# AN INTRODUCTION TO PYTHAGOREAN ARITHMETIC 

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#### Abstract

This thesis provides a look at some aspects of Pythagorean Arithmetic. The topic is introduced by looking at the historical context in which the Pythagoreans flourished, that is at the arithmetic known to the ancient Egyptians and Babylonians. The view of mathematics that the Pythagoreans held is introduced via a look at the extraordinary life of Pythagoras and a description of the mystical mathematical doctrine that he taught. His disciples, the Pythagoreans, and their school and history are briefly mentioned. Also, the lives and works of some of the authors of the main sources for Pythagorean arithmetic and thought, namely Euclid and the Neo-Pythagoreans Nicomachus of Gerasa, Theon of Smyrna, and Proclus of Lycia, are looked at in more detail. Finally, an overview of the content of the arithmetic of the Pythagoreans is given, with particular attention paid to their relationship to incommensurable or irrational numbers.

With this overview in hand, the topics of Perfect and Friendly Numbers, Figurate Numbers, Relative Numbers (the Pythagorean view of ratios and fractions), and Side and Diagonal numbers are explored in more detail. In particular, a selection of the works of Nicomachus, Theon, and Proclus that deal with these topics are analyzed carefully, and their content is reformulated and commented upon using clearer and more modern terminology.


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## Foreword

This foreword is intended to clarify the organization of the thesis so that the reader will not be confused'by the layout, or think it disjointed and incoherent. The main chapters are really analyses of the closest English (or French) translations of the original Greek texts in question, along with some comments. I have attempted to read the texts carefully and then express to the reader, using modern language and notation, my understanding of what the original author was trying to say in a (hopefully!) clearer fashion than he was able.

The Introduction is extensive because in it I have tried to give: (A) an overview of the contributions of Pythagorean arithmetic to mathematics, and (B) a short look at the historical context within which the Pythagoreans made their discoveries - that is, where their ideas came from, and what they did with them. At various points in the Introduction, I have mentioned where the works that are looked at more closely in the main body fit into the grander scheme of things. I hope this will give the reader a better frame of reference from which to understand the works that are looked at, and the mathematics that they contain.

## Introduction

### 0.1 The Origins of Pythagorean Arithmetic

### 0.1.1 The Life of Pythagoras

In any work on Pythagorean Arithmetic, it would seem inappropriate not to begin by looking at the life and works of the namesake of this area of knowledge, Pythagoras himself.

Not much is known about him, but some details of his life are given to us by Iamblichus in his work On the Pythagorean Life ${ }^{1}$. He was born Pythagoras son of Mnesarchos in the city of Sidon in Phoenicia (on the shores of the Mediterranean Sea in what would now be called Lebanon) around 550 B.C., but grew up on the isle of Samos. When he was around eighteen years old, having earned a reputation for being exceptionally intelligent and beautiful, he is said to have left Samos to escape the growing tyranny of its ruler Polykrates. He traveled to see and study with the great sages of the time that resided nearby; among them Pherekydes of Syros and Anaximander who discovered the inclination of the ecliptic, but most notably Thales at Miletus. He is said to have impressed each of them with his natural abilities; so much so, in fact, that Thales urged him to travel to Egypt to continue his studies. Indeed, according to Thales, he would have taught Pythagoras himself, but claimed he couldn't due to his old age and weakness.

So Pythagoras sailed to Egypt by way of Sidon where he stopped to visit his birthplace and to study with the philosophers and prophets there. In particular, Iamblichus tells us that, "... he met the descendants of Mochos the natural philosopher and prophet, and the other Phoenician hierophants, and was initiated into all the rites peculiar to Byblos, Tyre and'other districts of Syria. He did not, as one might unthinkingly suppose, undergo this experience from superstition, but far more from a passionate desire for knowledge, and as a precaution lest something worth learning should elude him by being kept secret in the mysteries or rituals of

[^0]the gods. ${ }^{2}{ }^{2}$ This may, perhaps, give a clue as to the mind set and thinking of Pythagoras; an analytic, detached, unbiased, open-minded, and even scientific view of knowledge.

After arriving in Egypt, he continued his studies of every kind of wisdom, spiritual and scientific (including astronomy and geometry), for twenty-two years until he was, "... captured by the expedition of Kambyses and taken to Babylon. There he spent time with the Magi, to their mutual rejoicing, learning what was holy among them, acquiring perfected knowledge of the worship of the gods and reaching the heights of their mathematics and music and other disciplines." ${ }^{3}$. He is purported to have spent twelve years among the Magi, returning to Samos at the age of 56 .

Once home, he began attempting to teach his ideas, but there was little interest in them among the people of Samos (although according to Iamblichus they did attract students from elsewhere; "all Greece admired him and all the best people, those most devoted to wisdom, came to Samos on his account, wanting to share in the education he gave." ${ }^{4}$ ). So, on account of this, (and since, "His fellow Samians dragged him into every embassy and made him share in all their civic duties." ${ }^{5}$ which left him little time for philosophy), he left for the city of Croton in Italy where he established a school and began teaching what became known as the Pythagorean doctrine.

### 0.1.2 The Pythagorean Doctrine

From the description of his life and travels, it is fairly clear that the teachings of Pythagoras covered much more than just mathematics. In fact, according to B.L. Van der Waerden, "Pythagoras himself was looked upon by his contemporaries in the very first place as a religious prophet." ${ }^{6}$. He goes on to mention that, "Pythagoras was also known as a performer of miracles. All kinds of wonderful tales concerning him were in circulation, as, e.g., that the calf

[^1]of one of his legs was of gold, and that he was seen at two places at the same time. When he crossed a small stream, the river rose out of its bed, greeted him and said: 'Hail, Pythagoras." ${ }^{7}$.

The school that he founded, along with its associated brotherhood the Order of Pythagoreans, was a mystical one, with its goal being the elevation of the soul towards the divine. At this point, one may ask, 'what connection do the mystic Pythagoras and the brotherhood of Pythagoreans have with mathematics?' One answer is as follows:

The Pythagoreans thus have purification and initiation in common with several other mystery-rites. Ascetic, monastic living, vegetarianism and common ownership of goods occur also in other sects. But, what distinguishes the Pythagoreans from all others, is the road along which they believe the elevation of the soul and the union with God to take place, namely by means of mathematics. Mathematics formed a part of their religion. Their doctrine proclaims that God has ordered the universe by means of numbers. God is unity, the world is plurality and it consists of contrasting elements. It is harmony which restores unity to the contrasting parts and which moulds them into a cosmos. Harmony is divine, it consists of numerical ratios. Whosoever acquires full understanding of this number-harmony, he becomes himself divine and immortal. ${ }^{8}$

Thus it is not just the physical universe that is ordered by numbers. All aspects of life are so modeled. Aristotle expresses this fact by saying that,
... the so-called Pythagoreans, who were the first to take up mathematics, ... thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being - more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity - and similarly almost all other things being numerically expressive); since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. ${ }^{9}$

In this way, numbers were lifted from the status of being mere descriptions of the environment to being regarded as abstract things in of themselves; ideal forms on which reality is modeled.

As Plato puts it:

[^2]And do you not know also that although they [the Pythagoreans] make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; ... they are really seeking to behold the things themselves, which can only be seen with the eye of the mind" ${ }^{10}$.

To put it simply, the core of the Pythagorean doctrine was that, "Not only do all things possess numbers; but, in addition, all things are numbers" ${ }^{11}$, and all in this case means the entire physical, mental, emotional and spiritual universe.

The importance of this in mathematical terms is that for the first time, mathematics was studied as being something in of itself not linked to its more concrete usages; to wit, "... Pythagoras freed mathematics from these practical applications" ${ }^{12}$. Thus Pythagoras himself is credited with beginning the study of the science of mathematics as we know it today. The use of mathematics as an ideal representation of the real world, a model with all unnecessary information pared away, has enabled science to solve many problems that would otherwise have been deemed too difficult.

As was mentioned above, more than the numbers themselves being the essence of things, the relationships between the numbers, their ratios or harmonies, and the laws that govern these relationships, were the manifestation of God for the Pythagoreans. As one modern author puts it, " ... for the Pythagoreans, mathematics was more than a science: God manifests in the mathematical laws which govern everything, and the understanding of these laws, and even simply doing mathematics, could bring one closer to God." ${ }^{13}$. The fact that when a string or a flute is shortened to half its length it produces a tone that is an octave higher than it was originally, and similarly shortening or lengthening by other numerical ratios produces corresponding harmonic intervals (for instance, lengthening by a ratio of $3: 2$ produces an increase in tone of a fifth, and a $4: 3$ increase produces a fourth) was an important confirmation of their views ${ }^{14}$.

[^3]Thus, in pursuit of the understanding of the divine laws of the universe, the Pythagoreans taught four "mathemata" (i.e. subjects of study) ${ }^{15}$ to enhance their students' understanding of numbers and their harmonies. These are: Arithmetic, Geometry, Music, and Astronomy. Here Arithmetic is really the Theory of Numbers, being derived from the Greek word Arithmos meaning quantity, or whole number ${ }^{16}$. However, among these four (known also as the quadrivium of subjects), arithmetic was considered more fundamental than the rest. As Nicomachus puts it:

Which then of these four methods must we first learn? Evidently, the one which naturally exists before them all, is superior and takes the place of origin and root and, as it were, of mother to the others. And this is arithmetic, not solely because we said that it existed before all the others in the mind of the creating God like some universal and exemplary plan, relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends; but also because it is naturally prior in birth, inasmuch as it abolishes other sciences with itself, but is not abolished together with them. For example, "animal" is naturally antecedent to "man," for abolish "animal" and "man" is abolished; but if "man" be abolished, it no longer follows that "animal" is abolished at the same time. ${ }^{17}$

So the disciples of the Pythagorean school studied arithmetic, and in the process of becoming closer to God made great advances in the Theory of Numbers.

### 0.1.3 Egyptian and Babylonian Mathematics

The Pythagoreans did not simply create the arithmetic they studied out of nothing (as some authors claim). Rather, the foundations of their arithmetic and mathematics came from Egypt and Babylon. Moreover, Pythagoras' reputed travels to Egypt and Babylon, and the time he spent there studying all forms of wisdom certainly fits the conjecture that he learned what they knew of numbers, and brought it back with him to Greece.

In any case, to put Pythagorean mathematics in some sort of historical context and to understand the contributions they made more fully, we must survey what mathematics, and more

[^4]importantly for the present purposes what arithmetic, was known to the Egyptians and Babylonians.

Most of what is known about Egyptian mathematics ${ }^{18}$ comes from the various papyri that have been discovered, the most well known of these being the Rhind papyrus which dates back to somewhere between 1800 and 2000 B.C.. From these papyri, it is clear that the Egyptians knew how to multiply and divide whole numbers. They also had the concept of 'natural fractions', that is fractions of the form $1 / n$, and knew how to add these together. Using this concept, they were able to perform divisions of one number by another, obtaining an answer in the form of a whole number quotient plus a remainder expressed as a sum of natural or unit fractions. They also had a concept of how to solve what we would call in modern terminology a linear equation in one unknown. Knowing this, B. L. Van der Waerden concludes that, "It is certain that from the Egyptians, the Greeks learned their multiplication and their computations with unit-fractions, which they then developed further;" ${ }^{19}$.

It seems, however, that the Egyptians were interested in mathematics only for its use in solving applied, real world, problems; for example, how many bricks would be needed to build a structure such as a ramp. They left no record of any attempts to explore mathematics for its own sake, such as proofs of their propositions. They left "... only rules for calculation without any motivation." ${ }^{20}$. In Van der Waerden's opinion, Egyptian mathematics, "... can not serve as a basis for higher algebra," ${ }^{21}$ and thus Egypt cannot serve as the only place of origin of Greek mathematics.

Babylonian mathematics, on the other hand, was much more advanced and theoretical. The cuneiform texts of the Babylonians (which were only translated in the early part of this century by O. Neugebauer) tell us that they knew how to solve linear, and also some types of quadratic and even cubic equations in one unknown, as well as various systems of equations in two

[^5]unknowns. They also knew how to find the sum of certain arithmetical progressions, and knew such formulas as:
\[

$$
\begin{align*}
& a^{2}-b^{2}=(a+b)(a-b)  \tag{0.1}\\
& (a+b)^{2}=a^{2}+2 a b+b^{2}  \tag{0.2}\\
& (a-b)^{2}=a^{2}-2 a b+b^{2} \tag{0.3}
\end{align*}
$$
\]

Van der Waerden speculates ${ }^{22}$ that they may have derived such formulas using diagrams like:

for the proof of $0.1^{23}$. Moreover, he claims that although the proof is a geometric one, "... we must guard against being led astray by the geometric terminology. The thought processes of the Babylonians were chiefly algebraic. It is true that they illustrated unknown numbers by means of lines and areas, but they always remained numbers." ${ }^{24}$ From this, we see how the abstract, proof oriented mathematics of the Pythagoreans may have had much of its basis and origin in Babylon. The fact that the Babylonians knew and used the so-called "Theorem of Pythagoras" again attests to this ${ }^{25}$. In fact, the translator of the cuneiform texts Neugebauer

[^6]even goes so far as to conjecture that, "... we would more properly have to call "Babylonian" many things which the Greek tradition had brought down to us as "Pythagorean" ${ }^{26}$.

### 0.2 The Pythagoreans, The Neo-Pythagoreans, and Sources of Pythagorean Thought

The Pythagoreans, or those who followed the teachings of Pythagoras, flourished for a few generations after the passing of their founder, Pythagoras, but began to dwindle in popularity, their philosophies giving way to those of the great philosophers that were starting to write at that time (ca. 300-200 B.C.) such as Plato and Aristotle. There is, however, a notable Pythagorean influence that may be seen in the philosophies of these writers. Plato praises them repeatedly and has many mathematical references (both mystical and more scientific) in his works that may be traced back to the Pythagoreans. Aristotle also mentions the Pythagoreans repeatedly, although he was quite critical of their philosophies. In any case, the writings of Plato and Aristotle are good sources for parts of Pythagorean thought.

Due to the mystical nature of the original teachings of Pythagoras (which were regarded by some as being divine revelations), they were only divulged to initiates of the school of Pythagoras, and even then only to those who were sufficiently purified and prepared. Moreover, as Heath puts it, "The fact appears to be that oral communication was the tradition of the school ..." "27, and so the Pythagoreans never publicly shared their doctrines or mathematics and made no written record of their findings. However, Pythagoras called himself a philosopher or 'lover of wisdom' - a scientist using reason to uncover the truth, and this contradiction to the "divine revelation" view of his teachings created a conflict in the school after his death. One of the chief disciples of Pythagoras, named Hippasus, " ... made bold to add several novelties to the doctrine of Pythagoras and to communicate his views to others. ... [These] indiscretions caused a split: Hippasus was expelled. Later on he lost his life in a shipwreck, as a punishment for his sacrilege, according to his opponents." ${ }^{28}$. In any case, through incidents like this, the

[^7]Pythagorean teachings became more generally known.

Euclid is one of our chief sources for the Pythagorean teachings. According to Van der Waerden, the three arithmetical books of the Elements (books VII, VIII, and IX) are Pythagorean ${ }^{29}$, and in particular, "Book VII was a textbook on the elements of the Theory of Numbers, in use in the Pythagorean school." ${ }^{30}$ Heath also claims that, "The Pythagoreans, before the next century was out (i.e. before, say, 450 B.C.) had practically completed the subject matter of Books I-II, IV, VI (and perhaps III) of Euclid's Elements ... "31. Apart from Euclid, the chief sources of for the Pythagorean Theory of Numbers, and indeed of Pythagoreanism in general, are writers who lived quite a bit later.

Starting from about 100 A.D., a small but notable resurgence of Pythagoreanism occurred. According to some sources, the resurgence was an attempt (however unsuccessful) to challenge the domination by Christianity of the pagan religions. These later followers of Pythagoras are known and will be referred to henceforth as the Neo-Pythagoreans.

Among the most prominent Neo-Pythagoreans (that are referred to in this work) are Nicomachus of Gerasa (ca. 100 A.D.), Iamblichus of Chalcis (ca. 300 A.D.), Proclus of Lycia (ca. 400-450 A.D.), and Theon of Smyrna (ca. 450 A.D.).

Nicomachus, it is said ${ }^{32}$, flourished around the end of the first century of our era. He is said to be of Gerasa, a city in what is now Palestine, but he was most likely educated in Alexandria, which was the center of mathematical studies of the time. It was also, interestingly enough, a center of Neo-Pythagoreanism. He wrote many works, most notably an Introduction to Harmonics, and the Introduction to Arithmetic which is one of the best sources for Pythagorean mathematics. His approach to arithmetic in the Introduction to Arithmetic differs from Euclid's rather dry, scientific style of presentation. As Heath puts it, "Probably Nicomachus, who was not really a mathematician, intended his Introduction to be, not really a scientific treatise, but a popular

[^8]treatment of the subject calculated to awaken in the beginner an interest in the theory of numbers by making him acquainted with the most noteworthy results obtained up to date"33. The method he used to make it more accessible to the general public was to include more of the mystical side of the Theory of Numbers. Van der Waerden sums this up by saying that, "Although Nicomachus lived four centuries after Euclid, he makes, nevertheless a much more primitive impression. He is much closer to the original number mysticism of Pythagoras and his school." ${ }^{34}$ Nicomachus also wrote two books which have not survived: an Introduction to Geometry and even an Introduction to Astronomy (although evidence for this latter work is slight), which would have completed an overall introduction to the Pythagorean quadrivium of subjects. He also wrote a Life of Pythagoras, and is purported to have written a book on the mystical doctrine of numbers called the Theologoumena Arithmeticae in two volumes; again, neither of which have survived. In any case, Nicomachus was a writer of great fame in his day, and was considered to be one of the "golden chain" of true philosophers (whose members' works were said to have divine origin), as well as being one of the first 'popular' science writers.

Iamblichus, like Nicomachus, was much more interested in the mystical side of numbers than was Euclid. He was a student of Anatolius, Bishop of Laodicea, and also of the great polymath Porphyry of Tyre. He was not very highly regarded by his peers. By all accounts he was a, " ... notoriously unclear writer" ${ }^{35}$, and Van der Waerden even goes so far as to call him, "... fanciful and muddle-headed" ${ }^{36}$. However, he did manage to write a nine (or by some accounts ten) volume treatise on the Pythagoreans, only four of which have survived. Among these are his On the Pythagorean Life detailing the life and some of the teachings of Pythagoras, and his Theology of Arithmetic which gives the mystical meanings of the first ten numbers. This latter work is not to be confused with the work of the same name by Nicomachus. For all of his faults, though, he was ranked by the Emperor Julian the Apostate as a philosopher of the same caliber as Plato. His mathematics, however, is in general lacking in content.

[^9]Proclus was a fifth century Neo-Pythagorean (who was also known as being a Platonist) whose main interest here is in his two commentaries on The Republic of Plato and on The first Book of Euclid's Elements. He mentions some important facts about Pythagoras and the Pythagoreans in the latter work, and provides details on some of the arithmetic of the Pythagoreans that is mentioned in Plato. Likewise, Theon of Smyrna wrote an entire book, "... purporting to be a manual of mathematical subjects such as a student would require to enable him to understand Plato." ${ }^{37}$ In it is contained, among other things, a reasonably detailed presentation of Pythagorean Arithmetic similar to the Introduction to Arithmetic of Nicomachus, although much shorter. Theon's exposition is much less clear than Nicomachus's; he repeats himself, and offers very few examples to help the reader understand what he is talking about. One cannot be sure if this is a reflection of the poor mathematical skills of Theon himself, or of the very low mathematical knowledge and skill of his readers (who only wanted to know enough mathematics to be able to read Plato!). It is to be hoped that it is the latter, however Nicomachus's Introduction was written for virtually the same audience.

These writers are the main sources of Pythagorean arithmetic that are used here, although there exist many others which are more scarce and have more scattered material.

### 0.3 Pythagorean Arithmetic

### 0.3.1 Classification of Numbers

Given the metaphysical nature of the Pythagorean view of numbers and the universe described above, it is not surprising then that their arithmetic stems from a concept of divine or mystical oneness. Theon ${ }^{38}$ calls this abstract indivisible oneness the monad, and distinguishes it from the more concrete concept of 'the one', or unity, which is used to describe things in the real world such as one horse or one man. Number is then defined by Theon as a collection of monads ${ }^{39}$. In the same vein, Aristotle tells us that, "... 'the one' means the measure of some plurality, and 'number' mans a measured plurality and a plurality of measures. (Thus it is natural that

[^10]one is not a number; for the measure is not measures, but both the measure and the one are starting points.) ${ }^{40}$ So unity or one was not considered to be a number by the Pythagoreans. Nicomachus describes this rather succinctly.

Unity, then, occupying the place and character of a point, will be the beginning of intervals and of numbers, but not itself an interval or a number, just as the point is the beginning of a line, or an interval, but is not itself a line or an interval ${ }^{41}$.

Thus, the unit is not a number, but is nevertheless the root of all numbers. From this definition, it is easy to see why the Pythagoreans only considered whole numbers (what we would in modern terminology call positive integers or natural numbers) to be numbers.

Using this definition of number, the Pythagoreans proceeded to classify the numbers. The first and most important classification is identified by Nicomachus early in his Introduction to Arithmetic:

Number is a limited multitude or a combination of units or a flow quantity made up of units; and the first division of number is even and odd. ${ }^{42}$

Nicomachus goes on to define even and odd, saying that," ... by the Pythagorean doctrine ... the even number is that which admits of division into the greatest and the smallest parts at the same operation ... and the odd is that which does not allow this to be done to it, but is divided into two unequal parts. ${ }^{43}$ This first division of number into even and odd played a fundamental role in the Pythagorean metaphysics. ${ }^{44}$ The whole universe, according to them, was divided into antithetical pairs. As Aristotle puts it, the Pythagoreans, "... say there are ten principles, which they arrange in two columns of cognates - limited and unlimited, odd and even, one and plurality, right and left, male and female, resting and moving, straight and curved, light and darkness, good and bad, square and oblong." 45 So important to their metaphysics was this duality, in fact, that Nicomachus mentions that by the 'ancient' definition,

[^11]"... the even is that which can be divided alike into two equal and two unequal parts, except that the dyad, which is its elementary form, admits but one division, that into equal parts;" ${ }^{46}$. From this we can see that the original conception of two, or the dyad, was not as a number, but the principle or beginning of the even numbers, similar to the monad not being itself a number, but rather the principle of all numbers. ${ }^{47}$

The Pythagoreans went on to classify numbers in even more detail defining further subdivisions of the even and odd numbers. We will not explore these subdivisions here, although a complete description is given by Nicomachus in his Introduction to Arithmetic ${ }^{48}$. They were aware, however, of prime numbers, and although he came later than the original Pythagoreans, Eratosthenes developed a method of generating prime numbers, the 'Sieve of Eratosthenes', which is described by Nicomachus ${ }^{49}$. The method of the Sieve being to list as many odd numbers as desired beginning with 3 , and then to take the first number (3) and cross all multiples of 3 (not including 3 itself) off the list. The next step is to take the second number of the list (5) and cross all its multiples (again, not including 5 itself) off the list, and so on. The numbers that remain at the end of the process are the primes.

Stemming from the mystical roots of Pythagoreanism, the Neo-Pythagoreans spoke of some other interesting types of numbers. Most notable among these are the so-called perfect numbers and the related friendly or amicable numbers. These are discussed in detail in Chapter 1 below.

In keeping with the fact that the Pythagoreans only recognized whole numbers greater than 1 , and since paper was expensive at that time, they performed most of their calculations using pebbles on counting boards. In fact, "It is also significant that the common [Greek] verb for "to calculate" is Psephizein derived from the word Psephos the counting pebble."50 In the same vein, the modern word calculation has the Latin word 'calculus', or stone, as its root. In any case, from this work with counting pebbles came a favorite study of the Pythagoreans

[^12]and Neo-Pythagoreans, the so-called figurate numbers - seeing numbers as the shapes that they made when represented by pebbles, such as triangles, squares, etc.. Within these figures they were able to identify various interesting patterns in the whole numbers which added to their ever expanding Theory of Numbers. We explore figurate numbers more fully in Chapter 2 below.

As to a concept of fractions stemming from the ideas of the Egyptians and Babylonians, again the Pythagoreans were not willing to recognize anything but whole numbers. This could have been a problem. However, instead of viewing fractions as a division of unity which was against their metaphysical doctrine, they worked with ratios of whole numbers, and were able to give an extensive (although cumbersome) classification of fractions on this basis. This classification is detailed in Chapter 3 below. Some of the Pythagorean development of the arithmetic of fractions may be found in Book VII of Euclid's Elements.

Moreover, from ratios of whole numbers, they developed an intricate theory of means, the arithmetic, geometric and harmonic means between two numbers $a$ and $b$ (respectively $\frac{a+b}{2}$, $\sqrt{a b}$, and $\frac{2 a b}{a+b}$ ), being the most well known. Although the means of the Pythagoreans and NeoPythagoreans are not explored in detail here, they had ten means defined in total, and these means were of fundamental importance in the Pythagorean school. They were not only used in the theory of numbers, but also were crucial to the understanding of harmonies in music, and hence the harmonies of the universe. As Van der Waerden puts it: "Music, harmony and numbers - these three are indissolubly united according to the doctrine of the Pythagoreans. ${ }^{51}$

And finally, although it is really more a geometrical than an arithmetic result, the Pythagoreans knew the famous "Pythagorean Theorem" that the sum of the squares of the sides of a right triangle equals the square of the hypotenuse. Indeed, although the theorem predates the Greeks, Pythagoras himself is credited with finding a formula that gives a triple of whole numbers ( $a, b, c$ ) satisfying $a^{2}+b^{2}=c^{2}$ if one starts with a side length (not the hypotenuse) of any odd number

[^13]$a$. The formula is then as follows:
\[

$$
\begin{equation*}
a^{2}+\left(\frac{a^{2}-1}{2}\right)^{2}=\left(\frac{a^{2}+1}{2}\right)^{2}, a=3,5,7, \ldots . .^{52} \tag{0.4}
\end{equation*}
$$

\]

This gives an overview of how the Pythagoreans classified numbers, and the main areas of arithmetic that they studied. The only thing that is missing is a discussion of the relationship of the Pythagoreans to what we in modern language call irrational numbers.

### 0.3.2 Incommensurable numbers

In one part of the dialogue of the seventh book of Plato's Laws, a stranger berates the Hellenes (or Greeks, in particular a man named Cleinias) for their ignorance of incommensurable numbers; something, he says, that is common knowledge to children in Egypt. ${ }^{53}$ Whether this should be taken as evidence that knowledge of irrational numbers came to Greece from Egypt is highly debatable. However, it does point out some of the controversy surrounding the purported discovery of irrational numbers by the Pythagoreans.

According to Proclus, it was Pythagoras himself who discovered the "doctrine of proportionals" (i.e. of irrational numbers). ${ }^{54}$ This claim is not universally acknowledged, but in any event, most scholars would agree with Heath in saying that at the very least, there is, "... no reason to doubt that the irrationality of $\sqrt{2}$ was discovered by some Pythagorean at a date appreciably earlier than that of Democritus" ${ }^{55}$ (who lived around 430 B.C.).

It is thought that in investigating the ratios between the sides of various geometric figures, the Pythagoreans attempted to determine the ratio of the length of the diagonal of a square to the length of its side. This ratio, of course, is the irrational number $\sqrt{2}$. The problem it posed was that, "... if Pythagoras discovered even this, it is difficult to see how the theory that number is the essence of all existing things, or that all things are made of number, could have held its ground for any length of time." ${ }^{56}$ That is, since $\sqrt{2}$ cannot be expressed as a ratio of whole

[^14]numbers, its existence would have undermined the very foundation of the Pythagorean doctrine that 'all is number'.

Needless to say, this seems as though it would have created a problem for the Pythagoreans. According to some, they kept the discovery secret and that Hippasus, who was mentioned earlier as being expelled from the school of Pythagoras for divulging Pythagorean doctrine and killed in a shipwreck as penance, was actually killed (struck down by the very Hand of God as it were) for making public the discovery of the irrationals ${ }^{57}$ However, Van der Waerden submits that the Pythagoreans solution to (or, some would say, avoidance of) this problem was just simply to not consider $\sqrt{2}$ as a number since it was not a whole number or a ratio of whole numbers. They instead went on to develop what is known as geometric algebra.

In the domain of numbers, the equation $x^{2}=2$ can not be solved, not even in that of ratios of numbers. But it is solvable in the domain of segments: indeed the diagonal of the unit square is a solution. Consequently, in order to obtain exact solutions of quadratic equations, we have to pass from the domain of numbers to that of geometric magnitudes. Geometric algebra is valid also for irrational segments and is nevertheless an exact science. It is therefore logical necessity, not the mere delight in the visible, which compelled the Pythagoreans to transmute their algebra into a geometric form." ${ }^{58}$

Van der Waerden is quite strong in his support of the geometric algebra thesis, however, not all academics agree with him, and 'geometric algebra' has recently been much attacked. We discuss irrational or incommensurable numbers in the context of the so-called side and diagonal numbers in Chapter 4 below.

In any case, the discovery of incommensurable numbers changed the face of Pythagoreanism forever, and cast serious doubts on their 'all is number' doctrine.

Finally, given the fundamental role that the even-odd antithesis plays in the Pythagorean doctrine, the following proof from Heath ${ }^{59}$ of the irrationality of $\sqrt{2}$ (which is alluded to by Aristotle), is of interest.

[^15]Suppose AC, the diagonal of a square, to be commensurable with AB , its side; let $\alpha: \beta$ be their ratio expressed in the smallest

possible numbers. Then $\alpha>\beta$, and therefore $\alpha$ is necessarily $>1$. Now

$$
\begin{equation*}
A C^{2}: A B^{2}=\alpha^{2}: \beta^{2} \tag{0.5}
\end{equation*}
$$

and, since

$$
\begin{equation*}
A C^{2}=2 A B^{2}, \alpha^{2}=2 \beta^{2} \tag{0.6}
\end{equation*}
$$

Hence $\alpha^{2}$, and therefore $\alpha$, is even. Since $\alpha: \beta$ is in lowest terms, it follows that $\beta$ must be odd. Let $\alpha=2 \gamma$; therefore $4 \gamma^{2}=2 \beta^{2}$, or $2 \gamma^{2}=\beta^{2}$, so that $\beta^{2}$, and therefore $\beta$, is even.

But $\beta$ was also odd: which is impossible. Therefore the diagonal AC cannot be commensurable with the side AB .

This proof, of which a version is found in book X of Euclid's Elements, and is (according to Van der Waerden) the, "... only place at which the theory of the even and the odd is applied in the Elements themselves," ${ }^{60}$ becomes even more interesting when it is noted that it is highly probable that it is Pythagorean in origin. ${ }^{61}$ Thus, it turns out that inherent in their

[^16]doctrine, the Pythagoreans found the tools necessary to debunk it! And so the development of mathematics by the Pythagoreans came to a close, leaving behind a legacy that would influence all future developments in mathematics and science.
only place in Euclid's Elements where the even-odd antithesis is used. Euclid spends, for instance, a considerable part of Book IX (in particular propositions 21 to 34 ) discussing properties of even and odd numbers.

# Chapter 1 <br> Perfect and Friendly Numbers 

### 1.1 Nicomachus On Perfect Numbers

As we saw in the introduction (section 0.2.1), according to Nicomachus, the first classification of whole numbers recognized by the Pythagoreans was the division into even and odd. With this division defined, Nicomachus goes on to spend a fair portion of the first book of the Introduction To Arithmetic describing the various varieties of even and odd numbers. To end his discourse, he tells us of the breakdown of the 'simple even numbers' into three types: Superabundant, Deficient, and Perfect ${ }^{1}$.

Nicomachus gives these types of number and this division of even numbers a more esoteric, metaphysical meaning. He explains

Those which are said to be opposites to one another, the superabundant and deficient, are distinguished from one another in the relation of inequality in the directions of the greater and the less; for apart from these no other form of inequality could be conceived, nor could evil, disease, disproportion, unseemliness, nor any such thing, save in terms of excess or deficiency. For in the realm of the greater there arise excesses, overreaching, and superabundance, and in the less need, deficiency, privation, and lack; but in that which lies between the greater and the less, namely, the equal, are virtues, wealth, moderation, propriety, beauty, and the like, to which the aforesaid form of number, the perfect, is most akin.

Mathematically, the superabundant, deficient, and perfect numbers are defined in terms of their proper divisors or factors (or, as some writers put it, in terms of their aliquot parts. ${ }^{2}$ In

[^17]modern notation, given a number $X$ with proper divisors $x_{0}=1, x_{1}, x_{2}, x_{3}, \ldots, x_{n}, X$ is said to be superabundant if
\[

$$
\begin{equation*}
\sum_{i=0}^{n} x_{i} \geq X \tag{1.1}
\end{equation*}
$$

\]

deficient if

$$
\begin{equation*}
\sum_{i=0}^{n} x_{i} \leq X \tag{1.2}
\end{equation*}
$$

and perfect if

$$
\begin{equation*}
\sum_{i=0}^{n} x_{i}=X \tag{1.3}
\end{equation*}
$$

An example of a superabundant number is the number 12 since its proper divisors are $1,2,3$, 4 , and 6 , and their sum is

$$
1+2+3+4+6=16 \geq 12
$$

Nicomachus also mentions 24 as being superabundant, the sum of its proper divisors $1,2,3,4$, 6,8 , and 12 being

$$
1+2+3+4+6+8+12=36 \geq 24
$$

To hammer home the metaphysical point, he tells us that, "... the superabundant number is one which has, over and above the factors which belong to it and fall to its share, others in addition, just as if an animal should be created with too many parts of limbs, with ten tongues, as the poet says, and ten mouths, or with nine lips, or three rows of teeth, or a hundred hands or too many fingers on one hand. ${ }^{3}$ - the poet he refers to being Homer.

Examples of deficient numbers given by Nicomachus are 8 and 14 since their proper divisors are $\{1,2,4\}$ and $\{1,2,7\}$ respectively, and

$$
1+2+4=7 \leq 8 \text { and } 1+2+7=10 \leq 14 .
$$

Nicomachusi has this to say about the metaphysical role of deficient numbers:

[^18]It is as if some animal should fall short of the natural number of limbs or parts, or as if a man should have but one eye, as in the poem, "And one round orb was fixed in his brow"; or as though one should be one-handed, or have fewer than five fingers on one hand, or lack a tongue, or some such member. Such a one would be called deficient and so to speak maimed ... "4

Finally, he comes to the perfect numbers which he views as being the mean variety of number between the extremes of the superabundant and deficient types of number. He shows us that 6 is a perfect number since its proper divisors are 1,2 , and 3 , and their sum is

$$
1+2+4=6
$$

and also 28 is perfect since its proper divisors $1,2,4,7$, and 14 have as sum

$$
1+2+4+7+14=28
$$

The number 6 , he says, is the only perfect number in the units, 28 is the only perfect number in the tens. He goes on to state without justification that 496 is the only perfect number in the hundreds, and that 8,128 is the only perfect number in the thousands. The reason, according to Nicomachius, for the sparse population and regular ordering of the perfect numbers is,
... that even as fair and excellent things are few and easily enumerated, while ugly and evil ones are widespread, so also the superabundant and deficient numbers are found in great multitude and irregularly placed - for the method of their discovery is irregular - but the perfect numbers are easily enumerated and arranged with suitable order ... "5

He then goes on to explain a method for producing the perfect numbers, "... neat and unfailing, which neither passes by any of the perfect numbers nor fails to differentiate any of those that are not such ... "6. It is interesting to note that Nicomachus describes this method as a step by step process that the reader can follow and produce perfect numbers. It is truly presented as an algorithm, perhaps the first one of its kind!

[^19]The algorithm, in modern notation, may be expressed as follows. Perfect numbers are of the form:

$$
\begin{equation*}
\left(1+2+2^{2}+2^{3}+\cdots+2^{n}\right) \cdot 2^{n} \tag{1.4}
\end{equation*}
$$

where $1+2+2^{2}+2^{3}+\cdots+2^{n}$ is a prime.

Thus, since $1+2=3$ is a prime, we see that $6=(1+2) \cdot 2$ is a perfect number according to the algorithm. Similarly, $1+2+4=7$ is prime, so $(1+2+4) \cdot 4=28$ is perfect as well.

Using the algorithm, Nicomachus easily establishes that 496 is also a perfect number as he claimed before. He observes that:

$$
1+2+2^{2}+2^{3}+2^{4}=1+2+4+8+16=31
$$

is prime, and so

$$
\left(1+2+2^{2}+2^{3}+2^{4}\right) \cdot 2^{4}=(1+2+4+8+16) \cdot 16=31 \cdot 16=496
$$

is perfect.

Similarly, since $1+2+4+8+16+32+64=127$ is a prime, by the algorithm:

$$
(1+2+4+8+16+32+64) \cdot 64=127 \cdot 64=8,128
$$

shows that 8,128 is a perfect number.

Nicomachus offers no proof of the validity of the algorithm other than the fact that it produces the first four consecutive perfect numbers, and no others. Euclid was the first to offer a proof that every number of the form given in 1.4 above is indeed perfect. Much later in history it was proven by Euler that the converse is also true: Every even perfect number has the form given in 1.4. To Nicomachus, perfect numbers were defined to be even, so the result of Euler would have pleased him immensely. However, modern mathematicians have asked whether or not there are any odd perfect numbers. The answer does not seem to be as simple as for the even numbers, and so this remains an open problem in the Theory of Numbers.

Nicomachus concludes his discourse on perfect numbers with a discussion of whether or not unity is a perfect number.

Now unity is potentially a perfect number, but not actually; for taking it from the series as the very first I observe what sort it is, according to the rule, and find it prime and incomposite; for it is so in very truth, not by participation like the rest, but it is the primary number of all, and alone incomposite. I multiply it, therefore, by the last term taken into the summation, that is, by itself, and my result is 1 ; for 1 times 1 equals 1 . Thus unity is perfect potentially; for it is potentially equal to its own parts, the others actually. ${ }^{7}$

### 1.2 Theon on Perfect Numbers

Theon's discourse on perfect numbers is much less detailed than that of Nicomachus, and contains virtually the same information ${ }^{8}$.

He describes briefly the definitions of superabundant, deficient, and perfect numbers. The definitions are identical to those outlined in equations 1.1, 1.2, and 1.3 above.

And, like Nicomachus, he gives examples of each type of number. He tells his reader that the numbers 6 and 28 are perfect since $6=1+2+3$ and $28=1+2+4+7+14$, that the number 12 is superabundant since $12 \leq 1+2+3+4+6=16$, and that the number 8 is deficient since $8 \geq 1+2+4=7$.

In contrast to the exposition of Nicomachus, though, Theon mentions two interesting things: The first is that according to him, the number 3 is perfect because it is the first to have a beginning, a middle, and an end; and also because it is the first which is both a line and a surface. This latter reason makes sense graphically since it takes a minimum of three points to describe a plane. Finally, he tells the reader that 3 is the first link to the idea of a solid since solids exist in 3 dimensions. The second interesting thing mentioned by Theon is that the Pythagoreans considered the number 10 to be the perfect number. He doesn't give any details on this, but'according to Heath,

[^20]10 is the sum of the numbers $1,2,3,4$ forming the $\tau \epsilon \tau \rho \alpha \kappa \tau \dot{v} \varsigma$ [tetraktys] ('their greatest oath', alternatively called the 'principle of health'). These numbers include the ratios corresponding to the musical intervals discovered by Pythagoras, namely 4:3 (the fourth), 3:2 (the fifth), and 2:1 (the octave). Speusippus observes further that 10 contains in it the 'linear', 'plane' and 'solid' varieties of number; for 1 is a point, 2 is a line, 3 is a triangle, and 4 a pyramid. ${ }^{9}$

### 1.3 Friendly or Amicable Numbers

As a final note of interest, we explore briefly the so-called amicable numbers of the Pythagoreans.
These are pairs of numbers that have the property that they are each equal to the sum of the proper divisors of the other. The most well known example of a pair of amicable numbers are the numbers 220 and 284. The proper divisors (or aliquot parts, as the ancients put it) of 284 are the numbers $1,2,4,71$, and 142 , and those of 220 are $1,2,4,5,10,11,20,22,44,55$, and 110. When we add these together, we see that:

$$
\begin{array}{ll}
1+2+4+5+10+11+20+22+44+55+110 & =284 \\
1+2+4+71+142 & =220
\end{array}
$$

This is all we will say about amicable numbers, except to mention an interesting story about Pythagoras and the genesis of the name 'friendly' for this type of number. According to Heath, "Iamblichus attributes the discovery of such numbers to Pythagoras himself, who, being asked 'what is a friend?' said 'Alter ego' ["a second I"], and on this analogy applied the term 'friendly' to two numbers the aliquot parts of either of which make up the other." ${ }^{10}$

[^21]
## Chapter 2 Nicomachus On Figurate Numbers

### 2.1 Numbers And Figures

Nicomachus ${ }^{1}$ begins his discourse on figurate numbers in chapter VI of the second book of his Introduction to Arithmetic by telling his reader that he has, up to this point, sufficiently discussed what he calls relative number. Relative number, he says, has as its elementary principle equality, as opposed to absolute number whose elementary principles are unity and the dyad ${ }^{2}$. He tells us it is time to discuss, "... certain subjects which involve a more serviceable inquiry, having to do with the properties of absolute number ... "3.

His first task in discussing absolute number is to describe the various types of linear, plane, and solid numbers. These are more closely related to geometry, he notes, and are discussed more fully in his Introduction to Geometry (now lost). He will, however, explore them here since the, "... germs" of these ideas are taken over into arithmetic, as the science which is the mother of geometry and more elementary than it." ${ }^{4}$.

Earlier in the Introduction (Book I, Chapter 4, section 4), Nicomachus mentions his rationale for concluding that arithmetic is the mother of geometry. He says, basically, that in order to be able to describe anything geometrical, such as a triangle, or an octahedron, or even speak the terms which are used in geometry such as double, or one and one-half times, we must first have numbers to do so. How, he asks, can, "... 'triple' exist without the number 3 existing

[^22]beforehand, or 'eightfold' without $8 ?$ " 5 . However, we may have numbers alone with out the figures that they are associated with such as having the number 4 without having the square. Thus, " ... arithmetic abolishes geometry along with itself, but is not abolished by it, and while it is implied by geometry, it does not itself imply geometry." ${ }^{6}$.

The way in which these germs of arithmetical ideas are captured by geometry is by designating the unit by a single $\alpha$, the number 2 by putting two $\alpha$ 's side by side, and so on, thereby introducing a schematic representation for the numbers. (As was discussed in the Introduction above (section 0.3.1), these $\alpha$ 's may be thought of as representing the pebbles with which the Pythagoreans did their calculating).

Seen in this way, the unit is clearly the beginning of all numbers and intervals. However, says Nicomachus, the unit is not itself a number or an interval in the same way that a point is the beginning of a line or an interval, but not itself a line or an interval. He goes on to expound that a point is non-dimensional, and so adding it to another point will give something which is again non-dimensional since nothing added to nothing is again nothing. And to make the point even clearer to the reader, he mentions that unity is the only number which when multiplied by itself remains itself. All this to emphasize that unity is something apart from all the other numbers - their origin, but not of them.

Thus, since unity is elementary and non-dimensional, the first dimension comes about from the number 2, formed by 2 points. Intuitively, these two points define a direction, and so it makes sense when Nicomachus defines a 'line' to be, "... that which is extended in one direction." ${ }^{7}$ He thus defines a linear numbers to be, "... all those which begin with 2 and advance by the addition of 1 in one and the same direction." ${ }^{8}$, again making the connection between numbers and figures.

Nicomachus, continues this construction by analogy noting that, "... the line is the beginning

[^23]of surface, but not surface; and the beginning of the two-dimensional, but not itself extended in two dimensions." 9 Thus he defines two dimensions as being 'surface', and 'a surface' as that which is extended in two directions. He also connects the number 3 with surface, the triangle being the first and most elementary plane figure (formed, as it were, from three dots (or $\alpha \mathrm{s}$ )). Hence he defines plane numbers to be, "... those that begin with 3 as their most elementary root and proceed through the next succeeding numbers." ${ }^{10}$ These numbers are connected to figures in the plane in the obvious way: 3 with the triangle, 4 with the square, 5 with the pentagon, and so on. To show his reader the elementary nature of the triangle in the plane, Nicomachus notes that given any rectilinear plane figure (presumably he means convex, regular polygons), connecting each vertex to the center of the figure resolves it into triangles - as many as there are sides to the figure.

Finally, to complete the construction, Nicomachus notes that, "... surface is the beginning of body, but not itself body, and likewise the beginning of the three-dimensional, but not itself extended in three directions." 11 He defines three dimensions to be 'solid', and 'a solid' to be that which is extended in three dimensions. He makes a vague connection of three dimensions .with the number 4 by stating that, "... dimension first is found and seen in 2 , then in 3 , then in $4, \ldots{ }^{12}$, but does not explicitly connect 4 to the four vertices of the tetrahedron. He also connects numbers to solid figures defining the 'solid numbers', but his discussion of these is more complex and will be investigated further below.

He also points out some relations between the dimensions, noting that they are mutually exclusive; for example, a surface is not a solid, and a solid cannot be a surface. He also states that the point falls short of the line by one dimension, the line falls short of the surface by one dimension, and the surface falls short of the solid by one dimension.

[^24]
### 2.2 Plane Numbers

### 2.2.1 Triangular numbers

Nicomachus begins his discussion of plane numbers with the most elementary plane number, the triangular number. This type of number, he says, "... is one which, when it is analyzed into units, shapes into triangular form the equilateral placement of its parts in a plane." ${ }^{13}$ The numbers $3,6,10,15,21,28$ and so on, are examples of these numbers, and they are obtained, he states, "... from the natural series of number set forth in a line, and by the continued addition of successive terms, one by one, from the beginning" ${ }^{14}$. That is, in modern notation, the $n^{\text {th }}$ triangular number is obtained by summing up the first $n$ consecutive natural numbers ${ }^{15}$, $1+2+3+\cdots+n$. Graphically, it becomes quite clear why these are called triangular numbers. The first triangular number is unity (which, it may be noted, is potentially any type of plane number), and is represented graphically as a single $\alpha$.


1

The second triangular number is $3=1+2$, and is obtained graphically by adding another row of $\alpha$ 's underneath the first one, the second row containing two $\alpha$ 's side by side representing the linear number 2 being added to the original 1.

[^25]

3

Adding 3 to the second triangular number above gets the third triangular number (which Nicomachus notes is really only the second triangle); and it is the number $6=1+2+3$. Again, this is represented graphically by adding the linear number 3 consisting of three $\alpha$ 's in a row onto the abọve triangle as the bottom row.


6

Nicomachus then proceeds to show graphic representations of the fourth, fifth, and sixth triangular numbers, obtained by the same process.


### 2.2.2 Square Numbers

The next plane numbers investigated by Nicomachus are the square numbers: $1,4,9,16,25$, $36, \ldots$. These have obvious graphic representations as squares (Nicomachus refers to them as equilateral squares) with, respectively, $1,2,3,4,5,6, \ldots \alpha$ 's on a side. As with the triangular numbers, unity has the potential to be considered a square number, and as such it would give a 1 by 1 square composed of a single $\alpha$.


To end chapter IX of the second book of the Introduction To Arithmetic, Nicomachus makes a few interesting notes about both triangular and square numbers. He first mentions that, for both types of number, " . . . the advance in their sides progresses with the natural series." ${ }^{16}$; that is, each successive member of either kind of plane number has side length one greater than the previous one. He then tells us that, as the triangular numbers were formed by adding the natural numbers successively beginning with 1 , the square numbers are obtained by adding the odd numbers successively beginning with 1 . Almost as an afterthought, he notes that, "... it is a fact that the side of each consists of as many units as there are numbers taken into the sum to produce it." ${ }^{17}$. For example, the third triangular number is the sum of the first three natural numbers in order beginning with 1 , and the fifth square number is the sum of the first five successive odd numbers.

[^26]An elegant way of showing that squares are produced when the odd numbers are added together is to introduce the notion of a gnomon. These are carpenter's square shaped parts of one of the square number figures that encompass two adjacent sides and the corner in between them. Thus, in order to add a gnomon onto a square of side length $n$, the gnomon would have to have $2 n+1 \alpha$ 's in it for the two sides and intervening corner. However, since the outer gnomon in the $n \times n$ square is of size $2(n-1)+1=2 n-1 \alpha$ 's, adding a gnomon is the same as adding the next odd number to the square. The first square number is the number 1 , and adding the gnomon $2 \cdot 1+1=3$ clearly gives the next square number 4 . This process continues, showing (by induction as it were), that:

$$
\begin{equation*}
1+3+5+7+9+\cdots+(2 n-1)=n^{2} \tag{2.1}
\end{equation*}
$$

The beauty of this view of the proof is that graphically, the gnomons are clearly odd numbers and they are stacked together such that their sum is a square (see below):


Gnomonic Proof that every square
is a sum of consecutive
odd numbers.

### 2.2.3 Pentagonal Numbers

In chapter X , Nicomachus discusses pentagonal numbers, which analogously to the above mentioned plane numbers are represented graphically as equilateral pentagons. He goes on to state that the numbers $1,5,12,22,35,51$, and 70 are examples of pentagonal numbers, and that they form pentagons of side lengths $1,2,3,4,5,6$, and 7 respectively. As with the triangular
and square numbers, he tells us that (in modern notation) the $n^{\text {th }}$ pentagonal number, $P_{n}$, is formed by summing the series

$$
P_{n}=1+4+7+\cdots+(3 n-2) .
$$

That is, $P_{n}$ is the sum of the first $n$ numbers of the sequence

$$
1,4,7, \ldots,(3 n-2) ;
$$

or the sum, as Nicomachus puts it, of, "... the terms beginning with 1 to any extent whatever that are two places apart, that is, those that have a difference of $3 .{ }^{18}$ It is again noted by Nicomachus that, "... the side contains as many units as are the numbers that have been added together to produce the pentagon' $\ldots{ }^{19}$; making our notation of $P_{n}$ for the $n^{\text {th }}$ pentagonal number an appropriate one. The graphic representations given by Nicomachus of the first few pentagonal numbers are:


1


5


12


22
although we would probably imagine pentagonal numbers to look like

[^27]

1


5


12
even though the pentagonal numbers are not as easily or clearly shown in equilateral pentagons.

### 2.2.4 Higher Plane Figures

Nicomachus rounds out his discussion of plane numbers by describing the hexagonal, heptagonal, octagonal, and succeeding numbers. They are obtained in the same manner as the preceding triangular, square, and pentagonal numbers; by summing all the natural numbers in order to obtain the triangular numbers, summing every second number (i.e. the odd numbers) to obtain the square numbers, every third number to obtain the pentagonal numbers, and so on.

Thus the hexagonal numbers are obtained by summing every fourth number in the natural series; that is, the $n^{\text {th }}$ hexagonal number $X_{n}$ is given by

$$
X_{n}=1+5+9+13+\cdots+(4 n-3),
$$

and corresponds to a hexagon composed of $\alpha$ 's having $n \alpha$ 's on each side.

The heptagonal numbers come from summing every fifth term in the natural series, and the $n^{\text {th }}$ heptagonal number $H_{n}$ is

$$
H_{n}=1+6+11+16+\cdots+(5 n-4) .
$$

Of the octagonal numbers, clearly assuming that the reader has grasped the pattern at this point, Nicomachus says only, "The octagonals increase after the same fashion, with a difference of 6 in their root numbers and corresponding variation in their total constitution." ${ }^{20}$

[^28]For the record, and also for later use, Nicomachus lists the first ten of each type of (regular) plane number as follows:

| Triangles | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 54 | 58 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| Squares | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
| Pentagonals | 1 | 5 | 12 | 22 | 35 | 51 | 70 | 92 | 11 | 145 |
| Hexagonals | 1 | 6 | 15 | 28 | 45 | 66 | 91 | 120 | 153 | 190 |
| Heptagonals | 1 | 7 | 18 | 34 | 55 | 81 | 112 | 148 | 189 | 235 |

Table 2.1: Some Polygonal Numbers

I include it here for the sake of completeness, and since it will be referenced in the next subsection.

Nicomachus ends his discussion of plane polygonal numbers with a general 'rule of thumb'.
In order that, as you survey all cases, you may have a rule generally applicable, note that the root-numbers of any polygonal differ by 2 less than the number of the angles shown by the name of the polygonal - that is, by 1 in the triangle, 2 in the square, 3 in the pentagon, 4 in the hexagon, 5 in the heptagon, and so on, with similar increase. ${ }^{21}$

### 2.2.5 A Few More Facts Concerning Plane Numbers

One interesting observation (of the Pythagoreans) that is pointed out by Nicomachus is that: "Every square figure diagonally divided is resolved into two triangles and every square number is resolved into two consecutive triangular numbers, and hence is make up of two successive triangular numbers." ${ }^{22}$ Arithmetically this can easily be seen by simply adding any two consecutive triangular numbers from the first row of Table 2.1 above to get the square number below the second triangular number. Graphically, it is elegantly seen as follows:

[^29]

Nicomachus then goes on to say that, "... any triangle joined to any square figure makes a pentagon, ... ${ }^{23}$, a statement that is obviously false. What he really means to say is that adding the $n^{\text {th }}$ triangular number to the $(n+1)^{s t}$ square number gives the $(n+1)^{\text {st }}$ pentagonal number (i.e. adding a given triangular number to the square number one column to the right in Table 2.1 gives the pentagonal number directly below the square number. For example, $1+4=5,3+9=12$, etc..

Furthermore, he observes that adding the $n^{t h}$ triangle to the $(n+1)^{s t}$ member of any type of regular polygon will produce the $(n+1)^{s t}$ member of the succeeding type of regular polygon. That is, "... if the triangles are added to the pentagons, following the same order, they will produce the hexagonals in due order, and again the same triangles with the latter will make the heptagonals in order, the octagonals after the heptagonals, and so on to infinity." ${ }^{24} \mathrm{He}$ goes on to give quite a few examples of this for pentagons, hexagons, heptagons, and octagons. However, I think the pattern is clear from Table 2.1, that taking any triangle and adding it to any regular polygon in the next column gives the regular polygon below the one you chose in the same column.

This then, says Nicomachus, is the reason that the triangle is the, "... element of the polygon both in figures and in numbers, and we say this because in the table, [Table 2.1] ... the successive numbers in the rows are discovered to have as differences the triangles in regular

[^30]order." ${ }^{25}$

### 2.3 Solid Numbers

Having thus completed his survey of plane figurate numbers, Nicomachus proceeds without pause to the next logical type of figurate number: The Solid Number. This is, fairly obviously, a number that has a graphic representation in three dimensions as some sort of regular geometric object.

### 2.3.1 Pyramidal Numbers

The first type of solid numbers discussed by Nicomachus are the so-called pyramidal numbers. They are representations in points (or $\alpha$ 's) of regular pyramids (that is pyramids with all vertices the same length) with regular polygons for bases.

The pyramids or pyramidal numbers, he states, merely represent the next step in the process he went through to obtain the linear numbers and then the polygonal and plane numbers from unity. 'A single $\alpha$ represents unity, a string of $\alpha$ 's in a line represents a linear number, and a collection of linear numbers from the natural series of numbers represents a plane or polygonal number; "the triangles by the combination of root-numbers [i.e. natural numbers] immediately adjacent, the square by adding every other term, the pentagons every third term, and so on." ${ }^{26}$. In the same manner, the pyramidal numbers are constructed by taking the plane polygonal numbers and piling them on top of one another.

The simplest of these are the pyramids with triangular bases, according to Nicomachus, and so he discusses them first. They are formed by taking the series of triangular numbers,

$$
1,3,6,10,15,21,28, \ldots
$$

and adding them as consecutive layers in a pyramid beginning with 1 which represents the apex of the pyramid. In more mathematical notation, if $T_{n}=1+2+\cdots+n$ represents the $n^{\text {th }}$

[^31]triangular number, then the $n^{\text {th }}$ triangle-based pyramidal number, $P_{n}^{T}$, is given by
$$
P_{n}^{T}=\sum_{i=1}^{n} T_{i} .
$$

Thus the first few triangle-based pyramidal numbers are:

$$
\begin{aligned}
1 & =1 \\
4 & =1+3 \\
10 & =1+3+6 \\
20 & =1+3+6+10 \\
35 & =1+3+6+10+15 \\
56 & =1+3+6+10+15+21 \\
& \vdots
\end{aligned}
$$

Nicomachus then describes the next pyramids in the sequence - the square-based pyramids. In the same way as the triangle-based pyramids, these are formed by stacking successive square numbers of the form $S_{n}=1+3+5+\cdots+(2 n-1)$ one below the other beginning with $S_{1}=1$ on the top. The $n^{\text {th }}$ square-based pyramidal number, $P_{n}^{S}$, is then given by:

$$
P_{n}^{S}=\sum_{i=1}^{n} S_{i} .
$$

Nicomachus then says that the pentagonal-based, hexagonal-based, etc. pyramids are formed in the same way, but does not explicitly describe them.

He ends his description of the pyramidal numbers by defining the various types of truncated pyramids, "... the names of which we are sure to encounter in scientific writings ... "27. He calls a pyramid with any type of polygon as a base truncated if the top layer of the pyramid (that is the unit polygon) is removed, bi-truncated if the top two layers (the unit polygon and the first non-trivial polygon) are removed, tri-truncated if the top three layers are removed, etc.. Nicomachus also tells us, for reference and completeness, that we may carry on the nomenclature by referring to the pyramids as four times truncated if the top four layers are removed, five times truncated if another layer is removed, and so on.

[^32]
### 2.3.2 Other Solid Numbers

The other types of solid numbers that Nicomachus discusses are what he calls: Cubes, "Beams", "Bricks", "Wedges", Parallelepidons, and Spheres.

The cubes are defined as may be expected. Take a square number of side length $n$ (graphically represented by $n^{2} \alpha$ 's in a square form as described above), and proceed to make a cube of side length $n$ by piling $n$ such squares one on top of the other. Thus the cubic numbers are: $1,8,27,64,125, \ldots$, a series again beginning with unity (1), which in this case is looked upon as a cube of side length 1 .

Nicomachus also gives us some facts concerning cubes. In the same way as a square has four equal sides and four equal plane angles, a cube made from a given square will always have six plane surfaces identical to the square, twelve edges each of which is the same length as those in the original square, and eight what he calls 'solid angles' each bounded by three edges. This passage is interesting as it shows the level of the book and the audience that Nicomachus must have had in mind; one that does not have much mathematical knowledge.

In defining the subsequent types of solid numbers, Nicomachus does not differentiate between the solid figure represented by a number (that is, made up of layers of $\alpha$ 's), and the actual solid, continuous, figure of certain side length. Presumably he is confident at this point that his reader is fluent enough with the material not to need a laborious exposition of the details.

With this in mind, Nicomachus proceeds to define scalene numbers as numbers which when represented graphically are solid rectangular figures with all sides unequal, such as 2 times 3 times 4. He puts these forth as the opposite extreme from the cube numbers, all of whose sides have equal length.

He elaborates on the scalene numbers by saying that they are often referred to as "wedges" since, ". . . carpenters', house-builders' and blacksmiths' wedges and those used in other crafts, having unequal sides in every direction, are fashioned so as to penetrate; they begin with a
sharp end and continually broaden out unequally in all the dimensions." ${ }^{28}$. He also tells us that they are sometimes called sphekiskoi (i.e. "wasps"), "... because wasps' bodies also are very like them, compressed in the middle and showing the resemblance mentioned." ${ }^{29}$. Nicomachus then practices a little etymology by musing that, "From this [sphekiskoi] also the sphekoma, "point of the helmet," must derive its name, for where it is compressed it imitates the waist of the wasp. ${ }^{30}$ Speculative illustrations of these three types of figures are as follows:


A "Wedge", "Wasp" or "Sphekiskoi"


A Scalene Solid Figure


A "Sphekoma" or "Point of the Helmet" (Very Speculative!)

Finally, he mentions that others call these scalene solid numbers "alters" since, "... the altars of ancient style, particularly the Ionic, do not have the breadth equal to the depth, nor either of these equal to the length, not the base equal to the top, but are of varied dimensions everywhere." ${ }^{31}$ Thus and alter would perhaps look like:

[^33]

## An "Alter"

Again, these are all examples of Nicomachus' writing the Introduction To Arithmetic for a more general audience, and making it more accessible by including 'real life' references. He wanted to write a book that would teach mathematics to the public at large, and it was, in fact, a bestseller in its day!

With the cube and scalene numbers marking the extreme cases of how rectangular solid numbers may appear; Nicomachus tells us about numbers that are means between the two - the so-called parallelepidons. These are again rectangular solids, but the faces of the solids are heteromecic numbers. Nicomachus then proceeds to define what these 'heteromecic numbers' are.

In modern notation, heteromecic numbers are numbers of the form $n(n+1)$ where $n$ is an integer greater than or equal to 1 . Examples of the heteromecic numbers are then:

$$
2=1 \times 2, \quad 6=2 \times 3, \quad 12=3 \times 4, \quad 20=4 \times 5, \quad 30=5 \times 6, \quad 42=6 \times 7, \ldots
$$

Here Nicomachus digresses from his exposition of solid numbers to briefly explain the importance of heteromecic numbers in Pythagorean thought.

For the ancients of the school of Pythagoras saw "the other" and "otherness" primarily in 2, and "the same"' and "sameness" in 1, as the two beginnings of all things, and these two are found to differ from each other only by 1 . Thus "the other" is fundamentally "other" by 1 , and by no other number, ...
${ }^{32}$ The odd numbers, which are given their form by unity according to the Pythagoreans, have the nature of "the same", and since the even numbers come from 2, they have the nature of "the other". In the same way, says Nicomachus, squares have the nature of "the same" since their,

[^34]"... sides display the same ratio, alike, unchanging and firmly fixed in equality, to themselves ... "33, and the heteromecic numbers and/or figures have the nature of "the other" since, "... just as 1 is differentiated from 2 , differing by 1 alone, thus also the sides of every heteromecic number differ from one another, one differing from the other by 1 alone." ${ }^{34}$.

Thus, although Nicomachus does not point this out explicitly, parallelepidons have to have two sides equal and the other side differing from the others by only one. That is, it is a scalene number of the form $a \times a \times(a \pm 1)$; for example a scalene number of dimensions $2 \times 2 \times 1$. Otherwise, if all three numbers are distinct, such as with $2 \times 3 \times 4$, there will be two faces having the non-heteromecic dimension $2 \times 4$. It then becomes clearer why he views parallelepidons as being the means between the cubes and scalene numbers

In any case, Nicomachus proceeds to define "bricks" as being rectangular solid figures with two sides of equal length and the other of a shorter length (he uses $8 \times 8 \times 2$ and $8 \times 8 \times 3$ as examples), and "beams" as rectangular solid figures with two sides equal and the other side longer ( $3 \times 3 \times 7,3 \times 3 \times 8$, and $3 \times 3 \times 9$ are given as examples). Given our discussion in the preceding paragraph, we can only suppose that in defining bricks and beams, Nicomachus means to say also that the unequal side has length differing from the length of the other two sides by more than 1 .

He ends his discourse on solid numbers with a description of spherical (or recurrent) numbers. These are cubic numbers which, "... have the further property of ending at every multiplication in the same number as that from which they began;" ${ }^{35}$. He gives 5 and 6 as examples of numbers that when multiplied by themselves continue ending in the same number from which they began. That is $5 \times 5=25$ ends in a 5 , as does $5 \times 5 \times 5=125$, and so on through all the powers of 5. In particular, since 125 is a cube also, it is a spherical number. In the same way, $6 \times 6=36$ ends in a 6 , as does $6 \times 6 \times 6=216$ making 216 a spherical number also. In the same way, a square number that ends in the same number from which it began, like 25 or 36 , is called a

[^35]circular number. It is also noted by Nicomachus that the number 1 is potentially circular and spherical being both the square and the cube of itself. Furthermore, even though he does not state it explicitly, this shows that unity is the beginning of the circles and spheres, and adds to his belief in unity as the beginning of all things.

Nicomachus claims that the numbers 1,5 , and 6 are, "... the only ones of the products of equal factors to return to the same starting point from which they began, in the course of all their increases." ${ }^{36}$; meaning that they are the only numbers with this property. This is not strictly true since the number 25 satisfies the condition given for all powers of itself, but perhaps he is only be referring to numbers below 10 with this property.

Another ambiguous passage comes about right at the end of Nicomachus' discussion of circular or spherical when he says, "... if they have three dimensions, or are multiplied still further than this, they are called spherical solid numbers, for example, $1,125,216$, or, again, 1,625 , $1,296 .{ }^{37}$. It is not clear whether the fourth or higher powers of 5 and 6 are also called spherical numbers - perhaps calling them hyper-spherical numbers would be the appropriate modern nomenclature.

[^36]
## Chapter 3

## Nicomachus on Relative Numbers

### 3.1 Introduction to Relative Numbers

After spending most of the first part of the first book of the Introduction To Arithmetic discussing what he calls absolute quantity (that is the nature and classification of the whole numbers), Nicomachus now takes a look at relative quantity ${ }^{1}$.

By relative quantity, he means the study of what occurs when one quantity is compared with another. He, gives examples of numbers compared with numbers, of course, but he is not limited by that. The quantities that he compares range to such things as a "friend" and a "neighbor", or a "father" and a "son". These two comparisons are examples given by Nicomachus of the first 'generic division' of the topic of relative quantity into the categories of equality and inequality. To Nicomachus this 'generic division' of the topic of relative quantity into the categories of equality and inequality is a comprehensive one as may be seen in his statement that," ... everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these." ${ }^{2}$

He proceeds to discuss the concept of equality a bit further, concluding that, "... there is no such thing as this kind of equality and that kind, but the equal exists in one and the same manner." ${ }^{3}$. For Nicomachus, equality was simple.

The concept of inequality, however, posed more of a concern. The unequal for him, " $\ldots$ is

[^37]split up by subdivisions, and one part of it is the greater, the other the less ... "4. Moreover, the greater is split into five 'species', "... one kind is the multiple, another the superparticular, another the superpartient, another the multiple superparticular, and another the multiple superpartient." ${ }^{5}$. Accompanying each of these subdivisions is the corresponding subdivision of the lesser into the, " ... submultiple, subsuperparticular, subsuperpartient, submultiplesuperparticular, and submultiple-superpartient ... " ${ }^{6}$.

As we shall see, these divisions of the greater and the lesser give a classification of the types of ratios of whole numbers. As was seen in the Introduction (section 0.1.2) the Pythagorean doctrine that 'all is number', or more precisely 'all is whole number', forbade them from having rational numbers as part of their theory of numbers. However, they were able to work around this constraint by doing all of their rational number theory using ratios of whole numbers. The drawback of this approach, at least from a modern perspective, is that the Pythagoreans do not seem to have an all-encompassing concept of a ratio $m: n$ (that is of a general rational number $m / n$ ), but rather have the five types of subdivisions mentioned above which are much more laborious to work with and name. We investigate these subdivisions below.

### 3.2 Multiples And Submultiples

The multiple and submultiple types of the greater and the lesser are fairly self-explanatory.

The multiple is, "... a number which, when it is observed in comparison with another, contains the whole of that number more than once." ${ }^{7}$. Nicomachus gives the examples of 2 being the double of 1,3 being its triple, 4 being its quadruple, and so on, explaining that, "... "more than once" means twice, or three times, and so on in succession as far as you like." ${ }^{8}$.

The submultiple, then is defined inversely to the multiple as being, "... the number which, when it is compared with a larger, is able to measure it completely more than once, and "more

[^38]than once" starts with twice and goes on to infinity." ${ }^{9}$. Thus, 1 is the subdouble of 2 , the subtriple of 3 , the subquadruple of 4 , and so on. In modern notation, the submultiples may be taken to be fractions of the form $1 / n$ for $n>1$, where $n$ is a whole number.

Nicomachus notes that the species of multiples and submultiples form infinite sequences, or series as he calls them, which take their definition from the 'natural series': $1,2,3,4, \ldots$. For example, $2,4,6,8, \ldots$ is the series representing the double compared with the natural series. Nicomachus observes that the double numbers are those that are one place apart in the natural series. Similarly $3,6,9,12, \ldots$ are the triples, and are found two places apart in the natural series. The quadruples $4,8,12,16, \ldots$ are three places apart in the natural series, and also may be regarded as the numbers one place apart in the double or even series. In the same manner, the sextuples are given by those numbers two places apart in the even series, the octuples three places apart, and so on. He mentions also that the quintuples are four places apart in the natural series, and that, like the triples, they alternate between odd and even numbers. These patterns noted by Nicomachus are examples given to show the general behavior of multiple numbers.

### 3.3 Superparticular and Subsuperparticular Numbers

With this section, Nicomachus begins naming the types of what we would call fractions, but which he views as ratios of whole numbers. The first type that he focuses on is the superparticular which in modern notation would be fractions (in lowest terms) of the form

$$
\begin{equation*}
1+\frac{1}{n}=\frac{n+1}{n} \tag{3.1}
\end{equation*}
$$

which is described without fractions as being, "... a number that contains within itself the whole of the number compared with it, and some one factor of it besides." ${ }^{10}$. From this definition, we may also state the definition of superparticular numbers as being fractions of the

[^39]form
$$
1+\frac{\mathbf{1}}{n}=\frac{n+d}{n}, \text { where } d \text { divides } n
$$

This definition turns out to be the same as the one given in 3.1, since if $d$ divides $n$, then $n=d \cdot m$ for some whole number $m$, and so

$$
\frac{n+d}{n}=\frac{d m+d}{d m}=\frac{d(m+1)}{d m}=\frac{m+1}{m}
$$

which is of the required form.

It is important to keep in mind that according to Nicomachus, the superparticular is not a fraction. It is, rather, a specific whole number. However, inherent in its definition is a comparison to another whole number, so it cannot be looked upon as a whole number by itself, but rather must be seen relative to another number. This is why Nicomachus refers to these numbers as being relative numbers. The same will be true of the other types of relative number that we explore below; they are whole numbers, but are defined in comparison with another whole number, and this number compared to must be kept in mind at all times. The two numbers are paired together.

The first few such pairs of relative numbers, or ratios, have specific names. The ratio 3:2 corresponding to the fraction $1+1 / 2=3 / 2$ is called the sesquialter, the whole in this case being 2 along with a factor of a half of 2 (that is 1 ) the total being 3 , all compared to the whole which is 2 . The notation of a ratio of sesquialter:whole is to be taken as two whole numbers connected in the manner described above. The ratios $6: 4,9: 6,12: 8$, etc. are also sesquialters.

The modern viewpoint on fractions like $6 / 4,9 / 6,12 / 8, \ldots$ is that they are simply not reduced to the essential 'fraction in lowest terms' $3 / 2$. Nicomachus, however, does not view the ratio, or pair of relative numbers $6: 4$ in quite the same way as, say, the ratio $3: 2$. They are of the same form (sesquialter), but they spring from different wholes, 4 and 2 in particular, and thus are different situations entirely. The one points out the relation between the numbers 6 and 4 when one views 4 as the whole, the other situation looks at how the numbers 3 and 2 relate taking 2 to be the whole. The idea of reduction to lowest terms was completely alien to Nicomachus's
view of ratios because for him the defining of the relation of a number to a whole was what was important, and the whole could be any number you wished, or that suited your purpose.

As to other examples, the ratio $4: 3$ is named the sesquitertian, and the ratio $5: 4$ the sesquiquartan, with the nomenclature extending on in the same way for similar types of ratios.

The subsuperparticular numbers are those whose fractional form in modern notation would be

$$
\begin{equation*}
\frac{1}{1+1 / n}=\frac{n}{n+1} . \tag{3.2}
\end{equation*}
$$

In Nicomachus' terms, seen as a a pair of relative numbers, or a ratio, and not a fraction, the definition of the subsuperparticular (analogous to the definition of the superparticular) is that it is a number that is contained along with one part of itself in the number compared with it. Thus the subsesquialter is the number 2 such that 2 along with one part (i.e. a half) of itself (that is 1 ) is contained in the number compared to it, which is 3 , giving the ratio $2: 3$. In this case, the notation of the ratio is subsesquialter:whole, and the ratios $4: 6,6: 9,9: 12$, etc. share this relationship relative to one another. Similarly, of course, $3: 4$ is the subsesquitertian, $4: 5$ is the subsesquiquartan, etc.

Nicomachus adds at the end of his discussion of these numbers, that the first forms of each type of ratio, the 'root numbers' as he puts it, only differ by 1 - that is they have the form $n+1: n$ or $n: n+1$, and that the other forms are built up from these using the series of multiple numbers described in"the previous section. He also notes that, "... the fraction after which each of the superparticulars is named is seen in the lesser of the root numbers, never in the greater." ${ }^{11}$ meaning that, for example, the suffix -alter in sesquialter comes from the the number 2 which is the lesser of the two numbers in the root-form ratio 3:2. Thus the nomenclature for the superparticular numbers is defined.

### 3.3.1 Some Interesting Observations By Nicomachus

In order to convince his reader that the multiple form of relative number is older and more elementary than the superparticular, Nicomachus gives us a demonstration. He sets out the

[^40]different types of multiples up to the tenfold multiple giving the first 10 members of each type in a table (see below), and proceeds to observe various numerical patterns:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

The first thing he notes is that the array is identical if one looks at either the rows or the columns, and so when speaking of a row, he means either the row or the column (or both together which have the form of a $\Gamma$ ). The second row consists of the doubles of the first row, and the successive entries in the second row differ from the corresponding entries in the first row by amounts increasing as the natural series. That is, $2-1=1,4-2=2,6-3=3$, $8-4=4$, and so on.

Similarly, he notes that the third row consists of the triples, which differ from corresponding numbers in the first row by successive even numbers; $3-1=2,6-2=4,9-3=6,12-4=8$, and so on. And finally, the fourth row is the quadruples, with differences in terms from the first row being the triples; $4-1=3,8-2=6,12-3=9,16-4=12$, etc.. He doesn't list anymore examples explicitly, but states that, "... in subsequent forms of the multiple the analogy will hold throughout." ${ }^{12}$

Nicomachus then proceeds to note that if one compares the second and third rows in the array, one sees the various forms of the first species of superparticular, the sesquialter; that is, 3:2 in the first position, 6:4 in the second, 9:6 in the third, and so on. In the same way exactly, comparing the fourth and third rows gives the various forms of the sesquitertian, 4:3, 8:6, 12:9, $16: 12$, etc. Since the rows of multiples had to precede the comparison of rows which produced

[^41]the superparticulars in the way described, Nicomachus draws the conclusion that, '... by divine nature, not by our convention or agreement, the superparticulars are of later origin than the multiples." ${ }^{13}$, thereby answering the question he posed at the outset. He also notes for the reader that, "... in the other multiple and superparticular relations you will see that the results are in harmony and not by any means inconsistent as you go on to infinity." ${ }^{14}$, meaning, presumably, that all the other superparticular and multiple relationships may be found in the table (or an extension of the table with more rows and columns) in the same manner as the examples described above.

### 3.3.2 More Interesting Observations By Nicomachus

The rest of the observations made by Nicomachus of patterns among the numbers in the above table have nothing to do with superparticular or multiple numbers.

The first observation is that, "The terms at the corners are units; the one at the beginning a simple unit, that at the end the unit of the third course, and the other two units of the second course appearing twice; so that the product (of the first two) is equal to the square (of the last)." ${ }^{15}$.

According to Heath, the reasons for calling 10 the unit of the second course and 100 the unit of the third course are given by Iamblichus.

The first is the view of a square number as a race-course formed of successive numbers from 1 (as start) up to $n$, the side of the square, which is the turningpoint, and then back again through $(n-1),(n-2)$, etc., to 1 (the goal), thus:

$$
\begin{aligned}
& 1+2+3+4 \ldots+(n-2)+(n-1)+\searrow_{n} \\
& 1+2+3+4 \ldots+(n-2)+(n-1)+\swarrow \quad .
\end{aligned}
$$

... He observes that it was on this principle that, after 10 , which was called the unit of the second course, the Pythagoreans regarded $100=10 \cdot 10$ as the unit of

[^42]the third course, $1000=10^{3}$ as the unit of the fourth course, and so on, since
\[

$$
\begin{align*}
1+2+3+\cdots+10+9+8+\cdots+2+1 & =10 \cdot 10  \tag{3.3}\\
10+20+30+\ldots+100+90+80+\ldots+20+10 & =10^{3}  \tag{3.4}\\
100+200+300+\ldots+1000+900+\ldots+200+100 & =10^{4}
\end{align*}
$$
\]

and so on. Iamblichus sees herein the special virtue of 10: but of course the same formulae would hold in any scale of notation as well as the decimal." ${ }^{16}$

So, equation 3.3 represents the first course whose length, or unit of measure, is 10 , and similarly equation 3.4 represents the second course whose length, or unit of measure, is 100 , and so on. The use of the number 10 specifically probably had to do with the Pythagorean view of 10 as being the perfect number (see Chapter 1, Section 2 above for more details).

Back to the table of numbers, Nicomachus remarks that the terms on the diagonal are squares, and further, "... those flanking them on either side are all heteromecic, unequal, and the products of sides of which one is greater than the other by unity;" ${ }^{17}$. That is to say that the numbers on the super and subdiagonals of the square are numbers of the form $n(n+1)$, where $n$ is the length of the side of the square number above the heteromecic number ${ }^{18}$, and $n+1$ is the side length of the square number below the heteromecic number in the arrangement

$$
\begin{array}{lccc}
\ldots & n^{2} & n(n+1) & n(n+2) \ldots \\
\ldots & (n+1) n & (n+1)^{2} & (n+1)(n+2) \ldots \\
\ldots & (n+2) n & (n+2)(n+1) & (n+2)^{2} \ldots
\end{array}
$$

He goes on to say that, "... the sum of two successive squares and twice the heteromecic numbers between them is always a square, and conversely a square is always produced from the two heteromecic numbers on the sides and twice the square between them." ${ }^{19}$, which amounts

[^43]to saying, in modern terms, that
$$
n^{2}+2 n(n+1)+(n+1)^{2}=(2 n+1)^{2}
$$
and that
$$
n(n+1)+(n+1)(n+2)+2(n+1)^{2}=[2(n+1)]^{2}
$$

These are both identities that are easily verified.
With this, Nicomachus ends his discussion of the multiple and superparticular numbers, and the interesting numerical relations and patterns that may be obtained from the array of multiple numbers. However, he leaves us with the assurance that if one was so inclined, "An ambitious person might find many other pleasing things displayed in this diagram, upon which it is not now the time to dwell ... "20.

### 3.4 Superpartients And Subsuperpartients

Nicomachus promptly gives this definition of superpartients: "It is the superpartient relation when a number contains within itself the whole of the number compared and in addition more than one part of it; and "more than one" starts with 2 and goes on to all the numbers in succession." ${ }^{21}$. Thus, in modern notation, the superpartient is a ratio of two numbers represented by the fraction

$$
\begin{equation*}
1+\frac{m}{n}=\frac{n+m}{n}, 1<m<n . \tag{3.5}
\end{equation*}
$$

Specifically, if $m=2$ in equation 3.5 above, the ratio is called a superbipartient, and similarly it is called a supertripartient if $m=3$, a superquadripartient if $m=4$, a superquintipartient if $m=5$, and so on.

As with the superparticulars above, we may write the definition of a superpartient more generally as a fraction of the form

$$
\begin{equation*}
\frac{n+m d}{n}, \text { where } d \text { divides } n . \tag{3.6}
\end{equation*}
$$

[^44]Thus 3.5 gives the root form of the superpartients, and 3.6 gives the general form. Hereafter we will only discuss the reduced form, it being understood that there exists a more general form.

The first example of a superpartient, then, is the superbipartient $1+2 / 3=5 / 3$, which Heath notes was also known by the name superbitertius ${ }^{22}$. Nicomachus notes that in order to obtain a superpartient and not a superparticular, we cannot use the number 2 as number compared to (the denominator), since 2 halves is, of course, a whole giving a multiple number (namely $1+2 / 2=2$ ). The first example of a supertripartient would then be the number $1+3 / 4=7 / 4$, also called the supertriquartus by Heath. Nicomachus again notes that using the number 4 as the denominator (that is with $n=4$ in equation 3.5), the first superpartient is when $m=3$ since $m=1$ gives the sesquitertius form of the superparticular, and $m=2$ gives $1+2 / 4=$ $1+1 / 2=3 / 2$ which is the sesquialter. Thus one must be careful in constructing superpartients that one is not constructing some other form of number (a multiple, superparticular, or even another form of superpartient) instead.

The subsuperpartient has the corresponding definition of being the number formed, "... whenever a number is completely contained in the one compared with it, and in addition several parts of it, $2,3,4$, or 5 , and so on." ${ }^{23}$. The modern form of the root form of the subsuperpartient is then

$$
\begin{equation*}
\frac{1}{1+\frac{m}{n}}=\frac{n}{n+m}, 1<m<n . \tag{3.7}
\end{equation*}
$$

Nicomachus does not give any specific examples or nomenclature for the subsuperpartients, but we may safely assume that, since the prefix sub- gives the reciprocal of the fraction in question, the names of the various subsuperparticulars correspond to the names of the superparticulars that they are the reciprocals of with the prefix sub- attached.

With this definition of a superpartient in hand, Nicomachus goes on to present a method of

[^45]generating each type of superpartient, in particular superpartients of the form
\[

$$
\begin{equation*}
1+\frac{n-1}{n}=\frac{2 n-1}{n}, n \geq 3 \tag{3.8}
\end{equation*}
$$

\]

The way he states this construction is that,
... we set forth the successive even and odd numbers [that is, the natural series of numbers], beginning with 3 , and compare with them simple series of odd numbers only, from 5 in succession, first to first - that is 5 to 3 , - second to second - that is, 7 to 4 , - third to third - that is 9 to 5 , - fourth to fourth - that is, 11 to 6 , - and so on in the same order as far as you like. ${ }^{24}$

In this way he produces a table of these kinds of superpartients with the ratios found for each $n$ in equation 3.8 being the 'root forms' of each type of superpartient which he then proceeds to multiply by $2,3,4$, etc. to show the different ways that these particular superpartients may appear. In modern terminology, he lists equivalent fractions to the root forms that have not been reduced. The table is as follows,

| Root-Forms | 5 | 3 | 7 | 4 | 9 | 5 | 11 | 6 | 13 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 10 | 6 | 14 | 8 | 18 | 10 | 22 | 12 | 26 | 14 |
|  | 15 | 9 | 21 | 12 | 27 | 15 | 33 | 18 | 39 | 21 |
|  | 20 | 12 | 28 | 16 | 36 | 20 | 44 | 24 | 52 | 28 |
|  | 25 | 15 | 35 | 20 | 45 | 25 | 55 | 30 | 65 | 35 |
|  | 30 | 18 | 42 | 24 | 54 | 30 | 66 | 36 | 78 | 42 |
|  | 35 | 21 | 49 | 28 | 63 | 35 | 77 | 42 | 91 | 49 |
|  | 40 | 24 | 56 | 32 | 72 | 40 | 88 | 48 | 104 | 56 |
|  | 45 | 27 | 63 | 36 | 81 | 45 | 99 | 54 | 117 | 63 |

the first column being forms (multiples) of the first type of superbipartient 5:3, the second column being forms of the first type of supertripartient 7:4, the third column being forms of the superquadripartient 9:5, and so on.

[^46]
### 3.5 Multiple Superparticulars And Submultiple Superparticulars

As the name implies, the multiple superparticular numbers and their corresponding inverses designated by the prefix sub- are a combination of multiple numbers and superparticular numbers. Nicomachus gives this definition: "Now the multiple superparticular is a relation in which the greater of the compared terms contains within itself the lesser term more than once and in addition some one part of it, whatever this may be." ${ }^{25}$. In modern (fractional) notation, we may express these numbers (or ratios of numbers) as being of the form

$$
\begin{equation*}
m+\frac{1}{n}=\frac{m n+1}{n} . \tag{3.9}
\end{equation*}
$$

They are very similar to the simple superparticulars except that in this case a multiple of the whole $(m \cdot 1=m)$ instead of just a whole is added to a single part of the number compared (which is $n$ ).

The nomenclature is straightforward putting first the name of the multiple, $m$, followed by the name of the adjoined superparticular. For example, the ratio 5:2 (or equivalently the fraction $2+1 / 2=5 / 2$ would be known as the double sesquialter ratio; so, formally speaking, 5 would be the double sesquialter of 2 . Heath gives the name duplex sesquialter for this ratio as an alternative. ${ }^{26}$ Similarly, 7 would be the double sesquitertian of 3 , and, to give a slightly more involved example, 13 would be the triple sesquiquartan of 4 .

Nicomachus' again notes for his reader that for each type of multiple superparticular there are many forms, corresponding, in modern terms, to multiplying both terms of the ratio by the same number to produce a fraction that is not in lowest terms. He does this by means of comparing series. "The successive terms beginning with 5 and differing by 5 will be without exception double sesquialters of all the successive even numbers from 2 on, when terms in the same position in the series are compared;" ${ }^{27}$, meaning that comparing the series $5,10,15,20$,

[^47]$25, \ldots$ term by term with the series $2,4,6,8,10, \ldots$ will give you the double sesquialter ratio in each successive comparison. He then goes on to say the same thing for the double sesquitertian $7: 3$ (how it has equivalent but distinct forms $7 n: 3 n, n>1$ ), and also for the double sesquiquartan $9: 4$, the triple sesquialter $7: 2$, and the triple sesquitertian 10:3. These forms of the multiple superparticulars may again be seen by comparing various rows in the multiplication table 3.3.1 given above.

As for the submultiple superparticulars, these are simply fractions of the form

$$
\begin{equation*}
\frac{1}{m+\frac{1}{n}}=\frac{n}{m n+1} \tag{3.10}
\end{equation*}
$$

the reciprocals once again of the corresponding multiple superparticulars. Nicomachus does not give any explicit definition, nomenclature or examples of the submultiple superparticulars, and merely says that, "It is plain that here too the smaller terms have names corresponding to the larger ones, with the prefix sub-, according to the nomenclature given them all." ${ }^{28}$, following the pattern established with the previously discussed types of ratios.

### 3.6 Multiple Superpartients and Submultiple Superpartients

These are the final types of ratio or fraction that are discussed by Nicomachus. He defines them as follows:

The multiple superpartient is the remaining relation of number. This, and the relation called by a corresponding name with the prefix sub-, exist when a number contains the whole of the number compared more than once (that is twice, thrice, or any number of times) and certain parts of it, more than one, either two, three, or four, and so on besides. ${ }^{29}$

The modern interpretation of these definitions is that the multiple superpartients are fractions of the form

$$
\begin{equation*}
k+\frac{m}{n}=\frac{k n+m}{n}, \text { for some } 1<m<n \tag{3.11}
\end{equation*}
$$

[^48]and the submultiple superpartients are the reciprocals of these having the form
\[

$$
\begin{equation*}
\frac{1}{k+\frac{m}{n}}=\frac{n}{k n+m}, \text { for some } 1<m<n \tag{3.12}
\end{equation*}
$$

\]

where in each case $k$ is the multiple of the whole, $n$ is the number compared, and $m$ is the number (more than 1) of parts of $n$ added to the multiple of the whole. Again $m$ is not a number such that the fraction $m / n$ may be reduced to a superparticular form $1 / n$, or to another, more reduced type of superpartient. In other words $m$ and $n$ must be relatively prime as in the case of the superpartients.

Nicomachus gives as examples that 8 is the double superbipartient of 3 , and that again we may produce equivalent forms of this by comparing term by term the series $8,16,24,32,40, \ldots$ with the series $3,6,9,12,15, \ldots$; that is comparing the eighth and third rows of the array of multiples (table 3.3.1) given above. To give a more involved example of the nomenclature, 19 would be the quadruple supertripartient of 4 . The nomenclature for the submultiple superpartients is again noted by Nicomachus to be the same as that for the corresponding multiple superpartients, but with the prefix sub- attached.

### 3.7 A Method Of Production Of All Ratios

To conclude his discourse on relative number, Nicomachus presents us with,
... a method very exact and necessary for all discussion of the nature of the universe which very clearly and indisputably presents to us the fact that that which is fair and limited, and which subjects itself to knowledge, is naturally prior to the unlimited, incomprehensible and ugly, and furthermore that the parts and varieties of the infinite and unlimited are given shape and boundaries by the former, and through it attain to their fitting order and sequence, and like objects brought beneath some seal or measure, all gain a share of likeness to it and similarity of name when they fall under its influence. ${ }^{30}$

Clearly this method has great metaphysical significance for Nicomachus, but he brings it down to the realm of relative numbers by saying that, in particular, the method points out that, " ... the rational part of the soul will be the agent which puts in order the irrational part, and

[^49]passion and appetite, which find their places in the two forms of inequality, will be regulated by the reasoning faculty as though by a kind of equality and sameness." ${ }^{31}$. By this he means that he views the concept of equality as being more primal or fundamental than that of inequality. He then goes on to use the method he discusses to show this fact clearly to the reader using relative numbers.

What he does is to generate, by algorithmic means beginning with equality, first the multiple ratios, then the superparticular (and multiple superparticular) ratios, and finally the superpartient (and multiple superpartient) ratios, thereby showing that all forms of relative number," $\ldots$ are produced out of equality, first and alone, as from a mother and root." ${ }^{32}$.

The algorithm he uses to do this, "... following which the whole aforesaid advance and progress from equality may go on with out failure." ${ }^{33}$, is as follows: He works with ordered triples of numbers going either from lowest to highest or highest to lowest, but in either case having the ratio between the first and second number equal to the ratio between the second and third number. He then produces a second ordered triple from the first using the mapping:

$$
\begin{equation*}
(a, b, c) \rightarrow(a, a+b, a+2 b+c) . \tag{3.13}
\end{equation*}
$$

Beginning with equality and applying algorithm 3.13, he produces the various multiple ratios. For example, beginning with ( $1,1,1$ ), he obtains through successive applications of 3.13:

$$
(1,1,1) \rightarrow(1,2,4) \rightarrow(1,3,9) \rightarrow(1,4,16) \rightarrow \cdots .
$$

Beginning with any other triple representing equality, say (3,3,3), he gets the same results:

$$
(3,3,3) \rightarrow(3,6,12) \rightarrow(3,9,27) \rightarrow(3,12,48) \rightarrow \cdots
$$

So, in short, this is proof for Nicomachus that the concept of equality precedes the concept of a multiple.

[^50]Since the above also shows that multiples only produce other multiples of higher order, to get another kind of ratio, Nicomachus applies the algorithm to one of the multiple triples, but with the entries in reverse order, such as $(4,2,1)$. This gives:

$$
(4,2,1) \rightarrow(4,6,9) \rightarrow(4,10,25) \rightarrow(4,14,49) \rightarrow \cdots
$$

which is the sesquialter ratio $3: 2$ after one iteration, the double sesquialter ratio $5: 2$ after two iterations, the triple sesquialter ratio 7:2 after three iterations, and so on. The same process beginning with a different reversed multiple triple, say ( $9,3,1$ ), gives:

$$
(9,3,1) \rightarrow(9,12,16) \rightarrow(9,21,49) \rightarrow(9,30,118) \rightarrow \cdots ;
$$

the sesquitertian ratio $4: 3$ after one iteration, the double sesquitertian ratio $7: 3$ after two iterations, the triple sesquitertian ratio 10:3 after three iterations, and so on. In general, reversing the $n^{\text {th }}$ multiple triple of numbers to get ( $n^{2}, n, 1$ ), and then applying algorithm 3.13 to it gives,

$$
\begin{equation*}
\left(n^{2}, n, 1\right) \rightarrow\left(n^{2}, n(n+1),(n+1)^{2}\right) \rightarrow\left(n^{2}, n(2 n+1),(2 n+1)^{2}\right) \rightarrow \cdots ; \tag{3.14}
\end{equation*}
$$

getting the general superparticular ratio $n+1: n$ after the first iteration, and getting the ratio $2 n+1: n$, which in fractional form is $2 n+1 / n=2+1 / n$, and thus is easily seen to be the double superparticular ratio after the second iteration, and so on in successive iterations. So in this way, the multiple ratios produce all different types of both the superparticular and multiple superparticular ratios, making the multiple ratios the more fundamental type of ratio.

Since iterating the superparticular ratio triples gives multiple superparticular ratios, in order to produce the superpartient and multiple superpartient ratios we must apply algorithm 3.13 to some other type of triple. The obvious candidates for this are the superparticular ratios in reverse order, and indeed they do produce superpartient and multiple superpartient ratios. For example, reversing the order of the sesquialter ratio triple $(4,6,9)$ from 3.7 above to get the triple $(9,6,4)$, and applying the algorithm to it successively, we get:

$$
\begin{equation*}
(9,6,4) \rightarrow(9,15,25) \rightarrow(9,24,64) \rightarrow(9,33,121) \rightarrow \cdots, \tag{3.15}
\end{equation*}
$$

which gives us the superbipartient ratio 5:3 after the first iteration, the double superbipartient ratio 8:3 after the second iteration, the triple superbipartient ratio 11:3 after the third iteration,
and so on. Similarly, using the reverse of the sesquitertian ratio from 3.7, we get:

$$
\begin{equation*}
(16,12,9) \rightarrow(16,28,49) \rightarrow(16,44,121) \rightarrow \cdots, \tag{3.16}
\end{equation*}
$$

which is the supertripartient ratio 7:4 after one iteration, the double supertripartient ratio 11:4 after two iterations, and so on. Again, looking at the situation in general, the reverse of the $n^{\text {th }}$ superparticular ratio triple $\left((n+1)^{2}, n(n+1), n^{2}\right)$ from 3.14 iterates to give:

$$
\begin{align*}
\left((n+1)^{2}, n(n+1), n^{2}\right) & \rightarrow\left((n+1)^{2},(n+1)(2 n+1),(2 n+1)^{2}\right) \\
& \rightarrow\left((n+1)^{2},(n+1)(3 n+2),(3 n+2)^{2}\right) \rightarrow \cdots . \tag{3.17}
\end{align*}
$$

After the first iteration, we have the ratio $2 n+1: n+1$, which in fractional notation is

$$
\frac{2 n+1}{n+1}=\frac{(n+1)+n}{n+1}=1+\frac{n}{n+1}
$$

which is the $n^{\text {th }}$ superpartient ratio. The second iteration, in a similar manner, gives the ratio $3 n+2: n+1$, which in fractional notation is

$$
\frac{3 n+2}{n+1}=\frac{(2 n+2)+n}{n+1}=2+\frac{n}{n+1},
$$

the double superpartient ratio of the $n^{\text {th }}$ degree. So again, the superpartient and multiple superpartient ratios are produced from the superparticular ratios, supporting Nicomachus' theory that superparticular and multiple superparticular ratios are more fundamental than superpartient and multiple superpartient ratios.

The construction in 3.17 above does not give all types of superpartient and multiple superpartient ratios. However, taking a triple of numbers produced by iterating 3.14 or 3.173 or 4 or more times, reversing the order, and then applying algorithm 3.13 to it, and even repeating this process with the triples that we obtain, we may produce many more kinds or ratios. In fact, by using this method, Nicomachus claims (without proof) that all ratios may be produced.

To demonstrate this, Nicomachus applies the algorithm to the reverse of the first superbipartient from 3.15 above to obtain:

$$
(25,15,9) \rightarrow(25,40,64) \rightarrow(25,65,169) \rightarrow \cdots,
$$

which is the supertripartient ratio 8:5 after one iteration, the double supertripartient ratio 13:5 after two iterations, and so on. The point being that from superbipartient ratios, supertripartient ratios are produced, and moreover these supertripartient ratios differ from the ones in 3.16 above in terms of the numbers that are compared.

An interesting fact noted by Nicomachus concerning algorithm 3.13 is that no matter what triple of numbers one begins with, as long as the numbers are in a continued proportion with the first number being a square, when one applies the algorithm to this triple, "... the first term becomes the smallest, and invariably the extremes are squares." ${ }^{34}$. To see this, take the triple with first term $m^{2}$ and common ratio $n: m^{2}$, and apply the algorithm to it as follows:

$$
\left(m^{2}, n, \frac{n^{2}}{m^{2}}\right) \rightarrow\left(m^{2}, n+m^{2}, m^{2}+2 n+\frac{n^{2}}{m^{2}}\right)=\left(m^{2}, n+m^{2},\left(\frac{m^{2}+n}{m}\right)^{2}\right)
$$

Nicomachus' conclusion is then clear since

$$
m^{2}<n+m^{2}<\left(\frac{m^{2}+n}{m}\right)^{2}
$$

[^51]
## Chapter 4 Side and Diagonal Numbers

### 4.1 Proclus on Side and Diagonal Numbers

One of the most famous appearances of side and diagonal numbers occurs in Plato's discussion of the 'Geometrical' or 'Nuptial' Number in the eighth book of the Republic where he refers to, "... a hundred numbers squared upon rational diameters of a square, the side of which is five, each of them being less by one or less by two perfect squares of irrational diameters..."1. According to Plato, this number, " . . . represents a geometrical figure which has control over the good and evil of births." ${ }^{2}$ The passage is extremely vague and cryptic, and historians today are still unclear as to its meaning. Not even the mathematics used by Plato makes any sense. One possible explanation for this may be that through the course of the passage being translated numerous times, the mathematics was distorted beyond recognition by translators who were not very mathematically aware. Also, Plato may have written the passage cryptically so as to keep the information from falling into the wrong hands and being abused. There is also the speculation that Plato, for whatever reason, was simply speaking intelligent nonsense!

In any case, as part of his extensive explanation and interpretation of the meaning of the Nuptial Number in the 'Commentary on the Republic', the Neo-Platonist Proclus explains more fully the what these 'Side and Diagonal' numbers are. It is on this explanation that our attention lies.

[^52]He states as a known fact that it is impossible for the diagonal of a square to be rational when the side is rational, since (in modern notation) there do not exist numbers $a$ and $b$ such that $a^{2}=2 b^{2}$.

This, he claims, demonstrates the error of Epicurus in making the atom the measure of all things and the error of Xenocrates in postulating the existence of an indivisible line that is the measure of all lines. His reasoning in this is, presumably, that if there was an atom that was the measure of all things (or an indivisible line that was the measure of all lines), there would be a finite (albeit large) number of atoms (or indivisible lines) making up both the side and the diagonal, and so the ratio between them would be a rational number. In a modern context, one might question this refutation of Proclus on the grounds that, as far as we know Epicurus was correct - things are composed of atoms. However, we also know that these atoms are divisible, and, more fundamentally, via the results of modern physics, we are theoretically unable to measure how many atoms are in certain length (of, say, a ruler) since the atoms are constantly moving. Given this, the question of determining whether or not the ratio of the length of the diagonal of a square to the length of its side is rational is meaningless since there are no fixed lengths to take the ratios of.

One might argue that although in the measuring of 'real' objects the ratio is meaningless, the results of Proclus are still true for 'ideal' geometrical objects. This is true. However, if this is the case, Proclus is guilty of confusing the abstraction of an ideal square and the 'real' things that are being measured by Epicurus and Xenocrates.

All in all, though, since the Pythagoreans were interested in 'ideal' forms, the non-existence of rational numbers $a$ and $b$ such that $a^{2}=2 b^{2}$, where $a$ and $b$ can be thought of as being the diagonal and side lengths of an 'ideal' square, effectively demonstrated to the Pythagoreans that there exist incommensurable magnitudes - in direct opposition to their doctrine that all is number.

He goes on to say that the Pythagoreans and Plato knew that given a square of rational side length, the diagonal is not rational, but the square of the so called 'rational diagonal'
approximating the diagonal by an integer is either one unit greater or one unit less than the sum of the squares of the sides (which he calls the double square of the side).

Again he reminds us that the double square of the side should in fact be equal to the square of the diagonal (the 'irrational diagonal'). Plato was certainly aware of this fact since in the Meno ${ }^{4}$, he proves to a young serving boy that the square of the diagonal is equal to the double square of the side in the particular case of a square of side length two.

Proclus then tells us that the Pythagoreans cited a square of side length 2 as an example of the case where the square of the rational diagonal exceeds the double square of the side by one unit - that is $(3)^{2}-2(2)^{2}=1$ - and a square of side length 5 as an example of the case where the square of the rational diagonal is one unit less than the double square of the side that is $(7)^{2}-2(5)^{2}=-1$.

He then tells us that the Pythagoreans had an elegant proposition concerning these side and diagonal numbers. This proposition (in modern notation, of course) is as follows: If $s$ denotes the rational side length of a square and $d$ the corresponding rational diagonal, then we may produce a new rational side $S$ and a new rational diagonal $D$ using

$$
\begin{align*}
& S=d+s  \tag{4.1}\\
& D=d+2 s . \tag{4.2}
\end{align*}
$$

This proposition is referenced by Proclus to Euclid's Elements, Book II, Proposition 10 where, he says, it is proved by means of figures. ${ }^{5}$ However, Proclus seems to be a little confused here. The proposition of Euclid proves that given the actual side and diagonal lengths of a square (not the rational approximations), call them $s_{a}$ and $d_{a}$, then equations above, 4.1 and 4.2 , give new lengths $S_{a}=d_{a}+s_{a}$ and $D_{a}=d_{a}+2 s_{a}$ which are the actual side and diagonal lengths of a bigger square. In more technical language, Euclid proves that if $d_{a}^{2}=2 s_{a}^{2}$, then

$$
\left(d_{a}+2 s_{a}\right)^{2}=2\left(d_{a}+s_{a}\right)^{2}
$$

[^53]So Proclus is referencing the wrong result. The result that Proclus gives is still valid, just not proven in Euclid. The result that Proclus has in mind is explored more fully below where he tells us of an 'arithmetic' method for deducing it - in contrast to the 'geometric' method which he gives here to prove proposition II. 10 of Euclid's Elements.

Proposition II. 10 of the Elements states that:
If a straight line be bisected and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line ${ }^{6}$.

The 'geometric' proof of this result, given by Proclus, is as follows. In order to deduce the desired result, Proclus uses the above proposition applied to a specific line segment $A B \Gamma \triangle$ with $A B=B \Gamma$ and $\Gamma \Delta$ corresponding to the diagonal of the square of side $A B$. From these definitions, we know that

$$
\begin{equation*}
\Gamma \Delta^{2}=2 A B^{2} \tag{4.3}
\end{equation*}
$$

Then, the proposition of Euclid says that

$$
\begin{equation*}
A \Delta^{2}+\Gamma \Delta^{2}=2\left[A B^{2}+B \Delta^{2}\right] \tag{4.4}
\end{equation*}
$$

Subtracting (4.3) from (4.4), Proclus shows that

$$
\begin{equation*}
A \Delta^{2}=2 B \Delta^{2} \tag{4.5}
\end{equation*}
$$

In the notation of (4.1) above, letting $A B=B \Gamma=s$ and $\Gamma \Delta=d$, we have $A \Delta=A B+B \Gamma+$ $\Gamma \Delta=2 s+d$ and $B \Delta=B \Gamma+\Gamma \Delta=s+d$ so that equation (4.5) gives us

$$
\begin{equation*}
(2 s+d)^{2}=2(s+d)^{2} \tag{4.6}
\end{equation*}
$$

That is, $S$ and $D$ from equations (4.1) and (4.2) above are the rational side and rational diagonal lengths of a new square. That is, given the side length of a square and a rational

[^54]approximation to the diameter of that square, we obtain a new side length of a different, bigger square and a rational approximation to the diameter of this new square. The advantage of this being that, because the new side and rational diagonal lengths are bigger than the old ones, the new rational diagonal is a closer approximation to the actual diagonal length. A geometric representation of this proof is as follows:


Proclus now proceeds to give the 'arithmetic' method for demonstrating the same proposition. He starts by taking a diagonal length of 1 and also a side length of 1 as a starting point in the algorithm given above (equations (4.1) and (4.2)). He does this because the method is one used by the Pythagoreans. Moreover, since, he notes earlier, as part of their number mysticism they claimed that the unit contains the essence of all things, it was evident to them that the unit was at once both a side and a diagonal ${ }^{7}$.

Proclus then restates that this diagonal of length 1 is a 'rational diagonal' if its square is one unit inferior to the sum of the squares of the sides; that is the double square on the side. He then redundantly says that if we add, in effect, one unit to the square of the diagonal, which is itself a unit in length, we do in fact obtain the double square on the unit which we took as the length of the side. I believe that Proclus is belaboring this point in order to ensure that his audience is convinced that the starting point in the algorithm for calculating successive side

[^55]and rational diagonal numbers is a valid rational diagonal number. That is, a unit length for both the side and the diagonal satisfies (in modern notation),
\[

$$
\begin{equation*}
d^{2}-2 s^{2}= \pm 1 \tag{4.7}
\end{equation*}
$$

\]

which it does since $(1)^{2}-2 \cdot(1)^{2}=-1$. (This equation (4.7) is of course known to modern day number theorists as an example of a Pell Equation). I am reminded of the scrutiny one applies to checking the validity of the first step in a modern day induction argument, and so maybe Proclus was aware of the Principle of Induction and was really presenting such an argument, although in a fairly loose (by modern standards) form. However, this is probably not the case. It is much more likely that Proclus belabors the point because he is aware of the Pythagorean ambivalence towards using unity as a number, and he wishes to ensure that everything is properly justified with regards to it.

Proclus goes on to apply the algorithm presented in equations (4.1) and (4.2) to the starting point of $s=1$ and $d=1$. By giving $s$ and $d$ the names $s_{1}$ and $d_{1}$ respectively, we may reformulate equations (4.1) and (4.2) using more modern notation:

$$
\begin{align*}
& s_{n+1}=d_{n}+s_{n}  \tag{4.8}\\
& d_{n+1}=d_{n}+2 s_{n} . \tag{4.9}
\end{align*}
$$

In an earlier passage ${ }^{8}$, Proclus explicitly computes the first three iterates of this algorithm getting:

$$
\begin{array}{lll}
s_{1}=1 & s_{2}=2 & s_{3}=5 \\
d_{1}=1 & d_{2}=3 & d_{3}=7
\end{array}
$$

and verifies that these are all pairs of side and diagonal numbers by checking that the square of the diagonals are within one unit of the double squares of the sides:
$(1)^{2}-2(1)^{2}=1-2(1)=-1$
$(3)^{2}-2(2)^{2}=9-2(4)=+1$
$(7)^{2}-2(5)^{2}=49-2(25)=-1$

[^56]The last verification that 49 is one unit smaller than the double of 25 is, he claims, the reason that Plato said that the number 48 was one unit less than the square of the rational diagonal of a square of side length 5 , and also two units less than the square of the irrational diagonal of the same square. The latter fact, Proclus states (again), is because the square of the diagonal equals the double square of the side.

Proclus also calculates that $s_{4}=12$ and $d_{4}=17$, and again checks that

$$
(17)^{2}-2(12)^{2}=289-2(144)=+1,
$$

remarking that the process continues on indefinitely. One may, perhaps, interpret this last statement as Proclus concluding his rather informal induction argument.

As an addendum to his earlier exposition on the side and diagonal numbers, Proclus notes that if one takes the sum of the squares of all of the diagonal numbers produced by the algorithm (4.8), it will exactly equal the double of the sum of the squares of all of the corresponding side numbers produced by (4.9). He then gives the example that $9+49=2(25+4)$, and states that it is this that gave the Pythagoreans confidence in this method. Although this is literally what Theon said, we cannot be sure this is exactly what he meant. It is questionable whether he truly meant ;'all' of the squares of the diagonal numbers since that would mean summing up an infinite series of numbers, which Theon was probably not able to do. It is more likely that he simply observed that the squares of the diagonals alternate between being one greater and one less than the double squares of the corresponding sides, and so, adding them in pairs will give an exact result similar to his example. He then simply extrapolated this result to 'all' squares since he saw no reason that the pattern would change (see the next section for more on this pattern).

### 4.2 Theon on Side and Diagonal Numbers

Theon is rather vague in his introduction to Side and Diagonal numbers ${ }^{9}$. He states (cryptically) that in the same way that numbers in general are potentially seen as having the form of triangles,

[^57]tetragons, pentagons, etc. via the production of figurate numbers by adding gnomons of various sizes to unity (this being the interpretation of this writer, the actual translation saying (as far as I can tell: in the same way that numbers in general are potentially seen as ratios(?) of(in?) triangles, tetragons, pentagons, and other figures), the ratio between side and diagonal numbers manifests in numbers for generative reasons since they are the numbers that harmonize figures. Thus, since unity is the principle of all the figures by the supreme generative reason described above (that all figures can be obtained by adding a sequence of gnomons to unity), in the same way the ratio between the diagonal and the side (of a square) has its essence in unity. Theon is quite unclear here, but it is to be guessed that his talk the relation of triangular numbers, etc., to side and diagonal numbers is merely some form of transition paragraph in his text (seeing as he has just finished discussing figurate numbers in the previous section). His main point being that both the figurate numbers and the side and diagonal numbers have their root in the monad.

Given this, he proceeds to give the 'generative reasons' by which side and diagonal numbers come about. Suppose two units, he says, for example, such that one is the diagonal and the other is the side (since unity is the principle of all things, and so must potentially by both the side and diagonal); then to the side add the diagonal and to the diagonal add two sides, since what the diagonal can do once it takes two sides to do. (The translator notes on this last that it comes from the fact that, by Pythagoras' theorem, the square of the diagonal is twice the square of the side). So, he says, through the application of the process, the diagonal has become bigger than the side. This last statement by theon, however, is somewhat unclear. He may also be saying that the new diagonal is larger than it should be for a square of the new side length (the square of it is one unit greater than the double square of the side length). We cannot be sure what exactly was meant.

Theon then proceeds to very carefully calculate the first few iterations of the algorithm that was mentioned by Proclus (equations (4.8) and (4.9)). He begins by showing that the square of the original diagonal $\left(d_{1}=1\right)$ is one unit less than the double square of the original side $\left(s_{1}=1\right)$. He then proceeds to show all his steps in applying the algorithm, and after each
iteration checks that twice the square of the side is alternately one unit greater or less than the square of the diagonal; that is:

$$
\begin{equation*}
2 s_{n}^{2}=d_{n}^{2} \pm 1 \tag{4.10}
\end{equation*}
$$

In fact, he calculates, as did Proclus, up to the fourth iteration; that is $s_{4}=12$ and $d_{4}=17$ so that:

$$
d_{4}^{2}-2 s_{4}^{2}=(17)^{2}-2(12)^{2}=289-288=1
$$

After the fourth step, he merely states that the process continues in the same manner. He seems to assume without justification that the process will continue to give similar results.

He then mentions that the proportion alternates: the square of the diagonal will be one unit greater and then one unit less than the double square of the corresponding side in successive iterations. He these points out that these diagonals and sides will always be integers (that is, expressible in terms of some fundamental beginning unit with which we started the process).

He then clarifies that, inversely to the way he presented it above (that is in (4.10)), one may compare the length of the diagonal to the length of the side, obtaining:

$$
d_{n}^{2}=2 s_{n}^{2} \pm 1
$$

that is, the square of the diagonal is alternately one unit greater or less than the double of the square of the side. He goes on to point out that because of this, in effect, that which lacks in the preceding square of the diagonal is found in excess (in theory) in the following one. In modern notation, he is saying that, given some integer $k>1$ :

$$
d_{k}^{2}=2 s_{k}^{2} \pm 1 \quad \Rightarrow \quad d_{k+1}^{2}=2 s_{k+1}^{2} \mp 1
$$

and so adding these together gives:

$$
d_{k}^{2}+d_{k+1}^{2}=2\left(s_{k}^{2}+s_{k+1}^{2}\right)
$$

This concludes Theon's discourse on the side and diagonal numbers.

It is interesting to note that, although Theon and Proclus do not mention it, many modern authors refer to the above described algorithm for finding side and diagonal numbers as an early attempt to approximate $\sqrt{2}$. The reason for this is that by dividing both sides of equation 4.10 above by $s_{n}^{2}$, we obtain

$$
2=\frac{d_{n}^{2} \pm 1}{s_{n}^{2}}=\left(\frac{d_{n}}{s_{n}}\right)^{2} \pm \frac{1}{s_{n}^{2}}
$$

As the numbers $d_{n}$ and $s_{n}$ increase in size through iterating the algorithm given by equations (4.8) and (4.9), the number $\frac{1}{s_{n}^{2}}$ becomes smaller and smaller, and consequently $\left(\frac{d_{n}}{s_{n}}\right)^{2}$ becomes closer and closer to 2 . This, however, is equivalent to saying that $\frac{d_{n}}{s_{n}}$ gets closer and closer to $\sqrt{2}$, and we see how the algorithm may be used as an approximation to $\sqrt{2}$.

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[Van61] B.L. Van der Waerden. Science Awakening. P. Noordhoff, Ltd., Groningen, Holland, 1961. English translation by Arnold Dresden with additions of the author.


[^0]:    ${ }^{1}$ Iamblichus[Iam89], pp. 2-8.

[^1]:    ${ }^{2}$ Iamblichus[Iam89], p. 6.
    ${ }^{3}$ Iamblichus[Iam89], p. 8.
    ${ }^{4}$ Iamblichus[Iàm89], p. 11.
    ${ }^{5}$ Iamblichus[Iám89], p. 11.
    ${ }^{6}$ Van der Waerden[Van61], p. 92.

[^2]:    7 Van der Waerden[Van61], p. 92-93.
    ${ }^{8}$ Van der Waerden[Van61], p. 93.
    ${ }^{9}$ Aristotle[Ari52], Metaphysics $9855^{b} .22-986{ }^{a} .3$, pp. 503-504.

[^3]:    ${ }^{10}$ Plato[Pla52], Republic VI 510 DE, p. 387.
    ${ }^{11}$ Heath[Hea21], p. 67.
    ${ }^{12}$ Van der Waerden[Van61], p. 105.
    ${ }^{13}$ Keith Crichlow in the introduction to The Theology of Arithmetic attributed to Iamblichus[Tam88], p. 25.
    ${ }^{14}$ See, for instance,Nicomachus[Nic94], p. 73.

[^4]:    ${ }^{15}$ Van der Waerden[Van61], p. 108.
    ${ }^{16}$ Van der Waerden[Van61], p. 125.
    ${ }^{17}$ Nicomachus[Nic52], I.4.1-2. p. 813.

[^5]:    ${ }^{18}$ See for instance Van der Waerden[Van61], pp. 15-36.
    ${ }^{19}$ Van der Waerden[Van61], p. 36.
    ${ }^{20}$ Van der Waerden[Van61], p. 35.
    ${ }^{21}$ Van der Waerden[Van61], p. 36.

[^6]:    ${ }^{22}$ Van der Waerden[Van61], p. 72.
    ${ }^{23}$ See Van der Waerden[Van61], p. 72.
    ${ }^{24}$ Van der Waerden[Van61], p. 72.
    ${ }^{25}$ See Van der Waerden[Van61], pp. 76-77.

[^7]:    ${ }^{26}$ Van der Waerden[Van61], p. 77.
    ${ }^{27}$ Heath[Hea21], p. 66.
    ${ }^{28}$ Van der Waerden[Van61], p. 107.

[^8]:    ${ }^{29}$ Van der Waerden[Van61], p. 97.
    ${ }^{30}$ Van der Waerden[Van61], p. 115.
    ${ }^{31}$ Heath[Hea21], p. 2.
    ${ }^{32}$ See Nicomachus[Nic52], p. 807.

[^9]:    ${ }^{33}$ Heath[Hea21], p. 98.
    ${ }^{34}$ Van der Waerden[Van61], p. 97.
    ${ }^{35}$ Introduction to Iamblichus[Iam89], p. xi.
    ${ }^{36}$ Van der Waerden[Van61], p. 91.

[^10]:    ${ }^{37}$ Heath[Hea21], p. 112.
    ${ }^{38}$ Théon[The66], p. 31.
    ${ }^{39}$ Théon[The66], p. 29.

[^11]:    ${ }^{40}$ Aristotle[Ari52], Metaphysics $1088^{a} .4-10$, p. 620.
    ${ }^{41}$ Nicomachus[Nic52], II.6.3, p. 832.
    ${ }^{42}$ Nicomachus[Nic52], I.7.1, p. 814.
    ${ }^{43}$ Nicomachus[Nic52], I.7.3, p. 814.
    ${ }^{44}$ Van der Waerden[Van61], p. 109.
    ${ }^{45}$ Aristotle[Ari52], Metaphysics $986{ }^{a} .22-28$, p. 504.

[^12]:    ${ }^{46}$ Nicomachus[Nic52], I.7.4, p. 814.
    ${ }^{47}$ Heath[Hea21], p. 71.
    ${ }^{48}$ See Nicomachus[Nic52], I.8.1 to I.13.13, pp.814-819.
    ${ }^{49}$ Nicomachus[Nic52], I.13.2 to I.13.8, pp.818-819.
    ${ }^{50}$ Van der Waerden[Van61], p. 48.

[^13]:    ${ }^{51}$ Van der Waerden[Van61], pp. 93-94.

[^14]:    ${ }^{53}$ See Plato[Plá52], Laws VII 819-820, p. 729.
    ${ }^{54}$ Proclus[Pro70a], p. 65.
    ${ }^{55}$ Heath[Hea21], p. 157.
    ${ }^{56}$ Heath[Hea21], p. 155.

[^15]:    ${ }^{57}$ See Heath[Hea21], p. 65 and p. 154.
    ${ }^{58}$ Van der Waerden[Van61], pp. 125-126.
    ${ }^{59}$ See Heath[Hea21], p. 91.

[^16]:    ${ }^{60}$ Van der Waerden[Van61], p. 110.
    ${ }^{61}$ Some would disagree with Van der Waerden on this point, but only with regards to this being the

[^17]:    ${ }^{1}$ Most of the information contained in this section comes from in Nicomachus[Nic52], I.14.1 to I.14.8, pp. 820-821.
    ${ }^{2}$ A proper divisor (or aliquot part) of a whole number $n$ is a number $k$ such that $k$ divides evenly into $n$,

[^18]:    and $k \neq n$. For example, the proper divisors of the number 8 are the numbers 1,2 , and 4 .
    ${ }^{3}$ Nicomachus[Nic52], I.14.3, p. 820.

[^19]:    ${ }^{4}$ Nicomachus[Nic52], I.15.1, p. 820.
    ${ }^{5}$ Nicomachus[Nic52], I.16.3, p. 821.
    ${ }^{6}$ Nicomachus[Nic52], I.16.4, p. 821.

[^20]:    ${ }^{7}$ Nicomachus[Nic52], I.16.8-I.16.10, p. 821.
    ${ }^{8}$ Most of the information in this section comes from Théon[The66], pp. 75-76.

[^21]:    ${ }^{9}$ Heath[Hea21], p. 75-76.
    ${ }^{10}$ Heath[Hea21], p. 75.

[^22]:    ${ }^{1}$ Most of the information in this chapter comes from Nicomachus[Nic52], II.6.1 to II.17.7, pp. 831-839.
    ${ }^{2}$ See Nicomachus[Nic52], II.1.1, p. 829.
    ${ }^{3}$ Nicomachus[Nic52], II.6.1, p. 831.
    ${ }^{4}$ Nicomachus[Nic52], II.6.1, p. 831.

[^23]:    ${ }^{5}$ Nicomachus[Nic52], I.4.4, p. 813.
    ${ }^{6}$ Nicomachus[Nic52], I.4.5, p. 813.
    ${ }^{7}$ Nicomachus[Nic52], II.6.4, p. 832.
    ${ }^{8}$ Nicomachus[Nic52], II.7.3, p. 833.

[^24]:    ${ }^{9}$ Nicomachus[Nic52], II.7.1, p. 832.
    ${ }^{10}$ Nicomachus[Nic52], II.7.3, p. 833.
    ${ }^{11}$ Nicomachus[Nic52], II.7.2, p. 832.
    ${ }^{12}$ Nicomachus[Nic52], II.6.3, p. 832.

[^25]:    ${ }^{13}$ Nicomachus[Nic52], II.8.1, p. 833.
    ${ }^{14}$ Nicomachus[Nic52], II.8.3, p. 833.
    ${ }^{15}$ The natural numbers, or the natural series of numbers, are the positive whole numbers greater than zero, $1,2,3,4, \ldots$.

[^26]:    ${ }^{16}$ Nicomachus[Nic52], II.9.2, p. 834.
    ${ }^{17}$ Nicomachus[Nic52], II.9.4, p. 834.

[^27]:    ${ }^{18}$ Nicomachus[Nic52], II.10.1, p. 834.
    ${ }^{19}$ Nicomachus[Nic52], II.10.1, p. 834.

[^28]:    ${ }^{20}$ Nicomachus[Nic52], II.11.3, p. 835.

[^29]:    ${ }^{21}$ Nicomachus[Nic52], II.11.4, p. 835.
    ${ }^{22}$ Nicomachus[Nic52], II.12.1, p. 835.

[^30]:    ${ }^{23}$ Nicomachus[Nic52], II.12.2, p. 835.
    ${ }^{24}$ Nicomachus[Nic52], II.12.3, p. 835.

[^31]:    ${ }^{25}$ Nicomachus[Nic52], II.12.8, p. 836.
    ${ }^{26}$ Nicomachus[Nic52], II.13.6, p. 836.

[^32]:    ${ }^{27}$ Nicomachus[Nic52], II.14.5, p. 837.

[^33]:    ${ }^{28}$ Nicomachus[Nic52]s, II.16.2, p. 838.
    ${ }^{29}$ Nicomachus[Nic52], II.16.2. p. 838.
    ${ }^{30}$ Nicomachus[Nic52], II.16.2, p. 838.
    ${ }^{31}$ Nicomachus[Nic52], II.16.2, p. 838.

[^34]:    ${ }^{32}$ Nicomachus[Nic52], II.17.1, p. 838.

[^35]:    ${ }^{33}$ Nicomachus[Nic52], II.17.3, p. 838.
    ${ }^{34}$ Nicomachus[Nic52], II.17.3, p. 838.
    ${ }^{35}$ Nicomachus[Nic52], II.17.7, p. 839.

[^36]:    ${ }^{36}$ Nicomachus[Nic52], II.17.7, p. 839.
    ${ }^{37}$ Nicomachus[Nic52], II.17.7, p. 839.

[^37]:    ${ }^{1}$ Most of the information in this chapter comes from Nicomachus[Nic52], I.17.1 to I.23.16, pp. 821-828.
    ${ }^{2}$ Nicomachus[Nic52], I.17.2, p. 822.
    ${ }^{3}$ Nicomachus[Nic52], I.17.4, p. 822.

[^38]:    ${ }^{4}$ Nicomachus[Nic52], I.17.6, p. 822.
    ${ }^{5}$ Nicomachus[Nic52], I.17.7, p. 822.
    ${ }^{6}$ Nicomachus[Nic52], I.17.8, p. 822.
    ${ }^{7}$ Nicomachus[Nic52], I.18.1, p. 822.
    ${ }^{8}$ Nicomachus[Nic52], I.18.1, p. 822.

[^39]:    ${ }^{9}$ Nicomachus[Nic52], I.18.2, p. 822.
    ${ }^{10}$ Nicomachus[Nic52], I.19.1, pp. 822-823.

[^40]:    ${ }^{11}$ Nicomachus[Nic52], I.19.7, p. 823.

[^41]:    ${ }^{12}$ Nicomachus[Nic52], I.19.13, p. 824.

[^42]:    ${ }^{13}$ Nicomachus[Nic52], I.19.14, p. 824.
    ${ }^{14}$ Nicomachus[Nic52], I.19.14, p. 824.
    ${ }^{15}$ Nicomachus[Nic52], I.19.17, p. 824.

[^43]:    ${ }^{16} \mathrm{Heath}[\mathrm{Hea} 21]$, p. 114.
    ${ }^{17}$ Nicomachus[Nic52], I.19.19, p. 824.
    ${ }^{18}$ See section 2!3.2, p. 40 for more on heteromecic numbers.
    ${ }^{19}$ Nicomachus[Nic52], I.19.19, p. 824.

[^44]:    ${ }^{20}$ Nicomachus[Nic52], I.19.20, p. 824.
    ${ }^{21}$ Nicomachus[Nic52], I.20.1, p. 824.

[^45]:    ${ }^{22}$ Heath[Hea21], p. 102.
    ${ }^{23}$ Nicomachus[Nic52], I.20.3, p. 824.

[^46]:    ${ }^{24}$ Nicomachus[Nic52], I.21.1, p. 824.

[^47]:    ${ }^{25}$ Nicomachus[Nic52], I.22.1, p. 825.
    ${ }^{26}$ Heath[Hea21], p. 103.
    ${ }^{27}$ Nicomachus[Nic52], I.22.3, p. 826.

[^48]:    ${ }^{28}$ Nicomachus[Nic52], I.22.7, p. 826.
    ${ }^{29}$ Nicomachus[Nic52], I.23.1, p. 826.

[^49]:    ${ }^{30}$ Nicomachus[Nic52], I.23.4, p. 826.

[^50]:    ${ }^{31}$ Nicomachus[Nic52], I.23.4, p. 827.
    ${ }^{32}$ Nicomachus[Nic52], I.23.6, p. 827.
    ${ }^{33}$ Nicomachus[Nic52], I.23.6, p. 827.

[^51]:    ${ }^{34}$ Nicomachus[Nic52], I.23.15, p. 828.

[^52]:    ${ }^{1}$ Plato, The Republic VIII 546D[Pla52], p. 403.
    ${ }^{2}$ Plato, The Republic VIII 546D[Pla52], p. 403.
    ${ }^{3}$ Most of the information in this section comes from Proclus[Pro70b], pp. 133-135.

[^53]:    ${ }^{4}$ Plato, The Meno[Pla52], sections [84] and [85], p. 182.
    ${ }^{5}$ See the Introduction, section 0.3.2, for a little more information on the reason for the proof using figures - a case of the 'geometric algebra' of the Pythagoreans.

[^54]:    ${ }^{6}$ Euclid[Euc52], Book II, p. 37.

[^55]:    ${ }^{7}$ Proclus[Pro70b], p. 130.

[^56]:    ${ }^{8}$ Proclus[Pro70b], p. 130.

[^57]:    ${ }^{9}$ Most of the information in this section comes from Théon[The66], pp. 71-75.

