On Square Summability and Uniqueness Questions
Concerning
Nonstationary Stokes Flow in an Exterior Domain

by

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ABSTRACT

In this thesis we investigate the square summability, uniqueness, and convergence to steady state of solutions of the nonstationary Stokes equations in an exterior domain.

A class of generalized solutions (which will be called class $H_0$ solutions), whose members are required a priori to have finite Dirichlet integrals but not necessarily to have finite $L^2$ norms, has been introduced by J.G. Heywood for the purpose of studying the convergence of nonstationary solutions to stationary ones as time $t \to \infty$. In our present work, we prove that, in the case of an exterior domain $\Omega$ of $\mathbb{R}^n (n > 2)$, such solutions are necessarily square-summable if both the initial data and the force are square-summable. We give a partial result for $\Omega$ in $\mathbb{R}^2$. Furthermore, we prove that if $\Omega \subset \mathbb{R}^3$ the unique class $H_0$ solution is identical with the unique finite energy solution (i.e. $L^2(\Omega)$) of various classes when the data permits existence of both types of solutions. This has enabled us to show that the finite energy solutions of a particular nonstationary Stokes problem converge to solutions of steady state as $t \to \infty$. We have also succeeded in extending the definition of class $H_0$ solutions
to nonstationary Stokes problems with general nonhomogeneous boundary values in such a way that the uniqueness theorem for such solutions is preserved.
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1. Introduction

This thesis deals with square summability questions associated with a class of solutions of the initial boundary value problem for the nonstationary Stokes equations. If $u(x,t)$ is the velocity field of a fluid, if $p(x,t)$ is the pressure, and if $f(x,t)$ is a given external force density, then the equations for nonstationary Stokes flow in a space-time region $\Omega \times (0,\infty)$ are

$$\frac{\partial u}{\partial t} - \Delta u = - \nabla p + f$$

(1)

$$\nabla \cdot u = 0$$

Here $\Omega$ is a domain in $\mathbb{R}^n (n \geq 2)$. Solutions of (1) are sought to satisfy prescribed initial and boundary conditions; the general initial boundary value problem (Section 5) can be reduced to that with homogeneous boundary values:

(2) $u(x,0) = a(x), x \in \Omega$

(3) $u(x,t) = 0, (x,t) \in \partial \Omega \times (0,\infty)$

(4) $u(x,t) \neq 0$ as $|x| \to \infty$ if $\Omega$ is unbounded.
We will be particularly interested in the case of an exterior domain $\Omega$ in $\mathbb{R}^n$ with $n > 2$. Because equations (1) - (4) admit a formal energy identity, it is natural to anticipate that solutions of (1) - (4) will belong to $L^2(\Omega)$ for every $t < T$ if the initial data $a \in L^2(\Omega)$ and if the prescribed force $f \in L^2(\Omega \times (0,T))$.

Our main objective is to prove this square summability for a class of solutions of (1) - (4) which are defined without being required, a priori, to have finite $L^2$ norms in $\Omega$.

In order to study the convergence of nonstationary solutions to stationary ones as $t \to \infty$ in the case of an exterior spatial domain, Heywood [2] has recently introduced a class of solutions which are required, a priori, to have finite Dirichlet integrals but not necessarily to have finite $L^2$ norms. Solutions of this class are characterized chiefly by membership in a certain function space $H^0_0$, and we follow Heywood in calling them class $H^0_0$ solutions. For some classes of initial data and forces which need not be square-summable, he proved the existence and uniqueness of a class $H^0_0$ solution of problem (1) - (4) without appealing to energy estimates. This class of solutions proved useful in treating problems of convergence to steady state principally because solutions of the exterior stationary Stokes problem possess finite Dirichlet integrals but not always finite $L^2$ norms.
Since class $H^2_0$ solutions of (1) - (4) were studied in [2] without use of any energy estimates, it is not clear whether class $H^2_0$ solutions will be square-summable if the initial data and the force are square-summable. In fact, Heywood has given a negative answer to this question in the case of a two-dimensional exterior domain. He has shown in [3] that there are choices of square-summable data for which the unique class $H^2_0$ solution tends, after a finite time, continuously to a nonzero limit at infinity and is thus not square-summable. The case of an exterior domain in $\mathbb{R}^n$ with $n > 2$, nevertheless, contrasts sharply with the case of $n = 2$, because when $n > 2$ the class $H^2_0$ solution tends to zero at infinity in a generalized sense which precludes the possibility of its tending continuously to a nonzero limit at infinity. It is therefore reasonable to expect that the class $H^2_0$ solution of problem (1) - (4) will possess a finite $L^2$ norm if the initial data and the force do. In this thesis, we will prove this conjecture in the case of an exterior domain in $\mathbb{R}^n$ with $n > 2$. For arbitrary domains in $\mathbb{R}^n$ with $n > 2$, the problem is still open. In the case of an exterior domain in $\mathbb{R}^2$, there remains a question of what conditions on the forces will ensure that class $H^2_0$ solutions are square-summable. It is important to determine such conditions for the forces in order to clarify the physical significance, in the case of a two-dimensional exterior domain, of a theorem of Heywood [3] which
states the attainability of stationary solutions as limits of nonstationary solutions. We do give a partial result for the case of a two-dimensional exterior domain; we show that if the force is square-summable then the time derivative of the class $H_0$ solution tends to a limit at infinity in a generalized sense for each fixed time.

Our method of proving that the class $H_0$ solution of (1) - (4) is square-summable, if the data is, involves proving that $u_\tau$ can be expressed in the form $u_\tau = \nabla q + g$, where $g$ is square-summable in $\Omega \times (0,T)$, and where $q$ is harmonic in the space variables. By expanding $q$ in a neighborhood of infinity as a series in spherical harmonics, and by using $L^2$ estimates for $g$ and $\nabla u_\tau$, we show that $\nabla q$ behaves like $\frac{1}{|x|^{n-1}}$ as $|x| \to \infty$, and this implies that $\nabla q$ is square-summable over $\Omega$ for every $t \in (0,T)$. The fact that $\nabla q$, and thus $u_\tau$, is square-summable over $\Omega \times (0,T)$ is then shown by obtaining a uniform estimate for $||\nabla q(\cdot, t)||_{L^2(\Omega)}$ in $0 < t < T$. The proof of this estimate is based in part on an inequality of Payne and Weinberger [8]. From the fact that $u_\tau \in L^2(\Omega \times (0,T))$, our result that $u$ is square-summable in $\Omega \times (0,T)$ is easily proved. Details are contained in Section 3.

Under the hypotheses of $L^2$ data, it is known that
problem (1) - (4) possesses a unique finite energy (i.e. $L^2(\Omega)$) solution in various classes. Solutions of a class introduced by Ladyzhenskaya [6], which we call class J solutions, belong to the function space $J$ defined as the completion in $L^2$ norm $||\cdot||$ of the set of all smooth solenoidal $\mathbb{R}^n$-valued functions with compact supports. She has studied the class J solutions in detail only in the case of a bounded spatial domain. We have extended the definition of class J solutions to cases of unbounded spatial domains.

Another class of finite energy solutions studied in [6] is characterized chiefly by membership in the space $J_1$ defined as the completion in norm $(||\cdot||^2 + ||\nabla\cdot||^2)^{1/2}$ of the set of all smooth solenoidal $\mathbb{R}^n$-valued functions with compact supports. It is natural to ask whether the class $H_0$ solution is identical with the solutions of these finite energy classes. For an exterior domain in $\mathbb{R}^2$ the answer is negative because, as mentioned above, there are forces for which the class $H_0$ solution fails to be square-summable. For an exterior domain in $\mathbb{R}^3$, however, our demonstration that the class $H_0$ solution is square-summable, combined with a recent characterization of the space $J_1$ by Heywood [5], proves that the class $H_0$ solution and these several finite energy solutions are necessarily all identical. The proof is in Section 4.

It remains unknown, for arbitrary unbounded domain in $\mathbb{R}^3$, whether these solution classes are all identical for smooth $L^2$ data.
Heywood has shown in [2] that solutions of the exterior stationary problem occur as limits of nonstationary class $H_0$ solutions. In particular, he established the convergence to steady state of the class $H_0$ solution of the problem which models the following physical experiment. An object is initially at rest in a three dimensional space filled with a Stokesian fluid. It is then smoothly accelerated until a given velocity is attained, after which it is kept in motion with the same velocity. Because of the uniqueness of solution classes of (1) - (4), we conclude in Section 5 that the finite energy solution of this nonstationary flow problem converges as $t \to \infty$ to the solution of the exterior stationary problem.

In order to define the class $H_0$ solution for problems with general nonhomogeneous boundary values in such a way as to preserve the uniqueness theorem, it is necessary to define a class of admissible extensions of the initial and boundary values into the space-time region, and to prove that the difference between any two such extensions belongs to the space $H_0$. The study of class $H_0$ solutions in [2] was limited to problems with constant prescribed boundary values; the method in [2] of defining extensions of constant boundary values does not extend to more general boundary values. Our result that the class $H_0$ solution is identical with the class $J_1$ solution has enabled us to define a reasonable class of such extensions for
much more general boundary values, at least for the case of an exterior domain in $\mathbb{R}^3$. 
2. Preliminaries

By an exterior domain \( \Omega \) in \( \mathbb{R}^n \) (\( n \geq 2 \)), we mean an open set which contains a complete neighborhood of infinity \( \{ x \in \mathbb{R}^n : |x| > R > 0 \} \). Throughout this thesis, we assume \( \Omega \) to be an exterior domain in \( \mathbb{R}^n \) with \( n > 2 \) unless otherwise stated. The space-time region \( \Omega \times (0,T) \) is denoted by \( Q_{\Omega,T} \) and simply by \( Q_T \) if \( \varepsilon = 0 \). The closure, boundary, and complement of a set \( S \) are denoted by \( \overline{S} \), \( \partial S \), and \( S^c \) respectively. The interior of a sphere of radius \( r \) centered at the origin \( 0 \) is denoted by \( S_r \); its boundary is denoted by \( \partial S_r \) and its exterior by \( E_r \).

All functions in this thesis are either \( \mathbb{R} \) or \( \mathbb{R}^n \) - valued. We use letters \( h, k, p, q, \alpha, \beta \), etc., to denote \( \mathbb{R} \) - valued functions and letters \( u, v, w, \phi, \psi, \xi, \eta, f, g \), etc., to denote \( \mathbb{R}^n \) - valued functions. We will use the same symbol to denote a function space of either \( \mathbb{R} \) - valued or \( \mathbb{R}^n \) - valued functions. The distinction will be clear from context. Thus, \( L^2(\Omega) \) denotes the space of all \( \mathbb{R} \) - valued or \( \mathbb{R}^n \) - valued functions which are square-summable in \( \Omega \), \( L^2_{\text{loc}}(\Omega) \) denotes the space of all \( \mathbb{R} \) - valued or \( \mathbb{R}^n \) - valued functions which are square-summable in compact subsets of \( \Omega \), and \( C^0(\Omega) \) denotes the space of all smooth
R-valued or $\mathbb{R}^n$-valued functions with compact supports in $\Omega$.

Spaces $L^2(Q_T)$, $L^2_{loc}(\tilde{Q}_\infty)$, and $C^\infty_0(\Omega \times [0,T])$, etc., over space-time regions are defined similarly.

We employ the usual notation of vector analysis and in addition the following notation:

\[
(u,v) = \int_\Omega u \cdot v \, dx = \int_\Omega \sum_{i=1}^n u_i v_i \, dx, \quad ||u|| = (u,u)^{1/2};
\]

\[
(\nabla u, \nabla v) = \int_\Omega \nabla u : \nabla v \, dx = \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx, \quad ||\nabla u|| = (\nabla u, \nabla u)^{1/2};
\]

\[
(u,v)_1 = (u,v) + (\nabla u, \nabla v), \quad ||u||_1 = (u,u)^{1/2};
\]

\[
(u,v)_{Q_\varepsilon,T} = \int_0^T (u,v) \, dt, \quad ||u||_{Q_\varepsilon,T} = (u,u)^{1/2};
\]

\[
(u,v)_{1,Q_\varepsilon,T} = \int_0^T (u,v)_1 \, dt, \quad ||u||_{1,Q_\varepsilon,T} = (u,u)_1^{1/2};
\]

We will also use extensively the following definitions of $\mathbb{R}^n$-valued function spaces:

\[
D(\Omega) = \{ \phi(x) : \phi \in C^\infty_0(\Omega) \text{ and } \nabla \cdot \phi = 0 \} ;
\]

\[
J(\Omega) = \text{Completion of } D(\Omega) \text{ in the norm } ||\cdot|| ;
\]

\[
J_0(\Omega) = \text{Completion of } D(\Omega) \text{ in the norm } ||\nabla \cdot \cdot|| ;
\]

\[
J_1(\Omega) = \text{Completion of } D(\Omega) \text{ in the norm } ||\cdot||_1 .
\]

Let $D(Q_T) = \{ \phi(x,t) : \phi \in C^\infty_0(\Omega \times [0,T]) \text{ and } \nabla \cdot \phi = 0 \}$. The
spaces \( J(Q_T) \), \( J_0(Q_T) \), and \( J_1(Q_T) \) are defined to be the completions of \( D(Q_T) \) in the norms \( \| \cdot \|_{Q_T} \), \( \| \nabla \cdot \|_{Q_T} \), and \( \| \cdot \|_{1,Q_T} \) respectively.

Below, we state without proofs three well-known lemmas which will be used throughout the latter sections. We refer the reader to [6, pp.16-31] for proofs of Lemmas 1 and 2. Lemma 3 follows from Lemma 2.

**Lemma 1.** Let \( \Omega \) be any domain in \( \mathbb{R}^n \) with \( n \geq 2 \). Then the orthogonal complement of \( J(\Omega) \) in \( L^2(\Omega) \) is \( G(\Omega) = \{ \phi(x) : \phi \in L^2(\Omega) \) and \( \phi = \nabla p \) for some \( p \in L^2_{loc}(\Omega) \). That is, every function \( u \) in \( L^2(\Omega) \) can be expressed uniquely as \( u = v + \nabla p \), where \( v \in J(\Omega) \), \( \nabla p \in G(\Omega) \), and \( \int_{\Omega} v \cdot \nabla p \, dx = 0 \).

**Lemma 2.** Suppose \( \Omega \) is a domain in \( \mathbb{R}^n \) with \( n \geq 2 \). Then for any \( \phi \in J_0(\Omega) \),

\[
\int_{\Omega} \frac{\phi(x)^2}{|x-y|^2} \, dx \leq \left( \frac{2}{n-2} \right)^2 \int_{\Omega} (\nabla \phi)^2 \, dx
\]

where \( y \) is an arbitrary point in \( \mathbb{R}^n \).

If \( \Omega \subset \mathbb{R}^2 \) is a domain such that \( \Omega^c \) contains a disk \( \{ x : |x| \leq r_0 \} \), then every \( \phi \in J_0(\Omega) \) satisfies

\[
\int_{\Omega} \frac{\phi^2(x) \, dx}{|x|^{2\log^2|x/r_0^c|}} \leq 4 \int_{\Omega} (\nabla \phi)^2 \, dx
\]
Lemma 3. Suppose $\Omega$ is either a domain in $\mathbb{R}^n$ with $n > 2$ or a two-dimensional domain whose complement contains a disk $\{x : |x| \leq r_0\}$. If $\Omega'$ is a bounded subset of $\Omega$ there exists a constant $C_\Omega'$ such that $\int_{\Omega'} \phi^2 \,dx \leq C_\Omega'^2 \|\nabla \phi\|^2$ holds for all $\phi \in J_0(\Omega)$.

The following two lemmas will be needed in Section 3. Lemma 5 is due to Payne and Weinberger [8]. We include its proof here because it plays a fundamental role in our work.

Lemma 4. Let $\Omega$ be a domain as in Lemma 3. Suppose either $u \in J(\Omega)$ or $u \in J_0(\Omega)$. Then $\int_{\Omega} u \cdot \nabla h \,dx = 0$ holds for all $h \in C^\infty(\Omega)$.

Proof. Suppose $u \in J(\Omega)$. It is evident from Lemma 1 that $\int_{\Omega} u \cdot \nabla h \,dx = 0$ for all $h \in C^\infty(\Omega)$ because $\nabla h \in C(\Omega)$ whenever $h \in C^\infty(\Omega)$.

Suppose $u \in J_0(\Omega)$. The integral $\int_{\Omega} u \cdot \nabla h \,dx$ with $h \in C^\infty_0(\Omega)$ exists since $u \in L^2_{loc}(\Omega)$ by Lemma 3. For any $\varepsilon > 0$, there exists $\phi \in D(\Omega)$ such that $\|\nabla u - \nabla \phi\| < \varepsilon$. Thus

$$\|\nabla u\| = \|\nabla u - \nabla \phi\| \leq C \|\nabla u - \nabla \phi\| < C\varepsilon$$

for some $C > 0$ and this implies that $\nabla u = 0$ in $\Omega$ since $\varepsilon$ is arbitrary. Hence we have, upon an integration by parts, $\int_{\Omega} u \cdot \nabla h \,dx = -\int_{\Omega} (\nabla u) h \,dx = 0$ for any $h \in C^\infty_0(\Omega)$.

Lemma 5. Let $E_R$ be the exterior of a sphere $S_R$ of
radius $R$ centered at $0$ in $\mathbb{R}^n (n > 2)$. Suppose that $\phi$ is continuously differentiable in $E_R$ and that $\nabla \phi \in L^2(E_R)$. Then there is a constant function $\phi_o$ such that

$$\frac{(n-2)}{R} \int_{\partial S_R} |\phi - \phi_o|^2 \, d\sigma \leq \int_{E_R} |\nabla \phi|^2 \, dx,$$

where $d\sigma$ denotes the area of a surface element.

Proof. We consider the spherical coordinates $x = (r, \theta)$ where $r = |x|$ and $\theta$ stands for the angular coordinates $(\theta_1, \theta_2, \ldots, \theta_{n-1})$. Let $\nabla_\theta \phi$ denote the vector obtained by multiplying the angular component of $\nabla \phi$ by $r$. Then $(\nabla \phi)^2 = (\frac{\partial \phi}{\partial r})^2 + \frac{1}{r^2} (\nabla_\theta \phi)^2$. For illustration, let us consider the case of $n = 3$. We introduce spherical coordinates $x_1 = r \cos \theta_1 \sin \theta_2$, $x_2 = r \sin \theta_1 \sin \theta_2$, $x_3 = r \cos \theta_2$. Let $a, b, c$ be a right-handed orthogonal system of unit vectors at $x = (r, \theta_1, \theta_2)$ in the directions of the coordinates $r, \theta_2, \theta_1$. Then the gradient of $\phi$ may be expressed as

$$\frac{\partial \phi}{\partial r} a + \frac{1}{r} \frac{\partial \phi}{\partial \theta_2} b + \frac{1}{r \sin \theta_2} \frac{\partial \phi}{\partial \theta_1} c.$$

Thus, $\nabla_\theta \phi = \frac{\partial \phi}{\partial \theta_2} b + \frac{1}{\sin \theta_2} \frac{\partial \phi}{\partial \theta_1} c$ and $(\nabla_\theta \phi)^2 = (\frac{\partial \phi}{\partial \theta_2})^2 + \frac{1}{\sin^2 \theta_2} (\frac{\partial \phi}{\partial \theta_1})^2$. Now integrating $(\nabla \phi)^2$ over
the annular domain $S_{R,\rho} = \{ x \in \mathbb{R}^n : R < |x| < \rho \}$, we obtain, in spherical coordinates, the expression

$$\int_{S_{R,\rho}} |\nabla \phi|^2 \, dx = \int_{S_1} \left[ \int_{R}^{\rho} \left( \frac{\partial \phi}{\partial r} \right)^2 r^{n-1} \, dr \right] \, d\sigma + \int_{R}^{\rho} r^{n-3} \left( \int_{S_1} (\nabla_\theta \phi)^2 \, d\sigma \right) \, dr,$$

where $S_1$ is the surface of the unit sphere $S$. By an inequality of Wirtinger (a sketch of proof is given in [1, pp. 273-274]) and by the Schwartz inequality, we have

$$\int_{S_1} |\nabla_\theta \phi|^2 \, d\sigma \geq (n-1) \left( \int_{S_1} |\phi|^2 \, d\sigma - \frac{1}{\omega_n} \left( \int_{S_1} \phi \, d\sigma \right)^2 \right) \geq 0,$$

where $\omega_n$ is the area of the spherical surface $S_1$. Using the Schwartz inequality again, we have

$$\int_{R}^{\rho} \left( \frac{\partial \phi}{\partial r} \right)^2 r^{n-1} \, dr \geq \left( \int_{R}^{\rho} \frac{\partial \phi}{\partial r} \, dr \right)^2 / \left( \int_{R}^{\rho} r^{n-1} \, dr \right)$$
Thus,

\[
\int_{S_{R,\rho}} |\nabla \phi|^2 \, dx \geq \int_{\partial S_1} \left( \int_{\partial S_1} \frac{\partial \phi}{\partial r} \, dr \right)^2 \left\{ \int_{R}^{\rho} \frac{dr}{r^{n-1}} \right\} \, d\sigma +
\]

\[
(n-1) \int_{R}^{\rho} r^{n-3} \left( \int_{\partial S_1} \phi^2 \, d\sigma - \frac{1}{\omega_n} (\int_{\partial S_1} \phi \, d\sigma)^2 \right) \, dr.
\]

(5)

Since both integrals on the right of (5) are nonnegative, each is bounded uniformly in \( \rho \) by \( \int_{E_R} |\nabla \phi|^2 \, dx \). Now the first integral is \( (n-2) R^{n-2} \int_{\partial S_1} (\phi(\rho) - \phi(R))^2 \, d\sigma \) and so \( \int_{\partial S_1} (\phi(\rho) - \phi(R))^2 \, d\sigma \)
must converge to zero as \( R \to \infty \). Thus \( \phi \), as a function of \( \theta \) on \( \partial S_1 \), converges in the \( L^2 \) norm over \( \partial S_1 \) as \( R \to \infty \). It follows from the Schwartz inequality that \( B = \lim_{R \to \infty} \int_{\partial S_1} \phi(R) \, d\sigma \) exists. By adding \( -B \omega_n \) to \( \phi \), we can make the limit of \( \int_{\partial S_1} (\phi(R) - \frac{R}{\omega_n}) \, d\sigma \) as \( R \to \infty \) to be zero. Denoting \( \frac{B}{\omega_n} \) by \( \phi_0 \) and replacing \( \phi \) in (5) by \( \overline{\phi} = \phi - \phi_0 \), we obtain

\[
\int_{S_{R,\rho}} |\nabla \overline{\phi}|^2 \, dx \geq (n-2) R^{n-2} \int_{\partial S_1} (\overline{\phi}(\rho) - \overline{\phi}(R))^2 \, d\sigma +
\]

\[
(n-1) \int_{R}^{\rho} r^{n-3} \left( \int_{\partial S_1} |\overline{\phi}|^2 \, d\sigma - \frac{1}{\omega_n} (\int_{\partial S_1} \overline{\phi} \, d\sigma)^2 \right) \, dr.
\]

(6)

Since the second integral on the right of (6) converges as \( \rho \to \infty \), and since we have shown that its integrand converges as \( r \to \infty \),
its integrand must tend to zero as \( r \to \infty \). This implies that

\[
\lim_{r \to \infty} \int_{\mathbb{S}^1} (\phi(r) - \phi_0)^2 \, d\sigma = 0 \quad \text{since} \quad \lim_{r \to \infty} \int_{\mathbb{S}^1} (\phi(r) - \phi_0) \, d\sigma = 0.
\]

Neglecting the second term on the right of (6) and letting \( \rho \to \infty \), we obtain our desired result.

Next, we investigate the space \( H_0(\Omega) \), \( \Omega \) being any domain in \( \mathbb{R}^n \) with \( n \geq 2 \). For reference, see Heywood [2].

We denote by \( K_0(\Omega) \) the set of all \( u \in J_0(\Omega) \) such that

\[
(\nabla u, \nabla \phi) = (-f, \phi) \quad \text{for some} \quad f \in D(\Omega) \quad \text{and all} \quad \phi \in J_0(\Omega).
\]

The map \( \tilde{\Delta} : K_0(\Omega) \to J(\Omega) \) is defined by setting \( \tilde{\Delta}u = f \). The space \( H_0(\Omega) \) is defined to be the completion of \( K_0(\Omega) \) in the norm

\[
(\| \nabla \cdot \|^2 + \| \tilde{\Delta} \cdot \|^2)^{1/2}
\]

and it can be identified as a subset of \( J_0(\Omega) \). The map \( \tilde{\Delta} \) can be extended to the completion \( H_0(\Omega) \). The spaces \( K_0(Q_{\varepsilon,T}) \) and \( H_0(Q_{\varepsilon,T}) \) are defined similarly by substituting \( Q_{\varepsilon,T} \) for \( \Omega \), \((\cdot,\cdot)_{Q_{\varepsilon,T}} \) for \((\cdot,\cdot)\), and \( \| \cdot \|_{Q_{\varepsilon,T}} \) for \( \| \cdot \| \) in the corresponding definitions of the spaces over \( \Omega \).

The next two lemmas may also be found in [2] and they will be referred to in our later sections.
Lemma 6. The equation \((\nabla \psi, \nabla \phi) = (\psi, -\Delta \phi)\) holds under either one of the following conditions:

(i) \(\phi \in K_0(\Omega)\) and \(\psi \in J_0(\Omega)\), or

(ii) \(\phi \in H_0(\Omega)\) and \(\psi \in J_0(\Omega) \cap L^2(\Omega)\).

Lemma 7. If \(u \in H_0(\Omega)\), then \(u\) has second order derivatives \(u_{x_i x_j} \in L^2_{\text{loc}}(\Omega)\) and \(\Delta u = \Delta u + \nabla p\) for some \(p \in L^2_{\text{loc}}(\Omega)\) with \(\nabla p \in L^2_{\text{loc}}(\Omega)\).

Finally in this section, we state a result for harmonic functions which is important in the work of Section 3. We refer the reader to Poincaré [10], du Plessis [9], and Hochstadt [12] for proofs.

Lemma 8. Suppose \(q\) is harmonic in an open set containing the closure of an annular domain \(S_{\rho, R} = \{x \in \mathbb{R}^n : \rho < |x| < R\}\). If \(n \geq 3\), \(q\) has a series expansion in \(S_{\rho, R}\) in spherical harmonics of the form.

\[
q(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{N(n,k)} \left( C_{k,m} r^k + \tilde{C}_{k,m} r^{-(n-2+k)} \right) Y_{k,m}(\xi).
\]

Here \(\xi = \frac{x}{|x|}\), \(C_{k,m}\) and \(\tilde{C}_{k,m}\) are constants, \(N(n,k) = 2k+n-2\left(k+n-3\right)\) for \(k \geq 1\) and \(N(n,0) = 1\), and \(Y_{k,m}\) is a spherical harmonic.
function of order \( k \). The spherical harmonics satisfy the orthonormality conditions

\[
\int_{S^1} Y_{k,\ell}(\xi) Y_{j,m}(\xi) \, d\sigma(\xi) = \delta_{k,j} \delta_{\ell,m}
\]

where \( \delta_{k,j} \) denotes the Kronecker delta which is equal to 1 if \( k = j \) and equal to 0 if \( k \neq j \).

If \( n = 2 \), \( q \) has an expansion in \( S^1, R \) in spherical harmonics of the form

\[
q(r, \phi) = q_o + q_o \log r + \sum_{k=1}^{\infty} \left[ (a_k r^{-k} + a_k r^k) \cos k\phi + (b_k r^{-k} + b_k r^k) \sin k\phi \right]
\]

where \( a_k, b_k, a_k, b_k, q_o \), and \( q_o \) are all constants.
3. Square Summability of Class $H_0$ Solutions

Let $\Omega$ be a domain in $\mathbb{R}^n$ with $n \geq 2$. Suppose $f \in L^2_{\text{loc}}(\overline{\Omega}_\infty)$ and $a \in J_0(\Omega)$. By a class $H_0$ solution of equations (1) - (4), we mean a function $u$ which satisfies the conditions:

(8) $u \in H_0(Q_T)$ and $u_t \in J_0(Q_{\varepsilon,T})$ for all $0 < \varepsilon < T < \infty$,

(9) $\|\nabla u(t) - \nabla a\| \to 0$ as $t \to 0^+$, and

(10) $u_{x_i x_j} \in L^2_{\text{loc}}(\Omega_\infty)$ and there exists a scalar function $p \in L^2_{\text{loc}}(\Omega_\infty)$ with $\nabla p \in L^2_{\text{loc}}(\Omega_\infty)$ so that $u_t - \Delta u = -\nabla p + f$ holds almost everywhere (a.e.) in $Q_\infty$.

The existence and uniqueness of class $H_0$ solutions of (1) - (4) has been studied by Heywood [2]. If the initial data $a \in H_0(\Omega)$ and if the force $f \in J_0(Q_T)$ for all $T > 0$, the class $H_0$ solution $u$ satisfies the a priori inequality $\int_0^T \|\nabla u_t(t)\|^2 dt \leq \|\Delta a\| + \int_0^T \|\nabla f\|^2 dt$ for all $T > 0$; see Theorem 2 of [2]. On the other hand, if $a \equiv 0$ and if $f, f_t \in L^2(Q_T)$ for all $T > 0$ with $f(x,0) \equiv 0$, $u$ satisfies the a priori inequality $\|\nabla u_t(t)\|^2 \leq \int_0^T \|f_t\|^2 dt$; see Theorem 3 of [2]. In view of these a priori inequalities, we
may suppose that $u_t \in J_0(Q_T)$ for all $T > 0$ if we assume that $a \in H_0(\Omega)$ and $f = f_1 + f_2$ where $f_1 \in J_0(Q_T)$ for all $T > 0$ and $f_2, f_{2t} \in L^2(Q_T)$ for all $T > 0$. In fact, this is the case because, as demonstrated in the proof of Theorem 3 of [2], $u$ can be obtained as a sum $u_1 + u_2 + u_3$, where (i) $u_1$ is a solution on $[0,\infty)$ subject to the force $f_1$, and equal to $a(x)$ at $t = 0$, (ii) $u_2$ is a solution on $[-1,\infty)$, equal to zero at $t = -1$, and subject to the force $f_2$ extended to be defined on $[-1,\infty)$ in such a way that $f_2(x, -1) = 0$ and $f_2, f_{2t} \in L^2(\Omega \times [-1, T])$ for all $T > -1$, and (iii) $u_3$ is a solution on $[0,\infty)$ subject to zero force, and equal to $-u_2(x, 0)$ at $t = 0$.

Now suppose that $f \in L^2(Q_T)$ for all $T > 0$ and that there exists a class $H_0$ solution $u$ of (1) - (4). According to Lemmas 1 and 7, the functions $f$ and $\Delta u$ in (10) may be decomposed as $f = F + Vp_1, \Delta u = \tilde{\Delta} u + Vp_2$ where $F \in J(Q_T), \tilde{\Delta} u \in J(Q_T)$, $Vp_1 \in L^2(Q_T)$ for all $T > 0$, and where $p_1, p_2$ and $Vp_2$ belong to $L^2_{\text{loc}}(Q_\infty)$. The equation $u_t - \Delta u = -Vp + f$ in (10) thus becomes

\[(11) \quad \nabla q = u_t - g ,\]

where $\nabla q = V(p_1 + p_2 - p)$ and $g = \tilde{\Delta} u + F \in J(Q_T)$ for all $T > 0$. Our main effort is devoted to showing that $\nabla q \in L^2(Q_T)$ for all $T > 0$ if $\Omega$ is an exterior domain in $R^n$ with $n > 2$, from which
it follows that $u_t$ and hence $u$ belong to $L^2(Q)$. To accomplish this, we begin by showing that, for fixed $t$, the term $q$ in (11) is harmonic in $\Omega$ and that $\nabla q$ behaves like $\frac{1}{|x|^{n-1}}$ at infinity as a function of the spatial variables $x$.

**Lemma 9.** Suppose $\Omega$ is either a domain in $\mathbb{R}^n$ with $n > 2$ or a two-dimensional domain whose complement contains a disk $\{x : |x| \leq r_0\}$. Then the function $q(x,t)$ in equation (11) is harmonic in the spatial variables $x$ in $\Omega$ for almost all $t > 0$. If $\Omega$ is an exterior domain in $\mathbb{R}^n$ and if $n > 2$, then, for each $t > 0$, $\nabla q = 0$ in $\Omega$ in a neighborhood of infinity; further $\nabla q \in L^2(\Omega)$ for each fixed $t > 0$.

**Proof.** By Lemma 4, we have $\int_\Omega u_t \cdot \nabla h \, dx = 0$ and $\int_\Omega g \cdot \nabla h \, dx = 0$ for all $h \in C^\infty_0(\Omega)$ and almost all $t > 0$ since $u_t(t) \in L^2(\Omega)$ and $g(t) \in L^2(\Omega)$ for almost all $t > 0$. Hence for almost all $t > 0$,

$$\int_\Omega \nabla q \cdot \nabla h \, dx = 0$$

holds for all $h \in C^\infty_0(\Omega)$. Given any bounded domain $\Omega''$ such that $\overline{\Omega''}$ is compact and $\overline{\Omega''} \subset \Omega$, we let $\Omega'$ be a subdomain of $\Omega$ such that $\overline{\Omega'}$ is compact and $\overline{\Omega''} \subset \overline{\Omega'} \subset \Omega$. Let $\zeta$ be an $\mathbb{R}$-valued function in $C^\infty_0(\Omega')$ such that $\zeta = 1$ on $\overline{\Omega''}$. Put the function $h = (\varepsilon^2 \Delta q)_\rho$ in (12) where the subscript $\rho$ denotes an averaging convolution $q_\rho(x) = \int q(x-\rho y)k(y) \, dy$ with kernel $k \in C^\infty_0(|x| < 1)$ satisfying $\int k(x) \, dx = 1$. The support of the function $h$ is
contained in $\Omega'$ if $\rho$ is small enough. Using the well-known
identities $\frac{\partial}{\partial x_i}(\phi_\rho) = \left[ \frac{\partial \phi}{\partial x_i} \right]_\rho$ and $(\phi_\rho, \psi) = (\phi, \psi_\rho)$ for convolution,
we get from (12)

(13) \[ \int_\Omega (\zeta \Delta q_\rho)^2 \, dx = 0 \]

through integrations by parts. But since $\Delta(\zeta q_\rho) = \zeta \Delta q_\rho + 2\nabla \zeta \cdot \nabla q_\rho + (\Delta \zeta) q_\rho$, we have, in virtue of (13), the inequality

(14) \[ ||\Delta(\zeta q_\rho)||_{\Omega''} \leq C'_\zeta ||\nabla q_\rho||_{\Omega'} + C''_\zeta ||q_\rho||_{\Omega'} \]

where $C'_\zeta$ and $C''_\zeta$ are constants depending only on $\zeta$. From the
fact that $q$ and $\nabla q$ are locally square-summable, the right side
of (14) will be bounded by some constant $C$ depending only on $\zeta$
and $\Omega'$. Thus, $||q_{x_i x_j}||_{\Omega''} \leq ||(\zeta q_\rho)_{x_i x_j}||_{\Omega'} \leq ||\Delta(\zeta q_\rho)||_{\Omega'} \leq C$.

Taking the limit as $\rho \to 0$ yields $q_{x_i x_j} \in L^2(\Omega'')$. Therefore,
$q_{x_i x_j} \in L^2_{\text{loc}}(\Omega)$. Furthermore, it is clear that $\int_\Omega \nabla q_{x_i} \cdot \nabla h \, dx = \int_\Omega \nabla q_{x_i} \cdot \nabla h \, dx = 0$ for all $h \in C^\infty_0(\Omega)$ since the derivatives of
functions $h \in C^\infty_0(\Omega)$ also belong to $C^\infty_0(\Omega)$. By what we have just
shown, $q$ has locally square-summable third derivatives. An
induction argument shows that $u$ has locally square-summable
derivatives of all orders. Therefore, $q \in C^\infty(\Omega)$ by a well-
known theorem of Sobolev (see for example [7]). Finally, we
note that \( \int_{\Omega} (\Delta q) h \, dx = -\int_{\Omega} \nabla q \cdot \nabla h \, dx = 0 \) for all \( h \in C^\infty_0(\Omega) \); thus \( \Delta q = 0 \).

Suppose now that \( \Omega \) is an exterior domain in \( \mathbb{R}^n \) with \( n > 2 \).
Without loss of generality, we may assume that \( q \) is harmonic in \( \Omega \) for every time \( t > 0 \). Let \( E_R, R > 1 \), be such that \( \overline{E_R} \subseteq \Omega \).
According to Lemma 8, for a fixed \( t > 0 \), \( q \) as a function of the space variables has an expansion in \( E_R \) of the form.

\[
q(x) = \sum_{k=0}^{\infty} \sum_{m=1}^{N(n,k)} \left( C_{k,m} r^k + C_{k,m} r^{-(n-2+k)} \right) Y_{k,m}(\xi)
\]

where \( \xi = \frac{x}{|x|} \) and \( Y_{k,m} \) is a spherical harmonic of order \( k \). This series may be differentiated term by term because it and its differentiated series are uniformly convergent in every compact subset of \( E_R \). Thus, differentiating (15) with respect to \( r \) gives

\[
\frac{\partial q}{\partial r} = \sum_{k=0}^{\infty} \sum_{m=1}^{N(n,k)} \left( k C_{k,m} r^{k-1} - (n-2+k) C_{k,m} r^{-(n-1+k)} \right) Y_{k,m}(\xi).
\]

By the orthonormality of \( Y_{k,m}(\xi) \) over the surface of the unit sphere, we have
\[
R < \left| x \right| < R_1 \quad \Rightarrow \int \frac{3q_r}{r^2} \, dx = \sum_{k=0}^{\infty} \sum_{m=1}^{m=l} \int_{R}^{R_1} \frac{1}{r^2} \left( kC_{k, m} r^{k-1} - (n-2+k)C_{k, m} r^{(n-1+k)} \right) r^{n-1} \, dr
\]

where \( R_1 > R \). Now since \( u \in J_0(\Omega) \), Lemma 2 implies that \( \int_{\Omega} \frac{|u|}{r^2} \, dx \) is finite. This together with the fact that \( \int_{E_R} \frac{|q|}{r^2} \, dx \) \( < \infty \) implies that \( \int_{E_R} \frac{|Vq|^2}{r^2} \, dx \) \( < \infty \). In particular, the integral \( \int_{E_R} \frac{(\partial q)}{r^2} \, dx \) is finite. It then follows that the coefficients \( C_{k, m} \) must vanish for \( k \geq 1 \). Thus the expansion for \( q \) becomes

(16) \( q(x) = q_0 + \sum_{k=0}^{\infty} \sum_{m=1}^{m=l} C_{k, m} r^{-(n-2+k)} Y_{k, m}(\xi) \)

where we denote \( C_{0,1} \) by \( q_0 \). Because the series in (16) is uniformly convergent on the unit sphere, and because for \( r = \left| x \right| > 1 \) for \( x \in E_1 \), it is not hard to see from (16) that \( q_0 - q_\infty = o(r^{2-n}) \) and thus

\( Vq = o(r^{1-n}) \) in \( E \) since \( q \) is harmonic. Hence \( Vq \in L^2(E) \). On the other hand, since \( Vq = u_t - g \) and since \( u_t \in J_0(\Omega) \) and \( g \in J(\Omega) \),
it follows from Lemma 3 that \( \nabla q \in L^2 \) in \( \Omega \cap S_R \). Consequently, \( \nabla q \in L^2(\Omega) \) and our lemma is proved.

If \( \Omega \) is an exterior domain in \( \mathbb{R}^n \) with \( n > 2 \), we will replace \( q \) in (11) by \( q - q_o \) and will denote \( q - q_o \) by \( q \) for simplicity. In order to show that \( \nabla q \) is square-summable over the space-time region \( Q_{\varepsilon,T} \) for all \( 0 < \varepsilon < T < \infty \), we will obtain an estimate for \( \| \nabla q(\cdot,t) \|_\Omega \) uniformly in \( [\varepsilon,T] \). More precisely, we will prove

**Lemma 10.** Suppose \( \Omega \) is an exterior domain in \( \mathbb{R}^n \) with \( n > 2 \). Then there is a constant \( C > 0 \) such that, for all \( 0 < \varepsilon < T < \infty \),

\[
(17) \quad \| \nabla q \|_{Q_{\varepsilon,T}} \leq C \left( \| \nabla u_t \|_{Q_{\varepsilon,T}} + \| g \|_{Q_{\varepsilon,T}} \right).
\]

If \( u_t \in J_0(Q_T) \) for all \( T > 0 \), then \( \nabla q \in L^2(Q_T) \) for all \( T > 0 \).

**Proof.** Let \( S_{R_1,R_2} = \{ x \in \mathbb{R}^n : R_1 < |x| < R_2 \} \) be an arbitrary fixed annular domain such that \( S_{R_1,R_2} \subset \Omega \). We have shown in Lemma 9 that \( q \) is harmonic in \( \Omega \) and thus \( \nabla q \) is uniformly continuous in \( S_{R_1,R_2} \). Clearly, \( I(r) = \int_{\partial S_r} |\nabla q(x)|^2 \, d\sigma \) is continuous in \( [R_1,R_2] \) as a function of \( r \) and is nonnegative. By the well-known Mean Value Theorem there exists some \( R_o \in [R_1,R_2] \) such that
Observe that for any $R > R_o$

\begin{equation}
\int_{\partial S_{R_o}} q \nabla q \cdot \nu \, d\sigma = \int_{\partial S_{R_o}} q \nabla q \cdot \nu \, d\sigma + \int_{\partial S_{R}} q \nabla q \cdot \nu \, d\sigma ;
\end{equation}

$v$ being the unit outward normal vector. This identity is obtained through an integration by parts and by using the fact that $\Delta q = 0$.

The second integral on the right of (19) tends to zero as $R \to \infty$ since $q$ behaves like $1/r^{n-2}$ as $r \to \infty$. The first integral on the right of (19) can be estimated by using the Schwartz inequality and (18):

\begin{equation}
\left| \int_{\partial S_{R_o}} q \nabla q \cdot \nu \, d\sigma \right| \leq \left\{ \int_{\partial S_{R_o}} |q|^2 \, d\sigma \right\}^{1/2} \left\{ \int_{\partial S_{R_o}} |\nabla q|^2 \, d\sigma \right\}^{1/2} \leq \left\{ \frac{1}{R_2-R_1} \right\}^{1/2} \left\{ \int_{\partial S_{R_o}} |q|^2 \, d\sigma \right\}^{1/2} \left\{ \int_{S_{R_1}, R_2} |\nabla q|^2 \, dx \right\}^{1/2}
\end{equation}

In the last integral on the right of (20), we use (11), Minkowski's inequality and Lemma 3 step by step to get

\begin{equation}
\left\{ \int_{S_{R_1}, R_2} |q|^2 \, dx \right\}^{1/2} \leq C_1 \left\{ \int_{\Omega} |\nabla u| \, dx \right\}^{1/2} + \left\{ \int_{\Omega} |g|^2 \, dx \right\}^{1/2}
\end{equation}
where $C_1$ is some constant depending only on $R_1$ and $R_2$. Applying Lemma 5 and inequality (21) to the integrals on the right of (20), one obtains

\begin{equation}
\int_{\overline{S}_{R_o}} q |vq| \cdot v \omega \, \text{d}x \leq \left( \frac{1}{R_2 - R_1} \right)^{\frac{1}{2}} \left[ \frac{R_o}{n - 2} \right]^{\frac{1}{2}} \left( \frac{R}{\Omega} \right)^{\frac{1}{2}} \left( \frac{\int \|vq\|^2 \, \text{d}x}{\Omega} \right)^{\frac{1}{2}} \left[ C_1 \left( \int \|v_t\|^2 \, \text{d}x \right)^{\frac{1}{2}} + \left( \int \|g\|^2 \, \text{d}x \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}
\end{equation}

\begin{equation}
\leq K \|vq\| \left( \|v_t\| + |g| \right)
\end{equation}

where $K$ is a suitable constant independent of time $t$. Thus, letting $R \to \infty$ in (19) and applying inequality (22) gives

\begin{equation}
\int_{E_{R_0}} |vq|^2 \, \text{d}x \leq K \|vq\| \left( \|v_t\| + |g| \right)
\end{equation}

On the other hand, it is evident from (11) and Lemma 3 that

\begin{equation}
\int_{\Omega \cap \overline{S}_{R_0}} |vq|^2 \, \text{d}x \leq L \|vq\| \left( \|v_t\| + |g| \right)
\end{equation}

where $L$ is some constant independent of $t$. We can now combine (23) with (24) to obtain $|vq|_{\Omega} \leq C \left( \|v_t\|_{\Omega} + |g| \right)$, where $C = K + L$. Our assertion follows.
Now if \( u_t \in J_0(Q_T) \) for all \( T > 0 \), then \( \forall q \in L^2(Q_T) \)
for all \( T > 0 \). It follows from (11) that \( u_t \in L^2(Q_T) \) for all \( T > 0 \). This enables us to show that \( u \) possesses a finite \( L^2 \)

norm over \( Q_T \) for all \( T > 0 \) and that \( u \) assumes the initial data \( a \) in a \( L^2 \) sense by the same arguments as those in Lemma 3 of Heywood [4]. We give the proof here.

**Theorem 1.** Suppose that \( \Omega \) is an exterior domain in \( \mathbb{R}^n \) with \( n > 2 \) and that \( u \) is the class \( H_0 \) solution of equations (1) - (4) in which the initial data \( a \in H_0(\Omega) \cap L^2(\Omega) \) and the force \( f = f_1 + f_2 \) where \( f_1 \in J_0(Q_T) \cap L^2(Q_T) \) and \( f_2, f_2t \in L^2(Q_T) \) for all \( T > 0 \). Then \( u \) is square-summable over the space-time region \( Q_T \) and \( \lim_{t \to 0} ||u(\cdot, t) - a|| = 0 \).

**Proof.** Under the above hypotheses on the prescribed data, \( u_t \in J_0(Q_T) \) for all \( T > 0 \) (see the remark at the beginning of this section). We see readily from Lemma 10 that \( u_t \in L^2(Q_T) \) for all \( T > 0 \). It is enough to show \( ||u(\cdot, t) - a||_\Omega \leq t ||u_t||_{L^2(Q_T)} \)

for all \( 0 < t < T \). According to the definition of class \( H_0 \) solutions and Lemma 3, we have \( u \in L^2(\Omega' \times [0, T)) \) and \( \lim_{t \to 0} ||u(\cdot, t) - a||_{\Omega'} = 0 \)

for every bounded subset \( \Omega' \) of \( \Omega \). Hence for almost every \( x \in \Omega \) there holds \( u(x, t) - a(x) = \int_0^t u_t(x, \tau) \, d\tau \). Now, using the inequality \( ab \leq \frac{1}{2}(a^2 + b^2) \),
\[(u(x,t) - a(x))^2 = \int_0^t \int_0^t u_t(x,\tau) u_t(x,\alpha) \, d\alpha \, d\tau \]
\[\leq \frac{1}{2} \int_0^t \int_0^t [u_t^2(x,\tau) + u_t^2(x,\alpha)] \, d\alpha \, d\tau \]
\[= \int_0^t \int_0^t u_t^2(x,\tau) \, d\alpha \, d\tau;\]

thus \[||u(\cdot,t) - a||_{L^2}^2 \leq t||u_t||_{L^2(Q_t)}^2.\]

The above demonstration shows that the class \(H_0\) solution of equations (1) - (4) in an exterior domain \(\Omega \subset \mathbb{R}^n\) for \(n > 2\) does in fact satisfy the boundary condition at infinity in a \(L^2\) generalized sense. If \(\Omega \subset \mathbb{R}^2\), however, Heywood [3] showed that the boundary condition at infinity is not satisfied by the unique class \(H_0\) solution for some forces. The rest of this section is devoted to proving that the behavior at infinity of the class \(H_0\) solution in the case of a two-dimensional exterior domain is restricted in such a way that its time derivative necessarily tends to a definite limit as \(|x| \to \infty\) in a \(L^2\) generalized sense for each fixed time.

Lemma 11. Let \(\Omega\) be an exterior domain in \(\mathbb{R}^n\), \(n \geq 2\). Suppose that \(w \in C^1(\Omega)\), \(\nabla \cdot w = 0\) and that there exists a sequence of functions \(\{\phi_k\}\) in \(D(\Omega)\) such that \[||\phi_k - w||_{L^2(\Omega')} \to 0\] as \(k \to \infty\) for every bounded subset \(\Omega'\) of \(\Omega\). Then for any spherical surface \(\partial S_\rho\) of radius \(\rho\) enclosing \(\overline{\Omega}\), \(\phi \in D(\mathbb{R}^n)\) with \(\partial S_\rho \cap \text{circlediff}(n' = 2)\), we have \[\int_{\partial S_\rho} w \cdot \nu \, d\sigma = 0,\] where \(\nu\) is the unit outward normal and \(d\sigma\) denotes \(d\sigma_{\partial S_\rho}\).
the area of a surface element (if \( n = 2 \), \( d\sigma \) denotes the length of a line element).

**Proof.** Let the annular domain \( \{ x \in \mathbb{R}^n : \rho < |x| < \rho + 1 \} \) be denoted by \( A \). If \( n = 2 \), we introduce polar coordinates
\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta.
\]
The radial and angular components of \( w \), \( w_r \) and \( w_\theta \), are related to the Cartesian components by
\[
w_1 = w_r \cos \theta - w_\theta \sin \theta, \quad w_2 = w_r \sin \theta + w_\theta \cos \theta
\]
and it follows that \( |w|^2 = |w_r|^2 + |w_\theta|^2 \).

For the case of \( n > 2 \), we introduce spherical coordinates \((r, \theta_1, \ldots, \theta_{n-1})\) and we can check easily that \( |w|^2 \geq |w_r|^2 \). Thus, by hypothesis, one can find a sequence \( \{ \phi_k \} \) in \( D(\Omega) \) such that \( \| (\phi_k)_r - w_r \|_A \to 0 \) as \( k \to \infty \); therefore
\[
\lim_{k \to \infty} \int_A (\phi_k)_r \, dx = \int_A w_r \, dx
\]
by the Schwartz inequality. Furthermore, for each \( k = 1, 2, \ldots \), \( \phi_k \) is divergence free and is compactly supported, thus the integral
\[
\int_{\partial S^{\rho+\alpha}} (\phi_k)_r \, d\sigma = 0
\]
by the divergence theorem, where \( S_{\rho+\alpha} \) is a sphere (a disk, if \( n = 2 \)) centered at the origin of radius \( \rho + \alpha \), \( \alpha > 0 \). Hence
\[
\int_A (\phi_k)_r \, dx = \int_0^1 \int_{\partial S^{\rho+\alpha}} (\phi_k)_r \, d\sigma d\alpha = 0
\]
and it follows that \( \int_A w_r \, dx = 0 \). But \( w \in C^1(\Omega) \) and \( \nabla \cdot w = 0 \), one has, again by the divergence theorem,
\[
\int_{\partial S^\rho} w_r \, d\sigma = \int_{\partial S^{\rho+\alpha}} w_r \, d\sigma \quad \text{for any } \alpha > 0.
\]
Thus,
\[
\int_A w_r \, dx = \int_0^1 \int_{\partial S^{\rho+\alpha}} w_r \, d\sigma d\alpha
\]
and our result follows readily from the above arguments.
Lemma 12. Let $\Omega$ be a two-dimensional exterior domain whose complement $\Omega^c$ contains $S_{r_0}$ for some $r_0 > 0$ and let $t > 0$ be fixed. Then, in a neighborhood of infinity, the function $q(\cdot, t)$ in equation (11) can be expressed as a series in spherical harmonics which has the form

$$q = q_0 + (a_1 \cos \theta + b_1 \sin \theta) r + \sum_{k=1}^{\infty} (a_k \cos k \theta + b_k \sin k \theta) r^{-k}$$

where $q_0$, $a_1$, $b_1$, $a_k$, $b_k$, $k = 1, 2, \ldots$, are all constants.

Proof. With $u$ a class $H_0$ solution of (1) - (4), it has been shown in Lemma 9 that, for fixed $t > 0$, the function $q(\cdot, t)$ in equation (11) is harmonic in $\Omega$. Thus, in a neighborhood of infinity, say $E \subset \subset \Omega$, $q$ has a series expansion of the form

$$q(r, \theta) = q_0 + \tilde{q}_0 \log r + \sum_{k=1}^{\infty} \left[(a_k r^{-k} + \tilde{a}_k r^{-k}) \cos k \theta + (b_k r^{-k} + \tilde{b}_k r^{-k}) \sin k \theta\right]$$

which may be differentiated term by term. We will now show that the coefficients $\tilde{a}_k$ and $\tilde{b}_k$ vanish for all $k > 1$. It is clear that

$$\int_{|x| \leq R} \frac{\left(\frac{\partial q}{\partial r}\right)^2}{r^2 \log^2 (r/r_0)} dx = 2\pi \int_{r_0}^{R} \frac{q_0^2}{r^3 \log^2 (r/r_0)} dr$$

$$+ \pi \sum_{k=1}^{\infty} \int_{\rho}^{R} \frac{(-k \tilde{a}_k r^{-k-1} + k \tilde{a}_k r^{-k-1})^2}{\log^2 (r/r_0)} dr$$
\[ + \pi \sum_{k=1}^{\infty} \int_{\rho} \frac{(-k b_k r^{-k-1} + k b_k r^{k-1})^2}{r \log^2(r/r_o)} \, dr. \]

Now since \( r_o > r \), it is easy to see that the integral

\[ \int_{E_{\rho}} \frac{(\frac{3q}{\rho})^2}{r^2 \log^2(r/r_o)} \, dx \] is finite in virtue of \( |\nabla q|^2 \geq (\frac{3q}{\rho})^2 \); \( \nabla q = u_t - g \), and Lemma 2. Thus, \( a_k \) and \( b_k \) must be zero for all \( k > 1 \) and the expansion for \( q \) is reduced to

\[ q(r, \theta) = q_o + \tilde{q}_o \log r + (a_1 \cos \theta + b_1 \sin \theta)r + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta)r^{-k}, \]

where the coefficient \( \tilde{q}_o \) is given by \( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\partial q}{\partial \theta}(\rho, \theta) \rho \, d\theta = \frac{1}{2\pi} \int_{S_{\rho}} \frac{3q}{\partial \nu} \, d\sigma. \)

It remains to show that \( \tilde{q}_o = 0 \). Evidently \( \nabla q \in C^\infty(\Omega) \) and \( \nabla \cdot \nabla q = 0 \). We will show that there exists a sequence \( \{\phi_k\} \) in \( D(\Omega) \) such that \( ||\phi_k - \nabla q||_{\Omega'} \to 0 \) for every bounded subset \( \Omega' \) of \( \Omega \). Since \( u_t \in J_0(\Omega) \) and \( g \in J(\Omega) \), there are sequences \( \{\xi_k\} \) and \( \{\eta_k\} \) in \( D(\Omega) \) such that \( ||\nabla \xi_k - \nabla u_t||_{\Omega} \to 0 \) and \( ||\eta_k - g||_{\Omega} \to 0 \) as \( k \to \infty \). Because of Lemma 3, \( ||\xi_k - u_t||_{\Omega'} \to 0 \) as \( k \to \infty \); thus \( ||\xi_k - \eta_k - \nabla q||_{\Omega'} \to 0 \) as \( k \to \infty \) since \( \nabla q = u_t - g \). Hence the functions \( \phi_k = \xi_k - \eta_k \), \( k = 1, 2, \ldots \), form the desired sequence.

In view of Lemma 11, we have thus \( \tilde{q}_o = \frac{1}{2\pi} \int_{S_{\rho}} \nabla q \cdot \nu d\sigma = \frac{1}{2\pi} \int_{S_{\rho}} \frac{3q}{\partial \nu} \, d\sigma = 0. \)

This completes the proof.
From Lemma 12, we see that
\[ \nabla q = w_o + \nabla \left( \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) r^{-k} \right) \]
where \( w_o = (a_1, b_1) \). Since the second term on the right behaves like
\[ \frac{1}{r^2} \]
as \( r \to \infty \), the integral \( \int_{|x|>\rho} |\nabla q - w_o|^2 \, dx \) is convergent. On the other hand, it is clear that \( \int_{\Omega \cap \{|x| \leq \rho\}} |\nabla q - w_o|^2 \, dx \) is finite.

Thus, \( \int_{\Omega} |\nabla q - w_o|^2 \, dx < \infty \) which in turn implies that \( \int_{\Omega} |u_t - w_o|^2 \, dx < \infty \).

We have proved

**Theorem 2.** Let \( \Omega \) be an exterior domain in \( \mathbb{R}^2 \) such that \( \Omega^c \) contains \( \overline{S}_{r_o} \) for some \( r_o > 0 \). Suppose \( u \) is a class \( H_o \) solution of (1) - (4) with \( a \in J_o(\Omega) \) and \( f \in L^2(\Omega_T) \) for all \( T > 0 \). Then for each \( t > 0 \), there exists a constant vector \( w_o \) depending on \( t \) such that
\[ \int_{\Omega} (u_t - w_o)^2 \, dx < \infty. \]
4. Uniqueness of Solution Classes

If $\Omega$ is an exterior domain in $\mathbb{R}^n$ with $n > 2$, it was shown in Section 3 that the unique class $H_0$ solution of the nonstationary Stokes problem (1) - (4) is square-summable, if the prescribed data are, despite the fact that the class $H_0$ theory is developed without using energy estimates. In this section, we show further that, if $\Omega$ is an exterior domain in $\mathbb{R}^3$, the class $H_0$ solution is identical with the solutions of various finite energy classes.

We first consider a class of finite energy solutions treated by Ladyzhenskaya in [6, pp. 81-104]. To define such a class of solutions for (1) - (4) in the case of an unbounded spatial domain $\Omega \subset \mathbb{R}^n$ ($n > 2$), it is necessary to introduce an auxiliary linear operator $\Delta_1$ related to the following linear stationary problem is related to the following linear stationary problem:

\begin{align*}
\Delta u - u &= \nabla p + f \quad \text{in } \Omega \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
u &\to 0 \quad \text{as } |x| \to \infty.
\end{align*}
Given \( f \in L^2(\Omega) \); we say that \( u \) is a generalized solution of equations (26) - (29) if and only if \( u \in J_1(\Omega) \) and \( \int_\Omega (\nabla u : \nabla \phi + u \cdot \phi + f \cdot \phi) \, dx = 0 \) holds for all \( \phi \in J_1(\Omega) \). It is not hard to see that \( \int_\Omega f \cdot \phi \, dx \) defines a bounded linear functional on all \( \phi \in J_1(\Omega) \). An application of the Riesz representation theorem (see for example Taylor [11,p.245]) proves the existence of a unique generalized solution for (26) - (29).

Now, we define the operator \( \tilde{\Delta}_1 \) as follows. Let \( \mathcal{D} \) denote the set of all generalized solutions of (26) - (29) corresponding to \( f \)'s which belong to \( J(\Omega) \). Define a map \( \tilde{\Delta}_1 : \mathcal{D} \to J(\Omega) \) by setting \( \tilde{\Delta}_1 u = f \) where \( u \in \mathcal{D} \) is the solution of (26) - (29) corresponding to \( f \in J(\Omega) \). It is clear that \( \tilde{\Delta}_1 \) is well defined, one-one, and \( \mathcal{D} \subseteq J_1(\Omega) \).

Moreover, \( \tilde{\Delta}_1 \) is a closed operator on \( \mathcal{D} \). In fact, if \( \{u_j\} \) is a sequence in \( \mathcal{D} \) such that \( u_j \) tends to \( u \) in \( J_1(\Omega) \) and that \( \tilde{\Delta}_1 u_j \) tends to \( f \) in \( L^2(\Omega) \) as \( j \to \infty \), then for each \( j = 1, 2, \ldots \) the equation \( \int_\Omega (\nabla u_j : \nabla \phi + u_j \cdot \phi) \, dx + \int_\Omega \tilde{\Delta}_1 u_j \cdot \phi \, dx = 0 \) holds for all \( \phi \in J_1(\Omega) \).

By taking the limits as \( j \to \infty \), one obtains \( \int_\Omega (\nabla u : \nabla \phi + u \cdot \phi + f \cdot \phi) \, dx = 0 \) for all \( \phi \in J_1(\Omega) \). This implies that \( u \in \mathcal{D} \) and that \( \tilde{\Delta}_1 u = f \).

Set \( \mathcal{D}_1 = \{ \xi(x,t) : \xi(x,t) = \phi(x) + \int_0^t \psi(x,t) \, dt, \text{ where} \phi, \psi(\cdot,t) \in \mathcal{D} \text{ for every t and where } \psi, \tilde{\Delta}_1 \psi, \text{and } \nabla \psi \text{ depend continuously on t as elements in } L^2(\Omega) \} \). It is not hard to see that \( \mathcal{D}_1 \) is a subset of \( J_1(Q_T) \) for all \( T > 0 \). Given \( a \in L^2(\Omega) \) and \( f \in L^2(Q_T) \) for all \( T > 0 \); we call \( u \) a class J solution of equations (1) - (4) if,
for all $T > 0$, $u \in J(Q_T)$ and $v = u e^{-t}$ satisfies the identity

\[ \int_{Q_T} v \cdot (\phi_t + \Delta_x \phi) \, dx \, dt + \int_{Q_T} f e^{-t} \cdot \phi \, dx \, dt + \int_{\Omega} \phi(x,0) a(x) \, dx = 0 \]

for all $\phi \in D_1$ with $\phi(\cdot,T) = 0$. This identity can be obtained formally by substituting $v = u e^{-t}$ in equation (1), multiplying the resulting equation by $\phi(x,t) \in D_1$ with $\phi(\cdot,T) = 0$, integrating over $Q_T$, and carrying out integrations by parts with respect to $t$ and $x$.

Whether the class $J$ solutions are unique is not readily seen from the above definition. The uniqueness proof below follows the same idea as that for the case of a bounded domain presented in [6].

**Proposition 1.** Let $a \in L^2(\Omega)$ and $f \in L^2(Q_T)$ for all $T > 0$. Equations (1) - (4) have at most one class $J$ solution.

**Proof.** Suppose that $u_1$ and $u_2$ are two class $J$ solutions of (1) - (4). Let $w = u_1 e^{-t} - u_2 e^{-t}$. It follows from the definition of class $J$ solutions that $w \in J(Q_T)$ for all $T > 0$ and that

\[ \int_{Q_T} w \cdot \phi_t \, dx \, dt = 0 \quad \text{for all } \phi \in D_1 \text{ with } \phi(\cdot,T) = 0. \]

We assert that if $g(x,t) \in D$ and if $\int_0^t g(x,\tau) \, d\tau \in D$ for every $t$, then
(32) \[ \int_0^t \Delta_1 g(x, \tau) \, d\tau = \int_0^t \Delta_1 g(x, \tau) \, d\tau. \]

Indeed for any \( \eta \in D(\Omega) \),

\[
\int_\Omega \Delta_1 \int_0^t g(x, \tau) d\tau \cdot \eta(x) \, dx = -\int_\Omega (\nabla \int_0^t g(x, \tau) d\tau \cdot \nabla \eta(x) + \int_0^t g(x, \tau) d\tau \cdot \eta(x)) \, dx
\]

\[
= \int_\Omega \int_0^t \Delta_1 g(x, \tau) d\tau \cdot \eta(x) \, dx,
\]

and our assertion follows. Now set \( \psi = \Delta_1^{-1} (\int_0^t w(x, \tau) \, d\tau) \). For each \( t > 0 \), \( \psi(x, t) \in D \) since \( \int_0^t w(x, \tau) \, d\tau \in L(\Omega) \). So does \( \int_0^t \Delta_1^{-1} w(x, \tau) \, d\tau \) belong to \( D \). It is then seen from equation (32) that

\[
\int_0^t \Delta_1^{-1} w(x, \tau) \, d\tau = \Delta_1^{-1} \int_0^t w(x, \tau) \, d\tau = \psi(x, t).
\]

Hence \( \psi_t = \Delta_1^{-1} w \) and \( \Delta_1(\psi_t) = w = (\Delta_1 \psi)_t \). Substituting \( \phi(x, t) = \int_0^t \psi(x, \tau) \, d\tau \) into equation (31), we get

\[
\int_Q \Delta_1 \psi_t (\psi + \int_0^T \Delta_1 \psi(x, \tau) \, d\tau) \, dx \, dt =
\int_Q \Delta_1 \psi_t \, \psi \, dx \, dt + \int_Q \Delta_1 \psi_t \, \int_0^T \Delta_1 \psi(x, \tau) \, d\tau \, dx \, dt = 0.
\]

Applying the definition of \( \Delta_1 \) to the first integral on the right and integrating by parts with respect to \( t \) in the second integral on the right, one obtains
\begin{equation}
0 = -\int_{0}^{T} (\nabla \psi_t : \nabla \psi + \psi_t \cdot \psi) dx dt + \int_{\Omega} (\tilde{\Delta} \psi, \int_{T}^{T} \tilde{\Delta} \psi d\tau) |_{0}^{T} dx - \int_{0}^{T} (\tilde{\Delta} \psi)^2 dt dx
\end{equation}

\begin{equation}
= -\frac{1}{2} (||\psi(x,T)||^2 + ||\psi(x,T)||^2) - \int_{0}^{T} ||\tilde{\Delta} \psi||_x^2 dt .
\end{equation}

It follows that \( \tilde{\Delta} \psi(x,t) = 0 \) for all \( t \in (0,T) \) and for all \( x \in \Omega \); this in turn implies that \( w = 0 \). Our proof is complete.

Let \( \Omega \) be an exterior domain in \( \mathbb{R}^3 \). Suppose that the initial data \( a \) belongs to \( H_0(\Omega) \cap L^2(\Omega) \) and that the prescribed force \( f \) equals \( f_1 + f_2 \) where \( f_1 \in J_0(Q_T) \cap L^2(Q_T) \) and \( f_2, f_2t \in L^2(Q_T) \) for all \( T > 0 \). We will show that the class \( H_0 \) solution of (1) - (4), whose existence and uniqueness are guaranteed by our hypotheses, is identical with the class \( J \) solution of (1) - (4). Instead of proving directly that these solutions are identical, it is convenient to introduce another class of finite energy solutions for (1) - (4), which we call class \( J_1 \) solutions, that are chiefly characterized by membership in the function space \( J_1 \) (see [4] or [6]). We will prove that every class \( J_1 \) solution is a class \( J \) solution, and that every class \( H_0 \) solution is a class \( J_1 \) solution under the above-mentioned hypotheses. The latter result is verified by combining our result on the square-summability of a class \( H_0 \) solution with a recent characterization of the function space \( J_1 \) obtained by Heywood [5].
Let \( \Omega \) be any domain in \( \mathbb{R}^n \) with \( n \geq 2 \). Given \( a \in L^2(\Omega) \) and \( f \in L^2(Q^*_T) \) for all \( T > 0 \). We say that \( u \) is a class \( J_1 \) solution of equations (1) - (4) if and only if for all \( T > 0 \), the following conditions are satisfied:

\[
\begin{align*}
(33) \quad u &\in J_1(Q_T) \quad \text{and} \quad u_t \in J(Q_T) \\
(34) \quad ||u(\cdot,t) - a|| &\to 0 \quad \text{as} \quad t \to 0^+ \\
(35) \quad u_{x_1x_j} &\in L^2_{\text{loc}}(Q_\infty) \quad \text{and} \quad u_t - \Delta u = -\nabla p + f \quad \text{holds a.e. for some scalar function} \quad p \in L^2_{\text{loc}}(Q_\infty). 
\end{align*}
\]

Condition (35) holds if and only if

\[
(35)' \quad \int_{Q_T} (u_t \phi + \nabla u : \nabla \phi - f\phi) \, dx \, dt = 0
\]

holds for all \( \phi \in J_1(Q_T) \) and all \( T > 0 \). For if (35)' holds, then \( u \) has second order derivatives \( u_{x_1x_j} \in L^2_{\text{loc}}(Q_\infty) \) (see for example Lemma 3 of [5]) and we get from (35)', through an integration by parts, \( \int_{Q_T} (u_t - \Delta u - f)\phi \, dx \, dt = 0 \) for all \( \phi \in D(Q_T) \). Thus (35) is obtained. On the other hand, it is an easy matter to verify the reverse implication.

If \( a \in J_1(\Omega) \) and \( f \in L^2(Q^*_T) \) for all \( T > 0 \), one can show that equations (1) - (4) possess a class \( J_1 \) solution (and thus a class \( J \)
solution by the following proposition) by the method of Galerkin's approximation. Details are omitted. It is also easy to prove directly that the class \( J_1 \) solutions of (1) - (4) are unique.

**Proposition 2.** Let \( \Omega \) be an unbounded domain in \( \mathbb{R}^n \), \( n \geq 2 \).

Suppose that \( a \in L^2(\Omega) \), that \( f \in L^2(Q_T) \) for all \( T > 0 \), and that \( u \) is a class \( J_1 \) solution of (1) - (4). Then \( u \) is also a class \( J \) solution of (1) - (4).

**Proof.** For all \( T > 0 \), \( u \in J(Q_T) \) since \( u \in J_1(Q_T) \) and since \( J_1(Q_T) \subset J(Q_T) \). Substituting \( ve^t \) for \( u \) in the equation of (35), multiplying the resulting equation by a function \( \phi \in D(Q_T) \), \( T > 0 \), and integrating over \( Q_T \), we obtain

\[
\int_{Q_T} (v_t \phi - (\Delta v - v) \phi) \, dx \, dt = \int_{Q_T} fe^{-t} \phi \, dx \, dt
\]

Integrating by parts with respect to \( x \) yields

\[
(36) \quad \int_{Q_T} \left( v_t \phi + \nabla v : \nabla \phi + v \phi \right) \, dx \, dt = \int_{Q_T} fe^{-t} \phi \, dx \, dt
\]

Identity (36) holds for all \( \phi \in J_1(Q_T) \) since \( D(Q_T) \) is dense in \( J_1(Q_T) \) in the norm \( || \cdot ||_{1,Q_T} \). In particular, it holds for all \( \phi \in D_1 \). Applying the identity \( (\nabla v, \nabla \phi) = -(v, \Delta_1 \phi) \) and integrating by parts with respect to \( t \) in (36), we obtain
for all \( \phi \in \mathcal{D}_1 \) with \( \phi(\cdot, T) = 0 \). The proposition is proved.

From now on, we assume that the spatial domain \( \Omega \) satisfies the following condition:

\( \Omega \) is an exterior domain in \( \mathbb{R}^3 \) which has an exterior subdomain \( D \) with a class \( C^2 \) boundary \( \partial D \subset \Omega \), such that

\[(*)\] the region \( \Omega - D \) is covered by nonintersecting normals to \( \partial \Omega \) (we assume that the normals do not intersect at points of \( \partial \Omega \)).

The following characterization of the function spaces \( J_1(\Omega) \) and \( J_0(\Omega) \) for such \( \Omega \) is due to Heywood [5].

**Proposition 3.** If \( \tilde{W}_1(\Omega) \) denotes the completion of \( C_0^\infty(\Omega) \) in the norms \( ||\cdot||_1 \), and if \( J_1^*(\Omega) \) denotes the collection of all \( \phi \) in \( \tilde{W}_1(\Omega) \) for which \( \nabla \cdot \phi = 0 \), then \( J_1(\Omega) = J_1^*(\Omega) \). If \( W_o(\Omega) \) denotes the completion of \( C_0^\infty(\Omega) \) in the norm \( ||\nabla \cdot \cdot \cdot || \) and if \( J_0^*(\Omega) \) denotes the collection of all \( \phi \) in \( W_o(\Omega) \) for which \( \nabla \cdot \phi = 0 \), then \( J_0(\Omega) = J_0^*(\Omega) \).

Now suppose that \( a \in H_0(\Omega) \cap L^2(\Omega) \) and that \( f = f_1 + f_2 \)

where \( f_1 \in J_0(Q_T) \cap L^2(Q_T) \) and \( f_2, f_{2t} \in L^2(Q_T) \) for all \( T > 0 \). Let \( u \) be a class \( H_0 \) solution of equations (1) - (4) with these prescribed
data. We have proved in Section 3 that \( u, u_t \in L^2(Q_T) \) for all \( T > 0 \), thus \( u \) is continuous in \( L^2(\Omega) \) as a function of \( t \) after redefinition on a set of \( t \) measure zero. Since, for every \( t > 0 \), \( u(\cdot, t) \in J_0(\Omega) \cap L^2(\Omega) \), we have \( u(\cdot, t) \in W^1_2(\Omega) \) by Lemma 2 of Heywood [4] which states that \( W_0(\Omega) \cap L^2(\Omega) \subset W^1(\Omega) \). If \( \Omega \) is an exterior domain. Furthermore, \( \nabla \cdot u = 0 \) and \( u \) is continuous in the norm \( || \cdot ||_1 \) as a function of \( t \). Hence, it follows from Proposition 3 that \( u \in J_1(Q_T) \) for all \( T > 0 \). The remaining conditions for \( u \) to be a class \( J_1 \) solution are easily verified.

We have therefore proved

**Theorem 3.** Let \( \Omega \) be an exterior domain in \( \mathbb{R}^3 \) described above. Suppose that \( a \in H^1_0(\Omega) \cap L^2(\Omega) \) and that \( f = f_1 + f_2 \) where \( f_1 \in J_0(Q_T) \cap L^2(Q_T) \) and \( f_2, f_2t \in L^2(Q_T) \) for all \( T > 0 \). Then the unique class \( H^1_0 \) solution of equations (1) - (4) is a class \( J_1 \) solution and hence also a class \( J \) solution of (1) - (4).
5. The Case of Nonhomogeneous Boundary Values

In this section, we assume that \( \Omega \) is the exterior of a finite object in \( \mathbb{R}^3 \) satisfying condition (*) (Section 4), and that there exists an inertial reference frame in which the fluid velocities tend to zero far from the object. If the object moves with a velocity \( -b_\infty(t) \) relative to the inertial frame, a fictitious force \( b_\infty(t) \) will appear in the equations of motions when written in a coordinate frame attached to the object, and a fluid velocity \( b_\infty(t) \) will be imposed at infinity relative to this noninertial frame. Thus, the equations for this nonstationary Stokes flow are given by

\[
\begin{aligned}
(36) & \quad u_t - \Delta u = -\nabla p + b_\infty \quad \text{in } Q_\infty \\
(37) & \quad \nabla \cdot u = 0 \quad \text{in } Q_\infty \\
(38) & \quad u(x,0) = a(x) \quad \forall x \in \Omega \\
(39) & \quad u(x,t) \to b_\infty(t) \quad \text{as } |x| \to \infty.
\end{aligned}
\]

We will define a class \( \mathcal{H}_0 \) solution for (36) - (39) in such a way that the uniqueness theorem holds.
We assume that \( b_0(x,t) \in C^2(\partial \Omega \times [0,\infty)) \), that \( b_\infty(t) \in C^2[0,\infty) \), and that the prescribed data \( a, b_0, \) and \( b_\infty \) permit the boundary values to be extended into \( Q_\infty \) as a solenoidal function \( b(x,t) \) which satisfies:

\[
\begin{align*}
(40) & \quad b \in C^2(\bar{Q}_\infty), \quad \Delta b_t \text{ exists and } \Delta b_t \in L^2(Q_T) \text{ for all } T > 0, \\
(41) & \quad \text{a number } R > 0 \text{ exists such that } b(x,t) = b_\infty(t) \text{ for all } x \text{ with } |x| > R \text{ and all } t \in [0,\infty), \text{ and} \\
(42) & \quad a - b(\cdot,0) \in \mathcal{H}_0^1(Q) .
\end{align*}
\]

We call such extension \( b \) of the boundary values admissible.

Because \( b \in C^2(\bar{Q}_\infty) \) and because \( \nabla b, \Delta b, \nabla b_t \) vanish for \( |x| > R \), one finds readily that \( \nabla b, \Delta b, \nabla b_t \in L^2(\Omega \times [0,T)) \) for all \( T > 0 \).

We call \( u \) a class \( H_0^1 \) solution of (36) - (39) if \( u = v + b \) where \( b \) is an admissible extension of the boundary values into \( Q_\infty \) as described above and \( v \) is a class \( H_0^1 \) solution of equations

(1) - (4) with initial data \( a(x) - b(x,0) \) for all \( x \in \Omega \) and force

\( f \equiv \Delta b - b_t + b_\infty \). That is, \( v \) satisfies the conditions

\[
\begin{align*}
(43) & \quad v \in H_0^1(Q_T), \text{ and } v_t \in J_0^1(Q_{\varepsilon,T}) \text{ for all } 0 < \varepsilon < T < \infty \\
(44) & \quad |\nabla v(\cdot,t) - \nabla(a(\cdot) - b(\cdot,0))| \to 0 \text{ as } t \to 0^+ \\
(45) & \quad v_{x_j x_k} \in L^2_{\text{loc}}(Q_\infty) \text{ and for some scalar function } p \in L^2_{\text{loc}}(Q_\infty) \\
\text{with } \nabla p \in L^2_{\text{loc}}(Q_\infty), \text{ there holds } v_t - \Delta v = -\nabla p + \Delta b - b_t + b_\infty \text{ a.e.} \text{ in } Q_\infty .
\end{align*}
\]
Since \( b \in C^2(\mathbb{Q}_\infty) \) and \( b(x,t) = b_\infty(t) \) for all \( |x| > R \) and all \( t \in [0,\infty) \), it is easy to see that \( \zeta = -b + b_\infty \) satisfies \( \zeta, \zeta_t \in L^2(\mathbb{Q}_T) \) for all \( T > 0 \). Further, the force \( f \) satisfies \( f, f_t \in L^2(\mathbb{Q}_T) \) for all \( T > 0 \) because \( b \) satisfies \( \Delta b_t \in L^2(\mathbb{Q}_T) \) for all \( T > 0 \). It then follows from Theorem 3 of [2] that there exists a class \( H_0 \) solution \( \zeta \) of equations (36) - (39).

The following lemma, needed to prove the uniqueness theorem, provides a partial characterization of the space \( H_0(\Omega) \).

Lemma 13. If \( \zeta \in W^1_2(\Omega) \) and \( \nabla \cdot \zeta = 0 \) (or equivalently, \( \zeta \in J_1(\Omega) \) by Proposition 3) and if \( \zeta_{x_1} \chi_{x_1} \in L^2(\Omega) \), then \( \zeta \in H_0(\Omega) \). If, for all \( T > 0 \), \( b, b_t \in J_1(\mathbb{Q}_T) \), \( b_{x_1} \chi_{x_1} \), \( b_t \chi_{x_1} \chi_{x_1} \) \( \in L^2(\mathbb{Q}_T) \), and if \( b(x,0) \in H_0(\Omega) \cap L^2(\Omega) \), then \( b \in H_0(\mathbb{Q}_T) \) for all \( T > 0 \).

Proof. Let \( \alpha(t) \) be a smooth function on \([0,\infty)\) such that \( \alpha(0) = 0 \) and \( \alpha(t) = 1 \) for \( t \geq 1 \). We consider the function \( u(x,t) = \alpha(t)\zeta(x) \). Clearly, \( ||u(\cdot,t)|| \to 0 \) as \( t \to 0^+ \). It is also clear from our hypotheses that \( u \) and \( u_t \) belong to \( J_1(\mathbb{Q}_T) \) for all \( T > 0 \). Thus \( u \) is a class \( J_1 \) solution of equations (1) - (4) with zero initial data and force \( f = \alpha'(t)\zeta(x) - \alpha(t)\Delta\zeta(x) \). Since the force \( f \) satisfies the condition \( f, f_t \in L^2(\mathbb{Q}_T) \) for all \( T > 0 \), it follows from Theorem 3 that \( u \) is also a class \( H_0 \) solution of (1) - (4). This implies that \( \alpha(t)\zeta(x) \in H_0(\mathbb{Q}_T) \) for all \( T > 0 \); thus \( \zeta(x) \in H_0(\Omega) \).
The second assertion follows readily from Theorem 3 as well if we observe that $b$ is a class $J_1$ solution of (1) - (4) in which the initial value $b(x,0)$ and force $b_t - \Delta b$ satisfy the assumptions of Theorem 3.

Before we proceed to prove the uniqueness theorem, we state a lemma (see Heywood [5, Lemma 11]) which will be needed in our proof.

**Lemma 14.** Let $\Omega$ be an arbitrary open set of $\mathbb{R}^n$. Suppose that $u \in C(\overline{\Omega})$, that $u = 0$ on $\partial \Omega$, that $u$ has generalized first derivatives, and that the integrals $\int_{\Omega} u^2 \, dx$ and $\int_{\Omega} (\nabla u)^2 \, dx$ are finite. Then $u \in W^{1,2}(\Omega)$.

**Theorem 4.** Suppose the prescribed data $a$, $b$, $b_\infty$ permit an admissible extension $b$ of the boundary values into $Q_\infty$. Then equations (36) - (39) can have at most one class $H_0$ solution.

**Proof.** Let $b$ and $\overline{b}$ be any two admissible extensions of the boundary values into $Q_\infty$. We first note that $\Delta (b - \overline{b})_t \in L^2(Q_T)$ for all $T > 0$. Since $b - \overline{b} \in C^2(\overline{Q})$ and since $b - \overline{b}$ and $(b - \overline{b})_t$ vanish in a neighborhood of infinity for all $t \in [0,\infty)$, we see that, for every $t \in [0,\infty)$, $(b - \overline{b})(\cdot, t)$, $\nabla (b - \overline{b})(\cdot, t)$, $\Delta (b - \overline{b})(\cdot, t)$ belong to $L^2(\Omega)$, and that $b - \overline{b}$, $\nabla (b - \overline{b})$, $(b - \overline{b})_t$ and $\nabla (b - \overline{b})_t$ all belong to $L^2(\Omega \times [0,T))$ for all $T > 0$. By Lemma 14, $b - \overline{b}$, $(b - \overline{b})_t \in W^{1,2}(\Omega \times [0,T))$ for all $T > 0$ because they are continuous on
$\overline{\Omega}$, equal to zero on $\partial \Omega \times [0,\infty)$, and are bounded in norm $||\cdot||_1$ uniformly in $0 \leq t \leq T$. In virtue of Proposition 3, $b - \overline{b}$ and $(b - \overline{b})_t$ belong to $J^1(Q_T)$ for all $T > 0$. Moreover, we have, by Lemma 14 and the first part of Lemma 13, $(b - \overline{b})(\cdot,t) \in \mathbb{H}^0(\Omega)$ for every $t \geq 0$ and in particular $(b - \overline{b})(\cdot,0) \in \mathbb{H}^0(\Omega)$. Thus the second part of Lemma 13 implies that $b - \overline{b} \in \mathbb{H}^0(Q_T)$ for all $T > 0$.

Now let $u = v + b$ be a class $\mathbb{H}^0$ solution of (36) - (39) where $b$ is an admissible extension of the boundary values into $Q_\infty$ and where $v$ satisfies conditions (43) - (45). Let $\overline{b}$ be any admissible extension of the boundary values and set $\overline{v} = u - \overline{b} = v + b - \overline{b}$. We assert that $\overline{v}$ and $\overline{b}$ satisfy conditions (43) - (45).

Indeed, we have shown in the previous paragraph that $b - \overline{b} \in \mathbb{H}^0(Q_T)$ and $(b - \overline{b})_t \in J^1(Q_T)$ for all $T > 0$, and thus $\overline{v} \in \mathbb{H}^0(Q_T)$ and $\overline{v}_t \in J^0(Q_T)$ for all $0 < \varepsilon < T < \infty$. Moreover, it is easy to verify that $||\nabla \overline{v}(\cdot,t) - \nabla v + \nabla \overline{b}(\cdot,0)|| \to 0$ as $t \to 0^+$ and that $\overline{v}_t - \Delta \overline{v} = -\nabla p + \Delta b - \frac{b_t}{t} + b_{\infty t}$ holds a.e. for some scalar function $p \in L^2_{\text{loc}}(Q_{\infty})$ with $\nabla p \in L^2_{\text{loc}}(Q_{\infty})$. This proves our assertion. Next suppose that $u$ and $\overline{u}$ are two class $\mathbb{H}^0$ solutions of (36) - (39) and that $b$ is any admissible extension of the boundary values. By what we have just shown, $v = u - b$ and $\overline{v} = \overline{u} - b$ both are class $\mathbb{H}^0$ solution of equations (1) - (4) with force $b_{\infty t} - b_t + \Delta b$ and initial value $a - b(\cdot,0)$. Uniqueness of solutions of equations (1) - (4) implies that $v \equiv \overline{v}$ and thus $u \equiv \overline{u}$. 
Suppose that the prescribed initial and boundary values \( a, b_0, b_\infty \) admit an extension \( b \) into \( Q_\infty \) satisfying (40), (41) such that \( a - b(\cdot,0) \in L^2(\Omega) \). We call \( u \) a class \( J_1 \) solution of (36) - (39) if \( u = v + b \) in which \( v \) is a class \( J_1 \) solution of (1) - (4) with force \( -b_{\infty t} - b_t + \Delta b \) and initial value \( a - b(\cdot,0) \). It can be deduced from the proof of Theorem 4 that if \( u - b \in J_1(Q_T) \) for all \( T > 0 \) for some extension \( b \) then \( u - b \in J_1(Q_T) \) for all \( T > 0 \) for every such extension \( b \). Again, following the same method of proof as in Theorem 4, one can show that the class \( J_1 \) solutions of (36) - (39) are unique. We have

**Theorem 5.** If the prescribed initial and boundary values \( a, b_0, b_\infty \) admit an extension \( b \) into \( Q_\infty \) satisfying (40), (41) such that \( a - b(\cdot,0) \in L^2(\Omega) \), then equations (36) - (39) have a unique class \( J_1 \) solution. If, in addition, the initial value \( a \) satisfies \( a \in H^1(\Omega) \) \( \cap L^2(\Omega) \), then equations (36) - (39) have a unique class \( H^1_0 \) solution and it is identical with the unique class \( J_1 \) solution.

**Proof.** The last part of this theorem follows from Theorem 3 whereas the remaining assertions are obvious.

In [2, Theorems 4, 5 and 6] Heywood has studied the convergence of solutions of some nonstationary flow problems to the solution of a steady flow problem which describes the physical
situation of a fluid occupying the exterior of an object, adhering to the object's boundary, with its velocity tending to a constant prescribed vector at infinity. Similar to his method, we study the attainability as limits of nonstationary solutions of solutions of the steady flow problem

\begin{align}
\Delta w &= \nabla p \quad \text{in } \Omega \\
\nabla \cdot w &= 0 \quad \text{in } \Omega \\
w(x) &= w_o \quad \text{for } x \in \partial \Omega \\
w(x) &\to w_\infty \quad \text{as } |x| \to \infty
\end{align}

in which $w_o$ and $w_\infty$ are constant prescribed vectors. We first define the generalized solutions of equations (46) - (48). Let $\xi$ be a function defined in $\Omega$ such that

\begin{equation}
\xi \text{ is smooth, solenoidal, equal to } w_o \text{ in a neighborhood of } \partial \Omega, \text{ and equal to } zw_\infty \text{ in a neighborhood of infinity.}
\end{equation}

We call $w$ a generalized solution of (46) - (48) if and only if

(i) $w = w - \xi \in H_0^1(\Omega)$ for some $\xi$ with the properties just described, and (ii) $\Delta w = \nabla p$ holds a.e. for some scalar $p \in L^2_{\text{loc}}(\Omega)$ with $\nabla p \in L^2_{\text{loc}}(\Omega)$.

We claim that generalized solutions are unique. Clearly the difference of any two functions which satisfy (49) belongs to $D(\Omega)$. Thus, if $w_1$ and $w_2$ are any two generalized solutions of (46) - (48) and if $\xi$ is any
function satisfying (49), then \( w_1 - \xi \in H_0(\Omega) \) and \( w_2 - \xi \in H_0(\Omega) \).

It follows that \( w_1 - w_2 \in H_0(\Omega) \) and \( \Delta (w_1 - w_2) = \nabla p \) for some \( p \in L^2_{\text{loc}}(\Omega) \) with \( \nabla p \in L^2_{\text{loc}}(\Omega) \). Hence \( (\nabla (w_1 - w_2), \nabla \phi) = 0 \) for all \( \phi \in D(\Omega) \), and there follows \( w_1 \equiv w_2 \) by taking a limit for \( \phi = w_1 - w_2 \). Existence of a solution is also easily shown; since \( \Delta \xi \in D(\Omega) \), one can choose \( \bar{w} \in K_0(\Omega) \) such that \( \Delta \bar{w} = -\Delta \xi \).

We now show that the generalized solutions of (46) - (48) can be obtained as limits as \( t \to \infty \) of solutions of the nonstationary flow problem (36) - (39) in which the initial and boundary values are prescribed to satisfy:

\[
\tag{50}
\text{a number } T_0 > 0 \text{ exists such that, for all } t > T_0, \ b_0(x,t) = w_0 \text{ for all } x \in \partial \Omega \text{ and } b_\infty(t) = w_\infty, \text{ and}
\]

\[
\tag{51}
\text{the initial and boundary values admit an admissible extension } b \text{ defined in } \overline{Q}_\infty.
\]

We first state a result (Theorem 4 of [2]) which concerns with the behavior of class \( H_0 \) solutions as \( t \to \infty \).

**Proposition 4.** If \( a \in J_0(\Omega) \) and if \( f = f_1 + f_2 + f_3 \) where

(i) \( f_1 \in J_0(Q_T) \) for all \( T > 0 \), (ii) \( f_2, f_2t \in L^2(Q_T) \) for all \( T > 0 \), and (iii) \( f_3 = \nabla q \) for some \( q \in L^2_{\text{loc}}(Q_\infty) \) with \( \nabla q \in L^2_{\text{loc}}(\overline{Q}_\infty) \), then the class \( H_0 \) solution of (1) - (4) converges to zero in \( L^2_{\text{loc}}(\Omega) \) as \( t \to \infty \) provided \( \int_0^\infty \|\nabla f_1\|^2 \, dt \) and \( \int_0^\infty \|f_2\|^2 \, dt \) are finite. (It is worth noting that this proposition is valid for any domain \( \Omega \subset \mathbb{R}^n(n \geq 2) \).)
Next, suppose that \( \xi \) is a function which satisfies (49) and that \( b \) is an admissible extension of the boundary values \( b_0, b_\infty \) into \( Q_\infty \). It is evident that \( \tilde{b}(x,t) = (1-\alpha(t))b(x,t) + \alpha(t)\xi(x) \) is also an admissible extension of \( b_0, b_\infty \) into \( Q_\infty \); here \( \alpha \) is a twice continuously differentiable real-valued function defined for all \( t \geq 0 \) which is equal to 0 for \( t \leq T_0 \) and equal to 1 for \( t \geq T_0 + 1 \). Now suppose \( u \) is the class \( H_0 \) solution of (36) - (39). Setting \( \tilde{v} = u - \tilde{b} \), we see that \( \tilde{v} \) and \( \tilde{b} \) satisfy (43) - (45). If \( w = \overline{w} + \xi \) is the generalized solution of (46 - 48), then \( \tilde{v} - \overline{w} \) is a class \( H_0 \) solution of (1) - (4) with initial value \( (a(x) - b(x,0)) - \overline{w}(x) \) for all \( x \in \Omega \) and force \( f = \Delta(b - \xi) - \tilde{b}_t + b_\infty \). We assert that \( f \) fulfills the hypotheses of Proposition 6, from which it follows that \( \tilde{v}(\cdot,t) \rightarrow \overline{w} \) in \( L^2(\Omega) \) as \( t \rightarrow \infty \).

First, we note that \( \Delta(b - \xi), \Delta(\tilde{b} - \xi) \in L^2(Q_T) \) for all \( T > 0 \) and that \( \int_0^\infty \|\Delta(b - \xi)\|^2 \, dt < \infty \) because \( b(x,t) = \xi(x) \) for all \( x \in \Omega \) and all \( t \geq T_0 + 1 \). Let \( \xi = \tilde{b}_t + b_\infty \). From the expression for \( \tilde{b} \), one readily finds that \( \xi(x,t) \) is equal to \( -\tilde{b}_t(x,t) + b_\infty(t) \) if \( t \leq T_0 \), equal to \( \alpha'(t)b(x,t) + (1-\alpha(t))b_\infty(x,t) \) if \( T_0 < t < T_0 + 1 \), and equal to 0 if \( t \geq T_0 + 1 \). It is then clear that \( \xi, \xi_t \in L^2(Q_T) \) for all \( T > 0 \) and that \( \int_0^\infty \|\xi\|^2 \, dt < \infty \) because \( \tilde{b}_t = b_\infty = 0 \) for all \( t \geq T_0 + 1 \). Thus our assertion is proved.

Further, since \( \tilde{b}(x,t) = \xi(x) \) for all \( x \in \Omega \) and all \( t \geq T_0 + 1 \), we can deduce that \( \|u(\cdot,t) - \overline{w}\|_{\Omega'} \rightarrow 0 \) as \( t \rightarrow \infty \) for every bounded subset \( \Omega' \) of \( \Omega \). We have therefore proved
Theorem 6. If \( a, b_0 \), and \( b_\infty \) are prescribed to satisfy (50), (51), then equations (36) - (39) have a unique class \( H_0 \) solution \( u \) and it converges to the generalized solution \( w \) of (46) - (48) as \( t \to \infty \) in the sense that \( ||u(\cdot,t) - w||_{\Omega'} \to 0 \) for every bounded subset \( \Omega' \) of \( \Omega \).

Corollary. If, in addition to the hypotheses of Theorem 6, the initial value \( a \) satisfies also the condition \( a - b(\cdot,0) \in H_0(\Omega) \cap L^2(\Omega) \), then the nonstationary problem (36) - (39) possesses a unique class \( J_1 \) solution and it converges in \( L^2_{loc}(\Omega) \) as \( t \to \infty \) to the generalized solution of the stationary problem (46) - (48).

Proof. This Corollary is a direct consequence of Theorem 5 and Theorem 6.

As a particular case, Theorem 6 and its corollary model the following physical experiment. Suppose an object is initially at rest in a three dimensional space filled with a Stokesian fluid. It is then smoothly accelerated until a given velocity is attained, after which it is kept in motion with the same velocity. Let the object's velocity relative to the inertial frame of initial rest be \( -\alpha(t)w_\infty \), where \( w_\infty \) is a constant vector and \( \alpha \) is a smooth function defined on \([0,\infty)\) such that \( \alpha(0) = 0 \) and \( \alpha(t) = 1 \) for all sufficiently
large t. Then equations (36) - (39) with \( a(x) \equiv 0 \), \( b_0(x,t) \equiv 0 \), and \( b_\infty(t) = \alpha(t)w_\infty \), describe the motion of the fluid in a reference frame attached to the object. This nonstationary problem has a unique class \( J_\perp \) solution, which is identical with the unique class \( H_0 \) solution, and it converges in \( L^2_{\text{loc}}(\Omega) \) as \( t \to \infty \) to the solution of the stationary problem (46) - (48) with boundary values 0 on \( \partial \Omega \) and \( w_\infty \) at infinity. Indeed, the prescribed values \( a \), \( b_0 \) and \( b_\infty \) satisfy conditions (50) and (51) if we construct an admissible extension as \( b(x;t) = \alpha(t)\xi(x) \), where \( \xi \) is smooth, solenoidal, equal to zero in a neighborhood of \( \partial \Omega \), and equal to \( w_\infty \) in a neighborhood of infinity. Moreover, we see that \( a - b(\cdot,0) \equiv 0 \). Thus our assertion follows from the corollary of Theorem 6.
References


