A STUDY OF THE SEQUENCE CATEGORY

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ABSTRACT

For a given abelian category $\mathcal{A}$, a category $\mathcal{E}$ is formed by considering exact sequences of $\mathcal{A}$. If one imposes the condition that a split sequence be regarded as the zero object, then the resulting sequence category $\mathcal{E}/\mathcal{S}$ is shown to be abelian. The intrinsic algebraic structure of $\mathcal{E}/\mathcal{S}$ is examined and related to the theory of coherent functors and functor rings. $\mathcal{E}/\mathcal{S}$ is shown to be the natural setting for the study of pure and copure sequences and the theory is further developed by introducing repure sequences. The concept of pure semi-simple categories is examined in terms of $\mathcal{E}/\mathcal{S}$. Localization with respect to pure sequences is developed, leading to results concerning the existence of algebraically compact objects. The final topic is a study of the simple sequences and their relationship to almost split exact sequences.

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INTRODUCTION

Category theory has drastically altered the face of ring theory. However there is still a great resistance to category theory as a bona fide branch of mathematics. One even has the algebraist (mis)using Mitchell's [21] embedding theorems of abelian categories into module categories to dismiss abelian category theory as esoteric ring theory.

Given suitable knowledge of abelian categories, the study of rings by examining the module category gives a firm foundation and structure to much of the existing theory and stimulates further research. However if abelian category theory were only slightly generalized ring theory, then this approach would be putting the cart before the horse.

There are two key features to this thesis which the reader should keep in mind. The first is to regard the thesis as a demonstration of applied elementary abelian category theory. A specific category, the sequence category, is introduced, and is examined as an abelian category. To some extent, module techniques can be mimicked, however attempts to represent this category as a module category fail. Indeed, the sequence category will not have a generator and will not be locally small (i.e., objects will not have just a set of subobjects) except in very special circumstances; in particular, it is not a Grothendieck category. The second feature will be the direct application of the results, gleaned from the sequence category, to the study of rings, accompanied by indirectly revealing that a suitable framework has been established into which various ring theoretical problems may be posed and solved. Hopefully fertile ground has been exposed.

The following is a quick breakdown of the contents. Chapter one is a short intuitive introduction to the object of study : the sequence category $\mathcal{E}/\mathcal{S}$. In chapter two, chain homotopy between exact sequences becomes the crucial factor in defining morphisms for the sequence category. The major result is to demonstrate explicitly that $\mathcal{E}/\mathcal{S}$ is an abelian category. The chapter concludes with propositions intended to give the 'flavour' or 'feel' for the algebra involved in working with sequences as objects, so that the reader will be comfortable with the mechanisms of this specific abelian category. In chapter three, the basic link between the underlying category and the sequence category is established by examination of the projectives in $\mathcal{E}/\mathcal{S}$.

Chapter four places the sequence category within the familiar ground of coherent functors. The equivalences established in this chapter should be kept in mind, so that any result concerning sequences can be formulated into functors. A torsion theory for $\mathcal{E}/\mathcal{S}$ is introduced in chapter five. The familiar concept of purity enters as the torsion free part of this theory. In chapter six the problem of characterizing those module categories, in which every object is a direct sum of finitely generated objects, is examined in the context of $\mathcal{E}/\mathcal{S}$.

The repure category is introduced and studied in chapter seven, as the torsion free part of a torsion theory, now using pure sequences as torsion.

(Co)localization is the major topic of chapter eight using the torsion theories of chapters five and seven. The major result shows the existence of the category of additive fractions with respect to pure sequences. This category turns out to be a functor category, and consequences of this fact are examined.

Chapter nine is somewhat of a diversion, relating the theory of functor rings to the functor category arising in Chapter eight. In chapter ten, the simple objects of $\mathcal{E}/\mathcal{S}$ are characterized and compared to almost split exact sequences.

CHAPTER 1

THE SEQUENCE CATEGORY

In the study of an arbitrary abelian category $\mathcal{A}$ the subcategories $\mathcal{P}$ and $\mathcal{I}$, projective and injective objects, figure prominently. If one establishes an adequate knowledge of either class, for example in certain module categories every projective is free, then in studying the structure of general objects one would like to dispense with 'projectiveness' or 'injectiveness' (these terms to be taken intuitively for the moment). The method of this disposal will be to pass to the additive quotient categories $\mathcal{A}/\mathcal{P}$ and $\mathcal{A}/\mathcal{I}$ (Chapter 3). However these quotient categories are rarely abelian (3.9), and this is a major stumbling block.

Another way to study $\mathcal{A}$ is to consider the category $\mathcal{E}$ of exact sequences of $\mathcal{A}$. $\mathcal{E}$ has objects exact sequences and a morphism is a triple of morphisms of $\mathcal{A}$ making the following diagram commutative :

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & = & E_1 \arrow[d] \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 & = & E_2
\end{tikzcd}$$

Although $\mathcal{E}$ naturally inherits an additive structure, it is never abelian except trivially when $\mathcal{A} = 0$. We will elaborate on a proof of this statement (Maclane [19], page 375). because it will give some insight into what follows.

<u>1.1</u> $\mathcal{E}$ is not abelian, unless $\mathcal{A} = 0$.

<u>Proof</u> a map of the form

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

will be epi in $\mathcal{E}$ if $B \twoheadrightarrow B'$ and $C \twoheadrightarrow C'$ are epi in $\mathcal{A}$. For suppose

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, two heads] & C \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] & B'' \arrow[r] & C'' \arrow[r] & 0
\end{tikzcd}$$

gives the zero map. Then since both $C \twoheadrightarrow C' \rightarrow C''$ and $B \twoheadrightarrow B' \rightarrow B''$ are zero, the epis in $\mathcal{O}\!\!/$ cancel to give $C' \rightarrow C'' = 0$ and $B' \rightarrow B'' = 0$. But then

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & B' \arrow[d] \\
A'' \arrow[r, tail] & B''
\end{tikzcd} \quad 0 \qquad \text{gives} \qquad \begin{tikzcd}
A' \arrow[d] & \\
A'' \arrow[r, tail] & B''
\end{tikzcd}$$ is the zero map,

and the monic can be cancelled, so $A' \rightarrow A''$ is also the zero map.

Dually, a map of the form

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, tail] & B \arrow[r] \arrow[d, tail] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

is monic in $\mathcal{E}$.

Hence the map $$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d, tail] & B' \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & B \arrow[r] & B \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

is both monic and epi. But there is only one map from the lower sequence to the upper, the zero morphism

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "0"] & B \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & B \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

so there can be no inverse map. Hence $\mathcal{E}$ cannot be abelian.//

The proof suggests both sequences of the form

$$0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow B \longrightarrow B \longrightarrow 0$$

be considered as zero objects, and so the natural sum $0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$, the canonical split sequence, should also be the zero object. Thus one is led to consider the quotient category $\mathcal{E}/\mathcal{S}$, where $\mathcal{S}$ is the subcategory of split sequences. Objects of $\mathcal{E}/\mathcal{S}$ are those of $\mathcal{E}$, but $\mathrm{HOM}_{\mathcal{E}/\mathcal{S}}(E_1,E_2) = \mathrm{HOM}_{\mathcal{E}}(E_1,E_2)/\mathcal{S}(E_1,E_2)$ where $\mathcal{S}(E_1,E_2)$ is the subgroup of morphisms that factor through a split sequence (the zero object). Properties of this subgroup will be given in Chapter 2 (Prop. 2.4).

$\mathcal{E}/\mathcal{S}$ is an abelian category (Chapter 2, Thm. 2.5), and if $\mathcal{A}$ has sufficient projectives (injectives) then there is a full embedding $\mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{E}/\mathcal{S}$ $(\mathcal{A}/\mathcal{I} \hookrightarrow \mathcal{E}/\mathcal{S})$ assigning to each object X a projective presentation $0 \to K \to P \to X \to 0$ (injective co-presentation $0 \to X \to I \to N \to 0$) (Thm. 3.6). Under this embedding, $\mathcal{A}/\mathcal{P}$ $(\mathcal{A}/\mathcal{I})$ becomes a (co)-resolving class of projectives (injectives) (see Chapter 3). One then has this curious process of eliminating projectiveness from $\mathcal{A}$ via the passage $\mathcal{A} \to \mathcal{A}/\mathcal{P}$, but then embedding $\mathcal{A}/\mathcal{P}$ as a resolving class of projectives in the larger abelian category $\mathcal{E}/\mathcal{S}$. Thus in some ways the difficulty of non-abelianness of $\mathcal{A}/\mathcal{P}$ is somewhat overcome in the embedding, and the embedding is quite efficient because the study of projective objects is a tractable one.

CHAPTER 2

BASIC FACTS CONCERNING $\mathcal{E}/\mathcal{S}$

In this chapter we show $\mathcal{E}/\mathcal{S}$ is abelian and investigate various consequences of this. The aim is to work in $\mathcal{E}/\mathcal{S}$ and translate results to $\mathcal{A}$, so we will develop the algebra of $\mathcal{E}/\mathcal{S}$, explicitly illustrating the abelian concepts of kernels, cokernels, sums, products, intersections, etc.

The following three lemmas are recorded for reference (they arose in the study of homological algebra, but in essence are statements reflecting the abelian structure of $\mathcal{E}/\mathcal{S}$).

<u>Lemma 2.1</u> (Hilton and Stammbach [15], page 83)

Given

$$\begin{tikzcd} C \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & A \arrow[d, "\beta_2"] \\ B \arrow[r, "\beta_1"'] & D \end{tikzcd}$$

then $(0 \longrightarrow)\ C \xrightarrow{(\alpha_1,\alpha_2)} A \oplus B \xrightarrow{\binom{\beta_1}{-\beta_2}} D\ [\longrightarrow 0]$

is exact iff the square is a (pull-back),[push-out].

<u>Lemma 2.2</u> (Hilton and Stammbach [21], page 84)

If

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows, then the right-hand square is a pull-back <u>and</u> a push-out.

<u>Lemma 2.3</u> (Mitchell [ ], page 163)

Any $\underline{A} \to \underline{B}$ in $\mathcal{E}$ has a factorization

$$\begin{tikzcd} 0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & 0 & & \underline{A} \arrow[dd] \\ 0 \arrow[r] & B'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 & & \\ 0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0 & & \underline{B} \end{tikzcd}$$

.

That is, $\underline{A} \to \underline{B}$ can be factored as a push-out (of $\begin{tikzcd}[row sep=small] A'' \arrow[r] \arrow[d] & A \\ B'' \end{tikzcd}$)

followed by a pull-back (of $\begin{tikzcd}[row sep=small] & A' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$). This will be the epi-monic factorization of A given morphism in $\mathcal{E}/\mathcal{S}$.

Recall that $\mathrm{HOM}_{\mathcal{E}/\mathcal{S}}(\underline{A},\underline{B}) = \mathrm{HOM}_{\mathcal{E}}(\underline{A},\underline{B}) \,/\, \mathcal{S}(\underline{A},\underline{B})$

where $\mathcal{S}(\underline{A},\underline{B})$ was the subgroup of morphisms factoring through split sequences. The next proposition gives the basic facts concerning such morphisms. Since the objects of $\mathcal{E}$, being exact sequences, can be thought of as (short) chain complexes, the notion of homotopy naturally arises.

<u>Proposition 2.4</u>

Given

$$\begin{tikzcd} \underline{A} \arrow[d, "\underline{f}"'] \\ \underline{B} \end{tikzcd} : \begin{tikzcd} 0 \arrow[r] & A'' \arrow[r, "\alpha"] \arrow[d, "f''"'] & A \arrow[r, "\alpha'"] \arrow[d, "f"'] \arrow[dl, dashed, "g"'] & A' \arrow[r] \arrow[d, "f'"] \arrow[dl, dashed, "h"'] & 0 \\ 0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0 \end{tikzcd} ,$$

the following are equivalent :

(i) there exists g such that $g\alpha = f''$

(ii) there exists h such that $\beta' h = f'$

(iii) there exists g and h such that $\beta g + h\alpha' = f$

(iv) $\underline{f}$ factors through a split exact sequence

(v) $\underline{F}$ is chain homotopic to zero ($\underline{f} \sim 0$).

<u>NOTE</u> : (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) Fieldhouse [6].

(i) $\Leftrightarrow$ (iv) Freyd [8] .

<u>Proof</u> It suffices to show (i) $\Leftrightarrow$ (iv) for then (ii) $\Leftrightarrow$ (iv) is proved dually, and (i) with (ii) $\Leftrightarrow$ (iii) is clear, finally (i), (ii), (iii) constitute (v).

(iv) $\Rightarrow$ (i)

$$\begin{tikzcd}[row sep=small] 0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r] \arrow[d] & C \oplus D \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0 \end{tikzcd}$$

The required g is achieved via the projection $C \oplus D \to C$.

(i)$\Rightarrow$(iv) Consider

$$\begin{tikzcd}
A'' \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left, "g"] & \\
B'' \arrow[r] \arrow[drr, equal] & E \arrow[dr, dashed, "K"] & \\
& & B''
\end{tikzcd}$$

where the square is a push-out. K exists to give a commutative diagram. Hence $B'' \longrightarrow E$ is split monic and result now follows by Lemmas 2.2 and 2.3 .//

The following theorem is the major result of this section. The proof is adapted from Freyd [8], Thm. 3.3. However, we wish to work internally in $\mathcal{E}/\mathcal{S}$, and for our purposes we need the explicit calculation of the kernel and cokernel of a morphism and its canonical factorization for further propositions.

Theorem 2.5 $\mathcal{E}/\mathcal{S}$ is abelian.

Proof $\mathcal{E}/\mathcal{S}$ is additive because $\mathcal{E}$ is additive (additivity easily seen to be preserved under quotients). Hence it will suffice to prove that every morphism $\underline{f}$ has a kernel and cokernel, and a factorization $\underline{f} = \underline{gh}$ where $\underline{h}$ is a cokernel and $\underline{g}$ a kernel (Stenstrom [25], page 87).

Given $\underline{f} : \underline{A} \to \underline{B}$ we will show

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d, equal] & B'' \oplus A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 & \arrow[d, phantom, "\underline{k}"] \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & 0 & \arrow[d, phantom, "\underline{h}"] \\
0 \arrow[r] & B'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 & \arrow[d, phantom, "\underline{g}"] \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d, equal] & 0 & \arrow[d, phantom, "\underline{l}"] \\
0 \arrow[r] & E \arrow[r] & B \oplus A' \arrow[r] & B' \arrow[r] & 0 &
\end{tikzcd}$$

represents $0 \to \ker \underline{f} \rightarrowtail A \twoheadrightarrow \operatorname{im} \underline{f} \rightarrowtail B \twoheadrightarrow \operatorname{coker} \underline{f} \longrightarrow 0$.
By Lemma 2.3, $\underline{f} = \underline{g}\,\underline{h}$. The exact sequences at top and bottom result from Lemma 2.1, using Lemma 2.2 and its dual to show they are exact.

We prove (a) $\underline{k} = \ker \underline{f}$ (b) $\underline{g} = \ker \underline{l}$

then dually (a') $\underline{l} = \operatorname{coker} \underline{f}$ (b') $\underline{h} = \operatorname{coker} \underline{k}$.

(a) (i) $\underline{k}$ is monic

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[ddl, dashed, "\theta"'] & X' \arrow[d] & \underline{x} \\
A'' \arrow[r] \arrow[d, equal] & B''\oplus A \arrow[r] \arrow[d] & E \arrow[d] & \\
A'' \arrow[r] & A \arrow[r] & A' & \underline{k}
\end{tikzcd}$$

If $\underline{k}\,\underline{x} = 0$ then θ exists by Proposition 2.4. The same θ then shows $\underline{x} \sim 0$.

(ii) $\underline{h}\,\underline{k} = 0$

$$\begin{tikzcd}
A'' \arrow[r] \arrow[d, equal] & B''\oplus A \arrow[ddl, dashed, "\psi"] \\
A'' \arrow[d] & \\
B'' &
\end{tikzcd}$$

Take ψ to be the projection.

(iii) Suppose $\underline{h}\,\underline{x} = 0$

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[ddl, dashed, "\theta"'] & X' \arrow[d] \\
A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[d] \\
B'' \arrow[r] & B \arrow[r] & B'
\end{tikzcd}$$

so that θ exists with the properties of Prop. 2.4. Then

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X' \arrow[d] \\
A'' \arrow[r] \arrow[d, equal] & B''\oplus A \arrow[r] \arrow[d] & E \arrow[d] \\
A'' \arrow[r] & A \arrow[r] & A'
\end{tikzcd}$$

gives a factorization of $\underline{x}$ through $\underline{k}$.

(i), (ii) and (iii) establish (a) $\underline{k} = \ker \underline{f}$.

(b) (i) $\underline{g}$ is monic, proof same as for $\underline{k}$.

(ii) $\underline{l}\,\underline{g} = 0$

$$\begin{tikzcd}
B'' \arrow[r] \arrow[d, equal] & E \arrow[ddl, dashed, "\theta"] \\
B'' \arrow[d] & \\
E &
\end{tikzcd}$$

, take θ to be identity.

(iii) Suppose $\underline{l}\ \underline{x} = 0$.

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[ddl, dashed, "\theta'"'] & X' \arrow[d] \\
B'' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B' \arrow[d, equal] \\
E \arrow[r] & B \oplus A' \arrow[r] & B'
\end{tikzcd}$$

Then θ exists as in Prop. 2.4.
Let $\underline{x}^{o}$ be the composite in $\mathcal{E}$.

$$\underline{g} \qquad \begin{tikzcd}
X'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\theta"'] & X' \arrow[d, dotted] \\
B'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A' \arrow[d] \\
B'' \arrow[r] & B \arrow[r] & B'
\end{tikzcd}$$

Then $(\underline{x} - \underline{x}^{o}) \sim 0$ because left side of $\underline{x} - \underline{x}^{o}$ is the zero map. Hence in $\mathcal{E}/\mathcal{S}$, $\underline{x} = \underline{x}^{o}$ and $\underline{x}$ can be factored through $\underline{g}$.

(i), (ii) and (iii) establish (b) $\underline{g} = \ker \underline{l}$ .//

2.6 Subobjects and Quotients

We use the factorization of a morphism, and the construction of cokernel and kernel to next investigate the concepts of subobject and quotient object.

Suppose
$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}$$
represents a monic. Factoring this monic as in the theorem

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.O.}"] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B'' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'' \arrow[r] & B \arrow[r] & B' \arrow[r] & 0
\end{tikzcd}$$

establishes an isomorphism

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B'' \arrow[r] & E \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

So, without loss of generality, one can assume each subobject of a sequence results from a pull-back, and dually each quotient from a push-out.

2.7 Kernels and Cokernels

Given a map of sequences $\underline{E}_1 \to \underline{E}_2$, the kernel $\underline{\mathbf{K}}$ is determined by a push-out as follows :

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & A_2 \oplus B_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathbf{K}} \arrow[d] \\ \underline{E}_1 \arrow[d] \\ \underline{E}_2 \end{tikzcd}$$

where E is the push-out of $\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B_1 \\ A_2 \end{tikzcd}$.

Dually, the cokernel $\underline{\mathbf{L}}$ is also determined by E as a pull-back :

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E \arrow[r] & C_1 \oplus B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{E}_1 \arrow[d] \\ \underline{E}_2 \arrow[d] \\ \underline{\mathbf{L}} \end{tikzcd}$$

where $\underline{E}$ is also the pull-back of $\begin{tikzcd} & C_1 \arrow[d] \\ B_2 \arrow[r] & C_2 \end{tikzcd}$.

Suppose $\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$ is monic.

Then in factoring this map into epi-monic the epi is an isomorphism : $\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & E \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd}$.

We exhibit the inverse as an illustration of techniques used in $\mathcal{E}/\mathcal{S}$: one has the kernel

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & A_2 \oplus B_1 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd}$$

but this splits, so by Prop. 2.4 $E \to C_1$ factors over B_1 (the map $E \to C_1$ is the composite $E \to A_2 \oplus B_1 \to B_1$ the splitting map followed by the projection). Thus one can form the diagram :

$$\begin{tikzcd} 0 \arrow[r] & A_2 \arrow[r] \arrow[d, dotted] & E \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_1 \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd},$$

where $A_2 \longrightarrow A_1$ is the map induced on kernels. This gives the required inverse of the sequence morphism. We remark that $A_2 \longrightarrow A_1$ is not the component of the splitting $A_2 \oplus B_1 \longrightarrow A_1$.

Dually, if
$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$
is epi, then
$$\begin{tikzcd} 0 \arrow[r] & A_2 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$
is an isomorphism. For the inverse, form the cokernel
$$\begin{tikzcd} 0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] \arrow[dl, dashed] & C_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E \arrow[r] & C_1 \oplus B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$
which splits so $A_2 \longrightarrow E$ factors over B_2 ($B_2 \longrightarrow E$ is the composite $B_2 \rightarrow C_1 \oplus B_2 \longrightarrow E$ the natural injection followed by the splitting). So forming the diagram
$$\begin{tikzcd} 0 \arrow[r] & A_2 \arrow[r] \arrow[d, equal] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & E \arrow[r] & C_1 \arrow[r] & 0 \end{tikzcd},$$
where $C_2 \longrightarrow C_1$ is map induced on cokernels, gives the inverse.

If $\underline{E}_1 \rightarrow \underline{E}_2$ is an isomorphism. the two inverses above can be combined to give an explicit inverse, However there is another way of viewing this isomorphism which lacks rigour but gives some insight into the character of $\mathcal{E}/\mathcal{S}$, and how it differs from $\mathcal{E}$ by regarding split sequences as zero. If $\underline{E}_1 \rightarrow \underline{E}_2$ is an isomorphism then both $0 \rightarrow A_1 \rightarrow B_1 \oplus A_2 \rightarrow E \rightarrow 0$ and $0 \rightarrow E \rightarrow C_1 \oplus B_2 \rightarrow C_2 \rightarrow 0$ split, so

$$0 \rightarrow A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow 0 \quad = \quad 0 \rightarrow A_1 \oplus E \rightarrow B_1 \oplus (C_1 \oplus B_2) \rightarrow C_1 \oplus C_2 \rightarrow 0$$

and

$$0 \rightarrow A_2 \rightarrow B_2 \rightarrow C_2 \rightarrow 0 \quad = \quad 0 \rightarrow A_2 \oplus B_1 \rightarrow B_2 \oplus (B_1 \oplus C_1) \rightarrow C_1 \oplus C_2 \rightarrow 0$$

and these sequences are isomorphic in the category $\mathcal{E}$. That is, by adding suitable split sequences an isomorphism can be lifted to $\mathcal{E}$.

<u>Proposition 2.8</u> If $\mathcal{A}$ is (co-)complete, then so is $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> Since $\mathcal{A}$ is abelian, it suffices to show direct sums exist for co-completeness. The obvious choice works. Given $0 \to A_i \to B_i \to C_i \to 0$, form $0 \to \oplus A_i \to \oplus B_i \to \oplus C_i \to 0$. This sequence will have the universal property; the only non-triviality is uniqueness.

Suppose
$$\begin{array}{ccccccccc} 0 & \to & \oplus A_i & \to & \oplus B_i & \to & \oplus C_i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0 \end{array}$$
has the property that
$$\begin{array}{ccccccccc} 0 & \to & A_i & \to & B_i & \to & C_i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \oplus A_i & \to & \oplus B_i & \to & \oplus C_i & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0 \end{array}$$
is zero. Then each $C_i \to \oplus C_i \to Z$ factors over Y and hence $\oplus C_i \to Z$ factors over Y by taking the sum of the individual factorizations. Thus the lower sequence map is zero.

Using additivity, this will imply uniqueness of the two maps induced from this sum sequence agreeing on the natural injections.//

<u>2.9 Example</u>

For direct sums and products, the procedure is to form them in $\mathcal{E}$ and pass to $\mathcal{E}/\mathcal{S}$. This method fails to form general limits and colimits. To illustrate the difficulties, let $\mathcal{A}$ = Ab, abelian groups, and consider the non-split sequence $0 \to K \to F \to \mathbb{Q} \to 0$ where F is a free abelian group, $\mathbb{Q}$ the rationals. Now $\mathbb{Q}$ is a direct limit of its finitely generated subgroups G_i. Forming pull-backs

$$\begin{array}{ccccc} 0 \to & K & \to & E_i & \to & G_i \\ & \| & & \downarrow & & \downarrow \\ 0 \to & K & \to & F & \to & \mathbb{Q} \end{array},$$

one has that in $\mathcal{E}$ the sequence $0 \to K \to F \to \mathbb{Q} \to 0$ is a direct limit of $0 \to K \to E_i \to G_i \to 0$. But all finitely generated subgroups of Q are isomorphic to $\mathbb{Z}$. Hence each $0 \to K \to E_i \to G_i \to 0$ splits and is zero in $\mathcal{E}/\mathcal{S}$ and the direct limit in $\mathcal{E}/\mathcal{S}$ will then be zero.

2.10 Sums of Subobjects

Let $X_i \hookrightarrow X$ be subobjects in an abelian category. Then the sum of these subobjects in X is the image of the induced map $\oplus X_i \to X$. Applying this procedure to $\mathcal{E}/\mathcal{S}$, let

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B_i \arrow[r] \arrow[d] & C_i \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

represent a set of subobjects. The sum map is

$$\begin{tikzcd} 0 \arrow[r] & \oplus A \arrow[r] \arrow[d] & \oplus B_i \arrow[r] \arrow[d] & \oplus C_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

and its image is
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \oplus C_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \quad ;$$

that is, the sum is achieved by taking the pull-back of the sum map $\oplus C_i \to C$ with the given epi $B \to C \to 0$.

2.11 Intersection of Subobjects

If $X_i \hookrightarrow X$ then the intersection of the X_i equals the kernel of the map $X \to \pi X/X_i$.

In $\mathcal{E}/\mathcal{S}$ the quotients are $0 \to B_i \to C_i \oplus B \to C \to 0$, where B_i is the pull-back of
$$\begin{tikzcd} & C_i \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$
. So the map to the product is

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi B_i \arrow[r] & \pi(C_i \oplus B) \arrow[r] & \pi C \arrow[r] & 0 \end{tikzcd}$$

which has kernel $0 \to A \to B \oplus \pi B_i \to N \to 0$.

There is another more intuitive way to construct the intersection : one has

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \pi B_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & A \arrow[r] & B_i \arrow[r] & C_i \arrow[r] & 0 \end{tikzcd} \quad ,$$

where M = cokernel, and $M \to C_i$ induced from M. So $0 \to A \to \pi B_i \to M \to 0$ is contained in the intersection.

But also,
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \oplus (\pi B_i) \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & A \arrow[r] & \pi B_i \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$
middle map projection

shows that the intersection is contained in $0 \to A \to \pi B_i \to M \to 0$. Hence this is the intersection. To exhibit an explicit inverse to this isomorphism :

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \pi B_i \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & A \arrow[r] & B \oplus (\pi B_i) \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

where middle map is the sum of the identity on πB_i and $\pi B_i \to B_j \to B$ for any B_j, where $B_j \to B$ result from formation of B_j as pull-back of
$$\begin{tikzcd}
B_j \arrow[r] \arrow[d] & C_j \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}$$
.

CHAPTER 3

EMBEDDING OF $\mathcal{A}$ INTO $\mathcal{E}/\mathcal{S}$ AND PROJECTIVES

We now investigate the intimacy of $\mathcal{A}$ with its associated sequence category. In some respects, the situation is similar to the Yoneda embedding $A \longrightarrow \mathrm{HOM}(-,A)$, which embeds $\mathcal{A}$ as a resolving set of projectives in the functor category.

For each A, choose a projective presentation $0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0$ (we assume $\mathcal{A}$ has sufficient projectives).

<u>Proposition 3.1</u> The assignment of projective presentations constitutes a functor $\Pi : \mathcal{A} \longrightarrow \mathcal{E}/\mathcal{S}$. Any two such functors are naturally equivalent.

<u>Proof</u> Given a morphism $f : A \longrightarrow B$, there is an induced morphism between projective presentations

$$\Pi(f) \qquad \begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & Q \arrow[r] & B \arrow[r] & 0 \end{tikzcd} .$$

If two different sequence maps both induce $A \xrightarrow{f} B$, then the difference is homotopic to zero since right side is the zero map $A \xrightarrow{0} B$.

Hence by Prop. 2.4 this constitutes the zero map in $\mathcal{E}/\mathcal{S}$. Hence this is a well-defined assignment which is then clearly a functor $\mathcal{A} \longrightarrow \mathcal{E}/\mathcal{S}$.

If Π' were defined using different presentations then

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, shift left] & P \arrow[r] \arrow[d, shift left] & A \arrow[d, equal] \\ 0 \arrow[r] & K' \arrow[r] \arrow[u, shift left] & P' \arrow[r] \arrow[u, shift left] & A \end{tikzcd} \qquad \begin{tikzcd} \Pi(A) \arrow[d, shift left, "\Theta_A"] \\ \Pi'(A) \arrow[u, shift left, "\Psi_A"] \end{tikzcd}$$

constitutes natural transformations Ψ, Θ, and $(1 - \Psi_A \Theta_A) \sim 0$ since the right side of $(1- \Psi_A \Theta_A)$ is the zero map $A \longrightarrow A$. Hence in $\mathcal{E}/\mathcal{S}$, Ψ_A and Θ_A are mutual inverses determining a natural equivalence between Π and Π'(in particular different projective presentations of A are isomorphic in $\mathcal{E}/\mathcal{S}$). //

<u>Lemma 3.2</u> A projective presentation $0 \to K \to P \to A \to 0$ is a projective object of $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> Let
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$
be a given morphism.

If $\underline{E} \twoheadrightarrow \underline{B} \to 0$ is epi, without loss of generality assume $\underline{B}$ is a quotient of $\underline{E}$. Then the given morphism can be factored as

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{E} \arrow[d, two heads] \\ \underline{B} \end{tikzcd}$$

Hence $0 \to K \to P \to A \to 0$ has the projective property. //

<u>Proposition 3.3</u> The image of $\pi : \mathcal{A} \to \mathcal{E}/\mathcal{S}$ is a resolving set of projectives.

<u>Proof</u>
$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C \arrow[r] & B \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$
is epi , so result follows with Lemma 3.2. //

<u>3.4 The Projective Homotopy Category</u>

The additive functor $\pi : \mathcal{A} \to \mathcal{E}/\mathcal{S}$ has kernel those objects A of $\mathcal{A}$ whose projective presentation is zero in $\mathcal{E}/\mathcal{S}$, i.e. it splits and so A itself is projective. Given $f : A \to B$, then $\pi(f) = 0$ if

$$\begin{tikzcd} K \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & A \arrow[d] \\ L \arrow[r] & Q \arrow[r] & B \end{tikzcd}$$
is the zero map, and this occurs

if and only if $A \to B$ factors over $Q \twoheadrightarrow B$ by Prop. 2.4.

<u>Proposition 3.5</u> For $f : A \to B$, the following are equivalent. (Hilton [13], page 131) :

(i) f can be factored through some projective.

(ii) f can be factored through a projective Q, such that $Q \twoheadrightarrow B$ ($\pi(f) = 0$).

(iii) f can be factored through any $C \twoheadrightarrow B$.

<u>Proof</u> (i) $\Rightarrow$ (ii) If $f : A \to B = A \to P \to B$ then $\pi(f) : \pi(A) \to \pi(P) \to \pi(B)$ is zero since

$\pi(\mathcal{P}) = 0$.

(ii)$\Rightarrow$(iii) Let $C \twoheadrightarrow B$. Then there is a map

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & A \arrow[d] \\
0 \arrow[r] & L \arrow[r] & C \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

$\pi(A) \to \underline{E}$

$= \underline{E}$

which factors as

$$\pi(A) \xrightarrow{\pi(f)} \pi(B) \twoheadrightarrow \underline{E}$$

and hence is zero, so by Prop. 2.4 $A \to B$ factors over C. //

We have now characterized the kernel of π, and can form a quotient category $\mathcal{A}/\mathcal{P}$. The objects of $\mathcal{A}/\mathcal{P}$ are those of $\mathcal{A}$, but $\mathrm{HOM}_{\mathcal{A}/\mathcal{P}}(A,A') = \mathrm{HOM}_{\mathcal{A}}(A,A') \,/\, \mathcal{P}(A,A')$

where $\mathcal{P}(A,A')$ is the subgroup of morphisms factoring over a Projective. Combining results gives

<u>Theorem 3.6</u>

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\pi"] \arrow[dr] & & \mathcal{E}/\mathcal{S} \\
& \mathcal{A}/\mathcal{P} \arrow[ur] &
\end{tikzcd}$$

is a full embedding of $\mathcal{A}/\mathcal{P}$ as a resolving category of projectives.

We will now denote $\mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{E}/\mathcal{S}$ by π.

<u>Proposition 3.7</u> $\mathcal{A}/\mathcal{P}$ has weak kernels.

<u>Proof</u> Let $A \to B$ in $\mathcal{A}/\mathcal{P}$, pass to $\mathcal{E}/\mathcal{S}$, and let $\underline{K} \rightarrowtail \pi(A) \to \pi(B)$ be exact in $\mathcal{E}/\mathcal{S}$. If $\pi(K) \twoheadrightarrow \underline{K}$ then $\pi(K) \to \pi(A) \to \pi(B)$ is a weak kernel, for if $\pi(X) \to \pi(A) \to \pi(B)$ is zero

$$\begin{tikzcd}
& & \pi(X) \arrow[dll, dashed] \arrow[dl, dashed] \arrow[d] & \\
\pi(K) \arrow[r, two heads] & \underline{K} \arrow[r, tail] & \pi(A) \arrow[r] & \pi(B)
\end{tikzcd}$$

there is an induced map into the kernel $\underline{K}$, and since $\pi(X)$ is projective, this factors over $\pi(K) \twoheadrightarrow \underline{K}$. //

<u>Proposition 3.8</u> If $0 \to P \to A \to B \to 0$ is exact in $\mathcal{A}$, and P projective, then $A \to B$ is monic in $\mathcal{A}/\mathcal{P}$.

<u>Proof</u> Let $Q \twoheadrightarrow B$, Q projective, forming pull-back

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d] & Q \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & P \arrow[r] & A \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

.

Top row splits since Q is projective, hence E is projective.

If $X \to A \to B$ is zero in $\mathcal{A}/\mathcal{P}$, it factors over $Q \twoheadrightarrow B$ by Prop. 3.4; hence

$$\begin{tikzcd} X \arrow[drr, bend left] \arrow[dr, dashed] \arrow[ddr, bend right] & & \\ & E \arrow[r, two heads] \arrow[d] & Q \arrow[d] \\ & A \arrow[r, two heads] & B \end{tikzcd}$$

$X \dashrightarrow E$ induced into pull-back

so $X \to A$ factors over the projective E and is zero in $\mathcal{A}/\mathcal{P}$. So $X \to A \to B$ zero implies $X \to A$ is zero and by definition $A \to B$ is monic. //

3.9 Example

$\mathcal{A}/\mathcal{P}$ will not, in general, be abelian. Take $\mathcal{A}$= Ab, $\mathcal{F} = \mathcal{P}$ = frees. Then in Ab/$\mathcal{F}$, $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic and epi but not an isomorphism.

Proof $HOM_{Ab}(\mathbb{Q}, \mathbb{Z}) = 0$ implies $HOM_{Ab}(\mathbb{Q}, F) = 0$ for F free. Suppose $\mathbb{Q} \twoheadrightarrow X$ in Ab, then this remains epi in Ab/$\mathcal{F}$. For if $\mathbb{Q} \to X \to A$ is zero in Ab/$\mathcal{F}$, it factors over some free $\begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[dr] & X \arrow[r] & A \\ & F \arrow[ur] & \end{tikzcd}$ but $\mathbb{Q} \to F = 0$ implies $\mathbb{Q} \twoheadrightarrow X \to A$ is zero, further implying $X \to A$ is zero. In particular $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is epi and by Prop. 3.8 it is monic, but this could not be an isomorphism because $HOM_{Ab}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$ implies also $HOM_{Ab/\mathcal{F}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$. //

Proposition 3.10 Let $\mathcal{B}$ be a full subcategory of resolving projectives of an abelian category $\mathcal{C}$. Then the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ preserves kernels.

Proof Suppose $K \to C$ is the kernel of $C \to D$ in $\mathcal{B}$. Claim- $K \to C$ is monic in $\mathcal{C}$. For if $N \to K \to C$ is zero let $B \twoheadrightarrow N$, B in $\mathcal{B}$, then $B \twoheadrightarrow N \to K \to C = 0$ implies $B \twoheadrightarrow N \to K = 0$ further implying $N \to K = 0$.

Let $L \to C$ be the kernel of $C \to D$ in $\mathcal{C}$ and let $B \twoheadrightarrow L$, B in $\mathcal{B}$.

$$\begin{tikzcd} & K \arrow[r] \arrow[d, dashed, "g"] & C \arrow[r] & D \\ B \arrow[ur, dotted, "h"] \arrow[r, tail] & L \arrow[ur] & & \end{tikzcd},$$

g exists since $L = \ker C \to D$ in $\mathcal{C}$, and $K \to C$ monic implies g is monic. h exists since $K = \ker C \to D$ in $\mathcal{B}$.

$$\begin{tikzcd} & K \arrow[d] & C \\ B \arrow[ur] & L \arrow[ur] & \end{tikzcd} = \begin{tikzcd} & K \arrow[r] & C \\ B \arrow[ur] & & \end{tikzcd} = \begin{tikzcd} & & C \\ B \arrow[r] & L \arrow[ur] & \end{tikzcd}$$

and the monic $L \to C$ can be cancelled hence $\begin{tikzcd} & K \arrow[d] \\ B \arrow[ur] \arrow[r] & L \end{tikzcd}$

is commutative, implying that g is also epic, g is then an isomorphism and $K \to C$ is also the kernel of $C \to D$ in $\mathcal{C}$. //

<u>Corollary</u> : $\pi : \mathcal{A}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ preserves kernels.

<u>Proposition 3.11</u> (A remark of Freyd [8], page 88)
If $\mathcal{A}/\mathcal{P}$ has kernels, the projective dimension of $\mathcal{E}/\mathcal{S} \leq 2$.

<u>Proof</u> For N in $\mathcal{E}/\mathcal{S}$, choose $\pi(B) \to \pi(C) \to N \to 0$ exact, B,C in $\mathcal{A}$, by Prop. 3.10,

$$0 \to \pi(A) \to \pi(B) \to \pi(C) \to N \to 0$$

for some A in $\mathcal{A}$, since $\pi(A)$ is projective, p.d.N ≤ 2. //

<u>Lemma 3.12</u> If $\pi(A) \cong \pi(B) \oplus X$ then $X \cong \pi(B')$ for some B'.

<u>Proof</u> If $0 \to X \to \pi(A) \to \pi(B) \to 0$ is an exact splitting, then the map $\pi(A) \to \pi(B)$ arises from a map $A \to B$, and corresponds to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & Q \arrow[r] & B \arrow[r] & 0 \end{tikzcd}.$$

Forming the pull-back $\begin{tikzcd} B' \arrow[r] \arrow[d] & A \arrow[d] \\ Q \arrow[r] & B \end{tikzcd}$, the sequence $0 \to B' \to A \oplus Q \to B \to 0$ is the coker of $\pi(A) \to \pi(B)$, which is zero, and hence splits.

Then

$$X \cong \ker : \pi(A) \to \pi(B) = \ker \pi(A \oplus Q) \to \pi(B) = \pi(B')$$

since $\pi(A) = \pi(A \oplus Q)$. //

The image of π is a resolving class of projectives, but are there others ? To answer this, we mimic a result of H. Bass, replacing free modules by elements of the image of π. Assume $\mathcal{O}$ is co-complete, if X is projective in $\mathcal{E}/\mathcal{S}$ then $X \oplus X' = \pi(A)$.(*) Then if I is a countable index set

$$X \oplus \pi(\bigoplus_I A) \cong X \oplus \bigoplus_I \pi(A) \qquad \text{by 2.8}$$
$$= X \oplus (X' \oplus X) \oplus (X' \oplus X) \oplus \quad \ldots$$
$$\cong (X \oplus X') \oplus (X \oplus X') \oplus \ldots$$
$$= \bigoplus_I \pi(A) \quad \cong \quad \pi(\bigoplus_I A) \qquad \text{by 2.8}$$

<u>Proposition 3.13</u> If $\mathcal{O}$ is co-complete, then every projective is of the form $\pi(A)$ for some A.

<u>Proof</u> Given X projective, one can determine a C such that $X \oplus \pi(C) \cong \pi(C)$ by above. Now apply Lemma 3.12. //

<u>3.14 The Syzygy Functor</u>

The functor π was defined by choosing specific projective presentations for each object of $\mathcal{O}$, different choices giving rise to a functor naturally equivalent to π. Associated with π, define Z(A) by $0 \to Z(A) \to P \to A \to 0 = \pi(A)$. Z is not a functor from $\mathcal{O}$ to $\mathcal{O}$; however if the target is $\mathcal{O}/\mathcal{P}$, then Z <u>is</u> a functor.

Defining Z on morphisms by

$$\begin{array}{ccccccccc} 0 & \to & Z(A) & \to & P & \to & A & \to & 0 \\ & & \downarrow{\scriptstyle Z(f)} & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \to & Z(B) & \to & Q & \to & B & \to & 0 \end{array} ,$$

if Z(f) is well-defined, then it will clearly be an additive functor $\mathcal{O} \to \mathcal{O}/\mathcal{P}$. To do this, it will suffice to consider the case f=0, and prove Z(f) = 0; that is, Z(f) factors over a projective. However even more is true : if f factors

(*for some A since image π resolves.)

over a projective ($f = 0$ in $\mathcal{A}/\mathcal{P}$) then so does the induced map $Z(A) \longrightarrow Z(B)$. In fact, if f factors over Q in the above diagram, then by Prop. 2.4 $Z(A) \longrightarrow Z(B)$ factors through P . Hence Z is a functor and factors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "Z"] \arrow[dr] & \mathcal{A}/\mathcal{P} \\ & \mathcal{A}/\mathcal{P} \arrow[u] \end{tikzcd} .$$

It will be more convenient to identify Z with the functor $\mathcal{A}/\mathcal{P} \longrightarrow \mathcal{A}/\mathcal{P}$.

Suppose Z and Z' are different Syzygy functors, arising from different presentations chosen. Consider

$$\begin{tikzcd} & K' \arrow[r, equal] \arrow[d] & K' \arrow[d] \\ K \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & P' \arrow[d] \\ K \arrow[r] & P \arrow[r] & A \end{tikzcd} .$$

Let $h_A : K \longrightarrow K \oplus P' \cong K' \oplus P \twoheadrightarrow K'$. Then h_A is an isomorphism in $\mathcal{A}/\mathcal{P}$ and determines a natural equivalence between Z and Z'.

The n^{th} Syzygy functor is then defined by

$$Z_n(A) = Z(Z_{n-1}(A)) ,$$

where Z is now regarded as a functor $\mathcal{A}/\mathcal{P} \longrightarrow \mathcal{A}/\mathcal{P}$. One can now extend Z to a functor $\mathcal{E}/\mathcal{S} \longrightarrow \mathcal{E}/\mathcal{S}$ using

<u>3.15</u> (Freyd [8], Prop. 1.2)

For any abelian category $\mathcal{C}$, if $\mathcal{B}$ is a full subcategory of resolving prejectives, then any functor $G : \mathcal{B} \longrightarrow \mathcal{D}$, $\mathcal{D}$ abelian, has a unique right exact expansion $\overline{G}: \mathcal{C} \longrightarrow \mathcal{D}$. Explicitly for each $C \in \mathcal{C}$, choose $B' \longrightarrow B \longrightarrow C \longrightarrow 0$, B, B' in $\mathcal{B}$ and define

$$\tilde{G}(C) = \text{coker} : G(B') \longrightarrow G(B).$$

<u>Theorem 3.16</u> Let $\underline{A} = 0 \longrightarrow A'' \longrightarrow A \longrightarrow A' \longrightarrow 0$ be exact in $\mathcal{A}$. Then there exists an exact sequence in $\mathcal{E}/\mathcal{S}$ (columns are elements of $\mathcal{E}/\mathcal{S}$), where P, P', P" are projective :

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r, equal] \arrow[d, tail] & K'' \arrow[r] \arrow[d, tail] & K \arrow[r] \arrow[d, tail] & K' \arrow[r, "w"] \arrow[d, tail] & A'' \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d, two heads] & P'' \arrow[r] \arrow[d, two heads] & P \arrow[r] \arrow[d, two heads] & P' \arrow[r] \arrow[d, two heads] & A \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K' \arrow[r, "-w"'] & A'' \arrow[r] & A \arrow[r] & A' \arrow[r, equal] & A' \arrow[r] & 0
\end{tikzcd} \qquad (\#)$$

Before the proof, some corollaries.

__Corollary 3.17__ If $\underline{A} = 0 \to A'' \to A \to A' \to 0$ is exact in $\mathcal{A}$ then $0 \to Z(A'') \to Z(A) \to Z(A') \to 0$ is exact in $\mathcal{E}/\mathcal{S}$ (using the embedding $\mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{E}/\mathcal{S}$).

__Proof__ Using notation of theorem, $K \doteq Z(A)$, $K' \doteq Z(A')$ and $K'' = Z(A'')$ in $\mathcal{A}/\mathcal{P}$, hence in $\mathcal{E}/\mathcal{S}$ .//

__Corollary 3.18__ (Remark of Freyd [8], page 109)

(i) The extension of the Syzygy functor to $\mathcal{E}/\mathcal{S}$ is given by $\tilde{Z}(A) = 0 \to Z(A'') \to Z(A) \to Z(A') \to 0$.

(ii) $0 \to \tilde{Z}(A) \to \pi(A'') \to \pi(A) \to \pi(A') \to \underline{A} \to 0$ is exact in $\mathcal{E}/\mathcal{S}$.

__Proof__ From Thm. 3.16 $\pi(A'') \to \pi(A) \to \pi(A') \to \underline{A} \to 0$ is exact for any exact sequence $\underline{A}$. By definition of the extension functor

$$\begin{aligned}
\tilde{Z}(\underline{A}) &= \operatorname{coker}\ (\pi(Z(A)) \longrightarrow \pi(Z(A'))) \\
&= \operatorname{coker}\ (\pi(K) \longrightarrow \pi(K')) \\
&= 0 \to K'' \to K \to K' \to 0 \quad \text{(by theorem applied to } \underline{K} = 0 \to K'' \to K \to K' \to 0\text{)} \\
&= 0 \to Z(A'') \to Z(A) \to Z(A') \to 0 \quad \text{in } \mathcal{E}/\mathcal{S}.\ //
\end{aligned}$$

__Corollary 3.19__ (i) π is a half-exact functor $\mathcal{A} \to \mathcal{E}/\mathcal{S}$.

(ii) π is right exact iff $\mathcal{E}/\mathcal{S}=0$ iff all short exact sequences split.

(iii) If π is left exact then proj. dim $\mathcal{E}/\mathcal{S} \leq 2$.

__Corollary 3.20__ (Remark of Freyd [8], page 109)

$\tilde{Z}$ is an exact functor $\mathcal{E}/\mathcal{S} \to \mathcal{E}/\mathcal{S}$.

__Proof__ $\tilde{Z} : \mathcal{E}/\mathcal{S} \to \mathcal{E}/\mathcal{S}$ is the unique right exact extension of $Z : \mathcal{A}/\mathcal{P} \to \mathcal{A}/\mathcal{P}$, hence it suffices to prove $\tilde{Z}$ preserves monics.

Let $\underline{f} : \underline{A} \to \underline{B}$ be monic in $\mathcal{E}/\mathcal{S}$:

$$\underline{f}\qquad \begin{array}{ccccccccc} 0 & \to & A'' & \to & A & \to & A' & \to & 0 \\ & & \downarrow{\scriptstyle f''} & & \downarrow & & \downarrow & & \\ 0 & \to & B'' & \to & B & \to & B' & \to & 0 \end{array}\quad .$$

Using the canonical factorization of $\underline{f}$ given in Theorem 2.7, one can assume $A'' = B''$ and f'' is the identity. But then

$$\begin{array}{ccccccc} 0 & \to & \tilde{Z}(\underline{A}) & \to & \pi(A'') & \to & \pi(A) \\ & & \downarrow & & \| & & \downarrow \\ 0 & \to & \tilde{Z}(\underline{B}) & \to & \pi(A'') & \to & \pi(B) \end{array}$$

implies $\tilde{Z}(\underline{A}) \to \tilde{Z}(\underline{B})$ is monic. //

<u>Corollary 3.21</u> Following is a projective resolution of $\underline{A}$:

$$\to \pi(Z_n(A)) \to \pi(Z_n(A')) \to \pi(Z_{n-1}(A'')) \to \cdots \pi(Z(A')) \to \pi(A'') \to \pi(A) \to \pi(A') \twoheadrightarrow \underline{A}.$$

<u>Remarks</u> (a) The extension of the Syzygy functor $\mathcal{O}\!\mathcal{l}/\mathcal{P} \to \mathcal{O}\!\mathcal{l}/\mathcal{P}$ is not the Syzygy associated with $\mathcal{E}/\mathcal{S}$ but the 3rd Syzygy functor.

(b) Starting with the diagram

$$\begin{array}{ccccccccc} & & Z(A') & \to & P' & \to & A' & & \\ & & \downarrow{\scriptstyle w} & & \downarrow & & \| & & \\ 0 & \to & A'' & \to & A & \to & A' & \to & 0 \end{array}\quad ,$$

let $f_1 = A \to A'$, $f_2 = A'' \to A$, $f_3 = -w : Z(A') \to A''$. This gives rise to an infinite sequence

$$\ldots Z_n(A) \to Z_n(A') \xrightarrow[f_{3n}]{} Z_{n-1}(A'') \to \cdots \to Z(A) \xrightarrow[f_4]{} Z(A') \xrightarrow[f_3]{} A'' \xrightarrow[f_2]{} A \xrightarrow[f_1]{} A' \to 0 \quad .$$

<u>Corollary 3.22</u> (i) If f_m factors over a projective then p.d. $\underline{A} \leq m-1$.

(ii) In particular, if f_1 factors over a projective then $\underline{A} \cong \pi(A')$, and is projective.

<u>Proof</u> In the extension of (#) in Thm. 3.16 to the projective resolution given in Cor. 3.21, the sequence of maps f_m is formed from the bottom row. If f_m factors over a projective then the corresponding map between exact sequences is zero. //

(c) If $f : A \to B$ in $\mathcal{O}\!\mathcal{l}$, let $g : Q \twoheadrightarrow B$, Q projective. Then $\begin{pmatrix} f \\ -g \end{pmatrix} : A \oplus Q \twoheadrightarrow B$, and one can define the projective dimension of f as the projective dimension of

$$0 \longrightarrow K \longrightarrow A \oplus Q \longrightarrow B \longrightarrow 0$$

in $\mathcal{E}/\mathcal{S}$.

<u>Corollary 3.23</u> (a) If p.d.$A'' \leq n$ then p.d. $\underline{A} \leq 3n + 1$, $n \geq 0$

p.d.$A \leq n$ then p.d.$\underline{A} \leq 3n$, $n \geq 0$

p.d.$A' \leq n$ then p.d.$\underline{A} \leq 3n-1$, $n \geq 1$

p.d.$A' = 0$ then $\underline{A} = 0$.

(b) If p.d. $\mathcal{A} \leq n$ then p.d. $\mathcal{E}/\mathcal{S} \leq 3n-1$, $n \geq 1$

p.d. $\mathcal{A} = 0$ <u>iff</u> $\mathcal{E}/\mathcal{S} = 0$.

<u>Proof</u> π kills projectives, apply resolution of Cor. 3.21.//

<u>Remark</u> All results of this section dualize for injectives, resulting in a functor $\Psi : \mathcal{A}/\mathcal{J} \longrightarrow \mathcal{A}/\mathcal{J}$ using injective co-presentations.

We are now ready to prove Thm. 3.16. The proof commences exactly as the construction of the long Ext homology sequence, and in fact Thm. 3.16 could be proved using the long Ext sequence, but we prefer to work within the category $\mathcal{E}/\mathcal{S}$.

<u>Proof of Theorem 3.16</u> Let $P' \longrightarrow A'$ and $P'' \twoheadrightarrow A''$, P', P'' projective. Set $P = P' \oplus P''$, then form (+)

(+)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K'' \arrow[r, "k"] \arrow[d, "\beta''"'] & K \arrow[r, "k'"] \arrow[d, "\beta"] \arrow[dl, "u"'] & K' \arrow[r] \arrow[d, "\beta'"] \arrow[dl, "v"'] & 0 & {} \arrow[d, phantom, "\underline{\beta}"] \\
0 \arrow[r] & P'' \arrow[r, "i"] \arrow[d, "\alpha''"'] & P \arrow[r, "p"] \arrow[d, "\alpha"] \arrow[dl, "r"'] & P' \arrow[r] \arrow[d, "\alpha'"] \arrow[dl, "s"'] & 0 & {} \arrow[d, phantom, "\underline{\alpha}"] \\
0 \arrow[r] & A'' \arrow[r, "f"] \arrow[d] & A \arrow[r, "f'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 & {} \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

The maps r,s,u,v result from the splitting of $P'' \longrightarrow P \longrightarrow P'$, via canonical projections and injections and give the properties of Prop. 2.4, so that $\underline{\alpha}$ and $\underline{\beta}$ as sequence maps are homotopic to zero.

(I) Making use of Thm. 2.7, one starts a projective resolution of $\underline{A}$.

$$\begin{tikzcd}
K' \arrow[r, "{(w,\beta')}"] \arrow[d, equal] & A'' \oplus P' \arrow[r, "\binom{-f}{s}"] \arrow[d, two heads] & A \arrow[d, "f'"] \\
K' \arrow[r, "\beta'"] \arrow[d, "w"] & P' \arrow[r, "\alpha'"] \arrow[d, "s"] & A' \arrow[d, equal] \\
A'' \arrow[r, "f"] & A \arrow[r, "f'"] & A'
\end{tikzcd},$$

w the induced map on kernels, since the lower right square commutes by (+). Top sequence is the kernel; to continue the resolution, use Thm. 2.7 again to find an epi from a projective sequence to the kernel.

(II)
$$\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & K' \oplus P \arrow[r] \arrow[d, two heads] & A'' \oplus P' \arrow[d] \\
K \arrow[r, "\beta"] \arrow[d, "k'"'] & P \arrow[r, "\alpha"] \arrow[d, "{(-r,p)}"'] & A \arrow[d, equal] \\
K'' \arrow[r, "{(\omega,\beta')}"'] & A'' \oplus P' \arrow[r, "\binom{-f}{s}"] & A
\end{tikzcd}$$

One need verify the bottom squares commute. For lower left, one needs $k'\beta' = \beta p$ which is clear from (+) , and $k'\omega = -\beta r$. For the second equality apply the monic f.

$$\begin{aligned}
(k'w + \beta r)f &= k'wf + \beta rf \\
&= k'\beta's + \beta(\alpha - ps) \\
&= k'\beta's - \beta ps \\
&= (k'\beta' - \beta p)s \\
&= 0
\end{aligned}$$

A similar calculation for lower right square.

(III) Continuing to project on the kernel, the obvious choice is

$$\begin{tikzcd}
K'' \arrow[r, "{(0,\beta'')}"] \arrow[d, "k"'] & P \arrow[r, "{(p,r)}"] \arrow[d, tail] & P' \oplus A'' \arrow[d, equal] \\
K \arrow[r] & K' \oplus P \arrow[r] & P' \oplus A''
\end{tikzcd}.$$

However $(p,r) : P' \oplus P'' = P \longrightarrow P' \oplus A''$,

$$(p,r) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha'' \end{pmatrix} = 1 \oplus \alpha''$$

(this follows by definition of r, and s from projections and injections, and diagram chase). So the top row is isomorphic in $\mathcal{E}/\mathcal{S}$ to

$0 \longrightarrow K'' \longrightarrow P'' \longrightarrow A'' \longrightarrow 0$. We use this representation, and then compute the kernel

$$\begin{tikzcd}
K'' \arrow[r, "{(k,\beta'')}"] \arrow[d, equal] & K \oplus P'' \arrow[r, "{\begin{pmatrix} k' & \beta \\ 0 & -i \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"] & K' \oplus P \arrow[d, "{\begin{pmatrix} -w \\ -r \end{pmatrix}}"] \\
K'' \arrow[r, "\beta''"] \arrow[d, "k"'] & P'' \arrow[r, "\alpha''"] \arrow[d, "{(0,i)}"] & A'' \arrow[d, "{(0,-1)}"] \\
K \arrow[r, "{(k',\beta)}"'] & K' \oplus P \arrow[r, "{\begin{pmatrix} -\beta' & -w \\ p & -r \end{pmatrix}}"'] & P' \oplus A''
\end{tikzcd}$$

Lower left commutes by (+). For lower right,

$$(0,i) \begin{pmatrix} -\beta' & -w \\ p & -r \end{pmatrix} = (ip, -ir) = (0, -\alpha'') \quad \text{(by (+))}.$$

For upper right,

$$\begin{pmatrix} k' & \beta \\ 0 & -i \end{pmatrix} \begin{pmatrix} -w \\ -r \end{pmatrix} = (-k'w - \beta r\ ,\ ir) = (0, \alpha'')$$

where $k'w = -\beta r$ is proved in (II).

(IV) The kernel of (III) is isomorphic to $K'' \to K \to K'$ via

$$\begin{tikzcd}
\underline{K} \arrow[d, "\theta"'] \\
\underline{K}^{\circ} \arrow[d, "\Psi"'] \\
\underline{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
K'' \arrow[r, "k"] \arrow[d, equal] & K \arrow[r, "k'"] \arrow[d, "{(1,u)}"'] & K' \arrow[d, "{(1,v)}"] \\
K'' \arrow[r, "{(k,\beta'')}"'] \arrow[d, equal] & K \oplus P'' \arrow[r, "{\begin{pmatrix} k', \beta \\ 0, i \end{pmatrix}}"'] \arrow[d] & K' \oplus P \arrow[d] \\
K'' \arrow[r] & K \arrow[r] & K'
\end{tikzcd}$$

.

All squares commute, the only non-trivial one being the upper right

$$(1,u)\begin{pmatrix} k' & \beta \\ 0 & i \end{pmatrix} = (k', \beta + ui) = (k',v) = k'(1,v).$$

That $(1-\theta\psi) \sim 0$ and $(1-\psi\theta) \sim 0$ is clear because left sides for both are the zero map and so are trivially homotopic to zero. I,II,III,IV establish (#) of the theorem. //

CHAPTER 4

THE FUNCTORIAL APPROACH

Another method of studying $\mathfrak{A}$ is to study the associated category of additive covariant functors from $\mathfrak{A}$ to Ab (Abelian groups) (contravariant). These functor categories inherit most of the properties pointwise from $\mathfrak{A}$, and using the Yoneda lemma, the assignment $A \longrightarrow (-,A)$ $[A \longrightarrow (A,-)]$ is a full-embedding of $\mathfrak{A}$ as a class of resolving projectives. However if $\mathfrak{A}$ is not small, these functor categories are too large to manipulate. To make the embedding 'tighter', one can consider the sub-category of coherent functors. F is coherent if it is the cokernel of a transformation between representable functors (which are small projectives in the functor category, so coherent functors are analogues of finitely presented modules in the module category). The full subcategory of coherent functors is abelian and has projective dimension at most two (a quick proof : if F is cokernel of $(-,B) \longrightarrow (-,C)$, by Yoneda this arises from a morphism $B \longrightarrow C$, if $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact then $0 \longrightarrow (-,A) \longrightarrow (-,B) \longrightarrow (-,C) \longrightarrow F \longrightarrow 0$ is a projective resolution of F).

Now the notion of killing projectives, by the passage $\mathfrak{A} \longrightarrow \mathfrak{A}/\mathcal{P}$, can be combined with the study of coherent functors by considering the full subcategory of coherent functors that factor through $\mathfrak{A}/\mathcal{P}$; that is, those coherent functors that vanish on projectives. With each coherent functor, one can associate a left exact sequence $0 \to A \to B \to C$ where $F = \operatorname{coker} (-,B) \longrightarrow (-,C)$.

Proposition 4.1 If F is contravariant coherent, and $F = \operatorname{coker} (-,B) \longrightarrow (-,C)$, then F factors through $\mathfrak{A}/\mathcal{P}$ iff $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact.

Proof ($\Rightarrow$) Let $B \longrightarrow C \longrightarrow D \longrightarrow 0$ be exact. Consider

$$\begin{tikzcd} & & P \arrow[dl, dashed, "\psi"'] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & D \arrow[r] & 0 \end{tikzcd}$$

, P projective (we assume sufficient projectives).

There is an induced ψ since P is projective. But

$0 \to (P,A) \to (P,B) \to (P,C) \to F(P) \to 0$ is exact

$\Rightarrow$ $0 \to (P,A) \to (P,B) \to (P,C) \to 0$ is exact, since $F(P)=0$

$\Rightarrow$ ψ can be factored over B

$$\begin{tikzcd} & & P \arrow[dll, dashed, "\psi'"'] \arrow[dl, dashed, "\psi"'] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & D \arrow[r] & 0 \end{tikzcd}$$

but then

$$\begin{tikzcd} P \arrow[d] \\ D \end{tikzcd} = \begin{tikzcd} & & P \arrow[dll] \\ B \arrow[r] & C \arrow[r] & D \end{tikzcd} = 0,$$

$\Rightarrow D=0$.

$\Leftarrow$ Given $0 \to A \to B \to C \to 0$ then

$0 \to (-,A) \to (-,B) \to (-,C) \to F \to 0$ is exact, so

for any P, $0 \to (P,A) \to (P,B) \to (P,C) \to F(P) \to 0$ is exact.

But any $P \to C$ can be factored through the epi $B \twoheadrightarrow C$, so

$(P,B) \twoheadrightarrow (P,C)$, implying $F(P) = 0$. //

<u>Theorem 4.2</u> The assignment $F \longmapsto 0 \to A \to B \to C \to 0$ establishes an equivalence between $\hat{\mathcal{O}\!\!/}$ (the category of coherent functors vanishing on projectives), and $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> To make this a functor we must first define it in morphisms. Suppose $F \to F'$ is a natural transformation

$$\begin{tikzcd} 0 \arrow[r] & (-,A) \arrow[r] \arrow[d, dashed] & (-,B) \arrow[r] \arrow[d, dashed] & (-,C) \arrow[r] \arrow[d, dashed] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (-,A') \arrow[r] & (-,B') \arrow[r] & (-,C') \arrow[r] & F' \arrow[r] & 0 \end{tikzcd}$$

This induces a commutative diagram on the projective resolutions of F and F', and by Yoneda this arises from a commutative diagram in $\mathcal{O}\!\!/$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

and this can be considered as a morphism in $\mathcal{E}/\mathcal{S}$.

To check that this is well-defined, suppose

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f_i"'] & B \arrow[r] \arrow[d, "g_i"'] & C \arrow[r] \arrow[d, "h_i"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

both induce $F \to F'$, $i = 1,2$.

Then the difference will induce the zero transformation $F \to F'$; evaluate at C :

$$\begin{array}{ccccccc} & & (C,C) & \longrightarrow & F(C) & \longrightarrow & 0 \\ & & \downarrow h_1-h_2 & & \downarrow 0 & & \\ (C,B') & \longrightarrow & (C,C') & \longrightarrow & F'(C) & \longrightarrow & 0 \end{array}$$

Follow 1 : $C \to C$, implies h_1-h_2 is in the kernel of $(C,C') \to F'(C)$, $\Rightarrow$ by exactness that there exists ψ in (C,B') such that $\psi \mapsto (h_1-h_2)$ which means that h_1-h_2 factors over $B' \to C'$. So applying Prop. 2.4 the difference map on sequences is homotopic to zero, hence is zero in $\mathcal{E}/\mathcal{S}$.

The fact that the assignment is well-defined easily yields that it is also functorial.

For the inverse, given an exact sequence $0 \to A \to B \to C \to 0$, assign the cokernel, and any morphism of sequences induces a unique transformation on the cokernels:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$ in $\mathcal{E}/\mathcal{S}$ leads to

$$\begin{array}{ccccccccccc} 0 & \to & (-,A) & \to & (-,B) & \to & (-,C) & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (-,A') & \to & (-,B') & \to & (-,C') & \to & F' & \to & 0 \end{array}$$

This gives rise to a functor $\mathcal{E} \to \hat{\mathfrak{A}}$, and split sequences are assigned the zero functor, so it yields a functor $\mathcal{E}/\mathcal{S} \to \hat{\mathfrak{A}}$.

Now $\hat{\mathfrak{A}} \to \mathcal{E}/\mathcal{S} \to \hat{\mathfrak{A}}$ is the identity.

Consider $\mathcal{E}/\mathcal{S} \to \hat{\mathfrak{A}} \to \mathcal{E}/\mathcal{S}$,

say $0 \to A \to B \to C \to 0 \mapsto F \mapsto 0 \to A' \to B' \to C' \to 0$.

Then
$$\begin{array}{ccccccccccc} 0 & \to & (-,A') & \to & (-,B') & \to & (-,C') & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & (-,A) & \to & (-,B) & \to & (-,C) & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & (-,A') & \to & (-,B') & \to & (-,C') & \to & F & \to & 0 \end{array}$$.

Taking the difference of the identity map and the composition of these maps, results in the zero transformationn $F \to F$, so the induced map of differences between sequences is zero in $\mathcal{E}/\mathcal{S}$, and hence the map induced from $0 \to A' \to B' \to C' \to 0$ to itself is the identity in $\mathcal{E}/\mathcal{S}$, hence $\hat{\mathfrak{A}} \to \mathcal{E}/\mathcal{S} \to \hat{\mathfrak{A}}$ is naturally equivalent to the identity transformation.//

The following hold by duality.

Proposition 4.3 If F is a covariant coherent and $F = \text{coker} : (B,-) \to (A,-)$, then F factors through $\mathfrak{A}/\mathcal{J}$ iff $0 \to A \to B \to C \to 0$ is exact.

Theorem 4.4 The assignment $F \mapsto 0 \to A \to B \to C \to 0$ establishes a contravariant equivalence between $\check{\mathfrak{A}}$ (the category of covariant coherent functors vanishing on injectives) and $\mathcal{E}/\mathcal{S}$.

Corollary 4.5 There is a contravariant equivalence between $\hat{\mathfrak{A}}$ and $\check{\mathfrak{A}}$.

Remarks Auslander proves that $\hat{\mathfrak{A}}$ is abelian, so Thm. 4.1 would establish that $\mathcal{E}/\mathcal{S}$ is abelian (Thm. 2.7). The proof is easy once it has been established that the subcategory of coherent functors is abelian, but this is non-trivial (Auslander [2]).

We now examine the equivalences $\hat{\mathfrak{A}}$, $\mathcal{E}/\mathcal{S}$, $\check{\mathfrak{A}}^{op}$ and interpret results of Chapter 3 in terms of functors.

4.6 (a) Injectives in $\check{\mathfrak{A}}$

The sequences $0 \to K \to P \to A \to 0$ are projective in $\mathcal{E}/\mathcal{S}$, for P projective, and if $\mathfrak{A}$ is co-complete all such projectives are of this form (by 3.2 and 3.13). Under the contravariant equivalence $\mathcal{E}/\mathcal{S} \to \check{\mathfrak{A}}$, the resulting functor is

$$\text{coker} : (P,-) \longrightarrow (K,-) = \text{Ext}'(A,-),$$

and so these are injective in $\check{\mathfrak{A}}$.

The projective objects $\pi Z_n(A)$ correspond to the functors $\text{Ext}^n(A,-)$ and the projective resolution of Cor. 3.21 is the standard long Ext homology sequence, truncated of the first three terms

$0 \to \underline{A} \to \text{Ext}'(A',-) \to \text{Ext}'(A,-) \to \text{Ext}'(A'',-) \to \text{Ext}^2(A',-) \to \ldots$

This is an injective co-resolution in $\check{\mathfrak{A}}$.

<u>(b) Projectives in $\hat{\mathcal{A}}$</u>

The functors corresponding to sequences $0 \to K \to P \to A \to 0$ under the equivalence $\mathcal{E}/\mathcal{S} \to \hat{\mathcal{A}}$, are coker : $(-,P) \to (-,A)$. Now for fixed X, images of $(X,P) \to (X,A)$ are those morphisms which factor through P and hence by Prop. 3.4 those morphisms factoring through any projective and so

$$\begin{aligned} \operatorname{coker} (X,P) \to (X,A) &= (X,A) \,/\, \mathcal{P}(X,A) \\ &= \operatorname{Hom}_{\mathcal{A}/\mathcal{P}} (X,A) \quad \text{(Prop. 3.5)} \end{aligned}$$

Thus projectives are 'representable' functors $\operatorname{Hom}_{\mathcal{A}/\mathcal{P}}(-,A)$, following Hilton [ ], we denote these as $\pi(-,A)$ (it is for this reason we chose $\pi : \mathcal{A}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ as embedding functor).

Note that since $\pi : \mathcal{A}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ is full, $\operatorname{Hom}_{\mathcal{A}/\mathcal{P}} (X,A) \cong \operatorname{Hom}_{\mathcal{E}/\mathcal{S}} (\pi(X), \pi(A))$.

To carry the correspondance further, set $\pi_n(A) = \pi(Z_n(A))$, and $\pi_n(-,A) = \pi(-,Z_n(A))$. Then the projective resolution of Cor. 3.21 is

$$\ldots \to \pi_1(A') \to \pi_1(A) \to \pi_1(A') \to \pi(A'') \to \pi(A) \to \pi(A') \to \underline{A} \to 0$$

and correspondingly a long homology sequence (Hilton [ ])

$$\ldots \to \pi_1(-,A) \to \pi_1(-,A') \to \pi(-,A'') \to \pi(-,A) \to \pi(-,A') \to \underline{A} \to 0$$

which is a projective resolution in $\hat{\mathcal{A}}$.

<u>4.7</u> <u>(a) Injectives in $\hat{\mathcal{A}}$</u>

The sequence $0 \to A \to I \to N \to 0$ is injective in $\mathcal{E}/\mathcal{S}$, for I injective, and if $\mathcal{A}$ is complete all injectives are of this form (dual of 3.2 and 3.13).

The co-Syzygy functor W can be defined on $\mathcal{A}/\mathcal{J}$ by $0 \to A \to I \to W(A) \to 0$ and $W_n(A) = W_{n-1}(W(A))$.

Finally set

$$\psi(A) = 0 \to A \to I \to W(A) \to 0, \text{ and}$$
$$\psi_n(A) = \psi(W_n(A)) \text{ , so that}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{E}/\mathcal{S} \\ \mathcal{A}/\mathcal{J} \arrow[ur, "\psi"'] & \end{tikzcd}$$

is a full-embedding of as a co-resolving class of injectives.

Now examine the equivalence $\mathcal{E}/\mathcal{S} \to \hat{\mathcal{A}}$,

$\Psi(A) \longmapsto \text{coker} : (-,I) \to (-,N) \;=\; \text{Ext}'(-,A)$

and $\qquad \Psi_n(A) \longmapsto \text{Ext}^n(-,A)$.

The dual of Cor. 3.21 is an injective co-resolution of

$\underline{A} = 0 \to A'' \to A \to A' \to 0$,

$$0 \to \underline{A} \to \Psi(A'') \to \Psi(A) \to \Psi(A') \to \Psi_1(A'') \to \Psi_1(A) \to \Psi_1(A') \to \dots$$

and in $\hat{\mathcal{A}}$ this is the truncated long Ext sequence

$$0 \to \underline{A} \to \text{Ext}'(-,A'') \to \text{Ext}'(-,A) \to \text{Ext}'(-,A') \to \text{Ext}^2(-,A'') \to \dots$$

<u>(b) Projectives in $\check{\mathcal{A}}$</u>

With notation as above, these are of the form

$\text{coker} : (I,-) \to (A,-) = (A,-) \,/\, \mathcal{J}(A,-) \;=\; \Psi(A,-)$.

In analogy with the $\pi(-,A)$ functors, where

$\mathcal{J}(A,X)$ = maps $A \to X$ which factor through an injective,

and setting $\Psi_n(A,-) \;=\; \Psi(W_n(A),-)$, one gets another

homology sequence

$$\dots \to \Psi_1(A'',-) \to \Psi(A',-) \to \Psi(A,-) \to \Psi(A'',-) \to \underline{A} \to 0 \dots$$

which is a projective resolution in $\check{\mathcal{A}}$.

<u>NOTE</u> $\Psi(A,X) = \text{Hom}_{\mathcal{A}/\mathcal{J}}(A,X) \;=\; \text{Hom}_{\mathcal{E}/\mathcal{S}}(\Psi(A), \Psi(X))$

and $\Psi(A,-) \;=\; \text{Hom}_{\mathcal{A}/\mathcal{J}}(A,-)$ is representable.

We would now like to transfer some homological algebra into the category $\mathcal{E}/\mathcal{S}$.

<u>4.8</u> <u>Example</u> The functor $\text{Ext}'(-,\mathbb{Z})$ in $\widehat{Ab}$ corresponds to the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ in $\mathcal{E}/\mathcal{S}$, which then corresponds to $\Psi(\mathbb{Z},-)$ in $\check{Ab}$.

The Whitehead conjecture is $\text{Ext}'(A,\mathbb{Z}) \;=\; 0$

$\Rightarrow$ A is projective

A natural dual would then be $\Psi(\mathbb{Z},A) \;=\; 0$ implies that A is injective. This holds.

<u>Proof</u> We show A is divisible. Let $a \in A$ and n an integer.

We need to solve $nx = a$

$$\begin{array}{ccc} 1 & \mathbb{Z} & \xrightarrow{\cdot n} \mathbb{Q} \\ \downarrow & \downarrow & \\ a & A & \end{array}$$

Complete this diagram in $\mathcal{E}/\mathcal{S}$:

$$\begin{array}{ccccccccccc}
0 & \to & \mathbb{Z} & \xrightarrow{\cdot n} & \mathbb{Q} & \to & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 & = & \psi(Z) \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \longrightarrow & I & \longrightarrow & W(A) & \longrightarrow & 0 & = & \psi(A)\,.
\end{array}$$

By assumption $\psi(\mathbb{Z},A) = \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\psi(\mathbb{Z}),\psi(A)) = 0$, so there exists a factorization over $\mathbb{Q}$(Prop. 2.4). To solve $nx=a$, follow 1 from $\mathbb{Q}\to A$. //

<u>Proposition 4.9</u> (Hilton and Rees [14], Cor. to Thm. 1.3)
Every natural transformation

$$\Theta : \mathrm{Ext}'(B,-) \to \mathrm{Ext}'(A,-)$$

is induced by a map $f : A \to B$.

<u>Proof</u> Using the contravariant equivalence of $\mathcal{E}/\mathcal{S}$ and $\check{\mathcal{O}}$, Θ can be regarded as a morphism from $\pi(A)$ to $\pi(B)$, but $\pi : \mathcal{O}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ is full so this morphism is induced from a morphism $A \to B$ in $\mathcal{O}/\mathcal{P}$, and hence represents a morphism f in $\mathcal{O}$. //

<u>Proposition 4.10</u> A map $f : A \to B$ induces the zero map $\mathrm{Ext}'(B,-) \to \mathrm{Ext}'(A,-)$ <u>iff</u> f factors over a projective.

<u>Proof</u> The map $f : A \to B$ regarded in $\mathcal{O}/\mathcal{P}$ is zero <u>iff</u> f factors over a projective. Since $\mathcal{O}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ is full, the contravariant equivalence of $\mathcal{E}/\mathcal{S}$ and $\check{\mathcal{O}}$ gives the result. //

<u>Proposition 4.11</u> (Auslander and Bridger [4], Thm. 1.40)

$$0 \to P(A,B) \longrightarrow (A,B) \longrightarrow [\mathrm{Ext}'(B,-),\mathrm{Ext}'(A,-)] \to 0$$

is exact. $[\ ,\ ]$ = natural transformations
= Hom set in functor category

<u>Proof</u> $0 \to P(A,B) \longrightarrow (A,B) \to \pi(A,B) \to 0$ is exact by definition.

$$\begin{aligned}
\pi(A,B) &= \mathrm{Hom}_{\mathcal{O}/\mathcal{P}}(A,B) \\
&= \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\pi(A),\pi(B)) \quad , \quad \mathcal{O}/\mathcal{P} \to \mathcal{E}/\mathcal{S} \text{ is full} \\
&= \mathrm{Hom}_{\check{\mathcal{O}}}(\mathrm{Ext}'(B,-),\mathrm{Ext}'(A,-)) \quad \text{contravariant eq. of } \mathcal{E}/\mathcal{S} \text{ and } \check{\mathcal{O}} \\
&= [\,(\mathrm{Ext}'(B,-),\mathrm{Ext}'(A,-)\,] \quad \text{since the subcategory } \check{\mathcal{O}} \text{ is full in the functor category.//}
\end{aligned}$$

We also extend a result of Hilton and Rees [14], Thm.2.1.

<u>Theorem 4.12</u> For $f : A \to B$, the following are equivalent :

(i) $\mathrm{Ext}'(B,-) \rightarrowtail \mathrm{Ext}'(A,-)$ is monic.

(ii) $\mathrm{Ext}'(B,-) \rightarrowtail \mathrm{Ext}'(A,-)$ splits.

(iii) There exists B' with

$$\mathrm{Ext}'(B,-) \rightarrowtail \mathrm{Ext}'(A,-) \twoheadrightarrow \mathrm{Ext}'(B',-)$$

(split) exact.

(iv) $\pi(-,A) \twoheadrightarrow \pi(-,B)$ epi.

(V) $\pi(-,A) \twoheadrightarrow \pi(-,B)$ split epi.

(vi) There exists B' with

$$0 \to \pi(-,B') \rightarrowtail \pi(-,A) \to \pi(-,B) \to 0$$

(split)exact.

(vii) $\pi(A) \twoheadrightarrow \pi(B)$ epi (in $\mathcal{E}/\mathcal{S}$).

(viii) $\pi(A) \twoheadrightarrow \pi(B)$ split epi.

(ix) There exists B' with

$$0 \to \pi(B') \to \pi(A) \to \pi(B) \to 0$$

split exact.

(x) Given $Q \twoheadrightarrow B$, Q projective, $A \oplus Q \twoheadrightarrow B$ splits in $\mathcal{O}\!\iota$.

(xi) $A \to B$ is split epi in $\mathcal{O}\!\iota/\mathcal{P}$.

<u>Proof</u> If (vii),(viii) and (ix) are equivalent, then category equivalences handle (i) through (vi).

(ix) $\Rightarrow$ (viii) $\Rightarrow$ (vii) trivial

(vii) $\Rightarrow$ (viii) since $\pi(B)$ is projective in $\mathcal{E}/\mathcal{S}$.

(viii) $\Rightarrow$ (ix) by lemma 3.12.

(ix) $\Leftrightarrow$ (x) is done in proof of lemma 3.12.

(xi) $\Leftrightarrow$ (vii) since $\mathcal{O}\!\iota/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ is full. //

<u>Theorem 4.14</u> If $\mathcal{O}\!\iota$ is co-complete, then every direct summand of Ext'(A,-) is also of the form Ext'(B,-) for some object B. (Auslander [3]).

<u>Proof</u> This is a restatement of Prop. 3.13 using contravariant equivalence of $\mathcal{E}/\mathcal{S} \to \mathcal{F}$ .//

<u>Corollary 4.13</u> For $f : A \to B$ The following are equivalent :

(i) $\mathrm{Ext}'(B,-) \to \mathrm{Ext}'(A,-)$ is an isomorphism.

(ii) $\pi(-,A) \to \pi(-,B)$ is an isomorphism.

(iii) $\pi(A) \to \pi(B)$ is an isomorphism in $\mathcal{E}/\mathcal{S}$.

(iv) Given $Q \twoheadrightarrow B$, Q projective, then $Q \oplus A \twoheadrightarrow B$ splits and has a projective kernel.

(v) There exist projectives P,Q with an isomorphism $A \oplus Q \to B \oplus P$, where f is the component $A \to B$.

(vi) $A \to B$ is an isomorphism in $\mathcal{A}/\mathcal{P}$.

<u>Proof</u> Again (i),(ii) and (iii) are equivalent by category equivalences, and (iii) $\Leftrightarrow$ (vi) since $\tilde{\pi} : \mathcal{A}/\mathcal{P} \rightarrowtail \mathcal{E}/\mathcal{S}$ is a fully faithful embedding. (iii) $\Rightarrow$ (iv):By Thm. 4.12 $B' \rightarrowtail A \oplus Q \twoheadrightarrow B$ splits; then $\pi(B' \oplus B) \cong \pi(B') \oplus \pi(B) \cong \pi(A \oplus Q) = \pi(A)$ implying $\pi(B') = 0$ so B' is projective.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) trivial. //

<u>Remark</u> Condition (v) is the definition of stable isomorphism.

CHAPTER 5
PURE AND COPURE SUBCATEGORIES

We return to the internal structure of $\mathcal{E}/\mathcal{S}$ by considering the subcategory of pure and copure sequences. Before doing so, we establish a few lemmas.

An object in an abelian category is called small if any map into an arbitrary sum factors through a finite sum via the canonical map of the finite sum into the total sum. An object is finitely generated if an epimorphism to it from an arbitrary sum can be reduced to some finite sum and can remain an epi.

Lemma 5.1 For any abelian category, a quotient of a small object is small.

Proof Let B be small, and $C \to \oplus X_i$ where C is a quotient of B

$$\begin{tikzcd} & & \oplus_J X_i \arrow[d, tail] \\ B \arrow[urr] \arrow[r, two heads] & C \arrow[r] & \oplus X_i \arrow[d, two heads] \\ & & D \end{tikzcd}$$

. A finite sum $\oplus_J X_i$ exists which factors the composite map, since B is small. Taking coker D , then $B \twoheadrightarrow C \to \oplus X_i \twoheadrightarrow D$ is zero, and $B \twoheadrightarrow C$ is epi, so can be cancelled. Hence $C \to \oplus X_i$ factors through ker : $\oplus X_i \twoheadrightarrow D$, which is $\oplus_J X_i$.

//

Lemma 5.2 In any abelian category, a small projective is finitely generated.

Proof If $\oplus X_i \twoheadrightarrow P$, P small projective, it splits so $P \to \oplus X_i \to P$ the identity for some $P \to \oplus X_i$. But this factors through a finite subsum

$$P \to \oplus_J X_i \to \oplus X_i \to P \quad ,$$

and hence $\oplus_J X_i \to \oplus X_i \to P$ is epic. //

Lemma 5.3 If C is small in $\mathcal{O}$, then $0 \to A \to B \to C \to 0$ is small in $\mathcal{E}/\mathcal{S}$.

Proof $0 \to A \to B \to C \to 0$ is a quotient of $\pi(C)$, so by lemma 5.1 it suffices to show $\pi(C)$ is small. Given

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & C & \to & 0 \quad = \pi(C) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \oplus A_i & \to & \oplus B_i & \to & \oplus C_i & \to & 0 \end{array}$$

then $C \to \oplus C_i$ factors through a finite subsum and

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \oplus_J A & \to & \oplus_J B & \to & \oplus_J C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \oplus A_i & \to & \oplus B_i & \to & \oplus C_i & \to & 0 \end{array}$$

is a factorization of the sequence morphism through a finite subsum.//

Lemma 5.4 If C is finitely generated in $\mathcal{O}$ then $0 \to A \to B \to C \to 0$ is finitely generated in $\mathcal{E}/\mathcal{S}$.

Proof $0 \to A \to B \to C \to 0$ is a quotient of $\pi(C)$, which is small and projective, hence finitely generated by lemma 5.2. Hence also $0 \to A \to B \to C \to 0$ is finitely generated.//

Remark It is not true for abelian categories in general that finitely generated implies small or vice versa, and a direct proof of 5.4 avoiding smallness is non-trivial.

Assume $\mathcal{O}$ has a generating set of small projectives (in particular $\mathcal{O}$ will be locally small, i.e. every object has a set of subobjects). So one can consider the set of finite presentations $0 \to K \to P \to A \to 0$, P finitely generated projective, K finitely generated. Let $\mathcal{J}$ be the full subcategory generated by this set. Objects of $\mathcal{J}$ are quotients of direct sums of finite presentations. Define $\mathcal{S}$ such that $\mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\mathcal{J}, \mathcal{S}) = 0$, i.e. a sequence is in $\mathcal{S}$ if the only morphism from a sequence in $\mathcal{J}$ is zero.

For $\mathcal{O}$ = MOD R, $\mathcal{S}$ is the class of pure sequences (in the sense of Cohn, remaining exact under tensoring; for a proof, see my Masters Thesis, Gentle [12], or Fieldhouse[6]).

So we adopt this terminology and call $\mathcal{S}$ the category of pure sequences and $\mathcal{T}$ the category of copure sequences. Clearly $\mathcal{S}$ and $\mathcal{T}$ are additive. Much of what follows is standard 'torsion theory' simply applied to the pair $(\mathcal{T},\mathcal{S})$ but will be included for completeness, and for ease of reference.

<u>Proposition 5.5</u> (i) $\mathcal{T}$ is closed under quotients (taken in $\mathcal{E}/\mathcal{A}$).

(ii) If $\underline{T}_1 \to \underline{T}_2$ is epi in $\mathcal{T}$, then it is epi in $\mathcal{E}/\mathcal{A}$.

(iii) $\mathcal{T}$ is closed under colimits (which are taken in $\mathcal{E}/\mathcal{A}$).

<u>Proof</u> (i) Trivial by definition of $\mathcal{T}$ (this is non-trivial if one first defines purity in Cohn's sense).

(ii) Suppose $\underline{T}_1 \to \underline{T}_2$ is epi in $\mathcal{T}$, and let $\underline{T}_1 \to \underline{T}_2 \to \underline{X}$ be zero. Then $\underline{T}_1 \to \underline{T}_2 \twoheadrightarrow \underline{T} \rightarrowtail \underline{X}$ is also zero (factoring $\underline{T}_2 \to \underline{X}$ into epi-monic). This implies that $\underline{T}_1 \to \underline{T}_2 \to \underline{T}$ is zero. But $\underline{T}$ is in $\mathcal{T}$ by (i) so $\underline{T}_2 \to \underline{T}$ is zero since $\underline{T}_2 \to \underline{T}_1$ is epi in $\mathcal{T}$. Hence $\underline{T}_2 \to \underline{X}$ is zero.

(iii) $\mathcal{T}$ is closed under direct sums, so combined with (i) gives result. //

<u>Proposition 5.6</u> $\underline{T}_1 \to \underline{T}_2$ is monic in $\mathcal{T}$ <u>iff</u> its kernel is pure in $\mathcal{E}/\mathcal{A}$.

<u>Proof</u> $\Leftarrow$ $\mathcal{T}$ is additive, so we need only show $\underline{T} \to \underline{T}_1 \to \underline{T}_2$ zero implies $\underline{T} \to \underline{T}_1$ zero. But

$$\begin{tikzcd} \underline{T} \arrow[r] \arrow[dr, dashed] & \underline{T}_1 \arrow[r] & \underline{T}_2 \\ & \underline{K} \arrow[u] & \end{tikzcd}$$

factors through the pure kernel $\underline{K}$.
But then $\underline{T} \to \underline{K}$ is zero implying that $\underline{T} \to \underline{T}_1$ is also zero.

$\Rightarrow$ Form the kernel $\underline{K}$. If $\underline{T} \to \underline{K}$ with $\underline{T}$ in $\mathcal{T}$ then $\underline{T} \to \underline{K} \rightarrowtail \underline{T}_1 \to \underline{T}_2$ is zero, so $\underline{T} \to \underline{K} \rightarrowtail \underline{T}_1$ is zero.

Cancel the monic to get $\underline{T}\to\underline{K}$ is zero. Then by definition, $\underline{K}$ is pure. //

Proposition 5.7 $\mathcal{T}$ is closed under extensions.

Proof Suppose $0\to\underline{T}_1\to\underline{X}\to\underline{T}_2\to 0$ is exact. By definition of $\mathcal{T}$, one can choose projectives (also copure) with $\underline{P}_1 \twoheadrightarrow \underline{T}_1$, $\underline{P}_2 \twoheadrightarrow \underline{T}_2$. Since $\underline{X}\twoheadrightarrow\underline{T}_2$ the map $\underline{P}_2\to\underline{T}_2$ factors over $\underline{X}\twoheadrightarrow\underline{T}_2$. Then the sum map $\underline{P}_1 \oplus \underline{P}_2 \twoheadrightarrow \underline{X}$. Hence $\underline{X}$ is a quotient of $\underline{P}_1\oplus\underline{P}_2$ and is in $\mathcal{T}$. //

Theorem 5.8 For any sequence $\underline{E}$ there exists a subobject $\underline{T}$ in $\mathcal{T}$ with

$$0\to\underline{T}\to\underline{E}\to\underline{S}\to 0\quad,\quad \underline{S}\text{ in }\mathcal{S},$$

and $\underline{T}$ is unique with this property. (Characterized as being the largest copure subobject of $\underline{E}$.)

Proof Let $\underline{E} = 0\to A\to B\to C\to 0$, and $\{X_i\}$ be the set of finitely present objects of $\mathcal{O}\!\!/$. For each X_i, let

$$Y_i = \bigoplus_{g\in(X_i,C)} X_{i,g} \qquad \text{where } X_{i,g} \text{ is a copy of } X_i$$

for each g of $\mathrm{Hom}_{\mathcal{O}\!\!/}(X_i,C)$. Then there is a canonical map $Y_i = \bigoplus_{(X_i,C)} X_i \to C$, the image being the trace of X_i in C. Set $Y = \bigoplus Y_i$, and $Y\to C$ the sum map.

Now form the pull-back $\underline{T}$ from this map

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r, tail] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}.$$

$\underline{T}$ is a copure subobject of $\underline{E}$ (it is a quotient of $\pi(Y)$ and Y is a direct sum of finitely presented objects).

Claim $\underline{T}$ is the sum of all copure subobjects of $0\to A\to B\to C\to 0$. In fact, any copure subobject is generated by images of morphisms $\pi(X_i)\to\underline{E}$, which are of the form

$$\begin{tikzcd} K_i \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & X_i \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}.$$

Since $X_1 \to C$ factors naturally through $X_1 \to Y \to C$, there is a factorization $\pi(X_1) \to \pi(Y) \to \underline{E}$. But the map $\pi(Y) \to \underline{E}$ factors as $\pi(Y) \twoheadrightarrow \underline{T} \rightarrowtail \underline{E}$. Hence all copure subobjects are contained in $\underline{T}$.

To prove $\underline{S}$ is pure, it suffices by Prop. 5.5 to show it has no copure subobjects. Suppose $\underline{X} \rightarrowtail \underline{S}$, $\underline{X}$ copure. In $\mathcal{E}/\mathcal{S}$, form the pull-back

$$\begin{tikzcd} 0 \arrow[r] & \underline{T} \arrow[r] \arrow[d, equal] & \underline{E}' \arrow[r] \arrow[d, tail] & \underline{X} \arrow[r] \arrow[d, tail] & 0 \\ 0 \arrow[r] & \underline{T} \arrow[r] & \underline{E} \arrow[r] & \underline{S} \arrow[r] & 0 \end{tikzcd}.$$

$\underline{E}' \rightarrowtail \underline{E}$ is monic (pull-back of monic is monic). $\underline{E}'$ is copure by Prop. 5.7. But $\underline{T}$ is sum of all copure subobjects so $\underline{E}' \rightarrowtail \underline{E}$ factors over $\underline{T}$. Then

$$\begin{tikzcd} \underline{T} \arrow[r] & \underline{E}' \arrow[dl] \\ \underline{T} \arrow[r] & \underline{E} \end{tikzcd} = \begin{tikzcd} \underline{T} \arrow[r] & \underline{E}' \arrow[d] \\ & \underline{E} \end{tikzcd} = \begin{tikzcd} \underline{T} \arrow[d, equal] & \\ \underline{T} \arrow[r] & \underline{E} \end{tikzcd}.$$

Cancel monic to get $\begin{tikzcd} \underline{T} \arrow[r] \arrow[d, equal] & \underline{E}' \arrow[dl] \\ \underline{T} & \end{tikzcd}$. Hence $\underline{E}' \rightarrowtail \underline{T}$ is also epi and so an isomorphism, which then implies $\underline{T} \to \underline{E}'$ is an isomorphism and $\underline{X}$ its cokernel is zero.

For uniqueness, If $0 \to \underline{T}' \to \underline{E} \to \underline{S}' \to 0$ with $\underline{T}'$ copure, $\underline{S}'$ pure, then the maps induced on kernels and cokernels
$$\begin{tikzcd} \underline{T} \arrow[r] \arrow[d, dashed] & \underline{E} \arrow[r] \arrow[d, equal] & \underline{S} \arrow[d, dashed] \\ \underline{T}' \arrow[r] \arrow[d, dashed] & \underline{E} \arrow[r] \arrow[d, equal] & \underline{S}' \arrow[d, dashed] \\ \underline{T} \arrow[r] & \underline{E} \arrow[r] & \underline{S} \end{tikzcd}$$
show that $\underline{T} = \underline{T}'$ as subobjects of $\underline{E}$. //

<u>Corollary 5.9</u> $\underline{E}$ is copure <u>iff</u> Hom $(\underline{E}, \mathcal{S}) = 0$.

<u>Proof</u> $\Rightarrow$ By definition of $\mathcal{S}$

$\Leftarrow$ Form $0 \to \underline{T} \to \underline{E} \to \underline{S} \to 0$ as in theorem. $\underline{S}=0$ so $\underline{T} = \underline{E}$ is copure. //

An object X in $\mathfrak{A}$ is called pure projective if given

$$\begin{tikzcd} & & & X \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} = \underline{E}$$

with $\underline{E}$ pure, there is a factorization over B. This is equivalent to $\pi(X)$ being in $\mathcal{J}$. By construction of $\mathcal{S}'$, the set of finitely presented objects are pure-projective, and clearly direct sums of pure projectives are pure projective.

Suppose now X is pure projective, so that $0 \to L \to P \to X \to 0$, P projective is in $\mathcal{J}$. Then using the construction of the largest copure subobject,

$Y = \oplus Y_i \quad , \quad Y_i \quad = \bigoplus_{g \in (X_i, X)} X_{i,g} \qquad ,$

the subobject
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & P \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$
is actually the sequence itself, i.e. the cokernel is zero, i.e. it splits. The cokernel is

$$0 \to E \to P \oplus Y \to X \to 0.$$

Hence X is a direct summand of $P \oplus Y$. Now $\mathcal{O}\!\!/$ has a generating set of finitely generated projectives (by assumption), so P can be taken as a direct sum of finitely generated projectives, thus establishing

<u>Corollary 5.10</u> <u>(i)</u> An object X is pure projective <u>iff</u> it is a direct summand of a direct sum of finitely presented objects. //

For any C there is a pure sequence $0 \to N \to X \to C \to 0$ with X pure projective. In fact, take $X = P \oplus Y$ of the theorem, with P taken as a direct sum of finitely generated projectives. This property is usually stated as the property of 'sufficient pure projectives' in the literature. This is well-justified intuitively, but also in the following sense : that the sequence $0 \to N \to X \to C \to 0$ is projective in $\mathcal{S}$.

In fact, we have an exact sequence $0 \to \underline{T} \to \pi(C) \to \underline{S} \to 0$ where $\underline{S} = 0 \to N \to X \to C \to 0$ is pure and $\underline{T}$ is the maximal copure subobject of $\pi(C)$.

Suppose
$$\begin{tikzcd} 0 \arrow[r] & \underline{T} \arrow[r] \arrow[d, dotted] & \pi(C) \arrow[r] & \underline{S} \arrow[r] \arrow[d] \arrow[dll, dashed] & 0 \\ & \underline{S}_2 \arrow[rr, two heads] & & \underline{S}_1 & \end{tikzcd}$$

There is an induced map from $\pi(C)$ by projectivity, but this will factor through the cokernel of $0 \to \underline{T} \to \pi(C)$ since $\underline{T}$ is copure. Then

$$\begin{tikzcd} \pi(C) \arrow[r] & \underline{S} \arrow[dl] \\ \underline{S}_2 \arrow[r] & \underline{S}_1 \end{tikzcd} = \begin{tikzcd} \pi(C) \arrow[d] & \\ \underline{S}_2 \arrow[r] & \underline{S}_1 \end{tikzcd} = \begin{tikzcd} \pi(C) \arrow[r] & \underline{S} \arrow[d] \\ & \underline{S}_1 \end{tikzcd}.$$

Cancel epi to get $\underline{S} \to \underline{S}_1$ factoring over $\underline{S}_2 \to \underline{S}_1$. So $\underline{S}$ is projective in $\mathcal{S}'$.

Now suppose $\underline{E} = 0 \to A \to B \to C \to 0$ is pure. Then

$$\begin{tikzcd} \underline{T} \arrow[r] & \pi(C) \arrow[r] \arrow[d] & \underline{S} \arrow[dl, dashed] \\ & \underline{E} & \end{tikzcd} \quad \text{shows } \underline{S} \twoheadrightarrow \underline{E},$$

thus establishing

<u>Corollary 5.10 (ii)</u> Given C, there is a sequence $0 \to N \to X \to C \to 0$ in $\mathcal{S}$ which is projective as an object of $\mathcal{S}'$, and X can be taken to be pure projective. Every pure sequence $0 \to A \to B \to C \to 0$ is a quotient of this sequence. Hence as an abelian category, $\mathcal{S}$ has sufficient projectives. //

<u>Lemma 5.11</u> (i) $\mathcal{S}'$ is closed under subobjects (taken in $\mathcal{E}/\mathcal{S}$).

(ii) $\prod \underline{S}_i$ is pure <u>iff</u> all $\underline{S}_i$ are pure.

(iii) $\mathcal{S}'$ is closed under limits (taken in $\mathcal{E}/\mathcal{S}$).

<u>Proof</u> (i) $\underline{S}$ is pure <u>iff</u> $\underline{S}$ has no copure subobjects, so (i) is trivial.

(ii) $\underline{X} \to \prod \underline{S}_i$ is zero <u>iff</u> $X \to \prod \underline{S}_i \to \underline{S}_i$ is zero for all i, so (ii) follows from Hom $(\mathcal{T}, \mathcal{S}') = 0$.

(iii) follows from (i) and (ii). //

<u>Theorem 5.12</u> $\mathcal{S} \rightarrowtail \mathcal{E}/\mathcal{S}$ is a full exact embedding. (i.e. $\mathcal{S}$ is an abelian full subcategory of $\mathcal{E}/\mathcal{S}$ and the inclusion is exact).

<u>Proof</u> It will suffice to prove $\mathcal{S}$ is closed under quotients taken in $\mathcal{E}/\mathcal{S}$.

Let $\underline{S} \twoheadrightarrow \underline{S}'$. We need to show that $\pi(C) \to S'$ is zero for C finitely presented; but $\pi(C) \to S'$ factors over $\underline{S}$ by projectivity, and this must be zero since $\underline{S}$ is pure. //

<u>Proposition 5.13</u> $\mathcal{S}'$ is closed under co-limits.

<u>Proof</u> By Thm. 5.12, it will suffice to show $\mathcal{S}'$ is closed under direct sums.

If $\pi(C) \longrightarrow \bigoplus S_i$, then since $\pi(C)$ is small by lemma 5.3 (if $\mathcal{A}$ has a generating set of small projectives then a finitely generated object is a quotient of a small object, hence small; so C is small), we have $\pi(C) \rightarrow \bigoplus_J S_i \rightarrowtail \bigoplus S_i$ for some finite subsum, but a finite sum is also a finite product, so $\bigoplus_J S_i$ is pure by 5.11 and $\pi(C) \longrightarrow \bigoplus_J S_i$ is then zero. //

<u>Proposition 5.14</u> $\mathcal{S}'$ is dense in $\mathcal{E}/\mathcal{S}$ (Closed under sub-objects, quotients and extensions).

<u>Proof</u> All that is needed is extensions :

$$\begin{tikzcd} & & \underline{T} \arrow[dl, dashed] \arrow[d] & & \\ 0 \arrow[r] & \underline{S}_1 \arrow[r] & X \arrow[r] & \underline{S}_2 \arrow[r] & 0 \end{tikzcd}$$

Consider $0 \longrightarrow \underline{S}_1 \longrightarrow X \longrightarrow \underline{S}_2 \longrightarrow 0$, with $\underline{T}$ copure, Then $\underline{T} \rightarrow \underline{X}$ factors through the kernel of $\underline{X} \rightarrow \underline{S}_2$ since $\underline{T} \rightarrow \underline{X} \rightarrow \underline{S}_2$ is zero. But then also $T \rightarrow \underline{S}_1$ is zero, resulting in $\underline{T} \rightarrow \underline{X}$ zero. //

We now recall the definition of a torsion theory (Dickson [5], pages 223-235) for an abelian category $\mathcal{C}$: is a couple $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ satisfying

(i) $\mathcal{T} \cap \mathcal{F} = \{0\}$

(ii) If $T \rightarrow A \rightarrow 0$ is exact with $T \in \mathcal{T}$, then $A \in \mathcal{T}$.

(iii) If $0 \rightarrow A \rightarrow F$ is exact with $F \in \mathcal{F}$, then $A \in \mathcal{F}$.

(iv) For each object X of $\mathcal{C}$, there is an exact sequence $0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$ with $T \in \mathcal{T}$, $F \in \mathcal{F}$.

The pair $(\mathcal{T}, \mathcal{S}')$ is thus a torsion theory for $\mathcal{E}/\mathcal{S}$, and by theorem 5.12, $\mathcal{S}'$ is closed under quotients, so it is cohereditary.

We will return to the study of these subcategories as a torsion theory in a later chapter.

<u>Proposition 5.15</u> <u>(i)</u> The inclusion $\nu : \mathcal{T} \rightarrow \mathcal{E}/\mathcal{S}$ has a right adjoint.

(ii) The inclusion $\mu : \mathcal{S} \longrightarrow \mathcal{E}/\mathcal{S}$ has a left adjoint.

Proof Define $t(\underline{E})$ and $r(\underline{E})$ by

$$0 \longrightarrow t(\underline{E}) \longrightarrow \underline{E} \longrightarrow r(\underline{E}) \longrightarrow 0$$

with $t(\underline{E})$ copure and $r(\underline{E})$ pure (using Thm. 5.8) (for torsion theories t the radical, r the coradical). Then the uniqueness and Hom $(\mathcal{T}, \mathcal{S}) = 0$ easily shows t and r are functors

$$\mathrm{Hom}_{\mathcal{T}}(\underline{T}, t(\underline{E})) \cong \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\nu(\underline{T}), \underline{E})$$

is simply the statement that $\mathcal{T}$ is closed under quotients.

$$\mathrm{Hom}_{\mathcal{S}}(r(\underline{E}), \underline{S}) \cong \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\underline{E}, \mu(\underline{S}))$$

assigns $\underline{E} \longrightarrow \mu(\underline{S})$ the induced map
$$\begin{tikzcd} \underline{E} \arrow[r] \arrow[d] & \underline{S} \\ r(E) \arrow[ur, dashed] & \end{tikzcd}$$

out of the cokernel r(E), since
$$\begin{tikzcd} t(E) \arrow[d] & \\ \underline{E} \arrow[r] & \underline{S} \end{tikzcd}$$

is zero. //

Remarks $\mathcal{T}$ is generated by $\{\pi(X)\}$, X finitely presented, a set of small projectives. This would yield an abundance of results if $\mathcal{T}$ were abelian because then $\mathcal{T}$ would be equivalent to a functor category. Unfortunately $\mathcal{T}$ will rarely be abelian as is suggested by prop. 5.6. On the other hand, $\mathcal{S}$ is abelian, however it is doubtful that it will have a generating set of small projectives. Indeed, it will even likely not be locally small. That is, subobjects of a given object may not form a set.

As to throwing the pair $(\mathcal{T}, \mathcal{S})$ into the machinery of torsion theories and localization, the major obstruction is that all such literature on the subject imposes a minimum condition that the underlying category be locally small, and more usually that it is Grothendieck.

Just when is $\mathcal{E}/\mathcal{S}$ Grothendieck, or even just locally small ? The next section will take this subject up.

5.16 Finitely Presented Objects

(a) We assume $\mathcal{A}$ is equivalent to a functor category (co-complete with a set of generating small projectives). In particular, $\mathcal{A}$ is Grothendieck and every object is a direct limit of its finitely generated subobjects. Suppose X is finitely generated. Then X is a quotient of a finitely generated projective. Form $0 \to K \to P \to X \to 0$, then $K = \varinjlim K_i$, K_i finitely generated subobjects of K.

Then
$$\begin{tikzcd} 0 \arrow[r] & K_i \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & X_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & P \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$
.

The X_i are finitely presented and $X = \varinjlim X_i$. Every object will be a direct limit of its finitely generated subobjects, which in turn are direct limits of finitely presented objects. Combining these limits gives that every object is a direct limit of finitely presented objects.

(b) Suppose P is a small projective. Consider a map $P \to \varinjlim Y_i$ (direct limit over a directed set). Now we have an epi $\oplus Y_i \twoheadrightarrow \varinjlim Y_i$, so by projectivity

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ \oplus Y_i \arrow[r] & \varinjlim Y_i \end{tikzcd}$$
.

But P is small so this can be reduced to a finite subsum. Then since this is over a directed set, there exists a Y_j with
$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & \varinjlim Y_i \\ & Y_j \arrow[ur] & \end{tikzcd}$$
. This establishes

$$\varinjlim \mathrm{Hom}\,(P,Y_i) \cong \mathrm{Hom}\,(P,\varinjlim Y_i) ,$$

where $\varinjlim \mathrm{Hom}\,(P,Y_i) \to \mathrm{Hom}\,(P,\varinjlim Y_i)$ is the unique map out of the direct limit induced by the compatible maps $\mathrm{Hom}\,(P,Y_i) \to \mathrm{Hom}\,(P,\varinjlim Y_i)$ which arose from $Y_i \to \varinjlim Y_i$.

<u>Proposition 5.17</u> (stated without proof by Stenstrom [24], page 323)

If A is finitely presented, then for any direct system (Y_i), $\varinjlim \mathrm{Hom}(A,Y_i) \cong \mathrm{Hom}(A,\varinjlim Y_i)$.

Conversely, if $\varinjlim \mathrm{Hom}(A,Y_i) \twoheadrightarrow \mathrm{Hom}(A,\varinjlim M_i)$ for any directed system, then A is finitely presented.

<u>Proof</u> Let $P' \to P \to A \to 0$ be exact, P, P' small projectives.

$$\begin{array}{ccccccc}
0 \to \varinjlim \mathrm{Hom}(A,Y_i) & \to & \varinjlim \mathrm{Hom}(P,Y_i) & \to & \varinjlim \mathrm{Hom}(P',Y_i) \\
\downarrow & & \downarrow\cong & & \downarrow\cong \\
0 \to \mathrm{Hom}(A,\varinjlim Y_i) & \to & \mathrm{Hom}(P,\varinjlim Y_i) & \to & \mathrm{Hom}(P',\varinjlim Y_i)
\end{array}$$

implies that the left side is also an isomorphism.

Conversely, if $A = \varinjlim A_i$ for some directed system (A_i) of finitely presented objects, and if

$\varinjlim \mathrm{Hom}(A,A_i) \twoheadrightarrow \mathrm{Hom}(A,\varinjlim A_i) = \mathrm{Hom}(A,A)$

then the identity factors over some A_i :

$$\begin{tikzcd} & A_i \arrow[dr] & \\ A \arrow[ur] \arrow[rr, equal] & & A \end{tikzcd}$$

. Hence A is a direct summand of a finitely presented object and is also finitely presented.

//

<u>5.18 Construction of Pure Sequences</u>

Suppose $0 \to A \to B \to C \to 0$ is exact, and $C = \varinjlim C_i$, C_i finitely presented. Form the pull-backs

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B_i \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

.

Then in the category $\mathcal{E}$, $0 \to A \to B \to C \to 0$ is the direct limit of $0 \to A \to B_i \to C_i \to 0$.

Now if $0 \to A \to B \to C \to 0$ is also pure, then each $0 \to A \to B_i \to C_i \to 0$ splits since it is a copure subobject of a pure object, hence zero. So a pure sequence is a direct limit of split sequences, in $\mathcal{E}$.

Conversely, given such a direct limit and a map from a finitely presented object

$$\begin{array}{ccccccc} & & & & & & X \\ & & & & & & \downarrow \\ 0 & \longrightarrow & \varinjlim A_i & \longrightarrow & \varinjlim B_i & \longrightarrow & \varinjlim C_i \end{array} ,$$

X factors over C_j for some j, so

$$\begin{tikzcd} & & & X \arrow[d] & \\ 0 \arrow[r] & A_j \arrow[r] \arrow[d] & B_j \arrow[r, shift left] \arrow[d] & C_j \arrow[r] \arrow[l, shift left] \arrow[d] & 0 \\ 0 \arrow[r] & \varinjlim A_i \arrow[r] & \varinjlim B_i \arrow[r] & \varinjlim C_i & \end{tikzcd}$$

shows $X \longrightarrow \varinjlim C_i$ factors over $\varinjlim B_i$. This shows the limit sequence is pure, establishing

<u>Proposition 5.19</u> $0 \to A \to B \to C \to 0$ is pure <u>iff</u> it is a direct limit, in $\mathcal{E}$, of split sequences.//

<u>Corollary 5.20</u> $0 \to \bigoplus A_i \longrightarrow \prod A_i \longrightarrow E \longrightarrow 0$ is pure.

<u>Proof</u> This is the direct limit of sequences

$$0 \longrightarrow \bigoplus_J A_i \longrightarrow \prod A_i \longrightarrow E_J \longrightarrow 0 \text{ , J finite. //}$$

<u>Corollary 5.21</u> (of Prop. 5.17) The sequence

$$0 \longrightarrow K \longrightarrow \bigoplus A_i \longrightarrow \varinjlim A_i \longrightarrow 0$$

used in the construction of direct limits from the sum is pure.

<u>Proof</u> If X is finitely presented, then any $X \longrightarrow \varinjlim A_i$ factors through some A_j, hence through $\bigoplus A_i$ .//

CHAPTER 6

PURE SEMISIMPLE CATEGORIES

Rather than impose that $\mathcal{E}/\mathcal{S}$ be Grothendieck, we will find sufficient conditions on $\mathcal{A}$ to force $\mathcal{E}/\mathcal{S}$ to become Grothendieck.

Suppose $\mathcal{E}/\mathcal{S}$ has a generator $0 \to A \to B \to C \to 0$, so $\pi(C)$ also is a generator. C is an object of $\mathcal{A}$ and we can assume it generates $\mathcal{A}$. (If necessary, replace C by $C \oplus U$, with U generating $\mathcal{A}$, and $0 \to A \to B \to C \to 0$ with $0 \to A \to B \oplus U \to C \oplus U \to 0$.)

Given X in $\mathcal{A}$, form $\underline{E} = 0 \to K \to \oplus_I C \to X \to 0$ for some direct sum of C. Now $\pi(C)$ generates this sequence, so $\oplus_K \pi(C) \twoheadrightarrow \underline{E}$ for some index set K. Then the cokernel must split, and this is the sequence

$$0 \to L \to (\oplus_K C) \oplus (\oplus_I C) \to X \to 0$$

, L the kernel of the sum map. This gives the nontrivial part of

<u>Proposition 6.1</u> $\mathcal{E}/\mathcal{S}$ has a generator <u>iff</u> there exists an object C in $\mathcal{A}$ such that every object of $\mathcal{A}$ is a direct summand of a direct sum of copies of C. //

By Prop. 2.8, if $\mathcal{A}$ is (co-)complete, then so is $\mathcal{E}/\mathcal{S}$. This is condition Ab 3(*). Condition Ab 4 is : given a family of monics $\{A_i \to B_i\}$, then $\oplus A_i \to \oplus B_i$ is monic (Mitchell calls this condition C_1).

<u>Proposition 6.2</u> $\mathcal{A}$, Ab 4 $\Longrightarrow$ $\mathcal{E}/\mathcal{S}$, Ab 4.

<u>Proof</u> If $\underline{A}_i \to \underline{B}_i$ are monic, realize these as

$$\begin{array}{ccccccccc} 0 & \to & A_i'' & \to & A_i & \to & A_i' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B_i'' & \to & B_i & \to & B_i' & \to & 0 \end{array}$$

Then the kernel $0 \to A_i'' \to A_i \oplus B_i'' \to E_i \to 0$ splits.

The sum map is

$$\begin{array}{ccccccccc} 0 & \to & \oplus A_i'' & \to & \oplus A_i & \to & \oplus A_i' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \oplus B_i'' & \to & \oplus B_i & \to & \oplus B_i' & \to & 0 \end{array}$$

with kernel $0 \to \oplus A_i'' \to (\oplus B_i'') \oplus (\oplus A_i) \to E \to 0$.

But $(\oplus B_i'') \oplus (\oplus A_i) \cong \oplus (B_i'' \oplus A_i)$, so the sum of the splitting maps splits the kernel sequence.

Hence $\oplus \underline{A}_i \to \oplus \underline{B}_i$ is monic. //

A category is C_2 if for any direct sum $\oplus X_i$, the natural map $\oplus X_i \longrightarrow \prod X_i$ is monic. Module categories are trivially C_2, in fact most reasonable categories are C_2. However this is a very deep imposition on $\mathcal{E}/\mathcal{S}$.

Proposition 6.3 $\mathcal{E}/\mathcal{S}$ is C_2 iff given any set of monics $C_i \to D_i$ in $\mathcal{G}$, the map $\oplus C_i \to (\prod C_i) \oplus (\oplus D_i)$ splits.

Proof This is simply a restatement of the definition from $\mathcal{E}/\mathcal{S}$ to $\mathcal{O}\!\!\!\!/$, i.e. $\oplus C_i \rightarrowtail (\prod C_i) \oplus (\oplus D_i) \longrightarrow E \longrightarrow 0$ is the kernel of the map from sum to product. //

Lemma 6.4 Given $A \to B$ monic,and $A \to I$, I injective. If $A \rightarrowtail B \oplus I$ splits, then $A \rightarrowtail B$ splits.

Proof Lift $A \to I$ to $B \xrightarrow{\alpha} I$. Then

$$A \rightarrowtail B \xrightarrow{\binom{1}{\alpha}} B \oplus I \;=\; A \rightarrowtail B \oplus I \text{ splits and hence}$$

also $A \rightarrowtail B$ splits.

(Note In $\mathcal{E}/\mathcal{S}$ jargon, the kernel of

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.O.}"] & B \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & E \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

splits hence is zero, so this is an isomorphism, but the bottom row splits because I is injective, hence top row also splits.) //

A locally Noetherian category is a Grothendieck category having a set of Noetherian generators (i.e. Mod R is locally Noetherian iff R is Noetherian). When $\mathcal{O}\!\!\!\!/$ has a generating set of finitely generated objects, this is the equivalent to the condition that the direct sum of injectives is injective.

Corollary 6.5 If $\mathcal{O}\!\!\!\!/$ is locally Noetherian, $\mathcal{E}/\mathcal{S}$ is C_2 iff $\oplus A_i \longrightarrow \prod A_i$ splits for any direct sum.

Proof Take $A_i \longrightarrow I_i$, I_i injective, apply Prop. 6.3 and Lemma 6.4. //

At this point $\mathcal{A}$ will become a module category over a ring R, although much of what follows probably generalizes to functor categories.

Definition (a) M is Pure - injective if given any map

$$\begin{array}{c} 0 \to A \to B \to C \to 0 \\ \downarrow \quad\quad\quad\quad\quad \\ M \quad\quad\quad\quad\quad \end{array}$$

with $0 \to A \to B \to C \to 0$ pure, $A \to M$ factors through B. This is more frequently called algebraically compact.

(b) M is Σ-algebraically compact (Σ pure-injective) if any direct sum of copies of M is algebraically compact.

By a Theorem of Wolfgang Zimmermann [27], this is equivalent to $0 \to \bigoplus_I M \to \prod_I M$ splitting for arbitrary sums of copies of M.

(Note - Cor. 5.20 gives the implication one way.)

Theorem 6.6 The following are equivalent :

(i) All modules are algebraically compact (pure injective).

(ii) All modules are pure projective.

(iii) All pure sequences split.

(iv) All sequences are copure.

(v) $\mathcal{E}/\mathcal{S}$ is C_2 and $\mathcal{A}$ is locally Noetherian.

(vi) $\mathcal{E}/\mathcal{S}$ is C_2 and has a generator.

(vii) $\mathcal{E}/\mathcal{S}$ is Grothendieck.

(viii) $\mathcal{E}/\mathcal{S}$ is equivalent to a functor category.

Proof Equivalence of (i),(ii),(iii),and (iv) is playing with language.

(iv) $\Rightarrow$ (viii) The $\{K \to P \to A\} = \mathcal{L}$ of finite projective presentations, P finitely generated projective, K finitely generated, is a generating set of small projectives in the co-complete abelian category $\mathcal{E}/\mathcal{S}$. Hence $\mathcal{E}/\mathcal{S}$ is equivalent to $(\mathcal{L}^*, \mathrm{Ab})$.

(viii) $\Rightarrow$ (vii) $\Rightarrow$ (vi) trivial.

(vi) $\Rightarrow$ (v) By Prop. 6.1, every object of $\mathcal{O} =$ Mod R is a direct summand of a direct sum of copies of some fixed module (subobject would suffice). This implies that R is Noetherian (e.g. Fuller & Anderson [1], page 297 Cor. 26.3).

(v) $\Rightarrow$ (i) By Cor. 6.5, $\bigoplus M \to \prod M$ splits for arbitrary sums. Applying the Zimmermann result gives that every M is Σ algebraically compact.//

Corollary 6.7 A ring R satisfying the condition of the theorem is artinian.

Proof Since all pure sequences split, flat $\Rightarrow$ projective (M is flat iff $0 \to A \to B \to M \to 0$ pure for all such sequences). Hence R is both perfect and Noetherian, which implies Artinian. //

Corollary 6.8 The conditions of Thm. 6.6 are also equivalent to : every module is a direct sum of finitely generated modules.

Proof A module is pure-projective iff it is a direct summand of a direct sum of finitely presented modules by Cor, 5.10 (i).

Now each finitely presented module is a direct sum of indecomposables (necessarily finitely presented) with local endo morphism rings since R is Artinian. By a theorem of Crawley-Jonsson-Warfield and its corollary ([ ], pages 299-300), any direct summand of a direct sum of finitely presented modules with local endomorphism rings is again of this form.

Conversely, R is Noetherian (again see reference in (vi) $\Rightarrow$ (v) of theorem).

Hence we can assume finitely presented instead of finitely generated, and so Cor. 5.10 implies that all modules are pure-projective. //

6.9 Remarks

Can the condition of Cor. 6.8 be weakened to 'every module is a direct sum of indecomposables' ? For there is a striking similarity with the rings satisfying Thm.6.6 and semi-simple rings. For semi-simple rings, one has that

(i) all sequences split ($\mathcal{E}/\mathcal{S} = 0$) ; and

(ii) all modules are direct sums of simples

are equivalent statements. Replacing all sequences by pure sequences and simples by finitely generated indecomposables, the equivalence remains intact. Daniel Simson has coined (or at least promotes) the name pure semi-simple rings for the rings satisfying pure $\Rightarrow$ split. For semi-simple rings, one has the Wedderburn structure theorem, which uses matrix rings as building blocks. Is there a structure theorem for pure-semi-simple rings, and what is the suitable replacement for matrix rings (simple rings) ?

The Wedderburn theorem yields two important results : That right semi-simple $\Rightarrow$ left semi-simple, and a quick proof that there are only finitely many non-isomorphic simples. Even if there is no structure theorem akin to the Wedderburn, is it true that right pure semi-simple $\Rightarrow$ left pure semi-simple , and is there only a finite number of non-isomorphic (finitely generated) indecomposables ? Towards a solution of these problems, M. Auslander has shown that a ring is both right and left pure semi-simple <u>iff</u> it is of finite representation type (left Artinian, with a finite number of non-isomorphic finitely generated left decomposables). So the problem becomes one of showing that left pure semi-simple rings are of finite representation type. A great deal of effort has been put into this. I had the opportunity to talk to M.Auslander (and others at the Canadian Mathematical Conference, December, 1980) concerning this problem. M.Auslander at first thought the proof would be straight-forward, and in fact thought he had solved it, but caught his own error when writing it up. Kent Fuller was less fortunate and Auslander caught his mistake during Fuller's presentation at a ring theory conference. D. Simson even less fortunate, published an incorrect proof, again error was pointed put by Auslander, (see Simson [23]). L. Gruson also believed he had solved the problem but fell short of completion.

As of December 1980, Auslander still felt that left pure semi-simple $\Rightarrow$ finite representation type, but had stopped working on the problem. Kent Fuller also had given up, commenting that he felt it was 'undecidable', and that a solution would involve set theoretic considerations (akin to Martin's axiom for the solution of the Whitehead conjecture that $\mathrm{Ext}'(A, \mathbb{Z}) = 0 \Rightarrow A$ is free), and D. Simson now believes the conjecture is false. (see Simson [23]).

It is unfortunate that at present I cannot conquer the dragon. However I hope that the previous discussion indicates that this is an important area of investigation. So the following results may seem lacking in content standing on their own, but the hope is that they can be used as building blocks towards a solution.

The first move towards a solution will be to express the condition of finite representation type into a statement concerning $\mathcal{E}/\mathcal{S}$ (a more categorical condition).

<u>Proposition 6.10</u> A ring R is of finite representation type <u>iff</u> $\mathcal{E}/\mathcal{S}$ is equivalent to a module category.

<u>Proof</u> $\Rightarrow$ Rings of finite represented type satisfy the conditions of Cor. 6.8 (see for instance Fuller & Reiten [10]). So $\mathcal{E}/\mathcal{S}$ is a functor category with the set of $\{K \to P \to A\}$ of finite projective presentations as a set of generators. But if there is only a finite set of finitely generated indecomposables this can further be reduced to a finite set. So $\mathcal{E}/\mathcal{S}$ has a small projective generator and is co-complete, implying that it is equivalent to a module category (the ring being the endomorphism ring of the small projective generator).

$\Leftarrow$ By Thm. 6.6, $\mathcal{E}/\mathcal{S}$ is a functor category with $\{\pi(A)\} = \{K \to P \to A\}$, with A indecomposable finitely presented, as a set of small projective generators.

<u>Lemma 6.11</u> (i) If A has a local endomorphism ring, then $\pi(A)$ is indecomposable projective (also with local endomorphism)

(ii) $0 \neq \pi(A) \cong \pi(A')$ <u>iff</u> $A \approx A'$ (where A' has local endomorphism).

<u>Proof</u> (i) Trivial since End $(\pi(A)) \approx$ End A/P(A)

(P(A) = endomorphisms factoring over a projective)

(ii) If $\pi(A) \cong \pi(A')$ this isomorphism must arise from a map $A \to A'$ inducing the isomorphism

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & K' & \to & P' & \to & A' & \to & 0 \end{array}$$

But then the cokernel is zero, i.e. the sequence $0 \to E \to P' \oplus A \to A' \to 0$ splits. But since A' has a local endomorphism ring, $P' \oplus A \to A' \to 0$ splits, forcing $P' \to A' \to 0$ or $A \to A' \to 0$ to split (see Lemma 6.14 ahead). The first would imply $\pi(A') = 0$ (i.e. A' projective). Hence $A \to A' \to 0$ splits, and so must be an isomorphism.//

<u>Proof of Proposition 6.10 continued</u>

So $\{\pi(A)\}$, A indecomposable finitely generated, is a generating set of small indecomposable projectives with local endomorphism rings. Then $U = \bigoplus_{i \in I} \pi(A_i)$, taken over the set $\{A_i\}$ of nonisomorphic indecomposable finitely generated modules, is a generator. Now, by assumption, $\mathcal{E}/\mathcal{S}$ has a small projective generator, so there is an epi $\oplus U \twoheadrightarrow V$. But then the splitting monic $V \rightarrowtail \oplus U$ can be reduced to a finite subsum of copies of U, and then further reduced to a finite subsum of $\{\pi(A_i)\}$. That is, there is a splitting $V \rightarrowtail \bigoplus_{\text{finite}} \pi(A_i)$; this implies that $\bigoplus_{\text{finite}} \pi(A_i)$ is a generator. Now given any A_j , $\pi(A_j)$ is the (split) epi image of

$$\Big(\bigoplus_{\text{finite}} \pi(A_i)\Big)^n \twoheadrightarrow \pi(A_j)$$

for some n. But $\pi(A_j)$ has a local endomorphism ring so $\pi(A_i) \twoheadrightarrow \pi(A_j)$ splits for some i, by the Lemma 6.11, $A_i \cong A_j$. Hence there are only finitely many nonisomorphic A_i . Hence R is of finite representation type (R is Artinian by Cor. 6.7). //

<u>6.12 Remarks</u>

Thm. 6.6 was proved for $\mathcal{O}\!\!/$ a module category, the crucial step being the use of the Zimmerman result that M is Σ-algebraically compact <u>iff</u> $\bigoplus_I M \to \prod_I M$ splits for arbitrary index sets I. All the rest of the theorem is valid providing that $\mathcal{O}\!\!/$ has a generating set of small projectives and is co-complete (i.e. a functor category). The Zimmerman result probably holds for functor categories, but I have not proved it yet. This raises the question of generalization simply for the sake of getting new results. The theory surrounding Thm. 6.6 arises from pure-semi-simple rings which are important enough to neglect the added constraint of proving the result for functor categories. However one is faced with problems which can not be dismissed as just 'generalizing'. For the conjecture is now: $\mathcal{E}/\mathcal{S}$ functor category $\Rightarrow$ $\mathcal{E}/\mathcal{S}$ module category . If the problem is to be solved, a first battle plan would be to study $\mathcal{E}/\mathcal{S}$ as a functor category. To a great extent this will require verifying that certain ring theory results hold in functor categories, so we will assume for the next section that $\mathcal{O}\!\!/$ is a functor category with a generating set $\{P_\alpha\}$ of small projectives.

For the results concerning semi-perfect objects, the proofs in the ring theory case (module categories) can be found in Mares [18].

6.13 Baer-Injective Test

X is injective iff X is injective relative to $\{P_\alpha\}$.

Proof Consider $\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \\ X \end{tikzcd}$. Let $\bar{A}$ be a maximal extension in B. (Zorn's lemma can be used since there is only a finite set of non-isomorphic monics into B). Assume $\bar{A} \neq B$.

Form

$$\begin{tikzcd}
K \arrow[r] \arrow[d, dotted] & P_\alpha \arrow[r] \arrow[d, dotted] & C \arrow[d, equal] \\
\bar{A} \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, tail] & C \arrow[d, tail] \\
\bar{A} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dotted] & \bar{C} \arrow[d, dotted] \\
X \arrow[r] & I \arrow[r] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi(C) \arrow[d] \\
\underline{E'} \arrow[d, tail] \\
\underline{E} \arrow[d] \\
\mathcal{J}(X)
\end{tikzcd}$$

where C is a non-zero finitely generated subobject of $\bar{C}$, by assumption $K \longrightarrow \bar{A} \longrightarrow X$ can be lifted to P_α, i.e. $\pi(C) \twoheadrightarrow \underline{E'} \rightarrowtail \underline{E} \longrightarrow \mathcal{J}(X) = 0$ where $\mathcal{J}(X)$ is the injective sequence. Hence $\underline{E'} \rightarrowtail \underline{E} \longrightarrow \mathcal{J}(X) = 0$ which means $\bar{A} \rightarrow X$ can be extended to E (Prop. 2.4).

But $E \rightarrowtail B$ since it is a pull-back of the monic $C \rightarrowtail \bar{C}$. Then the maximality of $\bar{A}$ implies that $\bar{A} \rightarrowtail E$ is an isomorphism $\Rightarrow$ C = 0 a contradiction. //

Lemma 6.14 If C has a local endomorphism ring and $\overset{n}{\oplus} C_i \overset{\alpha}{\twoheadrightarrow} C$ split then $C_i \longrightarrow \overset{n}{\oplus} C_i \longrightarrow C$ splits for some i.

Proof Let $\varphi_i = C_i \rightarrowtail \overset{n}{\oplus} C_i \twoheadrightarrow C$ and

$\psi_i = C \rightarrowtail \overset{n}{\oplus} C_i \twoheadrightarrow C_i$ where ψ is the split.

Then $\sum \psi_i \varphi_i = 1_C$ in End C. So at least one of the $\psi_i \varphi_i$ is a unit, implying that φ_i is a split epi. //

Lemma 6.15 If P is indecomposable projective with a local endomorphism ring, then $P \twoheadrightarrow X$ is a projective cover. (That is, all subobjects are superfluous.)

<u>Proof</u> Let $0 \to K \to P \to X \to 0$. Assume $K \to P$ is not superfluous, then there exists $Y \hookrightarrow P$ with $K + Y = P$. Form

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, dashed] & P \arrow[r] \arrow[d, dashed] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K' \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] & P \arrow[r] & X \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \pi(X) \arrow[d, "\overline{\alpha}"] \\ \pi(X) \end{tikzcd} .$$

Let α be $P \to Y \to P$. Consider the difference $1-\overline{\alpha}$

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r, two heads] \arrow[d, "1-\alpha"] & X \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & K \arrow[r] & P \arrow[r, two heads] & X \arrow[r] & 0 \end{tikzcd} .$$

Commutativity implies $1-\alpha$ is not a unit in End P, hence α must be a unit since End P is local. Hence $Y \hookrightarrow P$ is also epi, implying that $Y = P$. //

<u>Remark</u> If C is finitely generated with local endomorphism ring, then $\pi(C)$ is a small projective with local endomorphism ring in $\mathcal{E}/\mathcal{S}$, and all its subobjects are superfluous. If $\pi(C)$ had only a set of subobjects, the total sum would be the unique maximal subobject of $\pi(C)$ (since $\pi(C)$ is a small object, a direct sum of proper subobjects is proper) , and so the quotient would be simple. Unfortunately, this quick way of producing simples will fail if there is more than a set of subobjects.

<u>Lemma 6.16</u> Let $0 \neq X$ be a small subobject of A, such that $X \nsubseteq N$, $N \subseteq A$. Then there is a subobject $\widetilde{N}$ containing N maximal with respect to not containing X.

<u>Proof</u> Order the subobjects of A containing N but not X. If $N_1 \hookrightarrow N_2 \hookrightarrow N_3 \hookrightarrow N_4 \ldots$ is an ascending chain, let $\overline{N} = \bigcup N_i$. Since X is small, $X \nsubseteq \overline{N}$. This is the essence of smallness, so we give a proof : let P be a small projective with $P \twoheadrightarrow X$. Form

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, two heads] & \bigoplus N_i \arrow[d, two heads] \\ X \arrow[r, hook] & \bigcup N_i = \overline{N} \end{tikzcd},$$

assuming $X \hookrightarrow \overline{N}$, P is small so there is a factorization through a finite subsum

$$\begin{tikzcd} P \arrow[r] \arrow[dd, two heads] & \overset{k}{\bigoplus} N_i \arrow[r, hook] \arrow[d, two heads] & \bigoplus N_i \arrow[d, two heads] & \\ & N_k \arrow[r, hook] & \overline{N} \arrow[r, two heads] & \overline{N} / N_k \\ X \arrow[urr, hook] & & & \end{tikzcd}.$$

By diagram chase, $\begin{tikzcd} P \arrow[d, two heads] & \overline{N} \arrow[r] & \overline{N} / N_k \\ X \arrow[ur] & & \end{tikzcd} = 0$,

and the epi $P \twoheadrightarrow X$ can be cancelled, so $X \to \overline{N} \to \overline{N}/N_k$ is zero and $X \longrightarrow \overline{N}$ can be factored through $N_k \hookrightarrow \overline{N}$, that is $X \hookrightarrow N_k$ contradiction.

Hence Zorn's lemma can be applied to achieve maximal elements. //

<u>Corollary 6.17</u> If $X + N = A$ and $\overline{N}$ is maximal with respect to containing N but not X, then $\overline{N}$ is a maximal subobject of A. //

<u>Corollary 6.18</u> Rad A = $\sum$ superfluous subobjects of A.

<u>Proof</u> Definition of Rad A = $\bigcap$ maximal subobjects.

If X is superfluous, and M is maximal then

$$X + M \neq A \Rightarrow X \subseteq M.$$

Since any object is the directed limit of its finitely generated subobjects, and finitely generated implies small (if there is a set of small projective generators, see Lemma 5.2) , it suffices to show that small subobjects X of Rad A are superfluous.

Suppose $X + N = A$. If $X \nsubseteq N$, there exists a maximal subobject not containing X, by Cor. 6.17, which contradicts $X \subseteq$ Rad A. //

G.M.Kelly [17] defined the radical of a category by

$$\mathcal{J}(\mathrm{Hom}_{\mathcal{C}}(A,B)) = \left\{ f \in \mathrm{Hom}(A,B) \;\middle|\; \begin{array}{l} 1-gf \text{ is a unit in End } A \\ \text{for all } g \in \mathrm{Hom}(B,A) \end{array} \right\}$$

Equivalently,

$$= \left\{ f \in \mathrm{Hom}(A,B) \;\middle|\; gf \in \mathrm{Rad}(\mathrm{End}\ A) \ \forall g \in \mathrm{Hom}(B,A) \right\}$$

Equivalently,

$$= \left\{ f \in \mathrm{Hom}(A,B) \;\middle|\; \begin{array}{l} 1-fg \text{ is a unit in End } B \\ \text{for all } g \in \mathrm{Hom}(B,A) \end{array} \right\}$$

Equivalently,

$$= \left\{ f \in \mathrm{Hom}(A,B) \;\middle|\; fg \in \mathrm{Rad}(\mathrm{End}\ B) \ \forall g \in \mathrm{Hom}(B,A) \right\} .$$

The next proposition is a simple extension of the fact that Rad (End P) is the set of morphisms with superfluous images. In fact, the proof is almost identical, but is included for completeness.

<u>Proposition 6.19</u> If P is projective, then $\mathcal{J}(Q,P)$ is the subgroup of morphisms with superfluous images.

<u>Proof</u> Suppose $\mathrm{Im}\,\varphi$ is superfluous, $\varphi : Q \longrightarrow P$.

Given any $\psi : P \longrightarrow Q$,

$$P = \mathrm{Im}\ 1_P = \mathrm{Im}\ ((1_P - \psi\varphi) + \psi\varphi)$$
$$\subseteq \mathrm{Im}\ (1_P - \psi\varphi) + \mathrm{Im}\ (\psi\varphi)$$

so $P = \mathrm{Im}\ (1_P - \psi\varphi) + \mathrm{Im}\ (\psi\varphi)$. But if $\mathrm{Im}\,\varphi$ is superfluous then so is $\mathrm{Im}\ \psi\varphi$, hence $\mathrm{Im}\ (1_P - \psi\varphi) = P$, $1_P - \psi\varphi$ is an epi, $P \rightarrow P$ hence splits implying it is a unit in End (P,P). So by definition φ is in Rad (Q,P).

Conversely, if φ is in $\mathcal{J}(Q,P)$ suppose $K + \mathrm{Im}\,\varphi = P$, then $Q \rightarrow P \longrightarrow P/K$ is epi , and

$$\begin{tikzcd} & & P \arrow[d, "h"] \arrow[lld, dashed, "s"'] \\ Q \arrow[r, "\varphi"'] & P \arrow[r, "h"'] & P/K \end{tikzcd}$$

gives $(1 - s\varphi)h = 0$.

But if $\varphi \in \mathcal{J}(Q,P)$, $1-s\varphi$ is invertible hence h is zero implying $K = P$, so $\operatorname{Im} \varphi$ is superfluous. //

Corollary 6.20 $\operatorname{Rad} P = \operatorname{Im}\left(\bigoplus_{\alpha} \left(\bigoplus_{\varphi \in \mathcal{J}(P_\alpha, P)} P_{\alpha,\varphi} \right)\right)$ where $P_{\alpha,\varphi}$ is an isomorphic copy of P_α for each $\varphi \in \mathcal{J}(P_\alpha, P)$ and $P_{\alpha,\varphi} \to P$ is the map φ.

Proof By Prop. 6.19, if $\varphi \in \mathcal{J}(P_\alpha, P)$ then $\operatorname{Im} \varphi$ is superfluous, hence the image of

$$\left(\bigoplus_{\alpha} \bigoplus P_{\alpha,\varphi} \right) \longrightarrow P$$

is contained in Rad P.

Conversely, let X be any small object of Rad P. Then X is superfluous in P. Letting $P_\alpha \twoheadrightarrow X$, by Prop. 6.19 again $P_\alpha \twoheadrightarrow X \hookrightarrow P$ lies in $\mathcal{J}(P_\alpha, P)$. Since Rad P is the sum of its small subobjects, this implies the inclusion the other way. //

Corollary 6.21 $\operatorname{Rad} P \neq P$ for P projective (an extension of a theorem of Bass, see Prop. 17.14 in Fuller & Anderson [1] for module case).

Proof If $\operatorname{Rad}(P) = P$, then $\bigoplus \left(\bigoplus P_{\alpha,\varphi} \right) \twoheadrightarrow P$ is epi, hence splits. For each $P_{\alpha,\varphi} \xrightarrow{\varphi} P$, let $\hat{\varphi}$ be $P \longrightarrow \bigoplus \left(\bigoplus P_{\alpha,\varphi} \right) \xrightarrow{\varphi} P$ (first map the split monic), i.e. the 'dual basis'. For X a small subobject of P, the split monic factors through a finite subsum of $\bigoplus \left(\bigoplus P_{\alpha,\varphi} \right)$, which means that there is a finite set φ_i with $\left(1 - \sum \hat{\varphi}_i \varphi_i\right)\Big|_X$ the zero map (on X).

But since the sum is finite, $\sum \hat{\varphi}_i \varphi_i \in \operatorname{Rad}(\operatorname{End} P)$, implying that $1 - \sum \hat{\varphi}_i \varphi_i$ is invertible, which is impossible, hence $\operatorname{Rad} P \neq P$. //

Corollary 6.22 If P is a projective summand of A and $P \subseteq \operatorname{Rad} A$ then $P = 0$.

Proof Let X be finitely generated, hence small subobject of P. Then $X \subseteq P \subseteq \operatorname{Rad} A = \sum$ superfluous $\Rightarrow$ by smallness, that X is superfluous in A. Since P is a direct summand, this implies X is also superfluous in P, hence $X \subseteq \text{Rad } P$. This implies that $P = \text{Rad } P$, hence $P = 0$. //

A semi-perfect object is defined to be a projective object such that all its quotients have projective covers.

<u>Proposition 6.23</u> P is semi-perfect <u>iff</u>

(i) Rad P is superfluous in P

(ii) P/Rad P is semi-simple

(iii) each simple component of P/Rad P has a projective cover.

<u>Proof</u> ($\Rightarrow$) To establish (i), let $Q \twoheadrightarrow P/\text{Rad } P$ be a projective cover, claim $Q \cong P$.

Form
$$\begin{tikzcd} & Q' \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \text{Rad } P \arrow[r] & P \arrow[r, two heads] & P/\text{Rad } P \arrow[r] & 0 \end{tikzcd}$$

The map $P \rightarrow Q$ resulting from projectivity of Q must be epi since $Q \twoheadrightarrow P/\text{Rad } P$ is a cover. Hence $P \twoheadrightarrow Q$ splits and Q' is a projective summand of P contained in Rad P, $\Rightarrow$ $Q' = 0$ by Cor. 6.22, hence $Q \cong P$ and Rad P is superfluous.

To establish (ii), let $P/\text{Rad } P \twoheadrightarrow V$. We must show that this splits. Let $Q \twoheadrightarrow V$ be a projective cover, and consider

$$\begin{tikzcd} & K \arrow[d, hook] \\ & Q \arrow[d, two heads] \arrow[dl] \\ P \arrow[r, two heads] \arrow[d, two heads] & V \arrow[d, equal] \arrow[dl, dashed] \\ P/\text{Rad } P \arrow[r, two heads] & V \end{tikzcd}$$

There is an induced map from $Q \rightarrow P$ which is a split monic since $Q \twoheadrightarrow V$ is the unique projective cover. Then K is superfluous in Q hence in P, and so $K \rightarrow P/\text{Rad } P = 0$ inducing a map out of $V = \text{coker} : K \hookrightarrow Q$. This is the required split since

$$\begin{tikzcd} & Q \arrow[d, two heads] \\ & V \arrow[dl] \\ P/\mathrm{Rad}\,P \arrow[r, two heads] & V \end{tikzcd} = \begin{tikzcd} & Q \arrow[dl] \\ P \arrow[d, two heads] & \\ P/\mathrm{Rad}\,P \arrow[r, two heads] & V \end{tikzcd} = \begin{tikzcd} & Q \arrow[dl] \\ P \arrow[r] & V \arrow[d, equal] \\ & V \end{tikzcd} = \begin{tikzcd} Q \arrow[d, two heads] \\ V \arrow[d, equal] \\ V \end{tikzcd}$$

Cancel the epi $Q \twoheadrightarrow V$.

(iii) is obvious.

($\Leftarrow$) Let $A \rightarrowtail P$, we want to construct a projective cover for $\bar{A} = \mathrm{coker} : A \rightarrowtail P$. W.l.o.g., $\mathrm{Rad}\,P \subseteq A$.

for consider

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, equal] & P/A \arrow[r] \arrow[d, two heads] & 0 \\ A + \mathrm{Rad}\,P \arrow[r] & P \arrow[r] & P/A + \mathrm{Rad}\,P \arrow[r] & 0 \end{tikzcd}$$

$P/A \twoheadrightarrow P/(A + \mathrm{Rad}\,P)$ is superfluous, for its kernel is $(A + \mathrm{Rad}\,P)/A$ which is superfluous in P/A, since Rad P is small in P. Hence a projective cover of $P/(A + \mathrm{Rad}\,P)$ will also be a projective cover of P/A.

So assuming $\mathrm{Rad}\,P \subseteq A$, consider

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Rad}\,P \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, equal] & P/\mathrm{Rad}\,P \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & A \arrow[r] & P \arrow[r] & P/A \arrow[r] & 0 \end{tikzcd}$$

Since $P/\mathrm{Rad}\,P$ is semi-simple, so is P/A.

Let $P/\mathrm{Rad}\,P \cong P/A \oplus X$ with $P/\mathrm{Rad}\,P \cong \bigoplus_{i \in I} S_i$

and $P/A \cong \bigoplus_{j \in J} S_j$, $X \cong \bigoplus_{k \in I \setminus J} S_k$, J a subset of I, S_i simple objects. Let $P_i \to S_i$ be projective covers, and consider

$$\begin{tikzcd} & & \bigoplus_I K_i \arrow[d, tail] \\ & & \bigoplus_I P_i \arrow[d] \arrow[dll, dashed, "\nu"'] \\ P \arrow[r] & P/\mathrm{Rad}\,P \cong & (\bigoplus_J S_i) \oplus (\bigoplus_{I \setminus J} S_k) \end{tikzcd}$$

ν exists since $\oplus P_i$ is projective, it is epi since $P \to P/\mathrm{Rad}\,P$ is superfluous, hence it splits because P is projective. But then Ker $\nu \subseteq \oplus K_i$ $\subseteq \sum$ (superfluous subobjects of $\oplus_I P_i$) = Rad ($\oplus_I P_i$) is a projective summand of the radical and is zero by Cor. 6.22. ν is thus an isomorphism, implying that $\oplus_I K_i$ is small in $\oplus_I P_i$, which implies $\oplus_J K_j$ is small in $\oplus_J P_j$, so $\oplus_J P_j \twoheadrightarrow \oplus_J S_j$ = P/A is a projective cover.//

Corollary 6.24 A finite direct sum of semi-perfect objects is semi-perfect.//

We return to the category $\mathcal{E}/\mathcal{S}$. By Lemma 6.15, indecomposable projectives with local endomorphism rings are semi-perfect. Suppose $C \in \mathcal{O}$ has a local endomorphism ring. Then $\pi(C)$ is projective with local endomorphism ring, hence semi-perfect. If

$$C = \overset{n}{\oplus} C_i ,$$

with each End (C_i) local, $\pi(C) \cong \overset{n}{\oplus} \pi(C_i)$ is then semi-perfect by Cor. 6.24.

Now if $\mathcal{E}/\mathcal{S}$ is a functor category , $\{\pi(C_\alpha)\}$ with C_α finitely presented and End (C_α) local is a generating set of small projectives. Hence every finitely generated object of $\mathcal{E}/\mathcal{S}$ is an epimorphic image of $\underset{\text{finite}}{\oplus} \pi(C_\alpha)$ for some finite set and thus has a projective cover ; this establishes

Proposition 6.25 If $\mathcal{E}/\mathcal{S}$ is a functor category, it is semi-perfect (finitely generated objects have projective covers).//

Remark For Grothendieck categories, the condition 'the direct sum of injectives is injective' is equivalent to locally Noetherian. The proof parallels the module proof using the Baer-Injective test as formulated in 6.13.

By Thm 6.6, $\mathcal{E}/\mathcal{S}$ a functor category implies $\mathcal{O}$ is locally Noetherian, which in turn implies the direct sum of injectives in $\mathcal{E}/\mathcal{S}$ is injective (since $\mathcal{J}(\oplus A_i) \cong \oplus \mathcal{J}(A_i)$, where $\mathcal{J}(A) = 0 \longrightarrow A \longrightarrow I \longrightarrow N \longrightarrow 0$ is an injective co-presentation). So since $\mathcal{E}/\mathcal{S}$ will in this case also be Grothendieck, this yields that $\mathcal{E}/\mathcal{S}$ is locally Noetherian. Thus if $\mathcal{O}$ is pure semi-simple, then $\mathcal{E}/\mathcal{S}$ is a semi-perfect locally Noetherian functor category. Next, we show $\mathcal{E}/\mathcal{S}$ is in fact perfect.

Theorem 6.26 If $\mathcal{O}$ is pure semi-simple, then $\mathcal{E}/\mathcal{S}$ is a perfect, locally Noetherian functor category.

Proof We need to show that each object has a projective cover. To do this, it suffices to show that every projective is semi-perfect. And since $\{\pi(C_\alpha)\}$, where C_α is finitely presented and indecomposable, is a generating set of projectives, we need only show that $\bigoplus_{\alpha \in I} \pi(C_\alpha)$ is semi-perfect for arbitrary index set I.

Now $\pi(C_\alpha)$ is semi-perfect by 6.25. Hence $\pi(C_\alpha)$ satisfies the three conditions of Prop. 6.23, and the last two will also hold for arbitrary sums of $\pi(C_\alpha)$ so it suffices to prove that the radical of $\bigoplus_{\alpha \in I} \pi(C_\alpha)$ is superfluous. The following lemmas hold for a general abelain category $\mathcal{O}$.

Let $\overset{n}{\oplus} A_i$ and $\overset{m}{\oplus} B_j$ be finite sums in $\mathcal{O}$. Then $\mathrm{Hom}\,(\overset{n}{\oplus} A_i , \overset{m}{\oplus} B_j) \cong \mathcal{M}_{n,m}\,(\mathrm{Hom}\,(A_i,B_j))$ [nxm matrices].

Lemma 6.27 $\mathcal{J}(\mathrm{Hom}\,(\overset{n}{\oplus} A_i , \overset{n}{\oplus} B_j)) \cong \mathcal{M}_{n,m}\,(\mathcal{J}(\mathrm{Hom}(A_i,B_j)))$

Proof This follows easily from

$$\mathcal{J}(\mathrm{Hom}\,(A_1 \oplus A_2 ,B)) \cong \mathcal{J}(\mathrm{Hom}(A_1,B)) \oplus \mathcal{J}(\mathrm{Hom}(A_2,B))$$

and

$$\mathcal{J}(\mathrm{Hom}\,(A,B_1 \oplus B_2)) \cong \mathcal{J}(\mathrm{Hom}(A,B_1)) \oplus \mathcal{J}(\mathrm{Hom}(A,B_2)).//$$

The essence of this lemma is that

$$f : \bigoplus^{n} A_i \longrightarrow \bigoplus^{m} B_j \quad \text{is in the radical}$$

if and only if the component maps

$$f_{i,j} : A_i \hookrightarrow \bigoplus A_i \longrightarrow \bigoplus B_j \twoheadrightarrow B_j$$

are in the radical. The extension to arbitrary sums is the following result.

<u>Lemma 6.28</u> Suppose $f : \bigoplus_{\alpha \in I} A_\alpha \longrightarrow \bigoplus_{\alpha \in I} A_\alpha$, I arbitrary, A_α objects of $\mathcal{A}$, has the property that each compnnent

$$f_{\alpha,\beta} : A_\alpha \hookrightarrow \bigoplus A_\alpha \xrightarrow[f]{} \bigoplus A_\alpha \twoheadrightarrow A_\beta$$

lies in $\mathcal{J}(\mathrm{Hom}(A_\alpha, A_\beta))$. Then 1-f is pure monic.

<u>Proof</u> $\bigoplus_{\alpha \in I} A_\alpha$ is the filtered direct limit of $\bigoplus_{\alpha \in J} A_\alpha$, J a finite set. SO

$$\begin{tikzcd} \bigoplus_J A_\alpha \arrow[r, "(1-f)|_J"] \arrow[d, hook] & \bigoplus_I A_\alpha \arrow[d, equal] \\ \bigoplus_I A_\alpha \arrow[r, "1-f"'] & \bigoplus_I A_\alpha \end{tikzcd}$$

gives 1-f as a direct limit of the maps $(1-f)|_J$. Hence it will suffice to prove that these are split monics.

Now

$$\bigoplus_J A_\alpha \xrightarrow{(1-f)|_J} \bigoplus_I A_\alpha \twoheadrightarrow \bigoplus_J A_\alpha$$

is the matrix map $(1 - (f_{\alpha,\beta}))$ where (α,β) range through J ; this is a unit by Lemma 6.27; hence $\bigoplus_J A_\alpha \longrightarrow \bigoplus_I A_\alpha$ is a split monic. //

We now return to the hypothesis that $\mathcal{A}$ is pure semi-simple.

<u>Corollary 6.29</u> If $\mathcal{A}$ is pure semi-simple, then Rad (End $(\bigoplus A_\alpha)$) consists of those maps whose components lie in $\mathcal{J}(A_\alpha, A_\beta)$ for each α, β.

Proof The set of such maps is an ideal containing the radical, hence it suffices to show that 1-f is a unit for such an f. But by Lemma 6.28, 1-f is pure-monic and hence splits. //

For objects A,B of $\mathcal{O}\!\!\!/$, the natural map $\pi : \mathrm{Hom}_{\mathcal{O}\!\!\!/}(A,B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\pi(A),\pi(B))$ sends $\mathcal{J}(A,B)$ into $\mathcal{J}(\pi(A),\pi(B))$. If A,B are indecomposable with local endomorphism rings, then $\mathcal{J}(A,B)$ equals Hom (A,B) for A,B non-isomorphic, and equals the unique maximal ideal of End (A) for $A \cong B$. So in these cases, $\pi(f)$ lies in the radical only if f does. (For indecomposables with local endomorphism rings,

$$A \cong B \quad \underline{\text{iff}} \quad \pi(A) \cong \pi(B) \quad .)$$

Corollary 6.30

Let $\oplus C_\alpha$ be a direct sum of indecomposables with local endomorphism rings in $\mathcal{O}\!\!\!/$. Then Rad (End $\pi(\oplus C_\alpha)$) consists of those maps whose components lie in $\mathcal{J}(\pi(C_\alpha),\pi(C_\beta))$ for each α,β.

Proof The implication is trivial one-way. So suppose $\pi(f)$ is a map in End $(\pi(\oplus C_\alpha))$ whose components lie in the radical. These components can be represented as $\pi(f_{\alpha\beta})$

$$f_{\alpha,\beta} : C_\alpha \rightarrowtail \oplus C_\alpha \xrightarrow[f]{} \oplus C_\alpha \twoheadrightarrow C_\beta \quad .$$

And since C_α, C_β have local endomorphism rings, $f_{\alpha,\beta}$ lie in $\mathcal{J}(C_\alpha,C_\beta)$. Hence by Cor. 6.29, f lies in the radical of End $(\oplus C_\alpha)$ and so $\pi(f)$ lies in the radical of End $(\pi \oplus C_\alpha)$. //

Corollary 6.31

$\mathcal{J}$(End P) = Hom (P,Rad P) for P any projective in $\mathcal{E}/\mathcal{S}$. ($\mathcal{O}\!\!\!/$ pure semi-simple)

Proof Any projective in $\mathcal{E}/\mathcal{S}$ is isomorphic to (a direct summand of) $\bigoplus_{\alpha\in I}\pi(C_\alpha)$ for a suitable index set I.

Now if $\pi(f) \in \mathcal{J}$(End P), its image is superfluous by Prop. 6.19. (holds in any abelian category), hence $\pi(f)$ can be regarded as an element of Hom (P,Rad P).

Conversely, if $\pi(f) : \oplus\pi(C_\alpha) \longrightarrow \oplus\pi(C_\alpha)$ factors through $\mathrm{Rad}\ (\oplus\pi(C_\alpha)) = \oplus\ \mathrm{Rad}\ \pi(C_\alpha)$, then

$$\pi(f_{\alpha,\beta}) : \pi(C_\alpha) \to \oplus\pi(C_\alpha) \longrightarrow \oplus\ \mathrm{Rad}\ \pi(C_\alpha) \hookrightarrow \oplus\ \pi(C_\alpha) \twoheadrightarrow \pi(C_\beta)$$

factors as $\pi(C_\alpha) \to \mathrm{Rad}\,\pi(C_\beta) \hookrightarrow \pi(C_\beta)$

and hence has a superfluous image since $\mathrm{Rad}\ \pi(C_\beta)$ is superfluous in $\pi(C_\beta)$ (It is a small projective). So by Prop. 6.19 each $\pi(f_{\alpha,\beta})$ lies in $\mathcal{J}\,\mathrm{Hom}\ (\pi(C_\alpha), \pi(C_\beta))$ and by Cor. 6.30 $\pi(f)$ is in the radical.//

<u>Corollary 6.32</u>

Rad P is superfluous in P, for P any projective in $\mathcal{E}/\mathcal{S}$ ($\mathcal{O}$ pure semi-simple).

<u>Proof</u> Again w.l.o.g. $P = \oplus\pi(C_\alpha)$. Suppose $N + \oplus\ (\mathrm{Rad}\ \pi(C_\alpha)) = \oplus\pi(C_\alpha)$. Then the epi $N \oplus (\oplus\ \mathrm{Rad}\,\pi(C_\alpha)) \twoheadrightarrow \oplus\pi(C_\alpha)$ splits, so the identity can be written as

$\oplus\pi(C_\alpha) \rightarrowtail N \oplus (\oplus\ \mathrm{Rad}\ \pi(C_\alpha)) \twoheadrightarrow \oplus\pi(C_\alpha)$,

that is, it is the sum of

$$\oplus\pi(C_\alpha) \longrightarrow \oplus\ \mathrm{Rad}\ \pi(C_\alpha) \hookrightarrow \oplus\pi(C_\alpha)$$

and

$$\oplus\pi(C_\alpha) \longrightarrow N \hookrightarrow \oplus\pi(C_\alpha)\ ;$$

but the first lies in $\mathcal{J}\mathrm{End}\ (\oplus\pi(C_\alpha))$ by Cor. 6.31, so this forces the second to be a unit, and so $N = \oplus\pi(C_\alpha)$.//

This also completes the proof of Theorem 6.26.//

One now has

<u>Corollary 6.33</u> $\mathfrak{A}$ is of finite representation type <u>iff</u> $\mathcal{E}/\mathcal{S}$ is equivalent to a module category over an Artinian ring.

<u>Proof</u> Prop. 6.10 and Thm. 6.26 yield result. //

CHAPTER 7

THE REPURE SUBCATEGORY

In Chapter 6 we concerned ourselves with the situation in which every sequence was copure (pure semi-simplicity). The other extreme is more familiar : "all sequences pure" is Von Neumann regularity. We return now to the general case and investigate the relationship between the pure and copure sequences.

The category $\mathcal{S}$ of pure sequences is a full exact abelian subcategory and is dense in $\mathcal{E}/\mathcal{S}$ (Thm.5.12, Prop.5.14). For the moment, we turn our attention to density. Let $\mathcal{C}$ be any class of sequences in $\mathcal{E}$, the sequence category. Denote by $\mathcal{C}_m$ (respectively, $\mathcal{C}_e$) the corresponding class of monics (epis). $\mathcal{C}$ is called a proper class if (Maclane [ ] page 367) :

P.1. Every split sequence is in $\mathcal{C}$.

P.2. If $\alpha,\beta \in \mathcal{C}_m$, then $\beta\alpha \in \mathcal{C}_m$ if defined.

P.2* if $\alpha,\beta \in \mathcal{C}_e$, then $\beta\alpha \in \mathcal{C}_e$ if defined.

P.3 If $\beta\alpha \in \mathcal{C}_m$, then $\alpha \in \mathcal{C}_m$.

P.3* If $\beta\alpha \in \mathcal{C}_e$ then $\beta \in \mathcal{C}_e$.

Denote by $\bar{\mathcal{C}}$ the class of representatives of $\mathcal{C}$ in $\mathcal{E}/\mathcal{S}$. Since we will always regard split sequences as zero, we tacitly assume P.1 is satisfied.

<u>Proposition 7.1</u> $\mathcal{C}$ is proper <u>iff</u> $\bar{\mathcal{C}}$ is dense in $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> <u>Remark</u> This is fairly routine , just a matter of reformulating the concept of proper in its 'proper' setting. That is to say, the axioms of properness seem more awkward than the concept of density. The details are as follows :

($\Leftarrow$) <u>P.2</u> Let α be $A \rightarrowtail B$, β be $B \rightarrowtail C$. Let $\underline{\alpha}$ be the corresponding sequence $A \rightarrowtail B \twoheadrightarrow A'$ and $\underline{\beta}$ be $B \rightarrowtail C \twoheadrightarrow B'$.

Consider the morphism in $\mathcal{E}/\mathcal{S}$:

$$\begin{tikzcd} A \arrow[r, tail, "\beta\alpha"] \arrow[d, tail, "\alpha"'] & C \arrow[r, two heads] \arrow[d, equal] & C' \arrow[d, two heads] \\ B \arrow[r, tail] & C \arrow[r] & B' \end{tikzcd} .$$

Taking the kernel and image in $\mathcal{E}/\mathcal{S}$ (using Thm.2.5) leads to

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \oplus C \arrow[r] \arrow[d, two heads] & D \arrow[d] \\ A \arrow[r] \arrow[d, tail] & C \arrow[r] \arrow[d] & C' \arrow[d, equal] \\ B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & C' \arrow[d, two heads] \\ B \arrow[r] & C \arrow[r] & B' \end{tikzcd} \qquad \begin{tikzcd} \underline{K} \arrow[d, tail] \\ \underline{\beta\alpha} \arrow[d, two heads] \\ \underline{L} \arrow[d, tail] \\ \underline{\beta} \end{tikzcd} .$$

Now $\underline{K}$ is also a subobject of $\underline{\alpha}$ since

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \oplus C \arrow[r] \arrow[d, two heads] & D \arrow[d, dotted] \\ A \arrow[r] & B \arrow[r] & A' \end{tikzcd} \qquad \begin{tikzcd} \underline{K} \arrow[d] \\ \underline{\alpha} \end{tikzcd}$$

represents a subobject in $\mathcal{E}/\mathcal{S}$. Hence $\underline{K}$ is in $\overline{\mathcal{C}}$ and $\underline{L}$ is a subobject of $\underline{\beta}$ hence also in $\overline{\mathcal{C}}$, but $0 \longrightarrow \underline{K} \longrightarrow \underline{\beta\alpha} \longrightarrow \underline{L} \longrightarrow 0$ is exact in $\mathcal{E}/\mathcal{S}$, so $\underline{\beta\alpha}$ is in $\overline{\mathcal{C}}$. This gives $\beta\alpha \in \mathcal{C}_m$ and P.2 is established. P.2* is established in a completely 'dual' fashion.

P.3 Again let $\underline{\alpha} = A \overset{\alpha}{\rightarrowtail} B \twoheadrightarrow A'$ and $\underline{\beta} = B \overset{\beta}{\rightarrowtail} C \twoheadrightarrow B'$ with $\beta\alpha$ in $\mathcal{C}_m$.

Forming

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r] \arrow[d, "\beta"] & B' \arrow[d, dotted] \\ A \arrow[r] & C \arrow[r] & C' \end{tikzcd} \qquad \begin{tikzcd} \underline{\alpha} \arrow[d] \\ \underline{\beta\alpha} \end{tikzcd}$$

shows that $\underline{\alpha}$ is a subobject of $\underline{\beta\alpha}$, so $\underline{\alpha}$ is in $\overline{\mathcal{C}}$. Thus α is in $\mathcal{C}_m$, establishing P.3. Again P.3* is a dual argument.

($\Rightarrow$) Let $\mathcal{C}$ be a proper class. First we show $\overline{\mathcal{C}}$ is closed under isomorphisms in $\mathcal{E}/\mathcal{S}$.

Let $\underline{E} \cong \underline{E'}$ with $\underline{E}$ in $\overline{\mathcal{C}}$. Factoring the isomorphism in canonical method (Thm. 2.5) :

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A' \oplus B \arrow[r] \arrow[d, two heads] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] & B' \oplus C \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{K} \arrow[d, tail] & (=0) \\
\underline{E} \arrow[d, two heads] & \\
\underline{D} \arrow[d, tail] & \cong \\
\underline{E}' \arrow[d, two heads] & \cong \\
\underline{N} & (=0)
\end{tikzcd}$$

Kernel and cokernel are both split sequences and $\underline{D}$ is also isomorphic to $\underline{E}$.

By P.3*, $D \twoheadrightarrow C$ is in $\mathcal{C}_e$, so $\underline{D}$ is in $\mathcal{C}$, hence $A' \rightarrowtail D$ is in $\mathcal{C}_m$. $\underline{N}$ splits so lies in $\mathcal{C}$. Then $A' \rightarrowtail D \rightarrowtail B' \oplus C$, which equals $A' \rightarrowtail B' \rightarrowtail B' \oplus C$, is in $\mathcal{C}_m$ by P.2. Then P.3 gives $A' \rightarrowtail B'$ in $\mathcal{C}_m$ so $\underline{E}'$ is in $\overline{\mathcal{C}}$.

Supposes now $0 \to \underline{E}_1 \to \underline{E}_2 \to \underline{E}_3 \to 0$ is exact in $\mathcal{E}/\mathcal{S}$. Since by the above $\overline{\mathcal{C}}$ is closed under isomorphisms, we can represent this exact sequence in $\mathcal{E}/\mathcal{S}$ as a quotient with corresponding kernel :

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A' \oplus B \arrow[r] \arrow[d, two heads] & B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{E}_1 \arrow[d, tail] \\
\underline{E}_2 \arrow[d, two heads] \\
\underline{E}_3
\end{tikzcd}$$

(a) If $\underline{E}_2$ is in $\overline{\mathcal{C}}$, P.3 gives $A \rightarrowtail A' \oplus B$ in $\mathcal{C}_m$ so $\underline{E}_1$ is in $\overline{\mathcal{C}}$, and P.3* gives $B' \to C$ in $\mathcal{C}_e$ so $\underline{E}_3$ is in $\overline{\mathcal{C}}$.

(b) If $\underline{E}_1$ and $\underline{E}_3$ are in $\overline{\mathcal{C}}$ then $A' \oplus B \to B' \to 0$ and $B' \to C \to 0$ are in $\mathcal{C}_e$ and by P.2* $A' \oplus B \to B' \to C \to 0 = A' \oplus B \to B \to C \to 0$ is in $\mathcal{C}_e$. Applying P.3* gives $B \to C$ in $\mathcal{C}_e$, hence $\underline{E}_2$ is in $\overline{\mathcal{C}}$. This establishes the density of $\overline{\mathcal{C}}$. //

<u>Remark</u> A dense subcategory of an abelian category will be a full abelian subcategory with exact inclusion.

Returning to the category of pure sequences, by Cor. 5.10 (iii), $\mathcal{S}$ has sufficient projectives. This result rests upon the fact that given an object C of $\mathcal{A}$, there is a pure epi $X \twoheadrightarrow C \longrightarrow 0$ with X pure projective.

If $\mathcal{A}$ is a module category, then given C there is a pure embedding $C \rightarrowtail Y$ with Y pure injective (= algebraically compact, Warfield). Stenström has extended this to functor categories. For the remainder of this chapter, $\mathcal{A}$ is a functor category.

<u>Lemma 7.2</u> If $0 \to C \to Y \to N \to 0$ is pure exact and Y is pure injective, then it is the unique maximal pure subobject of the injective $\mathcal{J}(C)$ in $\mathcal{E}/\mathcal{S}$ (contains all pure subobjects of $\mathcal{J}(C)$).

<u>Proof</u> Suppose
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r] & I \arrow[r] & C' \arrow[r] & 0 \end{tikzcd} \quad = \mathcal{J}(C)$$
is a pure subobject of $\mathcal{J}(C)$. Then
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d] & E \arrow[dl, dashed] \\ & Y & \end{tikzcd}$$
lifts to Y. Since Y is pure injective, this yields
$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & C \arrow[r] & Y \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$
, showing that $0 \to C \to E \to Z \to 0$ is a subobject of $0 \to C \to Y \to N \to 0$. //

<u>Corollary 7.3</u> (i) Every object $\underline{E}$ of $\mathcal{E}/\mathcal{S}$ has a unique maximal pure subobject $t'(\underline{E})$.

(ii) t' is an additive functor.

<u>Proof</u> Let $\underline{E} = 0 \to C \to D \to C' \to 0$. Then $\underline{E} \hookrightarrow \mathcal{J}(C)$ and, by Lemma 7.2 and Stenström's result, $\mathcal{J}(C)$ has a unique maximal pure subobject $\underline{S}$. The required maximal pure subobject of $\underline{E}$ is the intersection of $\underline{E}$ and $\underline{S}$. The uniqueness of the maximal pure subobject follows from the uniqueness of $\underline{S}$ and easily establishes that t' is an additive subfunctor of the identity. //

At this stage of the development, it will be useful to introduce a new subcategory $\mathcal{R}$; for want of a better term, $\mathcal{R}$ will be called the repure subcategory. Define $\underline{R} \in \mathcal{R}$ iff $\mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\mathcal{S}', \underline{R}) = 0$.

Since $\mathcal{S}'$ is closed under epis, this is equivalent to having no pure subobjects. $(\mathcal{S}', \mathcal{R})$ is a torsion theory since $\mathcal{S}'$ is closed under quotients and sums and $\mathcal{R}$ is its 'complement'.

Proposition 7.4 (i) $\mathcal{R}$ is closed under subobjects.

(ii) If $\underline{R}_1 \to \underline{R}_2$ is monic in $\mathcal{R}$, then it is monic in $\mathcal{E}/\mathcal{S}$.

(iii) $\mathcal{R}$ is closed under limits (which are taken in $\mathcal{E}/\mathcal{S}$).

Proof Dual to Prop. 5.5. //

Define $r'(\underline{E})$ by exactness of

$$0 \to t'(\underline{E}) \to \underline{E} \to r'(\underline{E}) \to 0 \quad .$$

Proposition 7.5 r' is an additive functor and $r'(\underline{E})$ lies in $\mathcal{R}$.

Proof This holds in a general torsion theory setting, but we supply proof in our setting. Functorially of r' follows from that of t'. Form the pull-back in $\mathcal{E}/\mathcal{S}$

$$\begin{tikzcd} 0 \arrow[r] & t'(\underline{E}) \arrow[r] \arrow[d, equal] & \underline{E}' \arrow[r] \arrow[d, hook] & \underline{S} \arrow[d, hook] & \\ 0 \arrow[r] & t'(\underline{E}) \arrow[r] & \underline{E} \arrow[r] & r'(\underline{E}) \arrow[r] & 0 \end{tikzcd}$$

where $\underline{S}$ is a pure subobject of $r'(\underline{E})$. Since $\mathcal{S}'$ is dense, $\underline{E}'$ is pure, so $\underline{E}' \hookrightarrow t'(\underline{E})$ since $t'(\underline{E})$ contains all pure subobjects. This forces $t'(\underline{E}) = \underline{E}'$ yielding $\underline{S} = 0$ hence $r'(\underline{E}) \in \mathcal{R}$. //

Corollary 7.6 $\underline{S}$ is pure iff $\mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\underline{S}, \mathcal{R}) = 0$.

Proof ($\Rightarrow$) by definition of $\mathcal{R}$.

($\Leftarrow$) Consider $\underline{S} \twoheadrightarrow r'(\underline{S})$. Since this must be zero, $t'(\underline{S}) = \underline{S}$. //

Proposition 7.7 $\underline{R}_1 \to \underline{R}_2$ is epi in $\mathcal{R}$ iff its cokernel is pure in $\mathcal{E}/\mathcal{S}$.

Proof the proof is almost 'dual' to Prop. 5.6 but using Cor. 7.6 in place of actual definition of purity.//

7.8 Characterization of t'(E)

The explicit construction of t'(E) is as follows : given $\underline{E} = 0 \to C \to D \to C' \to 0$, let $\underline{S} = 0 \to C \to Y \to N \to 0$ be pure exact with Y pure injective. $t'(\underline{E})$ is the intersection of $\underline{S}$ and $\underline{E}$ in $\mathcal{E}(C)$ which is $0 \to C \to D \oplus Y \to M \to 0$ (2.11) .

Proposition 7.9 $\mathcal{S}'$ has sufficient injectives.

Proof Pure sequences of the form $0 \to C \to Y \to N \to 0$, with Y pure injective, will be injective in $\mathcal{S}'$, and any pure sequence $0 \to C \to B \to A \to 0$ will be embedded in $0 \to C \to Y \to N \to 0$. Proof is entirely dual to proof of Cor. 5.10 (ii) which shows $\mathcal{S}'$ has enough projectives. //

Proposition 7.10 (i) $\mu : \mathcal{S}' \hookrightarrow \mathcal{E}/\mathcal{S}$ has a right adjoint t'.
(ii) $\eta : \mathcal{R} \hookrightarrow \mathcal{E}/\mathcal{S}$ has a left adjoint r'.

Proof Dual to Prop. 5.15, but we shall supply it :

(i) $\mathrm{Hom}_{\mathcal{S}'}(\underline{S}, t'(\underline{E})) \cong \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\mu(\underline{S}), \underline{E})$

follows from the fact that the image of $\underline{S} \to \underline{E}$ lies in $\mathcal{S}'$, and so can be regarded as a map to $t'(\underline{E})$.

(ii) $\mathrm{Hom}_{\mathcal{R}}(r'(\underline{E}), \underline{R}) \cong \mathrm{Hom}_{\mathcal{E}/\mathcal{S}}(\underline{E}, \eta(\underline{R}))$

assigns $\underline{E} \to \eta(\underline{R})$ the unique induced map

$$\begin{tikzcd} \underline{E} \arrow[r] \arrow[d, two heads] & \underline{R} \\ r'(\underline{E}) \arrow[ur, dashed] & \end{tikzcd}$$

out of the cokernel. //

Remarks The existence of the unique maximal pure subobject could have been established directly as the total sum of all pure subobjects, provided there were a set of pure subobjects. However this is doubtful. On the other hand, the maximal copure object can be explicitly calculated as the pull-back of $\underline{B} \to C$ and $\oplus X_f \to C$, the direct sum

of finitely presented objects, the sum taken over the set of morphisms from finitely presented objects to C.
(see 5.10)

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \bigoplus X_f \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} t(\underline{E}) \arrow[d, hook] \\ \underline{E} \end{tikzcd} .$$

The key ingredient is that $\mathcal{T}$ has as generator the set of finite presentations. Indeed, this was how $\mathcal{T}$ was defined for us. An alternative approach is to first define $\mathcal{S}$, purity in Cohn's sense, by means of the tensor product. Then $\mathcal{T}$ is defined by $\underline{T} \in \mathcal{T}$ $\underline{\text{iff}}$ Hom $(T, \mathcal{S}) = 0$. One must then show that $\mathcal{T}$ has finite presentations as generator which would then yield the maximal copure sub-object as above.

The advantage of this approach is that $\mathcal{T}$ and $\mathcal{R}$ can simultaneously be defined. However one needs to verify that the class $\mathcal{S}$ can act as both the torsion free part of a torsion theory, yielding $\mathcal{T}$ as torsion, and the torsion part of a torsion theory, yielding $\mathcal{R}$ as torsion free. On the whole I feel the approach is easier to define $\mathcal{T}$ first then $\mathcal{S}$ and finally $\mathcal{R}$. But now does duality really preside for $\mathcal{T}$ and $\mathcal{R}$? Most of Chapter 7 was a dualization of Chapter 5. However does the key fact concerning $\mathcal{T}$ dualize? Does $\mathcal{R}$ have a set of cogenerators? We investigate this topic further in the next chapter.

CHAPTER 8

LOCALIZATION AND COLOCALIZATION IN $\mathcal{E}/\mathcal{S}$

The triple $(\mathcal{J}, \mathcal{S}', \mathcal{R})$ is a T.T.F. theory for $\mathcal{E}/\mathcal{S}$ (torsion-torsion free). That is, $(\mathcal{J}, \mathcal{S}')$ and $(\mathcal{S}', \mathcal{R})$ are torsion theories.

We have an exact sequence

$$0 \longrightarrow t(\underline{E}) \longrightarrow \underline{E} \longrightarrow r(\underline{E}) \longrightarrow 0$$

where $t(\underline{E})$ is the unique maximal copure subobject of $\underline{E}$, and $r(\underline{E})$ is pure. t is the right adjoint of the inclusion $\mathcal{J} \hookrightarrow \mathcal{E}/\mathcal{S}$. r is the left adjoint of the inclusion $\mathcal{S}' \hookrightarrow \mathcal{E}/\mathcal{S}$.

We also have an exact sequence

$$0 \longrightarrow t'(\underline{E}) \longrightarrow \underline{E} \longrightarrow r'(\underline{E}) \longrightarrow 0$$

where $t'(\underline{E})$ is the unique maximal pure subobject of $\underline{E}$ and $r'(\underline{E})$ is repure. t' is the right adjoint of the inclusion $\mathcal{S}' \hookrightarrow \mathcal{E}/\mathcal{S}$. r' is the left adjoint of the inclusion $\mathcal{R} \hookrightarrow \mathcal{E}/\mathcal{S}$.

This holds for general T.T.F. theories in abelian categories, and most of what follows could be formulated as results for T.T.F. theory; at times one must impose the existence of enough projectives and injectives (which $\mathcal{E}/\mathcal{S}$ has) but reasonable abelian categories have these properties. However we stick to the notation of $\mathcal{E}/\mathcal{S}$ rather than attempt complete generality. Unfortunately a few definitions must follow ; we shall dispose of them immediately before applying them.

8.1 Category of (Additive) Fractions

Let $\mathcal{O}\!\!/$ be an (additive) category, and Σ a class of morphisms. The couple $(T, \mathcal{O}\!\!/_{\Sigma})$ is a category of (additive) fractions for $\mathcal{O}\!\!/$ and Σ, if $\mathcal{O}\!\!/_{\Sigma}$ is an (additive) category, T an (additive) functor $\mathcal{O}\!\!/ \longrightarrow \mathcal{O}\!\!/_{\Sigma}$, such that $T(s)$ is an isomorphism for any $s \in \Sigma$; and universal with this property. That is, if T' is an (additive) functor $\mathcal{O}\!\!/ \longrightarrow \mathcal{B}$, with $T'(s)$ isomorphism for $s \in \Sigma$, there is a unique (additive) functor T such that $\tilde{T}T$ is naturally equivalent to T'

$$\begin{tikzcd} \mathcal{A} \arrow[r, "T"] \arrow[dr, "T'"'] & \mathcal{A}_{\Sigma} \arrow[d, dashed, "\tilde{T}"] \\ & \mathcal{B} \end{tikzcd}$$

commutes in the category of (additive) categories and T is unique.

8.2 Divisible and Codivisible Objects

Let $(\mathcal{V},\mathcal{W})$ be a torsion theory, in an abelian category $\mathcal{B}$. $B \in \mathcal{B}$ is called divisible (codivisible) if $\mathrm{Hom}_{\mathcal{B}}(-,B)$ $(\mathrm{Hom}_{\mathcal{B}}(B,-))$ is exact on all short exact sequences $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{V}$ $(X' \in \mathcal{W})$.

i.e.
$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[dl, dashed] & X'' \arrow[r] & 0 \\ & B \end{tikzcd}$$

$X' \to B$ lifts to B provided X" is in $\mathcal{V}$.

(i.e.
$$\begin{tikzcd} & & & B \arrow[d] \arrow[dl, dashed] \\ 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0 \end{tikzcd}$$

$B \to X''$ factors over X provided X' is in $\mathcal{W}$).

8.3 Localization and Colocalization

notation as in 8.2

$g : A \to B$ is a localization

($f : B \to A$ is a colocalization)

if ker g and cok g $\in \mathcal{V}$, $B \in \mathcal{W}$ and B divisible

(if ker f and cok f $\in \mathcal{W}$, $B \in \mathcal{V}$ and B codivisible).

(Co)localizations are unique and if every object has a (co)localization, then (co)localization becomes an additive functor (Tachikawa & Ohtake [26]).

8.4 Category of Fractions Relative to a Dense Subcategory

Let $\mathcal{C}$ be an abelian category and $\mathcal{D}$ a dense subcategory. Define a class Σ to be those morphisms whose kernel and cokernel lie in $\mathcal{D}$. If $\mathcal{C}_{\Sigma}$ the additive category of fractions exists, it is usually denoted $\mathcal{C}/\mathcal{D}$ Popescu [22]).

8.5 (Co)Section Functor

Let $T : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{D}$ be the functor associated with the category of fractions.

If T has a left adjoint S, $\mathcal{D}$ is called a localizing subcategory and S is called a section functor (Popescu [22], page 174). (if T has a right adjoint R, $\mathcal{D}$ will be called colocalizing and R the cosection functor).

8.6 Outline of Remainder of Chapter 8

(i) For the torsion theory $(\mathcal{T},\mathcal{S}')$, localization exists, which will be denoted S(E).

(ii) For the torsion theory $(\mathcal{S}',\mathcal{R})$, colocalization exists, denoted R(E).

(iii) $\mathcal{S}'$ is a dense subcategory of $\mathcal{E}/\mathcal{S}$; the category of fractions $(\mathcal{E}/\mathcal{S})/\mathcal{S}'$ exists and is abelian, and in fact equals $\mathcal{T}\cap\mathcal{R}$.

(iv) $\mathcal{S}'$ is both localizing and colocalizing.

(v)
$$\begin{tikzcd}
\mathcal{E}/\mathcal{S} \arrow[rr, shift left, "t"] \arrow[dr, shift right, "T"'] & & \mathcal{T} \arrow[ll, shift left, "\nu"] \arrow[dl, shift right, "r'"'] \\
& \mathcal{T}\cap\mathcal{R} \arrow[ul, shift right, "R"'] \arrow[ur, shift right, "R"'] &
\end{tikzcd}$$

commutes; t is right adjoint to $\mathcal{T} \hookrightarrow \mathcal{E}/\mathcal{S}$

r' is right adjoint to colocalization

T = r't is right adjoint to colocalization

so R is left adjoint to T and is the cosection functor.

(vi)
$$\begin{tikzcd}
\mathcal{E}/\mathcal{S} \arrow[rr, shift left, "r'"] \arrow[dr, shift right, "T"'] & & \mathcal{R} \arrow[ll, shift left] \arrow[dl, shift right, "t"'] \\
& \mathcal{T}\cap\mathcal{R} \arrow[ul, shift right, "S"'] \arrow[ur, shift right, "S"'] &
\end{tikzcd}$$

also commutes : r' is left adjoint to $\mathcal{R} \hookrightarrow \mathcal{E}/\mathcal{S}$

t is left adjoint to localization

T = tr' is left adjoint to localization

so S is right adjoint to T, and is the section functor.

Note We will need to establish $T = tr' = r't$.

(vii) $\mathcal{T}\cap\mathcal{R}$ is a functor category.

(viii) Consequences of (vii)

Remarks Popescu handles the general theory of localization in his book, but we cannot appeal to the results (which would yield (iii) the existence) because of his tacit assumption that the underlying category be locally small (sets of subobjects) . We have the advantage of dealing with an abelian cateoory $\mathcal{E}/\mathcal{S}$ with enough projectives and injectives, but the disadvantage of being unable to assume locally small.

Lemma 8.7 (i) t preserves monics and epics.

(ii) r' preserves monics and epics.

Proof (i)

$$\begin{tikzcd} 0 \arrow[r] & t(\underline{E}) \arrow[r] \arrow[d, dashed] & \underline{E} \arrow[r] \arrow[d, "g"'] & r(\underline{E}) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & t(\underline{F}) \arrow[r] & \underline{F} \arrow[r] & r(\underline{F}) \arrow[r] & 0 \end{tikzcd}$$

and map $\underline{E} \to \underline{F}$ induces unique maps, $t(\underline{E}) \to t(\underline{F})$ (into kernel), and $r(\underline{E}) \to r(\underline{F})$ (out of cokernel) making diagram commute. These are the maps $t(g)$ and $r(g)$.

If g is monic, clearly $t(g)$ is monic.

If g is epi then the connecting morphism from ker $r(g)$ to coker $t(g)$ is epi, but ker $r(g)$ is a subobject of $r(\underline{E})$ which is pure hence ker $r(g)$ is also pure and then coker $t(g)$ is an epimorph of ker $r(g)$, and is also pure. But it is also an epimorph of $t(\underline{E})$ hence copure. Thus coker $t(g) = 0$.

(ii) Argument is dual (epi easy part, monic using connecting morphism). //

Lemma 8.8 (i) If $\underline{A} \to \underline{B}$ has pure kernel, then so does $t(\underline{A}) \to t(\underline{B})$.

(ii) If $\underline{A} \to \underline{B}$ has pure cokernel, then so does $r'(\underline{A}) \to r'(\underline{B})$.

<u>Proof</u> (i)
$$\begin{tikzcd} t(\underline{A}) \arrow[r] \arrow[d] & \underline{A} \arrow[r] \arrow[d] & r(\underline{A}) \arrow[d] \\ t(\underline{B}) \arrow[r] & \underline{B} \arrow[r] & r(\underline{B}) \end{tikzcd}$$

$\ker(t(A) \longrightarrow t(B)) \hookrightarrow \ker(\underline{A} \longrightarrow \underline{B})$ by ker-coker sequence.//

(ii) Dually.

<u>Proposition 8.9</u> $r't \cong tr'$ [hereafter define T to be r't.]

<u>Proof</u> Apply r' to the inclusion $t\underline{E} \hookrightarrow \underline{E}$ which has a pure cokernel. Note: $r't\underline{E}$ a quotient of $t\underline{E}$ is copure.

$0 \to r't\underline{E} \rightarrowtail r'\underline{E} \twoheadrightarrow \underline{N} \to 0$, $\underline{N}$ is pure by <u>8.8</u> .

Then
$$\begin{tikzcd} 0 \arrow[r] & r't\underline{E} \arrow[r] \arrow[d, dashed] & r'\underline{E} \arrow[r] \arrow[d, equal] & \underline{N} \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & tr'\underline{E} \arrow[r] & r'\underline{E} \arrow[r] & rr'\underline{E} \arrow[r] & 0 \end{tikzcd}$$

There is a unique map to the kernel $tr'\underline{E}$ (and out of cokernel $\underline{N}$). Uniqueness implies that it is natural and ker-coker sequence yields it is monic.

The connecting map from $\ker(\underline{N} \longrightarrow rr'\underline{E})$ to $\operatorname{coker}(r't\underline{E} \longrightarrow tr'\underline{E})$ is an isomorphism but the former is pure and the latter copure hence both are zero. So $r't\underline{E} \to tr'\underline{E}$ is a natural isomorphism. //

<u>Proposition 8.10</u> Localization exists for the torsion theory $(\mathcal{S}, \mathcal{R})$.

<u>Proof</u> <u>Sublemma 8.10(a)</u> Any repure sequence embeds in $\mathcal{J}(D)$ for a suitable pure injective D. ($\mathcal{J}(D)$ is then injective and repure.)

<u>Proof of Sublemma</u>

If $\underline{E}$ is repure, $\underline{E} = r'(\underline{E})$. To commute $r'(\underline{E})$ form the pushout of
$$\begin{tikzcd} \underline{E} = 0 \arrow[r] & A \arrow[r] \arrow[d, tail] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$
where D is pure injective and $A \rightarrowtail D$ is pure monic. So $\underline{E} = r'(\underline{E}) \hookrightarrow \mathcal{J}(D)$ and $\mathcal{J}(D)$ is repure by defintion of repure sequences and pure injectives.

Proof of Proposition

Given $\underline{M}$ any object of $\mathcal{E}/\mathcal{S}$, embed $r'(\underline{M})$ in an injective and repure sequence $\underline{I}$ (by sublemma). Form

$$\begin{tikzcd} & & r'(\underline{M}) \arrow[r, tail] \arrow[d, equal] & \widetilde{\underline{M}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "P.B."] & t'(\underline{N}) \arrow[d, hook] & \\ t'(\underline{M}) \arrow[r, tail] & \underline{M} \arrow[r, two heads] & r'(\underline{M}) \arrow[r, hook] & \underline{I} \arrow[r, two heads] & \underline{N} \arrow[r] & 0 \end{tikzcd}$$

by taking the pull-back of the cokernel map $\underline{I} \twoheadrightarrow \underline{N}$ and inclusion of the maximal pure subobject of $\underline{N}$.

Claim $0 \to t'(\underline{M}) \longrightarrow \underline{M} \longrightarrow \widetilde{\underline{M}} \twoheadrightarrow t'(\underline{N}) \longrightarrow 0$ is the localization of $\underline{M}$.

(i) By dropping the monic $r'(M) \rightarrowtail \widetilde{M}$,

$$\begin{aligned} \ker(\underline{M} \to \widetilde{\underline{M}}) &= \ker(M \twoheadrightarrow r'(M)) \\ &= t'(M) \quad \text{, which is also pure.} \end{aligned}$$

(ii) By dropping the epic $\underline{M} \twoheadrightarrow r'(\underline{M})$,

$$\begin{aligned} \text{cok}\,(\underline{M} \to \widetilde{\underline{M}}) &= \text{cok}\,(r'(M) \hookrightarrow \widetilde{M}) \\ &= t'(\underline{N}) \quad \text{, which is also pure.} \end{aligned}$$

(iii) $\widetilde{\underline{M}} \hookrightarrow \underline{I}$, $\underline{I}$ is repure so $\widetilde{\underline{M}}$ is repure also.

(iv) $\text{coker}\,(\widetilde{\underline{M}} \hookrightarrow \underline{I}) = \text{coker}\,(t'(\underline{N}) \hookrightarrow \underline{N}) = r'(\underline{N})$

is repure.

So given $0 \to \underline{X}' \to \underline{X} \to \underline{X}'' \to 0$ with $\underline{X}''$ pure, form

$$\begin{tikzcd} 0 \arrow[r] & \underline{X}' \arrow[r] \arrow[d] & \underline{X} \arrow[r] \arrow[d, dashed] & \underline{X}'' \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \widetilde{\underline{M}} \arrow[r] & I \arrow[r] & r'(\underline{N}) \arrow[r] & 0 \end{tikzcd}$$

$\underline{X} \to \underline{I}$ by injectivity, and $\underline{X}'' \to r'(\underline{N})$, out of cokernel. But if $\underline{X}''$ is pure, $\underline{X}'' \to r'(\underline{N})$ is the zero map. By Prop.2.4 this yields a map $\underline{X} \to \widetilde{\underline{M}}$ such that

$$\begin{tikzcd} 0 \arrow[r] & \underline{X}' \arrow[r] \arrow[d] & \underline{X} \arrow[dl, dashed] \\ & \widetilde{\underline{M}} & \end{tikzcd}$$

commutes. Hence $\widetilde{\underline{M}}$ is divisible. //

Proposition 8.11 Colocalization for the torsion theory $(\mathcal{T}, \mathcal{S}')$ exists.

<u>Proof</u> <u>Sublemma 8.12</u> Any copure sequence is the epimorphic image of $\pi(D)$ for a suitable pure projective (D) ($\pi(D)$ is then projective and copure).

<u>Proof</u> Dual to 8.10(a)

The proof of 8.11 is dual to 8.10 but we shall give the construction as reference.

Given $\underline{M}$, $t(\underline{M})$ is the epimorphic image $\underline{P} \twoheadrightarrow t(\underline{M})$ for some projective and copure $\underline{P}$. Form

$$\begin{tikzcd}
\underline{K} \arrow[r, tail] \arrow[d] & \underline{P} \arrow[r, two heads] \arrow[d] & t\underline{M} \arrow[r, hook] \arrow[d, equal] & \underline{M} \arrow[r, two heads] & r(\underline{M}) \\
r(\underline{K}) \arrow[r] & \hat{\underline{M}} \arrow[r] & t\underline{M} & &
\end{tikzcd}$$

then $0 \longrightarrow r(K) \longrightarrow \hat{\underline{M}} \longrightarrow \underline{M} \longrightarrow r(\underline{M}) \longrightarrow 0$ gives colocalization.//

For each $\underline{M}$, set $S(\underline{M})$ so that $\underline{M} \to S(\underline{M})$ is a localization. Then S is an additive functor (Tachikawa & Ohtake [26], Cor, 1.6), and also $R(\underline{M})$ by $R(\underline{M}) \longrightarrow \underline{M}$ a colocalization.

<u>Proposition 8.13</u>

$$\begin{tikzcd}
\mathcal{E}/\mathcal{S} \arrow[rr, shift left, "r'"] \arrow[dr, shift left, "\tau"] & & \mathcal{R} \arrow[ll, shift left, "n"] \arrow[dl, shift left, "t"] \\
& \mathcal{R} \cap \mathcal{T} \arrow[ul, shift left, "S"] \arrow[ur, shift left, "S"] &
\end{tikzcd}$$

r' is left adjoint to n

t is left adjoint to S (restricted to $\mathcal{R} \cap \mathcal{T} \to \mathcal{R}$)

Hence $T = r't$ is left adjoint to S.

<u>Proof</u> The objects $S(\underline{M})$ are repure and T is naturally equivalent to tr', so diagram commutes.

By Prop. 7.10 r' is left adjoint to n. To establish $\mathrm{Hom}_{\mathcal{R}\cap\mathcal{T}}(t(\underline{R}), \underline{E}) \cong \mathrm{Hom}_{\mathcal{R}}(\underline{R}, S(\underline{E}))$

since $\underline{E}$ is in $\mathcal{R}$, its localization is

$0 \longrightarrow \underline{E} \longrightarrow S(\underline{E}) \longrightarrow \underline{X} \longrightarrow 0$ where $\underline{X}$ is pure (i.e.

(i.e. $\ker(\underline{E} \longrightarrow S(\underline{E}) = t'(\underline{E}) = 0$).

Given $R \longrightarrow S\underline{E}$, form

$$\begin{tikzcd}
& tR \arrow[r, hook] \arrow[d, dashed] & R \arrow[d] & & \\
0 \arrow[r] & \underline{E} \arrow[r] & S\underline{E} \arrow[r] & \underline{X} \arrow[r] & 0
\end{tikzcd}$$

This assignment is the unique map into the kernel (where $t\underline{R} \longrightarrow \underline{X} = 0$ since it is copure to pure). This is natural by uniqueness.

For the inverse statement : given $t\underline{R} \longrightarrow \underline{E}$,

$$\begin{tikzcd}
0 \arrow[r] & t\underline{R} \arrow[r] \arrow[d] & \underline{R} \arrow[r] \arrow[d, dotted] & r\underline{R} \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \underline{E} \arrow[r] & S\underline{E} \arrow[r] & \underline{X} \arrow[r] & 0
\end{tikzcd}$$

By divisibility of $S\underline{E}$, there exists a map $\underline{R} \longrightarrow S\underline{E}$. But one must verify that it is unique. It suffices to show that the zero map $t\underline{R} \longrightarrow \underline{E}$ induces only the zero map $\underline{R} \longrightarrow S\underline{E}$.

$$\begin{tikzcd}
0 \arrow[r] & t\underline{R} \arrow[r] \arrow[d, "0"] & \underline{R} \arrow[r] \arrow[d, dotted] \arrow[dl, "0"'] & r\underline{R} \arrow[r] \arrow[d, dotted] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & \underline{E} \arrow[r] & S\underline{E} \arrow[r] & \underline{X} \arrow[r] & 0
\end{tikzcd}$$

By Prop. 2.4 there exists a map $r\underline{R} \longrightarrow S\underline{E}$ such that

$$\begin{tikzcd}
 & r\underline{R} \arrow[dl] \arrow[d] \\
S\underline{E} \arrow[r] & \underline{X}
\end{tikzcd}$$

commutes. But $r\underline{R}$ is pure and $S\underline{E}$ is repure, so this is the zero map. Hence $r\underline{R} \longrightarrow \underline{X}$ is also the zero map. The left and right sides are both zero mappings, forcing the middle to be the zero map also.

The assignments are clearly inverse to each other which establishes the adjoint relationship. //

<u>Proposition 8.14</u>

$$\begin{tikzcd}
\mathcal{E}/\mathcal{S} \arrow[rr, shift left, "t"] \arrow[dr, shift right, "T"'] & & \mathcal{T} \arrow[ll, shift left, "\nu"] \arrow[dl, shift right, "r'"'] \\
 & \mathcal{R} \cap \mathcal{T} \arrow[ul, shift right, "R"'] \arrow[ur, shift right, "R"'] &
\end{tikzcd}$$

r' is right adjoint to R

t is right adjoint to ν

Hence T=r't is right adjoint to R.

<u>Proof</u> Dual to 8.13. //

Theorem 8.15 $\mathcal{E}/\mathcal{S} \xrightarrow{T} \mathcal{R}\cap\mathcal{T}$ is the additive category of fractions for the dense subcategory $\mathcal{S}$.

Proof Let $0 \longrightarrow \underline{K} \rightarrowtail \underline{A} \longrightarrow \underline{B} \twoheadrightarrow \underline{N} \longrightarrow 0$ be exact with $\underline{K}$ and $\underline{N}$ pure. We must show $T(A) \xrightarrow{\cong} T(B)$.

Consider

$$\begin{tikzcd}
\underline{K}'' \arrow[r] \arrow[d] & \underline{K} \arrow[r] \arrow[d] & \underline{K}' \arrow[d] \\
t(\underline{A}) \arrow[r] \arrow[d] & \underline{A} \arrow[r] \arrow[d] & r(\underline{A}) \arrow[d] \\
t(\underline{B}) \arrow[r] \arrow[d] & \underline{B} \arrow[r] \arrow[d] & r(\underline{B}) \arrow[d] \\
\underline{N}'' \arrow[r] & \underline{N} \arrow[r, two heads] & \underline{N}'
\end{tikzcd}$$

The ker-coker sequence is

$$0 \to \underline{K}'' \to \underline{K} \to \underline{K}' \to \underline{N}'' \to \underline{N} \twoheadrightarrow \underline{N}' \to 0.$$

Now $\underline{N}$ is pure, and $\underline{K}'$ is pure, so

$$\begin{tikzcd}
\underline{K}' \arrow[rr] \arrow[dr, two heads] & & \underline{N}'' \arrow[rr] \arrow[dr, two heads] & & \underline{N} \\
& \underline{X} \arrow[ur, tail] & & \underline{Y} \arrow[ur, tail] &
\end{tikzcd}$$

, $\underline{X}$ and $\underline{Y}$ are pure.

Then by density of pure sequences, $\underline{N}''$ is also pure. But $\underline{N}''$ is also the epimorphic image of $t(\underline{B})$ hence copure. Therefore $\underline{N}'' = 0$. Also $\underline{K}'' \hookrightarrow \underline{K}$ is pure.

Now form

$$\begin{tikzcd}
0 \arrow[r] & \underline{L}' \arrow[r] \arrow[d] & \underline{K}'' \arrow[r] \arrow[d] & \underline{L} \arrow[d] & & \\
0 \arrow[r] & t't(\underline{A}) \arrow[r] \arrow[d] & t(\underline{A}) \arrow[r] \arrow[d] & r't(\underline{A}) & = & TA \arrow[r] & 0 \\
0 \arrow[r] & t't(\underline{B}) \arrow[r] \arrow[d] & t(\underline{B}) \arrow[r] & r't(\underline{B}) & = & TB \arrow[r] & 0 \\
& \underline{L}'' & & & &
\end{tikzcd}$$

The ker-coker sequence immediately gives $TA \twoheadrightarrow TB$. And then $0 \to \underline{L}' \to \underline{K}'' \to \underline{L} \to \underline{L}'' \to 0$ is exact with $\underline{K}''$ and $\underline{L}''$ pure, which again yields $\underline{L}$ pure by density of $\mathcal{S}$. But $\underline{L} \hookrightarrow r't(\underline{A})$ so $\underline{L}$ is also repure hence $\underline{L} = 0$.

This establishes $TA \xrightarrow{\cong} TB$.

Suppose now
$$\begin{tikzcd} \mathcal{E}/\mathcal{S} \arrow[r, "T"] \arrow[dr, "T'"'] & \mathcal{J}\cap\mathcal{R} \arrow[d, dashed, "\tilde{T}(?)"] \\ & \mathcal{B} \end{tikzcd}$$

T' is an additive functor with T'(g) an isomorphism if ker g and coker g are pure.

Now if $\tilde{T}$ exists to make $\tilde{T}T$ naturally equivalent to T', then it is obviously unique because T restricted to $\mathcal{J}\cap\mathcal{R}$ is the identity, so

$$\tilde{T}(\underline{E}) = \tilde{T}T(\underline{E}) = T'(\underline{E}) \quad , \quad \underline{E} \in \mathcal{J}\cap\mathcal{R} \quad .$$

We need only verify this works:

that is, for any $\underline{M}$ we need

$$\tilde{T}T(\underline{M}) \cong T'(\underline{M}) \quad \text{naturally.}$$

Now $tM \rightarrowtail M$ and

$tM \twoheadrightarrow r'tM$

have pure kernels and cokernels, so

$$T'(M) \cong T'(tM) \cong T'(r'tM) = \tilde{T}(r'tM) = \tilde{T}T(M) \quad .//$$

<u>Lemma 8.16</u> If $A \rightarrowtail B$ $[A \twoheadrightarrow B]$ then $TA \rightarrowtail TB$ $[TA \twoheadrightarrow TB]$ in $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> T = r't so this follows from Lemma 8.7. //

<u>8.17</u> <u>Remark</u>

Any functor having a left adjoint preserves limits, and any functor having a right adjoint preserves colimits (Maclane [20], page 114 Thm. 1). We note that kernels are a special limit and also cokernels (Maclane [20], page 64) are a colimit.

<u>Proposition 8.18</u> $T : \mathcal{E}/\mathcal{S} \to \mathcal{R}\cap\mathcal{T}$ is exact

(<u>Note</u> Regarded as a functor $\mathcal{E}/\mathcal{S} \to \mathcal{E}/\mathcal{S}$, T need not be exact.)

<u>Proof</u> T has a right and left adjoint, so it preserves kernels and cokernels. So given $0 \to A \to B \to C \to 0$, $T(A) \to T(B)$ is the kernel of $T(B) \to T(C)$, and $T(A) \to T(B)$ is monic and $T(B) \to T(C)$ epic in $\mathcal{E}/\mathcal{S}$ by Lemma 8.16, hence also in $\mathcal{R}\cap\mathcal{T}$.

So $0 \to T(A) \to T(B) \to T(C) \to 0$ is exact in $\mathcal{R}\cap\mathcal{J}$ .//

Proposition 8.19 $\mathcal{R} \cap \mathcal{T}$ is abelian.

Proof $\mathcal{R} \cap \mathcal{T}$ is clearly additive, so it suffices to prove that any map

$$\underline{A} \xrightarrow{f} \underline{B}$$

has a factorization $\underline{f} = \underline{gh}$ where $\underline{h}$ is a cokernel and $\underline{g}$ a kernel (Stenström [25], page 87). Let

$$\begin{tikzcd} \underline{A} \arrow[rr] \arrow[dr, two heads] & & \underline{B} \\ & \underline{D} \arrow[ur, hook] & \end{tikzcd}$$

be a factorization in $\mathcal{E}/\mathcal{S}$. Claim that this same factorization works in $\mathcal{R} \cap \mathcal{T}$:

$\underline{D} \hookrightarrow \underline{B}$ hence $\underline{D}$ is repure

$\underline{A} \twoheadrightarrow \underline{D}$ hence $\underline{D}$ is copure

so $\underline{D}$ lies in $\mathcal{R} \cap \mathcal{T}$.

Let $0 \to \underline{K} \rightarrowtail \underline{A} \to \underline{B} \twoheadrightarrow \underline{N} \to 0$ be exact in $\mathcal{E}/\mathcal{S}$. Then Prop. 8.18 and the fact that $T(\underline{A}) = \underline{A}$, $T(\underline{B}) = \underline{B}$, $T(\underline{D}) = \underline{D}$ yield that

$$\begin{tikzcd} 0 \arrow[r] & T(\underline{K}) \arrow[r] & \underline{A} \arrow[rr] \arrow[dr] & & \underline{B} \arrow[r] & T(\underline{N}) \arrow[r] & 0 \\ & & & \underline{D} \arrow[ur] & & & \end{tikzcd}$$

is exact in $\mathcal{R} \cap \mathcal{T}$, so $\underline{A} \to \underline{D}$ is coker of $T(\underline{K}) \to A$ and $\underline{D} \to \underline{B}$ is kernel of $\underline{B} \to T(\underline{N})$. //

8.20 Remarks

$\mathcal{R} \cap \mathcal{T}$ is not an exact subcategory of $\mathcal{E}/\mathcal{S}$. For example, to compute the kernel of a map $\underline{A} \to \underline{B}$, with $\underline{A}$, $\underline{B} \in \mathcal{R} \cap \mathcal{T}$, one applies the exact functor T to

$$0 \to \underline{K} \to \underline{A} \to \underline{B} \quad \text{giving} \quad T(\underline{K}) \to \underline{A} \to \underline{B} \quad .$$

Now $T(\underline{K}) = r't(\underline{K})$ and $t(\underline{K})$ is in $\mathcal{R}$ since $\underline{K}$ is in $\mathcal{R}$, so $T(\underline{K}) = t(\underline{K})$. So the kernel is the maximal copure subobject of the kernel in $\mathcal{E}/\mathcal{S}$. Similarly,

$$\operatorname{coker}_{\mathcal{R} \cap \mathcal{T}} (A \to B) = r' (\operatorname{coker}_{\mathcal{E}/\mathcal{S}} A \to B) \quad .$$

Corollary 8.21 $\mathcal{R} \cap \mathcal{T}$ is (co)complete.

Proof Since $\mathcal{R} \cap \mathcal{T}$ is abelian, to show cocompleteness, it suffices to show that arbitrary sums exist. Given

$\{\underline{E}_\alpha\}$ take $\oplus E_\alpha$ in $\mathcal{E}/\mathcal{S}$, then apply T. Since T preserves sums (8.17) $T(\oplus E_\alpha)$ is the direct sum in $\mathcal{R} \cap \mathcal{T}$. Completeness in dual manner. //

The subcategory $\mathcal{S}'$ is dense in $\mathcal{E}/\mathcal{S}$ and the inclusion $\mathcal{S}' \hookrightarrow \mathcal{E}/\mathcal{S}$ is exact. For $\mathcal{T} \hookrightarrow \mathcal{E}/\mathcal{S}$ we have :

<u>Proposition 8.22</u> The following are equivalent :

(i) $\mathcal{T}$ is an abelian category.

(ii) $\mathcal{T} \subset \mathcal{R}$.

(iii) $\mathcal{T}$ is hereditary.

(iv) $\mathcal{T}$ is a dense full exact subcategory of $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> Assuming (i), the inclusion functor $\mathcal{T} \hookrightarrow \mathcal{E}/\mathcal{S}$ preserves isomorphisms. Suppose $\underline{E}$ is copure. The epimorphism $\underline{E} \twoheadrightarrow r'(\underline{E})$ is also an epi in $\mathcal{T}$, its kernel is $t'(\underline{E})$ which is pure, so by Prop. 5.6 it is monic in $\mathcal{T}$. So if $\mathcal{T}$ is abelian, $\underline{E} \twoheadrightarrow r'(\underline{E})$ is an isomorphism in $\mathcal{T}$ hence also in $\mathcal{E}/\mathcal{S}$. But this forces $t'(\underline{E}) = 0$. So $\underline{E}$ is repure.

(ii) $\Rightarrow$ (iii) Let $\underline{X}$ be a subobject in $\mathcal{E}/\mathcal{S}$ of $\underline{T} \in \mathcal{T}$. Forming

$$\begin{tikzcd} t(\underline{X}) \arrow[r] \arrow[d, equal] & \underline{X} \arrow[r] \arrow[d, hook] & r(\underline{X}) \arrow[r] \arrow[d, hook] & 0 \\ t(\underline{X}) \arrow[r] & \underline{T} \arrow[r] & \underline{T}' \arrow[r] & 0 \end{tikzcd}$$

$\underline{T}'$ a quotient of $\underline{T}$ is copure, hence by assumption repure. But then $r(\underline{X}) \hookrightarrow \underline{T}'$ is the zero map, hence $r(\underline{X}) = 0$ and $\underline{X}$ is copure.

(iii) $\Rightarrow$ (iv) $\mathcal{T}$ is always closed under extensions (Prop. 5.7) and quotients (Prop. 5.5). So $\mathcal{T}$ hereditary implies $\mathcal{T}$ is dense, and then clearly $\mathcal{T} \hookrightarrow \mathcal{E}/\mathcal{S}$ is an exact embedding.

(iv) $\Rightarrow$ (i) Trivial. //

Dually,

<u>Proposition 8.23</u> The following are equivalent :

(i) $\mathcal{R}$ is an abelian category.

(ii) $\mathcal{R} \subset \mathcal{T}$

(iii) $\mathcal{R}$ is cohereditary.

(iv) $\mathcal{R}$ is a dense full exact subcategory of $\mathcal{E}/\mathcal{S}$. //

The adjoint pairs (T,S) and (R,T) have units and counits.

<u>8.24 Unit for (T,S) : $\eta : 1 \xrightarrow{\bullet} ST$</u>

Given $\underline{E}$, $T\underline{E}$ is repure, so localization yields an exact sequence $T\underline{E} \rightarrowtail S(T\underline{E}) \twoheadrightarrow \underline{X} \to 0$ with $\underline{X}$ pure. Form

$$\begin{tikzcd}
t\underline{E} \arrow[r] \arrow[d, two heads] & \underline{E} \arrow[r] \arrow[d, dashed] & r\underline{E} \arrow[d, dotted] \\
T\underline{E} \arrow[r, tail] & ST\underline{E} \arrow[r, two heads] & \underline{X}
\end{tikzcd} .$$

By divisibility of $ST\underline{E}$ a map is induced $\underline{E} \to ST\underline{E}$. It is unique. (Proof as in Prop. 8.13, the zero map $t\underline{E} \to T\underline{E}$ would induce a map $r\underline{E} \to \underline{X}$ out of the cokernel, which would factor over $ST\underline{E}$ by Prop. 2.4. This would necessarily be the zero map since $r\underline{E} \in \mathcal{S}$ and $ST\underline{E} \in \mathcal{R}$. This would then force $\underline{E} \to ST\underline{E}$ to be zero also.)

This is $\eta_{\underline{E}} : \underline{E} \to ST\underline{E}$.

Note also that from the diagram

$$\begin{tikzcd}
t't\underline{E} \arrow[r, tail] \arrow[d] & \underline{K} \arrow[r, two heads] \arrow[d, tail] & \underline{K}' \arrow[d, tail] \\
t\underline{E} \arrow[r] \arrow[d, two heads] & \underline{E} \arrow[r] \arrow[d, "\eta_E"] & r\underline{E} \arrow[d] \\
T\underline{E} \arrow[r] \arrow[d] & ST\underline{E} \arrow[r, two heads] \arrow[d] & \underline{X} \arrow[d] \\
0 \arrow[r] & \underline{L} \arrow[r, "\cong"] & \underline{L}'
\end{tikzcd} ,$$

$\underline{L} \cong \underline{L}' = \operatorname{coker}(r\underline{E} \to \underline{X})$ is pure and $\underline{K}$ is an extension of $t't\underline{E}$ and $\underline{K}'$ is also pure. Then

$$0 \to \underline{K} \to \underline{E} \xrightarrow{\eta_E} ST\underline{E} \to \underline{L} \to 0$$

$\eta_{\underline{E}}$ is a morphism of $\Sigma_{\mathcal{S}}$, the class of morphisms associated with the category of fractions with respect to $\mathcal{S}'$. Also since $ST\underline{E}$ is divisible and repure, η_E is the localization of $\underline{E}$.

8.25 Remark

We defined localization as the functor S (Prop. 8.10). Here, S is an additive functor $\mathcal{E}/\mathcal{S} \longrightarrow \mathcal{E}/\mathcal{S}$. The functor S was shown to be the right adjoint of T but to be precise, since $T : \mathcal{E}/\mathcal{S} \longrightarrow \mathcal{R} \cap \mathcal{T}$, the actual right adjoint is S restricted to $\mathcal{R} \cap \mathcal{T}$. However we wish to reserve S as the right adjoint rather than change notation, but the difference in usage should be noticed. (similarly for colocalization R). So if the insistence is that S be the right adjoint to T, then the actual localization functor is ST, not S as before.

8.26 Counit for (T,S): $\mathcal{E} : TS \xrightarrow{\bullet} 1$

For $\underline{E}$ in $\mathcal{R} \cap \mathcal{T}$ we have $\underline{E} \rightarrowtail S\underline{E} \longrightarrow \underline{Y} \longrightarrow 0$ where $\underline{Y}$ is pure. Since $\underline{E}$ lies in $\mathcal{T}$, this sequence yields $\underline{E} \cong t(S\underline{E})$, hence

$$\underline{E} = r'\underline{E} \cong r'(tS\underline{E}) = TS\underline{E}$$

so TS and 1 are naturally equivalent functors via the counit (that the above isomorphism is the counit map is 'easy' to establish).

Corollary 8.27 S is full and faithful.

Proof This is a general categorical theorem concerning counits : (F,G) adjoint pair, $\mathcal{E} : FG \xrightarrow{\bullet} 1$ is equivalence iff G is full and faithful. //

IN an adjoint situation, (F,G) with counit an equivalence F is said to be the 'left-adjoint-left-inverse' to G (Maclane [20], page 92). Then if $F : \mathcal{X} \to \mathcal{Y}$, G is an isomorphism of $\mathcal{Y}$ to a reflective subcategory $\mathcal{X}'$ of $\mathcal{X}$ (reflective means an inclusion has a left adjoint). Applying this to (T,S) gives

Corollary 8.28 S factors as $\mathcal{R}\cap\mathcal{T} \xrightarrow{\cong} \mathrm{Im}\, S \hookrightarrow \mathcal{E}/\mathcal{S}$ with Im S the class of divisible repure sequences (denote this as $\mathcal{D}\cap\mathcal{R}$). So $\mathcal{D}\cap\mathcal{R}$ is a reflective abelian subcategory of $\mathcal{E}/\mathcal{S}$.

Proof SE is divisible and repure by definition of colocalization. And if E is divisible and repure it is isomorphic to its localization. //

Turning now to the adjoint pair (R,T) :

8.29 Counit ε: RT $\xrightarrow{\cdot}$ 1

Procedure is dual to 8.24.

$$\begin{tikzcd} 0 \arrow[r] & \underline{X} \arrow[r] \arrow[d, dashed] & RT\underline{E} \arrow[r, two heads] \arrow[d, dashed] & T\underline{E} \arrow[d] \\ 0 \arrow[r] & t'\underline{E} \arrow[r] & \underline{E} \arrow[r, two heads] & r'\underline{E} \end{tikzcd}$$

$\varepsilon_{\underline{E}}$ will be the (unique) mapiinduced out of RT$\underline{E}$ using codivisibility. Again kernel and cokernel are pure (using ker-coker sequence) so that $\varepsilon_{\underline{E}}$: RT$\underline{E} \longrightarrow \underline{E}$ is colocalization.

8.30 Unit η: 1 $\longrightarrow$ TR

Dual to 8.26. For $\underline{E}$ in $\mathcal{R}\cap\mathcal{T}$ we have

$$0 \to \underline{Y} \to R\underline{E} \twoheadrightarrow \underline{E} \to 0 \quad \text{with } \underline{Y} \text{ pure.}$$

Since $\underline{E}$ lies in $\mathcal{R}$, this sequence yields $\underline{E} \cong r'(R\underline{E})$.

$$\underline{E} = t\underline{E} \cong tr'R\underline{E} \cong TR\underline{E}$$

and this natural equivalence is the unit transformation.

Corollary 8.31 R is full and faithful.

Proof Again general category theory, but the idea is simple so we sketch the proof :

(i) faithfulness is immediate from TR $\cong$ 1.

(ii) fullness : Any map in Hom (R$\underline{E}$, R$\underline{E}$') is naturally assigned by adjointness to a map in

Hom ($\underline{E}$,TR$\underline{E}$') $\cong$ Hom ($\underline{E}$,$\underline{E}$') [via $\Sigma^*_{\underline{E}'}$]

and then apply R to the map in Hom ($\underline{E}$,$\underline{E}$') yield original map. //

For the pair (R,T) , T is the 'right-adjoint-left-inverse' to R, and this yields :

Corollary 8.32 Factoring R as $\mathcal{R} \cap \mathcal{T} \xrightarrow{\cong} \text{Im } R \hookrightarrow \mathcal{E}/\mathcal{S}$ Im R is the class of codivisible copure sequences (denote this as $\mathcal{C} \cap \mathcal{T}$). $\mathcal{C} \cap \mathcal{R}$ is a coreflective abelian subcategory of $\mathcal{E}/\mathcal{S}$.

Proof Dual to 8.28. //

Corollary 8.33 The categories $\mathcal{D} \cap \mathcal{R}$ and $\mathcal{C} \cap \mathcal{T}$ are equivalent abelian.

Proof Both are equivalent to $\mathcal{R} \cap \mathcal{T}$.//

Proposition 8.34 (i) S preserves injectives.
(ii) R preserves projectives.

Proof (i) Again general theory : S is the right adjoint of the exact functor T.

[Details. Let I be injective in $\mathcal{R} \cap \mathcal{T}$, with $0 \longrightarrow \underline{E}' \longrightarrow \underline{E} \longrightarrow \underline{E}'' \longrightarrow 0$ exact in $\mathcal{E}/\mathcal{S}$. Apply T to get $0 \longrightarrow T\underline{E}' \longrightarrow T\underline{E} \longrightarrow T\underline{E}'' \longrightarrow 0$ exact, I injective gives

$0 \quad \text{Hom}(T\underline{E}'', I) \longrightarrow \text{Hom}(T\underline{E}, I) \longrightarrow \text{Hom}(T\underline{E}', I) \longrightarrow 0$

exact and adjointness gives

$0 \quad (\underline{E}'', SI) \longrightarrow (\underline{E}, SI) \longrightarrow (\underline{E}', SI) \longrightarrow 0$

exact which gives SI injective.]

(ii) Dually. //

Lemma 8.35 (i) If $0 \rightarrow \underline{X} \rightarrow \underline{R} \rightarrow \underline{S} \rightarrow 0$ is exact with $\underline{R} \in \mathcal{R}$ and $\underline{S} \in \mathcal{S}$, then $\underline{X} \rightarrow \underline{R}$ is an essential monic.
(ii) If $0 \rightarrow \underline{S} \rightarrow \underline{T} \rightarrow \underline{X} \rightarrow 0$ is exact with $\underline{S} \in \mathcal{S}$ and $\underline{T} \in \mathcal{T}$, then $\underline{T} \rightarrow \underline{X}$ is a superfluous epic.

Proof (i) Suppose $\underline{X}' \cap \underline{X} = \underline{0}$. Then

$$\underline{X}' \hookrightarrow \underline{R} \twoheadrightarrow \underline{S}$$

is monic, so $\underline{X}' \rightarrowtail \underline{S}$ gives $\underline{X}' \in \mathcal{S} \cap \mathcal{R} = \underline{0}$.

(ii) Dually. //

The inclusion $\mathcal{R}\cap\mathcal{T} \hookrightarrow \mathcal{E}/\mathcal{S}$ is not exact; however we have the following results :

Proposition 8.36 Given $\underline{A} \xrightarrow{f} \underline{B}$, $\underline{A},\underline{B} \in \mathcal{R}\cap\mathcal{T}$, $\underline{f}$ has property $\mathbb{P}$ in $\mathcal{R}\cap\mathcal{T}$ iff it has $\mathbb{P}$ in $\mathcal{E}/\mathcal{S}$, where $\mathbb{P}$ is any one of :

(i) monic

(ii) epic

(iii) isomorphism

(iv) essential monic

(v) superfluous epic.

Proof ($\Leftarrow$) Trivial in all cases since $\mathcal{R}\cap\mathcal{T}$ is full.

($\Rightarrow$) (i) If kernel of $\underline{A}\rightarrow\underline{B}$ equals $\underline{K}$ in $\mathcal{E}/\mathcal{S}$, kernel of $T(\underline{A})\rightarrow T(\underline{B})$ $(= A\rightarrow B)$ in $\mathcal{R}\cap\mathcal{T}$ is $T(\underline{K})$. But $T(\underline{K}) = 0$. Now $\underline{K} \hookrightarrow \underline{A} \in \mathcal{R}$ hence $\underline{K} \in \mathcal{R}$ and so $T(\underline{K}) = t(\underline{K})$. $t(\underline{K}) = 0$ means $\underline{K}$ is pure, but also repure. Hence $\underline{K} = 0$.

(ii) Dually.

(iii) Follows from (i) and (ii) and $\mathcal{R}\cap\mathcal{T}$ abelian.

(iv) Suppose $\underline{X} \cap \underline{A} = 0$ in $\mathcal{E}/\mathcal{S}$, then $t\underline{X} \cap \underline{A} = 0$. But $t\underline{X} \in \mathcal{R}\cap\mathcal{T}$ hence $t\underline{X} = 0$ and so $\underline{X}$ is pure but also repure implying $\underline{X} = 0$.

(v) Dual to (iv) .//

Proposition 8.37 (i) S preserves essential monics.

(ii) R preserves superfluous epics.

Proof (i) For $\underline{E} \in \mathcal{R}\cap\mathcal{T}$, $0 \rightarrow \underline{E} \rightarrow S\underline{E} \rightarrow \underline{X} \rightarrow 0$ with $\underline{X}$ pure. Hence $\underline{E} \rightarrow S\underline{E}$ is essential by Lemma 8.35.

The commutative diagram

$$\begin{array}{ccc} \underline{E} & \rightarrowtail & S\underline{E} \\ \downarrow & & \downarrow \\ \underline{E}' & \rightarrowtail & S\underline{E}' \end{array}$$

yields the result.

(ii) Dually. //

<u>Corollary 8.38</u> (i) S preserves injective hulls.
(ii) R preserves projective covers.

<u>Proof</u> By 8.37 and 8.34. //

We turn our attention now to the exact functor

$$T \; : \; \mathcal{E}/\mathcal{S} \longrightarrow \mathcal{R} \cap \mathcal{J}$$

<u>Proposition 8.39</u> (i) If $\underline{P}$ is projective in $\mathcal{E}/\mathcal{S}$ and $\underline{P} \in \mathcal{J}$ then $T(\underline{P})$ is projective in $\mathcal{R} \cap \mathcal{J}$.
(ii) If $\underline{I}$ is injective in $\mathcal{E}/\mathcal{S}$ and $\underline{I} \in \mathcal{R}$ then $T(\underline{I})$ is injective in $\mathcal{R} \cap \mathcal{J}$.

<u>Proof</u> Since $\underline{P} \in \mathcal{J}$, $T(\underline{P}) = r'(\underline{P})$.
So given

$$\begin{tikzcd} & T(\underline{P}) \arrow[d] \\ \underline{A} \arrow[r, two heads] & \underline{B} \end{tikzcd}$$

in $\mathcal{J} \cap \mathcal{R}$, by Prop. 8.36 $\underline{A} \twoheadrightarrow \underline{B}$ is epic in $\mathcal{E}/\mathcal{S}$. So

$$\begin{tikzcd} & \underline{P} \arrow[d] \arrow[ddl, dashed] \\ & T(\underline{P})=r'(\underline{P}) \arrow[d] \\ \underline{A} \arrow[r, two heads] & \underline{B} \end{tikzcd}$$

there is an induced map out of $\underline{P}$. But $\ker (\underline{P} \twoheadrightarrow r'(\underline{P})) = t'(\underline{P})$ is pure, and $\underline{A}$ is repure hence $\underline{P} \longrightarrow \underline{A}$ factors through the coker, $T(\underline{P})$. This is the required map showing $T(\underline{P})$ is projective in $\mathcal{J} \cap \mathcal{R}$.
(ii) Dual. //

<u>Lemma 8.40</u> If $\underline{P}$ is a small projective in $\mathcal{E}/\mathcal{S}$ with $\underline{P} \in \mathcal{J}$ then $T(\underline{P})$ is a small projective in $\mathcal{R} \cap \mathcal{J}$.

<u>Proof</u> By 8.39, $T(\underline{P})$ is projective. Given

$$T(\underline{P}) \longrightarrow \bigoplus_{i \in I} \underline{A}_i \quad ,$$

the direct sum is taken in $\mathcal{R} \cap \mathcal{J}$ where $\underline{A}_i \in \mathcal{R} \cap \mathcal{J}$. But this equals $T(\bigoplus_{i \in I} \underline{A}_i)$ where sum is taken in $\mathcal{E}/\mathcal{S}$ (Cor. 8.21). Now $\bigoplus_{i \in I} \underline{A}_i$ is copure, hence $T(\bigoplus \underline{A}_i) = r'(\bigoplus \underline{A}_i)$. This leads to

$$\begin{tikzcd} & t'(\underline{P}) \arrow[d] & \\ & \underline{P} \arrow[r, dashed] \arrow[d, two heads] & \bigoplus \underline{A}_i \arrow[d, two heads] \\ T(\underline{P}) = & r'(\underline{P}) \arrow[r] & r'(\bigoplus \underline{A}_i) \end{tikzcd}$$

where $\underline{P} \to \bigoplus \underline{A}_i$ is induced because $\underline{P}$ is projective.

Since $\underline{P}$ is small, this factors through a finite sum $\bigoplus_J \underline{A}_i$. But a finite sum of repure objects is repure so $\bigoplus_J A_i \in \mathcal{R} \cap \mathcal{J}$.

$$\begin{tikzcd} t'(\underline{P}) \arrow[d] & \bigoplus_J \underline{A}_i \arrow[d] \\ \underline{P} \arrow[ur] \arrow[r] \arrow[d] & \bigoplus \underline{A}_i \arrow[d] \\ r'(\underline{P}) \arrow[r] \arrow[uur, dashed] & r'(\bigoplus \underline{A}_i) \end{tikzcd}$$

and the map $\underline{P} \to \bigoplus_J \underline{A}_i$ factors over the coker of $(t'(\underline{P}) \to \underline{P}) = r'(\underline{P})$. A diagram chase, cancelling the epi $\underline{P} \twoheadrightarrow r'(\underline{P})$ shows the map $r'(\underline{P}) \to \bigoplus_J \underline{A}_i$ factors $r'(\underline{P}) \to r'(\bigoplus \underline{A}_i)$ through a finite subsum. So $r'(\underline{P}) = T(\underline{P})$ is small. //

<u>Theorem 8.41</u> $\mathcal{R} \cap \mathcal{J}$ is a functor category.

<u>Proof</u> $\mathcal{J}$ has $\{\underline{P}_\alpha\}$ where $\underline{P}_\alpha$ is a finite presentation, as a set of generators. $\underline{P}_\alpha$ are small and projective in $\mathcal{E}/\mathcal{S}$, hence $T(\underline{P}_\alpha)$ are small and projective in $\mathcal{R} \cap \mathcal{J}$ by Lemma 8.40, and clearly generate $\mathcal{R} \cap \mathcal{J}$. By Cor. 8.21 $\mathcal{R} \cap \mathcal{J}$ is complete and cocomplete. Hence result follows by characterization of functor categories.//

Since $T(\underline{P}_\alpha)$ is a small projective in an abelian category, it is also finitely generated and will have maximal subobjects. Hence there will be epimorphisms $T(\underline{P}_\alpha) \twoheadrightarrow \underline{S}$ in $\mathcal{R} \cap \mathcal{J}$ hence in $\mathcal{E}/\mathcal{S}$ (by Prop. 8.36) with $\underline{S}$ simple in $\mathcal{R} \cap \mathcal{J}$.

<u>Lemma 8.42</u> If $\underline{S}$ is simple in $\mathcal{R}\cap\mathcal{T}$, then it is simple as an object of $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> If $X \subsetneq S$ then $tX \subsetneq S$, But $tX \in \mathcal{R}\cap\mathcal{T} \Rightarrow tX = 0$. $\Rightarrow$ X is pure, but also repure $\Rightarrow$ $X = 0$. //

<u>Theorem 8.43</u> Given C finitely presented, there is an exact sequence $0 \to A \to B \to C \to 0$ which is simple as an object of $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> Follows from $T(\pi(C))$ being a small projective in $\mathcal{T}\cap\mathcal{R}$. //

We will return to the nature of simple sequences shortly, but first derive some further consequences of Thm. 8.41.

Let $\{\underline{S}_\beta\}$ be the set of simples for $\mathcal{R}\cap\mathcal{T}$. Then since $\mathcal{R}\cap\mathcal{T}$ has injective hulls (it is functor hence Grothendieck), $\{I(\underline{S}_\beta)\}$ is a cogenerating set of injectives (injective hulls of simples). By Prop. 8.34 $S(I(S_\beta))$ are injective in $\mathcal{E}/\mathcal{S}$ and also lie in $\mathcal{R}$.

<u>Theorem 8.44</u> $\{S(I(S_\beta))\}$ are a set of (indecomposable) injective cogenerators for $\mathcal{R}$.

<u>Proof</u> $I(S_\beta)$ is repure hence $I(S_\beta) \rightarrowtail SI(S_\beta) \longrightarrow \underline{X}_\beta \longrightarrow 0$ with $\underline{X}_\beta$ pure. Given $\underline{E} \in \mathcal{R}$, $T\underline{E} = t\underline{E}$ lies in $\mathcal{R}\cap\mathcal{T}$ so there exists an embedding

$$T\underline{E} \rightarrowtail \prod_{\mathcal{R}\cap\mathcal{T}} I(S_\beta)$$

for some product (taken in $\mathcal{R}\cap\mathcal{T}$) of the $\{I(S_\beta)\}$. This remains an embedding in $\mathcal{E}/\mathcal{S}$ by 8.36.

Now
$$\prod_{\mathcal{R}\cap\mathcal{T}} I(S_\beta) = t\Big(\prod_{\mathcal{E}/\mathcal{S}} I(S_\beta)\Big)$$

(i.e. the product of copure objects need not be copure). This leads to

$$\begin{tikzcd}
0 \arrow[r] & T\underline{E} = t\underline{E} \arrow[r] \arrow[d, tail] & \underline{E} \arrow[r] \arrow[dd, dashed] & r(\underline{E}) \arrow[r] \arrow[dd, dashed] & 0 \\
 & t(\prod I(S_\beta)) \arrow[d, tail] & & & \\
0 \arrow[r] & \prod I(S_\beta) \arrow[r] & \prod SI(S_\beta) \arrow[r] & \prod \underline{X}_\beta \arrow[r] & 0
\end{tikzcd}$$

$\prod SI(S_\beta)$ is injective in $\mathcal{E}/\mathcal{S}$ so the induced map $\underline{E} \longrightarrow \prod SI(S_\beta)$ exists. The kernel-cokernel sequence gives

$$\ker(\underline{E} \longrightarrow \prod SI(S_\beta)) \rightarrowtail \ker(r(\underline{E}) \longrightarrow \prod \underline{X}_\beta)$$

hence is pure, but also a subobject of $\underline{E}$ hence repure, thus zero.

So $\underline{E}$ embeds in a product of $\prod SI(S_\beta)$.

To show $SI(S_\beta)$ are indecomposable, suppose $\underline{X}_1 \oplus \underline{X}_2 = SI(S_\beta)$. Apply functor T to get $T(\underline{X}_1) \oplus T(\underline{X}_2) = TSI(S_\beta) \cong I(S_\beta)$ by 8.26.

But $I(S_\beta)$ is indecomposable in $\mathcal{R} \cap \mathcal{T}$ (hence also in $\mathcal{E}/\mathcal{S}$), being the injective hull of a simple. Now say $T(\underline{X}_1) = 0$, since $SI(S_\beta) \in \mathcal{R} \Rightarrow \underline{X}_1 \in \mathcal{R}$ so $T(\underline{X}_1) = t(\underline{X}_1) = 0$ implies $\underline{X}_1$ is pure also $\Rightarrow \underline{X}_1 = 0$. Similarly $T(\underline{X}_2) = 0 \Rightarrow \underline{X}_2 = 0$. //

The objects $SI(\underline{S}_\beta)$ are injective in $\mathcal{E}/\mathcal{S}$. Hence by the dual of Prop. 3.13 are of the form

$$\mathcal{J}(A_\beta) = 0 \longrightarrow A_\beta \longrightarrow I_\beta \longrightarrow Z_\beta \longrightarrow 0 .$$

But this is also an element of $\mathcal{R}$. Hence A_β is pure-injective.

<u>Theorem 8.45</u> Given $\mathcal{O}$ a functor category, there exists a set of algebraically compact (pure injective) objects $\{A_\alpha\}_{\alpha \in I}$ such that X is algebraically compact <u>iff</u> X is a direct summand of a direct product of copies of A_α.

<u>Proof</u> ($\Leftarrow$) Since products of pure injectives and direct summands of pure injectives are pure injective.

($\Rightarrow$) Given X algebraically compact, then $\mathcal{J}(X) \in \mathcal{R}$ hence embeds in a suitable product $\prod \mathcal{J}(A_\beta)$, where $\{A_\beta\}$ as above.

So there is a monic

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & I & \longrightarrow & X' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod A_\beta & \longrightarrow & \prod I_\beta & \longrightarrow & \prod X_\beta & \longrightarrow & 0 \end{array}$$

(I injective)

Since $\mathcal{O}\!\!/$ is a functor category, one can assume I is the product of injective hulls of simple objects of $\mathcal{O}\!\!/$, and injective objects are algebraically compact.

But being a monic simply means $X \longrightarrow I \oplus (\prod A_\beta)$ splits.

This gives the result. (Note that injective hulls of simples in $\mathcal{O}\!\!/$ have been thrown into the original set of A_β.) //

This completes the duality of $\mathcal{J}$ with $\mathcal{R}$, in the sense that $\mathcal{R}$ can be described as the set of objects cogenerated by the class of $\mathcal{J}(X)$ with X algebraically compact (pure injective), $\mathcal{J}$ the set of objects generated by the class of $\pi(X)$ with X pure projective. The structure theorem for pure projectives 5.10 (i) reduces this to the set $\{\pi(C_\alpha)\}$ over the $\{C_\alpha\}$ of finitely presented objects. Thm. 8.44 allows the reduction to a set $\{\mathcal{J}(A_\beta)\}$ for some (undetermined as yet) set $\{A_\beta\}$ of algebraically compact objects.

The category $\mathcal{E}/\mathcal{S}$ may not have injective hulls. However we have the following.

<u>Proposition 8.46</u> (i) Objects of $\mathcal{R}$ have injective hulls which are objects of $\mathcal{R}$ in the category $\mathcal{E}/\mathcal{S}$.

(ii) If $\mathcal{R} \cap \mathcal{J}$ is perfect, objects of $\mathcal{J}$ have projective covers (which are objects of $\mathcal{J}$) in the category $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> (i) Given $\underline{X} \in \mathcal{R}$, $t\underline{X} \in \mathcal{R} \cap \mathcal{J}$ has an injective hull in the category $\mathcal{R} \cap \mathcal{J}$ which is Grothendieck. If this is $t\underline{X} \hookrightarrow \underline{E}$ then by Cor. 8.38 $St\underline{X} \rightarrowtail S\underline{E}$ is an injective hull of $St(\underline{X})$ in $\mathcal{E}/\mathcal{S}$. But $St(\underline{X}) = ST(\underline{X})$ and since $\underline{X} \in \mathcal{R}$, $0 \to \underline{X} \to ST\underline{X} \to \underline{X}' \to 0$ is localization where $\underline{X}'$ is pure. By Lemma 8.35 $\underline{X} \rightarrowtail ST(\underline{X})$ is essential. Thus $\underline{X} \rightarrowtail ST(\underline{X}) \rightarrowtail S\underline{E}$ is

the injective hull of $\underline{X}$, and $S\underline{E} \in \mathcal{R}$.

(ii) Dual. //

<u>Lemma 8.47</u> (i) If $\underline{E}$ is injective and $\underline{E} \in \mathcal{R}$ then localization $\underline{E} \longrightarrow ST\underline{E}$ is an isomorphism.

(ii) If $\underline{P}$ is projective and $\underline{P} \in \mathcal{T}$ then the colocalization $RT\underline{P} \longrightarrow \underline{P}$ is an isomorphism.

<u>Proof</u> (i) Since $\underline{E} \in \mathcal{R}$, $\underline{E} \rightarrowtail ST\underline{E} \longrightarrow \underline{X}' \longrightarrow 0$ with $\underline{X}'$ pure, but $\underline{E}$ injective implies this splits, so $\underline{X}'$ is a direct summand of $ST\underline{E}$ which is repure, hence $\underline{X}' = 0$.

(ii) Dual. //

<u>Lemma 8.48</u> (i) $\underline{E} \in \mathcal{R}$ is injective in $\mathcal{R}$ <u>iff</u> it is injective in $\mathcal{E}/\mathcal{S}$.

(ii) $\underline{P} \in \mathcal{T}$ is projective in $\mathcal{T}$ <u>iff</u> it is projective in $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> (ii) By Prop. 5.5(ii) $\underline{T}_1 \longrightarrow \underline{T}_2$ is epi in $\mathcal{T}$ <u>iff</u> it is epi in $\mathcal{E}/\mathcal{S}$. So projective in $\mathcal{E}/\mathcal{S}$ implies projective in $\mathcal{T}$.

Conversely, suppose
$$\begin{tikzcd} & \underline{P} \arrow[d] \\ \underline{A} \arrow[r, two heads] & \underline{B} \end{tikzcd}$$
then $\underline{P}$ is copure so $\underline{P} \longrightarrow \underline{B}$ factors as $\underline{P} \longrightarrow t\underline{B} \hookrightarrow \underline{B}$; and by Lemma 8.7(i) t preserves epi, so

$$\begin{tikzcd} & \underline{P} \arrow[d] \arrow[dl, dashed] \\ t\underline{A} \arrow[r, two heads] \arrow[d, hook] & t\underline{B} \arrow[d, hook] \\ \underline{A} \arrow[r, two heads] & \underline{B} \end{tikzcd}$$

gives the required factorization through $\underline{A}$.

(i) Dual. //

<u>Proposition 8.49</u> (i) There is an equivalence of categories between the injective subcategory of $\mathcal{R}$ and the injective subcategory of $\mathcal{R} \cap \mathcal{T}$.

(ii) There is an equivalence of categories between the projective subcategory of $\mathcal{T}$ and the projective subcategory of $\mathcal{R} \cap \mathcal{T}$.

Proof (i) Regarding T as a functor $\mathcal{J} \to \mathcal{J} \cap \mathcal{R}$ (the restriction of T to $\mathcal{J}$), then by Cor. 8.27 the counit $TS \xrightarrow{\bullet} 1$ is an equivalence. But also, by Lemma 8.47 (i) on injectives, the unit $1 \xrightarrow{\bullet} ST$ is an equivalence. Hence $\mathcal{J} \underset{S}{\overset{T}{\rightleftarrows}} \mathcal{J} \cap \mathcal{R}$ restricts to the required equivalence (lemma 8.48 tacitly used in applying Lemma 8.47).

(ii) Dual. //

Now $\mathcal{J}$ has a canonical set of generating small projectives, the set of finite presentations $\{\pi(C_\alpha)\}$, C_α finitely presented. SO via the equivalence above, $\mathcal{R} \cap \mathcal{J}$ has $\{T\pi(C_\alpha)\} = \{r'\pi(C_\alpha)\}$ as a set of generating small projectives. Now $\mathcal{R} \cap \mathcal{J}$ is a functor category, hence by the fundamental characterization of functor categories

$$\mathcal{R} \cap \mathcal{J} \cong (\{T\pi(C_\alpha)\}^*, \mathrm{Ab})$$

(contravariant functors on the set of $T\pi(C_\alpha)$), hence this yields

Proposition 8.50 $\mathcal{R} \cap \mathcal{J} \cong (\{\pi(C_\alpha)\}^*, \mathrm{Ab})$ where $\{\pi(C_\alpha)\}$ is the set of finite presentations (of finitely presented objects C_α) in $\mathcal{E}/\mathcal{S}$. //

Remark Let $\mathcal{B}$ be the small additive category of finitely presented objects in the underlying category $\mathcal{O}$ upon which $\mathcal{E}/\mathcal{S}$ is established. And $\pi(\mathcal{B})$ its image under $\mathcal{O} \xrightarrow{\pi} \mathcal{E}/\mathcal{S}$. Prop. 8.50 gives $\mathcal{R} \cap \mathcal{J} \cong (\pi(\mathcal{B})^*, \mathrm{Ab})$. Now $\pi(\mathcal{B})$ can be described without referral to $\mathcal{E}/\mathcal{S}$ in the following fashion (recall $\mathcal{O} \xrightarrow{\pi} \mathcal{E}/\mathcal{S}$ factors as $\mathcal{O} \to \mathcal{O}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$ and $\mathcal{O}/\mathcal{P} \rightarrowtail \mathcal{E}/\mathcal{S}$ was a full embedding) : objects of $\pi(\mathcal{B})$ are finitely presented objects of $\mathcal{O}$ and

$$\begin{aligned} \mathrm{Hom}_{\pi(\mathcal{B})}(X,Y) &= \mathrm{Hom}_{\mathcal{O}}(X,Y) \,/\, \text{morphisms factoring through projectives} \\ &= \mathrm{Hom}_{\mathcal{O}}(X,Y) \,/\, \mathcal{P}(X,Y) \,. \end{aligned}$$

CHAPTER 9

FUNCTOR CATEGORY TECHNIQUES

In this chapter, we will be concerned with two functor categories and their relationship via adjunctions. One functor category will be $\mathcal{R}\cap\mathcal{T}$, and the other will be the module category ${}_R\mathcal{M}$ where R is the functor ring for the set of finitely presented objects of $\mathcal{A}$ (definition shortly). Much of this chapter will be of a peripheral nature to the theory of the sequence category $\mathcal{E}/\mathcal{S}$, however functor rings are a useful and important tool, so an examiniation of how this concept fits into the framework of the sequence category should be of some value, if not immediately then at least as the groundwork for future research.

Let $\mathcal{B}$ be the small additive category of finitely presented objects of $\mathcal{A}$. We use π to also denote the restriction of the full embedding $\pi: \mathcal{A}/\mathcal{P} \rightarrowtail \mathcal{E}/\mathcal{S}$ to the image of $\mathcal{B}$ in $\mathcal{A}/\mathcal{P}$. By Prop. 8.50, $\mathcal{R}\cap\mathcal{T} = (\pi(\mathcal{B})^*, \mathrm{Ab})$. Considering the functor category $(\mathcal{B}^*, \mathrm{Ab})$, one can form

$$\begin{tikzcd}
(\mathcal{B}^*, \mathrm{Ab}) \arrow[r, dashed, "\overline{\pi}"] & (\pi(\mathcal{B})^*, \mathrm{Ab}) \cong \mathcal{R}\cap\mathcal{T} \\
\mathcal{B} \arrow[u, hook] \arrow[r, "\pi"'] & \pi(\mathcal{B}) \arrow[u, hook]
\end{tikzcd}$$

The vertical inclusions are the Yoneda embeddings $X \mapsto (-,X)$. The map $\overline{\pi}$ is the unique colimit preserving extension of the map $\mathcal{B} \xrightarrow{\pi} \pi(\mathcal{B}) \hookrightarrow (\pi(\mathcal{B})^*, \mathrm{Ab})$ (see Mitchell [21], page 106 Thm. 5.2). There is also a natural functor induced by π, $\pi_*: (\pi(\mathcal{B})^*, \mathrm{Ab}) \longrightarrow (\mathcal{B}^*, \mathrm{Ab})$ where $\pi_*(F) = F\pi$.

<u>Proposition 9.1</u> π_* is the right adjoint of $\overline{\pi}$.

<u>Remark</u> This is a standard reuslt about functor categories in a more general situation, but we remain with our specific framework.

<u>Proof</u> First to establish $((-,X), \pi_* G) \cong (\overline{\pi}(-,X), G)$:

By Yoneda,

$$((-,X), \pi_* G) \cong \pi_* G(X) = (G\pi)(X) = G(\pi X).$$

By Yoneda again,

$$G(\pi X) \cong ((-,\pi X), G) = (\overline{\pi}(-,X), G) \quad .$$

Then for arbitrary sums $\bigoplus_I (-,X_\alpha)$,

$$\begin{aligned}(\bigoplus_I (-,X_\alpha), \pi_* G) &\cong \prod_I ((-,X_\alpha), \pi_* G) \\ &\cong \prod_I (\overline{\pi}(-,X_\alpha), G) \\ &\cong (\bigoplus_I (\overline{\pi}(-,X_\alpha)), G) \\ &\cong (\overline{\pi}(\bigoplus_I (-,X_\alpha), G)\end{aligned}$$

, since $\overline{\pi}$ is colimit preserving.

And finally, if F is in $(\mathcal{B}^*, \mathrm{Ab})$ since sums of representables are resolving, F is the cokernel $P_2 \longrightarrow P_1 \longrightarrow F \longrightarrow 0$ with P_1, P_2 'free'. Then

$$\begin{tikzcd} 0 \arrow[r] & (F, \pi_* G) \arrow[r] \arrow[d, dashed] & (P_1, \pi_* G) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & (P_2, \pi_* G) \arrow[d, phantom, "\cong" rotate=-90] \\ 0 \arrow[r] & (\overline{\pi} F, G) \arrow[r] & (\overline{\pi} P_1, G) \arrow[r] & (\overline{\pi} P_2, G) \end{tikzcd} .$$

The bottom row is exact because $\overline{\pi}$ is cokernel preserving, then apply $(-,G)$. The induced map is the adjunction isomorphism. //

<u>Proposition 9.2</u> π_* is (i) exact

(ii) faithful

(iii) limit and colimit preserving.

<u>Proof</u> (i) π_* is left exact since it is a right adjoint, so it suffices to show that π_* preserves epis. If $F \twoheadrightarrow G$ then $F(\pi(X)) \twoheadrightarrow G(\pi(X))$ for all X which becomes $(\pi_* F)(X) \twoheadrightarrow (\pi_* G)(X)$.

(ii) Suppose ν is a natural transformation $\nu : F \longrightarrow G$ such that $\pi_* \nu = 0$. Then $(\pi_* \nu)_X \ : \ (\pi_* F)(X) \longrightarrow (\pi_* G)(X)$ is the map

$$\nu_{\pi X} \ : \ F(\pi X) \longrightarrow G(\pi X)$$ and since

every object of $\pi(\mathcal{B})$ is of the form πX , ν must be the zero transformation.

(iii) π_* preserves limits because it is a right adjoint; and since π_* is exact, for colimit preserving one need only verify that sums are preserved :

$$\begin{aligned}(\pi_*(\oplus_I F_\alpha))(X) &= ((\oplus_I F_\alpha)\pi)(X) = (\oplus_I F_\alpha)(\pi X)\\ &= \oplus_I(F_\alpha(\pi X)) = \oplus_I(F_\alpha\pi)(X)\\ &= \oplus_I(\pi_* F_\alpha)(X) = (\oplus \pi_* F_\alpha)(X).//\end{aligned}$$

9.3 The Unit For $(\bar{\pi}, \pi_*)$: $\eta : 1 \longrightarrow \pi_*\bar{\pi}$

For $(-,X) \in (\mathcal{B}^*, Ab)$

$$\pi_*\bar{\pi}(-,X) = \pi_*(-,\pi X) = (\pi -, \pi X) .$$

This will yield the exact sequence

$$0 \longrightarrow P(-,X) \longrightarrow (-,X) \xrightarrow{\eta_{(-,X)}} (\pi -, \pi X) \longrightarrow 0$$

where $P(Y,X)$ = maps $Y \to X$ factoring over projectives. Since $\eta_{(-,X)}$ is epi, and representations generate, the fact that both $\bar{\pi}$ and π_* are colimit preserving implies η_F is epi for any F (if $P_2 \to P_1 \to F \to 0$, P_2, P_1 'free' then

$$\begin{tikzcd} P_2 \arrow[r] \arrow[d, "\eta_{P_2}"'] & P_1 \arrow[r] \arrow[d, "\eta_{P_1}"'] & F \arrow[r] \arrow[d, "\eta_F"] & 0 \\ \pi_*\bar{\pi}P_2 \arrow[r] & \pi_*\bar{\pi}P_1 \arrow[r] & \pi_*\bar{\pi}F \arrow[r] & 0 \end{tikzcd}$$

implies η_F epi) .

9.4 Counit for $(\bar{\pi}, \pi_*)$: $\varepsilon : \bar{\pi}\pi_* \longrightarrow 1$

The functor $\bar{\pi}$ is onto, for if G lies in $(\pi(\mathcal{B})^*, Ab)$ and

$$\oplus_{I_1}(-,\pi X_\alpha) \longrightarrow \oplus_{I_2}(-,\pi X_\beta) \longrightarrow G \longrightarrow 0$$

is exact then $G = \bar{\pi}\bar{G}$ where $\bar{G}$ = coker $(\oplus_{I_1}(-,X_\alpha) \to \oplus_{I_2}(-,X_\beta))$.

By the theory of adjoints,

$$\bar{\pi} \xrightarrow{\bar{\pi}\eta} \bar{\pi}\pi_*\bar{\pi} \xrightarrow{\varepsilon\bar{\pi}} \bar{\pi}$$

is the identity, so using the above notation

$$\bar{\pi}\,\bar{G} \xrightarrow{\bar{\pi}\eta_{\bar{G}}} \bar{\pi}\,\pi_*\bar{\pi}(\bar{G}) \xrightarrow{\varepsilon_{\bar{\pi}\bar{G}}} \bar{\pi}\,\bar{G}$$

is the identity, giving

$$G \xrightarrow{\bar{\pi}\eta_{\bar{G}}} \bar{\pi}\,\pi_* G \xrightarrow{\varepsilon_G} G$$

the identity. By 9.3, $\eta_{\bar{G}}$ is epi, and $\bar{\pi}$ preserves colimits so $\bar{\pi}\eta_{\bar{G}}$ is an isomorphism, hence also ε_G .

Proposition 9.5 The counit $\varepsilon\colon \bar{\pi}\pi_* \longrightarrow 1$ is an equivalence.//

Corollary 9.6 π_* is fully faithful.

Proof Fully faithful is equivalent (in adjoint situation) to components of counit being isomorphisms (Maclane [20], page 88 Thm. 1) . //

9.7 The Right Adjoint of π_*

Suppose a right adjoint (unique up to equivalence) π^* exists, then

$$(\pi_*(-,\pi X),F) \cong ((-,\pi X),\pi^*F) \underset{\text{Yoneda}}{\cong} (\pi^*F)(\pi X) \quad .$$

Since all objects of $\pi(\mathcal{B})$ are of the form πX, this isomorphism can actually serve as a definition :

$$\pi^*F(\pi X) \;=\; (\pi_*(-,\pi X),F) \;.$$

Using the fact that π_* is colimit preserving yields (proof similar to 9.1) that $(\pi_* G,F) \cong (G,\pi^*F)$.

9.8 Counit for (π_*,π^*): $\varepsilon\colon \pi_*\pi^* \longrightarrow 1$

To the exact sequence $0 \longrightarrow P(-,X) \longrightarrow (-,X) \longrightarrow (\pi-,\pi X) \longrightarrow 0$, apply $(-,F)$ to obtain

$0 \longrightarrow ((\pi-,\pi X),F) \longrightarrow ((-,X),F)$ which is

$0 \longrightarrow (\pi_*\pi^*F)(X) \longrightarrow F(X)$, this will be

the counit. Note that each component of ε is monic.

Lemma 9.9 For $\varphi\colon F \rightarrow G$, $\pi_*\varphi$ is an isomorphism iff φ is an isomorphism.

<u>Proof</u> Let $0 \to K \to F \to G \to N \to 0$ be exact. Then since π_* is exact, the sequence

$$0 \longrightarrow \pi_* K \longrightarrow \pi_* F \longrightarrow \pi_* G \longrightarrow \pi_* N \longrightarrow 0$$

is exact. If $\pi_* F \longrightarrow \pi_* G$ is an isomorphism, this forces $\pi_* K = \pi_* G = 0$, but evaluating

$$(\pi_* K)(X) = K(\pi X) = 0$$

for all X, so K = 0, similarly N = 0. //

<u>9.10</u> <u>Unit for (π_*, π^*) : $\eta : 1 \longrightarrow \pi^* \pi_*$</u>

$$\pi_* \xrightarrow{\pi_* \eta} \pi_* \pi^* \pi_* \xrightarrow{\varepsilon \pi_*} \pi_*$$

is the identity. By 9.8, $\varepsilon\pi_*$ is monic, hence an isomorphism, which implies $\pi_* \eta$ is also an isomorphism, so by Lemma 9.9 η is an isomorphism.

<u>Remark</u> That the components of the unit are isomorphisms can also be established by 'dualizing' Thm1 and its lemma, page 88 of Maclane [20], yielding for an adjoint pair (F,G) , F faithful <u>iff</u> unit is monic.

F full <u>iff</u> unit is split epi.

In our case, π_* is fully faithful by Cor. 9.6.

<u>9.11</u> <u>Tensor Product</u>

Before introducing the functor ring, we examine the functor $\overline{\pi}$ in another fashion which may be more familiar (once one swallows the elaborate definitions).

Let $\mathcal{C}$ be a small category. Then there is a unique (up to isomorphism) functor

$- \otimes - \;:\; (\mathcal{C}, Ab) \times (\mathcal{C}^*, Ab) \longrightarrow Ab$ with the following properties :

(a) $- \otimes B$ and $A \otimes -$ are right exact

(b) $- \otimes B$ and $A \otimes -$ preserve arbitrary sums

(c) $(C,-) \otimes B = B(C)$ and $A \otimes (-,C) = A(C)$ for any C in $\mathcal{C}$.

The existence of $A \otimes -$ and $- \otimes B$ is established as follows :

For fixed A in $(\mathcal{C}, Ab)$, then $A : \mathcal{C} \longrightarrow Ab$ can be interpreted via Yoneda as mapping $(-,C) \longmapsto A(C)$ and so has a unique right exact sum preserving extension $A \otimes -$ to $(\mathcal{C}^*, Ab)$. Similarly for $- \otimes B$. The uniqueness of extension yields that $(A \otimes -)(B)$ is naturally isomorphic to $(- \otimes B)(A)$.

9.12 The Left Adjoint $\overline{\pi}$ of π_* using Tensor Product

$(\overline{\pi}(F))(\pi X) \;=\; (\pi X, \pi -) \otimes F$

Under this light the computation of $\overline{\pi}$ is hidden away by the handy ' $\otimes$ ' symbol, which avoids the actual computation of $\overline{\pi}$ as the cokernel of applying π to a free presentation $P_2 \to P_1 \to F \to 0$ (see Mitchell [21], page 106).

One can also compute the unit (see 9.8) by applying $- \otimes F$ to the exact sequence

$$0 \to P(X,-) \to (X,-) \to (\pi X, \pi -) \to 0$$

to obtain

$$P(X,-) \otimes F \longrightarrow (X,-) \otimes F \longrightarrow (\pi X, \pi -) \otimes F \longrightarrow 0$$

which becomes

$$P(X,-) \otimes F \longrightarrow F(X) \xrightarrow{\eta_F(X)} (\pi_* \overline{\pi} F)(X) \longrightarrow 0$$

giving

$$\eta_F \;:\; F \twoheadrightarrow \pi_* \overline{\pi} F.$$

9.13 Counit Isomorphism Revisited

Consider the exact sequences

$$0 \longrightarrow P(X,-) \longrightarrow (X,-) \longrightarrow (\pi X, \pi -) \longrightarrow 0$$

and $\quad 0 \longrightarrow P(-,Y) \longrightarrow (-,Y) \longrightarrow (\pi -, \pi Y) \longrightarrow 0$, and form

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & P(X,-) \otimes P(-,Y) \arrow[r] \arrow[d] & (X,-) \otimes (-,Y) \arrow[r] \arrow[d] & (\pi X, \pi -) \otimes P(-,Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(X,-) \otimes (-,Y) \arrow[r] \arrow[d] & (X,-) \otimes (-,Y) \arrow[r] \arrow[d] & (\pi X, \pi -) \otimes (-,Y) \arrow[r] \arrow[d] & 0 \\
 & P(X,-) \otimes (\pi -, \pi Y) \arrow[r] \arrow[d] & (X,-) \otimes (\pi -, \pi Y) \arrow[r] \arrow[d] & (\pi X, \pi -) \otimes (\pi -, \pi Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The middle row and column are exact because $(-,Y)$ and $(-,X)$ are projective objects.

Using property 9.11 (c) , this becomes

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & P(X,-)\otimes P(-,Y) \arrow[r] \arrow[d] & P(X,Y) \arrow[r] \arrow[d] & (\pi X,\pi -)\otimes P(-,Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(X,Y) \arrow[r] \arrow[d] & (X,Y) \arrow[r] \arrow[d] & (\pi X,\pi Y) \arrow[r] \arrow[d] & 0 \\
 & P(X,-)\otimes(\pi -,\pi Y) \arrow[r] \arrow[d] & (\pi X,\pi Y) \arrow[r] \arrow[d] & (\pi X,\pi -)\otimes(\pi -,\pi Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Now the map $\begin{tikzcd} P(X,Y) \arrow[r] & (X,Y) \arrow[d] \\ & (\pi X,\pi Y) \end{tikzcd}$ of the lower left square is zero, hence

$$\begin{tikzcd}
P(X,Y) \arrow[d, two heads] & \\
P(X,-)\otimes(\pi -,\pi Y) \arrow[r, tail] & (\pi X,\pi Y)
\end{tikzcd}$$

is also zero, cancel the vertical epi, giving the horizontal monic the zero map. Hence $P(X,-)\otimes(\pi -,\pi Y) = 0$.

Similarly using the upper right square yields

$$\begin{tikzcd}
P(X,Y) \arrow[r, two heads] & (\pi X,\pi -)\otimes P(-,Y) \arrow[d] \\
 & (\pi X,\pi Y)
\end{tikzcd}$$

the zero map. Cancelling the epi gives vertical map zero. But this map is monic (use the fact that $P(X,-)\otimes(\pi -,\pi Y) = 0$ and Snake Lemma on top two rows). Hence also $(\pi X,\pi -)\otimes P(-,Y) = 0$, and from this, one also derives $(\pi X,\pi -)\otimes(\pi -,\pi Y) = (\pi X,\pi Y)$.

Now $P(X,-)\otimes(\pi -,\pi Y) = 0$ becomes

$$P(X,-)\otimes \pi_*(-,\pi Y) = 0$$

and π_* and $P(X,-)\otimes -$ are colimit preserving (for fixed object $P(X,-)$). Then since $P(X,-)\otimes \pi_*-$ kills the projective $(-,\pi Y)$, it must be the zero functor on $(\pi(\mathcal{B})^*, Ab)$.

Applying $- \otimes \pi_* G$ to the exact sequence

$$0 \longrightarrow P(X,-) \longrightarrow (X,-) \longrightarrow (\pi X, \pi -) \longrightarrow 0$$

then yields

$$(X,-) \otimes \pi_* G \xrightarrow{\cong} (\pi X, \pi -) \otimes \pi_* G$$

or

$$\pi_* G(X) \xrightarrow{\cong} \bar{\pi}(\pi_* G)(\pi X)$$

or

$$G(\pi X) \xrightarrow{\cong} \bar{\pi}\, \pi_* G(\pi X) \quad .$$

This is the counit isomorphism, $1 \longrightarrow \bar{\pi}\pi_*$.

9.14 The Functor Ring

Let $\mathcal{D}$ be an arbitrary abelian category and $\mathcal{U} = \{U_\alpha\}$ a set of small objects of $\mathcal{O}\!\!/$. Then the functor category $(\mathcal{U}^*, Ab)$ can be interpreted as a module category in the following manner (Gabriel [11]) :

Let $R = \{ \varphi \in \mathrm{Hom}_{\mathcal{D}}(\oplus U_\alpha, \oplus U_\alpha) \mid \varphi|_{U_\alpha} = 0 \text{ a.e.} \}$.

R usually does not have a unity (unless the set $\{U_\alpha\}$ is finite), but this is replaced by a 'complete' set of idempotents $\{e_\alpha\}$ where

$e_\alpha : \bigoplus_\gamma U_\gamma \twoheadrightarrow U_\alpha \hookrightarrow \bigoplus_\gamma U_\gamma$. By a left R-module M, one has the usual meaning but with the added property that RM=M, so that $M = \bigoplus_\alpha e_\alpha M$.

The category ${}_R\mathcal{M}$ of left R-modules is then a cocomplete abelian category with $\{Re_\alpha\}$ as a generating set of small projectives.

Setting $U = \oplus U_\alpha$, let

$$\overline{\mathrm{Hom}}_{\mathcal{D}}(U,M) = \{ \varphi \in \mathrm{Hom}_{\mathcal{D}}(U,M) \mid \varphi|_{U_\alpha} = 0 \text{ a.e.} \}$$

then $\overline{\mathrm{Hom}}_{\mathcal{D}}(U,-)$ is a functor $\mathcal{D} \longrightarrow {}_R\mathcal{M}$ which assigns $U_\alpha \longmapsto Re_\alpha$ and $U \longmapsto R$. Then the unique colimit preserving extension to $(\{U_\alpha\}^*, Ab)$ is an equivalence of categories :

$$\begin{array}{ccc} (\{U_\alpha\}^*, Ab) & \dashrightarrow & {}_R\mathcal{M} \\ \uparrow & & \uparrow \\ \{U_\alpha\} & \longrightarrow & \{Re_\alpha\} \end{array}$$

Yoneda embedding

9.15 The Finitely Present Functor Ring(s)

For our purposes, let $\mathcal{B} = \{C_\alpha\}$, the set of finitely presented objects of $\mathcal{O}\!\mathcal{l}$, and $\pi(\mathcal{B}) = \{\pi(C_\alpha)\}$ the set of finite presentations in $\mathcal{E}/\mathcal{S}$.

Let R be the functor ring $\overline{\mathrm{Hom}}_{\mathcal{O}\!\mathcal{l}}(\oplus C_\alpha, \oplus C_\alpha)$ and $\pi(R) = \overline{\mathrm{Hom}}_{\mathcal{E}/\mathcal{S}}(\oplus\pi(C_\alpha), \oplus\pi(C_\alpha))$.

Then ${}_R\mathcal{M} \cong (\mathcal{B}^*, Ab)$ and ${}_{\pi(R)}\mathcal{M} \cong (\pi(\mathcal{B})^*, Ab)$.

The natural functor $\pi: \mathcal{O}\!\mathcal{l}/\mathcal{P} \longrightarrow \mathcal{E}/\mathcal{S}$ induces a ring homomorphism $\pi: R \longrightarrow \pi(R)$ sending the complete set of idempotents $\{e_\alpha\}$ for R to a complete set $\{\pi(e_\alpha)\}$ for $\pi(R)$.

This ring homomorphism induces a change of ring functor $\pi_*: {}_{\pi(R)}\mathcal{M} \longrightarrow {}_R\mathcal{M}$ where each left $\pi(R)$-module is naturally a left R-module via π. π_* is the functor defined in 9.1.

Just as for rings with unity, π_* has a left and right adjoint.

9.16 Left Adjoint of π_* Revisited

The left adjoint is $\overline{\pi}(M) = \pi(R) \otimes_R M$. Only a few minor changes must be incorporated due to the lack of unity element.

$${}_R(M, \pi_* N) \cong {}_{\pi(R)}(\pi(R) \otimes {}_R M, N)$$

sends $\varphi \longmapsto \overline{\varphi}$ where $\overline{\varphi}(x \otimes m) = x\,\varphi(m)$ and conversely, $\overline{\varphi} \longmapsto \varphi$ where if $m = \sum_{\alpha \in J} e_\alpha m_\alpha$ (the sum is direct $M = \oplus e_\alpha M$) J finite then

$$\varphi(m) = \overline{\varphi}(\textstyle\sum e_\alpha \otimes m_\alpha).$$

9.17 The Right Adjoint of π_* (Once Again)

Just as for rings with unit, $\pi^*(M) = \overline{\mathrm{Hom}}_R(\pi(R),M)$, those $\varphi: \pi R \longrightarrow M$ with $\varphi(\pi e_\alpha) = 0$ a.e., and

$$\mathrm{Hom}_R(\pi_* N, M) \cong \mathrm{Hom}_{\pi(R)}(N, \pi^* M)$$

sends $\varphi \longmapsto \varphi^*$ where $\varphi^*(n)(x) = \varphi(xn)$ and conversely $\varphi^* \longmapsto \varphi$ where if $n = \sum_J \pi(e_\alpha)n_\alpha$ J a finite set,

$$\varphi(n) = \sum_J (\varphi^*(n_\alpha))(\pi e_\alpha) \quad .$$

9.18 Unit for $(\pi(R) \otimes -, \pi_*)$

This is the map $M \longrightarrow \pi_*(\pi(R) \otimes_R M)$ sending

$$m = \sum e_\alpha m_\alpha \longmapsto \sum \pi(e_\alpha) \otimes m_\alpha \quad .$$

9.19 Counit for $(\pi(R) \otimes -, \pi_*)$

The counit for $(\pi(R) \otimes -, \pi_*)$ is the isomorphism $\pi(R) \otimes_R \pi_* M \longrightarrow M$ sending

$$\sum \pi(e_\alpha) \otimes m_\alpha \longmapsto \sum \pi(e_\alpha) m_\alpha$$

(using the fact that ${}_{\pi(R)}M = \bigoplus \pi(e_\alpha)M$).

9.20 Unit for (π_*, π^*)

$$\eta_M : M \longrightarrow \pi^* \pi_*(M) = \overline{\mathrm{Hom}}_R(\pi(R), \pi_*(M))$$

where $(\eta_M(m))(x) = xm$ for $x \in \pi(R)$

with inverse $\varphi \longmapsto \sum_\alpha \varphi(\pi e_\alpha)$

where sum is actually finite since $\varphi(\pi e_\alpha) = 0$ a.e..

9.21 Counit for (π_*, π^*)

$$\varepsilon_M : \pi_* \pi^* M \longrightarrow M$$

$$\varepsilon_M : \pi_*(\overline{\mathrm{Hom}}_R(\pi(R), M)) \longrightarrow M$$

sends $\varphi \longrightarrow \sum_\alpha \varphi(\pi e_\alpha)$

Note that $\varphi(\pi e_\alpha) \in e_\alpha M$ since $\varphi(\pi e_\alpha) = \varphi(e_\alpha \cdot (\pi e_\alpha))$
$= e_\alpha \varphi(\pi e_\alpha)$

so the above sum is direct, implying ε_M is monic.

We will drop the topic of functor ring for the moment since it is not the tool that we wish to utilize in this thesis. However it will be convenient to call upon this theory when necessary.

Since π_* is fully faithful, any map $\pi_* F \longrightarrow \pi_* G$ arises uniquely from a map $F \to G$.

<u>Lemma 9.22</u> (i) $F \to G$ is epi in $(\pi(\mathcal{B})^*, Ab)$ <u>iff</u> $\pi_* F \longrightarrow \pi_* G$ is epi in $(\mathcal{B}^*, Ab)$.

(ii) $F \to G$ is superfluous in $(\pi(\mathcal{B})^*, Ab)$ <u>iff</u> $\pi_* F \longrightarrow \pi_* G$ is superfluous in $(\mathcal{B}^*, Ab)$.

<u>Proof</u> (i) Proof same as Lemma 9.9.

(ii)($\Rightarrow$) If $X \longrightarrow \pi_* F \longrightarrow \pi_* G$ is epi, first note

$$\begin{array}{ccc} X & \longrightarrow & \pi_* F \\ {\scriptstyle \eta_X}\downarrow & & \downarrow \cong \eta_{\pi_* F} \\ \pi_* \bar{\pi} X & \longrightarrow & \pi_* \bar{\pi}\, \pi_* F \end{array}$$

so we only need to verify that the lower map is epi. Applying $\bar{\pi}$,

$$\begin{array}{ccccc} \bar{\pi} X & \longrightarrow & \bar{\pi}\pi_* F & \longrightarrow & \bar{\pi}\pi_* G \\ & & \cong \downarrow \varepsilon_F & & \cong \downarrow \varepsilon_G \\ & & F & \longrightarrow & G \end{array}$$

Hence $\bar{\pi} X \longrightarrow \bar{\pi}\pi_* F$ is epi since $F \to G$ is superfluous. Now apply π_* to conclude $\pi_* \bar{\pi} X \longrightarrow \pi_* \bar{\pi} \pi_* F$ is epi.

($\Leftarrow$) If $Y \to F \to G$ is epi, apply π_* :

$\pi_* Y \longrightarrow \pi_* F \longrightarrow \pi_* G$ is also epi, so

$\pi_* Y \longrightarrow \pi_* F$ is epi, implying that

$$\begin{array}{ccc} \bar{\pi}\pi_* Y & \longrightarrow & \bar{\pi}\pi_* F \\ {\scriptstyle \varepsilon_Y}\wr\| & & \wr\| {\scriptstyle \varepsilon_F} \\ Y & \longrightarrow & F \end{array}$$

is epi gives $Y \to F$ epi. //

<u>Proposition 9.23</u> $(\mathcal{B}^*,Ab)$ perfect implies that $(\pi(\mathcal{B})^*,Ab)$ is perfect.

<u>Proof</u> Let G be in $(\pi(\mathcal{B})^*,Ab)$ and $P \twoheadrightarrow \pi_* G$ a projective cover. $\overline{\pi}$ is colimit preserving and takes the representable $(-,X)$ to $(-,\pi X)$ hence also preserves projectives.

Claim $\overline{\pi} P \longrightarrow \overline{\pi}\pi_* G \overset{\varepsilon_G}{\cong} G$ is a projective cover.

Form

$$\begin{tikzcd} P \arrow[r, two heads] \arrow[d, "\eta_P"'] & \pi_* G \arrow[d, "\eta_{\pi_* G}", "\cong"'] \\ \pi_*\overline{\pi} P \arrow[r, two heads] & \pi_*\overline{\pi}\pi_* G \end{tikzcd}$$

Lower map is superfluous since $P \longrightarrow \pi_* G$ is superfluous. Hence by Lemma 9.22, $\overline{\pi} P \longrightarrow G$ is superfluous.//

<u>Remark</u> This proposition is the functor version of 'factor rings of a perfect ring are perfect'. Here the functor ring $\pi(R)$ is a factor of the functor ring R.

<u>Proposition 9.24</u> If $\eta_Q : Q \twoheadrightarrow \pi_*\overline{\pi} Q$ is superfluous for all Q , and $(\pi(\mathcal{B})^*,Ab)$ is perfect then $(\mathcal{B}^*,Ab)$ is perfect.

<u>Proof</u> Let $M \in (\mathcal{B}^*,Ab)$ and $\overline{\pi} P \twoheadrightarrow \overline{\pi} M$ a projective cover , P projective (all projectives of $(\pi(\mathcal{B})^*,Ab)$ are of the form $\overline{\pi}P$, since $\overline{\pi}$ preserves sums and $\overline{\pi}(-,X)$ is the representable $(-,\pi X)$).

Form

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, "\eta_P"'] & M \arrow[d, two heads, "\eta_M"] \\ \pi_*\overline{\pi} P \arrow[r, two heads] & \pi_*\overline{\pi} M \end{tikzcd}$$

The induced map out of the projective P is epi since η_M is a superfluous epi, and is superfluous since η_P is superfluous and $\pi_*\overline{\pi} P \twoheadrightarrow \pi_*\overline{\pi} M$ is superfluous by Lemma 9.22. //

Remark For factors of rings with unity $R \to R/I$, the corresponding unit map is $Q \to R/I \otimes_R Q = Q/IQ$, the condition of the proposition is that IQ is superfluous in Q for any Q. This condition is equivalent to I being left T-nilpotent (Anderson & Fuller [1], Lemma 28.3).

For the next proposition, let $\mathcal{O}$ be an arbitrary abelian category and $\{P_\alpha\}$ a set of small projectives.

Proposition 9.25 The following are equivalent :

(i) The set of morphisms between $\{P_\alpha\}$ is left T-nilpotent with respect to the radical.

(ii) Rad $(\oplus P_\alpha, \oplus P_\alpha)$ is the set of morphisms with components $Q_{\alpha\beta} \in \mathcal{J}(P_\alpha, P_\beta)$.

(iii) Rad $\oplus P_\alpha$ is superfluous in $\oplus P_\alpha$.

Proof (i) $\Rightarrow$ (ii) The set of Q with $Q_{\alpha\beta} = P_\alpha \hookrightarrow \oplus P_\alpha \xrightarrow{Q} \oplus P_\alpha \to P_\beta$ in $\mathcal{J}(P_\alpha, P_\beta)$ is an ideal, hence it suffices to show that $1 - Q$ is a unit for any such Q. The Konig Graph Theorem and T-nilpotence implies that given any small object $X \subseteq \oplus P_\alpha$ (X= a finite sum of P_α would suffice), there exists an n with $Q^n(X) = 0$. Hence the infinite sum $1 + Q + Q^2 + Q^3 + \dots$ is well defined on $\oplus P_\alpha$ and is the required inverse to $1 - Q$.

(ii) $\Rightarrow$ (iii) (Remark Proof same as Cor. 6.32)
If $N + \mathrm{rad}(\oplus P_\alpha) = \oplus P_\alpha$ then the epi $N \oplus (\oplus \mathrm{rad}\, P_\alpha) \twoheadrightarrow \oplus P_\alpha$ splits so the identity can be written as

$$\oplus P_\alpha \to N \oplus (\oplus \mathrm{rad}\, P_\alpha) \to \oplus P_\alpha$$

the sum of

$$\oplus P_\alpha \to \oplus \mathrm{rad}\, P_\alpha \to \oplus P_\alpha$$

and

$$\oplus P_\alpha \to N \to \oplus P_\alpha .$$

But the first lies in rad $(\oplus P_\alpha, \oplus P_\alpha)$ by (ii). So the latter is a unit, implying $N = \oplus P_\alpha$.

(iii) $\Rightarrow$ (i) Given a sequence of maps $P_1 \xrightarrow{a_1} P_2 \xrightarrow{a_2} P_3 \to \cdots$ with each $a_i \in \mathcal{J}(P_i, P_{i+1})$ let

$$A = \begin{pmatrix} 0 & a_1 & & & & \\ 0 & 0 & a_2 & & 0 & \\ 0 & 0 & 0 & a_3 & & \\ 0 & 0 & 0 & 0 & a_4 & \\ & & 0 & & & \ddots \end{pmatrix} .$$

Then the image of $A : \oplus P_i \to \oplus P_i$ is contained in $\oplus \operatorname{Rad} P_\alpha$ which is superfluous. Hence a A lies in $\operatorname{Rad}(\oplus P_\alpha, \oplus P_\alpha)$ [for abelian categories, $\operatorname{Rad}(\operatorname{End} P) = \{ \alpha$ with superfluous images $\}$.]. Letting B be the right inverse of $(1 - A)$, choose n with

$$P_1 \to \oplus P_i \xrightarrow{B} \oplus P_i \to \bigoplus_{i=n+1}^{\infty} P$$

equal to zero (P_1 can be carried only so far by B since P_1 is small) .

Then

$$(1-A^{n+1})B = (1+A+\ldots+A^n)(1-A)B = 1+A+\ldots+A^n$$

so

$$P_1 \to \oplus P \xrightarrow{1-A^{n+1}} \oplus P \xrightarrow{B} \oplus P \to P_{n+1} = P_1 \to \oplus P \xrightarrow{1+A+A^2+\cdots+A^n} \oplus P \to P_{n+1}$$

But A^j maps $P_i \to P_{i+j}$ so this reduces to

$$P_1 \to \oplus P \xrightarrow{-A^{n+1}} \oplus P \xrightarrow{B} \oplus P \to P_{n+1} = P_1 \to \oplus P \xrightarrow{A^n} \oplus P \to P_{n+1}$$

which yields $-a_1 a_2 a_3 \ldots a_{n+1} b = a_1 a_2 \ldots a_n$ where b is the $(n+2, n+1)$th component of B.

This gives $(a_1 a_2 \ldots a_n)(1-a_{n+1}b) = 0$

But $a_{n+1} \in \mathcal{J}(P_{n+1}, P_{n+2})$

so $a_{n+1}b \in \mathcal{J}(P_{n+1}, P_{n+1})$ implying that

$1 - a_{n+1}b$ is a unit, hence $a_1 a_2 \ldots a_n = 0$. //

<u>Proposition 9.26</u> If $\mathcal{A}$ and $(\pi(\mathcal{B})^*, Ab)$ are perfect, then so is $(\mathcal{B}^*, Ab)$.

<u>Proof</u> It will follow from a later result (Prop. 10.4) that $(\mathcal{B}^*, Ab)$ is semi-perfect. Hence it suffices to prove that arbitrary sums of $(-, C_\alpha)$ are semi-perfect and, utilizing Prop. 6.23 since properties (ii) and (iii) are preserved in taking sums, one needs only show that the radical is superfluous.

By Prop. 9.24 one must show that $\{(-, C_\alpha)\}$ is T-nilpotent with respect to the radical.

We adopt the proof of Hullinger [16]. Let

$C_1 \xrightarrow{a_1} C_2 \xrightarrow{a_2} C_3 \ldots$ with $a_i \in \mathcal{J}(C_i, C_{i+1})$.

Then $\pi(a_i) \in \mathcal{J}(\pi(C_i), \pi(C_{i+1}))$, so there exists n_1 with $\pi(a_1)\pi(a_2) \ldots \pi(a_{n_1}) = 0$. That is,

$a_1 a_2 \ldots a_{n_1}$ factors over a (finitely generated) projective P_1.

Repeat this argument starting with $C_{n_1+1} \to C_{n_1+2} \to \ldots$

then

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[dr] & \cdots & C_{n_1} \arrow[r, "a_{n_1}"] & C_{n_1+1} \arrow[r] \arrow[dr] & \cdots & C_{n_2} \arrow[r, "a_{n_2}"] & C_{n_2+1} \arrow[dr] & \\
& P_1 \arrow[ur] \arrow[rrr, dashed] & & & P_2 \arrow[ur] \arrow[rrr, dashed] & & & P_3 \arrow[u]
\end{tikzcd}$$

The lower row are maps in $\mathcal{J}(P_i, P_{i+1})$ since they factor via $a_{n_i} \in \mathcal{J}(C_{n_i}, C_{n_i+1})$. But $\mathcal{A}$ is perfect hence

$\{P_i\}$ is a left T-nilpotent system in $\mathfrak{A}$. So composition of lower row becomes zero implying also upper composition becomes zero. //

9.27 Remarks

Under reasonable conditions $\mathfrak{A}$ will be perfect, i.e. if $\mathfrak{A} = {}_{\Lambda}\mathcal{M}$ for Λ a perfect ring. So that R is perfect iff $\pi(R)$ is perfect. In Fuller [9], the functor ring is based not on finitely presented modules but finitely generated modules. However this will be the same provided one imposes the Noetherian condition, in which case one then would like a perfect Noetherian ring, i.e. Artinian.

Theorem 9.28 (Fuller)

Let Δ be a ring with identity and R the functor ring from finitely generated left Δ-modules. Then R is left perfect iff every left Δ-module is a direct sum of finitely generated modules. //

Then using Thm. 6.25, Thm. 6.6 and 9.15, one has

Proposition 9.29 If $\mathfrak{A} = {}_{\Delta}\mathcal{M}$, Δ-Artinian, then $\mathcal{E}/\mathcal{S}$ is perfect iff $\mathcal{R} \cap \mathcal{T}$ is perfect iff $\mathfrak{A}$ is pure semi-simple. //

We summarize the various categories and adjoint pairings in a quick overview of the previous sections. $\mathcal{B}$ is the additive category of finitely presented objects.

$$(\mathcal{B}^*, \mathrm{Ab}) \rightleftarrows (\pi(\mathcal{B})^*, \mathrm{Ab}) \cong \mathcal{R} \cap \mathcal{T} \rightleftarrows \mathcal{E}/\mathcal{S}$$

9.30 Starting with $\mathcal{R} \cap \mathcal{T} \underset{S}{\overset{R}{\rightleftarrows}}_{T} \mathcal{E}/\mathcal{S}$

(R,T) is an adjoint pair, which factors as

$$\mathcal{R} \cap \mathcal{T} \underset{r'}{\overset{R}{\rightleftarrows}} \mathcal{T} \underset{t}{\leftrightarrows} \mathcal{E}/\mathcal{S}$$

again adjoint pairs.

(T,S) is an adjoint pair, which factors as

$$\mathcal{R}\cap\mathcal{T} \underset{s}{\overset{t}{\leftrightarrows}} \mathcal{R} \underset{}{\overset{r'}{\leftrightarrows}} \mathcal{E}/\mathcal{S}$$

again adjoint pairs.

(i) T is exact

(ii) $\mathcal{E}/\mathcal{S} \xrightarrow{T} \mathcal{R}\cap\mathcal{T}$ is the category of additive fractions with respect to the pure subcategory $\mathcal{S}$, which has ST as localization functor and RT as colocalization.

(iii) (a) The counit for (T,S) is an equivalence (so S is fully faithful), i.e. T is the left-adjoint-left-inverse to S. S then establishes an isomorphism of $\mathcal{R}\cap\mathcal{T}$ to the reflective subcategory of divisible repure objects $\mathcal{R}\cap\mathcal{D}$

$$S\ :\mathcal{R}\cap\mathcal{T} \xrightarrow{\cong} \mathcal{R}\cap\mathcal{D} \hookrightarrow \mathcal{E}/\mathcal{S}$$

(b) The unit for (R,T) is an equivalence (so R is fully faithful), i.e. T is the right-adjoint-left-inverse to R. R then establishes an isomorphism of $\mathcal{R}\cap\mathcal{T}$ to the coreflective subcategory of codivisible copure objects $\mathcal{C}\cap\mathcal{T}$.

$$R\ :\mathcal{R}\cap\mathcal{T} \xrightarrow{\cong} \mathcal{C}\cap\mathcal{T} \hookrightarrow \mathcal{E}/\mathcal{S}$$

(iv) (a) S preserves injectives and essential monics.

(b) R preserves projectives and superfluous epis.

<u>9.31</u>

$$(\mathcal{B}^*,\mathrm{Ab}) \overset{\bar{\pi}}{\underset{\pi^*}{\rightleftarrows}} \pi_*\ (\pi(\mathcal{B})^*,\mathrm{Ab})$$

$(\bar{\pi}, \pi_*)$ and (π_*, π^*) are adjoint pairs.

(i) π_* is exact and fully faithful, giving an exact embedding of $(\pi(\mathcal{B})^*,\mathrm{Ab})$ into $(\mathcal{B}^*,\mathrm{Ab})$.

(ii) (a) The counit for $(\bar{\pi}, \pi_*)$ is an equivalence, i.e. $\bar{\pi}$ is the left-adjoint-left-inverse to π_*. π_* then establishes an isomorphism of $(\pi(\mathcal{B})^*,\mathrm{Ab})$ to the reflective subcategory of contravariant functors on $\mathcal{B}$ which vanish on finitely generated projective objects of $\mathcal{O}\!\!/$.

(b) The unit for (π_*, π^*) is an equivalence, i.e. π^* is the right-adjoint-left inverse to π_*. π_* then establishes an isomorphism of $(\pi(\mathcal{B})^*, \mathrm{Ab})$ to the coreflective category of contravariant functors on $\mathcal{B}$ which vanish on finitely generated projective objects of $\mathcal{Q}$. Note this subcategory is both reflective and coreflective.

9.32 Using the equivalences $(\mathcal{B}^*, \mathrm{Ab}) \cong {}_R\mathcal{M}$ and $(\pi(\mathcal{B})^*, \mathrm{Ab}) \cong {}_{\pi(R)}\mathcal{M}$ where R [respectively $\pi(R)$] is the functor ring with respect to $\mathcal{B}$ [$\pi(\mathcal{B})$], then $\pi_* : (\pi(\mathcal{B})^*, \mathrm{Ab}) \longrightarrow (\mathcal{B}^*, \mathrm{Ab})$ is the change of ring functor ${}_{\pi(R)}\mathcal{M} \longrightarrow {}_R\mathcal{M}$ induced by the natural ring homomorphism $R \longrightarrow \pi(R)$. $\bar{\pi}$ and π^* are then the associated left and right adjoint as in 'standard' ring theory. In particular, $\bar{\pi}$ is tensoring over the ground ring

$$\bar{\pi}- = \pi(R) \otimes_R$$

9.33 The isomorphism $(\pi(\mathcal{B})^*, \mathrm{Ab}) \xrightarrow{\cong} \mathcal{R} \cap \mathcal{T}$ results from the equivalence of the subcategories $\pi(\mathcal{B})$ and $T\pi(\mathcal{B})$, where by $\pi(\mathcal{B})$ one means

$$\{(-, \pi(X)) \mid X \in \mathcal{B}\},$$

which is a generating set of small projectives for $(\pi(\mathcal{B})^*, \mathrm{Ab})$. $T\pi(\mathcal{B})$ is the set

$$\{T\pi(X) \mid X \in \mathcal{B}\}$$

where $\pi(X)$ is the image under $\pi\colon \mathcal{Q}/\mathcal{P} \to \mathcal{E}/\mathcal{S}$; this set is a generating set of small projectives for the functor category $\mathcal{R} \cap \mathcal{T}$.

CHAPTER 10

SIMPLE SEQUENCES

10.1
$$\begin{tikzcd} \mathcal{A} \arrow[r, two heads] & \mathcal{A}/\mathcal{P} \arrow[r, hook, "\pi"] & \mathcal{E}/\mathcal{S} \\ \mathcal{B} \arrow[u, hook] \arrow[r, two heads] & \mathcal{B}/\mathcal{P}\cap\mathcal{B} \arrow[u, hook] & \end{tikzcd}$$

We commence with a brief review. $\mathcal{A}/\mathcal{P}$ is the projective homotopy category (3.4). $\mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{P}$ identifies objects but assigns a morphism its class modulo maps factoring through projectives. $\pi : \mathcal{A}/\mathcal{P} \hookrightarrow \mathcal{E}/\mathcal{S}$ assigns to an object X a short exact sequence terminating in X, with middle term projective. π is a fully faithful embedding of $\mathcal{A}/\mathcal{P}$ (as a resolving set of projectives). Dually, one can consider the injective homotopy category $\mathcal{A}/\mathcal{I}$.

$\mathcal{B}$ is the subcategory of finitely presented objects. The importance of $\mathcal{B}$ is that it generates (via π) the copure subcategory $\mathcal{T}$ of $\mathcal{E}/\mathcal{S}$; every pure projective is a direct summand of a direct sum of finitely presented objects of $\mathcal{A}$. Note that $\mathcal{P}\cap\mathcal{B}$ is the category of finitely generated projectives.

One can dualize $\mathcal{B}$ as follows (8.34 and 8.44) : let $\mathcal{B}'$ be the set of pure-injectives resulting from taking injective hulls of simples of $\mathcal{R}\cap\mathcal{T}$, along with injective hulls of simples of $\mathcal{A}$ (this latter set is a suitable replacement for finitely generated projectives). By Thm. 8.45, $\mathcal{B}'$ cogenerates $\mathcal{R}$. Every pure-injective (algebraically compact) is a direct summand of a direct product of elements from $\mathcal{B}'$. This creation of $\mathcal{B}'$ is not very esoteric : the duality with $\mathcal{B}$ is imposed rather than arising naturally. The duality can be better illuminated if one imposes the following conditions :

(a) Every pure-projective is a direct summand of a direct sum of pure projectives with local endomorphism rings.

(a') Every pure-injective is a direct summand of a direct product of pure injectives with local endomorphism rings.

Now any simple of $\mathcal{R}\cap\mathcal{T}$ is an epimorph of $\pi(C)$ for some C finitely presented. However, assuming (a), one can further assume End C local. In this case, $\pi(C)$ is a projective with a local endomorphism ring, hence has a unique maximal subobject. Conversely, $\pi(C)$ is a small projective for C finitely presented and will thus have a maximal subobject. This yields a 1-1 correspondance between simples of $\mathcal{R}\cap\mathcal{T}$ and non-projective finitely presented objects with local endomorphism ring.

Suppose $\underline{S} = 0 \to A' \to B' \to C \to 0$ is the simple epimorph of $\pi(C)$. $\underline{S}$ is copure and simple hence must be repure, so $r'\underline{S} = \underline{S}$. To compute $r'\underline{S}$ take $A' \rightarrowtail A''$, pure monic with A'' pure-injective, and form push-out

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A'' \arrow[r] & B'' \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{S} \arrow[d, "\cong"] \\ r'\underline{S} \end{tikzcd}$$

so w.l.o.g. the first term of $\underline{S}$ is pure-injective. Now assuming condition (a'), one can further assume $A'' = \prod A_i$ with End A_i local. (i.e. $A'' \oplus X = \prod A_i$ for some X, but then $A' \to A'' \oplus X$ is still pure monic).

Now form quotients by taking push-outs

$$\begin{tikzcd} 0 \arrow[r] & \prod A_i \arrow[r] \arrow[d, two heads] & B'' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_i \arrow[r] & B_i \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{S} \arrow[d, two heads] \\ \underline{S}_i \end{tikzcd} .$$

Since $\underline{S}$ is simple, $\underline{S}_i = 0$ or $\underline{S} \cong \underline{S}_i$. But not all $\underline{S}_i = 0$ since one has an embedding

$$\begin{tikzcd} 0 \arrow[r] & \prod A_i \arrow[r] \arrow[d, equal] & B'' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \prod A_i \arrow[r] & \prod B_i \arrow[r] & \prod C_i \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{S} \arrow[d, tail] \\ \prod \underline{S}_i \end{tikzcd}$$

Thus $\underline{S}$ can be represented as a sequence $0 \to A \to B \to C \to 0$ with C pure-projective (in fact, finitely presented) and A pure-injective, and both End C and End A local. Noting that $\mathcal{J}(A)$ is an indecomposable injective and $\underline{S} \hookrightarrow \mathcal{J}(A)$ is the unique simple subobject, establishes

<u>Proposition 10.2</u> If $\mathcal{O}\!\!\!/$ satisfies conditions (a) and (a') then any simple $\underline{S}$ in $\mathcal{R} \cap \mathcal{T}$ has a unique representation as $0 \to A \to B \to C \to 0$ with End A and End C local, A pure-injective and C pure-projective. //

Before proceeding further into the topic of simples, we pause to investigate individually the conditions (a) and (a').

<u>Proposition 10.3</u> $\mathcal{O}\!\!\!/$ satisfies (a') <u>iff</u> given C finitely presented with local endomorphism ring, there exists a simple sequence $0 \to A \to B \to C \to 0$ with End A local.

<u>Proof</u> ($\Leftarrow$) As in Thm. 8.45, $\{\mathcal{J}(A)\}$ resulting from the simples of $\mathcal{R} \cap \mathcal{T}$ cogenerate $\mathcal{R}$, and this implies that if D is pure injective, $\mathcal{J}(D) \rightarrowtail \prod \mathcal{J}(A_\alpha)$ for some product. So $D \rightarrowtail I \oplus (\prod A_\alpha)$ splits where I is any injective containing D, but w.l.o.g. I is the product of injectives with local endomorphism rings (injective hulls of simples in $\mathcal{O}\!\!\!/$).

($\Rightarrow$) As in proof of Prop. 10.2. //

<u>Proposition 10.4</u> The following are equivalent for a functor category $\mathcal{O}\!\!\!/$:

(i) $\mathcal{O}\!\!\!/$ satisfies condition (a)

(ii) $\mathcal{B}$ is a Krull-Schmidt category (every object is a finite direct sum of objects with local endomorphism rings)

(iii) $(\mathcal{B}^*, Ab)$ is semi-perfect.

(iv) $\mathcal{O}\!\!\!/$ and $\mathcal{R} \cap \mathcal{T}$ are semi-perfect.

<u>Proof</u> Note first that a functor category $\mathcal{C}$ is semi-perfect (every finitely generated object has a projective cover) <u>iff</u> $\mathcal{C}$ has a generating set of small projectives with local endomorphism rings (achieved by taking projective covers of simples).

(ii) $\Rightarrow$ (i) Trivial

(i) $\Rightarrow$ (ii) Every finitely presented object will be a direct summand of a finite direct sum of objects with local endomorphism rings. But every direct summand of such an object is again of this form (this is a consequence of Azumaya's theorem, see Anderson and Fuller [1], Thm. 12.6, Cor. 12.7 and Lemma 12.3 ; the module techniques hold for functor categories).

(ii) $\Rightarrow$ (iii) $(\mathcal{B}^*,\mathrm{Ab})$ has $\{(-,X)\}$ with End X local as a set of small projective generators, hence is semi-perfect.

(iii) $\Rightarrow$ (i) If $(\mathcal{B}^*,\mathrm{Ab})$ is semi-perfect, then it has a set of small projective generators with local endomorphism rings. But any small projective in a functor category is representable (Freyd [7], page 119). Hence there is a set $\{(-,X)\}$ with End C local generating $(\mathcal{B}^*,\mathrm{Ab})$. And then any pure-projective will be a direct summand of a direct sum from $\{X\}$.

(ii) $\Rightarrow$ (iv) Every small projective in $\mathcal{O}\!\mathcal{l}$ is a finite sum of objects with local endomorphism rings, hence $\mathcal{O}\!\mathcal{l}$ is semi-perfect. Also $\mathcal{R}\cap\mathcal{J} \simeq (\pi(\mathcal{B})^*,\mathrm{Ab})$ has $\{(-,\pi X)\}$ with End πX local as a generating set of small projectives, hence is semi-perfect.

(iv) $\Rightarrow$ (i) For this implication, some preparatory results which can be found scattered throughout the literature in various disguises :

<u>Proposition 10.5</u> If $\mathcal{O}$ is semi-perfect and X has no projective summands, then $\mathcal{J}(X,P) = (X,P)$ and $\mathcal{J}(P,X) = (P,X)$ for P finitely generated projective.

<u>Proof</u> Since $\mathcal{O}$ is semi-perfect, then P is a direct sum of local projectives. Then using finite additivity of $\mathcal{J}(-,X)$ and $\mathcal{J}(X,-)$ (the Kelly radical), one can assume P is a local projective. Then every composition $P \longrightarrow X \longrightarrow P$ is a non-unit since X has no projective summands, thus lies in $\mathcal{J}(P,P)$. Result follows by definition of $\mathcal{J}$. //

<u>Corollary 10.6</u> If $\mathcal{O}$ is semi-perfect, and X finitely generated with no projective summands, then $P(-,X) \subseteq \mathcal{J}(-,X)$.

<u>Proof</u> Suppose $g \in P(Y,X)$, then g factors as $Y \longrightarrow P \xrightarrow{\varphi} X$ for some projective P which can be taken as finitely generated since X is finitely generated. But $\varphi \in \mathcal{J}(P,X)$ by 10.5 , hence $g \in \mathcal{J}(Y,X)$. //

<u>Proposition 10.7</u> If $\mathcal{O}$ is semi-perfect, every finitely generated object X has a decomposition $X = X' \oplus P$, with P projective and X' has no projective summands.

<u>Proof</u> Let $Q \twoheadrightarrow X$ be a projective cover. Then $Q/\mathcal{J}(Q) \cong X/\mathcal{J}(X)$ is a finite direct sum of simples (Prop. 6.22). Now any projective summand P' of X results in a non-zero summand $P'/\mathcal{J}(P')$ of $X/\mathcal{J}(X)$. Remove P' from X, and continue removing projective summands, since $X/\mathcal{J}(X)$ is the finite sum of simples, the process terminates. //

<u>Corollary 10.8</u> If $\mathcal{O}$ is semi-perfect, X, Y finitely generated with decompositions $X = X' \oplus P$, $Y = Y' \oplus Q$ as in 10.7. Then $\varphi : X \to Y$ is an isomorphism <u>iff</u> $X' \to X \to Y \to Y'$ and $P \to X \to Y \to Q$ are isomorphisms.

Proof Writing $\varphi: X' \oplus P \longrightarrow Y' \oplus Q$ as $\begin{pmatrix} \varphi_1 & \psi_1 \\ \psi_2 & \varphi_2 \end{pmatrix}$, by Prop. 10.5 $\begin{pmatrix} 0 & \psi_1 \\ \psi_2 & 0 \end{pmatrix} \in \mathcal{J}(X,Y)$, hence $\begin{pmatrix} \varphi_1 & \psi_1 \\ \psi_2 & \varphi_2 \end{pmatrix}$ is an isomorphism iff $\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$ is an isomorphism, iff φ_1 and φ_2 are isomorphisms. //

Recall that X and Y are stably isomorphic if there exist projectives P and Q, and an isomorphism $P \oplus X \xrightarrow{\cong} Q \oplus Y$ this is equivalent to $\pi(X) \cong \pi(Y)$ by Cor. 4.13.

Corollary 10.9 For $\mathfrak{A}$ semi-perfect, X,Y finitely generated, with no projective summands, then X is stably isomorphic to Y iff X is isomorphic to Y. //

Corollary 10.10 If $\mathfrak{A}$ is semi-perfect, X finitely generated with no projective summands, then End πX is local iff End X is local.

Proof Any element of End πX is of the form $\pi\varphi$. $\pi\varphi$ is a unit in End πX iff there is an isomorphism (Cor.4.13) $X \oplus P_1 \longrightarrow X \oplus P_2$ with P_1, P_2 projective and the component map $X \longrightarrow X$ the map φ. Then by Cor. 10.8, $\pi\varphi$ is a unit implies φ is a unit, but converse is trivial, and result follows readily. //

We return to the proof of (iv) $\Rightarrow$ (i) of PROP. 10.3. Just as in (iii) $\Rightarrow$ (i), $\mathcal{R} \cap \mathcal{J} \cong (\pi(\mathcal{B})^*, \mathrm{Ab})$ has $\{(-, \pi X)\}$ with End πX local as a set of generators. Now $\mathfrak{A}$ is semi-perfect so w.l.o.g. X has no projective summands, then by Cor. 10.10 End X is local. Then any pure projective is a direct summand of a direct sum from the $\{X\}$ and the set of small projectives with local endomorphism rings. //

10.11 Remark

Starting with C finitely presented non-projective with local endomorphism ring [if C is projective it will be the projective cover of a simple, associate C to the injective hull of this simple], one associates the unique simple epimorph of $\pi(C)$. This association is 1-1 and onto the simples of $\mathcal{R} \cap \mathcal{J}$. The simple then determines a unique pure-injective non-injective A with local endomorphism ring (provided condition (a') holds). The correspondence $C \mapsto A$ is 1-1, but is it onto?

We have the following proposition which holds in general, and whose proof is just a matter of definition of $\mathcal{J}(A)$ and characterization of subobjects in $\mathcal{E}/\mathcal{S}$.

Proposition 10.12 Given A pure injective non-injective then there exists a simple sequence $0 \to A \to B \to C \to 0$ iff $\mathcal{J}(A)$ has a minimal subobject. //

Corollary 10.13 For each A pure-injective non-injective, there exists a simple sequence $0 \to A \to B \to C \to 0$ iff $\mathcal{R} \cap \mathcal{J}$ is semi-Artinian (every object has a minimal subobject).

Proof ($\Leftarrow$) Trivial

($\Rightarrow$) Since $\{T\mathcal{J}(A)\}$ cogenerate $\mathcal{R} \cap \mathcal{J}$ (as in Thm. 8.45) and $T\mathcal{J}(A)$ has a minimal subobject iff $\mathcal{J}(A)$ has a minimal subobject. //

10.14 Remark

Returning to the situation for which (a) and (a') hold : in this case $\{T\mathcal{J}(A)\}$ with A pure-injective and End A local cogenerates $\mathcal{R} \cap \mathcal{J}$. In this case, the correspondance $C \mapsto A$ from the set of pure-projectives local endomorphisms to the set of pure-injectives with local endomorphisms is 1-1 and onto iff $\mathcal{R} \cap \mathcal{J}$ is semi-Artinian. Now $\mathcal{R} \cap \mathcal{J} \cong {}_{\pi(R)}\mathcal{M}$, and the functor ring $\pi(R)$ is semi-Artinian iff the radical is right T-nilpotent. But by assuming (a), $\pi(R)$ is semi-perfect which holds on both right and left. Hence $\pi(R)$ will be right perfect (the category $\mathcal{M}_{\pi(R)} \cong (\pi(\mathcal{B}), Ab)$ covariant functors on $\pi(\mathcal{B})$).

10.15 Remark

In attempting to analyze condition (a'), the dual of (a), for which every pure injective is a direct summand of a direct product of pure injectives with local endomorphism rings, the major stumbling block is the mysterious structure of pure injectives in the general case. The basic tool is Thm. 8.45, but this is essentially an existence theorem. One would like a closer dual to the set $\mathcal{B}$ in the general case.

The clue seems to be that in working with $\mathcal{B}$, projectives can be assumed to be finitely generated. As noted above, for $\mathfrak{A}$ semi-perfect it seems natural to associate the projective cover of a simple to its injective hull. We are thus led to consider finitely cogenerated injectives. For instance, results 10.5 and 10.10 have natural duals. The dual to 10.5 is

Proposition 10.16 If X has no injective summands and I is finitely cogenerated, then $\mathcal{J}(X,I) = (X,I)$ and $\mathcal{J}(I,X) = (I,X)$. //

One need not impose restrictions on $\mathfrak{A}$ since every finitely cogenerated injective is a finite direct sum of local injectives (dual statement requires $\mathfrak{A}$ to be semi-perfect).

One also has that any finitely cogenerated Y has a decomposition $Y = Y' \oplus I$, I injective, Y' no injective summands. In fact, working with the socle of Y, which is a finite sum of simples rather than $X/\mathcal{J}(X)$ as in Prop. 10.7 will give existence and will also yield that Y is a direct sum of indecomposables. Furthermore, if Y is also pure injective these indecomposables have local endomorphism rings (End A is local for A indecomposable pure injective, B. Zimmermann-Huisgen [28]). So the machinery is set to go, except for a major stumbling block, the natural dual of $\mathcal{B}$. $\mathcal{B}$ has the crucial property , upon which a great deal depends : that a finitely generated pure-projective is finitely presented, (since every pure-projective is a direct summand of a direct sum of finitely generated pure-projectives). A 'natural' dual of finitely presented could then be a finitely cogenerated pure-injective, rather than a cofinitely presented object! Consider then the condition (a") every pure-injective is a direct summand of a direct product of finitely cogenerated pure-injectives.

Note that (a") implies (a') (again using the Zimmermann-Huisgen result), so that one can further impose that the endomorphism rings are local.

If (a") is satisfied, then $\{\mathcal{J}(A)\}$ with A finitely cogenerated, pure-injective and local endomorphism ring cogenerates $\mathcal{R}$, hence for any given simple in $\mathcal{R} \cap \mathcal{T}$, there is a monic into $\mathcal{J}(A)$ for some A. Since subobjects of $\mathcal{J}(A)$ are of the form $0 \to A \to X \to Y \to 0$ for some X,Y, any given simple can be represented with first term finitely cogenerated pure-injective with local endomorphism ring. But then again this simple is copure, hence there is an epimorph of $\pi(C)$ for some C finitely presented

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & X \arrow[r] & Y \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \pi(C) \arrow[d, two heads] \\ \underline{S} \end{tikzcd}$$

But the image $\underline{S}'$ of this map can be computed by forming the pull-back

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & X \arrow[r] & Y \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} \underline{S}' \arrow[d, "\cong"] \\ \underline{S} \end{tikzcd}$$

Hence assuming condition (a") any given simple will have a representation $0 \to A \to B \to C \to 0$ with a finitely cogenerated pure-injective with local endomorphism ring and C finitely generated pure projective (finitely presented).

If one also imposes (a), one can further assume End C is local and in this case the representation is unique, by Prop. 10.2.

Recall from 10.1, the set $\mathcal{B}'$ of pure-injectives, which was to act as the dual of $\mathcal{B}$. This duality was imposed, and I feel that the 'natural' dual is the set of finitely cogenerated pure injectives. The above has shown that this is indeed the case if (a") is satisfied.

<u>10.17</u> <u>The Intrinsic Characterization of Simples</u>

An object $\underline{S}$ of $\mathcal{E}/\mathcal{S}$ is simple <u>iff</u> it has no proper subobjects <u>iff</u> it has no proper quotients, other than the zero object.

Given any object $\underline{S}$, let $\underline{S}_f$ denote the subobject resulting from the pull-back of a morphism $f : X \to C$, i.e.

$$\begin{array}{ccccccccc} 0 & \to & A & \to & Y & \to & X & \to & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array} \qquad \begin{array}{c} \underline{S}_f \\ \downarrow \\ \underline{S} \end{array}$$

Recall that any subobject of $\underline{S}$ can be represented as $\underline{S}_f$ for some f. Also let $\underline{S}_I$ be the sum of subobjects $\underline{S}_f$, $f \in I$ for any set of morphisms I. $\underline{S}$ is simple <u>iff</u> $\underline{S}_f = 0$ or $\underline{S}_f = \underline{S}$ for any morphism f. If $\underline{S}_f = \underline{S}$ the cokernel sequence splits; if $\underline{S}_f = 0$, f factors over $B \to C$. One then has

<u>Proposition 10.18</u> $0 \to A \to B \to C \to 0$ is a simple object of $\mathcal{E}/\mathcal{S}$ <u>iff</u> given any morphism $f : X \to C$, either f factors over $B \to C$, i.e.

$$\begin{array}{ccc} & \swarrow & X \\ & & \downarrow \\ B & \twoheadrightarrow & C \end{array}$$

or the sum map $X \oplus B \twoheadrightarrow C$ splits. //

<u>Proposition 10.19</u> $0 \to A \to B \to C \to 0$ is a simple object of $\mathcal{E}/\mathcal{S}$ <u>iff</u> given any morphism $g : A \to Y$, either g factors through B, i.e.
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\ Y & \end{tikzcd}$$
or the sum map $A \rightarrowtail B \oplus Y$ splits. //

The resemblance to Auslander's almost split exact sequences (a.s.e.s.) is immediate. I should mention that my initial study of $\mathcal{E}/\mathcal{S}$ resulted from piecing together some notions of Fieldhouse [6] on purity, Freyd [8], Maclane's [19] brief mention of a sequence category, and Auslander [3]. Characterizing the simples as above came quite naturally, it amazes me that Auslander pulled them out of partially clouded mid-air, but also saddens me not to be the creator for I had never heard of an a.s.e.s., having shied away from papers dealing with 'representation theory'. We proceed then with the concept of a.s.e.s. and demonstrate that it is not quite natural in the general setting of the sequence category .

<u>10.20 Almost Split Exact Sequences (a.s.e.s.)</u>

$0 \to A \to B \to C \to 0$ is an a.s.e.s. if A,B,C are finitely generated, End A and End C are local, such that either (in which case both)

(1) if $f : X \to C$ is not split epi, X finitely generated, then f factors over $B \to C$.

(1') if $g : A \to Y$ is not split monic, Y finitely generated, then g factors through $A \to B$.

Now this definition was used originally in the context of modules over an Artinian algebra, and has been adopted for the more general case of Artinian rings (so that in (1) and (1') X can be taken to be indecomposable). For Artinian algebras, the condition that X (Y) be finitely generated can be removed (one of Auslander's results).

It follows that an a.s.e.s. is a simple object of $\mathcal{E}/\mathcal{S}$. That is, if one considers the subcategory of finitely generated subobjects $\mathcal{O}'$ in $\mathcal{O}$, then forming $(\mathcal{E}/\mathcal{S})'$ using only objects from $\mathcal{O}'$, and one has that $(\mathcal{E}/\mathcal{S})'$ is a subcategory of $\mathcal{E}/\mathcal{S}$. Then if $\mathcal{O}$ is a module category over an Artinian algebra $(\mathcal{E}/\mathcal{S})'$ is abelian and by definition an a.s.e.s. is a simple sequence in $(\mathcal{E}/\mathcal{S})'$ in which beginning and end terms have local endomorphism rings. And it then follows that it is also simple as an object of $\mathcal{E}/\mathcal{S}$.

In easing the restrictions that $\mathcal{O}$ be a module category over an Artinian ring, should finitely generated be replaced by finitely presented?

For Artinian rings, the concepts coincide, and Thm. 8.43 suggests C should be finitely presented. However, it is too stringent to impose that A and B also be finitely presented, for we have observed that the existence of simples in $\mathcal{E}/\mathcal{S}$ leads to A being pure-injective, not finitely presented. Furthermore, if one drops conditions that End C and End A are local, one is dealing with the simples of $\mathcal{R} \cap \mathcal{T}$. However the complete generality achieved in just dealing with simples does not yield representation results. A compromise definition is as follows :

<u>10.21</u> <u>Locally Represented Simple Sequences</u>

$0 \to A \to B \to C \to 0$ is locally represented simple if

(1) End A and End C are local,

(2) either C is pure-projective <u>or</u> A is pure-injective, and

(3) either : if $f: X \to C$ is not split epi, then f factors over $B \to C$,

or : if $g: A \to Y$ is not split monic, then g factors through $A \to B$.

Either condition of (3) states that the sequence is simple, so if one holds then so does the other.

As for the conditions of (2), suppose C is pure-projective. Then this is a copure sequence which is simple, hence it is also repure. Consider $\underline{S} \rightarrowtail \mathcal{J}(A)$. Since End A is local, End $\mathcal{J}(A)$ is local, and so $\mathcal{J}(A)$ is indecomposable injective with $\underline{S}$ as a minimal subobject, i.e. $\mathcal{J}(A)$ is the injective hull of $\underline{S}$. This implies $\mathcal{J}(A)$ is also a repure sequence, for if $\mathcal{J}(A)$ has a pure subobject, this object would contain the repure object $\underline{S}$. But $\mathcal{J}(A)$ is repure iff A is pure-injective.

Similarly, if A is pure-injective, this implies C is pure-projective.

10.22 Remarks

The condition of Prop. 10.18 for a simple sequence, that the sum map $X \oplus B \twoheadrightarrow C$ splits, is equivalent to $X \twoheadrightarrow C$ splits, since End C is local.

We have attempted to give a self-dual definition, which is why C is assumed to be just pure-projective and not seemingly stronger 'finitely presented'.

However one has the following :

if End C is local, and if $\overset{n}{\oplus} X_i \to C$ splits, then $X_i \to C$ splits for some i. If C is finitely generated (which implies small) then restriction to finite sums is easily removed. Even more remarkable is that finitely generated can be dropped (again a folklore result).

Proposition 10.23 If End C is local, and $\oplus X_\alpha \to C$ splits, then for some α , $X_\alpha \to C$ splits.

Proof Consider maps $\varphi_\alpha : C \to \oplus X_\alpha \twoheadrightarrow X_\alpha \rightarrowtail \oplus X_\alpha \to C$

For any finitely generated subobject D of C, there is a finite set of φ_α such that $\sum_\alpha \varphi_\alpha$ is the identity on D. So $1 - \sum \varphi_\alpha$ is not a unit, hence at least one φ_α is not in $\mathcal{J}(C,C)$, and must be a unit. //

One is now faced with a problem of dualization. For pure projective + local endomorphism $\Rightarrow$ pure projective + finitely generated $\Leftrightarrow$ finitely presented. But when will pure injective + local endomorphism $\Rightarrow$ pure injective + finitely cogenerated ? If this implication holds, conditions (a') and (a") are equivalent. The implication does not hold in general : for example, if $\mathcal{O}\!\!\!/ =$ Ab, $\mathbb{Q}$ is not finitely cogenerated, but is pure injective with local endomorphism ring. Hence one is led to another generalization of the a.s.e.s., that of 'finitely locally represented simples' in which the final term is finitely presented with local endomorphism ring and the beginning term is finitely cogenerated pure-injective with local endomorphism ring.

Note that for Artinian algebra, finitely generated will imply both finitely presented and finitely cogenerated pure injective, so agreement is reached with a.s.e.s.

10.24 Existence Problems

Utilizing the proof of Prop. 10.2, one has

Proposition 10.25. Given C finitely presented with local endomorphism ring, there exists a [finitely] locally represented simple with C as final term if condition (a') [(a")] is satisfied. //

Corollary 10.26 If (a) holds, then condition (a') [(a")] is equivalent to positive solution of existence problem.

Proof For then any simple is the epimorph of $\pi(C)$ for some C finitely presented with local endomorphism ring and then the corresponding set $\{\mathcal{J}(A)\}$ cogenerate $\mathcal{R}$ (as in Thm. 8.45) which yields (a') [(a")]. //

The reverse procedure is to construct a [finitely] locally represented simple starting with a pure-injective [cofinitely generated] with local endomorphism ring. For the first step, it will be necessary to assume that $\mathcal{J}(A)$ has a simple subobject, which then represents a simple sequence $0 \to A \to X \to Y \to 0$. (This problem did not arise using C, for then $T\pi(C)$ was a small projective in $\mathcal{R} \cap \mathcal{T}$, hence has a simple epimorph.) The next step would then be to apply condition (a), to achieve the required [finitely] locally represented simple (dualizing proof of Prop. 10.2), one then has

Proposition 10.27 For $\mathcal{R} \cap \mathcal{T}$ semi-Artinian, then given A pure-injective [finitely cogenerated] with local endomorphism ring, there exists a [finitely] locally represented simple with A as the first term if condition (a) is satisfied. //

Corollary 10.28 If $\mathfrak{A}$ is semi-perfect and $\pi(R)$ right perfect, then the [finitely] locally represented simple existence problem of Prop. 10.27 has a solution.

Proof Condition (a) is equivalent to $\mathfrak{A}$ and $\pi(R)$ both semi-perfect; and right perfect is equivalent to semi-Artinian and semi-perfect. //

And a partial converse :

Proposition 10.29 If $\mathcal{R} \cap \mathcal{T}$ is semi-Artinian, $\mathfrak{A}$ semi-perfect and condition (a') [(a")] holds, then condition (a) is equivalent to a positive solution of the existence problem (of Prop. 10.27).

Proof ($\Rightarrow$) 10.27.

($\Leftarrow$) $\mathcal{R} \cap \mathcal{T}$ semi-Artinian and condition (a') [(a")] implies that each simple will be a subobject of $\mathcal{J}(A)$ for some A pure-injective [finitely cogenerated] with local endomorphism ring. The associated [finitely] locally represented simples yield a set $\{C\}$ with C finitely presented with local endomorphism rings. Then $\{T\pi(C)\}$ are projective covers of the simples of $\mathcal{R} \cap \mathcal{T}$ hence generate $\mathcal{R} \cap \mathcal{T}$, and hence $\{\pi(C)\}$ generate $\mathcal{T}$. Then given D pure projective, there is an epi $\bigoplus \pi(C) \twoheadrightarrow \pi(D)$. So if $P \twoheadrightarrow D$ is epi, with P projective, then $(\bigoplus C) \bigoplus P \twoheadrightarrow D$ splits. Now if $\mathfrak{A}$ is semi-perfect, then P itself is a direct sum of finitely generated objects with local endomorphism rings, which gives the result. //

10.30 Construction of Simples

Given C non-projective, finitely presented, by 10.2 there exists a simple $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. If End C is local, one has a procedure of constructing this simple :

Step 1 For each finitely presented X, let

$\overline{X} = \bigoplus_{g \in \mathcal{J}(X,C)} X_g$, where for each $g \in \mathcal{J}(X,C)$, $X_g = X$; one has a natural map $\overline{X} \to C$ with components $g : X_g \longrightarrow C$.

Note that since End C is local, $\mathcal{J}(X,C) = \{ g \in (X,C)$ which are not split epi$\}$. As before, let $\mathcal{B}$ be the set of representatives of finitely presented objects. Let $\overline{C} = \bigoplus_{X \in \mathcal{B}} \overline{X}$

then there is a natural map $\psi : \overline{C} \longrightarrow C$ with components $\overline{X} \longrightarrow C$ as above.

Step 2 Form the exact sequence in $\mathcal{E}/\mathcal{S}$, $0 \to \pi(C)_\psi \to \pi(C) \to \underline{E} \to 0$, that is

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & \overline{C} \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] & P \oplus \overline{C} \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi(C)_\psi \arrow[d, tail] \\
\pi(C) \arrow[d, two heads] \\
\underline{E}
\end{tikzcd}$$

Claim, $\pi(C)_\psi$ is the sum of all proper copure subobjects of $\pi(C)$. In fact, $\{\pi(X),\ X \in \mathcal{B}\}$ generates $\mathcal{T}$, and the image of

$$\pi(\varphi) \ : \pi(X) \longrightarrow \pi(C)$$

lies in $\pi(C)_\psi$ if $\varphi \in \mathcal{J}(X,C)$ by construction, i.e.

$$\begin{tikzcd}
 & X \arrow[dl, dashed] \arrow[d, "\varphi"] \\
\overline{C} \arrow[r, "\psi"'] & C
\end{tikzcd}$$

, φ factors over ψ if $\varphi \in \mathcal{J}(X,C)$, and if $\varphi \notin \mathcal{J}(X,C)$, φ is split epi, and then $\pi(X) \twoheadrightarrow \pi(C)$ is a split epi.

Furthermore, $\pi(C)_\psi \neq \pi(C)$, for if this were true, then $\underline{E} = 0$, which means $P \oplus \overline{C} \to C \to 0$ splits. But End C is local, which means $P \to C \to 0$ splits or $\overline{C} \to C \to 0$ splits. The first is not the case and if the latter holds, then some component map $X \to C \to 0$ would split. But all components are in the (Kelly) radical so this is not possible.

[Note A non-constructive approach of achieving $\pi(C)_\psi$, i.e. without mention of ψ or $\overline{C}$, is to show that the set of proper copure subobjects of $\pi(C)$ is closed under finite union. In fact, if $\pi(C)_{f_i}$, $i=1,\dots n$, then the sum of $\pi(C)_{f_i}$ is $\pi(C)_f$, where $f_i : X_i \to C$ and f is the sum of the f_i, $f : \oplus X_i \to C$. Then $\pi(C)_f = \pi(C)$ iff f is a split epi iff f_i is a split epi for some i, since End C is local, iff $\pi(C)_{f_i} = \pi(C)$ for some i. Now since $\pi(C)$ is a small projective, the total sum is a proper subobject of $\pi(C)$.]

Now every subobject of $\underline{E}$ is pure. In fact, suppose $\underline{E}'$ is a proper subobject. We show that $t\underline{E}' = 0$.

Form the pull-back in $\mathcal{E}/\mathcal{S}$:

$$\begin{tikzcd}
0 \arrow[r] & \pi(C)_\psi \arrow[r] \arrow[d, equal] & \underline{X} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "P.B."] & t\underline{E}' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \pi(C)_\psi \arrow[r] & \pi(C) \arrow[r] & \underline{E} \arrow[r] & 0
\end{tikzcd}$$

$\underline{X}$ is an extension of $\pi(C)_\psi$ and $t\underline{E}'$, so by Prop. 5.7, $\underline{X}$ is copure. But this implies $\underline{X} = \pi(C)$ or $\underline{X} \subseteq \pi(C)_\psi$. If the former holds, then $t\underline{E}' = \underline{E}$, so $\underline{E}' = \underline{E}$ contradiction. The latter implies $\underline{X} = \pi(C)_\psi$, forcing $t\underline{E}' = 0$.

Step 3 Form the sequence $r'\underline{E}$.

To form $r'\underline{E}$, find a pure monic $E \rightarrowtail A$ with A pure-injective and form pushout in $\mathcal{O}$.

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, tail] & P \oplus \overline{C} \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{E} \arrow[d, two heads] \\
r'\underline{E}
\end{tikzcd}$$

Now,

$$0 \to t'\underline{E} \to \underline{E} \to r'\underline{E} \to 0$$

is exact, where $t'\underline{E}$ is the unique maximal pure subobject of $\underline{E}$ (Cor. 7.3), but since all subobjects of $\underline{E}$ are pure, $t'\underline{E}$ is a maximal subobject, hence $r'\underline{E}$ is simple.

<u>10.31</u> <u>Remarks</u>

The object $\overline{C}$ formed in Step 1 is quite large (i.e. not finitely generated). Reducing $\overline{C}$ to a finitely generated object seems to be the crux of establishing the existence of an almost split exact sequence terminating in C. The following technical lemmas are useful in controlling the size of C. Let $\underline{E} = 0 \to A \to B \to C \to 0$ be an arbitrary sequence (dropping conditions on C).

<u>Lemma 10.32</u> If $h : X \to C$ factors as $X \xrightarrow{f} Y \xrightarrow{g} C$ then the subobject $\underline{E}_h \subseteq \underline{E}_g$.

<u>Proof</u> $\underline{E}_h$ results from the pull-back of h with $B \to C$. But this can be achieved first as pull-back with g then f.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{E}_h \arrow[d, hook] \\
\underline{E}_g \arrow[d, hook] \\
\underline{E}
\end{tikzcd}
\quad .//$$

<u>Lemma 10.33</u> If $g : \oplus X_i \to C$ has components g_i then

$$\underline{E}_g = \underline{E}_{\{g_i\}} \quad .$$

<u>Proof</u> Obvious from definitions. //

<u>Lemma 10.34</u> If $S = \{g_i\}_{i \in I}$ generate Hom (X,C) as an End X module, then $\underline{E}_S = \underline{E}_{\mathrm{Hom}\,(X,C)}$.

<u>Proof</u> If $h \in$ Hom (X,C), there exists a finite set $\{f_i\}$ in End X such that $h = \sum f_i g_i$. Then h factors

$$X \xrightarrow{f} \oplus X_i \xrightarrow{g} C \qquad \text{so } \underline{E}_h \subseteq \underline{E}_g = \underline{E}_{\{g_i\}} \subseteq \underline{E}_S$$

(equality step by 10.33). //

<u>Lemma 10.35</u> If $g : X \twoheadrightarrow C$ is epi, then coker $\pi(g) : \pi(X) \to \pi(C)$ is $\underline{E} = 0 \to K \to X \to C \to 0$.

<u>Proof</u> This follows from Cor. 3.21, for $\ldots \to \pi(X) \to \pi(C) \to \underline{E} \to 0$ is the start of a projective resolution for $\underline{E}$.

[For an explicit proof,

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, two heads, "g"] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] & X \oplus Q \arrow[r] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi(X) \arrow[d, "\pi(g)"] \\
\pi(C) \arrow[d, two heads] \\
\underline{E}
\end{tikzcd}$$

Then
$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, dashed] & X \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r] & X \oplus Q \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$
is an isomorphism

with inverse
$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, dashed] & X \oplus Q \arrow[r] \arrow[d, "\binom{1}{\nu}"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & X \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where ν is the map
$$\begin{tikzcd}
 & Q \arrow[dl, dashed, "\nu"'] \arrow[d, two heads] \\
X \arrow[r, two heads] & C
\end{tikzcd}$$
, using projectivity of Q.]

<u>10.36 Example</u>

Let $\mathcal{A}$ = Ab, the category of abelian groups. Then finitely presented is equivalent to finitely generated, and $\mathcal{B}$, the subcategory of finitely generated abelian groups, has the property that every object is a finite direct sum of cyclic groups $\mathbb{Z}(p^k)$, $k = 0,1,2,\ldots$ and p a prime. End $\mathbb{Z}(p^k)$ is local for $k \neq 0$, and for $k = 0$ one has the integers $\mathbb{Z}$. For Ab, G is pure projective iff G is a direct sum of cyclic groups (this follows from a classical theorem of Kulikov, that subgroups of direct sums of cyclic groups are again a direct sum of cyclic groups, or can be derived from the decomposition theorems in Fuller and Anderson [1].) If one ignores the projective (free) summand of a pure-projective object, then condition (a) holds, (i.e. modulo projectives (a) holds for Ab). $\mathcal{R} \cap \mathcal{J} \cong (\pi(\mathcal{B})^*, \mathrm{Ab})$ will have $\{(-, \pi(\mathbb{Z}(p^k))\}$ $k=1,2,\ldots$ as projective generators with local endomorphism rings, so $\mathcal{R} \cap \mathcal{J}$ is semi-perfect, but $\mathcal{O}\!\mathcal{l}$ = Ab is not (recall that (a) holds iff both $\mathcal{O}\!\mathcal{l}$ and $\mathcal{R} \cap \mathcal{J}$ are semi-perfect). Also for Ab, G is pure injective iff G is a direct summand of a direct product of cocyclic groups (see Fuchs [29], part of Thm. 38.1), $\mathbb{Z}(p^k)$, $k=1,2,\ldots$ and ∞.

Note $\mathbb{Z}(p^k) \hookrightarrow \mathbb{Z}(p^\infty)$ is an essential monic, and $\mathbb{Z}(p^\infty)$ is the injective hull of the simple $\mathbb{Z}(p)$, so all the cocyclic groups are finitely cogenerated. Also all have local endomorphism rings. So (a") (and hence (a')) holds for Ab. Hence every simple object of $\mathcal{R} \cap \mathcal{J}$ (which is also simple in $\mathcal{E}/\mathcal{S}$) has a unique representation $0 \to A \to B \to C \to 0$ with C cyclic (and not $\mathbb{Z}$) and A cocyclic (and not $\mathbb{Z}(p^\infty)$), yielding a 1-1 correspondence $C \longmapsto A$.

Set $C = \mathbb{Z}(p^k)$. We construct $\pi(C)_\psi$ as in Step 1 of 10.30. Given $X \to C$, X finitely presented, since every finitely presented object is a direct sum of cyclics, we can assume by Lemma 10.38 that X itself is cyclic. Then consider any map not split epi $g : X \to C$. If g is not epi, g factors as $X \longrightarrow \mathbb{Z}(p^{k-1}) \xhookrightarrow{\;i\;} \mathbb{Z}(p^k)$ (for the moment exclude the case $k = 1$). Hence by Lemma 10.32, $\pi(C)_g \subseteq \pi(C)_i$. If g is epi (but not split), then X is necessarily of the form $\mathbb{Z}(p^r)$, with $r > k$. In this

case, g factors as $X \longrightarrow \mathbb{Z}(p^{k+1}) \xrightarrow{\nu} \mathbb{Z}(p^k)$, with ν the canonical epi. Again by Lemma 10.32, $\pi(C)_g \subseteq \pi(C)_\nu$.
Hence $\pi(C)_\nu = \pi(C)_i + \pi(C)_\nu = \pi(C)_{i\oplus\nu}$
where $i\oplus\nu : \mathbb{Z}(p^{k-1}) \oplus \mathbb{Z}(p^{k+1}) \twoheadrightarrow \mathbb{Z}p^k$,
so by Lemma 10.37 the required copure simple of Step 2 is

$$\operatorname{coker} \pi(i\oplus\nu) = 0 \rightarrow A \rightarrow \mathbb{Z}(p^{k-1}) \oplus \mathbb{Z}(p^{k+1}) \rightarrow \mathbb{Z}(p^k) \rightarrow 0.$$

But then A is a finite abelian group, in particular A is pure-injective, so Step 3 factoring out the maximal pure subobject is not necessary. The simple has been achieved. A simple computation shows $A \cong \mathbb{Z}_{p^k}$, and that the required simple is

$$0 \longrightarrow \mathbb{Z}(p^k) \xrightarrow{(1,-p)} \mathbb{Z}(p^{k-1}) \oplus \mathbb{Z}(p^{k+1}) \xrightarrow{\binom{p}{1}} \mathbb{Z}(p^k) \longrightarrow 0 .$$

The case k=1 is even easier. In this case, non-zero maps are epi, which implies that $\pi(C)_g \subseteq \pi(C)_\nu$ for all g not split epi, where $\nu: \mathbb{Z}(p^2) \twoheadrightarrow \mathbb{Z}(p)$. The resulting simple is then

$$0 \rightarrow \mathbb{Z}(p) \xrightarrow{Xp} \mathbb{Z}(p^2) \longrightarrow \mathbb{Z}(p) \longrightarrow 0 \quad .$$

So for $\mathcal{O}\!\!/ =$ Ab, the correspondence $C \longmapsto A$ achieved by constructing simple sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is just the identity.

<u>10.37</u> <u>Remark</u>

By Cor. 3.23, if p.d.A = n , then p.d. $\mathcal{E}/\mathcal{S} \leq 3n-1$. Equality holds for Ab: for consider the simple $\underline{S} = 0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0$, we show that p.d.$(\underline{S}) = 2$, so p.d. $\mathcal{E}/\mathcal{S} = 2$.

By Cor. 3.21, one has a projective resolution

$$0 \rightarrow \pi(\mathbb{Z}_p) \longrightarrow \pi(\mathbb{Z}_{p^2}) \longrightarrow \pi(\mathbb{Z}_p) \longrightarrow \underline{E} \longrightarrow 0$$

If p.d.$(\underline{S}) < 2$, then $0 \rightarrow \pi(\mathbb{Z}_p) \longrightarrow \pi(\mathbb{Z}_{p^2})$ is a split monic, which is impossible since they are non-isomorphic indecomposable projectives.

10.38 The Existence of a.s.e.s.

The existence of a.s.e.s. with a given final term C (finitely presented, local endomorphism ring) is of a more difficult nature than the existence problem for locally represented simples. The difficulty arises from the imposition that the leading term be finitely generated. This seems to be a red herring : the basic justification is that the category $\mathcal{O}'$ of finitely generated objects is an abelian subcategory of $\mathcal{O}$. So one can form a sequence category $\mathcal{E}/\mathcal{S}'$, where $\underline{E} \in \mathcal{E}/\mathcal{S}'$ iff $\underline{E} \cong \underline{E}'$ where $\underline{E}'$ is an exact sequence with finitely generated terms. Then $\mathcal{E}/\mathcal{S}'$ is indeed a full exact abelian subcategory of $\mathcal{E}/\mathcal{S}$, but $\mathcal{E}/\mathcal{S}'$ is buried within $\mathcal{J}$ and does not play the role of finitely generated objects. It seems more natural in dealing with $\mathcal{E}/\mathcal{S}$ and its subcategories $\mathcal{R}, \mathcal{S}', \mathcal{J}$, to consider sequences $\underline{E} = 0 \to A \to B \to C \to 0$ with A pure injective and C pure projective, i.e. $\underline{E} \in \mathcal{R} \cap \mathcal{J}$ (and conversely, any object of $\mathcal{R} \cap \mathcal{J}$ has such a representation). These are precisely the sequences $\underline{E}$ such that Hom $(\underline{E}, \mathcal{S}')$ = Hom $(\mathcal{S}', \underline{E})$ = 0. Finite restrictions can then be imposed, for instance, final term finitely generated and/or leading term finitely cogenerated. One should note also that if one imposes the condition that the final term has a local endomorphism ring, then this is in fact a finite condition. For Prop. 10.23 implies $\pi(C)$ is finitely generated if End C is local, hence any quotient of $\pi(C)$ is also finitely generated. So that any sequence terminating in C will be will be finitely generated. Unfortunately the dual does not seem to hold; that is, End A local will not imply $\upsilon(A)$ finitely cogenerated.

Returning to a.s.e.s., to examine how the existence problem fits within the framework of the sequence category : as noted in 10.20, an a.s.e.s. is a simple object of $\mathcal{E}/\mathcal{S}'$, the sequence category using finitely generated objects, and every simple is uniquely represented as an a.s.e.s. (where

$\mathcal{A} = {}_{\Lambda}\mathcal{M}$, Λ Artinian. So condition (a) holds, so any simple can be represented with leading and final terms finitely generated with local endomorphism ring). Now given C finitely presented with local endomorphism ring, one can proceed as in 10.30 to construct the unique simple (in $\mathcal{E}/\mathcal{S}$) epimorph $\underline{S}$ of $\pi(C)$. The object $\underline{E}$ arising from the exact sequence $0 \longrightarrow \pi(C)_{\psi} \longrightarrow \pi(C) \longrightarrow \underline{E} \longrightarrow 0$ in Step 1 of 10.30 is copure simple. That is, $\underline{E}$ is copure but every proper subobject is pure. This is clearly equivalent to every map $\pi(X) \longrightarrow \underline{E}$, with X finitely presented, either zero or epi. If $\underline{E} = 0 \rightarrow A' \rightarrow B' \rightarrow C \rightarrow 0$, this is equivalent to the statement that any non split epi $X \rightarrow C$, X finitely presented, factors over $B' \rightarrow C$, i.e.

$$\begin{tikzcd} & & & X \arrow[d] \arrow[dl, dashed] & \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Suppose now $\underline{G} = 0 \rightarrow A'' \rightarrow B'' \rightarrow C \rightarrow 0$ is an a.s.e.s. Then $\pi(C)_{\psi} \hookrightarrow \pi(C) \twoheadrightarrow \underline{G}$ is either zero or epi, but $\pi(C)$ is a small local projective, so $\pi(C) \twoheadrightarrow \underline{G}$ is a superfluous epi, so the composition cannot be epi. Hence one has

$$\begin{tikzcd} 0 \arrow[r] & \pi(C)_{\psi} \arrow[r] \arrow[d, dotted, hook] & \pi(C) \arrow[r] \arrow[d, equal] & \underline{E} \arrow[r] \arrow[d, dotted, two heads] & 0 \\ 0 \arrow[r] & \underline{L} \arrow[r] & \pi(C) \arrow[r] & \underline{G} \arrow[r] & 0 \end{tikzcd}$$

i.e. $\pi(C)_{\psi} \hookrightarrow \underline{L}$. However the epi $\nu : B'' \rightarrow C$ factors through the map ψ since it is not split epi (see Step 2) so by Lemma 10.32 , $L = \pi(C)_{\nu} \subseteq \pi(C)_{\psi}$, which implies $\underline{E} = \underline{G}$. So the existence of an a.s.e.s. terminating in C implies $\underline{E} \in \mathcal{E}/\mathcal{S}'$. Conversely, if $\underline{E} \in \mathcal{E}/\mathcal{S}'$, it is a simple object of $\mathcal{E}/\mathcal{S}'$. Also $\pi(C) \twoheadrightarrow \underline{E}$, so w.l.o.g. $\underline{E}$ terminates in C (this is only a technicality but in detail, represent $\underline{E} = 0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0$ and $\pi(C) \twoheadrightarrow \underline{E}$ as

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 & & \pi(C) \arrow[d, two heads] \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 & & \underline{E} \end{tikzcd}$$

then $\underline{E} \cong \mathrm{Im}\,(\pi(C) \longrightarrow \underline{E})$ which is computed as the pushout of $K \longrightarrow P$ and $K \longrightarrow A'$, i.e. $\underline{E} \cong 0 \rightarrow K \rightarrow A' \oplus P \rightarrow C \rightarrow 0$

and all terms are finitely generated.) Set $E = 0 \to A' \to B' \to C \to 0$ then furthermore condition (a) holds so $A' = \bigoplus A_i$ each with local endomorphism ring, which implies $\underline{E} \to \underline{E}_i$ is an isomorphism for a unique i, where

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & B' & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & A_i & \to & B_i & \to & C & \to & 0 \end{array} \qquad \begin{array}{c} \underline{E} \\ \downarrow \\ \underline{E}_i \end{array}$$

(since $\underline{E} \rightarrowtail \prod \underline{E}_i$ and $\underline{E}$ is simple in $\mathcal{E}/\mathcal{S}'$).

Hence for the existence of the required a.s.e.s., it is necessary and sufficient that $\underline{E} \in \mathcal{E}/\mathcal{S}'$. To make this more tractible, we need the following technical lemmas :

<u>Lemma 10.39</u> $\mathcal{A} = {}_\Lambda\mathcal{M}$, Λ-Artinian, if $\underline{E}_1 \rightarrowtail \underline{E}_2$ and $\underline{E}_2 \in \mathcal{E}/\mathcal{S}'$ then $\underline{E}_1 \in \mathcal{E}/\mathcal{S}'$ <u>iff</u> $\underline{E}_1$ is copure and finitely generated as an object of $\mathcal{E}/\mathcal{S}$.

<u>Proof</u> ($\Rightarrow$) By Prop. 5.4 all objects of $\mathcal{E}/\mathcal{S}'$ are finitely generated; and in $\mathcal{A}$ finitely generated implies finitely presented so all objects of $\mathcal{E}/\mathcal{S}'$ are also copure.

($\Leftarrow$) For some X finitely presented, $\pi(X) \twoheadrightarrow \underline{E}_1$ then $\underline{E}_1 \cong \mathrm{Im}\,(\pi(X) \twoheadrightarrow \underline{E}_1 \rightarrowtail \underline{E}_2)$ so if

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \to & X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1 & \to & B_1 & \to & C_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_2 & \to & B_2 & \to & C_2 & \to & 0 \end{array} \qquad \begin{array}{c} \pi(X) \\ \downarrow \\ \underline{E}_1 \\ \downarrow \\ \underline{E}_2 \end{array}$$

then $\underline{E}_1 \cong 0 \to E \to X \oplus B_2 \to C_2 \to 0$ which has finitely generated terms. //

<u>Lemma 10.40</u> $\mathcal{A} = {}_\Lambda\mathcal{M}$, Λ- Artinian. If $0 \to \underline{E}_1 \to \underline{E}_2 \to \underline{E}_3 \to 0$ is exact in $\mathcal{E}/\mathcal{S}$ and $\underline{E}_2 \in \mathcal{E}/\mathcal{S}'$, then $\underline{E}_1 \in \mathcal{E}/\mathcal{S}'$ <u>iff</u> $\underline{E}_3 \in \mathcal{E}/\mathcal{S}'$.

<u>Proof</u> ($\Rightarrow$) If $\underline{E}_1 \to \underline{E}_2$ is represented as

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \to & B_1 & \to & C_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_2 & \to & B_2 & \to & C_2 & \to & 0 \end{array}$$

with all terms finitely generated then

$\underline{E}_3 \cong \operatorname{coker}(\underline{E}_1 \to \underline{E}_2) \;=\; 0 \to E \to C_1 \oplus B_2 \to C_2 \to 0.$

So $\underline{E}_3 \in \mathcal{E}/\mathcal{S}'$.

($\Leftarrow$) Dual. //

Returning to previous discussion, we have an exact sequence $0 \to \pi(C)_\psi \to \pi(C) \to \underline{E} \to 0$, and $\pi(C)_\psi$ is copure, so $\underline{E} \in \mathcal{E}/\mathcal{S}'$ iff $\pi(C)_\psi \in \mathcal{E}/\mathcal{S}'$ (Lemma 10.40) iff $\pi(C)_\psi$ is finitely generated (Lemma 10.39).

This establishes part of

<u>Proposition 10.41</u> Assume $\mathcal{O} = {}_\Lambda\mathcal{M}$, Λ Artinian. Then given C finitely presented with local endomorphism ring f.a.e.,

(i) there exists an a.s.e.s. terminating in C.

(ii) the unique maximal proper copure subobject of $\pi(C)$ is finitely generated.

(iii) the unique simple epimorph of $\pi(C)$ is finitely presented in $\mathcal{R}\cap\mathcal{T}$.

<u>Proof</u> (i)$\Leftrightarrow$(ii) by previous discussion

(ii)$\Rightarrow$(iii) One has the exact sequence $0 \to \pi(C)_\psi \to \pi(C) \to \underline{E} \to 0$, and by assumption, $\pi(X) \twoheadrightarrow \pi(C)_\psi$ for some X finitely presented. Then apply the exact functor T, $T\underline{E}$ is simple and

$0 \to T\pi(C)_\psi \to T\pi(C) \to T\underline{E} \to 0$ and $T\pi(X) \twoheadrightarrow T\pi(C)_\psi$, by Lemma 8.40 $T\pi(X)$ is a small projective in $\mathcal{R}\cap\mathcal{T}$ hence finitely generated so $T\pi(C)_\psi$ is also finitely generated.

(iii)$\Rightarrow$(ii) By assumption, $T\pi(X) \twoheadrightarrow T\pi(C)_\psi$ for some X finitely generated. Apply the right exact functor R, $RT\pi(X) \twoheadrightarrow RT\pi(C)_\psi$. But $RT\pi(X) \cong \pi(X)$ by Lemma 8.47, and the counit $RT\pi(C)_\psi \to \pi(C)_\psi$ is epi since $\pi(C)_\psi$ is copure (8.29).

So $\pi(X) \cong RT\pi(X) \twoheadrightarrow RT\pi(C)_\psi \twoheadrightarrow \pi(C)_\psi$, shows $\pi(C)_\psi$ is finitely generated. //

Corollary 10.42 For $\mathcal{O} = {}_\Lambda\mathcal{M}$, Λ Artinian, then given any C finitely generated with local endomorphism ring there exists an a.s.e.s. terminating in C iff the simples of $\mathcal{R} \cap \mathcal{T}$ are finitely presented. //

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