ALEXANDER INVIARANTS OF LINKS

by

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Abstract.

In the three main sections of this thesis (chapters II, III, and IV; chapter I consists of definitions) we explore three methods of studying Alexander polynomials of links which are alternatives to Fox's free differential calculus. In chapter II we work directly with a presentation of the link group and show how to obtain a presentation for the Alexander invariant. From this we deduce that the order ideal of the Alexander invariant is principal for links of two or three components (the case of one component is well known) but nonprincipal in general for links of four or more components. In any event we show that only one determinant is needed to obtain the Alexander polynomial.

In chapter III we use surgery techniques to characterize Alexander invariants of links of two components in terms of their presentation matrices. We then use this to show that the Torres conditions characterize link polynomials when the linking number of the two components is zero or both components are unknotted and the linking number is two.

Chapter IV uses Seifert surfaces to prove a generalization of a theorem of Kidwell which relates the individual degrees of the Alexander polynomial to the linking complexity, to present an algorithm for calculating the Alexander polynomial of a two-bridge link from a two-bridge diagram and to prove a conjecture of Kidwell in the special case of two-bridge links. These results are then used to generate link polynomials from allowable pairs (a concept introduced in chapter III) and these results in turn are used to produce methods of generating allowable pairs.
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Introduction.

In this thesis I had hoped to decide whether or not the Torres conditions characterize link polynomials, and if not to find added conditions which would characterize them. My plan of attack on this problem seemed quite reasonable: start with links of two components; find a characterization of the Alexander invariant; use this to characterize link polynomials; generalize the results to links of more than two components. My general feeling was that "what's good for two components must be good for more than two components." I knew that the order ideal of the Alexander invariant for links of two components is principal, so the Alexander invariant seemed simpler to work with than Fox' free differential calculus. Research proceeded smoothly to the characterization of the Alexander invariant, but the step from the invariant to the polynomial led to the problem of characterizing allowable pairs (chapter III section E) which has proved to be impossible (at least for me.) Consequently the problem of characterizing link polynomials remains unsolved.

Chapter I contains the definitions which are central to the thesis. Concepts which are important but standard (such as regular projection, Wirtinger presentation, linking number etc.) are omitted; the reader who is unfamiliar with knot theory is urged to refer to Rolfsen 2, especially chapters three, five, seven, and eight.

Chapter II presents an algorithm for calculating the Alexander polynomial from a presentation of the link group using the Alexander invariant as an alternative to Fox' free differential calculus. In sections A and B we present the theory for two and three components respectively and show that the order ideal in these cases is principal. In contrast to this is section C which deals with more than three components and is taken up with
a proof of the startling result that the order ideal in this case is not principal when it is nontrivial. This came as quite a surprise to me because I had always thought of "knots" and "links" as being the proper division while here the break occurs between links of three and four components. I do not know if this is related in any way to the fact that Burau and Gassner representations are known to be faithful for \( n < 3 \) while faithfulness is unknown if \( n > 3 \) (see Birman.)

Chapter III deals with surgery techniques for links of two components. Section A briefly reviews surgery techniques and contains a lemma which will be needed in section C. Section B contains an example. The main result of the chapter is in section C; here Alexander invariants of links of two components are characterized. This is used in section D to reprove the Torres conditions in the case of two components and to show that if there are restrictions on link polynomials other than the Torres conditions these will have to come from a study of allowable pairs. The concept of allowable pairs is introduced in section E where it is shown that the Torres conditions characterize link polynomials if the linking number of the two components zero or if both components are unknotted and the linking number is two.

Chapter IV explores the technique of Seifert surfaces in the study of link polynomials. In section A we prove a generalization of a result of Kidwell which relates the individual degrees of the Alexander polynomial to the linking complexity. Section B turns to the special case of two-bridge links which have long been known to be particularly suitable for analysis. In this section we prove that two bridge links are interchangable and present an algorithm for calculating the Alexander polynomial from a two-bridge presentation. As a corollary to this we prove a conjecture of Kidwell in the special case of two-bridge links. The work in sections C and
D was done in the hope of finding a way to characterize allowable pairs without working directly with the matrix in the characterization theorem. Unfortunately the results found are too complicated to do this. Section C gives methods of generating link polynomials from allowable pairs and section D uses these results as well as others in the thesis to compile a list of methods for generating allowable pairs without resorting to matrices.

I would like to thank several people for their help while I was working on this thesis. First my sincerest thanks to Dale Rolfsen who, besides being my supervisor has also become a close friend; to Kee Lam and Denis Sjerve for serving on my committee; to Roy Westwick and Ben Moyls for their helpful encouragement when I was feeling that matrices must be the most damnable things ever invented; to Mark Kidwell for a stimulating correspondence; and finally, special thanks to Ali Roth whose diagrams help to relieve the monotony of the text.
CHAPTER I: Definitions.

A link is a homeomorphic image of \( \mu \) (finitely many) disjoint oriented circles in \( S^3 \); we further assume that \( S^3 \) is oriented and the components of the link are indexed. We write \( L = \cup y_1 \cup y_2 \cdots \cup y_\mu \) where each \( y_i \) is homeomorphic to \( S^1 \) and \( \mu \) is the number of components or multiplicity of the link. Two links \( K \) and \( L \) are equivalent iff the components have the same index set (so \( K = \cup k_1 \cdots \cup k_\mu \) and \( L = \cup l_1 \cdots \cup l_\mu \)) and there is an orientation preserving homeomorphism \( S^3 \rightarrow S^3 \) which restricts to orientation preserving homeomorphisms \( k_i \rightarrow l_i \) on each component. A link is tame if it is equivalent to a polygonal (that is piecewise linear) link. All links will be tacitly assumed to be tame.

In studying links, link diagrams are often used. Roughly speaking, a link diagram is what you would get if you were to take a photograph of a link. We will further assume that there are at most double points and that line segments intersect transversally (in the literature this is called a regular projection; see Rolfsen 2 for details. Figure II.A.1 should make the concept clear enough.) If \( L \) is a link then \( X = S^3 - L \) is a link complement and \( \pi_1(X) \) is a link group. The Hurewicz homomorphism \( h:\pi_1(X) \rightarrow H_1(X) \) defines a regular covering space \( p:\tilde{X} \rightarrow X \) called the universal abelian covering space of \( X \), namely the covering space so that \( p_* (\pi_1(X)) = \ker h \). As an immediate consequence of this definition we see that a loop in \( X \) lifts to a loop in \( \tilde{X} \) iff its linking number with each component of the link is zero. This is usually a convenient way to check when a given cover is the universal abelian cover. Since \( \tilde{X} \) is a regular covering space its group of covering automorphisms is \( H_1(X) \). But the components of the link are indexed and oriented, so by Alexander
duality there is a canonical isomorphism $H_1(X) \cong \mathbb{Z}^\mu$. Let $x_1, x_2, \ldots, x_\mu$ be the generators for the group of covering automorphisms of $X$ corresponding to the canonical generators of $\mathbb{Z}^\mu$ and take $a \in H_1(X)$. We define $\tau_i H = \mathbb{Z}^\mu$ and take $a \in H_1(X)$. We define

$$x_1 x_2 \cdots x_\mu a = x_1 x_2 \cdots x_\mu(a)$$

where $x_\mu: H_1(X) \to H_1(X)$ is the homomorphism induced by the covering automorphism $x: X \to X$. Since we can also multiply by integers and add this allows us to define the action of a polynomial on an element of $H_1(X)$. In other words we have a $\mathbb{Z} \mathbb{Z}^\mu = \Lambda_{\mu}$-module structure on $H_1(X)$; this module will be called the Alexander invariant of the link.

A $\Lambda_{\mu}$ module will be called a link module provided it is isomorphic to the Alexander invariant of some link of multiplicity $\mu$.

Given a presentation of a $\Lambda$ module in terms of generators and relations

$$A = (a_1, a_2, \ldots, a_n : r_1, r_2, \ldots, r_m)$$

the corresponding presentation matrix is $(a_{ij})$ where $a_{ij} \in \Lambda$ is defined by $r_i = \sum_{j=1}^n a_{ij} a_j$. Given a presentation matrix the ideal $E^k_A$ generated by the determinants of all $(n-k) \times (n-k)$ submatrices is called the $k$-th elementary ideal and depends only on $A$ (see Zassenhaus page 90.) In case $k=0$, $E^0_A$ is called the order ideal of $A$. If $\Lambda$ is a unique factorization domain we let $\Delta^k_A$ be a generator of the minimal principal ideal containing $E^k_A$; it is defined up to units in $\Lambda$. Again the case $k=0$ is special: $\Delta^0_A$ is called the Alexander polynomial of $A$; $\Delta^0_{H_1(X)}$ is called the Alexander polynomial of the link; a link polynomial is a polynomial which is equal to the Alexander polynomial of some link.
$S^n$ the unit sphere in $\mathbb{R}^{n+1}$ or anything homeomorphic to it. It is sometimes thought of as the one point compactification of $\mathbb{R}^n$.

$\partial$ boundary, either in homology or topology.

$\overset{\circ}{T}$ the interior of $T$.

$\cong$ equivalence in the category in question, i.e. homeomorphism for topological spaces, isomorphism for groups, modules, etc.

$\mathbb{Z}$ the group of integers, written multiplicatively.

$\mathbb{Z}^\mu$ $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ ($\mu$ copies of $\mathbb{Z}$)

$\mathbb{Z}[G]$ the integral group ring of $G$.

$\Lambda^\mu$ $\mathbb{Z}[G]^\mu$. It is thought of as finite Laurent polynomials in $\mu$ commuting variables $x_2, x_3, \cdots, x_\mu$ with integral coefficients.

$[a, \beta]$ $\alpha \beta \alpha^{-1} \beta^{-1}$ when $\alpha$ and $\beta$ are elements of a group.

$[a, b]$ the closed interval between $a$ and $b$ when $a$ and $b$ are real numbers.

$\alpha \pm \beta$ $\beta \alpha \pm 1 \beta^{-1}$. $\alpha$ and $\beta$ are elements of a group.

$(\alpha, \beta, \gamma, \cdots)$ the ideal generated by $\alpha$, $\beta$, $\gamma$, $\cdots$.

$C^a_b$ $\frac{a!}{b!(a-b)!}$

$\Pi^N_{i=1} a_i$ $a_1 \cdot a_2 \cdot \cdots \cdot a_n$

$\gamma^N_{i=1} a_i$ $a_1 + a_2 + \cdots + a_n$

$\Lambda^{tr}$ the transpose of the matrix $\Lambda$.

$\text{diag}(a, b, \cdots)$ the diagonal matrix with $a$, $b$, $\cdots$ down the diagonal.

$1k(a, b)$ the linking number of the two cycles $a$ and $b$.

$\text{sgn } \sigma$ +1 if $\sigma$ is an even permutation and -1 if $\sigma$ is an odd permutation.

(2m, 2) torus link has 2m crossings

Table 1. A list of standard symbols.
CHAPTER II: computing the Alexander polynomial from the link group.

There are three sections in this chapter, one for each of two, three, and more than three components. In each of these is given an algorithm for calculation of the Alexander invariant as a $\mu$ module given a presentation of the fundamental group of the link, as well as the Alexander polynomial. We use this algorithm to show that the order ideal is principal in the case of links of two and three components (the case of one component is well known) but is nonprincipal in general for links of more than three components.
CHAPTER II: Computing the Alexander polynomial from the link group.

The Alexander polynomial of a link is usually defined using Fox' free differential calculus. What he calls the Alexander matrix is a $\Lambda_\mu$-module presentation matrix for the relative Alexander invariant $H_1(\tilde{X},\tilde{x}_0)$ where $x_0 \in X$ is a point and $\tilde{x}_0 = p^{-1}(x_0)$ (the notation is the same as in chapter I.) The Alexander polynomial is then defined to be $\Delta_{H_1(\tilde{X},\tilde{x}_0)}^1$. Although $E_{H_1(\tilde{X})}^k \neq E_{H_1(\tilde{X},\tilde{x}_0)}^{k+1}$ in general, Levine 2 shows that $\Delta_{H_1(\tilde{X})}^k = \Delta_{H_1(\tilde{X},\tilde{x}_0)}^{k+1}$ and so the two definitions of Alexander polynomial are equivalent. In this chapter we present a method of calculating the Alexander polynomial given a presentation of the link group as an alternative to Fox' free calculus; it is a generalization of the method used in Rolfsen 2 for calculating knot polynomials.

Section A: Links of two components.

We start this section with an example.

Example 1:

The link in figure 1 has fundamental group

![Figure 1](image-url)
\[ \pi_1(X) \cong (\xi, \zeta, \eta : \xi\eta\xi^{-1}\zeta^{-1}\eta^{-1}\zeta^{-1}\eta, \xi\zeta\xi^{-1}\zeta^{-1}\eta\xi^{-1}\zeta^{-1}\eta^{-1}\zeta^{-1}) . \]

If \( h: \pi_1(X) \to H_1(X) \cong \mathbb{Z}^2 \) is the Hurewicz homomorphism we have \( h(\xi) = x, \]
\[ h(\zeta) = h(\eta) = y \text{ where } x \text{ and } y \text{ are the first and second generators of } \]
\[ \mathbb{Z}^2 \text{ respectively. If we set } \alpha = \eta\xi^{-1} \text{ so that } h(\alpha) = 1, \]
this can be rewritten as
\[ \pi_1(X) \cong (\xi, \zeta, \alpha : \xi\alpha\zeta\xi^{-1}\zeta^{-1}\alpha^{-1}\zeta^{-1} \alpha^{-1}\zeta^{-1}, \xi\zeta\xi^{-1}\zeta^{-1}\alpha\xi^{-1}\alpha^{-1}\zeta^{-1}) . \]

This can be further rewritten as
\[ \pi_1(X) \cong (\xi, \zeta, \alpha : \alpha^\xi[\xi, \zeta]^{-\zeta}a^{-1}\alpha^{-1}\zeta^{-1}, [\zeta, \zeta]^{-\xi}\alpha\zeta[\zeta, \zeta]a^{-\xi} ) \]
where \([s, t] = sts^{-1}t^{-1} \text{ and } s^tt = ts^{-1}t^{-1} \).

To better visualize what is happening we replace \( X \) by a cell complex \( Y \) with the same fundamental group: \( Y \) is the one point union of three circles with two discs attached, the boundaries of the discs being attached to loops which represent the relators. We will continue to call the loops \( \xi, \zeta, \alpha \) and \( h: \pi_1(Y) \to \mathbb{Z}^2 \) will again be given by \( h(\xi) = x, h(\xi) = y, \) and \( h(\alpha) = 1; \) \( \tilde{Y} \) will be the covering space corresponding to \( \ker h \). For the sake of clarity let \( Z \) and \( \tilde{Z} \) be the 1-skeleton of \( Y \) and \( \tilde{Y} \) respectively. Then it is clear that
\[ \pi_1(\tilde{Z}) \cong (\gamma_{12}, \alpha : -) \]
where \( \gamma_{12} = [\xi, \zeta] \). A presentation for \( \pi_1(\tilde{Y}) \) and hence for \( \pi_1(\tilde{X}) \) is obtained by adding the lifts of words representing the relators in \( \pi_1(Z) \) to \( \pi_1(\tilde{Z}) \) as the relators of \( \pi_1(\tilde{Y}) \). This explains why we chose the presentation (*); in this form it is particularly easy to read off the new relators. For example \( \alpha^\xi[\gamma_{12}^{-1}a^{-1}\alpha^{-1}] \) becomes \( ya-yc_{12}^{-1}a+y^{-1}a \), where the Hurewicz homomorphism \( \tilde{h}: \pi_1(\tilde{Z}) \to H_1(\tilde{Z}) \) is given by \( \tilde{h}(\gamma_{12}) = c_{12} \) and \( \tilde{h}(\alpha) = a \). Thus we see that
\[ H_1(\tilde{X}) \cong (a, c_{12} : (y-1+y^{-1})a-yc_{12}, -(1-x)ya+(1-x)c_{12} ) \]
\[ = (a : (1-x)(1-y)(1+y^{-2})a) \]
presented as a $\Lambda_2$ module. Hence we have

$$E^0_{H_1}(\mathcal{X}) = (1-x)(1-y)(1+y^{-2}), \quad E^k_{H_1}(\mathcal{X}) = \Lambda_2 \quad \text{if } k>0$$

$$\Delta^0_{H_1}(\mathcal{X}) = (1-x)(1-y)(1+y^{-2}), \quad \Delta^k_{H_1}(\mathcal{X}) = 1 \quad \text{if } k>0$$

Notice that in this case the Alexander invariant is cyclic (this is not true in general) and the order ideal is principal (this is always true for two component links, as will be shown below.)

---

$\tilde{Z}$: $\tilde{Y}$ consists of $\tilde{Z}$ to which discs have been equivariantly attached to loops representing the lifts of the relators.

$Z$: $Y$ consists of $Z$ with discs attached to loops representing the relators.

Figure 2.
We summarize the above example as an algorithm for computing an
a presentation of the Alexander invariant as a $\Lambda_2$ module from a presentation
of a link group in the case of two component links.
1. find a presentation for $\pi_1(X)$ in the form
$$\left( \xi, \zeta, a_i, 1 \leq i \leq n : r_j, 1 \leq j \leq n+1 \right)$$
where the Hurewicz homomorphism $h: \pi_1(X) \to H_1(X) \cong \mathbb{Z}^2$ takes $h(\xi) = x,$
h($\zeta$) = $y,$ and $h(a_i) = 1$ if $1 \leq i \leq n$ where $x$ and $y$ are the two generators of
$\mathbb{Z}^2.$ Notice that there is one more generator than relator. Using the
Wirtinger presentation of a link it is possible to show that the defect of any
link group (that is, the number of generators minus the number of relators)
is at least one (see Rolfsen 2 for a more complete discussion). Hence
we can always assume that we have a presentation with defect one by adding
the trivial relator wherever necessary.
2. write the relators in terms of $\alpha_i, 1 \leq i \leq n,$ $\gamma_{12} = [\xi, \zeta]$ and their conjugates.
In the example this was done by inspection, but here we present an algorithm
for reducing a relator to this form. Since we will need this result later
we prove it for any number of components.

**Lemma 1:** Suppose that $r$ is a word in the symbols $\xi_i, 1 \leq i \leq \mu$ and $\alpha_j, 1 \leq j \leq n.$
Suppose further that the sum of the exponents of $\xi_i$ is zero for $1 \leq i \leq \mu.$ Then
$r$ can be written as a word in $\gamma_{ij} = [\xi_i, \xi_j], 1 \leq i, j \leq \mu, \alpha_i, 1 \leq i \leq n,$ and their conjugates.

**Proof:** Notice that
$$w\alpha = w\alpha w^{-1} \alpha$$
that is
(1.) $$w\alpha = \alpha \omega \omega^{-1} \omega.$$ We can apply (1.) whenever $r$ contains an $\alpha_i$ preceded by a $\xi_j$ to rewrite $r$
as a product of conjugates of the $\alpha_i$'s followed by a word $\omega$ in the $\xi_j$'s.
Next consider $\mathcal{W}$. We know that

\[(2.) \quad \zeta_1 \zeta_j = [\zeta_1, \zeta_j] \zeta_1 \zeta_j\]

where $\zeta_k = \xi_k^{\pm 1}$. In the event that we can find a $\xi_1^{\pm 1}$ followed by a $\xi_i^{\pm 1}, i \neq 1$, in $\mathcal{W}$ we can apply (2.) to it; that is if $\mathcal{W} = \mathcal{W}_1 \xi_1 \mathcal{W}_2$ we can write

$$\mathcal{W} = \mathcal{W}_1 \xi_1 \mathcal{W}_2 = \mathcal{W}_1 [\xi_1, \xi_1] \xi_1 \mathcal{W}_2 = [\xi_1, \xi_1] \mathcal{W}_1 \xi_1 \xi_1 \mathcal{W}_2.$$ 

Now replace $\mathcal{W}$ by $\mathcal{W}_1 \xi_1 \xi_1 \mathcal{W}_2$ and continue as above until no $\xi_1^{\pm 1}$ is followed by a $\xi_i^{\pm 1}, i \neq 1$. Then we have written $\mathcal{W}$ as a product of symbols of the form $[\xi_1, \xi_j]$ followed by $\mathcal{W}_1$ where $\mathcal{W}_1$ is $\mathcal{W}$ with the $\xi_1^{\pm 1}$'s omitted (all the $\xi_1^{\pm 1}$'s have been moved to the end, and since their exponent sum is zero they cancel each other out.) Now replace $\mathcal{W}$ by $\mathcal{W}_1$ and $\xi_1^{\pm 1}$ by $\xi_1^{\pm 2}$ and so on until $r$ has been written as a product of conjugates of the $\alpha_i$'s followed by a product of conjugates of $[\xi_1, \xi_j]$ where $1 \leq i < j \leq n$. The proof of the lemma is completed by noticing that

$$[\xi_1^{-1}, \xi_j] = [\xi_1, \xi_j]^{-1} = \gamma_{i,j}^{-1},$$

$$[\xi_1, \xi_j^{-1}] = [\xi_1, \xi_j]^{-1} = \gamma_{i,j}^{-1},$$

$$[\xi_1^{-1}, \xi_j^{-1}] = [\xi_1, \xi_j] \xi_1^{-1} \xi_j^{-1} = \gamma_{i,j}^{-1} \xi_1 \xi_j^{-1}$$

and $(a^b)^c = a^{bc}$.

3. We can now write down a presentation matrix for $H_1(\tilde{X})$. It is of the form

$$\left( \begin{array}{c} a_i \ 1 \leq i \leq n, \ c_{ij} : R_j \ 1 \leq j \leq n+1 \end{array} \right)$$

where $R_j$ is obtained by writing $r_j$ additively and replacing $\beta^{\pm w}$ by $\pm h(\omega) \tilde{h}(\beta)$ where $h : \pi_1(X) \to H_1(X)$ and $\tilde{h} : \pi_1(\tilde{X}) \to H_1(\tilde{X})$ are the Hurewicz homomorphisms ( $h$ was defined on the previous page and $\tilde{h}$ is defined by $\tilde{h}(\alpha_i) = a_i$ and $\tilde{h}(\gamma_{ij}) = c_{ij}$.)

As an immediate consequence of this algorithm we have
Proposition 2: For two component links the order ideal \( E^0_{H_1(\tilde{X})} \) is principal.

Proof: \( H_1(\tilde{X}) \) has a square presentation matrix ( \( n+1 \) generators and \( n+1 \) relators.)

Remark: For links of two or more components the ideal \( E^1_{H_1(\tilde{X}, \tilde{x}_0)} \) is the product of the nonprincipal ideal generated by the elements \( 1-x_i \) for \( 1 \leq i \leq \mu \) where \( \mu \) is the multiplicity of the link and a principal ideal generated by the Alexander polynomial. Hence in the case of two component links it is slightly simpler to work with the Alexander invariant rather than the Alexander matrix.
Section B: Links of three components.

The case of three components is not substantially different from the case of two components. Again we present an algorithm for calculating a presentation for the Alexander invariant as a $\Lambda_3$ module.

1. find a presentation for $\pi_1(X)$ of the form
   $$\pi_1(X) = (\xi_1, \xi_2, \xi_3, a_1, 1 \leq i \leq n : r_j, 1 \leq j \leq n+2)$$
   where $h: \pi_1(X) \to H_1(X) \cong \mathbb{Z}^3$ is the Hurewicz homomorphism which takes
   $$h(\xi_i) = x_i \text{ for } i = 1, 2, 3 \text{ where } x_i \text{ is the } i\text{-th generator of } \mathbb{Z}^3$$
   and $h(a_i) = 1$ for $1 \leq i \leq n$.

2. use lemma 1 to write the relators in terms of the $a_i$'s, the $\gamma_{ij}$'s where
   $$\gamma_{ij} = [\xi_i, \xi_j] \quad 1 \leq i < j \leq 3,$$
   and their conjugates.

3. in this case $Z$ is a wedge of $n+3$ circles and $\tilde{Z}$ is homeomorphic to the set
   $$\{ (x, y, z) \in \mathbb{R}^3 : \text{at most one of } x, y, z \text{ is not an integer} \}$$
   with a wedge of $n$ circles attached at each vertex. Again $\tilde{Z}$ has a $\mathbb{Z}^3$ action
   on it and $H_1(\tilde{Z})$ has a presentation
   $$(a_i, 1 \leq i \leq n, c_{12}, c_{13}, c_{23} : (1-x_3)c_{12} + (1-x_2)c_{13} + (1-x_1)c_{23})$$
   as a $\Lambda_3$ module. Here again we have the Hurewicz map $\tilde{h}: \pi_1(\tilde{Z}) \to H_1(\tilde{Z})$
   defined by $\tilde{h}(a_i) = a_i$ and $\tilde{h}(\gamma_{ij}) = c_{ij}$ where $1 \leq i < j \leq 3$. Hence a presentation
   for $H_1(\tilde{Z})$ as a $\Lambda_3$ module is
   $$(a_i, 1 \leq i \leq n, c_{12}, c_{13}, c_{23} : (1-x_3)c_{12} + (1-x_2)c_{13} + (1-x_1)c_{23}, R_i, 1 \leq i \leq n+2)$$
   where $R_i$ is obtained by writing $r_i$ additively and replacing $h^+\omega$ by $h(\omega)\tilde{h}(\beta)$.

   Since we again have an equal number of generators and relators we again have
   Proposition 3: For three component links the order ideal $E_0^{\tilde{Z}}$ is principal. \qed
Section C: Links of more than three components.

The method of calculating a presentation matrix for the Alexander invariant from a presentation for the fundamental group in the general case should now be clear, although it holds a bit of a surprise.

1. find a presentation for \( \pi_1(X) \) of the form

\[
\pi_1(X) = \langle \xi_1, 1 \leq i \leq u, a_j, 1 \leq j \leq n : r_k, 1 \leq k \leq n+u-1 \rangle
\]

where \( h: \pi_1(X) \rightarrow H_1(X) \cong \mathbb{Z}^u \) is the Hurewicz homomorphism which takes

\[
h(\xi_i) = x_i \text{ for } i = 1, 2, \ldots, u \text{ where } x_i \text{ is the } i\text{-th generator of } \mathbb{Z}^u \text{ and } \]

\[
h(a_i) = 1 \text{ for } i = 1, 2, \ldots, n.
\]

2. use lemma 1 to write the relators in terms of the \( a_i \)'s, the \( \gamma_{ij} \)'s where

\[
\gamma_{ij} = \frac{[\xi_i, \xi_j]}{1 \leq i < j \leq u}, \text{ and their conjugates.}
\]

3. in this case \( Z \) is a wedge of \( n+u \) circles and \( \tilde{Z} \) is homeomorphic to the set

\[
\{ \text{ points in } \mathbb{R}^u : \text{ at most one coordinate is not an integer } \}
\]

with a wedge of \( n \) circles attached at each vertex. Again \( \tilde{Z} \) has a \( \mathbb{Z}^u \) action on it and \( H_1(\tilde{Z}) \) has a presentation

\[
(a_1, 1 \leq i \leq n, c_{ij}, 1 \leq i < j \leq u : R_{ijk}, 1 \leq i < j < k \leq u)
\]

as a \( \Lambda^\mu \) module. Here \( R_{ijk} = (1-x_k)c_{ij} - (1-x_j)c_{ik} + (1-x_i)c_{jk} \) and the Hurewicz homomorphism \( \tilde{h}: \pi_1(\tilde{Z}) \rightarrow H_1(\tilde{Z}) \) is defined by \( \tilde{h}(a_i) = a_i \) and \( \tilde{h}(\gamma_{ij}) = c_{ij} \)

where \( 1 \leq i < j \leq u \). Hence a presentation for \( H_1(\tilde{Z}) \) as a \( \Lambda^\mu \) module is

\[
( a_1, 1 \leq i \leq n, c_{ij}, 1 \leq i < j \leq u : R_{ijk}, 1 \leq i < j < k \leq u, R_1, 1 \leq i \leq n+u-1 )
\]

where \( R_1 \) is obtained by writing \( r_1 \) additively and replacing \( \beta^{+\omega} \) by \( \pm h(\omega)h(\beta) \).

Notice that we have a new wrinkle: we have \( n + C_2^u \) generators and \( n + u - 1 + C_3^u \) relators and

\[
n + u - 1 + C_3^u - n - C_2^u = C_3^{u-1} > 0 \text{ if } u \geq 4.
\]

Hence the corresponding presentation matrix is not square.

Theorem 4: For links of more than three components the order ideal \( E_0^H(\tilde{X}) \)
is the product of the nonprincipal ideal \( I \) generated by elements of the form...
\[ n_i \begin{pmatrix} 1-x_1 \\ 1-x_2 \\ \vdots \\ 1-x_n \end{pmatrix} \] where \( n_i \geq 0 \) and \( \sum_{i=1}^{n} n_i = c^{u-2}_2 \), and a principal ideal (generated by the Alexander polynomial) and hence is not principal in general.

**Proof:** Let \( P \) be the presentation matrix corresponding to the presentation \((\star)\) for \( H_1(\tilde{X}; E_0^{0}) \) is generated by the determinants of all submatrices obtained by deleting \( c^{u-1}_3 \) rows from \( P \). Since submatrices with zero determinant do not effect this generating set, it is of value to notice that the rows of \( P \) are not independent; in fact, for any \( 1 \leq i < j < k < m \leq \mu \) we have

\[ R_{ijkm} : (1-x_m)R_{ijk} - (1-x_k)R_{ijm} + (1-x_j)R_{ikm} - (1-x_i)R_{jkm} = 0 \]

We can calculate how many rows must be deleted to obtain a linearly independent set by using vector spaces over the field of quotients of \( \Lambda^U \).

Let \( V_3 \) (respectively \( V_4 \)) be the vector space of dimension \( c^{u}_3 \) (respectively \( c^{u}_4 \)) with basis the set \{ \( R_{ijk} : 1 \leq i < j < k \leq \mu \) \} (respectively \{ \( R_{ijkm} : 1 \leq i < j < k < m \leq \mu \) \}).

Then we have a map \( V_4 \rightarrow V_3 \) determined by \((\star)\) and the number we want is \( \dim(\text{im}V_4) \).

**Lemma 5:** \( \dim(\text{im}V_4) = c^{u-1}_3 \).

**Proof:** The set \{ \( \text{im}R_{ijkm} \) \} is not independent. To see what is happening in general notice that we have a \( \mathbb{Z}^u \) action on \( \mathbb{R}^u \) which restricts to the \( \mathbb{Z}^u \) action on \( \tilde{Y} \) with fundamental region the hypercube \( \prod_{i=1}^{n} I_i \) where \( I_i = [0,1] \) for all \( i \).

The elements \( c_{ij} \) can be thought of either as the squares \( I_i \times I_j \) or their boundaries; the relators \( R_{ijk} \) correspond to the cubes \( I_i \times I_j \times I_k \) or their boundaries; the relations \( R_{ijkm} \) correspond to the hypercubes \( I_i \times I_j \times I_k \times I_m \) or their boundaries, and so on. If \( V_n \) is the \( c^{u}_n \) dimensional vector space with basis \{ \( R_{ij\ldots m} : 1 \leq i < j < \ldots < m \leq \mu, n \) subscripts \} there is a homomorphism\n
\[ V_{n} \rightarrow V_{n-1} \] defined by \( \Gamma^n \rightarrow \partial \Gamma^n \). Then

\[ \dim(\text{im}V_4) = \dim(V_4/\text{im}V_5) \]

\[ = \dim V_4 - \dim(\text{im}V_5/\text{im}V_6) \]
\[ \dim V = \dim V_4 - \dim V_5 + \dim V_6 - \cdots + (-1)\dim V_{\mu} \]

\[ = C_4^{\mu} - C_5^{\mu} + C_6^{\mu} - \cdots + (-1)C_{\mu}^{\mu} \]

But

\[ 0 = (1 - 1)\mu = C_0^{\mu} - C_1^{\mu} + C_2^{\mu} - \cdots + (-1)C_{\mu}^{\mu} \]

so

\[ \dim(\text{im} V_4) = -(C_0^{\mu} - C_1^{\mu} + C_2^{\mu} - C_3^{\mu}) \]

\[ = C_3^{\mu-1} \]

Hence we must delete from \( P \) at least \( C_3^{\mu-1} \) of the rows corresponding to the \( R_{ijk} \)'s to obtain a submatrix with nonzero determinant. Since this is exactly the number of extra rows we see that we can never delete a row corresponding to a \( R_i \) in (1) and get a maximal submatrix of \( P \) with nonzero determinant.

We now need some way to compare the determinants of an arbitrary submatrix \( M \) of \( P \) with independent rows with a fixed submatrix \( M_\mu \).

Definition: \( M_n \) is the submatrix of \( P \) obtained by deleting the rows corresponding to \( R_{ijk} \) where \( n \notin \{i, j, k\} \). The remaining rows are independent since \( c_{ij} \) \( i \neq n \neq j \) has nonzero coefficient exactly once. In the next lemma there is a problem with keeping track of the order of the subscripts; in order to get around this, let \( R_{\{i, j, k\}} = R_{\{a, b, c\}} \) where \( \{i, j, k\} = \{a, b, c\} \) and \( a < b < c \). Similarly for \( R_{\{a, j, k, m\}} \).

Lemma 6: \( \det M_\alpha = \pm \frac{(1-x_\alpha)}{C_2^{\mu-2}} \det M_\beta \)

Proof: The \( C_2^{\mu-2} \) relations \( R_{\{i, j, a, \beta\}} \) allow us to write \( (1-x_\beta)R_{\{i, j, \beta\}} \) in terms of \( (1-x_\beta)R_{\{i, j, a\}} \), \( (1-x_\beta)R_{\{j, a, \beta\}} \), and \( (1-x_\beta)R_{\{i, a, \beta\}} \) with \( \pm 1 \) coefficients. Put the relations \( R_{\{i, j, a, \beta\}} \) into some order, say \( R_{\{i_1, j_1, a, \beta\}} \), \( R_{\{i_2, j_2, a, \beta\}} \) and so on. Let \( M^{(1)} \) be the matrix obtained from \( M_\alpha \) by replacing
the row \(R_{\{i_1,j_1,\alpha\}}\) by \(R_{\{i_1,j_1,\beta\}}\). Because of \(R_{\{i_1,j_1,\alpha,\beta\}}\) we see that

\[
\det M_\alpha = \pm \frac{(1-x_\alpha)}{(1-x_\beta)} \det M^{(1)}.
\]

Now consider \(M^{(1)}\) and \(R_{\{i_2,j_2,\alpha,\beta\}}\) and proceed as before to obtain \(M^{(2)}\) with

\[
\det M^{(1)} = \frac{(1-x_\alpha)}{(1-x_\beta)} \det M^{(2)}
\]

that is

\[
\det M_\alpha = \pm \frac{(1-x_\alpha)^2}{(1-x_\beta)^2} \det M^{(2)}
\]

After \(C_2^{\mu-2}\) steps we obtain a matrix which differs from \(M_\beta\) only in the order of its rows, and hence

\[
\det M_\alpha = \pm \frac{(1-x_\alpha)^{C_2^{\mu-2}}}{(1-x_\beta)^{C_2^{\mu-2}}} \det M_\beta
\]

Corollary 7: \((1-x_\alpha)^2 \mid \det M_\alpha\)

Lemma 8: the polynomial in corollary 7 is maximal in the sense that if

\(p_\alpha \in \Lambda_\mu\) and \(p_\alpha \mid \det M_\alpha\) for all links and calculations of \(M_\alpha\) then \(p_\alpha \mid (1-x_\alpha)^{C_2^{\mu-2}}\).

Proof: By symmetry we need only consider the case \(\alpha = \mu\). Since we have \(p_\mu \mid \det M_\mu\) for all links, in particular \(p_\mu\) must be invariant up to units in \(\Lambda_\mu\) under permutations of \(\{1, 2, \cdots, \mu-1\}\). We complete the proof by exhibiting a family of links so that if \(p_\mu \mid \det M_\mu\) and \(p_\mu\) is invariant under permutations of \(\{1, 2, \cdots, \mu-1\}\) then \(p_\mu \mid (1-x_\mu)^{C_2^{\mu-2}}\).

Example: The daisy chain of length \(\mu\).
The fundamental group of this link is

$$( \xi_1, \xi_2, \cdots, \xi_\mu : [\xi_1, \xi_2], [\xi_2, \xi_3], \cdots, [\xi_{\mu-1}, \xi_\mu] )$$

and so its Alexander invariant has a presentation

$$( c_{ij} \mid 1 \leq i < j \leq \mu : R_1, R_2, \cdots, R_{\mu-1}, R_{ijk} \mid 1 \leq i < j < k \leq \mu )$$

where $R_i = c_{i,i+1}$. $M_\mu$ is the matrix corresponding to the rows $R_{ij\mu}$ and $R_i$.

Notice that the rows $R_i$ have a 1 in the $c_{i,i+1}$ column and 0 elsewhere, so we can omit the rows corresponding to the $R_i$'s and the columns corresponding to the $c_{i,i+1}$'s and obtain a matrix whose determinant differs from that of $M_\mu$ by a factor of $\pm 1$. Of the remaining columns, those corresponding to $c_{ij} \mid 1 \leq i < j < \mu$ have a $(1-x_\mu)$ in the row $R_{ij\mu}$ and zeroes elsewhere, so we can omit these $c_{ij}^{\mu-1} - (\mu-2) = c_{ij}^{\mu-2}$ rows and columns to obtain the matrix below whose determinant differs from that of $M_\mu$ by a factor of $\pm (1-x_\mu) c_{ij}^{\mu-2}$.
The determinant of this matrix is \(\pm (1-x_2)(1-x_3)(1-x_4) \cdots (1-x_{\mu-1})\) and so
\[
\det M_\mu = \pm (1-x_2)(1-x_3) \cdots (1-x_{\mu-1})(1-x_\mu)^{\binom{\mu-2}{2}}.
\] Since the factor \((1-x_1)\) is missing it is clear that these links have the required properties.

We are now in a position to compare the determinant of an arbitrary submatrix \(M\) of \(P\) with independent rows with that of \(M_\mu\). The construction used generalizes that of lemma 6. Put the rows \(R_{i_1j_1k_1}, R_{i_2j_2k_2}, \ldots, R_{i_{\mu}j_{\mu}k_{\mu}}\). The row \(R_{i_1j_1k_1}\) can be written as a linear combination of the rows of \(M_\mu\) with coefficients in \(A_\mu\). At least one row which takes part in this linear combination is not a row of \(M\), say \(R_{a_1b_1c_1}\). Let \(M^{(1)}\) be the matrix obtained from \(M_\mu\) by replacing this row with \(R_{i_1j_1k_1}\). By the construction it is clear that
\[
\det M_\mu = \frac{\text{coefficient of } R_{i_1j_1k_1}}{\text{coefficient of } R_{a_1b_1c_1}} \det M^{(1)}.
\]

We can replace \(M_\mu\) by \(M^{(1)}\) to obtain \(M^{(2)}\) and so on. After \(n\) stages we arrive at a matrix which differs from \(M\) by at most a permutation of the rows. Hence
\[
\det M_\mu = \pm \frac{\text{product of coefficients of rows added}}{\text{product of coefficients of rows deleted}} \det M.
\]

Now we need a better idea of what these coefficients look like. To start out with we have the set of relations
\[
S = \{ \prod_{\alpha \neq i_1, j_1, k_1, \mu} (1-x_\alpha) R_{i_1j_1k_1\mu}, \ldots, \prod_{\alpha \neq i_{\mu}, j_{\mu}, k_{\mu}, \nu} (1-x_\alpha) R_{i_{\mu}j_{\mu}k_{\mu}\nu} \}
\]
which has the following properties:
1. each row of \(M\) which is not a row of \(M^{(0)} = M_\mu\) is contained in a relation which is an element of \(S\) which can be used to express it in terms of the rows
which are in \( M(0) \).

2. the coefficient of \( R_{i,j,k} \) is always an integral multiple of \( \prod_{\alpha \neq i,j,k} (1-x_\alpha) \).

I want to modify this set at each stage so that at the \( m \)-th stage 1. and 2. continue to be true with \( M(0) \) replaced by \( M(m) \). This is done as follows:

the relation which was used to go from \( M^{(m-1)} \) to \( M^{(m)} \) (which must contain the row \( R_{i,j,k} \)) is removed from the set. Some of the relations which remain may contain \( R_{i,j,k} \), but by hypothesis 2. these can be multiplied by integers and added to an integral multiple of the newly removed relation to eliminate the \( R_{i,j,k} \) term; these linear combinations form my new set.

In view of 2., after cancelling like terms from the numerator and denominator we have

\[
\det M = \pm \frac{A \prod_{i=1}^{\mu} (1-x_i)^{m_i}}{B \prod_{i=1}^{\mu} (1-x_i)^{n_i}} \det M
\]

where \( A, B \in \mathbb{Z}, n_i > 0, m_i > 0, \) and \( \sum_{i=1}^{\mu} n_i = \sum_{i=1}^{\mu} m_i \). Since \( A \prod_{i=1}^{\mu} (1-x_i)^{m_i} \mid \det M \),

by lemma 8 we have

\[
A \prod_{i=1}^{\mu} (1-x_i)^{m_i} = \pm (1-x_1)^{2-n_\mu} C_2^{\mu-2} - n_\mu
\]

for some \( 0 \leq n_\mu \leq C_2^{\mu-2} \). Hence

\[
\frac{\det M}{(1-x_\mu)^{2-n_\mu}} = \pm \frac{\det M}{B \prod_{i=1}^{\mu} (1-x_i)^{n_i}} = \Delta
\]

is a polynomial where \( B \in \mathbb{Z}, n_i \geq 0, \sum_{i=1}^{\mu} n_i = C_2^{\mu-2} \). I do not know if the case \( B \neq \pm 1 \) can occur.

The difficult part of the proof will be finished if, given \( n_i \geq 0 \)

\( 1 \leq i \leq \mu \) with \( \sum_{i=1}^{\mu} n_i = C_2^{\mu-2} \) we can find a matrix \( M \) satisfying
Fortunately this is relatively easy. Look at lemma 6 and apply the proof with $\alpha = \mu$, $\beta = 1$, but stop after the $n_1$-st stage. This gives us a matrix $M^{(1)}$ with

$$\det M^{(1)} = \pm \frac{(1-x_1^{n_1})}{n_1} \det M(1)$$

Now look at the set of $C^{n_2}_{n_2}$ relations $\{ R_{i,j} \}$. At least $C^{n_2}_{n_2} - n_1 \geq n_2$ of these do not contain rows which have been eliminated, so use these to get $M^{(2)}$ with

$$\det M^{(2)} = \pm \frac{(1-x_1^{n_1+n_2})}{(1-x_1^{n_1})(1-x_2^{n_2})} \det M^{(2)}$$

It is clear that we can continue in this way; at the $m$-th stage there are at least $C^{n_2}_{n_2} - (n_1 + n_2 + \cdots + n_{m-1}) \geq n_m$ relations which do not contain eliminated rows. We eventually arrive at $M$ with

$$\det M^{(m)} = \pm \frac{n_1^{n_1+n_2+\cdots+n_{m-1}}}{n_1^{n_{m-1}}(1-x_1^{n_1})} \det M$$

which is equivalent to the required statement.

All that remains to complete the proof is to prove that $I$ is not principal. We do this by contradiction. We can define an epimorphism $\Lambda_\mu : \mathbb{Z} \to \mathbb{Z}$ by setting $x_i = -1$ for all $i$. Under this homomorphism the generators of $I$ are sent to $2^2$ so $I$ is sent to the ideal generated by $2^2$. On the other hand, if $I$ were principal it would have to be all of $\Lambda_\mu$. Since
Corollary 9: The Alexander polynomial of a link is given by

\[
\frac{\det M_\mu}{(1-x_2)(1-x_3)(1-x_4) \cdots (1-x_{\mu-1})}
\]
CHAPTER III: Surgery techniques.

This chapter deals almost exclusively with links of two components. Section A briefly reviews surgery techniques and contains a lemma which will be needed in section C. Section B contains an example. The main result of the chapter is in section C; here the Alexander invariants of links of two components are characterized in terms of their presentation matrices. This is used in section D to reprove the Torres conditions in the case of two components and to show that if there are restrictions on link polynomials other than the Torres conditions these will have to come from a study of allowable pairs. The concept of allowable pairs is introduced in section E where it is shown that the Torres conditions characterize link polynomials if the linking number of the two components is zero or if both components are unknotted and the linking number is two.
Chapter III: Surgery techniques.

Section A: Using surgeries to unknot knots.

Although surgery techniques had been used for some time in the study of 3-manifolds and even to untie knots (see for example Hempel), Levine was the first to use them to analyse knot and link complements. The idea has since proved useful to several authors including Rolfsen, Goldsmith, Shaneson, Bailey-Rolfsen, and Lickorish. We begin this section by quickly reviewing the ideas which will be used.

If we are using a link diagram to study a link $L$, we can change a crossing at the expense of adding a surgery torus as follows: encircle the crossing with a solid torus $T$ so that (i.) $T$ is unknotted, (ii.) $T$ lies in a ball which is disjoint from any other surgery tori which may be present, and (iii.) $lk( c, l_i ) = 0$ for $i = 1, 2, \cdots, u$ where $c$ is the (oriented) centreline of $T$ and $L = l_1 \cup l_2 \cup \cdots \cup l_u$. Then there is an autohomeomorphism $f : S^3 - T \to S^3 - T$ so that the projection of the resulting link $f(L)$ is the same as the original one except that the encircled crossing is reversed. See figure 1. The autohomeomorphism which reverses the crossing may be visualized as follows: $S^3 - T$ is again a solid torus; cut along a meridional disc, give a right or left hand twist as required and reattach along the meridional disc.

We will orient the centreline $c$, the longitude $l$, and the meridean $m$ of $T$ so that $c$, $l$, and $f(m)$ are all homotopic in $T$. Labelling $f(T)$ by $lk( f(m), c ) = \pm 1$ gives us enough information to reconstruct $f$ up to isotopy and hence is enough to recover the original link. More generally, an integer $n$ which labels a surgery torus (as in section B) will indicate
that a meridional disc \( f(m) \) which realizes the surgery travels once longitudinally around the torus in the direction of the centreline \( c \) and
\[ \text{lk} \left( f(m), c \right) = n. \]
See Rolfsen 2

\[ \text{Figure 1.} \]

**Lemma 7.** Given a two component link \( L = \lambda_1 \cup \lambda_2 \) there are finitely many disjoint solid tori \( T_i \), \( i = 1, 2, \ldots \) with centrelines \( c_i \) and an autohomeomorphism \( f: S^3 - (\bar{\lambda}_1 \cup \bar{\lambda}_2 \cup \cdots ) \rightarrow S^3 - (\bar{\lambda}_1 \cup \bar{\lambda}_2 \cup \cdots ) \) so that

(i.) the tori are unknotted and unlinked,
(ii.) \( \text{lk} \left( c_i, \lambda_j \right) = \text{lk} \left( c_i, f(\lambda_j) \right) = 0 \) \( i = 1, 2, \ldots \), \( j = 1, 2 \).
(iii.) \( f(\partial T_i) = \partial T_i \). Each torus is labeled by a \( \pm 1 \) as above.
(iv.) \( f(L) \) is a \( (2, 2m) \)-torus link pictured in the table of standard symbols.

**Proof:** Take a link diagram of \( L \). Since we can change some of the crossings...
of $\ell_2$ to unknot it, there are solid tori $T_i$ for $i = 1, 2, \cdots, n_1$ as in the previous discussion and a homeomorphism $f_1: S^3 - \bigcup_{i=1}^{n_1} T_i \to S^3 - \bigcup_{i=1}^{n_1} T_i$ so that $f(\ell_2)$ is unknotted. We can now take a homeomorphism $h_1: S^3 \to S^3$ which is isotopic to the identity and so that there is a regular projection of $h_1 \circ f_1(L)$ in which $h_1 \circ f_1(\ell_2)$ is the standard projection of the unknot. We further insist that there be a short distance where the two components are close to and parallel to each other which we engulf in a ball $B$ (see figure 2.)

![Figure 2.](image)

Next, follow $h_1 \circ f_1(\ell_1)$ from where it leaves $B$ to its first crossing with $h_1 \circ f_1(\ell_2)$. We want to change crossings so that this part of $h_1 \circ f_1(\ell_1)$ lies entirely over / under the rest of $h_1 \circ f_1(\ell_1)$ depending on whether that first crossing has $h_1 \circ f_1(\ell_1)$ over / under $h_1 \circ f_1(\ell_2)$ respectively. To do this we place solid tori $h_1(T_i)$ for $i = n_1 + 1, n_1 + 2, \cdots, n_2$ around the crossings to be reversed and let $f_2: S^3 - \bigcup_{i=n_1+1}^{n_2} h_1(T_i) \to S^3 - \bigcup_{i=n_1+1}^{n_2} h_1(T_i)$ which simultaneously reverses the crossings as required. We are now in a position to find an autohomeomorphism $h_2$ of $S^3$ which is isotopic to the identity, leaves $B$ fixed as a set, and so that there is one fewer crossing between the components of the image of $L$ outside of $B$ and one more inside $B$. We can
further insist that the only part of the projection of the link which changes lies on the image of \( l_1 \) from just inside \( B \) to just past the first crossing. Figure 3 should help clarify these ideas.

![Figure 3](image)

(The surgery tori have been omitted from the middle and right diagrams.)

This last step can be repeated until the images of \( l_1 \) and \( l_2 \) have no more crossings outside \( B \), say after \( m \) steps. \( l_1 \) may still be knotted, but we can encircle crossings with surgery tori \( h \circ \cdots \circ h_1(T_{i_1}) \) \( i = m+1, \ldots \) and apply an autohomeomorphism \( f_{m+1} \) of \( S^3 \cup_{i > m} h \circ \cdots \circ h_1(T_{i_1}) \) so that \( f_{m+1} \circ h \circ \cdots \circ h_1(T_{i_1}) \) is unknotted. The required homeomorphism is then

\[
f = h_1^{-1} \circ \cdots \circ h_1^{-1} \circ h \circ \cdots \circ h_1 \circ f_{m+1} : S^3 \cup_{i_1} S^3 \to S^3 \cup_{i_1} S^3.
\]

**Remark:** the proof of this lemma generalizes to more than two components, the result being a pure braid rather than a \((2, 2m)\) torus link. (The \((2, 2m)\) torus links are the pure 2-braids.)
Section B: An example.

Surgery techniques are usually too cumbersome for routine calculations of Alexander invariants; the following example is to illustrate the techniques used in the next section. The reader should be familiar with the examples presented in the chapter on surgery calculations of Alexander polynomials of knots found in Rolfsen 2.

In order to obtain an interesting presentation matrix we take the link $L$ obtained by performing the surgeries as instructed in figure 4 and sewing a thickened disc across $B$ to a regular neighbourhood of $\mu_0$ (this produces a link with linking number two between the components.) Figure 5 shows the universal abelian cover $\tilde{Y}$ of $Y = S^3 \setminus (L \cup B)$ together with the lifts of the surgery tori and one lift of $\mu_0$, labelled $\tilde{\mu}_0$. The full lift of $\mu_0$ is obtained by taking the translates of $\tilde{\mu}_0$. The surgery instructions (that is $-1$ on $T_1$ and $+1$ on $T_2$) lift to surgery instructions $-3$ on $\tilde{T}_1$ and $+1$ on $\tilde{T}_2$.

If $\tilde{Z} = \tilde{Y} - (\text{the lifts of the surgery tori})$ then it is clear that $H_1(\tilde{Z})$ is a free $\Lambda_2$ module of rank 3, the generators being $\alpha_0$ (this is the lift of the commutator of the two most obvious generators of $\pi_1(Y)$), $\alpha_1$ and $\alpha_2$ which are Alexander dual to the centrelines of $T_1$ and $T_2$ respectively. Sewing meridional discs to the surgery tori as instructed gives

$$R_1: -x\alpha_0 + (y-3+y^{-1})\alpha_1 + (x^{-1}y^{-1}-y^{-1})\alpha_2 = 0$$

$$R_2: y\alpha_0 + (xy-y)\alpha_1 + \alpha_2 = 0$$

while sewing a thickened disc to a neighbourhood of $\tilde{\mu}_0$ gives

$$R_0: (1+xy)\alpha_0 + (1-x)(1-y)x^{-1}\alpha_1 - (1-x)(1-y)y^{-1}\alpha_2 = 0.$$
Notice that if we ignore the upper left hand entry and the factor \(-(1-x)(1-y)\) in the first row, we get a matrix with an Hermitian symmetry. This will be explained in the next section.
Section C: The characterization theorem.

We now restrict our attention to links of two components. This section is devoted to a theorem which characterizes link modules as $\Lambda_2$ modules which have a presentation matrix with a symmetry condition.

**Theorem 2:** A $\Lambda_2$ module is a link module iff it has a presentation matrix of the form

$$
\begin{pmatrix}
\frac{1-(xy)}{1-xy} & -(1-x)(1-y)\frac{1-(xy)}{1-xy} & B(x,y) \\
B(x^{-1},y^{-1})^T & A(x,y)
\end{pmatrix}
$$

where $A(x,y)$ is a square matrix, $B(x,y)$ is a row matrix, both with entries in $\Lambda_2$, satisfying $A(x,y) = A(x^{-1},y^{-1})^T$ and $A(1,1) = \text{diag}(\pm 1, \pm 1, \cdots, \pm 1)$.

Further, $A(x,1)$ (respectively $A(1,y)$) is a presentation matrix for the first (respectively second) component of the link and $L$ is the linking number of the two components.

Here "tr" means transpose and $\text{diag}(\pm 1, \pm 1, \cdots, \pm 1)$ is a diagonal matrix with $\pm 1$'s down the diagonal.

**Proof:** Suppose $L = \ell_1 \cup \ell_2$ is a link. By lemma 1 there are surgery tori $T_i$ $i = 1, 2, \cdots, n$ and an autohomeomorphism $f$ of $S^3 - \cup T_i$ so that $f(L)$ is a $(2, 2m)$ torus link. Let $B$ be as in the proof of lemma 1, and let $Y = S^3 - (B \cup f(L))$. Then $Y$ is an open genus 2 handlebody and there is a canonical isomorphism $H_1(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$, the two generators being Alexander dual to $f(\ell_1)$ and $f(\ell_2)$ respectively. As in the case of links, the universal abelian cover $\tilde{Y}$ of $Y$ has $\mathbb{Z} \oplus \mathbb{Z}$ as its group of covering automorphisms, and we see that $H_1(\tilde{Y})$ also has a $\Lambda_2$ module structure on it. In fact $\pi_1(\tilde{Y}) = \mathbb{Z} \ast \mathbb{Z}$ so it is clear that $H_1(\tilde{Y})$ is the free $\Lambda_2$ module generated by $\tilde{h}(\xi \zeta \xi^{-1} \zeta^{-1}) = \delta_0$ where $\tilde{h} : \pi_1(\tilde{Y}) \to H_1(\tilde{Y})$ is the Hurewicz homomorphism, and $\xi$ and $\zeta$ are elements of $\pi_1(Y)$ which are sent to the first and second generators of $H_1(Y) = \mathbb{Z} \oplus \mathbb{Z}$.
respectively by the Hurewicz homomorphism \( h: \pi_1(Y) \to H_1(Y) \). Here I am thinking of \( \pi_1(\tilde{Y}) \) as a subgroup of \( \pi_1(Y) \), so \( \xi \xi^{-1} \in \pi_1(\tilde{Y}) \). We can visualize \( \tilde{Y} \) as follows: situate \( f(L)-B \) in \( S^3-B \) so as to have apparent linking number +1.

Next, imagine \( f(L)-B \) being completed to a \((2,2)\) torus link by adding line segments in \( B \). The complement of a \((2,2)\) torus link is \( S^1 \times S^1 \times \mathbb{R} \) so its universal abelian cover is \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 \) (in the diagrams we have drawn \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) as if it were \( \mathbb{R} \times \mathbb{R} \times (0,1) \). If we remove the lift of \( B \) from this cover we obtain \( \tilde{Y} \). This means that \( \tilde{Y} \) is embedded in \( S^3 \) and we can use a linking number argument. Notice that the \( \mathbb{Z}^2 \) action on \( \tilde{Y} \) extends to a \( \mathbb{Z}^2 \) action on \( \mathbb{R}^3 \) and hence to a \( \mathbb{Z}^2 \) action on \( S^3 \) with \( \infty \) as a fixed point.

The cycle \( \tilde{c}_o \) in \( H_1(S^3-\tilde{Y}) \) which is Alexander dual to \( \tilde{a}_o \) runs from \( \infty \) up through the pillar which \( \tilde{a}_o \) is situated on (linking \( \tilde{a}_o \) once) and continuing up to \( \infty \).

Let \( Z = Y - \bigcup_{i=1}^n T_i \) and \( \tilde{Z} \) the lift of \( Z \) to \( \tilde{Y} \). Since the linking number of each surgery torus with the components of the link is zero, the preimage of a surgery torus consists of a torus which covers it homeomorphically plus all the translates of it under the \( \mathbb{Z}^2 \) action. From this we see that \( H_1(\tilde{Z}) \) is a free \( \mathbb{Z} \) module of rank \( n+1 \).

We are now in a position to recover the universal abelian cover of the link. First, notice that each time a meridional disc is sewn to a torus in \( Z \) a family of meridional discs is sewn equivariantly to the corresponding family of tori in \( \tilde{Z} \), giving one relator for \( H_1(\tilde{X}) \). These relators can be calculated explicitly as follows: let \( c_i \) be the centreline of the \( i \)-th surgery torus \( T_i \), \( \mu_i = f(m_i) \) where \( m_i \) is a meridian of \( T_i \) (oriented as in section A) and \( \tilde{c}_i \) and \( \tilde{\mu}_i \) some lift of \( c_i \) and \( \mu_i \) respectively to \( \tilde{X} \). An explicit set of generators for \( H_1(\tilde{Z}) \) is the set \( \{ \tilde{a}_i \}_{i=0}^n \) where \( \tilde{a}_i \) is Alexander dual to \( \tilde{c}_i \) (that is,
The relator $R_i$ obtained by sewing a meridional disc to $\mu_i$ $i = 1, 2, \cdots, n$ is

$$R_i = \varepsilon_{j,k,m} r_{ijkm} x^j y^k \alpha_m$$

where $r_{ijkm} = \text{lk}(\tilde{\mu}_i, x^j y^k \tilde{\alpha}_m)$. There is another relator, $R_0$, obtained by sewing a disc across $E$, the cycle $\mu_0$ in $S^3_B$ to which it is sewn being determined by the linking number of the two components of the link. $\mu_0$ is the image of $\alpha_0$ under a homeomorphism $g:S^3 = S^3$ which consists of $\ell-1$ Lickorish twists about $t$ (see figure 6). The lift $\tilde{\mu}_0$ of $\mu_0$ can be described as follows: start at the basepoint and follow $\tilde{\alpha}$ until $\tilde{t}$ is reached; follow $\tilde{t}$ until you have covered $t \ell-1$ times; continue along the lift of $\alpha_0$ which you are at (namely $x^{\ell-1} y^{\ell-1} \tilde{\alpha}_0$) until $\tilde{t}$ is reached again; follow $\tilde{t}$ until $t$ has been covered $-\ell$ times; continue along $\alpha_0$ to the basepoint. From this we see that the coefficient of $\alpha_0$ in $R_0$ will be $1+xy+\cdots+(xy)^{\ell-1} \frac{1-(xy)^\ell}{1-xy}$.

Since $\tilde{\mu}_0$ is homotopic to $-(1-x)(1-y) \frac{1-(xy)}{1-xy} \tilde{c}_0$ in $S^3-\tilde{y}$ (see figure 6 for the case $\ell = 3$) we see that the coefficient of $\tilde{\alpha}_m$ in $R_0$ will be

$$-\varepsilon_{j,k,(1-x)(1-y) \frac{1-(xy)}{1-xy} \tilde{c}_0} r_{ijkm} x^j y^k$$

where $r_{ijkm} = \text{lk}(\tilde{c}_0, x^j y^k \tilde{c}_m)$.

We can now write down a presentation matrix for $H_1(\tilde{X})$: the entry in the $i$-th row $j$-th column is the coefficient of $\tilde{\alpha}_j$ in $R_i$ $i,j=0,1,2,\cdots,n$. This is the matrix promised by the theorem; $A(x,y)$ is the matrix obtained by omitting the row and column corresponding to $R_0$ and $\alpha_0$ respectively, and $B(x^{-1},y^{-1}) \text{tr}$ is the column matrix corresponding to $\alpha_0$ omitting the $R_0$ entry. We must show it has the required properties. First,
Figure 6
1k ( \tilde{\mu}_1, x^1 y^{j-1} c_m^{-1} ) = 1k ( \tilde{c}_1^k, x^1 y^{j-1} c_m^{-1} ) \quad \text{unless } i=m \text{ and } j=k=0

= 1k ( x^1 y^{j-1} c_1, c_m^{-1} )

= 1k ( c_m^{-1}, x^1 y^{j-1} c_1^{-1} )

and hence \( A(x,y) = A(x^{-1}, y^{-1})^{tr} \) and the top row (the row corresponding to \( R_0 \)) must be

\[
\frac{1-(xy)^L}{1-xy} \quad -(1-x)(1-y)\frac{1-(xy)^L-1}{1-xy} B(x,y).
\]

Once we have shown that \( A(x,1) \) is a presentation matrix for the Alexander module of the first component corresponding to the given surgeries as described in Rolfsen 1, it will also follow that \( A(1,1) = \text{diag}\ (\pm1, \pm1, \cdots) \).

First, notice that \( A(x,y) \) is a presentation matrix for the link \( L' \) obtained by sewing a disc across \( \alpha_o \) and preforming the given surgeries (so the linking number of the two components is 1). Sewing a disc across \( \alpha_o \) gives a \((2,2)\) torus link, and we can construct its universal abelian cover in two stages: the universal abelian cover of the first component is \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) and the lift of the second component together with the point at \( \infty \) is the unknot in \( \mathbb{R}^3+\infty = S^3 \). The universal abelian cover of this knot then gives the universal abelian cover of \( L' \) as the composite cover. This plus the lemma in Rolfsen 3 makes it clear that \( A(x,1) \) is a presentation matrix for the Alexander invariant of the first component of \( L \). The second component is symmetrical with the first, so \( A(1,y) \) is a presentation matrix for the Alexander invariant of the second component of \( L \).

To prove the converse we need only situate unknotted unlinked surgery tori in \( Y \) together with surgery coefficients which produce the relators \( R_i \) \( i = 1, 2, \cdots, n \). Here we follow Levine 2. Let

\[
D = [0, n+1] \times [0, 1] \times [0, 1]
\]

and let \( D_i = [i-1, i] \times [0, 1] \times [0, 1] \), \( i = 1, 2, \cdots, n+1 \).

Let \( T_i \subset D_i \) be an unknotted solid torus for \( i = 1, 2, \cdots, n \) and let
\( \iota : D \to S^3 - L' \) be an embedding where \( \alpha \subset (D \cap B) \subset (D_{n+1}) \) (so \( (D - D_{n+1}) \subset Y \)).

Follow the instructions given in Levine 2 pages 78 to 80 to modify \( T_1 \) to give the coefficient of \( \tilde{a}_1 \) in \( R_1 \). In order to avoid making the notation too cumbersome we continue to call the modified torus \( T_1 \); similarly for the other tori. But now we must further modify \( T_1 \) as follows: if the coefficient of \( \tilde{a}_0 \) in \( R_1 \) is \( \varepsilon_{(i,j) \in I} c_{ij} x^i y^j \) then situate disjoint 2-discs \( D_{ij} \) in \( (D_{n+1}) \) so that \( \partial D_{ij} = S_{ij} \) is homotopic to \( \tilde{a}_0 c_{ij} \) in \( (D_{n+1}) - B \) and use these and use these for further modifications of \( T_1 \) as above.

Next look at \( R_2 \) and proceed as for \( R_1 \). In addition we must take care of the coefficient of \( \tilde{a}_1 \) which we do as follows: let the coefficient of \( \tilde{a}_1 \) be \( \varepsilon_{(i,j) \in I} (x^i y^j)^{-1} c_{ij} \) where \( (0, 0) \notin I \). Situate \( S_{ij}^+ \) and \( S_{ij}^- \) in a small neighbourhood \( N \) of \( T_1 \) (\( N \) is chosen small enough that \( N \cap T_i = \emptyset \) for all \( i \) and \( N \cap (D_i) = N \) ) where \( S_{ij}^+ \) and \( S_{ij}^- \) bound disjoint discs in \( Y \) and \( \text{lk} (S_{ij}^+, T_1) = -\text{lk} (S_{ij}^-, T_1) = c_{ij} \). Let \( a_{ij} \) be an arc in \( N \) and let \( u_{ij} \) represent \( x^i y^j \) in \( H_1(X) \). Take the boundary connected sum of \( a_{ij} \) and \( u_{ij} \) along an arc between them to obtain \( a_{ij}' \), and let \( S_{ij} \) be the boundary connected sum of \( S_{ij}^+ \) and \( S_{ij}^- \) along \( a_{ij}' \), being careful not to introduce twists (so \( S_{ij} \) and \( T_1 \) are unlinked). Finally, a boundary connected sum of \( T_2 \) and \( S_{ij} \) gives the required torus. Notice that \( T_1 \) and \( T_2 \) are still unlinked. The way to proceed should now be clear.
Section D: The Torres conditions.

As a corollary to the characterization theorem we have

Corollary 3: The order ideal \( E_0^{H_1(\bar{X})} \) is generated by a polynomial of the form

\[
(*) \quad \Delta(x,y) = \frac{1-(xy)^L}{1-xy} A(x,y) - \frac{1-(xy)^{L-1}}{1-xy} B(x,y)
\]

where \( A(x,y) = A(x^{-1}, y^{-1}) \), \( B(x,y) = B(x^{-1}, y^{-1}) \), and \( A(x,1) = \Delta_1(x) \), \( A(1,y) = \Delta_2(y) \). \( \Delta_1(x) \) and \( \Delta_2(y) \) are Alexander polynomials for the first and second components respectively of the link and \( L \) is the linking number of the two components.

Proof: If we calculate the determinant of the matrix given in the characterization theorem by expanding along the first row we get the polynomial \( (*) \) where \( A(x,y) = \det A(x,y) \) and

\[
B(x,y) = \det \begin{pmatrix} 0 & B(x,y) \\ B(x^{-1}, y^{-1}) \text{tr} & A(x,y) \end{pmatrix}
\]

The symmetry conditions on \( A(x,y) \) and \( B(x,y) \) follow from the Hermitian character of \( A(x,y) \).

Corollary 4 (Torres): The Alexander polynomial \( \Delta(x,y) \) of a two component link satisfies

\[
(T1) \quad \Delta(x,1) = \frac{1-x^L}{1-x} \Delta_1(x)
\]
\[
(T2) \quad \Delta(1,y) = \frac{1-y^L}{1-y} \Delta_2(y)
\]
\[
(T3) \quad \Delta(x,y) = x^{L-1} y^{L-1} \Delta(x^{-1}, y^{-1})
\]

Because of corollary 4 it is natural to ask if \( (*) \) puts more restrictions on possible link polynomials than just \( (T1) \) and \( (T2) \). It turns out that it does not.

Proposition 5: A polynomial is in the form \( (*) \) iff it satisfies \( (T1) \) and \( (T2) \).
Proof: "only if" is corollary 4. For the "if" part, let \( \Delta_1(x) \) and \( \Delta_2(y) \) be defined by (T1) if \( \ell \neq 0 \) and let \( \Delta_1(x) = \Delta_2(y) = 1 \) if \( \ell = 0 \). (T2) forces \( \Delta_1(x) = \Delta_1(x^{-1}) \) and \( \Delta_2(y) = \Delta_2(y^{-1}) \). (T1) insures that \( \Delta(1,1) = \ell \Delta_1(1) = \ell \Delta_2(1) \) so \( \Delta_1(1) = \Delta_2(1) = \pm 1 \). Then

\[
P(x,y) = \Delta(x,y) - \frac{1-(xy)^\ell}{1-xy} \Delta_1(x) \Delta_2(y) \Delta_1(1)
\]
satisfies \( P(x,1) = P(1,y) = 0 \) and consequently has the form

\[
P(x,y) = (1-x)(1-y)Q(x,y)
\]
that is

\[
\Delta(x,y) = \frac{1-(xy)^\ell}{1-xy} \Delta_1(x) \Delta_2(y) \Delta_1(1) + (1-x)(1-y)Q(x,y)
\]
where by (T2) we know that \( Q(x,y) = (xy)^{\ell-2}Q(x^{-1},y^{-1}) \). It is easy to check that

\[
R(x,y) = \frac{1-(xy)^\ell}{1-xy} \frac{1-(xy)^\ell}{1-xy} (xy)^{1-\ell} - \frac{1-(xy)^\ell}{1-xy} \frac{1-(xy)^\ell}{1-xy} (xy)^{1-\ell}
\]

so

\[
\Delta(x,y) = \frac{1-(xy)^\ell}{1-xy} \Delta_1(x) \Delta_2(y) \Delta_1(1) + (1-x)(1-y)Q(x,y)R(x,y)
\]

\[
= \frac{1-(xy)^\ell}{1-xy} \{ \Delta_1(x) \Delta_2(y) \Delta_1(1) + (1-x)(1-y) \frac{1-(xy)^\ell}{1-xy} (xy)^{1-\ell}Q(x,y) \}
\]

\[
- (1-x)(1-y) \frac{1-(xy)^\ell}{1-xy} \frac{1-(xy)^\ell}{1-xy} (xy)^{1-\ell}Q(x,y)
\]

Let \( A(x,y) = \Delta_1(x) \Delta_2(y) \Delta_1(1) \) and \( B(x,y) = \frac{(xy)^{1-\ell}-(xy)^2}{1-xy} Q(x,y) \). Then

\[
A(x^{-1},y^{-1}) = \Delta_1(x^{-1}) \Delta_2(y^{-1}) \Delta_1(1) + (1-x^{-1})(1-y^{-1}) \frac{(xy)^{1-\ell}-(xy)^{-1}}{1-(xy)^{-1}} Q(x^{-1},y^{-1})
\]

\[
= \Delta_1(x) \Delta_2(y) \Delta_1(1) + (x-1)(y-1) \frac{(xy)^{1-\ell}}{1-xy} Q(x,y)
\]

\[
= \Delta_1(x) \Delta_2(y) \Delta_1(1) + (x-1)(y-1) \frac{xy-(xy)^{1-\ell}}{1-xy} Q(x,y)
\]
= A(x,y).

The symmetry condition on B(x,y) is handled similarly. The other conditions are obvious.
Section E: Characterizing link polynomials.

In corollary 4 we characterized Alexander polynomials of two component links as those polynomials of the form (*) where \( A(x,y) \) and \( B(x,y) \) were as in the proof of the corollary. Unfortunately this does not allow us to decide when a given polynomial is a link polynomial. We now examine this question.

**Definition:** An ordered pair \( (A(x,y), B(x,y)) \) of elements of \( \Lambda_2 \) is *allowable* if there is a square matrix \( A(x,y) \) and a row matrix \( B(x,y) \) both with entries in \( \Lambda_2 \) satisfying \( A(x,y) = A(x^{-1}, y^{-1})^{\text{tr}} \) and \( A(1,1) = \text{diag}(\pm 1, \ldots, \pm 1) \) so that

\[
A(x,y) = \det A(x,y)
\]

\[
B(x,y) = \det \begin{pmatrix} 0 & B(x,y) \\ B(x^{-1}, y^{-1})^{\text{tr}} & A(x,y) \end{pmatrix}
\]

We will use this notation (possibly with subscripts) in the remainder of this section.

The problem is now to characterize allowable pairs. It is clear that we must have \( A(x,y) = A(x^{-1}, y^{-1}) \), \( A(1,1) = \pm 1 \), and \( B(x,y) = B(x^{-1}, y^{-1}) \); I do not know if these conditions are sufficient. In this section we find a method for generating enough allowable pairs to prove that the Torres conditions characterize two component link polynomials provided the linking number of the two components is zero or both components are unknotted and the linking number is two.

**Notation:** if

\[
C_i = \begin{pmatrix} 0 & B_i(x,y) \\ B_i(x^{-1}, y^{-1})^{\text{tr}} & A_i(x,y) \end{pmatrix}
\]

then
Lemma 6: \( \det C_1 \cdot C_2 = A_1 B_2 + A_2 B_1 \)

Proof: Suppose that \( A_1(x,y) \) is \( n \times n \) and \( A_2(x,y) \) is \( m \times m \) and let \( C_1 \cdot C_2 = c_{ij} \) where \( i, j = 1, 2, \ldots, n+m \). Then

\[
\det C_1 \cdot C_2 = \sum_{i=0}^{m+n} \text{sgn} \prod_{i=0}^{n} c_{i \sigma(i)}.
\]

Let \( X = \{ \sigma \mid \sigma \text{ leaves } 0, 1, \ldots, n \text{ fixed as a set} \} \),
\( Y = \{ \sigma \mid \sigma \text{ leaves } 1, 2, \ldots, n \text{ fixed as a set} \} \),
\( Z = \{ \sigma \mid \text{there is an } i \text{ where } 1 < i < n \text{ so that } \sigma(i) > n \} \).

Notice that \( X \cap Y = \{ \sigma \mid \sigma(0) = 0 \} \) and hence \( \prod_{i=0}^{m+n} c_{i \sigma(i)} = 0 \) if \( \sigma \in X \cap Y \).

If \( 1 \leq i \leq n \) and \( \sigma(i) > n \) then \( c_{i \sigma(i)} = 0 \) and so \( \prod_{i=0}^{m+n} c_{i \sigma(i)} = 0 \) if \( \sigma \in Z \).

Also, every permutation is in at least one of the three sets. Hence

\[
\det C_1 \cdot C_2 = \sum_{\sigma \in X} \text{sgn} \prod_{i=0}^{n} c_{i \sigma(i)} + \sum_{\sigma \in Y} \text{sgn} \prod_{i=0}^{n} c_{i \sigma(i)} + \sum_{\sigma \in Z} \text{sgn} \prod_{i=0}^{n} c_{i \sigma(i)}.
\]

If \( \sigma \in X \) we can think of \( \sigma = \sigma_1 \sigma_2 \) where \( \sigma_1 \) is a permutation of the set \( \{ 0, 1, 2, \ldots, n \} \) and \( \sigma_2 \) is a permutation of \( \{ n+1, \ldots, n+m \} \). Then

\[
\sum_{\sigma \in X} \text{sgn} \prod_{i=0}^{n} c_{i \sigma(i)} = \sum_{\sigma_1} \sum_{\sigma_2} \text{sgn} \prod_{i=0}^{n} c_{i \sigma_1(i)} \prod_{i=n+1}^{m+n} c_{i \sigma_2(i)}
\]

\[
= A_2 B_1.
\]

Similarly \( \sum_{\sigma \in Y} \text{sgn} \prod_{i=0}^{n} c_{i \sigma(i)} = A_1 B_2 \).

Theorem 7: \( (1, B(x,y)) \) is an allowable pair for any \( B(x,y) \) satisfying

\( B(x,y) = B(x^{-1}, y^{-1}) \).

Proof: Write \( B(x,y) = a_{00} + \sum (i,j) \neq (0,0) a_{ij} (x_i y_j + x^{-1} y^{-j}) \) with a suitable restriction on the index set to insure that not both of \((i,j)\) and \((-i,-j)\) are present. Let
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\[
C_{ij} = \begin{cases}
0 & x^i y^j - a_{ij} & a_{ij} & 1 \\
-x^i y^j - a_{ij} & 1 & 0 & 0 \\
a_{ij} & 0 & -1 & 0 \\
1 & 0 & 0 & -1
\end{cases}
\text{if } (i,j) \neq (0,0)
\]

\[
C_{00} = \begin{cases}
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \ast \cdots \ast \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & -a_{00} \text{ times if } a_{00} < 0 \\
\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \ast \cdots \ast \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} & a_{00} \text{ times if } a_{00} > 0 \text{ and even} \\
\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \ast \cdots \ast \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \ast \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} & a_{00} + 1 \text{ terms if } a_{00} > 0 \text{ and odd}
\end{cases}
\]

Then \( A_{ij}(x,y) = 1 \) and

\[
B_{ij}(x,y) = \begin{cases}
a_{ij} \left( x^i y^j + x^{-i} y^{-j} \right) & (i,j) \neq (0,0) \\
a_{00} & (i,j) = (0,0)
\end{cases}
\]

so lemma 6 gives \( \det (A_{ij}) = B(x,y) \) and so \( C = \left( A_{ij} \right) \) realizes the allowable pair \( (1, B(x,y)) \).

Remark: Looking at \( C \) we see that \( A(x,y) = \text{diag}(\pm 1, \pm 1, \cdots, \pm 1) \). Retracing the steps in the characterization theorem we see that the surgery tori corresponding to \( C \) can be chosen to be unlinked from \( L' \); hence the link so obtained has both components unknotted.

Corollary 8: 1. The Torres conditions characterize link polynomials when the components have linking number zero; in fact any such polynomial can be realized by a link with two unknotted components.

2. The Torres conditions characterize link polynomials when the components have linking number two and both components are unknotted.

Proof: 1. A polynomial satisfies the Torres conditions with linking number zero iff it is of the form \((1-x)(1-y)B(x,y)\) where \(B(x,y) = B(x^{-1}, y^{-1})\). But for any such \(B(x,y)\) we know by theorem 7 that \( (1, B(x,y)) \) is allowable and the result follows from the above remark.
2. The Torres conditions for linking number two and both components unknotted force the polynomial to have the form \((1+xy) - (1-x)(1-y)B(x,y)\) where \(B(x,y) = B(x^{-1}, y^{-1})\). Again \((1, B(x,y))\) is allowable for any such \(B(x,y)\) and the conclusion follows from the above remark.

Remark: Levine 2 shows that the Torres conditions characterize link polynomials when the linking number of the two components is one. This also follows from our results since it is clear that \((A(x,y), 0)\) is allowable for any \(A(x,y) = A(x^{-1}, y^{-1})\) and \(A(1,1) = \pm 1\) (take \(A(x,y) = A(x,y)\) and \(B(x,y) = 0\)). Similarly Levine's theorem 2 in the case of two component links follows from our work by taking the direct sum of the given presentation matrix and the matrix with a single entry as the new presentation matrix.

We have also answered the question following theorem 2 affirmatively in the case of two component links: the polynomial of a link of two components with splittable linking matrix (which for links of two components simply means that the linking number of the components is zero) may be multiplied by any polynomial satisfying \(P(x,y) = P(x^{-1}, y^{-1})\) and the product realized as the Alexander polynomial of a link.
CHAPTER IV: Seifert surfaces.

In this chapter we explore the technique of Seifert surfaces in the study of link polynomials. In section A we prove a generalization of a result of Kidwell which relates the individual degrees of the Alexander polynomial to the linking complexity. Section B turns to the special case of 2-bridge links, which have long been known to be particularly suitable for analysis. In this section we prove that 2-bridge links are interchangable and present an algorithm for calculating the Alexander polynomial from a 2-bridge presentation. As a corollary to this we prove a conjecture of Kidwell in the special case of 2-bridge links. The work in sections C and D was done in the hope of finding a way to characterize allowable pairs without working directly with the matrix in the characterization theorem. Unfortunately the results found are too complicated to do this but may be of use in the future machine calculations of links. Section C gives methods of generating link polynomials from allowable pairs and section D uses these results as well as others in the thesis to compile a list of methods for generating allowable pairs without resorting to matrices.
Chapter IV. Seifert surfaces.

Seifert surfaces have been used extensively to study the Alexander invariants (and, more generally, the abelian invariants) of knots, but so far their use in studying link invariants has been restricted to boundary links (that is, links which bound disjoint Seifert surfaces) as in Gutierrez and reduced Alexander polynomials (see Torres and Kidwell.)

The construction for knots can be iterated to construct the universal abelian cover of a link as follows: let \( L = \bigcup_{i=1}^{n} \ell_i \) be a link.

Then \( \ell_1 \) bounds a Seifert surface which can be used to construct \( \tilde{\mathcal{X}}_1 \) where \( \mathcal{X}_1 = S^3 - \ell_1 \) (see Rolfsen 2 for details.) \( \ell_2 \) also bounds a Seifert surface in \( S^3 \), say \( \mathcal{S}_2 \). Let \( \tilde{\mathcal{S}}_2 \) be the preimage of \( \mathcal{S}_2 - \ell_1 \) in \( \tilde{\mathcal{X}}_1 \). We can cut along this surface and assemble countably many copies of \( \tilde{\mathcal{X}}_1 - \tilde{\mathcal{S}}_2 \) into a covering space \( \tilde{\mathcal{X}}_{12} \) of \( \tilde{\mathcal{X}}_1 \) so that a loop \( a \) in \( \tilde{\mathcal{X}}_1 - \tilde{\mathcal{S}}_2 \) lifts to a loop in \( \tilde{\mathcal{X}}_{12} \) iff \( a \cdot \tilde{\mathcal{S}}_2 = 0 \) (that is, the algebraic intersection of \( a \) and \( \tilde{\mathcal{S}}_2 \) is zero.) From this it is clear that \( \tilde{\mathcal{X}}_{12} \) is the universal abelian cover of \( \mathcal{X}_{12} = S^3 - \ell_1 \cup \ell_2 \). If this construction is continued it eventually yields the universal abelian cover of \( \mathcal{X} = S^3 - L \).

Section A: A theorem of Kidwell.

Definition: Let \( L = \ell_1 \cup \ell_2 \) be a link and \( \mathcal{S} \) be a Seifert surface for \( \ell_1 \) with \( \mathcal{S} \) and \( \ell_2 \) in general position. If

\[
\alpha_{\mathcal{S}} = 2(\text{genus of } \mathcal{S}) + \text{the number of times } \ell_2 \text{ intersects } \mathcal{S}
\]

then \( \alpha_{\ell_1} = \min_{\mathcal{S}} \alpha_{\mathcal{S}} \) is the linking complexity of \( \ell_2 \) with \( \ell_1 \). The linking complexity of the link \( L \) is the ordered pair \( (\alpha_{\ell_1}, \alpha_{\ell_2}) \).

Remark: If \( \ell_1 \) is unknotted and we insist that \( \mathcal{S} \) be a disc then we get the definition of order as found in Kidwell. In general the order is greater than the linking complexity; for example the n-th iterated double of the Whitehead link has order \( (2^n, 2^n) \) (see Kidwell) while it is clear that
they have linking complexity (2, 2). Consequently the following theorem
sharpens as well as generalizes the main result in Kidwell. First we need

**Definition:** If \( \Delta(x,y) = \sum_{i=0}^{m} p_i(y)x^i \) where \( p_i(y) \) is a polynomial in \( y \) and
\( p_n(y) \neq 0 \neq p_m(y) \) then \( \deg_\Delta(x,y) = m-n \).

**Theorem 1 (Kidwell):** If \( \Delta(x,y) \) is the Alexander polynomial of a link
\( L = \ell_1 \cup \ell_2 \) with linking complexity \( (\alpha_1, \alpha_2) \) then
\( \alpha_1 - 1 \geq \deg_\Delta(x,y) \).

**Proof:** Let \( X_2 = S^3 - \ell_2, \tilde{X}_2 \) its universal abelian cover, and \( \tilde{S} \) the lift of
\( S - \ell_2 \) to \( \tilde{X}_2 \) where \( S \) is a Seifert surface for \( \ell_1 \). If \( Y = \tilde{X}_2 - \tilde{S} \) we can construct
\( \tilde{X} \) by identifying \( \{ Y_1 = Y \}_{i \in \mathbb{Z}} \) and \( \{ N_i = \tilde{S} \times (-1,1) \}_{i \in \mathbb{Z}} \) appropriately. From
the Mayer–Vietoris sequence

\[
\cdots \to H_1(\bigcup N_i) \xrightarrow{\psi} H_1(\bigcup Y_i) \to H_1(\tilde{X}) \to H_0(\bigcup N_i) \xrightarrow{\phi} H_0(\bigcup Y_i) \to \cdots
\]

We obtain the short exact sequence

\[
0 \to \text{coker}\psi \to H_1(X) \to \text{ker}\phi \to 0
\]

By lemma 5 of Levine 2 we know that \( \Delta(x,y) = \Delta_{\text{coker}\psi} \cdot \Delta_{\text{ker}\phi} \) so we now
examine the latter two.

1. \( \Delta_{\text{coker}\psi} \). A presentation for \( \text{coker}\psi \) can be obtained from one for \( H_1(\bigcup Y_i) \)
by adding the relations \( \psi(\tilde{a}_i) \) where \( \{ \tilde{a}_i \} \) is a set of generators for \( H_1(\bigcup N_i) \).
These relations are of the form \( \psi_-(\tilde{a}_i) - \psi_+(\tilde{a}_i)x = 0 \); since the relations
for \( H_1(\bigcup Y_i) \) do not involve \( x \) we see that \( \deg_{x \text{coker}\psi} \leq \) the number of generators
of \( H_1(\bigcup N_i) \). \( H_1(S - \ell_2) \) is a free \( \mathbb{Z} \) module on \( 2g + h \) generators where
\( g = \text{genus of } S \) and \( h = \text{number of times } \ell_2 \text{ intersects } S \). Let this set of
generators be represented by cycles \( \{ a_i \}_{i=1}^{2g+h} \) in \( S - \ell_2 \) where we can assume
that \( 1k(a_i, \ell_2) = \ell > 0 \) and \( 1k(a_i, \ell_2) = 0 \) if \( i \neq 1 \). When \( \ell \neq 0 \) the
lifts \( \tilde{a}_i \) of \( a_i \) for \( i \geq 2 \) will generate \( H_1(\tilde{S}) \) and \( H_1(\bigcup N_i) \) will be a free \( A_2 \)
module on \( 2g + h - 1 \) generators, giving \( \deg_{x \text{coker}\psi} \leq \alpha_1 - 1 \). To finish
the proof of the theorem we show that \( \Delta_{\text{ker}\phi} = 1 + x + x^2 + \cdots + x^{\ell-1} \) if \( \ell > 0 \)
and \( \Delta_{\text{ker}\phi} = 0 \) if \( \ell = 0 \).
2. $\Delta_{\ker \phi}$. Notice that $\tilde{S}$ has $l$ components if $l > 0$ and infinitely many components if $l = 0$. In either case,

$$H_0( \cup N_1) \cong ( a ; ( y - 1 ) a = 0 )$$

as a $\Lambda_2$ module. It is easy to see that $b = (y-1)a$ will generate $\ker \phi$ so that we must have $\ker \phi \cong ( b ; \frac{1-y}{1-y} b = 0 )$. Hence $\Delta_{\ker \phi} = \frac{1-y}{1-y}$.  

Corollary 2: If $L$ is a boundary link then $\Delta(x,y) = 0$.

Proof: Boundary links have $l = 0$ in the above proof.  

The proof of this theorem generalizes to links of more than two components. This is found in the appendix.
Section B: Two-bridge links.

We now turn our attention to 2-bridge links and prove that order = linking complexity = $\deg_x \Delta(x, y)$ for this special class of links.

We start by proving that $\alpha_1 = \alpha_2$.

Proposition 4: 1. 2-bridge links are interchangeable. More precisely, there is an isotopy of $S^3$ which interchanges the two components, possibly also reversing the orientations of the two components.

2. There is an isotopy of $S^3$ which reverses the orientations of the two components.

Proof: Every 2-bridge link can be put in the form of figure 1.

If we remove $B_1$ (a 3-ball situated as in figure 1) then

i. $A_2 = L - \partial B_1 \subset S^3 - \partial B_1 = B_2$ is a pair of disjoint trivial spanning arcs (as in figure 2) and

ii. $L$ can be effectively recovered by sewing a thickened disc to a regular neighbourhood of $c$ in figure 1, that is $L$ is recovered by sewing a thickened
Figure 2.

disc to a regular neighbourhood of $h(c_2)$ in figure 2 where $H : B_2 \rightarrow B_2$ is the orientation preserving homeomorphism from figure 1 to figure 2 and $h = H|\partial B_2$. Up to isotopy leaving $\partial A_2$ fixed $h$ is a composite of the homeomorphisms which are described in figure 3.

Figure 3.
More precisely, if $\partial B_2$ is the unit sphere in $\mathbb{R}^3$, points on $\partial B_2$ are determined by the two angles ($\theta, \phi$) in spherical coordinates where $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. If $\partial A_2 = \{(0, \pm\frac{\pi}{4}), (\pi, \pm\frac{\pi}{4})\}$ then

$$h_1(\theta, \phi) = \begin{cases} (\theta + \pi, \phi) & \phi \geq \frac{\pi}{4} \\ (\theta + 2\phi + \frac{\pi}{2}, \phi) & -\frac{\pi}{4} \leq \phi \leq \frac{\pi}{4} \\ (\theta, \phi) & \phi \leq -\frac{\pi}{4} \end{cases}$$

$h_2$ is obtained by rigidly rotating $\mathbb{R}^3$ through $\pi$ about the line $\theta = 0$, $\phi = \frac{\pi}{4}$, applying $h_1$ and rotating back again. We also have a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action on $(B_2, A_2)$; one generator, $\iota_1$ (rotation through $\pi$ about a vertical axis) interchanges the two components, while the other $\iota_2$ (rotation through $\pi$ about a horizontal axis) reverses the orientation of the two components. (On $B_2$, $\iota_1(\theta, \phi) = (\theta + \pi, \phi)$ and $\iota_2(\theta, \phi) = (-\theta, -\phi)$.) Since $\iota_1 \circ h_1 = h_1 \circ \iota_1$ and $\iota_2 \circ h_1 = h_1 \circ \iota_2$ we see that if $C$ is any set invariant under the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action then so is $h_1(C)$; by symmetry the same is true of $h_2(C)$. The proof is completed by noticing that $c_2$ is invariant under the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action.

If you look at figure 1 you will see that any 2-bridge link can be built up in stages as is represented schematically in figure 4. We use this observation to calculate the Alexander polynomial of a 2-bridge link.

**Theorem 4:** The Alexander polynomial of a 2-bridge link is the determinant of an $m \times m$ matrix (where $m$ is the number of stages in figure 4) with diagonal entries

$$a_{ii}(x,y) = \begin{cases} n_i(x-1)(y-1) & n_i \neq 0 \\ -n_i(x-1)(y-1) - (1+xy) & n_i \geq 0 \\ n_i(x-1)(y-1) - (x+y) & n_i > 0 \end{cases}$$

superdiagonal entries

$$a_{i,i-1}(x,y) = xy$$
The link at left has (see corollary 5)

\[ P_1(x, y) = (x-1)(y-1) - (x+y) \quad \epsilon_1 = 1 \]

\[ \Delta_1(x, y) = (x-1)(y-1) - (x+y) = 1-2x-2y+xy \]

The link 222 (see Conway for notation) has this as link polynomial.

\[ P_2(x, y) = (x-1)(y-1) \quad \epsilon_2 = -1 \]

\[ \Delta_2(x, y) = (x-1)(y-1)(1-2x-2y+xy) + xy \]

The link 22112 has this as link polynomial.

\[ P_3(x, y) = -(x-1)(y-1) - (xy+1) \quad \epsilon_3 = 1 \]

\[ \Delta_3(x, y) = (-x)(y-1)-(xy+1)((x-1)(y-1)(1-2x-2y+xy)+xy) -xy(1-2x-2y+xy) \]

The link 22111122 has this as link polynomial.

Figure 4.
subdiagonal entries
\[
a_{i-1,1} = \begin{cases} 
1 & \text{if } a_{i1}(1,1) = -2 \\
-1 & \text{if } a_{i1}(1,1) = 0
\end{cases}
\]
and all other entries zero.

Proof: Let the link be \( L = \ell_1 \cup \ell_2 \), \( \tilde{X}_1 \) be the universal abelian cover of \( X_1 = S^3 - \ell_1 \), and \( \tilde{D} \) be the lift of the disc spanned by \( \ell_2 \) as in figure 4. We construct the universal abelian cover of \( X = S^3 - L \) by cutting along \( \tilde{D} \). Each stage in figure 4 adds a generator to \( H_1(\tilde{D}) \) as a \( \Lambda_1 \) module. There are 16 cases to consider. The first four are contained in figure 5; the corresponding relators are contained as the first four entries of table 1. The relators are obtained from the diagram as follows (the notation is the same as in Rolfsen 2, chapter 7, section B.)

Case 1.
\[
x \beta_i + n_i(x-1)\beta_i - x \beta_{i+1} \equiv b_i \pm \beta_{i-1} + n_i(x-1)\beta_{i-1} - \beta_i \\
\]
\( R_i: y \beta_{i-1} + \{n_i(x-1)(y-1) - (x+y)\beta_i + x \beta_{i+1}\} \)

Case 2.
\[
x \beta_i + n_i(x-1)\beta_i - x \beta_{i+1} \equiv b_i \pm x \beta_{i-1} + n_i(x-1)\beta_{i-1} - \beta_i \\
\]
\( R_i: xy \beta_{i-1} + \{n_i(x-1)(y-1) - (x+y)\beta_i + x \beta_{i+1}\} \)

Case 3.
\[
\beta_{i-1} + n_i(x-1)\beta_i - x \beta_{i+1} \equiv b_i \pm n_i(x-1)\beta_i \\
\]
\( R_i: -\beta_{i-1} + n_i(x-1)(y-1)\beta_{i-1} + x \beta_{i+1} \)

Case 4.
\[
x \beta_{i-1} + n_i(x-1)\beta_i - x \beta_{i+1} \equiv b_i \pm n_i(x-1)\beta_i \\
\]
\( R_i: -x \beta_{i-1} + n_i(x-1)(y-1)\beta_{i-1} + x \beta_{i+1} \)

The next four cases are obtained from the previous four by taking the mirror image of the diagram and replacing \( x \) by \( x^{-1} \), \( \beta_j \) by \( -\beta_j \), and \( b_i \) by \( -b_i \). This gives rise to the next four entries in the table. The last eight cases are obtained from the first eight by reversing the orientation of \( \tilde{D} \). This has the effect of replacing \( y \) by \( y^{-1} \) and accounts for the last eight entries in the table.

We can now build a presentation matrix for \( H_1(\tilde{X}) \). The entry in the
Case 1
\[ a_i = 2n_i, \; n_i > 0 \]

Case 2
\[ a_i = 2n_i, \; n_i > 0 \]
Figure 5.
i-th row, j-th column is the coefficient of $\beta_j$ in $R_1$. We immediately see that $a_{ij} = 0$ if $|i - j| > 1$. Notice that case 1 can be preceded by cases 1, 2, 3, and 4 and can be followed by cases 1, 2, 3, and 4. This plus the relevant observations for the other cases allows us to conclude that, around the i-th diagonal entry, the presentation matrix looks as in table 2.

By simultaneously multiplying the rows and columns by $x^m y^n$ as we move down the diagonal we can make every entry on the subdiagonal ±1 and every entry on the superdiagonal $xy$. Table 3 lists this set of possibilities. But this is the form promised in the theorem.

As a corollary to the proof we have

**Corollary 5:** Let $L$ be a link as in figure 4 where $\ell_2$ and hence $D$ have been oriented. Let $P_i(x,y)$ and $\epsilon_i \ i = 1, 2, \cdots, m$ be as in figure 6 and inductively define $\Delta_i(x,y)$ as follows:

$$
\Delta_1(x,y) = P_1(x,y) \\
\Delta_2(x,y) = P_2(x,y)\Delta_1(x,y) - \epsilon_2 xy \\
\Delta_i(x,y) = P_i(x,y)\Delta_{i-1}(x,y) - \epsilon_i xy \Delta_{i-2}(x,y) \quad i > 2
$$

Then $L$ has Alexander invariant isomorphic to $\frac{\Delta_2}{\Delta_m(x,y)}$.

**Proof:** It is clear that the Alexander invariant of a 2-bridge link is cyclic; to see that the polynomial is correct, take the presentation matrix in the above theorem and expand by the last row and column to obtain its determinant.

Kidwell has conjectured that for alternating links the inequality in theorem 1 (with linking complexity replaced by order) is an equality. We now prove this for 2-bridge links.

**Definition:** If $\Delta(x,y) = \sum_{i=n}^{m} p_i(y)x^i$ where $p_i(y)$ is a polynomial in $y$ and $p_n(y) \neq 0 \neq p_m(y)$ then $\min_{x} \Delta(x,y) = n$ and $\max_{x} \Delta(x,y) = m$. Hence
Figure 6

\[ P_i(x, y) = n_i(x-1)(y-1) - (x+y) \]
\[ \varepsilon_i = 1 \]

\{ \text{an even number of crossings} \} = 2n_i

\[ P_i(x, y) = -n_i(x-1)(y-1) - (xy+1) \]
\[ \varepsilon_i = 1 \]

\{ \text{an odd number of crossings} \} = 2n_i - 1

\[ P_i(x, y) = n_i(x-1)(y-1) \]
\[ \varepsilon_i = -1 \]

\[ P_i(x, y) = -n_i(x-1)(y-1) \]
\[ \varepsilon_i = -1 \]
\[ \text{deg}_x \Delta(x,y) = \max_x \Delta(x,y) - \min_x \Delta(x,y). \]

**Proposition 6:** \( \text{deg}_x \Delta(x,y) = a_1 - 1 \) for 2-bridge links.

**Proof:** by induction on the number of stages in figure 4. If figure 4 has \( m \) stages it is clear that \( a_1 \leq m+1 \) so by theorem 1 it suffices to show that \( \text{deg}_x \Delta^m(x,y) = m \). If \( m = 1 \) or 2 we see that \( \min_x \Delta^m(x,y) = 0 \) and \( \max_x \Delta^m(x,y) = m \); assume that this is true for \( m \leq i-1 \). Then we have

\[
\begin{align*}
\min_x P_i(x,y)\Delta^{i-1}(x,y) &= 0 \\
\max_x P_i(x,y)\Delta^{i-1}(x,y) &= i \\
\min_x xy\Delta^{i-2}(x,y) &= 1 \\
\max_x xy\Delta^{i-2}(x,y) &= i-1
\end{align*}
\]

Hence \( \min_x \Delta^i(x,y) = 0 \) and \( \max_x \Delta^i(x,y) = i \) and the result is proved. \( \Box \)
<table>
<thead>
<tr>
<th>Case</th>
<th>Relator</th>
<th>$n_i \geq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$y\beta_{i-1} + {n_i(x-1)(y-1) - (x+y)}\beta_i + x\beta_{i+1}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$xy\beta_{i-1} + {n_i(x-1)(y-1) - (x+y)}\beta_i + x\beta_{i+1}$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$-\beta_{i-1} + n_i(x-1)(y-1)\beta_i + x\beta_{i+1}$</td>
<td>$n_i \geq 1$</td>
</tr>
<tr>
<td>4</td>
<td>$-x\beta_{i-1} + n_i(x-1)(y-1)\beta_i + x\beta_{i+1}$</td>
<td>$n_i \geq 1$</td>
</tr>
<tr>
<td>1</td>
<td>$xy\beta_{i-1} - {n_i(x-1)(y-1) + (1+xy)}\beta_i + \beta_{i+1}$</td>
<td>$n_i \geq 0$</td>
</tr>
<tr>
<td>2</td>
<td>$y\beta_{i-1} - {n_i(x-1)(y-1) + (1+xy)}\beta_i + \beta_{i+1}$</td>
<td>$n_i \geq 0$</td>
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<td>$-\beta_{i-1} - n_i(x-1)(y-1)\beta_i + \beta_{i+1}$</td>
<td>$n_i \geq 1$</td>
</tr>
<tr>
<td>1</td>
<td>$\beta_{i-1} - {n_i(x-1)(y-1) + (1+xy)}\beta_i + xy\beta_{i+1}$</td>
<td>$n_i \geq 0$</td>
</tr>
<tr>
<td>2</td>
<td>$x\beta_{i-1} - {n_i(x-1)(y-1) + (1+xy)}\beta_i + xy\beta_{i+1}$</td>
<td>$n_i \geq 0$</td>
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<td>$n_i \geq 1$</td>
</tr>
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<td>$x\beta_{i-1} + {n_i(x-1)(y-1) - (x+y)}\beta_i + y\beta_{i+1}$</td>
<td>$n_i \geq 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\beta_{i-1} + {n_i(x-1)(y-1) - (x+y)}\beta_i + y\beta_{i+1}$</td>
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<td>$-y\beta_{i-1} + n_i(x-1)(y-1)\beta_i + y\beta_{i+1}$</td>
<td>$n_i \geq 1$</td>
</tr>
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</table>

Table 1.

Table 2 is on the next page.
Table 2.
<table>
<thead>
<tr>
<th>Cases</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2, $1_1$, $2_1$</td>
<td>$\begin{pmatrix} * &amp; xy &amp; 0 \ 1 &amp; n_1(x-1)(y-1)-(x+y) &amp; xy \ 0 &amp; \pm 1 &amp; * \end{pmatrix}$</td>
</tr>
<tr>
<td>1, 2, $1_1$, $2_1$</td>
<td>$\begin{pmatrix} * &amp; xy &amp; 0 \ 1 &amp; n_1(x-1)(y-1)-(x+y) &amp; xy \ 0 &amp; \pm 1 &amp; * \end{pmatrix}$</td>
</tr>
<tr>
<td>3, 4, $3_1$, $4_1$</td>
<td>$\begin{pmatrix} * &amp; xy &amp; 0 \ -1 &amp; n_1(x-1)(y-1) &amp; xy \ 0 &amp; \pm 1 &amp; * \end{pmatrix}$</td>
</tr>
<tr>
<td>3, 4, $3_1$, $4_1$</td>
<td>$\begin{pmatrix} * &amp; xy &amp; 0 \ -1 &amp; n_1(x-1)(y-1) &amp; xy \ 0 &amp; \pm 1 &amp; * \end{pmatrix}$</td>
</tr>
</tbody>
</table>

Table 3
Section C: Generating link polynomials from allowable pairs.

**Theorem 7:** If \((A(x,y), B(x,y))\) is an allowable pair and the polynomials \(\Delta^i(x,y)\) are as in corollary 5, then

\[
\Delta^i(x,y)A(x,y) \pm (1-x)(1-y)\Delta^{i-1}(x,y)B(x,y)
\]

and

\[
\Delta^i(x,y)A(x,y) \pm (1-x^{-1})(1-y^{-1})\Delta^{i+1}(x,y)B(x,y)
\]

are link polynomials.

**Proof:** The allowable pair is generated by a pair of matrices \(A(x,y)\) and \(B(x,y)\) which in turn determine a set of surgery instructions for a link. The relevant observation is that the cycle \(\mu_0\) in \(\partial B\) is restricted only by \(\mu_0 = 0 \in H_1(Y)\) and has no self intersections. Hence \(\mu_0\) can generate any 2-bridge link. The cycle \(\tilde{\mu}_0\) in \(\partial \tilde{B}\) determines the relator \(R_0\) and it is clear that the coefficient of \(\tilde{\alpha}_0\) in \(R_0\) may be taken to be the Alexander polynomial of any 2-bridge link, say \(\Delta^i(x,y)\) (the coefficient of \(\tilde{\alpha}_0\) in \(R_0\) is the Alexander polynomial of the link \(L_1\) obtained by ignoring all the surgery tori.) If \(T\) is as in figure 7 then the coefficient which multiplies \(B(x,y)\) to complete the presentation matrix is

1. \[
\sum_{i,j} 1 k(\tilde{\mu}_0, x^i y^j T ) x^i y^j
\]

The lower half of figure 7 is just a rearranging of the top half so that \(\tilde{B}\) is emphasized. \(H_1(\tilde{B})\) is a free \(\Lambda_2\) module on the generator \(\beta\) which is dual to the cycle \(\beta\) which starts at \(-\infty\), runs through the grid linking \(\beta\) once and continues to \(+\infty\); hence \(T = (1-x)(1-y)b\). If we sew the disc to \(\tilde{\mu}_0\) in \(\tilde{B}\) (rather than \(\tilde{Y}\)) the resulting polynomial is

2. \[
\sum_{i,j} 1 k(\tilde{\mu}_0, x^i y^j b ) x^i y^j
\]

This is the Alexander polynomial of link \(L_2\) obtained by sewing a disc to \(B\) along \(\mu_0\). Figure 8 shows the relationship between \(L_1\) and \(L_2\); we see that if \(a_1 = 0\) then \(L_1\) has one fewer stage than \(L_2\) so we let the Alexander
polynomial of $L_2$ be $\Delta^{i+1}(x,y)$, while in the other cases $L_1$ has one more stage than $L_2$ so we let the Alexander polynomial of $L_2$ be $\Delta^{i-1}(x,y)$. Comparing 1, 2, and knowing that $T = (1-x)(1-y)b$ we see that the required coefficient is $(1-x)(1-y)\Delta^{i-1}(x,y)$ or $(1-x^{-1})(1-y^{-1})\Delta^{i+1}(x,y)$.

Strictly speaking we have determined $\Delta^j(x,y)$ for $j = i-1, i, i+1$ only up to a factor of $\pm x^m y^n$. If we insist that $x^j y^j \Delta^j(x^{-1}, y^{-1}) = \Delta^j(x,y)$ (which we require by the Torres conditions) we have determined them up to a factor of $\pm 1$. That this is unimportant is seen in lemma 8 which follows.

The presentation matrix is

$$
\begin{pmatrix}
\Delta^i(x,y) & \pm(1-x)(1-y)\Delta^{i-1}(x,y)B(x,y) \\
B(x^{-1}, y^{-1}) tr & A(x,y)
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
\Delta^i(x,y) & \pm(1-x^{-1})(1-y^{-1})\Delta^{i+1}(x,y)B(x,y) \\
B(x^{-1}, y^{-1}) tr & A(x,y)
\end{pmatrix}
$$

whose determinants are those promised by the theorem.

\textbf{Lemma 8:} If $(A(x,y), B(x,y))$ is an allowable pair, then so is $(A(x,y), -B(x,y))$

\textbf{Proof:} Let $A(x,y)$ and $B(x,y)$ be the matrices which generate the allowable pair. We can assume that $A(x,y)$ is $n \times n$ where $n$ is even (otherwise replace it by $A(x,y)\Phi(1)$.) Then $\det(-A(x,y)) = A(x,y)$ and

$$
\begin{pmatrix}
0 & -B(x,y) \\
-B(x^{-1}, y^{-1}) tr & -A(x,y)
\end{pmatrix} = -B(x,y)
$$

\textbf{Remark:} This lemma can be realized geometrically by reversing the orientation of $S^3$.

The following generalizes theorem 7.

\textbf{Theorem 9:} Let $(A_i(x,y), B_i(x,y))$ be allowable pairs for $i = 1, 2$. Let
$L_1$ and $L_2$ are the same below the line

Figure 8.

1. reverse the orientations of $S^3$ and $L_2$.

2. glue via the identity on the boundary

Figure 9.
\[ \Delta_i(x, y) = \frac{1-(xy)^{l}}{1-xy} A_i(x, y) - (1-x)(1-y)\frac{1-(xy)^{l-1}}{1-xy} B_i(x, y) \quad i = 1, 2 \]

Then

\[ \Delta_{i+1}(x, y) \Delta_2(x, y) - \Delta_1(x, y) \Delta_{2+1}(x, y) \]

is a link polynomial.

**Proof:** Figure 9 shows the construction which realizes the promised polynomial. Let \( A_i(x, y) \) and \( B_i(x, y) \) be the matrices, \( L_i \) the tangles, \( Z_i \) the surgery complements, and \( \tilde{Z}_i \) their covers as in the characterization theorem which correspond to the allowable pair \( (A_i(x, y), B_i(x, y)) \) for \( i = 1, 2 \). A presentation matrix for the constructed link can be calculated from the Mayer–Vietoris sequence for \( \tilde{Z} = \tilde{Z}_1 \cup \tilde{Z}_2 \); it will have all the relators corresponding to the surgeries in \( Z_1 \) and \( Z_2 \) as well as the relators obtained when \( \tilde{Z}_1 \) is identified with \( \tilde{Z}_2 \). \( H_1(\tilde{Z}_1) \) has two generators: the lifts of \( \mu_0 \) and \( \nu_0 \) (see figure 10).

![Figure 10](image-url)
If $R_1, l$ is the relator obtained by sewing a disc across $\partial Z_1$ to produce a $(2l, 2)$ torus link as in the characterization theorem, then the relations obtained by identifying $\partial Z_1$ with $\partial Z_2$ are $R_0^{l+1} = R_0^{2, k+1}$ and $R_0^{l} = R_0^{2, k}$. In short, a presentation matrix is

$$
\begin{bmatrix}
A_1(x, y) & B_1(x^{-1}, y^{-1})_{tr} & 0 & 0 \\
P_1(x, y)B_1(x, y) & \frac{1-(xy)^{l+1}}{1-xy} & \frac{1-(xy)^{k+1}}{1-xy} & P_k(x, y)B_2(x, y) \\
P_{l-1}(x, y)B_1(x, y) & \frac{1-(xy)^{l}}{1-xy} & \frac{1-(xy)^{k}}{1-xy} & P_{k-1}(x, y)B_2(x, y) \\
0 & 0 & B_2(x^{-1}, y^{-1})_{tr} & A_2(x, y)
\end{bmatrix}
$$

where $P_1(x, y) = -\frac{1-(xy)^{l}}{1-xy} (1-x)(1-y)$. If $A_2(x, y)$ is $m \times m$ then I have multiplied the last $m+1$ columns and the last $m$ rows by $-1$. We want to calculate the determinant of this matrix. As usual, let the entry in the $i$-th row, $j$-th column be $a_{ij}$ so the required determinant is

$$\sum_{\sigma} \prod_{i} a_{i\sigma(i)} \text{sgn } \sigma$$

If $A_1(x, y)$ is $(n-1) \times (n-1)$ let

$$
\Gamma_1 = \{ \sigma : \sigma(1, 2, \ldots, n) = \{1, 2, \ldots, n\} \} \\
\Gamma_2 = \{ \sigma : \sigma(1, 2, \ldots, n-1, n+1) = \{1, 2, \ldots, n\} \} \\
\Gamma_3 = \{ \sigma : \text{either there is a } j \leq n-1 \text{ so that } \sigma(j) > n \\
or there is a } j > n-1 \text{ so that } \sigma(j) \leq n \} 
$$

Then $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ are pairwise disjoint and exhaust all permutations. Also, if $\sigma \in \Gamma_3$ then $\prod_{i} a_{i\sigma(i)} \text{sgn } \sigma = 0$. If $\sigma \in \Gamma_1$ then it is of the form $\sigma_1 \sigma_2$ where $\sigma_1$ is a permutation of $\{1, 2, \ldots, n\}$ and $\sigma_2$ is a permutation of $\{n+1, n+2, \ldots, n+m+1\}$ and hence

$$
\sum_{\sigma \in \Gamma_1} \prod_{i} a_{i\sigma(i)} = \sum_{\sigma_1} \prod_{i} a_{i\sigma_1(i)} \text{sgn } \sigma_1 \sum_{\sigma_2} \prod_{i} a_{i\sigma_2(i)} \text{sgn } \sigma_2 \\
= \Delta_1^{l+1}(x, y) \Delta_2^k(x, y).
$$
Similarly
\[ \sum_{\sigma \in \Gamma_2} \prod_{i \in \sigma} a_i \sigma(i) = -\Delta_1(x,y) \Delta_2^{k+1}(x,y). \]

There is another possibility for the construction in this theorem, namely that suggested by figure 11. If this is carried out we see that the resulting polynomial is \( \Delta_1^{L+1}(x,y) \Delta_2^k(x,y) - \Delta_1^L(x,y) \Delta_2^{k-1}(x,y). \)

I believe that the results in this and the preceding section will be useful in the machine calculation of Alexander polynomials of links of two components.

Figure 11.
Section D: Generating allowable pairs.

The work in this chapter was done because it was found that the presentation matrix in the characterization theorem was too difficult to work with; it was hoped that a method would be found which would generate enough allowable pairs to show that the restrictions pointed out in chapter III section E were also sufficient for a pair to be allowable. Unfortunately the methods discovered seem too complicated to allow this. For the sake of completeness we here present a list of methods for generating allowable pairs.

1. \( (\Delta^i(x,y), \frac{\Delta(x,y)}{(1-x)(1-y)}) \) is allowable where \( \Delta^i(x,y) \) is as in corollary 5 and \( \Delta^i(1,1) = 0 \).

2. \( (1, B(x,y)) \) is allowable whenever \( B(x,y) = B(x^{-1}, y^{-1}) \) (theorem III.E.7)

3. \( (A(x,y), P(x,y)P(x^{-1}, y^{-1})) \) is allowable whenever \( A(x,y) = A(x^{-1}, y^{-1}) \) and \( A(1,1) = 1 \).

The following are methods for generating allowable pairs from known allowable pairs.

4. \( (A(x,y), -B(x,y)) \) is allowable if \( (A(x,y), B(x,y)) \) is.

5. \( (A_1(x,y)A_2(x,y), A_1(x,y)B_2(x,y) + A_2(x,y)B_1(x,y)) \) is allowable (lemma III.E.6)

6. Theorem 9 and the remark after it give four ways of generating allowable pairs:

\[
\begin{align*}
(\Delta_1^0(x,y)\Delta_2^0(x,y) - \Delta_1^{-1}(x,y)\Delta_2^1(x,y), & \frac{\Delta_1^1(x,y)\Delta_2^0(x,y) - \Delta_1^0(x,y)\Delta_2^1(x,y)}{(x-1)(y-1)}) \\
(\Delta_1^2(x,y)\Delta_2^0(x,y) - \Delta_1^1(x,y)\Delta_2^1(x,y), & \frac{\Delta_1^1(x,y)\Delta_2^0(x,y) - \Delta_1^0(x,y)\Delta_2^1(x,y)}{(x-1)(y-1)}) \\
(\Delta_1^0(x,y)\Delta_2^0(x,y) - \Delta_1^{-1}(x,y)\Delta_2^{-1}(x,y), & \frac{\Delta_1^1(x,y)\Delta_2^0(x,y) - \Delta_1^0(x,y)\Delta_2^{-1}(x,y)}{(x-1)(y-1)}) \\
(\Delta_1^2(x,y)\Delta_2^0(x,y) - \Delta_1^1(x,y)\Delta_2^{-1}(x,y), & \frac{\Delta_1^1(x,y)\Delta_2^0(x,y) - \Delta_1^0(x,y)\Delta_2^{-1}(x,y)}{(x-1)(y-1)})
\end{align*}
\]
For anyone who may want to try his hand at characterizing allowable pairs, \((x-1+x^{-1}, 2)\) is the simplest pair which I have not been able to show is allowable. Also, Kidwell has a family of polynomials which he proves cannot be realized by links of linking number 3, order \((3, 3)\).

A family of pairs which realizes these polynomials is

\[
(1 - nx^{-1}y^{-1}(1-x)^2(1-y)^2, 1 - n(1-x)(1-y)(1+x^{-1}y^{-1})) \quad n > 0
\]

It would be interesting to know if any of these are allowable (I have worked on the case \(n = 1\)).
Bibliography


Appendix. The general case of Kidwell's theorem.

Since we have already introduced most of the concepts needed to generalize theorem IV.A.1 to more than two components it seems worthwhile to show how to modify the proof to obtain the more general result.

**Definition:** Let \( L = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_\mu \) be a link and \( S \) be a Seifert surface for \( \ell_1 \) with \( S \) and \( \ell_2 \cup \ell_3 \cup \cdots \cup \ell_\mu \) in general position. If

\[
\alpha_S = 2(\text{genus of } S) + \text{the number of times } \ell_2 \cup \ell_3 \cup \cdots \cup \ell_\mu \text{ intersects } S
\]

then \( \alpha_1 = \min_S \alpha_S \) is the linking complexity of \( \ell_2 \cup \ell_3 \cup \cdots \cup \ell_\mu \) with \( \ell_1 \).

The linking complexity of the link \( L \) is the ordered \( \mu \)-tuple \((\alpha_1, \alpha_2, \ldots, \alpha_\mu)\).

**Definition:** If \( \Delta(x_1, x_2, \ldots, x_\mu) = \sum_{i=n=0}^{m} |p_i(x_2, x_3, \ldots, x_\mu)x_1^i \) where the coefficients of the \( x_i \)'s are polynomials in the other variables and \( p_i(x_2, \ldots, x_\mu) \neq 0 \)

\[
p_m(x_2, \ldots, x_\mu) \neq 0 \text{ then } \deg_1 \Delta(x_1, x_2, \ldots, x_\mu) = m-n.
\]

**Theorem (Kidwell):** If \( \Delta(x_1, x_2, \ldots, x_\mu) \) is the Alexander polynomial of a link \( L = \ell_1 \cup \ell_2 \cup \cdots \cup \ell_\mu \) with linking complexity \((\alpha_1, \alpha_2, \ldots, \alpha_\mu)\) then

\( \alpha_1 - 1 \geq \deg_1 \Delta(x_1, x_2, \ldots, x_\mu) \).

**Proof:** Let \( \tilde{X}_{23\ldots \mu} = S^3 - (\ell_2 \cup \ell_3 \cup \cdots \cup \ell_\mu) \), \( \tilde{S}_{23\ldots \mu} \) be its universal abelian cover, and \( \tilde{S} \) the lift of \( S - (\ell_2 \cup \ell_3 \cup \cdots \cup \ell_\mu) \) to \( \tilde{X}_{23\ldots \mu} \) where \( S \) is a Seifert surface for \( \ell_1 \). If \( Y = \tilde{X}_{23\ldots \mu} - \tilde{S} \) we can construct \( \tilde{X} \) by identifying \( \{ Y_1 = Y \}_{i \in \mathbb{Z}} \) and \( \{ N_1 = \tilde{S} \times (-1, 1) \}_{i \in \mathbb{Z}} \) appropriately. From the Mayer–Vietoris sequence

\[
\cdots \to H_1(\bigcup N_i) \xrightarrow{\psi} H_1(\bigcup Y_i) \to H_1(\tilde{X}) \to H_0(\bigcup N_i) \xrightarrow{\phi} H_0(\bigcup Y_i) \to \cdots
\]

We obtain the short exact sequence

\[
0 \to \text{coker}\psi \to H_1(\tilde{X}) \to \ker\phi \to 0
\]

As before \( \Delta(x_1, x_2, \ldots, x_\mu) = \Delta_{\text{coker}\psi} \Delta_{\ker\phi} \).

1. \( \Delta_{\text{coker}\psi} \). A presentation for \( \text{coker}\psi \) can be obtained from one for \( H_1(\bigcup Y_i) \) by adding the relations \( \psi(\tilde{a}_i) \) where \( \{ \tilde{a}_i \} \) is a set of generators for \( H_1(\bigcup N_i) \). These relations are of the form \( \psi(\tilde{a}_i) - \psi(\tilde{a}_i)x = 0 \); since the relations for
$H_1(\cup Y_1)$ do not involve $x_1$ we see that $\deg_1^A \text{coker}\psi \leq$ the number of generators of $H_1(\cup N_1)$ as a $\Lambda_\mu$ module. $H_1(S-(l_2 \cup l_3 \cup \cdots \cup l_\mu))$ is a free $\mathbb{Z}$ module on $2g+h$ generators where $g =$ genus of $S$ and $h =$ number of times $l_2 \cup \cdots \cup l_\mu$ intersects $S$. Let this set of generators be represented by cycles $\{a_i\}_{i=1}^{2g+h}$ in $S-(l_2 \cup l_3 \cup \cdots \cup l_\mu)$. If we form a matrix of linking numbers where the entry in the $i$-th row $j$-th column is $lk(a_i, l_j)$ we can assume that the matrix has the following form:

$$\begin{pmatrix}
0 & \cdots & 0 & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
0 & \cdots & 0 & 0 & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
0 & \cdots & 0 & 0 & 0 & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \ast & \cdots & \ast & \cdots & \ast & \cdots & \ast
\end{pmatrix} = (b_{ij})$$

This matrix of linking numbers determines the lifts of the $a_i$'s to $\tilde{S}$; the lift of the loop $a_i$ to $\tilde{S}$ is a path from the basepoint to the basepoint shifted by $x_1 b_{11} x_2 b_{12} \cdots x_\mu b_{1\mu}$. If this matrix consists of zeroes then all the $a_i$'s lift to cycles and we see that $\deg_1^A \text{coker}\psi \leq a_1$; as before we will show that in this case $\Delta_{\text{ker}\phi} = 0$. If there is at least one non-zero entry in the matrix then there is at least one of these cycles which lifts to a path which is not a loop. We can further assume that such paths do not share both endpoints, since this would imply that two rows of the matrix were the same; we can multiply one of them by $-1$ if need be and add it to the other to obtain a row of zeroes which we move to the bottom of the matrix. If there is only one non-zero row we see immediately that $H_1(\cup N_1)$ will have $2g+h-1$ generators. If there is $n \geq 1$ non-zero rows then we must include commutators in the lifts of the generators corresponding to these rows, and hence in this case we have $2g+h+n+\frac{c_n}{2}$ generators for $H_1(\cup N_1)$. As in theorem II.C.4 these are not independent; in fact the techniques of lemma II.C.5 can be used to show that
at least $C_2^{n-1}$ must be omitted to obtain a linearly independent set. Hence
\[ \deg_1 \Delta(x_1, x_2, \ldots, x_\mu) \leq 2g + h - n + C_2^n - C_2^{n-1} = 2g + h - 1. \]

2. $\Delta_{\ker\phi}$. It is clear that $\Delta_{\ker\phi}$ cannot involve $x_1$: $H_0(\cup N_i)$ is cyclic as a $\Lambda_\mu$ module and the relators do not involve $x_1$. $\ker\phi$ is generated by the set
\begin{equation}
(1.) \quad b_2 = (x_2-1)a, \quad b_3 = (x_3-1)a, \quad \ldots, \quad b_\mu = (x_\mu-1)a
\end{equation}
where $a$ is the generator of $H_0(\cup N_i)$ with relators
\begin{equation}
(2.) \quad (x_i-1)b_j - (x_j-1)b_i \quad 2 \leq i < j \leq \mu
\end{equation}
and when we rewrite the relators for $H_0(\cup N_i)$ to obtain relators for $\ker\phi$ we can avoid using $x_1$. In case the matrix in the previous section consists entirely of zeroes, each component of $\mathcal{B}$ covers $\mathcal{B} - (\ell_2 \cup \ell_3 \cup \cdots \cup \ell_\mu)$ homeomorphically and consequently
\[ H_0(\cup N_i) = (a; -) \]
as a $\Lambda_\mu$ module. Then $\ker\phi$ has a presentation with generators (1.) and relators (2.). Again it is clear that the relators are not independent and the techniques of lemma II.C.5 can be used to show that $C_2^{\mu-2}$ must be omitted before the set can be independent. Since we started with $C_2^{\mu-1}$ this leaves $\mu-2$ relators. Since we have $\mu-1$ generators this proves that $\Delta_{\ker\phi} = 0$. \[ \square \]