A MODEL FOR OPTIMAL INFRASTRUCTURE INVESTMENT IN BOOM TOWNS

by

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ABSTRACT

A Model for Optimal Infrastructure Investment in Boomtowns

A linear model to determine the optimal policy for investment in social infrastructure is formulated and its solution is obtained using the Maximum Principle. The unique solution is characterized by a bang-bang control, with only one interval of investment in social capital, and the endpoints of this interval can be numerically determined, given values for the parameters of the model. A generalization of the model which allows instantaneous jumps in the level of social capital is also analyzed, and the solution to the modified problem is shown to be a uniquely determined impulse control. The final extension of the model allows us to determine an upper bound for the optimal time horizon.
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1. **INTRODUCTION**

An isolated town or community close to the site of a proposed mining or power-generating facility rarely has the social infrastructure needed to support the large influx of labour required during the construction phase of the facility. Therefore, this "boom" period, which is characterized by large increases in population, must be accompanied or preceded by some growth in the level of social capital. Health care, education, sanitation, serviced land, etc. must all be supplied in order to support the town's population. Once construction of the facility has been completed, however, the town typically experiences a drastic population decline, due to the fact that the labour force required for operation of the facility is usually much smaller than that required during construction. Thus the need for provision of the higher level of services is restricted to the construction phase of the project.

Determining the optimal investment in social infrastructure in boomtowns provides an excellent problem to which control theory can be applied. What makes the infrastructure investment decisions interesting is the fact that this investment is, for the most part, irreversible. Once in place, schools, hospitals, roads and sewage systems are, in the main, not easily moveable, so that capital
invested in them is often not recoverable through resale.\textsuperscript{1} Therefore, the decisions which face the boomtown community are important ones. Over-investment in social capital during the relatively short boom period can result in substantial idle stocks when the population settles down to its long run post-boom. Such a misallocation of capital, which is observed in many boomtowns is, of course, socially undesirable. This problem is of some interest, then, as a practical matter of public policy, and as a suitable application of control techniques.

The problem of determining the optimal investment strategy has been previously addressed by Cummings and Schulze [5]. Their mathematical formulation of the problem, however, appears to be too complex to permit a complete analytical solution; consequently, we need to find a simpler form of model for which more complete analytical results can be obtained. Simplification of a model, achieved by imposing additional assumptions or constraints, usually results, of course, in a greater departure from the real world. However, there is some justification for studying simpler models for which the solution can be fully specified, in that subsequent analysis can sometimes provide some insight for possible solutions of more complex models.

\textsuperscript{1}The "irreversibility" aspect of investment is not unique to social capital. This problem is also encountered to some degree in the investment of capital assets to exploit certain natural resource deposits. For example, harvesting a stock of fish requires a decision on the level of investment in a fishing fleet. For a general treatment of this problem, see Clark, Clarke, and Munro [3].
In this paper, then, we study a simplified version of the model formulated by Cummings and Schulze. The problem of determining the optimal investment strategy is posed as a linear optimal control problem, and a unique solution is obtained using the Maximum Principle. Our simplified model permits an easy characterization of the optimal time horizon for the construction project. This question was not considered by Cummings and Schulze.

The organization of the paper is as follows: in Section 2 we briefly discuss the Cummings and Schulze model and show how our model is derived from it; Section 3 contains the solution to the new optimal control problem; a qualitative analysis of the solution, illustrated by numerical results, is presented in Section 4; in Section 5, we solve a more general version of our model which extends the set of admissible controls to impulse investment policies; finally, in Section 6, we discuss the problem of extending the model to determine the optimal time horizon.
2. THE BASIC MODEL

The problem posed by Cummings and Schulze [5] centers around the construction of a large energy extraction/conversion facility near a small town. Construction of the facility requires high levels of labour, which must be "imported" from neighbouring communities, and the influx of labourers and their families necessitates investment in social infrastructure.

Determination of the correct level of infrastructure investment revolves around a simple "trade-off". On one hand, it is desirable to maintain a high level of per capita infrastructure (or, equivalently, a high infrastructure/labour ratio) throughout construction of the facility, because this will tend to decrease the high wage rate that incoming labour will otherwise demand. On the other hand, infrastructure is costly, and the labour force required to operate the facility after its completion is usually much smaller than the level required to complete the construction on time, so that, during the operational phase, a smaller amount of social capital is sufficient to maintain a reasonable per capita infrastructure level. Since investment in social capital is irreversible, large amounts of social infrastructure put in place during the boom period may be redundant when construction has been completed.

The problem, then, is one of choosing how much labour to use, and how much investment in infrastructure to make during the
construction phase so that the total costs of the project (wages plus social capital costs) will be minimized, in accordance with certain physical constraints. The earliest point in time at which construction of the facility may begin is taken as time $t = 0$, and the facility must be operational by a fixed time $T$. Construction of the facility is modelled by the equation

$$\dot{C} = f(L), \quad 0 \leq t \leq T. \quad \ldots (2.1)$$

where $C(t)$ is the level of construction completed at time $t$, $L(t)$ is the amount of labour used in construction at time $t$, and $f(L)$ is a concave production function. Of course, the amount of labour used at any time is non-negative, so

$$L(t) \geq 0, \quad 0 \leq t \leq T. \quad \ldots (2.2)$$

Initially, no construction has been done on the facility, so

$$C(0) = 0 \quad \ldots (2.3)$$

and if $C_T$ denotes the specified level of construction for the completed facility, then

$$C(T) = C_T. \quad \ldots (2.4)$$

Increases in the stock of social capital, $K(t)$, are determined by the equation

$$\dot{K} = 1, \quad 0 \leq t \leq T. \quad \ldots (2.5)$$
where \( I(t) \) is the rate of investment in social capital at time \( t \). The irreversibility of investment in infrastructure is enforced by the constraint

\[
I(t) > 0 , \quad 0 < t < T . \tag{2.6}
\]

It is assumed that before the project is started, there is some non-negative level of infrastructure \( K(0) \) at the site of the proposed facility.

The decision maker responsible for planning the project wants to minimize costs, subject to the constraints in (2.1) to (2.6). Most of the costs explicitly stated in the model are costs which are incurred during the construction phase; these include wage payments, \( W(K,L) \), for labour used in construction of the facility, investment \( I \), in social infrastructure, and maintenance costs, \( M(K) \), of existing infrastructure. In addition, there may be costs \( F(K(T)) \) which are incurred during the operational phase of the facility, but which are dependent only on the amount of infrastructure in place at the end of the construction phase. (For example, \( F \) could represent maintenance costs for infrastructure over the operational life of the facility.) Assuming that the decision maker will minimize the present value of these costs, his objective functional will be

\[
\text{minimize} \quad J(I,L) = \int_{0}^{T} e^{-\delta t} \left[ W(K,L) + I + M(K) \right] dt + F(K(T)) \tag{2.7}
\]

where \( \delta \) is the positive rate of discount.
The model posed by Cummings and Schulze [5] is the general model described above,\(^2\) in (2.1) to (2.7), with specific functional forms assumed for \(f, W, M,\) and \(F\). The production function \(f\) is defined as

\[
f(L) = L^p, \quad \text{for some constant } p \in (0,1). \quad \text{(2.8)}
\]

The wage function is also non-linear in both \(K\) and \(L\):

\[
W(K,L) = \frac{B}{(K/L)^\eta} \cdot L \quad \text{(2.9)}
\]

where \(B\) and \(\eta\) are positive constants. The wage rate \(B/(K/L)^\eta\) reflects the trade-off which is assumed to exist between wages and the infrastructure/labour ratio. The model abstracts from depreciation of the stock of social capital by assuming that with a rate of unit maintenance costs \(m\) per period, capital stocks do not deteriorate. Thus maintenance of existing infrastructure is given by

\[
M(K) = mK. \quad \text{(2.10)}
\]

Finally, the terminal function \(F\) is defined to include total wages paid to labour during the operational phase of the facility, plus maintenance costs for social capital. The rate of labour required to operate the

\(^2\)In their model, Cummings and Schulze explicitly allow for "front-end" investments, which increase the level of social capital from \(K(0^-)\) to \(K(0^+)\) at \(t = 0\). This instantaneous jump in the level of infrastructure is simply the result of an impulse control (or \(\delta\)-function) at \(t = 0\), which may or may not be allowed, depending on the control set for investment specified in the model.
facility is assumed to be fixed at \( L = \hat{L} \), so that

\[
F(K(T)) = \int_0^\infty e^{-\delta t} \left\{ W(K(T), \hat{L}) + mK(T) \right\} dt \\
= \frac{1}{\delta} e^{-\delta T} \left\{ W(K(T), \hat{L}) + mK(T) \right\} . \tag{2.11}
\]

Substitution of Equations (2.8) to (2.11) into the general model results in a nonlinear optimal control problem, with two state variables, \( C \) and \( K \), and two control variables, \( L \) and \( I \). Such problems are usually difficult to solve analytically and this one appears to be no exception. Cummings and Schulze [5] derive some necessary conditions for a solution, but these are by no means a complete specification of the solution.

The model solved in this thesis is derived from the model in Equations (2.1) to (2.11), by means of two additional assumptions:

(i) The production function \( f(L) \) is linear. (I.e., \( \rho = 1 \).)

(ii) The infrastructure/labour ratio is constant over the time interval \([0, T]\). I.e., \( L = \alpha K \), for some constant \( \alpha \), and by scaling the labour and construction variables, we may assume that \( \alpha = 1 \). Note that this results in fixing the wage rate \( \beta/(K/L)^\eta = \beta \).

With these assumptions, the state equation for construction, (2.1), becomes

\[
\dot{C} = K . \tag{2.12}
\]
From (2.9) and (2.10), the sum of wages and maintenance costs during construction reduces to

\[ W(K, L) + M(K) = \beta K + mK = YK \ldots (2.13) \]

with \( \hat{Y} = \beta + m \). Since the wage rate for construction labour is now fixed, it is somewhat inconsistent to have the wage rate for operation of the completed facility dependent on the level of infrastructure available. Therefore, we assume that the wage rate for operation of the facility is fixed, and with labour for the operational phase fixed at \( L = \hat{L} \), total wage payments made after time \( T \) are a specified constant and can be eliminated from the cost functional. The terminal function in Equation (2.11) is replaced by

\[ F(K(T)) = \frac{m}{\delta} e^{-\frac{\delta t}{\delta}} K(T) \ldots (2.14) \]

which is just the cost of maintaining (forever) the level of infrastructure which exists at the end of the construction phase.

Equations (2.12) to (2.14) show that in the simplified model, \( L \) has been eliminated, so there remains only one control variable, \( I \). For this model, we specify the control set for \( I \) to be the interval \([0, I_M]\), where \( I_M \) is a constant. (The case in which \( I_M = +\infty \) which makes impulse controls admissible, is discussed in Section 5). Finally, we shall replace the initial condition in Equation (2.3) with

\[ C(0) = C_0 \ldots (2.15) \]
where \( C_0 \) is a non-negative constant. This generalization simply includes the case in which construction of the facility has been started before the time from which optimization begins (i.e., \( t = 0 \)).

Incorporating the changes in Equation (2.12) to (2.15) into the original model thus yields the version which will be analyzed in this thesis. The essence of the problem can be viewed as follows. The level of infrastructure at any time specifies how much labour is hired, which in turn determines how quickly the facility is being constructed. Therefore, extra units of infrastructure allow the construction period to be compressed into a shorter length of time so that construction costs can be delayed, thus reducing the present value of those costs. On the other hand, an increased level of infrastructure will result in higher costs in the period in which investment occurs, and in all later periods, due to maintenance costs. Therefore, the level of infrastructure in any period should be increased until the present value of the cost of an extra unit is just equal to the reduction in discounted costs due to the delay in construction which that unit makes possible. This trade-off, then, is the fundamental determinant of the optimal investment strategy in this model. A statement of the model and its solution follow in Section 3.
3. **SOLUTION TO THE BASIC PROBLEM**

In this section, we find an optimal control for our basic problem and we show that, among the class of piecewise continuous \(^3\) (PWC) controls, it is unique. For reference, the problem as modified in Section 2 is stated below:

\[
\begin{align*}
\text{minimize} & \quad J\{l\} = \int_{0}^{T} e^{-\delta t} \left\{ \gamma K + l \right\} dt + F(K(T)) \quad \ldots \quad (3.1) \\
\text{subject to} & \quad \begin{cases} 
\dot{C} = K \\
\dot{K} = l \\
l(t) \in [0, l_M] 
\end{cases} \quad 0 \leq t \leq T. \quad \ldots \quad (3.2) \\
& \quad C(0) = C_0 \quad \ldots \quad (3.5) \\
& \quad C(T) = C_T \quad \ldots \quad (3.6) \\
& \quad K(0) = K_0 \quad \ldots \quad (3.7)
\end{align*}
\]

where \(\delta, \gamma, l_M, T, \text{ and } C_T\) are fixed positive constants and \(K_0, C_0\) are non-negative constants.

There are several points about the structure of the problem which should be noted at this stage. Firstly, the minimization is taken over all controls \(l\) which are PWC on \([0, T]\). Secondly, we will not

\(^3\)A function \(u(t)\) is PWC on \([0, T]\) if it is continuous at all but a finite number of points in \([0, T]\), and it admits finite limits from the left and right at each of these points of discontinuity.
initially specify the terminal function $F(K(T))$ as Equation (2.14); we
need only assume that $F$ is lower semi-continuous on $[0, \infty]$ to ensure
the existence of an optimal control. Thirdly, we also need to assume
that
\[ 0 \leq C_T - C_0 - K_0 \cdot T \leq \frac{l_M \cdot T^2}{2} \quad \ldots (3.8) \]
to guarantee that the set of feasible controls is non-empty. If the
second inequality in (3.8) is violated, then any admissible control $I$,
with state trajectories $C$ and $K$, satisfies
\[
C(T) = C_0 + \int_0^T K(t) \, dt \leq C_0 + \int_0^T (l_M t + K_0) \, dt \\
= C_0 + \frac{l_M \cdot T^2}{2} + K_0 \cdot T \\
< C_T.
\]
Thus there are no feasible controls because the time horizon is too
short to complete construction on time, even if investment and
construction proceed at the maximum possible rate. Similarly, if the
first inequality in (3.8) is violated, then every control $I$ with values
in $[0, l_M]$ results in $C(T) > C_T$, which means that the terminal
condition in Equation (3.6) cannot be satisfied. The first inequality
thus eliminates the trivial case in which the level of infrastructure in
place prior to the period of optimization is already greater than that

\[ ^{\text{4}} \text{The distinction between feasible and admissible controls, as defined in Clark [2, p. 89], will be followed in this thesis.} \]
required to ensure completion of the project by time $T$.

The solution to the problem stated above depends primarily on an application of the Pontryagin Maximum Principle (see [1]), which describes necessary conditions for an optimal control. Strictly speaking, in attempting to solve any control problem, one should use the Maximum Principle in conjunction with other theorems which establish the existence of an optimal control among the class of admissible controls. To ensure that the optimal investment strategy for this problem is one which can be physically implemented, we have restricted the class of admissible controls to the class of PWC functions which satisfy the control constraint in (3.4). By referring to well-known existence theorems, one can easily verify that there is an optimal control among the class of measurable controls. With this information, the Maximum Principle may be applied to identify candidates for an optimal control which is measurable, but not necessarily PWC. We shall see, however, that these measurable controls are equivalent, in the sense that they differ only on a set of measure zero, and therefore, they must all result in the same value for the objective functional. From this class of optimal measurable controls, one which is PWC shall be uniquely specified. This control must be optimal among the class of piecewise continuous controls, and therefore it will provide the solution to our problem.

---

5See, for example, Berkovitz [1, p. 61] or Lee and Markus [7, p. 233].
We proceed with the Maximum Principle. The Hamiltonian for this problem is

$$\mathcal{H}(t, K, l, \lambda_0, \mu, \lambda) = -\lambda_0 e^{-\delta t} (\gamma K + 1) + \mu K + \lambda l.$$ \ldots (3.9)

If \( I \) and \((C, K)\) are an optimal control and its response, respectively, then there exist a constant, \( \lambda_0 \), equal to 0 or 1, and functions \( \mu \) and \( \lambda \), absolutely continuous on \([0, T]\), such that

i) \( \mathcal{H}(t, K(t), l(t), \lambda_0, \mu(t), \lambda(t)) = \max \mathcal{H}(t, K(t), z, \lambda_0, \mu(t), \lambda(t)) \text{ a.e.} \quad z \in [0, l_M] \) \ldots \ldots (3.10)

ii) the vector \( (\lambda_0, \mu(t), \lambda(t)) \) is never 0 on \([0, T]\). \ldots \ldots (3.11)

iii) \[
\begin{align*}
\dot{\mu} &= -\frac{\partial \mathcal{N}}{\partial C} = 0 \\
\dot{\lambda} &= -\frac{\partial \mathcal{N}}{\partial K} = \lambda_0 \gamma e^{-\delta t} - \mu(t)
\end{align*}
\] \text{ a.e.} \ldots \ldots (3.12)

iv) \( -(\mu(T), \lambda(T)) = \lambda_0 \left( 0, F'(K(T)) \right) \in \{ C_T \} \times (0, \infty) \),

or \( \lambda(T) = -\lambda_0 F'(K(T)) \). \ldots \ldots (3.13)

Since \( \mu \) is absolutely continuous, (3.12) implies that

$$\mu(t) = \text{constant} = \mu, \quad 0 \leq t \leq T. \ldots \ldots (3.14)$$

The case in which \( \lambda_0 = 0 \) is referred to as the abnormal form of the problem. It can easily be shown that if \( \lambda_0 = 0 \), then one
of the inequalities in (3.8) must be an equality. This makes the problem trivial to solve because there is only one feasible control. (See Appendix A.) Clearly, the more interesting form of the problem is the normal one, when \( \lambda_0 = 1 \), and (3.8) holds with strict inequality.

In order to make the form of the solution more transparent, we consider first the case in which the initial levels of social capital and construction are both zero (i.e., \( K_0 = C_0 = 0 \)), and there is no terminal payoff (i.e., \( F = 0 \)). Then the transversality condition (3.13) is just

\[
\lambda(T) = 0.
\]  

(3.15)

From Equation (3.9), with \( \lambda_0 = 1 \), the Hamiltonian may be written as

\[
\mathcal{H}(t, K, l, \mu, \lambda) = (\lambda - e^{-\delta t})l + (\mu - \gamma e^{-\delta t})K. \quad (3.16)
\]

so by (3.10) a control \( l \) which is optimal must satisfy

\[
l(t) = \begin{cases} 
  l_M & \text{when } \sigma(t) > 0 \\
  0 & \text{when } \sigma(t) < 0 
\end{cases} \quad \text{a.e.} \quad (3.17)
\]

where

\[
\sigma(t) = \lambda(t) - e^{-\delta t} \quad (3.18)
\]

is the switching function, and \( \lambda(t) \), the adjoint variable for \( K(t) \), is determined by
\[ \dot{\lambda} = \gamma e^{-\delta t} - \lambda, \quad 0 \leq t \leq T \]  
\[ \lambda(T) = 0 \]  
\[ \ldots (3.19) \]

for some constant \( \mu \). Examination of (3.18) and (3.19) shows that there is no singular solution to the problem, because \( \sigma \) is strictly concave.

Therefore, for a given value of \( \mu \), Equation (3.17) determines a class of measurable controls which differ only on a set of measure zero, and consequently, each has the same cost. From this class of measurable controls, we shall work with the one which is PWC and takes the value \( l_M \) at roots of \( \sigma \), that is,

\[ l(t) = \begin{cases} 
  l_M \quad \text{when } \sigma(t) > 0 \\
  0 \quad \text{when } \sigma(t) < 0 
\end{cases}, \quad 0 \leq t \leq T \]  
\[ \ldots (3.20) \]

What remains to be shown, then, is how to determine the optimal value(s) of \( \mu \). In fact, there is only one value of \( \mu \) for which the corresponding PWC control specified by Equation (3.20) satisfies the terminal condition \( C(T) = C_T \). This becomes clear after studying the dependence of the adjoint variable \( \lambda(t, \mu) \) on the parameter \( \mu \).

From (3.19), \( \lambda(t, \mu) \) is concave and has a root at \( t = T \). Figure 1 divides the family of curves \( \{ \lambda(t, \mu) : \mu \in \mathbb{R} \} \) into three groups, according to the sign of the expression

\[ \dot{\lambda}(T; \mu) = \gamma e^{-\delta T} - \mu. \]
Figure 1  The Adjoint Function $\lambda(t; \mu)$ for Alternate Signs of $\dot{\lambda}(T; \mu)$
We can dispense with those values of \( \mu \) for which \( \lambda(T; \mu) \geq 0 \); for such values of \( \mu \), \( \lambda(t; \mu) \) is negative on \([0, T)\), and therefore, the corresponding switching function, \( \sigma(t) = \lambda(t; \mu) - e^{-\delta t} \) specifies a control which is not feasible.

Integration of (3.19) yields

\[
\lambda(t) - \lambda(0) = \frac{\gamma}{\delta} \left( 1 - e^{-\delta t} \right) - \mu t ,
\]

and since

\[
\lambda(T) = 0 , \quad \lambda(0) = \mu T - \frac{\gamma}{\delta} (1 - e^{-\delta T}) ,
\]

so

\[
\lambda(t; \mu) \equiv \lambda(t) = \frac{\gamma}{\delta} (e^{-\delta T} - e^{-\delta t}) + \mu (T - t) \quad \ldots \ldots (3.21)
\]

The curve \( \lambda(t; \mu) \) reaches its maximum at \( t = \hat{t}(\mu) \), where

\[
\hat{t}(\mu) = -\frac{1}{\delta} \left( \frac{\mu}{\delta} \right) \quad \ldots \ldots (3.22)
\]

The function \( \hat{t} \) is decreasing and for fixed \( t < T \), \( \lambda(t; \mu) \) is an increasing function of \( \mu \). Therefore, as \( \mu \) increases from \( \mu = \gamma e^{-\delta T} \), the curve \( \lambda(t; \mu) \) pivots clockwise about the point \((T, 0)\) and "straightens out", as shown in Figure 2. Comparing these curves with the graph of \( e^{-\delta t} \) shows how the zeros of the switching function \( \sigma(t) \) depend on \( \mu \). Let \( \mu = \mu_* \) be the value of \( \mu \) for which \( \lambda(t; \mu) \) is just tangent to the curve \( e^{-\delta t} \). Let the point of tangency occur at \( t = t_* \). (See curve C in Figure 3.) For \( \mu > \mu_* \), the curves \( \lambda(t; \mu) \) and \( e^{-\delta t} \) intersect at two points \( t_0 = t_0(\mu) \) and \( t_1 = t_1(\mu) \), with \( t_0 < t_* < t_1 < T \). As \( \mu \) increases, \( t_0 \) decreases and \( t_1 \) increases. The points \( t_0(\mu) \) and \( t_1(\mu) \) are the zeros of the switching function \( \sigma(t) \), so the control \( I \) specified by Equation (3.20) is
Figure 2  The Adjoint Function $\lambda(t;\mu)$ for Increasing Values of $\mu$

Figure 3  Curves $e^{-\delta t}$ and $\lambda(t;\mu)$ for Varying $\mu$
\[
I(t) = \begin{cases} 
I_M, & t_0 < t < t_i \\
0 & \text{otherwise} 
\end{cases} \quad \ldots \quad (3.23)
\]

where \[t_0 = \begin{cases} 
t_o(\mu), & 0 \end{cases} \quad \ldots \quad (3.24)
\]
\[t_i = t_i(\mu)\]

From the state Equations (3.2) and (3.3) with initial conditions \[K_0 = C_0 = 0\], the response trajectories for this control are:

\[
K(t) = \begin{cases} 
0, & 0 < t < t_0 \\
I_M(t - t_0), & t_0 < t < t_i \\
I_M(t_i - t_0) = \bar{K}, & t_i < t < T \end{cases} \quad \ldots \quad (3.25)
\]

\[
C(t) = \begin{cases} 
0, & 0 < t < t_0 \\
\int_{t_0}^{t} I_M(s - t_0) \, ds, & t_0 < t < t_i \\
C(t_i) + \int_{t_i}^{t} \bar{K} \, ds, & t_i < t < T \end{cases}
\]

and evaluation of the integrals gives us

\[
C(t) = \begin{cases} 
0, & 0 < t < t_0 \\
1/2 \cdot I_M(t - t_0)^2, & t_0 < t < t_i \\
I_M(t - t_0)(T - t_0 + t_i), & t_i < t < T \end{cases} \quad \ldots \quad (3.26)
\]

Since \(t_0\) and \(t_1\) are functions of \(\mu\), so is \(C(T)\), the level of construction at time \(T\). That is,

\[
C(T) = C(T, t_o(\mu), t_i(\mu)) = I_M(t_i - t_0)(T - t_0 + t_i). \quad \ldots \quad (3.27)
\]
Taking partial derivatives of Equation (3.27) with respect to \( t_0 \) and \( t_1 \), we have

\[
\frac{\partial C(T)}{\partial t_0} = -l_M(T - t_0) < 0. \\
\frac{\partial C(T)}{\partial t_1} = l_M(T - t_1) > 0.
\]

which is exactly what common sense would predict. It is clear from Figure 3 that

\[
\frac{\partial t_0}{\partial \mu} < 0 \quad \text{and} \quad \frac{\partial t_1}{\partial \mu} > 0,
\]

so we conclude that

\[
\frac{dC(T)}{d\mu} = \left( \frac{dC(T)}{dt_0} \cdot \frac{\partial t_0}{\partial \mu} + \frac{dC(T)}{dt_1} \cdot \frac{\partial t_1}{\partial \mu} \right) > 0.
\]

Therefore, \( C(T) \) is monotone in \( \mu \), so there can be only one value of \( \mu \) for which \( C(T) = C_T \). That is, for only one choice of \( \mu \) is the control described by Equation (3.23) an admissible control. Choosing \( \mu \) determines the switching function \( \sigma(t) \) (Equation (3.18)) and the switching times \( t_0 \) and \( t_1 \). These, in turn, specify the response trajectories for capital, \( K \), and construction, \( C \), which satisfy the terminal constraint \( C(T) = C_T \), for only one value of \( \mu \). This establishes the uniqueness of the optimal control.

The economic interpretation of the trajectory for social capital which corresponds to this control is clear from Equation (3.21),
which may be rearranged as

\[ \lambda(t) = \mu(T-t) - \frac{\gamma}{\delta} \left( e^{-\delta t} - e^{-\delta T} \right). \]

The adjoint variables \( \lambda \) and \( \mu \) can be interpreted as the present value shadow prices of social capital, \( K \), and construction, \( C \), along the optimal trajectory. The first term, \( \mu(T-t) \), represents the present value of the reduction in total costs due to the lower levels of infrastructure required in later periods if an extra unit of capital is invested at time \( t \). The second term,

\[ \frac{\gamma}{\delta} \left( e^{-\delta t} - e^{-\delta T} \right) = \int_t^T e^{-\delta y} \, dy \]

is the additional cost in maintenance and wages which would accrue over \([t,T]\) with that extra unit of infrastructure. The difference of these two terms, \( \lambda(t) \), is therefore, the marginal benefit of investment in social capital at time \( t \). Since the marginal discounted cost of investment at time \( t \) is \( e^{-\delta t} \), the decision of whether or not to invest in more capital at time \( t \) depends on whether or not \( \lambda(t) \) is greater than \( e^{-\delta t} \). This is precisely what is stated in Equations (3.17) and (3.18), which describe the optimal investment strategy.

It is obvious that the form of the solution does not change if we now consider the more general initial conditions

\[ K(0) = K_0 > 0 \]
\[ C(0) = C_0 > 0. \]
The response trajectories change from $K$ and $C$, defined in (3.25) and (3.26) to $\tilde{K}$ and $\tilde{C}$, where

\[
\begin{align*}
\tilde{K}(t) & \equiv K(t) + K_0 \quad \ldots \ldots (3.30) \\
\tilde{C}(t) & \equiv C(t) + K_0 t + C_0. \quad \ldots \ldots (3.31)
\end{align*}
\]

Thus, the level of construction at time $T$ is still a monotonically increasing function of $\mu$, and the optimal value of $\mu$ is the one which satisfies

\[
C(T,t_0(\mu),t_1(\mu)) = C_T - (C_0 + K_0 T).
\]

So far, the discussion of the solution to the basic problem has assumed that the terminal function is identically zero. The simplest form of terminal function is one which is linear in $K(T)$, because this makes the transversality condition for $\lambda(T)$ independent of $K(T)$ and allows the same method of solution as when $F \equiv 0$.

For example, we might consider the terminal function discussed in Section 2, Equation (2.14),

\[
F(K(T)) = \frac{m}{\delta} e^{-\delta t} K(T) \quad \ldots \ldots (3.32)
\]

and investigate the qualitative change that this terminal function induces in the optimal solution. It is easy to see that the optimal investment strategy is still to invest at the maximum rate, $I_M$, over a sub-interval $[t_0^m, t_1^m]$ of $[0,T]$. Let $[t_0, t_1]$ be the optimal investment period when $F \equiv 0$. We shall show that if $t_0 > 0$, then
\[ t_0^m < t_0 < t_1^m < t_1 \quad \cdots (3.33) \]

and
\[ t_1^m - t_0^m < t_1 - t_0 \quad \cdots (3.34) \]

That is, investment in infrastructure starts earlier, and does not last as long when the model includes the maintenance costs of infrastructure over the operational phase of the facility.

If we denote the adjoint variable for \( K \) by \( \lambda^m(t;\mu) \equiv \lambda^m(t) \), then the transversality condition (3.13) becomes
\[ \lambda^m(T) = -m \frac{\delta}{\delta^T} \equiv -d < 0 \quad \cdots (3.35) \]

Since the adjoint Equations (3.12) do not change, \( \lambda^m(t;\mu) \) and \( \lambda(t;\mu) \) (for the case \( F \equiv 0 \)) differ by the constant \( d \), for any given value of \( \mu \). That is,
\[ \lambda^m(t;\mu) = \lambda(t;\mu) - m \frac{\delta}{\delta^T} = \lambda(t;\mu) - d \quad \cdots (3.36) \]

For a given \( \mu \), let \( t_0^m(\mu) < t_1^m(\mu) < T \) be the solutions to the equation
\[ \lambda^m(t;\mu) = e^{-\delta t} \]
if these two curves intersect, and let \( t_0(\mu) < t_1(\mu) < T \) be the solutions to the equation
\[ \lambda(t;\mu) = e^{-\delta t} \]

By Equation (3.36), if the curves \( \lambda^m(t;\mu) \) and \( e^{-\delta t} \) cross, they must
Figure 4  Curves $\lambda(t; \mu)$, $\lambda^m(t; \mu)$ and $e^{-\delta t}$, for a given $\mu$
cross below the curve \( \lambda(t; \mu) \), as in Figure 4, so that
\[
t_0(\mu) < t_0^m(\mu) < t_1^m(\mu) < t_1(\mu) . \tag{3.37}
\]
Since the function \( C(T, t_0, t_1) \) is decreasing in \( t_0 \) and increasing in \( t_1 \), (3.37) implies that
\[
C(T, t_0^m(\mu), t_1^m(\mu)) < C(T, t_0(\mu), t_1(\mu)) . \tag{3.38}
\]
Let \( \mu^m \) and \( \mu^0 \) be the optimal values of \( \mu \) for the problem with and without the non-zero terminal function, respectively. Because the solutions to both problems must satisfy \( C(T) = C_T \),
\[
C(T, t_0^m(\mu^m), t_1^m(\mu^m)) = C_T = C(T, t_0(\mu^0), t_1(\mu^0)) . \tag{3.39}
\]
Recalling that \( C(T, t_0^m(\mu), t_1^m(\mu)) \) is increasing in \( \mu \), (3.38) and (3.39) imply that \( \mu^m > \mu^0 \). We shall therefore consider the curves \( \lambda^m(t; \mu) \) for \( \mu > \mu^0 \).

Let \( \mu^1 > \mu^0 \) be the value of \( \mu \) such that \( \lambda^m(t; \mu^1) \) crosses the curves \( e^{-\delta t} \) and \( \lambda(t; \mu^0) \) at \( t_0(\mu^0) \), as shown in Figure 5.

Then
\[
\lambda^m(t_0(\mu^0); \mu^1) = e^{-\delta t_0(\mu^0)}
= \lambda(t_0(\mu^0); \mu^0)
= \lambda^m(t_0(\mu^0); \mu^0) + d
\]
or,
\[
\lambda^m(t_0(\mu^0); \mu^1) - \lambda^m(t_0(\mu^0); \mu^0) = d .
\]
Figure 5  Optimal Curve $\lambda^m(t;\mu^m)$ is Trapped Between $\lambda^m(t;\mu^1)$ and $\lambda^m(t;\mu^2)$
Since
\[ \lambda^m(t;\mu^1) - \lambda^m(t;\mu^0) = (\mu^1 - \mu^0)(T-t), \]
which is decreasing on \([0,T]\), and \(t_0(\mu^0) < t_1(\mu^0)\), it must be true that
\[ \lambda^m(t_1(\mu^0);\mu^1) > \lambda^m(t_1(\mu^0);\mu^0) < d. \]

Therefore,
\[
\begin{align*}
\lambda^m(t_1(\mu^0);\mu^1) &< \lambda^m(t_1(\mu^0);\mu^0) + d \\
&= \lambda(t_1(\mu^0);\mu^0) \\
&= e^{-\delta t_1(\mu^0)}
\end{align*}
\]

so the curve \(\lambda^m(t;\mu^1)\) crosses the curve \(e^{-\delta t}\) at times to \(t_0^m(\mu^1)\) and \(t_1^m(\mu^1)\), where
\[ t_0^m(\mu^1) = t_0(\mu^0) < t_1^m(\mu^1) < t_1(\mu^0) . \quad \ldots \ldots (3.40) \]

Thus, \(C(T,t_0^m(\mu^1),t_1^m(\mu^1)) < C_T\), and we conclude that
\[ \mu^m > \mu^1 . \quad \ldots \ldots (3.41) \]

Similarly, if \(\mu^2 > \mu^1\) is the value of \(\mu\) such that \(\lambda^m(t;\mu^2)\) crosses \(e^{-\delta t}\) and \(\lambda(t;\mu^0)\) at \(t_1^m(\mu^2) = t_1(\mu^0)\), then the other intersection of \(\lambda^m(t;\mu^2)\) with \(e^{-\delta t}\) occurs at a point
\[ t_0^m(\mu^2) < t_0(\mu^0), \quad \text{so} \quad C(T,t_0^m(\mu^2),t_1^m(\mu^2)) > C_T, \]
which implies that
\[ \mu_1^m < \mu_2^1. \] \[ \text{... (3.42)} \]

From (3.41) and (3.42), and by monotonicity of \( t_{0}^{m}(\mu) \) and \( t_{1}^{m}(\mu) \), it follows that
\[ t_{0}^{m}(\mu^m) \in \left( t_{0}^{m}(\mu^1), t_{0}^{m}(\mu^1) \right) = \left( t_{0}^{m}(\mu^1), t_{0}^{m}(\mu^0) \right), \]
\[ \text{and} \]
\[ t_{1}^{m}(\mu^m) \in \left( t_{1}^{m}(\mu^1), t_{1}^{m}(\mu^0) \right). \]
\[ \text{... (3.43)} \]

as Figure 5 indicates. Simplifying notation to \( t_{i}^{m}(\mu^m) \equiv t_{i}^{0} \) and \( t_{i}^{m}(\mu^0) \equiv t_{i}^{1} \), where the \( t_{i}^{m} \) and \( t_{i}^{1} \) (i = 0, 1) are the optimal switching times, the conditions in (3.43) imply that \( t_{0}^{m} < t_{0}^{0} < t_{1}^{m} \leq t_{1}^{1} \), which is one of the results (Equation (3.33)) which we wanted to show.

The second result follows easily. From Equation (3.39) and (3.27),
\[ \left( t_{1}^{m} - t_{0}^{m} \right) \left( T - \frac{t_{0}^{m} + t_{1}^{m}}{2} \right) = \left( t_{1}^{m} - t_{0}^{m} \right) \left( T - \frac{t_{0}^{m} + t_{1}^{m}}{2} \right) \]
\[ \text{... (3.44)} \]

and by Equation (3.33),
\[ \frac{t_{0}^{m} + t_{1}^{m}}{2} \]
\[ \frac{t_{0}^{m} + t_{1}^{m}}{2} \]
\[ \text{... (3.45)} \]

Equations (3.44) and (3.45) imply that
\[ \left( t_{1}^{m} - t_{0}^{m} \right) = \left( t_{1}^{m} - t_{0}^{m} \right) \left( T - \frac{t_{0}^{m} + t_{1}^{m}}{2} \right) < \left( t_{1}^{m} - t_{0}^{m} \right) \]
\[ \left( T - \frac{t_{0}^{m} + t_{1}^{m}}{2} \right) \]
which is the inequality we claimed in (3.34).

Therefore, the investment intervals for the two problems overlap, but with the non-zero terminal function (3.30), the investment starts earlier, ends earlier, and lasts for a shorter time, than with the terminal function identically zero. This will be discussed further in Section 4.

The conditions in (3.33) and (3.34) have been shown to be true when \( t_0 > 0 \), i.e., when construction (and investment) doesn't start right at time \( t = 0 \). We should mention what happens when \( t_0 = 0 \). In this case, the optimal control (for \( F \equiv 0 \)) is

\[
I(t) = \begin{cases} 
  I_W, & 0 \leq t \leq t_1 \\
  0, & t_1 < t \leq T
\end{cases}
\]

because the smaller root, \( t_0(\mu^0) \), of the switching function is negative.

A similar argument to the one above shows that

\[
t_0^m(\mu^m) < t_0(\mu^0) < 0 < t_1^m(\mu^m) = t_1(\mu^0)
\]

In this case, therefore, the optimal control does not change when the terminal function (3.32) is included in the objective functional.

This concludes our description of the solution to the basic model. The following section extends this discussion by analyzing some qualitative aspects of the solution.
4. QUALITATIVE BEHAVIOUR OF THE SOLUTION

As we have already seen, the unique optimal solution to the basic problem is a bang-bang control, with only one interval, \([t_0, t_1]\), during which investment occurs. The investment may begin at time \(t = 0\), and it always ends before time \(T\).

The most obvious question which can be raised at this point is: how does this investment policy depend on the parameters of the model? In this section, we discuss the qualitative behaviour of the solution which results from changes in the parameter values. Our analytical results are numerically exemplified in the accompanying tables, which describe the optimal investment interval corresponding to particular parameter values.

The numerical method of solution and the computer program for it are summarized in Appendix C. Some attempt has been made to determine an initial set of parameter values which are realistic, based on data for boomtowns in the western United States. (See Mehr and Cummings [8]). Calculations to estimate reasonable ranges of parameter values are contained in Appendix B. The initial set of parameter values comprises the first line in Table I, and other combinations have been chosen to provide numerical examples of the qualitative changes in the solution. In all cases, we have assumed that the initial levels of social capital, \(K_0\), and construction \(C_0\), are both zero. This is not an unrealistic assumption, and as we point

\[^6\text{For example, the proposed location of a new mine is often too distant from any town or city to make use of existing social infrastructure, so that a while new town must be built to service the labour force associated with the construction and operation of the mine.}\]
out below, the qualitative change in the solution corresponding to a positive value for $K_0$ or $C_0$ is easy to predict analytically.

The parameters in the model can be separated into three groups, namely:

(i) those parameters which affect the response trajectories $C$ and $K$, but do not appear in the adjoint function $\lambda(t;\mu)$,
(ii) those which affect the adjoint function, but not the response trajectories, and
(iii) one parameter which appears in the response trajectories and the adjoint function.

The following discussion concentrates on each of these groups separately.

The parameters $K_0$, $C_0$, $I_M$, and $C_T$ do not affect the adjoint function $\lambda(t;\mu)$. They determine the initial and terminal conditions and control constraints, and thus enter the solution only through the response trajectories. Therefore, when one of these parameters is changed, the qualitative change in the solution is obvious from Equation (4.1)

$$\frac{C_T - (C_0 + K_0 T)}{I_M} = \frac{C(T, t_0(\mu), t_1(\mu))}{I_M} = (t_1 - t_0)(T - \frac{t_0 + t_1}{2}).$$

if we recall that

$$\frac{dc}{d\mu} > 0, \quad \frac{db}{d\mu} < 0, \quad \text{and} \quad \frac{dt_1}{d\mu} > 0.$$
If the ratio \( \frac{C_T - (C_0 + K_0 T)}{I_M} \) is decreased (by a change in any parameter except \( T \)), then \( (t_1 - t_0) \left[ T - \frac{t_0 - t_1}{2} \right] \) must also decrease, causing \( \mu^0 \), the optimal value of \( \mu \), to decrease, and \( t_0 \) and \( t_1 \) to increase and decrease, respectively.

This result is intuitively clear. If the level of completed construction, \( C_T \), is reduced, or the rate of investment in infrastructure, \( I_M \), is increased, costs can be reduced by starting the investment later, and by investing for a shorter period of time. This effect can be observed in Table I. Lines 1 and 4 differ only in the value of \( C_T \). For the smaller value of \( C_T \), the investment period \([t_0, t_1]\) is reduced to \([6.1, 6.4]\) from \([4.8, 7.3]\). Similarly, lines 6 and 1 show that increasing \( I_M \) from \( 2.0 \times 10^6 \) to \( 5.0 \times 10^6 \) shrinks the investment interval from \([2.7, 8.2]\) to \([4.8, 7.3]\).

This leads us to consider a "limiting" form of the problem, in which \( I_M = +\infty \). Successively larger, but finite values for \( I_M \) shrink the investment interval (as shown in Table II) so that when \( I_M = +\infty \), we expect that the investment "interval" should just be a point, \( t_0 = t_1 \). The corresponding control would have to be an impulse control, i.e., investment which occurs only at \( t = t_0 \), at which time the level of infrastructure jumps from \( K(0) \) to \( K(t_0) = K(T) \), in order to just complete construction by time \( T \). The control problem with \( I_M = +\infty \) is solved in Section 5.

A second group of parameters, which includes \( \delta \), \( \gamma \), and \( m \), appear in the solution only through the adjoint function \( \lambda(t; \mu) \).
<table>
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<tr>
<th>Set</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$C_T$</th>
<th>$I_M$</th>
<th>$T$</th>
<th>$m^i$</th>
<th>$\mu$</th>
<th>$t_0(\mu)$</th>
<th>$t_1(\mu)$</th>
<th>$t_1 - t_0$</th>
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## TABLE II

**Behaviour of Solution as \( I_M \) Increases**

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<th>( I_M )</th>
<th>( T )</th>
<th>( m )</th>
<th>( \mu )</th>
<th>( t_0(\mu) )</th>
<th>( t_1(\mu) )</th>
<th>( t_1-t_0 )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>2.0</td>
<td>1.0 ( \times 10^7 )</td>
<td>5.0 ( \times 10^5 )</td>
<td>10</td>
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<td>8.859900</td>
<td>5.286400</td>
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<td>2.0</td>
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<td>5.0 ( \times 10^6 )</td>
<td>10</td>
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<tr>
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<td>1.0 ( \times 10^7 )</td>
<td>10</td>
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<td>0.355563</td>
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<td>1.5 ( \times 10^7 )</td>
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<td>0</td>
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<td>7.071745</td>
<td>7.309025</td>
<td>0.237280</td>
</tr>
<tr>
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<td>0.10</td>
<td>2.0</td>
<td>1.0 ( \times 10^7 )</td>
<td>2.0 ( \times 10^7 )</td>
<td>10</td>
<td>0</td>
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<td>7.280389</td>
<td>0.178023</td>
</tr>
<tr>
<td>6</td>
<td>0.10</td>
<td>2.0</td>
<td>1.0 ( \times 10^7 )</td>
<td>3.0 ( \times 10^7 )</td>
<td>10</td>
<td>0</td>
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<td>7.132732</td>
<td>7.251444</td>
<td>0.118712</td>
</tr>
<tr>
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<td>0.10</td>
<td>2.0</td>
<td>1.0 ( \times 10^7 )</td>
<td>5.0 ( \times 10^7 )</td>
<td>10</td>
<td>0</td>
<td>1.022954</td>
<td>7.156834</td>
<td>7.228070</td>
<td>0.071236</td>
</tr>
<tr>
<td>8</td>
<td>0.10</td>
<td>2.0</td>
<td>1.0 ( \times 10^7 )</td>
<td>1.0 ( \times 10^8 )</td>
<td>10</td>
<td>0</td>
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<td>7.174795</td>
<td>7.210415</td>
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<tr>
<td>9</td>
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<td>2.0</td>
<td>1.0 ( \times 10^7 )</td>
<td>1.0 ( \times 10^{20} )</td>
<td>10</td>
<td>0</td>
<td>1.022931</td>
<td>7.192656</td>
<td>7.192657</td>
<td>0.000001</td>
</tr>
</tbody>
</table>
A change in one of these parameters will shift the interval over which investment occurs either towards \( T \), or away from \( T \), with an accompanying increase or decrease, respectively, in \( (t_1-t_0) \), the length of the interval, in order to meet the condition \( C(T) = C_T \).

The analysis in Section 3 on the terminal function

\[
F(K(T)) = \frac{m}{\delta} e^{-\delta T} K(T),
\]

where \( m \) represents unit maintenance costs for social capital, shows that increasing \( m \) from zero to some positive level moves the interval \([t_0, t_1]\) closer to \( t = 0 \), and decreases \( (t_1-t_0) \) (see (3.33) and (3.34)). This effect appears in Table I, lines 1 and 10, where setting \( m = 0.1 \) shifts \([t_0, t_1]\) from \([4.8, 7.3]\) to \([3.9, 5.8]\). Of course, if the root \( t_0(\mu^0) \) is negative for \( m = 0 \), then the first switching time is \( t_0 = 0 \), and increasing \( m \) to a positive level does not change the solution. This prediction is illustrated in lines 4 and 6 of Table III.

It is clear from the analysis in Section 3 that the same qualitative change in the solution occurs if \( m \) is increased from some positive level \( m_1 \) to a higher level \( m_2 \), and this result makes economic sense. When \( m \) is increased, the marginal cost of social capital is higher. Since \( K(T) \), the optimal level of social capital at time \( T \), is that level for which marginal costs are equal to marginal benefits, an upward shift in the marginal cost curve results in a lower optimal level for social capital. That is, \( K^m(T) < K^0(T) \), so from Equation (3.25), \((t_1^m-t_0^m) < t_1-t_0 \). Therefore, in order to meet the terminal
TABLE III
Sensitivity of Solution to Changes in $T$ and $m$, with Negative $t_0 (\mu)$

<table>
<thead>
<tr>
<th>Set</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$C_T$</th>
<th>$I_M$</th>
<th>$T$</th>
<th>$m$</th>
<th>$\mu$</th>
<th>$t_0 (\mu)$</th>
<th>$t_1 (\mu)$</th>
<th>$t_1 - t_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.02</td>
<td>1.0</td>
<td>$5.0 \times 10^7$</td>
<td>$5.0 \times 10^6$</td>
<td>10</td>
<td>0.0</td>
<td>1.00620</td>
<td>0.14710</td>
<td>1.2205</td>
<td>1.0734</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>1.0</td>
<td>$5.0 \times 10^7$</td>
<td>$5.0 \times 10^6$</td>
<td>20</td>
<td>0.0</td>
<td>0.82378</td>
<td>10.14710</td>
<td>11.2205</td>
<td>1.0734</td>
</tr>
<tr>
<td>3</td>
<td>0.02</td>
<td>1.0</td>
<td>$5.0 \times 10^7$</td>
<td>$5.0 \times 10^6$</td>
<td>10</td>
<td>0.1</td>
<td>1.46370</td>
<td>-35.01200</td>
<td>1.0557</td>
<td>1.0557</td>
</tr>
<tr>
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<td>1.0</td>
<td>$5.0 \times 10^7$</td>
<td>$2.0 \times 10^6$</td>
<td>10</td>
<td>0.0</td>
<td>1.01280</td>
<td>-2.17970</td>
<td>2.9289</td>
<td>2.9289</td>
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<tr>
<td>5</td>
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<td>1.0</td>
<td>$5.0 \times 10^7$</td>
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<td>10</td>
<td>0.1</td>
<td>1.59180</td>
<td>-43.81700</td>
<td>2.9289</td>
<td>2.9289</td>
</tr>
</tbody>
</table>
condition $C(T) = C_T$, the investment interval for the model which includes the terminal function (or has a larger value of $m$) must start earlier. I.e., $t_0^m < t_0$.

An increase in either of the other two parameters, $\delta$, and $\gamma$, appears to have the opposite effect on the solution from an increase in $m$. As $\delta$ or $\gamma$ increases, $[t_0, t_1]$ shifts towards $T$. This observation relies, however, on the numerical results presented in Table I. Lines 1 and 8 reflect the change in the solution corresponding to a change in $\delta$, and lines 1, 2, and 3 show the optimal investment interval for different values of $\gamma$. That these qualitative effects are exhibited for all positive values of $\delta$ and $\gamma$ is more difficult to prove than the corresponding result for the parameter $m$. The simple graphical analysis which was used to prove results for $m$ relies on the fact that $3A/3m$ is not a function of $t$, which is not the case for either $3A/3\delta$ or $3A/3\gamma$! Thus, the same method of proof cannot be applied in this case. The behaviour of the investment interval, corresponding to changes in $\delta$ and $\gamma$, can be intuitively motivated by appealing to the effect of discounting. As the discount rate $\delta$ increases, costs imposed in the future are weighted less, so that it pays to postpone investment until a time closer to $T$, in spite of the fact that this delay must be accompanied by a longer period of investment, and therefore, a larger final level of social capital. Similarly, an increase in $\gamma$, which we can think of as an increase in the wage rate for labour, means that the total wage bill is increased for a given investment--construction policy. Because costs are discounted at a
positive rate, it is therefore better to wait longer to start investment and construction, even though more social capital will ultimately be required.

The one parameter which has, so far, not been accounted for, is the time horizon $T$. Changes in this parameter affect both the terminal condition $C(T) = C_T$ and the adjoint function $\lambda(t; \mu)$, so one might expect that the corresponding changes in the optimal solution would be complicated. However, numerical results indicate a simple relationship between $T$ and the optimal investment interval, $[t_0, t_1]$. These results show that if $t_0 > 0$, then changing $T$ to $T + \Delta$ will change the investment interval to $[t_0 + \Delta, t_1 + \Delta]$, provided that $t_0 + \Delta > 0$.

For example, by comparing lines 8 and 9 of Table I, we see that decreasing $T$ by 14 units, from 20 to 6, simply shifts $[t_0, t_1]$ 14 units towards the origin, i.e., from $[14.79, 17.33]$ to $[0.79, 3.33]$. If $t_0 + \Delta < 0$, however, investment over $[0, t_1 + \Delta]$ is not a feasible control, since it results in $C(T) < C_T$, so the optimal investment interval extends from $t = 0$ past $t = t_1 + \Delta$. (Compare the value of $t_1 - t_0$ in lines 4 and 5, Table III).

Examination of the switching function allows us to verify these results analytically. Suppose that $\mu_0$ is the optimal value of $\mu$ for the problem with time horizon $T$, and let $t_0 = t_0(\mu_0)$ and $t_1 = t_1(\mu_0)$, with $t_0 > 0$, be the optimal switching times. If the time horizon is changed to $T + \Delta$, it suffices to show that
(i) $t_0^+ \Delta$ and $t_1^+ \Delta$ are roots of $\sigma(t; T^+ \Delta, \mu)$ for some value of $\mu = \mu_\Delta$. I.e., $\sigma(t_0^+ \Delta; T^+ \Delta, \mu_\Delta) = \sigma(t_1^+ \Delta; T^+ \Delta, \mu_\Delta) = 0$.

(ii) the switching times $t_0^+ \Delta$, $t_1^+ \Delta$ satisfy the terminal condition for construction with time horizon $T^+ \Delta$. I.e.,

$$C(T^+ \Delta, t_0^+ \Delta, t_1^+ \Delta) = C_T.$$ 

If, in addition to (i) and (ii), $t_0^+ \Delta \geq 0$, then the investment interval $[t_0^+ \Delta, t_1^+ \Delta]$ describes a feasible control, and since the value of $\mu$ which satisfies (i) and (ii) is unique, this control is optimal.

We claim that the value of $\mu$ which satisfies (i) is defined by

$$\mu_\Delta = \mu_0 e^{-\delta \Delta}.$$ 

To see this, we evaluate the switching function $\sigma(t; T^+ \Delta, \mu_\Delta)$ at $t = t_0^+ \Delta$ and $t = t_1^+ \Delta$. From Equation (3.18) and (3.21), for $i = 0, 1$,

$$\sigma'(t_i^+ \Delta; T^+ \Delta, \mu_\Delta) = \frac{\chi}{\delta} e^{-\delta (T^+ \Delta)} - (\frac{\chi}{\delta} + 1) e^{-\delta (t_i^+ \Delta)} + \mu_\Delta [T^+ \Delta - (t_i^+ \Delta)]$$

$$= e^{-\delta \Delta} \left\{ \frac{\chi}{\delta} e^{-\delta T} - (\frac{\chi}{\delta} + 1) e^{-\delta t_i} \right\} + \mu_\Delta (T^+ \Delta - (t_i^+ \Delta))$$

$$= e^{-\delta \Delta} \left\{ \frac{\chi}{\delta} e^{-\delta T} - (\frac{\chi}{\delta} + 1) e^{-\delta t_i} + \mu_0 (T^+ \Delta - t_i^+ \Delta) \right\}$$

by (4.2)

$$= e^{-\delta \Delta} \cdot \sigma'(t_i^+ \Delta; T^+ \Delta, \mu_0)$$

$$= 0 .$$

I.e., $t_0^+ \Delta$ and $t_1^+ \Delta$ are zeros of $\sigma(t; T^+ \Delta, \mu_\Delta)$. 


Claim (ii) is obvious from Equation (4.1) (with $K_0 = C_0 = 0$). If $t_0$ and $t_1$ are the optimal switching lines for time horizon $T$, then

$$C(T; t_0, t_1) = I_M(t_1 - t_0)\left(T - \frac{t_0 + t_1}{2}\right) = C_T.$$ 

This equation is also satisfied for switching times $t_0 + \Delta_0$, $t_1 + \Delta_1$, with time horizon $T + \Delta$.

This shift in the investment interval corresponding to a change in $T$ agrees with the predictions of a simple economic analysis. For a given size of facility, $C_T$, investment rate, $I_M$, "wage" rate, $\gamma$, and discount rate, $\delta$, there is an optimal length of time, $\Delta_2$, to complete construction of the facility. The total construction time determines the total amount of social capital required, or, the length of the investment interval, $\Delta_1$, since investment always occurs at a constant rate.

Since costs are discounted, the construction should not be started until $\Delta_2$ units before the end of the time horizon, $T$. This $(\Delta_1^*, \Delta_2^*)$ investment-construction policy remains optimal for any time horizon $T \geq \Delta_2^*$. However, if $T < \Delta_2^*$, the time horizon is too short to allow the optimal construction time, so $\Delta_2$ decreases to just cover the time horizon, and consequently, the length of the investment interval, $\Delta_1$, must increase. The exact relationship between $\Delta_1$ and $\Delta_2$ can be derived from the terminal condition on $C(T)$, in Equation (3.27), which relates $T$, $t_0$, and $t_1$. This equation, stated in terms of $\Delta_1 = t_1 - t_0$ and $\Delta_2 = T - t_0$ gives us
\[
\Delta_1 \left( \Delta_2 - \frac{\Delta_4}{2} \right) = \frac{C_T}{I_M} \quad . \quad \ldots \quad (4.3)
\]

Thus, the optimal investment-construction policy as a function of \( T \) can be summarized as

\[
\Delta_2 = \Delta_2(T) \equiv \begin{cases} T, & T \leq T < \Delta_2^*, \\ \Delta_2^*, & T \geq \Delta_2^* \end{cases}
\]

\[
\Delta_4 = \Delta_4(\Delta_2) = \Delta_2 - \left( \Delta_2^2 - \mathcal{T}^2 \right)^{1/2}, \quad \ldots \quad (4.4)
\]

\[
\mathcal{T}^2 = \frac{2C_T}{I_M} .
\]

The equation for \( \Delta_4 \) follows from (4.3), and the constant \( \tau \) is the smallest time horizon for which there exists a feasible control, given \( C_T \) and \( I_M \) (with \( C_0 = K_0 = 0 \)).

To sum up the thesis to this point, then, we have completely solved the basic model which we derived from Cummings' and Schulze's model, and have carried out a sensitivity analysis of that solution.

The remainder of the thesis considers two extensions which arise naturally from this basic model.
5. EXTENSION TO UNBOUNDED CONTROL SET

The first extension to the basic model which we shall discuss is one which has been suggested in Section 4, as a result of the sensitivity analysis on the parameter $l_M$. It was clear that successively larger values for $l_M$ caused the optimal investment interval $[t_0, t_1]$ to continuously shrink, and we hypothesized that in the limit, as $l_M$ approaches $+\infty$, the optimal investment policy should be an impulse control which occurs at a certain time $t$ in the interval $[0, T]$.

First, we shall define the extended control problem more clearly. The basic problem in Section 3 assumes that there is a maximum rate of investment allowed, i.e.,

$$0 < |l(t)| < l_M < +\infty, \quad 0 \leq t \leq T.$$  

If the problem is changed so that there is no upper bound on the rate of investment, the corresponding control constraint is

$$0 < |l(t)| < +\infty, \quad 0 \leq t \leq T.$$  

We shall refer to this modified control problem as $P_\infty$. The condition "$l(t) = +\infty$" signifies that $l$ is an impulse control which causes an instantaneous (but finite) jump in $K$ (the level of social capital) at time $t$. The class of admissible controls for $P_\infty$ includes all controls which

---

7To avoid unnecessary algebraic clutter, it will be assumed throughout the section that $C_0 = K_0 = 0$, and $F(K(T)) \equiv 0$. 

are PWC on \([0,T]\), plus the set of impulse controls which are PWC on
\([0,T]\), except at a finite number of points in \([0,T]\), at which finite
jumps in \(K\) occur.\(^8\)

As a first step in finding a solution for \(P_\infty\), we observe that
the Maximum Principle and, in fact, the existence theorems referred to
in Section 3 are not valid for this problem, because the control set has
changed from \([0,l_M]\) to \([0,\infty)\), and is therefore not compact. Fortunately,
an alternative method of solution is available. We begin by examining
more rigorously what happens to the optimal solution for the control
problem with finite-valued \(l_M\), as \(l_M \to \infty\).

**Proposition 1.** Let \(P_n\) be the basic control problem with \(l_M = n\),
for \(n = 1, 2, \ldots\). Let \(l_n\) be the optimal control for \(P_n\), with response
trajectory \((C_n, K_n)\), switching times \(t_0^{(n)}\) and \(t_1^{(n)}\), and cost

\[
J_n = J[l_n] = \int_0^T e^{-\delta t} \{ Y K_n + l_n \} \, dt.
\]

Then \(\exists\) a unique \(t_* \in [0,T)\) such that

\[
(i) \quad t_0^{(n)} \uparrow t_* \text{ and } t_1^{(n)} \uparrow t_* \text{ as } n \to \infty \quad \ldots \ldots (5.2)
\]

\[
(ii) \quad t_* \text{ solves the equation } e^{\delta (T-t_*) \left[ \delta (T-t_*) - 1 \right]} + \frac{Y}{\delta + Y} = 0. (5.3)
\]

\(^8\) Note that piecewise continuity of controls rules out functions \(l(\cdot)\) for which

\[
\lim_{t \to \tau} l(t) = +\infty \quad \text{or} \quad \lim_{t \to \tau} l(t) = +\infty, \quad \text{for some } \tau \in [0,T].
\]
(iii) $K_n(T) \rightarrow \frac{C_T}{T-t_1} = K_*$ \hspace{1cm} . . . . (5.4)

(iv) $J_n \downarrow K_* e^{-\delta t_*} \left\{ 1 + \frac{\gamma}{\delta} \left( 1 - e^{-\delta (T-t_*)} \right) \right\} = J_* \hspace{1cm} . . . . (5.5)$

**Proof**

(i) The existence of $t_*$ can be deduced by examining several equations from the solution to the basic control problem with $l_M < \infty$. From the analysis in Section 3, the construction response at time $T$, scaled by $l_M$ is given by

$$\tilde{C} \left( t_0(\mu), t_1(\mu) \right) = \frac{C(T)}{l_M} = \left( t_i - t_0 \right) \left( T - \frac{t_0 + t_1}{2} \right)$$ \hspace{1cm} . . . . (5.6)

Recalling that $\tilde{C}$ is a monotone increasing function of $\mu$, the optimal value of $\mu$ is the value for which

$$\tilde{C} \left( t_0(\mu), t_1(\mu) \right) = \frac{C_T}{l_M} \hspace{1cm} . . . . (5.7)$$

Therefore, if $l_M$ increases, the optimal value of $\mu$ must decrease. Since

$$\frac{\partial t_0}{\partial \mu} < 0 \text{ and } \frac{\partial t_1}{\partial \mu} > 0,$$

the decrease in $\mu$ implies that the switching times $t_0$ and $t_1$ must increase and decrease, respectively. For any finite value of $l_M$, $t_0 < t_1$. Therefore,
\[
\lim_{l_M \to +\infty} t_0 = t_L \quad \text{and} \quad \lim_{l_M \to +\infty} t_T = t_U
\]

both exist, and

\[
0 \leq t_L < t_T < T \tag{5.8}
\]

From Equations (5.6) and (5.7)

\[
(t_t - t_0)\left(T - \frac{t_0 + t_1}{2}\right) = \frac{C_T}{l_M} \tag{5.9}
\]

Taking limits of both sides of Equation (5.9) as \(l_M \to +\infty\),

\[
\lim_{l_M \to +\infty} (t_t - t_0)\left(T - \frac{t_0 + t_1}{2}\right) = 0 ,
\]

or

\[
(t_T - t_L)\left(T - \frac{t_L + t_U}{2}\right) = 0 . \tag{5.10}
\]

The second factor in Equation (5.10) is positive, by (5.8), so the first factor must be zero. I.e.,

\[
t_T = t_L = t_* \in \left[0, T\right)
\]

Thus, in terms of the sequence of problems \(\{P_n\}_{n=1}^{\infty}\), we have

\[
\lim_{n \to +\infty} t_0^{(n)} = \lim_{n \to +\infty} t_t^{(n)} = t_* ,
\]

which proves (i) .
(ii) As we noted previously, increasing values of $\mu = \mu(n)$. Since $t_0(n)$ and $t_1(n)$ converge monotonically to a single point $t_*$, the sequence $\{\mu(n)\}$ must converge monotonically to a value $\mu_*$ such that the curves $\lambda(t;\mu_*)$ and $e^{-\delta t}$ are just tangent\(^9\) at $t_*$, as shown in Figure 6. Therefore, $\mu_*,t_*$ is the unique solution of the equations

\[
\begin{align*}
\lambda(t_*,\mu_*) &= e^{-\delta t_*} \\
\dot{\lambda}(t_*,\mu_*) &= -\delta e^{-\delta t_*}
\end{align*}
\]

with $t_* < T$. Substituting Equation (3.21) and (3.19) for $\lambda$ and $\dot{\lambda}$, we have

\[
\begin{align*}
\frac{\gamma}{\delta} \left( e^{-\delta T} - e^{-\delta t_*} \right) + \mu_* (T - t_*) &= e^{-\delta t_*} \\
\gamma e^{-\delta t_*} - \mu_* &= -\delta e^{-\delta t_*}
\end{align*}
\]

Solving Equation (5.12) for $\mu_*$ and substituting in Equation (5.11) leaves an equation in $t_*$:

\[
\frac{\gamma}{\delta} \left( e^{-\delta T} - e^{-\delta t_*} \right) + (\gamma + \delta) e^{-\delta t_*} (T - t_*) = e^{-\delta t_*}
\]

which algebra will reduce to Equation (5.3), thus verifying part (ii) of the Proposition.

\(^9\)The curve $\lambda(t;\mu_*)$ also appears in a previous section; see Figure 3.
Figure 6  Adjoint curve $\lambda(t; \mu^{(n)})$ for increasing $n$
(iii) By Equation (3.25) the final optimal level of social capital for $P_n$ is

$$K_n(T) = n \left( t_1^{(n)} - t_0^{(n)} \right).$$

Equation (5.9) rewritten for $P_n$ says that

$$n \left( t_i^{(n)} - t_0^{(n)} \right) \left( T - \frac{t_0^{(n)} + t_1^{(n)}}{2} \right) = C_T,$$

so substituting (5.13) into (5.14) and solving for $K_n(T)$, we have

$$K_n(T) = \frac{C_T}{\left( T - \frac{t_0^{(n)} + t_1^{(n)}}{2} \right)}$$

and taking limits on each side, Equation (5.2) gives us

$$\lim_{n \to \infty} K_n(T) = \frac{C_T}{T - t_*}$$

which is what we wanted to show.

(iv) The last condition which we must check is the behaviour of the sequence of costs $\{J_n\}$. Because the set of feasible controls for $P_n$ is properly contained in the set of feasible controls for $P_{n+1}$, $J\{l_{n+1}\} \leq J\{l_n\}$. In fact, uniqueness of the solution to $P_{n+1}$ makes the inequality strict, so $\{J_n\} \equiv \{J\{l_n\}\}$ is a monotonically decreasing sequence of positive numbers, which must, therefore, have a limit.
The limit in Equation (5.5) can be verified by evaluating the cost integral for the optimal control $l_n$. Thus

$$J_n = \int_{t_0}^{T} e^{-\delta t} \left\{ YK_n + l_n \right\} dt$$

$$= \int_{t_0}^{t_1^{(n)}} e^{-\delta t} \left\{ Yn(t - t_0^{(n)}) + n \right\} dt + \int_{t_0}^{T} e^{-\delta t} YK_n(\tau) d\tau \ldots . . . (5.15)$$

by Equation (3.23) to (3.25) and definition of $K_n$ and $l_n$. Integrating the first integral by parts, we have

$$\int_{t_0}^{t_1^{(n)}} ne^{-\delta t} \left\{ Y(t - t_0^{(n)}) + 1 \right\} dt$$

$$= nY \left\{ \frac{1}{\delta} e^{-\delta t_0^{(n)}} \left[ 1 - e^{-\delta(t_1^{(n)} - t_0^{(n)})} \right] - \frac{1}{\delta} e^{-\delta t_1^{(n)}} \left( t_1^{(n)} - t_0^{(n)} \right) \right\}$$

$$+ \frac{n}{\delta} e^{-\delta t_0^{(n)}} \left[ 1 - e^{-\delta(t_1^{(n)} - t_0^{(n)})} \right]$$

$$= \left( \frac{Y + 1}{\delta} \right) \frac{n}{\delta} e^{-\delta t_0^{(n)}} \left[ 1 - e^{-\delta(t_1^{(n)} - t_0^{(n)})} \right] - \frac{Y}{\delta} K_n(T) e^{-\delta t_1^{(n)}} \ldots . . . (5.16)$$

by Equation (5.13). The second integral in (5.15) is simply

$$YK_n(T) \int_{t_0}^{T} e^{-\delta \tau} d\tau = \frac{Y}{\delta} K_n(T) \left[ e^{-\delta t_1^{(n)}} - e^{-\delta T} \right] \ldots . . . (5.17)$$
So from Equations (5.15) and (5.16),

\[
\lim_{n \to \infty} J_n = \lim_{n \to \infty} \left( \frac{Y + 1}{\delta} \right) \frac{n}{\delta} e^{-\delta t_0^{(n)}} \left[ 1 - e^{-\delta (t_1^{(n)} - t_0^{(n)})} \right] - \lim_{n \to \infty} \frac{Y}{\delta} K_n(T) e^{-\delta T} - \lim_{n \to \infty} \frac{Y}{\delta} K_\ast e^{-\delta T} \quad \ldots \quad (5.18)
\]

by Equation (5.4). To evaluate the limit on the right side of Equation (5.18), we expand \( e^{-\delta (t_1^{(n)} - t_0^{(n)})} \) in a Maclaurin series.

Thus

\[
\frac{n}{\delta} e^{-\delta t_0^{(n)}} \left[ 1 - e^{-\delta (t_1^{(n)} - t_0^{(n)})} \right]
\]

\[
= \frac{n}{\delta} e^{-\delta t_0^{(n)}} \left[ 1 - \left( 1 - \delta (t_1^{(n)} - t_0^{(n)}) + O((t_1^{(n)} - t_0^{(n)})^2) \right) \right]
\]

\[
= e^{-\delta t_0^{(n)}} n (t_1^{(n)} - t_0^{(n)}) \left[ 1 - O(t_1^{(n)} - t_0^{(n)}) \right]
\]

\[
= e^{-\delta t_0^{(n)}} K_n(T) \left[ 1 - O(t_1^{(n)} - t_0^{(n)}) \right] \longrightarrow K_\ast e^{-\delta t_\ast} \quad \text{as} \quad n \to \infty.
\]

Substituting this result into Equation (5.18) gives us the required expression for \( \lim_{n \to \infty} J_n \), which concludes the proof of the proposition.
A glance at the results in the preceding proposition suggests a candidate as a likely solution to $P_{\infty}$. As $n \to \infty$, the sequence of optimal controls $\{l_n\}$ approaches an impulse control $l_{t^*}$, determined by (5.3) and (5.4). It is only reasonable to expect that if $P_{\infty}$ has a solution, it should be the control $l_{t^*}$. Theorem 1 below will verify that this guess is correct. The proof of the theorem, however, requires a preliminary result which is stated in the following lemma.

**Lemma 1.** Given $\epsilon > 0$ and a control $l$ which is feasible for $P_{\infty}$, $\exists$ a control $\hat{l}$ and an integer $N$ such that $\forall n \geq N$, $\hat{l}$ is feasible for $P_n$, and $J(\hat{l}) \leq J(l) + \epsilon$.

The proof of the lemma is contained in Appendix D. Basically, the lemma shows that the cost of any impulse control can be approximated (with arbitrarily small error) by the cost of a control which is PWC and of finite value everywhere. We proceed now to our main result.

**Theorem 1.** The impulse control, $l_{t^*}$, defined below, is optimal for $P_{\infty}$.

$$l_{t^*}(t) = \begin{cases} +\infty, & t = t^* \\ 0, & \text{otherwise} \end{cases} \quad \ldots \ldots (5.19)$$

$$K_{t^*}(t) = \frac{C_T}{T - t^*_k} \equiv K_{t^*} \quad \ldots \ldots (5.20)$$

where $t^*$ is defined in Equation (5.2).

---

10This discussion does not address the question of uniqueness of the solution for $P_{\infty}$. 
Proof: It is clear that the control \( l_{t^*} \) is feasible for \( P_\infty \). In order to conclude optimality of \( l_{t^*} \), it suffices to show that
\[
J\{l_{t^*}\} = \inf \left\{ J\{l\} \mid l \text{ feasible for } P_\infty \right\}.
\]

The infimum does exist because the set \( \{ J\{l\} \mid l \text{ feasible for } P_\infty \} \) is bounded below by zero.

Straightforward calculation shows that
\[
J\{l_{t^*}\} = K_* e^{-\delta t^*} + \int_{t^*}^T e^{-\delta t} Y K_* \, dt
\]
\[
= K_* e^{-\delta t^*} + \frac{\gamma}{\delta} K_* e^{-\delta t^*} \left[ 1 - e^{-\delta(T-t^*)} \right]
\]
\[
= J_* = \lim_{n \to \infty} J_n
\]

by Equation (5.5), so Equation (5.21) is equivalent to the two inequalities
\[
J_* \geq \inf \left\{ J\{l\} \mid l \text{ feasible for } P_\infty \right\}.
\]

Since \( l_n \) is feasible for \( P_\infty \) for every \( n \),
\[
\left\{ J_n \right\}_{n=1}^\infty \subset \left\{ J\{l\} \mid l \text{ feasible for } P_\infty \right\}.
\]

Therefore,
\[
\left\{ J_n \right\}_{n=1}^\infty \supset \left\{ J\{l\} \mid l \text{ feasible for } P_\infty \right\},
\]
or from (5.5),
$$J^*_t \geq \left\{ J\{1\} \mid \text{feasible for } P_\infty \right\}.$$  

The second inequality in (5.22) follows from the Lemma. Given $\varepsilon > 0$ and any control $I$ feasible for $P_\infty$, $\exists$ a control $\hat{I}$ and an integer $N$ such that $\hat{I}$ is feasible for $P_n$, $\forall n \geq N$, and $J\{\hat{I}\} \leq J\{I\} + \varepsilon$. Since $I_n$ is optimal for $P_n$,

$$J_n = J\{I_n\} \leq J\{\hat{I}\} \leq J\{I\} + \varepsilon, \quad \forall n \geq N,$$

and by taking the limit of the sequence $J_n$, as $n \to \infty$

$$J^*_t = \lim_{n \to \infty} J_n \leq J\{I\} + \varepsilon. \quad \ldots \ldots (5.23)$$

The inequality above is true for any $\varepsilon > 0$, from which we may conclude that $J^*_t \leq J\{I\}$, for any control $I$ feasible for $P_\infty$. Thus, $J^*_t$ is a lower bound for $\left\{ J\{I\} \mid \text{feasible for } P_\infty \right\}$, so

$$J^*_t \leq \left\{ J\{I\} \mid \text{feasible for } P_\infty \right\}.$$  

This verifies the second inequality in (5.22), which implies the optimality of $I_{t^*_t}$ for $P_\infty$.

With the results proved in this section, we have solved the extended version of our basic control problem in which there is no upper bound on the rate of investment, and instantaneous jumps in the level of social infrastructure are permitted. Under these more
general conditions, we have shown that the optimal investment strategy is a "one-shot" instantaneous investment in social capital at a time $t^*$, which occurs before the end of the time horizon ($t=T$), and is uniquely determined by the discount rate, the wage rate, and the time horizon.
6. OPTIMIZATION OF THE TIME HORIZON

The second extension to the basic model which we discuss in this thesis is the optimization of the time horizon \( T \). It is not unreasonable to assume that the decision-maker faced with the job of planning the construction of the facility and investment in infrastructure may first have to determine when the facility should go into operation, in which case treating the time horizon as a variable to be optimized, rather than as a specified parameter, makes the model more realistic. An examination of the objective functional, however, shows that when \( T \) is optimized, the cost can be made arbitrarily small by choosing a large enough value of \( T \). To be specific, let \( J(T) \) be the cost of the optimal control for the basic model with fixed time horizon \( T \). This cost has been evaluated in Equation (5.15) to (5.17), and may be simplified to the form

\[
J(T) = \lim_{\delta \to 0} e^{-\delta T} \left\{ \left( \frac{\gamma + 1}{\delta} \right) e^{\delta \Delta_2} \left( 1 - e^{-\delta \Delta_1} \right) - \gamma \Delta_1 \right\}
\]

where \( \Delta_1 \equiv t_1 - t_0 \) and \( \Delta_2 \equiv T - t_0 \), defined by Equation (4.4), are the optimal lengths of time for investment and construction for time horizon \( T \). If \( (\hat{\Delta}_1, \hat{\Delta}_2) \) is the optimal policy for some time horizon \( \hat{T} \), then investment and construction are both started \( \hat{\Delta}_2 \) periods before \( \hat{T} \). This policy is obviously feasible (although perhaps not optimal) for all time horizons \( T > \hat{T} \), with cost
which approaches 0 as $T \to \infty$. Therefore, the objective functional for our basic model does not admit an optimal value of $T$ which is finite. The reason that the infinite time horizon is optimal is simply that the objective functional for the basic model includes only costs, and no benefits. Presumably, the reasons for constructing any facility include some form of benefits which accrue during the operational phase of the facility, after the completion of its construction. In this case, choosing an optimal value of $T$ is equivalent to balancing high construction costs (from small $T$) with a loss in benefits (from large $T$). For the purposes of this discussion, we assume that the benefits accrue at a fixed rate $\beta$ over time, as that the present value of the stream of benefits over $[T, \infty)$ is

$$B(T) = \int_T^\infty \beta e^{-\delta t} dt = \frac{\beta}{\delta} e^{-\delta T}.$$ 

(6.2)

The problem of optimizing the time horizon, then, is one of maximizing the difference between benefits and costs. Given a particular time horizon $T$, the decision-maker will choose an investment-construction strategy which minimizes his cost, so the appropriate cost function for this problem is $J(T)$, described in (6.1). Our problem, therefore is to determine a value (or values) $T = T^*$ which maximize $B(T) - J(T)$ over the domain $[T, \infty)$. 

$$\hat{J}(T) = e^{-\delta (T-T)} \cdot J(\hat{T})$$
The benefit function $B(T)$ has a very simple form which is easy to handle analytically, but the cost function, $J(T)$, is a complicated function of several parameters. From Equation (4.4), $J(T)$ can be explicitly expressed as a function of $T$ on each of the intervals $[\tau, \Delta_2^*)$ and $(\Delta_2^*, \infty)$ which, together, constitute its domain:

$$
J(T) = \begin{cases} 
  f(T) = \frac{1}{\delta} \left( \left( \frac{\gamma + 1}{\delta} \right) \left( 1 - e^{-\delta \Delta_1(T)} \right) - \gamma \Delta_1(T) e^{-\delta T} \right), & T \in [\tau, \Delta_2^*) \\
  g(T) = \frac{1}{\delta} \left( \left( \frac{\gamma + 1}{\delta} \right) e^{\Delta_2^*} \left( 1 - e^{-\delta \Delta_1^*} \right) - \gamma \Delta_1^* \right) e^{-\delta T}, & T \in [\Delta_2^*, \infty)
\end{cases}
$$

(6.3)

where $\tau$ is the smallest feasible time horizon, given other parameter values, $\Delta_2^*$ is the largest time horizon for which the optimal investment policy starts at $t_0 = 0$, and $\Delta_1^* = \Delta_1(\Delta_2^*)$. With $J(T)$ written as two different functions, $f$ and $g$, a very simple argument establishes the main result of this section, which is stated below in Proposition 2.

The proof of the proposition depends, in part, upon the fact that $J$ is differentiable at $\Delta_2^*$. Continuity of $J$ at $\Delta_2^*$ is obvious, since $f$ and $g$ are continuous, and $f(\Delta_2^*) = g(\Delta_2^*)$. Therefore, from the following lemma, which shows that $f'(\Delta_2^*) = g'(\Delta_2^*)$, we conclude that $J'(\Delta_2^*)$ exists.

**Lemma 2.** Let $\Delta_2^*$ be the largest time horizon for which the optimal investment policy starts at $t_0 = 0$, and let $f(T)$ and $g(T)$ be defined by (6.3), with domains $[\tau, \infty)$ and $(-\infty, \infty)$. Then $f'(\Delta_2^*) = g'(\Delta_2^*)$. 
Proof: Differentiation of f and g yields

\[ f'(T) \left|_{\frac{1}{M}/\delta} \right. = (\gamma + \delta) e^{-\delta A_1} \Delta_i'(T) + \gamma e^{-\delta T} (\delta A - \Delta_i'(T)) \]
\[ = \Delta_i'(T) \left\{ (\gamma + \delta) e^{-\delta A_1} - \gamma e^{-\delta T} \right\} + \gamma \delta \Delta_i e^{-\delta T} \quad \ldots (6.4) \]
\[ g'(T) \left|_{\frac{1}{M}/\delta} \right. = (\gamma + \delta) (e^{-\delta A_1} - 1) e^{\delta (\Delta_2^* - T)} + \gamma \delta \Delta_i^* e^{-\delta T} \quad \ldots (6.5) \]

At \( T = \Delta_2^* \), the second term in both equations if the same, so that

\[ \frac{f'(\Delta_2^*) - g'(\Delta_2^*)}{1/\delta} = \Delta_i'(\Delta_2^*) \left\{ (\gamma + \delta) e^{-\delta A_1} - \gamma e^{-\delta A_2^*} \right\} - (\gamma + \delta) (e^{-\delta A_1} - 1) \]
\[ = -\frac{\Delta_i}{\Delta_2^* - \Delta_i} \left\{ (\gamma + \delta) e^{-\delta A_1} - \gamma e^{-\delta A_2^*} \right\} + (\gamma + \delta) (1 - e^{-\delta A_1}) \]
\[ \ldots (6.6) \]

after differentiating \( \Delta_i(T) \) (Equation (4.4)). Therefore, our claim that \( f'(\Delta_2^*) = g'(\Delta_2^*) \) will follow if we show that the right side of Equation (6.6) is zero.

For this, we require an equation which implicitly defines \( \Delta_2^* \) in terms of the parameters of the model. Because \( \Delta_2^* \) is the largest value of \( T \) for which \( t_0 = 0 \), the first root, \( t_0(\mu) \), of the switching function \( \sigma(t;\mu) = \lambda(t;\mu) - e^{-\delta t} \) must be zero, and the second root, \( t_1(\mu) \) must be \( \Delta_1^* \), as shown in Figure 7. The two relevant equations are
Figure 7  Switching times for $T = \Delta_2^*$
\[ \Lambda(\Delta^*_1; \mu) = e^{-\delta \Delta^*_1} \]
\[ \Lambda(0; \mu) = 1 \]

which are equivalent to

\[ \frac{\gamma}{\delta} \left( e^{-\delta \Delta^*_2} - e^{-\delta \Delta^*_1} \right) + \mu \left( \Delta^*_2 - \Delta^*_1 \right) = e^{-\delta \Delta^*_1} \] \hspace{1cm} \ldots \ldots \text{(6.7)}
\[ \frac{\gamma}{\delta} \left( e^{-\delta \Delta^*_2} - 1 \right) + \mu \Delta^*_2 = 1 \] \hspace{1cm} \ldots \ldots \text{(6.8)}

Subtracting (6.8) from (6.7) results in the equation

\[ \frac{\gamma}{\delta} \left( 1 - e^{-\delta \Delta^*_1} \right) - \mu \Delta^*_1 = e^{-\delta \Delta^*_1} - 1 \] \hspace{1cm} \ldots \ldots \text{(6.9)}

Solving Equations (6.7) and (6.9) for \( \mu \), equating them, and multiplying by \( \delta \Delta^*_1 \), we are left with

\[ \delta \Delta^*_1 \mu = \Delta^*_1 \frac{\left\{ (\gamma + \delta) e^{-\delta \Delta^*_1} - \gamma e^{-\delta \Delta^*_2} \right\}}{\Delta^*_2 - \Delta^*_1} = (\gamma + \delta) \left( 1 - e^{-\delta \Delta^*_1} \right) \] \hspace{1cm} \ldots \ldots \text{(6.10)}

Since these two expressions for \( \delta \Delta^*_1 \mu \) correspond to the two terms in Equation (6.6), the right side of Equation (6.6) is zero, which proves our claim.

We proceed now to our main result.

**Proposition 2.** Let \( B(T) \) and \( J(T) \) be defined by Equations (6.2) and (6.3), respectively, and let \( T^* \) maximize \( B(T) - J(T) \) on \([\tau, \infty)\).

Then either (i) \( T^* \in (\tau, \Delta^*_2) \), or
(ii) Every $T \in [\Delta_2^*, \infty)$ maximizes $B(T) - J(T)$, and $\max \{B(T) - J(T)\} = 0$.

Proof: We shall compare the two exponential functions $B(T)$ and $g(T)$, which defines $J(T)$ on $[\Delta_2^*, \infty)$. From Equation (6.3), we note that

$$g(T) = g(0) e^{-\delta T}, \quad g(0) = \frac{\log M}{\delta} \left\{ \left( \frac{\gamma + 1}{\delta} \right) e^{\delta \Delta_2^*} (1 - e^{-\delta \Delta_2^*}) - \gamma \Delta_2^* \right\},$$

and from Equation (6.2),

$$B(T) = B(0) e^{-\delta T}, \quad B(0) = \frac{\Delta_2^*}{\delta}.$$

Therefore, depending on the sign of $B(0) - g(0)$, the curve $B(T)$ lies below $g(T)$ for all $T$, above $g(T)$ for all $T$, or $B$ is coincident with $g$. We show that the first two possibilities correspond to (i) in the proposition, and that (ii) may arise when $B$ and $g$ are identical.

If $B(0) < g(0)$, then

$$B'(T) - g'(T) = -\delta [B(0) - g(0)] e^{-\delta T} > 0 \quad \forall T$$

so $B(T) - g(T)$ has no maximum. Since $J(T) \equiv g(T)$ on $[\Delta_2^*, \infty)$, this means that $B(T) - J(T)$ has no maximum on $[\Delta_2^*, \infty)$. Thus, if there is a value $T^*$ which maximizes $B(T) - J(T)$ on $[\tau, \Delta_2^*)$, $T^* \in [\tau, \Delta_2^*)$.

On the other hand, if $B(0) > g(0)$ then $B'(T) - g'(T) < 0$ for all $T$. Therefore, by Equation (6.3),
\[ B'(T) - J'(T) \equiv B'(T) - g'(T) < 0, \quad T \in [\Delta_2^*, \infty) \quad \ldots \quad (6.11) \]

where equality at \( T = \Delta_2^* \) follows from Lemma 2. It is clear that \( B \) and \( J \equiv f \) have continuous derivatives on \((\tau, \Delta_2^*)\), so Equation (6.11) implies that \( B(T) - J(T) \) is decreasing on \([\Delta_2^*-\epsilon, \infty)\), for some \( \epsilon > 0 \). Thus, over its entire domain \([\tau, \infty)\), \( B(T) - J(T) \) is maximized for some \( T = T^* \in [\tau, \Delta_2^*) \).

For both of the above cases, we can eliminate the possibility that \( T^* = \tau \). From Equation (4.4),

\[ \Delta_4'(T) = 1 - \frac{T}{(T^2 - T^4)^{1/2}} \longrightarrow -\infty \quad \text{as} \quad T \to T^+ \]

so Equation (6.4) implies that \( f'(T) \to -\infty \) as \( T \to T^+ \). Therefore,

\[ \lim_{T \to T^+} B'(T) - J'(T) = -\delta B(0) e^{-\delta T} \quad \lim_{T \to T^+} f'(T) = +\infty \quad \]

regardless of how large or small \( B(0) \) is. This means that \( B(T) - J(T) \) is always increasing in \([\tau, \tau + \epsilon]\), for some \( \epsilon > 0 \), so if \( T^* \) is optimal, \( T^* > \tau \).

The preceding remarks show that if \( B(0) < g(0) \), then (i) is true; that is, if any optimal values of \( T \) exist, they must be in the interval \((\tau, \Delta_2^*)\). The remaining possibility is that \( B(0) - g(0) \), which means that

\[ B(T) - J(T) \equiv 0, \quad T \in [\Delta_2^*, \infty) \quad \ldots \quad (6.12) \]
so the optimal time horizon really depends on the function \( f(T) \equiv J(T) \), for \( T \in (\tau, \Delta^*_2) \). If \( B(T) > f(T) \) for some \( T \in (\tau, \Delta^*_2) \), then (i) is true. Otherwise, \( B(T) - F(T) \leq 0 \) on \( (\tau, \Delta^*_2) \), so Equation (6.12) implies case (ii). i.e., every \( T \in [\Delta^*_2, \infty) \) maximizes \( B(T) - J(T) \), and 
max \{B(T) - J(T)\} = 0. Q.E.D.

From a practical point of view, this proposition makes sense. Case (ii) may be viewed as a somewhat pathological situation; first of all, only a relatively small set of parameter values will satisfy \( B(0) - g(0) \), and secondly, since the maximum net discounted return \( B(T^*) - J(T^*) \) is always zero in this case, it is unlikely that any planner would even undertake the project. Case (i), however, tells us that if there is an optimal time horizon \( T^* \), over which the facility should be constructed, \( T^* \) will always be smaller than \( \Delta^*_2 \). Therefore, by Equation (4.4), the optimal policy corresponding to time horizon \( T^* \) will be to start the project at time \( t_0 = 0 \). This is really not surprising at all, for the following reason: if the decision-maker can achieve a positive net discounted return \( R \) using an investment interval \([t_0, t_1]\), with \( t_0 > 0 \) and time horizon \( T \), he can clearly increase his net return to \( e^{\delta t_0} \cdot R \) by starting investment at \( t = 0 \), investing for the same length of time as before, and completing the construction of the facility at \( t = T - t_0 \). Furthermore, if his original investment policy is optimal for the time horizon \( T \), then \( T - t_0 = \Delta^*_2 \), and 
\[
e^{\delta t_0} R = [B(0) - g(0)]e^{-\delta \Delta^*_2} > 0, \text{ so } B(0) > g(0), \text{ and}
\]
Thus, he can increase his return by choosing a time horizon shorter than $\Delta_2^*$, even though this will force him to increase his final level of investment in social infrastructure.

In order to obtain stronger results than those stated in Proposition 2, we need to know something about the cost function $J(T) \equiv f(T)$ on $[\tau, \Delta_2^*)$. It is easy to deduce that $J(T)$ is decreasing, using the fact that it is the cost of the optimal policy for time horizon $T$. Unfortunately, more useful results regarding the shape of this function are difficult to prove for all possible parameter values. However, the problem of optimizing $T$ has been solved numerically for several sets of parameter values, and for these particular cases, we found that the function $f$ is convex and dominates $g$, except at $T = \Delta_2^*$, where the two curves are tangent. Figure 8 illustrates $f$ and $g$ for one set of parameters. We see that for this cost function, the optimal time horizon $T^*$ has a very simple relationship with the benefit function $B(T)$. If $B(0) < g(0)$, then $B(T) < g(T) \leq f(T)$, $\forall T \in [\tau, \infty)$, so $B(T) - J(T)$ is negative and increases to zero as $T \to +\infty$; thus there is no optimal time horizon $T^* < \infty$. In the unlikely case that $B(0) = g(0)$, $B(T) \equiv J(T)$ on $[\Delta_2^*, \infty)$ and $B(T) < J(T)$ on $[\tau, \Delta_2^*)$, so the set of optimal time horizons is the interval $[\Delta_2^*, \infty)$. Of course, the only interesting case from the point of view of the decision-maker is the one for which $B(0) > g(0)$, because this provides
Parameter Values:

\[ \begin{align*}
\beta &= 0.10 \\
\gamma &= 1.0 \\
M &= 5.0 \times 10^4 \\
C_T &= 5.0 \times 10^7 \\
g(0) &= 8.065 \times 10^7 \\
B_1(0) &= 7.500 \times 10^7 \\
B_2(0) &= 8.200 \times 10^7 \\
B_3(0) &= 9.000 \times 10^7
\end{align*} \]

Figure 8: Cost Curves \( f \) and \( g \) with Alternate Benefit Curves \( B_1, B_2, \) and \( B_3 \).
a maximum net return greater than zero. In fact, there is a unique optimal time horizon $T^*$, and as we predicted $T^* \in (\tau_0, \Delta^*)$. (Check the graphs of $B_2$ and $B_3$ in Figure 8.) Furthermore, as $B(0)$ increases, the slope $B'(T) = -\delta B(0)e^{-\delta T}$ decreases for every $T$, so that the point where $B'(T) = J'(T)$ must shift to the left; i.e., $T^*$ decreases. This effect is illustrated by curves $B_2$ and $B_3$ in Figure 8, with $B_2(0) = 8.2$ and $B_3(0) = 9.0$. The corresponding values for $T^*$ are approximately 5.2 and 5.1. Although the analysis is restricted to values of the parameter which yield a cost curve like the one in Figure 8, it does correspond to what intuition would dictate a decision-maker to do: the greater the per period benefits from the facility, the sooner the facility should be in operation.

These results correspond to the basic model, in which investment occurs at a maximum rate $l_\infty < \infty$. However, the problem of optimizing the time horizon can also be considered for the extended model in Section 5, in which impulse controls are allowed, and as one might expect, the analysis is simpler in this case. The optimal control with time horizon $T$ is an impulse investment which occurs at $t = t_*$, and $t_*$ is uniquely determined by Equation (5.3),

$$e^{\delta (T-t_*)} \left[ \delta (T-t_*) - 1 \right] + \frac{Y}{\delta + Y} = 0$$

It is clear from this equation that as $T$ changes, the difference $T - t_*$ remains constant. Thus, the cost of this impulse control, given by
Equations (5.4) and (5.5), can be written as a simple function of $T$,

$$
J(T) = \frac{C_T}{T-t_k} \left\{ (1+\gamma) \cdot e^{\delta(T-t_k)} - \frac{T}{\delta} \right\} e^{-\delta T} \quad \ldots \ldots (6.13)
$$

for any time horizon $T > 0$. Since our benefit function $B(T) = B(0)e^{-\delta T}$ is of the same form, the difference $B(0) - \alpha$ determines the optimal values of $T$. If $B(0) < \alpha$, the $B(T)-J(T)$ increases to 0 as $T$ approaches $+\infty$, so there is no optimal value of $T$ which is finite. If $B(0) = \alpha$, then any $t$ is optimal, and $\max \{B(T)-J(T) \mid T > 0\} = 0$. Finally, if $B(0) > \alpha$, then $[B(0) - \alpha] e^{-\delta T}$ is positive and decreasing on $(-\infty, \infty)$ so $B(T)-J(T)$ has no maximum on $(0, \infty)$. Of course, the practical interpretation of this result is simply that if benefits are high enough to offset the cost of construction of the facility, the project should be completed as soon as possible.

This completes our discussion of the optimization of the time horizon. The treatment of the problem presented here is by no means comprehensive, but it does provide some general analytical results which make sense and can be used as a basis for numerical analysis of the problem.
7. CONCLUSION

The results presented in this thesis indicate the benefits which can be derived from keeping analytical models relatively simple in structure. In contrast with Cummings' and Schulze's model in [5], our linear model of infrastructure investment can be completely solved analytically. We were also able to carry out a sensitivity analysis of the solution and to extend the scope of the problem to take account of a broader class of circumstances. In particular, the linearity of the model facilitates analysis of the general project planning problem in which economic benefits, as well as costs, must be considered. The structure of the model was also shown to be amenable to numerical solution using parameter estimates derived from actual data. In conclusion, then, when compared to more complex models, such a model would seem to better satisfy the requirements of practical policy problems.
BIBLIOGRAPHY


APPENDIX A

THE ABNORMAL FORM OF THE CONTROL PROBLEM

We shall refer to the problem described in Equations (3.1) to (3.8). The conditions in Equations (3.9) to (3.14) are consequences of the Maximum Principle.

Assume that $\lambda_0 = 0$. Then from Equation (3.9) the Hamiltonian is simply

$$\mathcal{H}(t, K, l, \lambda) = \mu K + \lambda l \quad \ldots \ldots (A.1)$$

and the adjoint Equations (3.12) and transversality condition (3.13) imply that

$$\mu(t) = \mu \quad \begin{cases} \lambda(t) = \mu(T-t) \end{cases} \quad 0 \leq t \leq T \quad \ldots \ldots (A.2)$$

Substituting (2) into (1), we have

$$\mathcal{H}(t, K, l, \mu) = \mu K + \mu(T-t)l \quad \ldots \ldots (A.3)$$

By condition (3.11), $\mu \neq 0$, and from (3.10), if $l$ is an optimal control,

$$l(t) = \begin{cases} 0, & \text{if } \mu(T-t) < 0 \\ l_M, & \text{if } \mu(T-t) > 0 \end{cases} \quad \text{a.e.} \quad \ldots \ldots (A.4)$$
Therefore, if \( \mu < 0 \), \( l(t) = 0 \) a.e. This control is feasible only if the initial and terminal conditions are such that

\[
C_T - C_0 = K_0 T.
\]  

Similarly, if \( \mu > 0 \), the optimal control must be \( l(t) = l_M \) a.e., which is feasible only if

\[
C_T - C_0 = \frac{l_M T^2}{2} + K_0 T.
\]

Equations (5) and (6) correspond to equality in (3.8). In either case, there is only one feasible control; the constraints on the problem are too restrictive to allow any optimization.
APPENDIX B

SELECTING AN INITIAL SET OF PARAMETER VALUES

The initial set of parameter values, which appear in the first line of Table I, were chosen with the following points in mind. For the basic model specified in Equations (3.1) to (3.7), with $C_0 = K_0 = 0$, the parameters which had to be specified were: $\delta$, $y$, $I_M$, $C_T$, and $T$. The values had to satisfy inequality (3.8) which can be simplified to

$$C_T \leq \frac{I_M T^2}{2}$$

(when $C_0 = K_0 = 0$.)

Most of the data which have been used come from reports and articles concerned with boomtowns in the Rocky Mountain states of the United States (i.e., Colorado, Utah, New Mexico, Wyoming; see [2], [4], [6]). Since these data series are expressed for the most part in terms of 1975 dollars, this monetary unit was used for the parameter values. The following notes summarize how the initial set of parameter values was obtained.

$\gamma$: [0.5, 2.2] "wages" for labour plus unit maintenance costs for infrastructure. To "estimate" a value for $\gamma$, we had to refer back to the original model in which $L$ (labour) was a variable.
\( \gamma K = \text{wages for labour} + \text{maintenance costs for existing social capital} \)

\[
= \omega \cdot L + m \cdot K
\]

\[
= \frac{\omega}{k} \cdot K + m \cdot K
\]

where

\( \omega = \text{annual wage for construction labourers} \)

\( k \equiv \frac{K}{L} = \text{fixed social capital} / \text{labour ratio} \)

\( m = \text{annual maintenance cost per unit of social capital}. \)

These three variables were "estimated" as follows:

\( \omega: \ [5,000, 15,000] \ $ / \text{year}. \)

From Table I in [4], a weekly wage of $105.30 was cited. This is equivalent to approximately $5,500 per year, so $5,000 - $15,000 was set as a range for \( \omega \) in 1975 dollars.
**k**: \([7,000-10,000]\) $ infrastructure/labourer.

In [8], a range of 2,800 to 3,900 $ infrastructure per person has been suggested as a "norm" for per capita infrastructure levels \(\frac{K}{P}\) in non-boom communities, i.e., communities in which per capita infrastructure is fairly stable. To obtain a range for \(k = \frac{K}{L}\) for our model, this range for \(\frac{K}{P}\) was multiplied by the factor 2.5 persons per labourer (an estimate for average family size suggested in [6]). This gave a range of 7,000 to 9,750 for \(k\), which was rounded to \([7,000, 10,000]\).

**m**: \([0.01, 0.10]\) unit maintenance costs for infrastructure in [6], where \(m\) has been defined as maintenance and depreciation costs per unit of infrastructure, this parameter has been assigned a value of \(1/30 = .0333\), based on the assumption of a thirty year lifetime for social capital. The interval \([0.01, 0.10]\) contains values with the same order of magnitude.
With these ranges for $\omega, k,$ and $m$, $\gamma \in [0.5, 2.2]$. For the initial set of parameter values, we took $\dot{\gamma} = 1.0$.

$$C_T: [3,000, 8,000] \text{ man-years or } [21,80] \text{ million infrastructure dollar-years. In the original model, the state equation and terminal condition for construction were:}$$

$$\dot{C} = L$$
$$\hat{C}(T) = \hat{C}_T$$

so the size of the facility was measured in man-years. For example, [9] states that the manpower content for the construction of coal-mining facilities may run in the order of 3,000 to 5,000 man-years.

In our model, $k = \text{constant}$, and the state equation and terminal condition are

$$\dot{C} = K$$
$$C(T) = C_T$$

$$\dot{C} = k \dot{C}$$
$$C_T = k \hat{C}_T$$

So, for numerical solutions to our model, $C_T$ is measured in units of "infrastructure dollar-years." If $k \in [7,000, 10,000]$, then the widest range for $C_T = k \hat{C}_T$ is 21,000,000 to 80,000,000 infrastructure dollar-years. An initial value of $C_T = 50,000,000$ was chosen.
$l_M$: [150,000, 10,000,000] $/\text{year} - \text{maximum allowable rate of investment in social infrastructure.}$

This range was based on data in [6], on annual investments in infrastructure for each of twenty-six towns in the Rocky Mountain states. An "average" value of $5,000,000/\text{year}$ was used for the initial set of parameter values.

$T$: 

With $l_M = 5,000,000$, $C_T = 50,000,000$, $T$ had to be chosen so that

$$T \gg \left( \frac{2 C_T}{l_M} \right)^{1/2} = 20^{1/2} = 4.5 ;$$

the initial value for $T$ was $T = 10$ years.

$\delta$: 

A single "realistic" value for the social rate of discount would have been difficult to choose. One purpose of computing numerical solutions to the basic problem was, in fact, to test the sensitivity of the solution for different values of $\delta$ in a wide range. The initial value was arbitrarily set at $\delta = 0.10$. 
APPENDIX C

METHOD OF NUMERICAL SOLUTION FOR THE BASIC MODEL

A method for numerically solving the basic problem, for given parameter values, follows naturally from the analysis in Section 3 which shows that the optimal solution is unique.

All of the solutions were obtained using a computer program which finds roots $t_0(\mu)$ and $t_1(\mu)$ for the switching function

$$\sigma(t; \mu) = \lambda(t; \mu) - e^{-\delta t}$$

in a search for the optimal value of $\mu$.

In the program, the first trial value of $\mu$ is chosen so that the curve $\lambda(t; \mu)$ intersects the curve $e^{-\delta t}$ at $t = 0$, i.e., so that $\lambda(0) = 1$.

Since

$$\lambda(t; \mu) = \frac{\gamma}{\delta} (e^{-\delta T} - e^{-\delta t}) + \mu (T - t),$$

$$\lambda(0; \mu) = 1 \implies \frac{\gamma}{\delta} (e^{-\delta T} - 1) = 1$$

$$\implies \mu = \frac{1}{T} \left\{ 1 + \frac{\gamma}{\delta} (1 - e^{-\delta T}) \right\}.$$

For each trial value of $\mu$, the roots $t_0(\mu)$ and $t_1(\mu)$ are calculated using the subroutine RZFUN which is based on Mueller's
method. (The routine is available in the public file *NUMLIB at the UBC computing centre; documentation for the subroutine is given in UBC NLE Zeros of Nonlinear Equations, February, 1977). Convergence to the optimal value of \( \mu \) is tested by comparing construction at time \( T \), \( C(T; t_0(\mu), t_1(\mu)) \), with \( C_T \). (In the program, these two values are actually "normalized" by \( l_M \)).

In the subsequent trial, \( \mu \) is decreased or increased, according to the sign of \( C(T; t_0(\mu), t_1(\mu)) - C_T \). The step size by which \( \mu \) changes is decreased when a successive value of \( \mu \) gives a response error for construction which, in magnitude, is greater than or equal to the previous response error. This eliminates the possibility of "bouncing" back and forth on either side of the optimal \( \mu \), in a loop which doesn't converge.

A listing of the computer program follows.

```
1 IMPLICIT REAL*8(A-H,0-Z)
2 EXTERNAL SW
3 REAL*8 X(2),IMAX
4 COMMON DEL,U,A,B
5 NR=2
6 MAXIT=50
7 READ(5,10)E1,E2,E3,E4,E5,ERR,H
8 10 FORMAT(10D8.1)
9 WRITE(6,30)E1,E2,E3,E4,E5,ERR,H
10 20 FORMAT(//7D18.3)
11 TEST=CT/IMAX
12 30 FORMAT(///7D18.3)
13 EDT=DEXP(-DEL*T)
14 R=GAM/DEL
15 C=R*EDT
```
20 ULOW=GAM*EDT
21 WRITE(6,30)EDT,R,B,C,ULOW
22 U=(1 D0/T)*(1.D0+R*(1.D0-EDT))
23 40 ITER=ITER+1
24 IF(ITER .GT. 200)GO TO 150
25 A=C+U*T
26 WRITE(6,50)ITER,H,U
27 50 FORMAT(/15,D16.7,D25.16)
28 C-------------SEE IF U IS GT LOWER BOUND
29 IF(U.GT.ULOW)GO TO 53
30 U=(U+H+ULOW)/2.D0
31 H=(U+ULOW)/2.D0
32 C-------------SEE IF ROOTS EXIT
33 53 TS=(-1.D0/DEL)*DLOG(U/(GAM+DEL))
34 Y=SW(TS)
35 IF(Y.GE.0.D0)GO TO 55
36 H=H/2.D0
37 GO TO 120
38 55 X(1)=T
39 X(2)=-0.5D0
40 C-------------FIND ZEROS OF SWITCHING FUNCTION
41 CALL DRZFUN(SW,NR,MAXIT,X,IND,E1,E2,E3,E4)
42 IF(IND .EQ. 1)GO TO 120
43 T0=X(1)
44 T1=DMAX1(0.D0,X(2))
45 C CALCULATE C(T)/IMAX FOR THIS CONTROL
46 CI=(T0-T1)*(T-(T1+T0)/2.D0)
47 C-------------TEST FOR OPTIMALITY
48 ERL=ERR
49 ERR=CI-TEST
50 ABER=DABS(ERR)
51 IF(ABER .GE. DABS(ERL)) H=H/2.D0
52 IF(U/H .GT. 1.D16)GO TO 150
53 WRITE(6,60)X(2),T0 ,CI,ERR
54 60 FORMAT(46X, 201 9. 7,  2D24.16/)
55 IF (ERR.LE.E5) GO TO 120
56 IF (ERR.GE.E2) GO TO 100
57 STOP
58 C-------------DECREASE U
59 100 U=U-H
60 GO TO 40
61 C-------------INCREASE U
62 120 U=U+H
63 GO TO 40
64 150 STOP
65 END
66 DOUBLE PRECISION FUNCTION SW(X)
67 IMPLICIT REAL*8(A-H,0-Z)
68 COMMON DEL,U,A,B
69 SW=A-B*DEXP(-DEL*X)-U*X
70 RETURN
71 END
APPENDIX D

PROOF OF LEMMA 1

Lemma 1:  Given \( \varepsilon > 0 \) and a control \( I \) which is feasible for \( P_\infty \),

\[ \exists a control \hat{I} \text{ and an integer } N \text{ such that } V_n \geq N, \hat{I} \text{ is feasible for } P_n, \]

and \( J(\hat{I}) \leq J(I) + \varepsilon \).

Proof:  We first consider the case in which \( I \) is an impulse control at only

one instant, \( t = 0 \), and is PWC on \((0,T]\).  Let \((C,K)\) be the response

trajectory for \( I \), and Let \( \Delta K \) be the increase in \( K \) at \( t = 0 \).  Since

\( K \equiv I \geq 0 \), \( K \) is non-decreasing.

We want to define a PWC control \( \hat{I} \) which approximates \( I \) in

cost, so \( \hat{K} \), the response trajectory for \( \hat{I} \), must be "close" to the

trajectory \( K \).  In addition, \( \hat{I} \) must be feasible, so \( \hat{K} \) must satisfy

\[
\int_0^T \hat{K}(t) \, dt = C_T \quad \ldots \ldots (D.1)
\]

or, equivalently,

\[
\int_0^T (\hat{K} - K) \, dt = 0 \quad \ldots \ldots (D.2)
\]

We approach the feasibility requirement as follows.  Consider any number

\( \alpha \in (0,T) \) which is small enough so that the straight line through the

origin and \((\alpha,K(\alpha))\) crosses the trajectory \( K(t) \) only at \( t = \alpha \) and
\[ K(\alpha) \cdot T = \overline{T}_\alpha \cdot T > C_T, \quad \ldots \quad (D.3) \]

as shown in Figure 9. For each \( \alpha_1 \in [\alpha, T] \), define \( \alpha_2 \in [\alpha_1, T] \) as

\[
\alpha_2 = \begin{cases} 
\min \left\{ t \mid K(t) = \overline{T}_\alpha \cdot \alpha_1 \right\}, & \text{if } \overline{T}_\alpha \cdot \alpha_1 \leq K(T) \\
T, & \text{if } \overline{T}_\alpha \cdot \alpha_1 > K(T) 
\end{cases} \quad \ldots \quad (D.4)
\]

Let the functions \( K_{\alpha, \alpha_1}(t) \) and \( I_{\alpha, \alpha_1}(t) \) be defined by

\[
K_{\alpha, \alpha_1}(t) = \begin{cases} 
\overline{T}_\alpha \cdot t, & t \in [0, \alpha_1) \\
\overline{T}_\alpha \cdot \alpha_1, & t \in [\alpha_1, \alpha_2] \\
K(t), & t \in (\alpha_2, T]
\end{cases} \quad \ldots \quad (D.5)
\]

\[
I_{\alpha, \alpha_1}(t) = K(t), \quad t \in [0, T] \quad . \quad \ldots \quad (D.6)
\]

Then \( K_{\alpha, \alpha_1} \) is piecewise smooth, so \( I_{\alpha, \alpha_1} \) is PWC. When \( \alpha_1 = \alpha \),

\[
\int_0^T K_{\alpha, \alpha_1}(t) \, dt = \int_0^T K_{\alpha, \alpha}(t) \, dt = \int_0^\alpha \frac{K(\alpha)}{\alpha} \cdot t \, dt + \int_\alpha^T K(t) \, dt
\]

\[
\leq \int_0^T K(t) \, dt = C_T,
\]

and if \( \alpha_1 = T \),
Figure 9  Graph of $K_{\alpha, \alpha_1}$
\begin{equation}
\int_{0}^{T} K_{\alpha_1,\alpha_1}(t) \, dt = \int_{0}^{T} \frac{K(\alpha)}{\alpha} \cdot t \, dt = \frac{K(\alpha)}{\alpha} \cdot T \geq C_T,
\end{equation}

by (3). For fixed \( \alpha \), the integral \( \int_{0}^{T} K_{\alpha,\alpha_1}(t) \, dt \) changes continuously with \( \alpha_1 \), so there must be exactly one value of \( \alpha_1 \in (\alpha, T) \) for which

\begin{equation}
\int_{0}^{T} K_{\alpha,\alpha_1}(t) \, dt = C_T.
\end{equation}

I.e., for each \( \alpha \), there is a unique \( \alpha_1 = g(\alpha) \in (\alpha, T) \) with

\[ \alpha_2 = h(\alpha) \in [\alpha_1, T] \]

so that \( K_{\alpha, g(\alpha)} \) satisfies (7). This means that

\[ l_{\alpha, g(\alpha)} \]

is feasible for \( P_n \), for large enough \( n \). Henceforth, for a given \( \alpha \), we shall refer only to the feasible control \( l_{\alpha, g(\alpha)} \) and its response \( K_{\alpha} \equiv K_{\alpha, g(\alpha)} \). We claim that

\begin{equation}
\lim_{\alpha \to 0^+} J\{l_{\alpha}\} = J\{l\}. \tag{D.8}
\end{equation}

From the preceding definitions, it is clear that \( g \) and \( h \) are continuous and that

\begin{align}
g(\alpha) &\downarrow 0 \quad \text{as} \quad \alpha \downarrow 0, \tag{D.9} \\
\bar{l}_{\alpha} &\uparrow \infty \quad \text{as} \quad \alpha \downarrow 0, \tag{D.10} \\
\bar{l}_{\alpha} \cdot g(\alpha) &\equiv K_{\alpha}(g(\alpha)) \downarrow \Delta K \quad \text{as} \quad \alpha \downarrow 0. \tag{D.11}
\end{align}
The behaviour of $h(\alpha)$ as $\alpha \to 0$ depends on the control $l(t)$. If $l(t) > 0$ on $(0, \rho)$, for some $\rho > 0$, then $K(t)$ is strictly increasing on $(0, \rho)$, so by Equation (11) and (4), $h(\alpha) \to 0$ as $\alpha \to 0$, as shown in Figure 10(a). The other possibility is that $l(t) \equiv 0$ on the interval $(0, \rho)$ for some $\rho > 0$, and in this case, $h(\alpha) \to \rho$ as $\alpha \to 0$ (shown in Figure 10(b)). The limit in Equation (8) can be verified separately for each of these two possibilities.

Because the controls $l_\alpha$ and $l$ are identical on the interval $[h(\alpha), T]$,

$$\mathcal{J}\{l_\alpha\} - \mathcal{J}\{l\}$$

$$= \int_{0}^{h(\alpha)} e^{-\delta t} \{ yK + l_\alpha \} \, dt - \int_{0}^{h(\alpha)} e^{-\delta t} \{ yK + l \} \, dt - \Delta K$$

$$= \int_{0}^{\frac{q(\alpha)}{g(\alpha)}} e^{-\delta t} \{ yK + l_\alpha \} \, dt - \int_{0}^{\frac{q(\alpha)}{g(\alpha)}} e^{-\delta t} \{ yK + l \} \, dt$$

$$+ \int_{\frac{q(\alpha)}{g(\alpha)}}^{h(\alpha)} e^{-\delta t} \{ y \left( \frac{\gamma}{\alpha} \cdot g(\alpha) - K \right) - l \} \, dt - \Delta K$$

$$\ldots \ldots \text{(D.12)}$$

If the first term on the right side of Equation (12) is integrated by parts, we obtain

$$\int_{0}^{\frac{q(\alpha)}{g(\alpha)}} e^{-\delta t} \{ yK + l_\alpha \} \, dt = \int_{0}^{\frac{q(\alpha)}{g(\alpha)}} e^{-\delta t} \{ y \frac{\gamma}{\alpha} \cdot t + \frac{\gamma}{\alpha} \} \, dt$$
Figure 10  Trajectory $K(t)$ determines $\rho$ : (a) $\rho = 0$; (b) $\rho > 0$
\[
\frac{\gamma}{\delta^2} \overline{I}_\alpha (1 - e^{-\delta g(\alpha)}) - \frac{\gamma}{\delta} \overline{I}_\alpha g(\alpha) + \frac{\gamma}{\delta} (1 - e^{-\delta g(\alpha)}) \\
= \left( \frac{\gamma + 1}{\delta} \right) \overline{I}_\alpha (1 - e^{-\delta g(\alpha)}) - \frac{\gamma}{\delta} \overline{I}_\alpha g(\alpha) \quad \ldots \ldots (D.13)
\]

In order to take the limit of Equation (13), as \( \alpha \to 0 \), we expand 
\( e^{-\delta g(\alpha)} \) in a Maclaurin series, so the first term on the right side may be written

\[
\left( \frac{\gamma + 1}{\delta} \right) \overline{I}_\alpha [1 - e^{-\delta g(\alpha)}] = \left( \frac{\gamma + 1}{\delta} \right) \overline{I}_\alpha \left[ 1 - \left[ 1 - \delta g(\alpha) + O(g(\alpha))^2 \right] \right] \\
= \left( \frac{\gamma + 1}{\delta} \right) \overline{I}_\alpha g(\alpha) \left\{ 1 - O(g(\alpha)) \right\} \quad \ldots \ldots (D.14)
\]

Substituting this expression into Equation (13), we have

\[
\lim_{\alpha \to 0} \int_0^\infty e^{-\delta t} \left\{ \gamma K_\alpha + I_\alpha \right\} dt \\
= \lim_{\alpha \to 0} \overline{I}_\alpha \cdot g(\alpha) \left\{ \left( \frac{\gamma + 1}{\delta} \right) \left[ 1 - O(g(\alpha)) \right] - \frac{\gamma}{\delta} \right\} = \Delta K \cdot 1 \quad \ldots \ldots (D.15)
\]

by (9) and (11). The second integral on the right side of Equation (12) is easily disposed of:

\[
\lim_{\alpha \to 0} \int_0^\infty e^{-\delta t} \left\{ \gamma K + 1 \right\} dt = \lim_{g(\alpha) \to 0} \int_0^\infty e^{-\delta t} \left\{ \gamma K + 1 \right\} dt = 0 \quad \ldots \ldots (D.16)
\]
because the integrand is continuous (for small enough \( \alpha \)), and bounded on \((0, g(\alpha)]\). Similarly, the third integral in Equation (12) also approaches 0 as \( \alpha \to 0 \), although this should be verified for the case in which \( \lim h(\alpha) = \rho > 0 \). As we noted earlier, \( \rho > 0 \) iff \( l(t) \equiv 0 \) for \( t \in (0, \rho) \), in which case, \( K(t) \equiv \Delta K \) on \((0, \rho)\). Therefore,

\[
\int_{g(\alpha)}^{h(\alpha)} e^{-st} \left\{ Y \left[ I_{\alpha} \cdot g(\alpha) - K(t) \right] - l(t) \right\} \, dt
\]

\[
= Y \int_{g(\alpha)}^{h(\alpha)} e^{-st} \, dt - Y \int_{g(\alpha)}^{h(\alpha)} e^{-st} \cdot K(t) \, dt - \int_{g(\alpha)}^{h(\alpha)} e^{-st} \cdot l(t) \, dt
\]

\[
\to Y \Delta K \int_{0}^{\rho} e^{-st} \, dt - Y \int_{0}^{\rho} e^{-st} \cdot K(t) \, dt - \int_{0}^{\rho} e^{-st} \cdot l(t) \, dt
\]

\[
= Y \Delta K \int_{0}^{\rho} e^{-st} \, dt - Y \int_{0}^{\rho} e^{-st} \cdot \Delta K \, dt - 0
\]

\[
= 0 , \quad \text{as} \quad \alpha \to 0 . \quad \cdots \cdots \text{(D.17)}
\]

Combining Equation (12) to (17) verifies our claim in Equation (8), ie.

\[
J \left\{ I_{\alpha} \right\} - J \left\{ I \right\} \to 0 \quad \alpha \to 0 .
\]

Thus, given \( \varepsilon > 0 \), we may choose \( \alpha > 0 \) such that
Let $N = \text{smallest integer} \geq \sup_t\{l_\alpha(t)\} (< \infty \text{ because } l_\alpha \text{ is PWC})$. Then the control $\hat{l} = l_\alpha$ is feasible for $P_n$, $\forall n \geq N$, and $J(\hat{l}) \leq J(l) + \varepsilon$.

This completes the proof of the lemma for a control $l$ which has only one impulse, at $t = 0$. This proof can easily be extended to cover all other feasible controls for $P_\alpha$, by constructing a PWC control $l_{\alpha'}$, which approximates $l$ at each of its $k(<\infty)$ impulses,\(^{11}\) as indicated in Figure 11.

\(^{11}\) The proof can also be modified for the case of an impulse at $t = T$, by defining an approximating control with $h(\alpha) \leq g(\alpha) < \alpha < T$. However, any control with an impulse at $t = T$ cannot be optimal, for the same control without the jump in capital at $T$ is feasible and costs less. For our application of the lemma in Section 5, this case can be explicitly excluded from the lemma.
Figure 11 Construction of $K_\alpha$ when $K$ has more than one jump