CONSTRUCTION OF STRONG MARKOV PROCESSES THROUGH EXCURSIONS, AND A RELATED MARTIN BOUNDARY

By

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ABSTRACT

For certain Markov processes, K. Ito has defined the Poisson point process of excursions away from a fixed point. The law of this process is determined by a certain measure, called its Characteristic measure. He gives a list of conditions this measure must obey. I add to these conditions, obtaining necessary and sufficient conditions for a measure to arise in this way. The main technique is to use a 'last exit decomposition' related to those of Getoor and Sharpe. The more general problem of excursions away from a fixed set is treated using the Exit system of B. Maisonneuve.

This gives a useful technique for constructing new Markov processes from old ones. For example, we obtain a rigorous construction of the Skew Brownian motion of Ito and McKean, and another proof of results of Pittenger and Knight on excision of excursions.

A related question is that of determining whether an entrance point for a Markov process remains an entrance point for an h-transform of that process. Let E be an open subset of Euclidean space, with a Green function, and let λ be harmonic measure on the Martin boundary Δ of E. I show that, except for a $\lambda\otimes\lambda$ -null set of $(x,y)\in\Delta^2$, x is an entrance point for Brownian motion conditioned to leave E at y. R.S. Martin gave examples in dimension 3 or higher, for which there exist minimal accessible Martin boundary points $x\neq y$ for which this condition fails. I give a similar example in dimension 2. The argument uses recent results of M. Cranston and T. McConnell, together with Schwarz-Christoffel transformations.

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PART 1. EXCURSIONS

1. Introduction

In the theory of Markov chains, it is easily shown that the 'excursions' away from a fixed state are IID random variables, with values in path space. This fact us used extensively in the analysis of recurrence, and similar questions.

In his classic paper, "Poisson Point Processes attached to Markov Processes" [32], K. Itô obtains the corresponding property for the excursions of a continuous time standard process, away from a recurrent point a. He constructs a point process of excursions, and shows that it is a Poisson point process (PPP). This decomposition, of the original process into its excursions, has proved useful in analysing the original standard process. For example, J.B. Walsh has in [51] given a simple proof of the Ray-Knight theorems on the Markov property of local time, using the PPP of excursions of Brownian Motion away from 0. A different application lies in the results of P. Greenwood and E. Perkins [24] who analyse domain of attraction problems for random walk, using yet another type of excursion process.

We will be examining a converse problem; that of constructing a Strong Markov process from its excursions. In [32], Ito shows that the characteristic measure of the excursion process he constructs, satisfies certain conditions, and then indicates how to reverse his construction provided those conditions hold. That is, given a PPP whose characteristics measure obeys these conditions, it should be possible to construct a process whose excursion process is that PPP. Ito doesn't state this converse result as a formal theorem, and hence doesn't write out a proof.

Our first goal will be to make his argument rigorous.

This is perhaps more interesting in that the converse is not true as stated above. In section 4, examples will be given showing that unless we strengthen two of his conditions, the process constructed from the PPP may fail to be strong Markov or right continuous. This strengthening gives us necessary and sufficient conditions for the process obtained to be a right process.

The argument for sufficiency is presented in section 3, and that for necessity in section 5. Section 6 will examine the Ray property and 'hypothèses droites' of the constructed process. Section 2 will give notation and a statement of the results. Section 7 contains a variation on the main lemma used in the proof of sufficiency.

Section 9 will treat a generalization of Ito's result and of our converse, to the situation of excursions away from a set (rather than a point). This is done using the Exit System of B. Maisonneuve (generalizing the excursion measure of Ito's PPP). Section 8 contains some technical facts about the excursion process in this new situation. Section 10 gives applications to skew Brownian motion, and to the results of F. Knight and O. Pittenger on excision of excursions.

2. Notation and Results

E will be a seperable metric space, and $\mathcal{E}^0$ the σ-field of its Borel subsets. $\mathcal{E}$ will be the universal completion of $\mathcal{E}^0$. U will be the set of right-continuous E-valued paths, and, for $a \in E$, U^a will be the set of all such paths u ,

satisfying $u(0)=a$. $(W_t)_{t\geq 0}$ will be the coordinate process on U; $W_t(u) = u(t)$. For $t\geq 0$, $\mathcal{U}_t^o$ will be the smallest σ-field such that for every $s \in [0,t]$, W_s is a measurable function from $(U,\mathcal{U}_t^o)$ to $(E,\mathcal{E}^0)$. $\mathcal{U}_t$ will be the universal completion of $\mathcal{U}_t^o$, and $\mathcal{U}$ that of $\mathcal{U}_\infty^o$.

$(\Pi,\mathcal{P})$ will be the measurable space of $(U,\mathcal{U})$-valued point functions. That is, we adjoin a point δ to U, and let Π be the set of functions $p:[0,\infty) \to U\cup\{\delta\}$ such that $p(t) = \delta$ except for countably many t. $\mathcal{P}$ is the σ-field on Π generated by the functions $p \mapsto N(A,p)$, where $N(A,p)$ is the number of times t such that $(t,p(t)) \in A \subset [0,\infty) \times U$. Here A belongs to the product $\mathcal{B} \otimes \mathcal{U}$ of the Borel field $\mathcal{B}$ on $[0,\infty)$, and of $\mathcal{U}$. $(y_t)_{t\geq 0}$ will be the coordinate process on Π; $y_t(p) = p(t)$.

For $A \in \mathcal{B} \otimes \mathcal{U}$, we define the restriction of $p \in \Pi$ to A to be:

$$p|_A(t) = \begin{cases} p(t) \ , & \text{if } (t,p(t)) \in A \\ \delta \ , & \text{otherwise} \end{cases}$$

Special cases of this will be the killing operators:

$$\alpha_s(p) = p|_{[0,s] \times U}$$

$$k_s(p) = p|_{[0,s) \times U}$$

For $t\geq 0$, $\mathcal{P}_t$ will be the sub σ-field $\alpha_t^{-1}(\mathcal{P})$, of $\mathcal{P}$.

We define the shift operators of Π to be:

$$\Theta_s(p) = p(t+s), \text{ for } t\geq 0$$

$$\Theta_s^o(p) = \Theta_s(p)|_{(0,\infty)}$$

The same notation will be used for the corresponding shift operators on U.

For $u \in \dot{U}$ we define the hitting time and début of $\{a\} \subset E$ to be

$$\sigma_a(u) = \inf\{t > 0;\ u(t) = a\}$$

$$\tau_a(u) = \inf\{t \geq 0;\ u(t) = a\}.$$

It is well known that both σ_a and τ_a are $(\mathcal{U}_{t+})$ stopping times.

A <u>Poisson point process</u> on a probability space $(\Omega, \mathcal{F}, P)$, with values in $(U, \mathcal{U})$ is a measurable function $Y: (\Omega, \mathcal{F}) \to (\Pi, \mathcal{P})$, together with a filtration $(\mathcal{F}_t)_{t \geq 0}$ of $(\Omega, \mathcal{F})$ $(\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for $t \leq s)$, such that:

(a) $\alpha_t(Y) \in \mathcal{F}_t$ for $t \geq 0$.

(b) $\Theta_t^o(Y)$ is independent of $\mathcal{F}_t$ and has the same law as $Y|_{(0,\infty)}$, for $t \geq 0$.

(c) There exist sets $A_k \in \mathcal{U}$, $A_k \uparrow U$ such that $N([0,t] \times A_k, Y) < \infty$ a.s. for each k,t.

The special role of time $t = 0$ in (b) will be useful when we consider the point process of excursions away from a point $a \in E$, of a Markov process taking values in E. Time $t = 0$ corresponds to the first excursion, which is exceptional in that we will want to allow it to start in any initial distribution. In contrast, the other excursions will start in a manner dictated by the transition probabilities of the given Markov process.

Under conditions (a), (b), (c) above, there exists a σ-finite measure n on U (the <u>characteristic measure</u> of Y) such that

$$E[N((0,t) \times A, Y)] = t \cdot n(A) \quad \text{for } A \in \mathcal{U},\ t \geq 0.$$

This measure determines the law of $Y|_{(0,\infty)}$. In fact, if $(s_i, t_i) \times A_i$ are disjoint and $0 \leq s_i \leq t_i$, for $i = 1 .. k$, then the

$N((s_i,t_i)\times A_i,Y)$ are independent Poisson random variables, of means $(t_i-s_i)\cdot n(A_i)$.

In the theory of right processes, it is customary to equip a right continuous, E-valued process X based on $(\Omega,\mathcal{F})$, with laws $\hat{P}^b$ on $(\Omega,\mathcal{F})$, one for each $b \in E$. We will reserve the notation P^b for laws on the canonical space $(U,\mathcal{U})$. Later on we will work with a right process X, and the P^b will be the image laws under X of the $\hat{P}^b$.

In general, on a fixed measurable space $(\Omega,\mathcal{F})$, let $(X_t)_{t\geq 0}$ be a right continuous process, based on Ω, with values in E. We will say that $(X_t,\mathcal{F}_t,\mu,P^b)$ has the <u>strong Markov property</u> at T if the following situation holds; $(\mathcal{F}_t)_{t\geq 0}$ is a filtration of $(\Omega,\mathcal{F})$, $X^{-1}(\mathcal{U}_t) \subset \mathcal{F}_t$, μ is a σ-finite measure on $(\Omega,\mathcal{F})$, $(P^b)_{b\in E}$ is a family of probability measures on $(U,\mathcal{U})$ such that $b \mapsto P^b(A)$ is $\mathcal{E}$-measurable for each $A \in \mathcal{U}$, T is a stopping time for the filtration $(\mathcal{F}_{t+})$, and

$$\mu(X_{T+\cdot}\in A, T<\infty, B) = \int_{B\cap\{T<\infty\}} P^{X_T}(A)\,d\mu \quad \text{for} \quad A \in \mathcal{U},\ B \in \mathcal{F}_{T+} .$$

Except in Lemma 6 below, we will always take μ to be a probability measure.

For (X_t) right continuous with values in E, we will say that $(X_t,\mathcal{F}_t,\mu,P^b)$ is <u>strong Markov</u> if $(X_t,\mathcal{F}_t,\mu\ P^b)$ has the strong Markov property at each $(\mathcal{F}_{t+})$ stopping time. For $a\in E$, $(L_t)_{t\geq 0}$ is a <u>local time</u> at a, if $L_\cdot$ is continuous, nondecreasing, and adapted to $(\mathcal{F}_t)$, with set of increase exactly $\{t;X_t = a\}$, such that for every $(\mathcal{F}_{t+})$ stopping time T with $X_T = a$, $(X_{\cdot+T}, L_{\cdot+T}-L_T)$ is independent of $\mathcal{F}_{t+}$, with the same law as $(X_{\cdot+\sigma_a(X)}, L_{\cdot+\sigma_a(X)})$.

Ito performed the following construction (Actually, he considered only the case of a standard process, but as pointed out to me by J. Pitman, his arguments apply without change. Henceforth similar qualifications will be omitted.); Let P be a probability measure on $(\Omega,\mathcal{F})$ under which $\mathcal{F}$ is complete, and suppose the following conditions hold:

(2.1) (X_t) is right continuous with values in E, $(X_t,\mathcal{F}_t,P,P^b)$ is strong Markov, and each $\mathcal{F}_t$ contains all the P-null sets of $\mathcal{F}$.

(2.2) X is recurrent at a point $a \in E$. $(P^b(\sigma_a<\infty) = 1$ for $b \in E)$.

(2.3) If $P^a(\sigma_a=0) = 1$, then there is a local time (L_t) for X at a, which is canonical in the sense that it is normalized to make

$$E[e^{-\sigma_a(X)}] = E[\int_0^\infty e^{-t}\, dL_t].$$

(Condition (2.3) holds if X is a right process).

If $P^a(\sigma_a = 0) = 1$, let $S(s)$ be the right continuous inverse local time: $S(s) = \inf\{t \geq 0; L_t > s\}$, $S(0-) = 0$. Let

$$Y_s(t) = \begin{cases} X(S(s-)+t), & \text{if } 0 \leq t < S(s) - S(s-) \\ a, & \text{if } t \geq S(s) - S(s-) > 0 \end{cases}$$

$$Y_s = \delta, \quad \text{if } S(s) - S(s-) = 0.$$

Then Ito shows that Y is a $(U,\mathcal{U})$-valued PPP with respect to P and the filtration of Y.

If, on the contrary, $P^a(\sigma_a = 0) = 0$, then X visits a at a discrete set of times. In this case, let $S(k)$ be the k'th hitting time of a; $S(0) = 0$, $S(k+1) = \inf\{t > S(k); X_t = a\}$. Let

$$Y_k(t) = \begin{cases} X(S(k) + t), & \text{if } 0 \leq t < S(k+1) - S(k) \\ a, & \text{if } t \geq S(k+1) - S(k). \end{cases}$$

Then Ito shows that under P, the Y_k, $k \geq 1$ are IID, $(U,\mathcal{U})$-valued, $\mathcal{F}$-measurable random variables.

Let n be the characteristic measure of Y in the first case, and the common distribution of the $Y_k, k \geq 1$ in the second. We call n the <u>excursion measure</u> of X from a.

Ito is concerned with classifying all processes that agree up till the début of a point $a \in E$. Specifically, suppose that

(2.4) (P_o^b) is a family of probability measures on $(U,\mathcal{U})$ such that for each $c \in E$, the coordinate process $(W_t, \mathcal{U}_t, P_o^c, P_o^b)$ is strong Markov. (Note that here, c is fixed, and b ranges over E)

(2.5) $P_o^b\{u; \tau_a(u) < \infty, \text{ and } u(t) = a \text{ for } t \geq \tau_a(u)\} = 1$ for each $b \in E$.

The problem is to classify all families (P^b) of probability measures on $(U,\mathcal{U})$ for which there exists $(X_t, \mathcal{F}_t, P, P^b)$ on some probability space, which is a <u>recurrent extension</u> of (P_o^b) in the sense that (2.1), (2.2), (2.3) hold, and

(2.6) $P^b\{u; u(\cdot \wedge \tau_a(u)) \in A\} = P_o^b(A)$ for each $b \in E$, $A \in \mathcal{U}$.

Ito achieves this classification in terms of the excursion measure n of X from a. He shows that the P_o^b and n determine the P^b, and then derives a the following list of conditions that n obeys

<u>Theorem 1</u> (Ito) Let $(X_t, \mathcal{F}_t, P, P^b)$ satisfy (2.1), (2.2), (2.3). Let n be the excursion measure of X from a, and define $P_0^b(A)$ to be $P^b\{u; u(\cdot \wedge \tau_a(u)) \in A\}$. Then the following conditions are satisfied:

(i) n is concentrated on $\{u; 0 < \sigma_a(u) < \infty, u(t) = a$ for $t \geq \sigma_a(u)\}$.

(ii) $n\{u;u(0) \notin V\} < \infty$ for every open neighborhood V of a.

(iii) $\int (1 - e^{-\sigma_a})\, dn \leq 1$.

(iv) $n\{u;\sigma_a(u) > t,\ u \in \Lambda,\ \theta_t(u) \in M\} = \int_{\Lambda\cap\{\sigma_a > t\}} P_o^{u(t)}(M)\, n(du)$

for $t > 0$, $\Lambda \in \mathcal{U}_t$, $M \in \mathcal{U}$.

(v) $n\{u;u(0) \in B,\ u \in M\} = \int_{\{u;u(0) \in B\}} P_o^{u(0)}(M)\, n(du)$

for $M \in \mathcal{U}$, and $B \in \mathcal{E}$ such that $a \notin B$.

(vi) Either (a) n is a probability measure concentrated on $U^a = \{u;u(0) = a\}$ (discrete visiting case);

or (b) n is finite, $n(U^a) = 0$, and $\int (1-\exp(-\sigma_a))\, dn < 1$ (exponential holding case);

or (c) n is infinite and $n(U^a) = 0$ or ∞ (instantaneous case).

The main result of this paper is that if conditions (ii) and (vi) are strengthened, we obtain conditions that are necessary and sufficient for a σ-finite positive measure n to arise as the excursion measure of a recurrent extension of a family (P_o^b) satisfying (2.4) and (2.5).

The conditions are:

(ii') $n\{u;u \text{ leaves } V\} < \infty$ for every open neighbourhood V of a.

(vi') Either (a) n is a probability measure concentrated on U^a. If $n \geq n' \geq 0$ and n' satisfies (iv), then n' is a multiple of n ;

or (b) as in (vi b) ;

or (c) n is infinite. If $n \geq n' \geq 0$ and n' satisfies (iv) , then $n'(U^a) = 0$ or ∞.

The statement of necessity is:

<u>Proposition 1</u> under the conditions of Theorem 1, conditions (ii') and (vi') also hold.

A strong form of sufficiency is:

<u>Theorem 2</u> Assume that $(\Omega, \mathcal{F}, P)$ is complete, and that (P_o^b) satisfies (2.4) and (2.5).

a) If $(Y_t, \mathcal{F}_t)$ is a PPP with values in $(U, \mathcal{U})$ and with characteristic measure n, such that:

$(\mathcal{F}_t)$ is right continuous, and each $\mathcal{F}_t$ contains all the P-null sets of $\mathcal{F}$;

$$P(Y_o \in M) = \int P_o^{Y_o(0)}(M)\, dP \quad \text{for } M \in \mathcal{U};$$

(i), (ii'), (iii), (iv), (v) and either (b) or (c) of (vi') hold.

Then there is a right continuous strong Markov process $(X_t, \mathcal{G}_t, P, P^b)$ such that:

Y is the PPP constructed from X as above, P-a.s.;

$(X_t, \mathcal{G}_t, P, P^b)$ is a recurrent extension of (P_o^b);

$\mathcal{G}_{S(t+)} = \mathcal{F}_t$;

$(\mathcal{G}_t)$ is right continuous, and each $\mathcal{G}_t$ contains all the P-null sets of $\mathcal{F}$.

b) If $(\mathcal{F}_k)_{k\geq 0}$ is an increasing family of sub σ-fields of $\mathcal{F}$, each containing all the P-null sets of $\mathcal{F}$, and for each $k \geq 0$, Y_k is a measurable function from $(\Omega, \mathcal{F}_k)$ to $(U, \mathcal{U})$ such that:

For $k \geq 1$ the Y_k have a common distribution n;

$$P(Y_o \in M) = \int_{U \setminus U^a} P_o^{u(o)}(M) P(Y_o \in du) + n(M) P(Y_o \in U^a)$$

for $M \in \mathcal{U}$;

$\sigma(Y_i ; i>k)$ is independent of $\mathcal{F}_k$ for $k \geq 0$;

(i), (ii'), (iv) and (a) of (vi') hold.

Then there is a strong Markov right continuous process $(X_t, \mathcal{G}_t, P, P^b)$ such that :

The Y_k are the excursion random variables constructed from X as above (discrete visiting case), P-a.s. ;

$(X_t, \mathcal{G}_t, P, P^b)$ is a recurrent extension of (P_o^b) ;

$\mathcal{G}_{S(k+1)-} \subset \mathcal{F}_k \subset \mathcal{G}_{S(k+1)}$, for $k \geq 0$;

$(\mathcal{G}_t)$ is right continuous, and each $\mathcal{G}_t$ contains all the P-null sets of $\mathcal{F}$.

This result can be used to give a rigorous construction of processes such as skew Brownian motion (see J. Walsh [50]). Similar results in special cases, obtained by different techniques can be found in Blumenthal [4] and S. Watanabe [31],[54].

The key to the proof of Theorem 2 given here is to find an expression for conditioning Y_T on the strict past $\mathcal{F}_{T-}$, for T an $(\mathcal{F}_t)$ stopping time. This is done in Lemma 7, which is related to similar results of M. Weil [53], [54] as follows.

A version of Lemma 7 can be proven for $(Y_t, \mathcal{F}_t)$ now a càdlàg Markov process, and T an $(\mathcal{F}_t)$ stopping time such that

$$[\![T]\!] \subset \bigcup_{q \in \mathbb{Q}+} [\![q + S \circ \Theta_q]\!] ,$$

where S is a terminal time for the natural filtration of Y (the smallest right continuous filtration to which Y is adapted).

$P(Y_T \epsilon \cdot | F_{T-})$ is identified in terms of $P(Y_S \epsilon \cdot | F_{S-})$, and Weil's results identify the latter in terms of a Levy system for Y (assuming that Y is standard.) See section 7.

After completion of this work, it was pointed out to me that parts of it have appeared before; In his Thesis [34], S. Kabbaj obtained a result similar to Theorem 2. He shows that under Ito's conditions, and with (G_t^o) the minimal filtration to which the reconstructed process (X_t) is adapted, (X_t) is strong Markov at all (G_t^o) stopping times. His proof uses a weaker form of Lemma 7 below, and relies heavily on the powerful tools of martingale theory. The proof presented here works for T a (G_{t+}^o) stopping time, is more elementary, and applies in greater generality (we assume no compactness conditions on E!).

Also, when we are given a right process (X_t), and let (Y_t) be its excursion process, Lemma 7 is still of interest. In this context, it is closely related to Getoor and Sharpe's last exit decompositions, and a very similar result has appeared in Getoor and Sharpe [25]. Some of the methods used in the proof of Lemma 7 appear, both in [25] and in Pitman [45]. Part of the interest of the results presented here, however, lies in that they apply in a more elementary context than the theory of right processes. As pointed out in section 6, the proof of Theorem 2 can be greatly simplified in the case that (P_0^b) comes from a right process.

I would like to thank Yves LeJan for bringing [34] to my attention.

3. Proof of Theorem 2

Part (a) is shown first, and then the arguments are modified to show part (b). In part (a), conditions (i) and (iii) are used in the construction; the latter so that the normalization of local time agrees with (2.3). Condition (ii') appears in Lemma 3, in the proof of the right continuity of paths. Condition (vi'b) also appears in this lemma, and is used to make the " inverse local time " strictly increasing, so that local time will be continuous. Conditions (iv) and (v) are put in a more convenient form in Lemma 6, which, together with Lemma 7, yields Corollary 2. Lemma 7 is also used with conditions (vi'b) and (vi'c) to give Corollary 1. Note that these two lemmas essentially show the strong Markov property. In part (b), the conditions are put to the same uses, except that we use condition (vi'a) instead of conditions (vi'b) and (vi'c) .

We start the proof of part (a), by constructing X as an explicit measurable function of Y.

Put

$$m = 1 - \int (1-e^{-\sigma_a})\, dn$$

$$S^-(s,p) = ms + \sum_{r<s} \sigma_a(p(r)), \text{ for } s \geq 0,\ p \in \Pi$$

(with the convention that $\sigma_a(\delta) = 0$), and

$$S^+(s,p) = \lim_{r \downarrow s} S^-(r,p).$$

Then $S^-(\cdot,p)$, $S^+(\cdot,p)$ are nondecreasing, and respectively left and right continuous, with values in $[0,\infty]$. If $S^+(s,p) < \infty$,

then $S^{\tau}(s,p) = S^{-}(s,p) + \sigma_a(p(s))$. $S(s,\cdot) \in P_{s-}$ since it is left continuous, and $S^{-}(s,p) = S^{-}(s,\alpha_s(p))$. Thus also $S^{+}(s,\cdot) \in P_{s+}$.

Put

$$\ell_t(p) = \inf\{s \geq o;\ \infty > S^{+}(s) \geq t\},$$

with the usual convention that $\inf(\emptyset) = +\infty$. Then ℓ_t is a (P_{s+}) stopping time, and

$$S^{-}(\ell_t(\cdot),\cdot) \leq t \leq S^{+}(\ell_t(\cdot),\cdot)$$

(with the convention that $S^{+}(\infty,p) = \infty$). Put

$$x_t = \begin{cases} y_{\ell_t}(t - S^{-}(\ell_t,\cdot)), & \text{if } y_{\ell_t} \neq \delta, \ell_t \\ a & , \text{ otherwise}. \end{cases}$$

We will show next that x_t is measurable from P_{ℓ_t+} to E^0. Since $(u,r) \mapsto u(r)$ is measurable from $U \otimes B$ to E^0, and $S^{-}(\ell_t(\cdot),\cdot) \in P_{\ell_t}$ since S^{-} is predictable, we need only show that $y_{\ell_t} \in P_{\ell_t+}$.

We state this in a more general form, to be useful later, as ;

<u>Lemma 1</u>. Let (F_t) be a filtration of a measurable space (Ω,F), and let Y be a function which is measurable from (Ω,F_t) to (Π,P_t), for every $t \geq 0$. Let R be an (F_t) stopping time such that for each $\varepsilon > o$,

$$\{R < \infty\} \subset \{\sigma_a(Y_t) > \varepsilon \text{ for only finitely many times } t \text{ in any compact time set}\}.$$

Then $Y_R \in F_R$.

<u>Remark</u>: In the present situation, we apply the lemma with Y the identity map, and with $F_t = P_{t+}$.

<u>proof</u>: Let $A \in \mathcal{U}$, $s \geq o$. Then

$$\{Y_R \in A,\ R \leq s\} = \{Y_R \in A,\ R \leq s,\ \sigma_a(Y_R) = 0\} \cup$$

$$\cup \bigcup_{m \geq 1} \{Y_R \in A,\ R \leq s,\ \sigma_a(Y_R) \geq \frac{1}{m}\}\ .$$

Let B_n, $n = 1,2, \ldots$ be an open base in the space $[o,s]$. We can write

$$\{\sigma_a(Y_R) = o,\ R \leq s\} =$$

$$= \bigcap_{m \geq 1} \bigcup_{n \geq 1} [\{R \in B_n\} \cap \{N(B_n \times \{\sigma_a \geq \frac{1}{m}\}, Y) = 0\}]$$

and $\{Y_R \in A,\ R \leq s,\ \sigma_a(Y_R) \geq \frac{1}{m}\} =$

$$= \bigcap_{n \geq 1} [\{R \leq s\} \cap (\{R \notin B_n\} \cup \{N(B_n \times [A \cap \{\sigma_a \geq \frac{1}{m}\}], Y) \geq 1\})]$$

Thus $\{Y_R \in A,\ R \leq s\} \in \mathcal{F}_s$, as required. □

Put

$M = \{p; S^-(s,p) < \infty$ for each $s \geq o$,

$S^-(s,p) \to \infty$ as $s \to \infty$,

$S^-(\cdot,p)$ is strictly increasing, and for each open neighbourhood V of a, p(s) leaves V for only finitely many times s in any compact set of times$\}$.

Thus on M, $\ell_t < \infty$ for every t, $t \mapsto \ell_t$ is continuous, and

$$x_t = \begin{cases} y_s(t - S^-(s,\cdot)) & \text{if } S^-(s,\cdot) \leq t < S^+(s,\cdot) \\ a & \text{if } t = S^+(s,\cdot) \end{cases}$$

<u>Lemma 2</u> $t \mapsto x_t$ is right continuous on M.

<u>proof</u>: Fix some element p of M. By definition of U, $x_\cdot(p)$ is right continuous on each interval $[S^-(s,p), S^+(s,p))$. If t lies in no such interval, then it must equal $S^+(s,p)$ for some s, and hence $x_t(p) = a$. Let V be an open neighbourhood of a. Since $p \in M$, there is an $s' > s$ such that $y_r(p)$ remains in V for any $r \in (s,s')$. Thus $x_q(p) \in V$ for any $q \in (S^+(s,p), S^-(s',p))$. But by definition of M, $S^-(s',p)$ is strictly greater than $S^+(s,p)$, so that as V was arbitrary, $x_\cdot(p)$ must be right continuous at t. □

(The same argument would show that if we had taken U to be the space of càdlàg paths, then on M, $x_\cdot$ would be càdlàg as well.)

For convenience, we separate out the contribution of $p(o)$; we have shown that there is a measurable function

$$F: (U \times \Pi, \mathcal{U} \otimes \mathcal{P}) \longrightarrow (U, \mathcal{U})$$

such that $x_t = F(y_o, y|_{(o,\infty)})(t)$ on M.

Put

$$S^-(s) = S^-(s,Y)$$

$$S^+(s) = S^+(s,Y)$$

$$L_t = \ell_t(Y)$$

$$X_t = x_t(Y)$$

Since Y is, by the hypotheses of Theorem 2, a measurable function from $(\Omega, \mathcal{F}_s)$ to $(\Pi, \mathcal{P}_{s+})$, for each s, the above results imply that S^- and S^+ are adapted to $(\mathcal{F}_s)$, and that for each $t \geq o$, L_t is an $(\mathcal{F}_s)$ stopping time and X_t is measurable from $\mathcal{F}_{L_t}$ to $\mathcal{E}$ (as $\mathcal{F}_{L_t}$ is complete).

<u>Lemma 3</u>. $P(Y \in M) = 1$.

<u>proof</u>: For $f \in \mathcal{B}$ with $f(o) = o$,

$$\sum_{o<t<s} f(\sigma_a(Y_t)) = \int_{(o,\infty)} f(r) N(\{\sigma_a \in dr\} \times (o,s), Y)$$

is $\mathcal{F}$-measurable, and has expectation

$$s \int_{(o,\infty)} f(r)\, n\,(\sigma_a \in dr) = s \int f o \sigma_a \, dn.$$

In particular,

$$E[\sum_{\substack{o<t<s \\ \sigma_a(Y_t)\le 1}} \sigma_a(Y_t)\ ] = s \int_{\{\sigma_a \le 1\}} \sigma_a \, dn$$

$$\le \frac{s}{1-e^{-1}} \int (1-e^{-\sigma_a})\, dn$$

$$= \frac{s}{1-e^{-1}} \ (1-m) < \infty .$$

Also, there are only finitely many times $t \in [0,s]$ with $\sigma_a(Y_t)>1$, as

$$n(\sigma_a > 1) \le \frac{1}{1-e^{-1}} \int (1-e^{-\sigma_a})\, dn < \infty ,$$

so that $S^-(s) < \infty$ a.s., for each s.

If $n(U) = o$ (so that a is a trap), then $m = 1$ so that $S^-(s) = s$ a.s. If $n(U) \neq o$, then by (i) there is an $\varepsilon > o$ such that $n(\sigma_a > \varepsilon) > o$. Thus, there are a.s. infinitely many times s such that $\sigma_a(Y_s) > \varepsilon$, so that in either case, $S^-(s) \to \infty$ as $s \to \infty$, a.s.

If $m > o$, it is clear that S^- is strictly increasing. If $m = o$,

then by (vib') it follows that $n(U) = \infty$, and hence that $\{s;\sigma_a(Y_s) > o\}$ is dense in $[o,\infty)$, a.s.

The last condition is obtained from (ii'), letting V run through a countable base of open neighborhoods of a. □

Appealing to the completeness of F, we will without loss of generality assume that $Y \in M$ surely, hence that X is right continuous and by Lemma 1, that $Y_R \in F_R$ for every (F_s) stopping time R.

If we are given a filtration $(V_t)_{t\geq o}$, a σ-field $V_{o-} \subset V_o$, and a random variable R with values in $[o,\infty]$, recall that V_{R-} is defined to be the filtration generated by V_{o-} and by sets of the form $A \cap \{R > s\}$, for $A \in F_s$ and $s \geq o$. In our case, let F_{o-} be generated by all the P-null sets of F. For R a random time, and $r \geq o$, put

$$H^R_{o-} = F_{R-}$$

$$H^R_r = F_{R-} \vee \sigma(Y_R(s);\ o \leq s \leq r)$$

Put

$$G^o_t = H^{L_t}_{(t-S^-(L_t))-}$$

$$G_t = G^o_{t+}$$

$$G^o_{o-} = G_{o-} = F_{o-}$$

Lemma 4

(a) (G_t) is right continuous, increasing, and each G_t contains all the P-null sets of F. $X_t \in G_t$ for each $t \geq o$.

(b) If T is a $(\mathcal{G}_t)$ stopping time then L_T is an $(\mathcal{F}_s)$ stopping time, and

$$\mathcal{F}_{L_T-} \subset \mathcal{G}_{T-} \subset \mathcal{G}_T \subset \mathcal{F}_{L_T}$$

(c) $S^+(s)$ is a $(\mathcal{G}_t)$ stopping time, for $s \geq o$. If T is any $(\mathcal{G}_t)$ stopping time such that $S^+(L_T) = T$ then $\mathcal{G}_T = \mathcal{F}_{L_T}$.

(d) If R is an $(\mathcal{F}_s)$ stopping time, and T is a $(\mathcal{G}_t)$ stopping time such that $T < S^+(L_T)$ on $\{T < \infty\}$, put

$$V = \begin{cases} T - S^-(R) & , \text{ if } L_T = R < \infty \\ \infty & , \text{ otherwise} \end{cases}$$

Then V is an $(\mathcal{H}^R_{t+})$ stopping time.

<u>proof</u>: (a) We know that each L_t is an $(\mathcal{F}_s)$ stopping time. Thus, if $t < t'$ then

$$\begin{aligned} \mathcal{G}^o_t \cap \{L_t < L_{t'}\} &\subset \mathcal{H}^{L_t}_\infty \cap \{L_t < L_{t'}\} \\ &\subset \mathcal{F}_{L_t} \cap \{L_t < L_{t'}\} \quad (\text{as } Y_{L_t} \in \mathcal{F}_{L_t}) \\ &\subset \mathcal{F}_{L_{t'}-} \subset \mathcal{G}^o_{t'} , \end{aligned}$$

and $\mathcal{G}^o_t \cap \{L_t = L_{t'}\}$ is, by monotone class arguments, and Prop 18 of [7], generated by

$$\begin{aligned} \mathcal{H}^{L_t}_{o-} \cap \{L_t = L_{t'}\} &\subset \mathcal{H}^{L_{t'}}_{o-} \cap \{L_t = L_{t'}\} \\ &\subset \mathcal{G}^o_{t'} \end{aligned}$$

and by

$$H_r^{L_t} \cap \{t-S^-(L_t)>r\} \cap \{L_t=L_{t'}\} \subset H_r^{L_{t'}} \cap \{t-S^-(L_{t'})>r\} \cap \{L_t=L_{t'}\}$$

$$\subset H^{L_{t'}}_{(t-S^-(L_{t'}))-} \subset G^o_{t'} .$$

Thus, $G^o_t \subset G^o_{t'}$.

For $s > t$ we can write X_t as a measurable function of L_s, $k_{L_s}(Y)$, and $k_{s-S^-(L_s)}(Y_{L_s})$ (let

$$\hat{Y}_r = \begin{cases} k_{L_s}(Y)(r) , & \text{if } r < L_s \\ k_{s-S^-(L_s)}(Y_{L_s}), & \text{if } r = L_s \\ \delta , & \text{if } r > L_s . \end{cases}$$

Then $X_t = x_t(\hat{Y}_\cdot)$.)

As in [7] Prop. 25, the first two are F_{L_s-} measurable and the latter lies in $H^{L_s}_{(s-S^-(L_s))-}$

(b) For $r > t \geq 0$ we have that

$$F_{L_t-} \subset G_t \subset G^o_r \subset H_\infty^{L_r} \subset F_{L_r} .$$

By definition, $L_r \downarrow L_t$ as $t \downarrow r$, hence

$$F_{L_t} = \bigcap_{r>t} F_{L_r} ,$$

so that $G_t \subset F_{L_t}$.

If now T is a (G_t) stopping time, then by the right continuity of L,

$$\{L_t < \lambda\} = \bigcup_{q\in\mathbb{Q}} \{T < q\} \cap \{L_q < \lambda\} .$$

But $\{T < q\} \in G_q \subset F_{L_q}$, so that since L_q is an (F_s) stopping time,

$$\{L_T < \lambda\} \in F_\lambda .$$

Further, if $A \in G_T$, then

$$A \cap \{T < q\} \in G_q \subset F_{L_q}$$

so that

$$A \cap \{T < q\} \cap \{L_q < \lambda\} \in F_\lambda .$$

Taking the union over $q \in \mathbb{Q}$, we get that $A \cap \{L_T < \lambda\} \in F_\lambda$, hence that $A \in F_{L_T}$.

We will prove (c) before showing the remainder of (b); that $F_{L_{T-}} \subset G_{T-}$.

(c) Since S^- and S^+ are strictly increasing, and $S^-(L_t) \le t \le S^+(L_t)$, we see that

$$\{S^+(s) < t\} = \{s < L_t\} \in F_{L_{t-}} \subset G_t,$$

hence $S^+(s)$ is a (G_t) stopping time.

Similarly, if $S^+(L_T) = T$, then

$$\{L_T < L_t\} = \{T < t\} ,$$

and hence if $A \in F_{L_T}$, then by [7] Prop. 16,

$$A \cap \{T < t\} = A \cap \{L_T < L_t\} \in F_{L_{t-}} \subset G_t .$$

That is, $A \in G_T$. Since $F_{L_T} \supset G_T$ by the part of (b) already shown, we get that $G_T = F_{L_T}$.

To finish the proof of (b), let $A \in F_s = G_{S^+(s)}$. Then

$$A \cap \{L_T > s\} = A \cap \{T > S^+(s)\} \in G_{T-}$$

by [7], Prop. 16 again.

(d)

$$\{V < h\} = \{T < S^-(R) + h,\ L_T \le R < \infty\} \setminus \{L_T < R\}$$

$$= \{T < (S^-(R) + h) \wedge S^+(R)\} \setminus \{L_T < R\}$$

, as $T < S^+(L_T)$ on $\{T < \infty\}$. Because T is a (G^o_{t+}) stopping time,

$$\{L_T < R\} \in F_{R-} \subset H^R_h$$

and

$$\{T < (S^-(R) + h) \wedge S^+(R)\} \in G^o_{[(S^-(R) + h) \wedge S^+(R)]-} .$$

We must therefore show that the latter field lies in H^R_h. It is generated by $G^o_{o-} = F_{o-} \subset H^R_h$, and by

$$G^o_t \cap \{(S^-(R) + h) \wedge S^+(r) > t\} \quad , \quad \text{for} \quad t \ge 0.$$

By monotone class arguments, the latter is generated by

$$H^{L_t}_{o-} \cap \{(S^-(R) + h) \wedge S^+(R) > t\}$$

and by

$$H^{L_t}_r \cap \{t - S^-(L_t) > r\} \cap \{(S^-(R) + h) \wedge S^+(R) > t\}, \quad \text{for} \quad r \ge 0.$$

$\sigma_a(Y_R)$ is an (H^R_{s+}) stopping time, since σ_a is a (U_{s+}) stopping time , each σ-field H^R_s is complete, and $Y_R^{-1}(U_s) \subset H^R_s$. $(t - S^-(R)) \vee 0$ is also an (H^R_{s+}) stopping time, since $S^-(R) \in F_{R-}$ as S^- is predictable. Thus

$$\{(S^-(R) + h) \wedge S^+(R) > t\} = \{\sigma_a(Y_R) \wedge h > (t - S^-(R)) \vee 0\}$$

$$\in H^R_{(\sigma_a(Y_R) \wedge h)-} \subset H^R_h .$$

Also,

$$F_{L_{t-}} \cap \{L_t \le R\} \subset F_{R-} \cap \{L_t \le R\}$$

by [7] Prop 18. Therefore

$$H^{L_t}_{o-} \cap \{(S^-(R) + h) \wedge S^+(R) > t\}$$

$$\subset F_{R-} \cap \{(S^-(R) + h) \wedge S^+(R) > t\} \subset H^R_h ,$$

as $\{(S^-(R) + h) \wedge S^+(R) > t\} \subset \{L_t \le R\}$.

We argue as in part (a) to see that

$$H^{L_t}_r \cap \{t - S^-(L_t) > r\} \cap \{(S^-(R) + h) \wedge S^+(R) > t\} \cap \{L_t < R\}$$

$$\subset F_{R-} \cap \{(S^-(R) + h) \wedge S^+(R) > t\} \subset H^R_h,$$

and

$$H^{L_t}_r \cap \{t - S^-(L_t) > r\} \cap \{(S^-(R) + h) \wedge S^+(R) > t\} \cap \{L_t = R\}$$

$$\subset H^R_r \cap \{\sigma_a(Y_R) \wedge h > t - S^-(R) > r\}$$

$$\subset H^R_{(\sigma_a(Y_R) \wedge h)-} \subset H^R_h \qquad \square$$

<u>Lemma 5</u>. Let (Ω_1, F^1_t), (Ω_2, F^2_t) be right continuous filtered spaces. Let $Z: \Omega_1 \to \Omega_2$ be such that $Z^{-1} F^2_t \supset F^1_t$ for every t. Then for every (F^1_t) stopping time T_1, there is an (F^2_t) stopping time T_2 such that $T_2 \circ Z = T_1$.

<u>proof</u>: For $r \in \mathbb{Q}$, let $B_r = \{T_1 < r\}$, and find $A_r \in F_r^2$ such that $B_r = Z^{-1} A_r$. Set

$$\hat{A}_t = \bigcup_{\substack{r<t \\ r\in\mathbb{Q}}} A_r .$$

Thus, putting $T_2(\omega) = \inf\{t;\omega \in \hat{A}_t\}$, we have that

$$\{T_2 < t\} = \hat{A}_t \in F_t^2 .$$

Also,

$$Z^{-1} \hat{A}_t = \bigcup_{\substack{r<t \\ r\in\mathbb{Q}}} Z^{-1} A_r = \{T_1 < t\} , \quad \text{for every} \quad t \geq 0 ,$$

so that $T_2 \circ Z = T_1$. □

The following lemma would be much simplified if, instead of the conditions of Theorem 2, we had assumed that the coordinate process (W_t) was a right process under (P_o^b). (It would follow as usual from the right continuity of the $U^\alpha f(W_\cdot)$ at R ,(U^α) being the resolvant for (P_o^b) .)

<u>Lemma 6</u>. Let (P_o^b) satisfy (2.4) and (2.5), and suppose that n is a σ-finite positive measure on $(U,\mathcal{U})$ satisfying (i), (iv) and (v). Let R be a $(\mathcal{U}_{t+})$ stopping time such that $n(U^a, R=0) = 0$. Then the coordinate process $(W_t,\mathcal{U}_t,n,P_o^b)$ is strong Markov at R.

<u>Proof</u>: By replacing R by the $(\mathcal{U}_{t+})$ stopping times

$$R_B = \begin{cases} R & \text{on} \quad B \\ \infty & \text{off} \quad B \end{cases}$$

for $B \in \mathcal{U}_{R+}$, we see that it suffices to show that for $A \in \mathcal{U}$,

$$n(\Theta_R^{-1}A,\ R < \infty) = \int_{\{R<\infty\}} P_o^{u(R(u))}(A)\ n\ (du) .$$

Let $h > 0$. By (iv),

$$(3.1)\qquad \int_{\{\sigma_a>h\}} f(u,\Theta_h u)\ n\ (du) = \int_{\{\sigma_a>h\}} \left[\int f(u,v)\ P_o^{u(h)}\ (dv)\ \right] n\ (du)\ ,$$

for f of the form $1_{B\times C}$ where $B \in \mathcal{U}_h$, $C \in \mathcal{U}$. Thus, this holds for every $f \in \mathcal{U}_h \otimes \mathcal{U}$, $f \geq 0$.

Put

$$\hat{\mathcal{U}}_t = \mathcal{U}_h \otimes \mathcal{U}_t\ .$$

Since n is σ-finite, it follows as in Theorem 7.3 of Blumenthal and Getoor [5], that there is a $(\mathcal{U}^o_{t+})$ stopping time $\tilde{R}$ such that $n(R \neq \tilde{R}) = 0$. Define $Z:U \to U\times U$ by $Z(u) = (u,\theta_h u)$. Then $Z^{-1}(\hat{\mathcal{U}}_t) \supset \mathcal{U}^o_{(t+h)+}$. (Note that this might fail for the universal completion $\mathcal{U}_{(t+h)+}$.) Thus by Lemma 5, there is a $(\hat{\mathcal{U}}_{t+})$ stopping time $\hat{R}$ with

$$\hat{R}(u,\Theta_h u) = \begin{cases} \tilde{R}(u) - h\ , & \text{if } \tilde{R}(u) > h \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$f = 1_{\{(u,v);\ \hat{R}(u,v) < \infty, \Theta_{\hat{R}(u,v)}(v) \in A\}}.$$

Since $\hat{R}$ is a $(\hat{\mathcal{U}}_{t+})$ stopping time, it is immediate that $\hat{R}(u,\cdot)$ is a $(\mathcal{U}_{t+})$ stopping time, for each $u \in U$. Since also $(W_t, \mathcal{U}_t, P^c_o, P^b_o)$ was assumed to be strong Markov for $c\in E$, we can use (3.1) twice to get that

$$n(\sigma_a > h,\ h < R < \infty,\ \Theta_R^{-1}A)$$

$$= \int_{\{\sigma_a>h\}} f(u,\Theta_h u)\ n\ (du)$$

$$= \int_{\{\sigma_a>h\}} \left[\int P_o^{v(\hat{R}(u,v))}(A)\ 1_{\{\hat{R}<\infty\}}(u,v)\ P_0^{u(h)}(dv)\ \right] n(du)$$

$$= \int_{\{\sigma_a > h\}} P_o^{\theta_h u(\hat{R}(u,\theta_h u))}(A)\ 1_{\{\hat{R}<\infty\}}(u,\theta_h u)\ n(du)$$

$$= \int_{\{\sigma_a > h\}} P_o^{u(R(u))}(A)\ 1_{\{h<R<\infty\}}(u)\ n(du) \quad .$$

Letting $h \downarrow 0$, we get that

$$n(0<R<\infty, \Theta_R^{-1}A) = \int_{\{0<R<\infty\}} P_o^{u(R(u))}\ n\ (du) \ .$$

From (v) , we see that

$$(3.2) \qquad \int_{U\setminus U^a} f(u(0))g(u)n(du) = \int_{U\setminus U^a} f(u(o))E_o^{u(o)}(g)\ n(du)$$

for nonnegative $f \in \mathcal{E}$, $g \in \mathcal{U}$.

Let the set of branch points be

$$E_{br} = \{b \in E;\ P_o^b(W_o \neq b) > 0\} \in \mathcal{E}.$$

The strong Markov property of $(W_t, \mathcal{U}_t, P_o^c, P_o^b)$ shows that $P_o^c(W_o \in E_{br}) = 0$ for every $c \in E$. Thus,

$$n(R = 0,\ U\setminus U^a, A)$$

$$= \int_{U\setminus U^a} P_o^{u(o)}\ (R = 0, A)\ n\ (du)$$

$$= \int_{U\setminus U^a} [\int_{\{R=0\}} P_o^{v(0)}\ (A)\ P_o^{u(o)}(dv)]\ n\ (du) \qquad \text{(SMP)}$$

$$= \int_{U\setminus U^a} P_0^{u(o)}\ (A) P_o^{u(o)}(R = 0)\ n\ (du)$$

$$= \int_{U\setminus U^a} P_o^{u(o)}\ (A)\ 1_{\{R=o\}}(u)\ n\ (du) \quad . \qquad \text{(by 3.2))}$$

Since $n(\{R=0\} \cap U^a) = 0$ by hypothesis, we can see that (3.1) holds. □

The heart of the proof of Theorem 2 lies in the next result, whose proof we defer until later.

Lemma 7. Let T be a $(\mathcal{G}_t)$ stopping time such that $L_T > 0$ and $T < S^+(L_T)$ on $\{T < \infty\}$. Let $\mathcal{H}_t$ be the σ-field $\mathcal{F}_{L_T-} \otimes \mathcal{U}_t$ on $\Omega \times U$. Then there is an $(\mathcal{H}_{t+})$ stopping time R such that

(3.3) $P\{\omega; n\{u; R(\omega,u) < \infty\} < \infty\} = 1$

(3.4) $R(\omega, Y_{L_T}(\omega)) = (T-S^-(L_T))(\omega)$, if $T(\omega) < \infty$

(3.5) $R(\omega,u) = \infty$ for every $u \in U$, if $T(\omega) = \infty$

(3.6) $P(Y_{L_T} \in A,\ T-S^-(L_T) \in B,\ T < \infty \mid \mathcal{F}_{L_T-})\ (\omega)$

$$= \frac{n\{u; u\in A, R(\omega,u) \in B, R(\omega,u) < \infty\}}{n\{u; R(\omega,u) < \infty\}} \quad \text{for } P\text{-a.e.}\,\omega,$$

where $A \in \mathcal{U}$, $B \in \mathcal{B}$, and we take the convention that $\frac{0}{0} = 0 = \frac{\infty}{\infty}$.

Corollary 1. Let T be a $(\mathcal{G}_t)$ stopping time such that $X_T = a$. Then $T = S^+(L_T)$ a.s.

proof. By the strong Markov property of $(W_t, \mathcal{U}_t, P_o^c, P_o^b)$ for $c \in E$, and the hypothesis that $P_o^c(\tau_a < \infty,\ W_t = a$ for every $t \geq \tau_a) = 1$, we get $t \geq \tau_a) = 1$, we get that $P_0^a(\sigma_a = 0) = 1$, and hence that $P_0^c(W_0=a, \sigma_a>0) = 0$ $c \in E$. Since also $P(Y_o \in M) = \int P_o^{Y_o(o)}(M)\, dP$, it follows that $S^+(0) = 0$ a.s. on $\{X_o = 0\}$, hence that $T = S^+(L_T)$ a.s. on $\{L_T = 0\}$. Thus putting

$$B = \{L_T > 0 \text{ and } T < S^+(L_T)\},$$

we may replace T by T_B, to get that $L_T > 0$ and $T < S^+(L_T)$ on $\{T < \infty\}$. We need to show that $T = \infty$ a.s.

Apply Lemma 7 to T, to get an $(\mathcal{H}_{t+})$ stopping time R. For $\omega \in \Omega$, let

$$H(\omega) = \{u; u(o) = a,\ R(\omega,u) = 0\}.$$

Since R is an $(\mathcal{H}_{t+})$ stopping time, it is immediate that $R(\omega,\cdot)$ is a $(\mathcal{U}_{t+})$ stopping time, hence that $H(\omega) \in \mathcal{U}_{o+}$ for each $\omega \in \Omega$. Since also $H(\omega) \subset U^a$, we have that

$$n \geq n|_{H(\omega)} \geq 0,$$

and that $n|_{H(\omega)}$ satisfies (iv) and (v). If $n(U)<\infty$, then $n(H(\omega))\leq n(U^a)=0$ by (vi'b). If $n(U)=\infty$, then $n(H(\omega)) = 0$ or ∞ for each ω, by (vi'c) (this is the only place we use this condition!). But $n(H(\omega)) \leq n\{u; R(\omega,u) < \infty\}$, which is itself finite for P-a.e. ω. Thus in either case, $n(H)=0$ a.s., so

$$P(T < \infty) = E[\frac{n(H)}{n(R<\infty)}] = 0$$

by (3.6) (recall that $0/0 = 0$). □

Using these three results, we will show

<u>Corollary 2</u>. Let T be a $(\mathcal{G}_t)$ stopping time, and $A \in \mathcal{U}$. Then

$$P[Y_{L_T}(\cdot+T-S^-(L_T)) \in A,\ T < \infty] = E[P_o^{X_T}(A), T < \infty]$$

<u>proof</u>: Replacing T by various $(\mathcal{G}_t)$ stopping times T_B, it suffices to treat several distinct cases, namely that $X_T = a$ on $\{T < \infty\}$, that $X_T \neq a$ and $L_T > 0$ on $\{T < \infty\}$, and that $X_T \neq a$ and $L_T = 0$ on $\{T < \infty\}$.

In the first case, Corollary 1 shows that the conclusion is trivial.

In the second case, also $T < S^+(L_T)$ on $\{T < \infty\}$, so that we can apply Lemma 7 to obtain R as in that result. It follows that if $f \in \mathcal{F}_{L_T-} \otimes \mathcal{U}$ is bounded, then

$$(3.7) \quad E[f(\cdot, Y_{L_T}(\cdot)),\ T < \infty]$$

$$= \int \left[\frac{1}{n\{u; R(\omega,u) < \infty\}} \int_{\{R(\omega,\cdot) < \infty\}} f(\omega,u)\ n\ (du) \right] P(d\omega) \quad .$$

Take

$$f = 1_{\{(\omega,u); R(\omega,u) < \infty, \Theta_{R(\omega,u)} u \in A\}} ,$$

to obtain that

$$P(Y_{L_T}(\cdot + T - S^-(L_T)) \in A,\ T < \infty)$$

$$= E\left[\frac{n(\Theta_R^{-1}(A), R < \infty)}{n(R < \infty)}\right].$$

Since R is an $(\mathcal{H}_{t+})$ stopping time, it is immediate that $R(\omega,\cdot)$ is a $(\mathcal{U}_{t+})$ stopping time, for each $\omega \in \Omega$. Also, since $X_T \neq a$ on $\{T < \infty\}$ we get that $n(\mathcal{U}^a, R(\omega,\cdot) = 0) = 0$ for P-a.e.ω. Thus, by Lemma 6

$$n(\Theta_{R(\omega,\cdot)}^{-1}(A),\ R(\omega,\cdot) < \infty) = \int_{\{R(\omega,\cdot) < \infty\}} P_o^{u(R(\omega,u))}(A)\ n(du)$$

for P-a.e.ω. Therefore by (3.7) again,

$$P(Y_{L_T}(\cdot + T - S^-(L_T)) \in A,\ T < \infty)$$

$$= E[P_o^{Y_{L_T}(R(\cdot, Y_{L_T}(\cdot)))}(A),\ T < \infty]$$

$$= E[P_o^{X_T}(A),\ T < \infty] \ .$$

In the third case, that $L_T = 0$ and $X_T \neq a$ on $\{T < \infty\}$, then also $T < S^+(L_T)$ on $\{T < \infty\}$. Thus Lemma 4, part (d) applies, to

show that T is an $(\mathcal{H}^o_{t+})$ stopping time. Since $(\mathcal{H}^o_{t+})$ is a completion of the natural right continuous filtration of the process $X_{\cdot\wedge\sigma_a(X)}$, it is easy to use the hypotheses that $P(Y_o \in A) = \int P_0^{u(0)}(A)\, P(Y_o \in du)$ for $A \in \mathcal{U}$, and that for $c \in E$ the coordinate process $(W_t, \mathcal{U}_t, P_o^c, P_o^b)$ is strong Markov, to conclude that the process

$$(X_{t\wedge\sigma_a(X)}, \mathcal{H}^o_t, P, P_o^b)$$

is strong Markov. This suffices. □

The remainder of Theorem 2 part (a) now follows easily; For $A \in \mathcal{U}$ and $b \in E$ put

$$P^b(A) = \iint 1_A\,(F(u,v))\; P(Y|_{(o,\infty)} \in dv)\; P_o^b(du).$$

Since $b \mapsto P_o^b(A)$ is $\mathcal{E}$-measurable for $A \in \mathcal{U}$, the same holds for $b \mapsto P^b(A)$. If T is any $(\mathcal{G}_t)$ stopping time, then by the strong Markov property of PPP's (see Ito [32] Theorem 5.1), $\Theta^o_{L_T}(Y)$ is independent of $\mathcal{F}_{L_T}$, with the same law as $Y|_{(o,\infty)}$.

By replacing T with the $(\mathcal{G}_t)$ stopping times

$$T_B = \begin{cases} T & \text{on } B \\ \infty & \text{off } B\,, \end{cases}$$

where $B \in \mathcal{G}_T$, it follows from Corollary 2 that for $A \in \mathcal{U}$,

$$P(Y_{L_T}(\cdot+T-S^-(L_T)) \in A \mid \mathcal{G}_T) = P_o^{X_T}(A) \quad \text{on } \{T < \infty\}.$$

Since also $\mathcal{G}_T \subset \mathcal{F}_{L_T}$, $Y_{L_T}(\cdot+T-S^-(L_T)) \in \mathcal{F}_{L_T}$, and

$$X_{\cdot+T} = F(Y_{L_T}(\cdot+T-S^-(L_T)),\ \Theta^o_{L_T}\, Y)\ ,$$

this yields the strong Markov property of (X_t, G_t, P, P^b) at T.

By definition of (P^b), (X_t, G_t, P, P^b) is a recurrent extension of (P_o^b). The remaining points have already been dealt with, except for showing that Y is the PPP constructed from X as in Ito [32]. Since $P^a(\sigma_a = 0) = 1$, this will follow provided (L_t) satisfies (2.3).

The set of increase of (L_t) is exactly $\{t; X_t = a\}$, since $X_t = a$ only if $t = S^+(s)$ for some s. The normalization

$$E[e^{-\sigma_a(x)}] = E\left[\int_o^\infty e^{-t}\, dL_t\right]$$

follows easily from the PPP nature of Y, and Theorem 4.5 of Ito [32]. Finally, we can write

$$(X_{\cdot+T}, L_{\cdot+T} - L_T) = (F(Y_{L_T}(\cdot+T-S^-(L_T)), \Theta^o_{L_T} Y), \ell_\cdot(\Theta^o_{L_T} Y)) ,$$

so that by Corollary 1, if T is any (G_t) stopping time with $X_T = a$, then $(X_{\cdot+T}, L_{\cdot+T} - L_T)$ is independent of G_T with a law not depending on the choice of T, as required. Thus, except for the proof of Lemma 7, the proof of part (a) is complete.

<u>proof of Lemma 7</u>: Choose $\delta_k \downarrow 0$ with $n(\sigma_a > \delta_k) > 0$ for each k. For $q \in \mathbb{Q}$, $q \geq o$, let

$$S_q^k = \inf\{s > q;\ \sigma_a(Y_s) > \delta_k\}.$$

The S_q^k are (F_s) stopping times. By completeness of F, we may assume without loss of generality that $N(\{\sigma_a > \delta_k\} \times [0,s], Y(\omega)) < \infty$ for every ω, k and s, and that $N(\{\sigma_a > \delta_k\} \times [o,\infty), Y(\omega)) = \infty$ for every ω and k. Thus, each S_q^k is surely finite, and (writing $[\![V]\!]$ for the graph $\{(t,\omega); 0 \leq t < \infty,\ t = V(\omega)\}$)

$$[\![(L_T)_{\{\sigma_a(Y_{L_T}) > \delta_k\}}]\!] \subset \bigcup_{q \in \mathbb{Q}_+} [\![S_q^k]\!].$$

Also, for any k,k',q,q' we have that

$$S_q^k \le S_{q'}^{k'}, \text{ if } k \ge k' \text{ and } q \le q',$$

$$\{S_q^k = S_{q'}^{k'}\} \in F_{S_q^{k-}} \text{ if } k \le k'.$$

Put

$$R_q^k = (T - S^-(L_T))_{\{L_T = S_q^k\}}$$

$$H_t^k = H_t^{S_q^k}.$$

$$\hat{H}_t^{k,q} = F_{S_{q-}^k} \otimes U_t$$

By Lemma 4(d), R_q^k is an $(H_{t+}^{k,q})$ stopping time.

Use Lemma 5 with $\Omega_1 = \Omega$, $F_t^1 = H_{t+}^{k,q}$, $\Omega_2 = \Omega \times U$, $F_t^2 = \hat{H}_{t+}^{k,q}$, $Z(\omega) = (\omega, Y_{S_q^k}(\omega))$, to obtain an $(\hat{H}_{t+}^{k,q})$ stopping time $\tilde{R}_q^k$ with

$$\tilde{R}_q^k(\omega, Y_{S_q^k}(\omega)) = R_q^k(\omega) \text{ for each } \omega \in \Omega.$$

Let

$$\hat{R}_q^k(\omega,u) \begin{cases} \tilde{R}_{q'}^{k'}(\omega,u), \text{ if } k' \ge k,\ q' \in \mathbb{Q}_+,\ S_q^k(\omega) = S_{q'}^{k'}(\omega), \\ \qquad \text{and for every } k'' \ge k',\ q'' \in \mathbb{Q}_+ \\ \qquad \text{such that } S_{q''}^{k''}(\omega) = S_{q'}^{k'}(\omega), \text{ also} \\ \qquad R_{q''}^{k''}(\omega,u) = R_{q'}^{k'}(\omega,u). \\ \infty \qquad \text{, if for every } k' \ge k,\ q' \in \mathbb{Q} \text{ this fails to hold.} \end{cases}$$

Thus, for every $k, k' \in \mathbb{Z}_+$ and $q, q' \in \mathbb{Q}_+$ we have that

$$\hat{R}^k_q = \hat{R}^{k'}_{q'} \quad \text{on} \quad \{S^k_q = S^{k'}_{q'}\} \times U \text{ , and}$$

$$\hat{R}^k_q(\omega, Y_{S^k_q}(\omega)) = (T - S^-(L_T))(\omega) \quad \text{for} \quad \omega \in \{L_T = S^k_q\} \text{ .}$$

We now show that $\hat{R}^k_q$ is an $(\hat{\mathcal{H}}^{k,q}_{t+})$ stopping time. Let $t > 0$, and choose an open base $B_1, B_2, \ldots$ of the space $[o,t)$. Then

$$\{\hat{R}^k_q < t\} = \bigcup_{\substack{k' \geq k \\ q' \in \mathbb{Q}_+}} \bigcap_{\substack{k'' \geq k' \\ q'' \in \mathbb{Q}_+}} [(\{S^{k''}_{q''} \neq S^k_q\} \times U) \cup$$

$$\cup \bigcap_{i \geq 1} \bigcup_{j \geq i} ((\{\tilde{R}^{k''}_{q''} \in B_j\} \cap \{S^{k''}_{q''} = S^k_q\} \times U) \cap$$

$$\cap (\{\tilde{R}^{k'}_{q'} \in B_j\} \cap \{S^{k'}_{q'} = S^k_q\} \times U))] \text{ .}$$

Since $\{S^{k'}_{q'} = S^k_q\} \in \mathcal{F}_{S^k_q -}$ whenever $k' \geq k$, it will thus suffice to show that

$$\hat{\mathcal{H}}^{k',q'}_t \cap (\{S^{k'}_{q'} = S^k_q\} \times U) \subset \hat{\mathcal{H}}^{k,q}_t$$

for every $k' \geq k$, $q' \in \mathbb{Q}_+$. This holds, since by monotone class arguments,

$$(\mathcal{F}_{S^{k'}_{q'}-} \otimes \mathcal{U}_t) \cap (\{S^{k'}_{q'} = S^k_q\} \times U)$$

$$= (\mathcal{F}_{S^{k'}_{q'}-} \cap \{S^{k'}_{q'} = S^k_q\}) \otimes \mathcal{U}_t$$

$$= (\mathcal{F}_{S^k_q -} \cap \{S^{k'}_{q'} = S^k_q\}) \otimes \mathcal{U}_t$$

$$\subset \mathcal{F}_{S^k_q -} \otimes \mathcal{U}_t \text{ .}$$

Put

$$R = \bigwedge_{k,q} (\hat{R}_q^k)\{L_T = S_q^k\} \times U .$$

Then $R = \hat{R}_q^k$ on $\{L_T = S_q^k\} \times U$. We will show that R is an $(\mathcal{H}_{t+})$ stopping time. Let $t > 0$, $k \in \mathbb{Z}_+$. Then

$$\{R < t\} \cap \{\sigma_a(Y_{L_T}) > \delta_k\} \times U$$

$$= \bigcup_{q \in \mathbb{Q}_+} [\{\hat{R}_q^k < t\} \cap \{S_q^k = L_T\} \times U].$$

As before,

$$\hat{\mathcal{H}}_t^{k,q} \cap \{S_q^k = L_T\} \times U$$

$$= \mathcal{H}_t \cap \{S_q^k = L_T\} \times U.$$

Since also

$$\{S_q^k = L_T\} = \{q < L_T \le S_q^k\} \cap \{\sigma_a(Y_{L_T}) > \delta_k\} ,$$

we get that for each k,

$$\{R < t\} \cap \{\sigma_a(Y_{L_T}) > \delta_k\} \times U$$

$$= H_k \cap \{\sigma_a(Y_{L_T}) > \delta_k\} \times U ,$$

for some set $H_k \in \mathcal{H}_t$. Using that $\sigma_a(Y_{L_T}) > 0$ on $\{T < \infty\}$, it is easy to see that in fact

$$\{R < t\} = [\bigcap_{k \ge 1} \bigcup_{\ell \ge k} H_\ell]\backslash\{T = \infty\} \times U \in \mathcal{H}_t ,$$

as required.

Properties (3.4) and (3.5) are now immediate. (3.3) will follow

from (3.5), (3.6), and the convention that $\infty/\infty = 0$. Thus all that remains is to show (3.6).

Let $A \in \mathcal{F}_r$. By the Markov property of Y at r, and Theorem 4.4A of Ito [32],

$$P(Y_{S_q^k} \in B,\ S_q^k > r, A) = P(A, S_q^k > r) P(Y_{S^k_{(q-r)\vee 0}} \in B)$$

$$= P(A, S_q^k > r)\ \frac{n(B, \sigma_a > \delta_k)}{n(\sigma_a > \delta_k)} \quad ,$$

for $B \in \mathcal{U}$. That is, for $f = 1_{A \times B}$, $A \in \mathcal{F}_{S_q^{k-}}$, $B \in \mathcal{U}$ we have

$$E[f(\cdot, Y_{S_q^k}(\ ))] = \frac{1}{n(\sigma_a > \delta_k)} \int \Big(\int_{\{\sigma_a > \delta_k\}} f(\omega, u) n(du) \Big) P(d\omega)$$

Therefore this holds for $f \in \mathcal{F}_{S_{q-}^k} \otimes \mathcal{U}$, $f \geq 0$. Take

$$f(\omega, u) = 1_C(\omega) 1_A(u) 1_B(R_q^k(\omega, u)) 1_{\{R_q^k < \infty\}}(\omega, u) \quad ,$$

where $C \in \mathcal{F}_{S_q^{k-}}$, $A \in \mathcal{U}$, $B \in \mathcal{B}$. Then

$$P(C, Y_{L_T} \in A,\ T - S^-(L_T) \in B,\ L_T = S_q^k) = E[f(\cdot, Y_{S_q^k}(\cdot))]$$

$$= \int_C \frac{n(A, R_q^k(\omega, \cdot) \in B,\ R_q^k(\omega, \cdot) < \infty,\ \sigma_a > \delta_k)}{n(\sigma_a > \delta_k)}\ P(d\omega)$$

$$= \int_C P(L_T = S_q^k | \mathcal{F}_{S_q^{k-}})(\omega)\ \frac{n(A, R_q^k(\omega, \cdot) \in B, R_q^k(\omega, \cdot) < \infty, \sigma_a > \delta_k)}{n(R_q^k(\omega, \cdot) < \infty, \sigma_a > \delta_k)}\ P(d\omega)$$

$$= \int_{C \cap \{L_T = S_q^k\}} \frac{n(A, R_q^k(\omega, \cdot) \in B, R_q^k(\omega, \cdot) < \infty, \sigma_a > \delta_k)}{n(R_q^k(\omega, \cdot) < \infty, \sigma_a > \delta_k)}\ P(d\omega) \quad .$$

Enumerate $\mathbb{Q}_+$ as $q_1, q_2, \ldots$, and let $C \in F_{L_T-}$. Then there are $C_q^k \in F_{S_q^k-}$ such that

$$C \cap \{L_T = S_q^k\} = C_q^k \cap \{L_T = S_q^k\}.$$

Thus

$$P(C, Y_{L_T} \in A,\ T - S^-(L_T) \in B,\ \sigma_a(Y_{L_T}) > \delta_k,\ T < \infty)$$

$$= \sum_j P(C_j^k, Y_{L_T} \in A,\ T - S^-(L_T) \in B,\ L_T = S_{q_j}^k,\ S_{q_i}^k \neq S_{q_j}^k \quad \text{for } i<j)$$

$$= \sum_j \int_{C_j^k \cap \{S_{q_j}^k = L_T\} \cap \{S_{q_i}^k \neq S_{q_j}^k \text{ for } i<j\}} \frac{n(A, R_{q_j}^k(\omega,\cdot) \in B, R_{q_j}^k(\omega,\cdot) < \infty, \sigma_a > \delta_k)}{n(R_{q_j}^k(\omega,\cdot) < \infty, \sigma_a > \delta_k)} P(d\omega),$$

$$, \text{ as } \{S_{q_j}^k \neq S_{q_i}^k\} \in F_{S_{q_j}^k-}$$

$$= \sum_j \int_{C \cap \{S_{q_j}^k = L_T\} \cap \{S_{q_j}^k \neq S_{q_i}^k \text{ for } i<j\}} \frac{n(A, R(\omega,\cdot) \in B,\ R(\omega,\cdot) < \infty\ \sigma_a > \delta_k)}{n(R(\omega,\cdot) < \infty, \sigma_a > \delta_k)} P(d\omega)$$

$$= \int_{C \cap \{\sigma_a(Y_{L_T}) > \delta_k\}} \frac{n(A, R(\omega,\cdot) \in B, R(\omega,\cdot) < \infty, \sigma_a > \delta_k)}{N(R(\omega,\cdot) < \infty, \sigma_a > \delta_k)} P(d\omega).$$

If A is chosen so that $n(A) < \infty$, then the integrand converges boundedly to

$$\frac{n(A, R(\omega,\cdot) \in B, R(\omega,\cdot) < \infty)}{n(R(\omega,\cdot) < \infty)}.$$

Thus

$$P(C, Y_{L_T} \in A,\ T - S^-(L_T) \in B,\ T < \infty)$$

$$= \int_C \frac{n(A, R(\omega,\cdot) \in B,\ R(\omega,\cdot) < \infty)}{n(R(\omega,\cdot) < \infty)}\, P(d\omega)$$

whenever $n(A) < \infty$. Since n is σ-finite, this holds for every $A \in \mathcal{U}$, which yields (3.6). □

Part (a) is proven. The proof of part (b) is similar. For $u_0, u_1, u_2, \ldots \in U$, put

$$S_k(u_0, u_1, \ldots) = \sum_{i<k} \sigma_a(u_i),\ k \geq 0$$

$$F(u_0, u_1, \ldots)(t) = \begin{cases} u_k(t - S_k(u_0, u_1, \ldots)), & \text{if } S_k(u_0, u_1, \ldots) \leq t < S_{k+1}(u_0, u_1, \ldots). \\ a & \text{, if } t \geq S_\infty(u_0, u_1, \ldots). \end{cases}$$

Put

$$X_t = F(Y_0, Y_1, \ldots)(t),\ S(k) = S_k(Y_o, Y_1, \ldots)\ .$$

For $A \in \mathcal{U}$, put

$$P^b(A) = \begin{cases} \displaystyle\int_U \int_{U \times U \times \ldots} 1_A(F(u, v_1, v_2, \ldots)) P((Y_1, Y_2, \ldots) \in d(v_1, v_2, \ldots)) P_o^b(du), & \text{if } b \neq a \\ \displaystyle\int_U \int_{U \times U \times \ldots} 1_A(F(u, v_1, v_2, \ldots)) P((Y_1, Y_2, \ldots) \in d(v_1, v_2, \ldots)) n(du), & \text{if } b = a\ . \end{cases}$$

We let $\mathcal{F}_{-1}$ be the set of P-null sets of $\mathcal{F}$. For R a random variable with values in $\{-1, 0, 1, 2, \ldots\}$, $\mathcal{F}_R$ is defined to be the σ-field generated by sets in $\mathcal{F}_r \cap \{R = r\}$, for $-1 \leq r \leq \infty$. For R a non negative integer valued random time, we put

$$\mathcal{H}^R_{0-} = \mathcal{F}_{R-1},\ \mathcal{H}^R_t = \mathcal{F}_{R-1} \vee \sigma\ (Y_R(s);\ 0 \le s \le t).$$

For $r \ge 0$, let $L_r = \inf\{k; S(k+1) > r\}$. We put

$$\mathcal{G}^o_t = \mathcal{H}^{L_t}_{(t-S(L_t))-}\ ,\ \mathcal{G}_t = \mathcal{G}^o_{t+},\ \mathcal{G}_{o-} = \mathcal{G}^o_{o-} = \mathcal{F}_{-1}$$

Since $n(U \backslash U^a) = 0$, we have that $S(k) \to \infty$, a.s., and that for each $k \ge 1$, $S(k) < \infty$, $S(k-1) < S(k)$, and $X_{S(k)} = a$, a.s. Delete a null set of Ω to ensure that these statements hold surely. The discrete-time analogue to Lemma 4 is;

<u>Lemma 4'</u>

(a) $(\mathcal{G}_t)$ is right continuous, increasing, and each $\mathcal{G}_t$ contains all the P-null sets of $\mathcal{F}$. $X_t \in \mathcal{G}_t$ for each $t \ge 0$.

(b) If T is a $(\mathcal{G}_t)$ stopping time, then L_T is an $(\mathcal{F}_k)$ stopping time, and $\mathcal{F}_{L_T-1} \subset \mathcal{G}_T \subset \mathcal{F}_{L_T}$. If further $X_T \neq a$ on $\{T < \infty\}$, then $\mathcal{F}_{L_T-1} \subset \mathcal{G}_{T-}$.

(c) $S(k)$ is a $(\mathcal{G}_t)$ stopping time, for $k \ge 0$. If T is a $(\mathcal{G}_t)$ stopping time such that $X_T = a$ on $\{T < \infty\}$, then $\mathcal{F}_{L_T-1} \supset \mathcal{G}_{T-}$.

(d) If R is an $(\mathcal{F}_k)$ stopping time, and T is a $(\mathcal{G}_t)$ stopping time, put

$$V = \begin{cases} T - S(R) & \text{on } \{L_T = R < \infty\} \\ \infty & \text{otherwise} \end{cases}$$

Then V is an $(\mathcal{H}^R_{t+})$ stopping time.

<u>Proof</u>: The proofs of part (a), and that for T a $(\mathcal{G}_t)$ stopping time, L_T is an $(\mathcal{F}_k)$ stopping and $\mathcal{G}_T \subset \mathcal{F}_{L_T}$, are as in Lemma 4. Further,

$$\mathcal{F}_{L_T-1} \cap \{T < t\} = \mathcal{F}_{L_T-1} \cap \{L_T - 1 \le L_t - 1\} \cap \{T < t\}$$

$$\subset \mathcal{F}_{L_t-1} \cap \{L_T - 1 \le L_t - 1\} \cap \{T < t\}$$

$$= \mathcal{F}_{L_t-1} \cap \{T < t\} \subset \mathcal{G}_t,$$

so that $\mathcal{F}_{L_T-1} \subset \mathcal{G}_T$.

For $k \ge 0$ we have that

$$\{S(k) \le t\} = \{k \le L_t\} \in \mathcal{F}_{L_t-1} \subset \mathcal{G}_t,$$

so that $S(k)$ is a $(\mathcal{G}_t)$ stopping time. Applying the previous computation, we see that

$$\mathcal{F}_k \subset \mathcal{G}_{S(k+1)}.$$

Let T be a $(\mathcal{G}_t)$ stopping time such that $X_T \ne a$ on $\{T < \infty\}$, and let $A \in \mathcal{F}_k$. Then

$$A \cap \{L_T - 1 \ge k\} = A \cap \{L_T \ge k + 1\}$$

$$= A \cap \{T > S(k+1)\}$$

$$\in \mathcal{G}_{S(k+1)} \cap \{T > S(k+1)\} \in \mathcal{G}_{T-} .$$

Now let T be a $(\mathcal{G}_t)$ stopping time such that $X_T = a$ on $\{T < \infty\}$. Then $\{T > r\} = \{L_T > L_r\}$. By monotone class arguments, $\mathcal{G}_{T-}$ is generated by

$$\mathcal{G}_{0-} = \mathcal{F}_{-1} \subset \mathcal{F}_{L_T-1},$$

and by

$$\mathcal{G}_r \cap \{T > r\} \subset \mathcal{F}_{L_r} \cap \{L_T - 1 \ge L_r\} \subset \mathcal{F}_{L_T-1},$$

for $r \ge 0$. Thus part (c) is shown. Part (d) follows as in Lemma 4. □

The discrete time version of Lemma 7 is

<u>Lemma 7'</u> Let T be a $(\mathcal{G}_t)$ stopping time such that $X_0 = a$ on $\{L_T = 0,\ T < \infty\}$. Let $\mathcal{H}_t = \mathcal{F}_{L_T - 1} \otimes \mathcal{U}_t$. Then there is an $(\mathcal{H}_{t+})$ stopping time R such that

(3.8) $R(\omega, Y_{L_T}(\omega)) = (T - S(L_T))(\omega)$ if $T(\omega) < \infty$

(3.9) $R(\omega, u) = \infty$ for every $u \in U$ if $T(\omega) = \infty$

(3.10) $P(Y_{L_T} \in A,\ T - S(L_T) \in B,\ T < \infty \mid \mathcal{F}_{L_T - 1})(\omega)$

$$= \frac{n\{u; u \in A,\ R(\omega,u) \in B,\ R(\omega,u) < \infty\}}{n\{u; R(\omega,u) < \infty\}} \quad \text{for P-a.e. } \omega,$$

, where $A \in \mathcal{U}$, $B \in \mathcal{B}$, and we take the convention that $0/0 = 0$.

<u>proof</u>: For $k \geq 0$, let

$$R^k = (T - S(L_T))_{\{L_T = k\}}$$

$$\hat{\mathcal{H}}^k_t = \mathcal{F}_{k-1} \otimes \mathcal{U}_t$$

By Lemma 4'(d), R^k is an $(\mathcal{H}^k_{t+})$ stopping time. Use Lemma 5 with $\Omega_1 = \Omega$, $\mathcal{F}^1_t = \mathcal{H}^k_{t+}$, $\Omega_2 = \Omega \times u$, $\mathcal{F}^2_t = \hat{\mathcal{H}}^k_{t+}$ and $Z(\omega) = (\omega, Y_k(\omega))$, to obtain an $(\hat{\mathcal{H}}^k_{t+})$ stopping time $\hat{R}^k$ such that for every $\omega \in \Omega$,

$$\hat{R}^k(\omega, Y_k(\omega)) = R^k(\omega).$$

Let

$$R = \bigwedge_k (\hat{R}^k)_{\{L_T = k\} \times U}.$$

Thus for $t \geq 0$,

$$\{R < t\} = \bigcup_k \{\hat{R}^k < t\} \cap \{L_t = k\} \times U.$$

By monotone class arguments,

$$\hat{H}_t^k \cap \{L_T = k\} \times U = (F_{k-1} \cap \{L_T = k\}) \otimes U_t$$

$$\subset F_{L_T-1} \otimes U_t = H_t,$$

so that R is an (H_{t+}) stopping time. Properties (3.8) and (3.9) are immediate. For $k \geq 1$, Y_k is independent of F_{k-1} with law n, so that

$$E[f(\cdot, Y_K(\cdot))] = \int \left(\int f(\omega,u)\ n\ (du)\right) P\ (d\omega)$$

for $f = 1_{A\times B}$, $A \in F_{k-1}$, $B \in U$. Thus this holds for $f \in F_{k-1} \otimes U$, $f \geq 0$. As in the proof of Lemma 7 it follows from this that for $k \geq 1$, $C \in F_{k-1}$, $A \in U$, and $B \in B$, we have that

$$P(C, Y_{L_T} \in A,\ T - S(L_T) \in B,\ L_T = k,\ T < \infty)$$

$$= \int_{C\cap\{L_T=k\}} \frac{n(A, R^k(\omega,\cdot) \in B,\ R^k(\omega,\cdot) < \infty)}{n(R^k(\omega,\cdot) < \infty} P(d\omega)$$

We will show this for $k = 0$ as well, as then the argument of Lemma 7 will give (3.10). Let

$$E_{br} = \{b \in E\colon P_0^b(W_0 = b) \neq 1\}\ .$$

Since (W_t, U, P_o^c, P_o^b) is strong Markov for each $c \in E$, we get that

$$P_o^c(W_o \in E_{br}) = 0, \quad \text{for every} \quad c \in E.$$

By hypothesis,

$$P(Y_o \in M) = \int_{U\setminus U^a} P_o^{u(o)}\ (M)\ P(Y\epsilon du) + n(M)\ P(Y_o \epsilon U^a),$$

so that as $a \notin E_{br}$ and $n(U \setminus U^a) = 0$, we see that

$$P(Y_o(o) \in E_{br}) = 0$$

Thus,

$$P(Y_o \in M \cap U^a) = \int_{U \setminus U^a} P_o^{u(o)}(M \cap U^a)\ P(Y_o \in du) + n(M)\ P(Y_o \in U^a)$$

$$= n(M)\ P(Y_o \in U^a).$$

Since F_{-1} consists of sets of P-measure 0, it follows that for $f \in F_{-1} \otimes \mathcal{U}$, $f \geq 0$,

$$E[f(\cdot, Y_o(\cdot)),\ Y_o \in U^a] = E[\int f(\cdot, u)\ n(du)]\ P(Y_o \in U^a).$$

Taking f as before and using the hypothesis that $X_o = a$ if $L_T = 0$, we see that for $C \in F_{-1}$, $A \in \mathcal{U}$, $B \in \mathcal{B}$,

$$P(C, Y_{L_T} \in A,\ T - S(L_T) \in B,\ L_T = 0,\ T < \infty)$$

$$= E(C, n(A, R^o \in B, R^o < \infty)]\ P(Y_o \in U^a)$$

$$= \frac{E(C, n(A, R^o \in B, R^o < \infty)]}{E[n(R^o < \infty)]}\ P(L_T = 0)$$

$$= \int_{C \cap \{L_T = 0\}} \frac{n(A, R^o(\omega, \cdot) \in B,\ R^o(\omega, \cdot) < \infty)}{n(R^o(\omega, \cdot) < \infty)}\ P(d\omega)\ . \qquad \square$$

Once the following result is established, the remainder of part (b) follows immediately, as before.

<u>Corollary 2'</u> Let T be a (G_t) stopping time, and $A \in \mathcal{U}$. Then

$$P(Y_{L_t}(\cdot + T - S(L_T)) \in A,\ T < \infty)$$

$$= E[P_o^{X_T}(A),\ X_T \neq a,\ T < \infty] + n(A)\ P(X_T = a,\ T < \infty)\ .$$

<u>proof</u>: As before, it suffices to treat three cases; that on $\{T < \infty\}$, respectively $X_T = a$, $X_T \neq a$ and $L_T > 0$, or $X_T \neq a$ and $L_T = 0$. The second case is handled as before, using Lemmas 7' and 6. In the third case, we use Lemma 4'(d) to see that T is an $(\mathcal{H}^o_{t+})$ stopping time. Since $(\mathcal{H}^o_{t+})$ is a completion of the natural right continuous filtration of the process $X_{\cdot \wedge \tau_a(X)}$, and

$$P(Y_o \in M) = \int_{U \setminus U^a} P_o^{u(o)}(M)\, P(Y_o \in du), \quad \text{for } M \in \mathcal{U},\ M \cap U^a = \emptyset$$

it is simple to use the strong Markov property of the processes $(W_t, \mathcal{U}_t, P_o^c, P_o^b)$, for $c \in E$, to conclude that the process

$$(X_{t \wedge \tau_a(X)}, \mathcal{H}^o_t, P, P_o^b)$$

is strong Markov. This suffices.

In the third case, we apply Lemma 7' to obtain an R satisfying the conclusions of that result. Put $H(\omega) = \{u; R(\omega,u) = 0\}$. Then for $A \in \mathcal{U}$,

$$P(Y_{L_T} \in A,\ T < \infty) = \int \frac{n(H(\omega) \cap A)}{n(H(\omega))} P(d\omega)$$

, as $T - S(L_T) = 0$ on $\{T < \infty\}$. Since R is an $(\mathcal{H}_{t+})$ stopping time, also $R(\omega, \cdot)$ is a $(\mathcal{U}_{t+})$ stopping time for each $\omega \in \Omega$, so that $H(\omega) \in \mathcal{U}_{0+}$. Thus $n|_{H(\omega)}$ satisfies (iv), so that by (vi'a), it must be a multiple of n. Since

$$n(H(\omega)) = n|_{H(\omega)}(H(\omega)) \ ,$$

we must in fact have $n|_{H(\omega)} = n$ or 0, so that $n(H(\omega)) = 1$ or 0 for each ω. Thus

$$P(Y_{L_t} \in A,\ T < \infty) = n(A)\ P(T < \infty, n(H) = 1), \quad \text{for} \quad A \in \mathcal{U}.$$

Putting $A = \mathbf{U}$, we have that $n(H) = 1$ a.s. on $\{T < \infty\}$, so that

$$P(Y_{L_T} \in A,\ T < \infty) = n(A)\ P(T < \infty),$$

as required. □

4. Insufficiency of conditions (ii) and (vi)

First, we present some examples to show that the conditions (vi) and (ii) do not suffice.

Example 1: (vi a) is not sufficient:

Let P_0 correspond to uniform motion as indicated, with absorption at a.

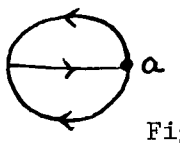

Fig. 1

There are two excursion measures n_1, n_2 corresponding to strong Markov processes visiting a discretely. $\frac{n_1 + n_2}{2}$ satisfies (vi a), and gives a Markov process which visits a discretely, but is not strong Markov.

Example 2: (vi c) is not sufficient:

Consider a Bessel process on (o,∞) (so that 0 is an entrance, non-exit point), and make the point 1 absorbing. Wrap $(0,1]$ around to make a circle E, and let P_o correspond to the resulting process on E. Let $a = \{1\}$. P_o corresponds to a continuous process that is absorbed at a, but which approaches a only from the counter clockwise direction

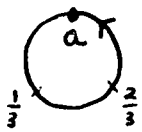

Fig. 2

We have strong Markov continuous recurrent extensions with a

instantaneous, corresponding to making the Bessel process (slowly) reflecting at 1, with various delay coefficients. Let the excursion measure with delay coefficient m be n_m, so that

$$m = 1 - \int (1 - e^{-\sigma_a})\, dn_m .$$

There are also continuous 'recurrent extensions' which are not strong Markov, corresponding to stopping the original process at a, holding it there an exponential time, and then making it enter $E\backslash\{a\}$ in the counterclockwise direction (this is possible, as 0 is an entrance point for the Bessel process). For $m \in (0,1)$, this gives an 'excursion measure' n'_m such that

$$m = 1 - \int (1 - e^{-\sigma_a})\, dn'_m .$$

(m determines the mean of the holding time at a).

Though condition (vi c) fails for the n'_m, it will hold for any measure $n_m + n'_q$ for which $m + q \geq 1$ and $m, q \in (0,1)$. These measures are ruled out by (vi'c).

<u>Example 3</u>: (ii) does not suffice:

Let E be the subset of $\mathbb{R}^2$ described in polar coordinates as

$$\bigcup_{R \in (1,2]} \{(r \cos \Theta,\ r \sin \Theta);\ r = R \cos \Theta \geq 0\}$$

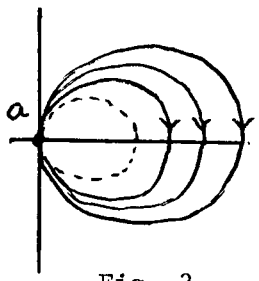

Fig. 3

Let a be the origin, and let (P_o^b) correspond to uniform clockwise motion around the circles $r = R \cos \Theta$, at speed $(\pi R(1-R))^{-1}$, with absorption at a.

For μ any σ-finite measure on $E\backslash\{a\}$, $n = P_o^\mu$ satisfies (i), (iv), (v), and (vi'c). Let n_o be the excursion measure of one dimensional Brownian motion, from 0, and let $M(u) = \max\{|u(t)|;\ 0 \le t < \infty\}$, $u \in U$. Let $f: E \to (0,1] \times (0,2\pi)$ be given by

$$f(r\cos\Theta,\ r\sin\Theta) = \left(\frac{r}{\cos\Theta} - 1,\ \frac{\pi}{2} - \Theta\right).$$

Then for $\mu(A) = n_o((\sigma_a, M) \in f(A))$, we get properties (ii) and (iii) for free, but (ii') fails, so that the resulting process is not right continuous.

We can replace (ii') by the more appealing condition (ii), provided we assume some regularity of (P_o^b);

<u>Proposition 2</u> (a) Suppose that (P_o^b) satisfies (2.4) and (2.5), and that n satisfies (i), (iii), (iv) and (v). Suppose also that

(4.1) for every open neighbourhood V of a, there is an open neighborhood V' of a, $V' \subset V$, such that

$$\sup_{b \in V'} \frac{P_o^b((W_t) \text{ leaves } V)}{E_o^b(1-e^{-\sigma_a})} < \infty$$

(with the convention that $\frac{0}{0} = 0$). Then (ii) holds if and only if (ii') does.

(b) Conversely, suppose that for every initial measure μ with $\mu\{a\} = 0$ and $E_o^\mu[1 - \exp(-\sigma_a)] < 1$, the measure P_o^μ is the excursion measure of a right continuous process. Then (4.1) holds.

<u>proof</u>: (a) Let V,V' be as above, and assume that (ii) holds. Then

$$n((W_t)\text{leaves } V) \le n(W_o \notin V') + n(W_o \notin V', (W_t) \text{ leaves } V).$$

The first term is finite by (ii), and the second term is

$$n(W_o \in V'\backslash\{a\}, (W_t) \text{ leaves } V)$$

$$+ \lim_{\delta\downarrow o} n(W_o = a,\ W_t \in V' \text{ for } t \in [o,\delta],\ \sigma_a > \delta,\ (W_t) \text{ leaves } V)$$

$$= \int_{\{W_o \in V'\backslash\{a\}\}} P_o^{u(o)}((W_t) \text{ leaves } V) n(du) + \lim_{\delta\downarrow o} \int_{\{W_o=a, W_t\in V' \text{ for } t\in[0,\delta],\ \sigma_a>\delta\}} P_o^{u(\delta)}((W_t) \text{ leaves } V)\ n(du)$$

$$\le \sup_{\substack{b\in V'\\ b\neq a}} \frac{P_o^b((W_t) \text{ leaves } V)}{E_o^b(1-e^{-\sigma_a})} \Big[\int_{\{W_o\in V'\backslash\{a\}\}} E_o^{u(o)}(1-e^{-\sigma_a})\ n(du) +$$

$$+ \liminf_{\delta\downarrow o} \int_{\{W_o=a, W_t\in V' \text{ for } t \in [o,\delta],\ \sigma_a > \delta\}} E_o^{u(\delta)}(1-e^{-\sigma_a})\ n(du) \Big]$$

$$= \sup_{\substack{b\in V'\\ b\neq a}} \frac{P^b((W_t) \text{ leaves } V)}{E_o^b(1-e^{-\sigma_a})} \Big[\int_{\{W_o\in V'\backslash\{a\}\}} (1-e^{-\sigma_a})\ dn + \liminf_{\delta\downarrow o} \int_{\substack{\{W_o=a, W_t\in V' \text{ for}\\ t \in [o,\delta],\ \sigma_a > \delta\}}} (1-e^{-\sigma_a-\delta})\ dn \Big]$$

$$\le \sup_{\substack{b\in V'\\ b\neq a}} \frac{P^b((W_t) \text{ leaves } V)}{E_o^b(1-e^{-\sigma_a})} \int (1-e^{-\sigma_a})\ dn < \infty.$$

(b) Assume (4.1) fails for some open neighborhood V of a. Then there are $b_k \in E\backslash\{a\}$, $b_k \to a$ such that

$$a_k = \frac{P_o^{b_k}((W_t) \text{ leaves } V)}{E_o^{b_k}(1-e^{-\sigma_a})} \to \infty.$$

By passing to a subsequence, we may assume that $a_k \ge k$ for each k.

Let

$$\lambda_k = (k^2 \, E_o^{b_k} \, (1-e^{-\sigma a}))^{-1}$$

$$\mu = \sum_{k=2}^{\infty} \lambda_k \, \varepsilon_{b_k}$$

(where ε_b is the point mass concentrated at b). Then

$$E_o^{\mu} \, (1-e^{-\sigma a}) = \sum_{k=2}^{\infty} k^{-2} < 1,$$

while

$$p_o^{\mu}((W_t) \text{ leaves } V) = \sum_{k=2}^{\infty} \lambda_k \, a_k \, E_o^{b_k} \, (1-e^{-\sigma a})$$

$$\geq \sum_{k=2}^{\infty} k^{-1} = \infty \qquad \square$$

The corresponding condition on (P_o^b) under which (vi') may be replaced by (vi) is that the class of positive measures n satisfying (i), (iii), (iv) and $n(U \backslash U^a) = 0$ consists of either multiples of a single probability measure, or consists completely of infinite measures. It is perhaps worth mentioning that though this condition fails for example 3, (4.1) is not in general a consequence of this condition. As example we can replace the space E of example 3 by

$$E' = E \cap \{(x,y);\ y \leq \sqrt{2x}\} \ .$$

In this case there are no measures satisfying (i) and (iv), and concentrated on U^a, since no path $r = R \cos \theta$, $R \in (1,2]$ lies entirely within E', whereas by Proposition 2(b), there do exist n satisfying (i), (ii), (iii), (iv), (v), (vi') but not (ii').

5. Proof of Proposition 1

Condition (ii') is clearly necessary. Assume that a is instantaneous $(n(U) = \infty)$, but (vi'c) fails. We will find a set $H \in \mathcal{U}_{o+}$ such that $o < n(H) < \infty$, and $H \subset U^a \cap \{\sigma_a > o\}$. Let $H^0 \in \mathcal{U}^0_{0+}$ satisfy $H^0 \subset H$ and $n(H \backslash H^0) = 0$. By completeness of each $\mathcal{F}_t$, we obtain easily that for $h > 0$, the $(U, \mathcal{U}^0_h)$ -valued process $(X_{t+\cdot})$ is progressively measurable, for the filtration $(\mathcal{F}_{t+h})_{t\ 0}$. Thus,

$$T = \inf \{t > o;\ X_{t+\cdot} \in H^0\}$$

is an $(\mathcal{F}_{t+h+})_{t \geq o}$ stopping time, for every $h > 0$, so that it is in fact an $(\mathcal{F}_{t+})$ stopping time. T is finite almost surely since $n(H^0) > 0$, and $X_{T+\cdot} \in H^0$ a.s., since $n(H^0) < \infty$. This contradicts the strong Markov property of $(X_t, \mathcal{F}_t, P, P^b)$, as a is instantaneous, yet $X_T = a$ and $\sigma_a(Y_{L_T}) > o$.

Similarly, in the case that X visits a discretely, but (vi'a) fails, we will obtain a contradiction to the strong Markov property of X by finding a set $H \in \mathcal{U}_{o+}$ such that $H \subset U^a \cap \{\sigma_a > o\}$ and $o < n(H) < 1$.

In both cases, the argument would be simplified if we had assumed that X was a right process, and we had the Ray-Knight compactification at our disposal.

In the first case, let D be the set of dyadic rational numbers. For $\varepsilon, \delta > o$ let

$$B_{\varepsilon,\delta} = \{b \in E;\ P^b_o\ (\sigma_a \geq \delta) \geq \varepsilon\} \in \mathcal{E}$$

$$H_{\varepsilon,\delta} = \bigcup_{\eta > o} \{W_o = a,\ W_t \in B_{\varepsilon,\delta} \text{ for } t \in D \cap (o,\eta),\ \sigma_a > 0\}.$$

We will show that for some $\varepsilon, \delta > o$ we have

$$o < n(H_{\varepsilon,\delta}) < \infty,$$

so that $H_{\varepsilon,\delta}$ satisfies the above conditions.

For ε and δ fixed, let

$$\tau(u) = \inf \{t > o;\ t \in D \text{ and } u(t) \notin B_{\varepsilon,\delta}\}.$$

Then τ is a $(\mathcal{U}_{t+})$ stopping time. For $\rho \in D \cap (o,\infty)$, we have $\{\tau > \rho\} \in \mathcal{U}_{\rho+}$, so that by Lemma 6,

$$\begin{aligned} n(\sigma_a \geq \delta) &\geq n(\tau > \rho,\ \sigma_a \geq \rho + \delta) \\ &= \int_{\{\tau > \rho\}} P_o^{u(\rho)}(\sigma_a \geq \delta)\ n(du) \\ &\geq \varepsilon\ n\ (\tau > \rho). \end{aligned}$$

Letting $\rho \downarrow o$, $\rho \in D$ we obtain that $n(\sigma_a \geq \delta) \geq \varepsilon n\ (H_{\varepsilon,\delta})$. Thus $n(H_{\varepsilon,\delta}) < \infty$ for every $\varepsilon, \delta > o$.

Conversely, since (vi'c) fails, there is some measure n' satisfying (iv) and (v) such that $n \geq n' \geq o$, and $o < n'\ (U^a) < \infty$. Fix $\varepsilon\ \delta > o$ such that $n(H_{\varepsilon,\delta}) = o$. Then also $n'(H_{\varepsilon,\delta}) = 0$. Thus, for $\eta > o$,

$$n'\ (W_o = a,\ W_t \in B_{\varepsilon,\delta} \text{ for every } t \in (o,\eta) \text{ such that } t = j\,2^{-k} \text{ for some } j\ ) \downarrow o \text{ as } k \to \infty.$$

That is,

$$\sum_{j=1}^{[2^k\eta-1]} n'\ (W_o = a,\ W_{j2^{-k}} \notin B_{\varepsilon,\delta},\ W_{i2^{-k}} \in B_{\varepsilon,\delta} \text{ for } 1 \leq i < j\ ) \uparrow n'(U^a) \text{ as } k \to \infty.$$

Thus, for $\eta \in (o,\delta)$,

$$n'(W_o = a,\ \sigma_a \geq 2\delta)$$

$$= \lim_{k\to\infty} \sum_{j=1}^{[2^k\eta-1} n'(W_o = a,\ \sigma_a \geq 2\delta,\ W_{j2^{-k}} \notin B_{\varepsilon,\delta},\ W_{i2^{-k}} \in B_{\varepsilon,\delta} \quad \text{for } 1 \leq i < j)$$

$$= \lim_{k\to\infty} \sum_{j=1}^{[2^k\eta-1]} \int_{\{W_o = a,\ \sigma_a \geq j2^{-k},\ W_{j2^{-k}} \notin B_{\varepsilon,\delta},\ W_{i2^{-k}} \in B_{\varepsilon,\delta} \quad \text{for } 1 \leq i < j\}} P_o^{u(j2^{-k})}(\sigma_a \geq 2\delta - j2^{-k})\ n(du)$$

$$\leq (1-\varepsilon)\ n'\ (U^a)\ .$$

Since $n'(U^a) \in (o,\infty)$, this cannot happen for every $\varepsilon,\delta>o$, so that indeed $n(H_{\varepsilon,\delta}) > o$ for some ε,δ.

In the second case, suppose that n is a probability measure concentrated on U^a, yet (vi'a) fails. Then we may find $\mu_1,\mu_2 \in (0,1)$ and probability measures n_1 and n_2, each concentrated on U^a and satisfying (i), (iv) and (v), such that $n_1 \neq n_2$ and $n = \mu_1 n_1 + \mu_2 n_2$. Since $n_1 \neq n_2$, we obtain from Lemma 6 that there is an open neighbourhood V of a, a set $A \in \mathcal{E}$ with $A \subset E\backslash V$; and numbers λ_1,λ_2 such that

$$n_1(W_\tau \in A) > \lambda_1 > \lambda_2 > n_2\ (W_\tau \in A),$$

where $\tau(u) = \inf\{t > o;\ u(t) \notin V\}$. Let V' be an open neighborhood of a, with $\overline{V'} \subset V$. Let

$$B = \{b \in V';\ P_o^b\ (W_\tau \in A) \geq \lambda_1\}\ .$$

Let D be the dyadic rationals, and put

$$H = \bigcup_{\eta>o} \{W_o = a,\ \sigma_a > o,\ W_t \in B \text{ for every } t \in D \cap (o,\eta)\}\ .$$

Then as before we obtain that

$$\lambda_2 > n_2\ (W_\tau \in A) \geq \lambda_1 n_2(H)\ ,$$

so that $n_2(H) < \lambda_2/\lambda_1 < 1$, and hence

$$n(H) = \mu_1 n_1(H) + \mu_2 n_2(H) < \mu_1 + \mu_2 = 1.$$

Conversely, if $n(H) = 0$ then also $n_1(H) = 0$, so that as before,

$$\lambda_1 < n_1\ (W_\tau \in A) \leq \lambda_1 n_1(U^a) = \lambda_1\ ,$$

which is impossible. □

6. Ray and Right Processes

Let (P^b) satisfy (2.4), and put

$$U^\alpha f(b) = E^b[\int_o^\infty e^{-\alpha t} f\ (W_t)\ dt],$$

for $f \in E$, $\alpha > o$. (P^b) is said to be Ray if E is compact, and if for each $\alpha > 0$, $U^\alpha f$ is continuous whenever f is.

Proposition 3 Let (P_o^b) be Ray, and satisfy (2.5). Let (P^b) satisfy (2.4) and (2.6). Then (P^b) is Ray.

proof:

$$U^\alpha f(b) = E^b[\int_o^{\tau_a} e^{-\alpha t} f\ (W_t)\ dt] + E^b[e^{-\alpha\tau_a}]\ U^\alpha f(a)$$

$$U_o^\alpha f(b) = E_o^b\ [\int_o^{\tau_a} e^{-\alpha t}\ f(W_t)\ dt] + \frac{1}{\alpha} E_o^b\ [e^{-\alpha\tau_a}]\ f(a).$$

Thus

$$U^\alpha f(b) = U_o^\alpha\ f(b) + E_o^b\ [e^{-\alpha\tau_a}](U^\alpha f(a) - \alpha^{-1}\ f(a)),$$

and so we need only show that $E_o^b\ [\exp(-\alpha\tau_a)]$ is continuous in b.

Using partitions of unity, choose continuous functions f_n on E such that $0 \le f_n \le 1$ and

$$1_{E\setminus\{a\}} = \sum_{n=1}^{\infty} f_n .$$

Each $U_0^\alpha f_n$ is continuous, so that by dominated convergence so is $U_o^\alpha 1_{E\setminus\{a\}}$, whenever $\alpha > 0$. But

$$U_o^\alpha 1_{E\setminus\{a\}}(b) = E_o^b \left[\int_o^{\tau_a} e^{-\alpha t}\, dt \right] = \frac{1 - E_o^b[e^{-\tau_a}]}{\alpha} \qquad \square$$

<u>Proposition 4</u> Let (P_o^b) be Ray, and satisfy (2.5). Suppose also that $P_o^b(U^a) < 1$ for each $b \neq a$. Let n be a positive measure on $(U,\mathcal{U})$ satisfying (i), (iv) and (v), and suppose that $n(\sigma_a \ge \delta) < \infty$ for every $\delta > o$. Then n satisfies (ii'). If $n(U^a) > o$ then n also satisfies (vi'c).

<u>Remark</u>: The condition that $n(\sigma_a \ge \delta) < \infty$ for $\delta > o$ will be satisfied provided (iii) holds. In particular, in the above situation, there are no 'discrete-visiting' extensions of (P_o^b).

<u>proof</u>: Suppose $n(U^a) > o$. If n is finite, or (vi'c) fails, then there is a finite non zero measure n' concentrated on U^a which satisfies (iv), so that

$$\mu_t(A) = n'\ (W_t \in A)$$

defines a bounded system of entrance laws for (P_o^b). Because (P_o^b) is Ray, there is a finite measure μ_o on E such that $\mu_t = P_0^{\mu_o}(W_t \in (\cdot))$. Thus $n' = P_o^{\mu_o}$, which is impossible, as n' is concentrated on U^a. Thus (vi'c) holds, and (vi'a) is vacuous.

To show (ii'), observe as in Proposition 3, that $E_o^b[1-\exp(-\sigma_a)]$ is continuous in b, hence

$$V_\varepsilon = \{b;\ E_o^b[1-e^{-\sigma_a}] < \varepsilon\},\ \varepsilon > o$$

define a nested family of open neighborhoods of a. Since $P_o^b(U^a) < 1$ for $b \neq a$, their intersection is $\{a\}$. Thus if W is any other open neighborhood of a,

$$\{W\} \cup \{E\backslash\overline{V}_\varepsilon\ ;\ \varepsilon > o\}$$

forms an open cover of E. Since E is compact, this shows that the V_ε form a base of open neighborhoods of a. Thus, it suffices to show that

$$n((W_t) \text{ leaves } V_\varepsilon) < \infty$$

for each $\varepsilon > o$. Fix $\varepsilon > o$, and let

$$\tau(u) = \inf\{t \geq o;\ u(t) \notin V_\varepsilon\}\ .$$

Then $W_\tau \notin V_\varepsilon$ on $\{\tau < \infty\}$, since V_ε is open. Also, for $b \in E\backslash V_\varepsilon$ and $\delta > o$ we have that

$$\varepsilon \leq E_o^b\ (1 - e^{-\sigma_a}) \leq 1 - e^{-\delta} + P_o^b\ (\sigma_a \geq \delta)\ ,$$

so that by lemma 6,

$$\infty > n(\sigma_a \geq \delta) \geq n(\tau < \infty,\ \sigma_a \geq \delta + \tau)$$

$$= \int_{\{\tau<\infty\}} P_o^{u(\tau(u))}\ (\sigma_a \geq \delta)\ n(du)$$

$$\geq (\varepsilon - (1 - e^{-\delta}))\ n\ (\tau < \infty)\ .$$

Choosing δ small, we obtain that $n(\tau < \infty) < \infty$ □

We could use this result to verify the conditions of Theorem 2 for the measures n considered in [4].

The condition that $P_o^b(U^a) < 1$ for each $b \neq a$ rules out the following pathology:

Example 4: Let $E = [0,1] \times \{0,1\}$, $a = (o,o)$. Make a absorbing, and on $[0,1] \times \{1\}$ let (P^b_o) correspond to Brownian motion, reflecting at $(1,1)$, with $(0,1)$ a branch point to a. On $(0,1] \times \{0\}$, let (P^b_o) correspond to a Brownian motion reflected at $(1,0)$, except that there is a jump from $(x,0)$ to $(x,1)$ at rate $g(x)$, where $g(x) \to \infty$ fast enough as $x \downarrow 0$ so that

$$P_o^b((W_t) \text{ hits } [0,1] \times \{1\}) = 1$$

for every $b \in E\backslash\{a\}$. Then we can find μ on $E\backslash\{a\}$ such that $n = P_o^\mu$ satisifes (i), (ii), (iii), (iv), (v), and (vi'), but not (ii'), even though (P^b_o) is Ray.

We now consider Right processes; Let E be a U-space, that is, a universally measurable subset of some compact metric space. Let $\mathcal{E}$ be the σ-field of its universally measurable subsets. Let $(\Omega,\mathcal{F})$ be a measurable space, and let $(\hat{P}^b)_{b\in E}$ be a family of probability measures on $(\Omega,\mathcal{F})$, such that $\hat{P}^{\cdot}(B)$ is $\mathcal{E}$-measurable for each $B \in \mathcal{F}$. Let $(\mathcal{F}_t)$ be a filtration of $(\Omega,\mathcal{F})$ which is right continuous, and satisfies

$$\mathcal{F}_t = \bigcap_\mu \mathcal{F}_t^\mu$$

where for each finite positive measure μ on $(\Omega,\mathcal{F})$, $\mathcal{F}_t^\mu$ denotes the σ-field obtained from $\mathcal{F}_t$ by adjoining all the P^μ- null sets of the $\hat{P}^\mu$ completion of $\mathcal{F}$. These conditions will be assumed throughout the

remainder of the section. If further

(6.1)
$$\begin{cases} X \text{ is a right continuous process with values in } E, \\ \quad \text{which is adapted to } (\mathcal{F}_t). \\ \hat{P}^b(X_o = b) = 1 \quad \text{for each } b \in E \\ \hat{P}^b(X_{t+h} \in B \mid \mathcal{F}_t) = \hat{P}^{X_t}(X_h \in B),\ \hat{P}^b\text{-a.s., for} \\ \quad \text{each } b \in E,\ B \in \mathcal{U} \text{ and } t, h \geq o \end{cases}$$

(6.2) $f(X.)$ is $\hat{P}^\mu$ - a.s. right continuous, for each $\alpha > o$, finite positive measure μ, and α-excessive function f.

Then $(X_t, \mathcal{F}, \mathcal{F}_t, \hat{P}^b)$ is called a <u>right process</u>.

Under condition (6.1), we will write

$$P^b(B) = \hat{P}^b(X.\in B)\ , \text{ for } b \in E,\ B \in \mathcal{U}.$$

Following Sharpe [46] we call a process (Z_t), adapted to $(\mathcal{F}_t)$, <u>nearly optional</u> if it is optional with respect to each filtration $(\mathcal{F}_t^\mu)$. A function $f \in \mathcal{E}$ is <u>nearly Borel</u> if for each μ there exist Borel functions f_1 and f_2 on E such that $f_1 \leq f \leq f_2$, and

$$\hat{P}^\mu(f_1(X_t) \neq f_2(X_t) \text{ for some } t \geq o) = 0\ .$$

Recall that in the presence of (6.1), condition (6.2) is equivalent to the conditions

(6.3) $(X_t, \mathcal{F}_t, \hat{P}^c, P^b)$ is strong Markov for each $c \in E$,

(6.4) $f(X.)$ is nearly optional, for each $\alpha > o$ and each α-excessive function f.

A stronger condition than (6.4) is

(6.5) For every $\alpha > o$, each α-excessive function is nearly Borel.

This property has the advantage of being invariant under choices of right continuous realizations, hence is a property of the transition laws (P^b). That is, if $(X_t, \mathcal{F}, \mathcal{F}_t, \hat{P}^b)$ satisfies (6.1), and the (P^b) are the transition laws of some right process with all α-excessive functions nearly Borel, for $\alpha > o$, then in fact $(X_t, \mathcal{F}, \mathcal{F}_t, \hat{P}^b)$ has nearly Borel α-excessive functions for $\alpha > o$ as well, so that it is a right process.

It is unknown (see Sharpe [46]) whether (6.4) (and hence (6.2)) are invariant in the same manner.

Suppose now that Y is a measurable function $(\Omega, \mathcal{F}) \to (\Pi, \mathcal{P})$ such that with respect to each $\hat{P}^b$, $(Y_t, \mathcal{F}_t)$ is a PPP with characteristic measure n not depending on b. Define (P_o^b) by

$$P_o^b(B) = \hat{P}^b\ (Y_o \in B)\ ,$$

for $B \in \mathcal{U}$, and suppose that (P_o^b) satisfies (2.4), (2.5) and

$$P_o^b(W_o = b) = 1\ ,$$

for each $b \in E$. Suppose finally that n satisfies (i), (ii'), (iii), (iv), (v), and (vi'). In the proof of theorem 2, we used the completeness of our filtration to discard certain subsets of Ω. Under the present conditions on $(\mathcal{F}_t)$, these sets will still lie in $\mathcal{F}_o$, so that the proof of theorem 2 will give a single process $(X_t, \mathcal{G}_t)$ and a family of laws (P^b), such that for every $c \in E$, $(X_t, \mathcal{G}_t, \hat{P}^c, P^b)$ is strong Markov, and

$$\hat{P}^c\ (X \in B) = P^c(B)$$

for each $B \in \mathcal{U}$. We can ask when $(X_t, \mathcal{F}, \mathcal{G}_t, \hat{P}^b)$ is a right process.

By the above discussion, the following gives a sufficient condition for this to hold .

<u>Proposition 5</u> Suppose that for each $\alpha > o$, the α-excessive functions for (P_o^b) are nearly Borel. Then so are those for (P^b).

<u>proof</u>: Let f be α-excessive for (P^b), for some $\alpha > o$. Then f is α-excessive for (P_0^b) , so that by hypothesis, it is also nearly Borel for (P_0^b) . Using (iv) and (v) we can find Borel functions f^1 and f^2 with $f^1 \le f \le f^2$ and

$$n(f^1(W_t) \neq f^2(W_t) \text{ for some } t \ge 0) = 0 .$$

It follows that f is nearly Borel for (P^b), as required . □

Until the question is settled, of whether (6.4) is invariant under choices of right continuous realizations, it will be impossible to show that $(X_t, \mathcal{F}, \mathcal{G}_t, \hat{P}^b)$ is in general a right process, even when the (P_o^b) arise from one. We can show, however, that this is the only obstruction to a proof of this result;

Assume that the (P_o^b) arise from a right process. Then there is a compact metric space $\hat{E}$ (the Ray-Knight compactification), containing E as a universally measurable subset, such that the resolvant of (P_o^b) extends to a Ray resolvant on $\hat{E}$, separating points thereon. Further, any right process on E with transition laws (P_o^b), is almost surely right continuous in the topology of $\hat{E}$.

Replace U by the set

$\tilde{U} = \{u;$ u is right continuous: $[0,\infty) \to E$, in the E-topology, and is right continuous in the $\hat{E}$ topology at all times t such that $u(t) \neq a.\}$.

$\tilde{\mathcal{U}}$ and $(\tilde{\mathcal{U}}_t)$ will be as before, with $\tilde{\mathcal{U}}$ replacing $\mathcal{U}$. The (P_o^b) induce laws $(\tilde{P}_o^b)$ $b \in \hat{E}$ on this new $\tilde{\mathcal{U}}$. Suppose we are given n and Y satisfying the condition preceding Proposition 5, but for our new objects $\tilde{\mathcal{U}}$ and $(\tilde{P}_o^o)$. Then the same argument applies to a give a $\tilde{\mathcal{U}}$ valued process (X_t), a filtration $(\mathcal{G}_t)$, and laws (P^b) such that for each $c \in E$, $(X_t, \mathcal{G}_t, \hat{P}^c, P^b)$ is strong Markov, and $\hat{P}^c(X.\epsilon B) = P^b(B)$ for $B \in \tilde{\mathcal{U}}$.

<u>Proposition 6</u> $(X_t, \mathcal{F}, \mathcal{G}_t, \hat{P}^b)$ is a right process.

proof: As in Sharpe [46] we need only show that $U^\alpha f(X_t)$ is nearly optional whenever $\alpha > o$ and f is positive, bounded and $\mathcal{E}$-measurable. Since E is universally measurable in $\hat{E}$, we need only check this for f the restriction to E of a positive, bounded, $\hat{E}$-Borel function, and hence by monotone class arguments, only for f the restriction to E of a positive, $\hat{E}$-continuous function. The argument of Proposition 3 shows that for such f, $U^\alpha f$ is the restriction to E of an $\hat{E}$-continuous function, so that in fact, we need only show that X is $\hat{E}$-right continuous, $\hat{P}^\mu$ - a.s., for μ finite and positive.

By the proof of Proposition 4, there is a base of $\hat{E}$-neighbourhoods V_ε of a, such that

(6.6) $n((W_t) \text{ leaves } V_\varepsilon) < \infty$

for each $\varepsilon > o$. (Note that since the resolvant of $(\tilde{P}_o^b)$ on $\hat{E}$ separates the points of $\hat{E}$, the condition that $\tilde{P}_o^b(U^a) < 1$ for each $b \in \hat{E}$ is satisfied). In case (vi'b), this suffices, as in Theorem 2. Suppose that (vi'c) holds. Then

$$\{W_o = a,\ W_{t_k} \notin V_\varepsilon \text{ for some sequence } t_k \downarrow o\} \in \tilde{\mathcal{U}}_{o+},$$

so that by (vi'c), it has n-measure zero. Thus

(6.7) $n(W_o = a \text{ but } W_t \not\to a \text{ in } \hat{E} \text{ as } t \downarrow o) = 0.$

As in Theorem 2, (6.6) and (6.7) suffice to show the result in case (vi'c).

Suppose finally, that (vi'a) holds. We use entrance laws, as in the proof of Proposition 4 to conclude the analogue of (6.7); that

(6.8) There is a point $C \in \hat{E}\backslash E$ such that

$$n(W_o = a,\ W_t \not\to c \text{ in } \hat{E} \text{ as } t \downarrow o) = 0.$$

Since $c \neq a$, (X_t) will not be $\hat{E}$-right continuous, however, the $U^\alpha f(X_t)$ will be, since a is an $\hat{E}$-branch point to c, so that $U^\alpha f(a) = U^\alpha f(c)$. □

We can reinterpret (vi') in terms of the Ray Knight compactification;

<u>Proposition 7</u> Let (P_o^b), E, and $(\tilde{P}_o^b)$ be as above. Let n be a positive measure on $(\tilde{U},\tilde{\mathcal{U}})$ satisfying (i), (iv), and (v) (for $\hat{E},\tilde{\mathcal{U}}$, and $(\tilde{P}_o^b)$), and suppose that $n(\sigma_a > \delta) < \infty$ for every $\delta > o$. Then

(a) $n\,(\hat{E} - \lim_{t\downarrow o} W_t \text{ does not exist}) = 0$

(b) Suppose n is a probability measure concentrated on $\tilde{U}^a = \{u \in \tilde{U};\ u(o) = a\}$. Then (vi'a) is equivalent to (6.8).

(c) Suppose n is an infinite measure, and $n(U^a) > o$. Then (vi'c) is equivalent to (6.7).

<u>proof:</u> (a) Clearly

$$n\,(W_o \neq a,\ \hat{E} - \lim_{t\downarrow o} W_t \neq W_o) = 0,$$

so that without loss of generality, we may assume that

$$n(\tilde{U}\backslash\tilde{U}^a) = 0.$$

As in the proof of Proposition 4, there is a base of open $\hat{E}$-neighborhoods V_ε of a, such that

$$n((W_t) \text{ leaves } V_\varepsilon) < \infty ,$$

for each $\varepsilon > o$. Fix $\varepsilon > o$, and let

$$B_\varepsilon = \{\text{for every } \delta > o,\ W_s \notin V_\varepsilon \text{ for some } s \in (o,\delta)\}.$$

Then

$$\mu_t\ (A) = n\ (W_t \in A,\ B_\varepsilon)$$

defines a bounded system of entrance laws for $(\tilde{P}_o^b)$, so that, as usual, there is a finite positive measure μ_o on $\hat{E}$ with

$$\mu_t\ (A) = \tilde{P}_o^{\mu_o}\ (W_t \in A) ,$$

for $t > o$. Thus

$$n|_{B_\varepsilon} = \tilde{P}_o^{\mu_o} ,$$

so that

$$n\ (B_\varepsilon,\ \hat{E} - \lim_{t \downarrow o} W_t \text{ does not exist}) = o,$$

for every $\varepsilon > o$. But

$$\tilde{U} \setminus \bigcup_{\varepsilon > o} B_\varepsilon = \{\hat{E} - \lim_{t \downarrow o} W_t = a\} .$$

so that (a) is proven.

(b) It was shown in the proof of Proposition 6, that (vi'a) implies (6.8). Conversely, if (6.8) holds for n, then whenever $n \geq n' \geq o$, it also holds for n'. Thus if n' satisifes (iv) as well, also

$$n' = n'\ (\tilde{U})\ \tilde{P}_o^c = n'\ (u)\ n.$$

(c): It was shown in the proof of Proposition 6, that (vi'c) implies (6.7). The converse is an immediate consequence of part (a), and Proposition 4. □

Note that the proof of Theorem 2 could be shortened under the conditions of Proposition 6; It is well known that a Markov process with a Ray resolvant, is in fact strong Markov. Thus, by the proof of Proposition 3, if we start with a (P_o^b) which is Ray, then it suffices to check the Markov property of the constructed process (X_t), at deterministic times t. This is simpler because no excursion can start at t with positive probability, but as well, because an explicit expression can be given for the times R_q^k. If instead, only the conditions of Proposition 6 are verified, we construct a right continuous process (X_t) as in Theorem 2. Proposition 7(a) shows that (X_t) has an $\hat{E}$-right continuous modification $(\hat{X}_t)$, to which the argument just given applies, showing that $(\hat{X}_t)$ is strong Markov. Then (6.7) and (6.8), which are equivalent to (vi'a) and (vi'c), are seen to be exactly the conditions required to infer the strong Markov property of (X_t), from that of $(\hat{X}_t)$.

7. Variations on Lemma 7

We turn to the variations of Lemma 7 alluded to in section 2. Let E be a topological space and $\mathcal{E}$ its Borel field. Recall that a function $K(x,dy)$ is called a kernel, if for each $x \in E$, $K(x,\cdot)$ is a probability measure on $(E,\mathcal{E})$, and if $K(\cdot,A) \in \mathcal{E}$ whenever $A \in \mathcal{E}$.

Proposition 8 Let $(\Omega,\mathcal{F},P)$ be a probability space. Let (Y_t) be a càdlàg process with values in E and suppose it is adapted to a filtration $(\mathcal{F}_t)$ of $(\Omega,\mathcal{F})$. Let $K(x,dy)$ be a kernel on E. Let Q be a countable ordered set, and for each $q \in Q$, let S_q be an $(\mathcal{F}_t)$ stopping time. Suppose that the following conditions hold:

If $q \le q'$ then $S_q \le S_{q'}$ and $\{S_q \le S_{q'}\} \in \mathcal{F}_{S_{q-}}$.

$$P(Y_{S_q} \in A | \mathcal{F}_{S_{q-}}) = K(Y_{S_{q-}}, A) \quad \text{a.s., for } q \in Q,\ A \in \mathcal{E}.$$

Let T be an $(\mathcal{F}_t)$ stopping time such that

$$[\![T]\!] \subset \bigcup_{q \in Q} [\![S_q]\!],$$

$$\{T = S_q\} \in \mathcal{F}_{S_{q-}} \vee \sigma(Y_{S_q}), \text{ for } q \in Q.$$

Then there is an $\mathcal{F}_{T-}$ measurable random subset H of E (that is, $\{(\omega,e)\ ;\ e \in H(\omega)\} \in \mathcal{F}_{T-} \otimes \mathcal{E}$), such that

$$P(Y_T \in A | \mathcal{F}_{T-}) = \frac{K(Y_{T-}, H \cap A)}{K(Y_{T-}, H)} \quad \text{a.s.}$$

for any $A \in \mathcal{E}$.

<u>proof:</u> Since

$$\{T \subset S_q\} \in \mathcal{F}_{S_{q-}} \vee \sigma(Y_{S_q}),$$

there is a function $\tilde{f}_q : \Omega \times E \to \{0,1\}$, measurable with respect to $\mathcal{F}_{S_{q-}} \otimes \mathcal{E}$, with

$$\{T = S_q\} = \{\omega;\ \tilde{f}_q(\omega, Y_{S_q}(\omega)) = 1\}.$$

Put

$$f_q = \bigvee_r [\tilde{f}_r \wedge 1_{\{S_q = S_r\}} \times E].$$

Then $f_q \in \mathcal{F}_{S_{q-}} \otimes \mathcal{E}$, since

$$\{\tilde{f}_r \wedge 1_{\{S_q = S_r\} \times E} = 1\} = \{\tilde{f}_r = 1\} \cap \{S_q = S_r\} \times E$$

$$\in (\mathcal{F}_{S_r-} \otimes \mathcal{E}) \cap (\{S_q = S_r\} \times E)$$

$$= (\mathcal{F}_{S_r-} \cap \{S_q = S_r\}) \otimes \mathcal{E}$$

$$= (\mathcal{F}_{S_q-} \cap \{S_q = S_r\}) \otimes \mathcal{E}$$

$$\subset \mathcal{F}_{S_q-} \otimes \mathcal{E} ,$$

as $\{S_q = S_r\} \in \mathcal{F}_{S_q-}$. Also, $f_q = f_r$ on $\{S_q = S_r\} \times E$, and

$$\{T = S_q\} = \{\omega; f_q(\omega, Y_{S_q}(\omega)) = 1\} .$$

Put

$$f = \bigvee_q [f_q \wedge 1_{\{S_q = T\} \times E}].$$

Because $f_q = f_r$ on $\{S_q = S_r\} \times E$, it follows that $f = f_q$ on $\{S_q = T\} \times E$. We will show that $f \in \mathcal{F}_{T-} \otimes \mathcal{E}$. This follows as above, once we show that $\{S_q = T\} \in \mathcal{F}_{T-}$. To show this, observe that since the S_q are increasing in $q \in Q$, and

$$[\![T]\!] \subset \bigcup_q [\![S_q]\!] ,$$

we can write

$$\{S_q = T\} = \bigcup_{r \le q} [\{S_r \ge T\} \cap \{S_r = S_q\} \cap \bigcap_{r' < r} \{S_{r'} < T\}] .$$

But

$$\{S_r \ge T\} \cap \bigcap_{r' < r} \{S_{r'} < T\} \in \mathcal{F}_{T-},$$

and

$$\{S_r = S_q\} \cap \{S_r \le T\} \in \mathcal{F}_{S_r-} \cap \{S_r \le T\} \subset \mathcal{F}_{T-} \cap \{S_r \le T\} ,$$

so that for $r \le q$,

$$\{S_r \ge T\} \cap \{S_r = S_q\} \cap \bigcap_{r'<r} \{S_{r'} < T\}$$

$$\in (\mathcal{F}_{T-} \cap \{S_r \le T\}) \cap (\{S_r \ge T\} \cap \bigcap_{r'<r} \{S_{r'} < T\})$$

$$= \mathcal{F}_{T-} \cap (\{S_r \ge T\} \cap \bigcap_{r'<r} \{S_{r'} < T\})$$

$$\subset \mathcal{F}_{T-} ,$$

as required.

Let

$$H_q(\omega) = \{e;\ f_q\ (\omega,e) = 1\}$$

$$H(\omega) = \ e;\ f(\omega,e) = 1\} .$$

We have assumed that

$$E\ [g(\cdot,X_{S_q}(\cdot))] = \int [\int g(\omega,x)\ K\ (X_{S_{q-}}(\omega),\ dx)]\ P(d\omega)$$

for g of the form $1_{A\times B}$, $A \in \mathcal{F}_{S_q-}$, $B \in \mathcal{E}$. This therefore extends to general $g \in \mathcal{F}_{S_q-} \otimes \mathcal{E}$, $g \ge 0$. Take

$$g = 1_{A\times B} \cdot f_q ,$$

where $A \in \mathcal{F}_{S_q-}$, $B \in \mathcal{E}$. Then

$$P(A,\ X_T \in B,\ S_q = T) = E\ (g(\cdot, X_{S_q}(\cdot))]$$

$$= \int_A [\int_B f_q\ (\omega,x)\ K\ (X_{S_q^-}(\omega)\ ,\ dx)]\ P(d\omega)$$

$$= \int_A P(S_q = T \mid F_{S_q^-})(\omega)\ \frac{K\ (X_{S_q^-}(\omega),\ B \cap H_q(\omega))}{K(X_{S_q^-}(\omega),\ H_q(\omega)}\ P(d\omega)$$

$$= \int_{A \cap \{S_q = T\}} \frac{K(X_{S_q^-},\ B \cap H_q\ )}{K(X_{S_q^-},\ H_q)}\ dP.$$

Enumerate Q as $q_1, q_2, \ldots\ ,$ and let $A \in F_{T-}$. Then there are $A_q \in F_{S_{q-}}$ such that

$$A \cap \{T = S_q\} = A_q \cap \{T = S_q\}\ .$$

Thus

$$P(A,\ X_T \in B) = \sum_j P\ (A_{q_j},\ X_T \in B,\ S_{q_j} = T,\ S_{q_i} \neq S_{q_j}\ \text{ for }\ i < j)$$

$$= \sum_j \int_{A_{q_j} \cap \{S_{q_j} = T,\ S_{q_i} \neq S_{q_j}\ \text{ for }\ i < j\}} K\ (X_{S_{q_j}^-}, B \cap H_{q_j})/K(X_{S_{q_j}^-},\ H_{q_j})\ dP$$

$$= \sum_j \int_{A \cap \{S_{q_j} = T,\ S_{q_i} \neq S_{q_j};\ \text{ for }\ i < j\}} K\ (X_{T-},\ B \cap H)/K\ (X_{T-},\ H)\ dp$$

$$= E\ [A,\ \frac{K(X_{T-},\ H \cap B}{K(X_{T-},\ H)}]$$

□

A similar result holds in discrete time. For clarity, the proof will be given with a discrete state space, but the general version can easily be obtained by modifying the preceding proof.

<u>Proposition 9</u> Let E be a countable set, and $(X_n)_{n\geq 1}$ an E-valued Markov chain with transition probabilities $P(x;dy)$. Let T be a stopping time with respect to the natural filtration of (X_n). Then there is an $\mathcal{F}_{T-1}$ measurable random subset H of E such that

$$P\ (X_T \in B | \mathcal{F}_{T-1}) = \frac{P\ (X_{T-1};\ B \cap H)}{P\ (X_{T-1};H)}\ , \quad \text{for} \quad B \subset E.$$

<u>proof</u>: Atoms of $\mathcal{F}_{T-1}$ are of the form

$$A = \{T = n,\ X_1 = a_1\ ,\ \ldots\ ,\ X_{n-1} = a_{n-1}\}\ .$$

Since T is a stopping time with respect to the natural filtration, it follows that for $a \in E$,

$$\{T = n,\ X_1 = a_1\ ..\ X_{n-1} = a_{n-1}\} \cap \{X = a\}$$

$$= \emptyset \quad \text{or} \quad \{X_1 = a_1,\ ..\ ,\ X_{n-1} = a_{n-1},\ X_n = a\}\ .$$

Put

$$H(\omega) = \{a\ ;\ A \cap \{X_T = a\} \neq \emptyset\}\ ,$$

where A is the atom of $\mathcal{F}_{T-1}$ containing ω. Thus, for every atom A of $\mathcal{F}_{T-1}$,

$$P\ (A,\ X_T = a) = P\ (X_1 = a_1\ \ldots\ X_{n-1} = a_{n-1})\ P\ (X_{T-1};\ H\cap\{a\}),\ \text{on } A$$

so that

$$\frac{P(A,\ X_T = a)}{P(A)} = \frac{P(X_{T-1};\ H\cap\{a\})}{P(X_{T-1};H)}\ ,\ \text{on } A$$

□

8. Regulation, Uniqueness and Construction of Point Processes

This section deals with some general results that will be needed later, when we discuss point processes of excursions away from sets (as opposed to points, as in the preceding sections). In general, these processes will not be PPP's, but will instead have property (8.7) below. The results we will discuss will be analogous to those found in Itô [32], in the PPP case.

Our point processes will be connected to a Markov process $(\hat{X}_t)$ (the extraneous " ^ " is for consistency with the next section). Throughout this section, we could have assumed that $(\hat{X}_t)$ was a right process with nearly Borel excessive functions (actually, we only need that the version of $(\hat{X}_t)$ on the canonical space be a right process). In applications it will in fact be a time change of such a process, and Gzyl [28] has shown that such objects are right processes (the nearly Borel proviso follows from his proof). Being unaware of this until just before completing the final draft, I have not made use of Gzyl's results. The relevant properties of right processes will thus be specified, and proved directly when needed. The net effect is that we show an apparently stronger form of Corollary 6, than is needed for applications, and we work harder than need be in Theorem 3. The two relevant properties are called 'regulation' and 'strong regulation' in the following. The latter plays a role only in uniqueness results, while the former is also used for existensc. See Weidenfeld [55] for an analysis of similar ideas appearing in a general study of time changes.

(A) Hypotheses. First, some general definitions: Given two general measurable spaces (Ω_i, F_i) $i=1,2$, an $(\Omega_1, F_1; \Omega_2, F_2)$ kernel is a function

$$n : \Omega_1 \times F_2 \to [0,\infty]$$

such that $n(\omega_1 , \cdot)$ is a measure on (Ω_2, F_2) for fixed $\omega_1 \in \Omega_1$,

and $n(\cdot, d\omega_2)$ is $\mathcal{F}_1$ measurable for fixed $d\omega_2 \in \mathcal{F}_2$.

If $(\Omega, \mathcal{F})$ is a measurable space, and $(\mathcal{F}_t)$ is a filtration thereof, recall that the <u>'usual conditions'</u> (with respect to a measure ν) state that $\mathcal{F}$ is ν-complete; $\mathcal{F}_{0-}$ consists of the ν-null sets of $\mathcal{F}$ together with their complements; $(\mathcal{F}_t)$ is right continuous; and $\mathcal{F}_0 \supset \mathcal{F}_{0-}$. If now

$$(\nu_a)_{a \in A}$$

is a family of measures on $(\Omega, \mathcal{F})$, we let $\mathcal{F}^a$ be the ν_a-completion of $\mathcal{F}$. We let $\mathcal{F}^a_{0-}$ consist of all the ν_a-null sets of $\mathcal{F}^a$, together with their complements, and then put

$$\mathcal{F}^a_t = \mathcal{F}_{t+} \vee \mathcal{F}^a_{0-} .$$

This lets us form the <u>augmented fields</u> (following Sharpe [46]);

$$\bar{\mathcal{F}} = \bigcap_a \mathcal{F}^a , \quad \bar{\mathcal{F}}_t = \bigcap_a \mathcal{F}^a_t .$$

If $\mathcal{F} = \bar{\mathcal{F}}$ and $\mathcal{F}_t = \bar{\mathcal{F}}_t$, we say that $(\mathcal{F}, \mathcal{F}_t)$ is <u>augmented</u> (with respect to the ν_a). We will in any case use the notation $\mathcal{F}^a$, $\mathcal{F}^a_t$, $\mathcal{F}^a_{0-}$ later, without further explanation. In this context, we will as usual say that a property holds <u>almost surely</u> (a.s.), if it holds ν_a -a.s. for each a. Note that $(\mathcal{F}^a, \mathcal{F}^a_t, \nu_a)$ satisfies the 'usual condition', for each a.

We will now fix a U-space F, together with a metric d thereon. We define objects

$$\mathcal{E}^{F,0}, \ \mathcal{E}^F, \ U^F, \ \mathcal{U}^F, \ W^F_t, \ \text{etc...}$$

as in section 2 (we will later have a second space E, and reserve the notation $\mathcal{E}^0$, U, etc... for the corresponding objects, with respect to E.) Now let $Q^x(du)$ be an

$$(F, \mathcal{E}^F;\ U^F,\ \mathcal{U}^F)$$

kernel, such that for every $c \in F$, Q^c is a probability measure, and

$$(W_t^F,\ \mathcal{U}_t^F,\ Q^c,\ Q^x)$$

is strong Markov. Let $\bar{\mathcal{U}}$, $\bar{\mathcal{U}}_t$ be the augmentations of $\mathcal{U}^F$, $\mathcal{U}_t^F$ under the Q^μ, μ positive and σ-finite. Recall that (K_t) is called a <u>perfect additive functional</u> (PAF), adapted to $(\bar{\mathcal{U}}_t)$, if it is right continuous and nondecreasing in $t \geq 0$, is adapted to $(\bar{\mathcal{U}}_t)$, and satisfies

$$K_{t+s} = K_t + K_s \circ \theta_t \quad \text{for every } t, s \geq 0 .$$

It is called <u>pure jump</u> if K_t is the sum of the jumps of $K.$ at times $s \leq t$. Regardless of whether this holds, we may always find $(\bar{\mathcal{U}}_t)$ stopping times $\tau_k = \tau_k^K$, such that

$$\{t > 0;\ K_{t-} \neq K_t\} = \{\tau_k ; k \geq 1\} \cap [0,\infty) .$$

Since F is a U-space, it follows from the proof of (13.4) of Getoor [23], that for every μ,

(8.1) $\{(t,u);\ W_{t-}^F(u)$ does not exist$\}$ is Q^μ-indistinguishable from the union of the graphs of countably many predictable $(\bar{\mathcal{U}}_t^\mu)$ stopping times.

Thus, for any totally inaccessible $(\bar{U}_t^\mu)$ stopping time τ, $W^F_{\tau-}$ exists Q^μ-a.s.

Our main definition can now be stated. We say that a pure jump PAF (K_t^0) <u>regulates</u> (Q^x), if there exists a strictly positive $b \in E^F \otimes E^F$ such that the following conditions hold for each μ:

(8.2) $K_t^0 < \infty$ a.s., for each t;

(8.3) $\tau_k = \tau_k^{K^0}$ is $(\bar{U}_t^\mu)$-totally inaccessible,

and $W^F_{\tau_k-} \neq W^F_{\tau_k}$, $K^0_{\tau_k} = K^0_{\tau_k-} + b(W^F_{\tau_k-}, W^F_{\tau_k})$ Q^μ-a.s. on $\{\tau_k<\infty\}$.

(8.4) If τ is any $(\bar{U}_t^\mu)$-totally inaccessible stopping time, such that $W^F_{\tau-} \neq W^F_\tau$ Q^μ-a.s. on $\{\tau<\infty\}$, then $K^0_{\tau-} \neq K^0_\tau$ Q^μ-a.s. on $\{\tau<\infty\}$.

(Q^x) is said to be <u>strongly regulated</u> if there exists a regulator (K_t^0) for it, and the following additional condition holds.

(8.5) If τ is any $(\bar{U}_t^\mu)$ totally inaccessible stopping time, then $W^F_{\tau-} \neq W^F_\tau$ Q^μ-a.s. on $\{\tau<\infty\}$.

(so that the jumps of (K_t^0) are exactly the totally inaccessible times with respect to the natural filtration).

Note that if

$$(W_t^F, \bar{U}, \bar{U}_t, Q^x)$$

is a right process, then (7.6) of Getoor [23] identifies the totally inaccessible times τ as times at which $\tilde{W}^F_{\tau-}$ exists, lies in F, and does

not equal W^F_τ (here $\tilde{W}^F_{t-}$ denotes the left limit at t, of W^F in the topology of its Ray-Knight compactification.) At such times, $\tilde{W}^F_{\tau-} = W^F_{\tau-}$, so that (3.3) of Benveniste et Jacod becomes the statement that (Q^x) is strongly regulated. Thus,

<u>Corollary 3</u> If $(W^F_t, \bar{U}, \bar{U}_t, Q^x)$ is a right process then it is strongly regulated.

<u>Lemma 8</u> Let (K^0_t) regulate (Q^x). Then there is a continuous PAF (K_t), and an $(F, \mathcal{E}^F; F, \mathcal{E}^F)$ kernel $N(x,dy)$ such that $N(x,F) \le 1$ for each x, and the following two conditions hold for each μ: (K_t) is a version of the $(\bar{U}^\mu_t, Q^\mu)$-dual predictable projection of (K^0_t); and for every positive $(\bar{U}^\mu_t)$ predictable process (Z_t), and every positive $f \in \mathcal{E}^F \otimes \mathcal{E}^F$,

$$E^\mu_Q[\int_0^\infty Z_s f(W^F_{s-}, W^F_s) dK^0_s]$$

$$= E^\mu_Q[\int^\infty Z_s \int F(W^F_s, y) N(W^F_s, dy) dK_s] .$$

<u>proof</u>: Theorem 2 of Benveniste et Jacod [2] applies to construct (K_t). It is continuous since the jumps of (K^0_t) are totally inaccessible. Mokobodski's absolute continuity argument ((2.5) of [2]) works in our case, so that as in Benveniste et Jacod [2], we may construct the $N(x,g)$, $g \in \mathcal{E}^F$. We make them into a kernel using the 'classical argument' at the end of Maisonneuve [38] □ .

<u>(B) Uniqueness</u>. Turning now to point processes let (V, ν^0) be a

separable measurable space and let $\mathcal{V}$ be its universal completion. In applications, $(V,\mathcal{V}^0)$ will be $(U,\mathcal{U}^0)$, where E is a separable metric space, so that no confusion will arise if we call the corresponding space of point functions, etc... by the names Π, $\mathcal{P}$, etc..., that were used in section 2. Let $n(x,dv)$ be an $(F, \mathcal{E}^F;\ V, \mathcal{V})$ kernel satisfying the (unnecessarily strong) condition that:

(8.6) There exist $V_k \in \mathcal{V}$, $V_k \uparrow V$ such that

$$\sup_x n(x, V_k) < \infty \quad \text{for each } k .$$

Let $Q^x(du)$ be an $(F, \mathcal{E}^F;\ U^F, \mathcal{U}^F)$ kernel consisting of probability measures, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space, with a filtration $(\mathcal{F}_t)$ satisfying the 'usual conditions'. Let $(\hat{X}_t)$ be a right continuous process with values in F and adapted to $(\mathcal{F}_t)$, such that

$$(\hat{X}_t, \mathcal{F}_t, P, Q^x)$$

is strong Markov. We will consider point processes (Y_t) adapted to $(\mathcal{F}_t)$ (for a point process, this means that $Y : \Omega \to \Pi$ satisfies $Y^{-1}(\mathcal{P}_t) \subset \mathcal{F}_t$) and satisfying the condition;

(8.7) For every positive $(\mathcal{F}_t)$ predictable process (Z_t), and every positive $f \in \mathcal{V}$,

$$E[\sum_{\substack{s>0 \\ Y_s \neq \delta}} Z_s f(Y_s)] = E[\int_0^\infty Z_s \int f(v) n(\hat{X}_s, dv) ds] .$$

We will determine to what degree (8.7) determines the joint law of

of $(\hat{X}., Y.)$, consider the strong Markov property of the pair $(\hat{X}., Y.)$, and then show how to construct such processes (Y_t). It turns out that solutions to (8.7) do not have unique conditional laws. The reason this is so is that (8.7) does not tell us how Y behaves at the totally inaccessible times for the minimal filtration of $(\hat{X}_t)$. We will soon impose conditions on the (Q^x) that give us information about the form of these times. To obtain uniqueness, we also fix a function G, measurable from $(F\times V, \mathcal{E}^F \otimes \mathcal{V})$ to $(F, \mathcal{E}^F)$, and we will impose the further condition that

(8.8) $\hat{X}_t = G(\hat{X}_{t-}, Y_t)$ for every $t > 0$ such that $Y_t \neq \delta$, a.s..

(Note that by (8.7), $Y_T = \delta$ a.s. for any $(\mathcal{F}_t)$ predictable time T, so that by (8.1), $\hat{X}_{t-}$ exists for every time t such that $Y_t \neq \delta$, P-a.s.). Let

$$G_0 = \{(x,v) ;\ G(x,v) = x\} .$$

Then for positive $f \in \mathcal{V}$, $g \in \mathcal{E}^F$,

$$x \to \int 1_{G_0^c}(x,v)g(G(x,v))F(v)n(x,dv)$$

is $\mathcal{E}^F$ - measurable, so that by (8.6), V.58 of Dellacherie et Meyer [14], and the seperability of $\mathcal{V}^0$, there exists a positive $\mathcal{E}^F \otimes \mathcal{E}^F$ measurable function

$$n(\cdot, \cdot; f) : F \times F \to [0,1] ,$$

for each $f \in \mathcal{V}$ with $\sup_v |f(v)| \leq 1$, such that

$$\int 1_{G_0^c}(x,v)\ 1_A(G(x,v))f(v)n(x,dv)$$

$$= \int 1_{A\setminus\{x\}}(G(x,v))n(x,G(x,v);f)n(x,dv)$$

for each $A \in \mathcal{V}$.

[Note: if V is itself a U-space, and $\mathcal{V}^0$ is its Borel field, then the 'classical argument' in the last section of Maisonneuve [38] shows that we can find $(F\times F,\ \mathcal{E}^F\otimes\mathcal{E}^F;\ V,\ \mathcal{V})$ kernels $n(x,y;\ dv)$ with $n(x,\ x;\ V) = 0$ and $n(x,\ y;\ V) \le 1$, such that

$$n(x,y;f) = \int f(v)n(x,y;dv)$$

will work in the above. This is the case when we take $(V,\mathcal{V}^0) = (U,\mathcal{U}^0)$, provided E is a U-space; see p.217 of Maisonneuve et Meyer [39]].

Lemma 9 Let (Y_t) be adapted to $(\mathcal{F}_t)$, and satisfy (8.7). Then for every positive $(\mathcal{F}_t)$ predictable process (Z_t), and positive $h \in \mathcal{E}^F\otimes\mathcal{E}^F$ we have that

$$E[\sum_{\substack{s>0\\ Y_s\neq\delta}} Z_s h(\hat{X}_{s-},\hat{X}_s)] = E[\int_0^\infty Z_s \int h(\hat{X}_s,v)n(\hat{X}_s,dv)ds]$$

proof: Let $h(x,v) = g(x)f(v)$ with $g \in \mathcal{E}^{F,0}$ positive and bounded and $f \in \mathcal{V}$ positive. The argument of (13.4) of Getoor [23] yields even more than (8.1), namely that

$$g(\hat{X}_{t-})1_{\{\hat{X}_{t-}\text{exists}\}}$$

is $(\mathcal{F}_t)$ predictable. Thus by (8.7),

$$E[\sum_{\substack{s>0\\ Y_s\neq\delta}} Z_s h(\hat{X}_{s-},\hat{X}_s)] = E[\int_0^\infty Z_s \int h(\hat{X}_{s-},v)1_{\{\hat{X}_{s-}\text{exists}\}}n(\hat{X}_s,dv)ds] .$$

But because F is a metric space and $(\hat{X}_t)$ is right continuous, $(\hat{X}_t)$ can have only countably many discontinuities. Thus, the right hand side equals

$$E[\int_0^\infty Z_s \int h(\hat{X}_s,v)n(\hat{X}_s,dv)ds] .$$

By monotone class and completion arguments, this extends to general positive $h \in \mathcal{E}^F \otimes \mathcal{V}$.

The final condition that we will assume, really belongs in a discussion of the problem of constructing (Y_t) as above. It is:

(8.9) For every $x \in F$, $t \geq 0$, and positive $f \in \mathcal{E}^F \otimes \mathcal{E}^F$, we have that

$$E_Q^x[\sum_{s\in(0,t]} f(W_{s-}^F, W_s^F)1_{\{K_{s-}^0 \neq K_s^0\}}]$$

$$\geq E_Q^x[\int_0^t \int 1_{G_0^c}(W_s^F,v)f(W_s^F,G(W_s^F,v))n(W_s^F,dv)ds] .$$

Observe that by (8.8), this would be a special case of Lemma 9, provided we assumed that for each $x \in F$, there exist Ω, $\mathcal{F}$, $(\mathcal{F}_t)$, P, $(\hat{X}_t)$, (Y_t) as above, with Q^x the image law of P under $\hat{X}$.

<u>Lemma 10</u> Let n satisfy (8.6, 7, 8, and 9), and suppose that (Q^x) is regulated by (K_t^0). Then

(a) Recall that (8.3) gave us a function $b \in \mathcal{E}^F \otimes \mathcal{E}^F$. There exists an $\mathcal{E}^F \otimes \mathcal{E}^F$ measurable function $J : F \times F \to [0,1]$, such that for each μ, $t \geq 0$, and positive $g \in \mathcal{E}^F \otimes \mathcal{E}^F$, we have that Q^μ-a.s.,

$$\int_0^t \int 1_{G_0^c}(W_s^F,v)g(W_s^F,G(W_s^F,v))n(W_s^F,dv)ds$$

$$= \int_0^t \int \frac{J(W_s^F,y)g(W_s^F,y)}{b(W_s^F,y)} 1_{\{b\neq 0\}}(W_s^F,y)N(W_s^F,dy)dK_s$$

(b) If $\varepsilon > 0$ and $r \geq 0$, put

$$R = \inf\{t>r; K_t^0(\hat{X}.) > K_{t-}^0(\hat{X}.) + \varepsilon\}$$

Then the inf is attained, and if J is any function as in (a), and $f \in \mathcal{V}$ is positive and bounded by 1, then

$$E[f(Y_R),\ Y_R \neq \delta \mid \mathcal{F}_{R-} \vee \sigma(\hat{X}_t;t\geq 0)]$$

$$= J(\hat{X}_{R-},\hat{X}_R)n(\hat{X}_{R-},\hat{X}_R;f),\ P\text{-a.s. on } \{R<\infty\} .$$

(c) If $r \geq 0$ and $k \geq 1$, set

$$T = \inf\{t>r;\ Y_T \in V_k,\ \hat{X}_{T-}=\hat{X}_T\} .$$

Then the infimum is attained, and for each positive $f \in \mathcal{V}$,

$$E[f(Y_T),\ T<\infty \mid \mathcal{F}_{T-} \vee \sigma(\hat{X}_t;t\geq 0)]$$

$$\frac{\int_{V_k} 1_{G_0}(\hat{X}_T,v)f(v)n(\hat{X}_T,dv)}{\int_{V_k} 1_{G_0}(\hat{X}_T,v)n(\hat{X}_T,dv)} \quad P\text{-a.s. on } \{\tau < \infty\}$$

(d) If in fact (Q^x) is strongly regulated, then

$$P(T > t \mid \mathcal{F}_r \vee \sigma(\hat{X}_s; s \geq 0))$$

$$= \exp\left(-\int_r^t \int_{V_k} 1_{G_0}(\hat{X}_s, v) n(\hat{X}_s, dv) ds\right)$$

for every $t \geq r$.

<u>proof</u>: Suppose that $f \in \mathcal{E}^F$ is positive, and

$$E_Q^x\left[\int_0^\infty f(W_s^F) dK_s\right] = 0 \quad \text{for each } x .$$

Then by Lemma 8,

$$E_Q^x\left[\int_0^\infty f(W_{s-}^F) dK_s^0\right] \leq E_Q^x\left[\int_0^\infty f(W_s^F) dK_s\right] = 0 .$$

for each x. Let

$$G_1 = \{x; \int 1_{G_0^c}(x,v) n(x,dv) > 0\} .$$

Thus by (8.9) we have that

$$E_Q^x\left[\int_0^\infty \int f(W_s^F) 1_{G_0^c}(W_s^F, v) n(W_s^F, dv) ds\right]$$

$$\leq E_Q^x\left[\sum_{s>0} f(W_{s-}^F) 1_{\{K_{s-}^0 \neq K_s^0\}}\right] = 0 ,$$

and hence

$$E_Q^x\left[\int_0^\infty f(W_s^F) 1_{G_1}(W_s^F) ds\right] = 0 .$$

As remarked in the proof of Lemma 8, (2.5) of Benveniste et Jacod [2] applies to our situation, and hence there is a positive $h \in \mathcal{E}^F$ such that for every μ,

$$\int_0^t 1_{G_1}(W_s^F)ds = \int_0^t h(W_s^F)dK_s \ , \quad Q^\mu\text{-a.s.} \ .$$

As before, we use V.58 of Dellacherie et Meyer [14] to obtain $\mathcal{E}^F \otimes \mathcal{E}^F$ measurable densities $J_1(x,y)$ and $J_2(x,y)$, for

$$n_1(x,dy) = 1_{\{x\neq y\}} n(x,\{v;\ G(x,v) \in dy\}) \ ,$$

and for $N(x,dy)$, each with respect to $n_1(x,dy) + N(x,dy)$. Let

$$J(x,y) = h(x)b(x,y)J_1(x,y)J_2^{-1}(x,y)1_{\{J_2\neq 0\}}(x,y) \ .$$

By (8.9),

$$E_Q^x[\int_0^\infty 1_{G_1}(W_s^F)\int 1_{\{J_2=0\}}(W_s^F,y)n_1(W_s^F,dy)ds] = 0$$

for every x, and hence similarly,

$$E_Q^\mu[\int_0^\infty Z_s \int 1_{G_0^c}(W_s^F,v)g(W_s^F,G(W_s^F,v))n(W_s^F,dv)ds]$$

$$= E_Q^\mu[\int_0^\infty Z_s 1_{G_1}(W_s^F)\int 1_{\{J_2\neq 0\}}(W_s^F,y)g(W_s^F,y)n_1(W_s^F,dy)ds]$$

$$= E_Q^\mu[\int_0^\infty Z_s \int \frac{J(W_s,y)g(W_s^F,y)}{b(W_s^F,y)} 1_{\{b\neq 0\}}(W_s^F,y)N(W_s^F,dy)dK_s]$$

for every μ, every positive $g \in \mathcal{E}^F \otimes \mathcal{E}^F$, and every positive $(\bar{\mathcal{U}}_t^\mu)$ predictable process (Z_t).

(b) Since $Y_R \in \mathcal{F}_R$, and X_{R+} is conditionally independent of $\mathcal{F}_R$ given X_R, it will suffice to show that the right hand side of (a) equals

$$E[f(Y_R),\ Y_R \neq \delta,\ R < \infty \mid \mathcal{F}_{R-} \vee \sigma(\hat{X}_R)] .$$

Let

$$R_0 = \inf\{t > 0;\ K_t^0 > K_{t-}^0 + \varepsilon\}.$$

Let $C \in \mathcal{F}_t$, $t \in r$, and let $g \in \mathcal{E}^F$ be positive. Then

$$E[C, t<R<\infty,\ Y_R \neq \delta,\ f(Y_R)g(\hat{X}_R)]$$

$$= E[C, \sum_{\substack{s \in (t,R] \\ Y_s \neq \delta}} f(Y_s)g(G(\hat{X}_{s-}, Y_s)) \cdot$$

$$1_{\{(x,v);G(x,v)\neq x, b(x,G(x,v))>\varepsilon\}}(\hat{X}_{s-}, Y_s)]$$

$$= E[C, \int_t^R \int f(v)g(G(\hat{X}_s, v))$$

$$1_{\{(x,v);G(x,v)\neq x, b(x,G(x,v))>\varepsilon\}}(\hat{X}_s, v)n(\hat{X}_s, dv)ds]$$

$$= E[C, t < R,\ E_Q^{\hat{X}_t}[\int_0^{R_0} \int$$

$$1_{\{(x,z);x\neq z, b(x,z)>\varepsilon\}}(W_s^F, G(W_s^F, v))$$

$$g(G(W_s^F, v))n(W_s^F, G(W_s^F, v); f)n(W_s^F, dv)ds]]$$

$$= E[C, t < R,\ E_Q^{\hat{X}_t}[\int_0^{R_0} \int \frac{J(W_s^F, y)g(y)n(W_s^F, y; f)}{b(W_s^F, y)}$$

$$1_{\{(x,z);x\neq z, b(x,z)>\varepsilon\}}(W_s^F, y)N(W_s^F, dy)dK_s]]$$

$$= E[C,\ t < R,\ E_Q^{\hat{X}_t}[\int_0^{R_0} \frac{J(W_{s-}^F, W_s^F)g(W_s^F)n(W_{s-}^F, W_s^F; f)}{b(W_{s-}^F, W_s^F)}$$

$$\cdot\ 1_{\{(x,z);x\neq z,b(x,z)>\varepsilon\}}(W^F_{s-},W^F_s)dK^0_s]]$$

$$= E[C,t<R,\ E_Q^{\hat{X}_t}[J(W^F_{R_0-},W^F_{R_0})g(W^F_{R_0})n(W^F_{R_0-},W^F_{R_0};f)]]$$

$$= E[C,t<R<\infty,\ J(\hat{X}_{R-},\hat{X}_R)g(\hat{X}_R)n(\hat{X}_{R-},\hat{X}_R;f)]\ ,$$

by Lemma 9 part (a), and (8.3), as required

(c) We argue as above, and use that $\hat{X}_T = \hat{X}_{T-} \in F_{T-}$, to see that it suffices to show that the right hand side of (c) equals

$$E[f(Y_T),\ T<\infty \mid F_{T-}]\ .$$

Let $C \in F_t$, $t \geq r$. Then by Lemma 9, applied twice,

$$E[C,t<T<\infty,\ f(Y_T)]$$

$$= E[C,\sum_{\substack{s\in(t,T]\\ Y_s\neq\delta}} f(Y_s)1_{V_k}(Y_s)1_{G_0}(\hat{X}_{s-},Y_s)]$$

$$= E[C,\int_t^T\int_{V_k} 1_{G_0}(\hat{X}_s,v)f(v)n(\hat{X}_s,dv)ds]$$

$$= E[C,\sum_{\substack{s\in(t,T]\\ Y_s\neq\delta}} \frac{\int_{V_k} 1_{G_0}(\hat{X}_{s-},v)f(v)n(\hat{X}_{s-},dv)}{\int_{V_k} 1_{G_0}(\hat{X}_{s-},v)n(\hat{X}_{s-},dv)}\ 1_{V_R}(Y_s)1_{G_0}(\hat{X}_{s-},Y_s)]$$

$$= E[C,t<T<\infty,\ \frac{\int_{V_k} 1_{G_0}(\hat{X}_{T-},v)f(v)n(\hat{X}_{T-},dv)}{\int_{V_k} 1_{G_0}(\hat{X}_{T-},v)n(\hat{X}_{T-},dv)}]\ ,$$

as required.

(d) For $t \geq r$, let T_t be a version of

$$P(T>t \mid \mathcal{F}_r \vee \sigma(\hat{X}_s;\ s \geq 0)$$

which is right continuous, decreasing, and equals 1 at $t = r$. Let

$$Z = Z^1 \cdot Z^2(\hat{X}_{.+r}) ,$$

where $Z^1 \in \mathcal{F}_r$ and $Z^2 \in \mathcal{U}^F$ is a function of the form

$$\prod_{i=1}^{k} f_i \circ W^F_{t_i} .$$

Write $({}^oZ_t)$ for a right continuous version of the $(P, \mathcal{F}_t)$-martingale $E[Z \mid \mathcal{F}_t]$. Write $({}^oZ^2_t)$ for the right continuous process that is a version of the Q^μ-martingale

$$E^\mu_Q[Z^2 \mid \bar{\mathcal{U}}^\mu_t] ,$$

for each μ (It exists by Lemma 1 of Benveniste et Jacod [3]). Then a direct computation shows that

$$\begin{aligned} {}^oZ_t &= Z^1 \cdot {}^oZ^2_{t-r}(\hat{X}_{r+\cdot}) \\ &= Z^1 \cdot {}^oZ^2_t(\hat{X}_\cdot) \quad \text{for every } t > r ,\ P\text{-a.s.} \end{aligned}$$

Because (Q^x) is strongly regulated, it follows that for each μ,

$$\{(t,u);\ {}^oZ^2_{t-}(u) \neq {}^oZ^2_t(u),\ K^0_{t-}(u) = K^0_t(u)\}$$

is Q^μ-indistinguishable from the union of the graphs of countably many $(\bar{\mathcal{U}}^\mu_t)$ accessible stopping times. Since T is totally inaccessible for $(\mathcal{F}_t)$, and $\hat{X}_{T-} = \hat{X}_T$, it follows that

$${}^oZ^2_{T-}(\hat{X}_\cdot) = {}^oZ^2_T(\hat{X}_\cdot) \quad P\text{-a.s.},$$

and hence also that ${}^oZ_{T-} = {}^oZ_T$ P-a.s. . But V.10 of Dellacherie [12] shows that

$$Z_t = {}^oZ_{t-}$$

is the $(P,\mathcal{F}_t)$ predictable projection of Z, so in fact,

$$Z_T = E[Z \mid \mathcal{F}_T] .$$

Thus, for positive bounded $g \in \mathcal{B}$,

$$E[Z\int_r^\infty g(t)dT_t] = -\hat{E}^\mu[Zg(T),\ r < T < \infty]$$

$$= -E[Z_T g(T),\ R < T < \infty]$$

$$= -E[\sum_{\substack{s\in(r,T] \\ Y_s\neq\delta}} Z_s g(s) 1_{V_k}(s) 1_{G_0}(\hat{X}_{s-},Y_s)]$$

$$= -E[Z\int_r^T g(s)\int_{V_k} 1_{G_0}(\hat{X}_s,v.)n(\hat{X}_s,dv)ds]$$

$$= E[Z(\int_r^\infty\int_r^t g(s)\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)dsdT_t -$$

$$- T_{\infty-}\int_r^\infty g(s)\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds)] .$$

Taking g in a countable dense subset of $C_0([r,\infty))$, we see that

$$\int_r^\infty g(t)dT_t = \int_r^\infty\int_r^t g(s)\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)dsdT_t$$

$$- T_{\infty-}\int_r^\infty g(s)\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds$$

for every bounded positive $g \in \mathcal{B}$, P-a.s.. We put

$$g(t) = 1_{[r,t_0]}(t)\ \exp(\int_r^t\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds)\ ,$$

for $t_0 \geq r$, to obtain that

$$\int_r^{t_0} \exp(\int_r^t\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds)dT_t$$

$$= \int_r^{t_0} (\exp(\int_r^t\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds) - 1)dT_t^+$$

$$+ (T_{\infty-} - T_{t_0})(\exp(\int_r^{t_0}\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds) - 1) -$$

$$- T_{\infty-}(\exp(\int_r^{t_0}\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds) - 1)\ ,$$

and hence that

$$T_{t_0} = \exp(-\int_r^{t_0}\int_{V_R} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds)$$

as required. □

We gather together the current hypotheses for the next result:

<u>Proposition 10</u> Let (Q^x) be strongly regulated, and let n satisfy (8.6) and (8.9). Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, with

a filtration $(\mathcal{F}_t)$ thereon, satisfying the 'usual conditions'. Let $(\hat{X}_t)$ be a right continuous process with values in F, which is adapted to $(\mathcal{F}_t)$ and is such that

$$(\hat{X}_t, \mathcal{F}_t, P, Q^x)$$

is strong Markov. Let G be fixed, and let (Y_t) be a point process, adapted to $(\mathcal{F}_t)$, satisfying (8.7) and (8.8). Let (K_t^0) regulate (Q^x), and let J be given by Lemma 10. Then

(a) For each $k \geq 1$, there exists (after enlarging Ω if necessary) a sequence $S_1, S_2, \ldots$ of exponentially distributed random variables with unit means, which are independent of each other and of $(\hat{X}_t)$, and satisfy the further conditions:

(b) If we put $T_0 = 0$ and

$$T_{i+1} = \inf\{t > T_i;\ Y_t \in V_k \text{ and } \hat{X}_{t-} = \hat{X}_t\},$$

then in fact,

$$T_i = \inf\{t > 0;\ \int_0^t \int_{V_k} 1_{G_0}(\hat{X}_s, v) n(\hat{X}_s, dv) ds \geq \sum_{j=1}^{i} S_j\}, \quad P\text{-a.s.};$$

(c) For $\varepsilon > 0$, we put $R_0 = 0$, and

$$R_{i+1} = \inf\{t > R_i;\ K_t^0(\hat{X}.) > K_{t-}^0(\hat{X}.) + \varepsilon\}.$$

Then for any positive $f_1 .. f_\ell, g_1 .. g_\ell \in \mathcal{V}$, the random variables

$$g_i(Y_{T_i}),\quad f_j(Y_{R_j}) 1_{\{Y_{R_j} \neq \delta\}},\quad i,j = 1..\ell$$

are conditionally independent given $(\hat{X}_t)$ and the $T_1 .. T_\ell$, with

conditional expectations respectively

$$\frac{\int_{V_k} 1_{G_0}(\hat{X}_{T_i},v)g_i(v)n(\hat{X}_{T_i},dv)}{\int_{V_k} 1_{G_0}(\hat{X}_{T_i},v)n(\hat{X}_{T_i},dv)} 1_{\{T_i<\infty\}}, \text{ and}$$

$$J(\hat{X}_{R_j-}, \hat{X}_{R_j})n(\hat{X}_{R_j-},\hat{X}_{R_j}; f_j)1_{\{R_j<\infty\}} .$$

<u>proof</u>: For $C \in F_{T_{\ell-1}} \vee \sigma(\hat{X}_t;\ t \geq 0)$

$$P(C,\int_{T_{\ell-1}}^{T_\ell}\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds > t)$$

$$= \lim_{i\to\infty}\sum_{j=1}^{\infty} P\Big(C,\int_{2^{-i}j}^{T_\ell}\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds > t,$$

$$T_\ell > \frac{j}{2^i} \geq T_{\ell-1} > \frac{j-1}{2^i}\Big)$$

$$= \lim_{i\to\infty}\sum_{j=1}^{\infty} e^{-t}P\Big(C,\int_{2^{-i}j}^{\infty}\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds > t,$$

$$T_\ell > \frac{j}{2^i} \geq T_{\ell-1} > \frac{j-1}{2^i}\Big)$$

$$= e^{-t}P\Big(C,\int_{T_{\ell-1}}^{\infty}\int_{V_k} 1_{G_0}(\hat{X}_s,v)n(\hat{X}_s,dv)ds\Big) > t) ,$$

by Lemma 10 (d). Thus, conditional on $F_{T_{\ell-1}} \vee \sigma(\hat{X}_t;\ t \geq 0)$,

$$\int_{T_{\ell-1}}^{T_\ell} \int_{V_k} 1_{G_0}(\hat{X}_s, v) n(\hat{X}_s, dv) ds$$

has the law of the minimum of a mean 1 exponential random variable and the number

$$\int_{T_{\ell-1}}^{\infty} \int_{V_k} 1_{G_0}(\hat{X}_s, v) n(\hat{X}_s, dv) ds .$$

Thus we can find an appropriate S_ℓ, P-independent of (X_t) and of $F_{T_{\ell-1}}$. Repeating the argument gives $S_{\ell-1}$, and since S_ℓ is independent of $F_{T_{\ell-1}}$, $S_{\ell-1}$ may be chosen independent of S_ℓ. By continuing this argument, we obtain a proof of (a) and (b). The proof of (c) is similar, now using (b) and (c) of Lemma 10 instead of (d).

Corollary 4 Under the conditions of Proposition 10, the joint law under P of $(\hat{X}., Y|_{(0,\infty)})$ may be expressed in terms of G, n, the Q^x, the initial law of $\hat{X}_0$ under P, and the function J obtained from an arbitrary regulator of (Q^x).

proof: All relevant probabilities may be calculated using Proposition 10. □

Note that even without our regulation hypotheses, there is much that (8.7) tells us about the distribution of (Y_t).

Lemma 11 Let (Ω, F, P) be a complete probability space, with a filtration (F_t) satisfying the usual conditions. Let $n(x, dv)$

and $Q^x(du)$ be $(F, \mathcal{E}^F; V, \mathcal{V})$ and $(F, \mathcal{E}^F; U^F, \mathcal{U}^F)$ kernels respectively, the Q^x being probability measures. Let $(\hat{X}_t)$ be a right continuous process with values in F, and adapted to $(\mathcal{F}_t)$, such that $(\hat{X}_t, \mathcal{F}_t, P, Q^x)$ is strong Markov. Let (Y_t) be a point process adapted to $(\mathcal{F}_t)$ and satisfying (8.7). Let $A \in \mathcal{V}$ satisfy

$$\sup_x n(x,A) < \infty ,$$

and let $r \geq 0$. Put

$$T = \inf\{t > r ;\ Y_t \in A\} .$$

Then: (a) There exists (upon enlarging Ω perhaps) an exponentially distributed random variable S, of unit mean, which is independent of $\mathcal{F}_r$ (but not necessarily of $\sigma(X_s;\ s \geq 0)$) and satisfies

$$T = \inf\{t;\ \int_r^t n(\hat{X}_s, A)ds \geq S\} .$$

(b) For $f \in \mathcal{V}$ positive,

$$E[f(Y_T) \mid \mathcal{F}_{T-}] = \frac{\int_A f(v)n(\hat{X}_{T-}, dv)}{n(\hat{X}_{T-}, A)} \quad P\text{-a.s. on } \{T<\infty\}.$$

(where again, $0/0 = 0$).

<u>proof</u>: (a) This is proved exactly as in Lemma 10 (d), except that since we no longer require S to be independent of $\sigma(\hat{X}_s;\ s \geq 0)$, we do not need to use the hypothesis of strong regulation.

(b) By (8.7), $\{s;\ Y_s \in A\}$ is P-a.s. discrete, so that $Y_T \in A$ on $\{T < \infty\}$. Let $B \in \mathcal{F}_t$. Then

$$E[B, t < T < \infty,\ f(Y_T)] = E[B, \sum_{\substack{s \in (r \vee t, T] \\ Y_s \neq \delta}} f(Y_s) 1_A(Y_s)]$$

$$= E[B, \int_{r \vee t}^{T} \int_A f(v) n(\hat{X}_s, dv) ds]$$

$$= E[B, \sum_{\substack{s \in (r \vee t, T] \\ Y_s \neq \delta}} \int_A \frac{f(v) n(\hat{X}_{s-}, dv)}{n(\hat{X}_{s-}, A)} 1_A(Y_s)]$$

(by Lemma 9)

$$= E[B,\ t < T < \infty\ ,\ \frac{1}{n(\hat{X}_{T-}, A)} \int f(v) n(\hat{X}_{T-}, dv)]$$

By monotone class arguments, this suffices. □

<u>(c) The Strong Markov Property</u>: To facilitate future applications, we will consider the following formulation. Let E be a separable metric space, with $\mathcal{E}$ its σ-field of universally measurable subsets. Let F be both a U-space and a universally measurable subset of E. Fix a single measurable space $(\Omega, \mathcal{F})$, a right continuous filtration $(\mathcal{F}_t)$ thereon, and an $(E, \mathcal{E}; \Omega, \mathcal{F})$ kernel $\hat{P}^x(d\omega)$ such that each $\hat{P}^x$ has unit mass, and $(\mathcal{F}, \mathcal{F}_t)$ is augmented with respect to the $\hat{P}^\mu$. As above, let $(V, \mathcal{V}^0)$ be a separable measurable space. Write $\mathcal{V}$ for the universal completion of $\mathcal{V}^0$, and let $n(x, dv)$ be an $(F, \mathcal{E}^F; V, \mathcal{V})$ kernel satisfying (8.6).

Let $(\hat{X}_t)$ be a right continuous F-valued process, adapted to $(\mathcal{F}_t)$, and write Q^x for the image law of $\hat{P}^x$ on $(U^F, \mathcal{U}^F)$ under $\hat{X}.$. If now (Y_t) is any $(V,\mathcal{V})$-valued point process adapted to $(\mathcal{F}_t)$, we will say that

$$(\hat{X}_t, Y_t, \mathcal{F}_t, \hat{P}^x)$$

is <u>strong Markov</u> if

$$\hat{P}^\mu((\hat{X}_{T+.}, \theta_T^0 Y) \in A | \mathcal{F}_T^\mu) = \hat{P}^{\hat{X}_T}((\hat{X}., Y|_{(0,\infty)}) \in A)$$

whenever μ is a probability on E, T is an $(\mathcal{F}_t^\mu)$ stopping time, and $A \in \mathcal{U}^F \otimes \mathcal{P}$.

<u>Corollary 5</u> Assume in addition to the above hypotheses, that each $(\hat{X}_t, \mathcal{F}_t^\mu, \hat{P}^\mu, Q^x)$ is strong Markov, and that (Q^x) is strongly regulated. Fix G, and assume that (Y_t) is a $(V,\mathcal{V})$-valued point process, adapted to $(\mathcal{F}_t)$ and satisfying (8.7) and (8.8). Then $(\hat{X}_t, Y_t, \mathcal{F}_t, \hat{P}^x)$ is strong Markov.

<u>proof</u>: Condition (8.9) follows immediately from Lemma 9, by definition of the Q^x. We apply Proposition 10 with the probability space $(\Omega, \mathcal{F}^\mu, \hat{P}^\mu)$, the filtration

$$(\mathcal{F}_t') = (\mathcal{F}_{T+t}^\mu), \quad \theta_T^0(Y)(t) .$$

and the processes

$$\hat{X}_t' = \hat{X}_{T+t}, \quad Y_t' = \theta_T^0(Y)(t) .$$

Our hypotheses guarantee that those of Proposition 10 are met, hence,

as in Corollary 4, the conditional law of $(\hat{X}'_\cdot, Y'_\cdot)$ given $\mathcal{F}'_0$ may be expressed using only n, G, the Q^x, a regulator for (Q^x), and the initial law of $\hat{X}'_0$, showing the result.

<u>(D) Existence</u> We now consider the problem of constructing point processes (Y_t) satisfying (8.7). With future applications in mind, we will do so in a context similar to that of (C). In addition to the general hypotheses of (C), assume that each $(\hat{X}_t, \mathcal{F}^\mu_t, \hat{P}^\mu, Q^x)$ is strong Markov, and that (Q^x) is regulated by a PAF (K^0_t). Fix G, and assume that n satisfies (8.9). We will further assume that an $(F \times F, \mathcal{E}^F \otimes \mathcal{E}^F; V, \mathcal{V})$ kernel $n(x,y;dv)$ may be found, with $n(x,y;V) \le 1$ and $n(x,x;V) = 0$, such that

$$\int 1_{G_0^c}(x,v)g(G(x,v))f(v)n(x,dv)$$

$$= \int g(G(x,v)) \int f(v')n(x,G(x,v);dv')n(x,dv)$$

for every positive $g \in \mathcal{E}$ and $f \in \mathcal{V}$. (As remarked before, such a kernel will exist, provided $(V, \mathcal{V}^0) = (U, \mathcal{U}^0)$, where E is a U-space.)

We may then perform the following construction; obtain a function $J(x,y)$ on $F \times F$ from Lemma 10(a). It is easily checked that the function

$$J(x,y)1_{\{(x,y);n(x,y;V) = 1\}}$$

satisfies the same condition as J, so that we may assume that $J(x,y) = 0$ whenever $n(x,y;V) \neq 1$.

Enlarge Ω by setting

$$(\Omega', F') = (\Omega \times \Omega_0^N, F \otimes C^N),$$

where Ω_0 is the space of all right continuous functions $[0,\infty) \to N$, and C is its cylinder σ-field. We retain the notation F for the preimage of F under the projection of Ω' on its first coordinate. Similarly, we write $\hat{X}_t$ for the composition of $\hat{X}_t$ and this projection. Let $B^k(t)$, $t \geq 0$ and $k \geq 1$, be the coordinate maps on the second factor. Then we can extend the $\hat{P}^x$ by the appropriate product measures, so as to make the $(B^k(t))$ Poisson processes of unit intensity, and independent of each other and of F, under each law $\hat{P}^\mu$. Set

$$T_j^k = \inf\{t;\ B^k(\int_0^t \int_{V_k \setminus V_{k-1}} 1_{G_0}(\hat{X}_s, v) n(\hat{X}_s, dv) ds) \geq j\} .$$

Let $R_0^k = 0$, and put

$$R_{j+1}^k = \inf\{t > R_j^k;\ K_t^0(\hat{X}.) - K_{t-}^0(\hat{X}.) \in [\tfrac{1}{k}, \tfrac{1}{k-1})\}$$

Now enlarge (Ω', F') again, to (Ω'', F''), and redefine the $\hat{P}^x$ so that there exist $(V \cup \{\delta\}, V \vee \sigma(\{\delta\}))$-valued random variables ${}^iY_j^k$; $i=1,2$ and $j,k \geq 1$; which are $\hat{P}^\mu$-conditionally independent given F and the T_j^k, for every μ, with conditional laws

$$\frac{n(\hat{X}_{T_j^k}, dv)}{\int_{V_k \setminus V_{k-1}} 1_{G_0}(\hat{X}_{T_j^k}, v') n(\hat{X}_{T_j^k}, dv')} 1_{G_0 \cap (F \times V_k \setminus V_{k-1})}(\hat{X}_{T_j^k}, v), \text{ if } i=1;$$

and

$$J(\hat{X}_{R_j^k-},\hat{X}_{R_j^k})n(\hat{X}_{R_j^k-},\hat{X}_{R_j^k};dv) + (1-J(\hat{X}_{R_j^k-},\hat{X}_{R_j^k}))\varepsilon_\delta(dv), \quad \text{if } i=2 .$$

(and while we're at it, enlarge the space enough, so that more independent random variables may be found in the future). Set

$$Y_t = \begin{cases} {}^1Y_j^k , & \text{if } t = T_j^k \\ {}^2Y_j^k , & \text{if } t = R_j^k \\ \delta , & \text{otherwise} , \end{cases}$$

$$\mathcal{H}^0 = \mathcal{F} \vee Y^{-1}(\mathcal{P}) , \quad \mathcal{H}_t^0 = \mathcal{F}_t \vee Y^{-1}(\mathcal{P}_t) ,$$

and augment $(\mathcal{H}^0, \mathcal{H}_t^0)$ with respect to the $\hat{P}^\mu$, to form $(\mathcal{H}, \mathcal{H}_t)$. Then

<u>Proposition 11</u>: On the enlarged space Ω'', $(\hat{X}_t, Y_t, \mathcal{H}_t, \hat{P}^x)$ is strong Markov, and satisfies (8.7) and (8.8).

<u>proof</u>: Let

$$T^k(r) = \inf\{t > r;\ Y_t \in V_k \backslash V_{k-1},\ \hat{X}_t = \hat{X}_{t-}\} ,$$

and write

$$n_s^k(A) = \int_{V_k \backslash V_{k-1}} 1_{G_0}(\hat{X}_s,v)1_A(v)n(\hat{X}_s,dv)$$

Then for $t > q > r$, $A \in \mathcal{V}$, and every probability μ, we have by definition, that

$$\hat{P}^{\mu}(T^k(r)\in(q,t], Y_{T^k(r)} \in A,\ T^k(T^k(r)) > t \mid \mathcal{F}\vee\mathcal{H}_r\vee\sigma(T^i_j; i\neq k))$$

$$= \sum_{\ell=1}^{\infty} \hat{P}^{\mu}(T^k_{\ell-1}\le r<T^k_{\ell}, T^k_{\ell}\in(q,t],\ Y_{T^k_{\ell}} \in A,\ T^k_{\ell+1} > t \mid \mathcal{F}\vee\mathcal{H}_r\vee\sigma(T^i_j; i\neq k))$$

$$= \sum_{\ell=1}^{\infty} 1_{\{T^k_{\ell-1}\le r<T^k_{\ell-1}\}} e^{-\int_r^q n^k_s(V)ds} \int_q^t n^k_s(V) e^{-\int_q^s n^k_{s'}(V)ds'}$$

$$\cdot \frac{n^k_s(A)}{n^k_s(V)} e^{-\int_s^t n^k_{s'}(V)ds'} ds$$

$$= e^{-\int_r^t n^k_s(V)ds} \int_q^t n^k_s(A)ds \ .$$

Similarly,

$$\hat{P}^{\mu}(T^k(r) > t \mid \mathcal{F} \vee \mathcal{H}_r \vee \sigma(T^i_j;\ i\neq k)) = e^{-\int_r^t n^k_s(V)ds} \ .$$

Now fix N, and let

$$T(r) = \bigwedge_{k\le N} T^k(r)\ ,\quad n_s(\cdot) = \sum_{k=1}^{N} n^k_s(\cdot)\ .$$

Then by independence,

$$\hat{P}^{\mu}(T(r) \in (q,t],\ Y_{T(r)} \in A,\ T(T(r)) > t \mid \mathcal{F} \vee \mathcal{H}_r)$$

$$= \sum_{k=1}^{N} \hat{P}^{\mu}(T^k(r)\in(q,t], Y_{T^k(r)} \in A,\ T^k(T^k(r)) > t,\ T^i(r)>t \text{ for } i\neq k$$

$$\mid \mathcal{F} \vee \mathcal{H}_r)$$

$$= e^{-\int_r^t n_s(V)ds} \int_q^t n_s(A)ds \ .$$

<u>Lemma 12</u> Let $V_N = \cup\{A_i;\ i=1..N''\}$, the A_i disjoint elements of V, and let $\varepsilon > 0$, $\varepsilon^{-1} \in N$. There exists processes $B_i(t)$, $i = 1..N''$, and random variables C_j, $j \geq 1$, such that under each $\hat{P}^{\mu}$:

(a) Each $B_i(t)$ is a Poisson process of unit intensity;

(b) Each C_j is uniformly distributed on $[0,1]$;

(c) The B_i and C_j are independent of each other and of F ;

(d) $$\sum_{\substack{s\in(r,t] \\ Y_s\neq\delta}} 1_{A_i}(Y_s)1_{G_0}(\hat{X}_{s-},Y_s) = B_i\left(\int_0^t n_s(A_i)ds\right) - B\left(\int_0^r n_s(A_i)ds\right)$$

for each i and each $r < t$, a.s.;

(d) Let $\rho_0 = 0$,

$$\rho_{j+1} = \inf\{t > \rho_j;\ K_t^0(\hat{X}.) \geq K_{t-}^0(\hat{X}.) + \varepsilon\} ,$$

$$a_j^i = n(\hat{X}_{\rho_j-}, \hat{X}_{\rho_j}; A_i) ,$$

$$b_j^i = \sum_{k=1}^{i} a_j^k \in [0,1] ,$$

and write $b_j^0 = 0$, $b_j^{N''+1} = 1$, $A_{N''+1} = V\cup\{\delta\}\backslash V_N$. Then

$$c_j \in (b_j^{i-1}, b_j^i] \text{ whenever } Y_{\rho_j} \in A_i .$$

<u>proof</u>: Let $\tau_0^i = 0$,

$$\tau_{k+1}^i = \inf\{t > \tau_k^i;\ Y_t \in A_i,\ \hat{X}_{t-} = \hat{X}_t\} \quad , \text{ and}$$

$$\gamma^i = \int_0^\infty n_s(A_i)ds .$$

For each i, pick a process B_0^i such that under each $\hat{P}^\mu$, B_0^i is a Poisson process of unit intensity, and the B_o^i are independent of each other and of H. Set

$$B^i(t) = \begin{cases} k, \text{ if } t \in [\int_0^{\tau_k^i} n_s(A_i)ds, \int_0^{\tau_{k+1}^i} n_s(A_i)ds), \ t < \gamma^i \\ B_i(\gamma^i -) + B_0^i(t - \gamma^i), \ t \geq \gamma^i \end{cases}$$

and that (d) holds by definition.

Choose C_j', $j \geq 1$ to be uniformly distributed on $[0,1]$, and independent of each other, the B^i and H, under each $\hat{P}^\mu$. Set

$$C_j = b_j^{i-1} + a_j^i C_j'$$

if $Y_{\rho_j} \in A_i$, $i = 1 \ldots, N'' + 1$. Then (e) holds by definition.

Since the T_j^k used to define (Y_t) are independent of the R_j^k, given F, we see that the B_i are independent of the C_j, given F. Let $t_0 = 0 < r_1 < t_1 < \ldots < r_{N'} < t_{N'}$, $i_1 .. i_{N'} \in \{1,\ldots,N''\}$, and $H \in F$. Set $j_0 = 0$,

$$j_k = j_\ell + 1,$$

where ℓ is the last number before k such that $i_\ell = i_k$ ($\ell = 0$ if no such number exists), and define $j(i)$ to be the largest of the j_k such that $i_k = i$ (and if there is no such k, we set it equal to 0). To show that the B_i are independent

of each other and of $\mathcal{F}$, it will suffice to check that the probability of the set

$$H \cap \{\tau_{j_k}^{i_k} \in (r_k, t_k],\ k = 1..N',\ \sup_i \tau_{j(i)+1}^{i} > t_{N'}\}$$

agrees with the result we would obtain under (a) and (c) (as finite unions of such sets form a Boolian algebra which generates a σ-field for which all the B_i are measurable). By the computational preceding this lemma (and similar arguments),

$$\hat{\hat{P}}^{\mu}(H;\ \tau_{j_k}^{i_k} \in (r_k, t_k],\ k = 1..N';\ \sup_i \tau_{j(i)+1}^{i} > t_k)$$

$$= \hat{P}^{\mu}(H;\ T(t_{k-1}) \in (r_k, t_k],\ Y_{T(t_{k-1})} \in A_{i_k},\ T(T(t_{k-1})) > t_k,$$

$$k = 1..N')$$

$$= \hat{E}^{\mu}[H,\ \prod_{k=1}^{N'} e^{-\int_{t_{k-1}}^{t_k} n_s(V)ds} \int_{r_k}^{t_k} n_s(A_{i_k})ds]$$

$$= \hat{E}^{\mu}[H,\ \prod_{i=1}^{N'} (\prod_{k=1}^{N'} [(1-\delta_{ii_k}) e^{-\int_{t_{k-1}}^{t_k} n_s(A_i)ds} +$$

$$+ \delta_{ii_k} e^{-\int_{t_{k-1}}^{t_k} n_s(A_i)ds} \int_{r_k}^{t_k} n_s(A_i)ds])]$$

which is the result we would have obtained under (a) and (c).

To show that the C_j are independent of each other and of $\mathcal{F}$, with uniform distributions, let $0 \le r_j < t_j \le 1$, $j = 1..N$; and $H \in \mathcal{F}$. It will again suffice to show that the probability of

$$H \cap \{C_j \in [r_j, t_j],\ j = 1..N\}$$

agrees with that calculated using (b) and (c).

$$\hat{P}^{\mu}(H;\ C_j \in [r_j, t_j],\ j = 1..N)$$

$$= \sum_{i_1..i_N \in \{1..,N''+1\}} \hat{P}^{\mu}(H; C_j \in [r_j, t_j],\ Y_{\rho_j} \in A_{i_j},\ j = 1.,N)$$

$$= \hat{E}^{\mu}[H, \sum_{i_1..i_N \in \{1..,N''+1\}} (\prod_{j=1}^{N} |[\frac{r_j - b_j^{i_j-1}}{a_j^{i_j}}, \frac{t_j - b_j^{i_j-1}}{a_j^{i_j}}] \cap [0,1]|)$$

$$(\prod_{j=1}^{N} a_j^{i_j})]$$

$$= \hat{E}^{\mu}[H, \sum_{i_1..i_N \in \{1..,N''+1\}} \prod_{j=1}^{N} |[r_j, t_j] \cap [b_j^{i_j-1}, b_j^{i_j-1} + a_j^{i_j}]|]$$

$$= \hat{E}^{\mu}[H, \prod_{j=1}^{N} (\sum_{i_j \in \{1..,N''+1\}} |[r_j, t_j] \cap [b_j^{i_j-1}, b_j^{i_j}]|)]$$

$$= \prod_{j=1}^{N} (t_j - r_j)\hat{P}^{\mu}(H),$$

as required. □

As an immediate consequence, we see that there is a σ-field $\mathcal{F}^{\perp}$ which is independent of $\mathcal{F}$ under each $\hat{P}^{\mu}$, and such that

$$\mathcal{H}_t^0 \subset \mathcal{F}_t \vee \mathcal{F}^{\perp}.$$

Thus, for every $(\mathcal{H}_{t+}^0)$ stopping time T, we can find (as in

Lemma 7) a stopping time $\hat{T}$ with respect to the filtration

$$\bigcap_{s>t} (\mathcal{F}_s \otimes \mathcal{F}^{\perp})$$

on $\Omega'' \times \Omega''$, such that $T(\omega) = \hat{T}(\omega,\omega)$ for every ω. Then $\hat{T}(\cdot,\omega)$ is an $(\mathcal{F}_t)$ stopping time for each $\omega \in \Omega''$, so that for $f \in \mathcal{U}^F$, f positive, we get that

$$\hat{E}^{\mu}[f(\hat{X}_{T+\cdot})] = \int\int f(\hat{X}(\omega_1)_{\hat{T}(\omega_1,\omega_2)+\cdot})\hat{P}^{\mu}(d\omega_1)\hat{P}^{\mu}(d\omega_2)$$

$$= \int\int Q^{\hat{X}(\omega_1)_{\hat{T}(\omega_1,\omega_2)}}(f)\hat{P}^{\mu}(d\omega_1)\hat{P}^{\mu}(d\omega_2)$$

$$= \hat{E}^{\mu}[Q^{\hat{X}_T}(f)] .$$

We now easily obtain that each

$$(\hat{X}_t,\ \mathcal{H}_t^{\mu},\ \hat{P}^{\mu},\ Q^x)$$

is strong Markov.

Similarly each $B^i(t)$ remains strong Markov when we adjoin to its minimal σ-field, the information in

$$\mathcal{F} \vee \sigma(B^k, {}^1Y_j^k, T_j^k; k\neq i, j\geq 1) \vee \sigma({}^2Y_j^k;\ k,j\geq 1) \vee \sigma({}^1Y_j^i;\ j\leq B^i(t)) ,$$

and hence, for any $(\mathcal{H}_t^{\mu})$ stopping time T,

$$B^i(\cdot + \int_0^T n_s^i(V)ds) - B^i(\int_0^T n_s^i(V)ds)$$

is a Poisson process, independent of $\mathcal{F} \vee \mathcal{H}_{T+}^{\mu} \vee \sigma(B^k; k \neq i)$, under

each $\hat{P}^{\mu}$.

Thus, $\theta_T^0(Y)$ is constructed from $(\hat{X}_{t+T})$ in exactly the same way that Y was constructed from $(\hat{X}_t)$, so that by the strong Markov property of each $(\hat{X}_t, \mathcal{H}_t^{\mu}, \hat{P}^{\mu}, Q^x)$, we see that

$$(\hat{X}_t, Y_t, \mathcal{H}_t, \hat{P}^x)$$

is strong Markov.

Condition (8.8) follows by definition. To show (8.7), it suffices to show the conclusion of Lemma 9, with (Z_t) of the form

$$Z_t = Z_{r+}1_{(r,s]}(t) \ , \ Z_{r+} \in \mathcal{H}_r^{\mu} ,$$

and $h(x,v)$ of the form

$$1_{A\cap V_N}(v)1_{G_0}(x,v) \ , \ A \in \mathcal{V} \ ; \text{ or}$$

$$f(v)1_{\{(y,z);y\neq z,b(y,z)\in[\frac{1}{k},\frac{1}{k-1})\}}(x,G(x,v)); \ f\in\mathcal{V} \text{ positive.}$$

In the first case, by Lemma 12, conditional on $\mathcal{H}_r^{\mu} \vee \sigma(\hat{X}_s;s\geq 0)$,

$$\sum_{\substack{s\in(r,t] \\ Y_s\neq\delta}} h(\hat{X}_{s-},Y_s)$$

is a Poisson random variable, of mean

$$\int_r^t n_s(A)ds ,$$

Showing the corresponding expression of Lemma 9.

In the second case,

$$\hat{E}^{\mu}[\sum_{\substack{s>0\\ Y_s\neq\delta}} Z_s h(\hat{X}_{s-},Y_s)]$$

$$= \hat{E}^{\mu}[Z_{r+}\sum_{j=1}^{\infty} 1_{(r,t]}(R_j^k)J(\hat{X}_{R_j^k-},\hat{X}_{R_j^k})\int f(v)n(\hat{X}_{R_j^k-},\hat{X}_{R_j^k};dv)]$$

$$= \hat{E}^{\mu}[Z_{r+}E_Q^{\hat{X}_r}[\int_0^{t-r}\int 1_{\{(x,z);x\neq z,b(x,z)\in[\frac{1}{k},\frac{1}{k-1})\}}(W_{s-}^F,W_s^F)\cdot$$

$$\cdot\frac{J(W_{s-}^F,W_s^F)}{b(W_{s-}^F,W_s^F)}\int f(v)n(W_{s-}^F,W_s^F;dv)dK_s^0]]$$

$$= \hat{E}^{\mu}[Z_{r+}E_Q^{\hat{X}_r}[\int_0^{t-r}\int 1_{\{(x,z);x\neq z,b(x,z)\in[\frac{1}{k},\frac{1}{k-1})\}}(W_s^F,y)\cdot$$

$$\cdot\frac{J(W_s^F,y)}{b(W_s^F,y)}\int f(v)n(W_s^F,y;dv)N(W_s^F,dy)dK_s]]$$

$$= \hat{E}^{\mu}[Z_{r+}E_Q^{\hat{X}_r}[\int_0^{t-r}\int 1_{\{(x,z);x\neq z,b(x,z)\in[\frac{1}{k},\frac{1}{k-1})}(W_s^F,G(W_s^F,v))\cdot$$

$$\cdot\int f(v')n(W_s^F,G(W_s^F,v);dv')n(W_s^F,dv)ds]]$$

$$= \hat{E}^{\mu}[\int_0^{\infty} Z_s\int h(\hat{X}_s,v)n(\hat{X}_s,dv)ds]. \quad \square$$

9. Excursions away from a set

The techniques of the foregoing sections can be used to examine the more general problem of constructing a strong Markov process

from a 'process on the boundary', together with the excursions away from that 'boundary'. In this section we will be content with modifying a result due to B. Maisonneuve [38] , proving a partial converse, along the lines of Theorem 2 and Proposition 5, and then showing some auxilliary results. Applications follow in Section 10.

Let E be a U-space. Fix a metric d on E, compatable with its topology. We will assume that some isolated point $\Delta \in E$ has been singled out, to function as a 'cemetary', and we will let

$$\partial(u) = \inf\{t \geq 0;\ u(t) = \Delta\}\ , \quad \text{for} \quad u \in U\ .$$

Let $P^x(du)$ be an $(E, \mathcal{E}; U, \mathcal{U})$ kernel such that $P^x(U) = 1$ for every x, and $P^\Delta(W_t = \Delta$ for every $t) = 1$. Augment $(\mathcal{U}, \mathcal{U}_t)$ with respect to the P^μ to form $(\bar{\mathcal{U}}, \bar{\mathcal{U}}_t)$. Assume that

$$(W_t, \bar{\mathcal{U}}, \bar{\mathcal{U}}_t, P^x)$$

is a right process satisfying the further condition (6.5) of near-Borel measurability of the excessive functions. Let $F \in \mathcal{E}$ be nearly Borel, with $\Delta \notin F$. We set

$$\rho_F = \{t \geq 0;\ W_t \in F\ \}, \quad M = \bar{\rho}_F$$
$$D_t = \inf\{s > t;\ s \in M\}, \quad \psi(x) = E^x[e^{-D_0}], \quad M_0 = \{t; D_t > D_{t-}\}.$$

Since F is nearly Borel, each D_s is a $(\bar{\mathcal{U}}_t)$ stopping time, and ψ is 1-excessive. Assume that

$$F = \{x \in E;\ \psi(x) = 1\ \}\ . \tag{9.1}$$

Then M has no isolated points, a.s.. Assume also that

(9.2) ψ is regular (see Blumenthal and Getoor [5]).

Then there exists a continuous PAF l_t, satisfying

(9.3) $E^x[\int_0^\infty e^{-t} dl_t] = \psi(x)$ for every $x \in E$,

whose set of increase is a.s. exactly M. A word of caution ; at this point we are treating Δ just as any other point of F, and hence <u>DO NOT TAKE THE CONVENTION THAT</u> $\psi(\Delta) = 0$. In fact, $\psi(\Delta) = 1$, so that instead of having $l_t = l_{t\wedge\partial}$, we have that $P_0^\Delta(l_t = t \text{ for every } t) = 1$. Of course, we could have instead stopped l_t at ∂ ; that is, we could have used the function $\psi(x) - E^x[e^{-\partial}]$ instead of $\psi(x)$.

Also, recall for later use, that a function $D : U \to [0,\infty]$ is called a <u>perfect terminal time</u> if

$$D = t + D \circ \theta_t \quad \text{on} \quad \{D > t\}, \quad \text{for every} \quad t \geq 0.$$

It is called <u>exact</u> if $D \circ \theta_t \to D$ as $t \downarrow 0$.

Now, let $(\Omega, \mathcal{G})$ be a measurable space, on which we are given a right continuous E-valued process (X_t), and a right continuous filtration $(\mathcal{G}_t)$ to which (X_t) is adapted. Suppose that $\hat{P}^x(d\omega)$ is an $(E, \mathcal{E}; \Omega, \mathcal{G})$ kernel such that P^b is the image law of $\hat{P}^b$ under $X.$. Suppose also that $(\mathcal{G}, \mathcal{G}_t)$ is augmented with respect to the $\hat{P}^\mu$, and that $(X_t, \mathcal{G}_t^\mu, \hat{P}^\mu, P^x)$ is strong Markov for each μ (so that also $(X_t, \mathcal{G}, \mathcal{G}_t, \hat{P}^x)$ is a right process). We set

$$L_t = l_t(X.), \quad S^+(s) = \inf\{t;\ L_t > s\}, \quad S^-(s) = \inf\{t;\ L_t \geq s\}.$$

Then S^+ and S^- are the right and left continuous inverses

of (L_t), respectively, and since (1_t) is continuous, they are strictly increasing whenever finite. Also, $S^+(s)$ is a $(\mathcal{G}_t)$ stopping time. Put

$$\zeta = \partial(X.) \; ; \; \hat{\zeta} = L_\zeta \; ; \; \hat{D}_t = D_t(X.) \; ;$$

$$(9.4) \quad \hat{X}_s = \begin{cases} X_{S^+(s)} \, , & \text{if } S^+(s) < \infty \\ \Delta \, , & \text{otherwise} \; ; \end{cases}$$

$$\mathcal{F}_s = \mathcal{G}_{S^+(s)} \, , \; \mathcal{F}^\mu_{0-} = \mathcal{G}^\mu_{0-} \; ;$$

$$(9.5) \quad Y_s = \begin{cases} X_{(S^-(s)+.)\wedge S^+(s)} \, , & \text{if } S^-(s) < S^+(s) \\ \delta \, , & \text{otherwise} \; ; \end{cases}$$

and let P^b_0 be the image law of $\hat{P}^b$ on $(U, \mathcal{U})$, under $X._{\wedge D_0}$. Parts of the following result are taken from El Karoui [20]. The reader is strongly encouraged to omit its proof, and pass directly to Theorem 3.

<u>Lemma 13</u> Fix a probability μ on E.

(a) If T is a $(\mathcal{G}^\mu_t)$ stopping time, then L_T is an $(\mathcal{F}^\mu_s)$ stopping time. If T is any random time, we have that

$$\mathcal{G}^\mu_{S^-(L_T)-} \subset \mathcal{G}^\mu_{T-} \; .$$

(b) If T is an $(\mathcal{F}^\mu_s)$ stopping time, then $S^+(T)$ is a $(\mathcal{G}^\mu_t)$ stopping time,

$$\mathcal{F}^\mu_{T-} = \mathcal{G}^\mu_{S^-(T)-} \; , \text{ and } \mathcal{F}^\mu_T = \mathcal{G}^\mu_{S^+(T)} \; .$$

(c) If (Z_t) is $(\mathcal{G}_t^\mu)$ predictable, then $(Z_{S^-(s)})$ is $(\mathcal{F}_s^\mu)$ predictable. If (Z_s) is $(\mathcal{F}_s^\mu)$ predictable, then (Z_{L_t}) is $(\mathcal{G}_t^\mu)$ predictable.

(d) $(\hat{X}_s)$ is adapted to $(\mathcal{F}_s)$, and takes values in F, a.s.. For $x \in F$, let Q^x be the image law of $\hat{P}^x$ on $(U^F, \mathcal{U}^F)$ under $\hat{X}$. . Then

$$(\hat{X}_t, \mathcal{F}_t^\mu, \hat{P}^\mu, Q^x)$$

is strong Markov, and $Q^x(W_0 = x) = 1$ for every $x \in F$.

(e) (Y_t) is adapted to $(\mathcal{F}_t)$

(f) The following hold a.s. :

$$M_0(X.) = \{S^-(s);\ s \in (0,\infty)\ ,\qquad S^-(s) < S^+(s)\}\ ;$$
$$\hat{D}_{S^-(s)} = S^+(s) \quad \text{for every} \quad s \geq 0\ ; \quad \text{and}$$
$$\hat{\zeta} < \infty \text{ if and only if } S^-(\hat{\zeta}) < \infty\ .$$

<u>proof</u>: (a) We already know that $S^+(s)$ is a $(\mathcal{G}_t^\mu)$ stopping time, for s fixed, and that so is $\hat{D}_T$, whenever T is a $(\mathcal{G}_t^\mu)$ stopping time. Since (1_t) increases exactly on M, and S^- is strictly increasing, it follows that $S^+(L_T) = \hat{D}_T$ a.s., and hence

$$\{L_T < s\} = \{S^+(L_T) < S^+(s)\} \in \mathcal{G}^\mu_{S^+(s)} = \mathcal{F}_s^\mu\ .$$

For T now a random time, $\mathcal{G}^\mu_{S^-(L_T)-}$ is generated by the

$\hat{P}^{\mu}$-null sets, and by sets

$$A \cap \{t < S^{-}(L_T)\} = A \cap \{\hat{D}_t < T\} ,$$

for $A \in \mathcal{G}_t^{\mu} \subset \mathcal{G}_{\hat{D}_t}^{\mu}$. Since $\hat{D}_t$ is a $(\mathcal{G}_t)$ stopping time, these sets lie in $\mathcal{G}_{T-}^{\mu}$.

(b) $\mathcal{F}_{T-}^{\mu}$ is generated by the $\hat{P}^{\mu}$-null sets, and by sets

$$A \cap \{s < T\} = A \cap \{S^{+}(s) < S^{-}(T)\} ,$$

for $A \in \mathcal{F}_s^{\mu} = \mathcal{G}_{S^{+}(s)}^{\mu}$. Because $S^{+}(s)$ is a $(\mathcal{G}_t^{\mu})$ stopping time, they lie in $\mathcal{G}_{S^{-}(T)-}^{\mu}$. Thus

$$\mathcal{F}_{T-}^{\mu} \subset \mathcal{G}_{S^{-}(T)-}^{\mu} .$$

Before proving the reverse inclusion, we will show the remainder of (b) .

$S^{+}(T)$ is a $(\mathcal{G}_t^{\mu})$ stopping time, since

$$\{S^{+}(T) < t\} = \{T < L_t\} \in \mathcal{F}_{L_t-}^{\mu}$$

(as T is an $(\mathcal{F}_s^{\mu})$ stopping time), and

$$\mathcal{F}_{L_t-}^{\mu} \subset \mathcal{G}_{S^{-}(L_t)-}^{\mu} \subset \mathcal{G}_t^{\mu} ,$$

by the above, and part (a). Similarly, if $A \in \mathcal{F}_T^{\mu}$, then

$$A \cap \{S^{+}(T) < t\} = A \cap \{T < L_t\} \in \mathcal{F}_{L_t-}^{\mu} \subset \mathcal{G}_t^{\mu} ,$$

so that

$$\mathcal{F}^{\mu}_{L_T} \subset \mathcal{G}^{\mu}_{S^+(T)} \quad .$$

Conversely, if $A \in \mathcal{G}^{\mu}_{S^+(T)}$, then

$$A \cap \{T < s\} = A \cap \{S^+(T) < S^+(s)\} \in \mathcal{G}^{\mu}_{S^+(s)} = \mathcal{F}^{\mu}_s \, .$$

Finally, $\mathcal{G}^{\mu}_{S^-(T)-}$ is generated by the $\hat{P}^{\mu}$-null sets, and by sets of the form

$$A \cap \{t < S^-(T)\} = A \cap \{L_t < T\} \quad , \text{ where}$$

$$A \in \mathcal{G}^{\mu}_t \subset \mathcal{G}^{\mu}_{S^+(L_t)} = \mathcal{F}^{\mu}_{L_t} \quad .$$

These sets lie in $\mathcal{F}^{\mu}_{T-}$ since L_t is an $(\mathcal{F}^{\mu}_s)$ stopping time.

(c) Let (Z_t) be left continuous and adapted to $(\mathcal{G}^{\mu}_t)$. Then $(Z_{S^-(s)})$ is also left continuous, and for each s,

$$Z_{S^-(s)} \in \mathcal{G}^{\mu}_{S^-(s)-} = \mathcal{F}^{\mu}_{s-}$$

by (b). Thus $(Z_{S^-(s)})$ is $(\mathcal{F}^{\mu}_s)$ predictable for such (Z_t), and hence by monotone class arguments, for every $(\mathcal{G}^{\mu}_t)$ predictable (Z_t). The 'converse' is proved similarly.

(d) By (6.2), ρ_F is a.s. closed on the left, so that the argument of Meyer [40], Proposition 1, shows that $M \backslash M_0 \subset \rho_F$ a.s.. Since S^+ is right continuous, strictly increasing, and (1_t) increases only on M, a.s., we get that

$$S^+(s) \in (M \backslash M_0)(X.) \cup \{\infty\} \quad \text{for every } s, \text{ a.s.},$$

showing that $\hat{X}_s \in F$ for every s, a.s.. Since S^+ is right

continuous, so is $(\hat{X}_s)$. It is clearly adapted to $(\mathcal{F}_s)$. If T is an $(\mathcal{F}_t^\mu)$ stopping time, $B \in \mathcal{E}^F \subset \mathcal{E}$, and $h > 0$, then

$$\hat{P}^\mu(\hat{X}_{h+T} \in B | \mathcal{F}_T^\mu) = \hat{P}^\mu(X_{S^+(h+T)} \in B | \mathcal{G}_{S^+(T)}^\mu)$$
$$= \hat{P}^{\hat{X}_T}(\hat{X}_h \in B) = Q^{\hat{X}_T}(W_h \in B), \quad \hat{P}^\mu\text{-a.s.}$$

by the strong Markov property of (X_t), showing (d).

(e) $S^-(s)$ and $S^+(s)$ are adapted to $(\mathcal{F}_s)$. Thus, for $\varepsilon > 0$,

$$T = \inf\{s \geq 0 ;\ S^+(s) > S^-(s) + \varepsilon\}$$

is an $(\mathcal{F}_s)$ stopping time. Because $\{s \leq t;\ S^+(s) > S^-(s) + \varepsilon\}$ is finite for every t, a.s., it therefore suffices to show that Y_T is measurable, from $(\Omega, \mathcal{F}_T)$ to $(U, \mathcal{U})$. Let $0 \leq t_1 < t_2 < \ldots < t_k$, and $B_1, \ldots, B_k \in \mathcal{E}^0$. Then

$$\{Y_T(t_i) \in B_i,\ i = 1..k\} = \{X_{(S^-(T) + t_i) \wedge S^+(T)} \in B_i,\ i = 1..k\}.$$

By (b), $S^+(T)$ is a $(\mathcal{G}_t)$ stopping time, so that

$$(t,\omega) \to X_{t \wedge S^+(T)}(\omega)$$

is a measurable function from $\mathcal{B} \otimes \mathcal{G}_{S^+(T)}$ to $\mathcal{E}^0$. Since

$$S^-(T) \in \mathcal{G}_{S^+(T)}$$

by (a), it follows that the above set lies in $\mathcal{G}_{S^+(T)} = \mathcal{F}_T$. By completeness, we can extend this to $\{Y_t \in A\}$, $A \in \mathcal{U}$.

(f) We first prove the third statement. For $x \in E$ we have that

$$e^t \geq e^t E^x[\int_0^t e^{-s} dl_s] \geq E^x[l_t],$$

so that $l_t < \infty$ for every t, a.s., and hence that a.s., $\hat{\zeta} < \infty$ whenever $S^-(\hat{\zeta}) < \infty$. Conversely, let

$$T = T_1 = \inf\{t > 1;\ W_t \in F \text{ and } l_t > 1\}$$

and

$$T_{k+1} = T_k + T \circ \theta_{T_k} .$$

For $x \in F$ we have that

$$E^x[l_T] \geq E^x[\int_0^T e^{-s} dl_s]$$

$$= \psi(x) - E^x[e^{-T}\psi(W_T)] \geq 1 - e^{-1} .$$

Thus,

$$P^x(l_T \geq (1 - e^{-1})) \geq \frac{1}{2}$$

for every $x \in F$. Because $W_T \in F$ a.s. on $\{T < \infty\}$, we see by the strong Markov property of (W_t), that

$$P^\mu(T_k < \infty \text{ and } l_{T_k} - l_{T_{k-1}} < 1 - e^{-1} \text{ for every } k \geq K)$$

$$= 0, \text{ for every } K \text{ and } \mu .$$

Since $T_k(X.) < \zeta = \infty$ a.s. on $\{S^-(\hat{\zeta}) = \infty\}$ it follows that

$$\hat{\zeta} = \lim_{t\to\infty} L_t = \infty \quad \text{a.s. on that set.}$$

Now let M_1 be the set on the right of the first statement

of (f) . For ω such that $L_\cdot(\omega)$ increases exactly on $M(X_\cdot)(\omega)$ and $S^-(\cdot)(\omega)$ is strictly increasing, we obtain that

$$D_{S^-(s)}(\omega) = S^+(s)(\omega) \quad ,$$

and hence $M_1(\omega) \subset M_0(X_\cdot)(\omega)$. Conversely, for $t \in M_0(X_\cdot)(\omega)$, we will have that $t \in M_1(\omega)$ provided $t = S^-(s)(\omega)$ for some $s > 0$. It follows by choice of ω , that $L_t(\omega) > 0$, and that $t = S^-(L_t)(\omega)$. We need therefore only show that $L_t(\omega) < \infty$. This follows from the third part of (f). □

<u>Theorem 3</u> (Q^x) is regulated. There exists an exact, perfect terminal $(\mathcal{U}_{t+})$ stopping time D , and an $(F, \mathcal{E}^F; U, \mathcal{U})$ kernel $n(x, du)$ such that the following conditions are satisfied

I $\quad n(x, \{D = 0 \text{ or } W_t \neq W_D \text{ for some } t \geq D\}) = 0$

for every $x \in F$.

II (a) $n(x, \{\sup_t d(x, W_t) > \varepsilon\}) < \infty$, for every $x \in F$ and $\varepsilon > 0$.

(b) $\lim_{h \downarrow 0} \int_0^h n(\hat{X}_s, \{\sup_t d(x, W_t) > \varepsilon\}) ds < \infty \quad \hat{P}^x$-a.s.,

for every $x \in F$ and $\varepsilon > 0$.

III $\quad \int (1 - e^{-D(u)}) n(x, du) \leq 1$ for every $x \in F$.

IV $\quad n(x, \{u; D(u) > t,\ u \in \Lambda,\ \theta_t(u) \in M\})$

$$= \int_{\Lambda \cap \{D > t\}} P_0^{u(t)}(M) n(x, du) ,$$

for $t > 0$, $\Lambda \in \mathcal{U}_t$, $M \in \mathcal{U}$, $x \in F$.

V $\quad n(x,\{u;\ u(0) \in B,\ u \in M\}) = \int_{\{u;u(0)\in B\}} P_0^{u(0)}(M)n(x,du)$

for $x \in F$ and $B \in \mathcal{E}$ such that $B \cap F = \emptyset$.

VI (a) Let

$$m(x) = 1 - \int (1 - e^{-D(u)})n(x,du),\ x \in F\backslash\{\Delta\};\ m(\Delta) = 0\ ;\ \text{and}$$

$$I = \{x \in F;\ \hat{P}^x(\int_0^t m(\hat{X}_s)ds > 0 \ \text{ for every } \ t > 0) = 0\}\ .$$

Then for each $x \in I$ and $h > 0$,

$$\int_0^h n(\hat{X}_s,U)ds = \infty \quad \hat{P}^x\text{-a.s.}\ .$$

(b) If $n'(du)$ is a positive measure on $(U,\mathcal{U})$ satisfying IV, such that

$$n'(du) \le n(x,du)$$

for some x, then $n'(W_0 \in F) = 0$ or ∞.

Further:

(9.6) $\hat{D}_s = D(X_{s+.})$ for every s, a.s.;

(8.7), (8.8) and (8.9) hold, with

$$G(x,u) = \begin{cases} u(D(u)), & D(u) < \infty \\ \Delta, & \text{otherwise}\ ; \end{cases}$$

$(\hat{X}_t,\ Y_t,\ \mathcal{F}_t,\ \hat{P}^x)$ is strong Markov; and

(9.7) $$S^+(t) = \int_0^t m(\hat{X}_s)ds + \sum_{\substack{s\in[0,t] \\ Y_s \neq \delta}} D(Y_s)$$

Note: 1. In the situation of Thereom 2, VI (b) states that if $n(U) = \infty$ then (vi'c) holds, and if $n(U) < \infty$ then $n(U^a) = 0$.

VI (a) becomes the remainder of (vi'b), that if $n(U) < \infty$, then $m > 0$.

2. As a consequence we conclude that conditions (9.1) and (9.2) rule out the discrete visiting behaviour of (vi'a).

3. Condition VI (b) rules out two distinct types of behaviour, one identical to that precluded by (vi'), and another not found in the situation of Thereom 2. In fact, using II(a), condition VI(b) is seen to be equivalent to the condition that for every $x \in F$, both:

$$n(x,\{W_0 \in F\backslash\{x\}\}) = 0 \text{ ; and}$$

$n'(W_0 = x) = 0$ or ∞, whenever $n'(du)$ is a positive measure on $(U,\mathcal{U})$ satisfying IV, and such that $n'(du) \leq n(x,du)$.

4. The proof that (Q^x) is regulated uses (6.5). As remarked in the introduction to section 8, Gzyl [28] shows a stronger result, namely that $(\hat{X}_t)$ is a right process.

proof: We can apply the perfection arguments in Meyer [40] to the right process $(W_t, \bar{\mathcal{U}}, \bar{\mathcal{U}}_t, P^x)$, to obtain an exact perfect terminal $(\mathcal{U}_{t+})$ stopping time D such that $P^x(D = D_0) = 1$ for

each $x \in F$. Thus

$$\hat{D}_q = D(X_{q+.}) \quad \text{a.s.},$$

for each $q \in Q$, so that by exactness, we obtain condition (9.6). Note the following, from Meyer [40]. For $h > 0$, $(h \wedge D \circ \theta_t)$ is a càdlàg process adapted to $(\mathcal{U}_{t+2h})$, so that it is $(\mathcal{U}_{t+2h})$ progressive. Thus so is

$$M' = \{(t,u);\ \lim_{\substack{s \to t \\ s<t}} D \circ \theta_s(u) = 0\} .$$

and hence M' is $(\mathcal{U}_{t+})$ progressive by Dellacherie et Meyer [18], IV. 14. Thus M is indistinguishable from a progressive set relative to the filtration $(\mathcal{U}_{t+})$ (rather than $(\bar{\mathcal{U}}_t)$.

We can now apply Proposition 9.2 of Maisonneuve [38] to $(W_t, \bar{\mathcal{U}}, \bar{\mathcal{U}}_t, P^x)$ and M'. Maisonneuve works with (W_t) killed at D, rather than stopped there, but his kernel clearly extends to one which, by Lemma 13(c) and the above discussion, satisfies (8.7). It follows from his proof, and that of his Theorem 4.1, that this kernel n satisfies III.

We will now show that (Q^x) is regulated. Let $\tilde{E}$ denote the Ray Knight compactification of E, and let g belong to the Ray cone of $\tilde{E}$. Thus g is continuous on $\tilde{E}$, and $g|_E$ is α-excessive with respect to the P^x, for some $\alpha > 0$. By (6.5), g is thus nearly Borel. Fix a probability μ on F. Then there exist $g_1, g_2 \in \mathcal{E}^0$, with

$$g_1 \le g|_E \le g_2 \,,\quad P^\mu(g_1(W_t) \ne g_2(W_t) \quad \text{for some} \quad t) = 0 \,.$$

Now

$$g_i|_F \in \mathcal{E}^{F,0} \,,$$

so that $g_i \circ W^F_\cdot$ is a $(\mathcal{U}^{F,0}_{t+})$ optional process. Thus

$$A_g = \{g_1(W^F_t) \ne g_2(W^F_t) \,,\ \text{or} \ \lim_{s\downarrow t} g_1(W^F_s) \ne g_1(W^F_t) \,,$$

$$\text{or} \ \lim_{\substack{s\to t\\ s<t}} g_1(W^F_s) \ \text{does not exist}\} \in \mathcal{U} \,.$$

Since Q^μ is the image law of $\hat{P}^\mu$ under $\hat{X}_\cdot$, and $g \circ X_\cdot$ is a.s. càdlàg ($g|_E$ being α-excessive), we see that also

$$Q^\mu(A_g) = 0 \,.$$

If we now let g range through a countable dense subset of the Ray cone, we get that

$$\{W^F_\cdot \ \text{is cadlag in the topology of} \ \tilde{E}\} \in \mathcal{U}^{F,\mu} \,,$$

and has Q^μ-measure 1, whenever μ is a probability on F. Augment $(\mathcal{U}^F, \mathcal{U}^F_t)$ with respect to the Q^μ, to form $(\bar{\mathcal{U}}^F, \bar{\mathcal{U}}^F_t)$, and write

$$\tilde{W}^F_{t-} \,,\ \tilde{X}^F_{t-} \,,\ \tilde{X}_{t-}$$

for the left limits of $W^F_\cdot$, $\hat{X}_\cdot$, and $X_\cdot$ at t, in the topology of $\tilde{E}$. We conclude from the above that for any positive

$h \in \mathcal{E}^F \otimes \mathcal{E}^F$,

$$K_t^h = \sum_{s\in(0,t]} h(\tilde{W}_{s-}^F, W_s^F) 1_{\{\tilde{W}_{s-}^F \neq W_s^F,\ \tilde{W}_{s-}^F \in F\}}$$

defines a PAF adapted to $(\bar{\mathcal{U}}_t^F)$ (possibly one which is identically infinite).

Since E is a U-space, we conclude from the discussion prior to Lemma 9, that there is an $(F\times F,\ \mathcal{E}^F \otimes \mathcal{E}^F;\ U,\ \mathcal{U})$ kernel $n(x,y;\ du)$, such that

$$\int 1_A(u(D(u)))f(u)n(x,du) = \iint 1_A(u(D(u)))f(v)n(x,u(D(u));dv)n(x,du)$$

for every positive $f \in \mathcal{U}$, and $A \in \mathcal{E}^F$ such that $x \notin A$. Set

$$h_0(x',y') = \int (1-e^{-D(u)})n(x',y';du) \vee 1_{\{(x,y);x\neq y,n(x,y;U)=0\}}(x',y').$$

By the argument of Getoor [23] (13.4), we see once again that $\tilde{X}_{s-}^F = \hat{X}_{s-}$ for every s such that $Y_s \neq \delta$, a.s., so that by Lemma 9 (which did not require (Q^X) to be regulated) and III,

$$\hat{E}^\mu[\sum_{\substack{s\in(0,t]\\ Y_s\neq\delta}} h_0(\tilde{X}_{s-}^F, \hat{X}_s)] = \hat{E}^\mu[\int_0^t \int h_0(\hat{X}_s, u(D(u)))n(\hat{X}_s, du)ds]$$

$$= \hat{E}^\mu[\int_0^t \int (1-e^{-D(u)})n(\hat{X}_s, du)ds] \leq t.$$

Also, A. Benveniste and J. Jacod show in [2], that there is a function $h_1 \in \mathcal{E}$ such that the functional

$$\sum_{s\in(0,t]} h_1(\tilde{X}_{s-})1_{\{\tilde{X}_{s-}\neq X_s,\ \tilde{X}_{s-}\in E\}}$$

has a bounded 1-potential. But if $Y_s = \delta$ and $s < \hat{\zeta}$, then by (f) of Lemma 13, $S^-(s) = S^+(s)$ and so

$$\tilde{X}^F_{s-} = \tilde{X}_{S^+(s)-} .$$

Thus, if we put

$$h(x,y) = h_1(x) \wedge h_0(x,y) , \quad \text{for} \quad x,y \in F ,$$

we have that $h(x,y) > 0$ whenever $x \neq y$, and $K^h_t < \infty$ for every t, a.s. .

It follows once more by the argument of (13.4) of Getoor [23] , that

$$\tilde{W}^F_{\tau-} = W^F_{\tau-} \quad Q^\mu\text{-a.s.} \quad \text{on} \quad \{\tau < \infty\} ,$$

for any $(\mathcal{U}^{F,\mu}_t)$ totally inaccessible stopping time. Thus, taking $b = h$, condition (8.3) will hold, provided

$$Q^\mu(K^h_{\tau-} \neq K^h_\tau , \ \tau > 0) = 0$$

for any $(\mathcal{U}^{F,\mu}_t)$ predictable stopping time τ. We may, without loss of generality, assume that $\tau > 0$, and that $\tau = \infty$ on $\{\tau \geq \hat{\zeta}\}$. Write $T = \tau(\hat{X}.)$. Then T is $(\mathcal{F}^\mu_t)$ predictable, so that $Y_T = \delta$ $\hat{P}^\mu$-a.s., and hence $S^-(T) = S^+(T) > 0$ $\hat{P}^\mu$-a.s. (we use here that $\hat{P}^\mu(T = \hat{\zeta}<\infty) = 0)$. Let

$$T_n \uparrow T , \ T_n < T \quad \hat{P}^\mu\text{-a.s.} .$$

Then

$$S^+(T_n) \uparrow S^-(T)\ ,\quad S^+(T_n) < S^-(T) \quad \hat{P}^\mu\text{-a.s.}$$

also, so that $S^-(T)$ is a $(\mathcal{G}_t^\mu)$ predictable stopping time. Thus

$$\tilde{X}^F_{T-} = \tilde{X}_{S^-(T)-}$$

either equals $X_{S^+(T)} = \hat{X}_T$, or does not lie in E, from which we conclude that

$$Q^\mu(K^h_{\tau-} \neq K^h_\tau) = 0\ .$$

Conversely, let τ be $(\mathcal{U}_t^{F,\mu})$ totally inaccessible, with

$$W^F_{\tau-} \neq W^F_\tau \quad Q^\mu\text{-a.s. on } \{\tau < \infty\}\ .$$

Then as above,

$$\tilde{W}^F_{\tau-} = W^F_{\tau-} \quad Q^\mu\text{-a.s.}\ ,$$

so that by definition, $K^h_{\tau-} \neq K^h_\tau$ Q^μ-a.s. on $\{\tau < \infty\}$. Thus (K^h_t) regulates (Q^x).

(8.9) now follows as a special case of Lemma 9. The strong Markov property of $(\hat{X}_t, Y_t)$ is immediate from that of (X_t). (8.8) follows from (9.6), and the definition of (Y_t). To show (9.7), put

$$A(t) = \int_0^t m(W_s)d\tilde{l}_s + \int_0^t 1_{E\setminus F}(W_s)ds\ .$$

Since $m \in \mathcal{E}$, $A(t)$ is adapted to $(\bar{\mathcal{U}}_t)$, hence is a continuous perfect additive functional of (W_t). It will suffice to show

that $A(t) = t$ for every t, a.s., and for this it suffices in turn that the corresponding 1-potentials agree.

$$E^x[\int_0^\infty e^{-t} dA_t]$$

$$= \hat{E}^x[\int_0^\infty e^{-t} dL_t] - \hat{E}^x[\int_0^\infty e^{-t}(1-m(X_t))dL_t] + \hat{E}^x[\int_0^\infty e^{-t} 1_{E\setminus F}(X_t)dt]$$

$$= \hat{E}^x[e^{-\hat{D}}] - \hat{E}^x[\int_0^\infty e^{-S^+(s)}(1-m(\hat{X}_s))ds] + \hat{E}^x[\sum_{t\in M_0\cup\{0\}} e^{-t}(1-e^{-\hat{D}\circ\theta_t})]$$

$$= \hat{E}^x[e^{-\hat{D}}] - \hat{E}^x[\sum_{\substack{s>0\\ Y_s\neq\delta}} e^{-S^-(s)}(1-e^{-D(Y_s)})] +$$

$$+ \hat{E}^x[\sum_{\substack{s\geq 0\\ Y_s\neq\delta}} e^{-S^-(s)}(1-e^{-D(Y_s)})] \qquad \text{(by (8.7))}$$

$$= \hat{E}^x[e^{-\hat{D}}] + \hat{E}^x[1-e^{-\hat{D}}] = 1 \text{ , as required.}$$

Turning to II, let x F, and

$$R = \inf\{t;\ d(x,X_t) > \varepsilon\} \quad , \quad T = L_R \ .$$

By right continuity of (X_t), R and hence T, are $\hat{P}^x$-a.s. strictly positive. Thus by (8.7),

$$\hat{E}^x[\int_0^T n(\hat{X}_s,\{\sup_t d(x,W_t) > \varepsilon\})ds]$$

$$= \hat{E}^x[\sum_{\substack{s\in(0,T]\\ Y_s\neq\delta}} 1_{\{\sup_t d(x,W_t)>\varepsilon\}}(Y_s)] \le 1 ,$$

showing II(b). We obtain II(a) by modifying n on a set of $\hat{X}$-potential zero, as follows. Let

$$B = \{x\in F;\ n(x,\{\sup_t d(x,W_t) > \varepsilon\}) = \infty\} \in \mathcal{E} .$$

Then for each $t \ge 0$, $(s,\omega) \to 1_B(\hat{X}_s(\omega))$ is measurable with respect to the $ds \otimes \hat{P}^\mu$ completion of $\mathcal{B}|_{[0,t]} \otimes \mathcal{F}_t$, as a function $[0,t] \times \Omega \to \mathbb{R}$. Thus

$$A_t = \int_0^t 1_B(\hat{X}_s)ds$$

is $\mathcal{F}_t^\mu$ measurable for each μ, and hence $T = \inf\{t;\ A_t > 0\}$ is an $(\mathcal{F}_t)$ stopping time. Let

$$R = \inf\{r > T;\ \int_T^r n(\hat{X}_s,\{\sup_t d(\hat{X}_T,W_t) > \tfrac{\varepsilon}{2}\})ds \ge 1 ,$$

$$\text{or } d(\hat{X}_r, \hat{X}_T) > \tfrac{\varepsilon}{2}\} .$$

Then

$$\int_T^R n(\hat{X}_s,\{\sup_t d(\hat{X}_s, W_t) > \varepsilon\})ds \le 1 ,$$

so that by definition of T, we must have $R = T$ a.s.. But by II(b) and the strong Markov property of $\hat{X}$ at T, we have that $R > T$ a.s. on $\{T < \infty\}$, and hence $T = \infty$ a.s.. Thus we may replace $n(x,\cdot)$ by 0 for $x \in B$, and leave the other properties unchanged.

Properties IV and V follow from Theorem 5.1 of Maisonneuve [38], if necessary modifying n on a set of $\hat{X}$-potential zero, as above. A similar modification, together with (8.7), gives us I.

Finally, consider VI. Let

$$T = \inf\{t; \int_0^t \bar{m}(\hat{X}_s)ds > 0\}, \text{ and}$$

$$T_k = \inf\{t; \int_0^t n(\hat{X}_s,U)ds \geq k\},$$

for $k > 0$. Then

$$\hat{E}^x[\sum_{\substack{0<s\leq T\wedge T_k\wedge 1 \\ Y_s\neq\delta}} 1] = \hat{E}^x[\int_0^{T\wedge T_k\wedge 1} n(\hat{X}_s,U)ds] \leq k,$$

so that S^- increases at only finitely many times in $(0,T\wedge T_k\wedge 1)$. Since S^- is a.s. strictly increasing, we must have that $T_k = 0$ $\hat{P}^x$-a.s., whenever $x \in I$. Since k was arbitrary, VI(a) holds.

As in Proposition 1, let C be the set of dyadic rational numbers, and for $\varepsilon, \eta > 0$, let

$$B_{\varepsilon,\eta} = \{b\in E;\ P_0^b(D \geq \eta) \geq \varepsilon\} \in \mathcal{E}, \text{ and}$$

$$H_{\varepsilon,\eta} = \bigcup_{\lambda>0}\{D>0,\ W_t \in B_{\varepsilon,\eta} \quad \text{for} \quad t \in C \cap (0,\lambda)\} .$$

We obtain as before, that for every $x \in F$,

$$n(x,H_{\varepsilon,\eta}) \leq \frac{1}{\varepsilon} n(x,D \geq \eta) \leq \frac{1}{\varepsilon(1-e^{-\eta})}\int (1-e^{-D})\,dn(x,\cdot)$$

$$\leq \frac{1}{\varepsilon(1-e^{-\eta})} .$$

Fix μ. Because $H_{\varepsilon,\eta} \in \mathcal{U}_{0+}$, we can find $H^0_{\varepsilon,\eta} \in \mathcal{U}^0_{0+}$ such that

$$H_{\varepsilon,\eta} \supset H^0_{\varepsilon,\eta} \ , \quad \text{and} \quad \hat{E}^{\mu}[\int_0^{\infty} n(\hat{X}_s,\ H_{\varepsilon,\eta} \backslash H^0_{\varepsilon,\eta})ds] = 0 .$$

We argue as before, that (as F is nearly Borel)

$$T = \inf\{t>0;\ X_t \in F,\ X_{t+.} \in H^0_{\varepsilon,\eta}\}$$

is a $(\mathcal{G}_t)$ stopping time. Since $H^0_{\varepsilon,\eta} \subset \{D>0\}$, and

$$\hat{E}^{\mu}[\sum_{\substack{s\in(0,t]\\ Y_s\neq\delta}} 1_{H^0_{\varepsilon,\eta}}(Y_s)] = \hat{E}^{\mu}[\int_0^t n(\hat{X}_s,\ H^0_{\varepsilon,\eta})ds]$$

$$\leq t/\varepsilon(1-e^{-\eta}) < \infty ,$$

for $t \geq 0$, we have that

$$Y_{L_T} \in H^0_{\varepsilon,\eta} \quad \text{and} \quad X_T \in F ,\ \hat{P}^{\mu}\text{-a.s.} .$$

But by the strong Markov property of (X_t), no excursion can start at a stopping time at which X lies in F. Thus $T = \infty$ $\hat{P}^{\mu}$-a.s., so that

$$\sum_{\substack{s>0 \\ Y_s \neq \delta}} 1_{H^0_{\varepsilon,\eta} \cap \{W_0 \in F\}}(Y_s) = 0 , \quad \hat{P}^\mu\text{-a.s.},$$

and hence

$$\hat{E}^\mu[\int_0^\infty n(\hat{X}_s, H_{\varepsilon,\eta} \cap \{W_0 \in F\})ds] = 0 ,$$

for each ε, η and μ. We can therefore modify n as before, to make

$$(9.8) \qquad n(x, H_{\varepsilon,\eta} \cap \{W_0 \in F\}) = 0$$

for every x, ε and η.

Now observe that if VI(b) fails at x, then there is a finite, positive, nonzero measure n' concentrated on $\{W_0 \in F\}$, for which IV holds, and also $n(x,du) \geq n'(du)$. The argument of Proposition 1 shows that

$$n'(H_{\varepsilon,\eta} \cap \{W_0 \in F\}) > 0$$

for some $\varepsilon, \eta > 0$, contradicting (9.8). Thus VI(b) holds. □

We now turn to the converse. Let E be a seperable metric space, with an isolated point Δ singled out. Let ∂ be as before. The main result of this section is:

<u>Theorem 4</u> (a) Let $F \subset E$, $\Delta \in F$, $F \in \mathcal{E}$, and assume that F is a U-space. Let (P_0^x) satisfy (2.4), and assume that F is nearly optional for (W_t) and the P_0^μ. Thus, for each μ, the first hitting time D_0 of F ;

$$D_0 = \inf\{t > 0;\ W_t \in F\}\ ;$$

is a stopping time with respect to the P_0^μ completion of $(\mathcal{U}_{t+})$ in $\mathcal{U}$. Assume that

$$P_0^\mu(W_t \neq W_{D_0} \quad \text{for some} \quad t \geq D_0 \quad) = 0$$

for each μ. Then

$$D = \inf\{t \in (0,\infty) \cap \mathbb{Q}\ ;\ W_t \in F \quad \text{and for some} \quad \varepsilon > 0\ ,$$

$$W_s = W_t \quad \text{for every} \quad s \in [t, t+\varepsilon) \cap \mathbb{Q}\}$$

is an exact perfect terminal $(\mathcal{U}_{t+})$ stopping time which is P_0^μ-indistinguishable from D_0 for each μ. Let $n(x,du)$ be an $(F, \mathcal{E}^F;\ U, \mathcal{U})$ kernel. Let $(\Omega, \mathcal{F})$ be a measurable space, with a right continuous filtration $(\mathcal{F}_t)$. Let $\hat{P}^x(du)$ be an $(E, \mathcal{E};\ \Omega, \mathcal{F})$ kernel consisting of probabilities, and assume that $(\mathcal{F}, \mathcal{F}_t)$ is augmented with respect to the $\hat{P}^\mu$. Let $(\hat{X}_t)$ be a right continuous process with values in F, which is adapted to $(\mathcal{F}_t)$, and write $\hat{\zeta} = \partial(\hat{X}.)$. For $x \in F$, we denote the image law of $\hat{P}^x$ under $\hat{X}.$, by Q^x. Assume that for each probability μ on E, the process

$$(\hat{X}_t,\ \mathcal{F}_t^{\hat{\mu}},\ \hat{P}^\mu,\ Q^x)$$

is strong Markov. Suppose that conditions I-VI hold for these objects.

Let $Y : \Omega \to \Pi$ be adapted to $(\mathcal{F}_t)$, and satisfy

(8.7) for this n and $(\hat{X}_t)$. Suppose also that:

$(\hat{X}_t, Y_t, F_t, \hat{P}^x)$ is strong Markov;

(9.9) If $Y_t \neq \delta$, then $\hat{X}_t = \begin{cases} Y_t(D(Y_t)) & , \text{ if } D(Y_t) < \infty \\ \Delta & , \text{ if } D(Y_t) = \infty \end{cases}$

(9.10) $\hat{P}^\mu(Y_0 \in du) = P_0^\mu(du)$ for every μ.

Define m and S^+ by VI(a) and (9.7), and let

$$L_t = \sup\{s;\ S^+(s) \leq t\}.$$

Then there is a right continuous filtration (G_t) of (Ω, F), a family $(P^x)_{x \in E}$ of probabilities on $(U, \mathcal{U})$, and a right continuous process (X_t) adapted to (G_t), such that:

Each $(X_t, G_t^\mu, \hat{P}^\mu, P^x)$ is strong Markov;

(G, G_t) is augmented with respect to the $\hat{P}^\mu$;

(9.11) $\hat{P}^\mu(X._{\wedge D_0} \in du) = P_0^\mu(du)$ for each μ;

(9.12) Each $S^+(s)$ is a (G_t) stopping time, $G_{S^+(s)} = F_s$, and (9.4) and (9.5) hold for every s, a.s.;

(9.13) $\hat{M} = \overline{\{t;\ X_t \in F\ \}}$ is (G_t)-progressive, satisfies (9.1) and (9.2),

and $\hat{M} = \overline{\{S^+(s);\ s>0\ ,\ S^+(s)<\infty\ \}}$.

(9.14) For every (G_t^μ) stopping time T, $(X._{+T}, L._{+T} - L_T)$ is conditionally independent of G_T^μ under $\hat{P}^\mu$, given X_T, with conditional law

(9.15) $\hat{P}^{X_T}((X.,L.) \in (\cdot))$; and

(L_T) satisfies (9.3).

(b) Moreover, if the (P_0^x) are the transition laws of a right process satisfying (6.5), and if from other considerations, we know that the excessive functions for the (P^x) are (Q^x)-nearly Borel, when restricted to F, then in fact

$$(X_t, F, G_t, \hat{P}^x)$$

is a right process satisfying (6.5). In this case, there is a continuous PAF (1_t) adapted to (U_{t+}), such that

$$L_t = 1_t(X.) \quad \text{for every } t, \text{ a.s. .}$$

<u>Note</u>: 1. We do not assume that (Q^x) is regulated. We can get away with this because (Y_t) is assumed to be given to us, and to be strong Markov. If we were given only n, we would of course not know how to produce such a (Y_t), unless (Q^x) was regulated (see Corollary 6). Also note that Theorem 4 uses only the strong Markov property of $(\hat{X}_t, Y_t)$, and not the knowledge of their exact joint distribution, which we could obtain from Corollary 4 provided (Q^x) was known to be strongly regulated. In this case, the assumption that $(\hat{X}_t, Y_t)$ be strong Markov could be dropped, using Corollary 5. Note that (9.9) gives (8.8).

2. The conditions of Theorem 4(a) are met in the (vi'b) and (vi'c) cases of Theorem 2, and in this situation, Theorem 4(b) contains Proposition 5. The proof of Theorem 4 is essentially the same as that of Theorem 2, the chief complication coming in the proof of right continuity.

3. The condition that F be a U-space is included in order that Lemma 11 hold. This result is needed for Lemma 15. We will use this condition again in Corollary 6. In that result, condition (9.9) again plays the role of (8.8), but for now we use it only to give (9.5).

<u>proof</u>: Recall that m and S^+ are defined as in VI and (9.7) Put

$$S^+(t,p,u) = \int_0^t m(u(s))ds + \sum_{\substack{0\le s\le t \\ p(s)\neq\delta}} \mathbb{D}(p(s))$$

so that $S^+(t) = S^+(t, Y, \hat{X})$. Let $S^-(t, p, u) = S^+(t-, p, u)$,

$$\mathcal{V}_t^0 = \mathcal{P}_t \otimes \mathcal{U}_t , \quad \mathcal{V}^0 = \mathcal{V}_\infty^0 ,$$

and let $\mathcal{V}$ be the universal completion of $\mathcal{V}^0$. As in the proof of Theorem 2, we obtain $(\mathcal{V}_{s+}^0)$ stopping times $\ell_t(p,u)$, and functions $x_t(p,u)$ measurable from $\mathcal{V}_{\ell_t+}^0$ to $\mathcal{E}^0$. The set M of Lemma 2 becomes

$0 = \{(p,u);\ u(s) = \Delta$ and $p(s) = \delta$ for every $s>\partial(u)$, $S^-(s,p,u) < \infty$ for every $s\in[0,\partial(u)]\cap[0,\infty)$ and is strictly increasing at all such s, $S^-(s,p,u) \to \infty$ as $s \to \infty$, and for every $\varepsilon > 0$ and $t\in[0,\partial(u))$ there is an $h > 0$ such that $\sup_r d(p(s)(r), u(t)) \le \varepsilon$ whenever $p(s) \neq \delta$ and $s \in (t, t+h)$.$\}$.

Because

$$x_t(p,u) = u(\ell_t(p,u)) \quad \text{whenever} \quad t = S^+(t,p,u) ,$$

the proof of Lemma 2 applies, showing that $t \to x_t$ is right continuous on 0.

<u>Lemma 14</u> $0 \in V$, and $(Y., \hat{X}.) \in 0$ a.s..

<u>proof</u>: Let $0'$ be the set of (p,u) satisfying all but the last condition in the definition of 0. We see easily that $0' \in V^0$. Thus consider

$$\hat{0} = \{(t,p,u);\ (p,u) \in 0' ,\ \text{and for every } \varepsilon > 0 \text{ there is an } h > 0 \text{ such that } \sup_r d(p(s)(r),u(t)) \le \varepsilon \text{ whenever } p(s) \neq \delta \text{ and } s \in (t,\ t+h).\} .$$

Let

$$\tau_{q,k}(p) = \inf\{s \ge q;\ p(s) \neq \delta \text{ and } D(p(s)) > \tfrac{1}{k}\} .$$

The $\tau_{q,k}$ are $(\mathcal{P}_{t+})$ stopping times, and

$$\{s;\ p(s) \neq \delta\} = \{\tau_{q,k}(p) ;\ q \in \mathbb{Q} ,\ k \ge 1\} ,$$

whenever $(p,u) \in 0'$ for some u. Thus

$$\hat{0} = \bigcap_i \bigcup_j \bigcap_{\substack{k \\ q \in \mathbb{Q}}} \{(t,p,u);\ (p,u) \in 0' \text{ and } 1_{(q-\frac{1}{j},q)}(t) 1_{(q,q+\frac{1}{j})}(\tau_{q,k}(p)) \sup_r d(p(\tau_{q,k}(p))(r),u(t)) \le \tfrac{1}{i}\} .$$

Since $(t,u) \to u(t)$ is measurable from $\mathcal{B} \otimes \mathcal{U}$ to $\mathcal{E}^0$, and

$$(v,x) \to \sup_r d(v(r),x)$$

is measurable, from $\mathcal{U} \otimes \mathcal{E}^0$ to $\mathcal{B}$, we see that $\hat{0} \in \mathcal{B} \otimes \mathcal{V}^0$, and hence that $0 \in \mathcal{V}$.

Since $\partial(\hat{X}.) = \hat{\zeta}$, the first condition of 0 follows from (8.7) (take $Z_s = 1_{(\hat{\zeta},\infty)}(s)$). It follows as in the proof of Lemma 3, that $S^-(s) < \infty$ for $s \le \hat{\zeta}$, $s < \infty$.

Let

$$T = \inf\{s;\ S^-(s) = S^-(s+h) \text{ for some } h > 0\}, \text{ and}$$

$$R = \inf\{s > T;\ Y_s \neq \delta\}.$$

They are both $(\mathcal{F}_s)$ stopping times, $\hat{X}_T \in I$ a.s. on $\{T < \hat{\zeta}\}$, and

$$1 \ge \hat{E}^\mu[\sum_{\substack{s \in (T,R] \\ Y_s \neq \delta}} 1] = \hat{E}^\mu[\int_T^R n(\hat{X}_s,U)ds]$$

for every μ. Thus, by VI(a), $R = T$ a.s. on $\{T < \hat{\zeta}\}$, so that $T = \hat{\zeta}$ a.s., and hence S^- is a.s. strictly increasing on $(0,\hat{\zeta}] \cap (0,\infty)$.

Assume that

$$\hat{p}^\mu(\hat{\zeta} = \infty,\ S^-(s) \not\to \infty \text{ as } s \to \infty) > 0,$$

and let

$$T_k = \inf\{t; \sum_{\substack{s\in(0,t] \\ Y_s\neq\delta}} D(Y_s) \geq k\} .$$

Then for some k,

$$\hat{p}^{\mu}(\hat{\zeta}=\infty=T_k, \int_0^{\infty} m(\hat{X}_s)ds < \infty) > 0 ,$$

and hence

$$k \geq \hat{E}^{\mu}[\int_0^{T_k} \int D(u)n(\hat{X}_s,du)ds]$$

$$\geq \hat{E}^{\mu}[\int_0^{T_k} (1-m(\hat{X}_s))ds] = \infty ,$$

which is impossible.

Finally, we turn to the last condition. We argue as on p.94 of Blumenthal and Getoor [5] . Fix $\varepsilon > 0$, let $T_0 = 0$, and define T_α for α a countable ordinal number, by

$$T_{\alpha+1} = \inf\{s > T_\alpha;\ d(\hat{X}_{T_\alpha}, \hat{X}_s) > \varepsilon ,\ \text{or}\ Y_s \neq \delta \text{ and}$$

$$\sup_t d(\hat{X}_{T_\alpha}, Y_s(t)) > \varepsilon\} ,\ \text{and}$$

$$T_\alpha = \sup_{\beta<\alpha} T_\beta ,\ \text{if } \alpha \text{ is a limit ordinal.}$$

Then T_α is an $(\mathcal{F}_s)$ stopping time for each α. Let

$$R = \inf\{s > 0;\ \int_0^s n(\hat{X}_s,\{\sup_t d(\hat{X}_0,W_t) > \varepsilon\})ds \geq 1\} \wedge 1 .$$

By II(b), $R > 0$ a.s., and by (8.7),

$$\hat{E}^{\mu}[\sum_{\substack{s\in(0,R] \\ Y_s\neq\delta}} 1_{\{\sup_t d(\hat{X}_0,W_t)>\varepsilon\}}(Y_s)] \leq 1 < \infty ,$$

for every μ. The integrand is thus a.s. finite, and so $T_1 > 0$ a.s. by right continuity of $\hat{X}.$. By the strong Markov property of $(\hat{X}_t)$, we therefore obtain that for each $\alpha < \beta$,

$$T_\alpha < T_\beta \quad \text{a.s. on} \quad \{T_\beta < \infty\} .$$

But for each μ,

$$h(\alpha) = \hat{E}^\mu[1 - e^{-T_\alpha}]$$

is increasing in α, so that as usual, there exists a countable ordinal α_0 such that

$$h(\alpha) = h(\alpha_0)$$

for every $\alpha \geq \alpha_0$. This is only possible if

$$\hat{P}^\mu(T_{\alpha_0} < \infty) = 0 .$$

But if $T_{\alpha_0}(\omega) = \infty$, then there is an $\alpha \leq \alpha_0$ such that $t \in [T_\alpha(\omega), T_{\alpha+1}(\omega))$, and by definition of T_α,

$$\sup_r d(Y_s(\omega)(r), \hat{X}_t(\omega)) \leq 2\varepsilon$$

whenever $s \in (t, T_{\alpha+1}(\omega))$. From this, we conclude that

$$\hat{P}^\mu((Y.,\hat{X}.) \in 0) = 1 ,$$

and as μ was arbitrary, the lemma is proven. □

Now, for each μ, let $\mathcal{F}^\mu_{0-}$ be generated by the $\hat{P}^\mu$-null sets of $\mathcal{F}^\mu$, and define

$$H_r^{R,\mu}, \quad G_t^{\mu}$$

and their ilk from the F_t^{μ}, as in Theorem 2. We define

$$G_t = \bigcap_{\mu} G_t^{\mu} .$$

The proof of Lemma 4 applies to the pair of filtrations (F_t^{μ}), (G_t^{μ}), and since (9.5) follows from (9.9), (9.12) is now immediate. Conditions (9.11) follows immediately from (9.10).

Turning to (9.13), let

$$M' = \overline{\{S^+(s);\ s>0\ ,\ S^+(s)<\infty\ \}}$$

Since $X_{S^+(s)} \in F$ when $S^+(s)<\infty$, it follows that $M' \subset \hat{M}$. Conversely, to show that $\hat{M} \subset M'$ a.s., it suffices to show that $Y_t(s) \notin F$ whenever $t \le \hat{\zeta}$ and $s < D(Y_t)$; a.s.. Fix μ , and let

$$\mu'(A) = \hat{E}^{\mu}[\int_0^{\infty} e^{-s} 1_A(\hat{X}_s)ds] .$$

By IV, V and the P_0^{ν} indistinguishability of D and D_0 for each ν , we see that there is an $A \in U$ such that $\{D_0 < D\} \subset A$, and

$$\int_F n(x,A)\mu'(dx) = 0 .$$

Thus by (8.7), $\hat{P}^{\mu}(Y_t \in A$ for some $t) = 0$, as required.

$S^-(L_t)$ is a (G_t) predictable process, hence

$$\overline{\{t;\ S^-(L_t) = t\}}$$

is (G_t) progressive. This set is identical to M', which in turn

equals $\hat{M}$ a.s., so that $\hat{M}$ is $(\mathcal{G}_t)$ progressive. Since S^+ is right continuous and strictly increasing, (9.1) is now immediate.

Let $\hat{M}_0$ be the set of points of $\hat{M}$, isolated on the right. By (8.7), we cannot have $\hat{P}^\mu(Y_R \neq \delta) > 0$ for any $(\mathcal{F}_s^\mu)$ predictable stopping time $R > 0$. Let (T_n) be a sequence of $(\mathcal{G}_t^\mu)$ stopping times such that $T_n \uparrow T$ and $T_n < T$ on $\{T > 0\}$. Then $S^+(T_n) \uparrow S^-(T)$, and $S^+(T_n) < S^-(T)$ $\hat{P}^\mu$-a.s. on $\{T = S^-(L_T) > 0\}$. Thus $Y_T = \delta$ $\hat{P}^\mu$-a.s. on this set, and hence $\hat{P}^\mu(T \in \hat{M}_0) = 0$. Condition (9.2) now follows, from which we conclude (9.13).

Thus, all that remains of part (a) is to show the strong Markov property of each

$$(X_t, \mathcal{G}_t^\mu, \hat{P}^\mu, P^x),$$

together with (9.14) and (9.15)

The analogue of Lemma 6 is proved as in that result. Thus, for each $x \in F$, the coordinate process $(W_t, \mathcal{U}_t, n(x,\cdot), P_0^x)$ is strong Markov at every $(\mathcal{U}_{t+})$ stopping time R satisfying

$$n(x,\{W_0 \in F,\ R = 0\}) = 0.$$

Because F is a U-space, the argument of (13.4) of Getoor [23] applies once more, to show that $\hat{X}_{R-}$ exists $\hat{P}^\mu$-a.s. on $\{R < \infty\}$ for any $(\mathcal{F}_t^\mu)$ stopping time R such that $Y_R \neq \delta$ $\hat{P}^\mu$-a.s. on $\{R < \infty\}$.

<u>Lemma 15</u> Let T be a $(\mathcal{G}_t)$ stopping time such that $L_T > 0$ and $T < S^+(L_T)$ on $\{T < \infty\}$. Let μ be a σ-finite positive measure

on E, and let $\mathcal{H}_t$ be the σ-field

$$\mathcal{F}^{\mu}_{L_T-} \otimes \mathcal{U}_t ,$$

on $\Omega \times U$. Then there is an $(\mathcal{H}_{t+})$ stopping time R such that

(9.16) $\hat{P}^{\mu}\{\omega;\ n(\hat{X}_{L_T-}(\omega),\ \{u;\ R(\omega,u) < \infty\}) = \infty,\ T < \infty\} = 0$

(9.17) $R(\omega,\ Y_{L_T}(\omega)) = (T - S^-(L_T))(\omega)$, if $T(\omega) < \infty$

(9.18) $R(\omega,\ u) = \infty$ for every $u \in U$, if $T(\omega) = \infty$

(9.19) $\hat{P}^{\mu}(Y_{L_T} \in A,\ T - S^-(L_T) \in B,\ T < \infty \mid \mathcal{F}_{L_T-})(\omega)$

$$= \frac{n(\hat{X}_{L_T-}(\omega),\ \{u; u \in A, R(\omega,u) \in B \cap [0,\infty)\})}{n(\hat{X}_{L_T-}(\omega),\ \{u; R(\omega,u) < \infty\})},\ \hat{P}^{\mu}\text{-a.s. on } \{T<\infty\}$$

where $A \in \mathcal{U}$, $B \in \mathcal{B}$, and we make the convention that

$$\frac{0}{0} = 0 = \frac{\infty}{\infty} .$$

<u>proof</u>: The construction of the stopping times R and R^k_q, and the proofs of (9.17) and (9.18) are as before, the only formal changes coming from the replacement of σ_a by D, and the fact that now the S^k_q may take on the value ∞. (9.16) will again follow from (9.18), (9.19), and our convention that $\infty/\infty = 0$. At this point in Theorem 2, we used a result from Itô [32]. The corresponding result here is Lemma 11, and from it we obtain as before that

$$(9.20)\qquad \hat{P}^{\mu}(C,\ Y_{L_T} \in A,\ T - S^{-}(L_T) \in B\ ,\ T < \infty\ )$$

$$= \int_{C\cap\{T<\infty\}} \frac{n(\hat{X}_{L_T-}(\omega), A\cap\{R(\omega,\cdot)\in B\cap[0,\infty)\})}{n(\hat{X}_{L_T-}(\omega),\ \{R(\omega,\cdot)<\infty\})}\ \hat{P}^{\mu}(d\omega)$$

whenever $C \in \mathcal{F}_{L_T-}$, $A \in \mathcal{U}$, $B \in \mathcal{B}$, and $n(x,A) < \infty$ for every $x \in F$. But $n(x,\ \{D > 1/k\}) < \infty$ for every x, so that the above applies with A replaced by $A_k = A\cap\{D > 1/k\}$. By our convention, the integrands for the A_k converge boundedly to the integrand for A, so that (9.20) and hence (9.19) hold, for every $A \in \mathcal{U}$. □

With these results, the analogues of Corollaries 1 and 2 are proven as before (the former of these becoming that $T = S^{+}(L_T)$ $\hat{P}^{\mu}$-a.s., whenever T is a $(\mathcal{G}_t^{\mu})$ stopping time such that $X_T \in F$ on $\{T < \infty\}$).

We now separate out the contribution of Y_0 to (X_t), and write $X.$ as a measurable function

$$X. = H(Y_0,\ Y|_{(0,\infty)},\ \hat{X}.)$$

as before, and let

$$P^x(A) = \iint 1_A(H(u,\ Y|_{(0,\infty)}(\omega),\ \hat{X}_.(\omega))\hat{P}^{u(D(u))}(d\omega)P_0^x(du)\ .$$

The strong Markov property of

$$(X_t,\ \mathcal{G}_t^{\mu},\ \hat{P}^{\mu},\ P^x)$$

will now follow as before, using now the strong Markov property of $(\hat{X}_t,\ Y_t,\ \mathcal{F}_t,\ \hat{P}^x)$.

Since

$$L_{\cdot+T} - L_T = \ell.(\theta_T^0(Y),\ \hat{X}_{\cdot+T}) \ ,$$

this strong Markov property is used again, to argue as in Theorem 2 that (9.14) holds.

Finally, to show (9.15), define $A(t)$ as in the proof of (9.7) in Theorem 3. Because of (9.13), we have by definition of S^+, that $A(t) = t$ for every t, a.s.. Thus we can reverse the argument in Theorem 3 (using (9.10)), to obtain (9.3), showing part (a).

To show (b), let f be α-excessive for (X_t). Then f is also α-excessive for the (P_0^x). As in the proof of (9.13) above, we fix μ, and let

$$\mu'(A) = \hat{E}^\mu[\int_0^\infty e^{-t} 1_A(\hat{X}_t)dt] \ .$$

Because (6.5) holds for (W_t) under the (P_0^x), and f is nearly Borel for the (Q^x), by hypothesis, we conclude from IV and V that there are $f_1, f_2 \in \mathcal{E}^0$ such that

$$f_1 \le f \le f_2 \ , \text{ and}$$

$$\int_F n(x,\{f_1(W_t) \ne f_2(W_t) \text{ for some } t\})\ \mu'(dx)$$

$$+ P_0^\mu(f_1(W_t) \ne f_2(W_t) \text{ for some } t) +$$

$$+ \hat{E}^\mu[\hat{P}^{Y_0(D(Y_0))}(f_1(\hat{X}_t) \ne f_2(\hat{X}_t) \text{ for some } t)] = 0 \ .$$

It follows immediately, that

$$\hat{P}^{\mu}(f_1(X_t) \neq f_2(X_t) \quad \text{for some} \quad t) = 0 ,$$

showing that (6.5) holds.

The same will hold for the canonical process (W_t) on U, under the P^x, so that as in the discussion preceding Lemma 13, applied to

$$M = \overline{\{t;\ W_t \in F\}},$$

there exists a continuous additive functional of (W_t) with 1-potential

$$\psi(x) = E^x[e^{-D_0}] .$$

By the perfection arguments in Meyer [40], we may choose a perfect version, adapted to $(\mathcal{U}_{t+})$. Call it $l_t(u)$. We need to show that

$$l_t(X.) = L_t \quad \text{for every} \quad t , \text{ a.s.} .$$

Let $\mathcal{U}^{\mu}$ be the P^{μ} completion of $\mathcal{U}$, and adjoin the P^{μ}-null sets of $\mathcal{U}^{\mu}$ to $\mathcal{U}_t$, to obtain $\mathcal{U}_t^{\mu}$. Let M_0 be the set of points of M that are isolated on the right. Then as in Maisonneuve [38], p.409, the process (l_t) is the $(\mathcal{U}_t^{\mu})$-dual predictable projection of

$$t \to \int_0^t 1_F(W_s)ds + \sum_{s \in M_0 \cap (0,t]} (1 - e^{-D_0 \circ \theta_s})$$

under the measure P^{μ}.

Now let

$$Z_s = Z\, 1_{(t_0,t_1]}(s)\ ,\quad Z \in \mathcal{F}^{\mu}_{t_0}\ .$$

Then

$$\hat{E}^{\mu}[\int_0^{\infty} Z_s 1_F(X_s)ds + \sum_{s\in M_0} Z_s(1-e^{-D(Y_{L_s})})]$$

$$= \hat{E}^{\mu}[Z\, E^{\hat{X}_{t_0}}[\int_0^{t_1-t_0} 1_F(W_s)ds + \sum_{s\in M_0\cap(0,t_1-t_0]}(1-e^{-D\circ\theta_s})]]$$

$$= \hat{E}^{\mu}[Z\, E^{\hat{X}_{t_0}}[\int_0^{t_1-t_0} d1_s]]$$

$$= \hat{E}^{\mu}[\int_0^{\infty} Z_s d(1_s(X.))\quad .$$

Thus, $(1_t(X.))$ is the $(\mathcal{F}^{\mu}_t)$-dual predictable projection of

$$t \to \int_0^t 1_F(X_s)ds + \sum_{s\in M_0\cap(0,t]}(1-e^{-D(Y_{L_s})})\ .$$

But a direct computation shows that (L_t) has the same property. Thus by uniqueness of dual predictable projections,

$$L_t = 1_t(X.) \quad \text{for every } t\ ,\ \text{a.s.},$$

as required. □

For completeness, we include the following immediate corollary to Proposition 11.

<u>Corollary 6</u> Assume all the conditions of Theorem 4(a), except those involving (Y_t). Assume that E is a U-space, and that (Q^x)

is regulated. Then in order that we may enlarge Ω, F and the F_t, and find a process (Y_t) such that all of the conditions of Theorem 4 are met, it is necessary and sufficient that (8.9) hold.

<u>proof</u>: (8.6) holds by III, and as remarked in the discussion prior to Proposition 11, the condition on the existence of kernels $n(x,y;\, du)$ is satisfied when E is a U-space. Thus Proposition 11 applies, giving (Y_t). Note that (9.9) implies condition (8.8), with

$$G(x,u) = \begin{cases} u(D(u)) \,, & D(u) < \infty \\ \Delta \,, & \text{otherwise} \,. \end{cases} \quad \square$$

Note that if we start with (X_t) and M as in Theorem 3, and form $(\hat{X}_t)$, n, and (Y_t) from them, then Theorem 4 will apply, and will reconstruct (X_t). Similarly, a new process (X_t') may be constructed, using Corollary 6 and then Theorem 4. As remarked in section 8, the results of Gzyl [28] show that $(\hat{X}_t)$ is a right process, and so strongly regulated. Applying Corollary 4, we see that (X_t') has the same law as (X_t).

It follows from a remark made following the proof of Lemma 2, that in the situation of Theorem 2, the constructed process (X_t)

will be a.s. càdlàg, provided that a.s., $Y_t(\cdot)$ has left limits whenever $Y_t \neq \delta$. The following example shows that in the present case, (X_t) need not be càdlàg, even when $(\hat{X}_t)$ and the Y_t are:

<u>Example 5</u>: Let E be the unit square, with the boundary partially removed;

$$E = ([0,1] \times [0,1]) \setminus (\{1\} \times (0,1]) .$$

Let $F = [0,1] \times \{0\}$, and let $r : [0,1) \to (0,\infty)$ be continuous. Let $P_0^{(x,y)}$ correspond to uniform downward motion on $\{x\} \times [0,1]$ at rate $r(x)$, started at (x,y) and stopped upon contact with F. Let $(\hat{X}_t)$ be uniform motion to the right on F, stopped at the point $(1,0)$. It is a right process, so that (Q^x) is regulated. Set

$$n(x,du) = \frac{1}{2(1-\exp(-1/r(x)))} P_0^{(x,1)}(du) .$$

We see that $m(x) = 1/2$ for $x \in [0,1)$, so that VI(a) is vacuous. Thus, conditions I-VI are met, so that we can construct (Y_t) by Corollary 6, and apply Theorem 4 to obtain a process (X_t). Let

$$T = \inf\{t;\ X_t = (1,0)\} .$$

We have that $T < \infty$ a.s., and $X_t = (1,0)$ for every $t \geq T$. There are only finitely many excursions away from F in any time interval $[0,T-h]$, $h > 0$, but by letting $r(x) \to \infty$ sufficiently fast, as $x \to 1$, we can ensure that there are infinitely many excursions in $[0,T]$, a.s., so that there is no left limit at T.

We can impose conditions to guarantee càdlàg paths, as in the next result;

<u>Proposition 12</u> (a) Suppose that in addition to the hypotheses of Theorem 3, we assume that (X_t) has left limits on $(0,\zeta)$, a.s.. Then

$$(9.21) \qquad \int_0^h n(\hat{X}_s,\{\sup_t d(\hat{X}_s, W_t) > \varepsilon\})ds < \infty \quad \text{for every } \varepsilon > 0,\ h<\hat{\zeta},\ \text{a.s..}$$

(b) Suppose that the hypotheses of Theorem 4(a) are verified, and that (9.21) holds. Suppose also that: $(\hat{X}_t)$ has left limits on $(0,\hat{\zeta})$ a.s.; $(\hat{X}_t)$ has a left limit at $\hat{\zeta}$ a.s. on $\{\hat{\zeta}<\infty,\ Y_{\hat{\zeta}} \neq \delta\}$; and Y_t has left limits on $(0,D(Y_t)] \cap (0,\partial(Y_t))$ whenever $Y_t \neq \delta$, a.s.. Then we conclude that (X_t) has left limits on $(0,\zeta)$, a.s..

<u>proof</u>: Let

$$T = T_1 = \inf\{t > 0;\ Y_t \neq \delta \text{ and } \sup_s d(\hat{X}_{t-},Y_t(s)) > \varepsilon\}\ .$$

Since $\{(t,\omega);\ Y_t(\omega) \neq \delta\}$ is indistinguishable from a countable union of graphs of $(\mathcal{F}_t)$ stopping times, as before, T must also be an $(\mathcal{F}_t)$ stopping time. By the right continuity of (X_t), $T > 0$ a.s.. Let

$$T_{k+1} = T_k + T \circ \theta_{T_k}$$

Since (X_t) is a.s. càdlàg, it has no oscillatory discontinuities a.s., showing that $\lim T_k \geq \hat{\zeta}$ a.s.. But

$$k \geq \hat{E}^{\mu}[\sum_{\substack{s\in(0,T_k] \\ Y_s\neq\delta}} 1_{\{(x,u);\sup_t d(x,u(t))>\varepsilon\}}(\hat{X}_{s-},Y_s)]$$

$$= \hat{E}^{\mu}[\int_0^{T_k} n(\hat{X}_s,\{\sup_t d(\hat{X}_s,W_t) > \varepsilon\})ds],$$

by Lemma 9, showing (9.21).

(b) We set

$$T_k = \inf\{t; \int_0^t n(X_s,\{\sup_t d(\hat{X}_s, W_t) > \varepsilon\})ds \geq k\} .$$

By Lemma 9,

$$\sup_t d(\hat{X}_{s-}, Y_s(t)) > \varepsilon \quad \text{and} \quad Y_s \neq \delta$$

for only finitely many $s \in (0,T_k]$, and by hypothesis, $\lim T_k \geq \hat{\zeta}$. As in Lemma 2, this shows that X_{t-} exists at all times $t = S^-(s)$ (and equals $\hat{X}_{s-}$), $s < \hat{\zeta}$, and the other conditions produce left limits at all other relevant times t. □

Another difference between the general situation and that of Theorem 2 lies in that in the latter we can have 'instantaneous' behaviour only when the measure n is infinite. The following is an example in which the conditions of Theorem 4 are satisfied, and $n(x,u) < \infty$ for each x, but yet $m \equiv 0$.

<u>Example 6</u>: Let $E = [0,1] \times [0,1]$ and $F = [0,1] \times \{0\}$. Let $P_0^{(x,y)}$ corresponds to uniform downward motion on $\{x\} \times [0,1]$, at rate 1, started at (x,y), and stopped upon reaching F. Let

$(\hat{X}_t)$ be uniform motion to the right on F, at rate 1, absorbed at the point $(1,0)$. Let $r : [0,1] \to (0,\infty)$ be Borel measurable, and set

$$n(x,du) = \frac{1}{1-\exp(-r(x))} P_0^{(x,r(x))}(du) .$$

Then $m \equiv 0$. We may choose r such that

$$\int_b^a \frac{dx}{r(x)} = \infty$$

for every $a, b \in [0,1)$, $a < b$. In this case, VI (a) is satisfied, so that we may apply Corollary 6 and then Theorem 4 to obtain the desired process (X_t).

10. Applications

(A). Skew Brownian Motion Let n be the characteristic measure of the PPP of excursions of Brownian motion on $\mathbb{R}$ from 0, and let P_0^b correspond to Brownian motion started at b, and absorbed at 0. For $\alpha \in [0,1]$, let

$$n^+ = 2n|_{\{W_t \geq 0 \text{ for every } t\}}$$

$$n^- = 2n_{\{W_t \leq 0 \text{ for every } t\}}$$

$$n_\alpha = \alpha n^+ + (1-\alpha)n^- .$$

Then for $\Lambda \in \mathcal{U}_t$, $A \in \mathcal{U}$ we have that

$$n_\alpha(\Lambda, \sigma_0 > t, \theta_t^{-1}A)$$

$$= 2\alpha n(\Lambda,\sigma_0>t,\ \theta_t^{-1}A,W_t>0) + 2(1-\alpha)n(\Lambda,\sigma_0>t,\theta_t^{-1}A,W_t<0)$$

$$= 2\alpha \int_{\Lambda\cap\{\sigma_0>t,W_t>0\}} P_0^{W_t}(A)\,dn + 2(1-\alpha)\int_{\Lambda\cap\{\sigma_0>t,W_t<0\}} P_0^{W_t}(A)\,dn$$

$$= \int_{\Lambda\cap\{\sigma_0>t\}} P_0^{W_t}(A)\,dn_\alpha ,$$

so that n_α satisfies (iv) for the P_0^b. Thus Theorem 2 produces a corresponding strong Markov process, which is easily seen to be the "skew Brownian motion" examined in Walsh [50] . This process was introduced by Ito and McKean [77] . They gave a construction in terms of excursions, and in this context, Theorem 2 becomes a proof that the process they construct is strong Markov.

We will now discuss several particular techniques ((B), (C), and (D) below) for producing objects n, $(\hat{X}_t)$, and (Y_t) , from which Theorem 4 will produce a strong Markov process. They will in general be techniques that transform one set of such objects into new ones. Each technique will apply to the following situation:

(a) Start with a right process (X_t) and a set M as in Theorem 3.

(b) Obtain a process on the boundary, $(\hat{X}_t)$, and a kernel n as in that result.

(c) Apply the technique under consideration to produce a new process $(\hat{X}_t')$ and a new kernel n' .

(d) The conditions of Corollary 6 and Thereom 4 will now be met, so that (Y_t') and hence the strong Markov process (X_t') , may be

constructed.

We will however show that the techniques work under more general conditions on $(\hat{X}_t)$ and n, than that they arise as in (b). This is not mere frivolous generality, as it will tell us that the techniques may be iterated, provided this is done in the order in which they are presented. That is, if we start with $(\hat{X}_t)$ and n as in (b), we may perform (B) to produce $(\hat{X}'_t)$ and n'. These will meet the conditions of (C), so that new $(\hat{X}''_t)$ and n'' may be found, which in turn meet the conditions of (D).

We can introduce yet another variation, in that in certain circumstances, there is a corresponding transformation that produces a new excursion process (Y'_t) directly from the old one, (Y_t) (without using Corollary 6). Using

these direct constructions of the excursion processes, the transformations given in (B), (C), and (D) may still be iterated (however, if Corollary 6 is used at any stage of the iteration, the end result will be as if it had been used at each stage). A particular example of this is the procedure of F. Knight and O. Pittenger, for excision of excursions, which we will obtain in (E), below.

<u>(B) h-transformation</u>: We will change (P_0^x) and n via h-transforms, while leaving $(\hat{X}_t)$ unchanged. I owe this to a

suggestion of Mike Cranston.

Let (P_0^x), $(\hat{X}_t)$, n and F be as in Theorem 4. Assume in addition that (8.9) holds, (Q^x) is regulated, E is a U-space, and that upon augmentation of $(\mathcal{U}, \mathcal{U}_t)$ with respect to the P_0^μ, the coordinate process (W_t) becomes a right process. (this latter condition will hold in the situation of Theorem 3; see Sharpe [46], p.).

Let $h : E \to [0,1]$ be 0-excessive for the (P_0^x) (with $h(\Delta)=0$), and let C be the set of dyadic rational numbers. Set

$$E_h = \{x \in E;\ h(x) \neq 0\} ,$$

$$U_h = \{u \in U;\ \liminf_{q \downarrow 0, q \in C} h(u(q)) > 0\} .$$

We let ${}_hP_0^x(du)$ be the transition laws for the h-transformed semigroup, as in Meyer [41]. That is, the ${}_hP_0^x$ are probability measures on $(U, \mathcal{U})$ satisfying ${}_hP_0^x(W_t \neq \Delta$ for some $t \geq \partial) = 0$, and such that for every $(\mathcal{U}_{t+})$ stopping time τ, we have that

$${}_hP_0^x(A \cap \{\partial > \tau\}) = \begin{cases} \frac{1}{h(x)} E_0^x[A \cap \{\partial > \tau\},\ h(X_\tau)] , & x \in E_h \\ 0 , & \text{otherwise} \end{cases}$$

whenever $A \in \mathcal{U}_\tau$. By I.24 of Meyer [41], the ${}_hP_0^x$ are the transition laws of a right process with values in E_h.

Suppose now that $n_0(du)$ is any positive measure on $(U, \mathcal{U})$ satisfying IV, and

(10.1) $\quad n_0(D > t) < \infty$ for every $t > 0$.

We will define a measure $n_1 = {}^h n_0$ such that (10.1) holds, $n_1(\partial = 0) = 0$, and

$$(10.2) \qquad n_1(A \cap \{\partial > t\} \cap \theta_t^{-1} A') = \int_{A \cap \{\partial > t\} \cap U_h} h(W_t)\, {}_h P_0^{W_t}(A')\, dn_0$$

for every $t > 0$, $A' \in \mathcal{U}$ and $A \in \mathcal{U}_t$. To do so, we use this formula to consistently define $n_1^{(k)}(B)$ (the putative.

$$n_1(B \cap \{D \in [\tfrac{1}{k}, \tfrac{1}{k-1})\})),$$

for $k \geq 1$ and B a cylinder set in the space of all functions $[0,\infty) \to E$. By (14.6) of Getoor [23], and III.52 of Dellacherie et Meyer [13], $n_1^{(k)}$ extends to a finite measure on the cylinder σ-field. The coordinate process is immediately seen to have a right continuous modification, so that $n_1^{(k)}$ pulls back to a finite measure on $(U, \mathcal{U})$. We let

$$n_1 = \sum_k n_1^{(k)} .$$

(The reason for insisting that n_1 be concentrated on U_h will emerge later, when we consider property VI(b).)

Since $h \circ W$ is P_0^μ-a.s. right continuous on $[0,\infty)$ for each μ, it follows that $h \circ W$ is n_0-a.s. right continuous on $(0,\infty)$. Thus we may use approximation by stopping times taking values in $\mathbb{Q}$, to show as usual, that

$$n_1(A \cap \{\tau < \partial\}) = \int_{A \cap \{\tau < \partial\} \cap U_h} h(W_\tau)\, dn_0$$

for every strictly positive $(\mathcal{U}_{t+})$ stopping time τ, and $A \in \mathcal{U}_{\tau+}$. Since $h \le 1$ we have that $n_1 \le n_0$. Thus, if $n_0(D=0) = 0$, we have in particular that for every positive $f \in \mathcal{U}$,

$$(10.3) \qquad \int_{\{D<\partial\}} f(W_\cdot)dn_1 = \int_{\{D<\partial\}\cap U_h} f(W_\cdot)h(W_D)dn_0 .$$

We wish to apply Corollary 6 and Theorem 4 to $(\hat{X}_t)$, the ${}_hP_0^x$, and the ${}^hn(x,\cdot)$. Condition IV holds by definition, and conditions I, II, III, and (8.9) follow from the corresponding properties of n, since $h \le 1$ implies that ${}^hn(x,\cdot) \le n(x,\cdot)$ for each $x \in F$. We must thus only show conditions V and VI.

Observe first that $h \circ W$ is P_0^x-a.s. right continuous at 0, so that

$$P_0^x(U\backslash U_h) = 0 \quad \text{for} \quad x \in E_h , \text{ and}$$

$$P_0^x(U_h) = 0 \quad \text{for} \quad x \in E\backslash E_h .$$

Thus

$$n(x,\{W_0\in E_h\backslash F\}\backslash U_h) = \int_{\{W_0\in E_h\backslash F\}} P_0^{u(0)}(U\backslash U_h)n(x,du) = 0 , \text{ and}$$

$$n(x,U_h\cap\{W_0\in E\backslash(E_h\cup F)\}) = \int_{\{W_0\in E\backslash(E_h\cup F)} P_0^{u(0)}(U_h)n(x,du) = 0 .$$

Now let $B \cap F = \emptyset$, $B \in \mathcal{E}$. Then

$${}^hn(x,\{W_0\in B\}) = \lim_{t\downarrow 0} {}^hn(x,\{W_0\in B,\ D_\cdot > t\})$$

$$= \lim_{t\downarrow 0} \int_{\{W_0 \in B, D' > t\} \cap U_h} h(u(t))n(x,du)$$

$$= \lim_{t\downarrow 0} \int_{\{W_0 \in B \cap E_h\}} E_0^{u(0)}[h(W_t), D' > t]n(x,du)$$

$$= \int_{\{W_0 \in B \cap E_h\}} h(u(0))n(x,du) ,$$

by monotone convergence. Further, if $A \in \mathcal{U}_t$, $t > 0$ and $B \cap F = \emptyset$, then we can use this to obtain that

$$^h n(x, A \cap \{W_0 \in B, D' > t\}) = \int_{\{W_0 \in B, D' > t\} \cap A \cap U_h} h(u(t))n(x,du)$$

$$= \int_{\{W_0 \in B \cap E_h\}} \int_{A \cap \{D > t\}} h(v(t))P_0^{u(0)}(dv)n(x,du)$$

$$= \int_{\{W_0 \in B \cap E_h\}} h(u(0))_h P_0^{u(0)}(A \cap \{D > t\})n(x,du)$$

$$= \int_{\{W_0 \in B\}} {}_h P_0^{u(0)}(A \cap \{D > t\})\, {}^h n(x,du) ,$$

showing V.

Since $^h n \le n$,

$$1 - {}^h m(x) = \int_0^\infty {}^h n(x, D > t) e^{-t} dt \,, \quad \text{and}$$

$$1 - m(x) = \int_0^\infty n(x, D > t) e^{-t} dt \,,$$

it follows that if ${}^h m(x) = 0$, then $m(x) = 0$ and ${}^h n(x, D > t) = n(x, D > t)$ for every t. Thus also ${}^h n(x, \cdot) = n(x, \cdot)$. If now $x \in {}^h I$, then there is $\hat{P}^x$-a.s. a $t > 0$ with

$$\int_0^t 1_{\{{}^h m = 0\}}(\hat{X}_s) ds = 0 \,.$$

Thus also $x \in I$, and for every $r > 0$,

$$\int_0^{r \wedge t} {}^h n(\hat{X}_s, U) ds = \int_0^{r \wedge t} n(\hat{X}_s, U) ds = \infty \,,$$

showing VI(a).

Fix x, and let n_0 be a positive measure on $(U, \mathcal{U})$ satisfying IV for the ${}_h P_0^y$, and such that

$$n_0(du) \le {}^h n(x, du) \,.$$

Assume that $n_0(W_0 \in F) > 0$. We must show that $n_0(W_0 \in F) = \infty$. Since $n_0(U \backslash U_h) = 0$, and

$$\{\liminf_{q \downarrow 0, q \in C} h(W_q) > \varepsilon\} \in \mathcal{U}_{0+}$$

for each $\varepsilon > 0$, we may fix $\varepsilon > 0$, and assume that n_0 is concentrated on this set. Let

$$g = 1_{E_h} .$$

Since h is 0-excessive for the P_0^y, so is g. By I.19 of Meyer [41], g/h is 0-excessive for the ${}_hP_0^y$, so that we may form

$$n_1 = {}^{g/h}n_0 .$$

But

$${}^{g/h}({}_hP_0^y) = {}_gP_0^y ,$$

so that as before, n_1 satisfies IV for the ${}_gP_0^y$. Also,

$$n_1(du) \le {}^{g/h}({}^hn(x,du)) = {}^gn(x,du) .$$

E_h is nearly optional, so that as in Theorem 3, there is an exact perfect terminal $(\mathcal{U}_{t+})$ stopping time D_h which is P_0^μ indistinguishable from

$$\inf\{t > 0;\ W_t \notin E_h\} ,$$

for each μ. We have that for $A \in \mathcal{U}_t$,

$${}_gP_0^y(A \cap \{\partial > t\}) = P_0^y(A \cap \{\partial \wedge D_h > t\}) .$$

Using this, the formulae $n_1^+(D_h = 0) = 0$, and

$$n_1^+(A \cap \theta_t^{-1}B \cap \{D_h > t\}) = \int_{A\cap\{D_h>t\}} P_0^{u(t)}(B)n_1(du)$$

$(A \in \mathcal{U}_t,\ B \in \mathcal{U})$ can be used to define a function n_1^+ on the field

of elements of $\mathcal{U}$, consisting of finite unions of sets of the form

$$\{W_{t_i} \in C_i,\ i=1..k;\ D_h \in (t_j,\ t_{j+1}]\},\quad \{W_{t_i} \in C_i,\ i=1..k,\ D_h=0\}$$

where $t_1 < \ldots < t_{k+1} \le \infty$, $j \le k$. On this field, we have that $n_1 \le n_1^+ \le n(x,\cdot)$, so that because $n(x,\cdot)$ is countably additive, and there exist U_i in the field with $n(x,U_i) < \infty$, $n(x,U\setminus\cup_i U_i) = 0$, we see that n_1^+ extends to a measure on $(U, \mathcal{U})$ such that $n_1 \le n_1^+ \le n(x,\cdot)$. This n_1^+ satisfies IV for (P_0^x) since n_1 did for $({}_gP_0^x)$, so that by VI(b),

$$n_1^+(W_0 \in F) = 0 \quad \text{or} \quad \infty .$$

But $n_1^+ \ge n_1 \ge {}^h n_1 = n_0$, so that $n_1^+(W_0 \in F) \ge n_0(W_0 \in F) > 0$, and hence

$$\infty = n_1^+(W_0 \in F) = \lim_{q\downarrow 0,\, q\in C} n_1(W_0 \in F,\ D_h \wedge D > q)$$

$$= \lim_{q\downarrow 0,\, q\in C} \int_{\{W_0\in F,\ D > t\}} \frac{g(W_t)}{h(W_t)} dn_0$$

$$\le \frac{1}{\varepsilon}\, n_0(W_0 \in F) ,$$

Thus $n_0(W_0 \in F) = \infty$, showing VI(b). Corollary 6 and Theorem 4 now apply, to produce a corresponding strong Markov process.

<u>Note</u> In one particular case, we can construct an excursion process $({}^hY_t)$ directly, without appealing to Corollary 6.

That is, suppose in addition to the hypotheses of (B), that (Y_t) is given as in Theorem 4 (we could also drop the assumptions that (8.9) hold and that (Q^x) be regulated). Let $F_1 \subset F$, $F_1 \in \mathcal{E}^F$, and let h be the 0-excessive function

$$h(x) = P_0^x(D<\partial\ ,\ W_D \in F_1\ ).$$

Then

$$h(x) = 1_{F_1\setminus\{\Delta\}}(x) \quad \text{for} \quad x \in F\ .$$

It follows by definition of ${}_hP_0^x$, that

$$\begin{aligned} {}_hP_0^x(D<\partial) &= 1_{E_h}(x)\frac{1}{h(x)}\ E_0^x[D<\partial\ ,\ h(W_D)\ ] \\ &= 1_{E_h}(x)\frac{1}{h(x)}\ P_0^x[D<\partial\ ,\ W_D \in F_1\ ] \\ &= 1_{E_h}(x)\ . \end{aligned}$$

Thus for each $x \in F$,

$$\begin{aligned} {}^hn(x,\ D=\partial\ ) &= \lim_{t\to 0}\ {}^hn(x,\{\ D=\partial>t\ \}) \\ &= \lim_{t\to 0}\int_{\{D>t\}\cap U_h} h(u(t))\ {}_hP_0^{u(t)}(D=\partial)\ n(x,du) \\ &= 0\ . \end{aligned}$$

Let

$$h_{Y_t} = \begin{cases} Y_t\ , & \text{if}\quad Y_t \in U_h \cap \{D<\partial\ ,\ W_D \in F_1\ \} \\ \delta\ , & \text{otherwise}\ . \end{cases}$$

Then $({}^hY_t)$ inherits the strong Markov property from (Y_t), and for each positive $f \in \mathcal{U}$, and positive (F_t^μ) predictable process (Z_t), we have that

$$\hat{E}^\mu[\sum_{\substack{s>0 \\ {}^hY_s \neq \delta}} Z_s f({}^hY_s)] = \hat{E}^\mu[\sum_{\substack{s>0 \\ Y_s \neq \delta}} Z_s f(Y_s) 1_{U_h \cap \{D<\partial, W_D \in F_1\}}(Y_s)]$$

$$= \hat{E}^\mu[\int_0^\infty Z_s \int_{\{D<\partial, W_D \in F_1\} \cap U_h} f(u) n(\hat{X}_s, du) ds]$$

$$= \hat{E}^\mu[\int_0^\infty Z_s \int_{\{D<\partial\} \cap U_h} f(u) h(u(D(u))) n(\hat{X}_s, du) ds]$$

$$= \hat{E}^\mu[\int_0^\infty Z_s \int f(u)\, {}^hn(\hat{X}_s, du) ds]$$

by (10.3) and the above. Therefore, $(\hat{X}_t)$, ${}^hn$, $({}^hY_t)$, $({}_hP_0^x)$, and F satisfy the conditions of Theorem 4, so that a new strong Markov process may be constructed.

<u>(C) Excision</u>: We will excise pieces of the path of $(\hat{X}_t)$, while leaving n unchanged.

Suppose that (X_t) and M are as in Theorem 3, and obtain $(\hat{X}_t)$ and F from them. Suppose that (P_0^x), n, and (F_t), together with $(\hat{X}_t)$ and F, satisfy the conditions of Corollary 6 and Theorem 4 [note: We do not assume that (P_0^x) and n are also constructed from (X_t) and M as in Theorem 3. Since the

procedure of (B) left $(\hat{X}_t)$ unchanged, these hypotheses will thus be satisfied by $({}_hP_0^x)$ and ${}^hn]$.

Assume now that we can write $F = F_0 \cup F_1$, where F_0 and F_1 have disjoint closures in the (X_t)-Ray Knight compactification $\tilde{E}$ of E , and $\Delta \in F_1$. Suppose further, that

$$(10.4) \qquad P_0^x(D < \infty \text{ , and } W_D^F \in F_0) = 0 \text{ for } x \in E\backslash F_0 .$$

Let

$$A(t) = \int_0^t 1_{F_1}(\hat{X}_s)ds \text{ , } A'(s) = \inf\{t;\ A(t) > s\}$$

$$\hat{X}'_t = \hat{X}_{A'(t)} \text{ , } \mathcal{F}'_t = \mathcal{F}_{A'(t)} .$$

Since $(\hat{X}_t)$ is $\tilde{E}$-right continuous with values in F , it follows that; for every t there is an $\varepsilon > 0$ such that

$$(10.5) \qquad \hat{X}'_{t+s} = \hat{X}_{A'(t)+s} \text{ for } s \in [0,\varepsilon); \text{ a.s. .}$$

Properties II(b) and VI(a) thus follow by the strong Markov property of $(\hat{X}_t)$ at $A'(0)$, and (8.9) follows from (10.4). The remainder of properties I-VI are of course still satisfied. Let Q'^x be the image law of $\hat{P}^x$ on $(U^F, \mathcal{U}^F)$ under $\hat{X}'_.$, so that as before, each

$$(\hat{X}'_t, \mathcal{F}^\mu_t, \hat{P}^\mu, Q'^x)$$

is strong Markov. Augment $(\mathcal{U}^F, \mathcal{U}^F_t)$ with respect to the Q'^μ to obtain $(\bar{\mathcal{U}}', \bar{\mathcal{U}}'_t)$. As in Theorem 3,

$$\{(W_t^F) \text{ is not càdlàg in } \tilde{E}\} \in \bar{U}'$$

and has Q'^{μ} measure zero for each μ, so that the regulator K_t^0 for (Q^x), defined in Theorem 3, will be adapted to $(\bar{U}_t')$. For every t,

$$K_t^0(\hat{X}.) \quad \text{and} \quad K_{A(t)}^0(\hat{X}'.)$$

differ by only a finite random variable, since $(\hat{X}_t)$ has no oscillatory discontinuities in $\tilde{E}$, a.s., so that $(\hat{X}_t)$ makes only finitely many jumps from F_1 to F_0 in any compact time interval. Thus (8.3) holds. Write

$$\tilde{X}'^F_{t-},\ \tilde{X}^F_{t-},\ \tilde{W}^F_{t-}$$

for the left limits at t in $\tilde{E}$ of $\hat{X}'.$, $\hat{X}.$, and $W^F.$. Then as before, if τ is a strictly positive $(\bar{U}_t'^{\mu})$ predictable stopping time, we write $T = \tau(\hat{X}'.)$, and have that $A'(T-)$ is an $(\mathcal{F}_t^{\mu})$ predictable stopping time. Thus either

$$\tilde{X}'^F_{T-} = \tilde{X}^F_{A'(T-)-} \notin F\ ; \text{ or}$$

$$\tilde{X}^F_{A'(T-)-} = \hat{X}^F_{A'(T-)}\ ;\ \hat{P}^{\mu}\text{-a.s. on } \{T<\infty\}\ .$$

But

$$\tilde{X}^F_{A'(T-)-}$$

lies in the $\tilde{E}$-closure of F_1, so that in the latter case, by (10.5), we have that $A'(T-) = A'(T)$, and hence

$$\tilde{X}^{'F}_{T-} = \hat{X}'_T \ .$$

Thus

$$Q^{'\mu}(K^0_{\tau-} \neq K^0_\tau) = 0 \ ,$$

showing (8.4). Conversely, if τ is $(\bar{U}^{'\mu}_t)$ totally inaccessible, we have as before, that

$$\tilde{W}^F_{\tau-} \doteq W^F_{\tau-} \ Q^{'\mu}\text{-a.s. on } \{\tau < \infty\} \ .$$

Thus if

$$W^F_{\tau-} \neq W^F_{\tau-} \ Q^{'\mu}\text{a.s. on } \{\tau < \infty\},$$

then also $K^0_{\tau-} \neq K^0_\tau$ by definition of (K^0_t). (K^0_t) is therefore a regulator for $(Q^{'x})$, so that Corollary 6 and Theorem 4 apply, to produce a strong Markov process (X'_t).

<u>Note</u>: 1. There is a direct construction of the excursion process available. That is, suppose that in addition, we are given (Y_t) as in Theorem 4 (condition (8.9) need no longer be assumed), and put

$$Y'_t = Y_{A'(t-)} \ .$$

As in Lemma 13(c), $(Z_{A(t)})$ is $(\mathcal{F}^\mu_t)$ predictable whenever (Z_t) is $(\mathcal{F}^{'\mu}_t)$ predictable, so that in this case,

$$\hat{E}^\mu[\sum_{\substack{s>0 \\ Y'_s \neq \delta}} Z_s f(Y'_s)] = \hat{E}^\mu[\sum_{\substack{s>0 \\ Y_s \neq \delta}} Z_{A(s)} f(Y_s) 1_{F_1}(\hat{X}_{s-})]$$

$$= \hat{E}^{\mu}[\int_0^{\infty} Z_{A(s)} 1_{F_1}(\hat{X}_s) \int f(u) n(\hat{X}_s, du) ds]$$

$$= \hat{E}^{\mu}[\int_0^{\infty} Z_s \int f(u) n(\hat{X}'_s, du) ds] \quad ,$$

showing (8.7). The strong Markov property of

$$(\hat{X}'_t, Y'_t, F'_t, \hat{P}^x)$$

is immediate from that of

$$(\hat{X}_t, Y_t, F_t, \hat{P}^x) ,$$

so that Theorem 4 applies.

2. Condition (10.4) is only included for simplicity. We can remove it by taking $\Delta \in F_1$, and changing (P_0^x) to (P'^x_0), where the P'^x_0 are obtained from the P_0^x by making points $x \in F_0$ into branch points, with

$$P'^x_0(X_0 \in B) = Q^x(\tau = \inf\{t \geq 0;\ W_t^F \in F_1\} < \infty ,\ W_\tau^F \in B) .$$

The $n(x,\cdot)$ are altered similarly, and we once again obtain (8.9).

3. If, instead of assuming that F_0 and F_1 have disjoint closures in $\tilde{E}$, we assume that this holds with the regular topology of E, and that $(\hat{X}_t)$ is càdlàg in this topology, then we would still get (10.5). If in addition we had assumed that (X_t) was a Hunt process, then by (13.8) of Getoor [23] , (X_{t-})

and $(\tilde{X}_{t-})$ are indistinguishable. In this case, the proof of regulation still works, so that we could apply the arguments of (C) to construct a strong Markov process.

4. Similarly, if we wish to use the direct construction of note 1, it is enough to assume that we are given $(\hat{X}_t)$, n, (Y_t), (P_0^x), F and $(\mathcal{F}_t)$ as in Theorem 4, together with F_0 and F_1 which have disjoint closures in the regular topology on E, such that $F = F_0 \cup F_1$, $(\hat{X}_t)$ is càdlàg in this topology, and (10.4) holds. The latter condition could be removed as in note 2. In this case, we need no longer show (8.9); the particular form of the branching behaviour on F_0 is now needed to guarantee (9.9).

<u>(D) Scaling and Time Change</u>: We scale n and time change $(\hat{X}_t)$.

Let $(\hat{X}_t)$, $(\mathcal{F}_t)$, (P_0^x) and n satisfy the conditions of Corollary 6 and Theorem 4. Let $f : F \to (0,\infty)$ and $g : F \to [0,\infty)$ be $\mathcal{E}^F$-measurable, and assume that f is bounded, and bounded away from 0. Assume also that for each x, $f(x)g(x) \le 1$, and that

$$(10.5) \qquad g(x) \int (1 - e^{-D})\, dn(x,\cdot) \le 1 ,$$

with equality only if $m(x) = 0$. Put

$$A(t) = \int_0^t f(\hat{X}_s)\, ds , \quad A'(s) = \inf\{t;\ A(t) > s\} ,$$

$$\hat{X}'_t = \hat{X}_{A'(t)} , \quad \mathcal{F}'_t = \mathcal{F}_{A'(t)} , \text{ and}$$

$$n'(x,du) = g(x)n(x,du) .$$

Then for Q'^x the image law of $\hat{P}^x$ under $\hat{X}'_\cdot$, we have as before, that $(\hat{X}'_t, F'^\mu_t, \hat{P}^\mu, Q'^x)$ is strong Markov. Since A is continuous and strictly increasing, the totally inaccessible stopping times for the completions of $(\mathcal{U}_t)$ with respect to the Q^μ and Q'^μ correspond exactly (under a time change), so that (Q'^x) is regulated by the appropriate transformation (K'^0_t) of (K^0_t). Condition (III) holds by (10.5), and except for II(b), VI(a), and (8.9), the remaining conditions of Corollary 6 and Theorem 4 are now immediate.

Let $h \in \mathcal{E}^F \otimes \mathcal{E}^F$ be positive. Then

$$E^x_{Q'}[\int_0^t \int_{\{D<\infty\}} 1_{\{(x,y);x\neq y\}}(W^F_s,W_D)h(W^F_s,W_D)dn'(W^F_s,\cdot)ds]$$

$$= \hat{E}^x[\int_0^{A'(t)} \int_{\{D<\infty\}} 1_{\{(x,y);x\neq y\}}(\hat{X}_s,W_D)h(\hat{X}_s,W_D)g(\hat{X}_s)dn(\hat{X}_s,\cdot) f(\hat{X}_s)ds]$$

$$\leq \hat{E}^x[\sum_{\substack{s\in(0,A'(t)]\\ Y_s\neq\delta}} h(\hat{X}_{s-},\hat{X}_s)1_{\{(x,y);x\neq y\}}(\hat{X}_{s-},\hat{X}_s)]$$

$$\leq E^x_{Q'}[\sum_{s\in(0,t]} h(W^F_{s-},W^F_s)1_{\{K'^0_{s-}\neq K'^0_s\}}],$$

since $fg \leq 1$, showing (8.9). Condition II(b) follows similarly. To show VI(a), observe that since equality holds in (10.5) only

if $m(x) = 0$, we have that $m'(x) = 0$ only if $m(x) = 0$ and $g(x) = 1$. Since f is never zero, we see that if $x \in I'$, then also $x \in I$, so that

$$\int_0^t n(\hat{X}_s, U)ds = \infty \quad \text{for every} \quad t > 0, \ \hat{P}^x\text{-a.s.}.$$

Because f is bounded from zero, and $g(x) = 1$ whenever $m'(x) = 0$, it follows that also

$$\int_0^t n'(\hat{X}'_s, U)ds = \infty \quad \text{for every} \quad t > 0, \ \hat{P}^x\text{-a.s.},$$

showing VI(a). Thus Corollary 6 and Theorem 4 apply.

<u>Note</u>: 1. Because $f(x)g(x)$ may be strictly less than 1, the function $J'(x,y)$ given by Lemma 10(a) may have values decreased from those of $J(x,y)$. Thus fewer of the jumps of $(\hat{X}'_s)$ will correspond to excursions of (X'_t) than was the case for $(\hat{X}_s)$ and (X_t). This will be the case when $f \equiv 1$, $g \lneq 1$, so that even if we start with a continuous process (X_t), the process (X'_t) will in general have many jumps, between points of F.

2. If on the other hand, f and g balance, so that $fg \equiv 1$, and $f \geq 1-m$ (with equality only if $m = 0$) then we can obtain (Y'_t) directly. That is, suppose (Y_t) satisfies the conditions of Theorem 4, and let

$$Y'_t = Y_{A'(t)} .$$

We see as in Note 1 to (C), that (Y'_t) satisfies the conditions

of Theorem 4, for (X'_t), n', etc..., so that a strong Markov process (X'_t) can be constructed.

Let (X_t) be the strong Markov process associated to $(\hat{X}_t)$, (Y_t) and n, and set

$$B(t) = \int_0^t 1_{E\setminus F}(X_s)ds + \int_0^t (f+m-1)(X_s)dL_s$$

$$B'(s) = \inf\{t;\ B(t) > s\} .$$

We will see that

$$X'_t = \begin{cases} X_{B'(t)} & , B'(t) < \infty \\ \Delta & , B'(t) = \infty \end{cases} \quad \text{for every } t, \text{ a.s.}$$

Since the excursions are left unchanged, it will suffice to show this for t of the form $S'^{+}(s)$. Since

$$X'_{S'^{+}(s)} = \hat{X}'_s = \hat{X}_{A'(s)} = X_{S^{+}(A'(s))} ,$$

it will suffice to show that $B'(S'^{+}(s)) = S^{+}(A'(s))$, or, by right continuity, that $S'^{+}(s) = B(S^{+}(A'(s)))$.

$$B(S^{+}(A'(s))) = \int_0^{S^{+}(A'(s))} 1_{E\setminus F}(X_r)dr + \int_0^{S^{+}(A'(s))} (f+m-1)(X_r)dL_r$$

$$= \sum_{\substack{r\in[0,A'(s)] \\ Y_r\neq\delta}} D(Y_r) + \int_0^{A'(s)} (f+m-1)(\hat{X}_r)dr$$

$$= \sum_{\substack{r\in[0,s] \\ Y'_r\neq\delta}} D(Y'_r) + \int_0^{s} \left(\frac{f+m-1}{f}\right)(\hat{X}'_r)dr = S'^{+}(s) ,$$

since $(f+m-1)/f = 1 + (m'-1) = m'$.

Thus (X_t') is a time change of (X_t) on F, and no new jumps as in note 1 are introduced.

We summarize in:

<u>Proposition 13</u> Let (X_t) and M be as in Theorem 3, and obtain F, (P_0^x) and n from them. Let $h : E \to [0,1]$ be 0-excessive for the P_0^x. Let $F = F_0 \cup F_1$, where F_0 and F_1 have disjoint closures in the (X_t)-Ray Knight compactification of E, $\Delta \in F_1$, and

$$ {}_hP_0^x(D < \infty \text{ and } W_D \in F_0) = 0 \quad \text{for} \quad x \in E\backslash F_0 . $$

Let $f : F \to [0,\infty)$, $g : F_1 \to [0,\infty)$ be $\mathcal{E}^F$-measurable and satisfy: f is bounded and bounded below on F_1; $f = 0$ on F_0; $fg \le 1$ on F_1; and $g(1-m) \le 1$ on F_1, with equality only where $m = 0$. Let $(\hat{X}_t')$ be the time change of $(\hat{X}_t)$ by f, as above, and let

$$ n'(x,du) = \begin{cases} g(x)\,{}^hn(x,du) , & x \in F_1 \\ 0 , & x \in F_0 . \end{cases} $$

Then the conditions of Corollary 6 are met, for $({}_hP_0^x)$, $(\hat{X}_t')$ and n', so that we may construct a strong Markov process (X_t') from them by Theorem 4. □

<u>(E) the Knight-Pittenger procedure</u>

Start with (X_t) and M as in Theorem 3, and obtain F, $(\hat{X}_t)$, (P_0^x), (Y_t), n and m as there. Assume that $(\hat{X}_t)$ is

càdlàg in E, and that $F = F_0 \cup F_1$ where F_0 and F_1 have disjoint closures in E, and $\Delta \in F_1$. Let h be as in the note to (B), and apply the direct construction in that note to obtain $({}^hY_t)$ and ${}^hm$. Apply the direct construction in Notes 1 and 4 of (C), to obtain $(\hat{X}_t')$, (Y_t') and n' (observe that (10.4) holds in this case by (10.3)). Apply the direct construction in Note 1 of (D) to $(\hat{X}_t')$, (Y_t'), (Y_t') and n', with

$$f = 1 - {}^hm + m \geq 1 - {}^hm$$

(note that if $f = 1 - {}^hm$, then $m = 0$ so that ${}^hm = 0$), to obtain $(\hat{X}_t'')$, (Y_t'') and n'', and then use Theorem 4 to reconstruct a strong Markov process (X_t''). Let

$$A(t) = \int_0^t f(\hat{X}_s) 1_{F_1}(\hat{X}_s) ds , \quad A'(s) = \inf\{t; A(t) > s\} .$$

By our choice of f, we have that $m''f = m$, so that

$$\int_0^t m''(\hat{X}_s'') ds = \int_0^t m''(\hat{X}_{A'(s)}) ds$$
$$= \int_0^{A'(t)} m''(\hat{X}_s) dA_s = \int_0^{A'(t)} m(\hat{X}_s) 1_{F_1}(\hat{X}_s) ds .$$

As in Note 1 of (D), it follows immediately that (X_t'') can be described as follows: Kill (X_t) at the last exit time from $F_1 \backslash \{\Delta\}$ (that is, after this time, it jumps to Δ), and then excise from the path of (X_t) all those excursions away from F_1, that either meet F_0 or lie in $U \backslash U_h$. By Lemma 16 below, there

are almost surely no excursions lying in $U\backslash U_h$ that come back to F other than at points of $F_0 \cup \{\Delta\}$. Since all the excised excursions occur before the last exit time from $F_1\backslash\{\Delta\}$, they must all come back to F, and hence all meet F_0. Thus we have shown that the process (X_t'') is obtained from (X_t) by exactly the procedure given by Knight and Pittenger in [35]. Our results therefore give another proof of their theorem, that the process constructed in this manner is strong Markov. Note that in order for our techniques to work, we have had to impose conditions $((X_t, \mathcal{F}, \mathcal{F}_t, \hat{P}^x)$ is a right process,

$$\overline{\{t;\ X_t \in F_0 \cup F_1\}}$$

satisfies (9.1) and (9.2)) guaranteeing the existence of a local time. They do not need to do so.

We close this section with the lemma used above.

<u>Lemma 16</u>

$$n(x,\ \{D < \partial, \{W_D \in F_1\}\backslash U_h) = 0$$

for every $x \in F_1$.

<u>proof</u>: Fix $x \in F_1$, and ε, $s > 0$. Then

$$n(x,\{D < \partial,\ W_D \in F_1\ \ ,\ h(W_{j/2^k}) < \varepsilon \text{ for some } j/2^k \in (0,s)\})$$

$$= \sum_{j=1}^{[2^k s]} n(x,\{D \in (j/2^k, \infty)\ ,\ W_D \in F_1\backslash\{\Delta\}\ ,\ h(W_{j/2^k}) < \varepsilon\ ,$$

$$h(W_{i/2^k}) \geq \varepsilon \quad \text{for} \quad i<j\})$$

$$= \sum_{j=1}^{[2^k s]} \int_{\{h(W_{j/2^k})<\varepsilon,\ h(W_{i/2^k})\geq\varepsilon \text{ for } i<j\}} P_0^{W_{j/2^k}}(D<\infty,$$

$$W_D \in F_1 \setminus \{\Delta\})dn(x,\cdot)$$

$$\leq \varepsilon\ n(x,\ h(W_{j/2^k}) < \varepsilon \quad \text{for some} \quad j/2^k \in (0,s)\}) .$$

Thus,

$$(1-\varepsilon)n(x,\ \{D<\partial\ ,\ W_D \in F_1\ ,\ h(W_{j/2^k}) < \varepsilon \text{ for some } j/2^k \in (0,s)\})$$

$$\leq \varepsilon\ n(x,\ \{D=\partial,\ \text{or } D<\partial \text{ and } W_D \in F_0\ \}) .$$

Letting $k \to \infty$ and then $s \to 0$, we see that

$$(1-\varepsilon)n(x,\ \{D<\partial\ ,\ W_D \in F_1\ \}\setminus U_h)$$

$$\leq \varepsilon\ n(x,\ \{D=\partial\ ,\ \text{or } D<\partial \text{ and } W_D \in F_0\ \})$$

for every $\varepsilon > 0$. The right hand side is finite by II(a) and III, so that we obtain the desired conclusion upon letting $\varepsilon \to 0$. □

PART 2. MARTIN BOUNDARIES

11. Introduction

Recall that using Theorem 4 and Corollary 6 of Part 1, we constructed a strong Markov process from a 'process on the boundary' $(\hat{X}_t)$, and a kernel $n(x,du)$. Our applications in section 10 consisted of constructing these objects from other similar ones, thereby obtaining a transformation that produced new strong Markov processes from old ones.

A more far reaching program would hope to build the process on the boundary, and the kernel n , from scratch, knowing only the laws (P_0^x) of the process stopped at the boundary. The problem of constructing a process on the boundary seems hard in general; Condition (8.9) gives information about what its Lévy system must be, and in a general state space, it seems difficult to produce a process with a given Lévy system (how does one 'compensate'?). For results in Euclidean space, see Bass [1], and Stroock [47].

As to the kernels $n(x, du)$, one might first try to construct the kernels $\acute{n}(x, y; du)$ of Section 8D. The natural approach is via h-transforms, the hope being that

$$n(x, y; \cdot) = {}_hP_0^x$$

for an appropriate (P_0^x) - excessive function h , depending on y . The first step in this approach would therefore be to determine those excessive functions h for which x is an entrance point for the h-transformed process.

In the second part of this thesis, we will approach this problem from a different perspective, in the particular case of Brownian motion. The

arguments will be completely independent of those given in Part 1, and except in this introduction, we will feel free to use completely different notation from that used in Part 1, in an attempt to conform to the notation used in the Martin boundary literature. The numbering of the results will start over again also. The major new results are Theorem 6 and Corollaries 1 and 2.

12. h-transforms and Martin boundaries

In this section, we will outline those parts of the theory of h-transforms and Martin boundaries that will be needed later. The results given below are taken from a set of notes by J. B. Walsh, but were originally proven almost entirely by J. L. Doob ([15], [17]), and by R. S. Martin ([37]). The arguments using time-reversal seem to be less well known than they should be, so proofs will at least be sketched. In their present form, these arguments are due to J. Walsh, but many of them derive from the paper [30] of G. A. Hunt. Any mistakes are my contribution. Many of the other results may be found in the introduction to Walsh [49]. In addition to the original articles, proofs may also be found in Meyer [41]. The definitive treatment will be (once it appears) J. L. Doob's book [19] on this whole subject.

Let E be a domain (an open connected subset of R^d), and let $\mathcal{E}$ be the σ-field of its Borel subsets. Let δ be some additional point which will act as a cemetary. Let Ω be the canonical space of functions $\omega : [0, \infty) \to E \cup \{\delta\}$ such that there exists $\zeta > 0$ with ω continuous on $[0, \zeta)$, and $\{t ; \omega(t) = \zeta\} = [\zeta, \infty)$. Let $X_t ; t \geq 0$ be the coordinate maps on Ω, and let

$$F_t = \sigma(X_s;\ s \leq t)\ ,\ F = F_\infty \quad .$$

We will sometimes need to work with the space $\hat{\Omega}$ of functions $\omega : (0, \infty) \to E \cup \{\delta\}$, which are continuous up to their lifetime, as above. By the appropriate abuse of notation, we will use the same symbols F_t , F, X_t, etc... for the corresponding objects onn $\hat{\Omega}$. Of course, in this context, X_t is only defined for $t > 0$. Similarly, $\mathcal{B}$ will denote the Borel σ-field on both $[0, \infty)$ and $(0, \infty)$. We follow the convention of making all functions vanish at δ unless otherwise specified.

Let $P_t(x, dy)$ be the transition function of Brownian motion on R^d , killed upon leaving E , and as usual, set $P_t(x, \delta) = 1 - P_t(x, E)$, $P_t(\delta, \{\delta\}) = 1$. If now h is excessive for (P_t) , we let

$$E_h = \{x \in E\ ;\ h(x) < \infty\}$$

$$_hP_t(x, dy) = \begin{cases} \dfrac{1}{h(x)}\, P_t(x, dy)\ h(y)\ , & x \in E_h \\ 0 & , \text{otherwise} \end{cases}$$

(note that unless $h \equiv 0$, $\{h = 0\} = \emptyset$) .

<u>Theorem 1</u>

(a) If $h > 0$ is excessive, then $(_hP_t)$ forms a sub-Markov semigroup on E .

Extend it as above to form a Markov semigroup. Then:

(b) For $x \in E$, there exists a probability measure $_hP^x$ on (Ω, F)

under which (X_t) is strong Markov for the semigroup $({}_hP_t)$ and the filtration (F_{t+}), and for which ${}_hP^x(X_0 = x) = 1$;

(c) If T is any (F_{t+}) stopping time, and $\Lambda \in F_{t+}$, then

$$ {}_hP^x(\Lambda, \zeta > T) = \frac{1}{h(x)} 1_{E_h}(x) E^x[\Lambda, h(W_T)] ; $$

(d) v is h-excessive iff $v = 0$ on $E \setminus E_h$, and there is an excessive function u with $u = vh$ on E_h ;

(e) Every excessive or h-excessive function is ${}_hP^x$ - a.s. continuous along the path of (X_t), up to its lifetime.

In general, if we are given a measurable space, a probability P thereon, and a random variable Y with values in (Ω, F), we say that Y is an <u>h-transform</u> under P, if it is strong Markov with respect to P and the filtration $Y^{-1}(F_{t+})$ with transition semigroup ${}_hP_t$. We use the same notation if Y takes values in $\hat{\Omega}$, but now the strong Markov property is at strictly positive stopping times.

Now let $(A, \mathcal{A})$ be a measurable space, and u a measurable function: $(A \times E, \mathcal{A} \otimes \mathcal{E}) \to ([0, \infty), \mathcal{B})$ such that each $u(a, \cdot)$ is excessive. Let ν be a measure on $(A, \mathcal{A})$, and set

$$ h(x) = \int u(a, x)\nu(dx) . $$

Then if h is not identically infinite, it is excessive, and

(f) $$ {}_hP^x(\Lambda) = \frac{1}{h(x)} \int u(a, x) \, {}_{u(a,\cdot)}P^x(\Lambda)\nu(da) $$

on E_h, for $\Lambda \in F$.

Recall that an excessive function h is called <u>minimal</u> if whenever $h = u + v$, where u and v are both excessive, then both u and v are multiples of h.

(g) If E has a Green function $G(x,y)$, then $G(\cdot, y)$ is a minimal excessive function.

Let the <u>invariant</u> σ-field be

$$\mathcal{I} = \{\Lambda \in \mathcal{F} \; ; \; \theta_t^{-1}(\Lambda \cap \{\zeta > 0\}) = \Lambda \cap \{\zeta > t\}\} .$$

(h) If h is minimal, then $\mathcal{I}$ is trivial under each ${}_hP^x$. If $h > 0$ is excessive and $\Lambda \in \mathcal{I}$, then ${}_hP^{\cdot}(\Lambda, \zeta > 0)$ is h-excessive.

(i) Let u and v be excessive, and let (Y_t) and P be a process and a probability such that (Y_t) is both a u and a v-transform under P. If $P(Y_0 \neq \delta) > 0$, then u and v are multiples of each other.

(j) Let U be a subdomain of E, and let h be excessive on E. Let (Y_t) be an h-transform under P. Then killing (Y_t) upon first leaving U, produces a process which is a transform by the restriction of h to U, under P.

If $A \subset E$ is Borel, write T_A and L_A for the first hitting time and last exit time of A, respectively;

$$T_A = \inf\{t > 0 \; ; \; X_t \in A\}$$

$$L_A = \sup\{t > 0 \; ; \; X_t \in A\} \quad (\sup(\emptyset) = 0) .$$

Also, for $h \geq 0$, let

$$R_A h(x) = E^x[h(X_{T_A})] .$$

(k) If h is excessive, then

$$R_A h(x) = {}_hP^x(T_A < \infty)\ h(x) .$$

$R_A h$ is also the excessive regularization of the infimum of all excessive functions majorizing h on A .

(l) If (X_t) is an h-transform under P , then killing (X_t) at L_A produces a process which is an $R_A h$-transform under P .

(m) If (X_t) is an h-transform under P , and $E \setminus A$ is a subdomain of E , then the process

$$(X_{L_A + t})$$

is a v-transform on $E \setminus A$, under P , where

$$v(x) = h(x)\,{}_hP^x(T_A = \infty)$$

[for this, see 5.1 of Meyer, Smythe and Walsh [43]] .

Note that the new processes of (l) and (m) may be identically equal to δ , with positive probability . □

Now suppose that E has a Green function $G(x,y)$. Let $t_0 > 0$ and let ϕ be bounded and increasing, with $\phi(t) = t$ for $t \in [0, t_0)$. Let $y_0 \in E$ be fixed, and suppose that ϕ is concave, so that

$$p(x) = \phi(G(x, y_0))$$

is excessive. Following Helms [29], let

$$\sum = \{x \in E \; ; \; G(x, y_0) > t_0\} \; .$$

It is an open neighbourhood of y_0 , with compact closure in E . Define the <u>Martin function</u> to be

$$K(x, y) = G(x, y)/p(x) \; , \; x, y \in E \; .$$

Let the Martin metric d on E , be

$$d(x, y) = \int_{\sum} |K(x, z) - K(y, z)|(1 + |K(x, z) - K(y, z)|)dz$$

Let $\overline{E}$ be the completion of E under d , and call $\Delta = \overline{E} \setminus E$ the <u>Martin boundary</u> of E .

<u>Theorem 2</u>

(a) $\overline{E}$ is compact, and E is a dense open subset. The topology induced on E by d is the usual Euclidean one.

(b) K extends to a continuous function from $\overline{E} \times E$ to $(0, \infty]$. It takes on the value ∞ only on $\{(x, y) \; ; \; x = y \in E\}$, and never vanishes.

(c) For $x \in \Delta$, $K(x, \cdot)$ is harmonic, hence excessive. We say that $x \in \Delta$ is <u>minimal</u> if $K(x, \cdot)$ is . Let Δ_0 be the set of minimal $x \in \Delta$.

(d) For $x \in \Delta$, x is minimal iff for every open neighbourhood A of x ,

$$R_{A \cap E} K(x, \cdot) = K(x, \cdot) \; .$$

(e) Δ_0 is Borel. For every excessive $h > 0$, there is a unique measure ν on $\Delta_0 \cup E$ such that

$$h = \int K(x, \cdot)\nu(dx) .$$

h is harmonic iff $\nu(E) = 0$.

(f) Let h be harmonic, and continuous on an open subset U of $\overline{E}$. Let ν represent h , and let μ_0 represent the function identically equal to 1 . Then

$$\nu(dy) = h(y)\mu_0(dy) \quad \text{on} \quad U . \qquad \square$$

If h is excessive, write $({}_hU^\lambda)_{\lambda>0}$ for the resolvant of $({}_hP_t)$. Let

$$\tilde{X}_t = \begin{cases} X_{\zeta-t} , & t \in (0, \zeta) \\ \zeta , & t \geq \zeta \end{cases}$$

be the reverse of (X_t) from its lifetime. Observe that $\tilde{X}_\cdot \in \hat{\Omega}$, and that this definition applies regardless of whether $X_\cdot$ is defined at 0 .

<u>Theorem 3</u>

(a) Let h be excessive, and assume that ${}_hP^x(\zeta < \infty) = 1$ for every $x \in E$ (as usual, it suffices that this holds for some $x \in E_h$). Let μ be a probability on E . Then under the probability ${}_hP^\mu$, the process $(\tilde{X}_t)_{t>0}$ is a v-transform, where

$$v(x) = \int \frac{G(z, x)}{h(z)} \mu(dz) .$$

(b) If P is any probability on $(\hat{\Omega}, F)$ under which $(X_t)_{t>0}$ is an h-transform, then there is some excessive function v such that $(\tilde{X}_t)_{t>0}$ is a v-transform under P.

<u>proof</u>: (a) First, we show that $v < \infty$ a.e. . Let

$$A_n = \{x \in E \; ; \; {}_hP^x(\zeta > n) < 1/2\} ,$$

$$B_t = \int_0^t 1_{A_n}(X_s)ds ,$$

$$T(j) = \inf\{t \; ; \; B_t > jn\} .$$

Since

$${}_hP^{X_t}(\zeta > n)$$

is ${}_hP^x$ - a.s. continuous in $t < \zeta$, it follows that for each $x \in E$,

$$ {}_hP^x(T(j+1) < \infty \mid T(j) < \infty \leq {}_hP^x(\zeta \circ \theta_{T(j)} > n \mid T(j) < \infty)$$

$$= {}_hE^x\,[{}_hP^{X_{T(j)}}(\zeta > n) \mid T(j) < \infty] \leq \frac{1}{2} .$$

Thus

$$\int_{A_n} \frac{1}{h(x)} G(x, y)h(y)dy = \int_0^\infty {}_hP_t(x, A_n)dt$$

$$= {}_hE^x[B_\infty]$$

$$\leq n \sum_{j=0}^{\infty} {}_hP^x(B_\infty > jn) \leq n \sum_{j=0}^{\infty} 2^{-j} = 2n .$$

Integrating with respect to μ, we see that

$$\int_{A_n} v(x)h(x)\,dx \leq 2n \quad \text{for each } n .$$

Since (P_t) is strong Feller, and h is l.s.c., we see that

$$A_n = \{E^x[h(X_n)] < h(x) / 2\}$$

is open. By hypothesis, $A_n \uparrow E$, so that not only is $v < \infty$ a.e. on E, but $v(x)h(x)dx$ is a Radon measure thereon (compact subsets of E have finite measure). The occupation time measure of (X_t) is

$$\nu(dx) = \int \mu(dz)\ {}_hU^0(z, dx) = v(x)h(x)dx ,$$

so that by the results of Cartier, Meyer, Weil [6], (a) will be shown, once we show that $({}_hU^\lambda)_{\lambda>0}$ and $({}_vU^\lambda)_{\lambda>0}$ are in duality with respect to ν.

Let $G^\lambda(x,y)$ be the λ-order Green function. Then if f and g are positive and Borel, we have that

$$\begin{aligned}
&< f, {}_vU^\lambda g >_\nu \\
&= \int f(x)v(x)h(x)\frac{1}{v(x)} \int G^\lambda(x,y)g(y)v(y)\,dydx \\
&= \int g(y)h(y)v(y)\frac{1}{h(y)} \int f(x)h(x)G^\lambda(x,y)\,dxdy \\
&= < {}_hU^\lambda f, g >_\nu ,
\end{aligned}$$

by symmetry of G^λ, as required.

(b) Let $(E(n))$ be a sequence of subdomains of E, such that $E(n)$ has compact closure in $E(n+1)$, and $\cup E(n) = E$.

Let $T(n) = \inf\{t;\ X_t \in E(n)\}$. Then

$$(X_{T(n)+t})_{t\geq 0}$$

(defined by right continuity at 0 if $T(n) = 0$) is still an h-transform under P , and it does have an initial distribution, so that (a) applies. Let $(\tilde{X}^n_t)$ be the reverse of $(X_{T(n)+t})$, so that there exists an excessive v_n such that $(\tilde{X}^n_t)$ is a v_n-transform under P . Let

$$\tilde{S}_n = \inf\{t > 0\ ;\ \tilde{X}^n_t \notin E(n)\}$$

$$\tilde{L}_n = \sup\{t > 0\ ;\ \tilde{X}^n_t \in E(n)\}\ .$$

Then $\tilde{X}^n_t = \tilde{X}^{n+1}_t$ if $t < \tilde{L}_n$, and in particular, if $t < \tilde{S}_n$. From (i) and (j) of Theorem 1, it follows that there is a constant $c_n > 0$ such that $v_{n+1} = c_n v_n$ on $E(n)$. We may multiply each v_n by a constant, to make each $c_n = 1$, so that an excessive function v on E may be defined to make $v = v_n$ on $E(n)$, for each n . Since each $(\tilde{X}^n_t)$ is a v_n-transform, it follows that $(\tilde{X}_t)$ is a v-transform under P . □

<u>Lemma 1</u>. Let $h > 0$ be harmonic in E , and let A be compact in E . Then

$${}_hP^x(L_A < \zeta) = 1 \quad \text{for each } x \in E\ .$$

<u>proof</u>: By (f) of Theorem 1, and (e) of Theorem 2, it suffices to show this for h minimal. The event $\{L_A = \zeta\}$ is invariant, so that in this case

$${}_hP^x(L_A = \zeta) = 0 \text{ or } 1 \quad \text{for each } x$$

(by (h) of Theorem 1). Since $E_h = E$, this same result shows that this function is h-excessive. Thus by (d) of Theorem 1, it is either identically zero or identically one. Assume the latter. Then for each $x \in E$, it follows from (k) of Theorem 1, that

$$h(x) = h(x)\ {}_hP^x(L_A = \zeta) \leq h(x)\ {}_hP^x(T_A < \infty)$$

$$= R_A h(x) .$$

Since A is compact, $R_A h$ is a potential. This is a contradiction, since the only harmonic function dominated by a potential is identically zero. □

We will use the notation

$${}_yP^x = {}_{K(y,\cdot)}P^x .$$

whenever $y \in \overline{E}$.

<u>Theorem 4</u> Let

$$h(x) = \int_{E \cup \Delta_0} K(z, x)\mu(dz)$$

be excessive. Then for each $x \in E_h$, $X_{\zeta-}$ exists in $\overline{E}$, ${}_hP^x$-a.s., and has distribution

$${}_hP^x(X_{\zeta-} \in dy) = \frac{1}{h(x)} K(y, x)\mu(dy) .$$

<u>proof</u>: By (f) of Theorem 1, it suffices to show that

$${}_yP^x(X_{\zeta-} = y) = 1 \quad \text{for} \quad y \in E \cup \Delta_0 .$$

Case 1: Suppose $y \in E$. By (1) of Theorem 1, and Lemma 1, there exists an excessive function u such that $y \in E_u$ and u-transforms have finite lifetimes. By Theorem 3, $(\tilde{X}_t)$ is a $G(y, \cdot)$ transform under ${}_uP^y$, so that there exists a probability ν such that

$$ {}_yP^{\nu}(\zeta \in (0, \infty) , X_{\zeta-} = y) = 1 . $$

This holds for every ν by minimality, as in Lemma 1.

Case 2: Suppose $y \in \Delta_0$. In this case, there is no reason why ζ should be finite. Let $\sum'$ be an open neighbourhood of $x \in E$, with compact closure in E. Let $(E(n))$ be a sequence of subdomains of E, with each $E(n)$ having compact closure contained in $E(n+1)$, and such that $E(1)$ contains the closures of $\sum$ and $\sum'$, and $\cup E(n) = E$. By Lemma 1,

$$ {}_yP^x(L_{E(n)} < \zeta) = 1 $$

for each n, so that we may form the reverse $\tilde{X}^n$ of X from $L_{E(n)}$ and obtain from Theorem 3(a) (and (1) of Theorem 1), that $\tilde{X}^n$ is a $G(x, \cdot)$ transform under ${}_yP^x$. Let

$$ k(w, z) = G(w, z)/G(w, x) , $$

for $z \in \sum$ and $w \in E' = E \setminus (\sum \cup \sum')$. Let

$$ T(n) = \inf\{t > 0; \tilde{X}^n_t \notin E'\} . $$

Since $k(\cdot, z)$ is bounded on E', for $z \in \sum$, it follows as usual, that

$$ k(\tilde{X}^n_{t \wedge T(n)}, z) $$

is a bounded martingale under ${}_yP^x$, for each $z \in \sum$. If U_n denotes the number of upcrossings it makes, of a fixed interval $[a, b]$, we therefore obtain that

$$ {}_yE^x[U_n] \leq \frac{|a| + \sup_{w \in E'} k(w, z)}{b - a} < \infty . $$

But U_n is also the number of upcrossings of $[a, b]$ by $k(X_t, z)$, for

$$ t \in (L_{\sum \cup \sum'} , L_{E(n)}] . $$

Thus $U_n \uparrow U_\infty$, and since the above bound did not depend on n , we see that $U_\infty < \infty$ ${}_yP^x$ -a.s. . Thus by the usual argument, it follows that

$$ \lim_{t \uparrow \zeta} k(X_t, z) $$

exists ${}_yP^x$ -a.s., for each $z \in \sum$. But $k(\cdot, z)$ is bounded away from zero on E' , and

$$ K(w, z) = k(w, z)/k(w, y_0) $$

for $w \notin \sum$, so that in fact

$$ \lim_{t \uparrow \zeta} K(X_t, z) $$

exists ${}_yP^x$ -a.s., for each $z \in \sum$. By Fubini's theorem, this holds ${}_yP^x$ -a.s., simultaneously for a.e. $z \in \sum$, so that in fact $d(X_s, X_t) \to 0$ as $s, t \uparrow \zeta$, ${}_yP^x$ -a.s. . Thus, the $\overline{E}$-limit $X_{\zeta-}$ exists.

We must now identify it as y .

Let $(A(n))$ be a decreasing sequence of neighbourhoods of y , with $\cap A(n) = \{y\}$. We must show only that

$$T_{A(n)} < \infty \quad {}_yP^x \text{-a.s.} \quad .$$

But

$$K(y, z) = R_{A(n)\cap E}K(y, \cdot)(x) = K(y, x)\,{}_yP^x(T_{A(n)} < \infty) \ ,$$

by (k) of Theorem 1, and (d) of Theorem 2, so that this holds. □

This proof traces its ancestry to Hunt [30]. The technique of killing at the last exit times of compact subsets of E, let us eliminate his use of 'approximate' Markov chains.'

We say that a point $y \in \Delta_0$ is <u>accessible</u> if ${}_yP^x(\zeta < \infty) = 1$ for some (and hence every) $x \in E$. Let Acc denote the set of all accessible points.

<u>Theorem 5</u> Let $u(x) = \int G(x, z)\nu(dz)$ be an a.e. finite potential, and let $y \in \text{Acc}$. Suppose that

$$\int K(y, z)\nu(dz) < \infty \ .$$

Then there exists a unique probability ${}_uP^y$ on $\hat{\Omega}$, such that (X_t) is u-transform under ${}_uP^y$, and

$${}_uP^y(X_t \to y \text{ in } \overline{E} \text{ as } t \downarrow 0) = 1 \ .$$

<u>proof</u>: Let

$$a = \int_E K(y, z)\nu(dz) \ , \ \mu(dz) = \frac{1}{a} K(y, z)\nu(dz) \ .$$

Because y is accessible, we can reverse from ζ to obtain that under ${}_yP^\mu$, $(\tilde{X}_t)$ is a transform by the function

$$\int \frac{1}{K(y,z)} G(z, x)\mu(dz) = \frac{1}{a} \int G(z,x)\nu(dx) = \frac{1}{a} u(x) \ .$$

Since

$$\lim_{t\downarrow 0} \tilde{X}_t = \lim_{t\uparrow\zeta} X_t = y \ , \quad {}_yP^\mu \text{-a.s.} \ ,$$

by Theorem 4, we have shown existence.

To show uniqueness, we will show that if P is a probability on $\hat{\Omega}$, as in the theorem, then $(\tilde{X}_t)$ is a $K(y, \cdot)$ transform, with initial measure μ , under P . Thus the law of $(\tilde{X}_t)$, and hence off (X_t) is determined.

It follows from Theorem 3(b), that since (X_t) is a u-transform under P , $(\tilde{X}_t)$ is an h-transform for some h . Write

$$h(z) = \int K(x, z)\eta(dx) \ .$$

By Theorem 4,

$$P(\tilde{X}_{\zeta-} \in dx \ , \ \zeta > t) = E[{}_hP^{\tilde{X}_t}(X_{\zeta-} \in dx)]$$

$$= E[\frac{1}{h(\tilde{X}_t)} K(x, \tilde{X}_t)\eta(dx) \ , \ \zeta > t] \ ,$$

for every $t > 0$. But $\tilde{X}_{\zeta-} = y$ P-a.s., so that η must be a point mass at y , and hence $h(z) = cK(y, z)$ for some c .

By Theorem 4, we use a similar argument to see that $X_{\zeta-} \in E$ P-a.s. . Let μ' be the distribution of $X_{\zeta-}$. It is then the initial distribution of $(\tilde{X}_t)$, so that by Theorem 3(a), (X_t) is a transform by

$$\int \frac{1}{K(y, z)} G(z, x)\mu'(dz) = v(x) .$$

It follows from (i) of Theorem 1, that $v = cu$ for some $c > 0$. Since a potential determines its measure, we conclude that

$$\frac{1}{K(y, z)} \mu'(dz) = c\ \nu(dz) ,$$

or in other words, $\mu' = ac\mu$, so that $c = 1/a$ and $\mu' = \mu$. □

We wish to examine the set

$$D = \{(x, y) \in (\underset{\sim}{Acc} \cup E) \times (\underset{\sim}{Acc} \cup E) \text{ ; there exists a probability } {}_yP^x \text{ on } \hat{\Omega} \text{ under which } (X_t) \text{ is a } K(y, \cdot) \text{ transform, and } X_{0+} = x \text{ a.s.}\}$$

Theorems 4 and 5 show that

$$D \supset (\underset{\sim}{Acc} \times E) \cup (E \times \underset{\sim}{Acc}) .$$

The arguments given above also suffice to show the following statements:

If $(x, y) \in D$ then there is only one law ${}_yP^x$ satisfying the required conditions; and

(12.1) D is symmetric (that is, $(x, y) \in D$ iff $(y, x) \in D$) , and if $(x, y) \in D$ and $\Lambda \in (\hat{\Omega}, F)$, then

$${}_xP^y(\Lambda) = {}_yP^x((\tilde{X}_t) \in \Lambda).$$

Recall that a function f on E is said to have <u>fine limit</u> a, at a point $y \in \Delta_0$, if

$$ {}_yP^x(\lim_{t\uparrow\zeta} f(X_t) = a) = 1 $$

for some (and hence by minimality, as in Lemma 1, for every) $x \in E$. By the construction of Theorem 5, this is equivalent to

$$ {}_xP^y(\lim_{t\downarrow 0} f(X_t) = a) = 1 $$

for some (and hence every) $x \in E$.

<u>Lemma 2</u> Let A be open in $\overline{E}$, with $y \in A \cap \Delta_0$. Then

$$ h = R_{E\setminus A}\, K(y, \cdot) $$

is a potential; that is, for some measure ν, we have

$$ h = \int_E G(z, \cdot)\nu(dz) = U^0\nu . $$

<u>proof</u>: Write

$$ h = \int_{E\cup\Delta_0} K(z, \cdot)\mu(dz) . $$

Since $K(z, \cdot)$ is proportional to $G(z, \cdot)$, it will suffice, by Theorem 4, to show that

$$ {}_hP^x(X_{\zeta-} \in \Delta_0) = 0 \quad \text{for} \quad x \in E . $$

But

$$ {}_yP^x(X_{\zeta-} = y) = 1 , $$

and so

$$ {}_yP^x(L_{E\setminus A} < \zeta) = 1 $$

as well. Thus by (1) of Theorem 1,

$$ {}_hP^x(X_{\zeta-} \in \Delta_0) = {}_yP^x(X_{L_{E\setminus A}} \in \Delta_0) = 0 . \quad \square $$

<u>Theorem 6</u> Let $x, y \in Acc$. Let A be an open subset of $\overline{E}$ such that $y \in A$ and x does not lie in the closure of A . Let $z \in A \cap E$, and set $f(w) = K(y, w)/G(z, w)$. Then the following conditions are equivalent

(a) $(x, y) \in D$

(b) Let

$$ R_{E\setminus A}\, K(y, \cdot) = U^0\nu . $$

Then $\int K(x, z)\nu(dz) < \infty$.

(c) f has a finite fine limit at x .

(d) For some $w \in E$,

$$ \liminf_{t \uparrow \zeta} f(X_t) < \infty \quad {}_xP^w \text{ -a.s. } . $$

<u>proof</u>: We will show that (a) <=> (b) , and (a) => (c) => (d) => (a) .

((a) => (b)) : Let (Y_t) be (X_t) killed at $L_{E\setminus A}$, so that (Y_t) is an $R_{E\setminus A}\, K(y, \cdot)$ transform under ${}_yP^x$. Let μ be the distribution of

$$ X_{L_{E\setminus A}} $$

under ${}_yP^x$. Let $(\tilde{Y}_t)$ be the reverse of (Y_t) from its lifetime. By (12.1), (y, x) also belongs to D , and

$$ {}_yP^x((\tilde{Y}_t)_{t>0} \in \Lambda) = {}_xP^y((X_{t+T_A})_{t>0} \in \Lambda) $$

$$ = {}_xP^\mu((X_t)_{t>0} \in \Lambda) \ , \text{ for } \Lambda \in (\hat{\Omega}, F) \ . $$

Thus, by Theorem 3(a), (Y_t) is also a transform by

$$ \int \frac{G(z, \cdot)}{K(x, z)} \mu(dz) \ . $$

It follows from (i) of Theorem 1, that there is a constant c with

$$ c \int \frac{G(z, \cdot)}{K(x, z)} \mu(dz) = \int G(z, \cdot)\nu(dz) \ , $$

and since a potential determines its measure, it follows that

$$ \int K(x, z)\nu(dz) = c < \infty \ . $$

((b) => (a)): Let

$$ h = R_{E\setminus A} K(y, \cdot) \ . $$

By Theorem 5, there is a probability ${}_hP^x$ on $\hat{\Omega}$ under which (X_t) is an h-transform started at x. Define a probability P' on $\hat{\Omega} \times \Omega$ by

$$ P'(A) = \int\int 1_A(\omega, \omega') \ {}_yP^{X_{T_A}(\omega)}(d\omega') \ {}_hP^x(d\omega) \ , $$

and let P be the image law of P' on $\hat{\Omega}$ under the random variable

$$ Y_t(\omega, \omega') = \begin{cases} X_t(\omega) & , \ t < T_A(\omega) \\ X_{t-T_A(\omega)}(\omega') & t \geq T_A(\omega) \end{cases} $$

Then for f positive and bounded, we have that

$$E[f(X_{t+s})\,|F_s]$$

$$= E[f(X_{t+s}),\ T_A \le s \mid F_s] + E[f(X_{t+s}),\ T_A > s + t \mid F_s]$$

$$+ E[f(X_{t+s}),\ T_A \in (s,\ s + t] \mid F_s]$$

$$= {}_yE^{X_s}[f(X_t)]\ 1_{\{T_A \le s\}} + {}_hE^{X_s}[f(X_t),\ T_A > t]\ 1_{\{T_A > s\}}$$

$$+ 1_{\{T_A > s\}} \int_0^t \int {}_yE^z[f(X_{t-r})]\ {}_hP^{X_s}(T_A \in dr,\ X_{T_A} \in dz)$$

$$= {}_yE^{X_s}[f(X_t)]\ 1_{\{T_A \le s\}} + {}_yE^{X_s}[f(X_t),\ T_A > t]\ 1_{\{T_A > s\}}$$

$$+ 1_{\{T_A > s\}} \int_0^t \int {}_yE^z[f(X_{t-r})]\ {}_yP^{X_s}(T_A \in dr,\ X_{T_A} \in dz)$$

$$= {}_yE^{X_s}[f(X_t)] .$$

Thus (X_t) is a $K(y,\ \cdot)$ transform under P, as required.

((a) => (c)): The Green function for a $K(y,\ \cdot)$ transform is

$${}_yG(w,z) = \frac{K(y,\ z)}{K(y,\ w)}\ G(w,\ z) .$$

We have that if $\sum'$ is a compact neighbourhood of z, contained in E, then for $w \notin \sum'$,

$${}_yG(w,\ z) = \int {}_yG(w',\ z)\ {}_yP^{w'}(T_{\sum'} \in dw')$$

$$\le \sup_{w' \in \partial \sum'} {}_yG(w',\ z) < \infty .$$

Then as usual,

$$Z_t = {}_yG(X_{t\wedge T_{\Sigma'}}, z)$$

is a bounded ${}_yP^w$ - martingale, for $w \in E\backslash\Sigma'$. Since $(x, y) \in D$, and (Z_t) is bounded, we obtain from the Markov property, that $(Z_t)_{t>0}$ is a bounded ${}_yP^x$ - martingale as well. Thus, it convergex ${}_yP^x$-a.s. to a random variable Z_{0+} as $t\downarrow 0$. Since $Z_t > 0$ for $t > 0$, Z_{0+} cannot be identically zero. Let $(E(n))$ be a sequence of open subsets of E such that $E(n)$ has compact closure in $E(n+1)$, and $\cup E(n) = E$. Let

$$\tilde{X}_t^n = \begin{cases} X_{L_{E(n)}-t} & , t < L_{E(n)} \\ \delta & , t \geq L_{E(n)} . \end{cases}$$

Then by Theorem 3, $(\tilde{X}_t^n)$ is a $K(x, \cdot)$ transform under ${}_yP^x$, and

$${}_yP^x(\tilde{X}_0^n = \delta) = {}_yP^x(L_{E(n)} = 0) \downarrow 0 .$$

By minimality of x, Z_{0+} is ${}_yP^x$ -a.s. constant on $\{\tilde{X}_0^n \neq \delta\}$, so that in fact, there is a constant $a \neq 0$, such that $Z_{0+} = a$ ${}_yP^x$ - a.s. .

Thus also

$$\begin{aligned} {}_yP^x(\tilde{X}_0^n \neq \delta) &= {}_yP^x(\tilde{X}_0^n \neq \delta, \lim_{t\uparrow\zeta} {}_yG(\tilde{X}_t, z) = a) \\ &= \int_E {}_xP^w(\lim_{t\uparrow\zeta} {}_yG(X_t, z) = a)\ {}_yP^x(\tilde{X}_0^n \in dw) , \end{aligned}$$

so that

$$ {}_xP^w(\lim_{t\uparrow\zeta} {}_yG(X_t, z) = a) > 0 $$

for some $w \in E$. By minimality, this probability equals 1, showing (c). The implication (c) => (d) is trivial.

((d) => (a)): By (d) of Theorem 1, f is $G(z, \cdot)$-excessive, so that $(f(X_t))$ is a ${}_zP^w$ - supermartingale, for each $w \in E$. Thus $(f(X_t) \wedge n)$ is a bounded ${}_zP^w$ - supermartingale for each w, so that by the Markov property, $(f(X_t) \wedge n)_{t>0}$ is a bounded ${}_zP^x$-supermartingale. Thus, it converges at 0, ${}_zP^x$-a.s., and hence in fact

$$ \lim_{t\downarrow 0} f(X_t) \text{ exists in } [0, \infty] , \quad {}_zP^x\text{-a.s.} . $$

By minimality and time reversal, this limit is a constant a (non zero, but a priori, possibly infinite). As in Lemma 1, also

$$ f(X_t) \to a \quad \text{as} \quad t \downarrow 0 , \quad {}_wP^x \text{-a.s.} $$

for each $w \in E$, so that by (d), we have that in fact, $a < \infty$. Therefore

$$ {}_zE^x[f(X_{T_A})] = \lim_{n\to\infty} {}_zE^x[f(X_{T_A}) \wedge n] \leq a < \infty . $$

Let $R_AK(x, \cdot) = U^0\nu$. Since $z \in A$, we have that $L_A > 0$ ${}_xP^z$ -a.s., and so

$$ {}_zE^x[f(X_{T_A})] = {}_xE^z[f(X_{L_A})] $$

$$ = \frac{1}{U^0\nu} E^z[f(X_{\zeta-})] $$

$$ = \frac{1}{U^0\nu(z)} \int f(w)\ G(w,\ z)\nu(dw) \qquad \text{(Theorem 4)} $$

$$ = \frac{1}{U^0\nu(z)} \int K(x,\ w)\nu(dw)\ . $$

Thus

$$ \int K(x,\ w)\nu(dw) < \infty\ , $$

and by the previous implication, that (b) => (a), we get that $(y,\ x) \in D$. Thus $(x,\ y) \in D$ by (12.1), as required. □

Note that the argument for ((d) => (a)) shows that if h is excessive and $x \in \mathrm{Acc}$, $z \in E$, then $h\ /\ G(\cdot,\ z)$ has a fine limit at x (possibly infinite). This was first shown by Naim [44]; she showed it more generally, for $x \in \Delta_0$. We could obtain this more general result by combining the above argument with the argument of Theorem 4, in which upcrossings are estimated by killing (X_t) at times $L_{E(n)}$, under the probability ${}_xP^z$, and then reversing time.

<u>Theorem 7</u> Let $r > 0$. Let E be a bounded domain such that for every point $x \in \partial E$, there are open balls B and C , of radii greater than r , both containing x in their closures, and such that $B \subset E$, $C \subset R^d \setminus E$. Then the Martin compactification is homeomorphic to the Euclidean closure of E , and moreover $D = \{(x,\ y) \in \partial E \times \partial E;\ x \neq y\}$.

proof: The first statement is a consequence of the results of de la Vallée Poussin [11], and these results, together with Lemma 5.1 of Martin [37] suffice to show (d) of Theorem 6. □

13. A Martin Boundary in the plane

Let

$$D' = \{(x, y);\ x, y \in \mathrm{Acc} \cup E \text{ and } x \neq y\} .$$

It is not in general true that $D = D'$. An example of a domain D for which this fails may be found in Example 1 of Martin [37]. Martin constructed a certain bounded domain E in R^3, together with a subset E_0 of its Martin boundary, and showed that

$$\limsup_{\substack{z \to y \\ z \in E}} K(x, z) = \infty$$

for $x, y \in E_0$. One can modify the parameters in his construction slightly, and obtain (as in Theorem 9(b) below), that also

$$\lim_{\substack{w \to y \\ w \in E}} K(x, w)/G(z, w) = \infty$$

for $z \in E$ and $x, y \in E_0$. Thus condition (d) of Theorem 6 fails, so that $(x, y) \notin D$ for any $x, y \in E_0$. Modulo showing that E_0 contains at least two distinct accessible points, this gives us our counterexample. This latter property is in fact not difficult to show. One embedds E in a product $E' = R \times E''$, where $E'' \subset R^2$ is a bounded domain. By a result of M. Cranston and T. McConnell ([10]; Theorem 8 below), all minimal Martin boundary points of E'' are accessible.

One can therefore construct a harmonic h on E' such that the h-transform $X_t = (Y_t, Z_t)$ has finite lifetime ζ ; (Y_t) is a Brownian motion up till time ζ , independent of (Z_t) given ζ ; and with positive probability, $X_t \in E$ for every $t < \zeta$, and X_t converges in E as $t \uparrow \zeta$, with limit in E_0 . From this, one sees immediately that $E_0 \cap \mathrm{Acc}$ has many points.

Rather than dwelling on this, we will consider an analogous example in R^2 , in detail. The two dimensional situation is distinguished from that of higher dimensions, by the availability of the Riemann mapping theorem. That is, if we are given a domain $E_1 \subset R^2$ whose complement (in the Riemann sphere) has n components $(n \geq 1)$, none of which is a single point, then there is a conformal equivalence Φ mapping E_1 onto a bounded domain E_2 with smooth boundaries. If (X_t) is a Brownian motion started at x and killed upon leaving E_1 , then we obtain a Brownian motion on E_2 by time changing $Y_t = \Phi(X_t)$ by the inverse $A'(t)$ to the additive functional

$$A_t = \int_0^t |\Phi'(\Phi^{-1}(Y_s))|^2 ds .$$

Thus, if G_1 and G_2 are the Green functions on E_1 and E_2 respectively, then for every Borel $f \geq 0$,

$$\int_{E_1} f(\Phi(z)) G_2(\Phi(x), \Phi(z)) \; | \Phi'(z) |^2 dz$$

$$= \int_{E_2} f(y) G_2(\Phi(x), y) dy$$

$$= E[\int_0^\infty f(Y_{A'(t)}) dt]$$

$$= E[\int_0^\infty f(\Phi(X_t)) \mid \Phi'(X_t) \mid^2 dt]$$

$$= \int_{E_1} f(\Phi(z)) G_1(x, z) \mid \Phi'(z) \mid^2 dz .$$

That is, $G_2(\Phi(x), \Phi(z)) = G_1(x, z)$. Thus, Φ will also preserve the Martin function, and hence extends to a homeomorphism of the Martin compactifications of E_1 and E_2 , taking minimal points to minimal points. There is of course no reason why Φ should behave so nicely on the Euclidean boundary of E_1 . In our case, all the Martin boundary points of E_2 are accessible (and minimal) by Theorem 7, so that by the criteria of Theorem 6, it follows that $D = D'$. In dimension ≥ 3 , we could obtain pathological domains homeomorphic to the unit ball, but this shows that to do the same in dimension 2, we must consider domains of infinite connectivity.

Unlike the case of minimality, the accessibility or inaccessibility of a Martin boundary point will not in general be preserved under a conformal map. This problem is resolved by the following result of Cranston and McConnell [10] .

<u>Theorem 8</u> There exists a constant C such that for each domain E in R^2 and each excessive function h on E , we have that

$${}_hE^x[\zeta] \leq C \cdot \text{Area of } E , \quad \text{for } x \in E . \quad \square$$

Thus, all minimal Martin boundary points of bounded domains in R^2 , are accessible.

We will now construct a domain $E \subset R^2$ for which $D \neq D'$. Let

$a(n)$, $b(n) \in (0, 1)$, $n = 1, 2, \ldots$. Set $s(0) = 0$ and

$$s(n) = \sum_{i=1}^{n} b(n) \quad , \quad n \geq 1 .$$

Assume that $a(n)$ and $b(n)$ are decreasing in n , and that $s(\infty) = 1$. Let

$$E = (-1, 1) \times (0, 1) \setminus \bigcup_{n=1}^{\infty} [a(n) - 1, 1-a(n)] \times \{s(n)\} .$$

We will show that for every fixed sequence $(b(n))$ as above, the $a(n)$ may be chosen to converge to zero sufficiently fast that $D \neq D'$.

Let $c(n) = (0, (s(n) + s(n-1))/2)$, $n \geq 1$

$$A(n) = \begin{cases} (1 - a(n), 1) \times \{s(n)\} , & n \geq 1 \\ (-1, a(n) - 1) \times \{s(n)\} , & n \leq -1 \end{cases}$$

$$A+ = \bigcup_{n \geq 1} A(n) \quad , \quad A- = \bigcup_{n \leq -1} A(n) ,$$

$A = A+ \cup A-$, and $B(n) = A(n) \cup A(-n)$.

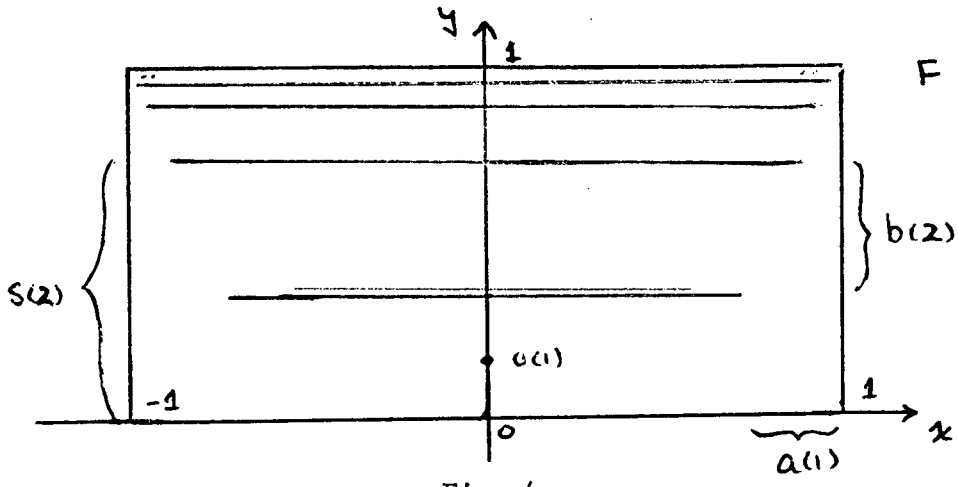

Fig. 4

Let x_0 be a limit point in $\overline{E}$ of the sequence $(c(n))$. Define

the Martin function relative to the point $c(1)$, so that $h = K(x_0, \cdot)$ is symmetric on E (that is, $h(-\bar{z}) = h(z)$; as the reader will have gathered, we are allowing ourselves to pass freely between notations used for R^2 and for C .) . We will write P^x for the law of Brownian motion, started at $x \in E$ and killed upon contact with ∂E . For ease of notation, we will write Q for the probability ${}_hP^{c(1)}$.

Martin's argument will give us the first part of the following.

<u>Theorem 9</u> Suppose that

$$\limsup_{n \to \infty} b(n) \log(1/a(n-1)) > 4\Pi \ . \quad \text{Then:}$$

(a) $h(c(n)) \to \infty$ as $n \to \infty$;

(b) Let $z \in E$. Then $h(w)/G(z,w) \to \infty$ as $\mathrm{Im}(w) \uparrow 1$.

The only reason this result does not immediately give us that $D \neq D'$, is that it is conceivable that the speed at which $a(n) \to 0$ forces that part $[-1, 1] \times \{1\}$ of the Euclidean boundary of E , to collapse to a single point of the Martin boundary. If we could show that h was not minimal, then there would have to be at least two minimal Martin boundary points x_1 and x_2 , such that $w \to x_i$ implies that $\mathrm{Im}(w) \to 1$, $i = 1, 2$. In this case, (b) of Theorem 9 together with criterion (d) of Theorem 6 would show that $D \neq D'$. Our main result will be:

<u>Theorem 10</u> Suppose that $a(n-1)/b(n) \to 0$. Then

$$Q(T_{A-} = \infty) > 0 \ .$$

By symmetry, also $Q(T_{A+} = \infty) > 0$, so that also $Q(T_{A(n)} < \infty$ for infinitely many $n \geq 1) \in (0, 1)$, and hence we conclude from (h) of Theorem 1, that h cannot be minimal. Thus we have

Corollary 1 Suppose that

$$\limsup_{n \to \infty} b(n) \log(1/a(n-1)) > 4\Pi .$$

Then $D \neq D'$.

Now to work. A common feature of many of the arguments is that we obtain estimates using the exact form of certain Schwarz-Christoffel transformations. To do so, we will extensively use elliptic integrals, for which we refer to Gradshteyn and Ryzhik [26]. The notation (GR8.1133) will refer to the appropriate formula in [26]. For ease of use, we will use a nonstandard notation, writing

$$F(\alpha, k) = \int_0^\alpha \frac{dx}{\sqrt{(1-x^2)(1-kx^2)}} , \quad \alpha, k \in [0, 1] .$$

For now, we will only recall that (GR8.1133)

(13.1) $F(1, 0) = \Pi/2$, $F(1, k) \sim \log(1/\sqrt{1-k})$ as $k \uparrow 1$.

proof of Theorem 9: Fix n , and consider $E' = E \cap (-1, 1) \times (0, s(n))$. For $x \in E'$, let P'^x be the law of Brownian motion started at x and stopped upon leaving E' . Also, let

$$W(n) = (-1, 1) \times (s(n-1), s(n)) \subset E' ,$$

$$C(n) = (-1, 1) \times \{(s(n) + s(n-1))/2\} , \text{ and}$$

$$\mu(x, dy) = P'^x(X_{T_{\partial W(n)}} \in dy) .$$

Let Φ be the Schwarz-Christoffel transformation mapping the unit disk to $W(n)$ and taking the points -1, 0, and 1, to $(-1, (s(n) + s(n-1)/2)$, $c(n)$, and $(1, (s(n) + s(n-1))/2)$. Since composing Φ with a Brownian motion on the disk produces a time changed Brownian motion on $W(n)$, it follows that Φ preserves hitting distributions. Thus, by the corresponding result for the disk, we see that for each $x \in W(n)$, there exists a density $u(x, y)$, continuous in y , for $\mu(x, dy)$ with respect to $\mu(c(n), dy)$, and that further, there exists a constant $k(n)$ with

$$\frac{u(x, y)}{u(x,y')} \leq k(n) \quad \text{for} \quad x \in C(n) \quad \text{and} \quad y, y' \in (-1, 1) \times \{s(n)\} .$$

Let

$$m(n, w) = k(n)^2/\mu(w, (-1, 1) \times \{s(n)\}) .$$

Thus, for $x, w \in C(n)$ and $y \in (-1, 1) \times \{s(n)\}$, we have that

$$\begin{aligned}
\mu(x, dy) &= \frac{u(x, y)}{u(w, y)} \mu(w, dy) \\
&\leq \frac{u(x, y)}{u(w, y)} \mu(w, dy) \\
&\qquad \frac{\displaystyle\int_{(-1,1) \times \{s(n)\}} \frac{u(x, y')}{u(w,y')} \mu(w, dy')}{\inf\{\frac{u(x,y')}{u(w,y')};\ y' \in (-1,1)\times\{s(n)\}\}\mu(w,(-1,1)\times\{s(n)\})} \\
&\leq m(n, w)\ \mu(x, (-1, 1) \times \{s(n)\})\mu(w, dy) .
\end{aligned}$$

Let $R(0) = 0$, and

$$R(i+1) = R(i) + (T_{B(n-1)} + T_{C(n)} \circ \theta_{T_{B(n-1)}}) \circ \theta_{R(i)} , \quad i \geq 0 .$$

Then for $x, w \in C(n)$ and $y \in (-1, 1) \times \{s(n)\}$,

$$P'^{x}(X_{T_{\partial E'}} \in dy)$$

$$= \sum_{i=0}^{\infty} E'^{x}[R(i) < \infty,\ P'^{X_{R(i)}}(R(1) = \infty\ ,\ X_{T_{\partial E'}} \in dy)]$$

$$= \sum_{i=0}^{\infty} E'^{x}[R(i) < \infty,\ \mu(X_{R(i)}, dy)]$$

$$\leq \sum_{i=0}^{\infty} m(n,w)\mu(w,dy)\ E'^{x}[R(i) < \infty,\ \mu(X_{R(i)}, (-1,1) \times \{s(n)\})]$$

$$= m(n,w)\mu(w,dy)\ P'^{x}(X_{T_{\partial E'}} \in (-1,1) \times \{s(n)\})$$

$$\leq m(n,w)\mu(w,dy)\ .$$

Also, one easily shows by the appropriate Schwarz-Christoffel transformation, that there is a constant M such that for every k, the first hitting distribution of Brownian motion started at $c(1)$, on the boundary of $(-1, 1) \times (0, s(k))$, has a density with respect to the arc length measure, which is bounded by M.

Now let h be the function of the theorem. Then for $w \in C(n)$,

$$h(c(1)) = \int\int h(y)P'^{x}(T_{B(n)} < \infty\ ,\ X_{T_{B(n)}} \in dy)\ .$$

$$\cdot\ P'^{c(1)}(T_{C(n)} < \infty\ ,\ X_{T_{C(n)}} \in dx)$$

$$\leq P'^{c(1)}(T_{B(n-1)} < \infty)\ m(n,\ w) \int h(y)\mu(w,\ dy)$$

$$\leq 2\ M\ a(n-1)m(n,\ w) \int h(y)\ P'^{w}(X_{T_{\partial E'}} \in dy)$$

$$= 2\ M\ a(n-1)\ m(n,\ w)h(w)\ .$$

Thus, provided $a(n-1)m(n,\ c(n)) \to 0$ as $n \to \infty$, it follows that

$h(c(n)) \to \infty$.

Now, consider $G(z, \cdot)$. Let z belong to the closure of $W(j-1)$, and suppose that the n used above satisfies $n > j$. $G(z, x)$ is bounded on $B(j)$ (say by M') , and converges to zero as x approaches ∂E . Thus, since

$$\mu(w, (-1, 1) \times \{s(n)\}) = \mu(w, (-1, 1) \times \{s(n-1)\})$$

$$\geq P'^{w}(T_{B(j)} < T_{\partial E}) \quad \text{for} \quad w \in C(n) ,$$

also

$$G(z, w) \leq M'\mu(w, (-1, 1) \times \{s(n)\}) \quad \text{for} \quad w \in C(n) .$$

Combining our two estimates, we see that

$$\frac{h(w)}{G(z,w)} \geq \frac{h(c(1))}{2\ MM'\ a(n-1)m(n,w)\mu(w,(-1,1) \times \{s(n)\})}$$

$$= \frac{h(c(1))}{2MM'} \cdot \frac{1}{a(n-1)k(n)^2} \quad , \text{for} \quad w \in C(n) .$$

Let $i(n)$ be the expression on the right hand side. Then if $a(n) \to 0$ sufficiently fast, also $i(n) \to \infty$.

Let $E(n) = (-1, 1) \times ((s(n-1) + s(n))/2, (s(n) + s(n+1))/2) \cap E$. Then $E(n)$ is regular for the Dirichlet problem, so that if $\nu(x, dy)$ is the hitting distribution of $\partial E(n)$ by Brownian motion started at x , then whenever g is harmonic on $E(n)$, with a continuous extension to its closure, we have that

$$g(x) = \int_{\partial E(n)} g(y)\nu(x, dy) .$$

These conditions hold for both h and $G(z, \cdot)$. Both functions vanish on $\partial E(n) \cap \partial E$, so that we have

$$h \geq i(n)\, G(z, \cdot) \quad \text{on} \quad \partial E(n) .$$

By the integral representation, this inequality holds throughout $E(n)$.

We have therefore shown that (a) and (b) hold, provided that both $m(n, c(n))a(n-1)$ and $k(n)^2 a(n-1)$ converge to zero. Since $\mu(c(n), (-1, 1) \times \{s(n)\}) \to 1/2$, the first of these is irrelevant, and hence the exact statement of the theorem will follow from the following estimate of $k(n)$. Theorem 9 will then be proven. □

<u>Lemma 3</u> $b(n) \log(k(n)) \to 2\Pi$ as $n \to \infty$.

<u>proof</u>: Write b for $b(n)$. Let Φ be the conformal map of the unit disk onto $(-1, 1) \times (0, b)$, mapping the points -1, 0, and 1, to $(-1, b/2)$, $(0, b/2)$, and $(1, b/2)$. Let

$$e^{i\theta} = \Phi^{-1}((1, b)) ; \quad \theta \in (0, \Pi/2) ,$$

and write dy for normalized arc length, on the boundary of the unit disk. The first hitting distribution of this boundary, by Brownian motion started at x is then

$$\frac{1 - |x|^2}{|x - y|^2}\, dy ,$$

so that since Φ preserves hitting distributions, we have that

$$k(n) = \sup \left\{ \frac{1 - |x|^2}{|x - e^{it}|} \Big/ \frac{1 - |x|^2}{|x - e^{is}|} \; ; \; IM(x) = 0 , \; s, t \in [\theta, \Pi - \theta] \right\}$$

By simple calculus, this becomes

$$k(n) = \frac{1 + \cos\theta}{1 - \cos\theta} \quad .$$

We must therefore examine the dependence of θ on b .

$$\frac{b}{2} = \int_{-\theta}^{\theta} \frac{d}{dt} |\Phi(e^{it})| dt \Big/ \int_{\theta}^{\Pi-\theta} \frac{d}{dt} |\Phi(e^{it})| dt \ ,$$

and

$$\Phi(w) = \phi \int_0^w [(z - e^{i\theta})(z + e^{i\theta})(z - e^{-i\theta})(z + e^{-i\theta})]^{-1/2} dz + \phi'$$

for some constants ϕ and ϕ' . Thus

$$\left|\frac{d}{dt} \Phi (e^{it})\right|$$

$$= |\phi \ i \ e^{it}[(e^{it} - e^{i\theta})(e^{it} + e^{i\theta})(e^{it} - e^{-i\theta})(e^{it} + e^{-i\theta})]^{-1/2}|$$

$$= 2|\phi||\cos 2t - \cos 2\theta|^{-1/2} \quad .$$

$$\int_{-\theta}^{\theta} (\cos 2t - \cos 2\theta)^{-1/2} dt = 2 \int_0^{\theta} (\cos 2t - \cos 2\theta)^{-1/2} dt$$

$$= \sqrt{2} \int_{\cos\theta}^{1} \frac{dx}{\sqrt{(1-x^2)(x^2-\cos^2\theta)}}$$

$$= \sqrt{2}\ F(1, \sin^2\theta) \qquad \text{(GR 3.1529)} \ ,$$

and similarly

$$\int_{\theta}^{\Pi-\theta} (\cos 2\theta - \cos 2t)^{-1/2} dt = \sqrt{2} \int_0^{\cos\theta} \frac{dx}{\sqrt{(1-x^2)(\cos^2\theta-x^2)}}$$

$$= \sqrt{2}\ F(1, \cos^2\theta) \qquad \text{(GR 3.1527)}$$

Thus by (13.1), as $\theta \to 0$ we have

$$b \sim \frac{\Pi}{\log(1/\theta)} .$$

Since $k(n) \sim (2/\theta)^2$, we are done. □

The proof of Theorem 10 will be given along the following lines: Because we know little about h (other than its 'symmetry'), we will reduce the problem to that of estimating certain quantities not depending on h . (An essential ingredient in this is Theorem 8, which estimates the expected lifetime of an h-transform, but in a manner independent of h .) These quantities are local, in that they involve potential theoretic properties of certain subsets of E . Schwarz-Christoffel transformations are used to relate potential theoretic questions about these subsets, to questions about the unit disk, and questions about the parameters describing the transformation. The answers to questions of the first type may be easily obtained, whereas the answers to the second type form the main non-probabilistic ingredient of our proof.

Before proceeding with the proof of Theorem 10, we will fix some notation. Let $S(0) = T(0) = T_A$. For $k \geq 0$, define

$$N(k) = n \text{ , if } X_{T(k)} \in A(n)$$

$$T(k+1) = T(k) + T_{A\setminus B(N(k))} \circ \theta_{T(k)}$$

$$S(k+1) = T(k) + T_{A\setminus A(N(k))} \circ \theta_{T(k)} .$$

Let $U(n) = E \cap (-1, 1) \times (s(n-1), s(n+1))$,

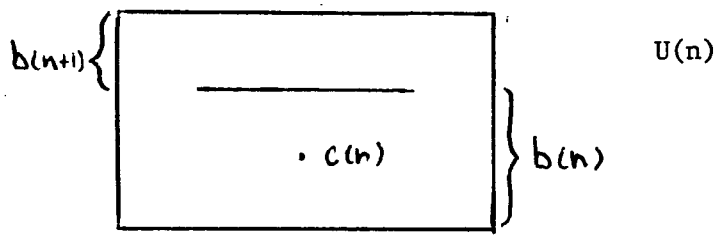

Fig. 5

and write P_n^x for the law of Brownian motion started at x and stopped upon leaving $U(n)$. Define the Martin function $K_n(x, y)$ on $U(n)$, relative to the base point $c(n-1)$, so that K_n is symmetric (that is,

$$K_n(-\bar{x}, -\bar{y}) = K_n(x, y)) .$$

As usual, ${}_hP_n^x$ and ${}_yP_n^x$ will denote the laws of the transformations by the functions

$$h\big|_{U(n)} , \; K_n(y, \cdot)$$

respectively, but we will also introduce the notation ${}_y\bar{P}_n^x$ for the transformation by the function

$$K_n(y, \cdot) + K_n(-\bar{y}, \cdot) .$$

Note that this object would be changed if we had used a base point other than $c(n\ )$. Set

$$\alpha(x, y) = \begin{cases} {}_y\bar{P}_n^x\,(\mathrm{Sgn}\ N(1) = \mathrm{Sgn}\ N(0)\ ,\ S(1) = T(1))\ , \text{ if} \\ \qquad x \in B(n) \text{ and } y \in B(n-1) \cup B(n+1) \\ \qquad \text{for some } n\ . \\ 0 \qquad , \text{ otherwise} . \end{cases}$$

$$\beta(n, m) = \inf\{\alpha(x, y);\ x \in B(n)\ ,\ y \in B(m)\}\ ;\quad n, m \geq 1$$

$$\gamma(x, y) = \begin{cases} {}_y\overline{E}_n^x\,[T(1)] & , \text{ if } x \in B(n) \text{ and } y \in B(n-1) \cup B(n+1) \\ & \quad \text{for some } n \\ 0 & , \text{ otherwise} \end{cases}$$

$$\lambda(n, m) = \inf\{\gamma(x, y) \; ; \; x \in B(n) \; , \; y \in B(m)\} \; ; \; n, m \geq 1 \; .$$

Note that $B(n, m) = \lambda(n, m) = 0$ unless $|m - n| = 1$. The estimates of these quantities that we will need are contained in the following result, whose proof we defer until the end of this section.

<u>Lemma 4</u> Let $a(n-1)/b(n) \to 0$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that for every $n \geq 1$,

(a) $(1 - \beta(n, n+1)) \vee (1 - \beta(n, n-1) \leq \exp(-C_1/b(n))$

(b) $\lambda(n, n+1) \wedge \lambda(n, n-1) \geq C_2 b(n)^2$.

Now, let I be the function that replaces the x-coordinate by its absolute value. That is,

$$I(z) = \begin{cases} z, & \mathrm{Re}(z) \geq 0 \\ -\bar{z}, & \mathrm{Re}(z) \leq 0 \; . \end{cases}$$

Recall that $\mathcal{F}_t = \sigma(X_s; \; s \leq t)$. We let

$$\mathcal{G} = \sigma(I(X_{T(k)}) \; ; \; k \geq 0) \; .$$

<u>Lemma 5</u>

(a) Let $H(x, y)$ be positive and jointly measurable in x and y .

Let T and S be $(\mathcal{F}_{t+})$ stopping times, and set

$$H'(x) = {}_hE^x[H(x, X_T)] .$$

Then

$$E_Q[H(X_S, X_{S+T\circ\theta_S})|\mathcal{F}_{S+}] = H'(X_S) \qquad Q\text{-a.s.} .$$

(b) $I(X_t)$ is strong Markov with respect to Q and $(\mathcal{F}_t)$.

(c) Let $Z \in \mathcal{F}$ be positive, and set

$$H(x, y) = {}_y\overline{E}^x_n[Z] \text{ , for } x \in B(n) \text{ , } y \in B(n-1) \cup B(n+1) .$$

Then ${}_hE^x_n[Z|I(X_{T(1)})] = H(x, I(X_{T(1)}))$ a.s.

(d) $Q(\text{Sgn } N(n) = \text{Sgn } N(n-1) \text{ , } T(n) = S(n)|\mathcal{F}_{T(n-1)+} \vee \mathcal{G})$

$$= \alpha(X_{T(n-1)}, I(X_{T(n)})) \text{ a.s., for every } n \geq 1 .$$

<u>proof</u>: (a): If $H_n \uparrow H$, then $H_n(x, \cdot) \uparrow H(x, \cdot)$, so that $H'_n \uparrow H'$. The class of functions H for which (a) holds is therefore a monotone class, and it contains functions $H(x, y) = H_1(x)\, H_2(y)$ by the strong Markov property.

(b): By symmetry of E and of Brownian motion, we have that

$$P^x(I(X.) \in \Lambda) = P^{I(x)}(I(X.) \in \Lambda)$$

for every $x \in E$ and $\Lambda \in \mathcal{F}$. We have chosen h so that $h(x) = h(I(x))$ for every x . Thus, if $f_1,\ldots,f_n \in \mathcal{E}$ are positive, T is an $(\mathcal{F}_{t+})$ stopping time, and $t_1 <..< t_n$, then

$$E_Q[\Pi f_i(I(X_{T+t_i}))\,|\,F_{T+}]$$
$$= {}_hE^{X_T}[\Pi f_i(I(X_{t_i}))]$$
$$= \frac{1}{h(X_T)}\, E^{X_T}[h(X_{T_n})\Pi f_i(I(X_{t_i}))]$$
$$= \frac{1}{h(I(X_T))}\, E^{I(X_T)}\,[h(I(X_{t_n}))\Pi f_i(I(X_{t_i}))]\ ,$$

so that $I(X_t)$ is strong Markov, with semigroup

$${}_hP_t(x,\ dy) + {}_hP_t(x,\ -\overline{dy})\ .$$

(c): Let the integral representation of h on $U(n)$ be

$$h = \int K_n(y,\ \cdot)\mu(dy)\ .$$

By uniqueness of the representation, and the fact that h and K_n are preserved by the transformation $x \to -\overline{x}$ of $U(n)$, it follows that $\mu(dy) = \mu(-\overline{dy})$. Thus, for $\Lambda \subset A(n+1) \cup A(n-1)$ and $Z \in F$ positive,

$${}_hE_n^x[Z,\ I(X_{T(1)}) \in \Lambda] = {}_hE_n^x[Z,\ X_{T(1)} \in \Lambda \cup -\overline{\Lambda}]$$
$$= \frac{1}{h(x)} \int_{\Lambda \cup -\overline{\Lambda}} K_n(x,\ y)\ {}_yE_n^x[Z]\mu(dy)$$
$$= \frac{1}{h(x)} \int_{\Lambda} (K_n(x,\ y)\ {}_yE_n^x[Z] + K_n(x,\ -\overline{y})\ {}_{-\overline{y}}E_n^x[Z])\mu(dy)$$
$$= \frac{1}{h(x)} \int_{\Lambda} (K_n(x,\ y) + K_n(x,\ -\overline{y}))\ {}_y\overline{E}_n^x[Z]\mu(dy)$$
$$= \frac{1}{h(x)} \int_{\Lambda \cup -\overline{\Lambda}} K_n(x,\ y)\ {}_y\overline{E}_n^x[Z]\mu(dy)$$

(by symmetry of μ)

$$= {}_hE_n^x[H(I(X_{T(1)})) \ , \ I(X_{T(1)}) \in \Lambda] \ ,$$

as required.

(d): First, observe that if $Z \in F_{T(n)}$, then

$$E_Q[Z|F_{T(n-1)+} \vee G] = E_Q[Z|F_{T(n-1)+} \vee \sigma(I(X_{T(n)}))] \ ,$$

by part (b). Now let $f \in E$ be positive, and set $H(x, y) = f(I(y))\ \alpha(x, y)$. Also put $H'(x) = {}_hE^x[H(x, X_{T(1)})]$, and

$$Z = 1_{\{\mathrm{Sgn}\ N(1) = \mathrm{Sgn}\ N(0),\ S(0) = T(0)\}} \ .$$

By (c), we have that

$$H'(x) = {}_hE^x[f(I(X_{T(1)}))\ \alpha(x, I(X_{T(1))}]$$

$$= {}_hE^x[f(I(X_{T(1)}))\ Z] \ .$$

Thus, if also $\Lambda \in F_{T(n-1)+}$, then

$$E_Q[\Lambda, f(I(X_{T(n)})), \mathrm{Sgn}\ N(n) = \mathrm{Sgn}\ N(n-1),\ T(n) = S(n)]$$

$$= E_Q[\Lambda, f(I(X_{T(n)})) \ , \ Z \circ \theta_{T(n-1)}]$$

$$= E_Q[\Lambda, {}_hE^{X_{T(n-1)}}[f(I(X_{T(1)}))Z]]$$

$$= E_Q[\Lambda, H'(X_{T(n-1)})]$$

$$= E_Q[\Lambda, H(X_{T(n-1)}, X_{T(n)})] \qquad \text{(by (a))}$$

$$= E_Q[\Lambda, f(I(X_{T(n)}))\ \alpha(X_{T(n-1)} \ , \ I(X_{T(n)}))]$$

as required. □

<u>proof of Theorem 10</u>:

For $n \geq 1$, and $m = n + 1$ or $n - 1$, let $N(n, m)$ be the number of $k \geq 0$ such that $|N(k)| = n$ and $|N(k+1)| = m$. Let $H(x, y) = \alpha(x, y)$, and apply (a) of Lemma 5. We have that

$$H'(x) = {}_hE^x[\gamma(x, X_{T(1)})] = {}_hE^x[T(1)]$$

(by (f) of Theorem 1), so that

$$\begin{aligned} E_Q[T(k+1) - T(k)] &= E_Q[H'(X_{T(k)})] \\ &= E_Q[\gamma(X_{T(k)}, X_{T(k+1)})] \\ &\geq E_Q[\lambda(|N(k)|, |N(k+1)|)] . \end{aligned}$$

By Theorem 8, we therefore have that

$$\begin{aligned} \infty > E_Q[\zeta] &= \sum_{k=0}^{\infty} E_Q[T(k+1) - T(k)] + E_Q[T(0)] \\ &\geq \sum_{n=1}^{\infty} [\lambda(n, n+1)\, E_Q[N(n, n+1)] + \\ &\qquad + \lambda(n, n-1)\, E_Q[N(n, n-1)]] . \end{aligned}$$

Since $\log(1/(1-x)) \leq 2x$ for $x \geq 0$ small, we have by Lemma 4, that there exists a constant C with

$$\log(1/\beta(n, m) \leq C\, \lambda(n, m) \quad \text{for} \quad n \geq 1 , \; m = n \pm 1 .$$

Thus also

$$\infty > \sum_{n=1}^{\infty} [\log(1/\beta(n, n+1))E_Q[N(n, n+1)] +$$

$$+ \log(1/\beta(n, n-1))E_Q[N(n, n-1)]]$$

$$= E_Q[-\log (\prod_{n=1}^{\infty} \beta(n, n+1)^{N(n, n+1)}\beta(n, n-1)^{N(n, n-1)})] .$$

The integrand is therefore finite almost surely, so that also

$$E_Q[N(0) = 1 , \prod_{k=0}^{\infty} \beta(|N(k)|, |N(k+1)|)]$$

$$= E_Q[N(0) = 1 , \prod_{n=1}^{\infty} \beta(n, n+1)^{N(n, n+1)}\beta(n, n-1)^{N(n, n-1)}]$$

$$> 0 .$$

But by (d) of Lemma 5,

$$E_Q[N(0) = 1 , \prod_{k=0}^{j} \beta(|N(k)|, |N(k+1)|)]$$

$$\leq E_Q[N(0) = 1 , E_Q[\mathrm{Sgn}\ N(1) = \mathrm{Sgn}\ N(0) ,$$

$$S(1) = T(1), E_Q[\ldots$$

$$\ldots E_Q[\mathrm{Sgn}\ N(j) = \mathrm{Sgn}\ N(j-1), S(0) = T(0)|F_{T(j-1)+} \vee G]\ldots$$

$$\ldots]|F_{T(0)+} \vee G]]$$

$$= Q(N(0) = 1, \mathrm{Sgn}\ N(k) = \mathrm{Sgn}\ N(k-1) \text{ and}$$

$$S(k) = T(k), k=1,\ldots j) = Q(T_{A-} > T(j))$$

Letting $j \to \infty$, we see that $Q(T_{A-} = \infty) > 0$, as required. □

Before proving Lemma 4, we will examine a slightly different situation; Consider a Schwartz-Christoffel transformation

$$\Phi(w) = \phi \int_0^w (z-e^{i\theta_0}) \prod_{j=1}^{6} (z-e^{i\theta_j})^{-1/2} \, dz + \phi' ,$$

where ϕ and ϕ' are constants, and $\theta_7 < \theta_6 < \theta_5 < \theta_4 < \theta_3 < \theta_2 < \theta_1 < \theta_0 = \theta_7 + 2\Pi$. Note that the effect of choosing different branches of the square root function, is to change the constant ϕ . Then Φ maps the unit disk O , to a region V with straight sides;

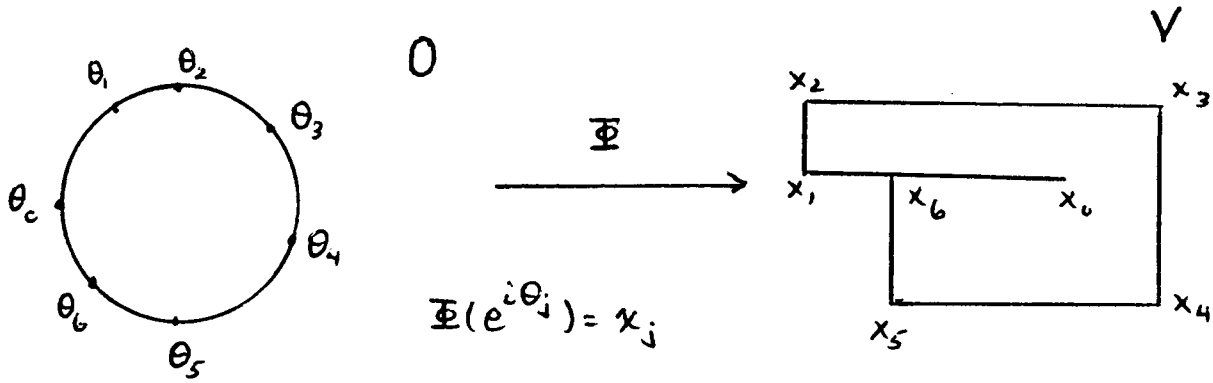

Fig. 6

For every such configuration of points $x_0 \ldots x_6$, there will exist such a map Φ , which is unique up to a Möbius transformation of the unit disk. Since such transformations are uniquely specified by the images of three distinct boundary points, we may always find a Φ with θ_0, θ_2, and θ_5 taking on some fixed values. We will take the image of Φ to be

$$V = [(-1, 1) \times (0, b+b')] \backslash [(-1, 1-a) \times \{b'\}] .$$

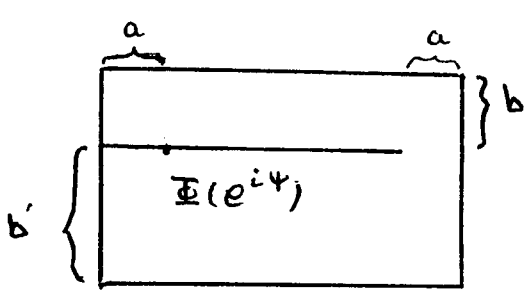

Fig. 7

and will consider ψ and ψ' such that

$$\Phi(e^{i\psi}) = \Phi(e^{i\psi'}) = (a-1,\ b')\ ,\ \psi \in (\theta_1,\ \theta_0),\ \psi' \in (\theta_7,\ \theta_6)\ .$$

We will also let L be the Martin function on V, defined with base point $\Phi(0)$. Thus,

$$L(\Phi(x),\ \Phi(y)) = (1-|x|^2)/|x - y|^2 .$$

<u>Lemma 6</u> Suppose that $b' \to 0$ and $a/b \to 0$ in such a way that $b \leq b'$. Then

(a) $\psi,\ \theta_1,\ \theta_3 \to \theta_2$ and $\psi',\ \theta_4,\ \theta_6 \to \theta_5$

(b) $\liminf\ b.\log\ (1/(\psi-\theta_1)) \geq 2\Pi$

$\liminf\ b'.\log\ (1/(\theta_6-\psi')) \geq 2\Pi$

(c) $\sup\{L(x,\ \Phi(e^{it}))/(L(x,\ \Phi(e^{is})) \wedge 1)\ ;\ x \in (1-a,\ 1) \times \{b'\}$, and

$t,\ s \in [\theta_3,\ \psi] \cup [\psi',\ \theta_4]\}$ remains bounded.

<u>proof</u>: (a): Let the distances from $e^{i\theta_0}$ to $e^{i\theta_2}$ and $e^{i\theta_5}$ be at least 2ε, and let 0_ε be the intersection of the unit disk 0 with a disk of radius ε and centre at $e^{i\theta_0}$. Then there is a constant $C > 0$ (we will use the notation C for any constant we do not wish to keep track of) such that no matter what a, b and b' are, we have that

$$|\Phi'(z)| \geq C|\phi| \qquad \text{for} \quad z \in 0\backslash 0_\varepsilon\ .$$

Now let $g_o : [0,\ r] \to 0$ be the unit speed curve whose image under Φ is $[1-a,\ 1] \times \{b'\}$. Since

$$\mathrm{Arg}\left(\frac{d}{dt}\,\Phi(g_o(t))\right)$$

is constant, we obtain that

$$a = |\Phi(g_o(r)) - \Phi(g_o(0))| = \left|\int_0^r \frac{d}{dt}\,\Phi(g_o(t))dt\right|$$

$$= \int_0^r \left|\frac{d}{dt}\,\Phi(g_o(t))\right|dt$$

$$\geq \varepsilon C|\phi| \ .$$

Let

$$|\phi|\ f(t) = \left|\frac{d}{dt}\,\Phi(e^{it})\right| \ .$$

Then

$$b = 2\int_{\theta_2}^{\theta_1} f(t)dt \ / \int_{\theta_3}^{\theta_2} f(t)dt$$

$$b' = 2\int_{\theta_6}^{\theta_5} f(t)dt \ / \int_{\theta_3}^{\theta_2} f(t)dt$$

$$\int_{\theta_3}^{\theta_2} f(t)dt = \int_{\theta_5}^{\theta_4} f(t)dt$$

$$2 - a = 2\int_{\theta_1}^{\theta_0} f(t)dt \ / \int_{\theta_3}^{\theta_2} f(t)dt \ .$$

Since

$$\frac{a}{b} = \frac{|\Phi(g(r)) - \Phi(g(0))|}{|\phi|\int_{\theta_2}^{\theta_1} f(t)dt} \geq \varepsilon\ C \ / \int_{\theta_2}^{\theta_1} f(t)dt \ ,$$

we must have that

$$\int_{\theta_2}^{\theta_1} f(t)dt \to \infty .$$

Thus, as $b \to 0$ and $2 - a \to 2$, also

$$\int_{\theta_3}^{\theta_2} f(t)dt \quad \text{and} \quad \int_{\theta_1}^{\theta_0} f(t)dt \to \infty .$$

Since

$$e^{it} - e^{i\theta} = 2i\, e^{i(t+\theta)/2} \sin((t-\theta)/2) ,$$

we have that

$$f(t) = \frac{1}{2} \left|\sin\left(\frac{t-\theta_0}{2}\right)\right| \left| \prod_{j=1}^{6} \sin\left(\frac{t-\theta_j}{2}\right)\right|^{-1/2} .$$

Thus, since $\sin((\theta_0-t)/2) \leq \sin((\theta_6-t)/2)$ whenever $t \in [\theta_1, \theta_0]$, we have for each $\eta > 0$, a constant C such that

$$\int_{\theta_1}^{\theta_0} f(t)dt \leq C \int_{\theta_1}^{\theta_0} (t-\theta_1)^{-1/2}\, dt$$

whenever $\theta_1 \in [\theta_2 + \eta, \theta_0]$. Since the right hand side is bounded as a function of θ_1 , we conclude that $\theta_1 \to \theta_2$.

Similarly, for each $\eta > 0$ there is a constant C such that

$$\int_{\theta_2}^{\theta_1} f(t)dt \leq C \int_{\theta_2}^{\theta_1} [(t-\theta_2)(\theta_1-t)]^{-1/2} dt$$

$$= C \int_0^1 [t(1-t)]^{-1/2} dt < \infty$$

whenever $\theta_3 \in [\theta_5, \theta_2 - \eta]$. Thus $\theta_3 \to \theta_2$. We have likewise, that

$\theta_6, \theta_4 \to \theta_5$.

It follows from the above estimates, that for $\eta > 0$ fixed,

$$\int_{\theta_1+\eta}^{\theta_0} f(t)dt$$

remains bounded, so that

$$\int_{\theta_1}^{\theta_{1+\eta}} f(t)dt \Big/ \int_{\theta_3}^{\theta_2} f(t)dt \to 1 \ .$$

Since

$$\int_{\theta_1}^{\psi} f(t)dt \Big/ \int_{\theta_3}^{\theta_2} f(t)dt = a/2 \ ,$$

we see that $\psi \to \theta_2$. Similarly, $\psi' \to \theta_5$.

(b): Let

$$C = \frac{1}{2} \sin\left(\frac{\theta_0-\theta_2}{2}\right) \sin^{-3/2}\left(\frac{\theta_2-\theta_5}{2}\right) \ ,$$

$$k = (\theta_2-\theta_3)/(\theta_1-\theta_3)$$

$$\rho = [(\psi-\theta_1)/(\psi-\theta_2)]^{1/2} \ .$$

By part (a), we have immediately that

$$\int_{\theta_3}^{\theta_2} f(t)dt \sim C \int_{\theta_3}^{\theta_2} [(\theta_1-t)(\theta_2-t)(t-\theta_3)]^{-1/2} \, dt$$

$$= 2C(\theta_1-\theta_3)^{-1/2} \, F(1, k) \qquad \text{(GR 3.1313)}$$

$$\int_{\theta_2}^{\theta_1} f(t)dt \sim C \int_{\theta_2}^{\theta_1} [(\theta_1-t)(t-\theta_2)(t-\theta_3)]^{-1/2} \, dt$$

$$= 2C(\theta_1-\theta_3)^{-1/2}F(1,\ 1-k) \qquad \text{(GR 3.1315)}$$

$$\int_{\theta_1}^{\psi} f(t)dt \sim C\int_{\theta_1}^{\psi} [(t-\theta_1)(t-\theta_2)(t-\theta_3)]^{-1/2}\ dt$$

$$= 2C(\theta_1-\theta_3)^{-1/2}F(\rho,\ k)\ .$$

Thus $b \sim 2F(1,\ 1-k)/F(1,k)$, so that $k \to 1$, and by (13.1), this becomes that $b \sim 2\Pi/\log((\theta_1-\theta_3)/(\theta_1-\theta_2))$. Similarly,

$$F(\rho,\ k)/F(1,\ 1-k) \sim \int_{\theta_1}^{\psi} f(t)dt\ /\ \int_{\theta_2}^{\theta_1} f(t)dt$$

$$= a/b \to 0\ ,$$

so that $\rho \to 0$. Thus also $(\psi-\theta_1)/(\theta_1-\theta_2) \to 0$, and hence

$$b\ \log(1/(\psi-\theta_1))$$

$$\geq b[\log(1/(\psi-\theta_1)) + \log((\psi-\theta_1)/(\theta_1-\theta_2)) +$$

$$+ \log(\theta_1-\theta_3)]$$

$$\to 2\Pi\ .$$

The second half of (b) follows similarly.

(c): Recall the path $t \longrightarrow g_o(t)$ of part (a). We had that

$$\left|\int_0^r \frac{d}{dt}\ \Phi(g_o(t))dt\right| = \int_0^r |\Phi'(g_o(t))|dt\ ,$$

from which it follows that g_o minimizes

$$\int_0^{r'} |\Phi'(g(t))|dt$$

over all $r' > 0$, and all unit speed curves $g : [0, r'] \to 0$ such that

$$g(o) = e^{i\theta_0} \text{ , and } g(r') = e^{it} \text{ for some } t \in (\theta_4, \theta_3) \text{ .}$$

In particular,

$$\int_0^{r} |\Phi'(g_o(t))|dt$$

remains bounded.

Suppose now that

$$\text{(13.2)} \quad \liminf_{\substack{b' \to 0 \\ a/b \to 0}} [\inf\{|g_o(t) - e^{i\theta_2}|;\ t \in [0, r]\}] = 0 \text{ .}$$

There are constants C and η_0 such that whenever $\varepsilon \in (0, \eta_0)$, and b' and a/b are so small that

$$|e^{i\theta_1} - e^{i\theta_2}|,\ |e^{i\theta_3} - e^{i\theta_2}| < \varepsilon \text{ ,}$$

then

$$|\Phi'(z)| \geq C(|z - e^{i\theta_2}| + 2\varepsilon)^{-3/2}$$

for $z \in 0$ with

$$|z - e^{i\theta_2}| \in (\varepsilon, \eta_0) \text{ .}$$

If

$$|g(t) - e^{i\theta_2}| < \varepsilon \text{ for some } t \text{ ,}$$

then since g is a unit speed curve, it follows that

$$\int_0^r 1_{\{z;\ |z - e^{i\theta_2}| \in (\varepsilon, \eta)\}} (g(t))dt \geq \eta - \varepsilon$$

for every $\eta \in (\varepsilon, \eta_0)$.

In particular, under condition (13.2), we have that

$$\limsup_{\substack{b' \to 0 \\ a/b \to 0}} \int_0^r |\Phi'(g_o(t))|dt \geq C(\eta - \varepsilon)(\eta + 2\varepsilon)^{-3/2}$$

for every ε and η with $0 < \varepsilon < \eta < \eta_0$. Let $\varepsilon \to 0$ and then $\eta \to 0$ to obtain that the left hand side is infinite; a contradiction.

Thus the path of g_o stays away from $\exp(i\theta_2)$ as $b' \to 0$ and $a/b \to 0$. By a similar argument, it stays away from $\exp(i\theta_5)$. The conclusion of part (c) now follows immediately from the definitions of L and g_o, and the result that for every ε and η with $0 < \varepsilon < \eta$,

$$\sup \left\{ \frac{1-|x|^2}{|x-e^{it}|^2} \Big/ \left(\frac{1-|x|^2}{|x-e^{is}|^2} \wedge 1\right) ;\ |x-e^{i\theta_2}| \wedge |x-e^{i\theta_5}| \geq \eta ,\right.$$

$$\left. \text{and } t, s \in \{r;\ |e^{ir}-e^{i\theta_2}| \wedge |e^{ir}-e^{i\theta_5}| \leq \varepsilon\} \right\}$$

is finite. □

<u>proof of Lemma 4</u>:

(a) We must show that there is a constant $C_1 > 0$ such that

$$_y\overline{P}^x_n(X_{S(1)} \in A-) \leq \exp(-C_1/b(n))$$

for $x \in A(n)$ and $y \in B(n-1) \cup B(n+1)$, $n \geq 1$. We will obtain this

in two parts; first exhibiting such a bound for

$$I_1(x, y) = {}_y\bar{P}_n^x(X_{S(1)} \in A(-n)) ,$$

and then for

$$I_2(x, y) = {}_y\bar{P}_n^x(X_{S(1)} \in A(-(n-1)) \cup A(-(n+1))) .$$

Let $V(n) = [(-1, 1) \times (s(n-1), s(n+1))] \setminus [(-1, 1-a(n)) \times \{s(n)\}]$. Write $\Delta(n)$ for the Martin boundary of $V(n)$, and write $\hat{P}_n^x$ for the law of Brownian motion started at x and killed upon leaving $V(n)$. Resume the notation of the last lemma, so that we have a canonical conformal equivalence Φ from the unit disk to $V(n)$, and we define the Martin function L_n on $V(n)$, with respect to the base point $\Phi(0)$. Also note that points of $A(-n) \subset \partial V(n)$ split into two points of the Martin boundary $\Delta(n)$ of $V(n)$; one being the limit of points above $A(-n)$, and the other from below. We let $A'(-n)$ be the collection of all points of the Martin boundary of $V(n)$, that are associated to a point of $A(-n)$ in this way.

Let

$$\mu(\Lambda) = \hat{P}_n^{\Phi(0)}(X_{\zeta-} \in \Lambda) \text{ , for } \Lambda \subset \Delta(n) \text{ measurable.}$$

Also, let C be the bound given by (c) of Lemma 6. Fix a point $y \in B(n-1) \cup B(n+1)$, and let

$$e(z) = K_n(y, z) + K_n(-\bar{y}, z) .$$

Let ν be the measure representing e on $\Delta(n)$, so that

$$e = \int_{\Delta(n)} L_n(w, \cdot)\nu(dw) .$$

By (f) of Theorem 2, and Theorem 4, we have that $\nu(dw)$ is the sum of the measure $e(w)\mu(dw \cap A'(-n))$, and of two point masses, one at y and the other at $-\bar{y}$. Thus, by (j) of Theorem 1,

$$I_1(x, y) = {}_e P_n^x(X_{S(1)} \in A(-n))$$

$$= {}_e \hat{P}_n^x(X_{\zeta-} \in A'(-n))$$

$$= \frac{1}{e(x)} \int_{A'(-n)} L_n(w, x) \; {}_w \hat{P}_n^x(X_{\zeta-} \in A'(-n))\nu(dw)$$

$$= \int_{A'(\bar{n}n)} \frac{L_n(w, x)}{e(x)} e(w)\mu(dw) \; .$$

Since

$$e(\Phi(e^{it})) = 0 \quad \text{unless} \quad t \in [\theta_3, \psi] \cup [\psi', \theta_4] \; ,$$

we have from (c) of Lemma 6, that

$$\frac{e(x')}{L_n(z,x')} = \int \frac{L_n(w, x')}{L_n(z, x')} e(w)\mu(dw)$$

$$\leq C^2 \int \frac{L_n(w, x)}{L_n(z, x)} e(w)\mu(dw)$$

$$= C^2 \frac{e(x)}{L_n(z, x)} \quad , \text{ for } x, x' \in A(n) \; , \; z \in A'(-n) \; .$$

Since $e(x') = e(-\bar{x}')$, we can choose $x' \in A(n)$ to maximize e over $A(n)$, and still have

$$e(w)/e(x') \leq 1 \quad \text{for} \quad w \in A'(-n) \; .$$

Thus

$$I_1(x, y) \leq C^2 \int_{A'(-n)} \frac{L_n(w, x')}{e(x')} e(w)\mu(dw)$$

$$\leq C^2 \mu(A'(-n)) \sup\{L_n(w, z);\ w \in A'(-n),\ z \in A(n)\}$$

$$\leq C^3[(\psi-\theta_1) + (\theta_5-\psi')]/2\Pi$$

We thus obtain a bound on $I_1(x, y)$ of the correct form, by (b) of Lemma 6.

Now consider $I_2(x, y)$. Suppose for now, that $y \in B(n-1)$, so that we may assume without logs of generality, that $y \in A(-\pi(n-1))$. Recall that $W(n) = (-1, 1) \times (s(n-1), s(n))$. Let $\tilde{P}_n^x$ be the law of Brownian motion started at x and killed upon leaving $W(n)$, and define the Martin function $\tilde{L}_n$ on $W(n)$ using the base point $c(n)$. Consider

$$Y_t = X_{t+L_{B(n)}} .$$

Then (Y_t) is a v-transform on $W(n)$, under ${}_eP_n^x$, where v is a harmonic function on $W(n)$, given by (m) of Theorem 1. Because e is symmetric (that is, $e(z) = e(-\bar{z})$ for $z \in W(n)$) , the same will be true of v . By Theorem 4 and our choice of base point, we therefore have that v is a multiple of

$$\tilde{L}_n(y, \cdot) + \tilde{L}_n(-\bar{y}, \cdot) .$$

Thus

$$I_2(x, y) \le {}_e P_n^x(Y_0 \in A(n),\ Y_{\zeta-} \in A(-(n-1)))$$

$$\le \sup_{z \in A(n)} {}_v\tilde{P}_n^z(X_{\zeta-} = y)$$

$$= \sup_{z \in A(n)} \frac{\tilde{L}_n(y, z)}{\tilde{L}_n(y, z) + \tilde{L}_n(-\bar{y}, z)}$$

To bound this, we basically repeat the proof of Lemma 6, but for a simpler transformation.

Let Φ be the conformal map from the unit disk to $W(n)$, mapping -1, 0, and 1 to $(-1, (s(n) + s(n-1))/2)$, $c(n)$, and $(1, (s(n) + s(n-1))/2)$. Then

$$\tilde{L}_n(\Phi(x), \Phi(z)) = (1 - |z|^2)/|x - z|^2 .$$

Let

$$e^{i\theta} = \Phi^{-1}((1, s(n))) , \quad e^{i\psi_1} = \Phi^{-1}((1-a(n), s(n))) ,$$

$$e^{i\psi_2} = \Phi^{-1}((1-a(n-1), s(n-1))) ,$$

where $-\Pi/2 < \psi_2 < -\theta \le \theta < \psi_1 < \Pi/2$. Thus

$$(13.3) \quad I_2(x, y) \le \sup_{\substack{s \in [\psi_2, -\theta] \\ t \in [\theta, \psi_1]}} \frac{|e^{is} - e^{it}|^2}{|e^{is} - e^{it}|^2 + |e^{i(\Pi - s)} - e^{it}|^2} .$$

We have that

$$\Phi(w) = \phi \int_0^w [(z-e^{i\theta})(z+e^{i\theta})(z-e^{-i\theta})(z+e^{-i\theta})]^{-1/2} dz + \phi'$$

for some ϕ and ϕ'. Let

$$2|\phi|f(t) = \left| \frac{d}{dt} \Phi(e^{it}) \right| ,$$

so that

$$b(n-1) = 2 \int_{-\theta}^{\theta} f(t)dt \Big/ \int_{\theta}^{\Pi-\theta} f(t)dt$$

$$a(n) = 2 \int_{\theta}^{\psi_1} f(t)dt \Big/ \int_{\theta}^{\Pi-\theta} f(t)dt$$

$$a(n-1) = 2 \int_{\psi_2}^{-\theta} f(t)dt \Big/ \int_{\theta}^{\Pi-\theta} f(t)dt .$$

Let

$$\rho_j = \frac{1}{\cos\theta} \quad \frac{\cos^2\theta - \cos^2\psi_j}{1 - \cos^2\psi_j} \quad , \quad j = 1, 2 \quad .$$

As in Lemma 3, we have that

$$\int_{-\theta}^{\theta} f(t)dt = \sqrt{2}\ F(1, \sin^2\theta)$$

$$\int_{\theta}^{\Pi-\theta} f(t)dt = \sqrt{2}\ F(1, \cos^2\theta) ,$$

and similarly,

$$\int_{\theta}^{\psi} f(t)dt = \sqrt{2} \int_{\cos\psi_1}^{\cos\theta} \frac{dx}{\sqrt{(1-x^2)(\cos^2\theta-x^2)}}$$

$$= \sqrt{2}\ F(\rho_1, \cos^2\theta) \ , \text{ and}$$

$$\int_{-\theta}^{\psi_2} f(t)dt = \sqrt{2}\ F(\rho_2, \cos^2\theta) \ .$$

Since $b(n-1) \to 0$, it follows that $\theta \to 0$ and

$$b(n-1) \sim \Pi/\log(1/\theta) \ .$$

Since $a(n-1)/b(n-1)$ and $a(n)/b(n-1) \to 0$, it follows in turn that $\rho_1, \rho_2 \to 0$, so that also $\psi_1, \psi_2 \to 0$. Thus

$$\rho_j \sim \left[\frac{(1 - \theta^2/2) - (1 - \psi_j^2/2)}{\psi_j^2/2}\right]^{1/2} = (1 - (\theta/\psi_j)^2)^{1/2} \ ,$$

and hence

$$b(n-1)\log(1/\psi_j) \sim b(n-1)(\log(1/\psi_j) + \log(\psi_j/\theta)) \sim \Pi \ .$$

By (13.3), there is a constant C with

$$I_2(x, y) \leq C(\psi_1-\psi_2)^2 \leq 2\ C(\psi_1^2+\psi_2^2) \ ,$$

and we obtain a bound of the desired form. The case $y \in B(n+1)$ is handled in the same manner, showing (a).

(b) By symmetry,

$$\lambda(n, n+1) = \inf\{{}_y E_n^x[T(1)] \ ; \ x \in B(n), \ y \in A(n+1)\} \ .$$

Consider the square $Z(n) = (1-b(n+1), 1) \times (s(n), s(n+1))$.

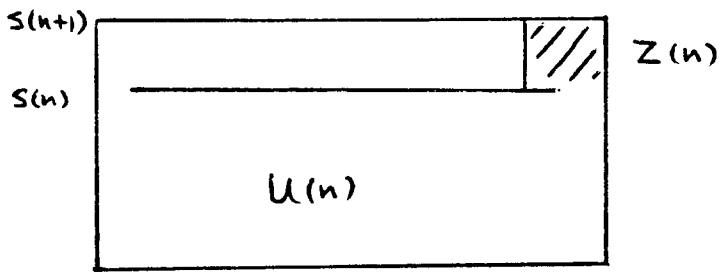

Fig. 8

Let

$$Y_t = X_{t+L_{U(n)\setminus Z(n)}} \ .$$

Under ${}_yP_n^x$, (Y_t) will be a v-transform for some v on $Z(n)$ (by (m) of Theorem 1), $Y_{\zeta-} = y$, and Y_0 lies on the lower or left sides of $Z(n)$. Let $Z = (0,1) \times (0,1)$, and define $\Phi : Z(n) \to Z$ by

$$\Phi(z) = [z - (1-b(n+1), s(n))]/b(n+1) .$$

Let

$$Y'_t = \Phi(Y_{tb(n+1)^2}) \in Z .$$

Because this scaling preserves Brownian motion, (Y'_t) will be a transform by $v \circ \Phi^{-1}$, under ${}_yP_n^x$. Let P_Z^z be the law of Brownian motion started at z and killed upon leaving Z . Thus

$$\lambda(n, n+1) \geq \inf\{{}_yE_n^x[\text{lifetime of } (Y_t)] \; ; \; y \in A(n+1), \; x \in B(n)\}$$

$$\geq b(n+1)^2 \inf\{{}_zE_Z^w[\zeta]; \; z \in (1 - \frac{a(n)}{b(n+1)}, 1) \times \{1\} ,$$

$$w \in [0, 1] \times \{0\} \cup \{0\} \times [0, 1]\} .$$

Since $a(n)/b(n+1) \to 0$, this infimum remains bounded away from 0 as $n \to \infty$. The corresponding estimate on $\lambda(n, n-1)$ follows similarly, showing (b). □

14. Smallness of $D'\backslash D$

The results of the last section show that even for planar Brownian motion, in general $D \neq D'$. The discussion of Section 11 suggests that in order to construct extensions of the process consisting

of Brownian motion on R^d , killed upon leaving a domain E , we should suppose that D is pretty well all of D' . We will in fact use the existence of such extensions (for example, unkilled Brownian motion on R^d) , to show that $D'\backslash D$ is small.

Let E be a bounded domain in R^d , and let U be a large open ball containing the closure of E . We retain the notation $(\hat{\Omega}, F)$ for paths with values in $E \cup \{\delta\}$, and write $(V, \mathcal{V})$ for the corresponding object, where now paths take values in U . The coordinate process on $\hat{\Omega}$ will still be (X_t) . We let the coordinate process on V be (W_t) . Write $G(x, y)$ for the Green function of U , and write P^x and Q^x for the laws of Brownian motion, started at x and killed upon leaving U in the first case, and E in the second. Let μ and ν be probabilities on U , such that a set is μ-null if and only if it is ν-null. Assume that neither one charges some neighbourhood of the closure of E . Write

$$h = \int G(z, \cdot)\nu(dz) ,$$

and $P = {}_hP^\mu$. Let $\tilde{P}$ be the law under P of the reverse of the coordinate process (X_t) from its lifetime. Then we have that

$$P(\Lambda) = \int\int \frac{G(x, y)}{h(x)} \; {}_yP^x(\Lambda)\mu(dx)\nu(dy)$$

$$\tilde{P}(\Lambda) = \int\int \frac{G(x, y)}{h(y)} \; {}_yP^x(\Lambda)\nu(dx)\mu(dy)$$

for $\Lambda \in \mathcal{V}$. Thus, since μ and ν share the same null sets, the same will be true for P and $\tilde{P}$.

Let

$$M = \{t > 0;\ W_t \notin E \cup \{\delta\}\ ,$$

and let M_0 be the set of points of M which are isolated on the right. Let

$$Y_t(s) = \begin{cases} W_{t+s} & , \quad s < T_{U\setminus E} \circ \theta_t \\ \delta & , \quad s \geq T_{U\setminus E} \circ \theta_t \end{cases} ,$$

and let

$$\rho(A) = E[\sum_{t\in M_0} 1_A(Y_t)] \quad , \quad \text{for} \quad A \in (\hat{\Omega}, F) .$$

Since every excessive function for Brownian motion is regular (see Blumenthal and Getoor [5]), the hypotheses of Maisonneuve [38], Proposition (9.2) are met. Thus, there exists a kernel $n(x, du)$, and an additive functional (L_t) such that

$$\rho(A) = E[\int_0^\zeta n(w_t, A)dL_t]$$

for every measurable subset A of $\hat{\Omega}$, and (X_t) is an h-transform under each $n(x, \cdot)$. Thus (X_t) is also an h-transform under ρ . It follows from Theorem 5 and a time reversal argument, that

$$Z = \lim_{t\uparrow\zeta} X_t \ , \text{ and } \ Z' = \lim_{t\downarrow 0} X_t$$

exist in the topology of the Martin compactification of E , ρ -a.s. . Because ν does not charge E , the limit actually lies in $Acc \subset \Delta$. Let η be the measure on $\Delta \times \Delta$, defined by

$$\eta(dx, dy) = \rho(Z \in dx, Z' \in dy) .$$

<u>Theorem 11</u> $\eta(D'\backslash D) = 0$.

Thus, with probability one, every excursion into E starts and finishes at points z, z' of the Martin boundary, for which $(z, z') \in D$.

<u>Lemma 7</u> η is σ-finite.

<u>proof</u>: By minimality, we have that for every $y \in \Delta_0$ and $x \in E$, either there exists a point $e(y) \in \partial E$ such that

$$_yQ^x(X_t \to e(y) \text{ in the Euclidean metric, as } t \uparrow \zeta) = 1 \text{ , or}$$

$$_yQ^x(X_t \text{ converges in the Euclidean metric, as } t \uparrow \zeta) = 0 \text{ .}$$

In the latter case, we write $e(y) = \delta$.

Since the Euclidean limits of X_t , as $t \uparrow \zeta$ and $t \downarrow 0$, both exist ρ -a.s. , it follows that

$$\rho(e(Z) = \delta \text{ or } e(Z') = \delta) = 0 \text{ .}$$

Since Brownian motion does not return to points, we have by the strong Markov property, that also

$$\rho(e(Z) = e(Z')) = 0 \text{ .}$$

Thus, it will suffice to show that

$$\rho(|e(Z) - e(Z')| > 2\varepsilon) < \infty \text{ for every } \varepsilon > 0 \text{ .}$$

Let $T(0) = 0$, $T(k+1) \doteq \inf\{t > T(k);\ |X_t - X_{T(k)}| > \varepsilon\}$.

Because h is bounded and bounded away from zero, on a neighbourhood of the closure of E , we obtain from (f) of Theorem 1, a constant $m > 0$

such that

$$ {}_hE^x[T(1)] \geq m $$

whenever x lies in E. Thus

$$ \infty > E[\zeta] \geq m\, E[\sum_{k=0}^{\infty} 1_E\,(X_{T(k)})] $$

$$ \geq m\, E[\sum_{t\in M_0} 1_{\{|X_t - X_{T_{\partial E}\circ\theta_t}| > 2\,\varepsilon\}}] $$

$$ = m\, \rho(|e(Z) - e(Z')| > 2\,\varepsilon) \ , $$

as required. □ .

<u>proof of Theorem 11</u>: Since η is σ-finite, we may use the 'classical argument' of the last section of Maisonneuve [38], to obtain a kernel $\rho(x, y; du)$ such that

$$ \rho = \int \rho(x,y;\ \cdot)\eta(dx,\ dy) \ . $$

Now, let K be the Martin function on E, and suppose that $A \in F$. It is easily checked (by (f) of Theorem 1), that

$$ {}_hE_Q^x[A|Z] = a(x,\ Z) \ , $$

where $a(x, y) = {}_yE_Q^x[A]$. Suppose in addition that f_0 and f_∞ are positive and measurable on Δ, and that $B \in F_s$. Then

$$ E_\rho[f_0(Z')f_\infty(Z),\ A\circ\theta_s,\ B] $$

$$ = E_\rho[f_0(Z')\ {}_hE_Q^{X_s}[A,\ f_\infty(Z)],\ B] $$

$$ = E_\rho[f_0(Z')\ f_\infty(Z)\ a(X_s,\ Z'),\ B] \ . $$

from which we see that (X_t) is a $K(y,\ \cdot)$ -transform under $\rho(x,y;\ \cdot)$

for η -a.e. (x, y) . Thus $(x, y) \in D$ for η -a.e. (x, y) , as required. □

Now fix $y_0 \in E$, and define <u>harmonic measure</u> λ on Δ by

$$\lambda(dz) = Q^{y_0}(Z \in dz) .$$

By (h) of Theorem 1, and (f) of Theorem 2, we have that for every $y \in E$, the measure

$${}_hQ^y(Z \in dz)$$

shares the same null sets as λ .

<u>Lemma 8</u> $\lambda \otimes \lambda$ is absolutely continuous with respect to η .

Thus we obtain a more potential theoretic version of Theorem 1;

<u>Corollary 2</u> $\lambda \otimes \lambda\ (D' \backslash D) = 0$.

<u>proof of Lemma 8</u>: Let A be a compact subset of E , with nonempty interior, and set

$$\eta_k(dx, dy) = \rho(Z' \in dx,\ Z \in dy\ ,\ T_A < \infty) .$$

We will actually show that $\lambda \otimes \lambda$ is absolutely continuous with respect to η_k .

We can write

$$\eta_k(\Lambda) = \int_{\{T_A<\infty\}} \int_{\Delta} 1_{\Lambda}(Z', y)\ {}_hQ^{X_{T_A}}(Z \in dy)d\rho .$$

If $\rho(Z' \in B) = 0$, then

$$\tilde{P}(Z \in B) = P(Z' \in B) = 0 ,$$

so that since P is absolutely continuous with respect to $\tilde{P}$, also

$$\eta(\Delta\times B) = P(Z \in B) = 0$$

(this is the reason for our choice of h) . Thus $\eta_k(\Delta\times B) = 0$, and hence

$$ {}_hQ^{X_{T_A}}(Z \in B) = 0 , \quad \rho\text{-a.s.} .$$

Because A has nonempty interior, we have $\rho(T_A < \infty) > 0$, so that by our above remark on the null sets of λ , also $\lambda(B) = 0$. That is, $\lambda(dz)$ is absolutely continuous with respect to

$$\rho(Z \in dz) .$$

If now $\eta_k(\Lambda) = 0$, then

$$\int 1_\Lambda(Z', y) \; {}_hQ^{X_{T_A}}(Z \in dy) = 0 \quad \rho\text{-a.s.} .$$

Thus

$$\int 1_\Lambda(Z', y) \; \lambda(dy) = 0 \quad \rho\text{-a.s.} ,$$

and so also

$$\int 1_\Lambda(x, y)\lambda(dy) = 0 \quad \text{for} \quad \lambda\text{-a.e.} \quad x .$$

This gives immediately that $\lambda \otimes \lambda(\Lambda) = 0$. $\square$

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