Equilibrium States of Two Stochastic Models in Mathematical Ecology

by

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Abstract

This work deals with two problems arising in mathematical ecology. The first problem is concerned with diploid branching particle models and its behavior when rapid stirring is added to the interaction. The particle models involve two types of particles, male and female, and branching can only occur when both types of particles are present. We show that if the branching rate is sufficiently large, this particle model has a nontrivial stationary distribution, i.e. one that does not concentrate all weight on the all-0 state, using a comparison argument due to R. Durrett. We also show extinction for small branching rates, thereby establishing the existence of a phase transition. We then add two different rapid stirring mechanisms to the interactions and show that for the particle models with rapid stirring, there also exist nontrivial stationary distribution(s); for this, we analyze the limiting PDE and establish a condition on the PDE that guarantees existence of nontrivial stationary distributions for sufficient fast stirring.

The second problem deals with a model of sympatric speciation, i.e. speciation in the absence of geographical separation, originally proposed by U. Dieckmann and M. Doebeli in 1999. We modify their original model to obtain several constant-population particle models. We concentrate on a continuous-time model that converges to a deterministic dynamical system as the number of particles becomes large. We establish various results regarding whether speciation occurs by studying the existence of bimodal stationary distributions for the limiting dynamical system.

Contents

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Chapter 1

Introduction and Overview

This work consists of two parts, each of which involves a class of models arising from a problem of mathematical ecology. In the first problem, we study diploid branching particle system models. This class of particle systems differs from the "usual" models that one normally finds in the literature, in that there are two types of particles, modelling the male and female populations, and branching (i.e. birth of new particles) requires the presence of both male and female particles. In the second problem, we study various particle models that are all related to a sympatric speciation model proposed in [Dieckmann and Doebeli 1999]. In both particle models, we are mainly concerned with the equilibrium behaviour. More specifically, we show that the stationary distributions of the particle models have desirable properties, e.g. nontriviality (i.e. does not concentrate all weight on the all-0 state) in the first problem and bimodality in the second problem.

Part I

Existence of Nontrivial Stationary Distribution for the Diploid Branching Particle System with Rapid Stirring

Chapter 2

The Particle Models

In this part of our work, we consider a type of particle systems that can be used to model sexual reproduction of a certain species. This work was inspired in part by [Dawson and Perkins 1998]. In that paper, the following system of stochastic partial differential equations is the object of study:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\Delta u(t,x) + (\gamma u(t,x)v(t,x))^{1/2}\dot{W}_1(t,x) \\ \frac{\partial v}{\partial t}(t,x) &= \frac{1}{2}\Delta v(t,x) + (\gamma u(t,x)v(t,x))^{1/2}\dot{W}_2(t,x), \end{aligned} \tag{2.1}$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplacian, $\gamma > 0$, and $\dot{W}_i(t,x)$ $(i = 1,2)$ are independent space-time white noises on $\mathbb{R}_+ \times \mathbb{R}$. One can associate $u(x,t)$ and $v(x,t)$ with the male and female populations of "particles" (respectively) at spatial location x and time t. Loosely speaking, (2.1) says that individual male or female particles moves around according to Brownian motion, but branching is only possible when both male and female particles are present at the same spatial location. Notice that at spatial locations where the female population is 0, the branching rate for the male population is also 0, therefore the male population does not "die" and the only effect on the male population at those spatial locations is the diffusive effect of the heat kernel Δ. This behaviour is not very realistic, since one would expect a "natural" death rate of male particles even without the presence of any female particles. In this work, we study a model involving a finite number of male and female particles with more (somewhat) "realistic" behaviour.

The model we study involves two types of particles, male and female, residing on the integer grid $S = \mathbb{Z}^d$ or $\epsilon\mathbb{Z}^d$. More specifically, each *site* $x \in S$ contains two *nests*, one for the male particle and the other for the female particle. Each nest can be inhabited by at most 1 particle, either male or female. Let $E = \{0,1\}$ and $F = E \times E$ be the set of possible states at each site in S. For $x \in S$, we write

$$\xi(x) = (\xi^1(x), \xi^2(x)),$$

where $\xi^1(x)$ denotes the number (0 or 1) of male particles at site x, and $\xi^2(x)$ denotes the number of female particles at site x. We define the interaction neighbourhood

$$\mathcal{N} = \{0, y_1, \ldots, y_N\},$$

and the neighbourhood of x

$$\mathcal{N}_x = x + \mathcal{N}.$$

For example, $\mathcal{N} = \{0, -1, 1\}$ if we have nearest-neighbour interaction on $\mathbb{Z}$. Let $c_i(x, m, \xi)$ denote the rate at which nest m ($m = 1, 2$) of site x flips to state i ($i = 0, 1$), and assume $c_i(x, m, \xi)$ depends only on the neighbourhood $\mathcal{N}_x$, i.e.

$$c_i(x, m, \xi) = h_{i,m}(\xi(x), \xi(x + y_1), \ldots, \xi(x + y_N))$$

for some function $h_{i,m} : F^{N+1} \to \mathbb{R}^+$. The death rate c_0 is constant,

$$c_0(x, m, \xi) = \begin{cases} \delta, & \text{if } \xi^m(x) = 1 \\ 0, & \text{otherwise} \end{cases}, \tag{2.2}$$

while the birth rate $c_1(x, m, \xi)$ is positive only if both male and female particles can be found in $\mathcal{N}_x$. For example, the diploid branching particle model we consider in Chapter 2.1 a bit later has

$$c_1(x, m, \xi) = \begin{cases} \lambda n_1(x, \xi) n_2(x, \xi), & \text{if } \xi^m(x) = 0 \\ 0, & \text{otherwise} \end{cases}, \tag{2.3}$$

where

$$n_{m'}(x, \xi) = |\{z \in \mathcal{N}_x : \xi^{m'}(x + z) = 1\}|,$$

i.e. at rate λ, each pair of male and female particles in $\mathcal{N}_x$ give birth to a particle at nest m of site x if that nest is not already occupied. A more stringent condition, as in the particle model with rapid stirring we consider in Chapter 2.2 a bit later, is to require both parent particles to reside at the same site, i.e.

$$c_1(x, m, \xi) = \begin{cases} \lambda n_{1+2}(x, \xi), & \text{if } \xi^m(x) = 0 \\ 0, & \text{otherwise} \end{cases}, \tag{2.4}$$

where

$$n_{1+2}(x, \xi) = |\{z \in \mathcal{N}_x : \xi^1(x + z) = 1 \text{ and } \xi^2(x + z) = 1\}|.$$

This more stringent condition should not alter the behaviour of the particle system if one allows a larger λ than in (2.3), but it does help to simplify the analysis somewhat.

2.1 Diploid Branching Particle Model

We first describe the model with birth and death rates as in (2.2) and (2.3), for which we will establish the existence of nontrivial stationary distribution(s) and consequently a phase transition later in Chapter 3. First, we restate the model in words:

1. **Birth.** For each nest (x, m) and each pair $(z_1, z_2) \in \mathcal{N}_x \times \mathcal{N}_x$ such that $\xi^1(z_1) = 1$ and $\xi^2(z_2) = 1$, where z_1 and z_2 need not be distinct, with rate λ, a child of (z_1, z_2) is born into nest m of site x if (x, m) is not already occupied.

2. **Death.** Each particle dies at rate δ.

We can think of this particle system as a generalized spin system, generalized in the sense that the phase space at each site is $\{0,1\}^2$ rather than $\{0,1\}$. One can refer to Chapter 3 of [Liggett 1985] for a detailed introduction on classic spin systems. We observe that the "all-0" state (i.e. $\xi^1(x) = \xi^2(x) = 0$ for all x) is an absorbing state, therefore the probability measure that concentrates only on the "all-0" state is a trivial stationary distribution. We say a stationary distribution is *nontrivial* if it does not concentrate only on the "all-0" state. A major goal of this work is to establish the existence of nontrivial stationary distributions for various particle systems.

This interacting particle system involving the birth and death mechanisms described above can be constructed using a countable number of Poisson processes [Durrett 1995]. Without any loss of generality, we assume λ and δ to be ≤ 1, since we can just slow down time by $\max(\lambda, \delta)$ if either $\lambda > 1$ or $\delta > 1$. Define

$$c^* = \sup_{\xi,m} \sum_i c_i(x, m, \xi).$$

We assume $c^* < \infty$. Let $\{T_n^{x,i,m} : n \geq 1\}$ be the arrival times of independent rate c^* Poisson processes, and $\{U_n^{x,i,m} : n \geq 1\}$ be independent uniform random variables on $[0,1]$. At time $t = T_n^{x,i,m}$, nest (x,m) flips to state i if $U_n^{x,i,m} \leq c_i(x, m, \xi_{t-})/c^*$, and stays unchanged otherwise.

Since the number of Poisson processes is infinite, there is no first flip and the existence and uniqueness of the process from this construction is not completely trivial. One can, however, use Theorem 2.1 of [Durrett 1995] to find a small t_0 such that the spatial grid S can be divided into an infinite number of components, each of which is finite and no two of which interact during time $[0, t_0]$. This allows construction of the process up to time t_0, and by iterating this procedure, we can construct the process for all t. One can also see easily that this construction is unique. Alternatively, one can explicitly write down the generators $\mathcal{G}^1$ and $\mathcal{G}^2$ associated with the particle system with death rates (2.2) and birth rates (2.3) and (2.4) respectively:

$$\begin{aligned}
\mathcal{G}^1 f(\xi) \quad = \quad & \sum_{(x,m)\in S\times\{1,2\}} \Bigg[\delta\xi^m(x)(f(\xi - \delta_{x,m}) - f(\xi)) \\
& + \sum_{y,z\in\mathcal{N}_x} \lambda\xi^1(y)\xi^2(z)(1-\xi^m(x))(f(\xi+\delta_{x,m}) - f(\xi)) \Bigg] \qquad (2.5)\\
\mathcal{G}^2 f(\xi) \quad = \quad & \sum_{(x,m)\in S\times\{1,2\}} \Bigg[\delta\xi^m(x)(f(\xi - \delta_{x,m}) - f(\xi)) \\
& + \sum_{y\in\mathcal{N}_x} \lambda\xi^1(y)\xi^2(y)(1-\xi^m(x))(f(\xi+\delta_{x,m}) - f(\xi)) \Bigg] \qquad (2.6)
\end{aligned}$$

where f has compact support and $\delta_{x,m}$ is a function on $S \times \{1,2\}$ that is one at (x,m) and zero everywhere else, and apply Theorem B3 in [Liggett 1999] (Theorem I.3.9 in [Liggett 1985] only gives the Markov property) to see that $\mathcal{G}$ is a Markov generator and therefore determines a unique $(\{0,1\}^2)^{\mathbb{Z}^d}$ Feller Markov process.

An important consequence of the construction using Poisson processes described in the last paragraph is that the semigroup $T_t f(\xi_0) = E^{\xi_0} f(\xi_t)$ corresponding to the particle system is a Feller semigroup. As in Corollary 2.3 of [Durrett 1995], one can show that if $\xi_0^n \to \xi_0$, then for $t \le t_0$, $E^{\xi_0^n} f(\xi_t^n) \to E^{\xi_0} f(\xi_t)$ since S consists of components that are finite and do not interact with each other during $[0, t_0]$. One can then iterate this for as many times as one likes. Summarizing results from the three previous paragraphs, we have the following theorem:

Theorem 2.1.1. *There exists a unique Feller process ξ_t constructed as before with generator (2.5) or (2.6).*

One can represent this construction graphically, for which we give an example with $S = \mathbb{Z}$ and $\mathcal{N} = \{-1,0,1\}$ in figure 2.1. Let $m \in \{1,2\}$, $x,y,z \in S$, $\{R_n^{x,m}, n \ge 1\}$ be independent Poisson processes with rate δ, and $\{T_n^{x,m,y,z}, n \ge 1\}$, with $y,z \in \mathcal{N}_x$, be independent Poisson processes with rate λ. At space-time points $((x,m), R_n^{x,m})$, we draw a symbol δ to indicate that the particle (if any) residing at (x,m) is killed at time $R_n^{x,m}$. At space-time points $((x,m), T_n^{x,m,y,z})$, we draw arrows from $((y,1), T_n^{x,m,y,z})$ and $((z,2), T_n^{x,m,y,z})$ to $((x,m), T_n^{x,m,y,z})$ to indicate that a birth event will occur at nest (x,m) if (x,m) is not already occupied and nests $(y,1)$ and $(z,2)$ are both occupied at time $T_n^{x,m,y,z}$.

In figure 2.1, the bottom line represent the state (occupied or empty) of nests at $t = 0$. We use thick lines to represent occupied (wet) nests, and thin lines to represent empty nests. Without any birth or death event, the state of a nest remains unchanged as t increases. At a death event, i.e. at points marked by δ, a thick line is changed to a thin line, while a thin line remains unchanged. And at a birth event, the state of nests at the origins of the two arrows pointing at (x,m) is checked – if they are both occupied, then a thin line at (x,m) is changed to a thick line, while a thick remains unchanged; otherwise, nothing happens.

In Chapter 3, we will use this graphical construction to establish the existence of a nontrivial stationary distribution for the diploid branching particle model if λ/δ is sufficiently large, and extinction if λ/δ is sufficiently small.

The particle system ξ with generator (2.5) or (2.6) is *attractive* in the sense that ξ is monotonic in initial conditions. One can check that if $\xi_0(x) \le \bar{\xi}_0(x)$ for all $x \in S$, where $(0,0) < (0,1) < (1,1)$ and $(0,0) < (1,0) < (1,1)$ but $(0,1) \not\le (1,0)$, then $\xi_t(x) \le \bar{\xi}_t(x)$ for all x and t. This is true since every birth or death event preserves the inequality $\le$. For example, if $\xi_{t-}(x) = (0,0)$ and $\bar{\xi}_{t-}(x) = (0,1)$, and at time t there is a male birth event at site x, then $\xi_t(x) = (1,0)$ and $\bar{\xi}_t(x) = (1,1)$, so the inequality $\xi_t(x) \le \bar{\xi}_t(x)$ has been maintained. Similarly, one can check that the particle system ξ is increasing in the birth rate λ and decreasing in the death rate δ, by coupling the random variables $T_n^{x,i;m}$ and $U_n^{x,i,m}$ involved in the constructions in the obvious way. Because of this monotonicity, along with the existence of nontrivial stationary distributions for sufficiently large λ/δ and extinction

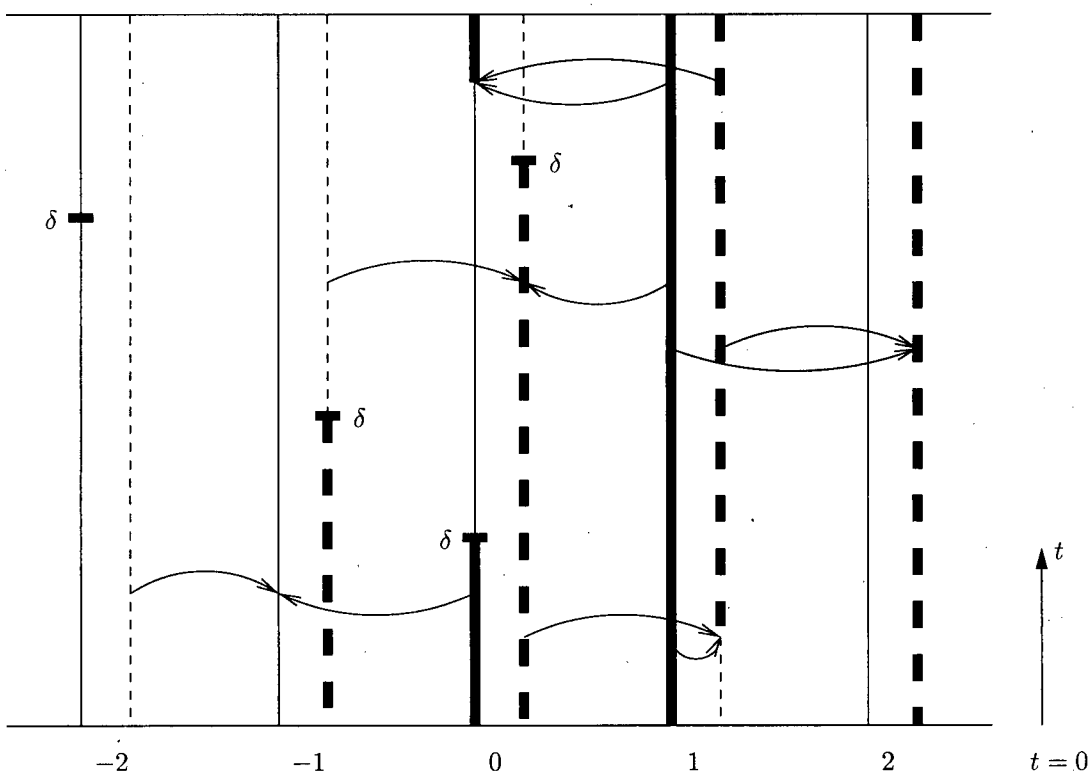


Figure 2.1: Graphical representation: the solid lines represent nests $(x, 1)$, while the dotted lines represent nests $(x, 2)$. Thick lines indicate occupied (wet) nests, while thin lines indicate empty nests.

for sufficiently small λ/δ which we will establish a bit later in this work, we may conclude that there is a phase transition in the behaviour of the particle system ξ.

2.2 Description of the Particle Model with Rapid Stirring

If we add rapid stirring to the particle system, i.e. we scale space by ϵ and "stir" neighbouring particles at rate ϵ^{-2} in addition to performing the birth and death mechanisms, then the particle system converges to the solution of a reaction-diffusion PDE as $\epsilon \to 0$ (see Theorems 8.1 and 8.2 in [Durrett 1995] and the beginning of Chapter 2.3). This PDE represents the "mean-field" behaviour of the particle system and is usually easier to analyze than the particle system itself. As promised earlier, we will establish later in Chapter 3 that there is a phase transition for the diploid branching particle model (i.e. without rapid stirring), but obtaining any reasonable estimates on exactly where this transition occurs seems to be difficult. One advantage of adding rapid stirring mechanisms is that one can get a pretty good idea where the phase transition occurs in the rapidly stirred particle model by

analyzing the limiting PDE, or simulating this PDE on a computer.

Moreover, this convergence establishes a connection between the particle model and PDE systems, which is of independent interest. Since many PDE's arise out of natural systems, this connection justifies the study of the PDE. The underlying stochastic system can also yield information about the PDE; for example, in our case, as we will see in Chapter 2.3, the monotonicity of the particle system will lead to the monotonicity of the PDE. Information about the PDE will similarly yield information about the particle model. In Chapter 5, we will establish condition (*) on the PDE (see page 31), which will tell us that there exist nontrivial stationary distributions for the particle system with sufficiently small ϵ.

For the particle models with rapid stirring, we work with $S = \epsilon\mathbb{Z}^d$, and denote the corresponding process by ξ^ϵ. We also assume the birth and death rates in (2.2) and (2.4) (i.e. generator $\mathcal{G}^2$ in (2.6)) with $\delta = 1$, while the neighbourhood $\mathcal{N}$ is nearest neighbour:

$$\mathcal{N} = \{y : \|y\| = 0 \text{ or } \epsilon\}.$$

Here we use the L^1-norm: $\|y\| = \sum_{k=1}^d y_k$. In addition to the transitions in the diploid branching model, we introduce spatial movement of particles between neighbouring sites called "rapid stirring". We consider two rapid stirring mechanisms in this work, one called "lily-pad" stirring, and the other called "individual" stirring:

- **Lily-pad Stirring.** For each $x, y \in \epsilon\mathbb{Z}^d$ with $\|x-y\|_1 = \epsilon$, $\xi^\epsilon(x) = (\xi^{\epsilon,1}(x), \xi^{\epsilon,2}(x))$ and $\xi^\epsilon(y) = (\xi^{\epsilon,1}(y), \xi^{\epsilon,2}(y))$ are exchanged at rate ϵ^{-2}.
- **Individual Stirring.** For each $i \in \{1,2\}$ and $x, y \in \epsilon\mathbb{Z}^d$ with $\|x-y\|_1 = \epsilon$, $\xi^{\epsilon,i}(x)$ and $\xi^{\epsilon,i}(y)$ are exchanged at rate ϵ^{-2}.

Just as in the particle model without rapid stirring described in 2.1, one can construct the particle model with either lily-pad stirring or individual stirring using a countable number of Poisson processes. Or alternatively, one can write down the generator explicitly and again apply Theorem B3 in [Liggett 1999] to establish:

Theorem 2.2.1. *There exists a unique Feller process ξ_t with generator $\mathcal{G}^L$ for the particle model with Lily-pad stirring or generator $\mathcal{G}^I$ for the particle model with individual stirring:*

$$\mathcal{G}^L f(\xi) = \mathcal{G}^2 f(\xi) + \sum_{x,y\in\mathbb{Z}^d, \|x-y\|_1=\epsilon} \epsilon^{-2}(f(\xi^{x\leftrightarrow y}) - f(\xi)) \tag{2.7}$$

$$\mathcal{G}^I f(\xi) = \mathcal{G}^2 f(\xi) + \sum_{m\in\{1,2\}, x,y\in\mathbb{Z}^d, \|x-y\|_1=\epsilon} \epsilon^{-2}(f(\xi^{(x,m)\leftrightarrow(y,m)}) - f(\xi)), \tag{2.8}$$

where

$$\xi^{x\leftrightarrow y}(z,m') = \begin{cases} \xi(z,m'), & \text{if } z \neq x, y \\ \xi(x,m'), & \text{if } z = y \\ \xi(y,m'), & \text{if } z = x \end{cases},$$

and

$$\xi^{(x,m)\leftrightarrow(y,m)}(z,m') = \begin{cases} \xi(z,m'), & \text{if } (z,m') \neq (x,m),(y,m) \\ \xi(x,m), & \text{if } (z,m') = (y,m) \\ \xi(y,m), & \text{if } (z,m') = (x,m) \end{cases}.$$

For lily-pad stirring, instead of thinking of a site that consists of two nests as in the diploid branching model, we can view each site as having 4 states in

$$F = \{0,1\}^2 = \{(0,0),(0,1),(1,0),(1,1)\}.$$

We restate the dynamics of the particle model in terms of these four states. At any site $x \in \epsilon\mathbb{Z}^d$, only the following transitions are possible: $(0,0) \leftrightarrow (0,1)$, $(0,1) \leftrightarrow (1,1)$, $(0,0) \leftrightarrow (1,0)$, and $(1,0) \leftrightarrow (1,1)$, i.e. only one particle is born or dies at a particular time. The rates of these transitions are as follows:

$$\begin{array}{ll}
c_{(0,0)}(x,\xi^\epsilon) = 1 & \text{if } \xi^\epsilon(x) = (0,1) \text{ or } \xi^\epsilon(x) = (1,0), \\
c_{(0,1)}(x,\xi^\epsilon) = c_{(1,0)}(x,\xi^\epsilon) = 1 & \text{if } \xi^\epsilon(x) = (1,1), \\
c_{(0,1)}(x,\xi^\epsilon) = c_{(1,0)}(x,\xi^\epsilon) = \lambda n_{1+2}(x,\xi^\epsilon) & \text{if } \xi^\epsilon(x) = (0,0), \\
c_{(1,1)}(x,\xi^\epsilon) = \lambda n_{1+2}(x,\xi^\epsilon) & \text{if } \xi^\epsilon(x) = (0,1) \text{ or } \xi^\epsilon(x) = (1,0).
\end{array}$$

In words, each particle, male or female, dies at rate 1. If site x is occupied by both a male and a female particle, then with rate λ, it gives birth to a male (respectively female) child-particle at a neighbouring site provided that the neighbouring site is not already occupied by a male (respectively female) particle.

For individual stirring, we still view the particle system with nests $(x,m) \in \epsilon\mathbb{Z}^d \times \{1,2\}$ and each nest assuming one of two states in $E = \{0,1\}$.

The difference between these two stirring mechanisms is that lily-pad stirring forces male and female particles at a site to move together, but individual stirring allows independent movement of male and female particles. Every exchange of particles, in both lily-pad stirring and individual stirring, is monotonicity preserving, thus neither stirring mechanism disrupts the monotonicity property of the particle system.

2.3 Convergence to a PDE for Lily-pad Stirring

Consider the particle system with lily-pad stirring with its generator given by (2.7). For $i \in F$, define

$$u_i^\epsilon(t,x) = P(\xi_t^\epsilon(x) = i)$$

then Theorem 8.1 in [Durrett 1995] shows that if $g_i(x)$ is continuous and $u_i^\epsilon(0,x) = g_i(x)$ for all i, then

$$u_i(t,x) = \lim_{\epsilon \to 0} u_i^\epsilon(t,x)$$

exists and satisfies the following system of PDE's:

$$\begin{aligned}
\frac{\partial u_{(0,0)}}{\partial t} &= \Delta u_{(0,0)} + u_{(0,1)} + u_{(1,0)} - 2\lambda d u_{(0,0)} u_{(1,1)} \\
\frac{\partial u_{(0,1)}}{\partial t} &= \Delta u_{(0,1)} + u_{(1,1)} - u_{(0,1)} + \lambda d(u_{(0,0)} - u_{(0,1)}) u_{(1,1)} \\
\frac{\partial u_{(1,0)}}{\partial t} &= \Delta u_{(1,0)} + u_{(1,1)} - u_{(1,0)} + \lambda d(u_{(0,0)} - u_{(1,0)}) u_{(1,1)} \\
\frac{\partial u_{(1,1)}}{\partial t} &= \Delta u_{(1,1)} - 2u_{(1,1)} + \lambda d(u_{(0,1)} + u_{(1,0)}) u_{(1,1)}.
\end{aligned} \tag{2.9}$$

Obviously, u_i must lie in $[0,1]$ for all i and t since it is a limit of probabilities. We want to study the long time behaviour of (2.9). The system (2.9) is 3-dimensional if one takes into account the condition $u_{(0,0)} + u_{(0,1)} + u_{(1,0)} + u_{(1,1)} = 1$. We first do two transformations on the 3-dimensional parameter space $(u_{(0,0)}, u_{(0,1)}, u_{(1,0)}, u_{(1,1)})$ to obtain a monotone 2-dimensional system, which will be easier to analyze. First, define $u_0 = u_{(0,0)}$, $u_1 = u_{(0,1)} + u_{(1,0)}$, and $u_2 = u_{(1,1)}$, then (u_0, u_1, u_2) satisfies:

$$\begin{aligned}
\frac{\partial u_0}{\partial t} &= \Delta u_0 + u_1 - 2\lambda d u_0 u_2 \\
\frac{\partial u_1}{\partial t} &= \Delta u_1 + 2u_2 - u_1 + \lambda d(2u_0 - u_1)u_2 \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - 2u_2 + \lambda d u_1 u_2. \qquad (2.10)
\end{aligned}$$

The above system can be written as the limiting PDE under rapid stirring of another particle system ζ^ϵ, still on $S = \epsilon\mathbb{Z}^d$, with state space $F = \{0,1,2\}$, and transitions $0 \leftrightarrow 1$ and $1 \leftrightarrow 2$ at rates

$$\begin{array}{ll}
c_0(x,\zeta^\epsilon) = 1 & \text{if } \zeta^\epsilon(x) = 1 \\
c_1(x,\zeta^\epsilon) = 2 & \text{if } \zeta^\epsilon(x) = 2 \\
c_1(x,\zeta^\epsilon) = 2\lambda n_2(x,\zeta^\epsilon) & \text{if } \zeta^\epsilon(x) = 0 \\
c_2(x,\zeta^\epsilon) = \lambda n_2(x,\zeta^\epsilon) & \text{if } \zeta^\epsilon(x) = 1,
\end{array}$$

where

$$n_2(x,\xi^\epsilon) = |\{z \in \mathcal{N} : \zeta^\epsilon(x+z) = 2\}|.$$

Under this model, monotonicity still holds: if $\zeta_0^\epsilon(x) \le \bar{\zeta}_0^\epsilon(x)$ for all x (here the ordering of F is the usual one: $0 < 1 < 2$), then $\zeta_t^\epsilon(x) \le \bar{\zeta}_t^\epsilon(x)$ for all x and t, since every transition still preserves the inequality $\le$. Let $(u_1^\epsilon(0,x), u_2^\epsilon(0,x)) = (g_1(x), g_2(x))$ and $(\bar{u}_1^\epsilon(0,x), \bar{u}_2^\epsilon(0,x)) = (\bar{g}_1(x), \bar{g}_2(x))$ be two sets of initial distributions such that $g_1 + g_2 \le \bar{g}_1 + \bar{g}_2$ and $g_2 \le \bar{g}_2$ everywhere. Then for all x,

$$\begin{aligned}
u_2^\epsilon(0,x) &\le \bar{u}_2^\epsilon(0,x) \\
u_1^\epsilon(0,x) + u_2^\epsilon(0,x) &\le \bar{u}_1^\epsilon(0,x) + \bar{u}_2^\epsilon(0,x),
\end{aligned}$$

so it is possible to set up two initial conditions ζ_0^ϵ and $\bar{\zeta}_0^\epsilon$, such that $P(\zeta_0^\epsilon(x) = i) = u_i^\epsilon(0,x)$ and $P(\bar{\zeta}_0^\epsilon(x) = i) = \bar{u}_i^\epsilon(0,x)$, $i = 1,2$, and $\zeta_0^\epsilon(x) \le \bar{\zeta}_0^\epsilon(x)$ holds for all x and ω. Since $\zeta_t^\epsilon(x) \le \bar{\zeta}_t^\epsilon(x)$ for all t and x, the monotonicity property of ζ implies

$$P(\zeta_t^\epsilon(x) \ge 1) \le P(\bar{\zeta}_t^\epsilon(x) \ge 1) \text{ and } P(\zeta_t^\epsilon(x) \ge 2) \le P(\bar{\zeta}_t^\epsilon(x) \ge 2),$$

i.e. for all t and x,

$$\begin{aligned}
u_2^\epsilon(t,x) &\le \bar{u}_2^\epsilon(t,x) \\
u_1^\epsilon(t,x) + u_2^\epsilon(t,x) &\le \bar{u}_1^\epsilon(t,x) + \bar{u}_2^\epsilon(t,x).
\end{aligned}$$

Now we transform the parameter space a second time by defining $(\alpha, \beta) = (u_1 + u_2, u_2)$ and writing $c = \lambda d$, then (α_t, β_t) is monotone in the initial condition since $u_i^\epsilon(t,x) \to u_i(t,x)$.

In particular, if $(1-\alpha, \alpha-\beta, \beta)$ and $(1-\bar{\alpha}, \bar{\alpha}-\bar{\beta}, \bar{\beta})$ are both solutions to (2.10) with $\alpha_0(x) < \bar{\alpha}_0(x)$ and $\beta_0(x) < \bar{\beta}_0(x)$ for all x, then $\alpha_t(x) < \bar{\alpha}_t(x)$ and $\beta_t(x) < \bar{\beta}_t(x)$ for all x and t. Straightforward calculation shows that (α, β) satisfies the following system:

$$\begin{aligned} \frac{\partial \alpha}{\partial t} &= \Delta\alpha + (2c - 2c\alpha + 1)\beta - \alpha \\ \frac{\partial \beta}{\partial t} &= \Delta\beta + (c(\alpha - \beta) - 2)\beta. \end{aligned}$$

Since $(u_1, u_2) \in [0,1]^2$, (α, β) lies in the triangular region

$$\mathcal{R} = \{(u,v) : 0 \le u, v \le 1, u \ge v\} \tag{2.11}$$

for all $t \ge 0$. We change variables from (α, β) to (u, v) and summarize this paragraph in the following lemma:

Lemma 2.3.1. *The PDE*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + (2c(1-u) + 1)v - u \\ \frac{\partial v}{\partial t} &= \Delta v + (c(u-v) - 2)v \end{aligned} \tag{2.12}$$

is monotone in initial conditions that lie in $\mathcal{R} = \{(u,v) : 0 \le u, v \le 1, u \ge v\}$, i.e. if there are two initial conditions $(u_0, v_0) \in \mathcal{R}$ and $(\bar{u}_0, \bar{v}_0) \in \mathcal{R}$, with $u_0 \le \bar{u}_0$ and $v_0 \le \bar{v}_0$ everywhere, then $u_t \le \bar{u}_t$ and $v_t \le \bar{v}_t$ everywhere, for all t; furthermore, both (u_t, v_t) and $(\bar{u}_t, \bar{v}_t)$ lie in $\mathcal{R}$ for all t.

In Chapter 5, we will analyze (2.12) to establish the following theorem:

Theorem 2.3.2. *If λ/δ is sufficiently large and ϵ is sufficiently small, then there exists a nontrivial translation invariant stationary distribution for the diploid branching particle model with lily-pad stirring with generator (2.7).*

2.4 Convergence to a PDE for Individual Stirring

Unlike lily-pad stirring, Theorem 8.1 in [Durrett 1995] cannot be directly applied to get convergence to a PDE system for individual stirring. We can, however, follow the ideas used in the proof of that theorem 8.1 to establish a corresponding result, Theorem 4.0.5. We consider the particle model with individual stirring described in Chapter 2.2. For $i \in E$, define

$$u^{\epsilon}_{i,m}(t,x) = P(\xi^{\epsilon}_t(x,m) = i).$$

Then Theorem 4.0.5 implies that if $g_{i,m} : \mathbb{R} \to [0,1]$ is continuous and $u^{\epsilon}_{i,m}(0,x) = g_{i,m}(x)$, then

$$u_{i,m}(t,x) = \lim_{\epsilon \to 0} u^{\epsilon}_{i,m}(t,x)$$

exists and satisfies the following system of PDE's:

$$\begin{aligned}
\frac{\partial u_{0,1}}{\partial t} &= \Delta u_{0,1} + u_{1,1} - 2cu_{0,1}u_{1,1}u_{1,2} \\
\frac{\partial u_{1,1}}{\partial t} &= \Delta u_{1,1} - u_{1,1} + 2cu_{0,1}u_{1,1}u_{1,2} \\
\frac{\partial u_{0,2}}{\partial t} &= \Delta u_{0,2} + u_{1,2} - 2cu_{0,2}u_{1,1}u_{1,2} \\
\frac{\partial u_{1,2}}{\partial t} &= \Delta u_{1,2} - u_{1,2} + 2cu_{0,2}u_{1,1}u_{1,2},
\end{aligned}$$

where we define

$$c = \lambda d^2.$$

Since $u_{0,1} + u_{1,1} = u_{0,2} + u_{1,2} = 1$, it suffices to study the PDE for $u_{1,1}$ and $u_{1,2}$:

$$\begin{aligned}
\frac{\partial u_{1,1}}{\partial t} &= \Delta u_{1,1} - u_{1,1} + 2c(1 - u_{1,1})u_{1,1}u_{1,2} \\
\frac{\partial u_{1,2}}{\partial t} &= \Delta u_{1,2} - u_{1,2} + 2c(1 - u_{1,2})u_{1,1}u_{1,2}.
\end{aligned} \tag{2.13}$$

Notice that if we start with a symmetric initial condition, i.e. $g_{i,1} = g_{i,2}$, then the solution to (2.13) is also symmetric. And if we define $u = u_{1,1} = u_{1,2}$, then we obtain the following PDE for u:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + f(u), \\
f(u) &= -u + 2c(1 - u)u^2.
\end{aligned} \tag{2.14}$$

This PDE has been analyzed in [Durrett and Neuhauser 1994] as their sexual reproduction model (example 3 on page 291). In fact, it is not difficult to see that if $u_{1,1} = u_{1,2}$ then choosing the "father" from the male population is exactly the same as choosing the "father" from the the female population, hence it is quite natural for this reduction to occur. Theorem 4 of [Durrett and Neuhauser 1994] states that if $c > 2.25$ then the sexual reproduction model of Durrett and Neuhauser has nontrivial stationary distribution(s). Although this theorem does not directly apply to our particle system ξ^ϵ with 2 types of particles because of the difference in stirring mechanisms, one can nevertheless trace through the proof of Lemma 3.3 of [Durrett and Neuhauser 1994] while making obvious changes, to establish a similar result:

- Let $0 < \rho_1 < \rho_0 < 1$ be the two nonzero roots of $f(u)$. Define $\beta = (\rho_0 - \rho_1)/10$ and $I_k = 2Lke_1 + [-L, L)^d$. If ϵ is small, L is large, and $\xi^\epsilon(0)$ has density at least $\rho_1 + \beta$ of both male particles and female particles in I_0, then for sufficiently large T, with high probability $\xi^\epsilon(T)$ will have density of at least $\rho_0 - \beta$ in I_1 and I_{-1}.

This result can then be fed into a comparison argument, comparing the particle system with oriented percolation, as on page 312 of [Durrett and Neuhauser 1994] or in the proof of Theorem 3.3.2 later on, to establish the existence of nontrivial stationary distribution(s) for the particle system ξ^ϵ under individual stirring with sufficiently small ϵ. Then we have the following theorem:

Theorem 2.4.1. *If λ/δ is sufficiently large and ϵ is sufficiently small, then there exists a nontrivial translation invariant stationary distribution for the diploid branching particle model with individual stirring with generator (2.8).*

We will not explicitly write down the details of the proof, but instead refer the interested reader to [Durrett and Neuhauser 1994] for details.

Chapter 3

Results on the Diploid Branching Particle Model

In this chapter, we assume the model with generator (2.5) described in Chapter 2.1, i.e. the particle system with birth and death mechanisms, but no stirring. We briefly restate the model to remind the reader: the rate at which nest m of site x flips to state i, $c_i(x, m, \xi)$, is

$$c_0(x,m,\xi) = \begin{cases} \delta, & \text{if } \xi^m(x) = 1 \\ 0, & \text{otherwise} \end{cases}, \quad c_1(x,m,\xi) = \begin{cases} \lambda n_1(x,\xi) n_2(x,\xi), & \text{if } \xi^m(x) = 0 \\ 0, & \text{otherwise} \end{cases},$$

where

$$n_{m'}(x,\xi) = |\{z \in \mathcal{N}_x : \xi^{m'}(x+z) = 1\}|,$$

and the $\mathcal{N}_x$ contains the site x and its $2d$ nearest neighbours. The goal is to establish the existence of a phase transition.

3.1 Existence of Stationary Distributions

We first establish that stationary distributions exist. Define

$$\overline{\xi}_0(x) = (1,1) \text{ for all } x.$$

Let $T_t f(\xi_0) = E^{\xi_0} f(\xi_t)$ be the semigroup corresponding to the particle system, then T_t is a Feller semigroup by Theorem 2.1.1. We begin with a lemma.

Lemma 3.1.1. *For any $A, B \subset S = \mathbb{Z}^d$, the function*

$$t \mapsto P\left(\overline{\xi}_t^1(x) = 0 \ \forall x \in A,\ \overline{\xi}_t^2(y) = 0 \ \forall y \in B\right) \tag{3.1}$$

is increasing.

Proof. Let $\alpha_0 = \overline{\xi}_s^1$ and $\beta_0 = \overline{\xi}_s^2$ for an arbitrary fixed s. Then $\overline{\xi}_0^1(x) \geq \alpha_0(x)$ and $\overline{\xi}_0^2(x) \geq \beta_0(x)$. Let (α_t, β_t) be the state at time t of the particle system that started

with initial condition (α_0, β_0). Then by the fact that the particle system is monotone in initial conditions, we have

$$\overline{\xi}_t^1(x) \geq \alpha_t(x) \text{ and } \overline{\xi}_t^2(x) \geq \beta_t(x)$$

for all t and x. Thus by the Markov property of ξ,

$$\begin{aligned} & P\left(\overline{\xi}_t^1(x) = 0 \ \forall x \in A,\ \overline{\xi}_t^2(y) = 0 \ \forall y \in B\right) \\ \leq\ & P\left(\alpha_t(x) = 0 \ \forall x \in A,\ \beta_t(y) = 0 \ \forall y \in B\right) \\ =\ & P\left(\overline{\xi}_{s+t}^1(x) = 0 \ \forall x \in A,\ \overline{\xi}_{s+t}^2(y) = 0 \ \forall y \in B\right). \end{aligned}$$

This implies that the function in (3.1) is increasing in t. □

Theorem 3.1.2. *As $t \to \infty$, $\overline{\xi}_t \Rightarrow \overline{\xi}_\infty$. The limit is a stationary distribution that stochastically dominates all other stationary distributions and called the* upper invariant measure.

Proof. For arbitrary subsets A, B, $C = \{x_1, \ldots, x_m\}$, and $D = \{y_1, \ldots, y_n\}$ of S, we write

$$\begin{aligned} & P\left(\overline{\xi}_t^1(z) = 0 \ \forall z \in A,\ \overline{\xi}_t^2(w) = 0 \ \forall w \in B,\ \overline{\xi}_t^1(x) = 1 \ \forall x \in C,\ \overline{\xi}_t^2(y) = 1 \ \forall y \in D\right) \\ =\ & P\left(\overline{\xi}_t^1(z) = 0 \ \forall z \in A,\ \overline{\xi}_t^2(w) = 0 \ \forall w \in B\right) - P\left(\bigcup_{i=1}^{m+n} E_i\right), \end{aligned}$$

where

$$E_i = \{\overline{\xi}_t^1(z) = 0 \ \forall z \in A \cup \{x_i\},\ \overline{\xi}_t^2(w) = 0 \ \forall w \in B\} \text{ if } i = 1, \ldots, m$$

and

$$E_i = \{\overline{\xi}_t^1(z) = 0 \ \forall z \in A,\ \overline{\xi}_t^2(w) = 0 \ \forall w \in B \cup \{y_{i-m}\}\} \text{ if } i = m+1, \ldots, m+n.$$

We can use the inclusion-exclusion formula on $P(\cup_{i=1}^{m+n} E_i)$, i.e.

$$P\left(\bigcup_{i=1}^{m+n} E_i\right) = \sum_{i=1}^{m+n} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \ldots + (-1)^{m+n+1} P(E_i \cap \ldots \cap E_j).$$

Every term in the above expansion is in the form of

$$P(\overline{\xi}_t^1(z) = 0 \ \forall z \in \cdot,\ \overline{\xi}_t^2(w) = 0 \ \forall w \in \cdot\cdot),$$

which is increasing in t by Lemma 3.1.1. Therefore

$$P(\overline{\xi}_t^1(z) = 0 \ \forall z \in A,\ \overline{\xi}_t^2(w) = 0 \ \forall w \in B,\ \overline{\xi}_t^1(x) = 1 \ \forall x \in C,\ \overline{\xi}_t^2(y) = 1 \ \forall y \in D)$$

converges for all A, B, C, and D, i.e. all finite dimensional distributions converge. Thus a weak limit $(\overline{\xi}_\infty^1, \overline{\xi}_\infty^2)$ exists and it follows from a standard result that since T_t is a Feller

semigroup, $(\bar{\xi}^1_\infty, \bar{\xi}^2_\infty)$ is a stationary distribution. We can also easily see that $(\bar{\xi}^1_\infty, \bar{\xi}^2_\infty)$ dominates all other stationary distributions: let $(\tilde{\xi}^1_0, \tilde{\xi}^2_0)$ be another stationary distribution and $(\tilde{\xi}^1, \tilde{\xi}^2)$ be the process with initial condition $(\tilde{\xi}^1_0, \tilde{\xi}^2_0)$, then $(\tilde{\xi}^1_t, \tilde{\xi}^2_t)$ has the same distribution as $(\tilde{\xi}^1_0, \tilde{\xi}^2_0)$ for all $t \geq 0$ and $(\bar{\xi}^1, \bar{\xi}^2)$ dominates $(\tilde{\xi}^1, \tilde{\xi}^2)$ because of monotonicity, therefore $(\bar{\xi}^1_\infty, \bar{\xi}^2_\infty)$ dominates $(\tilde{\xi}^1_0, \tilde{\xi}^2_0)$. □

Theorem III.2.3 in [Liggett 1985] establishes the previous theorem for spin systems with state space $\{0,1\}^S$, therefore it does not directly apply to our case. One can, however, easily adapt the proof of that theorem to this case, and obtain a slightly different proof.

3.2 Extinction for Sufficiently Small λ/δ

Theorem 3.2.1. *If $\lambda|\mathcal{N}|^2 < \delta$, then the particle system ξ has no nontrivial stationary distribution.*

Proof. We compare a modification of the particle system ξ with the contact process. We recall that the contact process ζ on $\mathbb{Z}^d$ has two states 0 and 1 at every site x, and has the following dynamics:

$$c_0(x,\zeta) = \begin{cases} \delta, & \text{if } \zeta(x) = 1 \\ 0, & \text{otherwise} \end{cases}, \quad c_1(x,\zeta) = \begin{cases} \alpha n(x,\zeta), & \text{if } \zeta(x) = 0 \\ 0, & \text{otherwise} \end{cases},$$

where $n(x,\zeta) = |\{z \in \mathcal{N}_x : \zeta(z) = 1\}|$. Theorem 2.6 of [Durrett 1995] states that if $\alpha|\mathcal{N}| < \delta$, then the contact process has no nontrivial stationary distribution.

We modify the mechanism of the particle model ξ as follows: for the males, when $(x,1) \in S \times \{1\}$ seeks out a pair of parents, say $(x_1,1)$ and $(x_2,2)$, it is no longer required that $\xi^2(x_2) = 1$, but $\xi^1(x_1)$ must still be 1. Correspondingly, when a female nest $(x,2) \in S \times \{2\}$ seeks out a pair of parents $(x_1,1)$ and $(x_2,2)$, it is only required that $\xi^2(x_2) = 1$. We denote this modified process $\tilde{\xi} = (\tilde{\xi}^1, \tilde{\xi}^2)$. The result of the modification is that $\tilde{\xi}^1$ and $\tilde{\xi}^2$ are now decoupled, and $\tilde{\xi}^i$ behaves exactly the same as the contact process with birth rate $\alpha = \lambda|\mathcal{N}|$. Furthermore, by Theorem III.1.5 in [Liggett 1985], the modified process $(\tilde{\xi}^1, \tilde{\xi}^2)$ stochastically dominates the original process (ξ^1, ξ^2). If $\alpha|\mathcal{N}| < \delta$, then $\tilde{\xi}$ has no nontrivial stationary distribution and $(\tilde{\xi}^1_t, \tilde{\xi}^2_t)$ converges weakly to the all-0 state as $t \to \infty$ for any initial condition. Thus (ξ^1_t, ξ^2_t) also converges to the all-0 state for any initial condition if $\alpha|\mathcal{N}| = \lambda|\mathcal{N}|^2 < \delta$, as required. □

The proof above also shows that if $\sum_x \xi^1_0(x) + \xi^2_0(x)$ is finite, then the population dies out in finite time a.s. if $\lambda|\mathcal{N}|^2 < \delta$, since the contact process $\tilde{\xi}^1$ or $\tilde{\xi}^2$ has this property.

3.3 Survival for Sufficiently Large λ/δ

We use the idea of Chapter 4 of [Durrett 1995], i.e. we compare the particle system to an oriented percolation process. First, we define the oriented percolation process W. Let

$$\mathcal{L}_0 = \{(x,n) \in \mathbb{Z}^2 : x + n \text{ is even}, n \geq 0\}$$

and make $\mathcal{L}_0$ into a graph by drawing oriented edges from (x,n) to $(x+1,n+1)$ and from (x,n) to $(x-1,n+1)$. Site (x,n) is said to be a parent of sites $(x+1,n+1)$ and $(x-1,n+1)$. Notice that any site (x,n) with $n \neq 0$ has two parents. We think of n as the time variable. Given random variables $\omega(x,n)$ that indicate whether site (x,n) is open (1) or closed (0), we say that (y,l) can be reached from (x,m) if there is sequence of points $x = z_m, \ldots, z_l = y$ such that $|z_k - z_{k-1}| = 1$ for $m < k \leq l$ and $\omega(z_k,k) = 1$ for $m \leq k \leq l$; in this case, we write

$$(x,m) \to (y,l).$$

We say that $\omega(x,n)$ $(n \geq 1)$ is "*M-dependent with density at least* $1-\gamma$" if whenever (x_i,n_i), $1 \leq i \leq I$, is a sequence with $\|(x_i,n_i)-(x_j,n_j)\|_\infty > M$ for $i \neq j$, we have

$$P(\omega(x_i,n_i) = 0 \text{ for } 1 \leq i \leq I) \leq \gamma^I. \tag{3.2}$$

Given an initial condition $W_0 \subset 2\mathbb{Z} = \{x : (x,0) \in \mathcal{L}_0\}$, the process

$$W_n = \{y : (x,0) \to (y,n) \text{ for some } x \in W_0\}.$$

gives all sites that can be reached from a site in W_0 at time n. We say sites in W_n are wet. Theorem 4.2 of [Durrett 1995] states:

Theorem 3.3.1. *Let W_n^p be an M-dependent oriented percolation with density at least $1-\gamma$ starting from the initial configuration W_0^p in which the events $\{x \in W_0^p\}$, $x \in 2\mathbb{Z}$, are independent and have probability p. If $p > 0$ and $\gamma \leq 6^{-4(2M+1)^2}$, then*

$$\liminf_{n\to\infty} P(0 \in W_{2n}^p) \geq 19/20. \tag{3.3}$$

This theorem shows that if the density of open sites $1-\gamma$ is sufficiently close to 1 and we start with a Bernoulli initial condition for W_0, then the probability that 0 is wet at time t does not go to 0 as $t \to \infty$. Notice that the right hand side of the estimate (3.3) is a constant that does not depend on p. We will construct an oriented percolation process that is stochastically dominated by the particle system ξ, such that existence of nontrivial stationary distribution for the oriented percolation process implies existence of nontrivial stationary distribution for the particle system.

Theorem 3.3.2. *If λ/δ is sufficiently large, then the particle system ξ with generator (2.5) has a nontrivial stationary distribution.*

Proof. We follow the method of proof as in Chapter 4 of [Durrett 1995]. Since scaling time by an factor of $1/\delta$ does not change the behaviour with respect to stationary distributions, we may assume without any loss of generality that $\delta = 1$. We will select an event G_{ξ_0} measurable with respect to the graphical representation in $[-1,1] \times \mathbb{Z}^{d-1} \times [0,T)$, i.e. measurable with respect to the filtration generated by all the Poisson arrivals $\{R_n^{x,m}\}$ and $\{T_n^{x,m,y,z}\}$ used to construct the particle model in Chapter 2.1 that arrive at any sites in $[-1,1] \times \mathbb{Z}^{d-1}$ during

the time interval $[0,t)$. For any $\gamma > 0$ no matter how small, there is λ and T and an event G_{ξ_0} with

$$P(G_{\xi_0}) > 1-\gamma,$$

so that on G_{ξ_0}, if $\xi_0(0,0,\ldots,0) = (1,1)$, then $\xi_T(1,0,\ldots,0) = \xi_T(-1,0,\ldots,0) = (1,1)$. One can achieve this by choosing T so small that the probability of any death occurring at any nests of sites $(-1,0,\ldots,0)$, $(0,0,\ldots,0)$, and $(1,0,\ldots,0)$ is less than $\gamma/2$; then one can choose λ large enough so that the probability of having birth events from $((0,0,\ldots,0),1)$ and $((0,0,\ldots,0),2)$ to each of the four nests at sites $(-1,0,\ldots,0)$ and $(1,0,\ldots,0)$ during $[0,T)$ is larger than $1-\frac{\gamma}{2}$. In other words, if we define the event

$$\begin{aligned} G_{\xi_0} \quad = \quad & \{\text{There are no death event during } [0,T) \text{ at sites } (-1,0,\ldots,0),\ (0,0,\ldots,0), \\ & \text{or } (1,0,\ldots,0); \text{ and there are birth events from } ((0,0,\ldots,0),1) \\ & \text{and } ((0,0,\ldots,0),2) \text{ to each of the four nests at sites } (-1,0,\ldots,0) \\ & \text{and } (1,0,\ldots,0) \text{ during } [0,T)\}, \end{aligned}$$

then G_{ξ_0} satisfies the requirement and $P(G_{\xi_0}) > 1-\gamma$ for some λ and T. G_{ξ_0} is the "good event" that will ensure male and female particles get born at sites $x-1$ and $x+1$ provided site x is inhabited by both a male and a female particle. See figure 3.1 for an illustration of this event.

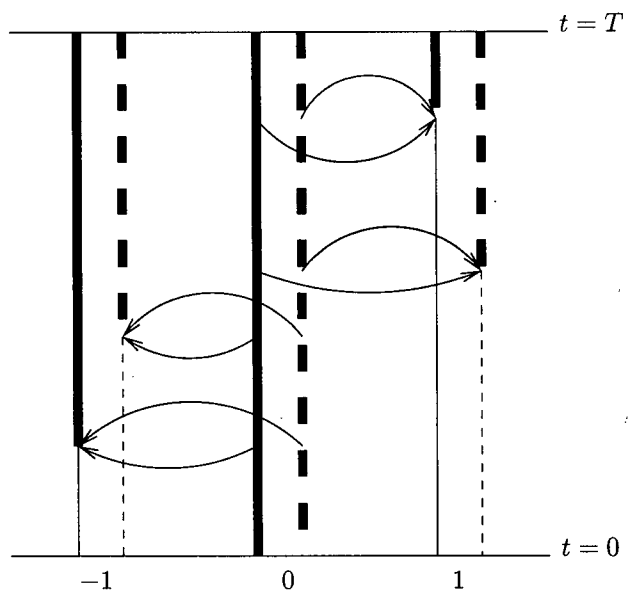


Figure 3.1: Graphical representation of the event G_{ξ_0}: at least 4 birth events and no death events.

We start with a configuration with events $\{\xi_0^1(x) = 1\}$ and $\{\xi_0^2(x) = 1\}$ all independent and having probability p_1 and p_2, respectively. Let

$$X_n = \{x : (x,n) \in \mathcal{L}_0, \xi_{nT}(x,0,\ldots,0) = (1,1)\}.$$

Before defining the oriented percolation process W_n, we first define V_n that will turn out to be slightly larger than W_n but nevertheless dominated by X_n.

We define V_n inductively. First, we set $V_0 = X_0$ and leave $\omega(\cdot, 0)$ undefined. Now assume that V_0, V_1, ..., V_n, and $\omega(x, l)$ with $l \leq n-1$ have been defined such that $V_0 \subset X_0$, ..., $V_n \subset X_n$. If $x \in V_n$ then we set

$$\omega(x,n) = \begin{cases} 1, & \text{if } G_{\sigma_{-xe_1}(\xi_{nT})} \text{ occurs in the graphical representation} \\ & (G_{\sigma_{-xe_1}(\xi_{nT})} \text{ is } G_{\xi_0} \text{ translated by } -xe_1 \text{ in space and} \\ & -nT \text{ in time}) \\ 0, & \text{otherwise} \end{cases}, \tag{3.4}$$

which ensures that $\omega(x, n) = 1$ with probability more than $1-\gamma$ if $x \in V_n$. For completeness, if $x \notin V_n$, then we set $\omega(x, n)$ equal to an independent random variable that is 1 with probability $1 - \gamma$ and 0 with probability γ. Now we define V_{n+1} to consist of all sites $(x, n+1)$ with either

$$x - 1 \in V_n \quad \text{and} \quad \omega(x-1, n) = 1 \tag{3.5}$$

or

$$x + 1 \in V_n \quad \text{and} \quad \omega(x+1, n) = 1. \tag{3.6}$$

For $(x', n) \in V_n \subset X_n$, the definition of X_n means that $\xi_{nT}(x', 0, \ldots, 0) = (1, 1)$; if $\omega(x', n) = 1$, then the "good event" occurs in the space-time rectangle

$$[x' - 1, x' + 1] \times \mathbb{Z}^{d-1} \times [nT, (n+1)T),$$

which, since $\xi_{nT}(x', 0, \ldots, 0) = (1, 1)$, implies that

$$\xi_{(n+1)T}(x' - 1, 0, \ldots, 0) = (1, 1) \text{ and } \xi_{(n+1)T}(x' + 1, 0, \ldots, 0) = (1, 1).$$

Therefore the conditions (3.5) and (3.6) ensure that any $(x, n+1) \in V_{n+1}$ is a member of X_{n+1}, hence $V_{n+1} \subset X_{n+1}$. By induction, for all $n \in \mathbb{Z}^+$, we have

$$V_n \subset X_n.$$

The $\omega(x, n)$ thus defined is a 2-dependent oriented percolation on $\mathcal{L}_0$ with density at least $1 - \gamma$; notice that $\{\omega(0, 0) = 0\}$ and $\{\omega(2, 0) = 0\}$ are dependent events since both require that no death occur during $[0, T)$ at nest $((1, 0, \ldots, 0), 1)$ or $((1, 0, \ldots, 0), 2)$, but $\{\omega(0, 0) = 0\}$ and $\{\omega(4, 0) = 0\}$ are clearly independent. Also, $\{\omega(x_1, n_1) = 0\}$ and $\{\omega(x_2, n_2) = 0\}$ are independent for any x_1 and x_2 provided $n_1 \neq n_2$. Thus if (x_i, n_i), $1 \leq i \leq I$, is a sequence with $\|(x_i, n_i) - (x_j, n_j)\|_\infty > 2$ for $i \neq j$, then the events $\{\omega(x_i, n_i) = 0\}$ are all independent, which implies

$$P(\omega(x_i, n_i) = 0 \text{ for } 1 \leq i \leq I) \leq \gamma^I,$$

as required by (3.2). Notice that even though V_{n+1} clearly depends on V_n, $\omega(x_1, n+1)$ and $\omega(x_2, n)$ as defined by (3.4) are indeed independent since they relate to independent Poisson arrivals in disjoint space-time rectangles.

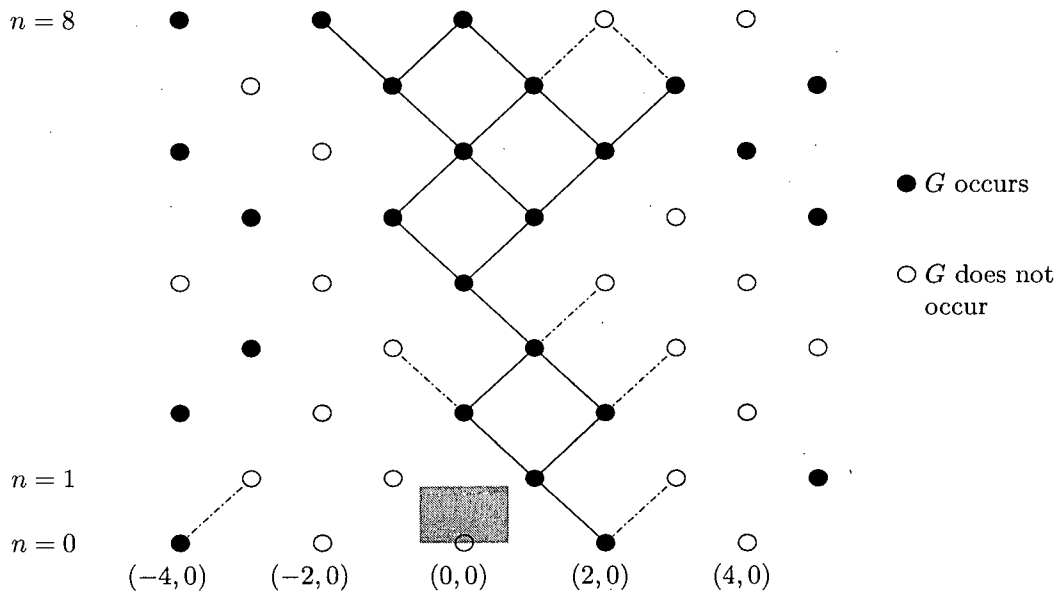


Figure 3.2: Illustration of V_n and W_n: all sites in W_n are connected to $(2,0)$ via a sequence of solid lines, while sites in $V_n \backslash W_n$ are connected to some site in W_n via a dotted line. The shaded rectangle indicates the space-time region that affects whether G_{ξ_0} occurs

Now we define

$$W_n = \{(x,n) : (x,n) \in V_n \text{ and } \omega(x,n) = 1\},$$

then $W_n \subset V_n$ and W_n is a 2-dependent oriented percolation with density at least $1-\gamma$. If γ is sufficiently small, then by Theorem 3.3.1,

$$\liminf_{n\to\infty} P(0 \in W_{2n}) \geq 19/20.$$

Since the particle system ξ dominates V_n, which in turn dominates W_n, we have

$$\liminf_{n\to\infty} P(\xi_{2nT}(0,0,\ldots,0) = (1,1)) \geq 19/20. \tag{3.7}$$

Thus if we start with initial configuration $\overline{\xi}_0(x) = (1,1)$ for all x (in this case, $p = 1$ for Theorem 3.3.1), then by Theorem 3.1.2, $\overline{\xi}_t \Rightarrow \overline{\xi}_\infty$. And (3.7) implies that

$$P(\overline{\xi}_\infty(x) = (1,1)) \geq 19/20 \quad \text{for any } x \in \mathbb{Z}^d,$$

i.e. the upper invariant measure $\overline{\xi}_\infty$ is nontrivial. □

Remark 3.3.3. *Using the same idea of comparing the particle system ξ to an oriented percolation process, one can show that if the initial condition is finite, e.g. $\xi_0(0,0,\ldots,0) = (1,1)$ and 0 everywhere else, then for sufficiently large λ,*

$$\liminf_{n\to\infty} P(\xi_{nT}(x,0,\ldots,0) = (1,1) \textit{ for some } x) \geq 19/20$$

for some T. Here we use the following fact (see Theorem 4.1 of [Durrett 1995]) about $C_0 = \{(y,n) : (0,0) \to (y,n)\}$:

$$\text{If } \gamma \leq 6^{-4(2M+1)^2}, \text{ then } P(|C_0| < \infty) \leq 1/20. \tag{3.8}$$

Remark 3.3.4. *In fact, the proof of Theorem 3.3.1 also establishes the following: if the initial population has a positive density of male and female particles everywhere and λ/δ is sufficiently large, then at sufficiently large times, the density of $(1,1)$-sites (i.e. sites with both a male and female particle) is larger than $9/10$. Similarly, using (3.8), the following is also true: if the initial population is nonzero and λ/δ is sufficiently large, then at sufficiently large times, the probability of survival (i.e. existence of a site with both a male and a female particle) is at least $9/10$.*

Chapter 4

Convergence Theorem for Individual Stirring

In this chapter, we establish the convergence result for the individual stirring model as promised in Chapter 2.4. We work in a slightly more general setting and consider random processes

$$\xi_t^\epsilon : \epsilon\mathbb{Z}^d \times \{1, 2, \ldots, M\} \to \{0, 1, \ldots, \kappa - 1\}.$$

We call each $x \in \epsilon\mathbb{Z}^d$ a site, and each $(x, m) \in \epsilon\mathbb{Z}^d \times \{1, 2, \ldots, M\}$ a nest. There are M nests at each site. We think of the set of spatial locations $\mathbb{Z}^d \times \{1, 2, \ldots, M\}$ as consisting of M "floors" of $\mathbb{Z}^d$. Let

$$\mathcal{N} = \{0, \epsilon y_1, \ldots, \epsilon y_N\}$$

be the interaction neighbourhood of site 0 and $r_{\mathcal{N}}^\epsilon = \max_{x \in \mathcal{N}} \|x\|_1$ be its radius. The process ξ_t^ϵ evolves as follows:

1. **Birth and Death.** The state of nest (x, m) flips to i, $i = 0, \ldots, \kappa - 1$, at rate

   $$c_i(x, m, \xi) = h_{i,m}(\xi(x, m), \xi(x + \epsilon z_1, m_1), \ldots, \xi(x + \epsilon z_L, m_L)),$$

   where L is a positive integer, $z_1, \ldots, z_L \in \mathcal{N}$, $m_1, \ldots, m_L \in \{1, 2, \ldots, M\}$, and

   $$h_{i,m} : \{0, 1, \ldots, \kappa - 1\}^{L+1} \to K \subset \mathbb{R}^+$$

   with $h_{i,m}(i, \ldots) = 0$ and K compact.

2. **Rapid Stirring.** For each $m \in \{1, 2, \ldots, M\}$ and $x, y \in \epsilon\mathbb{Z}^d$ with $\|x - y\|_1 = \epsilon$, $\xi^\epsilon(x, m)$ and $\xi^\epsilon(y, m)$ are exchanged at rate ϵ^{-2}.

This individual stirring model differs from the Lily-pad stirring model described in Chapter 2.2 in that the stirring action between corresponding nests at neighbouring sites are now independent. More specifically, exchanges are allowed between neighbouring nests on

the same floor only, i.e. between $(0,1)$ and $(\epsilon,1)$ but not between $(0,1)$ and $(\epsilon,2)$. We will show, in the theorem below, that this individual rapid stirring action between corresponding nests "decouples" (in the limit $\epsilon \to 0$) the dependence between all nests, at neighbouring sites and even at the same site.

As an example, for $d=1$, in the particle model with individual stirring with generator (2.8), we have $\kappa = 2$, $M = 2$, $L = 4$, $\mathcal{N} = \{0, -\epsilon, \epsilon\}$,

$$c_0(x,m,\xi) = \begin{cases} \delta, & \text{if } \xi(x,m) = 1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$c_1(x,m,\xi) = \begin{cases} \lambda(\xi(x-\epsilon,1)\xi(x-\epsilon,2) + \xi(x+\epsilon,2)\xi(x+\epsilon,1)), & \text{if } \xi(x,m) = 0 \\ 0, & \text{otherwise} \end{cases}.$$

In particular, we should define

$$(z_1,m_1) = (-1,1),\ (z_2,m_2) = (-1,2),\ (z_3,m_3) = (1,1),\ (z_4,m_4) = (1,2),$$

and $h_i = h_{i,m}$ as

$$h_0(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \delta\alpha_0,$$

and

$$h_1(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \lambda(\alpha_1\alpha_2 + \alpha_3\alpha_4)(1-\alpha_0),$$

where $\alpha_0 = \xi(x,m)$, $\alpha_1 = \xi(x+\epsilon z_1, m_1) = \xi(x-\epsilon,1)$, *etc.*

Theorem 4.0.5. *Suppose* $\{\xi_0^\epsilon(x,m),(x,m) \in \epsilon\mathbb{Z}^d \times \{1,2,\ldots,M\}\}$ *are independent and let* $u_{i,m}^\epsilon(t,x) = P(\xi_t^\epsilon(x,m) = i)$. *If* $u_{i,m}^\epsilon(0,x) = g_{i,m}(x) = g_i(x,m)$ *and* $g_i : \mathbb{R}^d \times \{1,2,\ldots,M\} \to [0,1]$ *is continuous, then for any smooth function* ϕ *with compact support, as* $\epsilon \to 0$,

$$\epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) 1_{\{\xi_t^\epsilon(y,m)=i\}} \to \int \phi(y) u_{i,m}(t,y)\, dy, \tag{4.1}$$

where $u_{i,m}(t,x)$ *is the bounded solution of*

$$\frac{\partial u_{i,m}}{\partial t} = \Delta u_{i,m} + f_{i,m}(u),\ u_{i,m}(0,x) = g_i(x,m),$$

$$f_{i,m}(u) = \langle c_i(0,m,\xi) 1(\xi(0,m) \neq i)\rangle_u - \sum_{j\neq i} \langle c_j(0,m,\xi) 1(\xi(0,m) = i)\rangle_u,$$

and $\langle \phi(\xi)\rangle_u$ *denotes the expected value of* $\phi(\xi)$ *under the product measure in which state* j *at nest* m *has density* $u_{j,m}$, *i.e.* $\xi(x,m)$, *with* $x \in \epsilon\mathbb{Z}^d$ *and* $1 \le m \le M$, *are independent with* $P(\xi(x,m) = j) = u_{j,m}$.

Proof. We follow the program used in the proof of Theorem 8.1 in [Durrett 1995]. We first define a dual process for the particle system in part (a), then in part (b) we show that the dual process is almost a branching random walk. We will not explicitly write down parts (c) and (d) of the proof, which almost exactly resemble parts (c) and (d) of the proof of Theorem 8.1 in [Durrett 1995]. But to summarize these two parts, part (c) establishes that the dual process converges to a branching Brownian motion as $\epsilon \to 0$ and defines a candidate limit $u_{i,m}(t,x)$ of $u^{\epsilon}_{i,m}(t,x)$, and part (d) shows that this candidate limit satisfies the PDE in the statement of the theorem. Finally, part (e), which we write out explicitly, shows that convergence described in (4.1) does occur, in addition to the convergence of the mean $u^{\epsilon}_{i,m}(t,x)$ to $u_{i,m}(t,x)$.

a. Defining the dual process. The dual process associated with nest $(\hat{x},\hat{m})$ and a fixed time t is a random process $I^{\hat{x},\hat{m},t}_{\epsilon}(s)$, $s \in [0,t]$, where

$$I^{\hat{x},\hat{m},t}_{\epsilon}(s) \in \bigcup_{k\in\mathbb{Z}^+} \{(x,m) : x \in \epsilon\mathbb{Z}^d, m \in \{1,2,\ldots,M\}\}^k \tag{4.2}$$

consists of a finite number of particles residing at nests $(x,m) \in \epsilon\mathbb{Z}^d \times \{1,2,\ldots,M\}$. The events that influence the behaviour of the dual process $I^{\hat{x},\hat{m},t}_{\epsilon}$ at time s are those Poisson arrivals in the original process ξ^{ϵ} that occur at time $t-s$. We start with

$$I^{\hat{x},\hat{m},t}_{\epsilon}(0) = \{(\hat{x},\hat{m})\}$$

and evolve $I^{\hat{x},\hat{m},t}_{\epsilon}(s)$ until $s=t$, which corresponds to rolling back the clock in the original process ξ^{ϵ} from time t back to time 0. The process $I^{\hat{x},\hat{m},t}_{\epsilon}$ is constructed such that: in order to know the value of $\xi^{\epsilon}(\hat{x},\hat{m})$ at time t, it suffices to do a computation using values of ξ^{ϵ} at time 0 and nests $(x,m) \in I^{\hat{x},\hat{m},t}_{\epsilon}(t)$. We call $I^{\hat{x},\hat{m},t}_{\epsilon}(s)$ the *influence set* of $I^{\hat{x},\hat{m},t}_{\epsilon}(0)$.

In order to define the dual process, we need to give a graphical construction for the particle system ξ^{ϵ} similar to the one given in Chapter 2.1. We define a family of uniform $[0,1]$ random variables $\{U^{x,m,i}_n : n \geq 1\}$ and two families of independent Poisson processes that respectively correspond to the birth/death flips and the rapid stirring mechanism:

$$\begin{aligned} &\{T^{x,m,i}_n : n \geq 1\} \text{ at rate } c^* = \sup_{\xi,m} \textstyle\sum_i c_i(x,m,\xi) \\ \text{and} \quad &\{S^{x,y,m}_n : n \geq 1\} \text{ at rate } \epsilon^{-2}, \end{aligned}$$

with $m \in \{1,2,\ldots,M\}$ and $\|x-y\|_1 = \epsilon$. Notice that $c^* < \infty$ since $\max_{i,m} \|h_{i,m}\|_\infty < \infty$. At $T^{x,m,i}_n$, we check all nests (x,m), $(x+\epsilon z_1, m_1), \ldots, (x+\epsilon z_L, m_L)$ and use $U^{x,m,i}_n$ to decide whether nest (x,m) should flip to state i based on the function $h_{i,m}$. And at $S^{x,y,m}_n$, we exchange values of $\xi^{\epsilon}(x,m)$ and $\xi^{\epsilon}(y,m)$, the corresponding nests at two neighbouring sites. The evolution of the dual process $I^{\hat{x},\hat{m},t}_{\epsilon}(s)$ depends on the Poisson arrival times $\{T^{x,m,i}_n : n \geq 1\}$ and $\{S^{x,y,m}_n : n \geq 1\}$ in the following way:

1. If $(x,m) \in I^{\hat{x},\hat{m},t}_{\epsilon}(s-)$ and $T^{x,m,i}_n = t-s$, then we set

$$I^{\hat{x},\hat{m},t}_{\epsilon}(s) = I^{\hat{x},\hat{m},t}_{\epsilon}(s-) \cup \{(x+\epsilon z_1, m_1), \ldots, (x+\epsilon z_L, m_L)\}. \tag{4.3}$$

Therefore each particle $(x,m) \in I^{\hat{x},\hat{m},t}_{\epsilon}(s)$ gives birth to L new particles at rate c^*.

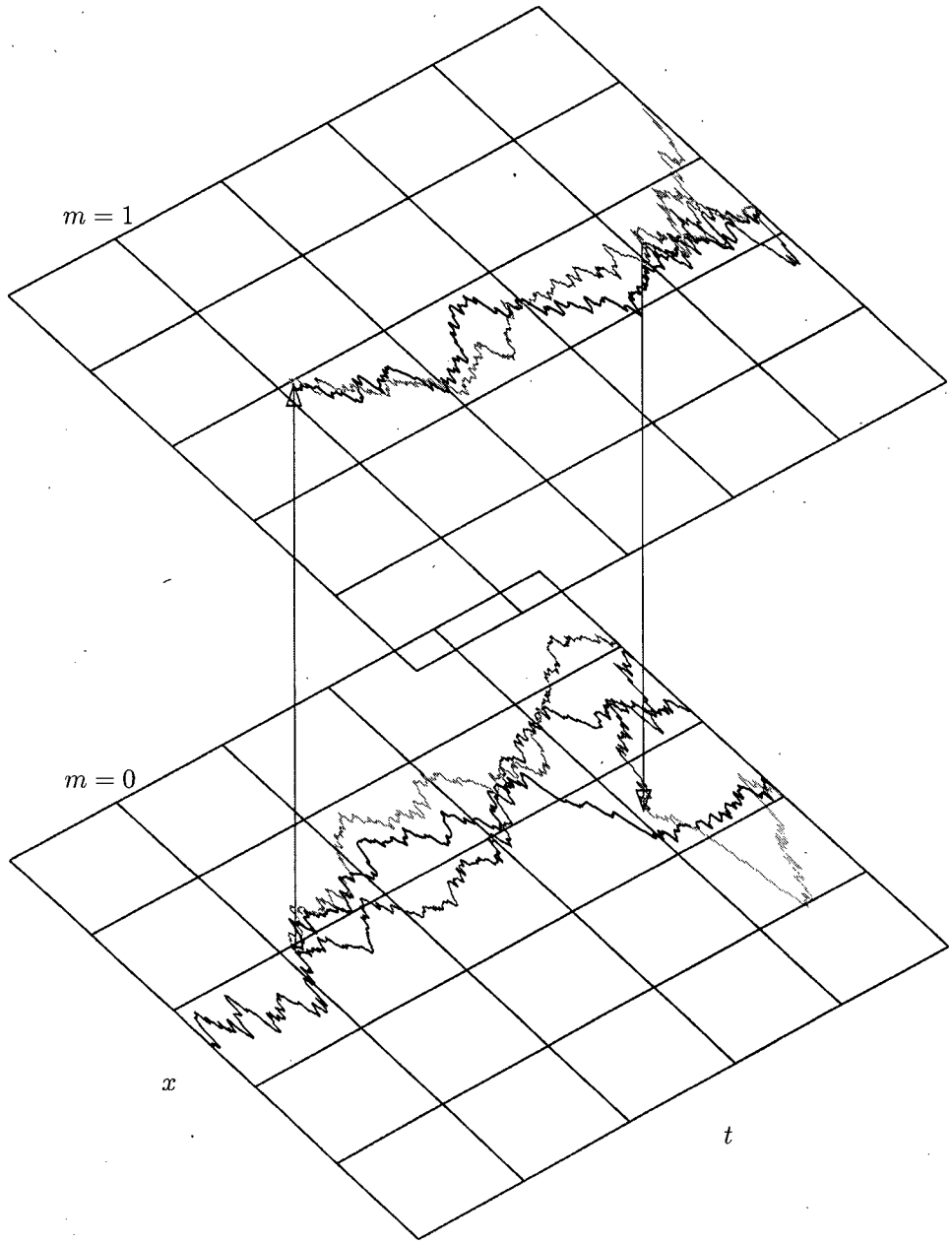


Figure 4.1: Illustration of the dual process: there are two birth/death events here, each giving birth to four additional particles, two on each floor; the first birth event is from a male ($m = 0$) nest, while the second birth event is from a female ($m = 1$) nest.

2. If $(x, m) \in I_\epsilon^{\hat{x},\hat{m},t}(s-)$ and either $S_n^{x,y,m} = t - s$ or $S_n^{y,x,m} = t - s$, then we set

$$I_\epsilon^{\hat{x},\hat{m},t}(s) = \left(I_\epsilon^{\hat{x},\hat{m},t}(s-) \backslash \{(x, m)\}\right) \cup \{(y, m)\}.$$

Thus two dual processes $I_\epsilon^{\hat{x},\hat{m},t}(s)$ and $I_\epsilon^{\hat{x}',\hat{m}',t}(s)$ evolve independently from each other except at time

$$s = \text{ some } T_n^{x,m,i}$$

where $(x,m) \in I_\epsilon^{\hat{x},\hat{m},t}(s) \cap I_\epsilon^{\hat{x}',\hat{m}',t}(s)$ or at time

$$s = \text{ some } S_n^{x,y,m}$$

where $(x,m) \in I_\epsilon^{\hat{x},\hat{m},t}(s)$ and $(y,m) \in I_\epsilon^{\hat{x}',\hat{m}',t}(s)$, or $(y,m) \in I_\epsilon^{\hat{x},\hat{m},t}(s)$ and $(x,m) \in I_\epsilon^{\hat{x}',\hat{m}',t}(s)$.

An equivalent way of describing the dual process is to define the pair $(X_\epsilon^{\hat{x},\hat{m}}(s), \mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s))$ for $0 \le s \le t$, where $X_\epsilon^{\hat{x},\hat{m}}(s)$ is the ordered set

$$X_\epsilon^{\hat{x},\hat{m}}(s) = (X_\epsilon^{\hat{x},\hat{m},0}(s), \ldots, X_\epsilon^{\hat{x},\hat{m},\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(0)-1}(s))$$

and each $X_\epsilon^{\hat{x},\hat{m},j}(s) \in \epsilon\mathbb{Z}^d \times \{1,2,\ldots,M\}$. For $s=0$, define the number of particles

$$\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(0) = n$$

for some $n \in \mathbb{Z}^+$ and the location of particles

$$X_\epsilon^{\hat{x},\hat{m},j}(0) = (x_j, m_j)$$

for $0 \le j < n$, such that

$$I_\epsilon^{\hat{x},\hat{m},t}(0) = \bigcup_{j=0}^{\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(0)-1} \{X_\epsilon^{\hat{x},\hat{m},j}(0)\}, \tag{4.4}$$

where $I_\epsilon^{\hat{x},\hat{m},t}(0)$ is defined in (4.2). Typically, $\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(0) = 1$ and $X_\epsilon^{\hat{x},\hat{m},0}(0) = \{(\hat{x},\hat{m})\}$, but we use the more general initial condition since we will define $X_\epsilon^{\hat{x},\hat{m},j}$ inductively. Each particle in $X_\epsilon^{\hat{x},\hat{m},j}$, $0 \le j < \mathcal{K}_\epsilon^{\hat{x},\hat{m}}$, jumps to a neighbouring nest on the same "floor" of $\epsilon\mathbb{Z}^d \times \{1,2,\ldots,M\}$ as dictated by the stirring mechanism (i.e. S-arrivals), until a T-arrival of type $T^{x,m,j}$ with $0 \le j < \mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s-)$ at some $(x,m) \in X_\epsilon^{\hat{x},\hat{m}}(s-)$. At time s of this T-arrival, we set

$$\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s) = L + \mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s-)$$

and

$$X_\epsilon^{\hat{x},\hat{m},k+\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s-)}(s) = (x + \epsilon z_k, m_k)$$

for $0 \le k < L$, while leaving all existing $X_\epsilon^{\hat{x},\hat{m},j}$'s unchanged. Observe that

$$I_\epsilon^{\hat{x},\hat{m},t}(s) = \bigcup_{j=0}^{\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s)-1} \{X_\epsilon^{\hat{x},\hat{m},j}(s)\},$$

therefore the relation (4.4) is maintained for all $s \in [0,t]$. Afterwards, the $\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s)$ particles jump according to the S-arrivals until the next T-arrival at some $(x,m) \in \{X_\epsilon^{\hat{x},\hat{m},j}(s'), 0 \leq j < \mathcal{K}_\epsilon^{\hat{x},\hat{m}}(s'-)\}$.

A new particle may be born at a nest where a particle already resides; if this happens, we say that a *collision* occurs, and call the new particle *fictitious*. We prescribe this fictitious particle to give birth to L particles at rate c^* and jump to a neighbouring nest on the same "floor" at rate ϵ^{-2}, independently from all other particles, fictitious or not; furthermore, all offsprings of a fictitious particle are defined to be fictitious as well. If the number of collisions is 0, then there are no fictitious particles. As will be seen in the next paragraph , this is in fact the case in the limit $\epsilon \to 0$. Note that the stirring mechanism does not cause particles at different nests to "collide", i.e. end up at the same nest, because only exchanges between nests occur under the stirring mechanism.

b. Characterizing the dual process. Having finished defining the dual process, we proceed to show that with high probability, the dual process can be coupled to a branching random walk and the movement of one dual process is independent from the movement of another dual. This part of the proof is quite similar to part (b) of the proof of Theorem 8.1 of [Durrett 1995].

First, we couple $X_\epsilon^{\hat{x},\hat{m},j}$ to independent random walks $Y_\epsilon^{\hat{x},\hat{m},j}$ that start at the same location at the time of birth of $X_\epsilon^{\hat{x},\hat{m},j}$ and jump to a randomly chosen neighbour at rate $2d\epsilon^{-2}$. We define the distance between two particles on two different "floors" to only depend on the site location:

$$\|(x_1,\hat{m}_1) - (x_2,\hat{m}_2)\|_1 = \|x_1 - x_2\|_1,$$

and say $X_\epsilon^{\hat{x},\hat{m},j}$ is *crowded* if for some $k \neq j$, $\|X_\epsilon^{\hat{x},\hat{m},j} - X_\epsilon^{\hat{x},\hat{m},k}\|_1 \leq r_{\mathcal{N}}^\epsilon$. Recall that $r_{\mathcal{N}}^\epsilon$ is the radius of the interaction neighbourhood $\mathcal{N}$. When $X_\epsilon^{\hat{x},\hat{m},j}$ is not crowded, we define the displacements of $Y_\epsilon^{\hat{x},\hat{m},j}$ to be equal to those of $X_\epsilon^{\hat{x},\hat{m},j}$. But when $X_\epsilon^{\hat{x},\hat{m},j}$ is crowded, we use independent Poisson processes to determine the jumps of $Y_\epsilon^{\hat{x},\hat{m},j}$. To estimate the difference between $X_\epsilon^{\hat{x},\hat{m},j}$ and $Y_\epsilon^{\hat{x},\hat{m},j}$, we need to estimate the amount of time $X_\epsilon^{\hat{x},\hat{m},j}$ is crowded. If $j \neq k$, then $V_s^\epsilon = X_\epsilon^{\hat{x},\hat{m},j}(s) - X_\epsilon^{\hat{x},\hat{m},k}(s)$ is stochastically larger than W_s^ϵ, a random walk that jumps to a randomly chosen neighbour in $\epsilon\mathbb{Z}^d \times \{1\}$ at rate $4d\epsilon^{-2}$ (see page 175 of [Durrett 1995] for details on how to couple these two processes so that one is stochastically larger than the other). Strictly speaking, V_s^ϵ is only defined after the jth and k^{th} particles of $X_\epsilon^{\hat{x},\hat{m}}$ are created, but as we shall see below, we are only interested in an upper bound on the occupation time of V_s^ϵ and W_s^ϵ in a ball, so the fact that these particles may only start to exist at some positive time only helps matters. Thus for any integer $M \geq 1$, $v_t^{M\epsilon} = |\{s \leq t : \|V_s^\epsilon\|_1 \leq M\epsilon\}|$ is stochastically smaller than $w_t^{M\epsilon} = |\{s \leq t : \|W_s^\epsilon\|_1 \leq M\epsilon\}|$. Well known asymptotic results [Durrett 1995] for random walks imply that if $t\epsilon^{-2} \geq 2$ then

$$Ew_t^{M\epsilon} \leq \begin{cases} CM\epsilon t^{1/2} & d=1 \\ CM^2\epsilon^2 \log(t\epsilon^{-2}) & d=2 \\ CM^d\epsilon^2 & d \geq 3 \end{cases} . \tag{4.5}$$

In the estimates above, $d = 1$ is the worst case, so we use $Ew_t^{M\epsilon} \leq CM\epsilon t^{1/2}$ if $t\epsilon^{-2} \geq 2$ from now on.

The amount of time $X_\epsilon^{\hat{x},\hat{m},j}$ is crowded in $[0,t]$, denoted $\chi_\epsilon^j(t)$, can be estimated as follows:

$$\begin{aligned} E[\chi_\epsilon^j(t)] &= \sum_{K=0}^{\infty} E[\chi_\epsilon^j(t)|\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t)=K]P(\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t)=K) \\ &\leq \sum_{K=0}^{\infty} KE[w_t^{M\epsilon}]P(\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t)=K), \end{aligned}$$

where we pick M large enough so that $M\epsilon \geq r_{\mathcal{N}}^\epsilon$, e.g. $M = \frac{1}{\epsilon}\max_{0<i\leq N}\|y_i\|_1$. In what follows, $C_{\ldots}$ is a constant whose value may change from line to line. Then

$$E[\chi_\epsilon^j(t)] \leq E[w_t^{M\epsilon}]E[\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t)] = e^{c^*Lt}E[w_t^{M\epsilon}] \leq C_M e^{c^*Lt}\epsilon t^{1/2}$$

if $t\epsilon^{-2} \geq 2$. To see the middle equality above, we observe that the branching mechanism of the dual process described in (4.3) occurs at rate c^* for both fictitious and nonfictitious particles, and every time a branching event occurs, one particle is replaced by L particles; therefore the mean number of branches at time t is e^{c^*Lt}. It follows that

$$E[\chi_\epsilon^j(t)] \leq C_M e^{c^*Lt}\epsilon t^{1/2} + 2\epsilon^2 \leq C_M e^{c^*Lt}\epsilon(1+t^{1/2}). \tag{4.6}$$

This means that the expected number of births from $X_\epsilon^{\hat{x},\hat{m},j}$ while there is some other $X_\epsilon^{\hat{x},\hat{m},k}$ in $\mathcal{N} + X_\epsilon^{\hat{x},\hat{m},j}$ is smaller than

$$C_{L,M}e^{c^*Lt}\epsilon(1+t^{1/2}). \tag{4.7}$$

Thus

$$\begin{aligned} &P(\text{at least one collision during } [0,t]) \\ &\leq P(\text{at least one collision during } [0,t]|\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) \leq \epsilon^{-0.5})P(\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) \leq \epsilon^{-0.5}) \\ &\qquad + P(\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) > \epsilon^{-0.5}) \\ &\leq E[\#\text{ of collisions during } [0,t]|\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) \leq \epsilon^{-0.5}]P(\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) \leq \epsilon^{-0.5}) \\ &\qquad + \epsilon^{0.5}E[\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t)] \qquad (4.8) \\ &\leq C_{L,M}e^{c^*Lt}\epsilon^{0.5}(1+t^{1/2}) + \epsilon^{0.5}e^{c^*Lt} \\ &\leq C_{L,M}e^{c^*Lt}\epsilon^{0.5}(1+t^{1/2}), \end{aligned}$$

which $\to 0$ as $\epsilon \to 0$. We use (4.7) and the condition $\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) \leq \epsilon^{-0.5}$ to bound the first expectation in (4.8), and we bound $P(\mathcal{K}_\epsilon^{\hat{x},\hat{m}}(t) \leq \epsilon^{-0.5})$ in (4.8) above by 1.

This shows that the probability of at least one collision within a single dual during time $[0,t]$ tends to 0 as $\epsilon \to 0$. Furthermore, the same argument shows that the probability of at least one collision between two different duals for nests $(\hat{x},\hat{m}) \neq (\hat{x}',\hat{m}')$ during time $[0,t]$ also tends to 0 as $\epsilon \to 0$. For that, we observe that the estimate (4.5) is independent of the initial condition W_0^ϵ; so in particular, this estimate still holds even if $\|W_0^\epsilon\|_1 = 0$, which is the case when for example one considers two duals for two nests $(\hat{x},\hat{m})$ and $(\hat{x},\hat{m}')$ at the same site x. Hence two different duals are asymptotically independent in the limit $\epsilon \to 0$.

The estimate (4.6) also leads to the following estimate on the difference between $X_\epsilon^{\hat{x},\hat{m},j}$ and $Y_\epsilon^{\hat{x},\hat{m},j}$ (see page 176 in [Durrett 1995] for details):

$$P\left(\max_{0\le s\le t} \|X_\epsilon^{\hat{x},\hat{m},j}(s) - Y_\epsilon^{\hat{x},\hat{m},j}(s)|_\infty \ge 2\epsilon^{0.3}\right) \le C\epsilon^{0.4}(1+t^{1/2})e^{c^*Lt}. \tag{4.9}$$

This shows that with high probability, the movements of all the particles in a dual can be coupled to independent random walks, in addition to being independent from the movement of any other dual.

c. and d. Defining a candidate limit and showing the limit satisfies the PDE. We will not write down the details of these two parts of the argument, since they are almost exactly the same as parts (c) and (d) of the proof of Theorem 8.1 in [Durrett 1995]. From estimate (4.9), it is not too difficult to see that the dual process converges to the branching Brownian motion as $\epsilon \to 0$. We can then define the candidate limit $u_{i,m}(t,x)$ (of $u_{i,m}^\epsilon(t,x) = P(\xi_t^\epsilon(x,m) = i)$) using the limiting branching Brownian motion as the dual process. Part (d) then establishes that the candidate limit $u_{i,m}(t,x)$ satisfies the integral from of the PDE in the statement of the theorem.

e. The particle systems converge. So far we have established that

$$u_{i,m}^\epsilon(t,x) \to u_{i,m}(t,x),$$

i.e. the expected value converges. It remains to establish (4.1). For this, we define for a bounded function ϕ with compact support supp(ϕ) of diameter D,

$$\langle \xi_t^\epsilon, \phi\rangle = \epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) 1_{\{\xi_t^\epsilon(y,m)=i\}}.$$

Then

$$\begin{aligned} E[\langle \xi_t^\epsilon, \phi\rangle] &= \epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) P(\xi_t^\epsilon(y,m)=i) = \epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) u_{i,m}^\epsilon(t,y) \\ &\to \int \phi(y) u_{i,m}(t,y)\, dy \end{aligned} \tag{4.10}$$

by bounded convergence. Now we compute the variance of $\langle \xi_t^\epsilon, \phi\rangle$:

$$\begin{aligned} &\mathrm{Var}[\langle \xi_t^\epsilon, \phi\rangle] \\ &= E\left[\epsilon^{2d}\left(\sum_{y\in\epsilon\mathbb{Z}^d} \phi(y)(1_{\{\xi_t^\epsilon(y,m)=i\}} - P(\xi_t^\epsilon(y,m)=i))\right)^2\right] \\ &= E\left[\epsilon^{2d}\sum_{y\in\epsilon\mathbb{Z}^d} \phi(y)^2(1_{\{\xi_t^\epsilon(y,m)=i\}} - P(\xi_t^\epsilon(y,m)=i))^2\right] + E\left[\epsilon^{2d}\sum_{y,z\in\epsilon\mathbb{Z}^d, y\neq z} \phi(y)\phi(z)\right. \\ &\qquad \left.\times (1_{\{\xi_t^\epsilon(y,m)=i\}} - P(\xi_t^\epsilon(y,m)=i))(1_{\{\xi_t^\epsilon(z,m)=i\}} - P(\xi_t^\epsilon(z,m)=i))\right] \\ &\le D^{2d}\|\phi\|_\infty^2 \sup_{y,z\in\epsilon\mathbb{Z}^d\cap\mathrm{supp}(\phi), y\neq z} \mathrm{Cov}[1_{\{\xi_t^\epsilon(y,m)=i\}}, 1_{\{\xi_t^\epsilon(z,m)=i\}}] \\ &\qquad + \epsilon^{2d}\|\phi\|_\infty^2 \sum_{y\in\epsilon\mathbb{Z}^d\cap\mathrm{supp}(\phi)} \mathrm{Var}[1_{\{\xi_t^\epsilon(y,m)=i\}}]. \end{aligned}$$

We observe that $1_{\{\xi_t^\epsilon(y,m)=i\}}$ is a random variable taking values in $\{0,1\}$ and therefore has variance $\leq 1/4$. Also, part (b) of the proof shows that

$$\mathrm{cov}_\epsilon = \sup_{y,z\in\epsilon\mathbb{Z}^d, y\neq z} \mathrm{Cov}[1_{\{\xi_t^\epsilon(y,m)=i\}}, 1_{\{\xi_t^\epsilon(z,m)=i\}}] \to 0$$

as $\epsilon \to 0$. The argument leading to the asymptotic independence of two duals in part (b) works for any two nests, so $\mathrm{Cov}[1_{\{\xi_t^\epsilon(y,m)=i\}}, 1_{\{\xi_t^\epsilon(z,m)=i\}}]$ goes to zero uniformly for all $y \neq z$. Now we have the following estimate on $\mathrm{Var}[\langle \xi_t^\epsilon, \phi\rangle]$:

$$\mathrm{Var}[\langle \xi_t^\epsilon, \phi\rangle] \leq D^{2d}\|\phi\|_\infty^2 \mathrm{cov}_\epsilon + \frac{D^d}{4}\epsilon^d \|\phi\|_\infty^2,$$

which $\to 0$ as $\epsilon \to 0$. Thus by (4.10) and Chebyshev's inequality, we have

$$\begin{aligned}
&P\left(\left|\langle \xi_t^\epsilon, \phi\rangle - \int \phi(y) u_{i,m}(t,y)\, dy\right| > \delta\right) \\
&\leq P\left(\left|\langle \xi_t^\epsilon, \phi\rangle - \epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) u_{i,m}^\epsilon(t,y)\right|\right. \\
&\qquad \left. + \left|\epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) u_{i,m}^\epsilon(t,y) - \int \phi(y) u_{i,m}(t,y)\, dy\right| > \delta\right) \\
&\leq P\left(\left|\langle \xi_t^\epsilon, \phi\rangle - \epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) u_{i,m}^\epsilon(t,y)\right| > \frac{\delta}{2}\right) \\
&\qquad + P\left(\left|\epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) u_{i,m}^\epsilon(t,y) - \int \phi(y) u_{i,m}(t,y)\, dy\right| > \frac{\delta}{2}\right) \\
&\leq \frac{4\mathrm{Var}[\langle \xi_t^\epsilon, \phi\rangle]}{\delta^2} + P\left(\left|\epsilon^d \sum_{y\in\epsilon\mathbb{Z}^d} \phi(y) u_{i,m}^\epsilon(t,y) - \int \phi(y) u_{i,m}(t,y)\, dy\right| > \frac{\delta}{2}\right) \to 0
\end{aligned}$$

as $\epsilon \to 0$, and the theorem follows. □

Chapter 5

Existence of Invariant Stationary Distribution For Lily-pad Stirring

In this chapter, we establish the existence of nontrivial stationary distribution of the particle system with lily-pad stirring as promised by Theorem 2.3.2. First, we rewrite (2.12) in the statement of Lemma 2.3.1:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + (2c(1-u)+1)v - u \\ \frac{\partial v}{\partial t} &= \Delta v + (c(u-v)-2)v. \end{aligned} \tag{5.1}$$

We show that for sufficiently large c, the solution to (5.1) with initial condition $u_0 = f, v_0 = g$, $f \geq g$ satisfies the following condition:

($*$) There are constants $0 < D_1 < d_1 < d_2 < D_2 < 1$, L, and T so that if $v_0(x) \in (D_1, D_2)$ for $x \in [-L, L]$ then $v_T(x) \in (d_1, d_2)$ for $x \in [-3L, 3L]$.

According to Chapter 9 of [Durrett 1995], this is a sufficient condition for the existence of nontrivial invariant stationary distribution for the particle system with sufficiently fast stirring, so Theorem 2.3.2 will follow once condition ($*$) is established. Recall that Theorem 3.3.2 establishes that the diploid particle model without rapid stirring has a nontrivial stationary distribution if the birth rate λ is sufficiently large. If one traces through the proof, however, one will find that "sufficiently large" in that argument means that λ is larger than a number on the order of 6^{100}, which is not too informative on where exactly the critical λ for the phase transition is. On the other hand, one can get a far better idea of exactly for what λ condition ($*$) holds.

For this proof, we also establish condition ($*$) for sufficiently large c (recall that $c = \lambda d$), but here "sufficiently large" means that c is "only" larger than a number on the order of 100. We assume dimension $d = 1$; extension to $d > 1$ is straightforward. The proof consists of two parts: the first part, Chapter 5.1, establishes the existence of constants d_1

and D_1, and the second part, Chapter 5.2, establishes the existence of constants d_2 and D_2; the second part will be easy once the first part has been established.

Theorem 9.2 in [Durrett 1995] establishes condition $(*)$ for a specific predator-prey system with phase space $\{0, 1, 2\}$ at each site. The critical fact used in the proof is that the associated ODE system (i.e. the dynamical system that results when one has constant initial conditions) has only one interior equilibrium point and has a *global* Lyapunov function. The phase portrait of the ODE associated with (5.1), however, shows that it has two interior equilibrium points, one of which is always a saddle point. See figure 5.3 for two examples. Thus it does not look likely that the ODE system associated with (5.1) has a readily identifiable global Lyapunov function. The method we use to establish condition $(*)$ for (5.1) is *ad hoc*, but does seem to apply to a wide variety of reaction-diffusion systems where the reaction part of the system is 2-dimensional (or even 3-dimensional), i.e. $\frac{\partial u}{\partial t} = \Delta u + f(u)$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$.

As established in Lemma 2.3.1, the PDE (5.1) is monotone in initial conditions that lie in $\mathcal{R} = \{(u, v) : 0 \leq u, v \leq 1, u \geq v\}$. This fact is critical for the proof of existence of constants d_1 and D_1 in condition $(*)$ above – it enables us to bound the initial condition $v_0(x)$ below by a function, say $\underline{v}_0(x)$, both v_0 and $\underline{v}_0$ having values $< D_1$ for $x \in [-L, L]$, such that if the solution $\underline{v}_t(x)$ to (5.1) with initial condition $\underline{v}_0(x)$ satisfies the condition

$$\underline{v}_T(x) > d_1 \ \forall x \in [-3L, 3L],$$

then $v_T(x) > d_1 \ \forall x \in [-3L, 3L]$ as well. We will also need results regarding the ODE associated with (5.1):

$$\begin{aligned} \frac{du}{dt} &= (2c(1-u)+1)v - u \\ \frac{dv}{dt} &= (c(u-v)-2)v. \end{aligned} \tag{5.2}$$

The above ODE system is also monotone in initial conditions, since if the initial condition for the PDE system in (5.1) is constant in x, then the solution (u_t, v_t) to (5.1) also remains constant in x for all time and therefore satisfies the ODE system in (5.2).

5.1 Lower Bounds: Existence of d_1 and D_1 in Condition $(*)$

First we recall the definition of the region $\mathcal{R}$ from (2.11):

$$\mathcal{R} = \{(u, v) : 0 \leq u, v \leq 1, u \geq v\}.$$

If the initial condition (u_0, v_0) lies in $\mathcal{R}$, then so does the solution (u_t, v_t). We will establish the existence of constants d_1, D_1, L, and T using the nonlinear Trotter product formula (Proposition 15.5.2 from [Taylor 1996]):

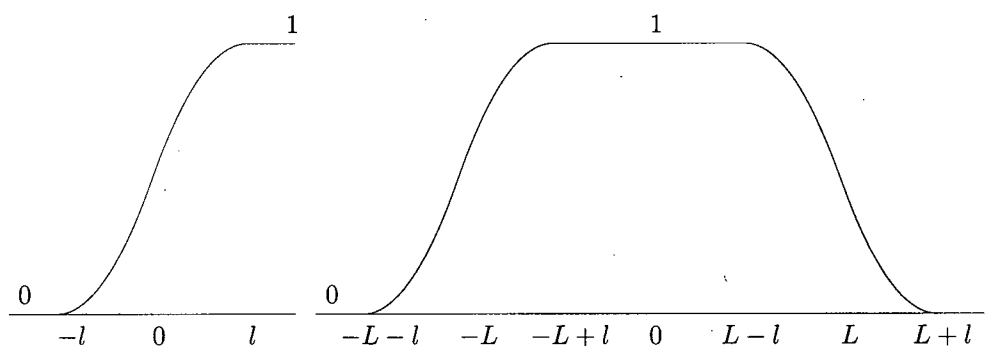
$$(u_t, v_t) = \lim_{n \to \infty} \left(e^{(t/n)\Delta} \mathcal{F}^{t/n} \right)^n (f, g). \tag{5.3}$$

Here the convergence occurs in the space $\mathcal{BC}^1(\mathbb{R})$, the space of functions whose first derivatives are bounded and continuous on $\mathbb{R}$ and extend continuously to the compactification $\widehat{\mathbb{R}}$ via the point at infinity; the norm used here is $\|\cdot\|_\infty + \|\frac{\partial}{\partial x}(\cdot)\|_\infty$. In (5.3), $e^{s\Delta}(f,g)$ gives the (independent) evolution of f and g for time s according the heat equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u \\ \frac{\partial v}{\partial t} &= \Delta v,\end{aligned}$$

and $\mathcal{F}^s(f,g)$ gives the pointwise evolution of (f,g) for time s according to the ODE in (5.2), i.e. for all x, if $(u_0(x), v_0(x)) = (f(x), g(x))$ then $\mathcal{F}^s(f,g)(x) = (u_s(x), v_s(x))$ where $(u(x), v(x))$ evolves according to (5.2). Note that both $e^{s\Delta}$ and $\mathcal{F}^s$ are monotone in initial conditions, therefore so is $e^{s\Delta}\mathcal{F}^s$.

To establish the existence of constants D_1 and d_1 in condition $(*)$, it suffices to show that for any initial condition (u_0, v_0) with v_0 dominating the function $D_1 I_{[-L,L]}(x)$, for sufficiently large T, (u_T, v_T) is such that v_T dominates the function $d_1 I_{[-3L,3L]}(x)$.



(a) The function h (b) The function f_0

Figure 5.1: The functions h and f_0.

Let $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \|f\| = \int(1+\xi^2)|\hat{f}(\xi)|^2 d\xi < \infty\}$ denote the Sobolev space with parameter 2, where $\hat{f}$ denotes the Fourier transform of f. Equivalently, $H^2(\mathbb{R})$ consists of L^2-functions with L^2-second derivatives. We first define $f_0 \in H^2(\mathbb{R})$ that will be the "shape" of the initial conditions (u_0, v_0):

IC 1. $f_0(x) = 1$ for $x \in [-L+l, L-l]$;

IC 2. $f_0(x) = 0$ for $x \in (\infty, -L-l] \cup [L+l, \infty)$;

IC 3. $f_0(x) = h(x+L)$ for $x \in [-L-l, -L+l]$ and $f_0(x) = h(L-x)$ for $x \in [L-l, L+l]$,

where $h \in H^2(\mathbb{R})$ is the following function:

$$h(x) = \begin{cases} 0, & x < -l \\ \frac{1}{2}(\frac{x+l}{l})^2, & -l \le x \le 0 \\ 1 - \frac{1}{2}(\frac{l-x}{l})^2, & 0 < x \le l \\ 1, & x > l \end{cases} . \tag{5.4}$$

In the above definition, the choice of L is arbitrary as long as $L > l$, but we will later on choose l small such that Δf_0 is large in $[-L-l, -L+l] \cup [L-l, L+l]$. We call the intervals $[-L-l, -L+l]$ and $[L-l, L+l]$ the "transition regions". We observe that h is continuous at $x = 0$, with

$$h''(x) = \begin{cases} \frac{1}{l^2} & \text{if } -l < x < 0 \\ -\frac{1}{l^2} & \text{if } 0 < x < l \end{cases} ,$$

so the graph of h in the plane is symmetric about the point $(0, \frac{1}{2})$ and also,

$$|\Delta f_0| \le \frac{1}{l^2} \tag{5.5}$$

everywhere.

We pick the initial condition to be $u_0 = a_0 f_0, v_0 = b_0 f_0$ with $(a_0 f_0(0), b_0 f_0(0)) = (a_0, b_0) \in \mathcal{R}_0$, where $\mathcal{R}_0$ is the region for the top tip of the line segment $\overline{O(a_0, b_0)}$ to be

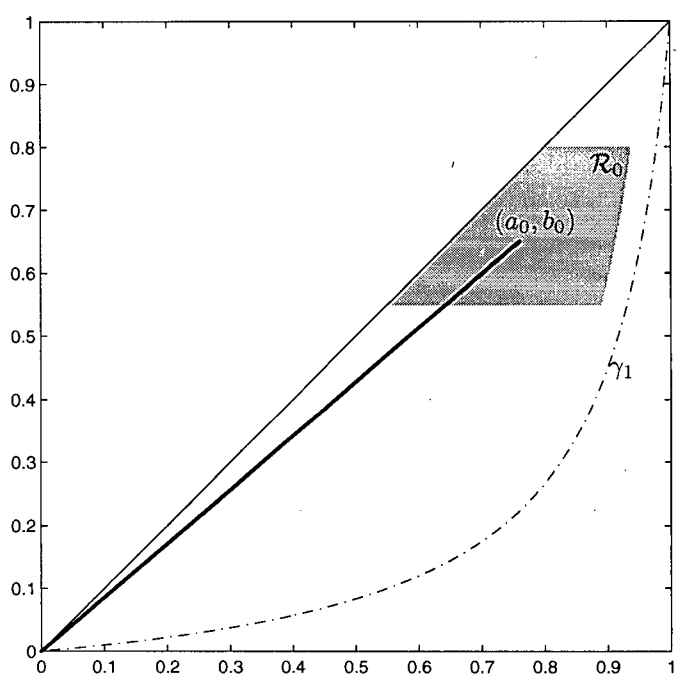


Figure 5.2: Shaded region is $\mathcal{R}_0$, and γ_1 will be defined in (5.9).

defined later in (5.19). See figure 5.2 for an illustration of $\mathcal{R}_0$ and the line segment $\overline{O(a_0, b_0)}$. We will define a family of parallel piecewise-linear curves $\overline{ABC}(u_0, v_0)$ (see figure 5.3) that satisfy the following requirements:

ABC 1. $\overline{ABC}(u_0, v_0)$ lies in $\mathcal{R}_0$;

ABC 2. $\overline{ABC}(u_0, v_0)$ passes through the points A, B, C, and (u_0, v_0);

ABC 3. A lies on the line $u = v$, B lies on the line $u = v + 0.1$, and C lies on the line $u = 1$;

ABC 4. $\overline{ABC}(u_0, v_0) = \overline{AB}(u_0, v_0) \cup \overline{BC}(u_0, v_0)$, where line segments $\overline{AB}(u_0, v_0)$ connects A and B, and $\overline{BC}(u_0, v_0)$ connects B and C;

ABC 5. $\overline{AB}(u_0, v_0)$ makes an angle of $-\theta$ $(0 < \theta < \arctan(0.05))$ with the positive u-axis and $\overline{BC}(u_0, v_0)$ is a horizontal line segment.

We will establish the following:

Proposition 5.1.1. *If c is sufficiently large and $(\tilde{a}_0, \tilde{b}_0) \in \{(u, v) \in \mathcal{R} : 0.55 \leq v \leq 0.8\}$, then for sufficiently small s, we have*

$$e^{s\Delta}\mathcal{F}^s(\tilde{a}_0 f_0, \tilde{b}_0 f_0) \geq (\tilde{a}_s f_s, \tilde{b}_s f_s),$$

where f_0 is defined in (IC 1-3) on page 34, $\geq$ means that $\geq$ holds in each component, and $\tilde{a}_s$, $\tilde{b}_s$, and f_s satisfy the following conditions:

1. *$\tilde{a}_s$ and $\tilde{b}_s$ are constants depending on s, such that the curve $\overline{ABC}(\tilde{a}_s, \tilde{b}_s)$ lies above the curve $\overline{ABC}(\tilde{a}_0, \tilde{b}_0)$, and the vertical distance separating them is at least $\delta_2 s$. In particular, since $\overline{ABC}(\tilde{a}_0, \tilde{b}_0)$ lies above the horizontal line $v = 0.5$, so does $\overline{ABC}(\tilde{a}_s, \tilde{b}_s)$.*

2. $$f_s(x) = \begin{cases} 1, & x \in [-L + l - \delta_1 s, L - l + \delta_1 s] \\ h(x + L + \delta_1 s), & x \in [-L - l - \delta_1 s, -L + l - \delta_1 s] \\ h(L + \delta_1 s - x), & x \in [L - l + \delta_1 s, L + l + \delta_1 s] \\ 0, & x \in (\infty, -L - l - \delta_1 s] \cup [L + l + \delta_1 s, \infty) \end{cases}, \tag{5.6}$$

   i.e. f_s is f_0 with each of the two transition regions is translated by $\delta_1 s$ away from the origin.

3. *δ_1 and δ_2 are positive constants independent of s and $(\tilde{a}_0, \tilde{b}_0)$.*

The proof of the above proposition requires a few lemmas and will be deferred until the end of Chapter 5.1.2. Proposition 5.1.1 states that $e^{s\Delta}\mathcal{F}^s(\tilde{a}_0 f_0, \tilde{b}_0 f_0)$ is bounded below by $(\tilde{a}_s f_s, \tilde{b}_s f_s)$, where $(\tilde{a}_s, \tilde{b}_s)$ moves $\delta_2 s$ above $\overline{ABC}(\tilde{a}_0, \tilde{b}_0)$. Furthermore, by (5.6), the region where the value of $(\tilde{a}_s f_s, \tilde{b}_s f_s)$ (and hence $e^{s\Delta}\mathcal{F}^s(\tilde{a}_0 f_0, \tilde{b}_0 f_0)$) is equal to or above $(\tilde{a}_s, \tilde{b}_s)$ has expanded by $\delta_1 s$, both to the left and to the right, while the transition regions of $(\tilde{a}_s f_s, \tilde{b}_s f_s)$ are shifted left or right by $\delta_1 s$ but maintains exactly the same profile as in

the initial condition. Thus by the monotonicity of $e^{s\Delta}\mathcal{F}^s$, we can iterate $e^{s\Delta}\mathcal{F}^s$ enough times and obtain information about the evolution of the PDE (5.1) for large time. From the construction of the piecewise linear curves $\overline{ABC}$, requirement (5) implies that $B(0.8, 0.8)$ on $\overline{AB}(0.8, 0.8)$ has v-coordinate ≥ 0.75. Therefore Corollary 5.1.2 below is an easy consequence of Proposition 5.1.1. Let

$$\pi_v(u_0, v_0) = v_0, \tag{5.7}$$

and $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$.

Corollary 5.1.2. *If c and T are sufficiently large, and $v_0(x) > 0.55$ for $x \in [-L+l, L-l]$ then*

$$\pi_v((e^{s\Delta}\mathcal{F}^s)^{\lfloor T/s \rfloor}(u_0, v_0)(x)) > 0.7$$

for $x \in [-3L, 3L]$ and sufficiently small s.

In other words, the constants D_1 and d_1 in condition $(*)$ are picked to be $D_1 = 0.55$ and $d_1 = 0.7$. Note that we restrict $v_0(x) = \tilde{b}_0 f_0(x)$ to be > 0.55 for $x \in [-L+l, L-l]$ in the above corollary because Prop 5.1.1 only works for $(\tilde{a}_0, \tilde{b}_0) \in \{(u, v) \in \mathcal{R} : 0.55 \leq v \leq 0.8\}$. Also note that the "$L$" in condition $(*)$ is picked to be $L - l$.

5.1.1 Analysis of the ODE (5.2)

We first characterize the phase portrait of the ODE. We carry this out for sufficiently large c. See figure 5.3 for phase portraits with $c = 5$ and $c = 25$. Define

$$\eta(u, v) = (\eta_1(u, v), \eta_2(u, v)) = ((2c(1-u)+1)v - u, (c(u-v)-2)v), \tag{5.8}$$

such that the solution to the ODE (5.2), $(u_s, v_s) = \mathcal{F}^s_\eta(u_0, v_0)$, flows along the vector field η. Define the curves γ_1, γ_2, and γ_3:

$$\gamma_1 = \left\{(u, v) : u \in [0, 1], v = \frac{u}{1 + 2c - 2cu}\right\}, \tag{5.9}$$

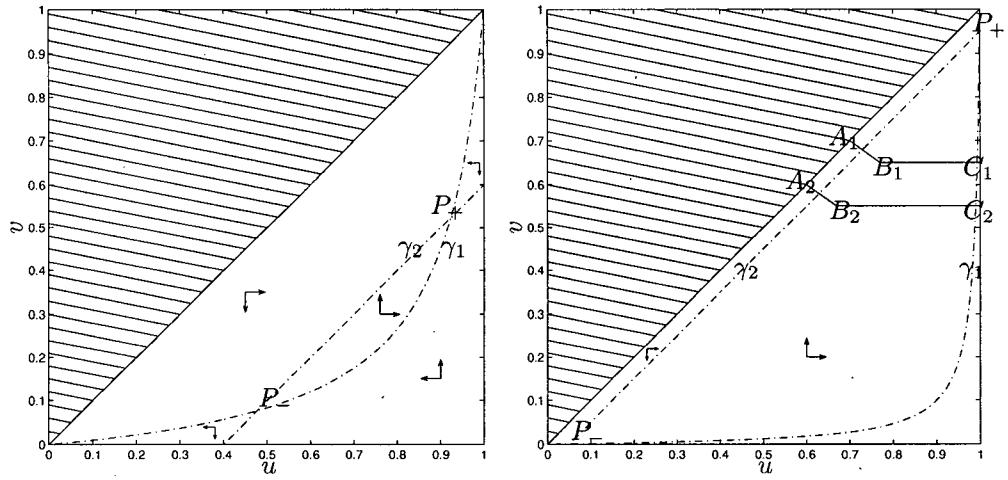
$$\gamma_2 = \left\{(u, v) : u \in [0, 1], v = u - \frac{2}{c}\right\}, \tag{5.10}$$

$$\gamma_3 = \{(u, v) : u \in [0, 1], u = v\}. \tag{5.11}$$

We have $\eta_1 = 0$ on γ_1 and $\eta_2 = 0$ on γ_2. An easy calculation shows that for all c, $(0, 0)$ and $(1, 1)$ pass through γ_1. We observe that η_1 is a linear function in u for fixed v, so $\eta_1 > 0$ to the left of γ_1 and $\eta_1 < 0$ to the right of γ_1. By similar reasoning, we also have $\eta_2 < 0$ to the left of γ_2, while $\eta_2 > 0$ to its right. The two intersection points of γ_1 and γ_2,

$$P_+ = \left(\frac{1}{2} + \frac{1}{c} + \sqrt{\frac{1}{4} - \frac{1}{c}}, \frac{1}{2} - \frac{1}{c} + \sqrt{\frac{1}{4} - \frac{1}{c}}\right),$$

$$P_- = \left(\frac{1}{2} + \frac{1}{c} - \sqrt{\frac{1}{4} - \frac{1}{c}}, \frac{1}{2} - \frac{1}{c} - \sqrt{\frac{1}{4} - \frac{1}{c}}\right),$$



(a) $c = 5$ (b) $c = 25$

Figure 5.3: Phase space of the ODE

are the only equilibrium points of η in the interior of $\mathcal{R}$, with $O = (0,0) \in \partial\mathcal{R}$ being the third equilibrium point. Elementary computation shows that O and P_+ are stable, but P_- is a saddle point, thus one would expect any point that lies significantly above P_- to flow toward P_+ under η. Elementary calculations also show the following:

$$\begin{aligned} P_+ &\to (1,1) \quad \text{as } c \to \infty, \\ P_- &\to (0,0) \quad \text{as } c \to \infty, \\ \tfrac{P_{-,u}}{P_{-,v}} &\to \infty \quad \text{as } c \to \infty, \end{aligned} \tag{5.12}$$

where $P_{-,u}$ and $P_{-,v}$ denote the u- and v- coordinates of P_-, respectively.

We will need some crude estimates of η_1 and η_2. First of all, since $u - v \geq 0$ everywhere in $\mathcal{R}$, we have

$$\eta_2 \geq -2v. \tag{5.13}$$

If the point (u,v) is at least δ to the right of γ_2, then since $\eta_2(u,v) = 0$ on γ_2,

$$\eta_2(u,v) > \delta c v, \tag{5.14}$$

Similar reasoning shows that if the point (u,v) is at least δ to the left of γ_1, then

$$\eta_1(u,v) > 2\delta c v. \tag{5.15}$$

Now the horizontal distance between γ_1 and γ_3 at a fixed v is

$$d(v) = \frac{(1+2c)v}{1+2cv} - v.$$

Simple calculations show that

$$d'(v) = -\frac{2c(2cv^2 + 2v - 1)}{(1+2cv)^2} \quad \text{and} \quad d''(v) = -\frac{4c(1+2c)}{(1+2cv)^3}.$$

Notice that $d''(v) < 0$ if $v, c > 0$, so for $v \in [0,1]$, $d(v)$ is a strictly concave function, and

$$\hat{v} = \frac{1}{2c}(\sqrt{1+2c} - 1)$$

is the unique point where $d(v)$ attains its maximum for $v \in [0,1]$. We observe that

$$\hat{v} \to 0 \text{ as } c \to \infty. \tag{5.16}$$

If c is sufficiently large such that the horizontal line $v = \epsilon$ lies above the line $v = \hat{v}$ but below the line $v = 0.8$, then for $v \in [\epsilon, 0.8]$, the minimum of $d(v)$ occurs at $v = 0.8$, and

$$d(0.8) = \frac{0.8(1+2c)}{1+2(0.8)c} - 0.8 = \frac{0.8 + 1.6c}{1+1.6c} - 0.8 \to 1 - 0.8 = 0.2 \tag{5.17}$$

as $c \to \infty$. This shows that for arbitrary $\epsilon > 0$ and sufficiently large c, the minimum horizontal distance between γ_1 and γ_3 for $v \in [\epsilon, 0.8]$, is larger than 0.19. This fact will be needed a bit later on in the proof of Lemma 5.1.3. We also observe that since $d(v)$ is strictly concave for $v \in [0,1]$, the curve γ_1 written as $u = u(v)$ is also strictly concave for $v \in [0,1]$; then (5.16) and the fact

$$u(\hat{v}) = \frac{1 + 2c - \sqrt{1+2c}}{2c} \to 1 \text{ as } c \to \infty$$

imply that

$$\gamma_1 \to \{(u,v) : v = 0, u \in [0,1]\} \cup \{(u,v) : u = 1, v \in [0,1]\} \text{ as } c \to \infty. \tag{5.18}$$

This finishes the characterization of the phase portrait of ODE (5.2). These facts, which are admittedly tedious and not much fun to establish, will all be required later in the proof of Lemmas 5.1.3 and 5.1.5.

We define the region (see figure 5.2)

$$\mathcal{R}_0 = \{(u,v) \in \mathcal{R} : u < \frac{(1+2c)v}{1+2cv} - 0.04, 0.55 \le v \le 0.8\}, \tag{5.19}$$

Recall that γ_1 defined in (5.9) is the curve $v = \frac{u}{1+2c-2cu}$ or $u = \frac{(1+2c)v}{1+2cv}$, and therefore the region $\mathcal{R}_0$ lies at least 0.04 to the left of the curve γ_1. By (ABC 5), the line segment AB forms a small negative angle with the positive u-axis, so by requiring $v \ge 0.55$ in the definition of $\mathcal{R}_0$, we can be sure that all of $\overline{ABC}(a_0, b_0)$ lies above the horizontal line $v = 0.5$ if $(a_0, b_0) \in \mathcal{R}_0$. For any point $(u,v) \in \mathcal{R}_0$, we have $u < 2v$ and (u,v) lies at least 0.04 to the left of γ_1. Since our initial condition has form $(a_0 f_0, b_0 f_0)$, the set of values in each "transition region"

$$\{(a_0 f_0(x), b_0 f_0(x)) : x \in [L - l, L + l]\},$$

forms a line segment with endpoints O and (a_0, b_0) in the (u, v)-plane. We require that the tip of this line segment (a_0, b_0) lies in the region $\mathcal{R}_0$. We do not need to worry about the case where the initial condition for the PDE (5.1) is such that $(a_0, b_0) \in \mathcal{R} \cap \{0.55 \leq v \leq 0.8\}$ lies to the right of $\mathcal{R}_0$. If we want to establish condition $(*)$ for that initial condition, then by the monotonicity of the PDE (5.1), it is sufficient to pick $a_0' < a_0$ such that $(a_0', b_0) \in \mathcal{R}_0$ and prove condition $(*)$ for the initial condition $(a_0' f_0, b_0 f_0)$. Therefore we only consider (a_0, b_0) in $\mathcal{R}_0$.

Assuming (a_0, b_0) lies in $\mathcal{R}_0$, a part of the line segment $\overline{O(a_0, b_0)}$ still lies below the horizontal line $v = 0.55$. To study the evolution of the whole line segment under $\mathcal{F}_\eta$, we will a bit later consider two cases: 1. $\epsilon \leq v \leq 0.8$, and 2. $0 \leq v < \epsilon$, where we will pick $\epsilon = 0.24$ in Chapter 5.1.2. We will construct piecewise linear curves $\overline{ABC}(v_0, v_0)$, $v_0 \in [0.55, 0.8]$, with $A = (v_0, v_0)$, B, and C satisfying the requirements laid down in (ABC 1-5) on page 35, in the proof of the following two lemmas. See figures 5.5 and 5.7 for an illustration of each lemma.

Lemma 5.1.3. *(Case 1) If (a_0, b_0) lies on $\overline{AB}(v_0, v_0) = \overline{AB}(a_0, b_0)$ with $a_0 - b_0 < 0.09$ and $0.55 < b_0 \leq 0.8$, then for sufficiently small s, there exist a_s and b_s with $b_s > 0.5$, and a positive number $\tilde{K}$ independent of s such that $\overline{AB}(a_s, b_s) = \overline{AB}(v_0, v_0)$ and*

$$\mathcal{F}_\eta^s(\alpha a_0, \alpha b_0) \geq \Big((1 + \tilde{K}s)\alpha a_s, (1 + \tilde{K}s)\alpha b_s\Big) \tag{5.20}$$

for all $\alpha \in [0, 1]$. Moreover, the constant $\tilde{K}$ can be chosen to be arbitrarily large if c is also allowed to be arbitrarily large.

Remark 5.1.4. *Using some easy geometric considerations, one can say the following: if ϵ is fixed and c is allowed to be arbitrarily large, then there exists an arbitrarily large positive number K depending on ϵ but independent of s, such that if $\alpha \in [\frac{\epsilon}{b_s}, 1]$ then*

$$((1 + \tilde{K}s)\alpha a_s, (1 + \tilde{K}s)\alpha b_s) - (\alpha a_s, \alpha b_s) > \left(\frac{a_s}{b_s} Ks, Ks\right), \tag{5.21}$$

and if $\alpha \in [0, \frac{\epsilon}{b_s})$ then

$$((1 + \tilde{K}s)\alpha a_s, (1 + \tilde{K}s)\alpha b_s) - (\alpha a_s, \alpha b_s) \geq (0, 0). \tag{5.22}$$

Lemma 5.1.5. *(Case 2) If (a_0, b_0) lies on $\overline{ABC}(v_0, v_0) = \overline{ABC}(a_0, b_0)$ with $0.08 + b_0 < a_0 < \frac{(1+2c)v}{1+2cv} - 0.04$ and $0.55 < b_0 \leq 0.8$, then for sufficiently small s, there exists a positive number K such that, if $\alpha \in [\frac{\epsilon}{b_0}, 1]$, then*

$$\mathcal{F}_\eta^s(\alpha a_0, \alpha b_0) - (\alpha a_0, \alpha b_0) \geq \left(\frac{a_0}{b_0} Ks, Ks\right), \tag{5.23}$$

and if $\alpha \in [0, \frac{\epsilon}{b_0})$, then

$$\mathcal{F}_\eta^s(\alpha a_0, \alpha b_0) - (\alpha a_0, \alpha b_0) \geq (-2\alpha a_0 s, -2\alpha b_0 s). \tag{5.24}$$

Moreover, the constant K can be chosen to be arbitrarily large if c is also allowed to be arbitrarily large.

In case 1 above (Lemma 5.1.3), (a_s, b_s) changes with s, but in case 2 (Lemma 5.1.5), (a_s, b_s) remains fixed and equal to (a_0, b_0), thus (a_s, b_s) is not explicitly defined. As will be seen later on, l is picked small so that the lower part of the "transition region" (of $(a_0 f_0, b_0 f_0)$) moves up at a sufficiently large speed under the heat kernel to cancel out the downward movement as described in (5.24). But the heat kernel pushes down the top part of the "transition region", so K (and thus c) is picked large to cancel out that effect. And finally s is picked small so that the movement caused by $\mathcal{F}_\eta^s$ is small.

In case 1, we assume (a_0, b_0), the top tip of the line segment formed by the "transition region", lies to the left of the line $u = v + 0.09$, while in case 2, we assume that (a_0, b_0) lies to the right of $u = v + 0.08$. There is a thin strip, i.e. $0.08 < u - v < 0.09$, where both cases apply, so we can apply either case 1 or case 2 there. Let (u_0, v_0) be a point on the line segment $\overline{O(a_0, b_0)}$. Intuitively, we would like to view "progress" as an increase in v-coordinate, which is measured by $\pi_v(\mathcal{F}_\eta^s(u_0, v_0) - (u_0, v_0))$. In case 1, however, it is not always possible for the v-coordinate to increase. So instead, we measure progress with respect to the family of parallel lines $\overline{AB}$, each of which makes a small negative angle with the positive u-axis, and thus allowing the v-coordinate to decrease slightly while still making "progress" with respect to $\overline{AB}$. More specifically, we compare $\mathcal{F}_\eta^s(u_0, v_0)$ not with $(u_0, v_0) = (\frac{u_0}{a_0} a_0, \frac{v_0}{b_0} b_0)$, but with $(u_s, v_s) = (\frac{u_0}{a_0} a_s, \frac{v_0}{b_0} b_s)$. We show that with respect to the lines $\overline{AB}$, the entire line segment makes progress with respect to the lines $\overline{AB}$ when moving under η. The u-coordinate actually increases very rapidly, so we move very quickly into where case 2 applies. In case 2, we compare $\mathcal{F}_\eta^s(u_0, v_0)$ directly with (u_0, v_0), and show that the part of the line segment with v-coordinate $> \epsilon$ makes progress, but the part of the line segment with v-coordinate $< \epsilon$ actually makes small negative progress. This negative progress will be compensated by positive progress made under evolution according to the heat kernel (to be shown in Chapter 5.1.2).

Proof of Lemma 5.1.3. We first define

$$\mathcal{R}_1 = \{(u, v) \in \mathcal{R} : u - v \in [0, 0.1], v \in [\epsilon, 0.8]\}, \tag{5.25}$$

$$\mathcal{R}_1' = \{(u, v) \in \mathcal{R}_1 : u - v \in [0, 0.09]\}. \tag{5.26}$$

$$\mathcal{R}_3' = \{(u, v) \in \mathcal{R} : u - v \in [0, 0.1], u \leq 2v, v \in [0, \epsilon)\}. \tag{5.27}$$

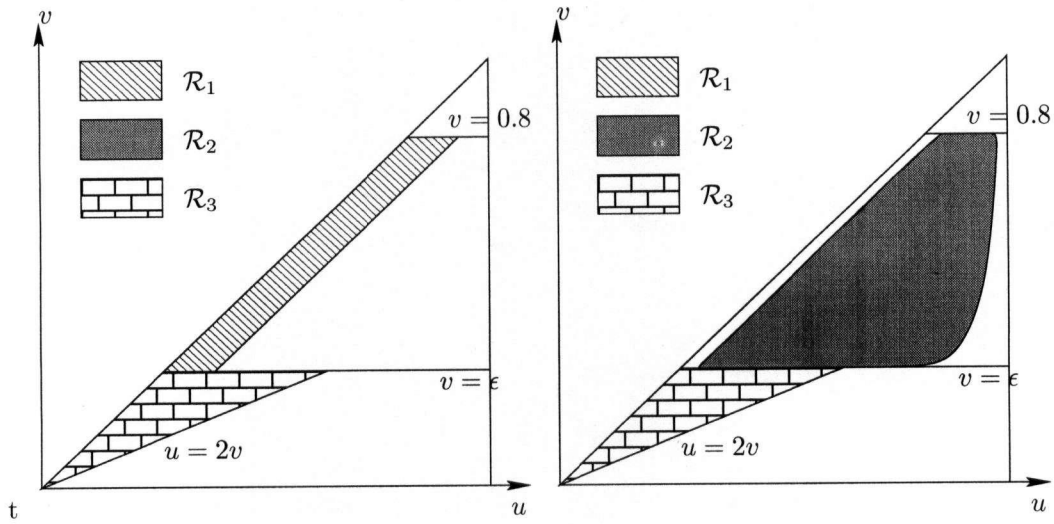
Later in the proof of Lemma 5.1.5, we will also define the follow three regions, which we include here for easy reference (See figure 5.4).

$$\begin{aligned}
\mathcal{R}_2 &= \{(u, v) \in \mathcal{R} : v + 0.02 < u < \frac{(1 + 2c)v}{1 + 2cv} - 0.04, v \in [\epsilon, 0.8]\}, \\
\mathcal{R}_2' &= \{(u, v) \in \mathcal{R}_2 : v + 0.08 < u < \frac{(1 + 2c)v}{1 + 2cv} - 0.04, v > 0.55\}, \\
\mathcal{R}_3 &= \{(u, v) \in \mathcal{R} : u \leq 2v, v \in [0, \epsilon)\}.
\end{aligned}$$

Notice that $\mathcal{R}_1' \subset \mathcal{R}_1$, $\mathcal{R}_2' \subset \mathcal{R}_2$, and $\mathcal{R}_3' \subset \mathcal{R}_3$.

To study the evolution of $\mathcal{F}_\eta$ of a line segment $\overline{OP}$ that passes through the origin O and has its top tip P in the region $\mathcal{R}_1' \cap \mathcal{R}_0$, we will define a new vector field $\xi = (\xi_1, \xi_2)$ for $(u, v) \in \mathcal{R}_1 \cup \mathcal{R}_3'$, such that $\xi_1 \leq \eta_1$ and $\xi_2 \leq \eta_2$ everywhere in $\mathcal{R}_1 \cup \mathcal{R}_3'$, which means that

$$\mathcal{F}_\xi^s(u, v) \leq \mathcal{F}_\eta^s(u, v) \tag{5.28}$$



(a) $\mathcal{R}_1$ and $\mathcal{R}_3$ (b) $\mathcal{R}_2$ and $\mathcal{R}_3$

Figure 5.4: Various regions.

in both u- and v- coordinates, for small s and all $(u, v) \in \mathcal{R}_1' \cup \mathcal{R}_3'$. Notice that $\mathcal{R}_1'$ is required to stay a finite distance left of the line $u - v = 0.1$, the right edge of the parallelogram $\mathcal{R}_1$; this is such that we can still control how much $(u, v) \in \mathcal{R}_1'$ moves under $\mathcal{F}_\xi$, even if it leaves $\mathcal{R}_1'$ and enters the strip $\mathcal{R}_1 \backslash \mathcal{R}_1'$. Also, we define ξ in $\mathcal{R}_1 \cup \mathcal{R}_3'$ because this is the region where the line segments $\overline{OP}$ lie. We only consider the region $\mathcal{R}_3'$ for (u, v) close to the origin, rather than the region

$$\{(u, v) \in \mathcal{R} : u - v \in [0, 0.1], v \in [0, \epsilon)\},$$

since top tip of the line segment b_s will be > 0.5, which implies that $u \leq 2v$ if $(u, v) \in \overline{OP}$.

Step 1: Defining ξ. We first define ξ on the diagonal line

$$\gamma_3 \cap \{(u, v) : v \leq 0.8\} = \{(u, v) : u = v, 0 \leq v \leq 0.8\},$$

where γ_3 is defined in (5.11). Let $\theta_0 \leq \frac{1}{2}\arctan(0.05)$ be a small angle such that a line passing through (v_0, v_0) with $v_0 > 0.55$ and making an angle of $-2\theta_0$ with the positive u-axis intersects the vertical line $u = 1$ above the horizontal line $v = 0.5$. Define

$$\delta = 0.1, \tag{5.29}$$

and pick F_1 to be large but

$$F_1 < \delta c. \tag{5.30}$$

We also define

$$\xi(0.8, 0.8) = (0.8F_1, 0.8(-2)), \tag{5.31}$$

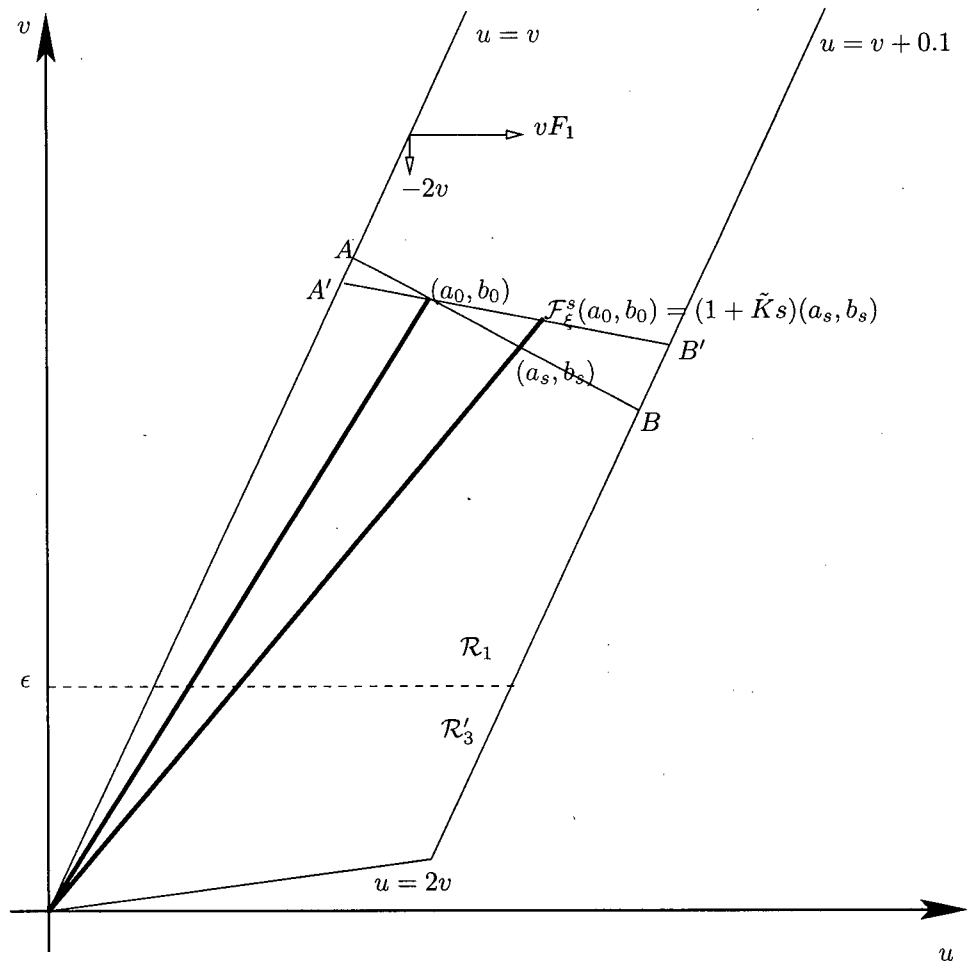


Figure 5.5: Illustration of Lemma 5.1.3 (Case 1)

where F_1 is large enough such that $\xi(0.8, 0.8)$ makes an angle of $-\theta_1$ with the positive u-axis, and $0 < \theta_1 < \theta_0$. This can be done for sufficiently large c. We define

$$\xi(v, v) = \frac{v}{0.8}\xi(0.8, 0.8) = (vF_1, -2v),$$

for $v \in [0, 0.8]$. This defines ξ on the line segment $\gamma_3 \cap (\mathcal{R}_1 \cup \mathcal{R}_3')$.

Finally, along straight lines that make angles of $-\theta_1$ with the positive u-axis, denoted $\overline{A'B'}(v, v)$, we define ξ to be equal to $\xi(v, v)$, i.e. for all $(u', v') \in \overline{A'B'}(v, v)$, we define

$$\xi(u', v') = \xi(v, v).$$

Here $A' = (v, v)$ is the point where $\overline{A'B'}(v, v)$ intersects the line $u = v$ and ξ has already been defined, and B' is the intersection point of $\overline{A'B'}(v, v)$ and the right/bottom edge of the 4-gon $\mathcal{R}_1 \cup \mathcal{R}_3'$, i.e. either the line $u = v + 0.1$ or the line $u = 2v$. Thus we have defined ξ on all points in $\mathcal{R}_1 \cup \mathcal{R}_3'$. To summarize, ξ in $\mathcal{R}_1$ is defined in a way such that:

1. On the line $u = v$ where $0 \le v \le 0.8$,

$$\xi = (vF_1, v(-2)), \tag{5.32}$$

such that ξ makes a small angle of $-\theta_1$ with the positive u-axis.

2. ξ is constant along lines that start at a point on the line $u = v$, and make angles of $-\theta_1$ with the positive u-axis, where

$$\theta_1 < \theta_0 < \frac{1}{2}\arctan(0.05). \tag{5.33}$$

Step 2: Verifying $\xi_1 \le \eta_1$. We divide into two sub-cases:

1. $\mathcal{R}_1$: $\epsilon \le v \le 0.8$;
2. $\mathcal{R}_3'$: $0 \le v < \epsilon$.

We first deal with sub-case 1. By the discussion following (5.17), the minimum horizontal distance between γ_1 and γ_3, defined in (5.9) and (5.11), is at least 0.19 for $v \in [\epsilon, 0.8]$ and sufficiently large c. Therefore the region $\mathcal{R}_1$ is more than $\delta = 0.1$ left of $\gamma_1 \cap \{(u,v) : \epsilon \le v \le 0.8\}$. Thus by (5.15),

$$\eta_1 > 2\delta c v \tag{5.34}$$

in $\mathcal{R}_1$. On the line segment $\gamma_3 \cap \mathcal{R}_1 = \{(u,v) : u = v, v \in [\epsilon, 0.8]\}$, ξ is defined by (5.32). Condition (5.30) then implies

$$\xi_1 = vF_1 < \delta c v.$$

Thus (5.34) shows that $\xi_1 \le \eta_1$ on the line segment $\gamma_3 \cap \mathcal{R}_1$.

We also need to verify that $\xi_1 \le \eta_1$ everywhere in $\mathcal{R}_1$. For that, recall that ξ in $\mathcal{R}_1$ is constant along line segments $\overline{A'B'}(v,v)$, each of which makes an angle of $-\theta_1$ with the positive u-axis, so we estimate how much the v-coordinate can decrease along the line segments $\overline{A'B'}$, to make sure that $\xi \le \eta$ even on the line $u = v + 0.1$. We observe the following: since $\tan\theta_1 < 0.05$, the amount by which the v-coordinate decreases, from the point $A'(v_0, v_0)$ on the line $u = v$ to the point B' on the line $u = v + 0.1$, is $0.1\tan\theta_1 \le \frac{\epsilon}{2} \le \frac{v_0}{2}$, where we will pick $\epsilon = 0.24$ in Chapter 5.1.2. Thus even for (u_0, v_0) lying on the line $u = v + 0.1$, we still have $\overline{A'B'}(u_0, v_0) = \overline{A'B'}(v_1, v_1)$ (i.e. (v_1, v_1) lies on the line $u = v$) for some v_1 with $v_1 < 2v_0$. Because ξ is constant along $\overline{A'B'}(u_0, v_0)$, we have

$$\xi_1(u_0, v_0) = v_1 F_1 < 2v_0 F_1 < 2\delta c v_0,$$

where we use the requirement (5.30) in the last inequality. Hence for all $(u_0, v_0) \in \mathcal{R}_1$,

$$\xi_1(u_0, v_0) < \eta_1(u_0, v_0)$$

by (5.34).

Now we deal with sub-case 2. For $v < \epsilon < \frac{1}{4}$ and $u \leq 2v$, we have

$$\begin{aligned} \eta_1 &= (2c(1-u)+1)v - u \geq (2c(1-2v)+1)v - 2v \\ &= (2c(1-2v)-1)v \geq cv \end{aligned} \tag{5.35}$$

if c is sufficiently large. This estimate applies to both Case 1 (this lemma) and a bit later on Case 2 (Lemma 5.1.5), and shows that on $\gamma_3 \cap \mathcal{R}_3' = \{(u,v) : u = v, v \in [0,\epsilon)\}$,

$$\xi_1 = vF_1 < \delta cv = 0.1cv < \eta_1,$$

where we use (5.29), (5.30), and (5.35) in the second, third, and fourth steps, respectively. For the rest of $\mathcal{R}_3'$, we make the observation that the v-coordinate of any point on $\overline{A'B'}(v_1,v_1)$ is larger than the v-coordinate v_2 of the intersection point of $\overline{A'B'}(v_1,v_1)$ and the line $u = 2v$, which we obtain by solving $v - v_1 = -\tan\theta_1(2v - v_1)$, i.e. $v_2 = \frac{1+\tan\theta_1}{1+2\tan\theta_1}v_1 > \frac{v_1}{2}$. Therefore for any $(u_0,v_0) \in \overline{A'B'}(v_1,v_1)$,

$$\xi_1(u_0,v_0) = v_1F_1 < 0.1cv_1 < 0.2cv_0 < \eta_1,$$

from which we conclude that $\xi_1 < \eta_1$ in $\mathcal{R}_3'$.

Step 3: Verifying $\xi_2 \leq \eta_2$. This is considerably easier than verifying $\xi_1 \leq \eta_1$. From (5.32), $\xi_2 = -2v$ on the line segment $\gamma_3 \cap (\mathcal{R}_1 \cup \mathcal{R}_3')$, so $\xi_2 \leq \eta_2$ by (5.13). Along $\overline{A'B'}(v_1,v_1)$, the v-coordinate decreases. So for any $(u_0,v_0) \in \overline{A'B'}(v_1,v_1)$,

$$\xi_2(u_0,v_0) = -2v_1 < -2v_0 < \eta_2.$$

Thus $\xi_2 < \eta_2$ in $\mathcal{R}_1 \cup \mathcal{R}_3'$.

Step 4: Defining a_s, b_s, and $\overline{AB}$. The vector field ξ is defined such that any point $(u_0,v_0) \in \mathcal{R}_1 \cup \mathcal{R}_3'$ moves under ξ at a constant speed (linear in v_0) along the line $\overline{A'B'}(u_0,v_0)$. Thus any line segment $\overline{OP}$ lying in $\mathcal{R}_1 \cup \mathcal{R}_3'$ remains a line segment (i.e. does not become a curve) under the flow ξ, and if (u_1,v_1) and (u_2,v_2) are two points on such a line segment, then the ratio $\frac{|\mathcal{F}_\xi^s(u_1,v_1)|}{|\mathcal{F}_\xi^s(u_2,v_2)|}$ remains constant. We define $\overline{AB}(v_1,v_1)$, $0 \leq v_1 \leq 0.8$, to be the line segment that makes an angle of $-2\theta_0$ with the positive u-axis and connects points $A = (v_1,v_1) \in \gamma_3$ and B, with B lying on the right/bottom boundary of the 4-gon $\mathcal{R}_1 \cup \mathcal{R}_3'$. Recall from (5.33) that θ_0 is a small angle and $\overline{A'B'}(a_0,b_0)$ makes an angle of $-\theta_1$ with the positive u-axis, where $0 < \theta_1 < \theta_0$. Thus the part of $\overline{A'B'}(a_0,b_0)$ to the right of the point (a_0,b_0) lies strictly above $\overline{AB}(a_0,b_0)$, and the angle between $\overline{A'B'}(a_0,b_0)$ and $\overline{AB}(a_0,b_0)$ is at least θ_0 by the choice of θ_0 and θ_1 in (5.33). Also, we define

$$(a_s,b_s) = \overline{AB}(a_0,b_0) \cap \overline{O\mathcal{F}_\xi^s(a_0,b_0)},$$

where $(a_0,b_0) \in \mathcal{R}_0 \cap \mathcal{R}_1'$ is the top tip of any line segment that we consider for this lemma. We collect various facts for later use:

1. $\mathcal{F}_\xi$ moves the point (a_0,b_0) to the right.
2. The part of $\overline{A'B'}(a_0,b_0)$ to the right of the point (a_0,b_0) lies strictly above $\overline{AB}(a_0,b_0)$.
3. (a_s,b_s) lies on $\overline{AB}(a_0,b_0)$.

4. $\mathcal{F}_\xi^s(a_0, b_0)$ lies on $\overline{A'B'}(a_0, b_0)$

5. (a_s, b_s) and $\mathcal{F}_\xi^s(a_0, b_0)$ both lie on $\overline{O\mathcal{F}_\xi^s(a_0, b_0)}$.

The above facts imply that b_s is a lower bound for $\pi_v(\mathcal{F}_\xi^s(a_0, b_0))$.

We now estimate the speed at which $\mathcal{F}_\xi^s(a_0, b_0)$ separates from (a_s, b_s). First of all, since $\overline{A'B'}(a_0, b_0)$ makes a negative angle with the positive u-axis, the intersection point (v_1, v_1) of $\overline{A'B'}(a_0, b_0)$ and the line $u = v$ must lie above (a_0, b_0), i.e.

$$v_1 \geq b_0.$$

This means that

$$|\xi(a_0, b_0)| = |\xi(v_1, v_1)| = |(v_1 F_1, -2v_1)| \geq b_0\sqrt{F_1^2 + 2^2}.$$

Let

$$F_2(b_0) = b_0\sqrt{F_1^2 + 2^2}.$$

Since $\overline{A'B'}(a_0, b_0)$ and $\overline{AB}(a_0, b_0)$ make angles of $-\theta_1$ and $-2\theta_0$ with the positive u-axis, respectively, where $0 < \theta_1 < \theta_0$, the angle θ_2 between $\overline{A'B'}(a_0, b_0)$ and $\overline{AB}(a_0, b_0)$ is larger than θ_0. Let $\alpha = sF_2(b_0)$ be the distance between (a_0, b_0) and $\mathcal{F}_\xi^s(a_0, b_0)$, and β be the distance between (a_0, b_0) and (a_s, b_s), then the Euclidean distance γ between $\mathcal{F}_\xi^s(a_0, b_0)$ and (a_s, b_s) (the thick line in figure 5.6) is $\gamma = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta\cos\theta_2}$, which attains the minimum $u\sin\theta_2$ when $\beta = \alpha\cos\theta_2$, therefore $\gamma \geq sF_2(b_0)\sin\theta_0$. Since s is small and (a_0, b_0) lies in $\mathcal{R}_0 \cap \mathcal{R}_1'$ (in particular, to the left of the line $u - v = 0.09$), $\mathcal{F}_\xi^s(a_0, b_0)$ lies in $\{(u, v) \in \mathcal{R} : u - v \in [0, 0.1], v \geq 0.5\}$, thus the smallest angle between $\overline{O\mathcal{F}_\xi^s(a_0, b_0)}$ (portion of which is $\overline{(a_s, b_s)\mathcal{F}_\xi^s(a_0, b_0)}$) and the positive u-axis is greater than $\arctan\frac{0.5}{0.6} > \frac{\pi}{6}$. Therefore the vertical distance between $\mathcal{F}_\xi^s(a_0, b_0)$ and (a_s, b_s) is at least $sF_2(b_0)\sin\theta_0\sin\frac{\pi}{6}$. Similarly, $\mathcal{F}_\xi^s(a_0, b_0)$ lies in $\mathcal{R}$, so the largest angle between $\overline{O\mathcal{F}_\xi^s(a_0, b_0)}$ and the positive u-axis is less than $\frac{\pi}{4}$, hence the horizontal distance between these two points is at least $sF_2(b_0)\sin\theta_0\cos\frac{\pi}{4}$, which is larger than $sF_2(b_0)\sin\theta_0\cos\frac{\pi}{3}$.

More precisely,

$$\begin{aligned}
\mathcal{F}_\xi^s(a_0, b_0) - (a_s, b_s) &\geq \left(sF_2(b_0)\sin\theta_0\cos\frac{\pi}{3}, sF_2(b_0)\sin\theta_0\sin\frac{\pi}{6}\right) \\
&= \left(\frac{s}{2}b_0\sqrt{F_1^2 + 2^2}\sin\theta_0, \frac{s}{2}b_0\sqrt{F_1^2 + 2^2}\sin\theta_0\right)
\end{aligned}$$

Since $b_s \leq b_0$, the above inequality implies

$$\mathcal{F}_\xi^s(a_0, b_0) - (a_s, b_s) \geq \left(\frac{s}{2}b_s\sqrt{F_1^2 + 2^2}\sin\theta_0, \frac{s}{2}b_s\sqrt{F_1^2 + 2^2}\sin\theta_0\right). \tag{5.36}$$

By the choice of a small θ_0 in (5.33), the entire line segment $\overline{AB}(a_0, b_0)$ lies above the horizontal line $v = 0.5$ if $(a_0, b_0) \in \mathcal{R}_0 \cap \mathcal{R}_1'$. Thus (a_s, b_s) lies above the horizontal line $v = 0.5$, which means that $a_s \leq 2b_s$. Thus (5.36) implies

$$\mathcal{F}_\xi^s(a_0, b_0) - (a_s, b_s) \geq \left(\frac{s}{4}a_s\sqrt{F_1^2 + 2^2}\sin\theta_0, \frac{s}{4}b_s\sqrt{F_1^2 + 2^2}\sin\theta_0\right). \tag{5.37}$$

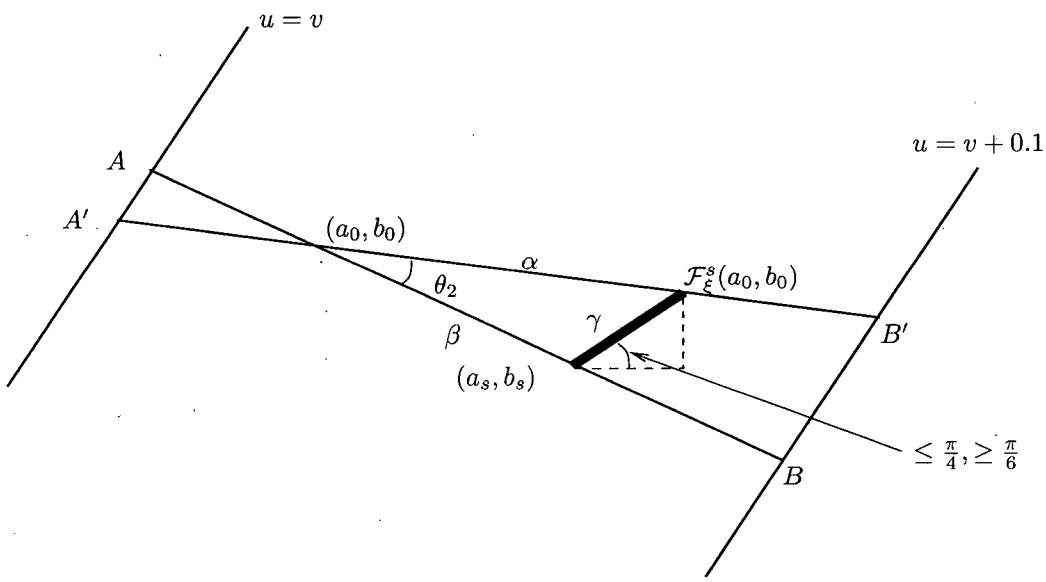


Figure 5.6: Another illustration of Lemma 5.1.3 (Case 1)

If we define

$$\tilde{K} = \frac{1}{4}\sqrt{F_1^2 + 2^2}\sin\theta_0,$$

then (5.28) and (5.37) verifies condition (5.20) for the point (a_0, b_0). We recall from (5.30) that F_1 can be chosen to be arbitrarily large if we allow c to be large. Therefore $\tilde{K}$ can also be chosen to be arbitrarily large if θ_0 is fixed.

For any other point $(\alpha a_0, \alpha b_0)$ on the line segment $\overline{O(a_0, b_0)}$, $\alpha \in [0, 1]$, notice that from (5.32), $\xi(\alpha a_0, \alpha b_0) = \alpha\xi(a_0, b_0)$, i.e. ξ is linear on the line $\overline{O(a_0, b_0)}$. Also, the entire line segment $\overline{O(a_0, b_0)}$ lies in $\mathcal{R}_1 \cup \mathcal{R}_3'$. Thus by (5.28), condition (5.20) holds for all $\alpha \in [0, 1]$. □

Proof of Lemma 5.1.5. We define

$$\mathcal{R}_2 = \{(u, v) \in \mathcal{R} : v + 0.02 < u < \frac{(1 + 2c)v}{1 + 2cv} - 0.04, v \in [\epsilon, 0.8]\}, \tag{5.38}$$

$$\mathcal{R}_2' = \{(u, v) \in \mathcal{R}_2 : v + 0.08 < u < \frac{(1 + 2c)v}{1 + 2cv} - 0.04, v > 0.55\}, \tag{5.39}$$

$$\mathcal{R}_3 = \{(u, v) \in \mathcal{R} : u \leq 2v, v \in [0, \epsilon)\}. \tag{5.40}$$

For this lemma, the region $\mathcal{R}_2'$ is the region for P, the top tip of the line segment $\overline{OP}$ that connects the origin O and the point P. Since we pick $\epsilon = 0.24$ later in Chapter 5.1.2, any line segment $\overline{OP}$, with $P \in \mathcal{R}_2'$, lies entirely in $\mathcal{R}_2 \cup \mathcal{R}_3$. We follow the same steps as in the proof of Lemma 5.1.3 (Case 1).

Step 1: Defining ζ. As in the the proof of Case 1, we will define a vector field ζ,

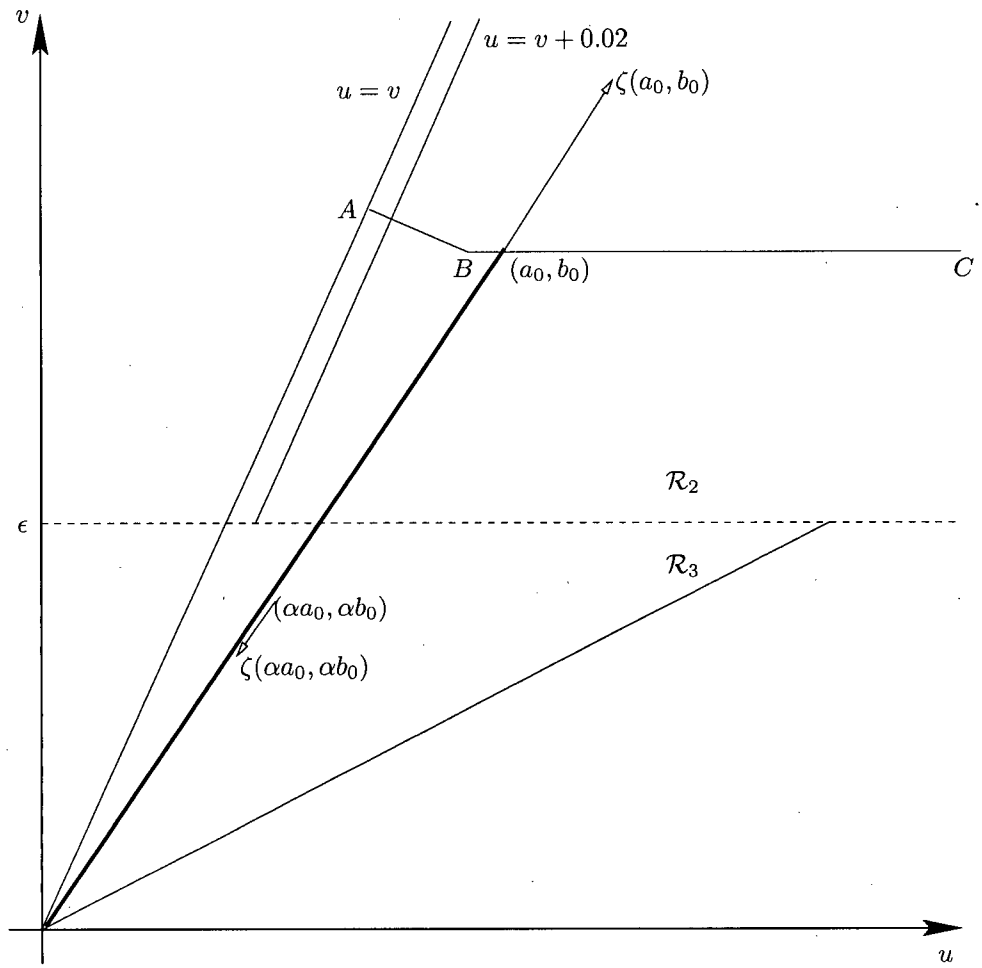


Figure 5.7: Illustration of Lemma 5.1.5 (Case 2): $\zeta(\alpha a_0, \alpha b_0) = (-2\alpha a_0, -2\alpha b_0)$ and $\zeta(a_0, b_0) = (a_0 F_3, b_0 F_3)$.

such that $\zeta_1 \leq \eta_1$ and $\zeta_2 \leq \eta_2$ everywhere in $\mathcal{R}_2 \cup R_3$. First, we define

$$\zeta(1,1) = (F_3, F_3),$$

where we pick F_3 large but with

$$0 < F_3 < 0.01c. \tag{5.41}$$

It is convenient to define ζ at the point $(1,1)$, even though this point is not even in $\mathcal{R}_2 \cup R_3$. For $(u,v) \in \mathcal{R}_2$, we define

$$\zeta(u,v) = (uF_3, vF_3). \tag{5.42}$$

And for $(u,v) \in \mathcal{R}_3$,

$$\zeta(u,v) = (u(-2), v(-2)). \tag{5.43}$$

Notice that ζ is discontinuous across the horizontal line $v = \epsilon$.

Step 2: Verifying $\zeta \leq \eta$. The region $\mathcal{R}_2$ is at least 0.01 to the right of γ_2 (the line $u = v + 2/c$) for sufficiently large c. By (5.14), we have

$$\eta_2 > 0.01cv \tag{5.44}$$

for all points in $\mathcal{R}_2$. The region $\mathcal{R}_2$ is also at least 0.04 left of γ_1, where γ_1 is the curve of $u = \frac{(1+2c)v}{1+2cv}$. Thus by (5.15), we have

$$\eta_1 > 0.08cv > 0.08c\epsilon = 0.08c(0.24) > 0.01c. \tag{5.45}$$

for all points in $\mathcal{R}_2$. By (5.41) and (5.42),

$$\zeta(u,v) < (0.01cu, 0.01cv) < (0.01c, 0.01cv),$$

which implies, by (5.44) and (5.45),

$$\zeta(u,v) < \eta(u,v)$$

for $(u,v) \in \mathcal{R}_2$.

Recall from (5.18) that the curve γ_1 approaches the degenerate curve

$$\{(u,v) : v = 0, u \in [0,1]\} \cup \{(u,v) : u = 1, v \in [0,1]\}$$

as $c \to \infty$. Therefore $\mathcal{R}_3$ stays to the left of γ_1 if c is sufficiently large, and by the discussion below (5.11) regarding the sign of η_1, $\eta_1 > 0$ for $(u,v) \in \mathcal{R}_3$. The definition of ζ in (5.43) says that $\zeta_1 < 0$ for $(u,v) \in \mathcal{R}_3$, therefore

$$\zeta_1(u,v) < \eta_1(u,v)$$

for $(u,v) \in \mathcal{R}_3$. Furthermore, (5.13) implies that $\zeta_2(u,v) < \eta_2(u,v)$. Thus for all $(u,v) \in \mathcal{R}_3$.

$$\zeta(u,v) < \eta(u,v).$$

Step 3: Defining $\overline{BC}$. We define $\overline{BC}(v_0, v_0)$ to be the *horizontal* line segment $v = v_1$ starting at point $B(v_0, v_0) = (v_1 + 0.1, v_1)$ on the line $u = v + 0.1$ and ending on the vertical line $u = 1$; notice that (v_0, v_0) itself does not lie on the line segment $\overline{BC}(v_0, v_0)$. This definition of $\overline{BC}(v_0, v_0)$ means that $B(v_0, v_0) = (v_1 + 0.1, v_1)$ is the right end point of the line segment $\overline{AB}(v_0, v_0)$, which was defined in Step 4 of the proof of Lemma 5.1.3. We also define

$$K = \frac{F_3}{2}. \tag{5.46}$$

Notice that K can be made arbitrarily large since F_3 (picked in (5.41)) is allowed to be arbitrarily large.

The vector field ζ is defined such that any point $(u,v) \in \mathcal{R}_2$ moves in the direction $\overrightarrow{O(u,v)}$, i.e. ζ is a dilation for points in $\mathcal{R}_2$. But any point $(u,v) \in \mathcal{R}_3$ moves in the direction of $\overrightarrow{(u,v)O}$, i.e. ζ is a contraction for points in $\mathcal{R}_3$. Thus any line segment $\overline{OP}$ with

$P \in \mathcal{R}_2'$ immediately splits into two line segments under ζ; the two line segments, however, lie on the same straight line through the origin O. For the top tip of the line segment $P = (a_0, b_0) \in \mathcal{R}_2'$, we have

$$\mathcal{F}_\zeta^s(a_0, b_0) - (a_0, b_0) \geq (sa_0F_3, sb_0F_3)\,, \tag{5.47}$$

since the fact that both u- and v- coordinates increases under ζ in $\mathcal{R}_2$ implies that sa_0F_3 and sb_0F_3 are lower bounds for the increase in u- and v- coordinates, respectively. Since $b_0 > 0.5$ if $(a_0, b_0) \in \mathcal{R}_2'$, (5.46) and (5.47) implies that

$$\mathcal{F}_\zeta^s(a_0, b_0) - (a_0, b_0) \geq \left(s\frac{a_0}{b_0}b_0F_3, sb_0F_3\right) \geq \left(\frac{a_0}{b_0}Ks, Ks\right). \tag{5.48}$$

This verifies (5.23) for $P = (a_0, b_0) \in \mathcal{R}_2'$. For any other point $(\alpha a_0, \alpha b_0)$ on $\overline{OP}$ that lies in $\mathcal{R}_2$ (i.e. $\alpha \in [\frac{\epsilon}{b_0}, 1]$), linearity in the definition $\zeta(\alpha a_0, \alpha b_0) = \alpha\zeta(a_0, b_0)$ in (5.42) implies (5.23).

The verification of (5.24) for points in $\mathcal{R}_3$ is similar. Recall from (5.43) the definition of ζ in $\mathcal{R}_3$:

$$\zeta(u, v) = (-2u, -2v).$$

Let $P = (a_0, b_0) \in \mathcal{R}_2'$ and $\alpha \in [0, \frac{\epsilon}{b_0})$. Then both αa_0 and αb_0 decrease under ζ, initially at speed $2\alpha a_0$ and $2\alpha b_0$, respectively. The speed of decrease immediately becomes smaller than $2\alpha a_0$ and $2\alpha b_0$ (respectively) after the initial movement. Thus $2\alpha a_0$ and $2\alpha b_0$ are upper bounds of the speed of decrease:

$$\mathcal{F}_\zeta^s(\alpha a_0, \alpha b_0) - (\alpha a_0, \alpha b_0) \geq (-2\alpha a_0 s, -2\alpha b_0 s)\,,$$

as required by (5.24). □

To summarize the results in Lemmas 5.1.3 (Case 1) and 5.1.5 (Case 2), if the top tip of the line segment $\overline{OP}$ at time 0, $P = (a_0, b_0)$ with $b_0 > 0.55$, lies to the left of the line $u = v + 0.09$, then we use Case 1 to define $(a_s, b_s) \in \overline{AB}(a_0, b_0)$ and $\xi \leq \eta$ where (a_s, b_s) is below but to the right of (a_0, b_0), such that ξ moves (a_0, b_0) at an arbitrarily large speed below and to the right of (a_0, b_0), but above (a_s, b_s). Once the top tip of the line segment has moved to the right of the line $u = v + 0.08$ but to the left of the curve $u = \frac{(1+2c)v}{1+2cv} - 0.04$, (or it lies between those two at time 0 to start with), then using Case 2, where $(a_s, b_s) \equiv (a_0, b_0)$, we define $\zeta \leq \eta$ such that ζ moves (a_0, b_0) above and to the right of (a_0, b_0), in fact, along the same direction as $\overrightarrow{O(a_0, b_0)}$, again at an arbitrarily large speed. Finally, if (a_0, b_0) lies to the right of the curve $u = \frac{(1+2c)v}{1+2cv} - 0.04$, then we move the initial condition to the left of this curve and apply Case 2.

5.1.2 Analysis of the PDE (5.1)

Now we use the results obtained in the previous section about the evolution of the ODE (5.2), together with some results on the heat equation, to study the evolution of the PDE (5.1). First, we need to characterize how values in the transition region evolve according to the heat equation. We will establish two technical lemmas to that end.

Lemma 5.1.6. *If l is fixed and $f = f_0$ is as defined in (IC 1-3) on page 34, then for*

$$x \in (-L-l-s, -L-\frac{l}{200}) \cup (L+\frac{l}{200}, L+l+s)$$

and s small, we have

$$e^{s\Delta} f(x) > f(x) + \frac{s}{5l^2}. \tag{5.49}$$

Proof. First, we shift f right by L such that $f(x) = h(x)$ for $x \in [-l, l]$ and $f(x) = h(2L-x)$ for $x \in [2L-l, 2L+l]$. Thus $\Delta f(x) = \Delta h(x)$ for $x \in (-l, l)$ and $\Delta f(x) = \Delta h(2L-x)$ for $x \in (2L-l, 2L+l)$, where

$$\Delta h(x) = \begin{cases} 0, & x < -l \\ \frac{1}{l^2}, & -l < x < 0 \\ -\frac{1}{l^2}, & 0 < x < l \\ 0 & x > l \end{cases}. \tag{5.50}$$

We make the following observation to aid our computation: if $u = e^{s\Delta} h$ gives the evolution of the heat equation with initial condition h, then $\Delta u = \Delta(e^{s\Delta} h)$ gives the evolution of the heat equation with initial condition Δh, i.e.

$$\Delta(e^{s\Delta} h) = e^{s\Delta}(\Delta h). \tag{5.51}$$

Define

$$k(x) = \Delta f(x),$$

then

$$k(x) = \begin{cases} \Delta h(x), & x \in (-l, l) \\ \Delta h(2L-x), & x \in (2L-l, 2L+l) \\ 0 & \text{otherwise} \end{cases}. \tag{5.52}$$

By equation (5.5.10) in [Taylor 1996], the solution of the heat equation can be expressed in terms of an integral.

$$e^{s\Delta} k(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi s}} e^{-\frac{y^2}{4s}} k(x-y) dy. \tag{5.53}$$

Using the above formula and the expression of k in (5.52), we can estimate $e^{s\Delta} k(x)$ for

$x \in (-\frac{5}{4}l, -l]$ and s small:

$$
\begin{aligned}
e^{s\Delta}k(x) &= \frac{1}{l^2}\left(\int_{-x-l}^{-x} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy - \int_{-x}^{-x+l} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy - \int_{-x-l+2L}^{-x+2L} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy \right.\\
&\qquad \left. + \int_{-x+2L}^{-x+l+2L} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy\right)\\
&\geq \frac{1}{l^2}\left(\int_{-x-l}^{\infty} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy - \int_{-x}^{\infty} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy - \int_{-x}^{-x+l} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy - \int_{-x-l+2L}^{-x+2L} \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy\right)\\
&\geq \frac{1}{l^2}\left(\int_{-x-l}^{\infty} \frac{1}{\sqrt{4\pi s}}e^{-\frac{y^2}{4s}}dy - 3\int_{-x}^{\infty} \frac{1}{\sqrt{4\pi s}}e^{-\frac{y^2}{4s}}dy\right)\\
&\geq \frac{1}{l^2}\left(\int_{|x|-l}^{\infty} \frac{1}{\sqrt{4\pi s}}e^{-\frac{y^2}{4s}}dy - 3\int_{l}^{\infty} \frac{1}{\sqrt{4\pi s}}e^{-\frac{y^2}{4s}}dy\right), \qquad (5.54)
\end{aligned}
$$

where in the last step we use the fact that $x \in (-\frac{5}{4}l, -l]$ implies $|x| = -x \geq l$. We can take s to be sufficiently small such that $\int_l^\infty \frac{e^{-\frac{y^2}{4s}}}{\sqrt{4\pi s}}dy < 10^{-5}/3$, then with a substitution of variable in the first integral in (5.54), we obtain

$$
\begin{aligned}
e^{s\Delta}k(x) &\geq \frac{1}{l^2}\left(\int_{(|x|-l)/\sqrt{s}}^{\infty} \frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}}dy - 10^{-5}\right),\\
&= \frac{1}{l^2}\left(\frac{1}{2} - 10^{-5} - \int_0^{(|x|-l)/\sqrt{s}} \frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}}dy\right). \qquad (5.55)
\end{aligned}
$$

If $x \in (-l-s, -l]$, then $|x| - l < s < \sqrt{s}$ if $s < 1$, and (5.55) implies

$$
e^{s\Delta}k(x) \geq \frac{1}{l^2}\left(\frac{1}{2} - 10^{-5} - \int_0^1 \frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}}dy\right) > \frac{1}{5l^2}. \qquad (5.56)
$$

On the other hand, for $x \in (-l, -\frac{l}{200})$ and s small, we also have

$$
e^{s\Delta}k(x) > \frac{1}{5l^2} \qquad (5.57)
$$

since for $x \in (-l, l)$, $k(x) = \Delta h(x)$ is a step function with discontinuity at 0, where Δh is given in (5.50).

Estimates (5.56) and (5.57) on the behaviour of k under the heat kernel implies that for s small and $x \in (-l-s, -\frac{l}{200})$,

$$
\frac{\partial e^{s\Delta}f(x)}{\partial s} = (\Delta(e^{s\Delta}f))(x) = (e^{s\Delta}(\Delta f))(x) > \frac{1}{5l^2},
$$

where we use (5.51) in the second equality. This establishes (5.49) for $x \in (-L-l-s, -L-\frac{l}{200})$. Verification of (5.49) for $x \in (L+\frac{l}{200}, L+l+s)$ is similar. □

Lemma 5.1.7. *Let $t > 0$ be fixed and $f_1 = f_0$ be as defined in (IC 1-3) on page 34, i.e.*

$$f_1(x) = \begin{cases} h(x+L), & x \in (-L-l, -L+l) \\ 1, & x \in [-L+l, L-l] \\ h(L-x), & x \in (L-l, L+l) \\ 0 & \textit{otherwise} \end{cases} . \tag{5.58}$$

Let

$$f_3(x) = \begin{cases} f_1(x) + mt, & -L-l-t < x < L+l+t \\ 0, & \textit{otherwise} \end{cases} \tag{5.59}$$

where $m > 0$. Then there exist positive constants δ_1 and δ_2 depending on m but independent of t such that if

$$f_2(x) = \begin{cases} (1+\delta_2 t)h(x+L+\delta_1 t), & x \in (-L-l-\delta_1 t, -L+l-\delta_1 t) \\ 1+\delta_2 t, & x \in [-L+l-\delta_1 t, L-l+\delta_1 t] \\ (1+\delta_2 t)h(L+\delta_1 t - x), & x \in (L-l+\delta_1 t, L+l+\delta_1 t) \\ 0, & \textit{otherwise} \end{cases} , \tag{5.60}$$

then $f_2 \le f_3$.

Proof. Without any loss of generality, assume $m < 1$. Let $M = 1 \wedge \sup_{x\in\mathbb{R}} |f_1'(x)|$, then $M = 1 \wedge 1/l = 1/l$ since l will be picked to be < 1 in (5.62) a bit later. Define

$$g_1(x) = \begin{cases} f_1(x + \frac{mt}{3M}), & x \in (-L-l-\frac{mt}{3M}, -L+l-\frac{mt}{3M}) \\ f_1(0), & x \in [-L+l-\frac{mt}{3M}, L-l+\frac{mt}{3M}] \\ f_1(x - \frac{mt}{3M}), & x \in (L-l+\frac{mt}{3M}, L+l+\frac{mt}{3M}) \\ 0, & \text{otherwise} \end{cases} ,$$

then any small piece of the curve of g_1 is f_1 shifted by either 0, $\frac{mt}{3M}$, or $-\frac{mt}{3M}$, with $\frac{mt}{3M} < t$. In particular, the two transition regions in g_1 are the two transition regions in f_1 shifted by $\frac{mt}{3M}$ or $-\frac{mt}{3M}$, and the "middle" region (i.e. the region sandwiched between the two transition regions) in g_1 is the middle region of f_1 expanded left and right by $\frac{mt}{3M}$. Since $M = \sup_{x\in\mathbb{R}} |f_1'(x)|$, we have

$$g_1(x) - f_1(x) \le \frac{mt}{3}$$

for all $x \in (-L-l-\frac{mt}{3M}, L+l+\frac{mt}{3M})$, therefore $f_3 \ge g_1$ everywhere and in particular, since $f_3(x) - f_1(x) = mt$ for $x \in (-L-l-t, L+l+t)$, we have

$$f_3(x) - g_1(x) \ge \frac{2mt}{3} \tag{5.61}$$

for $x \in (-L-l-\frac{mt}{3M}, L+l+\frac{mt}{3M})$.

Next we define

$$f_2(x) = \left(1 + \frac{mt}{3}\right) g_1(x).$$

Then

$$f_2(x) - g_1(x) = \frac{mt}{3} g_1(x) \leq \frac{mt}{3}$$

since $g_1(x) \leq 1$ everywhere. The above inequality and (5.61) imply that

$$f_2(x) < f_3(x)$$

for $x \in (-L - l - \frac{mt}{3M}, L + l + \frac{mt}{3M})$. Then $\delta_1 = \frac{m}{3M}$ and $\delta_2 = \frac{m}{3}$ satisfy the requirement of f_2 in (5.60), and the proof is complete. □

For the remainder of this section, we will establish Proposition 5.1.1. We assume that the initial condition of the PDE (5.1) is $(a_0 f_0, b_0 f_0)$, where f_0 is as defined in (IC 1-3) on page 34 and $(a_0, b_0) = (\tilde{a}_0, \tilde{b}_0)$ lies in the region $\mathcal{R}_0$ defined in (5.19). By Remark 5.1.4 and Lemma 5.1.5, we can pick $(a_s, b_s) \in \overline{ABC}(a_0, b_0)$, with $(a_s, b_s) = (a_0, b_0)$ if $a_0 - b_0 > 0.08$ (i.e. Lemma 5.1.5/Case 2), such that estimates (5.20)-(5.24) regarding the evolution of the ODE (5.2) are valid. We will use this, together with Lemma 5.1.6 at the beginning of this section, to show that there is a positive constant m such that for $x \in (-L - l - s, L + l + s)$ and sufficiently small s,

$$e^{s\Delta} \mathcal{F}_\eta^s (a_0 f_0, b_0 f_0)(x) - (a_s f_0, b_s f_0)(x) \geq (a_s ms, b_s ms).$$

Finally we will apply Lemma 5.1.7 to complete the proof of Proposition 5.1.1.

We divide this task into proving two lemmas, which correspond to the two cases in Lemmas 5.1.3 and 5.1.5, respectively. Before we proceed, we first pick

$$l = \sqrt{\frac{0.1}{3}} \tag{5.62}$$

and

$$\epsilon = 0.24 < 0.5h\left(-\frac{l}{100}\right). \tag{5.63}$$

Lemma 5.1.8. *(Case 1) Recall that*

$$\mathcal{R}_1' \cup \mathcal{R}_3' = \{(u, v) \in \mathcal{R} : u - v \in [0, 0.1] \text{ and } u \leq 2v \text{ for } v \in [0, \epsilon)\}$$

and

$$\mathcal{R}_1' \cap \mathcal{R}_0 = \{(u, v) \in \mathcal{R} : u - v \in [0, 0.09] \text{ and } v \in [0.55, 0.8]\}.$$

If $\{(a_0 f_0(x), b_0 f_0(x)) : x \in [-L - l, -L + l]\} \subset \mathcal{R}_1' \cup \mathcal{R}_3'$ and $(a_0, b_0) \in R_1' \cap \mathcal{R}_0$, where $\mathcal{R}_0$, $\mathcal{R}_1'$ and $\mathcal{R}_3'$ are defined in (5.19), (5.25), and (5.26) respectively, and f_0 is defined in (IC 1-3) on page 34, then the conclusion of Proposition 5.1.1 holds.

Proof. By Lemma 5.1.3 and Remark 5.1.4, for sufficiently small s, we can pick K and $\tilde{K}$ large enough such that

$$K > \frac{3}{l^2}, \tag{5.64}$$

and a point $(a_s, b_s) \in \mathcal{R}$ with $b_s > 0.5$ such that $(a_s f_0(0), b_s f(0)) \in \overline{AB}(a_0 f_0(0), b_0 f_0(0))$ and

$$\mathcal{F}_\eta^s(a_0 f_0(x), b_0 f_0(x)) \geq ((1+\tilde{K}s)a_s f_0(x), (1+\tilde{K}s)b_s f_0(x)) \tag{5.65}$$

for all x; furthermore, if $b_s f_0(x) \geq \epsilon$,

$$(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) > Ks, \tag{5.66}$$

and if $b_s f_0(x) \in [0, \epsilon)$,

$$(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) \geq 0. \tag{5.67}$$

Here, $(a_s f_0, b_s f_0)$ is the function to which we compare $\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)$ to see how much "progress" we are making in increasing the v-coordinate.

For $x \in \left[-L - \frac{l}{200}, L + \frac{l}{200}\right] = \left[-L - \frac{l}{200}, -L + l\right] \cup (-L+l, L-l) \cup \left[L - l, L + \frac{l}{200}\right]$, where the intervals $\left[-L - \frac{l}{200}, -L + l\right]$ and $\left[L - l, L + \frac{l}{200}\right]$ are in the transition region, we have

$$b_s f_0(x) \geq 0.5 f_0\left(-L - \frac{l}{200}\right) > 0.5 f_0\left(-L - \frac{l}{100}\right) = 0.5h\left(-\frac{l}{100}\right),$$

therefore by (5.63),

$$b_s f_0(x) > \epsilon. \tag{5.68}$$

Therefore by (5.66) we have, for $x \in [-L - \frac{l}{200}, L + \frac{l}{200}]$,

$$(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) > Ks.$$

For $x \in [-L - l, -L - \frac{l}{200}) \cup (L + \frac{l}{200}, L + l]$ where $b_s f_0$ is possibly smaller than ϵ, by (5.67), we have

$$(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) \geq 0.$$

To summarize, combining (5.65) and the two inequalities above, we have

$$\pi_v(\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))(x) \geq (1+\tilde{K}s)b_s f_0(x) \tag{5.69}$$

and

$$(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) \begin{cases} > Ks, & x \in [-L - \frac{l}{200}, L + \frac{l}{200}] \\ \geq 0, & x \in [-L - l, -L - \frac{l}{200}) \cup (L + \frac{l}{200}, L + l] \\ = 0, & x \notin [-L - l, L + l] \end{cases}. \tag{5.70}$$

As stated in (5.5), $|\Delta f_0| \leq \frac{1}{l^2}$. Therefore the heat operator $e^{s\Delta}$ applied to $b_s f_0$ may decrease its value by at most $\frac{b_s}{l^2}s$. More precisely,

$$e^{s\Delta} b_s f_0 - b_s f_0 \geq -\frac{b_s}{l^2}s. \tag{5.71}$$

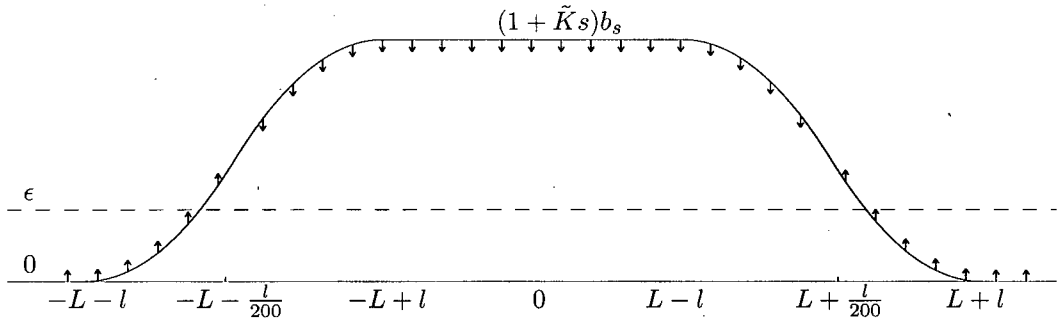


Figure 5.8: The effect of the heat kernel on the function $(1+\tilde{K}s)b_s f_0$. The arrows indicates whether $(1+\tilde{K}s)b_s f_0(x)$ increases or decreases. The effects illustrated here are lower bounds. In $[-L-\frac{l}{200}, L+\frac{l}{200}]$, the function decreases, which is why we need the first line (5.70) to be $> Ks$ to cancel out this decrease.

everywhere. We can use (5.70) and (5.71) to obtain estimates on $e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)$. We estimate the "progress" made after applying the heat kernel: by (5.69) and the monotonicity of the heat kernel $e^{s\Delta}$,

$$\pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0) - (a_s f_0, b_s f_0))(x) \geq e^{s\Delta}(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x).$$

For $x \in [-L-\frac{l}{200}, L+\frac{l}{200}]$,

$$\begin{aligned} & e^{s\Delta}(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) \\ = \quad & (1+\tilde{K}s)\left(e^{s\Delta}b_s f_0(x) - b_s f_0(x)\right) + \Big((1+\tilde{K}s)b_s f_0(x) - b_s f_0(x)\Big) \\ > \quad & -(1+\tilde{K}s)\frac{b_s}{l^2}s + \frac{3b_s}{l^2}s \end{aligned}$$

by (5.71), the first line of (5.70), and the fact $K > \frac{3}{l^2} > \frac{3b_s}{l^2}$. Therefore

$$\pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0) - (a_s f_0, b_s f_0))(x) > \frac{b_s}{l^2}s \tag{5.72}$$

for sufficiently small s. On the other hand, for $x \in (-L-l-s, -L-\frac{l}{200}) \cup (L+\frac{l}{200}, L+l+s)$, by Lemma 5.1.6 we have,

$$e^{s\Delta}(1+\tilde{K}s)b_s f_0(x) - (1+\tilde{K}s)b_s f_0(x) > (1+\tilde{K}s)\frac{b_s}{5l^2}s \geq \frac{b_s}{5l^2}s.$$

Therefore by (5.65) and the above inequality, for $x \in (-L-l-s, -L-\frac{l}{200}) \cup (L+\frac{l}{200}, L+l+s)$,

$$\begin{aligned} \pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0) - (a_s f_0, b_s f_0))(x) \quad & \geq \quad e^{s\Delta}(1+\tilde{K}s)b_s f_0(x) - b_s f_0(x) \\ & > \quad (1+\tilde{K}s)b_s f_0(x) + \frac{b_s}{5l^2}s - b_s f_0(x) \\ & > \quad \frac{b_s}{5l^2}s, \end{aligned}$$

where the last line is due to the second and third lines of (5.70). Hence for $x \in (-L-l-s, L+l+s)$,

$$\pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))(x) > b_s\left(f_0(x) + \frac{1}{5l^2}s\right).$$

Then Lemma 5.1.7 implies that there exist positive constants δ_1 and δ_2 independent of s such that

$$\pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0f_0, b_0f_0))(x) > b_sf_2(x), \tag{5.73}$$

where f_2 is defined in (5.60):

$$f_2(x) = \begin{cases} (1+\delta_2 s)h(x+L+\delta_1 s), & x \in (-L-l-\delta_1 s, -L+l-\delta_1 s) \\ 1+\delta_2 s, & x \in [-L+l-\delta_1 s, L-l+\delta_1 s] \\ (1+\delta_2 s)h(L+\delta_1 s - x), & x \in (L-l+\delta_1 s, L+l+\delta_1 s) \\ 0, & \text{otherwise} \end{cases}.$$

Similarly, the estimates in (5.72) to (5.73) also hold for the u-coordinate of $e^{s\Delta}\mathcal{F}_\eta^s(a_0f_0, b_0f_0) - (a_sf_0, b_sf_0)$, if b_s on the right hand side of each inequality is replaced by a_s. So for all $x \in (-L-l-s, L+l+s)$, we have

$$e^{s\Delta}\mathcal{F}_\eta^s(a_0f_0, b_0f_0)(x) > (a_sf_2(x), b_sf_2(x)),$$

as required. In particular, $(\tilde{a}_s, \tilde{b}_s)$ in the statement of Proposition 5.1.1 should be $((1+\delta_2 s)a_s, (1+\delta_2 s)b_s)$. □

Lemma 5.1.9. *(Case 2) Recall that*

$$\mathcal{R}_2' = \{(u,v) \in \mathcal{R}_2 : v + 0.08 < u < \frac{(1+2c)v}{1+2cv} - 0.04, v > 0.55\},$$

and

$$\mathcal{R}_2 \cup \mathcal{R}_3 = \{(u,v) \in \mathcal{R} : v + 0.02 < u < \frac{(1+2c)v}{1+2cv} - 0.04 \text{ for } v \in [\epsilon, 0.8] \text{ and } u \le 2v \text{ for } v \in [0,\epsilon)\}.$$

If $\{(a_0f_0(x), b_0f_0(x)) : x \in [-L-l, -L+l]\} \subset \mathcal{R}_2 \cup \mathcal{R}_3$ and $(a_0, b_0) \in R_2'$, where $\mathcal{R}_2$, R_2' and $\mathcal{R}_3$ are defined in (5.38), (5.39), and (5.40) respectively, and f_0 is defined in (IC 1-3) on page 34, then the conclusion of Proposition 5.1.1 holds.

Before we prove this lemma, we observe that the union of all the regions where (a_0, b_0) may lie is $(\mathcal{R}_1' \cap \mathcal{R}_0) \cup \mathcal{R}_2'$, which is exactly $\mathcal{R}_0$ as defined in (5.19). If $(a_0, b_0) \in \mathcal{R}_1' \cap \mathcal{R}_0$, then the part of the line segment $\overline{O(a_0, b_0)}$ ($\overline{O(a_0, b_0)}$ consists of values of (a_0f_0, b_0f_0)) above $y = \epsilon$ lies in $\mathcal{R}_1'$. On the other hand, if $(a_0, b_0) \in \mathcal{R}_2'$, then the part of the line segment $\overline{O(a_0, b_0)}$ above $y = \epsilon$ lies in $\mathcal{R}_2$. But in both these cases, for the part of the line segment $\overline{O(a_0, b_0)}$ below $y = \epsilon$, if suffices to consider $\mathcal{R}_3 = \{(u,v) \in \mathcal{R} : u \le 2v, v \in [0,\epsilon)\}$, because the top tip of $\overline{O(a_0, b_0)}$ in the (u,v)-plane lies above the horizontal line $v = 0.5$, where $u \le 1 \le 2v$. The sufficiency of restricting to $\{(u,v) \in \mathcal{R} : u < \frac{(1+2c)v}{1+2cv} - 0.04\}$ has been discussed below (5.19) on page 39.

Proof of Lemma 5.1.9. Under this case, the line segment formed by $\{(a_0f_0(x), b_0f_0(x)) : x \in [-L-l, -L+l]\}$ lies in $\mathcal{R}_2 \cup R_3$, i.e. the portion of the line segment

above the horizontal line $v = \epsilon$ lies in $\mathcal{R}_2$ and right of the line $u - v = 0.02$, and the portion below $v = \epsilon$ lies in $\mathcal{R}_3$. Furthermore, the top tip (a_0, b_0) lies in $\mathcal{R}_2'$, i.e. to the right of the line $u - v = 0.08$ and above the horizontal line $v = 0.55$. For $x \in [-L - \frac{l}{100}, L + \frac{l}{100}]$, we have

$$b_0 f_0(x) > 0.55 f_0\left(-L - \frac{l}{100}\right) > 0.5h\left(-\frac{l}{100}\right) > \epsilon$$

by (5.63). Therefore, by Lemma 5.1.5, we can construct functions g_2 and g_3:

$$g_2(x) = \begin{cases} b_0 f_0(x) + Ks, & x \in [-L - \frac{l}{100}, L + \frac{l}{100}] \\ b_0 f_0(x)(1-2s), & x \in (-L - l, -L - \frac{l}{100}) \cup (L + \frac{l}{100}, L + l) \\ 0, & \text{otherwise} \end{cases}, \tag{5.74}$$

and

$$g_3(x) = \begin{cases} b_0 f_0(x) + Ks, & x \in [-L - \frac{l}{200}, L + \frac{l}{200}] \\ b_0 f_0(x)(1-2s), & x \in (-L - l, -L - \frac{l}{200}) \cup (L + \frac{l}{200}, L + l) \\ 0, & \text{otherwise} \end{cases},$$

such that both $(\frac{a_0}{b_0} g_2, g_2)$ and $(\frac{a_0}{b_0} g_3, g_3)$ are lower bounds of $\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)$. Notice that $g_3 \leq g_2$ everywhere, and g_2 has discontinuities at $-L - l/100$ and $L + l/100$, while g_3 has discontinuities at $-L - l/200$ and $L + l/200$. See figure 5.9 for graphs of g_2 and g_3.

Now we construct the function g_4:

$$g_4(x) = \begin{cases} g_2(x), & x \in [-L - \frac{l}{200}, L + \frac{l}{200}] \\ g_3(x), & x \in (-\infty, -L - \frac{l}{100}] \cup [L + \frac{l}{100}, \infty) \end{cases}, \tag{5.75}$$

furthermore g_4 is required to be C^∞, monotone in $[-L - \frac{l}{200}, -L - \frac{l}{100}] \cup [L + \frac{l}{100}, L + \frac{l}{200}]$, and lying between g_2 and g_3, with

$$|\Delta g_4(x)| < \frac{2}{l^2} \tag{5.76}$$

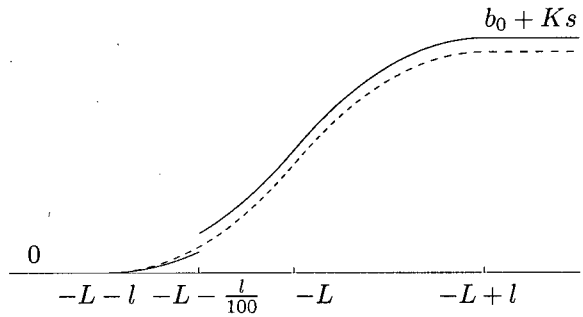
everywhere. For $x \notin [-L - \frac{l}{200}, -L - \frac{l}{100}] \cup [L + \frac{l}{100}, L + \frac{l}{200}]$, the last requirement is automatic by (5.5); but for $x \in [-L - \frac{l}{200}, -L - \frac{l}{100}] \cup [L + \frac{l}{100}, L + \frac{l}{200}]$, it can be achieved for sufficiently small s. Notice that since $g_4 \leq g_2$, $(\frac{a_0}{b_0} g_4, g_4)$ is a lower bound of $\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)$. Furthermore, for $x \in [-L - \frac{l}{200}, L + \frac{l}{200}]$,

$$g_4(x) - b_0 f_0(x) = Ks. \tag{5.77}$$

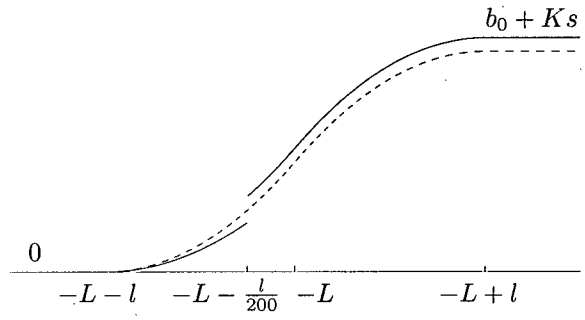
Here $(a_0 f_0, b_0 f_0)$ is the function to which we compare $\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)$ to see how much "progress" we are making increasing the v-coordinate.

We now turn to evolution according to the heat equation. First we deal with $x \notin [-L - \frac{l}{200}, L + \frac{l}{200}]$. For this, we use g_3 as the lower bound for $\pi_v(\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))$. We observe that g_3 dominates $(1 - 2s) b_0 f_0$, therefore monotonicity of the heat kernel implies

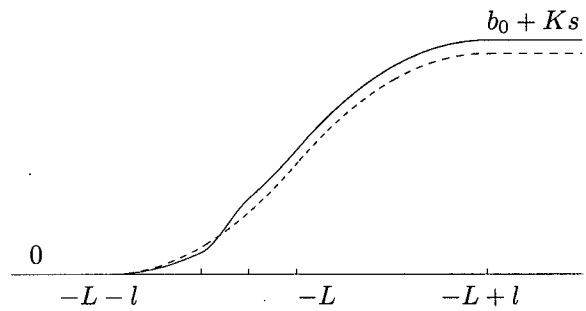
$$e^{s\Delta} g_3 \geq e^{s\Delta}((1 - 2s) b_0 f_0) = (1 - 2s) b_0 e^{s\Delta} f_0.$$



(a) The function g_2



(b) The function g_3



(c) The function g_4

Figure 5.9: The functions g_2, g_3, and g_4; dotted lines denote the function $b_0 f_0$.

By Lemma 5.1.6, $e^{s\Delta} f_0(x) > f_0(x) + \frac{s}{5l^2}$ for all $x \in (-L-l-s, -L-\frac{l}{200}) \cup (L+\frac{l}{200}, L+l+s)$, therefore

$$e^{s\Delta} g_3(x) > (1-2s) b_0 \left(f_0(x) + \frac{s}{5l^2} \right).$$

Since $b_0 > 0.55$, we have $(1-2s)b_0 > 0.5$ for sufficiently small s. Also recall that we pick

$l = \sqrt{\frac{3}{0.1}}$ in (5.62) such that $\frac{1}{l^2} = \frac{3}{0.1}$. Thus the inequality above can be written as

$$e^{s\Delta} g_3(x) > (1-2s)b_0 f_0(x) + 0.5\frac{3s}{0.5} = b_0 f_0(x) + (3 - 2b_0 f_0(x))s.$$

Finally, since $3 - 2b_0 f_0 > 3 - 2 = 1$, we have

$$e^{s\Delta} g_3(x) > b_0 f_0(x) + s. \tag{5.78}$$

for all $x \in (-L-l-s, -L-\frac{l}{200}) \cup (L+\frac{l}{200}, L+l+s)$.

For $x \in [-L-\frac{l}{200}, L+\frac{l}{200}]$, we use g_4 as the lower bound for $\pi_v(\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))$. By (5.76), the heat operator $e^{s\Delta}$ may decrease values of $g_4(x)$ by at most $\frac{2}{l^2}s$, i.e.

$$e^{s\Delta} g_4(x) - g_4(x) \geq -\frac{2}{l^2}s. \tag{5.79}$$

Therefore for $x \in [-L-\frac{l}{200}, L+\frac{l}{200}]$, we have

$$e^{s\Delta} g_4(x) - b_0 f_0(x) = (e^{s\Delta} g_4(x) - g_4(x)) + (g_4(x) - b_0 f_0(x)) \geq -\frac{2}{l^2}s + Ks,$$

where we apply (5.79) to $e^{s\Delta} g_4(x) - g_4(x)$ and (5.77) to $g_4(x) - b_0 f_0(x)$. Thus for $x \in [-L-\frac{l}{200}, L+\frac{l}{200}]$, we have

$$e^{s\Delta} g_4(x) - b_0 f_0(x) > \frac{1}{l^2}s$$

since K is chosen to be larger than $3/l^2$ in (5.64).

The estimates in (5.78) and (5.80), together with the fact that g_3 and g_4 are lower bounds of $\pi_v(\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))(x)$, imply that there is a positive constant m, such that for $x \in (-L-l-s, L+l+s)$,

$$\pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))(x) > b_0 f_0(x) + s \geq b_0(f_0(x) + s).$$

As in Lemma 5.1.8, we apply Lemma 5.1.7 to obtain the estimate

$$\pi_v(e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0))(x) > b_0 f_2(x), \tag{5.80}$$

where f_2 is defined in (5.60).

For the u-coordinate of $e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)$, we can obtain estimates (5.78) to (5.80) if we replace b_0 on the right hand side of each inequality by a_0. So we conclude that for all $x \in (-L-l-s, L+l+s)$,

$$e^{s\Delta}\mathcal{F}_\eta^s(a_0 f_0, b_0 f_0)(x) > (a_0 f_2(x), b_0 f_2(x)),$$

as required. In particular, $(\tilde{a}_s, \tilde{b}_s)$ in the statement of Proposition 5.1.1 should be $((1+\delta_2 s)a_0, (1+\delta_2 s)b_0)$. □

Proof of Proposition 5.1.1. The proposition follows from Lemmas 5.1.8 and 5.1.9, and the discussion below (5.19) on page 39 regarding the sufficiency of restricting the region for (a_0, b_0) to $\mathcal{R}_0$. □

5.2 Upper Bounds: Existence of d_2 and D_2 in Condition $(*)$

We establish the following proposition, which, together with Corollary 5.1.2, verifies condition $(*)$ on page 31. As in Corollary 5.1.2, the "L" in condition $(*)$ is picked to be $L - l$.

Proposition 5.2.1. *If c is sufficiently large, then there exist constants $d_2 < D_2 < 1$ and T such that if $v_0(x) < D_2$ for $x \in [-L + l, L - l]$ then $v_t(x) < d_2$ for $x \in [-3L, 3L]$, where (u_t, v_t) solves the PDE (5.1).*

Proof. Because of the monotonicity of the PDE (5.1), it suffices to pick a uniform initial condition

$$\begin{aligned} u_0 &\equiv \text{some } \tilde{u}, \\ v_0 &\equiv D_2, \end{aligned}$$

and show that at time T,

$$\begin{aligned} u_T &\equiv \text{some } \tilde{u}, \\ v_T &< d_2. \end{aligned}$$

Therefore we need only concern ourselves with the ODE (5.2). We can bound $\eta_2(u, v)$ defined in (5.8) for any $v > 1 - \frac{1}{c}$ as follows:

$$\eta_2(u, v) = (c(u - v) - 2)v \leq \left(c\left(1 - \left(1 - \frac{1}{c}\right)\right) - 2\right)v = -v < -\left(1 - \frac{1}{c}\right) < 0$$

if $c > 1$. Thus for any two numbers D_2 and d_2 that satisfy $1 > D_2 > d_2 > 1 - \frac{1}{c}$, there exists T, such that if $v_0 \equiv D_2$, then $v_T < d_2$. $\square$

Part II

Stationary Distributions of A Model of Sympatric Speciation

Chapter 6

A Model on Sympatric Speciation

6.1 Introduction

Understanding Speciation is one of the great problems in the field of evolution. According to Mayr [Mayr 1963], speciation means the splitting of a single species into several, that is, the multiplication of species. It is believed that many species originated through geographically isolated populations of the same ancestral species [Dieckmann and Doebeli 1999]. This phenomenon is relatively easy to understand. In contrast, sympatric speciation, in which new species arise without geographical isolation, is theoretically much more difficult.

6.1.1 The Dieckmann-Doebeli Model

Dieckmann and Doebeli [Dieckmann and Doebeli 1999] proposed a general model for sympatric speciation, for both asexual and sexual populations. We will describe their model for the asexual population first. Each individual in the population is assumed to have a quantitative character (phenotype) $x \in \mathbb{R}$ determining how effectively this individual can make use of resources in the surrounding environment. A typical example is the beak size of a certain bird species, which determines the size of seeds that can be consumed by an individual bird. The function $K : \mathbb{R} \to \mathbb{R}^+$ (carrying capacity) is associated with the surrounding environment, where K_x denotes the number of individuals of phenotype x that can be supported by the environment. For example, since birds with small beak size (say x_1) are more adapted to eating small seeds than birds with large beak size (say x_2, $x_2 > x_1$), K_{x_1} will be larger than K_{x_2} if the surrounding environment produces more small seeds than large seeds. In the Dieckmann-Doebeli model, K_x is taken to be $c\exp(-\frac{(x-\hat{x})^2}{2\sigma_K^2})$. Moreover, every pair of individuals compete at an intensity determined by the phenotypical distance of these two individuals. More specifically, an individual of phenotype x_1 competes with an individual of phenotype x_2 at intensity $C_{x_1-x_2}$, where $C_x = \exp(-\frac{x^2}{2\sigma_C^2})$. Therefore each individual in the population interacts with the environment via the carrying capacity K, and interacts with the population via the competition kernel C.

Let $N_x(t)$ denote the number of individuals with phenotype x at time t. At any time, an individual of phenotype x gives birth at a constant rate, and dies at a rate proportional to $\frac{(C*N.(t))_x}{K_x}$, i.e. inversely proportional to the x-carrying capacity, but proportional to the intensity of competition exerted by the population on phenotype x, the numerator $(C * N.(t))_x = \sum_y C_{x-y} N_y(t)$ being how much competition (from every individual in the population) individuals with phenotype x suffer. In addition, every time an individual gives birth, there is a small probability that a mutation occurs and the phenotype of the offspring is different from that of the parent; in this case, the phenotypical distance between the offspring and the parent is then random and assumed to have a Gaussian distribution.

Since the number of individuals of a certain phenotype increases via the birth mechanism at a linear rate, but decreases via the death mechanism at a quadratic rate, extinction of all phenotypes will occur in finite time with probability one, i.e. $N \equiv 0$ eventually. For large initial populations, however, extinction will happen far enough into the future that interesting behaviour does arise before the population becomes extinct.

Monte-Carlo simulations, shown in figure 6.1, give a fairly good idea of the behaviour of the Dieckmann-Doebeli model for asexual populations. If the initial population is monomorphic ($t = 1$ in figure 6.1), i.e. concentrated near a certain phenotype x_0 ($\frac{N.(0)}{\sum_x N_x(0)} \approx \delta_{x_0}$), then the entire population first moves ($t = 30, 100, 200$ in figure 6.1) toward $\hat{x}$, the phenotype with maximum carrying capacity. If $\sigma_C > \sigma_K$ (this includes the case $\sigma_C = \infty$, i.e. equal competition between all phenotypes), then the population stabilizes near phenotype $\hat{x}$. But if $\sigma_C < \sigma_K$, then the monomorphic population concentrated at phenotype $\hat{x}$ splits into two groups, one group concentrating on a phenotype $< \hat{x}$, while the other concentrating on a phenotype $> \hat{x}$ ($t = 330, 370, 400, 500$ in figure 6.1). In the latter case, one can say that one species has evolved into two distinct species.

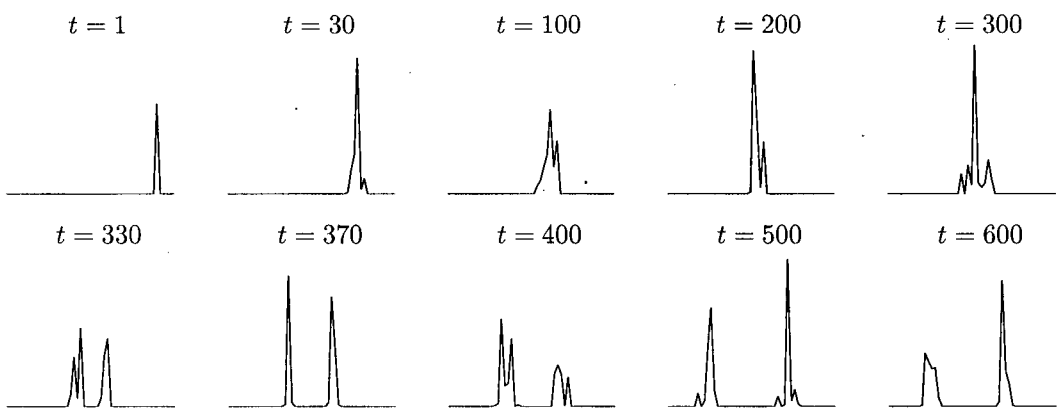


Figure 6.1: Simulation of the Dieckmann-Doebeli model with $E = [-50, 50] \cap \mathbb{Z}$, $\sigma_K = \sqrt{1000}$, and $\sigma_C = \sqrt{600}$.

We now give a qualitative description of the Dieckmann-Doebeli model for sexual populations. Each individual in a sexual population is assigned three diploid genotypes with (say) five diallelic loci each. The first set of loci determines the ecological character x (i.e. phenotype in the asexual model). The second set of loci determines the marker trait, which

is ecologically neutral, i.e. individuals with different marker traits but the same ecological character have exactly the same birth and death rates. The third set of loci determines mating probabilities m; if $m > 0$, then such individuals prefer to mate with individuals of similar phenotypes; if $m = 0$, then such individuals have no preference; and if $m < 0$, then such individuals prefer to mate with individuals of a distant phenotype. In addition, $|m|$ determines the strength of this preference.

The birth rates and the death rates are calculated the same way as in the asexual model; in particular, only information from the first set of loci is used to calculate these rates, as this is the only genotype that determines the phenotype of the individual. Dieckman and Doebeli considered two cases in their sexual model: 1. mating depends on the ecological character; and 2. mating depends on the ecologically neutral marker trait. For example, in the second case, individuals with $m > 0$ prefers to mate with individuals of similar marker traits.

Monte-Carlo simulations show that case 1 of the sexual model exhibits very similar behaviour to the asexual model, i.e. speciation if $\sigma_C < \sigma_K$ and no speciation if $\sigma_C > \sigma_K$. A caveat: if $\sigma_C < \sigma_K$, then only individuals, who prefer to mate with individuals of similar phenotypes, survive after the population splits into two groups. Hence in the end, there are two distinct groups of individuals who refuse to mate with individuals from the other group. For case 2 of the sexual model, Monte-Carlo simulations indicate that $\sigma_C < \sigma_K$ is not enough for speciation to occur. In this case, $\sigma_C < c\sigma_K$ is needed, where $c < 1$ is a constant.

As the sexual model exhibits similar behaviour to the asexual model, we will concentrate on the analysis of the simpler asexual model. It is our hope that understanding the asexual model will give some insights in explaining the behaviour of the sexual model as well.

6.1.2 A conditioned Dieckmann-Doebeli model

Although the Dieckmann-Doebeli model for asexual populations is considerably less complicated than their model for sexual populations, it still seems too complicated for rigorous analysis. Thus we will attempt to simplify the model while preserving its key ingredients. Henceforth we refer to the Dieckmann-Doebeli model for asexual populations simply as the Dieckmann-Doebeli model.

Before we describe our simplified Dieckmann-Doebeli model, we first introduce the concepts of *fitness* and *selection*. Selection occurs when individuals of different genotypes leave different numbers of offspring because their probabilities of surviving to reproductive age are different [Bürger 2000]. If we define fitness to be a measure of how likely a particular individual produces offspring that will survive to reproductive age, then individuals with higher fitness should have higher probability of being selected for reproduction. Along these lines, it is natural to define fitness of a phenotype as the difference between the birth rate and the death rate of individuals of this phenotype. It is also natural to require the fitness function to be bounded between 0 and 1.

The key feature of the Dieckmann-Doebeli model is that each individual has a fitness that depends on both the carrying capacity associated with its phenotype and the

configuration of the entire population. More specifically, the fitness of a phenotype x is an increasing function of K_x, the carrying capacity, but a decreasing function of $(C * N)_x$, the competition it suffers. Here N_x is the number of individuals of phenotype x.

In the Dieckmann-Doebeli model, the number of individuals can fluctuate with time. As mentioned before, since the birth rate is linear but the death rate is quadratic, extinction will occur in finite time with probability one, which makes it somewhat meaningless to analyze the equilibrium behaviour of the system. We make the assumption that the number of individuals N is fixed over time, reflecting a constant carrying capacity of the overall population. The mechanism by which we achieve this is to require that death of an individual and birth of its single offspring occur at the same time, called *replacement sampling* in Moran particle models [Dawson 1993, Chapter 2.5]. This way, the number of individuals remains constant, and analyzing the behaviour of the population is then equivalent to analyzing the empirical distribution

$$\pi^N = \frac{1}{N}\sum_{n=1}^{N} \delta_{x_n},$$

where x_n, $n = 1 \ldots N$, denotes the phenotype of the nth individual in a population of size N.

Before we describe our simplified Dieckmann-Doebeli model, we say a few words about our terminologies and notations: we refer to individuals in a population as "particles", and sometimes refer to a phenotype as a "site". We consider multiple models, both discrete-time and continuous-time; for discrete-time models, we use V to denote the fitness function; but for continuous-time models, we use m instead. Our simplified discrete-time Dieckmann-Doebeli model is as follows:

1. $E = [-L, L] \cap \mathbb{Z}$ is the phenotype space, and $\pi, \pi^N \in \mathcal{P}(E)$ is a probability measure on E;

2. $K : E \to [0, 1]$ is the carrying capacity, and $C : \mathbb{Z} \to \mathbb{R}^+$ is the competition kernel;

3. $V_x(\pi)$ is the fitness of phenotype x in a population with empirical distribution π (sometimes we notationally suppress the dependence on π); we define two possible fitness functions below;

4. A is a Markov transition matrix associated with mutation, with $A(y, x)$ denoting the probability of a particle of phenotype y mutating to a particle of phenotype x;

5. At every time step $t \in \mathbb{Z}^+$, the entire population is replaced by a new population of N particles, each particle chosen independently, according to the distribution $p.(t, \pi^N)$:

$$p_x(t, \pi^N) = \sum_y A(y, x) \frac{\pi_y^N(t) V_y(\pi^N(t))}{\sum_z \pi_z^N(t) V_z(\pi^N(t))} \tag{6.1}$$

In (6.1), the denominator $\sum_z \pi_z^N(t) V_z(\pi^N(t))$ is simply the normalization factor such that $\sum_x p_x(t, \pi^N) = 1$. In words, at every time step, the entire population dies and is replaced by

a new population, each individual x choosing an individual x' from the original population as its parent with a probability proportional to its fitness $V_{x'}$, after which the new individual x undergoes mutation according to A.

We consider two fitness functions:

$$\begin{aligned} V_x^{(1)}(\pi) &= 0 \vee \left(1 - \frac{\sum_z C_{x-z}\pi_z}{K_x}\right), \\ V_x^{(2)}(\pi) &= \frac{K_x}{\sum_z C_{x-z}\pi_z}. \end{aligned} \tag{6.2}$$

Each of the two fitness function defined above is an increasing function of K_x and a decreasing function of $(C * \pi)_x$. $V^{(1)}$ resembles more closely the original Dieckmann-Doebeli model, but it has the disadvantage of being in a more complicated form than $V^{(2)}$ and it is also not differentiable.

By Theorem 1 in [Del Moral 1998], which we state below, $\{\pi_t^N, t \in [0,T]\} \Rightarrow \{\pi_t, t \in [0,T]\}$ as $N \to \infty$, where $\Rightarrow$ denotes weak convergence and π_t evolves according to the following *deterministic* dynamical system:

$$\pi_x(t+1) = \sum_y A(y,x) \frac{\pi_y(t)V_y(\pi(t))}{\sum_z \pi_z(t)V_z(\pi(t))}. \tag{6.3}$$

Theorem 6.1.1. *Suppose E is compact and M is a Feller-Markov transition kernel on $\mathcal{P}(E)$, i.e. $M : \mathcal{P}(\mathcal{P}(E)) \to \mathcal{P}(\mathcal{P}(E))$. Let $M^{(N)} = MC_N$, where C_N is a Markov transition kernel on $\mathcal{P}(E)$ given by*

$$C_N F(\pi) = \int_{E^N} F\left(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\right) \pi(dx_1)\ldots\pi(dx_N),$$

i.e. a probability measure π replaced with an empirical measure π^N formed by N particles chosen independently according to π. Then

$$\left(M^{(N)}\right)^n \to M^n$$

as $N \to \infty$.

This theorem is easy to understand: take $n = 1$, then it says that the mapping MC_N converges to M, i.e. changing the input measure of M by an empirical measure of N particles makes almost no difference if N is large.

Analyzing the dynamical system (6.3) is not easy, partly because it is of a complicated form that is nonlinear in π, and we cannot find any Lyapunov function that associates with (6.3). A continuously differentiable function $V : U \to \mathbb{R}$ is called a Lyapunov function if V is nondecreasing (or nonincreasing) along orbits. For a discrete-time dynamical system such as (6.3), this means that $V(\pi(t+1)) - V(\pi(t)) \geq 0$ (or ≤ 0) for all $t \geq 0$. For a continuous-time dynamical system, it means that $\partial_t V(\pi(t)) \geq 0$ (or ≤ 0). Simulations of (6.3), however, seem to display some interesting behaviour, which we will describe after carrying out some non-rigorous analysis of (6.3).

Without mutation, any site x with $\pi_x = 0$ at any time τ will stay 0 for all $t \geq \tau$. Mutation enables individuals of phenotype x to be born in future generations even if there are no individuals of phenotype x in the present generation. But if we start with a polymorphic initial measure, i.e. $\pi_x(0) \neq 0$ for all x, then adding small mutation to the system should not cause significant changes in the behaviour of (6.3). Therefore we assume that $A = I$ and $\pi(0)$ is polymorphic. In this case, (6.3) can be simplified to

$$\pi_x(t+1) = \frac{\pi_x(t)V_x(\pi(t))}{\sum_z \pi_z(t)V_z(\pi(t))}.$$

Thus if $A = I$, then $\hat{\pi}$ is a stationary distribution of (6.3) if and only if

$$\hat{\pi}_x = \frac{1}{c}\hat{\pi}_x V_x(\hat{\pi}) \tag{6.4}$$

for some constant c. Condition (6.4) is equivalent to

$$V_x(\hat{\pi}) = c \text{ for all } x \text{ where } \hat{\pi}(x) \neq 0, \tag{6.5}$$

Let K and C be in the form considered by Dieckmann and Doebeli, i.e. $K_x = \exp(-x^2/2\sigma_K^2)$ and $C_x = \exp(-x^2/2\sigma_C^2)$. If $V = V^{(2)}$, then condition (6.5) means that

$$K_x = c(C * \hat{\pi})(x) \text{ for all } x \text{ where } \hat{\pi}(x) \neq 0,$$

which seems to indicate that if $\sigma_C < \sigma_K$, then $\hat{\pi}$ should be close to $\mathcal{N}(0, \sigma_K^2 - \sigma_C^2)$. On the other hand, if $V = V^{(1)}$, then $\hat{\pi}$ is a stationary distribution if $1 - \frac{\sum_z C_{x-z}\hat{\pi}_z}{K_x}$ is a strictly positive constant. Notice that if K and C are both Gaussian-shaped with $K_0 = C_0 = 1$ then $\hat{\pi} = \mathcal{N}(0, \sigma_K^2 - \sigma_C^2)$ makes $1 - \frac{\sum_z C_{x-z}\hat{\pi}_z}{K_x}$ constant; furthermore, this constant is strictly positive since $(C * \hat{\pi})(0) < K_0 = 1$ if $\sigma_C < \sigma_K$.

Therefore for both $V^{(1)}$ and $V^{(2)}$, the dynamical system (6.3) should have Gaussian-shaped stationary distributions if $\sigma_C < \sigma_K$. In simulations carried out by Dieckmann and Doebeli [Dieckmann and Doebeli 1999], however, $\sigma_C < \sigma_K$ is the case that leads to speciation, i.e. the stationary distribution supposedly has two sharp well-separated peaks, which contradicts the analysis carried out in the previous paragraph. Simulations of (6.3) with $V = V^{(1)}$, shown in Figure 6.2, reveal that if $\pi(0) \approx \delta_0$, initially the population does split into two groups and begins to move apart, but as $t \to \infty$, the empirical measure converges to a Gaussian-shaped hump. This suggests the possibility that in the original Dieckmann-Doebeli model, conditioning on the population surviving long enough for convergence to equilibrium to occur (recall that in the original Dieckman-Doebeli model, extinction occurs in finite time), speciation is also a transitory phenomenon, rather than an equilibrium phenomenon. Simulations of (6.3) with $V = V^{(2)}$, shown in Figure 6.3, does not even display transitory speciation behaviour. Instead, the initial spike at 0 simply widens to a Gaussian hump centred at 0. Hence the particular form of the dependence on K and $C * \pi$ seems to affect whether or not speciation occurs.

From the simulations and non-rigorous analysis above, it seems that the dynamical system in (6.3) does not have a bimodal stationary distribution if both K and C are taken to be Gaussian-shaped. If K and C are taken to be rectangular (i.e. $K_x = 1_{\{|x| \leq L\}}$ and

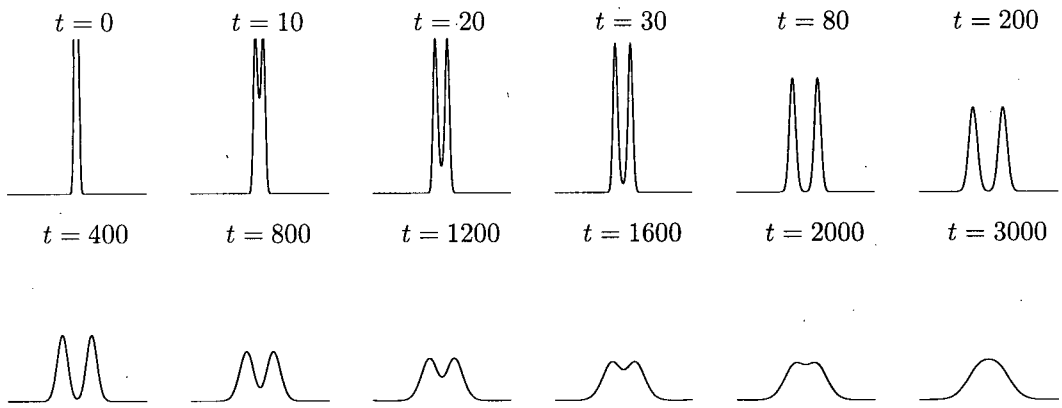


Figure 6.2: Simulation of (6.3) with $E = [-149, 149] \cap \mathbb{Z}$, $\sigma_K = 60$, $\sigma_C = 55$, and $V = V^{(1)}$.

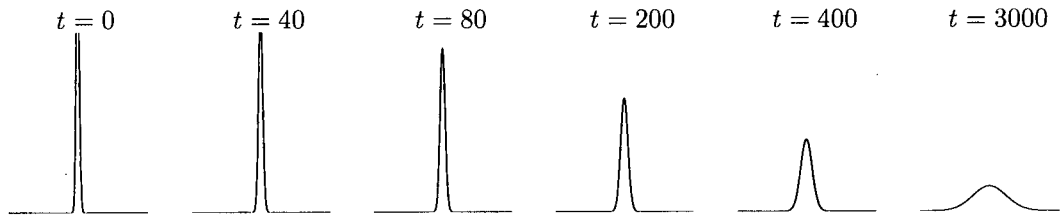


Figure 6.3: Simulation of (6.3) with $E = [-149, 149] \cap \mathbb{Z}$, $\sigma_K = 60$, $\sigma_C = 55$, and $V = V^{(2)}$.

$C_x = 1_{\{|x| \leq M\}}$ for some integers L and M), however, results from Appendix A shows that there exist bimodal stationary distributions. More specifically, Theorem A.0.14 says that if ν^n is a convergent sequence of *symmetric* stationary distributions for the conditioned Dieckmann-Doebeli model with mutation parameter μ^n with $\mu^n \to 0$, then $\nu^n_{[-l,l]} \to 0$, where $l = M - L + 1$; in words, the mass in the middle gets very small as the mutation parameter approaches zero.

6.1.3 A Moran Model with Competitive Selection

As discussed earlier, the dynamical system (6.3) cannot be easily associated with a Lyapunov function, which makes analyzing its behaviour difficult. Keeping in mind that the essential ingredient of the original Dieckmann-Doebeli model is that the fitness function is an increasing function of K_x and a decreasing function of $(C * \pi)_x$, we define the fitness $m_x(\pi)$ to have the following form:

$$m_x(\pi) = K_x \sum_z B_{x-z} K_z \pi_z, \tag{6.6}$$

where the "cooperation" kernel B can be taken to be $1 - C$. We assume B is symmetric. In the original Dieckmann-Doebeli model, pairs of individuals with small phenotypical distance *compete* at a higher intensity than pairs of individuals with large phenotypical distance; in our model, pairs of individuals with small phenotypical distance *cooperate* at a lower intensity than pairs of individuals with large phenotypical distance. To make our formulation

cleaner, we also adopt a continuous-time model. The advantage of adopting m_x in (6.6) as fitness and using a continuous time model is that the mean fitness of the population

$$\overline{m}_\pi = \sum_x \pi_x m_x = \sum_{x,z} \pi_x K_x B_{x-z} K_z \pi_z$$

is a Lyapunov function [Bürger 2000] for the dynamical system

$$\partial_t \pi_x = \pi_x (m_x - \overline{m}_\pi). \tag{6.7}$$

This assertion can be verified by the following calculation:

$$\begin{aligned}
\partial_t \overline{m}_\pi &= 2\sum_x m_x \partial_t \pi_x \\
&= 2\sum_x m_x \pi_x (m_x - \overline{m}_\pi) \\
&= 2\sum_x m_x \pi_x (m_x - \overline{m}_\pi) - 2\sum_x \overline{m}_\pi \pi_x (m_x - \overline{m}_\pi) \\
&= 2\sum_x \pi_x (m_x - \overline{m}_\pi)^2, \qquad (6.8)
\end{aligned}$$

where in the second line we use the fact.

$$\sum_x \overline{m}_\pi \pi_x (m_x - \overline{m}_\pi) = \overline{m}_\pi^2 - \overline{m}_\pi^2 \sum_x \pi_x = \overline{m}_\pi^2 - \overline{m}_\pi^2 = 0.$$

Since $\partial_t \overline{m}_\pi \geq 0$ for any π, $\overline{m}_\pi$ is a Lyapunov function for the dynamical system (6.7), and in particular, the mean fitness $\overline{m}_\pi$ increases at a rate proportional to the variance of the fitness. We call (6.7) the selection-only equation, as it does not have a part that corresponds to mutation. In Chapter 6.2.1, we will derive (6.7) as the deterministic limit of particle systems as the number of particles tends to infinity.

6.2 The Particle Model

We introduce two particle models, one with "strong selection" that yields a deterministic limit, and another with "weak selection" that yields a stochastic limit. We work on space $E = [-L, L] \cap \mathbb{Z}$. Let

$$\Delta = \{(p_{-L}, \ldots, p_0, \ldots, p_L) : p_i > 0 \ \forall i \text{ and } \sum_{i=-L}^{L} p_i = 1\}$$

be the space of probability measures on E, i.e. $\Delta = \mathcal{P}(E)$. Members of Δ are usually denoted by π, $\hat{\pi}$, π^N, etc. We endow Δ with the following metric:

$$d(\hat{\pi}, \tilde{\pi}) = \max_x |\hat{\pi}(x) - \tilde{\pi}(x)|.$$

Let $K : E \to [0,1]$ be the carrying capacity function, and $B : \mathbb{Z} \to [0,1]$ be the cooperation kernel, with $B_z = 0$ meaning that sites separated by phenotypical distance z do not cooperate at all (i.e. compete at full intensity), and $B_z = 1$ meaning that they cooperate at full

intensity (i.e. do not compete at all). We assume B to be symmetric. The fitness of site x in a population with distribution π is defined as

$$m_x(\pi) = K_x \sum_z B_{x-z} K_z \pi_z.$$

If one abuses notation by writing K as a diagonal matrix, B as a matrix, and π as a vector, then the vector formed by $m.(\pi)$ can be written as $KBK\pi$. The mean fitness of a population with distribution π is defined as

$$\overline{m}_\pi = \sum_{x,z} \pi_x K_x B_{x-z} K_z \pi_z.$$

If one abuses notation again, then $\overline{m}_\pi$ can be written as a quadratic form $\pi^t KBK\pi$.

Throughout this work, we will use symmetric or house-of-cards mutation, which means that the rate $\mu_{xy} = \mu_y$ at which phenotype x mutates to phenotype y depends on y only. This is a common assumption in population genetics [Bürger 2000], and it is precisely this assumption that allows one to explicitly write down a Lyapunov function for the selection-mutation equation (to be defined in (6.10)). As a further simplification, we assume that $\mu_y = \mu$ is constant in y, which makes the proofs a bit cleaner. Let $X^N(t) = (X_1^N(t), \ldots, X_N^N(t))$, $t \in \mathbb{R}^+$, be an N-particle system, with $X_i^N(t) \in E$ for all t and i. Define the empirical measure

$$\pi^N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^N(t)}.$$

6.2.1 The Strong Selection Model

For our model with strong selection, the particle system undergoes the following:

- Selection: At rate $N\overline{m}_{\pi^N}$, a particle, say X_i^N, is chosen at random from the N particles and killed; at the same time, a new particle is born at x with probability $\frac{m_x(\pi^N)\pi_x^N}{\overline{m}_{\pi^N}}$. Since $\sum_x \frac{m_x(\pi^N)\pi_x^N}{\overline{m}_{\pi^N}} = 1$, $\frac{m.(\pi^N)\pi^N}{\overline{m}_{\pi^N}}$ is a probability distribution;
- Mutation: At rate $N(2L+1)\mu$, a particle, say X_i^N, is chosen at random from the N particles and killed; at the same time, a new particle is born at a site y with probability $\frac{1}{2L+1}$.

A particle at x gets replaced via selection by a particle at y at rate $N\pi_x^N m_y(\pi^N)\pi_y^N$, and gets replaced via mutation by a particle at y at a rate of $N\mu\pi_x^N$. Let $\bar{K} = \sup_{x\in[-L,L]} K_x$, so that $m_x(\pi) = K_x \sum_z B_{x-z} K_z \pi_z \leq \bar{K} \sum_z \bar{K}\pi_z = \bar{K}^2$. Let $l(dx)$ denote the Lebesgue measure on $\mathbb{R}^+$. The process described above can be constructed using a Poisson point process $\Lambda^N(dt, dx, dy, d\xi, de)$ on

$$\mathbb{R}^+ \times \{(x,y) \in E^2 : x \neq y\} \times [0,1] \times \{1,2\},$$

with intensity measure

$$\lambda^N(A \times B \times C \times D) = l(A)(\#B)(\#C)k(D),$$

where # denotes the counting measure, $D \subset [0,1] \times \{1,2\}$, and $k = l \times (N\bar{K}^2\delta_1 + N\mu\delta_2)$. For all $x, y \in E^2$ with $x \neq y$, jumps of $\Lambda^N(dt, x, y, [0,1], \{1\})$ give possible times at which a particle at y may be replaced by a particle at x by the selection mechanism, while jumps of $\Lambda^N(dt, x, y, [0,1], \{2\})$ give possible times at which a particle at y may be replaced by a particle at x by the mutation mechanism. The strong selection model can be expressed in terms of the following formula for $\pi_x^N(t)$:

$$\begin{aligned}
\pi_x^N(t) &= \pi_x^N(0) \\
&+\frac{1}{N}\left[\int_0^t \int 1\left(\xi \leq \frac{\pi_y^N(s-)m_x(\pi^N(s-))\pi_x^N(s-)}{\bar{K}^2}\right)\Lambda^N(ds, x, dy, d\xi, 1)\right. \\
&\quad \left.-\int_0^t \int 1\left(\xi \leq \frac{\pi_x^N(s-)m_y(\pi^N(s-))\pi_y^N(s-)}{\bar{K}^2}\right)\Lambda^N(ds, dy, x, d\xi, 1)\right] \\
&+\frac{1}{N}\left[\int_0^t \int 1(\xi \leq \pi_y^N(s-))\Lambda^N(ds, x, dy, d\xi, 2)\right. \\
&\quad \left.-\int_0^t \int 1(\xi \leq \pi_x^N(s-))\Lambda^N(ds, dy, x, d\xi, 2)\right]. \qquad (6.9)
\end{aligned}$$

A solution to (6.9) exists because the total jump rate is finite for a fixed N. The two integrals inside the first set of brackets corresponds to selection, i.e. a particle at x gets replaced by a particle at y at rate $N\pi_y^N m_x(\pi^N)\pi_x^N$ due to selection. In particular, $\Lambda^N(ds, x, dy, d\xi, 1)$ in the first integral accounts for the killing of a particle at y and a new particle being born at x, and $\Lambda^N(ds, dy, x, d\xi, 1)$ in the second integral accounts for the killing of a particle at x and a new particle being born at y. The two integrals inside the second set of brackets corresponds to mutation, i.e. a particle at x gets replaced by a particle at y at rate $N\mu\pi_y^N$ due to mutation.

Proposition 6.2.1. *As $N \to \infty$, the processes π^N converge weakly to a deterministic process π that takes values in $\mathcal{P}(E)$ and obeys the following system of ODE's:*

$$\partial_t \pi_x = \pi_x(m(x, \pi) - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_x). \qquad (6.10)$$

Proof. First, we rewrite (6.9) by decomposing Λ^N into a martingale term $\tilde{\Lambda}^N$ and a deterministic drift term:

$$\begin{aligned}
\pi_x^N(t) &= \pi_x^N(0) + M_x^N(t) + \sum_{y=-L}^{L}\int_0^t [\pi_y^N(s-)m_x(\pi^N(s-))\pi_x^N(s-) \\
&\quad -\pi_x^N(s-)m_y(\pi^N(s-))\pi_y^N(s-)]\, ds + \mu\sum_{y=-L}^{L}\int_0^t [\pi_y(s-) - \pi_x(s-)]\, ds \\
&= \pi_x^N(0) + M_x^N(t) + \int_0^t \pi_x^N(s-)[m_x(\pi^N(s-)) - \overline{m}_{\pi^N(s-)}] \\
&\quad +\mu[1 - (2L+1)\pi_x(s-)]\, ds \qquad (6.11)
\end{aligned}$$

where we define $\tilde{\Lambda}^N = \Lambda^N - \lambda^N$ to be the martingale part of Λ^N and

$$\begin{aligned}
M_x^N(t) &= \frac{1}{N}\left[\int_0^t \int 1\left(\xi \le \frac{\pi_y^N(s-)m_x(\pi^N(s-))\pi_x^N(s-)}{\bar{K}^2}\right)\tilde{\Lambda}^N(ds,x,dy,d\xi,1)\right. \\
&\quad \left. - \int_0^t \int 1\left(\xi \le \frac{\pi_x^N(s-)m_y(\pi^N(s-))\pi_y^N(s-)}{\bar{K}^2}\right)\tilde{\Lambda}^N(ds,dy,x,d\xi,1)\right] \\
&\quad + \frac{1}{N}\left[\int_0^t \int 1(\xi \le \pi_y^N(s-))\tilde{\Lambda}^N(ds,x,dy,d\xi,2)\right. \\
&\quad \left. - \int_0^t \int 1(\xi \le \pi_x^N(s-))\tilde{\Lambda}^N(ds,dy,x,d\xi,2)\right].
\end{aligned}$$

We estimate the quadratic variation of the martingale term $M_x^N(t)$ and show that it converges to 0 as $N \to \infty$:

$$\begin{aligned}
\langle M_x^N \rangle_t &= \frac{1}{N^2}\sum_{y=-L}^{L}\left[\int_0^t 1\left(\xi \le \frac{\pi_y^N(s-)m_x(\pi^N(s-))\pi_x^N(s-)}{\bar{K}^2}\right) N\bar{K}^2\, ds\right. \\
&\quad \left. + \int_0^t 1\left(\xi \le \frac{\pi_x^N(s-)m_y(\pi^N(s-))\pi_y^N(s-)}{\bar{K}^2}\right) N\bar{K}^2\, ds\right] \\
&\quad + \frac{1}{N^2}\sum_{y=-L}^{L}\left[\int_0^t 1(\xi \le \pi_y^N(s-))N\mu\, ds + \int_0^t 1(\xi \le \pi_x^N(s-))N\mu\, ds\right]. \\
&\le \frac{2\bar{K}^2}{N}\sum_{y=-L}^{L}\int_0^t ds + \frac{2\mu}{N}\sum_{y=-L}^{L}\int_0^t ds \\
&\le \frac{2(\bar{K}^2+\mu)(2L+1)t}{N}, \qquad (6.12)
\end{aligned}$$

which $\to 0$ as $N \to \infty$. Since the maximum jump size in M_x^N is $\frac{1}{N}$, by Burkholder's inequality (see for example Theorem 21.1 in [Burkholder 1973]), we have

$$E\left(\left(\sup_{t\le T} M_x^N(t)\right)^2\right) \le C\left(E\langle M_x^N\rangle_T + \frac{1}{N^2}\right) \to 0 \qquad (6.13)$$

as $N \to 0$. The other two terms in (6.11), i.e. $\pi_x^N(0)$ and $\int_0^t \pi_x^N(s-)[m_x(\pi^N(s-)) - \overline{m}_{\pi^N(s-)}] + \mu[1-(2L+1)\pi_x(s-)]\, ds$, are both C-tight, $\pi_x^N(0)$ being constant and the integrand in the integral bounded uniformly between constants. Therefore π_x^N is C-tight for each x.

If there exists a sequence N_n such that π^{N_n} converges to π weakly and π is continuous, then by bounded convergence, $\int_0^t \pi_x^{N_n}(s-)[m_x(\pi^{N_n}(s-)) - \overline{m}_{\pi^{N_n}(s-)}] + \mu[1-(2L+1)\pi_x(s-)]\, ds$ converges to

$$\int_0^t \pi_x(s)[m_x(\pi(s)) - \overline{m}_{\pi(s)}] + \mu[1-(2L+1)\pi_x(s)]\, ds,$$

and since π^N has representation (6.11), π satisfies the following deterministic integral equation:

$$\pi_x(t) = \pi_x(0) + \int_0^t \pi_x(s)[m_x(\pi(s)) - \overline{m}_{\pi(s)}] + \mu[1 - (2L+1)\pi_x(s)]\, ds. \tag{6.14}$$

A continuous $(\pi(t), 0 \le t \le \infty)$ solves the integral equation (6.14) if and only if it satisfies the ODE system (6.10). By well-known results from ODE theory (e.g. Theorem 1.1.1 from [Wiggins 1988]), the solution to (6.10) is unique because its right-hand-side is C^∞ in π. Therefore solution to (6.14) is unique as well, and the proof is complete. □

6.2.2 The Weak Selection Model

For our model with weak selection, the particle system with N particles undergoes the following:

- Selection: A particle at x gets replaced by a particle at y at rate $N\pi_x^N m_y(\pi^N)\pi_y^N$;
- Mutation: A particle at x gets replaced by a particle at y at rate $N\mu\pi_x^N$;
- Replacement sampling: A particle at x gets replaced by a particle at y at rate $\frac{N^2}{2}\pi_x^N\pi_y^N$.

Just as in the strong selection model, this process can be constructed using a Poisson point process $\Lambda^N(dt, dx, dy, d\xi, de)$ on

$$\mathbb{R}^+ \times \{(x,y) \in E^2 : x \neq y\} \times [0,1] \times \{1,2,3\},$$

with intensity measure

$$\lambda^N(A \times B \times C \times D) = l(A)(\#B)(\#C)k(D),$$

where l denotes the Lebesgue measure on $\mathbb{R}^+$, $\#$ denotes the counting measure, $D \subset [0,1] \times \{1,2,3\}$, and $k = l \times (N\bar{K}^2\delta_1 + N\mu\delta_2 + \frac{N^2}{2}\delta_3)$. The weak selection model can then be expressed in terms of the following formula for $\pi_x^N(t)$:

$$\begin{aligned}
\pi_x^N(t) &= \pi_x^N(0) \\
&+ \frac{1}{N}\left[\int_0^t \int 1\left(\xi \le \frac{\pi_y^N(s-)m_x(\pi^N(s-))\pi_x^N(s-)}{\bar{K}^2}\right)\Lambda^N(ds, x, dy, d\xi, 1)\right. \\
&\quad \left. - \int_0^t \int 1\left(\xi \le \frac{\pi_x^N(s-)m_y(\pi^N(s-))\pi_y^N(s-)}{\bar{K}^2}\right)\Lambda^N(ds, dy, x, d\xi, 1)\right] \\
&+ \frac{1}{N}\left[\int_0^t \int 1(\xi \le \pi_y^N(s-))\Lambda^N(ds, x, dy, d\xi, 2)\right. \\
&\quad \left. - \int_0^t \int 1(\xi \le \pi_x^N(s-))\Lambda^N(ds, dy, x, d\xi, 2)\right] \\
&+ \frac{1}{N}\left[\int_0^t \int 1(\xi \le \pi_y^N(s-)\pi_x^N(s-))\Lambda^N(ds, x, dy, d\xi, 3)\right. \\
&\quad \left. - \int_0^t \int 1(\xi \le \pi_x^N(s-)\pi_y^N(s-))\Lambda^N(ds, dy, x, d\xi, 3)\right].
\end{aligned} \tag{6.15}$$

The two integrals inside the first set of brackets correspond to selection, those inside the second set of brackets correspond to mutation, and those inside the third set of brackets correspond to replacement sampling. By carrying out computations similar to those done in the proof of Proposition 6.2.1 (and the tightness proof in [Perkins 2002]), one can conclude the processes $\{\pi^N : N \in \mathbb{N}\}$ is C-tight, and that each weak limit point π satisfies the following martingale problem:

$$\pi_x(t) = \pi_x(0) + \int_0^t \pi_x(s)[m_x(\pi(s)) - \overline{m}_{\pi(s)}] + \mu[1 - (2L+1)\pi_x(s)]\, ds + M_x(t),$$

where M_x is a continuous $(\mathcal{F}_t^\pi)$-martingale such that $M_x(0) = 0$, and

$$\langle M_x, M_y \rangle_t = \int_0^t \delta_{xy}\pi_x(s) - \pi_x(s)\pi_y(s)\, ds. \tag{6.16}$$

Here $\delta_{xy} = 1(x = y)$. Ito's formula shows that for $F \in C^2(E)$,

$$F(\pi(t)) - F(\pi(0)) - \int_0^t \mathcal{G}F(\pi(s))\, ds$$

is a bounded continuous martingale, where

$$\begin{aligned} \mathcal{G}F(\pi) &= \sum_{x=-L}^{L} [\pi_x(m_x(\pi) - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_x)]\frac{\partial F}{\partial \pi_x} \\ &\quad + \frac{1}{2}\sum_{x=-L}^{L}\sum_{y=-L}^{L} \pi_x(\delta_{xy} - \pi_y)\frac{\partial^2 F}{\partial \pi_x \partial \pi_y}. \end{aligned}$$

In particular, if F takes on the form $F(\pi) = f(\langle \pi, \phi \rangle)$ where $\langle \pi, \phi \rangle = \sum_z \pi_z \phi_z$ and $\phi : E \to \mathbb{R}$ is a function, then

$$\begin{aligned} \mathcal{G}F(\pi) &= \sum_{x=-L}^{L} f'(\langle \pi, \phi \rangle)\phi_x[\pi_x(m_x(\pi) - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_x)] \\ &\quad + \frac{1}{2}\sum_{x=-L}^{L}\sum_{y=-L}^{L} f''(\langle \pi, \phi \rangle)\phi_x\phi_y\pi_x(\delta_{xy} - \pi_y). \end{aligned} \tag{6.17}$$

This is a special (finite E and symmetric mutation) case of generator for the Fleming-Viot process with selection. The martingale problem associated with $\mathcal{G}$ has a unique solution (see Chapter 10.1.1 from [Dawson 1993]), so π^N converges weakly to π on $D(\mathbb{R}^+, M_1(E))$.

For the process described by the martingale problem (6.16), Lemma 4.1 from [Ethier and Kurtz 1994] says that

$$\nu(d\pi) = C\left(\prod_{x=-L}^{L} \pi_x\right)^{\mu-1} e^{\overline{m}_\pi}\, d\pi_{-L} \cdots d\pi_L$$

is the unique stationary distribution. Here C is the normalizing constant such that ν is a probability measure on $\mathcal{P}(E)$. Notice that if $m_x(\pi) = 0$ for all x and π, then $\overline{m}_\pi = 0$ for

all π, and in that case, $\nu(d\pi) = C\left(\prod_{x=-L}^{L} \pi_x\right)^{\mu-1} d\pi_{-L} \cdots d\pi_L$ is the unique stationary distribution of the Fleming-Viot process on E with symmetric mutation [Ethier and Kurtz 1981]. Adding selection to the model has the effect of putting weight $e^{\overline{m}_\pi}$ (where $\overline{m}_\pi$ is the mean fitness of π) on the phenotypical distribution π. But even the least fit π has weight at least 1, since $\exp(\min_\pi \overline{m}_\pi) = e^0 = 1$, and the fittest π has weight at most e. The effect of fitness on the density at a particular phenotypical distribution π is thus only marginal. In contrast, as we shall see later, fitness will have a much more pronounced effect in the strong selection model. For the rest of this part of the thesis, we deal with the strong selection model outlined in Chapter 6.2.1.

Chapter 7

The Selection-Mutation Equation

As discussed in Chapter 6.2.1, the particle model with strong selection converges weakly to the selection-mutation equation as the number of particles N tends to infinity:

$$\partial_t \pi_x = \pi_x(m_x - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_x), \tag{7.1}$$

where, as before, $m_x = m_x(\pi)$ is the fitness of site x in population π, and $\overline{m}_\pi$ is the mean fitness of the population:

$$m_x = K_x \sum_z B_{x-z} K_z \pi_z \tag{7.2}$$

$$\overline{m}_\pi = \sum_x \pi_x m_x = \sum_{x,z} \pi_x K_x B_{x-z} K_z \pi_z. \tag{7.3}$$

First, we state a few assumptions on the parameters involved:

1. $x \in E = [-L, L] \cap \mathbb{Z} = \{-L, \ldots, -1, 0, 1, \ldots, L\}$,
2. $K : E \to (0, 1]$,
3. $B : \mathbb{Z} \to [0, 1]$.

We establish a few basic facts about the system (7.1). First of all, m_x (uniformly in x) and $\overline{m}_\pi$ both lie in $[0, 1]$, therefore $m_x - \overline{m}_\pi \geq -1$. Thus

$$\begin{aligned} \partial_t \pi_x &= \pi_x(m_x - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_x) \\ &\geq -\pi_x + \mu(1 - (2L+1)\pi_x) \\ &= \mu - (2L\mu + \mu + 1)\pi_x. \end{aligned} \tag{7.4}$$

If $\pi_x < \frac{\mu}{2L\mu+\mu+1}$, then $\partial_t \pi_x > 0$ regardless of π. This means that there can be no stationary points of the system (7.1) with any $\pi_x < \frac{\mu}{2L\mu+\mu+1}$, and furthermore, for those x where $\pi_x < \frac{\mu}{2(2L\mu+\mu+1)}$ initially, π_x will increase at a positive (bounded away from 0) speed and

eventually $\pi_x \geq \frac{\mu}{2(2L\mu+\mu+1)}$ for all x. Secondly, since the sum over all x of the right hand side of (7.1) is

$$\begin{aligned}
&\sum_x \pi_x(m_x - \overline{m}_\pi) + \mu(1-(2L+1)\pi_x) \\
&= \sum_x \pi_x m_x - \overline{m}_\pi \sum_x \pi_x + (2L+1)\mu - \mu(2L+1)\sum_x \pi_x \\
&= \overline{m}_\pi - \overline{m}_\pi + (2L+1)\mu - (2L+1)\mu \\
&= 0, \qquad (7.5)
\end{aligned}$$

we have

$$\partial_t \sum_x \pi_x = 0,$$

hence the total mass $\sum_x \pi_x$ remains constant. This, together with the first observation we just made, imply that if at $t=0$, $\pi(0)$ is a probability measure, i.e. $\sum_x \pi_x(0) = 1$, then $\pi(t)$ remains a probability measure for all t.

Since π_x will become instantly nonzero at x where $\pi_x = 0$ initially, we restrict our attention to polymorphic initial conditions, i.e. $\pi_x(0) > 0$ for all x, which is equivalent to saying that $\pi(0)$ lies in the interior $\mathring{\Delta}$ of the set Δ. For polymorphic initial conditions, $\pi_x(t) > 0$ for all x and t, and therefore (7.1) can be written as:

$$\partial_t \pi_x = \pi_x \left(m_x - \overline{m}_\pi + \frac{\mu}{\pi_x} - \mu(2L+1) \right). \qquad (7.6)$$

Furthermore, there is a Lyapunov function $V_\pi : \mathring{\Delta} \to \mathbb{R}$ for the dynamical system (7.6):

$$V_\pi = \frac{1}{2}\overline{m}_\pi + \mu \sum_x \log \pi_x. \qquad (7.7)$$

Notice that $\partial_{\pi_x} \overline{m}_\pi = 2\sum_z K_x B_{x-z} K_z \pi_z = 2m_x$. The assertion that V_π is a Lyapunov function for (7.6) can then be verified by the following simple calculus exercise:

$$\begin{aligned}
\partial_t V_\pi &= \sum_x (\partial_{\pi_x} V_\pi)(\partial_t \pi_x) \\
&= \sum_x \left(m_x + \frac{\mu}{\pi_x} \right) \pi_x \left(m_x + \frac{\mu}{\pi_x} - \overline{m}_\pi - \mu(2L+1) \right) \\
&= \sum_x \left(m_x + \frac{\mu}{\pi_x} \right) \pi_x \left(m_x + \frac{\mu}{\pi_x} - \overline{m}_\pi - \mu(2L+1) \right) \\
&\quad - (\overline{m}_\pi + \mu(2L+1)) \sum_x \pi_x \left(m_x + \frac{\mu}{\pi_x} - \overline{m}_\pi - \mu(2L+1) \right).
\end{aligned}$$

Notice that $\sum_x \pi_x \left(m_x + \frac{\mu}{\pi_x} - \overline{m}_\pi - \mu(2L+1) \right) = 0$ by (7.5), therefore

$$\partial_t V_\pi = \sum_x \pi_x \left(m_x + \frac{\mu}{\pi_x} - \overline{m}_\pi - \mu(2L+1) \right)^2 \geq 0,$$

hence V_π is a Lyapunov function for (7.6) as claimed.

In fact, according to Theorem A.9 of [Bürger 2000], (7.6) is a so-called Svirezhev-Shahshahani gradient system with potential V_π as defined in (7.7), i.e. $\partial_t \pi = \tilde{\nabla} V(\pi)$, where $\tilde{\nabla} V(\pi) = G_\pi \nabla V(\pi)$ and G_π is the matrix formed by entries $g^{xy} = \pi_x(\delta_{xy} - \pi_y)$. Any gradient system, such as (7.6), has the property that all orbits, regardless of initial condition, converge to some point in the ω-limit set

$$D_\omega = \{p : p \text{ is an accumulation point of } \pi_x(t) \text{ as } t \to \infty\}.$$

All points in D_ω are stationary points of (7.6). Since $\hat{\pi}$ is a stationary point of (7.6) if and only if

$$\hat{m}_x - \overline{m}_{\hat{\pi}} + \mu\left(\frac{1}{\hat{\pi}_x} - (2L+1)\right) = 0 \text{ for all } x \in E, \tag{7.8}$$

where we write $\hat{m}_x = m_x(\hat{\pi})$, all points in D_ω satisfies (7.8). We observe that if $\hat{\pi}$ is a stationary distribution and $\hat{m}_x > \hat{m}_y$, then condition (7.8) means that $\frac{1}{\hat{\pi}_x} - (2L+1) < \frac{1}{\hat{\pi}_y} - (2L+1)$, therefore $\hat{\pi}_x > \hat{\pi}_y$, i.e.

$$\hat{m}_x > \hat{m}_y \Longrightarrow \hat{\pi}_x > \hat{\pi}_y. \tag{7.9}$$

In words, fitter sites have more mass.

We will try to characterize the stationary points of the dynamical system (7.6) for K and B satisfying the following conditions:

$$\begin{aligned} &K \text{ is symmetric and unimodal with } K_0 = 1, \text{ and } B \text{ is of the form} \\ &B_x = b + (1-b)1_{\{|x| \geq M\}} \text{ with } b \in [0,1] \text{ and } L < M \leq 2L. \end{aligned} \tag{7.10}$$

Define $l = M - L$, then for $x \in [-l+1, l-1]$, the cooperation intensity between x and any other site $z \in [-L, L]$, B_{x-z}, is equal to b, which means that

$$m_x = K_x \sum_z B_{x-z} K_z \pi_z = bK_x \sum_z K_z \pi_z. \tag{7.11}$$

7.1 Mild Competition: b close to 1

If $b = 1$, then $B_x = b$ for all x, hence there is equal competition between all sites. This actually means that competition plays no part in how fit site x is and m_x is proportional to K_x. Therefore, since K_x is unimodal (hence K_x is strictly increasing in $[-L, 0]$ and strictly decreasing in $[0, L]$), the fitness should be unimodal, too. Recall from (7.9) that stationary distributions of (7.6) has the property of fitter sites having more mass, thus we expect the stationary distribution $\hat{\pi}$ to be unimodal as well. In particular, $\hat{\pi}$ should attain its maximum at $x = 0$. As $\mu \to 0$, we expect the "peak" of $\hat{\pi}$ concentrated around 0 to become sharper and sharper, approaching δ_0, the δ-measure concentrated at 0. In fact, as we shall see, b only needs to be somewhat close to 1 for this behaviour to occur.

We now show that for any stationary distribution $\hat{\pi}$ of (7.6), site 0 is fitter than any other site for $b \in (\frac{1}{2}, 1]$ sufficiently close to 1, i.e. $\hat{m}_0 > \hat{m}_x$ if $x \neq 0$. This will mean that as

$\mu \to 0$, any $\hat{\pi}$ approaches the δ-measure. Recall from (7.11) that for $x \in [-l+1, l-1]$, we have

$$m_x = bK_x \sum_z K_z \pi_z.$$

We recall that $K_0 = 1$ and K_x is assumed to be strictly increasing in $[-L, 0]$ and strictly decreasing in $[0, L]$, therefore K_x attains its maximum at $x = 0$, and thus for $x \in [-l+1, -1] \cup [1, l-1]$, $K_x - K_0 \leq K_1 - K_0 < 0$. Therefore for $x \in [-l+1, -1] \cup [1, l-1]$,

$$m_x - m_0 = b(K_x - K_0) \sum_z K_z \pi_z \leq -b(K_0 - K_1) \sum_z K_z \pi_z.$$

We now apply the bound $\sum_z K_z \pi_z \geq \inf_{\pi \in \Delta} \sum_z K_z \pi_z = K_L$ to the above inequality to obtain

$$m_x - m_0 \leq -b(K_0 - K_1) K_L.$$

Since b is assumed to be $> \frac{1}{2}$, we have for $x \in [-l+1, -1] \cup [1, l-1]$,

$$m_x - m_0 < -\frac{1}{2}(K_0 - K_1) K_L. \tag{7.12}$$

We also bound the fitness for sites x in $[l, L]$:

$$\begin{aligned}
m_x &= bK_x \sum_z K_z \pi_z + (1-b) K_x \sum_{|x-z| \geq M} K_z \pi_z \\
&= bK_x \sum_z K_z \pi_z + (1-b) K_x \sum_{x-z \geq M} K_z \pi_z \qquad \text{since } x \geq l \\
&\leq bK_x \sum_z K_z \pi_z + (1-b) K_x \sum_{z=-L}^{x-M} K_z \qquad \text{since } \pi_z \leq 1 \\
&\leq bK_x \sum_z K_z \pi_z + (1-b) K_x \sum_{z=-L}^{-l} K_z \\
&\leq bK_x \sum_z K_z \pi_z + (1-b)(L-l+1) K_x K_l, \qquad (7.13)
\end{aligned}$$

where in the last line, we use the following: for $z \in [-L, x-M] \subset [-L, -l]$, $K_z \leq K_{-l} = K_l$. Similarly, for $x \in [-L, -l]$, we have the same bound (7.13). Therefore for $x \in [-L, -l] \cup [l, L]$, we have

$$\begin{aligned}
m_x &\leq bK_x \sum_z K_z \pi_z + (1-b)(L-l+1) K_x K_l \\
&\leq bK_x \sum_z K_z \pi_z + (1-b)(L-l+1) K_l^2, \qquad (7.14)
\end{aligned}$$

where again we use the bound $K_x \leq K_l$ for $x \in [-L, -l] \cup [l, L]$. We use (7.11) and (7.14) to estimate $m_x - m_0$ for $x \in [-L, -l] \cup [l, L]$:

$$\begin{aligned}
m_x - m_0 &\leq bK_x \sum_z K_z \pi_z + (1-b)(L-l+1) K_l^2 - bK_0 \sum_z K_z \pi_z \\
&= b(K_x - K_0) \sum_z K_z \pi_z + (1-b)(L-l+1) K_l^2.
\end{aligned}$$

Since $K_x - K_0 < 0$ for $x \in [-L, -l] \cup [l, L]$, we have

$$\begin{aligned} m_x - m_0 &\leq -b \inf_{x \in [-L,-l] \cup [l,L]} (K_0 - K_x) \inf_{\pi \in \Delta} \sum_z K_z \pi_z + (1-b)(L-l+1)K_l^2 \\ &= -b(K_0 - K_l)K_L + (1-b)(L-l+1)K_l^2. \end{aligned}$$

Since b is assumed to be $> \frac{1}{2}$, the above bound can be simplified to:

$$m_x - m_0 \leq (1-b)(L-l+1)K_l^2 - \frac{1}{2}(K_0 - K_l)K_L. \tag{7.15}$$

Thus if b is so close to 1 that

$$1 - b \leq \frac{(K_0 - K_l)K_L}{4(L-l+1)K_l^2}, \tag{7.16}$$

then let $\delta_1 = \frac{1}{4}(K_0 - K_l)K_L$ be a positive constant and we have

$$(1-b)(L-l+1)K_l^2 \leq \frac{1}{4}(K_0 - K_l)K_L = \frac{1}{2}(K_0 - K_l)K_L - \delta_1. \tag{7.17}$$

Thus if condition (7.16) holds, then (7.15) and (7.17) imply that for $x \in [-L, -l] \cup [l, L]$,

$$m_x - m_0 \leq -\delta_1. \tag{7.18}$$

Define $\delta = \min\left(\delta_1, \frac{1}{2}(K_0 - K_1)K_L\right)$, then the estimates in (7.12) and (7.18) mean that for all $x \in E \backslash \{0\}$ and any π,

$$m_x - m_0 \leq -\delta. \tag{7.19}$$

This shows that for b satisfying condition (7.16) and for any π, site 0 is fitter than any other site. We will use this bound to establish the following.

Theorem 7.1.1. *If K is symmetric and unimodal with $K_0 = 1$, and $B_x = b + (1-b)1_{\{|x| \geq M\}}$ with $L < M \leq 2L$, $l = M - L$, and $b \in \left[1 - \frac{(K_0 - K_l)K_L}{4(L-l+1)K_l^2}, 1\right]$, then as $\mu \to 0$,*

$$\sup\{\|\hat{\pi}^\mu - \delta_0\|_\infty : \hat{\pi}^\mu \text{ is an stationary distribution for mutation parameter } \mu\} \to 0.$$

Proof. Recall condition (7.8): $\hat{\pi}^\mu$ is a stationary distribution of (7.6) if and only if for all $x \in [-L, L]$,

$$\hat{m}_x^\mu - \overline{m}_{\hat{\pi}^\mu} + \mu\left(\frac{1}{\hat{\pi}_x^\mu} - (2L+1)\right) = 0, \tag{7.20}$$

where we write $\hat{m}_x^\mu = m_x(\hat{\pi}^\mu)$. We make the following observations using condition (7.20): if $\hat{m}_x^\mu \geq \overline{m}_{\hat{\pi}^\mu}$, then $-\frac{1}{\hat{\pi}_x^\mu} + (2L+1) \geq 0$, which implies that $\hat{\pi}_x^\mu \geq 1/(2L+1)$; similarly, if $m_x < \overline{m}_{\hat{\pi}^\mu}$ then $\hat{\pi}_x^\mu < 1/(2L+1)$.

Since $1/\hat{\pi}_x^\mu \geq 1$, the following bound holds for all x:

$$\hat{m}_x^\mu - \overline{m}_{\hat{\pi}^\mu} + \mu(1 - (2L+1)) \leq 0,$$

which implies that

$$\hat{m}_x^\mu - \overline{m}_{\hat{\pi}^\mu} \leq 2L\mu. \tag{7.21}$$

We consider μ small enough such that $2L\mu < \frac{\delta}{2}$, where δ is defined right after (7.18). Then the estimate (7.21) applied to $x = 0$ means that

$$\hat{m}_0^\mu - \overline{m}_{\hat{\pi}^\mu} \leq 2L\mu < \frac{\delta}{2},$$

which implies that

$$\hat{m}_0^\mu < \overline{m}_{\hat{\pi}^\mu} + \frac{\delta}{2}.$$

Applying the above to estimate (7.19), we get, for all $x \neq 0$,

$$\hat{m}_x^\mu \leq \hat{m}_0^\mu - \delta < \overline{m}_{\hat{\pi}^\mu} - \frac{\delta}{2}, \tag{7.22}$$

In particular, the only x where $\hat{m}_x^\mu \geq \overline{m}_{\hat{\pi}^\mu}$ is $x = 0$.

Using the bound (7.22), condition (7.20) implies that for $x \neq 0$,

$$\mu\left(\frac{1}{\hat{\pi}_x^\mu} - (2L+1)\right) \geq \frac{\delta}{2}.$$

Therefore

$$\frac{1}{\hat{\pi}_x^\mu} \geq \frac{\delta}{2\mu} + (2L+1),$$

which $\to \infty$ as $\mu \to 0$. Hence $\hat{\pi}_x^\mu \to 0$ as $\mu \to 0$ for all $x \neq 0$. Notice that the proof does not depend on which $\hat{\pi}^\mu$ we pick, therefore we are done. □

Remark 7.1.2. *The crucial estimate for the above proof is (7.19). In the case of equal competition, i.e. $b = 1$, it is very easy to derive (7.19):*

$$\begin{aligned} m_x - m_0 = b(K_x - K_0)\sum_z K_z\pi_z &\leq -b\inf_{x\neq 0}(K_0 - K_x)\inf_{\pi\in\Delta}\sum_z K_z\pi_z \\ &= -b(K_0 - K_1)K_L, \end{aligned}$$

which is a positive constant independent of μ.

7.2 Intense Competition: b close to 0

Results from Chapter 7.1 show that if the competition between pairs of sites that are far away from each other is not intense, i.e. b is close to 1, then as $\mu \to 0$, the stationary distribution(s) converge to δ_0, and therefore, there is no speciation. In this section, we show that if there is the most intense competition between pairs of sites that are far away from each other, i.e. $b = 0$, then we do see speciation in the stationary distribution(s). More

interesting behaviour arises in the case of positive but small b. We will show that, in this case, if μ is small enough, then there are at least two vastly different stationary distributions, one resembling the δ-measure, the other bimodal and having almost zero mass in the middle; on the other hand, if μ is sufficiently large, then all stationary distributions are bimodal and have little mass in the middle. Thus for small μ, whether speciation occurs eventually in the dynamical system depends on the initial state of the system. But for large enough μ, speciation will occur eventually if one waits long enough. We first illustrate this behaviour in a system with 3 phenotypes $\{-1, 0, 1\}$, whose stationary points we can calculate explicitly.

7.2.1 Study of A One-dimensional System

Here we will take the simplest possible scenario, and show that the dynamical system (7.6) has exactly two stable stationary points. Let $L = 1$, $E = \{-1, 0, 1\}$, $K_0 = 1$, and $K_1 = K_{-1} = \frac{1}{2}$. Let $B_x = b + (1-b)1_{\{|x|\geq 2\}}$, i.e. phenotype -1 cooperates with phenotype 0 and itself at level b (i.e. some competition), and cooperates with phenotype 1 at level 1 (i.e. no competition). We only consider symmetric distributions, i.e. $\pi_{-1} = \pi_1$. Taking into account that $\pi_{-1} + \pi_0 + \pi_1 = 1$, the dynamical system (7.6) has only one variable, say π_0. The fitness of the 3 sites in E and the mean fitness are:

$$\begin{aligned}
m_0 &= K_0 b(K_0\pi_0 + 2K_1\pi_1) = b(\pi_0 + \pi_1) \\
m_1 &= K_1(bK_0\pi_0 + (1+b)K_1\pi_1) = \frac{1}{2}(b\pi_0 + \frac{1+b}{2}\pi_1) \\
\overline{m}_\pi &= b\pi_0(\pi_0 + \pi_1) + \pi_1(b\pi_0 + \frac{1+b}{2}\pi_1)
\end{aligned}$$

Thus the dynamical system (7.6) with variable π_0 can be written as a single ordinary differential equation:

$$\begin{aligned}
\partial_t \pi_0 &= \pi_0(m_0 - \overline{m}_\pi) + \mu(1 - 3\pi_0) \\
&= \pi_0\left(b(\pi_0 + \pi_1) - b\pi_0(\pi_0 + \pi_1) - \pi_1(b\pi_0 - \frac{1+b}{2}\pi_1)\right) + \mu(1 - 3\pi_0)
\end{aligned}$$

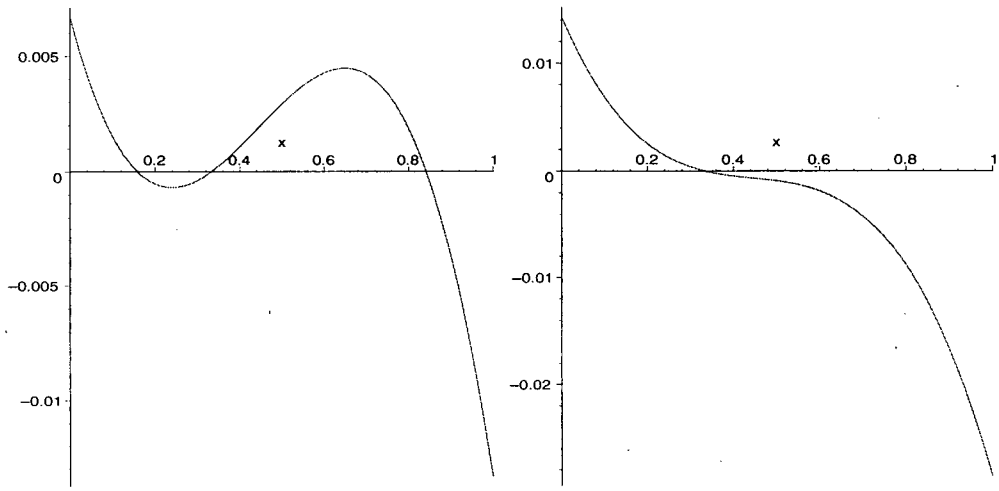
Substituting in $\pi_1 = \frac{1}{2}(1 - \pi_0)$, we get

$$\partial_t \pi_0 = -\frac{b+1}{8}\pi_0^3 + \frac{1-b}{4}\pi_0^2 + \frac{3b - 1 - 24\mu}{8}\pi_0 + \mu. \tag{7.23}$$

Now we take $b = \frac{1}{5}$, then (7.23) can be simplified:

$$\begin{aligned}
\partial_t \pi_0 &= -\frac{3}{20}\pi_0^3 + \frac{1}{5}\pi_0^2 - \left(\frac{1}{20} + 3\mu\right)\pi_0 + \mu \\
&= -\frac{3}{20}\left(\pi_0^3 - \frac{4}{3}\pi_0^2 + \left(\frac{1}{3} + 20\mu\right)\pi_0 - \frac{20}{3}\mu\right).
\end{aligned}$$

We define the polynomial $p(x) = x^3 - \frac{4}{3}x^2 + (\frac{1}{3} + 20\mu)x - \frac{20}{3}\mu$, then p has roots $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{2}(1 + \sqrt{1 - 80\mu})$, and $x_3 = \frac{1}{2}(1 - \sqrt{1 - 80\mu})$. Notice that the root x_1 does not depend on μ, although this is only true for $b = \frac{1}{5}$. For other b's, all roots depend on μ. Two plots of $-\frac{3}{20}p(x)$ are shown in figure 7.1.

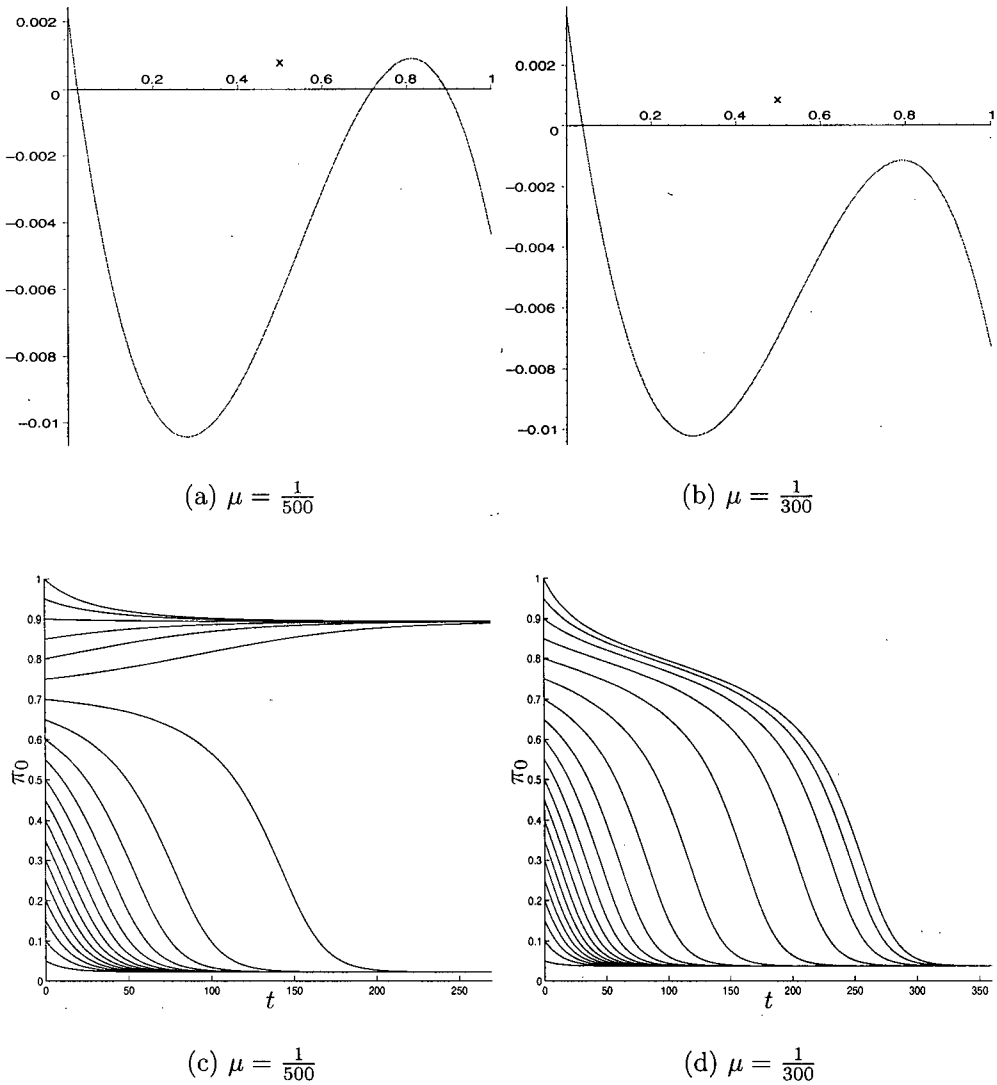


(a) $\mu = \frac{1}{150}$ (b) $\mu = \frac{1}{70}$

Figure 7.1: One dimensional case: $b = \frac{1}{5}$

For $\mu < \frac{1}{80}$, there are three real roots, as in figure 7.1(a). In this case, if $\pi_0(0) < \frac{1}{3}$, then $\pi_0(t) \to x_3 \approx 0.16$ as $t \to \infty$; since $\pi_{-1} = \pi_1$ and $\pi_{-1} + \pi_0 + \pi_1 = 1$, $\pi_{-1} = \pi_1 \to 0.42$ as $t \to \infty$. This distribution has much larger mass on sites -1 and 1 than on site 0, thus we can say that speciation occurs eventually. But if $\pi_0(0) > \frac{1}{3}$, then $\pi_0(t) \to x_2 \approx 0.84$ as $t \to \infty$, then $\pi_{-1} = \pi_1 \to 0.08$. This distribution has much larger mass on site 0 than on sites -1 and 1, and we say that speciation never occurs. On the other hand, if $\mu > \frac{1}{80}$, there is only one real root, as in figure (7.1(b). In this case, regardless of initial condition, $\pi_0(t) \to \frac{1}{3}$ as $t \to \infty$.

Two plots for $b = \frac{1}{10}$ are shown below. For sufficiently large μ, e.g. $\mu = \frac{1}{300}$ in figure 7.2(b), regardless of initial condition, π converges to a configuration with little mass in the middle (approximately 0.037), i.e. speciation occurs. But for μ sufficiently small, e.g. $\mu = \frac{1}{500}$ in figure 7.2(a), there may or may not be speciation depending on the initial condition. The time evolution of π_0 with different initial conditions for both $\mu = \frac{1}{300}$ and $\mu = \frac{1}{500}$ are also shown.



(a) $\mu = \frac{1}{500}$ (b) $\mu = \frac{1}{300}$

(c) $\mu = \frac{1}{500}$ (d) $\mu = \frac{1}{300}$

Figure 7.2: One dimensional case: $b = \frac{1}{10}$

7.2.2 Large enough μ

Analysis of the dynamical system (7.6) in its simplest form in the last subsection shows that if μ is small, there may be two vastly different types of stationary points for (7.6). In subsections 7.2.3 and 7.2.4, we establish this for (7.6) in its general form. But in this section, we examine the behaviour of (7.6) when $\mu \geq \frac{b}{4K_L^2(L-l)}$ and establish that all stationary points of (7.6) are bimodal.

We maintain the assumption in (7.10) that K is symmetric and unimodal with $K_0 = 1$, and B is of the form $B_x = b + (1-b)1_{\{|x| \geq M\}}$ with $b \in [0,1]$ and $L < M \leq 2L$. We will need a uniform (in μ) lower bound on $\overline{m}_{\hat{\pi}^\mu}$. We first establish a crude lower bound on $\overline{m}_{\hat{\pi}^\mu}$ that does depend on μ. Recall from (7.8) that $\hat{\pi}^\mu$ is a stationary distribution if and only if for all $x \in [-L, L]$,

$$\hat{m}_x^\mu - \overline{m}_{\hat{\pi}^\mu} + \mu\left(\frac{1}{\hat{\pi}_x^\mu} - (2L+1)\right) = 0. \tag{7.24}$$

Therefore

$$\mu\left(\frac{1}{\hat{\pi}_x^\mu} - (2L+1)\right) = \overline{m}_{\hat{\pi}} - \hat{m}_x^\mu \leq \overline{m}_{\hat{\pi}} \leq \sup_{x \in E, \pi \in \Delta} m_x(\pi).$$

Since K and B lie in $[0,1]$, $m_x = K_x \sum_z B_{x-z} K_z \pi_z \leq \sum_z \pi_z = 1$, and therefore

$$\mu\left(\frac{1}{\hat{\pi}_x^\mu} - (2L+1)\right) \leq 1,$$

which means that

$$\hat{\pi}_x^\mu \geq \frac{1}{2L + 1 + \frac{1}{\mu}} \geq \mu. \tag{7.25}$$

For $x \in [l, L]$, we have (recall the first three steps of (7.13)),

$$\begin{aligned} m_x(\pi) &= bK_x \sum_z K_z \pi_z + (1-b)K_x \sum_{z=-L}^{x-M} K_z \pi_z \\ &\geq bK_L \sum_z K_z \pi_z + (1-b)K_L \sum_{z=-L}^{x-M} K_z \pi_z \qquad \text{since } K_x \text{ is decreasing in } [0, L] \\ &\geq bK_L^2 + (1-b)K_L^2 \\ &= K_L^2. \end{aligned} \tag{7.26}$$

Since K and B are both symmetric, the same bound applies if $x \in [-L, -l]$. Therefore (7.25) and (7.26) imply the following estimate of the mean fitness $\overline{m}_\pi$:

$$\begin{aligned} \overline{m}_{\hat{\pi}^\mu} &= \sum_x m_x(\hat{\pi}^\mu)\hat{\pi}_x^\mu \geq \sum_{x=l}^{L} m_x(\hat{\pi}^\mu)\hat{\pi}_x^\mu + \sum_{x=-L}^{-l} m_x(\hat{\pi}^\mu)\hat{\pi}_x^\mu \\ &\geq 2\sum_{x=l}^{L} K_L^2 \mu = 2K_L^2 \mu (L-l). \end{aligned} \tag{7.27}$$

This crude lower bound on $\overline{m}_{\hat{\pi}^\mu}$ depends on μ, but in Lemma 7.2.1 we will improve it such that it does not depend on μ for sufficiently small b. For now, we use it to establish an estimate on $\hat{\pi}^\mu([-l+1, l-1])$ that will be needed for the proof of Lemma 7.2.1 below. Condition (7.24) implies that for $x \in [-l+1, l-1]$,

$$\mu\left(\frac{1}{\hat{\pi}^\mu_x} - (2L+1)\right) = \overline{m}_{\hat{\pi}^\mu} - \hat{m}^\mu_x. \tag{7.28}$$

For $x \in [-l+1, l-1]$, $\sum_z B_{x-z} K_z \pi_z = b \sum_z K_z \pi_z$ is constant, thus the maximum fitness is attained at $x = 0$, with

$$m_0(\pi) = bK_0 \sum_z K_z \pi_z \leq b, \tag{7.29}$$

since K_x is increasing in $[-L, 0]$ and decreasing in $[0, L]$. If $b \leq 2K_L^2 \mu(L-l)$, then we can apply the estimate (7.27) on $\overline{m}_\pi$, and bound the right hand side of (7.28):

$$\overline{m}_{\hat{\pi}^\mu} - \hat{m}^\mu_x \geq \overline{m}_{\hat{\pi}^\mu} - \hat{m}^\mu_0 \geq 2K_L^2 \mu(L-l) - b,$$

which is ≥ 0 if $b \leq 2K_L^2\mu(L-l)$. Thus for $b \leq 2K_L^2\mu(L-l)$, (7.28) implies that $\frac{1}{\hat{\pi}^\mu_x} - (2L+1) \geq 0$, i.e.

$$\hat{\pi}^\mu_x \leq \frac{1}{2L+1}.$$

Hence we can bound the mass in $[-l+1, l-1]$:

$$\hat{\pi}^\mu([-l+1, l-1]) \leq \frac{2l+1}{2L+1}. \tag{7.30}$$

Before we state the theorem of this section, we establish the following lemma, which improves upon the bound (7.27) such that it does not depend on μ:

Lemma 7.2.1. *For μ and b in the region $R_1 = \{(\mu, b) : 0 \leq b \leq \min(4\mu K_L^2(L-l), \frac{K_L^2(L-l)}{4(2L+1)^3})\}$, there is a positive constant c_1 that depends on L, l, and K but not on μ, such that $\overline{m}_{\hat{\pi}^\mu} > c_1$ for any stationary distribution $\hat{\pi}^\mu$.*

Proof. We notationally suppress the dependence of $\hat{\pi}^\mu$, $\hat{m}^\mu_x$, and $\overline{m}_{\hat{\pi}^\mu}$ on μ. Suppose that

$$\hat{\pi}_x \hat{\pi}_y < \delta \quad \text{for all } x \in [-L, -l] \text{ and } y \in [x+M, L], \tag{7.31}$$

where

$$\delta = \frac{K_L^2(L-l)}{2(2L+1)^3} - b \tag{7.32}$$

is a positive constant if $(\mu, b) \in R_1$. The pairs (x, y), with $x \in [-L, -l]$ and $y \in [x+M, L]$, are exactly those that contribute weight 1 to the calculation of the mean fitness $\overline{m}_{\hat{\pi}}$ as defined

in (7.3), while the pairs (x', y') with $x' \in [-L, -l]$ and $y' \in [-L, x+M-1]$ contribute only weight b. Condition (7.31) implies that

$$\begin{aligned}
\overline{m}_{\hat{\pi}} &= \sum_{|x-z|\geq M} K_x K_z \hat{\pi}_x \hat{\pi}_z + b \sum_{|x-z|<M} K_x K_z \hat{\pi}_x \hat{\pi}_z \\
&\leq K_0^2 \sum_{|x-z|\geq M} \delta + b \sum_{|x-z|<M} K_0^2 \\
&\leq 2(L-l+1)^2\delta + 2b(2L+1)^2 \\
&\leq 2(2L+1)^2(\delta+b). \qquad (7.33)
\end{aligned}$$

Recall from (7.30) that $\hat{\pi}([-l+1, l-1]) \leq \frac{2l+1}{2L+1}$, which is equivalent to $\hat{\pi}([-L,-l] \cup [l, L]) \geq \frac{2(L-l)}{2L+1}$. Therefore either $\hat{\pi}([-L,-l]) \geq \frac{L-l}{2L+1}$ or $\hat{\pi}([l,L]) \geq \frac{L-l}{2L+1}$. Suppose $\hat{\pi}([l,L]) \geq \frac{L-l}{2L+1}$, then we can bound the fitness of site $-L$:

$$\begin{aligned}
\hat{m}_{-L} &= K_{-L} \sum_{z=-L}^{L} B_{-L-z} K_z \hat{\pi}_z \geq K_{-L} \sum_{z=l}^{L} K_z \hat{\pi}_z \\
&\geq K_L^2 \sum_{z=l}^{L} \hat{\pi}_z = K_L^2 \hat{\pi}([l,L]) \geq K_L^2 \frac{L-l}{2L+1}. \qquad (7.34)
\end{aligned}$$

Since $\delta + b = \frac{K_L^2(L-l)}{2(2L+1)^3}$, (7.33) and (7.34) together imply that

$$\hat{m}_{-L} - \overline{m}_{\hat{\pi}} \geq K_L^2 \frac{L-l}{2L+1} - 2(2L+1)^2(\delta+b) = 0.$$

As a result of the inequality above, condition (7.24) implies that

$$\frac{1}{\hat{\pi}_{-L}} - (2L+1) \leq 0,$$

hence

$$\hat{\pi}_{-L} \geq \frac{1}{2L+1}. \qquad (7.35)$$

Combining the two bounds (7.34) and (7.35) on $\hat{\pi}_{-L}$ and $\hat{m}_{-L}$, we have

$$\overline{m}_{\hat{\pi}} = \sum_x \hat{\pi}_x \hat{m}_x \geq \hat{\pi}_{-L} \hat{m}_{-L} \geq K_L^2 \frac{L-l}{(2L+1)^2}. \qquad (7.36)$$

By similar reasoning, if $\hat{\pi}([-L,-l]) \geq \frac{L-l}{2L+1}$, then we have $\hat{m}_L \geq K_L^2 \frac{L-l}{2L+1}$ and $\hat{\pi}_L > \frac{1}{2L+1}$, which also implies that $\overline{m}_{\hat{\pi}} \geq K_L^2 \frac{L-l}{(2L+1)^2}$. Therefore condition (7.31) implies that $\overline{m}_{\hat{\pi}} \geq K_L^2 \frac{L-l}{(2L+1)^2}$. This would actually contradict (7.33) for sufficiently small δ and b, but it does not matter since in that case, it just says that condition (7.31) is impossible and the analysis below applies.

On the other hand, if condition (7.31) is not satisfied, i.e. there is at least one pair of phenotypes, say $(\tilde{x}, \tilde{y})$, with $\tilde{x} \in [-L, -l]$ and $\tilde{y} \in [\tilde{x}+M, L]$, such that $\hat{\pi}_{\tilde{x}} \hat{\pi}_{\tilde{y}} > \delta$. Then it is easy to see that

$$\overline{m}_{\hat{\pi}} \geq \sum_{|x-z|\geq M} K_x K_z \pi_x \pi_z \geq K_L^2 \hat{\pi}_{\tilde{x}} \hat{\pi}_{\tilde{y}} \geq K_L^2 \delta \geq \frac{K_L^4(L-l)}{4(2L+1)^3}, \qquad (7.37)$$

the last inequality due to (7.32) and the requirement $b \leq \frac{K_L^2(L-l)}{4(2L+1)^3}$. Therefore combining (7.36) and (7.37), we conclude that

$$\overline{m}_{\hat{\pi}} \geq \min\left(\frac{K_L^2(L-l)}{(2L+1)^2}, \frac{K_L^4(L-l)}{4(2L+1)^3}\right) > 0.$$

This bound is uniform for any positive (μ, b) satisfying the condition $b \leq 2K_L^2\mu(L-l)$ and $b \leq \frac{K_L^2(L-l)}{4(2L+1)^3}$ and the conclusion of the lemma follows. □

Theorem 7.2.2. *If K is symmetric and unimodal with $K_0 = 1$, and $B_x = b+(1-b)1_{\{|x|\geq M\}}$ with $L < M \leq 2L$, then with the constant $c_1 = c_1(K, L, l)$ defined in Lemma 7.2.1, we have $\hat{\pi}_x^\mu \leq \frac{2\mu}{c_1}$ for $x \in [-l+1, l-1]$ and (μ, b) lying in*

$$R = \left\{(\mu, b) : 0 \leq b \leq \min\left(4\mu K_L^2(L-l), \frac{c_1}{2}, \frac{K_L^2(L-l)}{4(2L+1)^3}\right)\right\},$$

Proof. Recall from (7.29) that for $x \in [-l+1, l-1]$, the maximum fitness is attained at $x = 0$, and $m_0(\pi) \leq b$. If $b \leq \frac{c_1}{2}$, with c_1 from Lemma 7.2.1, then $m_0(\pi) \leq \frac{c_1}{2}$. Hence for μ and b lying in $R \subset R_1$, where R_1 is defined in Lemma 7.2.1, Lemma 7.2.1 implies that

$$\overline{m}_{\hat{\pi}^\mu} - \hat{m}_0^\mu \geq \frac{c_1}{2}. \tag{7.38}$$

Condition (7.24) and (7.38) together imply that for $x \in [-l+1, l-1]$ and $(\mu, b) \in R$,

$$\mu\left(\frac{1}{\hat{\pi}_x^\mu} - (2L+1)\right) = \overline{m}_{\hat{\pi}^\mu} - \hat{m}_x^\mu \geq \frac{c_1}{2}.$$

Therefore for $x \in [-l+1, l-1]$ and $(\mu, b) \in R$,

$$\hat{\pi}_x^\mu \leq \frac{1}{\frac{c_1}{2\mu} + 2L + 1} \leq \frac{2\mu}{c_1},$$

and the proof is complete. □

Remark 7.2.3. *If $b = 0$, then Theorem 7.2.2 works for any μ, no matter how small.*

Remark 7.2.4. *The constant c_1 in Lemma 7.2.1 is small.*

7.2.3 Small μ: Existence of δ-like Stationary Measure

Results from Chapter 7.2.1 for one-dimensional systems indicate that when b and μ are sufficiently small, there should be at least two stationary distributions, one resembling the δ-measure, the other bimodal and having little mass in the middle. In this section, we show the existence of δ-like stationary distributions. More specifically, we establish the following:

Proposition 7.2.5. *Define $k = \min_x |K_x - K_{x-1}|$. If $\mu < \frac{bk}{8}\epsilon_1$, then the set A_1 is an invariant set for the dynamical system (7.6), where we define*

$$A_1 = \{\pi \in \Delta : \pi_x \leq \epsilon_1 \text{ for all } x \neq 0\},$$

with $\epsilon_1 \leq \min(\frac{1}{4L}, \frac{bk}{2K_l^2(L-l+1)}, \frac{k}{16LK_1})$.

Proof. Let $\mu < \frac{bk}{8}\epsilon_1$ be fixed. For $\pi \in A_1$, since $\pi_x < \epsilon_1 < \frac{1}{4L}$, we have $\pi_0 \geq 1 - 2L\epsilon_1 > \frac{1}{2}$. Recall that $l = M - L$ and that for $x \in [-l+1, l-1]$,

$$m_x = bK_x \sum_z K_z \pi_z.$$

Since K_x is increasing in $[-L, 0]$ and decreasing in $[0, L]$ with $K_0 = 1$, sites in $[0, l-1]$ have decreasing fitness, and sites in $[-l+1, 0]$ have increasing fitness, and we also have the following estimates:

$$m_0 - m_1 = b(K_0 - K_1)\sum_z K_z\pi_z \geq bkK_0\pi_0 \geq \frac{bk}{2} \quad \text{since } \pi_0 \geq \tfrac{1}{2}, \tag{7.39}$$

$$m_1 \leq bK_1K_0 \leq bK_1. \tag{7.40}$$

For $\pi \in A_1$ and $x \in [l, L]$,

$$\begin{aligned} m_l - m_x &= K_l b \sum_z K_z\pi_z - K_x\left[b\sum_z K_z\pi_z + (1-b)\sum_{z=-L}^{x-M} K_z\pi_z\right] \\ &= (K_l - K_x)b\sum_z K_z\pi_z - K_x(1-b)\sum_{z=-L}^{x-M} K_z\pi_z \\ &\geq kbK_0\pi_0 - K_l\sum_{z=-L}^{-l} K_z\pi_z \\ &\geq \frac{kb}{2} - K_l^2(L-l+1)\epsilon_1 \qquad \text{by (7.39)} \\ &\geq 0, \end{aligned} \tag{7.41}$$

since we assume $\epsilon_1 \leq \frac{kb}{2K_l^2(L-l+1)}$. By the same calculation as in (7.41), $m_{-l} - m_x \geq 0$ for $x \in [-L, -l]$ as well. Therefore sites in $[-L, -l]$ are less fit than site $-l$, and sites in $[l, L]$ are less fit than site l; furthermore, sites in $[-l, 0]$ have increasing fitness, while sites in $[0, l]$ have decreasing fitness. In particular, among sites in $[-L, -1] \cup [1, L]$, sites 1 and -1 have maximum fitness. Any measure $\pi \in A_1$ looks like the δ-measure. For such measures, the mean fitness $\overline{m}_\pi$ is close to but less than m_0. We estimate the difference between $\overline{m}_\pi$ and m_1:

$$\overline{m}_\pi - m_1 = \sum_x m_x\pi_x - m_1 \geq m_0\pi_0 - m_1 \geq m_0(1 - 2L\epsilon_1) - m_1,$$

since $\pi_0 \geq 1 - 2L\epsilon_1$ from the beginning of the proof. Now using estimates (7.39) and (7.40), we continue estimating $\overline{m}_\pi - m_1$:

$$m_0(1 - 2L\epsilon_1) - m_1 = (1 - 2L\epsilon_1)(m_0 - m_1) - 2L\epsilon_1 m_1 \geq (1 - 2L\epsilon_1)\frac{bk}{2} - 2L\epsilon_1 bK_1.$$

Recalling from the beginning of the proof that $1 - 2L\epsilon_1 > \frac{1}{2}$, we use the assumption $\epsilon_1 \leq \frac{k}{16LK_1}$ to estimate the right hand side of the above inequality:

$$(1 - 2L\epsilon_1)\frac{bk}{2} - 2L\epsilon_1 bK_1 \geq \frac{bk}{4} - 2LbK_1\frac{k}{16LK_1} \geq \frac{bk}{8}.$$

Therefore

$$\overline{m}_\pi - m_1 \geq \frac{bk}{8}. \tag{7.42}$$

Since among sites in $[-L,-1] \cup [1,L]$, sites 1 and -1 have maximum fitness, (7.42) implies that for all $x \neq 0$,

$$m_x - \overline{m}_\pi \leq -\frac{bk}{8}.$$

We write $\partial A_1 = B_1 \cup B_2 \cup B_3$, where

$$\begin{aligned} B_1 &= \{\pi \in \partial A_1 : \pi_x = 0 \text{ for some } x \text{ and } \pi_x \neq \epsilon_1 \text{ for all } x\}, \\ B_2 &= \{\pi \in \partial A_1 : \pi_x = \epsilon_1 \text{ for some } x \text{ and } \pi_x \neq 0 \text{ for all } x\}, \\ B_3 &= \{\pi \in \partial A_1 : \pi_x = \epsilon_1 \text{ for some } x \text{ and } \pi_y = 0 \text{ for some } y\}. \end{aligned}$$

For $\pi \in B_1$, we have shown following (7.4) that $\partial_t \pi_x > 0$ at x where $\pi_x = 0$. Therefore $\partial_t \pi$ points toward the interior of Δ. For $\pi \in B_2$, we have for x where $\pi_x = \epsilon_1$,

$$\begin{aligned} \partial_t \pi_x &= \pi_x(m_x - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_x) \\ &\leq -\epsilon_1 \frac{bk}{8} + \mu \\ &< 0, \end{aligned} \tag{7.43}$$

since $\mu < \frac{bk}{8}\epsilon_1$. Thus $\partial_t \pi$ also points toward the interior of Δ. For $\pi \in B_3$, we can apply (7.4) to sites x where $\pi_x = 0$ and (7.43) to sites y where $\pi_y = \epsilon_1$, and conclude that $\partial_t \pi$ also points toward the interior of Δ. Therefore the set A_1 is invariant for the dynamical system (7.6), as required. □

7.2.4 Small μ: Existence of Bimodal Stationary Measure

Results from Chapter 7.2.3 show that the set A_1, members of which resemble the δ-measure, is invariant for the dynamical system (7.6). In this section, we assume that $\frac{b}{1-b} \leq \frac{K_{M/2}^2}{8}$ and show that there is another invariant set

$$A_2 = \left\{\pi : \pi_x \leq \epsilon_2 \text{ for all } x \notin \{p, -p\}, \text{ and } |\log \tfrac{\pi_p}{\pi_{-p}}| \leq \epsilon_3\right\}, \tag{7.44}$$

where $p = M/2$ and $L < M \leq 2L$,

$$c_2 = \frac{1}{2}\min\left((1-b)\frac{K_p(K_p - K_{p+1})}{8}, b, \frac{K_p^2}{16}\right), \tag{7.45}$$

$$\begin{aligned} \epsilon_2 = \min\Bigg(&\frac{1}{8(2L+1)}, \frac{K_p(K_p - K_{p+1})}{8K_{p+1}^2(L-l+1)}, \frac{K_p^2}{16K_l(L-l+1)}, \\ &\frac{c_2}{16(2L+1)K_p}\Bigg), \end{aligned} \tag{7.46}$$

$$\epsilon_3 = \min\left(\log 2, \log\left(1 + \frac{c_2}{4K_p^2}\right)\right). \tag{7.47}$$

Apparently, M must be even; this is such that $B_{p-(-p)} = 1$ but $B_{(p-1)-(-p)} = b$. Notice that since $M \leq 2L$, $l = M - L \leq M - \frac{M}{2} = \frac{M}{2} = p$, and also $p = \frac{M}{2} \leq L$. Thus $l \leq p \leq L$. Members of A_2 are bimodal distributions with very sharp peaks at sites p and $-p$.

For $\pi \in A_2$, $\pi_p + \pi_{-p} = 1 - \sum_{x \neq p,-p} \pi_x \geq 1 - (2L-1)\frac{1}{8(2L+1)}$ by the condition $\pi_x \leq \frac{1}{8(2L+1)}$ for $x \notin \{p, -p\}$, therefore $\pi_p + \pi_{-p} \geq \frac{7}{8}$. Now since $|\log \frac{\pi_p}{\pi_{-p}}| \leq \log 2$, we have $\frac{1}{2} \leq \frac{\pi_p}{\pi_{-p}} \leq 2$. This means that

$$\min(\pi_p, \pi_{-p}) > \frac{1}{4}, \tag{7.48}$$

for otherwise, say $\pi_p \leq \frac{1}{4}$, then $\pi_{-p} > \frac{5}{8}$, which means that $\frac{\pi_p}{\pi_{-p}} < \frac{2}{5} < \frac{1}{2}$.

We use the same idea that we used to establish the invariance property of A_1 to show that A_2 is also an invariant set for the dynamical system, for $\mu < \frac{3c_2}{8}\epsilon_2$.

Lemma 7.2.6. *For any $\pi \in A_2$, the following estimate holds:*

$$\min(m_p, m_{-p}) - m_x > \frac{c_2}{2} \text{ for } x \in [-L, L] \backslash \{p, -p\}, \tag{7.49}$$

where A_2 and c_2 are defined in (7.44) and (7.45), respectively.

Proof. As in the proof of Proposition 7.2.5, we first establish a few bounds on fitness of various sites for $\pi \in A_2$. For $x = p$,

$$\begin{aligned} m_p &= K_p \left[b \sum_z K_z \pi_z + (1-b) \sum_{z=-L}^{-p} K_z \pi_z \right] \\ &\geq K_p (1-b) K_{-p} \pi_{-p} \geq \frac{(1-b)K_p^2}{4}, \end{aligned} \tag{7.50}$$

where we use (7.48) in the last inequality. For $x \in [p+1, L]$,

$$\begin{aligned} & m_p - m_x \\ &= K_p \left[b \sum_z K_z \pi_z + (1-b) \sum_{z=-L}^{-p} K_z \pi_z \right] - K_x \left[b \sum_z K_z \pi_z + (1-b) \sum_{z=-L}^{x-M} K_z \pi_z \right] \\ &= (K_p - K_x) b \sum_z K_z \pi_z + (K_p - K_x)(1-b) \sum_{z=-L}^{-p} K_z \pi_z - K_x (1-b) \sum_{z=-p+1}^{x-M} K_z \pi_z. \end{aligned}$$

Since $x \in [p+1, L]$ and K_x is decreasing in $[0, L]$, we have $K_p - K_x \geq K_p - K_{p+1} > 0$. We apply these facts, along with (7.48) and the requirements on π_x for $\pi \in A_2$, to the right hand side and obtain

$$\begin{aligned} m_p - m_x &\geq 0 + (K_p - K_{p+1})(1-b) K_{-p} \frac{1}{4} - K_{p+1}(1-b)(L-l+1) K_{-p+1} \epsilon_2 \\ &= (1-b) \left[\frac{K_p (K_p - K_{p+1})}{4} - K_{p+1} K_{p-1} (L-l+1) \epsilon_2 \right] \\ &\geq (1-b) \frac{K_p (K_p - K_{p+1})}{8} \end{aligned} \tag{7.51}$$

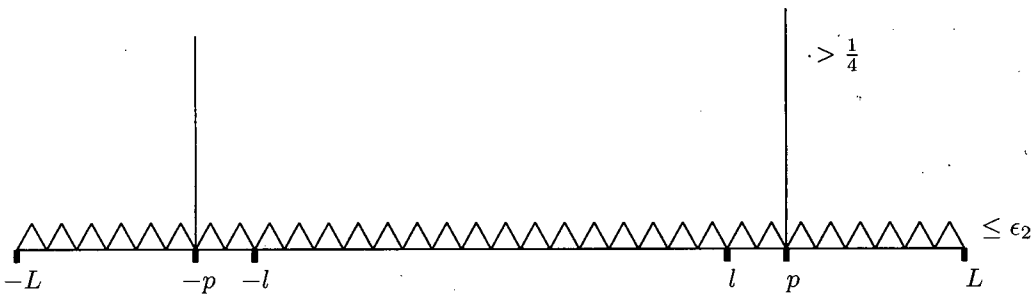


Figure 7.3: Illustration of A_2

for $x \in [p+1, L]$ since $\epsilon_2 \leq \frac{K_p(K_p - K_{p+1})}{8K_{p+1}K_{p-1}(L-l+1)}$. For $x \in [0, l-1]$,

$$m_x = bK_x \sum_z K_z \pi_z \leq bK_x K_0 \leq b,$$

therefore using (7.50) and the inequality above, we have

$$m_p - m_x \geq \frac{(1-b)K_p^2}{4} - b \geq b \tag{7.52}$$

since $\frac{b}{1-b} \leq \frac{K_p^2}{8}$. For $x \in [l, p-1]$,

$$\begin{aligned} m_x &= K_x \left[b \sum_z K_z \pi_z + (1-b) \sum_{z=-L}^{x-M} K_z \pi_z \right] \\ &\leq K_l (bK_0 \sum_z \pi_z + K_0 \sum_{z=-L}^{-l} \epsilon_2) \\ &\leq K_l (b + (L-l+1)\epsilon_2), \end{aligned}$$

therefore using (7.50) and the inequality above, we have

$$\begin{aligned} m_p - m_x &\geq \frac{(1-b)K_p^2}{4} - K_l(b + (L-l+1)\epsilon_2) \\ &= \frac{K_p^2}{4} - b\left(\frac{K_p^2}{4} + K_l\right) - K_l(L-l+1)\epsilon_2. \end{aligned}$$

Since $b \leq \frac{K_p^2}{2K_p^2 + K_l K_0}$, we have $\frac{K_p^2}{4} - b\left(\frac{K_p^2}{4} + K_l\right) \geq \frac{K_p^2}{8}$. Furthermore, since $\epsilon_2 \leq \frac{K_p^2}{16K_l(L-l+1)}$, we have

$$m_p - m_x \geq \frac{K_p^2}{16}. \tag{7.53}$$

The estimates (7.51), (7.52), and (7.53) compare m_p with m_x for $x \in [0, L]$. Similar calculations comparing m_{-p} with m_x for $x \in [-L, 0]$ yield similar results. Then recalling the

definition of c_2 in (7.45), we have

$$\begin{aligned} & m_p - m_x > c_2 \text{ for } x \in [0, L] \backslash \{p\}, \\ \text{and} \quad & m_{-p} - m_x > c_2 \text{ for } x \in [-L, 0] \backslash \{-p\}. \end{aligned} \tag{7.54}$$

To establish the lemma, it suffices to compare m_p and m_{-p}:

$$\begin{aligned} |m_p - m_{-p}| &= \left| K_p \left[b \sum_z K_z \pi_z + (1-b) \sum_{z=-L}^{-p} K_z \pi_z \right] \right. \\ &\quad \left. - K_{-p} \left[b \sum_z K_z \pi_z + (1-b) \sum_{z=p}^{L} K_z \pi_z \right] \right| \\ &= K_p (1-b) \left| \sum_{z=p}^{L} K_z (\pi_{-z} - \pi_z) \right| \\ &\leq K_p (1-b) \sum_{z=p}^{L} K_z |\pi_{-z} - \pi_z|. \end{aligned}$$

Since $K_z \leq K_p$ for $z \in [p, L]$, $K_z \leq K_{-p} = K_p$ for $z \in [-L, -p]$, and $\pi_z \leq \epsilon_2$ for $z \neq -p, p$, we have

$$\begin{aligned} |m_p - m_{-p}| &\leq K_p^2 (1-b)(|\pi_{-p} - \pi_p| + (L-p) 2\epsilon_2) \\ &\leq K_p^2 |\pi_{-p} - \pi_p| + 2K_p^2 (L-p)\epsilon_2. \end{aligned} \tag{7.55}$$

We treat the two terms in the above sum separately. We first deal with the second term $2K_p^2(L-p)\epsilon_2$. Since $L - p < 2L + 1$ and $K_p \geq K_p^2$, we have $\frac{c_2}{16(2L+1)K_p} \leq \frac{c_2}{8(L-p)K_p^2}$. Then the definition of ϵ_2 in (7.46) means that $\epsilon_2 \leq \frac{c_2}{16(2L+1)K_p} \leq \frac{c_2}{8(L-p)K_p^2}$, which implies

$$2K_p^2 (L-p)\epsilon_2 \leq \frac{c_2}{4}. \tag{7.56}$$

For the first term in (7.55), we can divide into two cases: $\pi_p \geq \pi_{-p}$ and $\pi_p \leq \pi_{-p}$. If $\pi_p \geq \pi_{-p}$, then

$$0 \leq \pi_p - \pi_{-p} = \pi_p - e^{\epsilon_3} \pi_{-p} + e^{\epsilon_3} \pi_{-p} - \pi_{-p} = (\pi_p - e^{\epsilon_3} \pi_{-p}) + (e^{\epsilon_3} - 1) \pi_{-p}.$$

The definition of ϵ_3 in (7.47) implies that $\pi_p \leq e^{\epsilon_3} \pi_{-p}$ and $e^{\epsilon_3} - 1 \leq \frac{c_2}{4K_p^2}$, so continuing the calculation in the line above, we obtain

$$|\pi_p - \pi_{-p}| = \pi_p - \pi_{-p} \leq 0 + \frac{c_2}{4K_p^2} \pi_{-p} \leq \frac{c_2}{4K_p^2}. \tag{7.57}$$

If $\pi_p \leq \pi_{-p}$, we get the same bound. Therefore applying (7.56) and (7.57) to (7.55), we have

$$|m_p - m_{-p}| \leq \frac{c_2}{4} + \frac{c_2}{4} = \frac{c_2}{2}.$$

This result means that the estimate in (7.54) can be generalized to

$$\min(m_p, m_{-p}) - m_x > \frac{c_2}{2} \text{ for } x \in [-L, L] \backslash \{p, -p\},$$

as required by the lemma. □

Proposition 7.2.7. *If $\mu < \frac{3c_2}{8}\epsilon_2$, then the set A_2 defined in (7.44) is an invariant set for the dynamical system (7.6).*

Proof. As in the proof of Proposition 7.2.5, it suffices to observe that $\partial_t \pi_x > 0$ where $\pi_x = 0$ as shown by the argument following (7.43), and check that the following inequalities hold:

$$(\partial_t \pi_x)|_{\pi_x = \epsilon_2, \pi \in A_2} < 0 \text{ for } x \neq p, -p, \tag{7.58}$$

$$\left(\partial_t \log \frac{\pi_p}{\pi_{-p}}\right)\Bigg|_{\log \frac{\pi_p}{\pi_{-p}} = \epsilon_3, \pi \in A_2} < 0, \tag{7.59}$$

$$\text{and} \quad \left(\partial_t \log \frac{\pi_{-p}}{\pi_p}\right)\Bigg|_{\log \frac{\pi_{-p}}{\pi_p} = \epsilon_3, \pi \in A_2} < 0. \tag{7.60}$$

The estimate (7.49) means that for $\pi \in A_2$ and $x \notin \{p, -p\}$, m_x is significantly smaller than $\min(m_p, m_{-p})$. But the mean fitness $\overline{m}_\pi$ cannot be much smaller than $\min(m_p, m_{-p})$:

$$\begin{aligned}
\min(m_p, m_{-p}) - \overline{m}_\pi &= \min(m_p, m_{-p}) - \sum_x m_x \pi_x \\
&\leq \min(m_p, m_{-p}) - m_p \pi_p - m_{-p} \pi_{-p} \\
&\leq \min(m_p, m_{-p}) - \min(m_p, m_{-p})(\pi_p + \pi_{-p}) \\
&= (1 - \pi_p - \pi_{-p}) \min(m_p, m_{-p}) \\
&\leq (2L - 1)\epsilon_2 \min(m_p, m_{-p})
\end{aligned}$$

by the requirement that $\pi_x \leq \epsilon_2$ for $x \neq p, -p$ and $\pi \in A_2$. Since $m_p = K_p \sum_z B_{p-z} K_z \pi_z \leq K_p K_0$ and similarly $m_{-p} \leq K_p K_0$, we have

$$\min(m_p, m_{-p}) - \overline{m}_\pi \leq (2L - 1)\epsilon_2 K_p K_0 = (2L - 1)\epsilon_2 K_p.$$

We use the definition of ϵ_2 in (7.46) to obtain $\epsilon_2 \leq \frac{c_2}{16(2L+1)K_p} < \frac{c_2}{8(2L-1)K_p}$, which, when applied to the estimate in the line above, implies that

$$\min(m_p, m_{-p}) - \overline{m}_\pi < \frac{c_2}{8}. \tag{7.61}$$

Now (7.49) and (7.61) imply that for $\pi \in A_2$ and $x \notin \{p, -p\}$,

$$m_x - \overline{m}_\pi < m_x + \frac{c_2}{8} - \min(m_p, m_{-p}) < -\frac{c_2}{2} + \frac{c_2}{8} = -\frac{3c_2}{8}.$$

Thus for $\pi \in \partial A_2 \cap \{\pi_z = \epsilon_2 \text{ for some } z \neq \{p, -p\}\}$, we have for x where $\pi_x = \epsilon_2$,

$$\begin{aligned}
(\partial_t \pi_x)|_{\pi_x = \epsilon_2} &= \pi_x(m_x - \overline{m}_\pi) + \mu(1 - (2L + 1)\pi_x)|_{\pi_x = \epsilon_2} \\
&\leq -\epsilon_2 \frac{3c_2}{8} + \mu \\
&< 0,
\end{aligned}$$

since $\mu < \frac{3c_2}{8}\epsilon_2$. This verifies (7.58).

Now we deal with $\pi \in \partial A_2 \cap \{\log \frac{\pi_p}{\pi_{-p}} = \epsilon_3\}$. By (7.6), we have

$$\begin{aligned}\partial_t \log \frac{\pi_p}{\pi_{-p}} &= (m_p - \overline{m}_\pi + \frac{\mu}{\pi_p} - \mu(2L+1)) - (m_{-p} - \overline{m}_\pi + \frac{\mu}{\pi_{-p}} - \mu(2L+1)) \\ &= m_p - m_{-p} + \mu\left(\frac{1}{\pi_p} - \frac{1}{\pi_{-p}}\right). \end{aligned} \tag{7.62}$$

If $\frac{\pi_p}{\pi_{-p}} = e^{\epsilon_3} > 1$, then $\pi_p > \pi_{-p}$, which means that $\frac{1}{\pi_p} - \frac{1}{\pi_{-p}} < 0$. Therefore it remains to check the sign of $m_p - m_{-p}$:

$$\begin{aligned} & m_p - m_{-p} \\ &= K_p\left[b\sum_z K_z\pi_z + (1-b)\sum_{z=-L}^{-p} K_z\pi_z\right] - K_{-p}\left[b\sum_z K_z\pi_z + (1-b)\sum_{z=p}^{L} K_z\pi_z\right] \\ &= K_p(1-b)\left(K_p(\pi_{-p} - \pi_p) + \sum_{z=p+1}^{L} K_z(\pi_{-z} - \pi_z)\right). \end{aligned} \tag{7.63}$$

If $\frac{\pi_p}{\pi_{-p}} = e^{\epsilon_3}$, then

$$\pi_{-p} - \pi_p = \pi_{-p}(1 - e^{\epsilon_3}) = \pi_{-p}\max(-1, -\frac{c_2}{4K_p^2}) \leq -\frac{c_2}{16K_p^2} \tag{7.64}$$

by the definition of ϵ_3 in (7.47) and the fact $\pi_{-p} > \frac{1}{4}$ established in (7.48). For $\pi \in A_2$ and $z \geq p+1$, $\pi_{-z} - \pi_z \leq 2\epsilon_2$, therefore

$$\sum_{z=p+1}^{L} K_z(\pi_{-z} - \pi_z) \leq (L-p)K_p 2\epsilon_2. \tag{7.65}$$

Applying (7.64) and (7.65) to (7.63), and using the requirement $\epsilon_2 \leq \frac{c_2}{16(2L+1)K_p}$ in (7.46), which implies $\epsilon_2 \leq \frac{c_2}{32K_p^2(L-p)}$, we conclude that $m_p - m_{-p} < 0$ if $\frac{\pi_p}{\pi_{-p}} = e^{\epsilon_3}$. Therefore (7.62) implies that

$$\left(\partial_t \log \frac{\pi_p}{\pi_{-p}}\right)\Bigg|_{\frac{\pi_p}{\pi_{-p}} = e^{\epsilon_3}} < 0,$$

which verifies (7.59). The verification of (7.60) is similar, and the proof is complete. □

Propositions 7.2.5 and 7.2.7 imply that if b is sufficiently small and $\mu < c_3 b^2$, where c_3 is a small constant dependent on K and L, then (7.6) has at least two stationary distributions, one resembling a δ-measure and the other bimodal and having little mass in the middle. But Theorem 7.2.2 imply that if b is sufficiently small and $\mu > c_4 b$, where c_4 is a large constant dependent on K and L, then all stationary distributions are bimodal and have very little mass in the middle. This phenomenon is illustrated in figure 7.4. We conjecture that there is a phase transition between a unique stationary distribution and two stationary distributions in the behaviour of (7.6) for small μ and b. But since we cannot come up with a straightforward comparison argument in either μ or b, this remains a conjecture.

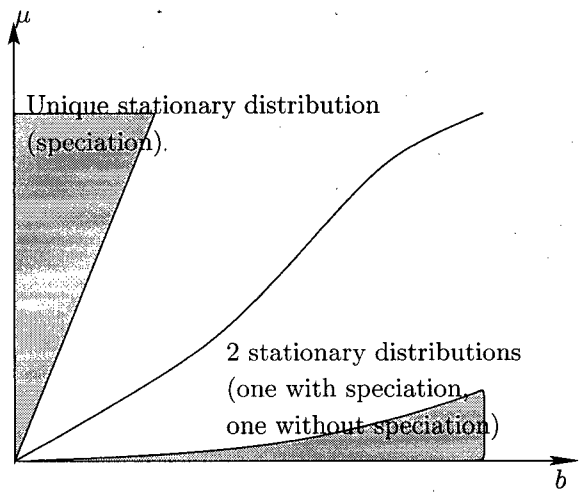


Figure 7.4: Conjecture of phase transition in (7.6) for small μ and b (Shaded region is where Theorem 7.2.2, Propositions 7.2.5 and 7.2.7 work)

Chapter 8

Stationary Distributions

In Chapter 7, we examined the large-time behaviour of the deterministic dynamical system (7.1), i.e. $\lim_{t\to\infty}\lim_{N\to\infty}\pi^N(t)$. In this chapter, however, we will take the limit $t\to\infty$ first and examine the behaviour of $\nu^{\mu,N}$, the stationary distribution of π^N; more specifically, we will do this by taking the limits $N\to\infty$ then $\mu\to 0$ and examine $\lim_{\mu\to 0}\lim_{N\to\infty}\nu^{\mu,N}$. For this, we consider the case of symmetric, strictly positive, and unimodal K, with K_x strictly decreasing for $x\in[0,L]$ and $K_0=1$, and $B_x=1_{\{|x|\geq M\}}$ with $L<M\leq 2L$. Define $l=-L+M$, then $m_x=0$ for $x\in[-l+1,l-1]$. Define $c_3=\sup_\pi \overline{m}_\pi$. For the strong selection model with N particles and mutation rate μ described in Chapter 6.2.1, we observe that this continuous-time finite-state Markov process has the property that all states communicate, and therefore it has a unique stationary distribution $\nu^{\mu,N}$ [Durrett 1991]. Let $(\mathcal{G}^N,\mathcal{D}(\mathcal{G}^N))$ denote the generator associated with this Markov process, then for all $F\in C^\infty(\mathcal{P}(E))\subset\mathcal{D}(\mathcal{G}^N)$, we have

$$\int \mathcal{G}^N F(\pi)\nu^{\mu,N}(d\pi)=0. \tag{8.1}$$

Let $(\mathcal{G},\mathcal{D}(\mathcal{G}))$ denote the generator associated with the deterministic process described by the ODE (6.10):

$$\partial_t\pi_x=\pi_x(m(x,\pi)-\overline{m}_\pi)+\mu(1-(2L+1)\pi_x).$$

We calculate the effect of the generator $\mathcal{G}^N$ on a C^∞-function $F(\pi^N)=F(\pi^N_{-L},\ldots,\pi^N_0,\ldots,\pi^N_L)$:

$$\begin{aligned}
\mathcal{G}^N F(\pi^N) &= \sum_x\sum_{y\neq x}\left[F\left(\pi^N-\frac{\delta_x}{N}+\frac{\delta_y}{N}\right)-F(\pi^N)\right]\left(N\pi^N_x m(y,\pi^N)\pi^N_y+N\mu\pi^N_x\right)\\
&= N\sum_x\sum_{y\neq x}\left[F\left(\pi^N_{-L},\ldots,\pi^N_x-\frac{1}{N},\ldots,\pi^N_y+\frac{1}{N},\ldots,\pi^N_L\right)-F(\pi^N)\right]\\
&\quad\times\pi^N_x\left(m(y,\pi^N)\pi^N_y+\mu\right).
\end{aligned}$$

Performing a Taylor expansion on F, we continue the above computation:

$$\begin{aligned}
&\mathcal{G}^N F(\pi^N) \\
&= N\sum_x\sum_{y\neq x}\left[-\frac{1}{N}\frac{\partial F}{\partial \pi_x^N}(\pi^N)+\frac{1}{N}\frac{\partial F}{\partial \pi_y^N}(\pi^N)+O(N^{-2})\right]\pi_x^N\left(m(y,\pi^N)\pi_y^N+\mu\right) \\
&= \sum_x\sum_{y\neq x}\left[\frac{\partial F}{\partial \pi_y^N}(\pi^N)-\frac{\partial F}{\partial \pi_x^N}(\pi^N)\right]\pi_x^N\left(m(y,\pi^N)\pi_y^N+\mu\right)+R_1(N,F)(\pi^N). \qquad (8.2)
\end{aligned}$$

Since $\sup_{x,y,\pi}\pi_x^N(m(y,\pi^N)+\mu)O(N^{-2})=O(N^{-2})$, we have

$$R_1(N,F)(\pi)\leq\frac{C\|F\|}{N},$$

where we suppress the dependence on L. Therefore $R_1(N,F)(\pi)$ is $O(N^{-1})$ uniformly for all $\pi\in\mathcal{P}(E)$. Here we use the norm

$$\|F\|=\int\left(1+\sum_{k=-L}^{L}\xi_k^2\right)|\hat{F}(\xi)|^2d\xi$$

associated with the Sobolev space $H^2(\mathbb{R}^E)$ for F, where $\hat{F}$ denotes the Fourier transform of F. The first term in (8.2) is in fact equal to $\mathcal{G}F(\pi^N)$ by the following computation:

$$\begin{aligned}
&\sum_x\sum_{y\neq x}\left[\frac{\partial F}{\partial \pi_y}(\pi)-\frac{\partial F}{\partial \pi_x}(\pi)\right]\pi_x(m(y,\pi)\pi_y+\mu) \\
&= \sum_x\sum_y\left[\frac{\partial F}{\partial \pi_y}(\pi)-\frac{\partial F}{\partial \pi_x}(\pi)\right]\pi_x(m(y,\pi)\pi_y+\mu) \\
&= \sum_x\pi_x\sum_y\frac{\partial F}{\partial \pi_y}(\pi)(m(y,\pi)\pi_y+\mu)-\sum_y(m(y,\pi)\pi_y+\mu)\sum_x\frac{\partial F}{\partial \pi_x}(\pi)\pi_x \\
&= \sum_x\frac{\partial F}{\partial \pi_x}(\pi)(m(x,\pi)\pi_x+\mu)-(\overline{m}_\pi+(2L+1)\mu)\sum_x\pi_x\frac{\partial F}{\partial \pi_x}(\pi) \\
&= \sum_x[\pi_x(m(x,\pi)-\overline{m}_\pi)+\mu(1-(2L+1)\pi_x)]\frac{\partial F}{\partial \pi_x}(\pi) \\
&= \mathcal{G}F(\pi).
\end{aligned}$$

Therefore (8.2) can be written in the following much-simplified form

$$\mathcal{G}^N F(\pi^N)=\mathcal{G}F(\pi^N)+R_1(N,F)(\pi^N).$$

Then (8.1) implies

$$\left|\int\mathcal{G}F(\pi)\nu^{\mu,N}(d\pi)\right|=\left|\int R_1(N,F)(\pi)\nu^{\mu,N}(d\pi)\right|\leq\frac{C\|F\|}{N}\left|\int\nu^{\mu,N}(d\pi)\right|=\frac{C\|F\|}{N},$$

i.e.,

$$\int\mathcal{G}F(\pi)\nu^{\mu,N}(d\pi)=O(N^{-1}). \qquad (8.3)$$

Since E is compact, so is $\mathcal{P}(E)$ and $\mathcal{P}(\mathcal{P}(E))$, therefore for each μ, we can take a sequence $N_k(\mu)$ such that $\nu^{\mu,N_k(\mu)}$ converges weakly to some $\nu^\mu \in \mathcal{P}(\mathcal{P}(E))$. By (8.3), ν^μ satisfies: for all $F \in C^\infty(\mathcal{P}(E))$,

$$\begin{aligned}\int \mathcal{G}F(\pi)\nu^\mu(d\pi) &= \int \sum_x [\pi_x(m(x,\pi) - \overline{m}_\pi) + \mu(1-(2L+1)\pi_x)]\frac{\partial F}{\partial \pi_x}(\pi)\nu^\mu(d\pi) \\ &= 0. \end{aligned} \tag{8.4}$$

Therefore ν^μ is an stationary distribution for the deterministic flow $\mathcal{G}$. Now we take a sequence $\mu_k \to 0$, such that ν^{μ_k} converges weakly to some $\nu^0 \in \mathcal{P}(\mathcal{P}(E))$, and by (8.4) and the following estimate: as $\mu \to 0$,

$$\left|\int \sum_x \mu(1-(2L+1)\pi_x)\frac{\partial F}{\partial \pi_x}(\pi)\nu^\mu(d\pi)\right| \le \mu C(F,L)\left|\int \nu^\mu(d\pi)\right| = \mu C(F,L) \to 0,$$

ν^0 satisfies:

$$\int \sum_x \pi_x(m(x,\pi) - \overline{m}_\pi)\frac{\partial F}{\partial \pi_x}(\pi)\nu^0(d\pi) = 0. \tag{8.5}$$

After establishing several lemmas, we will use the above characterization of ν^μ and ν^0 to prove the following:

Theorem 8.0.8. *Suppose K is symmetric, unimodal, and strictly decreasing for $x \in [0,L]$, with $K_0 = 1$, and $B_x = 1_{\{|x|\ge M\}}$ with $L < M \le 2L$. Define $l = -L + M$. If ν^μ is a weak limit point of $\nu^{\mu,N}$, then for any $z \in [-l+1, l-1]$, we have*

$$\nu^\mu\{\pi : \pi_z \ge \delta\} \le \frac{1}{\delta}\frac{\mu}{\frac{\delta_2}{2} + \mu(2L+1)},$$

where $\delta_2 = \min\left(\frac{K_L^2}{2(2L+1)}, \frac{K_L^4}{4(2L+1)K_l^2}, \frac{K_L^2}{2(2L+1)^2}\right)$.

Corollary 8.0.9. *Under the same assumption on K and B as in Theorem 8.0.8, we have*

$$\nu^0\{\pi : \pi_x = 0 \ \forall x \in [-l+1, l-1]\} = 1$$

if ν^0 is a weak limit point of ν^μ, and consequently, for any $\delta > 0$,

$$\nu^{\mu_i, N_j(\mu_i)}\{\pi : \pi_x < \delta \ \forall x \in [-l+1, l-1]\} \ge 1 - \delta$$

for some sufficiently large i and $j = j_i$.

Proof of Corollary 8.0.9. We recall that if a sequence of random variables $X_n \Rightarrow X_\infty$, then $\liminf_{n\to\infty} P(X_n \in A) \ge P(X_\infty \in A)$ for any open set A. Thus Theorem 8.0.8 implies that for any $z \in [-l+1, l-1]$ and sequence μ_i such that $\nu^{\mu_i} \Rightarrow \nu^0$,

$$\nu^0\{\pi : \pi_z > \delta'\} \le \liminf_{i\to\infty} \nu^{\mu_i}\{\pi : \pi_z > \delta'\} \le \liminf_{i\to\infty} \frac{1}{\delta'}\frac{\mu_i}{\frac{\delta_2}{2} + \mu_i(2L+1)} = 0.$$

This holds for any positive δ', so $\nu^0\{\pi : \pi_z > 0\} = 0$ and therefore,

$$\nu^0\{\pi : \pi_x = 0 \ \forall x \in [-l+1, l-1]\} = 1.$$

□

Lemma 8.0.10. *If $\mu < 1$, then*

$$\nu^\mu \left\{ \pi : \pi_x > \frac{\mu}{4(2L+2)} \ \forall x \in [-L, L] \right\} = 1.$$

Proof. Let $z \in [-L, L]$ be an arbitrary site. Let $\delta < \frac{\mu}{2(2L+2)}$ be a small positive constant and $f \in C^\infty(\mathbb{R})$ be a function that satisfies the following requirements:

(a) $f'(x) = 1$ for $x \leq \frac{\delta}{2}$;

(b) $f'(x) = 0$ for $x \geq \delta$;

(c) $f'(x) \in [0, 1]$ for $x \in [\frac{\delta}{2}, \delta]$; and

(d) $f(1) = 0$.

Define $F(\pi_{-L}, \ldots, \pi_L) = f(\pi_z)$. Then (8.4) implies:

$$\int [\pi_z(m_z - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_z)] f'(\pi_z) \nu^\mu(d\pi) = 0. \tag{8.6}$$

Since $m_z \geq 0$ and $\overline{m}_\pi$ is bounded above by 1 uniformly in π, we have $m_z - \overline{m}_\pi > -1$. The integrand in the above integral is nonzero only for $\pi_z \in [0, \delta]$, therefore (8.6) can be rewritten as:

$$\int_{\{\pi : \pi_z \leq \delta\}} [\pi_z(m_z - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_z)] f'(\pi_z) \nu^\mu(d\pi) = 0. \tag{8.7}$$

Furthermore, the integrand is bounded below:

$$\begin{aligned} \pi_z(m_z - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_z) &\geq -\pi_z + \mu(1 - (2L+1)\pi_z) \\ &= -(1 + \mu(2L+1))\pi_z + \mu. \end{aligned} \tag{8.8}$$

Since $\mu < 1$, we have $\delta < \frac{\mu}{2(1+2L+1)} < \frac{\mu}{2(1+\mu(2L+1))}$, and therefore if $\pi_z \in [0, \delta] \subset [0, \frac{\mu}{2(1+\mu(2L+1))}]$, then $(1 + \mu(2L+1))\pi_z < \frac{\mu}{2}$. Thus (8.8) implies that

$$\pi_z(m_z - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_z) > \frac{\mu}{2}. \tag{8.9}$$

Applying the estimate (8.9) to (8.7), we obtain

$$\begin{aligned} 0 &\geq \frac{\mu}{2} \int_{\{\pi : \pi_z \leq \delta\}} f'(\pi_z) \nu^\mu(d\pi) \\ &\geq \frac{\mu}{2} \int_{\{\pi : \pi_z \leq \delta/2\}} f'(\pi_z) \nu^\mu(d\pi) \qquad \text{since } f'(x) \geq 0 \text{ for } x \in [\delta/2, \delta] \\ &= \frac{\mu}{2} \int_{\{\pi : \pi_z \leq \delta/2\}} \nu^\mu(d\pi) \qquad \text{since } f'(x) = 1 \text{ for } x \in [0, \delta/2] \\ &= \frac{\mu}{2} \nu^\mu \left\{ \pi : \pi_z \leq \frac{\delta}{2} \right\}. \end{aligned}$$

Therefore

$$\nu^{\mu}\left\{\pi : \pi_z \leq \frac{\delta}{2}\right\} = 0$$

which implies that

$$\nu^{\mu}\left\{\pi : \pi_x \leq \frac{\delta}{2} \text{ for some } x \in [-L, L]\right\} \leq \sum_{x=-L}^{L} \nu^{\mu}\left\{\pi : \pi_x \leq \frac{\delta}{2}\right\} = 0.$$

Thus the lemma follows. □

Lemma 8.0.10 implies that ν^{μ}-a.s.,

$$\overline{m}_{\pi} \geq \pi_{-L} K_{-L} K_L \pi_L \geq \frac{\mu^2 K_L^2}{16(2L+2)^2}. \tag{8.10}$$

Recall that $\delta_2 = \min\left(\frac{K_L^2}{2(2L+1)}, \frac{K_L^4}{4(2L+1)K_l^2}, \frac{K_L^2}{2(2L+1)^2}\right)$ is independent of μ. We define $\delta_1 = \frac{\mu^2 K_L^2}{16(2L+2)^2}$ and

$$A = \{\pi : \overline{m}_{\pi} < \delta_2\}, \tag{8.11}$$

and the function $\psi : \mathcal{P}(E) \to \mathbb{R}$,

$$\psi(\pi) = \sum_x \pi_x (m_x - \overline{m}_{\pi})^2 + \mu(2L+1) \sum_x m_x \left(\frac{1}{2L+1} - \pi_x\right). \tag{8.12}$$

We recall from a formula following (7.7) that $\frac{\partial \overline{m}_{\pi}}{\partial \pi_x} = 2m_x$, and observe that

$$\begin{aligned}
\mathcal{G}\overline{m}_{\pi} &= \sum_x [\pi_x(m_x - \overline{m}_{\pi}) + \mu(1 - (2L+1)\pi_x)] \frac{\partial \overline{m}_{\pi}}{\partial \pi_x} \\
&= \sum_x [\pi_x(m_x - \overline{m}_{\pi}) + \mu(1 - (2L+1)\pi_x)] 2m_x \\
&= 2\sum_x \pi_x m_x (m_x - \overline{m}_{\pi}) + \mu(2L+1) \sum_x m_x \left(\frac{1}{2L+1} - \pi_x\right).
\end{aligned}$$

Adding $-2\sum_x \pi_x \overline{m}_{\pi}(m_x - \overline{m}_{\pi}) = 0$ to the right hand side, we continue as follows:

$$\begin{aligned}
\mathcal{G}\overline{m}_{\pi} &= 2\left[\sum_x \pi_x m_x (m_x - \overline{m}_{\pi}) - \sum_x \pi_x \overline{m}_{\pi}(m_x - \overline{m}_{\pi})\right. \\
&\qquad \left. + \mu(2L+1) \sum_x m_x \left(\frac{1}{2L+1} - \pi_x\right)\right] \\
&= 2\left[\sum_x \pi_x (m_x - \overline{m}_{\pi})^2 + \mu(2L+1) \sum_x m_x \left(\frac{1}{2L+1} - \pi_x\right)\right].
\end{aligned}$$

Therefore

$$\mathcal{G}\overline{m}_{\pi} = 2\psi(\pi). \tag{8.13}$$

We will establish in the following two lemmas that on the set A defined in (8.11), $\psi(\pi)$ is bounded below by $\frac{1}{4(2L+1)}\overline{m}_\pi^2$. We write A as a disjoint union of two sets A_1 and A_2, where

$$\begin{aligned} A_1 &= \left\{\pi \in A : \pi_x \le \frac{1}{2L+1} \quad \text{for all } x \text{ with } m_x \ne 0\right\}, \\ A_2 &= \left\{\pi \in A : \pi_x > \frac{1}{2L+1} \quad \text{for some } x \text{ with } m_x \ne 0\right\}, \end{aligned}$$

and prove a lemma for each case.

Lemma 8.0.11. *For any $\pi \in A_1$, we have $\psi(\pi) \ge \frac{1}{2L+1}\overline{m}_\pi^2$.*

Proof. For any $\pi \in A_1$, there are two cases:

Case 1 $\pi_x \le \frac{1}{2L+1}$ for all $x \in [-L,-l] \cup [l,L]$;

Case 2 $\pi_x > \frac{1}{2L+1}$ and $m_x = 0$ for some (possibly more than one) $x \in [-L,-l] \cup [l,L]$, but for any x with $m_x \ne 0$, π_x is still $\le \frac{1}{2L+1}$.

For Case 1, (8.12) implies

$$\psi(\pi) \ge \sum_{x=-l+1}^{l-1} \pi_x (m_x - \overline{m}_\pi)^2 + \mu(2L+1) \sum_{x:m_x \ne 0} m_x \left(\frac{1}{2L+1} - \pi_x\right).$$

Since $m_x = 0$ for $x \in [-l+1, l-1]$ and $\pi_x \le \frac{1}{2L+1}$ for all $x \notin [-l,l]$, the second sum on the right hand side is nonnegative, and therefore

$$\psi(\pi) \ge \sum_{x=-l+1}^{l-1} \pi_x (m_x - \overline{m}_\pi)^2 = \sum_{x=-l+1}^{l-1} \pi_x \overline{m}_\pi^2, \tag{8.14}$$

Using

$$\sum_{x=-l+1}^{l-1} \pi_x = 1 - \sum_{x=-L}^{-l} \pi_x - \sum_{x=l}^{L} \pi_x \ge \frac{2l-1}{2L+1},$$

(8.14) implies that

$$\psi(\pi) \ge \frac{2l-1}{2L+1}\overline{m}_\pi^2 \ge \frac{1}{2L+1}\overline{m}_\pi^2. \tag{8.15}$$

We now turn to Case 2. Suppose y is a site with $\pi_y > \frac{1}{2L+1}$ and $m_y = 0$. We bound $\psi(\pi)$ in (8.12) a bit differently from Case 1. Since $\pi_x \le \frac{1}{2L+1}$ for any x with $m_x \ne 0$, the second sum in (8.12) is nonnegative, and therefore

$$\psi(\pi) \ge \sum_x \pi_x (m_x - \overline{m}_\pi)^2 \ge \pi_y (m_y - \overline{m}_\pi)^2 \ge \frac{1}{2L+1}\overline{m}_\pi^2. \tag{8.16}$$

Therefore (8.15) and (8.16) imply the lemma. □

Lemma 8.0.12. *If $\mu < \frac{K_L^4}{8(2L+1)^4 K_l^2}$, then for any $\pi \in A_2$, we have $\psi(\pi) \geq \frac{\overline{m}_\pi^2}{4(2L+1)}$.*

Proof. For $\pi \in A_2$, the main inconvenience is that the second term in (8.12), i.e. the term involving $\frac{1}{2L+1} - \pi_x$, can be negative. We divide into three cases and show that in each case, we have

$$\psi(\pi) \geq \frac{\overline{m}_\pi^2}{4(2L+1)}.$$

Case 1. For all $x \in [-L, -l]$, $\pi_x \leq \frac{1}{2L+1}$, and $i \in [l, L]$ is the rightmost site with $\pi. > \frac{1}{2L+1}$, i.e. there is no x to the right of i with $\pi_x > \frac{1}{2L+1}$;

Case 2. For all $x \in [l, L]$, $\pi_x \leq \frac{1}{2L+1}$, and $i \in [-L, -l]$ is the leftmost site with $\pi. > \frac{1}{2L+1}$, i.e. there is no x to the left of i with $\pi_x > \frac{1}{2L+1}$;

Case 3. $i_1 \in [-L, -l]$ is the leftmost site with $\pi. > \frac{1}{2L+1}$, and $i_2 \in [l, L]$ is the rightmost site with $\pi. > \frac{1}{2L+1}$.

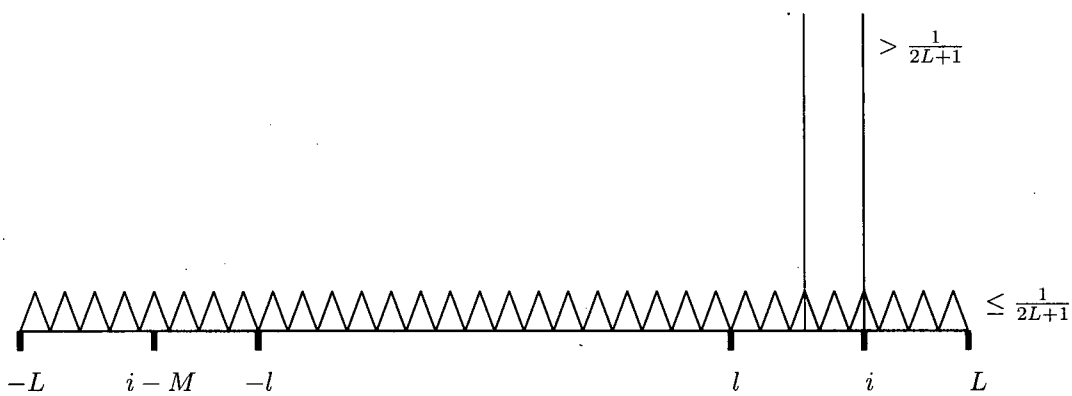


Figure 8.1: Illustration of Case 1

We first deal with Case 1. The terms in the first sum of the definition of ψ in (8.12) are squares and therefore nonnegative, hence we can throw some of them away and obtain the following:

$$\begin{aligned}\psi(\pi) \quad \geq \quad & \sum_{x=-L}^{i-M} \pi_x (m_x - \overline{m}_\pi)^2 + \mu(2L+1) \sum_{x=-L}^{L} m_x \left(\frac{1}{2L+1} - \pi_x \right) \\ & + \pi_i (m_i - \overline{m}_\pi)^2. \end{aligned} \tag{8.17}$$

For $x \in [-L, i-M]$, $B_{x-i} = 1$, therefore

$$m_x = K_x \sum_{z=x+M}^{L} B_{x-z} K_z \pi_z \geq K_L^2 \pi_i > \frac{K_L^2}{2L+1}.$$

Applying the above and the requirement $\overline{m}_\pi < \frac{K_L^2}{2(2L+1)}$ for $\pi \in A$ to (8.17), we obtain:

$$\begin{aligned}\sum_{x=-L}^{i-M} \pi_x (m_x - \overline{m}_\pi)^2 &\geq \sum_{x=-L}^{i-M} \pi_x \left(\frac{K_L^2}{2L+1} - \frac{K_L^2}{2(2L+1)} \right)^2 \\ &\geq \frac{K_L^4}{4(2L+1)^2} \sum_{x=-L}^{i-M} \pi_x. \end{aligned} \tag{8.18}$$

This deals with the first term in (8.17). For the second term in (8.17), we observe that $m_x = 0$ for $x \in [-l, l]$, and $\frac{1}{2L+1} - \pi_x \geq 0$ for $x \in [-L, -l] \cup [i+1, L]$, therefore

$$\sum_{x=-L}^{L} m_x \left(\frac{1}{2L+1} - \pi_x \right) \geq \sum_{x=l}^{i} m_x \left(\frac{1}{2L+1} - \pi_x \right).$$

Applying the universal bound $\frac{1}{2L+1} - \pi_x \geq -1$ to right hand side above, we obtain

$$\sum_{x=-L}^{L} m_x \left(\frac{1}{2L+1} - \pi_x \right) \geq -\sum_{x=l}^{i} m_x. \tag{8.19}$$

Now applying (8.18), (8.19), and the requirement $\pi_i > \frac{1}{2L+1}$ to (8.17), we obtain

$$\psi(\pi) \geq \frac{K_L^4}{4(2L+1)^2} \sum_{x=-L}^{i-M} \pi_x - \mu(2L+1) \sum_{x=l}^{i} m_x + \frac{1}{2L+1} (m_i - \overline{m}_\pi)^2. \tag{8.20}$$

We observe that

$$\begin{aligned}\sum_{x=l}^{i} m_x &= \sum_{x=l}^{i} K_x \sum_{y=-L}^{x-M} K_y \pi_y \leq K_l^2 \sum_{x=l}^{i} \sum_{y=-L}^{x-M} \pi_y \leq K_l^2 \sum_{x=l}^{i} \sum_{y=-L}^{i-M} \pi_y \\ &\leq K_l^2 (L - l + 1) \sum_{x=-L}^{i-M} \pi_x \leq K_l^2 (2L+1) \pi([-L, i-M]). \end{aligned} \tag{8.21}$$

Therefore the computation in (8.20) can be continued as follows:

$$\begin{aligned}\psi(\pi) &\geq \frac{K_L^4}{4(2L+1)^2} \pi([-L, i-M]) - \mu K_l^2 (2L+1)^2 \pi([-L, i-M]) \\ &\quad + \frac{1}{2L+1} (m_i - \overline{m}_\pi)^2 \\ &\geq \frac{K_L^4}{8(2L+1)^2} \pi([-L, i-M]) + \frac{1}{2L+1} (m_i - \overline{m}_\pi)^2 \end{aligned} \tag{8.22}$$

since $\mu < \frac{K_L^4}{8(2L+1)^4 K_l^2}$. The right hand side of the above is a sum of two nonnegative terms, both of which cannot be small at the same time. Indeed, notice that

$$m_i = K_i \sum_{x=-L}^{i-M} K_x \pi_x \leq K_l^2 \pi([-L, i-M]).$$

Therefore, if $\pi([-L, i-M]) < \frac{1}{2K_l^2}\overline{m}_\pi$, then $m_i < \frac{1}{2}\overline{m}_\pi$, and we have $(m_i - \overline{m}_\pi)^2 > \frac{1}{4}\overline{m}_\pi^2$, hence the second term in (8.22) alone implies $\psi(\pi) \geq \frac{\overline{m}_\pi^2}{4(2L+1)}$. Otherwise, $\pi([-L, i-M]) \geq \frac{1}{2K_l^2}\overline{m}_\pi$, then the first term in (8.22) implies $\psi(\pi) \geq \frac{K_L^4 \overline{m}_\pi}{16(2L+1)^2 K_l^2}$. So we have the following estimate on $\psi(\pi)$:

$$\psi(\pi) \geq \min\left(\frac{K_L^4 \overline{m}_\pi}{16(2L+1)^2 K_l^2}, \frac{\overline{m}_\pi^2}{4(2L+1)}\right). \tag{8.23}$$

Since $\delta_2 \leq \frac{K_L^4}{4(2L+1)K_l^2}$, on the set $A = \{\pi : \overline{m}_\pi < \delta_2\}$, we have $\frac{K_L^4 \overline{m}_\pi}{16(2L+1)^2 K_l^2} \geq \frac{\overline{m}_\pi^2}{4(2L+1)}$, therefore (8.23) implies

$$\psi(\pi) \geq \frac{\overline{m}_\pi^2}{4(2L+1)}.$$

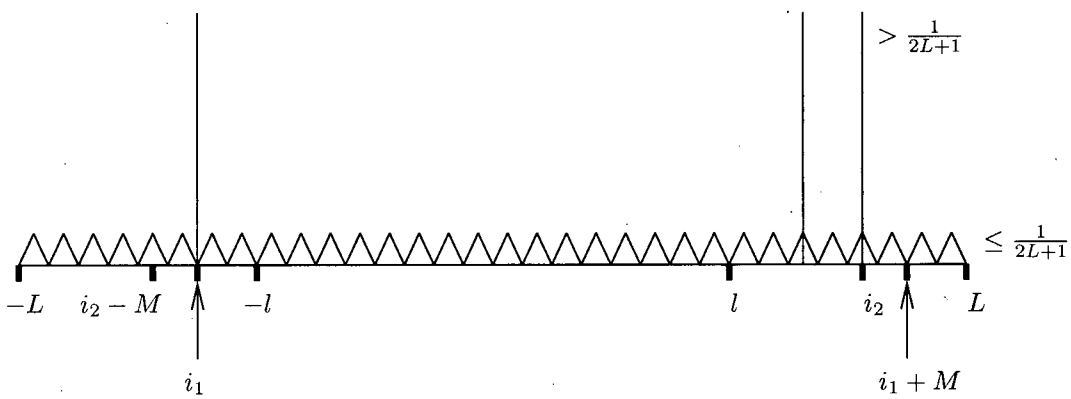


Figure 8.2: Illustration of Case 1

Case 2 follows by exactly the same argument. For Case 3, we first observe that if $i_2 - i_1 \geq M$ then $B_{i_2 - i_1} = 1$ and on $A = \{\pi : \overline{m}_\pi < \delta_2\}$, where by definition $\delta_2 \leq \frac{K_L^2}{2(2L+1)^2}$, we have $\delta_2 > \overline{m}_\pi \geq \pi_{i_1} K_{i_1} K_{i_2} \pi_{i_2} \geq \frac{K_L^2}{(2L+1)^2}$, which is impossible. Thus $i_2 - i_1 < M$, and

$$\begin{aligned}\psi(\pi) \quad \geq \quad & \sum_{x=-L}^{i_2-M} \pi_x (m_x - \overline{m}_\pi)^2 + \pi_{i_2}(m_{i_2} - \overline{m}_\pi)^2 + \sum_{x=i_1+M}^{L} \pi_x (m_x - \overline{m}_\pi)^2 \\ & + \pi_{i_1}(m_{i_1} - \overline{m}_\pi)^2 + \mu(2L+1) \sum_{x=-L}^{L} m_x \left(\frac{1}{2L+1} - \pi_x\right).\end{aligned} \tag{8.24}$$

We can use techniques similar to those used for Case 1 (leading to (8.18) and (8.21)) to obtain the following bounds:

$$\begin{aligned}& \sum_{x=-L}^{i_2-M} \pi_x (m_x - \overline{m}_\pi)^2 + \sum_{x=i_1+M}^{L} \pi_x (m_x - \overline{m}_\pi)^2 \\ & \geq \frac{K_L^4}{4(2L+1)^2}(\pi([-L, i_2 - M]) + \pi([i_1 + M, L]))\end{aligned}$$

and

$$\begin{aligned}
\sum_{x=-L}^{L} m_x\left(\frac{1}{2L+1}-\pi_x\right) &\geq \sum_{x=l}^{i_2} m_x\left(\frac{1}{2L+1}-\pi_x\right)+\sum_{x=i_1}^{-l} m_x\left(\frac{1}{2L+1}-\pi_x\right)\\
&\geq -\sum_{x=l}^{i_2} m_x-\sum_{x=i_1}^{-l} m_x\\
&\geq -K_l^2(2L+1)\pi([-L,i_2-M]\cup[i_1+M,L]).
\end{aligned}$$

Since $\mu<\frac{K_L^4}{8(2L+1)^4K_l^2}$, the above two estimates applied to (8.24) implies that

$$\begin{aligned}
\psi(\pi) &\geq \frac{K_L^4}{4(2L+1)^2}\pi([-L,i_2-M]\cup[i_1+M,L])\\
&\quad -\mu(2L+1)K_l^2(2L+1)\pi([-L,i_2-M]\cup[i_1+M,L])\\
&\quad +\frac{1}{2L+1}(m_{i_2}-\overline{m}_\pi)^2+\frac{1}{2L+1}(m_{i_1}-\overline{m}_\pi)^2\\
&\geq \frac{K_L^4}{8(2L+1)^2}\pi([-L,i_2-M])+\frac{1}{2L+1}(m_{i_2}-\overline{m}_\pi)^2\\
&\quad +\frac{K_L^4}{8(2L+1)^2}\pi([i_1+M,L])+\frac{1}{2L+1}(m_{i_1}-\overline{m}_\pi)^2.
\end{aligned}$$

We can now apply the technique leading to (8.23) to the sum of the first two terms above, and then to the sum of the last two terms, to obtain:

$$\psi(\pi)\geq 2\min\left(\frac{K_L^4\overline{m}_\pi}{16(2L+1)^2K_l^2},\frac{\overline{m}_\pi^2}{4(2L+1)}\right),$$

which by virtue of $\overline{m}_\pi<\delta_2\leq\frac{K_L^4}{4(2L+1)K_l^2}$ on A implies

$$\psi(\pi)\geq\frac{\overline{m}_\pi^2}{2(2L+1)}.$$

Thus we have established the necessary bound on $\psi(\pi)$ for $\pi\in A_2$ in all three cases. □

Lemma 8.0.13. *If* $\delta_1=\frac{\mu^2K_L^2}{16(2L+2)^2}$ *and* $\delta_2=\min(\frac{K_L^2}{2(2L+1)},\frac{K_L^4}{4(2L+1)K_l^2},\frac{K_L^2}{2(2L+1)^2})$, *then*

$$\nu^\mu\left\{\pi:\overline{m}_\pi\leq\frac{\delta_2}{2}\right\}=0.$$

Proof. Recall from (8.10) that ν^μ-a.s.,

$$\overline{m}_\pi\geq\delta_1. \tag{8.25}$$

Let $f\in C^\infty(\mathbb{R})$ to be a function that satisfies the following requirements:

(a) $f'(x)$ increases from 0 to $\frac{1}{\delta_1^2}$ for $x\in[0,\delta_1]$;

(b) $f'(x) = \frac{1}{x^2}$ for $\delta_1 \le x \le \frac{\delta_2}{2}$;

(c) $f'(x)$ decreases from $\frac{4}{\delta_2^2}$ to 0 for $x \in [\frac{\delta_2}{2}, \frac{2\delta_2}{3}]$;

(d) $f'(x) = 0$ for $x \ge \frac{2\delta_2}{3}$.

Define $F(\pi_{-L}, \ldots, \pi_L) = f(\overline{m}_\pi)$. Then (8.4) implies:

$$\int f'(\overline{m}_\pi)\mathcal{G}\overline{m}_\pi \nu^\mu(d\pi) = 0.$$

Substituting (8.13) into the above equation, we obtain

$$\int f'(\overline{m}_\pi)\psi(\pi)\nu^\mu(d\pi) = 0. \tag{8.26}$$

Lemmas 8.0.11 and 8.0.12 imply that $\psi(\pi)$ is bounded below by $\frac{1}{4(2L+1)}\overline{m}_\pi^2$ on the set $A = \{\pi : \overline{m}_\pi < \delta_2\}$, defined in (8.11). Applying this fact and (8.25) to (8.26), we obtain

$$\begin{aligned}
0 &= 2\int_{\{\pi:0\le\overline{m}_\pi\le\frac{2\delta_2}{3}\}} \psi(\pi)f'(\overline{m}_\pi)\nu^\mu(d\pi) && \text{since } f'(x) \text{ only nonzero for } x \text{ in } [0, \tfrac{2\delta_2}{3}] \\
&= 2\int_{\{\pi:\delta_1\le\overline{m}_\pi\le\frac{2\delta_2}{3}\}} \psi(\pi)f'(\overline{m}_\pi)\nu^\mu(d\pi) && \text{since } \nu^\mu\{\pi : \overline{m}_\pi < \delta_1\} = 0 \text{ by (8.10)} \\
&\ge 2\int_{\{\pi:\delta_1\le\overline{m}_\pi\le\frac{\delta_2}{2}\}} \psi(\pi)f'(\overline{m}_\pi)\nu^\mu(d\pi) && \text{since } \psi(\pi)f'(\overline{m}_\pi) > 0 \text{ if } \overline{m}_\pi \in [\tfrac{\delta_2}{2}, \tfrac{2\delta_2}{3}] \\
&\ge 2\int_{\{\pi:\delta_1\le\overline{m}_\pi\le\frac{\delta_2}{2}\}} \frac{\overline{m}_\pi^2}{4(2L+1)}\frac{1}{\overline{m}_\pi^2}\nu^\mu(d\pi) && \text{by the bound on } \psi(\overline{m}_\pi) \text{ for } \pi \in A \\
&= \frac{1}{2(2L+1)}\int_{\{\pi:\delta_1\le\overline{m}_\pi\le\frac{\delta_2}{2}\}} \nu^\mu(d\pi).
\end{aligned}$$

Therefore

$$\nu^\mu\left\{\pi : \overline{m}_\pi \le \frac{\delta_2}{2}\right\} = \nu^\mu\left\{\pi : 0 \le \overline{m}_\pi < \delta_1\right\} + \nu^\mu\left\{\pi : \delta_1 \le \overline{m}_\pi \le \frac{\delta_2}{2}\right\} = 0 + 0 = 0,$$

as required. □

Proof of Theorem 8.0.8. For an arbitrary site $z \in [-l+1, l-1]$, we have $m_z = 0$. If we take $F(\pi) = \pi_z$, then by (8.4), we have

$$\begin{aligned}
0 &= \int \pi_z(m(z,\pi) - \overline{m}_\pi) + \mu(1 - (2L+1)\pi_z)\nu^\mu(d\pi) \\
&= \int \mu - (\overline{m}_\pi + \mu(2L+1))\pi_z\nu^\mu(d\pi),
\end{aligned}$$

so

$$\begin{aligned}
\mu &= \int (\overline{m}_\pi + \mu(2L+1))\pi_z\nu^\mu(d\pi) \\
&= \int_{\{\pi:\overline{m}_\pi>\delta_2/2\}} (\overline{m}_\pi + \mu(2L+1))\pi_z\nu^\mu(d\pi) \\
&\quad + \int_{\{\pi:\overline{m}_\pi\le\delta_2/2\}} (\overline{m}_\pi + \mu(2L+1))\pi_z\nu^\mu(d\pi),
\end{aligned}$$

where δ_2 is as defined in Lemma 8.0.13. The same lemma shows that $\{\pi : \overline{m}_\pi \leq \delta_2/2\}$ has ν^μ-measure 0, so the second integral in the above equation is 0, thus

$$\mu = \int_{\{\pi:\overline{m}_\pi > \delta_2/2\}} (\overline{m}_\pi + \mu(2L+1))\pi_z \nu^\mu(d\pi) \geq \int \left(\frac{\delta_2}{2} + \mu(2L+1)\right) \pi_z \nu^\mu(d\pi),$$

i.e.

$$\int \pi_z \nu^\mu(d\pi) \leq \frac{\mu}{\frac{\delta_2}{2} + \mu(2L+1)}.$$

The observation $\nu^\mu\{\pi : \pi_z \geq \delta\} \leq \frac{1}{\delta} \int \pi_z \nu^\mu(d\pi)$ completes the proof. □

Appendix A

A Result on the Conditioned Dieckmann-Doebeli Model

In this section, we deal with a special case of the conditioned Dieckmann-Doebeli model described in (6.2) and (6.3), and show that in this special case, there exist symmetric bimodal stationary distributions. Let $E = [-L, L] \cap \mathbb{Z}$, A be a Markov transition matrix associated with mutation, $\pi(t) \in \mathcal{P}(E)$ for all $t \in \mathbb{Z}^+$, and M be an *even* constant such that $L - 1 \leq M < 2(L-1)$, then the equation of the discrete-time dynamical system is as follows:

$$\begin{aligned}
\pi_x(t+1) &= \sum_y A(y,x) \frac{\pi_y(t) V_y(\pi(t))}{\sum_z \pi_z(t) V_z(\pi(t))}, \\
\text{where } V_x(\pi) &= V_x^{(2)}(\pi) = \frac{K_x}{\sum_z C_{x-z} \pi_z}, \\
K_x &= 1_{\{|x| \leq L-1\}}, \\
\text{and } C_x &= 1_{\{|x| \leq M\}}.
\end{aligned} \tag{A.1}$$

Every step of (A.1) can be divided into three sub-steps:

$$\begin{aligned}
\text{Resampling} \quad & \pi'_x(t) = \pi_x(t) V_x(\pi(t)); \\
\text{Mutation} \quad & \pi''_x(t) = \sum_y A(y,x) \pi'_x(t); \\
\text{Normalization} \quad & \pi_x(t+1) = \frac{\pi''_x(t)}{\sum_y \pi''_y(t)}.
\end{aligned} \tag{A.2}$$

Notice that performing the normalization step before the mutation step does not change the model, but for this section, we will use the step order in (A.2).

If K, C, and $A(y,x) = A(y-x)$ are symmetric about 0, then the map $\pi(t) \mapsto \pi(t+1)$ maps the set of symmetric probability measures on $[-L, L]$ to itself. Therefore by Brouwer's fixed point theorem, there exists *symmetric* stationary distribution(s). We first derive a few simple facts about symmetric stationary distributions in the no-mutation case, i.e. when

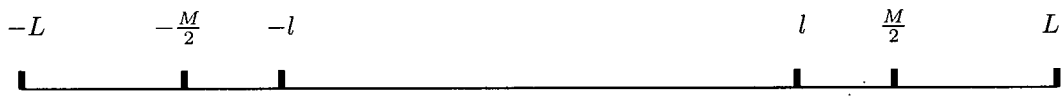


Figure A.1: Relative locations of various sites of interest

$A = I$. In this case, if ν^0 is a stationary distribution of (A.1), we must have

$$\text{if } \nu_x^0 \neq 0, \text{ then } V_x(\nu^0) = \frac{K_x}{\sum_z C_{x-z}\nu_z^0} = \frac{K_x}{\sum_{z=x-M}^{x+M} \nu_z^0} \text{ is a constant.} \tag{A.3}$$

This is the same condition as (6.5). Since $K_x = 0$ outside the interval $[-L+1, L-1]$, $V_x = 0$ outside that interval, i.e. at $x = \pm L$, therefore the support of any stationary distribution ν^0 must lie in $[-L+1, L-1]$. We restrict our attention to sites in $[-L+1, L-1]$. If the competition intensity function C is rectangular, as in (A.1), two sites either compete at intensity 1 or they do not compete against each other at all. If $M = L-1$, then 0 is the only site that competes with every other site in $[-L+1, L-1]$; but if $M = 2(L-1)-1$, then for any site $x \in [-L+2, L-2]$, x competes with every site in $[-L+1, L-1]$. Define $l = -(L-1-M)$, then $[-l, l]$ contains the sites that compete with every site in $[-L+1, L-1]$, therefore

$$V_x(\nu^0) = \frac{K_x}{\sum_{z=x-M}^{x+M} \nu_z^0} = \begin{cases} 1 & x \in [-l, l] \\ 0 & x \in (\infty, -L] \cup [L, \infty) \end{cases} . \tag{A.4}$$

We also observe that $\frac{M}{2} > l$ because $M - 2l = M + 2(L-1-M) = 2(L-1) - M > 0$. If there is some mass in $[-L+1, -\frac{M}{2}-1] \cup [\frac{M}{2}+1, L-1]$, then since no site in $[\frac{M}{2}+1, L-1]$ competes with any site in $[-L+1, -\frac{M}{2}-1]$, we have

$$\sum_{z=x-M}^{x+M} \nu_z^0 < 1$$

for $x \in [-L+1, -\frac{M}{2}-1] \cup [\frac{M}{2}+1, L-1]$, which by the definition of $V_x(\pi)$ in (A.1) implies the following:

$$V_x(\nu^0) > 1 \quad \text{for} \quad x \in \left[-L+1, -\frac{M}{2}-1\right] \cup \left[\frac{M}{2}+1, L-1\right]. \tag{A.5}$$

Combining the results on fitness $V_x(\pi)$ in (A.4) and (A.5) and condition (A.3), we conclude that stationary distributions with rectangular K and C as defined in (A.1) must have all the mass falling in either $\left[-L+1, -\frac{M}{2}-1\right] \cup \left[\frac{M}{2}+1, L-1\right]$ or $\left[-\frac{M}{2}, \frac{M}{2}\right]$.

Now we turn to the case with small mutations. We take A^μ to be an operator that corresponds to a small 1-step mutation, i.e. convolution with $\mu\delta_{-1} + (1-2\mu)\delta_0 + \mu\delta_1$, and define ν^μ to be a stationary distribution of (A.1) with mutation kernel A^μ. Any stationary distribution ν^μ of (A.1) satisfies the following condition:

$$\forall x \in [-L, L], \ \mu\nu_{x-1}^\mu V_{x-1} + (1-2\mu)\nu_x^\mu V_x + \mu\nu_{x+1}^\mu V_{x+1} = \overline{V}\nu_x^\mu, \tag{A.6}$$

where $V_y = V_y(\nu^\mu)$ and

$$\overline{V} = \overline{V}(\nu^\mu) = \sum_{z:\nu_z^\mu \neq 0} \nu_z^\mu V_z(\nu^\mu)$$

is the normalization constant. Condition (A.6) implies that ν^μ is nowhere zero in $[-L, L]$; otherwise, say $\nu_z^\mu = 0$, then then $\nu_{z-1}^\mu = \nu_{z+1}^\mu = 0$ as well, which by induction means that $\nu_x^\mu = 0$ for all x, a clear contradiction. Since the support of ν^μ has expanded on both sides each by 1 site compared to ν^0, the sites where $V_\cdot(\nu^\mu)$ is constant 1 should correspondingly contract by 1 site on each side:

$$V_x(\nu^\mu) = \frac{K_x}{\sum_{z=x-M}^{x+M} \nu_z^\mu} = \begin{cases} 1 & x \in [-l+1, l-1] \\ > 1 & x \in [-L+1, -l] \cup [l, L-1] \\ 0 & x \in (\infty, -L] \cup [L, \infty) \end{cases} \tag{A.7}$$

Therefore, for sites in the middle, i.e. $x \in [-l+2, l-2]$, ν^μ satisfies:

$$\mu\nu_{x-1}^\mu + (1-2\mu)\nu_x^\mu + \mu\nu_{x+1}^\mu = \overline{V}(\nu^\mu)\nu_x^\mu.$$

Since every site x in $[-L+1, L-1]$ competes with all sites lying on the same side (with respect to the origin) as x, and the stationary distributions we consider are symmetric, $\sum_{z=x-M}^{x+M} \nu_x^\mu$ is at least $\frac{1}{2}$ for $x \in [-L+1, L-1]$, therefore

$$V_x(\nu^\mu) \leq 2 \text{ for } x \in [-L+1, L-1], \tag{A.8}$$

i.e. the normalization constant $\overline{V}(\nu^\mu)$ is bounded above by 2. We also need a nontrivial lower bound of $\overline{V}(\nu^\mu)$ for *symmetric* ν^μ that is uniform for small μ for the proof of the upcoming theorem. We rewrite condition (A.6) for $x = L$ and $x = L-1$, taking into account the fact $V_L(\nu^\mu) = 0$ from (A.7):

$$\begin{aligned} \mu\nu_{L-1}^\mu V_{L-1}(\nu^\mu) &= \overline{V}(\nu^\mu)\nu_L^\mu, \\ \mu\nu_{L-2}^\mu V_{L-2}(\nu^\mu) + (1-2\mu)\nu_{L-1}^\mu V_{L-1}(\nu^\mu) &= \overline{V}(\nu^\mu)\nu_{L-1}^\mu. \end{aligned}$$

Dividing both sides of the above two equations, we get

$$\begin{aligned} \frac{\nu_L^\mu}{\nu_{L-1}^\mu} &= \frac{\mu\nu_{L-1}^\mu V_{L-1}(\nu^\mu)}{\mu\nu_{L-2}^\mu V_{L-2}(\nu^\mu) + (1-2\mu)\nu_{L-1}^\mu V_{L-1}(\nu^\mu)} \\ &\leq \frac{\mu\nu_{L-1}^\mu V_{L-1}(\nu^\mu)}{(1-2\mu)\nu_{L-1}^\mu V_{L-1}(\nu^\mu)} = \frac{\mu}{1-2\mu}, \end{aligned}$$

which is < 1 if $\mu < \frac{1}{3}$. This means that $\nu_L^\mu = \nu_{-L}^\mu \leq \frac{1}{4}$, for otherwise, $\nu_L^\mu = \nu_{-L}^\mu > \frac{1}{4}$ implies that $\nu_{L-1}^\mu = \nu_{-L+1}^\mu \leq \frac{1}{2} - \nu_L^\mu < \frac{1}{4}$, hence $\frac{\nu_L^\mu}{\nu_{L-1}^\mu} > 1$, a contradiction. Since $V_x(\nu^\mu) \geq 1$ for $x \in [-L+1, L-1]$ by (A.7), we have

$$\overline{V}(\nu^\mu) \geq \sum_{z=-L+1}^{L-1} \nu_z^\mu V_z(\nu^\mu) \geq \sum_{z=-L+1}^{L-1} \nu_x^\mu.$$

The fact $\nu_L^\mu = \nu_{-L}^\mu \leq \frac{1}{4}$ then implies

$$\overline{V}(\nu^\mu) \geq 1 - \nu_{-L}^\mu - \nu_L^\mu \geq \frac{1}{2}. \tag{A.9}$$

In particular, $\overline{V}(\nu^\mu)$ is bounded between $\frac{1}{2}$ and 2. We take a sequence $\mu_n \to 0$, such that $\nu^n = \nu^{\mu_n}$ converges to some ν^0, then since $\overline{V}(\nu)$ is a continuous function of ν, $\overline{V}_n = \overline{V}(\nu^n)$ also converges to a positive constant $\overline{V}$. We will prove the following:

Theorem A.0.14. *If $\nu^n = \nu^{\mu_n}$ is a convergent sequence of* symmetric *stationary distributions for the conditioned Dieckmann-Doebeli model in (A.1), then $\nu^n_{[-\frac{M}{2},\frac{M}{2}]} \to 0$ as $n \to \infty$.*

Proof. Since $\nu^n \to \nu^0$ and $\overline{V}(\nu^n) \to \overline{V}$ as $n \to \infty$, condition (A.6) converges to the following:

$$\forall x \in [-L, L],\ \nu_x^0 V_x(\nu^0) = \overline{V}\nu_x^0.$$

Therefore ν^0 is in fact an stationary distribution for the no-mutation case, i.e. (A.1) with $A = I$. If some mass of ν^0 lies in $\left[-L+1, -\frac{M}{2}-1\right] \cup \left[\frac{M}{2}+1, L-1\right]$, then $\nu^0_{[-\frac{M}{2},\frac{M}{2}]}$ must be zero, which means that $\nu^n_{[-\frac{M}{2},\frac{M}{2}]} \to 0$ when $n \to \infty$ as required. Therefore it suffices to show that $\nu^0_{[-L+1,-\frac{M}{2}-1]\cup[\frac{M}{2}+1,L-1]} > 0$.

We assume, toward a contradiction, that $\nu^0_{[-L+1,-\frac{M}{2}-1]} = \nu^0_{[\frac{M}{2}+1,L-1]} = 0$. Then for any positive δ, we have $\nu^n_{[-L+1,-\frac{M}{2}-1]} = \nu^n_{[\frac{M}{2}+1,L-1]} < \delta$ for sufficiently large n. We first derive more refined (than (A.9) and (A.8) respectively) lower and upper bounds for $\overline{V}_n$. Because of the supposition $\nu^n_{[\frac{M}{2}+1,L-1]} < \delta$, (A.6) and (A.7) with $x = L$ and the bound $\nu^n_{L-1} < \delta$ imply that

$$\nu_L^n < \frac{\delta\mu_n V_{L-1}(\nu^n)}{\overline{V}_n} \leq \frac{2\delta\mu_n}{\overline{V}_n} \tag{A.10}$$

by (A.8). Applying the estimate (A.9) to the right hand side, we have

$$\nu_L^n < \frac{2\delta\mu_n}{\overline{V}_n} \leq 4\delta\mu_n. \tag{A.11}$$

Therefore

$$\begin{aligned}
\overline{V}_n &= \sum_{z:\nu_z^n \neq 0} \nu_z^n V_z(\nu^n) \\
&= \sum_{z=-L+1}^{L-1} \nu_z^n V_z(\nu^n) \qquad \text{since } V_{-L}(\nu^n) = V_L(\nu^n) = 0 \text{ by (A.7)} \\
&\geq \sum_{z=-L+1}^{L-1} \nu_z^n \qquad \text{since } V_x(\nu^n) \geq 1 \text{ for } x \in [-L+1, L-1] \\
&= 1 - \nu_{-L}^n - \nu_L^n \\
&\geq 1 - 8\delta\mu_n,
\end{aligned} \tag{A.12}$$

by (A.11).

For $x \in \left[-\frac{M}{2}, \frac{M}{2}\right]$,

$$V_x(\nu^n) = \frac{K_x}{\sum_{z=x-M}^{x+M} \nu_z^n} \leq \frac{1}{\sum_{z=-\frac{M}{2}}^{\frac{M}{2}} \nu_z^n} = \frac{1}{1 - 2\nu^n_{\left[\frac{M}{2}+1, L\right]}} \leq \frac{1}{1 - 2(\delta + 4\delta\mu_n)} \tag{A.13}$$

by (A.11) and the supposition $\nu^n_{\left[\frac{M}{2}+1, L-1\right]} < \delta$. Therefore

$$\begin{aligned} \overline{V}(\nu^n) &= \sum_{x=-L}^{L} \nu_x^n V_x(\nu^n) = \sum_{x=-\frac{M}{2}}^{\frac{M}{2}} \nu_x^n V_x(\nu^n) + 2 \sum_{x=\frac{M}{2}+1}^{L-1} \nu_x^n V_x(\nu^n) \\ &\leq \frac{1}{1 - 2(\delta + 4\delta\mu_n)} + 4\delta \end{aligned} \tag{A.14}$$

using (A.13) for the first sum and (A.8) for the second.

Let $r = \nu^n_{\left[-\frac{M}{2}, -l-1\right]} = \nu^n_{\left[l+1, \frac{M}{2}\right]}$, then

$$\begin{aligned} V_{L-1}(\nu^n) &= \frac{K_{L-1}}{\sum_{z=L-1-M}^{L-1+M} \nu_z^n} = \frac{1}{\sum_{z=-l}^{L} \nu_z^n} = \frac{1}{1 - \nu^n_{\left[-\frac{M}{2}, -l-1\right]} - \nu^n_{\left[-L+1, -\frac{M}{2}\right]} - \nu^n_{-L}} \\ &\geq \frac{1}{1-r}. \end{aligned} \tag{A.15}$$

Condition (A.6) applied with $x = L - 1$ implies

$$(1 - 2\mu_n)\nu^n_{L-1} V_{L-1}(\nu^n) \leq \overline{V}(\nu^n)\nu^n_{L-1},$$

hence

$$V_{L-1}(\nu^n) \leq \frac{\overline{V}(\nu^n)}{1 - 2\mu_n}.$$

Then (A.14) and (A.15) imply

$$\frac{1}{1-r} \leq \frac{1}{1 - 2\mu_n}\left(\frac{1}{1 - 2(\delta + 4\delta\mu_n)} + 4\delta\right)$$

i.e.

$$1 - r \geq \frac{1 - 2\mu_n}{\frac{1}{1-2(\delta+4\delta\mu_n)} + 4\delta} = \frac{(1 - 2\mu_n)(1 - 2\delta - 8\delta\mu_n)}{1 + 4\delta - 8\delta^2 - 32\delta^2\mu_n} \geq \frac{1 - 3\delta}{1 + 4\delta} = 1 - \frac{7\delta}{1 + 4\delta}$$

for sufficiently small μ_n. Therefore

$$r = \nu^n_{\left[-\frac{M}{2}, -l-1\right]} = \nu^n_{\left[l+1, \frac{M}{2}\right]} \leq \frac{7\delta}{1 + 4\delta} \leq 7\delta.$$

The above inequality and the supposition $\nu^n_{\left[-L+1, -\frac{M}{2}-1\right]} = \nu^n_{\left[\frac{M}{2}+1, L-1\right]} < \delta$ imply that

$$\nu^n_{[-L+1, -l-1]} = \nu^n_{[l+1, L-1]} < 8\delta. \tag{A.16}$$

Let $p = \nu_l^n$. We bound $V_l(\nu^n)$. Since site $-L$ is the only site in the support of ν^n that does not compete with site l, we have

$$1 < V_l(\nu^n) = \frac{K_l}{\sum_{z=-L+1}^{L} \nu_z^n} = \frac{K_l}{1-\nu_{-L}^n} \leq \frac{1}{1-4\delta\mu_n} \tag{A.17}$$

by (A.11). We will establish the following lemma after the proof of the present theorem:

Lemma A.0.15. *Let $\nu^n = \nu^{\mu_n}$ be as in Theorem A.0.14, and suppose $\nu^n_{[-L+1,-l-1]} = \nu^n_{[l+1,L-1]} < 8\delta$, then*

1. *ν^n_{-l} and ν^n_l is bounded away from 0 as $n \to \infty$;*
2. *$\nu^n_{l-1} \leq \nu^n_l + \delta$ for sufficiently large n.*

Since $\nu^n_{[-L+1,-l-1]} < 8\delta$ by (A.16), we have from (A.11),

$$V_{l+1}(\nu^n) \leq \frac{1}{1-8\delta-4\delta\mu_n} \tag{A.18}$$

because sites $-L$ and $-L+1$ are the only sites in $[-L, L]$ that do not compete with site $l+1$. We estimate $\sum_y A(y,l)\nu_y^n V_y(\nu^n) - \nu_l^n$:

$$\begin{aligned}
&\sum_y A(y,l)\nu_y^n V_y(\nu^n) - \nu_l^n \\
&= \mu_n \nu_{l-1}^n V_{l-1}(\nu^n) + (1-2\mu_n)\nu_l^n V_l(\nu^n) + \mu_n \nu_{l+1}^n V_{l+1}(\nu^n) - \nu_l^n \\
&\leq \mu_n(p+\delta) + \frac{(1-2\mu_n)p}{1-4\delta\mu_n} + \frac{\mu_n\delta}{1-8\delta-4\delta\mu_n} - p,
\end{aligned}$$

using $\nu_{l-1}^n \leq p+\delta$, $V_{l-1}(\nu^n) = 1$, (A.17), $\nu_{l+1}^n \leq \delta$, and (A.18). Simplifying the right hand side of the above, we get

$$\begin{aligned}
\sum_y A(y,l)\nu_y^n V_y(\nu^n) - \nu_l^n &\leq \frac{-4\mu_n^2\delta p + p - \mu_n p}{1-4\delta\mu_n} + \mu_n\delta + \frac{\mu_n\delta}{1-8\delta-4\delta\mu_n} - p \\
&= \frac{-4\mu_n^2\delta p - \mu_n p + 4\mu_n\delta p}{1-4\delta\mu_n} + \mu_n\delta + \frac{\mu_n\delta}{1-8\delta-4\delta\mu_n} \\
&\leq -\frac{\mu_n p(1-4\delta)}{1-4\delta\mu_n} + \mu_n\delta + \frac{\mu_n\delta}{1-8\delta-4\delta\mu_n} \\
&\leq -\frac{\mu_n p}{2}
\end{aligned}$$

for sufficiently small δ and μ_n. This estimate means that after resampling and mutation (and before normalization), ν_l^n decreases by at least $\mu_n p/2$. On the other hand, ν_{l+1}^n can only increase: let $q = \nu_{l+1}^n$, then $q \leq \delta$ and

$$\begin{aligned}
&\sum_y A(y,l+1)\nu_y^n V_y(\nu^n) - \nu_{l+1}^n \\
&= \mu_n \nu_l^n V_l(\nu^n) + (1-2\mu_n)\nu_{l+1}^n V_{l+1}(\nu^n) + \mu_n \nu_{l+2}^n V_{l+2}(\nu^n) - \nu_{l+1}^n \\
&\geq \mu_n p + (1-2\mu_n)q - q,
\end{aligned}$$

since $V_l(\nu^n) \geq 1$ and $V_{l+1}(\nu^n) \geq 1$. Therefore

$$\sum_y A(y, l+1)\nu_y^n V_y(\nu^n) - \nu_{l+1}^n \geq \mu_n(p - 2q) \geq \mu_n(p - 2\delta) > 0$$

if δ is small enough. After normalization, i.e. dividing by $\overline{V}(\nu^n)$, ν_l^n and ν_{l+1}^n cannot possibly return to their original values. This contradicts the assumption that ν^n is an stationary distribution for (A.1) with mutation kernel A^{μ_n}, and the proof is complete. □

Proof of Lemma A.0.15. Define

$$\zeta_x = \nu_x^n V_x(\nu^n). \tag{A.19}$$

Notice that ζ_x depends on n, but notationally we suppress this dependence. For $x \in [-l+1, l-1]$, $\zeta_x = \nu_x^n$ since $V_x(\nu^n) = 1$ from (A.7). For $x \in [-l+1, l-1]$, we rewrite condition (A.6) as follows:

$$\mu_n \zeta_{x-1} + (1 - 2\mu_n - \overline{V}_n)\zeta_x + \mu_n \zeta_{x+1} = 0. \tag{A.20}$$

This is a recurrence relation with general solution $\zeta_x = A\beta_1^x + B\beta_2^x$, where β_1 and β_2 are the two roots of the quadratic polynomial $\mu_n + (1 - 2\mu_n - \overline{V}_n)r + \mu_n r^2$; or $\zeta_x = (A + Bx)\beta_1^x$, where β_1 is the double root of the polynomial. Elementary calculation shows that for the solution $\zeta_x = (A + Bx)\beta_1^x$ to satisfy the symmetry requirement for $L \geq 1$, either $B = 0$ or $\beta_1 = 0$; $\beta_1 = 0$ leads to the solution of $\zeta_x = 0$, and $B = 0$ leads to the conclusion $\beta_1 = 1$ and $\zeta_x = A$; both these two scenarios will be included in Case 2 below. For the solution $\zeta_x = A\beta_1^x + B\beta_2^x$, simple calculation leads to:

$$\beta_1, \beta_2 = \frac{1}{2\mu_n}\left(2\mu_n + \overline{V}_n - 1 \pm \sqrt{(\overline{V}_n - 1)^2 + 4\mu_n(\overline{V}_n - 1)}\right).$$

We divide into three cases:

1. If β_1 and β_2 are two real roots, then since ζ is symmetric, we must have $\beta_1 = 1/\beta_2$ with $\beta_1 > 0$, and the solution is $\zeta_x = A(\beta_1^x + \beta_1^{-x})$ for $x \in [-l, l]$.

2. If $\beta_1 = \beta_2$, then the solution is $\zeta_x = A$ for $x \in [-l, l]$.

3. If β_1 and β_2 are complex roots, then we write $\beta_1 = \gamma e^{i\theta}$ and $\beta_2 = \gamma e^{-i\theta}$, and the solution is $\zeta_x = A\gamma\cos(x\theta)$ for $x \in [-l, l]$. Define

$$\alpha_n = \overline{V}_n - 1, \tag{A.21}$$

then for β_1 and β_2 to be complex, $\alpha_n^2 + 4\mu_n\alpha_n = \alpha_n(\alpha_n + 4\mu_n) < 0$, which means that

$$\text{either} \quad \alpha_n < 0 \text{ and } \alpha_n + 4\mu_n > 0 \tag{A.22}$$

$$\text{or} \quad \alpha_n > 0 \text{ and } \alpha_n + 4\mu_n < 0. \tag{A.23}$$

Now (A.23) is clearly impossible since $\mu_n \geq 0$, and (A.22) implies that $\alpha_n < 0$. Furthermore,

$$\begin{aligned}
\tan\theta &= \sqrt{\frac{-\alpha_n^2 - 4\mu_n\alpha_n}{(\alpha_n + 2\mu_n)^2}} = \left(-\frac{\alpha_n^2 + 4\mu_n\alpha_n + 4\mu_n^2}{\alpha_n^2 + 4\mu_n\alpha_n}\right)^{-1/2} \\
&= \left(-1 - \frac{4\mu_n^2}{\alpha_n^2 + 4\mu_n\alpha_n}\right)^{-1/2} \\
&= \left(-1 + \frac{1}{-\frac{\alpha_n}{\mu_n} + (-\frac{\alpha_n}{2\mu_n})^2}\right)^{-1/2}. \qquad \text{(A.24)}
\end{aligned}$$

Hence (A.12) and (A.21) imply that $\alpha_n > -8\delta\mu_n$, and since $\alpha_n < 0$,we have

$$0 < -\alpha_n/\mu_n < 8\delta.$$

We conclude from (A.24) that for sufficiently small δ, $\tan\theta$ is also very small,

Note that $-l$ and l are the boundary sites for (A.20), therefore statements about ζ_x in the three cases above all hold for $x \in [-l, l]$, even though (A.20) holds for only $x \in [-l+1, l-1]$.

In case 1, ζ is a linear combination of two convex functions, therefore ζ is convex for $x \in [-l, l]$. In case 2, ζ is flat for $x \in [-l, l]$. And in case 3, ζ is concave for $x \in [-l, l]$, but since θ is small for small δ, it is close to being flat for small δ. Therefore recalling the definition of ζ_x in (A.19) and using (A.17), we have

$$(1 - 4\delta\mu_n)\zeta_l \leq \nu_l^n < \zeta_l.$$

In summary, for $x \in [-l, l]$, ζ is convex, or flat, or nearly flat for small enough δ; $\nu_x^n = \zeta_x$ for $x \in [-l+1, l-1]$ and $\nu_l^n = \nu_{-l}^n$ is smaller than but very close to $\zeta_l = \zeta_{-l}$; furthermore, by (A.11) and (A.16), we have

$$\nu_{[-l,l]}^n = 1 - \nu_{-L}^n - \nu_{[-L+1,-l-1]}^n - \nu_{[l+1,L-1]}^n - \nu_L^n \geq 1 - 8\delta\mu_n - 16\delta,$$

i.e. ν^n has almost all its mass on $[-l, l]$. We can then use the symmetry assumption on ν^n to arrive at the conclusion of the lemma. □

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