (8.9). Let \( (\Omega, F, P) \) be a complete probability space, with
a filtration \((F_t)\) thereon, satisfying the 'usual conditions'.

Let \((\hat{X}_t)\) be a right continuous process with values in \(F\), which is adapted to \((F_t)\) and is such that

\[
(\hat{X}_t, F_t, P, Q^X)
\]

is strong Markov. Let \(G\) be fixed, and let \((Y_t)\) be a point process, adapted to \((F_t)\), satisfying (8.7) and (8.8). Let \((K^0_t)\) regulate \((Q^X)\), and let \(J\) be given by Lemma 10. Then

(a) For each \(k \geq 1\), there exists (after enlarging \(\Omega\) if necessary) a sequence \(S_1, S_2, \ldots\) of exponentially distributed random variables with unit means, which are independent of each other and of \((\hat{X}_t)\), and satisfy the further conditions:

(b) If we put \(T_0 = 0\) and

\[
T_{i+1} = \inf\{t > T_i; Y_t \in V_k \text{ and } \hat{X}_{t-} = \hat{X}_t\},
\]

then in fact,

\[
T_i = \inf\{t > 0; \int \int 1_{G_0}(\hat{X}_s, v)n(\hat{X}_s, dv)ds \geq \sum_{j=1}^i S_j\}, \text{ P-a.s.}
\]

(c) For \(\epsilon > 0\), we put \(R_0 = 0\), and

\[
R_{i+1} = \inf\{t > R_i; K^0_t(\hat{X}_t) > K^0_{t-}(\hat{X}_t) + \epsilon\}.
\]

Then for any positive \(f_1, \ldots, f_{\ell}, g_1, \ldots, g_{\ell} \in V\), the random variables

\[
g_i(Y_{T_i}), f_j(Y_{R_j})1_{\{Y_{R_j} \neq \delta\}}, i, j = 1, \ldots, \ell
\]

are conditionally independent given \((\hat{X}_t)\) and the \(T_1, \ldots, T_{\ell}\), with
conditional expectations respectively

\[
\begin{align*}
&\int_{V_k} 1_{G_0} (\hat{X}_{T_i}, v) n(\hat{X}_{T_i}, dv) 1_{\{T_i < \infty\}}, \text{ and} \\
&\int_{V_k} 1_{G_0} (\hat{X}_{T_i}, v) n(\hat{X}_{T_i}, dv)
\end{align*}
\]

\[J(\hat{X}_{R_j}, \hat{X}_{R_j}) n(\hat{X}_{R_j}, \hat{X}_{R_j}; f_j) 1_{\{R_j < \infty\}}.\]

**proof:** For \( C \in F_{T_{\ell-1}} \), \( \forall \sigma(\hat{X}_t; t \geq 0) \)

\[
P(C, \int_{T_{\ell-1}}^{T_{\ell}} \int_{V_k} 1_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv) ds > t)
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{\infty} P\left(C, \int_{2^{-i} V_k} 1_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv) ds > t, \right.
\]

\[
T_{\ell} > \frac{1}{2^i} \geq T_{\ell-1} > \frac{1-1/2^i}{2^i}
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{\infty} e^{-t} P\left(C, \int_{2^{-i} V_k} 1_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv) ds > t, \right.
\]

\[
T_{\ell} > \frac{1}{2^i} \geq T_{\ell-1} > \frac{1-1/2^i}{2^i}
\]

\[
= e^{-t} P\left(C, \int_{T_{\ell-1} V_k} 1_{G_0} (\hat{X}_s, v)n(\hat{X}_s, dv) ds > t\right),
\]

by Lemma 10 (d). Thus, conditional on \( F_{T_{\ell-1}} \) \( \forall \sigma(\hat{X}_t; t \geq 0) \),
\[
\int_{T_{\ell-1} V_k}^{T_\ell} \int 1_{C_0} (\tilde{x}_s, v) n(\tilde{x}_s, dv) ds
\]

has the law of the minimum of a mean 1 exponential random variable and the number

\[
\int_{T_{\ell-1} V_k}^{\infty} \int 1_{C_0} (\tilde{x}_s, v) n(\tilde{x}_s, dv) ds .
\]

Thus we can find an appropriate \( S_\ell \), \( P \)-independent of \((X_t)\) and of \( F_{T_{\ell-1}} \). Repeating the argument gives \( S_{\ell-1} \), and since \( S_\ell \) is independent of \( F_{T_{\ell-1}} \), \( S_{\ell-1} \) may be chosen independent of \( S_\ell \).

By continuing this argument, we obtain a proof of (a) and (b). The proof of (c) is similar, now using (b) and (c) of Lemma 10 instead of (d).

**Corollary 4** Under the conditions of Proposition 10, the joint law under \( P \) of \((\tilde{X}_t, Y_t)_{0,\infty})\) may be expressed in terms of \( G, n, \) the \( Q^X \), the initial law of \( \tilde{X}_0 \) under \( P \), and the function \( J \) obtained from an arbitrary regulator of \( Q^X \).

**proof:** All relevant probabilities may be calculated using Proposition 10. □

Note that even without our regulation hypotheses, there is much that (8.7) tells us about the distribution of \((Y_t)\).

**Lemma 11** Let \((\Omega, F, P)\) be a complete probability space, with a filtration \((F_t)\) satisfying the usual conditions. Let \( n(x, dv) \)
and \( Q^X(du) \) be \((F, E^F; V, U)\) and \((F, E^F; U^F, U^F)\) kernels respectively, the \( Q^X \) being probability measures. Let \((\hat{X}_t)\) be a right continuous process with values in \( F \), and adapted to \((F_t)\), such that \((\hat{X}_t, F_t, P, Q^X)\) is strong Markov. Let \((Y_t)\) be a point process adapted to \((F_t)\) and satisfying (8.7). Let \( A \in \mathcal{V} \) satisfy

\[
\sup_X n(x, A) < \infty,
\]

and let \( r \geq 0 \). Put

\[
T = \inf\{ t > r; Y_t \in A \}.
\]

Then: (a) There exists (upon enlarging \( \Omega \) perhaps) an exponentially distributed random variable \( S \), of unit mean, which is independent of \( F_r \) (but not necessarily of \( \sigma(X_s; s \geq 0) \)) and satisfies

\[
T = \inf\{ t; \int_r^t n(\hat{X}_s, A) ds \geq S \}.
\]

(b) For \( f \in \mathcal{V} \) positive,

\[
E[f(Y_T) | F_{T^-}] = \frac{\int f(v) n(\hat{X}_{T^-}, A) dv}{n(\hat{X}_{T^-}, A)} \quad \text{P-a.s. on } \{T < \infty\}.
\]

(where again, \( 0/0 = 0 \)).

\textbf{proof}: (a) This is proved exactly as in Lemma 10 (d), except that since we no longer require \( S \) to be independent of \( \sigma(\hat{X}_s; s \geq 0) \), we do not need to use the hypothesis of strong regulation.
(b) By (8.7), \( \{s; Y_s \in A\} \) is \( P \)-a.s. discrete, so that \( Y_T \in A \) on \( \{T < \infty\} \). Let \( B \in F_t \). Then

\[
E[B,t < T < \infty, f(Y_T)] = E[B, \sum_{s \in (rvt,T] \setminus T} f(Y_s) 1_A(Y_s)]
\]

\[
= E[B, \int_{rvt}^T f(v)n(\hat{X}_s, dv) ds]
\]

\[
= E[B, \int_{rvt}^T \left( \sum_{s \in (rvt,T] \setminus T} \frac{f(v)n(\hat{X}_s, dv)}{n(\hat{X}_s, A)} \right) 1_A(Y_s)]
\]

(by Lemma 9)

\[
= E[B, t < T < \infty, \frac{1}{n(\hat{X}_{T-}, A)} f(v)n(\hat{X}_{T-}, dv)]
\]

By monotone class arguments, this suffices. □

(c) The Strong Markov Property: To facilitate future applications, we will consider the following formulation. Let \( E \) be a separable metric space, with \( E \) its \( \sigma \)-field of universally measurable subsets. Let \( F \) be both a U-space and a universally measurable subset of \( E \). Fix a single measurable space \((\Omega, F)\), a right continuous filtration \((F_t)\) thereon, and an \((E, E; \Omega, F)\) kernel \( \hat{P}^X(\omega) \) such that each \( \hat{P}^X \) has unit mass, and \((F, F_t)\) is augmented with respect to the \( \hat{P}^\mu \). As above, let \((V, V^0)\) be a separable measurable space. Write \( V \) for the universal completion of \( V^0 \), and let \( n(x, dv) \) be an \((F, E^F; V, V)\) kernel satisfying (8.6).
Let \((\hat{X}_t^X)\) be a right continuous \(F\)-valued process, adapted to \((F_t^X)\), and write \(Q^X\) for the image law of \(\hat{X}^X\) on \((U^F, U^F)\) under \(\hat{X}\). If now \((Y_t)\) is any \((V, V)\)-valued point process adapted to \((F_t)\), we will say that

\[ (\hat{X}_t, Y_t, F_t, \hat{X}^X) \]

is strong Markov if

\[ \hat{P}^\mu((\hat{X}_{T+t}, (\hat{X}_t^0)_{Y_t} \in A | F^\mu_T) = \hat{P}^{T}(\hat{X}_t, Y_t | (0, \infty) \in A) \]

whenever \(\mu\) is a probability on \(E\), \(T\) is an \((F^\mu_t)\) stopping time, and \(A \in U^F \otimes P\).

**Corollary 5** Assume in addition to the above hypotheses, that each \((\hat{X}_t, F^X_t, \hat{P}^X, Q^X)\) is strong Markov, and that \((Q^X)\) is strongly regulated. Fix \(G\), and assume that \((Y_t)\) is a \((V, V)\)-valued point process, adapted to \((F_t)\) and satisfying (8.7) and (8.8). Then \((\hat{X}_t, Y_t, F_t, \hat{P}^X)\) is strong Markov.

**proof:** Condition (8.9) follows immediately from Lemma 9, by definition of the \(Q^X\). We apply Proposition 10 with the probability space \((\Omega, F^\mu, \hat{P}^\mu)\), the filtration

\[ (F'_t) = (F^\mu_{T+t}), \theta^0_t(Y)(t), \]

and the processes

\[ \hat{X}'_t = \hat{X}_{T+t}, Y'_t = \theta^0_t(Y)(t). \]

Our hypotheses guarantee that those of Proposition 10 are met, hence,
as in Corollary 4, the conditional law of \((X', Y')\) given \(F_0\) may be expressed using only \(n, G, Q^X,\) a regulator for \((Q^X)\), and the initial law of \(\hat{X}_0'\), showing the result.

(D) Existence We now consider the problem of constructing point processes \((Y_t)\) satisfying (8.7). With future applications in mind, we will do so in a context similar to that of (C). In addition to the general hypotheses of (C), assume that each \((\hat{X}_t, F_t^0, \hat{F}^0, Q^X)\) is strong Markov, and that \((Q^X)\) is regulated by a PAF \((K^0_t)\).

Fix \(G\), and assume that \(n\) satisfies (8.9). We will further assume that an \((F \times F, \mathcal{E}^F \otimes \mathcal{E}^F; V, \mathcal{V})\) kernel \(n(x, y; dv)\) may be found, with \(n(x, y; \mathcal{V}) \leq 1\) and \(n(x, x; \mathcal{V}) = 0\), such that

\[
\int_{G^0} 1_{G(x,v)}(x,v) g(G(x,v)) f(v) n(x, dv) = \int g(G(x,v)) \int f(v') n(x, G(x,v); dv') n(x, dv)
\]

for every positive \(g \in E\) and \(f \in V\). (As remarked before, such a kernel will exist, provided \((V, \mathcal{V}_0) = (U, \mathcal{U}_0)\), where \(E\) is a \(U\)-space.)

We may then perform the following construction; obtain a function \(J(x, y)\) on \(F \times F\) from Lemma 10(a). It is easily checked that the function

\[
J(x, y) 1_{\{ (x, y); n(x, y; \mathcal{V}) = 1 \}}
\]

satisfies the same condition as \(J\), so that we may assume that \(J(x, y) = 0\) whenever \(n(x, y; \mathcal{V}) \neq 1\).
Enlarge $\Omega$ by setting

$$(\Omega', F') = (\Omega \times \Omega_0^N, F \otimes \mathcal{C}^N),$$

where $\Omega_0$ is the space of all right continuous functions $[0, \infty) \to \mathbb{N}$, and $\mathcal{C}$ is its cylinder $\sigma$-field. We retain the notation $F$ for the preimage of $\mathcal{F}'$ under the projection of $\Omega'$ on its first coordinate. Similarly, we write $\hat{X}_t$ for the composition of $\hat{X}_t$ and this projection. Let $B^k(t), t \geq 0$ and $k \geq 1$, be the coordinate maps on the second factor. Then we can extend the $\hat{P}^X$ by the appropriate product measures, so as to make the $(B^k(t))$ Poisson processes of unit intensity, and independent of each other and of $F$, under each law $\hat{P}^\mu$. Set

$$T^k_j = \inf \{ t; B^k \left( \int_0^t 1_{G_0} (\hat{X}_s, v) n(\hat{X}_s, dv) ds \right) \geq j \}.$$

Let $R^k_0 = 0$, and put

$$R^k_{j+1} = \inf \{ t > R^k_j; K^0_t(\hat{X}_t) - K^0_t(\hat{X}_t) \in \left[ \frac{1}{k}, \frac{1}{k} \right] \}.$$

Now enlarge $(\Omega', F')$ again, to $(\Omega'', F'')$, and redefine the $\hat{P}^X$ so that there exist $(\mathbb{V}_u \{ \delta \}, \mathbb{V}_\sigma(\{ \delta \}))$-valued random variables $\tilde{Y}^k_j; i=1,2$ and $j,k \geq 1$; which are $\hat{P}^\mu$-conditionally independent given $F$ and the $T^k_j$, for every $\mu$, with conditional laws

$$1_{G_0} \cap (F \otimes \mathbb{V}_k \backslash \mathbb{V}_k - 1) (\hat{X}_{T^k_j}, v), \text{ if } i=1;$$

and

$$\int_{\mathbb{V}_k \backslash \mathbb{V}_k - 1} 1_{G_0} (\hat{X}_{T^k_j}, v') n(\hat{X}_{T^k_j}, dv')$$
and while we're at it, enlarge the space enough, so that more independent random variables may be found in the future). Set

\[
J(\hat{X}^k, \hat{X}^k)n(\hat{X}^k, \hat{X}^k; dv) + (1 - J(\hat{X}^k, \hat{X}^k))\epsilon_d(dv), \text{ if } i = 2.
\]

Then for \( t > q > r \), \( A \in \mathcal{V} \), and every probability \( \mu \), we have by definition, that

\[
H^0 = F \cup Y^{-1}(P), \quad H^0_t = F_t \cup Y^{-1}(P_t),
\]

and augment \( (H^0, H^0_t) \) with respect to the \( \hat{p}^\mu \), to form \( (H, H_t) \).

**Proposition 11**: On the enlarged space \( \Omega''_t, (\hat{X}^k, Y_t, H_t, \hat{p}^\mu) \) is strong Markov, and satisfies (8.7) and (8.8).

**proof**: Let

\[
T^k(r) = \inf\{t > r; Y_t \in V_k \setminus V_{k-1}, \hat{X}_t = \hat{X}_t^r \},
\]

and write

\[
n^k_s(A) = \int_{V_k \setminus V_{k-1}} 1_{G_0}(\hat{X}_s, v)1_A(v)n(\hat{X}_s, dv).
\]

Then for \( t > q > r \), \( A \in \mathcal{V} \), and every probability \( \mu \), we have by definition, that
\begin{multline*}
\hat{P}^\mu(T^k(r) \in (q,t], Y_{T^k(r)} \in A, T^k(T^k(r)) > t \mid F \lor H_r \lor \sigma(T^i_j; i \neq k)) \\
= \sum_{\ell=1}^\infty \hat{P}^\mu(T^k_{\ell-1} \leq r < T^k_{\ell}, T^k_{\ell} \in (q,t], Y_{T^k_{\ell}} \in A, T^k_{\ell+1} > t \mid F \lor H_r \lor \sigma(T^i_j; i \neq k)) \\
= \sum_{\ell=1}^\infty \mathbb{1}_{\{T^k_{\ell-1} \leq r < T^k_{\ell}\}} e^{-\int_{r}^{T^k_{\ell}} n^k_s(V)ds} e^{-\int_{q}^{\infty} n^k_s(V)ds'} \frac{n^k_s(A)}{n^k_s(V)} \int_{q}^{t} n^k_s(V)ds' \\
= e^{-\int_{r}^{t} n^k_s(V)ds} \int_{q}^{t} n^k_s(A)ds .
\end{multline*}

Similarly,

\begin{equation*}
\hat{P}^\mu(T^k(r) > t \mid F \lor H_r \lor \sigma(T^i_j; i \neq k)) = e^{-\int_{r}^{t} n^k_s(V)ds} .
\end{equation*}

Now fix $N$, and let

\begin{equation*}
T(r) = \bigwedge_{k=1}^{N} T^k(r), \quad n_s(.) = \sum_{k=1}^{N} n^k_s(.) .
\end{equation*}

Then by independence,

\begin{multline*}
\hat{P}^\mu(T(r) \in (q,t], Y_{T(r)} \in A, T(T(r)) > t \mid F \lor H_r) \\
= \sum_{k=1}^{N} \hat{P}^\mu(T^k(r) \in (q,t], Y_{T^k(r)} \in A, T^k(T^k(r)) > t, T^i(r) > t \text{ for } i \neq k} \\
\quad \mid F \lor H_r) \\
= e^{-\int_{r}^{t} n_s(V)ds} \int_{q}^{t} n_s(A)ds .
\end{multline*}
Lemma 12  Let $V_N = u\{A_i; i=1..N''\}$, the $A_i$ disjoint elements of $V$, and let $\varepsilon > 0$, $\varepsilon^{-1} \in N$. There exists processes $B_i(t)$, $i=1..N''$, and random variables $C_j$, $j \geq 1$, such that under each $\hat{P}^j$:

(a) Each $B_i(t)$ is a Poisson process of unit intensity;
(b) Each $C_j$ is uniformly distributed on $[0,1]$;
(c) The $B_i$ and $C_j$ are independent of each other and of $F$;
(d) \[
\sum_{s \in [r,t]} 1_A_i(Y_s) - 0^0 \hat{X}_s - Y_s = B_i(\int_0^t n_s(A_i)ds) - B_r(\int_0^r n_s(A_i)ds) \]
for each $i$ and each $r < t$, a.s.;

(d) Let $\rho_0 = 0$,
\[
\rho_{j+1} = \inf\{t > \rho_j; K_t^0(\hat{X}_t) \geq K_{\rho_j}^0(\hat{X}_\rho_j) + \varepsilon\},
\]
\[
a_j^i = n(\hat{X}_\rho_j; \hat{X}_\rho_j; A_i),
\]
\[
b_j^i = \sum_{k=1}^i a_j^k \in [0,1],
\]
and write $b_j^0 = 0$, $b_j^{N''+1} = 1$, $A_{N''+1} = V \cup \{\delta\} \setminus V_N$. Then
\[
c_j \in (b_j^{i-1}, b_j^i) \text{ whenever } Y_{\rho_j} \in A_i.
\]

proof: Let $\tau_0^i = 0$,
\[
\tau_{k+1}^i = \inf\{t > \tau_k^i; Y_t \in A_i, \hat{X}_t = \hat{X}_t\}, \text{ and}
\]
\[
\gamma^i = \int_0^\infty n_s(A_i)ds.
\]
For each \( i \), pick a process \( B^i_0 \) such that under each \( \hat{\mathbb{P}}^\mu \), \( B^i_0 \) is a Poisson process of unit intensity, and the \( B^i_0 \) are independent of each other and of \( H \). Set

\[
B^i(t) = \begin{cases} 
  k, & \text{if } t \in \left[ \int_0^{
\gamma_i}
 n_s(A_i)ds, \int_0^{t+k+1} n_s(A_i)ds \right), \ t < \gamma_i \\
  B^i_1(t - \gamma_i) + B^i_0(t - \gamma_i), & \text{if } t \geq \gamma_i 
\end{cases}
\]

and that (d) holds by definition.

Choose \( C_j, j \geq 1 \) to be uniformly distributed on \([0,1]\), and independent of each other, the \( B^i \) and \( H \), under each \( \hat{\mathbb{P}}^\mu \).

Set

\[
C_j = b_{j-1}^i + a_j^i C_j
\]

if \( Y_{\rho_j} \in A_i, i = 1 \ldots, N'' + 1 \). Then (e) holds by definition.

Since the \( T_j^k \) used to define \( (Y_t) \) are independent of the \( R_j^k \), given \( F \), we see that the \( B^i \) are independent of the \( C_j \), given \( F \). Let \( t_0 = 0 < r_{1_1} < t_1 < \ldots < r_{1_{N'}} < t_{N'}, \ i_{1_1} \ldots i_{1_{N'}} \in \{1, \ldots, N''\} \), and \( H \in F \). Set \( j_0 = 0 \),

\[
j_k = j_{\ell} + 1,
\]

where \( \ell \) is the last number before \( k \) such that \( i_{\ell} = i_k \) \((\ell = 0 \text{ if no such number exists})\), and define \( j(i) \) to be the largest of the \( j_k \) such that \( i_k = i \) (and if there is no such \( k \), we set it equal to 0). To show that the \( B^i \) are independent
of each other and of \( F \), it will suffice to check that the probability of the set

\[
H \cap \{ \tau_{j,k}^i \in (r_k, t_k], \ k = 1..N', \ \sup_{i} \tau_{j}(i) + 1 > t_N, \}
\]

agrees with the result we would obtain under (a) and (c) (as finite unions of such sets form a Boolean algebra which generates a \( \sigma \)-field for which all the \( B_i \) are measurable). By the computational preceding this lemma (and similar arguments),

\[
\hat{\mu}^H(H; \tau_{j,k}^i \in (r_k, t_k], \ k = 1..N'; \ \sup_{i} \tau_{j}(i) + 1 > t_k)
\]

\[
= \hat{\mu}^H(H; T(t_{k-1}) \in (r_k, t_k], Y_{T(t_{k-1})} \in A_k, \ T(T(t_{k-1})) > t_k, \ k = 1..N')
\]

\[
= \hat{\mu}^H[H, \prod_{k=1}^{N'} e^{-\int_{t_{k-1}}^{t_k} n_s(V)ds} \int_{r_k}^{t_k} n_s(A_{i_k})ds]
\]

\[
= \hat{\mu}^H[H, \prod_{i=1}^{N'} \prod_{k=1}^{N'} [(1-\delta_{i,i_{k-1}}) e^{-\int_{t_{k-1}}^{t_k} n_s(A_{i})ds} + \delta_{i,i_{k-1}} e^{-\int_{t_{k-1}}^{t_k} n_s(A_{i})ds} \int_{r_k}^{t_k} n_s(A_{i})ds)]
\]

which is the result we would have obtained under (a) and (c).

To show that the \( C_j \) are independent of each other and of \( F \), with uniform distributions, let \( 0 \leq r_j < t_j \leq 1, \ j = 1..N \); and \( H \in F \). It will again suffice to show that the probability of
agrees with that calculated using (b) and (c).

\[ \hat{P}^\mu (H; C_j \in [r_j, t_j], j = 1..N) \]

\[ = \sum_{i_1..i_N \in \{1..N'+1\}} \hat{P}^\mu (H; C_j \in [r_j, t_j], Y_{\rho_j} \in A_j, j = 1..N) \]

\[ = \hat{E}[H, \sum_{i_1..i_N \in \{1..N'+1\}} \prod_{j=1}^N [r_j - b_j^{i_j-1} t_j - b_j^{i_j-1} + a_j^{i_j}] \cap [0,1]) \]

\[ = \hat{E}[H, \prod_{j=1}^N ([r_j, t_j] \cap [b_j^{i_j-1}, b_j^{i_j-1} + a_j^{i_j}]) \]

\[ = \hat{E}[H, \prod_{j=1}^N ([r_j, t_j] \cap [b_j^{i_j-1}, b_j^{i_j}])] \]

\[ = \prod_{j=1}^N (t_j - r_j) \hat{P}^\mu (H), \]

as required. \( \square \)

As an immediate consequence, we see that there is a \( \sigma \)-field \( \bar{F}^\perp \) which is independent of \( F \) under each \( \hat{P}^\mu \), and such that

\[ H^0_t \subset F_t \lor \bar{F}^\perp. \]

Thus, for every \( (H^0_{t+}) \) stopping time \( T \), we can find (as in
Lemma 7) a stopping time \( \hat{T} \) with respect to the filtration

\[
\cap_{s>t} (F_s \otimes F_{\hat{T}+})
\]

on \( \Omega^s \times \Omega^s \), such that \( T(\omega) = \hat{T}(\omega, \omega) \) for every \( \omega \). Then \( \hat{T}(\cdot, \omega) \) is an \( (F_t) \) stopping time for each \( \omega \in \Omega^s \), so that for \( f \in \mathcal{U}_F \), \( f \) positive, we get that

\[
\mathbb{E}^\mu[f(\hat{X}_{T+})] = \int \int f(\hat{X}(\omega_1, \omega_2)) \hat{P}^\mu(d\omega_1) \hat{P}^\mu(d\omega_2)
\]

\[
= \int \int Q(\omega_1, \omega_2) \hat{P}^\mu(d\omega_1) \hat{P}^\mu(d\omega_2)
\]

\[
= \mathbb{E}^\mu[Q^T(f)].
\]

We now easily obtain that each

\( (\hat{X}_t, H_t^\mu, \hat{P}^\mu, Q^\mu) \)

is strong Markov.

Similarly each \( B^i(t) \) remains strong Markov when we adjoin to its minimal \( \sigma \)-field, the information in

\[
F \vee \sigma(B^k, Y^k_j, T^k_j; k \neq i, j \geq 1) \vee \sigma(Y^k_j; k, j \geq 1) \vee \sigma(Y^1_j; j \leq B^i(t)),
\]

and hence, for any \( (H_t^\mu) \) stopping time \( T \),

\[
B^i(\cdot + \int_0^T n_s^i(Y)ds) - B^i(\int_0^T n_s^i(Y)ds)
\]

is a Poisson process, independent of \( F \vee H_t^\mu \vee \sigma(B^k; k \neq i) \), under
Thus, $\theta^0_t(Y)$ is constructed from $(\hat{X}_{t+T})$ in exactly the same way that $Y$ was constructed from $(\hat{X}_t)$, so that by the strong Markov property of each $(\hat{X}_t, H^\mu_t, \hat{P}^\mu, Q^X)$, we see that

$$(\hat{X}_t, Y_t, H_t, \hat{P}^X)$$

is strong Markov.

Condition (8.8) follows by definition. To show (8.7), it suffices to show the conclusion of Lemma 9, with $(Z_t)$ of the form

$$Z_t = Z_{r+1}(t), Z_{r+} \in H^\mu_r,$$

and $h(x,v)$ of the form

$$1_{A \cap V}(v)1_{C_0} (x,v), A \in V \; \text{or} \; f(v)1_{\{ (y,z); y \neq z, b(y,z) \in [\frac{1}{k}, \frac{1}{k-1}] \}} (x, G(x,v)), f \in V \text{ positive.}$$

In the first case, by Lemma 12, conditional on $H^\mu_r \nu \sigma(\hat{X}_s; s \geq 0)$,

$$\sum_{s \in (r,t]} h(\hat{X}_{s-}, Y_s)$$

is a Poisson random variable, of mean

$$\int_{r}^{t} n_s(A) ds,$$

Showing the corresponding expression of Lemma 9.

In the second case,
The techniques of the foregoing sections can be used to examine the more general problem of constructing a strong Markov process.
from a 'process on the boundary', together with the excursions away from that 'boundary'. In this section we will be content with modifying a result due to B. Maisonneuve [38], proving a partial converse, along the lines of Theorem 2 and Proposition 5, and then showing some auxiliary results. Applications follow in Section 10.

Let \( E \) be a \( U \)-space. Fix a metric \( d \) on \( E \), compatible with its topology. We will assume that some isolated point \( \Delta \in E \) has been singled out, to function as a 'cemetery', and we will let

\[
\vartheta(u) = \inf\{t \geq 0; u(t) = \Delta\}, \text{ for } u \in U.
\]

Let \( P^X(du) \) be an \((E, E; U, U)\) kernel such that \( P^X(U) = 1 \) for every \( x \), and \( P^\Delta(W_t = \Delta \text{ for every } t) = 1 \). Augment \((U, U^c)\) with respect to the \( P^\mu \) to form \((\bar{U}, \bar{U}_t)\). Assume that

\[
(W_t, \bar{U}, \bar{U}_t, P^X)
\]

is a right process satisfying the further condition (6.5) of near-Borel measurability of the excessive functions. Let \( F \in \bar{E} \) be nearly Borel, with \( \Delta \in F \). We set

\[
\rho_F = \{t \geq 0; W_t \in F\}, \quad M = \bar{\rho}_F,
\]

\[
D_t = \inf\{s > t; s \in M\}, \quad \psi(x) = E^x[e^{-D_0}], \quad M_0 = \{t; D_t > D_{t^-}\}.
\]

Since \( F \) is nearly Borel, each \( D_s \) is a \((\bar{U}_t)\) stopping time, and \( \psi \) is \( 1 \)-excessive. Assume that

\[
(9.1) \quad F = \{x \in E; \psi(x) = 1\}.
\]
Then $M$ has no isolated points, a.s.. Assume also that

$$(9.2) \psi \text{ is regular (see Blumenthal and Getoor [5] ).}$$

Then there exists a continuous PAF $1_t$, satisfying

$$(9.3) \mathbb{E}^x[\int_0^\infty e^{-t} dt] = \psi(x) \text{ for every } x \in E,$$

whose set of increase is a.s. exactly $M$. A word of caution; at this point we are treating $\Delta$ just as any other point of $F$, and hence DO NOT TAKE THE CONVENTION THAT $\psi(\Delta) = 0$. In fact, $\psi(\Delta) = 1$, so that instead of having $1_t = 1_{t\wedge \Delta}$, we have that $\mathbb{P}_0^\Delta(1_t = t \text{ for every } t) = 1$.

Of course, we could have instead stopped $1_t$ at $\Delta$; that is, we could have used the function $\psi(x) - E^x[e^{-\Delta}]$ instead of $\psi(x)$.

Also, recall for later use, that a function $D: U \to [0,\infty]$ is called a perfect terminal time if

$$D = t + D \circ \theta_t \text{ on } \{D > t\}, \text{ for every } t \geq 0.$$ 

It is called exact if $D \circ \theta_t \to D$ as $t \downarrow 0$.

Now, let $(\Omega, \mathcal{G})$ be a measurable space, on which we are given a right continuous $E$-valued process $(X_t)$, and a right continuous filtration $(\mathcal{G}_t)$ to which $(X_t)$ is adapted. Suppose that $\hat{P}^X(\omega)$ is an $(E, E; \Omega, \mathcal{G})$ kernel such that $\hat{P}^b$ is the image law of $\hat{P}^b$ under $X$. Suppose also that $(G, \mathcal{G}_t)$ is augmented with respect to the $\hat{P}^\mu$, and that $(X_t, \mathcal{G}_t, \hat{P}^\mu, \hat{P}^X)$ is strong Markov for each $\mu$ (so that also $(X_t, G, \mathcal{G}_t, \hat{P}^X)$ is a right process). We set

$$L_t = l_t(X.), \ S^+(s) = \inf\{t; L_t > s\}, \ S^-(s) = \inf\{t; L_t \geq s\}.$$ 

Then $S^+$ and $S^-$ are the right and left continuous inverses
of \((L_t)\), respectively, and since \((L_t)\) is continuous, they are strictly increasing whenever finite. Also, \(S^+(s)\) is a \((G^+_t)\) stopping time. Put

\[
\zeta = \mathbb{L}(X.) ; \quad \hat{\zeta} = L_t ; \quad \hat{D}_t = D_t(X.) ;
\]

\[
(9.4) \quad \hat{\chi}_s = \begin{cases} 
X_{S^+(s)}, & \text{if } S^+(s) < \infty \\
\Delta, & \text{otherwise} ;
\end{cases}
\]

\[
F_s = G_{S^+(s)} ; \quad F_{0-} = G_{0-} ;
\]

\[
(9.5) \quad \gamma_s = \begin{cases} 
X_{(S^-(s)+) \wedge S^+(s)}, & \text{if } S^-(s) < S^+(s) \\
\delta, & \text{otherwise} ;
\end{cases}
\]

and let \(p^b_0\) be the image law of \(\hat{p}^b\) on \((U, U)\), under \(X. \wedge D_0\). Parts of the following result are taken from El Karoui [20]. The reader is strongly encouraged to omit its proof, and pass directly to Theorem 3.

**Lemma 13** Fix a probability \(\mu\) on \(E\).

(a) If \(T\) is a \((G^\mu_t)\) stopping time, then \(L_T\) is an \((F^\mu_s)\) stopping time. If \(T\) is any random time, we have that

\[
G^\mu_{S^-(L_T)^-} \subset G^\mu_{T^-} .
\]

(b) If \(T\) is an \((F^\mu_s)\) stopping time, then \(S^+(T)\) is a \((G^\mu_t)\) stopping time,

\[
F^\mu_{T^-} = G^\mu_{S^-(T)^-} , \quad \text{and } F^\mu_T = G^\mu_{S^+(T)} .
\]
(c) If \((Z_t)\) is \((G^\mu_t)\) predictable, then \((Z_{S^-}(s))\) is \((F^\mu_s)\) predictable. If \((Z_s)\) is \((F^\mu_s)\) predictable, then \((Z_{L_t})\) is \((G^\mu_t)\) predictable.

(d) \((\hat{X}_s)\) is adapted to \((F^\mu_s)\), and takes values in \(F\), a.s.

For \(x \in F\), let \(Q^X\) be the image law of \(\hat{P}^X\) on \((U^F, U^F)\) under \(\hat{X}\). Then

\[
(\hat{X}_t, F^\mu_t, \hat{P}^\mu, Q^X)
\]

is strong Markov, and \(Q^X(W_0 = x) = 1\) for every \(x \in F\).

(e) \((Y_t)\) is adapted to \((F^\mu_t)\)

(f) The following hold a.s.:

\[
M_0(X.) = \{S^-(s); s \in (0, \infty)\} , \quad S^-(s) < S^+(s)\};
\]

\[
\hat{D}_{S^-}(s) = S^+(s) \text{ for every } s \geq 0; \quad \text{and}
\]

\[
\hat{\zeta} < \infty \text{ if and only if } S^-(\hat{\zeta}) < \infty.
\]

proof: (a) We already know that \(S^+(s)\) is a \((G^\mu_t)\) stopping time, for \(s\) fixed, and that so is \(\hat{D}_T\), whenever \(T\) is a \((G^\mu_t)\) stopping time. Since \((L_t)\) increases exactly on \(M\), and \(S^-\) is strictly increasing, it follows that \(S^+(L_T) = \hat{D}_T\) a.s., and hence

\[
\{L_T < s\} = \{S^+(L_T) < S^+(s)\} \in G^\mu_{S^+(s)} = F^\mu_s.
\]

For \(T\) now a random time, \(G^\mu_{S^-(L_T)}\) is generated by the
\( \hat{\mathcal{P}}^\mu \)-null sets, and by sets

\[
A \cap \{ t < S^-(L_T) \} = A \cap \{ \hat{D}_t < T \},
\]

for \( A \in G^\mu_T \cap G^\mu_{\hat{D}_t} \). Since \( \hat{D}_t \) is a \((G_t)\) stopping time, these sets lie in \( G^\mu_{\hat{D}_t} \).

(b) \( F^\mu_{T^-} \) is generated by the \( \hat{\mathcal{P}}^\mu \)-null sets, and by sets

\[
A \cap \{ s < T \} = A \cap \{ s^+(s) < S^-(T) \},
\]

for \( A \in F^\mu_s = G^\mu_{s^+(s)} \). Because \( s^+(s) \) is a \((G^\mu_s)\) stopping time, they lie in \( G^\mu_{s^+(s)} \). Thus

\[
F^\mu_{T^-} \subseteq G^\mu_{S^-(T)}.
\]

Before proving the reverse inclusion, we will show the remainder

of (b).

\( S^+(T) \) is a \((G^\mu_s)\) stopping time, since

\[
\{ S^+(T) < t \} = \{ t < L_T \} \in F^\mu_{L_T^-}
\]

(as \( T \) is an \((F^\mu_s)\) stopping time), and

\[
F^\mu_{L_T^-} \subseteq G^\mu_{S^-(L_T)} \subseteq G^\mu_T,
\]

by the above, and part (a). Similarly, if \( A \in F^\mu_T \), then

\[
A \cap \{ S^+(T) < t \} = A \cap \{ t < L_T \} \in F^\mu_{L_T^-} \subseteq G^\mu_T,
\]

so that
\[ F_{L_T}^\mu \subset G_{S^+(T)}^\mu. \]

Conversely, if \( A \in G_{S^+(T)}^\mu \), then
\[
A \cap \{T < \hat{s}\} = A \cap \{S^+(T) < S^+(s)\} \in G_{S^+(s)}^\mu = \hat{F}_s^\mu.
\]

Finally, \( G_{S^-}(T)^- \) is generated by the \( \hat{F}_s^\mu \)-null sets, and by sets of the form
\[
A \cap \{t < S^-(T)\} = A \cap \{L_t < T\}, \quad \text{where} \quad A \in G_t^\mu \subset G_{S^+(L_t)}^\mu = F_{L_t}^\mu.
\]

These sets lie in \( F_{L_t}^\mu \) since \( L_t \) is an \( (F_s^\mu) \) stopping time.

(c) Let \((Z_t)\) be left continuous and adapted to \((G_t^\mu)\). Then \((Z_{S^-}(s))\) is also left continuous, and for each \( s \),
\[
Z_{S^-}(s) \in G_{S^-}(s)^- = F_{S^-}^\mu
\]
by (b). Thus \((Z_{S^-}(s))\) is \((F_s^\mu)\) predictable for such \((Z_t)\), and hence by monotone class arguments, for every \((G_t^\mu)\) predictable \((Z_t)\). The 'converse' is proved similarly.

(d) By (6.2), \( \rho_F \) is a.s. closed on the left, so that the argument of Meyer [40], Proposition 1, shows that \( M \setminus M_0 \subset \rho_F \) a.s.. Since \( S^+ \) is right continuous, strictly increasing, and \( (L_t) \) increases only on \( M \), a.s., we get that
\[
S^+(s) \in (M \setminus M_0)(X_\omega) \cup \{\infty\} \quad \text{for every } s, \text{ a.s.}
\]
showing that \( \hat{X}_s \in F \) for every \( s \), a.s.. Since \( S^+ \) is right
continuous, so is \((\hat{X}_s)\). It is clearly adapted to \((F_s)\). If \(T\) is an \((F_T^\mu)\) stopping time, \(B \in \mathcal{F}^T \subset \mathcal{E}\), and \(h > 0\), then
\[
\hat{\mu}^\mu(\hat{X}_{h+T} \in B | F_T^\mu) = \hat{\mu}^\mu(X^{S^+(T)+h} \in \mathcal{E}^G_{S^+(T)} | B) = \hat{\mu}^\mu(\hat{X}_{h} \in B) = Q^T(\mathcal{W}_h \in B), \hat{\mu}^\mu\text{-a.s.}
\]
by the strong Markov property of \((X^T)\), showing (d).

(e) \(S^-(s)\) and \(S^+(s)\) are adapted to \((F_s)\). Thus, for \(\epsilon > 0\),
\[
T = \text{inf}\{s \geq 0; S^+(s) > S^-(s) + \epsilon\}
\]
is an \((F_s)\) stopping time. Because \(\{s \leq t; S^+(s) > S^-(s) + \epsilon\}\)
is finite for every \(t\), a.s., it therefore suffices to show that
\(Y_T\) is measurable, from \((\Omega, F_T)\) to \((U, \mathcal{U})\). Let
\(0 \leq t_1 < t_2 < \ldots < t_k\), and \(B_1, \ldots, B_k \in \mathcal{E}^0\). Then
\[
\{Y_T(t_i) \in B_i, i=1..k\} = \{X^{S^-(T) + t_i} \wedge S^+(T) \in B_i, i=1..k\}.
\]
By (b), \(S^+(T)\) is a \((G_T)\) stopping time, so that
\[
(t, \omega) \rightarrow X_T \wedge S^+(T)(\omega)
\]
is a measurable function from \(\mathcal{B} \otimes G_{S^+(T)}\) to \(\mathcal{E}^0\). Since
\[
S^-(T) \in G_{S^+(T)}
\]
by (a), it follows that the above set lies in \(G_{S^+(T)} = F_T\). By completeness, we can extend this to \(\{Y_T \in A\}, A \in \mathcal{U}\).

(f) We first prove the third statement. For \(x \in E\) we have that
\[ e^t \geq e^{\mathbb{E}_x[\int_0^t e^{-s} \, ds]} \geq \mathbb{E}_x[1_t], \]

so that \( 1_t < \infty \) for every \( t \), a.s., and hence that a.s., \( \hat{\zeta} < \infty \) whenever \( S^{-}(\hat{\zeta}) < \infty \). Conversely, let

\[ T = T_1 = \inf\{t > 1; W_t \in F \text{ and } 1_t > 1\} \]

and

\[ T_{k+1} = T_k + T \circ \theta_{T_k} \cdot \]

For \( x \in F \) we have that

\[ \mathbb{E}_x[1_T] \geq \mathbb{E}_x[\int_0^t e^{-s} \, ds] \]

\[ = \psi(x) - \mathbb{E}_x[e^{-T_\psi(W_t)}] \geq 1 - e^{-1}. \]

Thus,

\[ P_x^x(1_T \geq (1 - e^{-1})) \geq \frac{1}{2} \]

for every \( x \in F \). Because \( W_T \in F \) a.s. on \( \{T < \infty\} \), we see by the strong Markov property of \( (W_t) \), that

\[ P^x(T_k < \infty \text{ and } 1_{T_k} - 1_{T_{k-1}} < 1 - e^{-1} \text{ for every } k \geq K) \]

\[ = 0, \text{ for every } K \text{ and } \mu. \]

Since \( T_k(X) < \zeta = \infty \) a.s. on \( \{S^{-}(\zeta) = \infty\} \) it follows that

\[ \hat{\zeta} = \lim_{t \to \infty} L_t = \infty \text{ a.s. on that set.} \]

Now let \( M_1 \) be the set on the right of the first statement
of (f). For \( \omega \) such that \( L_t(\omega) \) increases exactly on \( M(X_t)(\omega) \) and \( S^-(\cdot)(\omega) \) is strictly increasing, we obtain that

\[
D_{S^-}(s)(\omega) = S^+(s)(\omega),
\]

and hence \( M_1(\omega) \subset M_0(X_t)(\omega) \). Conversely, for \( t \in M_0(X_t)(\omega) \), we will have that \( t \in M_1(\omega) \) provided \( t = S^-(s)(\omega) \) for some \( s > 0 \). It follows by choice of \( \omega \), that \( L_t(\omega) > 0 \), and that \( t = S^-(L_t)(\omega) \). We need therefore only show that \( L_t(\omega) < \infty \). This follows from the third part of (f). \( \square \)

**Theorem 3** \((Q^X)\) is regulated. There exists an exact, perfect terminal \((U_{t+})\) stopping time \( D \), and an \((F, F^F; U, U)\) kernel \( n(x, du) \) such that the following conditions are satisfied

I. \( n(x, \{D = 0 \text{ or } W_t \neq W_D \text{ for some } t \geq D \}) = 0 \)
for every \( x \in F \).

II. (a) \( n(x, \{\sup_{t} d(x, W_t) > \epsilon \}) < \infty \), for every \( x \in F \) and \( \epsilon > 0 \).

(b) \( \lim_{h \to 0} \int_0^h n(\hat{X}_s, \{\sup_{t} d(x, W_t) > \epsilon \}) ds < \infty \text{ \( \hat{P}^X\)-a.s.} \),
for every \( x \in F \) and \( \epsilon > 0 \).

III. \( \int (1 - e^{-D(u)}) n(x, du) \leq 1 \) for every \( x \in F \).

IV. \( n(x, \{u; D(u) > t, u \in A, \theta_t(u) \in M\}) \)

\( = \int_{A \cap \{D > t\}} p^{u(t)}_0(M) n(x, du), \)

for \( t > 0, A \in \mathcal{U}_t, M \in \mathcal{U}, x \in F. \)
\[
V \quad n(x,\{u; u(0) \in B, u \in M\}) = \int_{\{u; u(0) \in B\}} P^U_0(M)n(x,du)
\]

for \( x \in F \) and \( B \in \mathcal{E} \) such that \( B \cap F = \emptyset \).

VI (a) Let

\[
m(x) = 1 - \int (1 - e^{-D(u)})n(x,du), \; x \in F \setminus \{\Delta\}; \; m(\Delta) = 0; \; \text{and}
\]

\[
I = \{x' \in F; \; \int_0^t m(X_s')ds > 0 \; \text{for every} \; t > 0\} = 0.
\]

Then for each \( x \in I \) and \( h > 0 \),

\[
\int_0^h n(X_s, u)ds = a.s.
\]

(b) If \( n'(du) \) is a positive measure on \((U, \mathcal{U})\) satisfying IV, such that

\[
n'(du) \leq n(x,du)
\]

for some \( x \), then \( n'(W_0 \in F) = 0 \) or \( \infty \).

Further:

(9.6) \( \hat{D}_s = D(X_{s+}) \) for every \( s \), a.s.;

(8.7), (8.8) and (8.9) hold, with

\[
G(x,u) = \begin{cases} 
  u(D(u)), & D(u) < \infty \\
  \Delta, & \text{otherwise};
\end{cases}
\]

\((\hat{X}_t, Y_t, F_t, \hat{P}^X)\) is strong Markov; and

(9.7) \( S^+(t) = \int_0^t m(X_s)ds + \int_{s \in [0,t]} D(Y_s) \) for \( Y_s \neq \delta \).
Note: 1. In the situation of Thereom 2, VI (b) states that if 
\( n(U) = \infty \) then (vi'c) holds, and if \( n(U) < \infty \) then \( n(U^a) = 0 \).

VI (a) becomes the remainder of (vi'b), that if \( n(U) < \infty \), then 
\( m > 0 \).

2. As a consequence we conclude that conditions (9.1) and (9.2) 
rule out the discrete visiting behaviour of (vi'a).

3. Condition VI (b) rules out two distinct types of behaviour, 
one identical to that precluded by (vi'), and another not found 
in the situation of Thereom 2. In fact, using II(a), condition 
VI(b) is seen to be equivalent to the condition that for every 
\( x \in F \), both:

\[
\begin{align*}
& n(x, \{W_0 \in F \setminus \{x\}\}) = 0; \text{ and} \\
& n'(W_0 = x) = 0 \text{ or } \infty, \text{ whenever } n'(du) \text{ is a positive }
\end{align*}
\]

measure on \((U, \mathcal{U})\) satisfying IV, and such 
that \( n'(du) \leq n(x, du) \).

4. The proof that \((Q^X)\) is regulated uses (6.5). As remarked 
in the introduction to section 8, Czy1 [28] shows a stronger result, 
namely that \((\hat{X}_t)\) is a right process.

**proof:** We can apply the perfection arguments in Meyer [40] 
to the right process \((W_t, \bar{U}, \bar{U}_t, P^X)\), to obtain an exact perfect 
terminal \((\bar{U}_{t+}) stopping time \( D \) such that \( P^X(D = D_0) = 1 \) for
each \( x \in F \). Thus

\[
\hat{D}_q = D(X_{q+}) \quad \text{a.s.,}
\]

for each \( q \in Q \), so that by exactness, we obtain condition (9.6).

Note the following, from Meyer [40]. For \( h > 0 \), \((h \wedge D \circ \theta_t)\) is a càdlàg process adapted to \((U_{t+2h})\), so that it is \((U_{t+2h})\) progressive. Thus so is

\[
M' = \{(t,u) ; \lim_{s \to t} D \circ \theta_s(u) = 0\}.
\]

and hence \( M' \) is \((U_{t+})\) progressive by Dellacherie et Meyer [13], IV. 14. Thus \( M \) is indistinguishable from a progressive set relative to the filtration \((U_{t+})\) (rather than \((\tilde{U}_t)\)).

We can now apply Proposition 9.2 of Maisonneuve [38] to \((W_t, \tilde{U}_t, \tilde{U}_t, p^X)\) and \( M' \). Maisonneuve works with \((W_t)\) killed at \( D \), rather than stopped there, but his kernel clearly extends to one which, by Lemma 13(c) and the above discussion, satisfies (8.7). It follows from his proof, and that of his Theorem 4.1, that this kernel \( n \) satisfies III.

We will now show that \((Q^X)\) is regulated. Let \( \tilde{E} \) denote the Ray Knight compactification of \( E \), and let \( g \) belong to the Ray cone of \( \tilde{E} \). Thus \( g \) is continuous on \( \tilde{E} \), and \( g|_E \) is \( \alpha \)-excessive with respect to the \( P^X \), for some \( \alpha > 0 \). By (6.5), \( g \) is thus nearly Borel. Fix a probability \( \mu \) on \( F \). Then there exist \( g_1, g_2 \in E^0 \), with
\[ g_1 \leq g_1|_E \leq g_2, \quad P^\mu(g_1(W_t) \neq g_2(W_t)) \text{ for some } t = 0. \]

Now

\[ g_1|_F \in \mathcal{F}^F,0, \]

so that \( g_1 \circ W^F \) is a \((\mathcal{F}^F,0)\) optional process. Thus

\[ \mathbb{A}_g = \{ g_1(W^F_t) \neq g_2(W^F_t), \text{ or } \lim_{s \downarrow t} g_1(W^F_s) \neq g_1(W^F_t), \text{ or } \lim_{s \uparrow t} g_1(W^F_s) \text{ does not exist} \} \subseteq \mathcal{U}. \]

Since \( Q^\mu \) is the image law of \( \hat{P}^\mu \) under \( \hat{X}_t \), and \( g \circ X \) is a.s. càdlàg (\( g|_E \) being \( \alpha\)-excessive), we see that also

\[ Q^\mu(\mathbb{A}_g) = 0. \]

If we now let \( g \) range through a countable dense subset of the Ray cone, we get that

\[ \{ W^F \text{ is càdlàg in the topology of } \tilde{E} \} \in \mathcal{U}^{F,\mu}, \]

and has \( Q^\mu \)-measure 1, whenever \( \mu \) is a probability on \( F \).

Augment \((\mathcal{U}^F, \mathcal{U}_t^F)\) with respect to the \( Q^\mu \), to form \((\check{\mathcal{U}}^F, \check{\mathcal{U}}_t^F)\),

and write

\[ \tilde{W}_t^F, \tilde{X}_t^F, \tilde{X}_t^F \]

for the left limits of \( W_t^F, \hat{X}_t \), and \( X \) at \( t \), in the topology of \( \tilde{E} \). We conclude from the above that for any positive
h \in E^F \otimes E^F,

\sum_{s \in (0, t]} h(\tilde{\omega}_{s-}, \tilde{W}_{s}) \mathbb{1}_{\{\tilde{\omega}_{s-} \neq \tilde{W}_{s}, \tilde{W}_{s} \in E\}}

defines a PAF adapted to (U^E_L) (possibly one which is identically infinite).

Since E is a U-space, we conclude from the discussion prior to Lemma 9, that there is an \((F \times F, E^F \otimes E^F; U, \mu)\) kernel \(n(x, y; du)\), such that

\[\int_{A}(u(D(u)))f(u)n(x, du) = \int_{A}(v(D(u)))f(v)n(x, u(D(u)); dv)n(x, du)\]

for every positive \(f \in U\), and \(A \in E^F\) such that \(x \notin A\). Set

\[h_0(x', y') = \int (1 - e^{-u})n(x', y'; du) \mathbb{1}_{\{(x, y); x \neq y, n(x, y; U) = 0\}}(x', y').\]

By the argument of Getoor [23] (13.4), we see once again that

\[\hat{x}_{s-} = \hat{x}_{s-}\]

for every \(s\) such that \(Y_s \neq \delta\), a.s., so that by Lemma 9 (which did not require \((Q^\infty)\) to be regulated) and III,

\[\hat{\mu}^U[\sum_{s \in (0, t]} h_0(\hat{x}_{s-}, \hat{x}_{s})] = \hat{\mu}^U[\int_{Y_s \neq \delta} h_0(\hat{x}_{s}, u(D(u)))n(\hat{x}_{s}, du)ds\]

\[= \hat{\mu}^U[\int_{0}^{t} (1 - e^{-u})n(\hat{x}_{s}, du)ds] \leq t.\]

Also, A. Benveniste and J. Jacod show in [2], that there is a function \(h_1 \in E\) such that the functional

\[\sum_{s \in (0, t]} h_1(\tilde{x}_{s-}) \mathbb{1}_{\{\tilde{x}_{s-} \neq x_{s-}, \tilde{x}_{s-} \in E\}}\]
has a bounded 1-potential. But if \( Y_s = \delta \) and \( s < \zeta \), then by (f) of Lemma 13, \( S^- (s) = S^+ (s) \) and so

\[
\tilde{X}^F_s = \tilde{X}^F_{s+} (s^-) .
\]

Thus, if we put

\[
h(x, y) = h_1 (x) \wedge h_0 (x, y), \text{ for } x, y \in F,
\]

we have that \( h(x, y) > 0 \) whenever \( x \neq y \), and \( K_t^h < \infty \) for every \( t \), a.s.

It follows once more by the argument of (13.4) of Getoor [23], that

\[
\tilde{W}^F_{\tau-} = \tilde{W}^F_{\tau-} \text{ for } \{ \tau < \infty \},
\]

for any \( (\mu^F, \mu) \) totally inaccessible stopping time. Thus, taking \( b = h \), condition (8.3) will hold, provided

\[
Q^\mu (K^h_{\tau-} \neq K^h_{\tau+}, \tau > 0) = 0
\]

for any \( (\mu^F, \mu) \) predictable stopping time \( \tau \). We may, without loss of generality, assume that \( \tau > 0 \), and that \( \tau = \infty \) on \( \{ \tau \geq \zeta \} \). Write \( T = \tau (X) \). Then \( T \) is \( (F^\mu_t) \) predictable, so that \( Y_T = \delta \hat{P}^\mu \text{-a.s.} \), and hence \( S^- (T) = S^+ (T) > 0 \hat{P}^\mu \text{-a.s.} \). (we use here that \( \hat{P}^\mu (T = \zeta < \infty) = 0 \). Let

\[
T_n \uparrow T, T_n < T \hat{P}^\mu \text{-a.s.}
\]

Then
\begin{align*}
S^+(T_n) & + S^-(T_n), \quad S^+(T_n) < S^-(T) \quad \mathbb{P}^\mu\text{-a.s.}
\end{align*}
also, so that \( S^-(T) \) is a \((\mathbb{G}^\mu_t)\) predictable stopping time. Thus
\begin{align*}
\tilde{X}^F_{T^-} &= \tilde{X}_{S^-(T)^-}
\end{align*}
either equals \( X_{S^+(T)} = \tilde{X}_T \), or does not lie in \( E \), from which
we conclude that
\begin{align*}
Q^\mu(K^h_{\tau^-} \neq K^h_{\tau}) &= 0.
\end{align*}
Conversely, let \( \tau \) be \((\mathbb{U}^F_t, \mathbb{\mu})\) totally inaccessible, with
\begin{align*}
\tilde{W}^F_{\tau^-} &\neq W^F_{\tau^-} \quad Q^\mu\text{-a.s. on } \{ \tau < \infty \}.
\end{align*}
Then as above,
\begin{align*}
\tilde{W}^F_{\tau^-} &= W^F_{\tau^-} \quad Q^\mu\text{-a.s.},
\end{align*}
so that by definition, \( K^h_{\tau^-} \neq K^h_{\tau} \quad Q^\mu\text{-a.s. on } \{ \tau < \infty \} \). Thus
\( (K^h_{\tau}) \) regulates \((Q^X)\).
\begin{align*}
\text{(8.9) now follows as a special case of Lemma 9. The strong Markov property of } (\tilde{X}_t, \tilde{Y}_t) \text{ is immediate from that of } (X_t).
\end{align*}
(8.8) follows from (9.6), and the definition of \((Y_t)\). To show (9.7), put
\begin{align*}
A(t) &= \int_0^t m(W_s)d\tilde{I}_s + \int_0^t 1_{E\setminus F}(W_s)ds.
\end{align*}
Since \( m \in E \), \( A(t) \) is adapted to \( (\bar{U}_t) \), hence is a continuous perfect additive functional of \( (\bar{W}_t) \). It will suffice to show
that \( A(t) = t \) for every \( t \), a.s., and for this it suffices in turn that the corresponding \( l \)-potentials agree.

\[
\mathbb{E}^{x}[\int_{0}^{\infty} e^{-t} dA_t] = \mathbb{E}^{x}[\int_{0}^{\infty} e^{-t} (1-m(X_t)) dL_t] + \mathbb{E}^{x}[\int_{0}^{\infty} e^{-t} 1_E \mathbb{P}(X_t) dt]
\]

\[
= \mathbb{E}^{x}[e^{-D}] - \mathbb{E}^{x}[\int_{0}^{\infty} e^{-S^+(s)} (1-m(X_s)) ds] + \mathbb{E}^{x}[\sum_{t \in M_0 \cup \{0\}} e^{-t} (1-e^{-t})] + \mathbb{E}^{x}[\int_{0}^{\infty} e^{-S^-(s)} (1-e^{-S^-(s)}) ds] + \mathbb{E}^{x}[\sum_{s \geq 0} e^{-S^-(s)} (1-e^{-S^-(s)}) ds] + \mathbb{E}^{x}[\sum_{s \geq 0} e^{-S^-(s)} (1-e^{-S^-(s)}) ds] \quad \text{(by (8.7))}
\]

\[
= \mathbb{E}^{x}[e^{-D}] + \mathbb{E}^{x}[1-e^{-D}] = 1, \text{ as required.}
\]

Turning to II, let \( x \in F \), and

\[
R = \inf \{ t; d(x, X_t) > \varepsilon \}, \quad T = L_R.
\]
By right continuity of \((X_t)\), \(R\) and hence \(T\), are \(\mathbb{P}^x\)-a.s. strictly positive. Thus by (8.7),
\[
\mathbb{E}^x \left[ \int_0^T n(X_s, \{\sup_t d(X_t, W_t) > \varepsilon\}) \, ds \right] \\
= \mathbb{E}^x \left[ \sum_{s \in (0,T]} \mathbb{1}_{\{\sup_t d(X_t, W_t) > \varepsilon\}}(Y_s) \right] \leq 1,
\]
showing II(b). We obtain II(a) by modifying \(n\) on a set of \(\hat{X}\)-potential zero, as follows. Let
\[
B = \{ x \in F; n(x, \{ \sup_t d(x, W_t) > \varepsilon\}) = \infty \} \in \mathcal{F}.
\]
Then for each \(t \geq 0\), \((s, \omega) \mapsto \mathbb{1}_B(\hat{X}_s(\omega))\) is measurable with respect to the \(\mathbb{Q} \circ \mathbb{P}^u\) completion of \(\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t\), as a function \([0,t] \times \Omega \to \mathbb{R}\). Thus
\[
A_t = \int_0^t \mathbb{1}_B(\hat{X}_s) \, ds
\]
is \(\mathcal{F}^u_t\) measurable for each \(u\), and hence \(T = \inf\{t; A_t > 0\}\) is an \((\mathcal{F}^u_t)\) stopping time. Let
\[
R = \inf\{ r > T; \int_T^r n(\hat{X}_s, \{ \sup_t \hat{d}(\hat{X}_s, W_t) > \varepsilon/2\}) \, ds \geq 1, \\
or \hat{d}(\hat{X}_T, X_T) > \varepsilon/2 \}
\]
Then
\[
\int_T^R n(\hat{X}_s, \{ \sup_t \hat{d}(\hat{X}_s, W_t) > \varepsilon\}) \, ds \leq 1,
\]
so that by definition of $T$, we must have $R = T$ a.s. But by II(b) and the strong Markov property of $\hat{X}$ at $T$, we have that $R > T$ a.s. on $\{T < \infty\}$, and hence $T = \infty$ a.s. Thus we may replace $n(x, \cdot)$ by 0 for $x \in B$, and leave the other properties unchanged.

Properties IV and V follow from Theorem 5.1 of Maisonneuve [38], if necessary modifying $n$ on a set of $\hat{X}$-potential zero, as above. A similar modification, together with (8.7), gives us I.

Finally, consider VI. Let

$$T = \inf\{t; \int_0^t m(\hat{X}_s)ds > 0\} , \text{ and}$$

$$T_k = \inf\{t; \int_0^t n(\hat{X}_s, U)ds \geq k\} ,$$

for $k > 0$. Then

$$\hat{E}^X[\sum_{0 < S \leq T \wedge k} 1] = \hat{E}^X[\int_0^{T \wedge k} n(\hat{X}_s, U)ds] \leq k ,$$

so that $S^-$ increases at only finitely many times in $(0, T \wedge k)$. Since $S^-$ is a.s. strictly increasing, we must have that $T_k = 0$ $\hat{P}^X$-a.s., whenever $x \in I$. Since $k$ was arbitrary, VI(a) holds.

As in Proposition 1, let $C$ be the set of dyadic rational numbers, and for $\varepsilon, \eta > 0$, let

$$B_{\varepsilon, \eta} = \{ b \in E; \hat{P}^b_0 (D \geq \eta) \geq \varepsilon \} \in E ,$$

and
\[ H_{\varepsilon, \eta} = \lambda \{ D > 0, W_t \in B_{\varepsilon, \eta} \text{ for } t \in \mathbb{C} \cap (0, \lambda) \} . \]

We obtain as before, that for every \( x \in F \),

\[ n(x, H_{\varepsilon, \eta}) \leq \frac{1}{\varepsilon} n(x, D > \eta) \leq \frac{1}{\varepsilon (1 - e^{-\eta})} \int (1 - e^{-D})dn(x, \cdot) \]

\[ \leq \frac{1}{\varepsilon (1 - e^{-\eta})} . \]

Fix \( \mu \). Because \( H_{\varepsilon, \eta} \in U_{0^+} \), we can find \( H_{\varepsilon, \eta}^0 \in U_{0^+} \) such that

\[ H_{\varepsilon, \eta} \geq H_{\varepsilon, \eta}^0 \text{, and } \frac{\mu}{\varepsilon} \int_0^\infty n(\hat{X}_s, H_{\varepsilon, \eta} \setminus H_{\varepsilon, \eta}^0)ds = 0 . \]

We argue as before, that (as \( F \) is nearly Borel)

\[ T = \inf\{ t > 0; X_t \in F, X_{t^+} \in H_{\varepsilon, \eta}^0 \} \]

is a \((G_\varepsilon)\) stopping time. Since \( H_{\varepsilon, \eta}^0 \in \{ D > 0 \} \), and

\[ \frac{\mu}{\varepsilon} \int_0^t \frac{1}{H_{\varepsilon, \eta}^0} (Y_s) ds = \frac{\mu}{\varepsilon} \int_0^t n(\hat{X}_s, H_{\varepsilon, \eta}^0)ds \]

\[ \leq \frac{t}{\varepsilon (1 - e^{-\eta})} < \infty , \]

for \( t \geq 0 \), we have that

\[ Y_{L_T} \in H_{\varepsilon, \eta}^0 \text{ and } X_T \in F, \hat{P}^\mu - \text{a.s.} . \]

But by the strong Markov property of \((X_t)\), no excursion can start at a stopping time at which \( X \) lies in \( F \). Thus \( T = \infty \)

\( \hat{P}^\mu - \text{a.s.} \), so that
and hence
\[ \sum_{s>0} \int_{H_{\varepsilon, \eta} \cap \{W_0 \in F\}} (Y_s \neq 0, \hat{P}^\mu \text{-a.s.}) \]
\[
= 0,
\]
for each \( \varepsilon, \eta \) and \( \mu \). We can therefore modify \( n \) as before, to make
\[ n(x, H_{\varepsilon, \eta} \cap \{W_0 \in F\}) = 0 \]
for every \( x, \varepsilon \) and \( \eta \).

Now observe that if VI(b) fails at \( x \), then there is a finite, positive, nonzero measure \( n' \) concentrated on \( \{W_0 \in F\} \), for which IV holds, and also \( n(x, du) > n'(du) \). The argument of Proposition 1 shows that
\[ n'(H_{\varepsilon, \eta} \cap \{W_0 \in F\}) > 0 \]
for some \( \varepsilon, \eta > 0 \), contradicting (9.8). Thus VI(b) holds. \( \square \)

We now turn to the converse. Let \( E \) be a separable metric space, with an isolated point \( \Delta \) singled out. Let \( \delta \) be as before. The main result of this section is:

Theorem 4 (a) Let \( F \subset E, \Delta \in F, F \in E \), and assume that \( F \) is a U-space. Let \( (P_x^\delta) \) satisfy (2.4), and assume that \( F \) is nearly optional for \( (W_\varepsilon^\delta) \) and the \( P_0^\mu \). Thus, for each \( \mu \), the first hitting time \( D_0 \) of \( F \) ;
\[ D_0 = \inf\{t > 0; W_t \in F\} \]

is a stopping time with respect to the completion of \((U_{t+})\) in \(U\). Assume that

\[ P^\mu_0(W_t \neq W_{D_0} \text{ for some } t \geq D_0) = 0 \]

for each \(\mu\). Then

\[ D = \inf\{t \in (0, \infty) \cap \mathbb{Q}; W_t \in F \} \text{ and for some } \varepsilon > 0, \]

\[ W_s = W_t \text{ for every } s \in [t, t + \varepsilon) \cap \mathbb{Q} \]

is an exact perfect terminal \((U_{t+})\) stopping time which is \(P^\mu_0\)-indistinguishable from \(D_0\) for each \(\mu\). Let \(n(x, du)\) be an \((F, \mathbb{E}; U, U)\) kernel. Let \((\Omega, F)\) be a measurable space, with a right continuous filtration \((F_t)\). Let \(\hat{P}^X(du)\) be an \((E, \mathbb{E}; \Omega, F)\) kernel consisting of probabilities, and assume that \((F, F_t)\) is augmented with respect to the \(\hat{P}^\mu\). Let \((\hat{X}_t)\) be a right continuous process with values in \(F\), which is adapted to \((F_t)\), and write \(\hat{\zeta} = \hat{\rho}(\hat{X}_t)\). For \(x \in F\), we denote the image law of \(\hat{P}^X\) under \(X_t\), by \(Q^X\). Assume that for each probability \(\mu\) on \(E\), the process

\[ (\hat{X}_t, F^\mu_t, \hat{P}^\mu_t, Q^X) \]

is strong Markov. Suppose that conditions I-VI hold for these objects.

Let \(Y: \Omega \to \Pi\) be adapted to \((F_t)\), and satisfy
(8.7) for this n and \((\hat{X}_t)\). Suppose also that:

\((\hat{X}_t, Y_t, F_t, \hat{P}^X)\) is strong Markov;

(9.9) If \(Y_t \neq \delta\), then \(\hat{X}_t = \begin{cases} Y_t(D(Y_t)) & \text{if } D(Y_t) < \infty \\ \Delta & \text{if } D(Y_t) = \infty \end{cases}\)

(9.10) \(\hat{P}^\mu(Y_0 \in \text{du}) = p^\mu_0(\text{du})\) for every \(\mu\).

Define \(m\) and \(S^+\) by VI(a) and (9.7), and let

\[ L_t = \sup\{s; S^+(s) \leq t\}. \]

Then there is a right continuous filtration \((G_t)\) of \((\Omega, F)\), a family \((P^X)_{x \in E}\) of probabilities on \((U, \mathcal{U})\), and a right continuous process \((X_t)\) adapted to \((G_t)\), such that:

Each \((X_t, G_t, \hat{P}_t, P^X)\) is strong Markov;

\((G, G_t)\) is augmented with respect to the \(\hat{P}^\mu\);

(9.11) \(\hat{P}^\mu(X_{\cdot \wedge D_0} \in \text{du}) = p^\mu_0(\text{du})\) for each \(\mu\);

(9.12) Each \(S^+(s)\) is a \((G_t)\) stopping time, \(G_{S^+(s)} = F_s\), and (9.4) and (9.5) hold for every \(s\), a.s.;

(9.13) \(\hat{M} = \{t; X_t \in \mathcal{F}_t\}\) is \((G_t)\)-progressive, satisfies (9.1) and (9.2),

and \(\hat{M} = \{S^+(s); s > 0, S^+(s) < \infty\}\).

(9.14) For every \((G_t^\mu)\) stopping time \(T\), \((X_{\cdot + T}, L_{\cdot + T} - L_T)\)

is conditionally independent of \(G_T^\mu\) under \(\hat{P}^\mu\), given \(X_T\),

with conditional law

(9.15) \(\hat{P}^\mu((X_.L_.) \in (\cdot)); \text{ and} \)

\((L_T)\) satisfies (9.3).
(b) Moreover, if the \( (P^X_t) \) are the transition laws of a right process satisfying (6.5), and if from other considerations, we know that the excessive functions for the \( (P^X) \) are \( (Q^X) \)-nearly Borel, when restricted to \( F \), then in fact

\[
(X_t, F, G_t, P^X_t)
\]

is a right process satisfying (6.5). In this case, there is a continuous PAF \( (1_t) \) adapted to \( (U_{t+}) \), such that

\[
L_t = 1_t(X_t) \quad \text{for every } t, \text{ a.s.}
\]

**Note:** 1. We do not assume that \( (Q^X) \) is regulated. We can get away with this because \( (Y_t) \) is assumed to be given to us, and to be strong Markov. If we were given only \( n \), we would of course not know how to produce such a \( (Y_t) \), unless \( (Q^X) \) was regulated (see Corollary 6). Also note that Theorem 4 uses only the strong Markov property of \( (\hat{X}_t, Y_t) \), and not the knowledge of their exact joint distribution, which we could obtain from Corollary 4 provided \( (Q^X) \) was known to be strongly regulated. In this case, the assumption that \( (\hat{X}_t, Y_t) \) be strong Markov could be dropped, using Corollary 5. Note that (9.9) gives (8.8).

2. The conditions of Theorem 4(a) are met in the (vi'b) and (vi'c) cases of Theorem 2, and in this situation, Theorem 4(b) contains Proposition 5. The proof of Theorem 4 is essentially the same as that of Theorem 2, the chief complication coming in the proof of right continuity.
3. The condition that \( F \) be a U-space is included in order that Lemma 11 hold. This result is needed for Lemma 15. We will use this condition again in Corollary 6. In that result, condition (9.9) again plays the role of (8.8), but for now we use it only to give (9.5).

**Proof:** Recall that \( m \) and \( S^+ \) are defined as in VI and (9.7) Put

\[
S^+(t,p,u) = \int_0^t m(u(s))ds + \sum_{0 \leq s \leq t \atop p(s) \neq \delta} D(p(s))
\]

so that \( S^+(t) = S^+(t, Y, \tilde{X}) \). Let \( S^-(t, p, u) = S^+(t-, p, u) \),

\[
V_0 = p_t \boxtimes u_t, \quad V^0 = V_0^t,
\]

and let \( V \) be the universal completion of \( V^0 \). As in the proof of Theorem 2, we obtain \( (V^0_s) \) stopping times \( \ell_{\mathcal{L}}(p,u) \), and functions \( x_t(p,u) \) measurable from \( V^0_{\mathcal{L}} \) to \( E^0 \). The set \( M \) of Lemma 2 becomes

\[
0 = \{(p,u); u(s) = \Delta \text{ and } p(s) = \delta \text{ for every } s \geq \delta(u), S^-(s,p,u) < \infty \text{ for every } s \in [0, \delta(u)] \cap [0, \infty) \}
\]

and is strictly increasing at all such \( s \), \( S^-(s,p,u) \to \infty \) as \( s \to \infty \), and for every \( \varepsilon > 0 \) and \( t \in [0, \delta(u)) \) there is an \( h > 0 \) such that \( \sup \limits_{r} d(p(s)(r), u(t)) \leq \varepsilon \)
whenever \( p(s) \neq \delta \) and \( s \in (t, t+h) \).
Because
\[ x_t(p,u) = u(\ell_t(p,u)) \] whenever \( t = S^+(t,p,u) \),
the proof of Lemma 2 applies, showing that \( t \to x_t \) is right continuous on 0.

**Lemma 14** \( 0 \in \mathcal{V} \), and \((Y., \hat{X}.) \in 0 \) a.s.

**proof:** Let \( 0' \) be the set of \((p,u)\) satisfying all but the last condition in the definition of 0. We see easily that \( 0' \in \mathcal{V}^0 \).

Thus consider
\[
\hat{0} = \{(t,p,u); (p,u) \in 0', \text{ and for every } \varepsilon > 0 \text{ there is an } h > 0 \text{ such that } \sup_r d(p(s)(r), u(t)) < \varepsilon \text{ whenever } p(s) \neq \delta \text{ and } s \in (t, t+h)\}.
\]

Let
\[
\tau_{q,k}(p) = \inf\{s \geq q; p(s) \neq \delta \text{ and } D(p(s)) > \frac{1}{k}\}.
\]
The \( \tau_{q,k} \) are \((P^+_{t+})\) stopping times, and
\[
\{s; p(s) \neq \delta\} = \{\tau_{q,k}(p); q \in \mathbb{Q}, k \geq 1\},
\]
whenever \((p,u) \in 0'\) for some \( u \). Thus
\[
\hat{0} = \bigcup_{q \in \mathbb{Q}} \bigcup_{k=1}^{\infty} \{ (t,p,u); (p,u) \in 0' \text{ and } \tau_{q,k}(p) \text{ exists} \}.
\]
Since \((t,u) \to u(t)\) is measurable from \(B \otimes U\) to \(E^0\), and
\[
(v,x) \to \sup_{r} d(v(r),x)
\]
is measurable, from \(U \otimes E^0\) to \(B\), we see that \(0 \in B \otimes V^0\), and hence that \(0 \in V\).

Since \(\delta(\hat{X}.)) = \zeta\), the first condition of \(0\) follows from (8.7) (take \(Z_s = 1(\zeta,\infty)(s)\)). It follows as in the proof of Lemma 3, that \(S^- (s) < \infty\) for \(s \leq \zeta\), \(s < \infty\).

Let \[
T = \inf \{s; S^- (s) = S^- (s+h) \text{ for some } h > 0\}, \text{ and} \\
R = \inf \{s > T; \forall \delta\}
\]
They are both \((F_s)\) stopping times, \(\hat{X}_T \in I\) a.s. on \(\{T < \zeta\}\), and
\[
1 \geq \mathbb{E}^\mu[\sum_{s \geq T} 1] = \mathbb{E}^\mu[\int_{\delta}^{R} \mathbf{1}(\hat{X}_s, U)ds] \\
n_{\delta} \geq \mathbb{E}^\mu[\int_{\delta}^{R} \mathbf{1}(\hat{X}_s, U)ds]
\]
for every \(\mu\). Thus, by VI(a), \(R = T\) a.s. on \(\{T < \zeta\}\), so that \(T = \zeta\) a.s., and hence \(S^-\) is a.s. strictly increasing on \((0, \zeta) \cap (0, \infty)\).

Assume that
\[
\mathbb{P}^\mu(\zeta = \infty, S^- (s) \neq \infty \text{ as } s \to \infty) > 0,
\]
and let
$$T_k = \inf\{t; \sum_{s \in (0,t]} D(Y_s) \geq k\}.$$ 

Then for some $k$,

$$\hat{p}^\mu(\hat{\xi} = \omega = T_k, \int_0^\infty m(\hat{X}_s)ds < \infty) > 0,$$

and hence

$$k \geq \hat{E}^\mu[\int_0^{T_k} D(u)n(\hat{X}_s,du)ds] \geq \hat{E}^\mu[\int_0^{T_k} (1-m(\hat{X}_s))ds] = \infty,$$

which is impossible.

Finally, we turn to the last condition. We argue as on p.94 of Blumenthal and Getoor [5]. Fix $\epsilon > 0$, let $T_0 = 0$, and define $T_\alpha$ for $\alpha$ a countable ordinal number, by

$$T_{\alpha+1} = \inf\{s > T_\alpha; d(\hat{X}_{T_\alpha}, \hat{X}_s) > \epsilon, \text{ or } Y_s \neq \delta \text{ and }$$

$$\sup_t d(\hat{X}_{T_\alpha}, Y_s(t)) > \epsilon\} \text{, and}$$

$$T_\alpha = \sup_{\beta < \alpha} T_\beta, \text{ if } \alpha \text{ is a limit ordinal.}$$

Then $T_\alpha$ is an $(F_s)$ stopping time for each $\alpha$. Let

$$R = \inf\{s > 0; \int_0^s n(\hat{X}_s,\sup_t d(\hat{X}_0, W_t) > \epsilon)ds \geq 1\} \land 1.$$ 

By II(b), $R > 0$ a.s., and by (8.7),

$$\hat{E}^\mu\left[\sum_{s \in (0,R]} \frac{1}{N(0,R]} \{\sup_{t \geq T} d(\hat{X}_0, W_t) > \epsilon, Y_s \neq \delta\} \right] \leq 1 < \infty,$$
for every \( \mu \). The integrand is thus a.s. finite, and so \( T_1 > 0 \) a.s. by right continuity of \( \hat{X} \). By the strong Markov property of \( (\hat{X}_t) \), we therefore obtain that for each \( \alpha < \beta \),

\[
T_\alpha < T_\beta \text{ a.s. on } \{T_\beta < \infty\}.
\]

But for each \( \mu \),

\[
h(\alpha) = \mathbb{E}^\mu[1 - e^{-T_\alpha}]
\]

is increasing in \( \alpha \), so that as usual, there exists a countable ordinal \( \alpha_0 \) such that

\[
h(\alpha) = h(\alpha_0)
\]

for every \( \alpha \geq \alpha_0 \). This is only possible if

\[
\hat{P}_Y(T_{\alpha_0} < \infty) = 0.
\]

But if \( T_{\alpha_0}(\omega) = \infty \), then there is an \( \alpha \leq \alpha_0 \) such that

\[
t \in [T_\alpha(\omega), T_{\alpha+1}(\omega)) \text{, and by definition of } T_\alpha,
\]

\[
\sup_r d(Y_s(\hat{\omega})(r), \hat{X}_t(\omega)) \leq 2\varepsilon
\]

whenever \( s \in (t, T_{\alpha+1}(\omega)) \). From this, we conclude that

\[
\hat{P}_Y((Y, \hat{X}) \in 0) = 1,
\]

and as \( \mu \) was arbitrary, the lemma is proven. \( \square \)

Now, for each \( \mu \), let \( \mathcal{F}_0^\mu \) be generated by the \( \hat{P}_Y^\mu \)-null sets of \( \mathcal{F}_0^\mu \), and define
and their ilk from the $F^\mu_t$, as in Theorem 2. We define

$$G_t = \cap_\mu G_t^\mu.$$ 

The proof of Lemma 4 applies to the pair of filtrations $(F^\mu_t)$, $(G^\mu_t)$, and since (9.5) follows from (9.9), (9.12) is now immediate. Conditions (9.11) follows immediately from (9.10).

Turning to (9.13), let

$$M' = \{S^+(s); s>0, S^+(s) < \infty\}$$

Since $X_{S^+(s)} \in F$ when $S^+(s) < \infty$, it follows that $M' \subset \hat{M}$. Conversely, to show that $\hat{M} \subset M'$ a.s., it suffices to show that $Y_t(s) \in F$ whenever $t \leq \hat{\zeta}$ and $s < \mathcal{D}(Y_t)$; a.s.. Fix $\mu$, and let

$$\mu'(A) = E^\mu\left[ e^{-S_A(X_s)}ds \right].$$

By IV, V and the indistinguishability of $\hat{\mathcal{D}}$ and $\mathcal{D}_0$ for each $\nu$, we see that there is an $A \in \mathcal{U}$ such that $\{D_0 < D\} \subset A$, and

$$\int_{\mathcal{F}} n(x,A)\mu'(dx) = 0.$$ 

Thus by (8.7), $\hat{P}^\mu(Y_t \in A$ for some $t) = 0$, as required.

$S^{-}(L^-) \in (G_t)$ is a predictable process, hence

$$\{t; S^{-}(L^-) = t\}$$

is $(G_t)$ progressive. This set is identical to $M'$, which in turn
equals \( \hat{M} \) a.s., so that \( \hat{M} \) is \((G_t)\) progressive. Since \( S^+ \) is right continuous and strictly increasing, (9.1) is now immediate.

Let \( \hat{M}_0 \) be the set of points of \( \hat{M} \), isolated on the right. By (8.7), we cannot have \( \hat{P}^\mu(Y_R \neq \delta) > 0 \) for any \((G_t)\) predictable stopping time \( R > 0 \). Let \( (T_n) \) be a sequence of \((G_t)\) stopping times such that \( T_n \uparrow T \) and \( T_n < T \) on \( \{T > 0\} \). Then \( S^+(T_n) \uparrow S^-(T) \), and \( S^+(T_n) < S^-(T) \) \( \hat{P}^\mu \)-a.s. on \( \{T = S^-(L_T) > 0\} \). Thus \( Y_T = \delta \) \( \hat{P}^\mu \)-a.s. on this set, and hence \( \hat{P}^\mu(T \in \hat{M}_0) = 0 \).

Condition (9.2) now follows, from which we conclude (9.13).

Thus, all that remains of part (a) is to show the strong Markov property of each

\[(X_t, G^\mu_t, \hat{P}^\mu, \hat{P}^X),\]

together with (9.14) and (9.15).

The analogue of Lemma 6 is proved as in that result. Thus, for each \( x \in F \), the coordinate process \((W_t, U_t, n(x, \cdot), F_0^x)\) is strong Markov at every \((U_t)\) stopping time \( R \) satisfying

\[n(x, [W_0 \in F, R = 0]) = 0.\]

Because \( F \) is a U-space, the argument of (13.4) of Getoor [23] applies once more, to show that \( \hat{X}_{R^-} \) exists \( \hat{P}^\mu \)-a.s. on \( \{R < \infty\} \) for any \((F_t)\) stopping time \( R \) such that \( Y_R \neq \delta \) \( \hat{P}^\mu \)-a.s. on \( \{R < \infty\} \).

**Lemma 15** Let \( T \) be a \((G_t)\) stopping time such that \( L_T > 0 \) and \( T < S^+(L_T) \) on \( \{T < \infty\} \). Let \( \mu \) be a \( \sigma \)-finite positive measure
on $E$, and let $H_t$ be the $\sigma$-field

$$\mathcal{F}_{t+}^L \otimes \mathcal{U}_t,$$

on $\Omega \times U$. Then there is an (\(H_{t+}\)) stopping time $R$ such that

(9.16) $\mathbb{P}_t^\mu(\omega; n(\hat{X}_{LT^-}(\omega), \{u; R(\omega,u) < \infty\}) = \infty, T < \infty) = 0$

(9.17) $R(\omega, Y_{LT}(\omega)) = (T - S^-)(\omega)$, if $T(\omega) < \infty$

(9.18) $R(\omega, u) = \infty$ for every $u \in U$, if $T(\omega) = \infty$

(9.19) $\mathbb{P}_t^\mu(Y_{LT} \in A, T - S^- \in B, T < \infty \mid \mathcal{F}_{LT^-})(\omega) = n(\hat{X}_{LT^-}(\omega), \{u; u \in A, R(\omega,u) \in B \cap [0,\infty]\})$

$$\frac{n(\hat{X}_{LT^-}(\omega), \{u; R(\omega,u) < \infty\})}{n(\hat{X}_{LT^-}(\omega), \{u; R(\omega,u) < \infty\})}, \mathbb{P}_t^\mu\text{-a.s. on } \{T < \infty\}$$

where $A \in \mathcal{U}$, $B \in \mathcal{B}$, and we make the convention that

$$\frac{0}{0} = 0 = \frac{\infty}{\infty}.$$

proof: The construction of the stopping times $R$ and $R_q^k$, and the proofs of (9.17) and (9.18) are as before, the only formal changes coming from the replacement of $\sigma_a$ by $D$, and the fact that now the $S_q^k$ may take on the value $\infty$. (9.16) will again follow from (9.18), (9.19), and our convention that $\frac{\infty}{\infty} = 0$. At this point in Theorem 2, we used a result from Itô [32]. The corresponding result here is Lemma 11, and from it we obtain as before that
(9.20) \( \hat{P}^\mu(C, Y_{LT} \in A, T - S^-(L_T) \in B, T < \omega) \) 

\[
\frac{n(\hat{X}_{LT}^-(\omega), A \cap \{ R(\omega, \cdot) \in B \cap [0, \omega) \})}{n(\hat{X}_{LT}^-(\omega), \{ R(\omega, \cdot) < \omega \})} \hat{P}^\mu(du)
\]

whenever \( C \in F_{LT-}, A, B \in \mathcal{B}, \) and \( n(x, A) < \infty \) for every \( x \in F. \)

But \( n(x, \{ D > 1/k \}) < \infty \) for every \( x, \) so that the above applies with \( A \) replaced by \( A_k = A \cap \{ D > 1/k \}. \) By our convention, the integrands for the \( A_k \) converge boundedly to the integrand for \( A, \) so that (9.20) and hence (9.19) hold, for every \( A \in \mathcal{U}. \) □

With these results, the analogues of Corollaries 1 and 2 are proven as before (the former of these becoming that \( T = S^+(L_T) \) \( \hat{P}^\mu \)-a.s., whenever \( T \) is a \( (G^\mu_t) \) stopping time such that \( X_T \in F \) on \( \{ T < \infty \} \)).

We now separate out the contribution of \( Y_0 \) to \( (X_t), \) and write \( X. \) as a measurable function

\[ X. = H(Y_0, Y|_{(0, \omega)}, \hat{X}.). \]

as before, and let

\[ P^X(A) = \int 1_A(H(u, Y|_{(0, \omega)}, \hat{X}.(\omega))P^\mu(D(u))(du)P^X_0(du). \]

The strong Markov property of

\( (X_t, G^\mu_t, \hat{P}^\mu, P^X) \)

will now follow as before, using now the strong Markov property of \( (\hat{X}_t, Y_t, F_t, \hat{P}^X). \)
Since
\[ L_{T+T} - L_T = \ell_\mathbb{T}^0(\theta_\mathbb{T}^0(Y), \hat{X}_{T+T}) , \]
this strong Markov property is used again, to argue as in Theorem 2 that (9.14) holds.

Finally, to show (9.15), define \( A(t) \) as in the proof of (9.7) in Theorem 3. Because of (9.13), we have by definition of \( S^+ \), that \( A(t) = t \) for every \( t \), a.s. Thus we can reverse the argument in Theorem 3 (using (9.10)), to obtain (9.3), showing part (a).

To show (b), let \( f \) be \( \alpha \)-excessive for \( (X_t) \). Then \( f \) is also \( \alpha \)-excessive for the \( (P_0^X) \). As in the proof of (9.13) above, we fix \( \mu \), and let
\[
\mu'(A) = \hat{E}^\mu \left[ \int_0^\infty e^{-t} I_{A}(\hat{X}_t) dt \right].
\]
Because (6.5) holds for \( (W_t) \) under the \( (P_0^X) \), and \( f \) is nearly Borel for the \( (Q^X) \), by hypothesis, we conclude from IV and V that there are \( f_1, f_2 \in \mathcal{E}^0 \) such that
\[
f_1 \leq f \leq f_2 , \quad \text{and}
\]
\[
\int_F n(x, \{f_1(W_t) \neq f_2(W_t) \text{ for some } t \}) \mu'(dx)
\]
\[
+ P^\mu_0(f_1(W_t) \neq f_2(W_t) \text{ for some } t) + \hat{E}^\mu_0(D(Y_0))
\]
\[
+ \hat{E}^\mu_0(\hat{X}_t, \{f_1(\hat{X}_t) \neq f_2(\hat{X}_t) \text{ for some } t \}) = 0.
\]
It follows immediately, that

\[ \hat{P}^\mu(f_1(x_t) \neq f_2(x_t) \text{ for some } t) = 0, \]

showing that (6.5) holds.

The same will hold for the canonical process \( (W_t) \) on \( \mathcal{U} \), under the \( P^X \), so that as in the discussion preceding Lemma 13, applied to

\[ M = \{ t; W_t \in \mathcal{F} \}, \]

there exists a continuous additive functional of \( (W_t) \) with 1-potential

\[ \psi(x) = E^x[e^{-D_0}]. \]

By the perfection arguments in Meyer [40], we may choose a perfect version, adapted to \( (\mathcal{U}_{t+}) \). Call it \( l_t(u) \). We need to show that

\[ l_t(x.) = L_t \text{ for every } t, \text{ a.s.} \]

Let \( \mathcal{U}^\mu \) be the \( P^\mu \) completion of \( \mathcal{U} \), and adjoin the \( P^\mu \)-null sets of \( \mathcal{U}^\mu \) to \( \mathcal{U}_t \), to obtain \( \mathcal{U}_t^\mu \). Let \( M_0 \) be the set of points of \( M \) that are isolated on the right. Then as in Maisonneuve [38], p.409, the process \( (l_t) \) is the \( (\mathcal{U}_t^\mu) \)-dual predictable projection of

\[ t \rightarrow \int_0^t l_F(W_s)ds + \sum_{s \in M_0 \cap (0,t]} (1-e^{-D_0 s}) \]

under the measure \( P^\mu \).
Now let

\[ Z_s = Z\mathbb{1}_{(t_0, t_1]}(s), \quad Z \in F^\mu_{t_0} \]

Then

\[
\begin{align*}
\hat{E}^\mu &\left[ \int_0^\infty Z_s 1_F(X_s) ds + \sum_{s \in M_0} Z_s (1 - e^{-D(Y_L s)}) \right] \\
&= \hat{E}^\mu[Z \hat{X}_{t_0}^t - t_0 \int_0^{1-t_0} 1_F(W_s) ds + \sum_{s \in M_0 \cap (0, t_1-t_0]} (1 - e^{-D(Y_L s)})] \\
&= \hat{E}^\mu[Z \hat{X}_{t_0}^t - t_0 \int_0^{1-t_0} d1_s] \\
&= \hat{E}^\mu[Z \mathbb{d}(1_s(X_s))].
\end{align*}
\]

Thus, \((1_t(X_s))\) is the \((F^\mu_t)\)-dual predictable projection of

\[
t \mapsto \int_0^t 1_F(X_s) ds + \sum_{s \in M_0 \cap (0, t]} (1 - e^{-D(Y_L s)}).
\]

But a direct computation shows that \((L_t)\) has the same property. Thus by uniqueness of dual predictable projections,

\[ L_t = 1_t(X_s) \text{ for every } t, \text{ a.s.,} \]

as required. \(\square\)

For completeness, we include the following immediate corollary to Proposition 11.

Corollary 6 Assume all the conditions of Theorem 4(a), except those involving \((Y_t)\). Assume that \(E\) is a \(U\)-space, and that \((Q^X)\)
is regulated. Then in order that we may enlarge \( \Omega, F \) and the
\( F_t \), and find a process \(( Y_t )\) such that all of the conditions
of Theorem 4 are met, it is necessary and sufficient that (8.9)
hold.

**proof**: (8.6) holds by III, and as remarked in the discussion prior
to Proposition 11, the condition on the existence of kernels
\( n(x,y; du) \) is satisfied when \( E \) is a U-space. Thus Proposition
11 applies, giving \(( Y_t )\). Note that (9.9) implies condition (8.8),
with
\[
G(x,u) = \begin{cases} 
    u(D(u)), & D(u) < \infty \\
    \Delta, & \text{otherwise}.
\end{cases}
\]

Note that if we start with \(( X_t )\) and \( M \) as in Theorem 3,
and form \(( \hat{X}_t ), n, \) and \(( Y_t )\) from them, then Theorem 4 will apply,
and will reconstruct \(( X_t )\). Similarly, a new process \(( \hat{X}_t )\) may
be constructed, using Corollary 6 and then Theorem 4.

As remarked in section 8, the results of Gzyl [28] show that \(( \hat{X}_t )\)
is a right process; and so strongly regulated. Applying
Corollary 4, we see that \(( X'_t )\) has the same law as \(( X_t )\).

It follows from a remark made following the proof of Lemma 2,
that in the situation of Theorem 2, the constructed process \(( X_t )\)
will be a.s. càdlàg, provided that a.s., \( Y_t(\cdot) \) has left limits whenever \( Y_t \neq \delta \). The following example shows that in the present case, \( (X_t) \) need not be càdlàg, even when \( (\hat{X}_t) \) and the \( Y_t \) are:

**Example 5:** Let \( E \) be the unit square, with the boundary partially removed;

\[
E = ([0,1] \times [0,1]) \backslash \{(1) \times (0,1] \}.
\]

Let \( F = [0,1] \times \{0\} \), and let \( r: [0,1) \to (0,\infty) \) be continuous. Let \( P_0(x,y) \) correspond to uniform downward motion on \( \{x\} \times [0,1] \) at rate \( r(x) \), started at \((x,y)\) and stopped upon contact with \( F \). Let \( (\hat{X}_t) \) be uniform motion to the right on \( F \), stopped at the point \((1,0)\). It is a right process, so that \( (Q^X) \) is regulated. Set

\[
n(x,du) = \frac{1}{2(1-\exp(-1/r(x)))} P_0(x,1)(du).
\]

We see that \( m(x) = 1/2 \) for \( x \in [0,1) \), so that VI(a) is vacuous. Thus, conditions I-VI are met, so that we can construct \( (Y_t) \) by Corollary 6, and apply Theorem 4 to obtain a process \( (X_t) \). Let

\[
T = \inf\{t; X_t = (1,0)\}.
\]

We have that \( T < \infty \) a.s., and \( X_t = (1,0) \) for every \( t \geq T \). There are only finitely many excursions away from \( F \) in any time interval \([0,T-h]\), \( h > 0 \), but by letting \( r(x) \to \infty \) sufficiently fast, as \( x \to 1 \), we can ensure that there are infinitely many excursions in \([0,T]\), a.s., so that there is no left limit at \( T \).
We can impose conditions to guarantee càdlàg paths, as in the next result;

**Proposition 12** (a) Suppose that in addition to the hypotheses of Theorem 3, we assume that \((X_t)\) has left limits on \((0, \zeta)\), a.s.

Then

\[
\int_0^h \mathbb{1}(X_s, \{(\sup_{t} d(\hat{X}_s, W_t) > \varepsilon)\}) ds < \infty \text{ for every } \varepsilon > 0, \; h < \hat{\zeta}, \text{ a.s.}.
\]

(b) Suppose that the hypotheses of Theorem 4(a) are verified, and that (9.21) holds. Suppose also that: \((\hat{X}_t)\) has left limits on \((0, \hat{\zeta})\) a.s.; \((\hat{X}_t)\) has a left limit at \(\hat{\zeta}\) a.s. on \(\{\zeta < \infty, Y_{\zeta} \neq \delta\}\); and \(Y_t\) has left limits on \((0, D(Y_t)) \cap (0, \delta(Y_t))\) whenever \(Y_t \neq \delta\), a.s.. Then we conclude that \((X_t)\) has left limits on \((0, \xi)\), a.s..

**proof:** Let

\[
T = T_1 = \inf\{t > 0; Y_t \neq \delta \text{ and } \sup_s d(\hat{X}_{t-}, Y_t(s)) > \varepsilon\}.
\]

Since \(\{(t, \omega); Y_t(\omega) \neq \delta\}\) is indistinguishable from a countable union of graphs of \((F_t)\) stopping times, as before, \(T\) must also be an \((F_t)\) stopping time. By the right continuity of \((X_t)\), \(T > 0\) a.s. Let

\[
T_{k+1} = T_k + T \circ \theta_{T_k}.
\]

Since \((X_t)\) is a.s. càdlàg, it has no oscillatory discontinuities a.s., showing that \(\lim T_k \geq \hat{\zeta}\) a.s.. But
\[ k \geq E^H \left( \int_{s \in (0,T_k]} 1 \{ (x,u) ; \sup_t d(x,u(t)) > \varepsilon \} \{ \hat{X}_s = \hat{Y}_s \} \right) \]

\[ = E^H \left( \int_0^{T_k} n(\hat{X}_s, \{ \sup_t d(\hat{X}_s, W_t) > \varepsilon \}) ds \right), \]

by Lemma 9, showing (9.21).

(b) We set

\[ T_k = \inf \{ t ; \int_0^t n(\hat{X}_s, \{ \sup_t d(\hat{X}_s, W_t) > \varepsilon \}) ds \geq k \} . \]

By Lemma 9,

\[ \sup_t d(\hat{X}_s, Y_s(t)) > \varepsilon \quad \text{and} \quad Y_s \not\in \delta \]

for only finitely many \( s \in (0,T_k] \), and by hypothesis \( \lim T_k \geq \zeta \).

As in Lemma 2, this shows that \( X_{t^-} \) exists at all times \( t = S^{-}(s) \)
(and equals \( \hat{X}_{s^-} \), \( s < \zeta \)), and the other conditions produce left
limits at all other relevant times \( t \).

Another difference between the general situation and that of
Theorem 2 lies in that in the latter we can have 'instantaneous'
behaviour only when the measure \( n \) is infinite. The following is
an example in which the conditions of Theorem 4 are satisfied,
and \( n(x,u) < \infty \) for each \( x \), but yet \( m \equiv 0 \).

Example 6: Let \( E = [0,1] \times [0,1] \) and \( F = [0,1] \times \{0\} \). Let
\( \hat{p}_0(x,y) \) corresponds to uniform downward motion on \( \{x\} \times [0,1] \), at
rate 1, started at \( (x,y) \), and stopped upon reaching \( F \). Let
(\dot{X}_t) be uniform motion to the right on F, at rate 1, absorbed at the point (1,0). Let \( r : [0,1] \rightarrow (0,\infty) \) be Borel measurable, and set

\[ n(x,du) = \frac{1}{1 - \exp(-r(x))} p_0^{(x,r(x))}(du). \]

Then \( m = 0 \). We may choose \( r \) such that

\[ \int_a^b \frac{dx}{r(x)} = \infty \]

for every \( a, b \in [0,1) \), \( a < b \). In this case, VI (a) is satisfied, so that we may apply Corollary 6 and then Theorem 4 to obtain the desired process \((X_t)\).

10. Applications

(A). Skew Brownian Motion Let \( n \) be the characteristic measure of the PPP of excursions of Brownian motion on \( \mathbb{R} \) from 0, and let \( P^b_0 \) correspond to Brownian motion started at \( b \), and absorbed at 0. For \( \alpha \in [0,1] \), let

\[ n^+ = 2n|\{W_t \geq 0 \text{ for every } t\} \]

\[ n^- = 2n|\{W_t \leq 0 \text{ for every } t\} \]

\[ n_\alpha = \alpha n^+ + (1 - \alpha)n^- . \]

Then for \( \Lambda \in \mathcal{U}_t \), \( A \in \mathcal{U} \) we have that

\[ n_\alpha(\Lambda, \sigma_0 > t, \theta^{-1}_t A) \]
\[ = 2\alpha n(A, \sigma_0 > t, \theta_t^{-1}A, W_t > 0) + 2(1-\alpha)n(A, \sigma_0 > t, \theta_t^{-1}A, W_t < 0) \]
\[ = 2\alpha \int_{\Lambda \cap \{\sigma_0 > t, W_t > 0\}} W_t^t(A) dn + 2(1-\alpha) \int_{\Lambda \cap \{\sigma_0 > t, W_t < 0\}} W_t^t(A) dn \]
\[ = \int_{\Lambda \cap \{\sigma_0 > t\}} W_t^t(A) dn, \]

so that \( n_\alpha \) satisfies (iv) for the \( P^b_0 \). Thus Theorem 2 produces a corresponding strong Markov process, which is easily seen to be the "skew Brownian motion" examined in Walsh [50]. This process was introduced by Ito and McKean [77]. They gave a construction in terms of excursions, and in this context, Theorem 2 becomes a proof that the process they construct is strong Markov.

We will now discuss several particular techniques ((B), (C), and (D) below) for producing objects \( n, (X_t) \), and \( (Y_t) \), from which Theorem 4 will produce a strong Markov process. They will in general be techniques that transform one set of such objects into new ones. Each technique will apply to the following situation:

(a) Start with a right process \( (X_t) \) and a set \( M \) as in Theorem 3.
(b) Obtain a process on the boundary, \( (\hat{X}_t) \), and a kernel \( n \) as in that result.
(c) Apply the technique under consideration to produce a new process \( (\hat{X}_t') \) and a new kernel \( n' \).
(d) The conditions of Corollary 6 and Theorem 4 will now be met, so that \( (Y_t') \) and hence the strong Markov process \( (X_t') \), may be
constructed.

We will however show that the techniques work under more general conditions on \((X_t^X)\) and \(n\), than that they arise as in (b). This is not mere frivolous generality, as it will tell us that the techniques may be iterated, provided this is done in the order in which they are presented. That is, if we start with \((X_t^X)\) and \(n\) as in (b), we may perform (B) to produce \((X_t'^X)\) and \(n'\). These will meet the conditions of (C), so that new \((X_t''^X)\) and \(n''\) may be found, which in turn meet the conditions of (D).

We can introduce yet another variation, in that in certain circumstances, there is a corresponding transformation that produces a new excursion process \((Y_t^Y)\) directly from the old one, \((Y_t^X)\) (without using Corollary 6). Using these direct constructions of the excursion processes, the transformations given in (B), (C), and (D) may still be iterated (however, if Corollary 6 is used at any stage of the iteration, the end result will be as if it had been used at each stage). A particular example of this is the procedure of F. Knight and O. Pittenger, for excision of excursions, which we will obtain in (E), below.

(B) \(h\)-transformation: We will change \((F_0^X)\) and \(n\) via \(h\)-transforms, while leaving \((X_t^X)\) unchanged. I owe this to a
suggestion of Mike Cranston.

Let \((P^X_0, \hat{X}_t)\), \(\pi\) and \(F\) be as in Theorem 4. Assume in addition that (8.9) holds, \((Q^X)\) is regulated, \(E\) is a \(U\)-space, and that upon augmentation of \((U, U_t)\) with respect to the \(P^X_0\), the coordinate process \((W_t)\) becomes a right process. (this latter condition will hold in the situation of Theorem 3; see Sharpe [46], p. ).

Let \(h : E \to [0,1]\) be 0-excessive for the \((P^X_0)\) (with \(h(\Delta) = 0\)), and let \(C\) be the set of dyadic rational numbers. Set

\[
E_h = \{x \in E; h(x) \neq 0\},
\]
\[
U_h = \{u \in U; \liminf_{q \to 0, q \in C} h(u(q)) > 0\}.
\]

We let \(P^X_0(du)\) be the transition laws for the h-transformed semigroup, as in Meyer [41]. That is, the \(P^X_0\) are probability measures on \((U, U)\) satisfying \(P^X_0(W_t \neq \Delta\) for some \(t \geq 3) = 0\), and such that for every \((U_{t+})\) stopping time \(\tau\), we have that

\[
P^X_0(\tau > \tau) = \begin{cases} \frac{1}{h(x)} \int E_0(\Delta \geq \tau, h(X_\tau)), x \in E_h \setminus \{0\} \\ 0 \end{cases}, \text{otherwise}
\]

whenever \(A \in U_{t+}\). By I.24 of Meyer [41], the \(P^X_0\) are the transition laws of a right process with values in \(E_h\).

Suppose now that \(n_0(du)\) is any positive measure on \((U, U)\) satisfying IV, and

\[
n_0(D > t) < \infty \quad \text{for every} \quad t > 0.
\]
We will define a measure \( n_1 = h_n_0 \) such that (10.1) holds, 
\( n_1(\emptyset = 0) = 0 \), and

\[
(10.2) \quad n_1(A \cap \{\theta > t\} \cap \theta^{-1}(A')) = \int_{An(\{\theta > t\}nU_h)} h(W_t) P_0 A' d\theta \\
A \in \sigma(U, E),
\]

for every \( t > 0 \), \( A' \in U \) and \( A \in U_1 \). To do so, we use this formula to consistently define \( n_1^{(k)}(B) \) (the putative).

\[
n_1(B \cap \{D < [\frac{1}{k}, \frac{1}{k-1}]\}) ,
\]

for \( k \geq 1 \) and \( B \) a cylinder set in the space of all functions \([0, \infty) \rightarrow E\). By (14.6) of Getoor [23], and III.52 of Dellacherie et Meyer [13], \( n_1^{(k)} \) extends to a finite measure on the cylinder \( \sigma \)-field. The coordinate process is immediately seen to have a right continuous modification, so that \( n_1^{(k)} \) pulls back to a finite measure on \((U, E)\). We let

\[
n_1 = \sum_{k=1}^{\infty} n_1^{(k)} .
\]

(The reason for insisting that \( n_1 \) be concentrated on \( U_h \) will emerge later, when we consider property VI(b).)

Since \( h \circ W \) is \( P_0^{(\mu)} - \text{a.s.} \) right continuous on \([0, \infty)\) for each \( \mu \), it follows that \( h \circ W \) is \( n_0 - \text{a.s.} \) right continuous on \((0, \infty)\). Thus we may use approximation by stopping times taking values in \( Q \), to show as usual, that

\[
n_1(A \cap \{\tau < \emptyset\}) = \int_{An(\{\tau < \emptyset\}nU_h)} h(W_t) d\theta \\
A \in \sigma(U, E).
\]
for every strictly positive \((U_{\tau^+})\) stopping time \(\tau\), and \(A \in U_{\tau^+}\). Since \(h \leq 1\) we have that \(n_1 \leq n_0\). Thus, if \(n_0(D=0) = 0\), we have in particular that for every positive \(f \in U\),

\[
(10.3) \quad \int_{\{D < \partial\}} f(W.)dn_1 = \int_{\{D < \partial\}} f(W.)h(W_0)dn_0.
\]

We wish to apply Corollary 6 and Theorem 4 to \((\hat{X}_t)\), the \(h^P_0\), and the \(h^n(x,\cdot)\). Condition IV holds by definition, and conditions I, II, III, and (8.9) follow from the corresponding properties of \(n\), since \(h \leq 1\) implies that \(h^n(x,\cdot) \leq n(x,\cdot)\) for each \(x \in F\). We must thus only show conditions V and VI.

Observe first that \(h \circ W\) is \(P_0^x\)-a.s. right continuous at 0, so that

\[
P_0^x(U \setminus U_h) = 0 \text{ for } x \in E_h, \text{ and } B \in E.
\]

\[
P_0^x(U_h) = 0 \text{ for } x \in E \setminus E_h.
\]

Thus

\[
n(x, \{W_0 \in E_h \setminus F\} \setminus U_h) = \int_{\{W_0 \in E_h \setminus F\}} P_0^{(0)}(U \setminus U_h)n(x, du) = 0, \text{ and }
\]

\[
n(x, U_h \cap \{W_0 \in E \setminus (E_h \cup F)\}) = \int_{\{W_0 \in E \setminus (E_h \cup F)\}} P_0^{(0)}(U_h)n(x, du) = 0.
\]

Now let \(B \cap F = \emptyset\), \(B \in E\). Then

\[
h^n(x, \{W_0 \in B\}) = \lim_{t \to 0} h^n(x, \{W_0 \in B, D > t\})
\]
\[= \lim_{t \rightarrow 0^+} \int_{\{W_0 \in B, D > t\} \cap U_h} h(u(t)) n(x, du)\]

\[= \lim_{t \rightarrow 0^+} \int_{\{W_0 \in B \cap E_h\}} E_0^{u(0)}[h(W_t), D > t] n(x, du)\]

\[= \int_{\{W_0 \in B \cap E_h\}} h(u(0)) n(x, du),\]

by monotone convergence. Further, if \( A \in U_t, \ t > 0 \) and \( B \cap F = \emptyset \), then we can use this to obtain that

\[h_n(x, A \cap \{W_0 \in B, D > t\}) = \int_{\{W_0 \in B, D > t\} \cap A \cap U_h} h(u(t)) n(x, du)\]

\[= \int_{\{W_0 \in B \cap E_h\}} \int_{A \cap \{D > t\}} h(v(t)) P_0^{u(0)}(dv) n(x, du)\]

\[= \int_{\{W_0 \in B \cap E_h\}} h(u(0)) P_0^{u(0)}(A \cap \{D > t\}) n(x, du)\]

\[= \int_{\{W_0 \in B \cap E_h\}} P_0^{u(0)}(A \cap \{D > t\}) h n(x, du),\]

showing \( V \).

Since \( h_n \leq n \),
\[ 1 - \frac{\mathbf{m}(x)}{\mathbf{n}(x,D > t)} e^{-t} \mathbf{d}t, \text{ and} \]
\[ 1 - \mathbf{m}(x) = \int_0^\infty n(x,D > t)e^{-t} dt, \]

it follows that if \( \mathbf{m}(x) = 0 \), then \( \mathbf{m}(x) = 0 \) and \( \frac{\mathbf{n}(x,D > t)}{\mathbf{n}(x,D > t)} = n(x, D > t) \) for every \( t \). Thus also \( \frac{\mathbf{n}(x,\cdot)}{\mathbf{n}(x,\cdot)} = n(x,\cdot) \). If \( x \in \mathbb{I} \), then there is \( \mathbb{P}^x \text{-a.s.} \) a \( t > 0 \) with

\[ \int_0^t \{ h_{m=0} \} (\hat{X}_s) ds = 0. \]

Thus also \( x \in \mathbb{I} \), and for every \( r > 0 \),

\[ \int_0^{r \wedge t} \mathbf{n}(\hat{X}_s,U) ds = \int_0^{r \wedge t} n(\hat{X}_s,U) ds = \infty, \]

showing VI(a).

Fix \( x \), and let \( n_0 \) be a positive measure on \( (U, U) \) satisfying IV for the \( \mathbb{P}_0^Y \), and such that

\[ n_0(du) \leq h_n(x,du). \]

Assume that \( n_0(W_0 \in F) > 0 \). We must show that \( n_0(W_0 \in F) = \infty \).

Since \( n_0(U \setminus U_0) = 0 \), and

\[ \{ \lim \inf_{q \to 0, q \in C} h(W_q) > \varepsilon \} \in U_{0+} \]

for each \( \varepsilon > 0 \), we may fix \( \varepsilon > 0 \), and assume that \( n_0 \) is concentrated on this set. Let
Since \( h \) is 0-excessive for the \( p_0^y \), so is \( g \). By I.19 of Meyer [41], \( g/h \) is 0-excessive for the \( h_0^y \), so that we may form

\[
n_1 = \frac{g}{h} n_0.
\]

But

\[
\frac{g}{h}(p_0^y) = g_0^y,
\]

so that as before, \( n_1 \) satisfies IV for the \( g_0^y \). Also,

\[
n_1(du) \leq \frac{g}{h}n(x,du) = g_n(x,du).
\]

\( E_h \) is nearly optional, so that as in Theorem 3, there is an exact perfect terminal \( (U_{t+}) \) stopping time \( D_h \) which is \( p_0^l \) indistinguishable from

\[
inf\{t > 0; W_t \notin E_h\},
\]

for each \( \mu \). We have that for \( A \in U_t \),

\[
g_0^y(A \cap \{\exists t\}) = p_0^y(A \cap \{\exists A \wedge D_h > t\}).
\]

Using this, the formulae \( n_1^+(D_h = 0) = 0 \), and

\[
n_1^+(A \cap B \cap \{D_h > t\}) = \int \left. p_0^u(t)(B)n_1(du) \right|_{A \cap \{D_h > t\}}
\]

\((A \in U_t, B \in \mathcal{U})\) can be used to define a function \( n_1^+ \) on the field
of elements of $U$, consisting of finite unions of sets of the form

$$\{W_{t_1} \in C_i, i=1..k; D_h \in (t_j, t_{j+1})\}, \{W_{t_1} \in C_i, i=1..k, D_h=0\}$$

where $t_1 < \ldots < t_{k+1} \leq \infty$, $j \leq k$. On this field, we have that $n_1 \leq n_1^+ \leq n(x,\cdot)$, so that because $n(x,\cdot)$ is countably additive, and there exist $U_1$ in the field with $n(x,U_1) < \infty$, $n(x,U \cup U_1) = 0$, we see that $n_1^+$ extends to a measure on $(U, U)$ such that $n_1 \leq n_1^+ \leq n(x,\cdot)$. This $n_1^+$ satisfies IV for $(P_0^X)$ since $n_1$ did for $(\gamma P_0^X)$, so that by VI(b),

$$n_1^+(W_0 \in F) = 0 \text{ or } \infty.$$  

But $n_1^+ \geq n_1 \geq h_1 = n_0$, so that $n_1^+(W_0 \in F) \geq n_0(W_0 \in F) > 0$, and hence

$$\infty = n_1^+(W_0 \in F) = \lim_{q \to 0, q \in C} n_1(W_0 \in F, D_h \wedge D > q)$$

$$= \lim_{q \to 0, q \in C} \int_{\{W_0 \in F, D \geq t\}} \frac{g(W_t)}{h(W_t)} dn_0$$

$$\leq \frac{1}{c} n_0(W_0 \in F),$$

Thus $n_0(W_0 \in F) = \infty$, showing VI(b). Corollary 6 and Theorem 4 now apply, to produce a corresponding strong Markov process.

**Note** In one particular case, we can construct an excursion process $(hY_t)$ directly, without appealing to Corollary 6.
That is, suppose in addition to the hypotheses of (B), that
\( Y_t \) is given as in Theorem 4 (we could also drop the assumptions that (8.9) hold and that \( (Q^X) \) be regulated). Let \( F_1 \subset F \), \( F_1 \in \mathcal{E}^F \), and let \( h \) be the 0-excessive function

\[
    h(x) = P_0^X(D<\emptyset, W_D \in F_1).
\]

Then

\[
    h(x) = \mathbf{1}_{F_1 \setminus \{\emptyset\}}(x) \quad \text{for} \quad x \in F.
\]

It follows by definition of \( P^X_0 \), that

\[
    P^X_0(D<\emptyset) = \lim_{h \to 0} \left( E^X_0(D<\emptyset, h(W_D)) \right) = \lim_{h \to 0} \left( E^X_0(D<\emptyset, W_D \in F_1) \right) = \lim_{h \to 0} \left( E^X_0(D<\emptyset) \right) = 1_{E_X}(x).
\]

Thus for each \( x \in F \),

\[
    h_n(x, D=\emptyset) = \lim_{t \to 0} h_n(x, \{ D=\emptyset > t \}) = \lim_{t \to 0} \int_{\{D=\emptyset > t \} \cap U_h} h(u(t)) P^{u(t)}_0(D=\emptyset) n(x, du) = 0.
\]

Let

\[
    h_{Y_t} = \begin{cases} Y_t, & \text{if } Y_t \in U_h \cap \{ D<\emptyset, W_D \in F_1 \} \\ \delta, & \text{otherwise}. \end{cases}
\]
Then \((h_{Y_t})\) inherits the strong Markov property from \((Y_t)\), and for each positive \(f \in U\), and positive \((F_t^h)\) predictable process \((Z_t)\), we have that

\[
\hat{E}^h \left[ \sum_{s > 0} Z_s f(h_{Y_s}) \right] = \hat{E}^h \left[ \sum_{s > 0} Z_s f(Y_s) 1_{U \cap \{D < \delta, W \in F_1\}}(Y_s) \right] \\
= \hat{E}^h \left[ \int_0^\infty Z_s \int_{\{D < \delta, W \in F_1\} \cap U} f(u) n(\hat{X}_s, du) ds \right] \\
= \hat{E}^h \left[ \int_0^\infty Z_s \int_{\{D < \delta\} \cap U} f(u) h_n(u(D(u))) n(\hat{X}_s, du) ds \right] \\
= \hat{E}^h \left[ \int_0^\infty Z_s \int_{\{D < \delta\} \cap U} f(u) h_n(\hat{X}_s, du) ds \right]
\]

by (10.3) and the above. Therefore, \((\hat{X}_t)\), \(h_n\), \((h_{Y_t})\), \((h_{P_t^X})\), and \(F\) satisfy the conditions of Theorem 4, so that a new strong Markov process may be constructed.

(\(\mathcal{C}\)) Excision: We will excise pieces of the path of \((\hat{X}_t)\), while leaving \(n\) unchanged.

Suppose that \((\hat{X}_t)\) and \(M\) are as in Theorem 3, and obtain \((\hat{X}_t)\) and \(F\) from them. Suppose that \((P_0^X)\), \(n\), and \((F_t)\), together with \((\hat{X}_t)\) and \(F\), satisfy the conditions of Corollary 6 and Theorem 4 [note: We do not assume that \((P_0^X)\) and \(n\) are also constructed from \((X_t)\) and \(M\) as in Theorem 3. Since the
procedure of (B) left \((\hat{X}_t)\) unchanged, these hypotheses will thus be satisfied by \((p_0^x, h^m_0)\).

Assume now that we can write \(F = F_0 \cup F_1\), where \(F_0\) and \(F_1\) have disjoint closures in the \((X_t)\)-Ray Knight compactification \(\tilde{E}\) of \(E\), and \(\Delta \in F_1\). Suppose further, that

\[
(10.4) \quad P_0^x(D < \infty, \text{ and } W_0^F \in F_0) = 0 \quad \text{for } x \in E \setminus F_0.
\]

Let

\[
A(t) = \int_0^t 1_{F_1}(\hat{X}_s) \, ds, \quad A'(s) = \inf\{t; A(t) > s\}
\]

\[
\hat{X}'_t = \hat{X}_{A'(t)}, \quad F'_t = F_{A'(t)}.
\]

Since \((\hat{X}_t)\) is \(\tilde{E}\)-right continuous with values in \(F\), it follows that; for every \(t\) there is an \(\epsilon > 0\) such that

\[
(10.5) \quad \hat{X}'_{t+s} = \hat{X}_{A'(t)+s} \quad \text{for } s \in [0, \epsilon); \ a.s.
\]

Properties II(b) and VI(a) thus follow by the strong Markov property of \((\hat{X}_t)\) at \(A'(0)\), and (8.9) follows from (10.4). The remainder of properties I-VI are of course still satisfied. Let \(Q^x\) be the image law of \(\hat{P}^x\) on \((U^F, U^F)\) under \(\hat{X}'\), so that as before, each

\[
(\hat{X}'_t, F'_t, \hat{P}^x, Q^x)
\]

is strong Markov. Augment \((U^F, U'_t)\) with respect to the \(Q'^u\) to obtain \((\bar{U}'_t, \bar{U}'_t)\). As in Theorem 3,
\{(W^F_t) \text{ is not càdlàg in } \tilde{E}\} \in \mathcal{U}'

and has \(Q^\mu\) measure zero for each \(\mu\), so that the regulator \(K^0_t\) for \((Q^X)\), defined in Theorem 3, will be adapted to \((\mathcal{U}'_t)\). For every \(t\),

\[K^0_t(\hat{X}_t) \text{ and } K^0_{A(t)}(\hat{X}_t)\]

differ by only a finite random variable, since \((\hat{X}_t)\) has no oscillatory discontinuities in \(\tilde{E}\), a.s., so that \((\hat{X}_t)\) makes only finitely many jumps from \(F^1\) to \(F^0\) in any compact time interval. Thus (8.3) holds. Write

\[\tilde{X}^F_{t-}, X^F_{t-}, W^F_{t-}\]

for the left limits at \(t\) in \(\tilde{E}\) of \(\hat{X}^\prime\), \(\hat{X}_t\), and \(W^F\). Then as before, if \(\tau\) is a strictly positive \((\mathcal{U}'_t)\) predictable stopping time, we write \(T = \tau(\hat{X}^\prime)\), and have that \(A'(T-)\) is an \((F^\mu_t)\) predictable stopping time. Thus either

\[\tilde{X}^F_{T-} = X^F_{A'(T-)} \notin F; \text{ or}\]

\[\tilde{X}^F_{A'(T-)} = X^F_{A'(T-)} \text{; } \hat{F}^\mu\text{-a.s. on } \{T < \infty\}.\]

But

\[\tilde{X}^F_{A'(T-)}\]

lies in the \(\tilde{E}\)-closure of \(F^1\), so that in the latter case, by (10.5), we have that \(A'(T-) = A'(T)\), and hence
Thus

$$Q'_{\tau}^u(K^0_{\tau-} \neq K^0_{\tau}) = 0,$$

showing (8.4). Conversely, if $\tau$ is $(\bar{U}'_{\tau})$ totally inaccessible, we have as before, that

$$\hat{W}'_{\tau}^F = W_{\tau}^F - Q'_{\tau}^u - a.s. \text{ on } \{\tau < \infty\}.$$ 

Thus if

$$W_{\tau}^F \neq W_{\tau}^F - Q'_{\tau}^u a.s. \text{ on } \{\tau < \infty\},$$

then also $K^0_{\tau-} \neq K^0_{\tau}$ by definition of $(K^0_{\tau})$. $(K^0_{\tau})$ is therefore a regulator for $(Q'_{X})$, so that Corollary 6 and Theorem 4 apply, to produce a strong Markov process $(X'_{\tau})$.

Note: 1. There is a direct construction of the excursion process available. That is, suppose that in addition, we are given $(Y_{\tau})$ as in Theorem 4 (condition (8.9) need no longer be assumed), and put

$$Y'_{\tau} = Y_{A'(\tau-)}.$$ 

As in Lemma 13(c), $(Z_{A(t)})$ is $(F'_\tau)$ predictable whenever $(Z_t)$ is $(F'_\tau)$ predictable, so that in this case,

$$\hat{E}'^{H}[\sum_{s>0} Z_s f(Y'_s)] = \hat{E}'^{H}[\sum_{s>0} Z_{A(s)} f(Y'_s)1_F(\hat{X}'_{s-})].$$ 

$$Y'_s \neq \delta \quad Y_s \neq \delta$$
\[
\begin{align*}
&= E^\mu \left[ \int_0^\infty Z_{A(s)} \int_{F_1} f(u)n(\hat{X}_s, du) ds \right] \\
&= E^\mu \left[ \int_0^\infty f(u)n(\hat{X}_s', du) ds \right],
\end{align*}
\]
showing (8.7). The strong Markov property of
\[(\hat{X}_t', Y_t', F_t', \hat{P}^X)\]
is immediate from that of
\[(\hat{X}_t, Y_t, F_t, \hat{P}^X),\]
so that Theorem 4 applies.

2. Condition (10.4) is only included for simplicity. We can remove it by taking \(A \in F_1\), and changing \((P^X_0)\) to \((P'^X_0)\), where the \(P'^X_0\) are obtained from the \(P^X_0\) by making points \(x \in F_0\) into branch points, with
\[
P'^X_0(X_0 \in B) = Q^X(\tau = \inf\{t \geq 0; W^F_t \in F_1\} < \infty, W^F_\tau \in B).
\]
The \(n(x, \cdot)\) are altered similarly, and we once again obtain (8.9).

3. If, instead of assuming that \(F_0\) and \(F_1\) have disjoint closures in \(\tilde{E}\), we assume that this holds with the regular topology of \(E\), and that \((\hat{X}_t)\) is càdlàg in this topology, then we would still get (10.5). If in addition we had assumed that \((X_t)\) was a Hunt process, then by (13.8) of Getoor [23], \((X_{t-})\)
and \((\tilde{X}_{t^-})\) are indistinguishable. In this case, the proof of regulation still works, so that we could apply the arguments of (C) to construct a strong Markov process.

4. Similarly, if we wish to use the direct construction of note 1, it is enough to assume that we are given \((\tilde{X}_t)\), \((n, (Y_t))\), \((P_0^X)\), \(F\) and \((F_t)\) as in Theorem 4, together with \(F_0\) and \(F_1\) which have disjoint closures in the regular topology on \(E\), such that \(F = F_0 \cup F_1\), \((\tilde{X}_t)\) is càdlàg in this topology, and (10.4) holds. The latter condition could be removed as in note 2. In this case, we need no longer show (8.9); the particular form of the branching behaviour on \(F_0\) is now needed to guarantee (9.9).

(D) Scaling and Time Change: We scale \(n\) and time change \((\tilde{X}_t)\).

Let \((\tilde{X}_t), (F_t), (P_0^X)\) and \(n\) satisfy the conditions of Corollary 6 and Theorem 4. Let \(f: F \rightarrow (0, \infty)\) and \(g: F \rightarrow [0, \infty)\) be \(E^F\)-measurable, and assume that \(f\) is bounded, and bounded away from 0. Assume also that for each \(x\), \(f(x)g(x) \leq 1\), and that

\[
g(x) \int (1 - e^{-D})dn(x, \cdot) \leq 1,
\]

with equality only if \(m(x) = 0\). Put

\[
A(t) = \int_0^t f(\tilde{X}_s)ds, \quad A'(s) = \text{inf}\{t; A(t) > s\},
\]

\[
\hat{X}_t = \tilde{X}_{A'(t)}, \quad F'_t = F_{A'(t)}, \quad \text{and}
\]

\[
n'(x, du) = g(x)n(x, du).
\]
Then for $Q^X$ the image law of $\hat{P}^X$ under $\hat{X}'$, we have as before, that $(\hat{X}'_t, F^\mu_t, \hat{P}^\mu_t, Q'^X)$ is strong Markov. Since $A$ is continuous and strictly increasing, the totally inaccessible stopping times for the completions of $(U_t)$ with respect to the $Q^\mu$ and $Q'^\mu$ correspond exactly (under a time change), so that $(Q'^X)$ is regulated by the appropriate transformation $(K'_0)$ of $(K'_t)$. Condition (III) holds by (10.5), and except for II(b), VI(a), and (8.9), the remaining conditions of Corollary 6 and Theorem 4 are now immediate.

Let $h \in E^F \varotimes E^F$ be positive. Then

$$E^X_Q \left[ \int_0^{A'(t)} \int \left\{ (x,y); x \neq y \right\} (W^F_s, W^D_s) h(w^F_s, w^D_s) d\eta'(w^F_s, \cdot) ds \right]$$

$$= E^X Q^{A'(t)} \left[ \int_0^{A'(t)} \int \left\{ (x,y); x \neq y \right\} (\hat{X}'_s, W^D_s) h(\hat{X}'_s, W^D_s) d\eta(\hat{X}'_s, \cdot) ds \right]$$

$$\leq E^X \left[ \sum_{s \in (0, A'(t)]} h(\hat{X}'_s, \hat{X}'_s) \left\{ (x,y); x \neq y \right\} (\hat{X}'_s, \hat{X}'_s) \right]$$

$$\leq E^X_Q \left[ \sum_{s \in (0, t]} h(w^F_s, w^F_s) 1_{\left\{ (x,y); x \neq y \right\} (\hat{X}'_s, \hat{X}'_s)} ds \right]$$

since $f_g \leq 1$, showing (8.9). Condition II(b) follows similarly.

To show VI(a), observe that since equality holds in (10.5) only
if \( m(x) = 0 \), we have that \( m'(x) = 0 \) only if \( m(x) = 0 \) and \( g(x) = 1 \). Since \( f \) is never zero, we see that if \( x \in I' \), then also \( x \in I \), so that
\[
\int_0^t n(\hat{X}_s, U) ds = \infty \quad \text{for every } t > 0, \ P^X\text{-a.s.}.
\]

Because \( f \) is bounded from zero, and \( g(x) = 1 \) whenever \( m'(x) = 0 \), it follows that also
\[
\int_0^t n'(\hat{X}'_s, U) ds = \infty \quad \text{for every } t > 0, \ P^X\text{-a.s.},
\]
showing VI(a). Thus Corollary 6 and Theorem 4 apply.

Note: 1. Because \( f(x)g(x) \) may be strictly less than 1, the function \( J'(x,y) \) given by Lemma 10(a) may have values decreased from those of \( J(x,y) \). Thus fewer of the jumps of \( (\hat{X}'_s) \) will correspond to excursions of \( (X'_t) \) than was the case for \( (\hat{X}_s) \) and \( (X_t) \). This will be the case when \( f \equiv 1 \), \( g < 1 \), so that even if we start with a continuous process \( (X_t) \), the process \( (X'_t) \) will in general have many jumps, between points of \( F \).

2. If on the other hand, \( f \) and \( g \) balance, so that \( fg \equiv 1 \), and \( f \geq 1-m \) (with equality only if \( m = 0 \)) then we can obtain \( (Y'_t) \) directly. That is, suppose \( (Y_t) \) satisfies the conditions of Theorem 4, and let
\[
Y'_t = Y_{A'}(t).
\]
We see as in Note 1 to (C), that \( (Y'_t) \) satisfies the conditions
of Theorem 4, for \((X'_t, n')\), etc... so that a strong Markov process \((X'_t)\) can be constructed.

Let \((X'_t)\) be the strong Markov process associated to \((\hat{X}'_t, n)\), \((Y_t)\) and \(n\), and set

\[
B(t) = \int_0^t 1_{E \setminus F}(X_s) ds + \int_0^t (f+m-1)(X_s) dL_s
\]

\[
B'(s) = \inf\{t; B(t) > s\}.
\]

We will see that

\[
X'_t = \begin{cases} X'B'(t), & B'(t) < \infty \\ \Delta, & B'(t) = \infty \end{cases}
\]

for every \(t\), a.s.

Since the excursions are left unchanged, it will suffice to show this for \(t\) of the form \(S'^+(s)\). Since

\[
X'_s = \hat{X}'_s = \hat{X}'_A'(s) = X''(A'(s)),
\]

it will suffice to show that \(B'(S'^+(s)) = S'^+(A'(s))\), or, by right continuity, that \(S'^+(s) = B(S'^+(A'(s)))\).

\[
B(S'^+(A'(s))) = \int_0^{S'^+(A'(s))} 1_{E \setminus F}(X_r) dr + \int_0^{S'^+(A'(s))} (f+m-1)(X_r) dL_r
\]

\[
= \sum_{r \in [0,A'(s)], Y_r \neq \delta} D(Y_r) + \int_0^{A'(s)} (f+m-1)(\hat{X}_r) dr
\]

\[
= \sum_{r \in [0,S], Y_r \neq \delta} D(Y'_r) + \int_0^S \frac{f+m-1}{f}(\hat{X}'_r) dr = S'^+(s),
\]
since \((f + m - 1)/f = 1 + (m' - 1) = m'\).

Thus \((X_t')\) is a time change of \((X_t)\) on \(F\), and no new jumps as in note 1 are introduced.

We summarize in:

**Proposition 13** Let \((X_t)\) and \(M\) be as in Theorem 3, and obtain \(F\), \((p^X_0)\) and \(n\) from them. Let \(h : E \rightarrow [0,1]\) be 0-excessive for the \(p^X_0\). Let \(F = F_0 \cup F_1\), where \(F_0\) and \(F_1\) have disjoint closures in the \((X_t)\)-Ray Knight compactification of \(E\), \(\Delta \in F_1\), and

\[
\frac{p^X_0}{h} \text{ for } D < \infty \text{ and } W_D \in F_0 = 0 \text{ for } x \in E \setminus F_0.
\]

Let \(f : F \rightarrow [0,\infty)\), \(g : F \rightarrow [0,\infty)\) be \(E\)-measurable and satisfy:

- \(f\) is bounded and bounded below on \(F_1\); \(f = 0\) on \(F_0\); \(fg \leq 1\) on \(F_1\); and \(g(1 - m) \leq 1\) on \(F_1\), with equality only where \(m = 0\).

Let \((\hat{X}_t)\) be the time change of \((\hat{X}_t)\) by \(f\), as above, and let

\[
n'(x, du) = \begin{cases} 
g(x)h(x, du), & x \in F_1 \\
0, & x \in F_0. \end{cases}
\]

Then the conditions of Corollary 6 are met, for \((\hat{p}^X_0)\), \((\hat{X}_t)\) and \(n'\), so that we may construct a strong Markov process \((X'_t)\) from them by Theorem 4. \(\square\)

(E) the Knight-Pittenger procedure

Start with \((X_t)\) and \(M\) as in Theorem 3, and obtain \(F\), \((\hat{X}_t)\), \((\hat{p}^X_0)\), \((Y_t)\), \(n\) and \(m\) as there. Assume that \((\hat{X}_t)\) is
cadlag in $E$, and that $F = F_0 \cup F_1$ where $F_0$ and $F_1$ have disjoint closures in $E$, and $\Delta \in F_1$. Let $h$ be as in the note to (B), and apply the direct construction in that note to obtain $(Y_t^h)$ and $h'$. Apply the direct construction in Notes 1 and 4 of (C), to obtain $(X_t'), (Y_t')$ and $n'$ (observe that (10.4) holds in this case by (10.3)). Apply the direct construction in Note 1 of (D) to $(X_t'), (Y_t'), (Y_t')$ and $n'$, with

$$f = 1 - h' + m \geq 1 - h$$

(note that if $f = 1 - h$, then $m = 0$ so that $h = 0$), to obtain $(X_t''), (Y_t'')$ and $n''$, and then use Theorem 4 to reconstruct a strong Markov process $(X_t'')$. Let

$$A(t) = \int_0^t f(\hat{X}_s)1_{\hat{F}_1}(\hat{X}_s)ds, \quad A'(s) = \inf\{t; A(t) > s\}.$$ 

By our choice of $f$, we have that $m''f = m$, so that

$$\int_0^t m''(\hat{X}_s)ds = \int_0^t m''(\hat{X}_{A'}'(s))ds$$

$$= \int_0^{A'(t)} m''(\hat{X}_s)dA_s = \int_0^{A'(t)} m(\hat{X}_s)1_{\hat{F}_1}(\hat{X}_s)ds.$$

As in Note 1 of (D), it follows immediately that $(X_t'')$ can be described as follows: Kill $(X_t)$ at the last exit time from $F_1 \backslash \{\Delta\}$ (that is, after this time, it jumps to $\Delta$), and then excise from the path of $(X_t)$ all those excursions away from $F_1$, that either meet $F_0$ or lie in $U \backslash U_h$. By Lemma 16 below, there
are almost surely no excursions lying in \( U \setminus U_h \) that come back to \( F \) other than at points of \( F_0 \cup \{\Delta\} \). Since all the excised excursions occur before the last exit time from \( F_1 \setminus \{\Delta\} \), they must all come back to \( F \), and hence all meet \( F_0 \). Thus we have shown that the process \( (X''_t) \) is obtained from \( (X'_t) \) by exactly the procedure given by Knight and Pittenger in [35]. Our results therefore give another proof of their theorem, that the process constructed in this manner is strong Markov. Note that in order for our techniques to work, we have had to impose conditions

\((X, F, F, P) \) is a right process,

\[ \{ t; X_t \in F_0 \cup F_1 \} \]

satisfies \((9.1) \) and \((9.2) \) ) guaranteeing the existence of a local time. They do not need to do so.

We close this section with the lemma used above.

**Lemma 16**

\[ n(x, \{ D < \Delta, W_D \in F_1 \} \setminus U_h ) = 0 \]

for every \( x \in F_1 \).

**proof:** Fix \( x \in F_1 \), and \( \epsilon, s > 0 \). Then

\[ n(x, \{ D < \Delta, W_D \in F_1 \}, h(W_{j/2^k}) < \epsilon \text{ for some } j/2^k \epsilon (0, s) ) \]

\[ = \sum_{j=1}^{[2^k s]} n(x, \{ D \in (j/2^k, \infty) , W_D \in F_1 \setminus \{\Delta\} , h(W_{j/2^k}) < \epsilon , h(W_{i/2^k}) \geq \epsilon \text{ for } i < j ) \} \]
\[
\sum_{j=1}^{2^k s} \int \mathbb{P}_{0}^{W_j/2^k} (D<\infty, \{h(W_j/2^k)<\varepsilon, h(W_i/2^k)\geq \varepsilon \text{ for } i<j\})
\]

\[
W_D \in F_1 \setminus \{\Delta\} \text{dn}(x, \cdot) \leq \varepsilon n(x, h(W_j/2^k) < \varepsilon \text{ for some } j/2^k \in (0,s)).
\]

Thus,

\[
(1-\varepsilon)n(x, \{D<\emptyset, W_D \in F_1 \}, h(W_j/2^k) < \varepsilon \text{ for some } j/2^k \in (0,s))
\]

\[
\leq \varepsilon n(x, \{D=\emptyset, \text{ or } D<\emptyset \text{ and } W_D \in F_0 \}).
\]

Letting \( k \to \infty \) and then \( s \to 0 \), we see that

\[
(1-\varepsilon)n(x, \{D<\emptyset, W_D \in F_1 \} \setminus U_h)
\]

\[
\leq \varepsilon n(x, \{D=\emptyset, \text{ or } D<\emptyset \text{ and } W_D \in F_0 \})
\]

for every \( \varepsilon > 0 \). The right hand side is finite by II(a) and III, so that we obtain the desired conclusion upon letting \( \varepsilon \to 0 \). \( \square \)
PART 2. MARTIN BOUNDARIES

11. Introduction

Recall that using Theorem 4 and Corollary 6 of Part 1, we constructed a strong Markov process from a 'process on the boundary' \((\check{X}_t^x)\), and a kernel \(n(x, du)\). Our applications in section 10 consisted of constructing these objects from other similar ones, thereby obtaining a transformation that produced new strong Markov processes from old ones.

A more far reaching program would hope to build the process on the boundary, and the kernel \(n\), from scratch, knowing only the laws \((P^x_0)\) of the process stopped at the boundary. The problem of constructing a process on the boundary seems hard in general; Condition (8.9) gives information about what its Lévy system must be, and in a general state space, it seems difficult to produce a process with a given Lévy system (how does one 'compensate'?). For results in Euclidean space, see Bass [1], and Stroock [47].

As to the kernels \(n(x, du)\), one might first try to construct the kernels \(\hat{n}(x, y; du)\) of Section 8D. The natural approach is via h-transforms, the hope being that

\[ n(x, y; \cdot) = h^{P^x_0} \]

for an appropriate \((P^x_0)\) - excessive function \(h\), depending on \(y\). The first step in this approach would therefore be to determine those excessive functions \(h\) for which \(x\) is an entrance point for the h-transformed process.

In the second part of this thesis, we will approach this problem from a different perspective, in the particular case of Brownian motion. The
arguments will be completely independent of those given in Part 1, and except in this introduction, we will feel free to use completely different notation from that used in Part 1, in an attempt to conform to the notation used in the Martin boundary literature. The numbering of the results will start over again also. The major new results are Theorem 6 and Corollaries 1 and 2.

12. $h$-transforms and Martin boundaries

In this section, we will outline those parts of the theory of $h$-transforms and Martin boundaries that will be needed later. The results given below are taken from a set of notes by J. B. Walsh, but were originally proven almost entirely by J. L. Doob ([15], [17]), and by R. S. Martin ([37]). The arguments using time-reversal seem to be less well known than they should be, so proofs will at least be sketched. In their present form, these arguments are due to J. Walsh, but many of them derive from the paper [30] of G. A. Hunt. Any mistakes are my contribution. Many of the other results may be found in the introduction to Walsh [49]. In addition to the original articles, proofs may also be found in Meyer [41]. The definitive treatment will be (once it appears) J. L. Doob's book [19] on this whole subject.

Let $E$ be a domain (an open connected subset of $\mathbb{R}^d$), and let $E$ be the $\sigma$-field of its Borel subsets. Let $\delta$ be some additional point which will act as a cemetery. Let $\Omega$ be the canonical space of functions $\omega: [0, \infty) \to E \cup \{\delta\}$ such that there exists $\zeta > 0$ with $\omega$ continuous on $[0, \zeta)$, and $\{t; \omega(t) = \zeta\} = [\zeta, \infty)$: Let $X_t; t \geq 0$ be the coordinate maps on $\Omega$, and let
We will sometimes need to work with the space \( \hat{\Omega} \) of functions \( \omega : (0, \infty) \to E \cup \{\delta\} \), which are continuous up to their lifetime, as above. By the appropriate abuse of notation, we will use the same symbols \( F_t, F, X_t \), etc... for the corresponding objects on \( \hat{\Omega} \). Of course, in this context, \( X_t \) is only defined for \( t > 0 \). Similarly, \( \mathcal{B} \) will denote the Borel \( \sigma \)-field on both \( [0, \infty) \) and \( (0, \infty) \). We follow the convention of making all functions vanish at \( \delta \) unless otherwise specified.

Let \( P_t(x, dy) \) be the transition function of Brownian motion on \( \mathbb{R}^d \), killed upon leaving \( E \), and as usual, set \( P_t(x, \delta) = 1 - P_t(x, E) \), \( P_t(\delta, \{\delta\}) = 1 \). If now \( h \) is excessive for \( (P_t) \), we let

\[
E_h = \{ x \in E ; h(x) < \infty \}
\]

\[
hP_t(x, dy) = \begin{cases} 
\frac{1}{h(x)} P_t(x, dy) h(y), & x \in E_h \\
0, & \text{otherwise}
\end{cases}
\]

(note that unless \( h \equiv 0 \), \( \{h = 0\} = \emptyset \)).

Theorem 1

(a) If \( h > 0 \) is excessive, then \( (hP_t) \) forms a sub-Markov semigroup on \( E \).

Extend it as above to form a Markov semigroup. Then:

(b) For \( x \in E \), there exists a probability measure \( hP_t^x \) on \( (\Omega, F) \)
under which \((X_t)\) is strong Markov for the semigroup \((hP_t)\) and the filtration \((F_{t+})\), and for which \(hP^X(X_0 = x) = 1\);

(c) If \(T\) is any \((F_{t+})\) stopping time, and \(\Lambda \in F_{t+}\), then

\[
hP^X(\Lambda, \zeta > T) = \frac{1}{h(x)} \int_{E_h} \mathbb{E}_h^X(\Lambda, h(W_T)) \, (x) \, d\mu(x);
\]

(d) \(\nu\) is \(h\)-excessive iff \(\nu = 0\) on \(E \setminus E_h\), and there is an excessive function \(u\) with \(u = \nu h\) on \(E_h\);

(e) Every excessive or \(h\)-excessive function is \(h\)-a.s.

continuous along the path of \((X_t)\), up to its lifetime.

In general, if we are given a measurable space, a probability \(P\)-thereon, and a random variable \(Y\) with values in \((\Omega, \mathcal{F}, \mathbb{P})\), we say that \(Y\) is an \(h\)-transform under \(P\), if it is strong Markov with respect to \(P\) and the filtration \(Y^{-1}(F_{t+})\) with transition semigroup \(hP_t\). We use the same notation if \(Y\) takes values in \(\hat{\Omega}\), but now the strong Markov property is at strictly positive stopping times.

Now let \((A, A)\) be a measurable space, and \(u\) a measurable function: \((A \times E, A \otimes E) \to ([0, \infty), \mathcal{B})\) such that each \(u(a, \cdot)\) is excessive. Let \(\nu\) be a measure on \((A, A)\), and set

\[
h(x) = \int u(a, x) \nu(dx).
\]

Then if \(h\) is not identically infinite, it is excessive, and

\[
hP^X(\Lambda) = \frac{1}{h(x)} \int \int u(a, x) \, u(a, \cdot) \, P^X(\Lambda) \, \nu(da)
\]

on \(E_h\), for \(\Lambda \in F\).
Recall that an excessive function $h$ is called minimal if whenever $h = u + v$, where $u$ and $v$ are both excessive, then both $u$ and $v$ are multiples of $h$.

(g) If $E$ has a Green function $G(x,y)$, then $G(\cdot, y)$ is a minimal excessive function.

Let the invariant $\sigma$-field be

$$I = \{ A \in F; \theta^{-1}_t( A \cap \{ \zeta > 0 \}) = A \cap \{ \zeta > t \} \}.$$ 

(h) If $h$ is minimal, then $I$ is trivial under each $P^X$. If $h > 0$ is excessive and $A \in I$, then $P^X(A, \zeta > 0)$ is $h$-excessive.

(i) Let $u$ and $v$ be excessive, and let $(Y_t)$ and $P$ be a process and a probability such that $(Y_t)$ is both a $u$ and a $v$-transform under $P$. If $P(Y_0 \neq \delta) > 0$, then $u$ and $v$ are multiples of each other.

(j) Let $U$ be a subdomain of $E$, and let $h$ be excessive on $E$. Let $(Y_t)$ be an $h$-transform under $P$. Then killing $(Y_t)$ upon first leaving $U$, produces a process which is a transform by the restriction of $h$ to $U$, under $P$.

If $A \subset E$ is Borel, write $T_A$ and $L_A$ for the first hitting time and last exit time of $A$, respectively;

$$T_A = \inf\{ t > 0 ; X_t \in A \}$$

$$L_A = \sup\{ t > 0 ; X_t \in A \} \quad (\sup(\emptyset) = 0).$$
Also, for $h \geq 0$, let

$$R_A h(x) = E^X[h(X_{T_A})] .$$

(k) If $h$ is excessive, then

$$R_A h(x) = h P^X(T_A < \infty) h(x) .$$

$R_A h$ is also the excessive regularization of the infimum of all excessive functions majorizing $h$ on $A$.

(l) If $(X_t)$ is an $h$-transform under $P$, then killing $(X_t)$ at $L_A$ produces a process which is an $R_A h$-transform under $P$.

(m) If $(X_t)$ is an $h$-transform under $P$, and $E^\times A$ is a subdomain of $E$, then the process

$$(X_{L_A^+ t})$$

is a $v$-transform on $E \setminus A$, under $P$, where

$$v(x) = h(x) P^X(T_A = \infty)$$

[for this, see 5.1 of Meyer, Smythe and Walsh [43]] .

Note that the new processes of (l) and (m) may be identically equal to $\delta$, with positive probability. □

Now suppose that $E$ has a Green function $G(x,y)$. Let $t_0 > 0$ and let $\phi$ be bounded and increasing, with $\phi(t) = t$ for $t \in [0, t_0)$. Let $y_0 \in E$ be fixed, and suppose that $\phi$ is concave, so that

$$p(x) = \phi(G(x, y_0))$$

is excessive. Following Helms [29], let
\[ \sum = \{ x \in E ; G(x, y^0) > t_0 \} . \]

It is an open neighbourhood of \( y^0 \), with compact closure in \( E \). Define the **Martin function** to be

\[ K(x, y) = \frac{G(x, y)}{p(x)} , \quad x, y \in E . \]

Let the Martin metric \( d \) on \( E \), be

\[ d(x, y) = \int |K(x, z) - K(y, z)| (1 + |K(x, z) - K(y, z)|) \, dz \]

Let \( \overline{E} \) be the completion of \( E \) under \( d \), and call \( \Delta = \overline{E} \setminus E \) the **Martin boundary** of \( E \).

**Theorem 2**

(a) \( \overline{E} \) is compact, and \( E \) is a dense open subset. The topology induced on \( E \) by \( d \) is the usual Euclidean one.

(b) \( K \) extends to a continuous function from \( \overline{E} \times E \) to \( (0, \infty] \). It takes on the value \( \infty \) only on \( \{(x, y); x = y \in E\} \), and never vanishes.

(c) For \( x \in \Delta \), \( K(x, \cdot) \) is harmonic, hence excessive. We say that \( x \in \Delta \) is **minimal** if \( K(x, \cdot) \) is. Let \( \Delta_0 \) be the set of minimal \( x \in \Delta \).

(d) For \( x \in \Delta \), \( x \) is minimal iff for every open neighbourhood \( \Delta \) of \( x \),

\[ R_{\Delta \cap E} K(x, \cdot) = K(x, \cdot) . \]

(e) \( \Delta_0 \) is Borel. For every excessive \( h > 0 \), there is a unique measure \( \nu \) on \( \Delta_0 \cup E \) such that
h = \int K(x, \cdot)\nu(dx).

h is harmonic iff \( \nu(E) = 0 \).

(f) Let \( h \) be harmonic, and continuous on an open subset \( U \) of \( \overline{E} \).
Let \( \nu \) represent \( h \), and let \( \mu_0 \) represent the function identically equal to 1. Then

\[ \nu(dy) = h(y)\mu_0(dy) \text{ on } U. \]

If \( h \) is excessive, write \( (h^\lambda)_{\lambda>0} \) for the resolvant of \((h^t_p)\). Let

\[ X_t = \begin{cases} x - t, & t \in (0, \zeta) \\ \zeta, & t \geq \zeta \end{cases} \]

be the reverse of \((X_t)\) from its lifetime. Observe that \( \tilde{X} \in \hat{N} \), and that this definition applies regardless of whether \( X \) is defined at 0.

Theorem 3

(a) Let \( h \) be excessive, and assume that \( h^X(\zeta < \infty) = 1 \) for every \( x \in E \) (as usual, it suffices that this holds for some \( x \in \mathcal{E} \)). Let \( \mu \) be a probability on \( E \). Then under the probability \( h^\mu \), the process \((\tilde{X}_t)_{t>0}\) is a \( \nu \)-transform, where

\[ \nu(x) = \int \frac{G(z, x)}{h(z)} \mu(dz). \]
(b) If $P$ is any probability on $(\mathcal{F}, F)$ under which $(X_t)_{t>0}$ is an $h$-transform, then there is some excessive function $v$ such that $(\tilde{X}_t)_{t>0}$ is a $v$-transform under $P$.

**proof:** (a) First, we show that $v < \infty$ a.e. Let

$$A_n = \{x \in E ; \quad h^P_X(\zeta > n) < 1/2\},$$

$$B_t = \int_0^t 1_{A_n}(X_s)ds,$$

$$T(j) = \inf\{t ; B_t > jn\}.$$ 

Since

$$h^P_X(\zeta > n)$$

is $h^P_X$-a.s. continuous in $t < \zeta$, it follows that for each $x \in E$,

$$h^P_X(T(j+1) < \infty \mid T(j) < \infty) \leq h^P_X(X_T(j) > n \mid T(j) < \infty) \leq \frac{1}{2}.$$ 

Thus

$$\int_{A_n} \frac{1}{h(x)} X_T(j)^n h(y)dy = \int_0^\infty h_t(x, A_n)dt$$

$$= E^X[B_\infty]$$

$$\leq n \sum_{j=0}^{\infty} h^P_X(B_\infty > jn) \leq n \sum_{j=0}^{\infty} 2^{-j} = 2n.$$ 

Integrating with respect to $\mu$, we see that
\[ \int_{A_n} v(x)h(x)dx \leq 2n \text{ for each } n. \]

Since \((P_t)\) is strong Feller, and \(h\) is l.s.c., we see that
\[ A_n = \{ E^X[h(X_n)] < h(x)/2 \} \]
is open. By hypothesis, \(A_n \uparrow E\), so that not only is \(v < \infty \text{ a.e. on } E\), but \(v(x)h(x)dx\) is a Radon measure thereon. (Compact subsets of \(E\) have finite measure). The occupation time measure of \((X_t)\) is
\[ \nu(dx) = \int \mu(dz) \frac{U^0(z, dx)}{h} = v(x)h(x)dx, \]
so that by the results of Cartier, Meyer, Weil \([6]\), (a) will be shown, once we show that \((U^\lambda_h)_{\lambda > 0}\) and \((U^\lambda_v)_{\lambda > 0}\) are in duality with respect to \(v\).

Let \(G^\lambda(x,y)\) be the \(\lambda\)-order Green function. Then if \(f\) and \(g\) are positive and Borel, we have that
\[ < f, vU^\lambda g >_v = \int f(x)v(x)h(x)\frac{1}{v(x)} \int g^\lambda(x,y)g(y)v(y)dydx \]
\[ = \int g(y)h(y)v(y)\frac{1}{h(y)} \int f(x)h(x)G^\lambda(x,y)dxdy \]
\[ = < hU^\lambda f, g >_v, \]
by symmetry of \(G^\lambda\), as required.

(b) Let \((E(n))\) be a sequence of subdomains of \(E\), such that \(E(n)\) has compact closure in \(E(n+1)\), and \(\cup E(n) = E\).
Let \( T(n) = \inf\{t; X_e \in E(n)\} \). Then

\[
(X_{T(n)} + t)_{t \geq 0}
\]

(defined by right continuity at 0 if \( T(n) = 0 \)) is still an \( h \)-transform under \( P \), and it does have an initial distribution, so that (a) applies.

Let \( (X_T^n) \) be the reverse of \( (X_{T(n)} + t) \), so that there exists an excessive \( v_n \) such that \( (X_T^n) \) is a \( v_n \)-transform under \( P \). Let

\[
\hat{S}_n = \inf\{t > 0; X_T^n \notin E(n)\}
\]

\[
\hat{L}_n = \sup\{t > 0; X_T^n \notin E(n)\}.
\]

Then \( X_T^n = X_T^{n+1} \) if \( t < \hat{L}_n \), and in particular, if \( t < \hat{S}_n \). From (i) and (j) of Theorem 1, it follows that there is a constant \( c_n > 0 \) such that \( v_{n+1} = c_n v_n \) on \( E(n) \). We may multiply each \( v_n \) by a constant, to make each \( c_n = 1 \), so that an excessive function \( v \) on \( E \) may be defined to make \( v = v_n \) on \( E(n) \), for each \( n \). Since each \( (X_T^n) \) is a \( v_n \)-transform, it follows that \( (X_T^n) \) is a \( v \)-transform under \( P \). □

Lemma 1. Let \( h > 0 \) be harmonic in \( E \), and let \( A \) be compact in \( E \).

Then

\[
P^x(L_A < \zeta) = 1 \text{ for each } x \in E.
\]

proof: By (f) of Theorem 1, and (e) of Theorem 2, it suffices to show this for \( h \) minimal. The event \( \{L_A = \zeta\} \) is invariant, so that in this case

\[
P^x(L_A = \zeta) = 0 \text{ or } 1 \text{ for each } x.
\]
(by (h) of Theorem 1). Since \( E_h = E \), this same result shows that this function is \( h \)-excessive. Thus by (d) of Theorem 1, it is either identically zero or identically one. Assume the latter. Then for each \( x \in E \), it follows from (k) of Theorem 1, that

\[
h(x) = h(x) \int_{L_A = \zeta} P^x(L_A = \zeta) = h(x) \int_{T_A < \infty} P^x(T_A < \infty)
\]

\[= R_A h(x).\]

Since \( A \) is compact, \( R_A h \) is a potential. This is a contradiction, since the only harmonic function dominated by a potential is identically zero. \( \square \)

We will use the notation

\[
y^P = K(y, \cdot) P^x.
\]

whenever \( y \in \overline{E} \).

**Theorem 4** Let

\[
h(x) = \int_{E \cup \Delta_0} K(z, x) \mu(dz)
\]

be excessive. Then for each \( x \in E_h \), \( X_\zeta^- \) exists in \( \overline{E} \), \( h^P \)-a.s., and has distribution

\[
h^P(X_\zeta^- \in dy) = \frac{1}{h(x)} K(y, x) \mu(dy).
\]

**proof:** By (f) of Theorem 1, it suffices to show that

\[
y^P(X_\zeta^- = y) = 1 \text{ for } y \in E \cup \Delta_0.
\]
Case 1: Suppose \( y \in E \). By (1) of Theorem 1, and Lemma 1, there exists an excessive function \( u \) such that \( y \in E_u \) and \( u \)-transforms have finite lifetimes. By Theorem 3, \( (\tilde{X}_t) \) is a \( G(y, \cdot) \) transform under \( u \), so that there exists a probability \( \nu \) such that

\[
P_y^\nu(\zeta \in (0, \infty), X_{\zeta^-} = y) = 1.
\]

This holds for every \( \nu \) by minimality, as in Lemma 1.

Case 2: Suppose \( y \in \Delta_0 \). In this case, there is no reason why \( \zeta \) should be finite. Let \( \bigcup \gamma \) be an open neighbourhood of \( x \in E \), with compact closure in \( E \). Let \( (E(n)) \) be a sequence of subdomains of \( E' \), with each \( E(n) \) having compact closure contained in \( E(n+1) \), and such that \( E(1) \) contains the closures of \( \bigcup \gamma \) and \( \bigcup' \gamma \). By Lemma 1,

\[
P_y^X(L_{E(n)} < \zeta) = 1
\]

for each \( n \), so that we may form the reverse \( \tilde{x}^n \) of \( x \) from \( L_{E(n)} \) and obtain from Theorem 3(a) (and (1) of Theorem 1), that \( \tilde{x}^n \) is a \( G(x, \cdot) \) transform under \( y \). Let

\[
k(w, z) = G(w, z)/G(w, x),
\]

for \( z \in \bigcup \gamma \) and \( w \in E' = EX(\bigcup \gamma \cup \bigcup' \gamma) \). Let

\[
T(n) = \inf\{t > 0; \tilde{x}^n_t \notin E'\}.
\]

Since \( k(\cdot, z) \) is bounded on \( E' \), for \( z \in \bigcup \gamma \), it follows as usual, that

\[
k(\tilde{x}^n_{t \land T(n)}, z)
\]
is a bounded martingale under $\mathbb{P}^X_y$, for each $z \in \mathcal{Z}$. If $U_n$ denotes the number of upcrossings it makes, of a fixed interval $[a, b]$, we therefore obtain that

$$|a| + \sup_{y \in E'} \frac{\mathbb{E}^X[U_n]}{b - a} < \infty.$$ 

But $U_n$ is also the number of upcrossings of $[a, b]$ by $k(X_t, z)$, for

$$t \in (L_u \cup L_v, L_{E(n)}) .$$

Thus $U_n \uparrow U_\infty$, and since the above bound did not depend on $n$, we see that $U_\infty < \infty \mathbb{P}^X_y$-a.s. Thus by the usual argument, it follows that

$$\lim_{t \uparrow \zeta} k(X_t, z)$$

exists $\mathbb{P}^X_y$-a.s., for each $z \in \mathcal{Z}$. But $k(\cdot, z)$ is bounded away from zero on $E'$, and

$$K(w, z) = \frac{k(w, z)}{k(w, y_0)}$$

for $w \notin \mathcal{Z}$, so that in fact

$$\lim_{t \uparrow \zeta} K(X_t, z)$$

exists $\mathbb{P}^X_y$-a.s., for each $z \in \mathcal{Z}$. By Fubini's theorem, this holds $\mathbb{P}^X_y$-a.s., simultaneously for a.e. $z \in \mathcal{Z}$, so that in fact

$$d(X_s, X_t) \to 0 \text{ as } s, t \uparrow \zeta, \mathbb{P}^X_y \text{-a.s.}.$$ Thus, the $\mathcal{E}$-limit $X_{\zeta-}$ exists.
We must now identify it as \( y \).

Let \((A(n))\) be a decreasing sequence of neighbourhoods of \( y \), with \( nA(n) = \{y\} \). We must show only that

\[
T_{A(n)} < \infty \quad \text{\( y \)-a.s.}
\]

But

\[
K(y, z) = \lim_{n \to \infty} K(y, \cdot)(x) = K(y, x) \, P^x(T_{A(n)} < \infty),
\]

by (k) of Theorem 1, and (d) of Theorem 2, so that this holds. □

This proof traces its ancestry to Hunt [30]. The technique of killing \( \text{at the last exit times of compact subsets of } E \), let us \( \cdot \) and eliminate his use of \text{approximate Markov chains}.

We say that a point \( y \in \Delta_0 \) is accessible if \( P^x(\zeta < \omega) = 1 \) for some (and hence every) \( x \in E \). Let \( \text{Acc} \) denote the set of all accessible points.

**Theorem 5** Let \( u(x) = \int G(x, z)\nu(dz) \) be an a.e. finite potential, and let \( y \in \text{Acc} \). Suppose that

\[
\int K(y, z)\nu(dz) < \infty.
\]

Then there exists a unique probability \( P^y \) on \( \hat{\Omega} \), such that \((X_t)\) is \( u \)-transform under \( P^y \), and

\[
P^y(X_t \to y \text{ in } E \text{ as } t \downarrow 0) = 1.
\]

**proof:** Let
\[ a = \int K(y, z) \nu(dz) \, , \mu(dz) = \frac{1}{a} K(y, z) \nu(dz) . \]

Because \( y \) is accessible, we can reverse from \( \zeta \) to obtain that under \( y^P \), \((X_t)\) is a transform by the function

\[
\int \frac{1}{K(y, z)} G(z, x) \mu(dz) = \frac{1}{a} \int G(z, x) \nu(dx) = \frac{1}{a} u(x) .
\]

Since

\[
\lim_{t \to 0} X_t = \lim_{t \to \zeta} X_t = y , \quad y^P \text{-a.s. ,}
\]

by Theorem 4, we have shown existence.

To show uniqueness, we will show that if \( P \) is a probability on \( \hat{\Omega} \), as in the theorem, then \((\tilde{X}_t)\) is a \( K(y, \cdot) \) transform, with initial measure \( \mu \), under \( P \). Thus the law of \((\tilde{X}_t)\), and hence \( f \) (\( X_t \)) is determined.

It follows from Theorem 3(b), that since \((X_t)\) is a \( u \)-transform under \( P \), \((\tilde{X}_t)\) is an \( h \)-transform for some \( h \). Write

\[
h(z) = \int K(x, z) \eta(dx) .
\]

By Theorem 4,

\[
P(\tilde{X}_{\zeta^+} \in dx , \zeta > t) = E[\frac{1}{h(X_t)}(\tilde{X}_{\zeta^+} \in dx)]
\]

\[
= E[\frac{1}{h(X_t)} K(x, \tilde{X}_t) \eta(dx) , \zeta > t] ,
\]

for every \( t > 0 \). But \( \tilde{X}_{\zeta^+} = y \) \( P \)-a.s., so that \( \eta \) must be a point mass at \( y \), and hence \( h(z) = cK(y, z) \) for some \( c \).
By Theorem 4, we use a similar argument to see that $X_{\zeta^-} \in E$ P-a.s. Let $\mu'$ be the distribution of $X_{\zeta^-}$. It is then the initial distribution of $(\tilde{X}_t^\zeta)$, so that by Theorem 3(a), $(\tilde{X}_t^\zeta)$ is a transform by

$$\int \frac{1}{K(y, z)} G(z, x) \mu'(dz) = v(x).$$

It follows from (i) of Theorem 1, that $v = cu$ for some $c > 0$. Since a potential determines its measure, we conclude that

$$\frac{1}{K(y, z)} \mu'(dz) = c \; v(dz),$$

or in other words, $\mu' = ac\mu$, so that $c = 1/a$ and $\mu' = \mu$. □

We wish to examine the set

$$D = \{(x, y) \in (\mathcal{A} \times E) \times (\mathcal{A} \times E) ; \text{there exists a probability } y^P_X \text{ on } \hat{\Omega} \text{ under which } (X_t^\zeta) \text{ is a } K(y, \cdot) \text{ transform, and } X_{0+} = x \text{ a.s.}\}$$

Theorems 4 and 5 show that

$$D \supset (\mathcal{A} \times E) \cup (E \times \mathcal{A}).$$

The arguments given above also suffice to show the following statements:

If $(x, y) \in D$ then there is only one law $y^P_X$ satisfying the required conditions; and

(12.1) $D$ is symmetric (that is, $(x, y) \in D$ iff $(y, x) \in D$), and if $(x, y) \in D$ and $\Lambda \in (\hat{\Omega}, F)$, then

$$x^P_y(\Lambda) = y^P_X((\tilde{X}_t^\zeta) \in \Lambda).$$
Recall that a function \( f \) on \( E \) is said to have **fine limit** \( a \), at a point \( y \in \Delta_0 \), if

\[
y_P^X(\lim_\limits_{t \to \xi} f(X_t) = a) = 1
\]

for some (and hence by minimality, as in Lemma 1, for every) \( x \in E \).

By the construction of Theorem 5, this is equivalent to

\[
x_P^y(\lim_\limits_{t \to 0} f(X_t) = a) = 1
\]

for some (and hence every) \( x \in E \).

**Lemma 2** Let \( A \) be open in \( E \), with \( y \in A \cap \Delta_0 \). Then

\[
h = \frac{R}{E \setminus A} K(y, \cdot)
\]

is a potential; that is, for some measure \( \nu \), we have

\[
h = \int_E G(z, \cdot) \nu(dz) = U^0 \nu
\]

**proof:** Write

\[
h = \int_{E \setminus \Delta_0} K(z, \cdot) \mu(dz)
\]

Since \( K(z, \cdot) \) is proportional to \( G(z, \cdot) \), it will suffice, by

Theorem 4, to show that

\[
h_P^x(X_{\xi^-} \in \Delta_0) = 0 \text{ for } x \in E
\]

But

\[
y_P^x(X_{\xi^-} = y) = 1
\]
and so

\[ \mathbb{P}_y^{X}(L_{E \setminus A} < \zeta) = 1 \]

as well. Thus by (1) of Theorem 1,

\[ \mathbb{P}_y^{X}(X_{\zeta -} \in \Delta_0) = \mathbb{P}_y^{X}(X_{L_{E \setminus A}} \in \Delta_0) = 0 . \]

**Theorem 6** Let \( x, y \in \mathbb{R}^n \). Let \( A \) be an open subset of \( \mathbb{R}^n \) such that \( y \in A \) and \( x \) does not lie in the closure of \( A \). Let \( z \in A \cap E \), and set \( f(w) = K(y, w)/G(z, w) \). Then the following conditions are equivalent

(a) \((x, y) \in D\)

(b) Let \( R_{E \setminus A} K(y, \cdot) = U_0^y \).

Then \( \int K(x, z)v(dz) < \infty \).

(c) \( f \) has a finite fine limit at \( x \).

(d) For some \( w \in E \),

\[ \liminf_{t \to \zeta} f(X_t) < \infty \quad \mathbb{P}_y^{X} \text{-a.s.} \]

**proof:** We will show that \((a) \iff (b)\), and \((a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)\).

\((a) \Rightarrow (b)) : \) Let \((Y_t^e)\) be \((X_t)\) killed at \( L_{E \setminus A} \), so that \((Y_t)\) is an \( R_{E \setminus A} K(y, \cdot) \) transform under \( \mathbb{P}_y^{X} \). Let \( \mu \) be the distribution of \( X_{L_{E \setminus A}} \) under \( \mathbb{P}_y^{X} \). Let \((\tilde{Y}_t)\) be the reverse of \((Y_t)\) from its lifetime.

By (12.1), \((y, x)\) also belongs to \( D \), and
\[ y^{P^X}_t(\gamma_{t>0} \in \Lambda) = x^{P^Y}_t((X_{t+T_A})_{t>0} \in \Lambda) = x^{P^H}_t((X_{t>0} \in \Lambda), \text{ for } \Lambda \in \hat{\hat{\Omega}}) \]

Thus, by Theorem 3(a), \((Y^\cdot_t)\) is also a transform by
\[
\int \frac{G(z, \cdot)}{K(x, z)} \mu(dz).
\]

It follows from (i) of Theorem 1, that there is a constant \(c\) with
\[
c \int \frac{G(z, \cdot)}{K(x, z)} \mu(dz) = \int G(z, \cdot) \nu(dz),
\]
and since a potential determines its measure, it follows that
\[
\int K(x, z) \nu(dz) = c < \infty.
\]

\((b) \Rightarrow (a):\) Let
\[
h = R_{E \setminus A} K(y, \cdot).
\]

By Theorem 5, there is a probability \(p^X_h\) on \(\hat{\hat{\Omega}}\) under which \((X^\cdot_t)\) is an \(h\)-transform started at \(x\). Define a probability \(P'\) on \(\hat{\Omega} \times \Omega\) by
\[
P'(A) = \int \int 1_A(\omega, \omega') P^Y_{T_A}(d\omega') X^{T_A}(\omega),
\]
and let \(P\) be the image law of \(P'\) on \(\hat{\hat{\Omega}}\) under the random variable
\[
Y^\cdot_t(\omega, \omega') = \begin{cases} X^\cdot_t(\omega) & \text{if } t < T_A(\omega) \\ X^\cdot_{t-T_A}(\omega) & \text{if } t \geq T_A(\omega) \end{cases}
\]

Then for \(f\) positive and bounded, we have that
\[ E[f(X_{t+s}) \mid F_s] \]

\begin{align*}
= \ & E[f(X_{t+s}), T_A \leq s \mid F_s] + E[f(X_{t+s}), T_A > s + t \mid F_s] \\
  & + E[f(X_{t+s}), T_A \epsilon (s, s + t) \mid F_s] \\
= \ & s_X f(X_t) \mathbf{1}_{\{T_A < s\}} + \sum_X f(X_t) \mathbf{1}_{\{T_A > t\}} \\
  & \quad + \mathbf{1}_{\{T_A > s\}} \int_0^t \int y^Z[f(X_{t-r})] \sum_X y^S(T_A \in dr, X_{T_A} \in dz) \\
= \ & s_X f(X_t) \mathbf{1}_{\{T_A < s\}} + \sum_X f(X_t) \mathbf{1}_{\{T_A > t\}} \\
  & \quad + \mathbf{1}_{\{T_A > s\}} \int_0^t \int y^Z[f(X_{t-r})] \sum_X y^S(T_A \in dr, X_{T_A} \in dz) \\
= \ & s_X f(X_t) \\
\end{align*}

Thus \((X_t)\) is a \(K(y, \cdot)\) transform under \(P\), as required.

\((a) \Rightarrow (c)\): The Green function for a \(K(y, \cdot)\) transform is

\[
y G(w, z) = \frac{K(y, z)}{K(y, w)} G(w, z) .
\]

We have that if \(\Omega'\) is a compact neighbourhood of \(z\), contained in \(E\), then for \(w \notin \Omega'\),

\[
y G(w, z) = \int y G(w', z) y^P(T_{\Omega'} \in dw') \\
\leq \sup_{w' \in \delta \Omega'} y G(w', z) < \infty.
\]
Then as usual,

$$Z_t = G_{t \wedge T^*}, z$$

is a bounded $P^w$-martingale, for $w \in E \setminus E'$. Since $(x, y) \in D$, and $(Z_t)_{t>0}$ is a bounded $P^x$-martingale as well. Thus, it converges $P^x$-a.s. to a random variable $Z_{0+}$ as $t \to 0$. Since $Z_t > 0$ for $t > 0$, $Z_{0+}$ cannot be identically zero. Let $(E(n))$ be a sequence of open subsets of $E$ such that $E(n)$ has compact closure in $E(n+1)$, and $\cup E(n) = E$. Let

$$\bar{X}^n_t = \begin{cases} X^E_{t \wedge E(n)-}, & t < L^{E(n)} \\ \delta, & t \geq L^{E(n)} \end{cases}$$

Then by Theorem 3, $(\bar{X}^n_t)$ is a $K(x, \cdot)$ transform under $P^x$, and

$$P^x(\bar{X}^n_0 = \delta) = P^x(L^{E(n)} = 0) + 0.$$

By minimality of $x$, $Z_{0+}$ is $P^x$-a.s. constant on $\{\bar{X}^n_0 \neq \delta\}$, so that in fact, there is a constant $a \neq 0$, such that $Z_{0+} = a$ $P^x$-a.s.

Thus also

$$P^x(\bar{X}^n_0 \neq \delta) = P^x(\bar{X}^n_0 \neq \delta, \lim_{t \uparrow \zeta} G(\bar{X}_t, z) = a) \leq \int_{E \times X} P^w(\lim_{t \uparrow \zeta} G(X_t, z) = a) P^x(\bar{X}^n_0 \in dw),$$
so that

\[ P^w \left( \lim_{t \to \xi} G(X_t, z) = a \right) > 0 \]

for some \( w \in E \). By minimality, this probability equals 1, showing (c). The implication \((c) \Rightarrow (d)\) is trivial.

((d) \Rightarrow (a)): By (d) of Theorem 1, \( f \) is \( G(z, \cdot) \)-excessive, so that \( (f(X_t)) \) is a \( P^w \)-supermartingale, for each \( w \in E \). Thus

\( (f(X_t)) \times n \) is a bounded \( P^w \)-supermartingale for each \( w \), so that by the Markov property, \( (f(X_t)) \times n \) is a bounded \( P^x \)-supermartingale. Thus, it converges at 0, \( x \)-a.s., and hence in fact

\[ \lim_{t \to 0} f(X_t) \text{ exists in } [0, \infty] , \quad P^x \text{-a.s.} \]

By minimality and time reversal, this limit is a constant \( a \) (non zero, but a priori, possibly infinite). As in Lemma 1, also

\[ f(X_t) \to a \quad \text{as} \quad t \to 0 , \quad P^x \text{-a.s.} \]

for each \( w \in E \), so that by (d), we have that in fact, \( a < \infty \).

Therefore

\[ z E^x[f(X_n)] = \lim_{n \to \infty} E^x[f(X_n) \wedge n] \leq a < \infty . \]

Let \( R_A(x, \cdot) = U^0 \nu \). Since \( z \in A \), we have that \( L_A > 0 \)

\( x P^z \)-a.s., and so
\[ E^X = \mathbb{E}^X_X = \mathbb{E}^Z_{X_A} \]

\[ = \int_0^\infty \mathbb{E}^Z_{X_{\leq t}} \]

\[ = \frac{1}{u_0 \nu(z)} \int f(w) G(w, z) \nu(dw) \quad \text{(Theorem 4)} \]

\[ = \frac{1}{u_0 \nu(z)} \int K(x, w) \nu(dw) . \]

Thus

\[ \int K(x, w) \nu(dw) < \infty , \]

and by the previous implication, that (b) \(\Rightarrow\) (a), we get that \((y, x) \in D\). Thus \((x, y) \in D\) by (12.1), as required. \(\square\)

Note that the argument for (d) \(\Rightarrow\) (a) shows that if \(h\) is excessive and \(x \in \text{Acc}, z \in E\), then \(h / G(\cdot, z)\) has a fine limit at \(x\) (possibly infinite). This was first shown by Nairn [44]; she showed it more generally, for \(x \in \Delta_0\). We could obtain this more general result by combining the above argument with the argument of Theorem 4, in which upcrossings are estimated by killing \((X_t)\) at times \(L_{E(n)}\), under the probability \(x^{p^Z}\), and then reversing time.

Theorem 7 Let \(r > 0\). Let \(E\) be a bounded domain such that for every point \(x \in \partial E\), there are open balls \(B\) and \(C\), of radii greater than \(r\), both containing \(x\) in their closures, and such that \(B \subset E\), \(C \subset \mathbb{R}^d \setminus E\). Then the Martin compactification is homeomorphic to the Euclidean closure of \(E\), and moreover \(D = \{(x, y) \in \partial E \times \partial E; x \neq y\}\).
proof: The first statement is a consequence of the results of de la Vallée Poussin [11], and these results, together with Lemma 5.1 of Martin [37] suffice to show (d) of Theorem 6. □

13. A Martin Boundary in the plane

Let

\[ D' = \{(x, y); x, y \in \text{Acc} \cup E \text{ and } x \neq y\}. \]

It is not in general true that \( D = D' \). An example of a domain \( D \) for which this fails may be found in Example 1 of Martin [37]. Martin constructed a certain bounded domain \( E \) in \( \mathbb{R}^3 \), together with a subset \( E_0 \) of its Martin boundary, and showed that

\[ \lim \sup_{z \to y} K(x, z) = \infty \]

for \( x, y \in E_0 \). One can modify the parameters in his construction slightly, and obtain (as in Theorem 9(b) below), that also

\[ \lim_{w \to y} \frac{K(x, w)}{G(z, w)} = \infty \]

for \( z \in E \) and \( x, y \in E_0 \). Thus condition (d) of Theorem 6 fails, so that \( (x, y) \notin D \) for any \( x, y \in E_0 \). Modulo showing that \( E_0 \) contains at least two distinct accessible points, this gives us our counterexample. This latter property is in fact not difficult to show. One embeds \( E \) in a product \( E' = \mathbb{R} \times E'' \), where \( E'' \subset \mathbb{R}^2 \) is a bounded domain. By a result of M. Cranston and T. McConnell ([10]; Theorem 8 below), all minimal Martin boundary points of \( E'' \) are accessible.
One can therefore construct a harmonic \( h \) on \( E' \) such that the h-transform \( X_t = (Y_t, Z_t) \) has finite lifetime \( \zeta \); \((Y_t)\) is a Brownian motion up till time \( \zeta \), independent of \((Z_t)\) given \( \zeta \); and with positive probability, \( X_t \in E \) for every \( t < \zeta \), and \( X_t \) converges in \( E \) as \( t \to \zeta \), with limit in \( E_0 \). From this, one sees immediately that \( E_0 \cap \text{Acc} \) has many points.

Rather than dwelling on this, we will consider an analogous example in \( \mathbb{R}^2 \), in detail. The two dimensional situation is distinguished from that of higher dimensions, by the availability of the Riemann mapping theorem. That is, if we are given a domain \( E_1 \subset \mathbb{R}^2 \) whose complement (in the Riemann sphere) has \( n \) components (\( n \geq 1 \)), none of which is a single point, then there is a conformal equivalence \( \phi \) mapping \( E_1 \) onto a bounded domain \( E_2 \) with smooth boundaries. If \((X_t)\) is a Brownian motion started at \( x \) and killed upon leaving \( E_1 \), then we obtain a Brownian motion on \( E_2 \) by time changing \( Y_t = \phi(X_t) \) by the inverse \( A'(t) \) to the additive functional

\[
A_t = \int_0^t |\phi'(\phi^{-1}(Y_s))|^2 ds.
\]

Thus, if \( G_1 \) and \( G_2 \) are the Green functions on \( E_1 \) and \( E_2 \) respectively, then for every Borel \( f \geq 0 \),

\[
\int_{E_1} f(\phi(z))G_2(\phi(x), \phi(z)) |\phi'(z)|^2 dz = \int_{E_2} f(y)G_2(\phi(x), y) dy = E[\int_0^\infty f(Y_{A'(t)}) dt]
\]
That is, \( G_2(\phi(x), \phi(z)) = G_1(x, z) \). Thus, \( \phi \) will also preserve the Martin function and hence extends to a homeomorphism of the Martin compactifications of \( E_1 \) and \( E_2 \), taking minimal points to minimal points. There is of course no reason why \( \phi \) should behave so nicely on the Euclidean boundary of \( E_1 \). In our case, all the Martin boundary points of \( E_2 \) are accessible (and minimal) by Theorem 7, so that by the criteria of Theorem 6, it follows that \( D = D' \). In dimension \( \geq 3 \), we could obtain pathological domains homeomorphic to the unit ball, but this shows that to do the same in dimension 2, we must consider domains of infinite connectivity.

Unlike the case of minimality, the accessibility or inaccessibility of a Martin boundary point will not in general be preserved under a conformal map. This problem is resolved by the following result of Cranston and McConnell [10].

**Theorem 8** There exists a constant \( C \) such that for each domain \( E \) in \( \mathbb{R}^2 \) and each excessive function \( h \) on \( E \), we have that

\[
h \leq C \cdot \text{Area of } E, \text{ for } x \in E. \quad \Box
\]

Thus, all minimal Martin boundary points of bounded domains in \( \mathbb{R}^2 \), are accessible.

We will now construct a domain \( E \subset \mathbb{R}^2 \) for which \( D \neq D' \). Let

\[
= E[ \int_0^\infty f(\phi(X_t)) | \phi'(X_t) |^2 dt] = \int_{E_1} f(\phi(z)) G_1(x, z) | \phi'(z) |^2 dz.
\]
a(n), b(n) ∈ (0, 1), n = 1, 2, ... . Set s(0) = 0 and
\[ s(n) = \sum_{i=1}^{n} b(n), \quad n \geq 1. \]

Assume that a(n) and b(n) are decreasing in n, and that s(∞) = 1.

Let
\[ E = (-1, 1) \times (0, 1) \setminus \bigcup_{n=1}^{\infty} [a(n) - 1, 1-a(n)] \times \{s(n)\}. \]

We will show that for every fixed sequence (b(n)) as above, the a(n) may be chosen to converge to zero sufficiently fast that D ⊄ D'.

Let \( c(n) = (0, (s(n) + s(n-1))/2), \ n \geq 1 \)

\[ A(n) = \begin{cases} 
(1 - a(n), 1) \times \{s(n)\}, & \text{if } n \geq 1 \\
(-1, a(n) - 1) \times \{s(n)\}, & \text{if } n \leq -1 
\end{cases} \]

\[ A^+ = \bigcup_{n \geq 1} A(n), \quad A^- = \bigcup_{n \leq -1} A(n), \]

\[ A = A^+ \cup A^-, \text{ and } B(n) = A(n) \cup A(-n). \]

Let \( x_0 \) be a limit point in \( E \) of the sequence (c(n)). Define
the Martin function relative to the point \( c(1) \), so that \( h = \mathcal{K}(x_0, \cdot) \) is symmetric on \( E \) (that is, \( h(-z) = h(z) \); as the reader will have gathered, we are allowing ourselves to pass freely between notations used for \( \mathbb{R}^2 \) and for \( \mathbb{C} \)). We will write \( P^X \) for the law of Brownian motion, started at \( x \in E \) and killed upon contact with \( \partial E \). For ease of notation, we will write \( Q \) for the probability \( h^P_c(1) \).

Martin's argument will give us the first part of the following.

**Theorem 9** Suppose that

\[
\lim \sup_{n \to \infty} b(n) \log(1/a(n-1)) > 4\pi.
\]

Then:

(a) \( h(c(n)) \to \infty \) as \( n \to \infty \); 
(b) Let \( z \in E \). Then \( h(w)/G(z, w) \to \infty \) as \( \text{Im}(w) \to 1 \).

The only reason this result does not immediately give us that \( D \neq D' \), is that it is conceivable that the speed at which \( a(n) \to 0 \) forces that part \([-1, 1] \times \{1\} \) of the Euclidean boundary of \( E \), to collapse to a single point of the Martin boundary. If we could show that \( h \) was not minimal, then there would have to be at least two minimal Martin boundary points \( x_1 \) and \( x_2 \), such that \( w \to x_1 \) implies that \( \text{Im}(w) \to 1 \), \( i = 1, 2 \). In this case, (b) of Theorem 9 together with criterion (d) of Theorem 6 would show that \( D \neq D' \). Our main result will be:

**Theorem 10** Suppose that \( a(n-1)/b(n) \to 0 \). Then

\[ Q(T_{A-} = \infty) > 0 \]
By symmetry, also \( Q(T_{A^+} = \infty) > 0 \), so that also \( Q(T_{A(n)} < \infty \) for infinitely many \( n \geq 1 \) \( \in (0, 1) \), and hence we conclude from (h) of Theorem 1, that \( h \) cannot be minimal. Thus we have

**Corollary 1** Suppose that

\[
\limsup_{n \to \infty} b(n) \log(1/a(n-1)) > 4\pi.
\]

Then \( D \neq D' \).

Now to work. A common feature of many of the arguments is that we obtain estimates using the exact form of certain Schwarz-Christoffel transformations. To do so, we will extensively use elliptic integrals, for which we refer to Gradshteyn and Ryzhik [26]. The notation (GR8.1133) will refer to the appropriate formula in [26]. For ease of use, we will use a nonstandard notation, writing

\[
F(a, k) = \int_0^\alpha \frac{dx}{\sqrt{(1-x^2)(1-kx^2)}} , \quad a, k \in [0, 1].
\]

For now, we will only recall that (GR8.1133)

\[
(13.1) \quad F(1, 0) = \pi/2 , \quad F(1, k) \sim \log(1/\sqrt{1-k}) \quad \text{as } k \uparrow 1 .
\]

**proof of Theorem 9**: Fix \( n \), and consider \( E' = E \cap (-1, 1) \times (0, s(n)) \). For \( x \in E' \), let \( P'^X \) be the law of Brownian motion started at \( x \) and stopped upon leaving \( E' \). Also, let

\[
W(n) = (-1, 1) \times (s(n-1), s(n)) \subset E' ,
\]

\[
C(n) = (-1, 1) \times \{(s(n) + s(n-1))/2\} , \quad \text{and}
\]

\[
\mu(x, dy) = P'^X(X_{T_{W(n)}} \in dy) .
\]
Let $\phi$ be the Schwarz-Christoffel transformation mapping the unit disk to $W(n)$ and taking the points $-1$, $0$, and $1$, to $(-1, (s(n) + s(n-1))/2)$, $c(n)$, and $(1, (s(n) + s(n-1))/2)$. Since composing $\phi$ with a Brownian motion on the disk produces a time changed Brownian motion on $W(n)$, it follows that $\phi$ preserves hitting distributions. Thus, by the corresponding result for the disk, we see that for each $x \in W(n)$, there exists a density $u(x, y)$, continuous in $y$, for $\mu(x, dy)$ with respect to $\mu(c(n), dy)$, and that further, there exists a constant $k(n)$ with

$$\frac{u(x, y)}{u(x, y')} \leq k(n) \text{ for } x \in C(n) \text{ and } y, y' \in (-1, 1) \times \{s(n)\}.$$ 

Let

$$m(n, w) = k(n)^2/\mu(w, (-1, 1) \times \{s(n)\}).$$

Thus, for $x, w \in C(n)$ and $y \in (-1, 1) \times \{s(n)\}$, we have that

$$\mu(x, dy) = \frac{u(x, y)}{u(w, y)} \mu(w, dy) \leq \frac{u(x, y)}{u(w, y)} \mu(w, dy) \leq \frac{u(x, y')}{u(w, y')} \mu(w, dy') \inf\{\frac{u(x, y')}{u(w, y')}; y' \in (-1, 1) \times \{s(n)\}\} \mu(w, (-1, 1) \times \{s(n)\}) \leq m(n, w) \mu(x, (-1, 1) \times \{s(n)\}) \mu(w, dy).$$

Let $R(0) = 0$, and

$$R(i+1) = R(i) + (T_{B(n-1)} + T_{C(n)} \circ \theta_{T_{B(n-1)}(R(i))} \circ \theta_{R(i)}) \circ \theta_{R(i)}, i \geq 0.$$ Then for $x, w \in C(n)$ and $y \in (-1, 1) \times \{s(n)\}$,
\[
\begin{align*}
P'_{X_{T}} & \in \partial E' \\
& = \sum_{i=0}^{\infty} E^X_{R(i)}[R(i) < \infty, \ P'_{X_{R(i)}}(R(1) = \infty, \ X_{T} \not\in \partial E')] \\
& = \sum_{i=0}^{\infty} E^X_{R(i)}[R(i) < \infty, \ \mu(X_{R(i)}, \ dy)] \\
& \leq \sum_{i=0}^{\infty} m(n, w) \mu(w, dy) \ E^X_{R(i)}[R(i) < \infty, \ \mu(X_{R(i)}, (-1,1) \times \{s(n)\})] \\
& = m(n, w) \mu(w, dy) \ P'_{X_{T}}(X_{T} \not\in (-1,1) \times \{s(n)\}) \\
& \leq m(n, w) \mu(w, dy).
\end{align*}
\]

Also, one easily shows by the appropriate Schwarz-Christoffel transformation, that there is a constant \( M \) such that for every \( k \), the first hitting distribution of Brownian motion started at \( c(l) \), on the boundary of \((-1,1) \times (0, s(k))\), has a density with respect to the arc length measure, which is bounded by \( M \).

Now let \( h \) be the function of the theorem. Then for \( w \in C(n) \),
\[
\begin{align*}
h(c(l)) &= \int \int h(y) P'X_{T}(T_{B(n)} < \infty, \ X_{T_{B(n)}} \in \partial E', \ dy) \\
& \quad \cdot P'c(l)(T_{C(n)} < \infty, \ X_{T_{C(n)}} \in \partial E', \ dx) \\
& \leq P'\xi(1)(T_{B(n-1)} < \infty) m(n, w) \int h(y) \mu(w, dy) \\
& \leq 2 M a(n-1)m(n, w) \int h(y) P'w(X_{T} \not\in \partial E', \ dy) \\
& = 2 M a(n-1) m(n, w) h(w).
\end{align*}
\]

Thus, provided \( a(n-1)m(n, c(n)) \to 0 \) as \( n \to \infty \), it follows that
h(c(n)) \to \infty.

Now, consider G(z, \cdot). Let z belong to the closure of W(j-1), and suppose that the n used above satisfies n > j. G(z, x) is bounded on B(j) (say by M'), and converges to zero as x approaches \partial E. Thus, since

\begin{align*}
\mu(w, (-1, 1) \times \{s(n)\}) &= \mu(w, (-1, 1) \times \{s(n-1)\}) \\
&\geq p^w(T_{B(j)} < T_{\partial E}) \quad \text{for } w \in C(n),
\end{align*}

also

G(z, w) \leq M' \mu(w, (-1, 1) \times \{s(n)\}) \quad \text{for } w \in C(n).

Combining our two estimates, we see that

\begin{align*}
\frac{h(w)}{G(z,w)} &\geq \frac{h(c(l))}{2 MM' a(n-1)m(n,w)\mu(w,(-1,1) \times \{s(n)\})} \\
&= \frac{h(c(l))}{2 MM'} \cdot \frac{1}{a(n-1)k(n)^2}, \quad \text{for } w \in C(n).
\end{align*}

Let i(n) be the expression on the right hand side. Then if a(n) \to 0 sufficiently fast, also i(n) \to \infty.

Let E(n) = (-1, 1) \times ((s(n-1) + s(n))/2, (s(n) + s(n+1))/2) \cap E. Then E(n) is regular for the Dirichlet problem, so that if \nu(x, dy) is the hitting distribution of \partial E(n) by Brownian motion started at x, then whenever g is harmonic on E(n), with a continuous extension to its closure, we have that

\begin{align*}
g(x) = \int_{\partial E(n)} g(y) \nu(x, dy).
\end{align*}
These conditions hold for both $h$ and $G(z, \cdot)$. Both functions vanish on $\partial E(n) \cap \partial E$, so that we have
\[ h \geq i(n) \; G(z, \cdot) \quad \text{on} \quad \partial E(n). \]

By the integral representation, this inequality holds throughout $E(n)$.

We have therefore shown that (a) and (b) hold, provided that both $m(n, c(n))a(n-1)$ and $k(n)^2 a(n-1)$ converge to zero. Since $\mu(c(n), (-1, 1) \times \{s(n)\}) \to 1/2$, the first of these is irrelevant, and hence the exact statement of the theorem will follow from the following estimate of $k(n)$. Theorem 9 will then be proven.

Lemma 3 \( b(n) \log(k(n)) \to 2\pi \) as $n \to \infty$.

proof: Write $b$ for $b(n)$. Let $\phi$ be the conformal map of the unit disk onto $(-1, 1) \times (0, b)$, mapping the points $-1, 0, 1$, to $(-1, b/2), (0, b/2), (1, b/2)$. Let
\[ e^{i\theta} = \phi^{-1}(1, b); \quad \theta \in (0, \pi/2), \]
and write $dy$ for normalized arc length, on the boundary of the unit disk. The first hitting distribution of this boundary, by Brownian motion started at $x$ is then
\[ \frac{1 - |x|^2}{|x - y|^2} dy, \]
so that since $\phi$ preserves hitting distributions, we have that
\[ k(n) = \sup \left\{ \frac{1 - |x|^2}{|x - e^{it}|} / \frac{1 - |x|^2}{|x - e^{is}|} ; \right. \]

\[ \left. \text{IM}(x) = 0, \ s, t \in [\theta, \pi - \theta] \right\}. \]
By simple calculus, this becomes

\[ k(n) = \frac{1 + \cos \theta}{1 - \cos \theta} \, . \]

We must therefore examine the dependence of \( \theta \) on \( b \).

\[ \frac{b}{2} = \int_{-\theta}^{\theta} \frac{d}{dt} |\phi(e^{it})|dt / \int_{\theta}^{\pi-\theta} \frac{d}{dt} |\phi(e^{it})|dt \, , \]

and

\[ \phi(w) = \phi \int_{0}^{w} (z - e^{i\theta})(z + e^{i\theta})(z - e^{-i\theta})(z + e^{-i\theta})^{-1/2} dz + \phi' \]

for some constants \( \phi \) and \( \phi' \). Thus

\[
\left| \frac{d}{dt} \phi(e^{it}) \right| \\
= \left| \phi e^{it} [(e^{it} - e^{i\theta})(e^{it} + e^{i\theta})(e^{it} - e^{-i\theta})(e^{it} + e^{-i\theta})]^{-1/2} \right| \\
= 2 |\phi||\cos 2t - \cos 2\theta|^{-1/2} \, .
\]

\[
\int_{-\theta}^{\theta} (\cos 2t - \cos 2\theta)^{-1/2} dt = 2 \int_{0}^{\theta} (\cos 2t - \cos 2\theta)^{-1/2} dt \\
= \sqrt{2} \int_{\cos \theta}^{1} \frac{dx}{\sqrt{(1-x^2)(x^2-\cos^2 \theta)}} \\
= \sqrt{2} \, F(1, \sin^2 \theta) \quad \text{(GR 3.1529)} \, ,
\]

and similarly

\[
\int_{\theta}^{\pi-\theta} (\cos 2\theta - \cos 2t)^{-1/2} dt = \sqrt{2} \int_{0}^{\cos \theta} \frac{dx}{\sqrt{(1-x^2)(\cos^2 \theta-x^2)}} \\
= \sqrt{2} \, F(1, \cos^2 \theta) \quad \text{(GR 3.1527)}.
\]
Thus by (13.1), as $\theta \to 0$ we have

$$b \sim \frac{\prod}{\log(1/\theta)}.$$

Since $k(n) \sim (2/\theta)^2$, we are done. □

The proof of Theorem 10 will be given along the following lines: Because we know little about $h$ (other than its 'symmetry'), we will reduce the problem to that of estimating certain quantities not depending on $h$. (An essential ingredient in this is Theorem 8, which estimates the expected lifetime of an $h$-transform, but in a manner independent of $h$.) These quantities are local, in that they involve potential theoretic properties of certain subsets of $E$. Schwarz-Christoffel transformations are used to relate potential theoretic questions about these subsets, to questions about the unit disk, and questions about the parameters describing the transformation. The answers to questions of the first type may be easily obtained, whereas the answers to the second type form the main non-probabilistic ingredient of our proof.

Before proceeding with the proof of Theorem 10, we will fix some notation. Let $S(0) = T(0) = T_A$. For $k \geq 0$, define

$$N(k) = n, \text{ if } X_{T(k)} \in A(n)$$

$$T(k+1) = T(k) + T_A \setminus B(N(k)) \circ \theta T(k)$$

$$S(k+1) = T(k) + T_A \setminus A(N(k)) \circ \theta T(k).$$

Let $U(n) = E \cap (-1, 1) \times (s(n-1), s(n+1))$, 

$$U(n) = E \cap (-1, 1) \times (s(n-1), s(n+1)),$$
and write $P^x_n$ for the law of Brownian motion started at $x$ and stopped upon leaving $U(n)$. Define the Martin function $K_n(x,y)$ on $U(n)$, relative to the base point $c(n-1)$, so that $K_n$ is symmetric (that is, $K_n(-x,-y) = K_n(x,y)$).

As usual, $h_n^x$ and $y_n^x$ will denote the laws of the transformations by the functions
$$h \mid_{U(n)}, K_n(y, \cdot)$$
respectively, but we will also introduce the notation $y_n^x$ for the transformation by the function
$$K_n(y, \cdot) + K_n(-y, \cdot).$$

Note that this object would be changed if we had used a base point other than $c(n)$. Set
$$\alpha(x,y) = \begin{cases} P^x_n (\text{Sgn } N(1) = \text{Sgn } N(0), S(1) = T(1)) , & \text{if } x \in B(n) \text{ and } y \in B(n-1) \cup B(n+1) \\ 0, & \text{otherwise} \end{cases}$$
for some $n$.

$$\beta(n,m) = \inf\{\alpha(x,y); x \in B(n), y \in B(m)\}; \ n, m \geq 1$$
\[ \gamma(x, y) = \begin{cases} \mathbb{E}^x_T [1] & \text{if } x \in B(n) \text{ and } y \in B(n-1) \cup B(n+1) \\ 0 & \text{for some } n \leq N, \text{ otherwise} \end{cases} \]

\[ \lambda(n, m) = \inf \{ \gamma(x, y) ; x \in B(n), y \in B(m) \} ; n, m \geq 1 . \]

Note that \( B(n, m) = \lambda(n, m) = 0 \) unless \( |m - n| = 1 \). The estimates of these quantities that we will need are contained in the following result, whose proof we defer until the end of this section.

**Lemma 4** Let \( a(n-l)/b(n) \rightarrow 0 \). Then there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for every \( n \geq 1 \),

\( a(n-l)/b(n) \rightarrow 0 \). Then there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for every \( n \geq 1 \),

(a) \( (1 - \beta(n, n+1)) \lor (1 - \beta(n, n-1)) \leq \exp(-C_1/b(n)) \)

(b) \( \lambda(n, n+1) \lor \lambda(n, n-1) \geq C_2 \beta(n)^2 \).

Now, let \( I \) be the function that replaces the \( x \)-coordinate by its absolute value. That is,

\[ I(z) = \begin{cases} z, \Re(z) \geq 0 \\ -z, \Re(z) < 0 . \end{cases} \]

Recall that \( F_t = \sigma(X_s; s \leq t) \). We let

\[ G = \sigma(I(X_T(k)) ; k \geq 0) . \]

**Lemma 5**

(a) Let \( H(x, y) \) be positive and jointly measurable in \( x \) and \( y \).
Let $T$ and $S$ be $(\mathcal{F}_{t+})$ stopping times, and set 

$$H'(x) = h E^x_\cdot [H(x, X_T)].$$

Then 

$$E_Q[H(X_S, X_{S+T}) | F_{S+}] = H'(X_S) \quad Q\text{-a.s.}.$$ 

(b) $I(X_t)$ is strong Markov with respect to $Q$ and $(\mathcal{F}_t)$. 

(c) Let $Z \in \mathcal{F}$ be positive, and set 

$$H(x, y) = E^x_\cdot Z, \quad \text{for } x \in B(n), \ y \in B(n-1) \cup B(n+1).$$

Then $E^x_{\cdot n}[Z | I(X_{T(1)})] = H(x, I(X_{T(1)}))$ a.s.

(d) $Q(\text{Sgn } N(n) = \text{Sgn } N(n-1), T(n) = S(n) | F_{T(n-1)+} \vee G)$

$$= \alpha(X_{T(n-1)}, I(X_{T(n)})) \ a.s., \text{ for every } n \geq 1.$$ 

proof: (a): If $H_n \uparrow H$, then $H_n(x, \cdot) \uparrow H(x, \cdot)$, so that $H_n' \uparrow H'$. 

The class of functions $H$ for which (a) holds is therefore a monotone class, and it contains functions $H(x, y) = H_1(x) H_2(y)$ by the strong Markov property.

(b): By symmetry of $E$ and of Brownian motion, we have that 

$$P^x(I(X.) \in A) = P^I(x)(I(X.) \in A)$$

for every $x \in E$ and $A \in \mathcal{F}$. We have chosen $h$ so that 

$h(x) = h(I(x))$ for every $x$. Thus, if $f_1, \ldots, f_n \in E$ are positive, 

$T$ is an $(\mathcal{F}_{t+})$ stopping time, and $t_1 < \ldots < t_n$, then
\[ E_Q[\Pi fj(I(X_{T+t1}^1)) | F_{T+1}^T] \]
\[ = \frac{1}{h(I(X_{T1}^1))} E_{X_{T1}^1} \left[ h(I(X_{T_n}^n)) \Pi fj(I(X_{T1}^1)) \right] \]
\[ = \frac{1}{h(I(X_{T1}^1))} E_{X_{T1}^1} \left[ h(I(X_{T_n}^n)) \Pi fj(I(X_{T1}^1)) \right] , \]

so that \( I(X_{T1}^1) \) is strong Markov, with semigroup

\[ h^P_t(x, dy) + h^P_t(x, -dy) . \]

(c): Let the integral representation of \( h \) on \( U(n) \) be

\[ h = \int K_n(y, \cdot) \mu(dy) . \]

By uniqueness of the representation, and the fact that \( h \) and \( K_n \) are preserved by the transformation \( x \rightarrow -x \) of \( U(n) \), it follows that \( \mu(dy) = \mu(-dy) \). Thus, for \( A \subset A(n+1) \cup A(n-1) \) and \( Z \in F \) positive,

\[ E_n^X[Z, I(X_{T(1)}^1) \in A] = E_n^X[Z, X_{T(1)}^1 \in A \cup -A] \]
\[ = \frac{1}{h(x)} \int_{A \cup -A} K_n(x, y) E_n^X[Z] \mu(dy) \]
\[ = \frac{1}{h(x)} \int_{A} (K_n(x, y) E_n^X[Z] + K_n(x, -y) -y E_n^X[Z]) \mu(dy) \]
\[ = \frac{1}{h(x)} \int_{A} (K_n(x, y) + K_n(x, -y)) E_n^X[Z] \mu(dy) \]
\[ = \frac{1}{h(x)} \int_{A \cup -A} K_n(x, y) E_n^X[Z] \mu(dy) \]
(by symmetry of $u$)

$$= E_n^X[H(I(X_{T(1)})), I(X_{T(1)}) \in \Lambda]$$

as required.

(d): First, observe that if $Z \in F_{T(n)}$, then

$$E_Q[Z | F_{T(n-1)+ \vee G}] = E_Q[Z | F_{T(n-1)+ \vee \sigma(I(X_{T(n)}))}],$$

by part (b). Now let $f \in E$ be positive, and set $H(x, y) = f(I(y)) \alpha(x, y)$. Also put $H'(x) = h^X[H(x, X_{T(1)})]$, and

$$Z = 1\{\text{Sgn } N(1) = \text{Sgn } N(0), S(0) = T(0)\}.$$

By (c), we have that

$$H'(x) = h^X[f(I(X_{T(1)})) \alpha(x, I(X_{T(1)}))]$$

$$= h^X[f(I(X_{T(1)}))] Z.$$

Thus, if also $\Lambda \in F_{T(n-1)+ \circ}$, then

$$E_Q[\Lambda, f(I(X_{T(n)})), \text{Sgn } N(n) = \text{Sgn } N(n-1), T(n) = S(n)]$$

$$= E_Q[\Lambda, f(I(X_{T(n)}) \circ \theta_{T(n-1)}]$$

$$= E_Q[\Lambda, h^X_{T(n-1)}[f(I(X_{T(1)}))] Z]$$

$$= E_Q[\Lambda, H'(X_{T(n-1)})]$$

$$= E_Q[\Lambda, H(X_{T(n-1)}, X_{T(n)})] \quad \text{(by (a))}$$

$$= E_Q[\Lambda, f(I(X_{T(n)})) \alpha(X_{T(n-1)}), I(X_{T(n)})]$$
as required. □

**proof of Theorem 10:**

For \( n \geq 1 \), and \( m = n + 1 \) or \( n - 1 \), let \( N(n, m) \) be the number of \( k \geq 0 \) such that \( |N(k)| = n \) and \( |N(k+1)| = m \). Let 
\[ H(x, y) = \alpha(x, y) , \]
and apply (a) of Lemma 5. We have that 
\[ H'(x) = h \mathbb{E}_x [\gamma(x, X_{T(1)})] = h \mathbb{E}_x [T(1)] \]
(by (f) of Theorem 1), so that 
\[ \mathbb{E}_Q [T(k+1) - T(k)] = \mathbb{E}_Q [H'(X_{T(k)})] \\
= \mathbb{E}_Q [\gamma(X_{T(k)}, X_{T(k+1)})] \\
\geq \mathbb{E}_Q [\lambda(|N(k)|, |N(k+1)|)] . \]

By Theorem 8, we therefore have that 
\[ \sum_{k=0}^{\infty} \mathbb{E}_Q [T(k+1) - T(k)] + \mathbb{E}_Q [T(0)] \\
\geq \sum_{n=1}^{\infty} [\lambda(n, n+1) \mathbb{E}_Q [N(n, n+1)] + \\
\quad + \lambda(n, n-1) \mathbb{E}_Q [N(n, n-1)]] . \]

Since \( \log(1/(1-x)) \leq 2x \) for \( x \geq 0 \) small, we have by Lemma 4, that there exists a constant \( C \) with 
\[ \log(1/\beta(n, m) \leq C \lambda(n, m) \] for \( n \geq 1 \), \( m = n + 1 \).

Thus also
\[
\sum_{n=1}^{\infty} [\log(1/\beta(n, n+1))E_Q[N(n, n+1)] + \\
+ \log(1/\beta(n, n-1))E_Q[N(n, n-1)]]
\]

\[
= E_Q[\log \left( \prod_{n=1}^{\infty} \frac{\beta(n, n+1)N(n, n+1)\beta(n, n-1)N(n, n-1)}{\beta(n, n-l)N(n, n-l)} \right)].
\]

The integrand is therefore finite almost surely, so that also

\[
E_Q[N(0) = 1, \prod_{k=0}^{\infty} \beta(|N(k)|, |N(k+1)|)]
\]

\[
= E_Q[N(0) = 1, \prod_{n=1}^{\infty} \beta(n, n+1)N(n, n+1)\beta(n, n-1)N(n, n-1)]
\]

\[
> 0 .
\]

But by (d) of Lemma 5,

\[
E_Q[N(0) = 1, \prod_{k=0}^{j} \beta(|N(k)|, |N(k+1)|)]
\]

\[
\leq E_Q[N(0) = 1, E_Q[Sgn N(1) = Sgn N(0), S(1) = T(1)E_Q[... \\
... E_Q[Sgn N(j) = Sgn N(j-1), S(0) = T(0)|F_{T(j-1)} + G]|F_{T(0)} + G]]
\]

\[
= Q(N(0) = 1, Sgn N(k) = Sgn N(k-1) \text{ and } \\
S(k, k = 1, \ldots, j) = Q(T_{A-} > T(j))
\]

Letting \( j \to \infty \), we see that \( Q(T_{A-} = \infty) > 0 \), as required. \( \Box \)

Before proving Lemma 4, we will examine a slightly different situation; Consider a Schwartz-Christoffel transformation
\[ \Phi(w) = \phi \int_{0}^{w} \left( z - e^{i \theta} \right)^{6} \prod_{j=1}^{6} \frac{1}{(z - e^{i \theta}) - 1/2} \, dz + \phi' \],

where \( \phi \) and \( \phi' \) are constants, and \( \theta_1 < \theta_6 < \theta_5 < \theta_4 < \theta_3 < \theta_2 \) < \( \theta_1 < \theta_0 = \theta_7 + 2\pi \). Note that the effect of choosing different branches of the square root function, is to change the constant \( \phi \).

Then \( \phi \) maps the unit disk \( 0 \), to a region \( V \) with straight sides;

![Diagram](image)

**Fig. 6**

For every such configuration of points \( x_0 \ldots x_6 \), there will exist such a map \( \Phi \), which is unique up to a Möbius transformation of the unit disk. Since such transformations are uniquely specified by the images of three distinct boundary points, we may always find a \( \Phi \) with \( \theta_0, \theta_2, \) and \( \theta_5 \) taking on some fixed values. We will take the image of \( \Phi \) to be

\[ V = \left[ (-1, 1) \times (0, b+b') \right] \setminus \left[ (-1, 1-a) \times \{b'\} \right]. \]
and will consider \( \psi \) and \( \psi' \) such that

\[
\Phi(e^{i\psi}) = \Phi(e^{i\psi'}) = (a - 1, b') , \; \psi \in (\theta_1, \theta_0), \; \psi' \in (\theta_1, \theta_6) .
\]

We will also let \( L \) be the Martin function on \( V \), defined with base point \( \Phi(0) \). Thus,

\[
L(\Phi(x), \Phi(y)) = (1 - |x|^2)/|x - y|^2 .
\]

**Lemma 6** Suppose that \( b' \to 0 \) and \( a/b \to 0 \) in such a way that \( b \leq b' \). Then

(a) \( \psi, \theta_1, \theta_3 \to \theta_2 \) and \( \psi', \theta_4, \theta_6 \to \theta_5 \).

(b) \( \lim \inf b . \log (1/(\psi - \theta_1)) \geq 2\pi \)

\[
\lim \inf b' . \log (1/(\theta_6 - \psi')) \geq 2\pi
\]

(c) \( \sup \{L(x', \Phi(e^{i\theta}))/(L(x, \Phi(e^{i\theta})) \wedge 1) ; \; x \in (1 - a, 1) \times \{b'\} , \; \text{and} \; t, s \in [\theta_3, \psi] \cup [\psi', \theta_4]\} \) remains bounded.

**proof:** (a): Let the distances from \( e^0 \) to \( e^2 \) and \( e^5 \) be at least \( 2\varepsilon \), and let \( O_x \) be the intersection of the unit disk \( O \) with a disk of radius \( \varepsilon \) and centre at \( e^{i\theta_0} \). Then there is a constant \( C > 0 \) (we will use the notation \( C \) for any constant we do not wish to keep track of) such that no matter what \( a, b \) and \( b' \) are, we have that

\[
|\Phi'(z)| \geq C|\Phi| \quad \text{for} \; z \in O \setminus O_x .
\]

Now let \( g_0 : [0, r] \to 0 \) be the unit speed curve whose image under \( \Phi \) is \([1 - a, 1] \times \{b'\}\). Since
Arg\left(\frac{d}{dt} \phi(g_0(t))\right)

is constant, we obtain that

\begin{align*}
a &= \left| \phi(g_0(r)) - \phi(g_0(0)) \right| = \left| \int_0^r \frac{d}{dt} \phi(g_0(t)) dt \right| \\
&= \int_0^r \left| \frac{d}{dt} \phi(g_0(t)) \right| dt \\
&> \varepsilon C \phi \\
\end{align*}

Let

\begin{align*}
|\phi| f(t) &= \left| \frac{d}{dt} \phi(e^{it}) \right| .
\end{align*}

Then

\begin{align*}
b &= 2 \int_{\theta_2}^{\theta_1} f(t) dt / \int_{\theta_3}^{\theta_2} f(t) dt \\
b' &= 2 \int_{\theta_6}^{\theta_5} f(t) dt / \int_{\theta_3}^{\theta_2} f(t) dt \\
\int_{\theta_3}^{\theta_2} f(t) dt &= \int_{\theta_5}^{\theta_4} f(t) dt \\
2 - a &= 2 \int_{\theta_1}^{\theta_0} f(t) dt / \int_{\theta_3}^{\theta_2} f(t) dt .
\end{align*}

Since

\begin{align*}
\frac{a}{b} &= \frac{\left| \phi(g(r)) - \phi(g(0)) \right|}{\int_{\theta_2}^{\theta_1} f(t) dt} > \varepsilon C / \int_{\theta_2}^{\theta_1} f(t) dt , \\
|\phi| \int_{\theta_2}^{\theta_1} f(t) dt
\end{align*}

we must have that
Thus, as $b \to 0$ and $2 - a \to 2$, also
\[
\int_{\theta_2}^{\theta_1} f(t) \, dt \text{ and } \int_{\theta_3}^{\theta_2} f(t) \, dt \to \infty.
\]
Since
\[
e^{it} - e^{i\theta} = 2i \cos((t+\theta)/2) \sin((t-\theta)/2),
\]
we have that
\[
f(t) = \frac{1}{2} \left| \sin \left( \frac{t-\theta}{2} \right) \right| \prod_{j=1}^{\theta_1} \sin \left( \frac{t-j}{2} \right).^{-1/2}
\]
Thus, since $\sin((\theta_0-t)/2) \leq \sin((\theta_2-t)/2)$ whenever $t \in [\theta_1, \theta_0]$, we have for each $\eta > 0$, a constant $C$ such that
\[
\int_{\theta_1}^{\theta_0} f(t) \, dt \leq C \int_{\theta_1}^{\theta_0} (t-\theta_1)^{-1/2} \, dt
\]
whenever $\theta_1 \in [\theta_2 + \eta, \theta_0]$. Since the right hand side is bounded as a function of $\theta_1$, we conclude that $\theta_1 \to \theta_2$.

Similarly, for each $\eta > 0$ there is a constant $C$ such that
\[
\int_{\theta_2}^{\theta_1} f(t) \, dt \leq C \int_{\theta_2}^{\theta_1} [(t-\theta_2)(\theta_1-t)]^{-1/2} \, dt
\]
\[
= C \int_{0}^{\theta_1} [t(1-t)]^{-1/2} \, dt < \infty
\]
whenever $\theta_3 \in [\theta_5, \theta_2 - \eta]$. Thus $\theta_3 \to \theta_2$. We have likewise, that
\( \theta_6, \theta_4 \to \theta_3 \).

It follows from the above estimates, that for \( \eta > 0 \) fixed,
\[
\int_{\theta_1 + \eta}^{\theta_0} f(t) dt
\]
remains bounded, so that
\[
\int_{\theta_1}^{\theta_1 + \eta} f(t) dt / \int_{\theta_1}^{\theta_2} f(t) dt + 1.
\]

Since
\[
\int_{\theta_1}^{\psi} f(t) dt / \int_{\theta_1}^{\theta_2} f(t) dt = a/2,
\]
we see that \( \psi \to \theta_2 \). Similarly, \( \psi' \to \theta_5 \).

(b): Let
\[
C = \frac{1}{2} \sin(\frac{\theta_0 - \theta_2}{2}) \sin^{-3/2}(\frac{\theta_2 - \theta_5}{2}),
\]
\[
k = (\theta_2 - \theta_3) / (\theta_1 - \theta_3),
\]
\[
\rho = [(\psi - \theta_1) / (\psi - \theta_5)]^{1/2}.
\]

By part (a), we have immediately that
\[
\int_{\theta_3}^{\theta_2} f(t) dt \sim C \int_{\theta_3}^{\theta_2} [(\theta_1 - t)(\theta_2 - t)(t - \theta_3)]^{-1/2} dt
\]
\[= 2C(\theta_1 - \theta_3)^{-1/2} F(1, k) \quad \text{(GR 3.1313)}
\]
\[
\int_{\theta_2}^{\theta_1} f(t) dt \sim C \int_{\theta_2}^{\theta_1} [(\theta_1 - t)(t - \theta_2)(t - \theta_3)]^{-1/2} dt
\]
\[
\int_{\theta_1}^{\psi} f(t) \, dt \sim C \int_{\theta_1}^{\psi} [(t-\theta_1)(t-\theta_2)(t-\theta_3)]^{-1/2} \, dt
\]

\[= 2C(\theta_1-\theta_3)^{-1/2}F(\rho, k) \, .\]

Thus \( b \sim 2F(1, 1-k)/F(1, k) \), so that \( k \to 1 \), and by (13.1), this becomes that \( b \sim 2\pi/\log((\theta_1-\theta_3)/(\theta_1-\theta_2)) \). Similarly,

\[F(\rho, k)/F(1, 1-k) \sim \int_{\theta_1}^{\psi} f(t) \, dt / \int_{\theta_2}^{\psi} f(t) \, dt \]

\[= a/b \to 0 \, ,\]

so that \( \rho \to 0 \). Thus also \( (\psi-\theta_1)/(\theta_1-\theta_2) \to 0 \), and hence

\[b \log(1/(\psi-\theta_1)) \]

\[\geq b[16\log(1/(\psi-\theta_1)) + \log((\psi-\theta_1)/(\theta_1-\theta_2)) + \]

\[+ \log(\theta_1-\theta_3)] \]

\[\to 2\pi \, .\]

The second half of (b) follows similarly.

(c): Recall the path \( t \to g_0(t) \) of part (a). We had that

\[\int_{0}^{\tau} \phi(g_0(t)) \, dt = \int_{0}^{\tau} |\phi'(g_0(t))| \, dt \, ,\]

from which it follows that \( g_0 \) minimizes
\[ \int_0^{t'} |\phi'(g(t))| \, dt \]

over all \( r' > 0 \), and all unit speed curves \( g : [0, r'] \to 0 \) such that \( g(0) = e^{i \theta_0} \), and \( g(r') = e^{i t} \) for some \( t \in (\theta_4, \theta_3) \).

In particular,

\[ \int_0^r |\phi'(g_0(t))| \, dt \]

remains bounded.

Suppose now that

\[ (13.2) \quad \lim \inf \left[ \inf \{|g(t) - e^{i \theta_2}|; t \in [0, r]\} \right] = 0 . \]

There are constants \( C \) and \( \eta_0 \) such that whenever \( \varepsilon \in (0, \eta_0) \), and \( b' \) and \( a/b \) are so small that

\[ |e^{i \theta_1} - e^{i \theta_2}|, \quad |e^{i \theta_3} - e^{i \theta_2}| < \varepsilon , \]

then

\[ |\phi'(z)| \geq C(|z - e^{i \theta_2}| + 2\varepsilon)^{-3/2} \]

for \( z \in \Omega \) with

\[ |z - e^{i \theta_2}| \in (\varepsilon, \eta_0) . \]

If

\[ |g(t) - e^{i \theta_2}| < \varepsilon \]

for some \( t \),

then since \( g \) is a unit speed curve, it follows that
\[
\int_0^1 \frac{i\theta_2}{\{z; |z - e^2| \in (\varepsilon, \eta)\}} (g(t)) \, dt \geq \eta - \varepsilon
\]

for every \( \eta \in (\varepsilon, \eta_0) \).

In particular, under condition (13.2), we have that

\[
\lim_{b' \to 0} \sup_{a/b' \to 0} \int_0^\Gamma |\phi'(g_0(t))| \, dt \geq C(n - \varepsilon)(n + 2\varepsilon)^{-3/2}
\]

for every \( \varepsilon \) and \( n \) with \( 0 < \varepsilon < n < \eta_0 \). Let \( \varepsilon \to 0 \) and then \( n \to 0 \) to obtain that the left hand side is infinite; a contradiction.

Thus the path of \( g_0 \) stays away from \( \exp(i\theta_2) \) as \( b' \to 0 \) and \( a/b' \to 0 \). By a similar argument, it stays away from \( \exp(i\theta_5) \).

The conclusion of part (c) now follows immediately from the definitions of \( L \) and \( g_0 \), and the result that for every \( \varepsilon \) and \( n \) with \( 0 < \varepsilon < n \),

\[
\sup \left\{ \frac{1 - |x|^2}{|x - e^{i\theta_2}|^2} / \left( \frac{1 - |x|^2}{|x - e^{i\theta_5}|^2} + 1 \right); \frac{i\theta_2}{|x - e^{i\theta_2}|^2} \wedge \frac{i\theta_5}{|x - e^{i\theta_5}| \geq n} \right\}
\]

and \( t, s \in \{ r; |e^{ir} - e^{i\theta_2}| \wedge |e^{ir} - e^{i\theta_5}| \leq \varepsilon \} \)

is finite. \( \square \)

**proof of Lemma 4:**

(a) We must show that there is a constant \( C_1 > 0 \) such that

\[
\mathbb{P}_y^n(X_{n_S(1)} \in A-) \leq \exp(-C_1/b(n))
\]

for \( x \in A(n) \) and \( y \in B(n-1) \cup B(n+1) \), \( n \geq 1 \). We will obtain this
in two parts; first exhibiting such a bound for
\[ I_1(x, y) = \bar{P}_n^X(x, s(1) \in A(-n)) , \]
and then for
\[ I_2(x, y) = \bar{P}_n^X(x, s(1) \in A(-(n-1)) \cup A(-(n+1))) . \]

Let \( V(n) = \left[ (-1, 1) \times (s(n-1), s(n+1)) \right] \backslash \left[ (-1, 1-a(n)) \times \{ s(n) \} \right] . \)

Write \( \Delta(n) \) for the Martin boundary of \( V(n) \), and write \( \bar{P}_n^X \) for the law of Brownian motion started at \( x \) and killed upon leaving \( V(n) \).

Resume the notation of the last lemma, so that we have a canonical conformal equivalence \( \Phi \) from the unit disk to \( V(n) \), and we define the Martin function \( L_n \) on \( V(n) \), with respect to the base point \( \Phi(0) \). Also note that points of \( A(-n) \subset \partial V(n) \) split into two points of the Martin boundary \( \Delta(n) \) of \( V(n) \); one being the limit of points above \( A(-n) \), and the other from below. We let \( A'(n) \) be the collection of all points of the Martin boundary of \( V(n) \), that are associated to a point of \( A(-n) \) in this way.

Let
\[ \mu(\Lambda) = P_n^X(x, \xi_y \in \Lambda) , \text{ for } \Lambda \subset \Delta(n) \text{ measurable}. \]

Also, let \( C \) be the bound given by (c) of Lemma 6. Fix a point \( y \in B(n-1) \cup B(n+1) \), and let
\[ e(z) = K_n(y, z) + K_n(-y, z) . \]

Let \( \nu \) be the measure representing \( e \) on \( \Delta(n) \), so that
\[ e = \int_{\Delta(n)} L_n^w(w, \cdot) \nu(dw) . \]
By (f) of Theorem 2, and Theorem 4, we have that \( \nu(dw) \) is the sum of the measure \( e(w)u(dw \cap A'(-n)) \), and of two point masses, one at \( y \) and the other at \( -\bar{y} \). Thus, by (j) of Theorem 1,

\[
I_1(x, y) = \frac{p^X(X, S(1) \in A(-n))}{e_n \zeta^- \in A'(-n))}
\]

\[
= \frac{1}{e(x)} \int_{A'(-n)} L_n(w, x) \frac{p^X(X, \zeta^- \in A'(-n))}{w} \nu(dw)
\]

\[
= \int_{A'(\pm n)} \frac{L_n(w, x)}{e(x)} e(w) \mu(dw).
\]

Since

\[
e(\Phi(e^{it})) = 0 \text{ unless } t \in [\theta_3, \psi] \cup [\psi', \theta_4],
\]

we have from (c) of Lemma 6, that

\[
\frac{e(x')}{L_n(z, x')} = \int \frac{L_n(w, x')}{L_n(z, x')} e(w) \mu(dw)
\]

\[
\leq C^2 \int \frac{L_n(w, x)}{L_n(z, x)} e(w) \mu(dw)
\]

\[
= C^2 \frac{e(x)}{L_n(z, x)}, \text{ for } x, x' \in A(n), z \in A'(-n).
\]

Since \( e(x') = e(-\bar{x}') \), we can choose \( x' \in A(n) \) to maximize \( e \) over \( A(n) \), and still have

\[
e(w)/e(x') \leq 1 \text{ for } w \in A'(-n).
\]

Thus
\[ I_1(x, y) \leq C^2 \int_{A'(-n)} \frac{L_n(w, x')}{e(x')} e(w) \mu(dw) \]

\[ \leq C^2 \mu(A'(-n)) \sup \{ L_n(w, z); w \in A'(-n), z \in A(n) \} \]

\[ \leq C^3 [ (\psi \Theta_1) + (\Theta_5 - \psi') ] / 2 \pi \]

We thus obtain a bound on \( I_1(x, y) \) of the correct form, by (b) of Lemma 6.

Now consider \( I_2(x, y) \). Suppose for now, that \( y \in B(n-1) \), so that we may assume without loss of generality, that \( y \in A(-n(n-1)) \).

Recall that \( W(n) = (-1, 1) \times (s(n-1), s(n)) \). Let \( P^X_n \) be the law of Brownian motion started at \( x \) and killed upon leaving \( W(n) \), and define the Martin function \( \tilde{L}_n \) on \( W(n) \) using the base point \( c(n) \).

Consider

\[ Y_t = X_{t+L_B(n)} \cdot \]

Then \( (Y_t) \) is a \( \nu \)-transform on \( W(n) \), under \( P^X_n \), where \( \nu \) is a harmonic function on \( W(n) \), given by (m) of Theorem 1. Because \( e \) is symmetric (that is, \( e(z) = e(-z) \) for \( z \in W(n) \)), the same will be true of \( \nu \). By Theorem 4 and our choice of base point, we therefore have that \( \nu \) is a multiple of

\[ \tilde{L}_n(y, \cdot) + \tilde{L}_n(-\bar{y}, \cdot) \]

Thus:
\[ I_2(x, y) \leq \frac{P_n^*(Y_0 \in A(n), Y \zeta_\tau \in A(-n(\text{}}

\[ \leq \sup_{z \in A(n)} \frac{\hat{P}^Z_n(X \zeta_\tau = y)}{\hat{L}_n(y, z) + \hat{L}_n(-\bar{y}, z)} \]

To bound this, we basically repeat the proof of Lemma 6, but for a simpler transformation.

Let \( \phi \) be the conformal map from the unit disk to \( W(n) \), mapping \(-1, 0, 1\) to \((-1, (s(n) + s(n-1))/2), c(n), \) and \((1, (s(n) + s(n-1))/2)\). Then

\[ \tilde{I}_n(\phi(x), \phi(z)) = (1 - |z|^2)/|x - z|^2. \]

Let

\[ e^{i\theta} = \phi^{-1}((1, s(n))) \quad \text{and} \quad e^{i\psi_1} = \phi^{-1}((1-a(n), s(n))) \quad \text{and} \quad e^{i\psi_2} = \phi^{-1}((1-a(n-a(n)), s(n))) \]

where \(-\pi/2 < \psi_2 < -\theta < \theta < \psi_1 < \pi/2\). Thus

\[ (13.3) \quad I_2(x, y) \leq \sup_{s \in [\psi_2 - \theta, \psi_2 - \theta]} \frac{|e^{is} - e^{it}|^2}{|e^{is} - e^{it}|^2 + |e^{i(\pi-s)} - e^{it}|^2}. \]

We have that

\[ \phi(w) = \phi \int_0^w [z e^{i\theta} (z + e^{i\theta}) (z - e^{-i\theta}) (z + e^{-i\theta})]^{-1/2} \, dz + \phi' \]

for some \( \phi \) and \( \phi' \). Let
\[ 2 \phi_f(t) = \left| \frac{d}{dt} \phi(e^{it}) \right|, \]

so that

\[ b(n-1) = 2 \int_{-\theta}^{\theta} f(t) \, dt / \int_{\theta}^{\Pi-\theta} f(t) \, dt, \]

\[ a(n) = 2 \int_{\theta}^{\psi_1} f(t) \, dt / \int_{\theta}^{\Pi-\theta} f(t) \, dt, \]

\[ a(n-1) = 2 \int_{\psi_2}^{-\theta} f(t) \, dt / \int_{\theta}^{\Pi-\theta} f(t) \, dt. \]

Let

\[ \rho_j = \frac{1}{\cos \theta} \frac{\cos^2 \theta - \cos^2 \psi_j}{1 - \cos^2 \psi_j}, \quad j = 1, 2. \]

As in Lemma 3, we have that

\[ \int_{-\theta}^{\theta} f(t) \, dt = \sqrt{2} \mathcal{F}(1, \sin^2 \theta), \]

\[ \int_{\theta}^{\Pi-\theta} f(t) \, dt = \sqrt{2} \mathcal{F}(1, \cos^2 \theta), \]

and similarly,

\[ \int_{\theta}^{\psi_1} f(t) \, dt = \sqrt{2} \int_{\cos \psi_1}^{\cos \theta} \frac{dx}{\sqrt{(1-x^2)(\cos^2 \theta-x^2)}}, \]

\[ = \sqrt{2} \mathcal{F}(\rho_1, \cos^2 \theta), \quad \text{and} \]

\[ \int_{\psi_2}^{-\theta} f(t) \, dt = \sqrt{2} \mathcal{F}(\rho_2, \cos^2 \theta). \]

Since \( b(n-1) \to 0 \), it follows that \( \theta \to 0 \) and
Since \( \frac{a(n-l)}{b(n-l)} \) and \( \frac{a(n)}{b(n-l)} \to 0 \), it follows in turn that \( \rho_1, \rho_2 \to 0 \), so that also \( \psi_1, \psi_2 \to 0 \). Thus

\[
\rho_j \sim \left( \frac{1 - \theta_j^2/2}{2} - \frac{1 - \psi_j^2/2}{2} \right)^{1/2} = \left( 1 - \left( \frac{\theta_j}{\psi_j} \right)^2 \right)^{1/2},
\]

and hence

\[
b(n-l) \log(1/\psi_j) \sim b(n-l)(\log(1/\psi_j) + \log(\psi_j/\theta)) \sim \Pi.
\]

By (13.3), there is a constant \( C \) with

\[
I_2(x, y) \leq C(\psi_1 - \psi_2)^2 \leq 2 C(\psi_1^2 + \psi_2^2),
\]

and we obtain a bound of the desired form. The case \( y \in B(n+1) \) is handled in the same manner, showing (a).

(b) By symmetry,

\[
\lambda(n, n+1) = \inf_{y \in B(n)} \{ E^x[T(1)] ; x \in B(n), y \in A(n+1) \}.
\]

Consider the square \( Z(n) = (1-b(n+1), 1) \times (s(n), s(n+1)) \).

Let

\[
Y_t = X_{t+1} \mathbb{1}_{U(n) \setminus Z(n)}.
\]
Under $p^{x}_{n}$, $(Y_{t})$ will be a $v$-transform for some $v$ on $Z(n)$ (by (m) of Theorem 1), $Y_{t} = y$, and $Y_{0}$ lies on the lower or left sides of $Z(n)$. Let $Z = (0,1) \times (0,1)$, and define $\phi : Z(n) \rightarrow Z$ by

$$\phi(z) = [z - (1-b(n+1), s(n))]/b(n+1).$$

Let

$$Y_{t}' = \phi(Y_{tb(n+1)}) \in Z.$$

Because this scaling preserves Brownian motion, $(Y_{t}')$ will be a transform by $v \circ \phi^{-1}$ under $p^{x}_{n}$. Let $P^{Z}_{Z}$ be the law of Brownian motion started at $z$ and killed upon leaving $Z$. Thus

$$\lambda(n, n+1) \geq \inf_{y_{n}} \{E^{x}_{n}[\text{lifetime of } (Y_{t})]; y \in A(n+1), x \in B(n)\} \geq b(n+1)^{2} \inf_{z} \{E^{w}_{Z}[\tau]; z \in (1 - \frac{a(n)}{b(n+1)}, 1) \times \{1\}, w \in [0, 1] \times \{0\} \cup \{0\} \times [0, 1]\}.$$

Since $a(n)/b(n+1) \rightarrow 0$, this infimum remains bounded away from 0 as $n \rightarrow \infty$. The corresponding estimate on $\lambda(n, n-1)$ follows similarly, showing (b).[]

14. Smallness of $D' \setminus D$

The results of the last section show that even for planar Brownian motion, in general $D \neq D'$. The discussion of Section 11 suggests that in order to construct extensions of the process consisting.
of Brownian motion on $\mathbb{R}^d$, killed upon leaving a domain $E$, we should suppose that $D$ is pretty well all of $D'$. We will in fact use the existence of such extensions (for example, un killed Brownian motion on $\mathbb{R}^d$), to show that $D' \setminus D$ is small.

Let $E$ be a bounded domain in $\mathbb{R}^d$, and let $U$ be a large open ball containing the closure of $E$. We retain the notation $(\Omega, F)$ for paths with values in $E \cup \{\delta\}$, and write $(V, U)$ for the corresponding object, where now paths take values in $U$. The coordinate process on $\Omega$ will still be $(x_t)$. We let the coordinate process on $V$ be $(W_t)$. Write $G(x, y)$ for the Green function of $U$, and write $P^x$ and $Q^x$ for the laws of Brownian motion, started at $x$ and killed upon leaving $U$ in the first case, and $E$ in the second. Let $\mu$ and $\nu$ be probabilities on $U$, such that a set is $\mu$-null if and only if it is $\nu$-null. Assume that neither one charges some neighbourhood of the closure of $E$. Write

$$h = \int G(z, \cdot)\nu(dz),$$

and $P = hP^\mu$. Let $\tilde{P}$ be the law under $P$ of the reverse of the coordinate process $(x_t)$ from its lifetime. Then we have that

$$P(\Lambda) = \int \frac{G(x, y)}{h(x)} P^x(\Lambda) \mu(dx) \nu(dy)$$

$$\tilde{P}(\Lambda) = \int \frac{G(x, y)}{h(y)} P^x(\Lambda) \nu(dx) \mu(dy)$$

for $\Lambda \in \mathcal{V}$. Thus, since $\mu$ and $\nu$ share the same null sets, the same will be true for $P$ and $\tilde{P}$.

Let
\[ M = \{ t > 0; W_t \notin E \cup \{ \delta \} \}, \]
and let \( M_0 \) be the set of points of \( M \) which are isolated on the right. Let
\[
Y_t(s) = \begin{cases} 
W_{t+s}, & s < T_{U \setminus E \circ t} \\
\delta, & s \geq T_{U \setminus E \circ t}
\end{cases}
\]
and let
\[
\rho(A) = E\left[ \sum_{t \in M_0} 1_A(Y_t) \right], \text{ for } A \in (\hat{\Omega}, F).
\]
Since every excessive function for Brownian motion is regular (see Blumenthal and Getoor [5]), the hypotheses of Maisonneuve [38], Proposition (9.2) are met. Thus, there exists a kernel \( n(x, du) \), and an additive functional \( (L_t) \) such that
\[
\rho(A) = E\left[ \int_0^\xi n(w_t, A) dL_t \right]
\]
for every measurable subset \( A \) of \( \hat{\Omega} \), and \( (X_t) \) is an \( h \)-transform under each \( n(x, \cdot) \). Thus \( (X_t) \) is also an \( h \)-transform under \( \rho \).

It follows from Theorem 5 and a time reversal argument, that
\[
Z = \lim_{t \to \xi} X_t, \text{ and } Z' = \lim_{t \to 0} X_t
\]
exist in the topology of the Martin compactification of \( E \), \( \rho \)-a.s.

Because \( \nu \) does not charge \( E \), the limit actually lies in \( \text{Acc} \subset \Delta \).

Let \( \eta \) be the measure on \( \Delta \times \Delta \), defined by
\[
\eta(dx, dy) = \rho(Z \in dx, Z' \in dy).
\]
Theorem 11 \[ \eta(D' \setminus D) = 0. \]

Thus, with probability one, every excursion into \( E \) starts and finishes at points \( z, z' \) of the Martin boundary, for which \( (z, z') \in D \).

Lemma 7 \( \eta \) is \( \sigma \)-finite.

Proof: By minimality, we have that for every \( y \in \Delta_0 \) and \( x \in E \), either there exists a point \( e(y) \in \partial E \) such that

\[
y^{X}(X_t \to e(y) \text{ in the Euclidean metric, as } t \to \zeta) = 1, \text{ or}
\]

\[
y^{X}(X_t \text{ converges in the Euclidean metric, as } t \to \zeta) = 0.
\]

In the latter case, we write \( e(y) = \delta \).

Since the Euclidean limits of \( X_t \), as \( t \to \zeta \) and \( t \to 0 \), both exist \( \rho \)-a.s. , it follows that

\[
\rho(e(Z) = \delta \text{ or } e(Z') = \delta) = 0.
\]

Since Brownian motion does not return to points, we have by the strong Markov property, that also

\[
\rho(e(Z) = e(Z')) = 0.
\]

Thus, it will suffice to show that

\[
\rho(|e(Z) - e(Z')| > 2\varepsilon) < \infty \text{ for every } \varepsilon > 0.
\]

Let \( T(0) = 0 \), \( T(k+1) = \inf\{t > T(k); |X_t - X_{T(k)}| > \varepsilon\} \).

Because \( h \) is bounded and bounded away from zero, on a neighbourhood of the closure of \( E \), we obtain from (f) of Theorem 1, a constant \( m > 0 \).
such that

$$h^X \mathbf{E}^X[T(1)] \geq m$$

whenever $x$ lies in $E$. Thus

$$\infty > E[\zeta] \geq m \mathbf{E}\left[ \sum_{k=0}^{\infty} 1_E(X_{T(k)}) \right]$$

$$\geq m \mathbf{E}\left[ \sum_{t \in \mathbb{M}_0} 1\{|X_t - X_{T_3 \theta t} | > 2 \varepsilon\} \right]$$

$$= m \rho(\{|e(Z) - e(Z')| > 2 \varepsilon\}) ,$$

as required.  

**proof of Theorem 11:** Since $\eta$ is $\sigma$-finite, we may use the 'classical argument' of the last section of Maisonneuve [38], to obtain a kernel $\rho(x, y; du)$ such that

$$\rho = \int \rho(x, y; \cdot) \eta(dx, dy) .$$

Now, let $K$ be the Martin function on $E$, and suppose that $A \in F$. It is easily checked (by (f) of Theorem 1), that

$$h^X \mathbf{E}[A|Z] = a(x, Z) ,$$

where $a(x, y) = E^X_y[A]$. Suppose in addition that $f_0$ and $f_{\infty}$ are positive and measurable on $\Delta$, and that $B \in F_s$. Then

$$E_\rho[f_0(Z')f_{\infty}(Z), A \circ \theta_s, B]$$

$$= E_\rho[f_0(Z') h^{X_s}[A, f_{\infty}(Z)], B]$$

$$= E_\rho[f_0(Z') f_{\infty}(Z) a(X_s, Z'), B] .$$

from which we see that $(X_{\zeta})$ is a $K(y, \cdot)$ -transform under $\rho(x, y; \cdot)$.
for $\eta$-a.e. $(x, y)$. Thus $(x, y) \in D$ for $\eta$-a.e. $(x, y)$, as required. □

Now fix $y_0 \in E$, and define harmonic measure $\lambda$ on $\Delta$ by

$$\lambda(dz) = Q_{y_0}(Z \in dz).$$

By (h) of Theorem 1, and (f) of Theorem 2, we have that for every $y \in E$, the measure

$$h^Y_{y_0}(Z \in dz)$$

shares the same null sets as $\lambda$.

**Lemma 8** $\lambda \otimes \lambda$ is absolutely continuous with respect to $\eta$.

Thus we obtain a more potential theoretic version of Theorem 1;

**Corollary 2** $\lambda \otimes \lambda (D' \setminus D) = 0$.

**proof of Lemma 8**: Let $A$ be a compact subset of $E$, with nonempty interior, and set

$$\eta_k(dx, dy) = \rho(Z' \in dx, Z \in dy, T_A < \infty).$$

We will actually show that $\lambda \otimes \lambda$ is absolutely continuous with respect to $\eta_k$.

We can write

$$\eta_k(A) = \int_{\{T_A < \infty\}} \int_{\Delta} 1_{\lambda}(Z', y) h^Y_{T_A}(Z \in dy) dp .$$

If $\rho(Z' \in B) = 0$, then

$$\tilde{P}(Z \in B) = P(Z' \in B) = 0,$$
so that since \( P \) is absolutely continuous with respect to \( \tilde{P} \), also

\[
\eta(\Delta \times B) = P(Z \in B) = 0
\]

(this is the reason for our choice of \( h \)). Thus \( \eta_k(\Delta \times B) = 0 \), and hence

\[
\int_{A} X_T^{Q} h Q (Z \in B) = 0, \ \rho \text{-a.s.}
\]

Because \( A \) has nonempty interior, we have \( \rho(T_A < \infty) > 0 \), so that by our above remark on the null sets of \( \lambda \), also \( \lambda(B) = 0 \). That is, \( \lambda(dz) \) is absolutely continuous with respect to

\[
\rho(Z \in dz).
\]

If now \( \eta_k(A) = 0 \), then

\[
\int 1_A(Z', y) h Q A(Z \in dy) = 0 \ \rho \text{-a.s.}.
\]

Thus

\[
\int 1_A(Z', y) \lambda(dy) = 0 \ \rho \text{-a.s.},
\]

and so also

\[
\int 1_A(x, y) \lambda(dy) = 0 \text{ for } \lambda \text{-a.e. } x.
\]

This gives immediately that \( \lambda \otimes \lambda(A) = 0 \). \( \square \)
Bibliography


   (unpublished).


