EINSTEIN METRICS OF RANDERS TYPE

by

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Abstract

This thesis presents a study of Einstein Randers metrics. Initially introduced within the context of relativity, Randers metrics have a strong presence in both the theory and applications of Finsler geometry. The starting point is a new characterization of Einstein metrics of Randers type by three conditions. The conditions form a coupled, highly non-linear (due to the presence of a Riemannian Ricci tensor), second order system of partial differential equations. The equations are polynomial in the unknowns; a Riemannian metric $\tilde{a}$ and differential 1-form $\tilde{b}$.

Recently Z. Shen has generalized Zermelo's problem of navigation on the plane to arbitrary Riemannian manifolds. (The goal is to identify the paths of shortest time on a Riemannian manifold $(M, \breve{a})$ under the influence of an external force $W = W^i \partial_{x^i}$.) In this context, Randers metrics may be viewed as solutions to Zermelo's problem. The navigation structure yields the main result of the thesis, a succinct geometric description of Einstein metrics of Randers type. Explicitly, the Randers metric arising as the solution to Zermelo's problem on $(\breve{a}, W)$ is Einstein if and only if the Riemannian metric $\breve{a}$ is Einstein itself, and W is an infinitesimal homothety of $\breve{a}$.

The navigation description quickly yields a Schur lemma for the Ricci curvature of Randers metrics. It is a testament to the navigation description that this result, the first Schur lemma for Ricci curvature in (non-Riemannian) Finsler geometry, is obtained with relative ease. An extension of Matsumoto's Identity for Randers metrics of constant flag curvature to the Einstein setting then follows.

Having established these general results, I then explore three scenarios: Einstein metrics on surfaces of revolution, constant flag curvature metrics, and Einstein metrics on closed manifolds. The thesis closes with a collection of open questions.

Contents

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David is a seemingly endless source of guidance and insight. He presents an ethic and standard of scholarship I may only aspire to. Beyond these virtues, I appreciate David most for introducing me to the joys of collaboration. The excitement and pleasure I found in our work set the standard by which I shall compare future joint ventures.

Dedication

For Abuela,
our family's first student of mathematics.

CHAPTER 1

Introduction

1. Preliminaries

This thesis is about Finsler metrics. A Finsler space is a manifold M equipped with a family of smoothly varying Minkowski norms; one on each tangent space. Riemannian metrics are examples of Finsler norms that arise from an inner-product. Indeed, when introducing the concept of a manifold and its structures, Riemann acknowledged that quadratic differentials comprise only a special case. Unfortunately, the door closed on Finsler geometry when Riemann claimed "The study of the metric which is the fourth root of a quartic differential form is quite time-consuming and does not throw new light to the problem." It was not until Paul Finsler's 1918 thesis under Carathéodory that the field was resurrected.

Riemann's comment illuminates the two major obstacles facing the development of Finsler geometry. The first is the ubiquitous computational difficulty associated with the field. As challenging as Riemannian geometry is, the computations in Finsler geometry are considerably more daunting. This serves to frustrate and discourage many mathematicians. Nonetheless, recent advances have brought the computations of Finsler geometry to a more accessible level. The geometric description of Einstein Randers metrics, based on Zermelo's problem of navigation, I shall present offers a substantial improvement in this arena. Indeed, it is fair to say the computations of Einstein metrics of Randers type are now comparable to their Riemannian counterparts.

The computational difficulty contributes directly to the second challenge to Finsler geometry. This is the lack of meaningful examples. As evidenced by Riemann's comment that "The study of the metric ... does not throw new light to the problem", the prevailing belief is that Finsler metrics do not capture geometric phenomena omitted by their Riemannian counterparts. Until recently this belief has been supported by a dearth of explicit examples illustrating non-Riemannian behavior. In 2002 D. Bao and Z. Shen [**BS02**] constructed a Finsler metric on the 3-sphere that is of constant flag curvature, but not projectively flat. The example breaks the rigidity of Beltrami's theorem which states that a Riemannian metric is of constant flag (read "sectional") curvature if and only if it is projectively flat. Illustrating a distinctly non-Riemannian geometry, the S^3 example shows Finsler geometry may indeed "throw new light to the problem." The need for explicit examples motivates me to present several in §4.

2. Brief summary of results

In this short review of the contents I postpone defining terms and notation to subsequent sections.

Randers metrics are constructed from two familiar objects; a Riemannian metric $\tilde{a}_{ij}(x)$ and a 1-form $\tilde{b}_i(x)$ on the manifold M. The norm of a vector $y = y^i \partial_i \in T_x M$ in the tangent space at $x \in M$ is then defined to be

$$F(x,y) = \alpha(x,y) + \beta(x,y)\,,$$

where

$$\alpha(x,y) = \sqrt{\tilde{a}_{ij} y^i y^j} \qquad \text{and} \qquad \beta(x,y) = \tilde{b}_i y^i\,.$$

Essentially we alter a Riemannian metric by adding a linear term.

Conceptually the Ricci curvature is the average of the flag curvatures. (For the moment it suffices to think of flag curvature as the Finslerian analog of Riemannian sectional curvature.) A Finsler metric is Einstein if the Ricci curvature is a function of $x \in M$ alone, rather than the a priori $(x,y) \in TM$.

Theorem 2.3 provides the foundation for our analysis of Einstein Randers metrics. The theorem characterizes Einstein metrics of Randers type as solutions to a system of partial differential equations. The equations are polynomial in the unknowns: the Riemannian metric $\tilde{a}$ and its Ricci tensor; the 1-form $\tilde{b}$ and its first and second order covariant derivatives. The theorem is joint work with David Bao **[BR03a]**.

Recently, Shen **[She02a]** has shown that Randers metrics arise as solutions to Zermelo's problem on navigation on a Riemannian manifold $(M, \breve{a})$ under an external force W. The navigation structure is an enormously valuable tool for constructing examples of Einstein metrics. This utility is the motivation to rephrase the characterization of Einstein Randers metrics in terms of the underlying metric $\breve{a}$ and vector field W. We are rewarded by a succinct geometric description of Einstein metrics of Randers type. Explicitly, the Randers metric F is Einstein if and only if the underlying Riemannian metric $\breve{a}$ is Einstein, and W is an infinitesimal homothety of $\breve{a}$.

The navigation description yields two elegant results. First a Schur lemma, which states that the Ricci curvature of an Einstein Randers metrics must be constant in dimension greater than two. The second result is a generalization of Matsumoto's identity for Randers metrics of constant flag curvature to the Einstein setting. The identity further illuminates the relationship between the Ricci scalars of F and $\breve{a}$.

With the Einstein navigation description in hand I turn to three case studies. First, Einstein metrics are constructed on surfaces of revolution. The Ricci scalar of the non-Riemannian metrics is none other than the Gaussian curvature induced from Euclidean $\mathbb{R}^3$. Second, I discuss a recent classification of constant flag curvature Randers spaces. Lastly, I consider closed manifolds. The results here include rigidity theorems akin to Akbar-Zadeh's for constant flag curvature spaces. For example, I show that any closed Einstein Randers space with negative Ricci scalar is necessarily Riemannian.

3. Outline

The remainder of this chapter is concerned with introducing Finsler metrics, and their spray and flag curvatures. I then focus on Finsler metrics of Randers type. The underlying Riemannian metric $\tilde{a}$ of a Randers space permits an expression, known as Berwald's formula, of the spray curvature in terms of $\tilde{a}$, its curvature, and covariant derivatives of $\tilde{b}$. Berwald's formula is integral to our treatment of the flag curvature. The material of this chapter is treated briefly, without the detailed derivations that may be found elsewhere.

Chapter Two is devoted to a characterization theorem for Einstein metrics of Randers type. While refining the characterizing equations, I establish the constancy of an essential scalar. The material of Chapter Two is joint work with David Bao [**BR03b**].

The heart of the thesis is Chapter Three. Via Zermelo's problem of navigation, the characterization result of Chapter Two is parlayed into a breviloquent description of Einstein Randers metrics. At this point the Schur lemma and Matsumoto's Identity fall out neatly.

Chapter 4 is devoted to the three case studies outlined above. The closing chapter details some open questions.

4. Finsler metrics and flag curvature

The material in the remainder of the chapter is but a sketch of the elements of Finsler geometry I find useful. I shall not attempt a detailed treatment of this material which has been handled adroitly elsewhere. See, for example, [**BCS00, Run59, She01a, She01b**].

Let's begin with notation and some definitions. The following abbreviations are introduced to reduce some of the notational clutter associated with differential geometry:

- The Einstein convention : repeated up-down pairs of indices imply a summation. For example, $y^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n y^i \frac{\partial}{\partial x^i}$.
- The partial derivatives $\frac{\partial f}{\partial x^i}$ and $\frac{\partial f}{\partial y^i}$ are given by f_{x^i} and f_{y^i}, respectively.

Throughout the thesis M shall denote a manifold of dimension n. Points on M are denoted by x, and the tangent space to M at x is T_xM. The canonical coordinates on the tangent bundle TM are given by (x, y), where $y = y^i \frac{\partial}{\partial x^i} \in T_xM$.

A Finsler space (M, F) is a manifold M equipped with a smoothly varying family of Minkowski norms F – one on each tangent space – defined in the following way.

DEFINITION 1.1. *A Finsler metric is a continuous function* $F : TM \to [0, \infty)$ *with the following properties,*

- *Regularity: F is smooth on* $TM\backslash 0 := \{(x, y) \in TM | y \neq 0\}$.
- *Positive homogeneity:* $F(x, \lambda y) = \lambda F(x, y)$ *for all* $\lambda > 0$.
- *Strong convexity: the fundamental tensor*

$$g_{ij}(x, y) := \left(\tfrac{1}{2}F^2\right)_{y^i y^j}$$

is positive definite for all $(x, y) \in TM\backslash 0$.

Certainly, Riemannian metrics $\alpha = \sqrt{\tilde{a}_{ij} y^i y^j}$ are Finsler. Notice however, Riemannian metrics are absolutely homogeneous, while Finsler metrics need only be positively homogeneous. Loosening absolute homogeneity to positive homogeneity allows such exciting examples as the Finslerian Poincaré disc [**BCS00, Oka83**] for which the travel time from the center to the rim is infinite, while the return trip takes only log(2) seconds.

Also, the fundamental tensor $\tilde{a}_{ij}$ of a Riemannian metric is a function of x alone. In general the fundamental tensor of a Finsler metric is a function of $(x, y) \in TM$. Indeed, a Finsler metric is Riemannian if and only if the the fundamental tensor is a function of x alone.

As in Riemannian geometry, the formal Christoffel symbols of F are given by

$$\gamma^i{}_{jk} = \tfrac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right) ,$$

where (g^{ij}) is the inverse of the fundamental tensor. Similarly, the **geodesic spray coefficients** are

$$G^i = \tfrac{1}{2} \gamma^i{}_{jk} y^j y^k .$$

The **spray curvature** of F is then defined by

$$K^i{}_k = 2 \left(G^i\right)_{x^k} - \left(G^i\right)_{y^j} \left(G^j\right)_{y^k} - y^j \left(G^i\right)_{y^k x^j} + 2G^j \left(G^i\right)_{y^k y^j} .$$

The expression above is known as **Berwald's formula** [**Run59**].

Let us take a moment to consider the case of a Riemannian metric. Note that if $F = \alpha = \sqrt{\tilde{a}_{ij} y^i y^j}$ is Riemannian, then $K^i{}_k = y^h \tilde{R}_h{}^i{}_{kj} y^j$, where $\tilde{R}_h{}^i{}_{kj}$ is the Riemann curvature tensor. The sectional curvature of the plane spanned by $0 \neq w, y \in T_x M$ is given by

$$\frac{w^i K_{ik}(x, y) w^k}{\tilde{a}(y, y)\tilde{a}(w, w) - \tilde{a}(y, w)^2} ,$$

where $\tilde{a}(\cdot, \cdot)$ denotes the inner product associated to $\tilde{a}$. Because the Christoffel symbols are functions of x alone, the spray curvature $K^i{}_k(x, y)$ is quadratic in y and $w^i K_{ik}(x, y) w^k = y^i K_{ik}(x, w) y^k$.

On the other hand, the formal Christoffel symbols of a general Finsler metric depend on both x and y. So, while $K^i{}_k(x,y)$ is homogeneous of degree two in y, it is not quadratic. In particular, it will no longer be true that $w^i K_{ik}(x,y)w^k = y^i K_{ik}(x,w)y^k$. This necessitates the generalization of sectional curvature to Finslerian flag curvature.

Begin by planting a flag pole $y \neq 0$ in T_xM. To this flag pole y we associate the spray curvature $K^i{}_k = K^i{}_k(x,y)$. The flag – a two-dimensional subspace of T_xM – is then specified by selecting $0 \neq w \in T_xM$ transverse to y. The **flag curvature** is given by

$$K(x,y,w) = \frac{w^i K_{ik}(x,y) w^k}{g(y,y)g(w,w) - g(y,w)^2}\,,$$

where $K_{ik} = g_{ij}K^j{}_k$ and g is the fundamental tensor. Explicitly,

$$\begin{aligned} g(y,y) &= g_{ij}(x,y)y^i y^j\,, \\ g(w,w) &= g_{ij}(x,y)w^i w^j\,, \text{ and} \\ g(y,w) &= g_{ij}(x,y)y^i w^j\,. \end{aligned}$$

Notice that both K_{ik} and g_{ij} are evaluated at the flag pole y, *not* the transverse edge w. Allow me to reiterate that in general $K(x,y,w) \neq K(x,w,y)$.

Though it is by no means obvious from the definition, the tensor K_{ik} is symmetric,

$$K_{ik} = K_{ki}\,.$$

The verification is tedious; details may be found in [**BCS00**].

The flag curvature is invariant under positive re-scaling of y. To see this first note that the formal Christoffel symbols $\gamma^i{}_{jk}$ and the geodesic spray coefficients $G^i = \frac{1}{2}\gamma^i{}_{jk}y^j y^k$ are positive homogeneous of degree 0 and 2, respectively. Hence, the spray curvature $K^i{}_k$ is positive homogeneous of degree 2 in y. Now suppose $\lambda > 0$. Then

$$\begin{aligned} K_{ik}(x,\lambda y) &= \lambda^2 K_{ik}(x,y)\,, \\ g(\lambda y, \lambda y) &= \lambda^2 g(y,y)\,, \\ g(\lambda y, w) &= \lambda g(y,w)\,. \end{aligned}$$

Hence

$$K(x,\lambda y,w) = K(x,y,w)\,,$$

as claimed. This says the flag curvature depends only on the *direction* of the flag pole y, not its length. We would also like to see that K is determined by the flag alone, and not the transverse edge w. That is, if both $\{y,w\}$ and $\{y,v\}$ determine the same flag (i.e. subspace) of T_xM, then $K(x,y,w) = K(x,y,v)$. Happily, this is the case, as I shall illustrate in the next section.

4.1. Euler's Theorem. Many of the functions and tensors I introduced above are positively homogeneous in y. We may take advantage of this structure with Euler's theorem for homogeneous functions.

THEOREM 1.2 (Euler, [**BCS00**]). *Assume the function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is differentiable away from the origin. The following two statements are equivalent:*

- *Φ is positively homogeneous of degree m. That is,*

$$\Phi(\lambda y) = \lambda^m \Phi(y) \qquad \forall\, \lambda > 0\,.$$

- *The radial directional derivative of Φ is $m\,\Phi$. Explicitly,*

$$y^i \Phi_{y^i}(y) = m\,\Phi(y)\,.$$

Since F is positively homogeneous of degree 1, Euler's theorem gives

$$y^i F_{y^i} = F\,,$$

Also,

$$g_{ij} y^i = F F_{y^j} \qquad \text{and} \qquad g_{ij} y^i y^j = F^2\,.$$

Let's turn to the spray curvature . Recall the geodesic spray coefficients $G^i = \frac{1}{2}\gamma^i{}_{jk} y^j y^k$ are homogeneous of degree two. By Euler's theorem ,

$$\begin{aligned} y^k G^j_{y^k} &= 2G^j\,, \\ y^k (G^i)_{y^k x^j} &= 2(G^i)_{x^j}\,, \\ y^k (G^i)_{y^k y^j} &= (G^i)_{y^j}\,. \end{aligned}$$

These equalities, and the symmetry of K_{ik}, may be used to show that

$$K_{ik}(x,y) y^k = 0 = y^i K_{ik}(x,y)\,.$$

This last equality implies the flag curvature $K(x,y,w)$ depends on the flag alone, and not the transverse edge w. To see this, suppose both $\{y,w\}$ and $\{y,v\}$ determine the same flag in T_xM. Then $v = \mu y + \tau w$, for some $\mu, \tau \in \mathbb{R}$, and

$$\begin{aligned} g(v,v) &= \mu^2 g(y,y) + 2\mu\tau g(y,w) + \tau^2 g(w,w)\,, \\ g(y,v) &= \mu g(y,y) + \tau g(y,w)\,, \\ v^i K_{ik} v^k &= \tau^2 w^i K_{ik} w^k\,. \end{aligned}$$

It follows that

$$K(x,y,v) = K(x,y,w)\,.$$

Let me summarize our discussion by reiterating that the flag curvature depends only on

- the *direction* of the flag pole y, not its length, and
- the flag, or subspace, determined by the transverse edge w, but not w itself.

5. Randers metrics

Randers metrics were introduced by G. Randers [**Ran41**] in 1941 within the context of relativity. They offer a smooth transition from Riemannian to Finsler metrics as they are built from objects well known to the geometer: a Riemannian metric $\tilde{a}$ and a drift 1-form $\tilde{b}$.

DEFINITION 1.3. *A Randers metric is a Finsler function*

$$F = \alpha + \beta\,,$$

where

$$\alpha(x,y) = \sqrt{\tilde{a}_{ij}y^i y^j} \quad , \qquad \beta(x,y) = \tilde{b}_i y^i$$

and $\|\tilde{b}\|^2 := \tilde{b}_i \tilde{a}^{ij} \tilde{b}_j < 1$.

Let's consider the requirements in Definition 1.1 for the norm F to be Finsler. First, $\|\tilde{b}\| < 1$ is necessary if F is to be positive. As it happens, the condition is also necessary and sufficient for the fundamental tensor g_{ij} to be positive definite [**BCS00**]. Second, the Randers metric is positively homogeneous of degree one in y. The metric will be absolutely homogeneous (and Riemannian) if and only if $\tilde{b} = 0$. I shall say a Randers metric with nonzero $\tilde{b}$ is *non-Riemannian.*

Randers metrics are ubiquitous in Finsler geometry. Recently Mo and Shen [**MS02b**] have shown that if a compact Finsler manifold with $n > 2$ has flag curvature $K(x,y) \leq -1$, depending on position x and flag pole y alone (i.e. no dependence on the flag $\{y,w\}$), then F is Randers.

Randers metrics also describe the solutions to Zermelo's problem of navigation.

5.1. Zermelo's problem of navigation. Introduced and solved by Zermelo in 1931 [**Zer31, Car99**], the problem may be posed in the following way.

> Consider a ship moving with constant speed on the open sea in calm waters. Imagine a breeze comes up. How must the ship be steered in order to reach a given destination in the shortest time?

If the wind is time-independent, then the paths of shortest time are geodesics of a Randers metric.

Shen [**She02a, She02b**] has generalized Zermelo's problem to Riemannian spaces. Consider a manifold M with Riemannian metric $\breve{\alpha} = \sqrt{\breve{a}_{ij}y^i y^j}$. If a ball rolls about M with constant speed 1, then any geodesic is a path of shortest time. Next suppose a 'wind' $W = W^i \partial_{x^i}$ blows over M. The wind represents an external force acting on the ball. Assume additionally that $\breve{\alpha}(W) < 1$. Shen has shown [**She02a**] any path of shortest time for the ball is a geodesic of the Randers metric $F = \alpha + \beta$ given by

$$\tilde{a}_{ij} = \left(\tfrac{1}{2}\alpha^2\right)_{y^i y^j} = \frac{\breve{a}_{ij}\left[1 - \breve{\alpha}^2(W)\right] + \breve{W}_i \breve{W}_j}{\left[1 - \breve{\alpha}^2(W)\right]^2}\,, \qquad \tilde{b}_i = \beta_{y^i} = \frac{-\breve{W}_i}{1 - \breve{\alpha}^2(W)}\,,$$

where $\breve{W}_i := \breve{a}_{ij} W^j$. The assumption $\breve{\alpha}(W) < 1$ implies that α is indeed strongly convex and a Riemannian metric. The condition also guarantees that $\|\tilde{b}\| < 1$, and F is strongly convex.

We say the Randers metric F *solves* Zermelo's problem of navigation on the Riemannian manifold $(M, \breve{a})$ under the external force W. As we will see in Chapter Three, the navigation structure underlying a Randers metric is ideally suited to the study of Einstein metrics.

5.2. Some special tensors. In this section I define a few important tensors and establish some notation particular to Randers metrics.

The Christoffel symbols $\tilde{\gamma}^i{}_{jk}$ of the Riemannian metric $\tilde{a}$ define a covariant differentiation on M. Covariant derivatives are denoted by a vertical slash. For instance, the covariant derivative of the drift 1-form $\tilde{b}$ is given by

$$\tilde{b}_{j|k} := \tilde{b}_{j,x^k} - \tilde{b}_i \, \tilde{\gamma}^i{}_{jk} \,,$$

where $\tilde{b}_{j,x^k}$ means $\partial_{x^k} \tilde{b}_j$.

There are three ubiquitous objects in this thesis:

$$\begin{aligned} \mathrm{lie}_{ij} &:= \tilde{b}_{i|j} + \tilde{b}_{j|i} \,, \\ \mathrm{curl}_{ij} &:= \tilde{b}_{i|j} - \tilde{b}_{j|i} \,, \\ \Theta_i &:= \tilde{b}^h \mathrm{curl}_{hi} \,. \end{aligned}$$

Note that lie_{ij} is the Lie derivative of the metric $\tilde{a}$ along the vector field

$$\tilde{b}^\sharp := \tilde{b}^i \partial_{x^i} := \tilde{a}^{ij} \tilde{b}_j \partial_{x^i} \,,$$

whereas curl_{ij} is so named because

$$\begin{aligned} d(\tilde{b}_i dx^i) &= -(\tilde{b}_{i,x^j} - \tilde{b}_{j,x^i}) \, \tfrac{1}{2} \, dx^i \wedge dx^j \\ &= -(\tilde{b}_{i,x^j} - \tilde{b}_{j,x^i}) \, dx^i \otimes dx^j \end{aligned}$$

and

$$\tilde{b}_{i|j} - \tilde{b}_{j|i} = \tilde{b}_{i,x^j} - \tilde{b}_{j,x^i} \,.$$

Observe lie is symmetric, curl is skew-symmetric, and

$$\mathrm{lie}_{ij} + \mathrm{curl}_{ij} = 2\tilde{b}_{i|j} \,.$$

The third tensor Θ denotes a contraction of curl with $\tilde{b}$. Indices on $\tilde{b}$, lie, curl and Θ are lowered and raised by $\tilde{a}_{ij}$ and its inverse $\tilde{a}^{ij}$. Contraction of any tensor index with the vector y is indicated by a 0 subscript. For example:

$$\begin{aligned} \mathrm{lie}_{i0} &:= \mathrm{lie}_{ij} \, y^j \\ \mathrm{lie}_{00} &:= \mathrm{lie}_{ij} \, y^i y^j \\ \mathrm{curl}_{i0} &:= \mathrm{curl}_{ij} \, y^j \\ \mathrm{curl}^i{}_0 &:= \mathrm{curl}^i{}_j \, y^j \\ \Theta_0 &:= \Theta_i y^i \,. \end{aligned}$$

5.3. The spray curvature of a Randers metric. In §1.4 I introduced the spray curvature of a Finsler metric

$$K^i{}_k = 2\left(G^i\right)_{x^k} - \left(G^i\right)_{y^j}\left(G^j\right)_{y^k} - y^j\left(G^i\right)_{y^k x^j} + 2G^j\left(G^i\right)_{y^k y^j}\,.$$

Following Bao and Shen [**BS02, She01a**] I make two adjustments to this formula:

- adapt it to the Randers setting, by expressing the data in terms of the Riemannian metric $\tilde{a}$ and 1-form $\tilde{b}$, and
- covariantize the expression. As a whole the formula is tensorial. However, individually the four terms on the right hand side are not. We want an expression that is manifestly tensorial in each term on TM.

Recollect $G^i = \frac{1}{2}\gamma^i{}_{jk}y^j y^k$ are the geodesic spray coefficients of F. Similarly, let $\tilde{G}^i := \frac{1}{2}\tilde{\gamma}^i{}_{jk}y^j y^k$ denote the geodesic spray coefficients of α. The two geodesic spray coefficients differ by a perturbation term ζ [**BCS00**]:

$$G^i = \tilde{G}^i + \zeta^i\,,$$

with

$$\zeta^i = \frac{y^i}{2F}\left(\tfrac{1}{2}\mathrm{lie}_{00} - \alpha\,\Theta_0\right) + \tfrac{1}{2}\,\alpha\,\mathrm{curl}^i{}_0\,.$$

By substituting this decomposition into Berwald's formula we may express the spray curvature as $K^i{}_k = \tilde{K}^i{}_k + B^i{}_k$. This fulfills our first goal by expressing $K^i{}_k$ as the Riemannian spray curvature $\tilde{K}^i{}_k$, plus a perturbation.

The perturbation term $B^i{}_k$ involves x-partial derivatives of ζ. These are the terms we wish to covariantize. (The y-partial derivatives already transform tensorially. See [**BCS00**].) Notice ζ^i is a tensor over TM, not M. In order to covariantize the x-partial derivatives *on* TM we horizontally lift the vector $\frac{\partial}{\partial x^i} \in TM$ to

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - (\tilde{G}^h)_{y^i}\,\frac{\partial}{\partial y^h} \quad \in\; T(TM)\,.$$

We may now define horizontal covariant differentiation on TM by lifting the Riemannian covariant derivative operator on M. The action on ζ^i is given by

$$\zeta^i{}_{|j} := \frac{\delta}{\delta x^j}\zeta^i + \zeta^h\tilde{\gamma}^i{}_{hj}\,,$$

and is tensorial on TM. The action has two properties worth noting.

- First, when a tensor on M (e.g. $\tilde{b}_i$) is lifted to TM, the action of the horizontal covariant derivative on TM agrees numerically with the Riemannian covariant differentiation on M. So there is no great abuse in notation in referring to both actions as "|".
- Second, y is horizontally covariantly constant. That is, $y^i{}_{|j} = 0$. In particular, we may contract a tensor with y either before or after covariant differentiation, with the same result.

Substituting the formula

$$(\zeta^i)_{x^j} = \zeta^i{}_{|j} + (\zeta^i)_{y^h}(\tilde{G}^h)_{y^j} - \zeta^h(\tilde{G}^i)_{y^h y^j}$$

into the expression for $B^i{}_k$ we obtain **Berwald's formula in split and covariantized form**:

$$K^i{}_k = \tilde{K}^i{}_k + \left\{ 2\,\zeta^i{}_{|k} - y^j(\zeta^i{}_{|j})_{y^k} - (\zeta^i)_{y^j}(\zeta^j)_{y^k} + 2\,\zeta^j(\zeta^i)_{y^j y^k} \right\} ,$$

where $\tilde{K}^i{}_k$ is the spray curvature of α. The split and covariantized form of Berwald's formula is enormously useful, playing a substantial role in both the characterization theorem of the next chapter, and the navigation description of §3.

We now have all the tools necessary to begin the thesis proper.

CHAPTER 2

Einstein metrics of Randers type

The two objects under consideration, the Randers metric and its Ricci tensor, are related by their histories in physics. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's theory of gravitation. (Details and references may be found in [**Bou79**].)

Almost thirty years later the physicist Gunnar Randers remarked that:

> ...the Riemannian metric has one property which does not seem quite appropriate for the application to physical space-time, and that is the perfect symmetry between opposite directions for any coordinate interval. Perhaps the most characteristic property of the physical world is the uni-direction of time-like intervals [**Ran41**].

It was Randers aim to introduce asymmetry to the metric while retaining a quadratic indicatrix (the set of unit vectors in the tangent space). This is done by displacing the center of a Riemannian indicatrix; the result is a Randers metric.

It is the aim of this chapter to characterize Einstein metrics of Randers type. Einstein metrics are defined in the following section. Loosely, though, we will say a Finsler metric F is Einstein if the average of its flag curvatures at a flag pole y is a function of position x alone, rather than the a priori position x and flag pole y.

After definitions I begin the derivation of a characterization theorem for Einstein metrics of Randers type. The theorem characterizes Einstein Randers metrics by three conditions – the Basic, E_{23} and Curvature Equations – which form a tensorial system of partial differential equations, polynomial in the unknowns, the Riemannian metric $\tilde{a}$ and the 1-form $\tilde{b}$, with independent variable x. The Curvature Equation is of particular interest because it describes the Riemannian Ricci tensor $\widetilde{\mathrm{Ric}}_{ij}$ in terms of $\tilde{a}$, $\tilde{b}$ and their derivatives.

The material in this chapter is joint work with David Bao [**BR03b**].

1. Ricci curvature and Einstein metrics

Let's begin with a discussion of the Ricci tensor and Einstein metrics.

The **Ricci scalar** of F is given by the function

$$Ric(x,y) := \frac{1}{F^2} K^s{}_s \,.$$

In §1.4 we saw that the spray curvature $K^i{}_k$ is positive homogeneous of degree 2 in y. Therefore, the Ricci scalar is positive homogeneous of degree 0 in y. This means $Ric(x,y)$ depends on the direction of the flag pole y, but not its length. The **Ricci tensor** of a Finsler metric F is defined in [**BCS00**] as

$$\mathrm{Ric}_{ij} := \left[\tfrac{1}{2} K^s{}_s\right]_{y^i y^j} = \left[\tfrac{1}{2} F^2\, Ric(x,y)\right]_{y^i y^j} \,.$$

When $F = \alpha$ is Riemannian, the curvature tensor depends on x alone and the definition is equivalent to the familiar $\widetilde{\mathrm{Ric}}_{ij} = \tilde{R}_i{}^s{}_{sj}$.

Definition 2.1. ([**BCS00**]) *A Finsler metric is Einstein if the Ricci scalar Ric is a function of x alone. Equivalently,*

$$Ric_{ij} = Ric(x) g_{ij} \quad or, \quad Ric_{00} = Ric(x) F^2 \,.$$

Recollect (§1.5.2) the 0 subscript denotes contraction with y. This means $\mathrm{Ric}_{00} = \mathrm{Ric}_{ij} y^i y^j$. When $F = \alpha$ is Riemannian the definition yields $\tilde{R}_i{}^s{}_{sj} = Ric(x)\tilde{a}_{ij}$.

Notice that our Einstein metrics are distinguished from constant Ricci curvature metrics, characterized by $\mathrm{Ric}_{00} = \lambda F^2$, where λ is a constant. However, the Schur lemma of §3.7 says the Ricci scalar Ric of an Einstein metric of Randers type in dimension $n > 2$ must be constant. So, among Randers spaces of dimension greater than two, there is no distinction between Einstein metrics and constant Ricci curvature metrics.

The Ricci scalar $Ric(x,y)$ of a Finsler metric may be realized as the sum of $n-1$ appropriately chosen flag curvatures. To see how this is done, pick a flag pole $0 \neq y \in T_xM$. If necessary normalize y to have norm $F(y) = 1$. The pole determines an inner product on T_xM via the fundamental tensor of F, $g_{ij}(x,y)$. Recall (§1.4.1), $g_{ij} y^i y^j = F^2(y)$. This means y has norm 1 with respect to the inner product. Use the inner product to select an orthonormal basis $\{e_i\} \in T_xM$ with $e_n = y$. It follows from our discussion of flag curvature in §1.4.1 that

$$K(x,y,e_i) = \begin{cases} K_{ii} & \text{if } i \neq n \\ 0 & \text{if } i = n \,, \end{cases}$$

since $K_{nn} = y^i K_{ij}\, y^j = 0$ (§1.4.1). With respect to the basis $\{e_i\}$ the fundamental tensor is given by $g_{ij}(x,y) = \delta_{ij}$, the Kronecker delta, at (x,y). Therefore, the spray curvature $K^i{}_k = g^{ij} K_{jk}$ is numerically equal to K_{ik}. In particular,

$$K^s{}_s = \sum_{i=1}^{n-1} K_{ii} \,.$$

We may conclude that the Ricci scalar $Ric(x,y) = \frac{1}{F^2} K^s{}_s = K^s{}_s$ is the sum of $n-1$ flag curvatures.

2. The Einstein criterion

With this section I begin to derive necessary conditions for a Randers metric (§1.5) to be Einstein. The first step is to re-express the Einstein criterion via Berwald's split and covariantized formula (§1.5.3). Assume the Ricci scalar Ric of the Randers metric $F = \alpha + \beta$ is a function of x alone. We have $F^2 Ric(x) = K^i{}_i$. With Berwald's formula this is re-written as

$$\begin{aligned}
0 &= K^i{}_i - F^2 Ric(x) \\
&= \tilde{K}^i{}_i + \left\{ 2\,\zeta^i{}_{|i} - y^j (\zeta^i{}_{|j})_{y^i} - (\zeta^i)_{y^j} (\zeta^j)_{y^i} + 2\,\zeta^j (\zeta^i)_{y^j y^i} \right\} - F^2 Ric(x) \\
&= \widetilde{\mathrm{Ric}}_{00} \;+\; \alpha \mathrm{curl}^i{}_{0|i} \;+\; \tfrac{1}{2}(n-1)\frac{\alpha}{F}\Theta_{0|0} \;-\; \tfrac{1}{4}(n-1)\frac{1}{F}\mathrm{lie}_{00|0} \\
&\quad + \tfrac{1}{2}(n-1)\frac{\alpha}{F}\mathrm{curl}^i{}_0 \mathrm{lie}_{i0} \;-\; \tfrac{1}{2}(n-1)\frac{\alpha^2}{F}\Theta^i \mathrm{curl}_{i0} \;+\; \tfrac{1}{2}\mathrm{curl}^i{}_0 \mathrm{curl}_{i0} \\
&\quad + \tfrac{1}{4}\alpha^2 \mathrm{curl}^{ij} \mathrm{curl}_{ij} \;+\; \tfrac{3}{16}(n-1)\frac{1}{F^2}(\mathrm{lie}_{00})^2 \;-\; \tfrac{3}{4}(n-1)\frac{\alpha}{F^2}\mathrm{lie}_{00}\Theta_0 \\
&\quad + \tfrac{3}{4}(n-1)\frac{\alpha^2}{F^2}(\Theta_0)^2 \;-\; F^2 Ric(x)\,.
\end{aligned}$$

Recollect $\tilde{K}^i{}_k$ is the spray curvature of the Riemannian metric $\tilde{a}_{ij}$. The relation $\tilde{K}^i{}_i = \widetilde{\mathrm{Ric}}_{00}$ is a consequence of the definition of $\widetilde{\mathrm{Ric}}_{ij}$ and Euler's theorem. The 0 subscripts indicate contraction with y^i. Since y is horizontally covariantly constant (§1.5.2), those contractions can be carried out either before or after the covariant differentiations (vertical slash), with the same result.

Multiplying the re-expressed Einstein criterion by F^2 removes y from the denominators. The criterion for a Randers metric to be Einstein then takes the form

$$\mathrm{Rat} + \alpha\,\mathrm{Irrat} \;=\; 0\,,$$

where Rat and Irrat are, respectively, degree 4 and degree 3 polynomials in y, whose coefficients are functions of x. We think of Rat as the rational part of the equation, and Irrat as the irrational part of the equation.

Analogous to complex numbers $a + \mathbf{i}b$, which vanish if and only if both the real, a, and imaginary, b, parts vanish, we have the following lemma.

LEMMA 2.2. *A Randers metric is Einstein if and only if both Rat* $= 0$ *and Irrat* $= 0$ *hold.*

PROOF. In view of homogeneity, it suffices to prove this for all $y \neq 0$. First note that α can never be polynomial in y. Otherwise the quadratic $\tilde{a}_{ij(x)} y^i y^j = \alpha^2$ would have been factored into two (identical) linear terms. Its zero set would then consist of a hyper-plane, contradicting the positive definiteness of $\tilde{a}_{ij}$. Now suppose the polynomial Rat were not zero. The displayed equation would imply that it is the product of a polynomial Irrat with a non-polynomial factor α. This is not possible. So Rat must vanish and, since α is positive at all $y \neq 0$, we see that Irrat must be zero as well. □

The formulas for Rat and Irrat are,

$$\begin{aligned}\mathrm{Rat} &= (\alpha^2+\beta^2)\,\widetilde{\mathrm{Ric}}_{00} + 2\alpha^2\beta\mathrm{curl}^i{}_{0|i} + \tfrac{1}{2}\left(\alpha^2+\beta^2\right)\mathrm{curl}^i{}_0\mathrm{curl}_{i0} \\ &\quad + \tfrac{1}{4}\alpha^2\left(\alpha^2+\beta^2\right)\mathrm{curl}^{ij}\mathrm{curl}_{ij} - \left(\alpha^4+6\alpha^2\beta^2+\beta^4\right)Ric(x) \\ &\quad + \tfrac{1}{2}(n-1)\{\ \alpha^2\Theta_{0|0} - \tfrac{1}{2}\beta\mathrm{lie}_{00|0} + \alpha^2\mathrm{curl}^i{}_0\mathrm{lie}_{i0} \\ &\qquad - \alpha^2\beta\Theta^i\mathrm{curl}_{i0} + \tfrac{3}{8}(\mathrm{lie}_{00})^2 + \tfrac{3}{2}\alpha^2(\Theta_0)^2\}\end{aligned}$$

and,

$$\begin{aligned}\mathrm{Irrat} &= 2\beta\widetilde{\mathrm{Ric}}_{00} + \left(\alpha^2+\beta^2\right)\mathrm{curl}^i{}_{0|i} + \beta\mathrm{curl}^i{}_0\mathrm{curl}_{i0} \\ &\quad + \tfrac{1}{2}\alpha^2\beta\mathrm{curl}^{ij}\mathrm{curl}_{ij} - 4\beta\left(\alpha^2+\beta^2\right)Ric(x) \\ &\quad + \tfrac{1}{2}(n-1)\{\ \beta\Theta_{0|0} - \tfrac{1}{2}\mathrm{lie}_{00|0} + \beta\mathrm{curl}^i{}_0\mathrm{lie}_{i0} \\ &\qquad - \alpha^2\Theta^i\mathrm{curl}_{i0} - \tfrac{3}{2}\mathrm{lie}_{00}\Theta_0\}\,.\end{aligned}$$

From these two expressions we will derive the preliminary form of the three necessary conditions for a Randers metric to be Einstein.

3. Necessary conditions for Einstein: Preliminary form

For convenience I abbreviate $Ric(x)$ by Ric.

3.1. The Basic Equation. Assume F is Einstein. That is, Rat $= 0$ and Irrat $= 0$. Hence,

$$\begin{aligned}0 &= \mathrm{Rat} - \beta\,\mathrm{Irrat} \\ &= (\alpha^2-\beta^2)\Big\{\widetilde{\mathrm{Ric}}_{00} + \beta\mathrm{curl}^i{}_{0|i} + \tfrac{1}{2}\mathrm{curl}^i{}_0\mathrm{curl}_{i0} \\ &\qquad + \tfrac{1}{4}\alpha^2\mathrm{curl}^{ij}\mathrm{curl}_{ij} - (\alpha^2+3\beta^2)Ric \\ &\qquad + \tfrac{1}{2}(n-1)\left[\mathrm{curl}^i{}_0\mathrm{lie}_{i0} + \tfrac{3}{2}(\Theta_0)^2 + \Theta_{0|0}\right]\Big\} \\ &\quad + \tfrac{3}{16}(n-1)\left(\mathrm{lie}_{00} + 2\beta\Theta_0\right)^2.\end{aligned}$$

Fix x. Considering the above expression as a polynomial in y, we see that $\alpha^2-\beta^2$ divides $(\mathrm{lie}_{00}+2\beta\Theta_0)^2$. The polynomial $\alpha^2-\beta^2$ is irreducible. To see why it can not be factored, recollect that $\|\tilde{b}\| < 1$, an assumption that holds for all our Randers metrics. This implies the non-negative $\alpha^2-\beta^2$ is zero only at the origin. Were the polynomial to factor non-trivially, it must do so as two linear terms. In this case the zero set would contain a hyper-plane; a contradiction.

Because $\alpha^2-\beta^2$ is irreducible it must divide, not just the square, but $\mathrm{lie}_{00}+2\beta\Theta_0$ itself. Namely, there exists a scalar function $\sigma(x)$ on M such that

$$\mathrm{lie}_{00} + 2\beta\,\Theta_0 = \sigma(x)(\alpha^2-\beta^2)\,.$$

This is our Basic Equation. Differentiating with respect to y^i and y^k gives an equivalent version:

$$\mathrm{lie}_{ik} + \tilde{b}_i\,\Theta_k + \tilde{b}_k\,\Theta_i = \sigma(x)(\tilde{a}_{ik} - \tilde{b}_i\tilde{b}_k)\,.$$

To recover the original version, just contract this with y^iy^k.

3.2. The Curvature Equation. Return to the expression for $0 = \mathrm{Rat} - \beta\,\mathrm{Irrat}$. Use the Basic Equation to replace $\mathrm{lie}_{00} + 2\beta\Theta_0$ with $\sigma(x)(\alpha^2 - \beta^2)$. We may divide off by a uniform factor of $\alpha^2 - \beta^2$. The result reads

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{00} \;=\;& \left(\alpha^2 + 3\,\beta^2\right) Ric \;-\; \beta\,\mathrm{curl}^j{}_{0|j} \;-\; \tfrac{1}{4}\alpha^2\mathrm{curl}^{hj}\mathrm{curl}_{hj} \;-\; \tfrac{1}{2}\mathrm{curl}^j{}_0\mathrm{curl}_{j0}\\ & -\tfrac{1}{2}(n-1)\left\{\tfrac{3}{8}\sigma^2(x)(\alpha^2-\beta^2) \;+\; \mathrm{curl}^j{}_0\mathrm{lie}_{j0} \;+\; \tfrac{3}{2}(\Theta_0)^2 \;+\; \Theta_{0|0}\right\}.\end{aligned}$$

This is the Curvature Equation, so named because it describes the Ricci tensor of $\tilde{a}$. We obtain the indexed form by differentiating with respect to y^i and y^k, and making use of the symmetry of $\widetilde{\mathrm{Ric}}_{ik}$.

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{ik} \;=\;& (\tilde{a}_{ik} + 3\,\tilde{b}_i\,\tilde{b}_k)Ric \;-\; \tfrac{1}{2}(\tilde{b}_i\,\mathrm{curl}^j{}_{k|j} + \tilde{b}_k\,\mathrm{curl}^j{}_{i|j})\\ & -\tfrac{1}{4}\tilde{a}_{ik}\,\mathrm{curl}^{hj}\mathrm{curl}_{hj} \;-\; \tfrac{1}{2}\mathrm{curl}^j{}_i\,\mathrm{curl}_{jk}\\ & -\tfrac{1}{2}(n-1)\Big\{\tfrac{3}{8}\sigma^2(x)(\tilde{a}_{ik} - \tilde{b}_i\,\tilde{b}_k) \;+\; \tfrac{1}{2}(\mathrm{curl}^j{}_i\,\mathrm{lie}_{jk} + \mathrm{curl}^j{}_k\,\mathrm{lie}_{ji})\\ & \qquad + \tfrac{3}{2}\Theta_i\Theta_k \;+\; \tfrac{1}{2}(\Theta_{i|k} + \Theta_{k|i})\Big\}.\end{aligned}$$

3.3. The E_{23} Equation. Here I derive the final characterizing equation, the E_{23} Equation. (The number 23 is chosen because it is of some chronological significance in our research notes.) Two pieces of information from the Basic Equation are required. To reduce clutter, I abbreviate $\sigma(x)$ as σ. First, differentiate to obtain

$$\mathrm{lie}_{00|0} = \sigma_{|0}(\alpha^2 - \beta^2) \;-\; \mathrm{lie}_{00}(\sigma\,\beta + \Theta_0) \;-\; 2\beta\Theta_{0|0}\,.$$

Next, contract the indexed form of the Basic Equation with $y^i\mathrm{curl}^k{}_0$ for

$$\mathrm{curl}^j{}_0\mathrm{lie}_{j0} = -\beta\,\Theta^j\mathrm{curl}_{j0} \;-\; (\Theta_0)^2 \;-\; \sigma\,\beta\,\Theta_0\,.$$

Return to the expression $0 = \mathrm{Irrat}$. Replace the term $\widetilde{\mathrm{Ric}}_{00}$ with the expression given by the Curvature Equation. Then, wherever possible, insert the expressions for lie_{00}, $\mathrm{lie}_{00|0}$ and $\mathrm{curl}^j{}_0\mathrm{lie}_{j0}$ given by the Basic Equation. After dividing off a factor of $\alpha^2 - \beta^2$ we have the E_{23} Equation:

$$\mathrm{curl}^j{}_{0|j} = 2Ric\,\beta + (n-1)\left\{\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\sigma\Theta_0 + \tfrac{1}{2}\Theta^j\mathrm{curl}_{j0} + \tfrac{1}{4}\sigma_{|0}\right\}.$$

Again, differentiating by y^i produces the indexed form of the E_{23} Equation

$$\mathrm{curl}^j{}_{i|j} = 2Ric\,\tilde{b}_i + (n-1)\left\{\tfrac{1}{8}\sigma^2\tilde{b}_i + \tfrac{1}{2}\sigma\Theta_i + \tfrac{1}{2}\Theta^j\mathrm{curl}_{ji} + \tfrac{1}{4}\sigma_{|i}\right\}.$$

4. The constancy of σ

In this section I will show

> If the Basic, E_{23} and Curvature Equations are satisfied, then σ must be constant on each connected component of the manifold M.

I do so by deriving a second formula for ${\rm curl}^i{}_{0|i}$ from the Curvature Equation. A comparison of this expression with that given by the E_{23} Equation will show σ must be constant.

Utilizing Ricci identities and the definition of ${\rm lie}_{ij}$ we have

$$\begin{aligned}
\tilde{b}_{i|j|k} - \tilde{b}_{i|k|j} &= \tilde{b}^s \tilde{R}_{isjk}\\
\tilde{b}_{i|k|j} + \tilde{b}_{k|i|j} &= {\rm lie}_{ik|j}\\
-\tilde{b}_{k|i|j} + \tilde{b}_{k|j|i} &= -\tilde{b}^s \tilde{R}_{ksij}\\
-\tilde{b}_{k|j|i} - \tilde{b}_{j|k|i} &= -{\rm lie}_{kj|i}\\
\tilde{b}_{j|k|i} - \tilde{b}_{j|i|k} &= \tilde{b}^s \tilde{R}_{jski}\,.
\end{aligned}$$

Summing the five equalities above and applying the first Bianchi identity produces

$${\rm curl}_{ij|k} = -2\tilde{b}^s \tilde{R}_{ksij} + {\rm lie}_{ik|j} - {\rm lie}_{kj|i}\,.$$

Contracting with $\tilde{a}^{ik} y^j$ we have

$${\rm curl}^i{}_{0|i} = 2\tilde{b}^i \widetilde{\rm Ric}_{i0} + {\rm lie}^i{}_{i|0} - {\rm lie}^i{}_{0|i}\,.$$

The second and third terms on the right are computed with the Basic Equation. The formulas are

$$\begin{aligned}
{\rm lie}^i{}_{i|0} &= (n - \|\tilde{b}\|^2)\,\sigma_{|0} - \sigma\,(1 - \|\tilde{b}\|^2)\,(\sigma\beta + \Theta_0)\,,\\
{\rm lie}^i{}_{0|i} &= \sigma_{|0} - \beta\,\tilde{b}^i \sigma_{|i} - \tfrac{1}{2}\sigma^2(n - 2\|\tilde{b}\|^2 + 1)\,\beta + \tfrac{1}{2}\sigma(2\|\tilde{b}\|^2 - n)\Theta_0\\
&\quad + \tfrac{1}{2}\beta\Theta_i\Theta^i + \tfrac{1}{2}\Theta^i{\rm curl}_{i0} - \beta\Theta^i{}_{|i} - \tilde{b}^i\Theta_{0|i}\,.
\end{aligned}$$

The Curvature Equation produces

$$\begin{aligned}
2\tilde{b}^i\widetilde{\rm Ric}_{i0} &= \Theta^i{\rm curl}_{i0}\\
&\quad - (n-1)\left(\tfrac{1}{4}\|\tilde{b}\|^2\sigma_{|0} + \tfrac{1}{2}\tilde{b}^i(\Theta_{i|0} + \Theta_{0|i})\right)\\
&\quad + \beta\left\{2(1 + \|\tilde{b}\|^2){\rm Ric} - \tfrac{1}{2}{\rm curl}^{ij}{\rm curl}_{ij}\right.\\
&\qquad \left. -(n-1)\left[\tfrac{1}{8}\sigma^2(3 - \|\tilde{b}\|^2) + \tfrac{1}{4}\tilde{b}^i\sigma_{|i}\right]\right\}.
\end{aligned}$$

At this point it is convenient to compute $\tilde{b}^i\Theta_{i|0}$, $\tilde{b}^i\Theta_{0|i}$ and $\Theta^i{}_{|i}$.

Three intermediary computations. In these computations I make frequent use of the facts

$$\tilde{b}_{i|j} = \tfrac{1}{2}(\text{lie}_{ij} + \text{curl}_{ij}) \quad \text{and} \quad \tilde{b}_{i|j}\text{curl}^{ij} = \tfrac{1}{2}\text{curl}_{ij}\text{curl}^{ij}\,.$$

Notice that $\tilde{b}^i\Theta_i = \tilde{b}^i\tilde{b}^j\text{curl}_{ij} = 0$, because curl_{ij} is skew-symmetric. Whence, the first term is given by the Basic Equation as

$$\tilde{b}^i\Theta_{i|0} = \tfrac{1}{2}\Theta_i\Theta^i\beta \;-\; \tfrac{1}{2}\sigma\Theta_0 \;-\; \tfrac{1}{2}\Theta^i\text{curl}_{i0}\,.$$

The computation of the second term $\tilde{b}^i\Theta_{0|i}$ is involved. We require both the Basic Equation and the expression for $\text{curl}_{ij|k}$ derived at the beginning of this section. I will make use of the facts $\|\tilde{b}\| < 1$ and $\tilde{b}^h\tilde{b}^i\tilde{R}_{hijk} = 0$. (The last equality is a result of the skew-symmetry of the Riemann curvature tensor in (h,i).) The result is

$$\tilde{b}^i\Theta_{0|i} = \tfrac{1}{2}\Theta_i\Theta^i\beta \;-\; \tfrac{1}{2}\Theta^i\text{curl}_{i0} \;+\; \tfrac{1}{2}\sigma\Theta_0 \;+\; \|\tilde{b}\|^2\sigma_{|0} \;-\; \tilde{b}^i\sigma_{|i}\beta\,.$$

With the E_{23} equation we may express the final term as

$$\begin{aligned}\Theta^i{}_{|i} \;&=\; \tfrac{1}{2}\text{curl}_{ij}\,\text{curl}^{ij} \;-\; \{2Ric \;+\; \tfrac{1}{8}(n-1)\,\sigma^2\}\,\|\tilde{b}\|^2 \\ &\quad + \tfrac{1}{2}(n-1)\,\Theta_i\,\Theta^i \;-\; \tfrac{1}{4}(n-1)\,\tilde{b}^i\sigma_{|i}\,.\end{aligned}$$

The final maneuver. Substituting the three intermediary computations into the expressions for $\tilde{b}^i\widetilde{\text{Ric}}_{i0}$, $\text{lie}^i{}_{i|0}$ and $\text{lie}^i{}_{0|i}$ produces

$$\begin{aligned}\text{curl}^i{}_{0|i} \;&=\; 2\tilde{b}^i\widetilde{\text{Ric}}_{i0} \;+\; \text{lie}^i{}_{i|0} - \text{lie}^i{}_{0|i} \\ &=\; 2\text{Ric}\,\beta \;+ \\ &\quad\;(n-1)\left[\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\Theta^i\text{curl}_{i0} + \tfrac{1}{2}\sigma\Theta_0 + \sigma_{|0} - \tfrac{3}{4}\|\tilde{b}\|^2\sigma_{|0}\right]\,.\end{aligned}$$

A comparison of this expression with that given by the E_{23} equation indicates $\frac{3}{4}(1-\|\tilde{b}\|^2)\sigma_{|0} = 0$. Since the norm of $\tilde{b}$ is strictly less than 1, we must have $\sigma_{|0} = 0$. In particular, all covariant derivatives $\sigma_{|i}$ vanish. Since σ is a scalar (i.e. a function of x), this means all the partial derivatives of σ are zero. Therefore σ is constant on each connected component of M.

5. Necessary conditions for Einstein: Final form

We may now state the final form of the three necessary conditions for a Randers metric to be Einstein.

5.1. The Basic Equation. The Basic Equation undergoes little cosmetic alteration. I emphasize below the constancy of σ:

$$\text{lie}_{00} + 2\beta\,\Theta_0 = (\text{const.}\sigma)(\alpha^2 - \beta^2)\,.$$

Equivalently,

$$\text{lie}_{ik} + \tilde{b}_i\,\Theta_k + \tilde{b}_k\,\Theta_i = (\text{const.}\sigma)(\tilde{a}_{ik} - \tilde{b}_i\tilde{b}_k)\,.$$

5.2. The Curvature Equation. To derive the final form of the Ricci Curvature Equation we replace all appearances of lie_{00}, $\mathrm{lie}_{00|0}$ and $\mathrm{curl}^j{}_0\mathrm{lie}_{j0}$ with the formulas given by the Basic Equation in §2.3.1 and §2.3.3; and $\mathrm{curl}^j{}_{0|j}$ with the expression given by the E_{23} equation of §2.5.3.

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{00} &= \left(\alpha^2+\beta^2\right) Ric - \tfrac{1}{4}\alpha^2 \mathrm{curl}^{hj}\mathrm{curl}_{hj} - \tfrac{1}{2}\mathrm{curl}^j{}_0\mathrm{curl}_{j0} \\ &\quad -(n-1)\left\{\tfrac{1}{16}\sigma^2\left(3\alpha^2-\beta^2\right)+\tfrac{1}{4}(\Theta_0)^2+\tfrac{1}{2}\Theta_{0|0}\right\}.\end{aligned}$$

The indexed form of the final version of the Curvature Equation is

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{ik} &= (\tilde{a}_{ik}+\tilde{b}_i\tilde{b}_k) Ric - \tfrac{1}{4}\tilde{a}_{ik}\,\mathrm{curl}^{hj}\mathrm{curl}_{hj} - \tfrac{1}{2}\mathrm{curl}^j{}_i\mathrm{curl}_{jk} \\ &\quad -(n-1)\left\{\tfrac{1}{16}\sigma^2(3\tilde{a}_{ik}-\tilde{b}_i\tilde{b}_k)+\tfrac{1}{4}\Theta_i\Theta_k+\tfrac{1}{4}\left(\Theta_{i|k}+\Theta_{k|i}\right)\right\}.\end{aligned}$$

5.3. The E_{23} Equation. The constancy of σ updates the E_{23} Equation to

$$\mathrm{curl}^j{}_{0|j} = 2Ric\,\beta + (n-1)\left\{\tfrac{1}{8}\sigma^2\beta+\tfrac{1}{2}\sigma\Theta_0+\tfrac{1}{2}\Theta^j\mathrm{curl}_{j0}\right\}$$

or ,

$$\mathrm{curl}^j{}_{i|j} = 2Ric\,\tilde{b}_i + (n-1)\left\{\tfrac{1}{8}\sigma^2\tilde{b}_i+\tfrac{1}{2}\sigma\Theta_i+\tfrac{1}{2}\Theta^j\mathrm{curl}_{ji}\right\}.$$

6. Characterization of Einstein metrics of Randers type

I have shown that the Basic, E_{23} and Curvature Equations are necessary for a Randers metric to be Einstein. The are also sufficient, as we shall see in the following section.

6.1. The three necessary conditions are also sufficient. Recollect the Randers metric F is Einstein if and only if both Rat and Irrat vanish (§2.2). Assume the Basic, E_{23} and Curvature Equations hold as given in §2.5.1-2.5.3. I will show these three conditions imply Rat $= 0 =$ Irrat.

First, note that the preliminary (§2.3) and final (§2.5) forms of the three equations are equivalent. To see this, it suffices to show the final forms of the Basic, E_{23} and Curvature Equations imply the preliminary forms.

- Thanks to the constancy of σ the final forms of the Basic and E_{23} Equations are immediately equivalent to their preliminary forms.
- The final form of the Curvature Equation was deduced from the preliminary form by replacing the terms lie_{00}, $\mathrm{lie}_{00|0}$, $\mathrm{curl}^j{}_0\mathrm{lie}_{j0}$ and $\mathrm{curl}^j{}_{0|j}$ with the expressions given by the Basic and E_{23} Equations. Certainly, we may reverse this algebraic substitution to resurrect the preliminary form of the Curvature Equation from its final form.

Having seen that the preliminary and final forms of the necessary equations are equivalent, it remains to show that the preliminary forms imply Rat $= 0 =$ Irrat. To that end, assume the preliminary forms of the Basic, E_{23} and Curvature equations hold. Recall, from §2.3.3, that we deduced the preliminary E_{23} Equation from Irrat $= 0$ by:

- replacing $\widetilde{\mathrm{Ric}}_{00}$ with the expression given by the preliminary form of the Curvature Equation (§2.3.2),
- substituting the three terms lie_{00}, $\mathrm{lie}_{00|0}$ and $\mathrm{curl}^j{}_0\mathrm{lie}_{j0}$ with the formulas given by the Basic Equation,
- and dividing by a uniform factor of $\alpha^2 - \beta^2$.

By reversing the three algebraic steps above we may derive $\mathrm{Irrat} = 0$ from the E_{23} Equation.

Similarly, we derived the preliminary form of Curvature Equation from $\mathrm{Rat} - \beta\mathrm{Irrat} = 0$ by:

- using the Basic Equation to replace $\mathrm{lie}_{00} + 2\beta\Theta_0$ with $\sigma(\alpha^2 - \beta^2)$,
- and dividing by $\alpha^2 - \beta^2$.

Again, reversing the two step above will give us $\mathrm{Rat} - \beta\mathrm{Irrat} = 0$. Hence, $\mathrm{Rat} = 0$ since $\mathrm{Irrat} = 0$.

It follows that the Basic, E_{23} and Curvature Equations of §2.5.1-2.5.3 are necessary and sufficient conditions for $\mathrm{Rat} = 0 = \mathrm{Irrat}$. The vanishing of Rat and Irrat characterizes Einstein metrics, allowing us to formalize the previous three sections in the following theorem.

6.2. The theorem.

THEOREM 2.3 (Einstein Characterization). *Let $F = \alpha + \beta$ be a Randers metric on a smooth manifold M of dimension $n \geq 2$, with $\alpha = \sqrt{\tilde{a}_{ij\,(x)} y^i y^j}$ and $\beta = \tilde{b}_{i\,(x)} y^i$. Then (M, F) is Einstein with Ricci curvature $Ric(x)$ if and only if the Basic Equation, the Curvature Equation, and the E_{23} Equation of §2.5.1-2.5.3 are satisfied.*

Explicitly, a Randers metric is Einstein with Ricci curvature $Ric(x)$ if and only if there exists a constant σ, such that the following three equations are satisfied:

$$\mathrm{lie}_{ik} + \tilde{b}_i\,\Theta_k + \tilde{b}_k\,\Theta_i = \sigma(\tilde{a}_{ik} - \tilde{b}_i\tilde{b}_k)$$

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{ik} \;&=\; (\tilde{a}_{ik} + \tilde{b}_i\,\tilde{b}_k)\,Ric - \tfrac{1}{4}\tilde{a}_{ik}\,\mathrm{curl}^{hj}\mathrm{curl}_{hj} - \tfrac{1}{2}\mathrm{curl}^j{}_i\mathrm{curl}_{jk}\\ &\quad -(n-1)\left\{\tfrac{1}{16}\sigma^2\left(3\tilde{a}_{ik} - \tilde{b}_i\,\tilde{b}_k\right) + \tfrac{1}{4}\Theta_i\Theta_k + \tfrac{1}{4}\left(\Theta_{i|k} + \Theta_{k|i}\right)\right\}\end{aligned}$$

$$\mathrm{curl}^j{}_{i|j} = 2Ric\,\tilde{b}_i + (n-1)\left\{\tfrac{1}{8}\sigma^2\tilde{b}_i + \tfrac{1}{2}\sigma\Theta_i + \tfrac{1}{2}\Theta^j\mathrm{curl}_{ji}\right\}.$$

Tracing the Basic Equation tells us σ takes the geometrically significant value

$$\sigma = \frac{2\tilde{b}^i{}_{|i}}{n - \|\tilde{b}\|^2}.$$

(Recall, $\tilde{b}^i{}_{|i}$ is the divergence of the vector field $\tilde{b}^i\partial_{x^i}$.)

I would like to emphasize a few properties of the Basic, Curvature, and E_{23} Equations.

- The equations are tensorial and highly non-linear (see $\widetilde{\mathrm{Ric}}_{ik}$) second order partial differential equations.
- They are polynomial in the tangent space coordinates y^i, whereas the Einstein criterion is not (unless $\tilde{b} = 0$). The polynomial structure of the three characterizing equations substantially reduces the computational complexity. Recall the Einstein criterion (§2.2) is ${K^i}_i = F^2 Ric(x)$. While testing Einstein examples we found cases in which Maple (the mathematical software) was unable to complete computations of ${K^i}_i$. So for these examples we could not verify the Einstein criterion. Maple was able, however, to verify the three characterizing equations.
- The E_{23} Equation is redundant. That is, given the Basic and Curvature Equations, the E_{23} Equation necessarily holds. The redundancy is almost evident from the proof that σ is constant. In §2.4 we derive a second expression for ${\mathrm{curl}^i}_{0|i}$ which, with the constancy of σ, is identical to the E_{23} Equation.

  During the computation we use the E_{23} Equation only to compute ${\Theta^i}_{|i}$. Now that we know σ is constant, it is possible to recompute ${\Theta^i}_{|i}$ via the formula ${\tilde{b}^i}_{0|i} = 2\tilde{b}^i\widetilde{\mathrm{Ric}}_{i0} + {\mathrm{lie}^i}_{i|0} - {\mathrm{lie}^i}_{0|i}$, introduced at the beginning of §2.4, and requiring only the Basic and Curvature Equations. With this alteration, the argument that illustrates the constancy of σ, now shows the Basic and Curvature Equations imply the E_{23} Equation.

  Although superfluous, the E_{23} Equation is enormously useful in computations. For this reason it is included in the Einstein characterization theorem.
- A similar characterization for those special Einstein Randers metrics of constant flag curvature has also been derived. The interested reader might consult [**BR03a, MS02a**].

The next step is to parley the characterization into an explicit geometric description of Einstein metrics of Randers type. This is done in Chapter Three, where we re-express the characterizing equations within the context of the perturbation technique based on Zermelo's problem of navigation. It happens that this is the ideal setting to discuss Einstein metrics of Randers type.

CHAPTER 3

The Einstein navigation description

Given the straightforward construction of a Randers metric as the sum of a Riemannian metric $\tilde{a}$ and a linear term $\tilde{b}$, it seems most natural to derive an Einstein characterization based on $\tilde{a}$ and $\tilde{b}$. Such reasoning motivates the characterization theorem of Chapter 2. However, an overwhelming majority of the Einstein examples known today are constructed as solutions to Zermelo's problem of navigation (§1.5.1). This suggests the structure underlying the perturbation technique may be well-suited to a study of Einstein metrics of Randers type. So guided, I rewrite the characterizing equations of §2 in terms of the Zermelo data $(\breve{a}, W)$. I find that a Randers metric is Einstein with Ricci curvature $Ric = (n-1)K$ if and only if the Riemannian $\breve{a}$ is Einstein with Ricci scalar $(n-1)(K + \frac{1}{16}\sigma^2)$, and the Lie derivative of $\breve{a}$ by W is $-\sigma\breve{a}$.

With the navigation description in hand a Schur lemma and Matsumoto's Identity follow quickly. The Schur lemma constrains the geometry in dimensions three and higher by declaring the Ricci scalar of an Einstein Randers metric to be constant. The Matsumoto Identity, describing the constant σ, generalizes a result of Matsumoto for Randers metrics of constant flag curvature [**Mat01**], to the Einstein setting.

1. Randers metrics as solutions to Zermelo's problem

We know solutions to Zermelo's problem are given by Randers metrics (§1.5.1). The converse must be checked before deriving a classification based on $\breve{a}$ and W: can every Randers metric $F = \alpha + \beta$ be realized as a solution to Zermelo's problem of navigation on a Riemannian manifold $(M, \breve{a})$ under an external force W? Happily, the answer is 'yes'. The desired Riemannian metric $\breve{a}$ and vector field W may be constructed as follows.

Given $\alpha^2 = \tilde{a}_{ij}y^iy^j$ and $\beta = \tilde{b}_iy^i$, let $\Delta := 1 - \|\tilde{b}\|^2$. Consider the Riemannian metric and vector field defined by

$$\breve{a}_{ij} := \Delta(\tilde{a}_{ij} - \tilde{b}_i\tilde{b}_j)\,, \qquad W^i = -\frac{\tilde{b}^i}{\Delta}\,.$$

Recall $\|\tilde{b}\| < 1$ is required of all our Randers metrics. This means $\Delta > 0$, and W is well-defined.

Also a consequence of $\|\tilde{b}\| < 1$, we have

$$\breve{a}_{ij}y^iy^j = \Delta(\alpha^2 - \beta^2) \geq 0\,,$$

with equality only when $y = 0$. Hence $\breve{a}$ is positive definite, and is therefore a genuine Riemannian metric.

After checking $W^i \breve{a}_{ij} W^j =: \|\breve{W}\|^2 = \|\tilde{b}\|^2$, a straightforward computation confirms $F = \alpha + \beta$ solves Zermelo's problem for $(\breve{a}, W)$. Explicitly,

$$\tilde{a}_{ij} = \frac{\breve{a}_{ij}\,[1 - \breve{\alpha}^2(W)] + \breve{W}_i \breve{W}_j}{[1 - \breve{\alpha}^2(W)]^2}, \qquad \tilde{b}_i = \frac{-\breve{W}_i}{1 - \breve{\alpha}^2(W)},$$

where $\breve{W}_i = \breve{a}_{ij} W^j$. Equivalently,

$$\alpha^2(y) = \frac{\breve{\alpha}^2(y)\,[1 - \breve{\alpha}^2(W)] + \breve{W}_0{}^2}{[1 - \breve{\alpha}^2(W)]^2}, \qquad \beta = \frac{-\breve{W}_0}{1 - \breve{\alpha}^2(W)},$$

with $\breve{\alpha}^2(y) := \breve{a}_{ij} y^i y^j$. In particular, F solves Zermelo's problem for $\breve{a}$ and W.

Finally, with some linear algebra, the inverse $\breve{a}^{ij}$ of $\breve{a}$ is given by

$$\breve{a}^{ij} = \frac{\tilde{a}^{ij} \Delta + \tilde{b}^i \tilde{b}^j}{\Delta^2}.$$

Again, $\|\tilde{b}\| < 1$ guarantees $\breve{a}^{ij}$ is well-defined.

2. Covariant differentiation with respect to $\breve{a}$

This section is the first of four containing the computations to express the characterizing Basic, E_{23} and Curvature Equations in terms of the Riemannian metric $\breve{a}$, the vector field W, and covariant derivatives of W with respect to $\breve{a}$.

Let the colon denote covariant differentiation with respect to $\breve{a}$. For example, the covariant derivative of W is given by

$$\breve{W}^i{}_{:j} := W^i{}_{,x^j} + W^s \breve{\gamma}^i{}_{sj}.$$

Here, $\breve{\gamma}^i{}_{jk}$ denotes the Christoffel symbols of $\breve{a}$. Indices on W^i, $\breve{W}_i$ and their covariant derivatives are lowered and raised by the metric $\breve{a}_{ij}$ and its inverse $\breve{a}^{ij}$.

Because the covariant derivatives of the characterizing equations are with respect to $\tilde{a}$ we need to understand the relationship between the Christoffel symbols of $\tilde{a}$ and $\breve{a}$. To wit, the connection coefficients of $\tilde{a}$ and $\breve{a}$ are related by

$$\tilde{\gamma}^i{}_{jk} = \breve{\gamma}^i{}_{jk} + \eta^i{}_{jk},$$

with

$$\begin{aligned}
\eta^i{}_{jk} \;:=\;& \frac{1}{\Delta}\left(\delta^i{}_k W^s \breve{W}_{s:j} + \delta^i{}_j W^s \breve{W}_{s:k}\right) + \tfrac{1}{2} W^i \left(\breve{W}_{j:k} + \breve{W}_{k:j}\right) \\
&+ \left(\frac{\breve{a}_{jk}}{\Delta} + 2\frac{\breve{W}_j \breve{W}_k}{\Delta^2}\right)\left\{W^i W^s W^t \breve{W}_{s:t} - \breve{W}_s \breve{W}^{s:i}\right\} \\
&+ \frac{1}{2\Delta} W^i \breve{W}_j W^s \left(\breve{W}_{k:s} + \breve{W}_{s:k}\right) + \frac{1}{2\Delta} W^i \breve{W}_k W^s \left(\breve{W}_{j:s} + \breve{W}_{s:j}\right) \\
&+ \frac{1}{2\Delta}\left(\breve{W}^i{}_{:k} - \breve{W}_k{}^{:i}\right)\breve{W}_j + \frac{1}{2\Delta}\left(\breve{W}^i{}_{:j} - \breve{W}_j{}^{:i}\right)\breve{W}_k .
\end{aligned}$$

3. The Basic Equation

In this section I shall re-express the Basic Equation of §2.5.1

$$\mathrm{lie}_{00} + 2\beta\Theta_0 = \sigma(\alpha^2 - \beta^2)\,,$$

in terms of $\breve{a}$ and W. The expressions for α and β are given in §3.1; we need only compute lie_{00} and Θ_0.

First, consider $\mathrm{lie}_{00} = 2\tilde{b}_{0|0}$. With the formula for $\tilde{b}_i$ in §3.1 we have

$$\begin{aligned}
\tilde{b}_{i|j} &= \tilde{b}_{i,x^j} - \tilde{b}_s \tilde{\gamma}^s_{ij} \\
&= \tilde{b}_{i:j} - \tilde{b}_s \eta^s{}_{ij} \\
&= -\tfrac{1}{2}\frac{1+\Delta}{\Delta}\breve{W}_{i:j} + \tfrac{1}{2}\frac{1-\Delta}{\Delta}\breve{W}_{j:i} \\
&\quad - \left(\frac{2}{\Delta^2}\breve{W}_i \breve{W}_j + \frac{1}{\Delta}\breve{a}_{ij}\right) W^s W^t \breve{W}_{s:t} \\
&\quad - \frac{1}{2\Delta}\breve{W}_i \left(W^s \breve{W}_{s:j} + W^s \breve{W}_{j:s}\right) \\
&\quad + \left(\frac{4-\Delta}{\Delta^2} W^s \breve{W}_{s:i} - \frac{1}{2\Delta} W^s \breve{W}_{i:s}\right)\breve{W}_j .
\end{aligned}$$

Now lie_{00} is given by contracting this expression with $2y^i y^j$.

Moving on to Θ_0, a quick calculation yields

$$\begin{aligned}
\mathrm{curl}_{ij} &= \tilde{b}_{i|j} - \tilde{b}_{j|i} \\
&= -\frac{1}{\Delta}\left(\breve{W}_{i:j} - \breve{W}_{j:i}\right) \\
&\quad - \frac{2}{\Delta^2}\left(\breve{W}_i W^s \breve{W}_{s:j} - W^s \breve{W}_{s:i} \breve{W}_j\right) .
\end{aligned}$$

Whence $\Theta_0 = \tilde{b}^i \mathrm{curl}_{i0}$ is given by

$$\Theta_0 = \frac{1 + \|\breve{W}\|^2}{\Delta} W^s \breve{W}_{s:0} - W^s \breve{W}_{0:s} - \frac{2}{\Delta} W^s W^t \breve{W}_{s:t} \breve{W}_0 .$$

It follows now, that the Basic Equation is equivalent to a *preliminary expression*

$$-2\breve{W}_{0:0} - \frac{2}{\Delta}\breve{\alpha}^2 W^s W^t \breve{W}_{s:t} = \frac{\sigma}{\Delta}\breve{\alpha}^2 .$$

This formula may be refined. First differentiate by y^i and y^j for

$$-\breve{W}_{i:j} - \breve{W}_{j:i} - \frac{2}{\Delta}\breve{a}_{ij} W^s W^t \breve{W}_{s:t} = \frac{\sigma}{\Delta}\breve{a}_{ij} \, .$$

Contract this with $W^i W^j$ to obtain

$$W^s W^t \breve{W}_{s:t} = -\frac{\sigma}{2}\|\breve{W}\|^2 \, .$$

Substituting this equality into the preliminary expression yields

$$\breve{W}_{0:0} = -\frac{\sigma}{2}\breve{\alpha}^2 \, .$$

The indexed form, obtained via differentiation by y^i and y^j, describes the Lie derivative of $\breve{a}_{ij}$ by W

$$\breve{W}_{i:j} + \breve{W}_{j:i} = -\sigma \breve{a}_{ij} \, .$$

This the **Lie$_W$ Equation**, so named because $\breve{W}_{i:j}+\breve{W}_{j:i}$ is the Lie derivative of $\breve{a}$ by W. It says W is an infinitesimal homothety of $\breve{a}$. When σ vanishes, W is Killing (an infinitesimal isometry).

A quick calculation shows the Lie$_W$ Equation is equivalent to the preliminary expression above. This means,

The Lie$_W$ Equation is equivalent to the Basic Equation.

The Lie$_W$ Equation provides a number of useful identities that I shall apply to the derivation of the perturbation versions of E_{23} and Curvature Equations. These include

$$\begin{aligned}
\breve{W}^s{}_{:s} &= -\tfrac{1}{2}n\sigma \\
W^s W^t \breve{W}_{s:t} &= -\tfrac{1}{2}\sigma\|\breve{W}\|^2 \\
\breve{W}^{s:t}\breve{W}_{s:t} + \breve{W}^s{}_{:t}\breve{W}^t{}_{:s} &= \tfrac{1}{2}n\sigma^2 \\
W^s\left(\breve{W}_{0:s} + \breve{W}_{s:0}\right) &= -\sigma\breve{W}_0 \\
\breve{W}^s{}_{:0}\left(\breve{W}_{s:0} + \breve{W}_{0:s}\right) &= \tfrac{1}{2}\sigma^2\breve{\alpha}^2 \\
\breve{W}_{0:s}\breve{W}_{0:}{}^s - \breve{W}^s{}_{:0}\breve{W}_{s:0} &= 0 \\
W^s\breve{W}_{s:t}\left(\breve{W}^t{}_{:0} + \breve{W}_{0:}{}^t\right) &= -\sigma W^s\breve{W}_{s:0} \\
W^s\breve{W}^t{}_{:s}\left(\breve{W}_{t:0} + \breve{W}_{0:t}\right) &= -\sigma W^s\breve{W}_{0:s} \\
W^s\left(\breve{W}_{s:t} + \breve{W}_{t:s}\right)\breve{W}^t{}_{:0} &= -\sigma W^s\breve{W}_{s:0} \\
\breve{W}_{i:j:k} + \breve{W}_{j:i:k} &= 0 \\
W^s W^t \breve{W}_{s:t:k} &= 0 \, .
\end{aligned}$$

4. The E_{23} Equation

Next we turn our attention to the E_{23} Equation of the Einstein characterization (§2.5.3)

$$\mathrm{curl}^j{}_{0|j} = 2Ric\,\beta + (n-1)\left\{\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\sigma\Theta_0 + \tfrac{1}{2}\Theta^j\mathrm{curl}_{j0}\right\} .$$

Two terms require consideration: $\mathrm{curl}^j{}_{0|j}$ and $\Theta^j\mathrm{curl}_{j0}$. Let's start with

$$\mathrm{curl}^j{}_{0|j} = \tilde{a}^{ij}\mathrm{curl}_{i0|j} = \tilde{a}^{ij}\left(\mathrm{curl}_{i0:j} - \mathrm{curl}_{s0}\eta^s{}_{ij} - \mathrm{curl}_{is}\eta^s{}_{0j}\right) .$$

The tensors $\tilde{a}^{ij}$ and curl_{ij} are evaluated in §3.1 and §3.3, respectively. After simplifying with the identities derived from the Lie_W Equation we have

$$\begin{aligned}\mathrm{curl}^j{}_{0|j} \;&=\; \breve{W}_{0:}{}^{s}{}_{:s} - \breve{W}^s{}_{:0:s} + \frac{2}{\Delta}\breve{W}_0 W^s \breve{W}_{s:}{}^{t}{}_{:t} \\ &\quad + \frac{(n-1)}{\Delta^2}\Big(-\tfrac{1}{2}\sigma^2\,\|\breve{W}\|^2\,\breve{W}_0 + \Delta\,\sigma\,W^s\breve{W}_{s:0} \\ &\qquad\qquad + 2\,\Delta\,W^s\breve{W}^t{}_{:s}\breve{W}_{t:0} + 2\,\breve{W}_0\,W^sW^t\breve{W}^r{}_{:s}\breve{W}_{r:t}\Big) .\end{aligned}$$

Much of the work for the second term $\Theta^j\mathrm{curl}_{j0} = \Theta_i\tilde{a}^{ij}\mathrm{curl}_{j0}$ has been done: $\tilde{a}^{ij}$ is given in §3.1; Θ_j and curl_{j0} are computed in §3.3. The Lie_W Equation identities imply

$$\Theta^j\mathrm{curl}_{j0} = \frac{1}{\Delta^2}\left(4\breve{W}_0\,W^sW^t\breve{W}^r{}_{:s}\breve{W}_{r:t} + 4\Delta W^s\breve{W}^t{}_{:s}\breve{W}_{t:0} - \sigma^2\breve{W}_0\right) .$$

Now the E_{23} Equation is expressed in the *preliminary form*

$$\Delta\left(\breve{W}^s{}_{:0:s} - \breve{W}_{0:}{}^{s}{}_{:s}\right) = (n-1)\left(2K + \tfrac{1}{8}\sigma^2\right)\breve{W}_0 + 2\breve{W}_0\,W^s\breve{W}_{s:}{}^{t}{}_{:t} ,$$

where

$$K := \tfrac{1}{n-1}Ric .$$

The algebraic derivation may be reversed to recover the E_{23} Equation from the Lie_W Equation and the preliminary form above.

As was in the case in the computation of the Lie_W Equation, the preliminary expression may be refined. To do so differentiate by y^t and contract with W^t to obtain

$$-2W^s\breve{W}_{s:}{}^{t}{}_{:t} = (n-1)\left(2K + \tfrac{1}{8}\sigma^2\right)\|\breve{W}\|^2 .$$

Then, a straightforward computation shows the preliminary form is equivalent to

$$\breve{W}^s{}_{:0:s} - \breve{W}_{0:}{}^{s}{}_{:s} = (n-1)\left(2K + \tfrac{1}{8}\sigma^2\right)\breve{W}_0 .$$

This is the $\breve{\mathbf{E}}_{23}$ **Equation** , the navigation version of the E_{23} Equation.

To ensure the navigation description we are deriving is both necessary *and* sufficient for a Randers metric to be Einstein, it is important that we keep track of the relationships between the characterizing equations (which are necessary and sufficient by Theorem 2.3) and their navigation forms. Assume the Lie_W Equation holds. Above I mentioned that the E_{23} Equation and the preliminary form are equivalent. Since the preliminary form holds if and only if the $\breve{\mathrm{E}}_{23}$ Equation does, we see the E_{23} Equation is equivalent to its navigation version.

At the end of §3.3 I observed that the Basic and Lie_W Equations are equivalent conditions. Taken together these remarks imply

> The Basic and E_{23} Equations hold if and only if the Lie_W and $\breve{\mathrm{E}}(23)$ Equations do.

I close the section with an identity derived from the Lie_W and $\breve{\mathrm{E}}(23)$ Equations

$$\breve{W}^s{}_{:0:s} = (n-1)\left(K + \tfrac{1}{16}\sigma^2\right)\breve{W}_0\,.$$

This formula will come in handy during our computation of the Curvature Equation.

5. The Curvature Equation

In this section I turn to the characterizing Curvature Equation of §2.5.2

$$\begin{aligned}\widetilde{\mathrm{Ric}}_{00} &= \left(\alpha^2+\beta^2\right)Ric - \tfrac{1}{4}\alpha^2\mathrm{curl}^{ij}\mathrm{curl}_{ij} - \tfrac{1}{2}\mathrm{curl}^j{}_0\mathrm{curl}_{j0}\\ &\quad -(n-1)\left\{\tfrac{1}{16}\sigma^2\left(3\alpha^2-\beta^2\right)+\tfrac{1}{4}{\Theta_0}^2+\tfrac{1}{2}\Theta_{0|0}\right\}\,.\end{aligned}$$

Four terms of the term above require our attention. They are $\mathrm{curl}^{ij}\mathrm{curl}_{ij}$, $\mathrm{curl}^j{}_0\mathrm{curl}_{j0}$, $\Theta_{0|0}$, and $\widetilde{\mathrm{Ric}}_{00}$. Throughout these computations I assume that the Lie_W Equation, $\breve{\mathrm{E}}_{23}$ Equation, and related identities hold (§3.3, 3.4). Let's begin with $\mathrm{curl}^{ij}\mathrm{curl}_{ij}$. In §3.1 and §3.3 we computed $\tilde{a}^{ij}$ and curl_{ij}, respectively. We have

$$\begin{aligned}\mathrm{curl}^{ij} &= \tilde{a}^{ih}\tilde{a}^{jk}\mathrm{curl}_{hk}\\ &= \Delta\left(\breve{W}^{j:i}-\breve{W}^{i:j}\right)\,,\end{aligned}$$

and

$$\mathrm{curl}^{ij}\mathrm{curl}_{ij} = \frac{1}{\Delta}\left\{(n+2)\,\sigma^2\Delta - 2\,\sigma^2 - 4\,\Delta\,\breve{W}^s{}_{:t}\breve{W}^t{}_{:s} + 8\,W^sW^t\breve{W}^r{}_{:s}\breve{W}_{r:t}\right\}\,.$$

Similarly,

$$\begin{aligned}\mathrm{curl}^j{}_0\mathrm{curl}_{j0} &= \frac{1}{\Delta^3}\Big\{-\sigma^2\left({\breve{W}_0}^2+\Delta^2\breve{\alpha}^2\right) + 4\,\Delta\left(W^s\breve{W}_{s:0}\right)^2\\ &\quad + 4\,\Delta^2\,\breve{W}^s{}_{:0}\breve{W}_{s:0} + 4\,\sigma\,\Delta\,\breve{W}_0\,W^s\breve{W}_{s:0}\\ &\quad + 4\,{\breve{W}_0}^2\,W^sW^t\breve{W}^r{}_{:s}\breve{W}_{r:t} + 8\,\Delta\,\breve{W}_0\,W^s\breve{W}^t{}_{:s}\breve{W}_{t:0}\Big\}\,.\end{aligned}$$

This takes care of the curl terms. Next up is the covariant derivative of Θ. With the expression for Θ computed in §3.3 we have

$$\begin{aligned}\Theta_{0|0} &= \Theta_{0:0} - \Theta_s\eta^s{}_{00}\\ &= \frac{1}{\Delta^3}\Big\{\tfrac{1}{2}\left(4\,W^sW^t\breve{W}^r{}_{:s}\breve{W}_{r:t} - \sigma^2\right)\left(\Delta\,\breve{\alpha}^2 + 2\,{\breve{W}_0}^2\right)\\ &\quad 2\,\Delta^2\,\breve{W}^s{}_{:0}\breve{W}_{s:0} + 4\,\Delta\,\breve{W}_0\,W^s\breve{W}^t{}_{:s}\breve{W}_{t:0} + 2\,\Delta^2\,W^s\breve{W}_{s:0:0}\Big\}\,.\end{aligned}$$

The fourth and final term of the Curvature Equation is more involved: to compute $\widetilde{\mathrm{Ric}}_{00}$ with respect to $\breve{a}$ and W I return to Berwald's formula (in split and covariantized form) of §1.5.3. Recall, Berwald's formula allowed us to compare the spray curvature $K^i{}_k$ of F to the spray curvature $\tilde{K}^i{}_k$ of the 'background' metric α. This is but one case of a more general comparison formula (see [**BR03b**]). The general comparison formula allows us to compare the spray curvature $\tilde{K}^i{}_j$ of α to the spray curvature $\breve{K}^i{}_k$ of the 'background' metric $\breve{\alpha}$. First define

$$\xi^i := \tfrac{1}{2}\eta^i{}_{jk}y^j y^k\,.$$

Then the geodesic spray coefficients are related by

$$\tilde{G}^i = \breve{G}^i + \xi^i\,.$$

Berwald's formula tells us the Ricci tensor $\widetilde{\mathrm{Ric}}_{ij}$ of $\tilde{a}$ is related to the Ricci tensor $\breve{\mathrm{Ric}}_{ij}$ of $\breve{a}$ by

$$\begin{aligned}
\widetilde{\mathrm{Ric}}_{00} &= \breve{\mathrm{Ric}}_{00} + \left\{2\,\xi^i{}_{:i} - y^j(\xi^i{}_{:j})_{y^i} - (\xi^i)_{y^j}(\xi^j)_{y^i} + 2\,\xi^j(\xi^i)_{y^j y^i}\right\} \\
&= \quad \breve{\mathrm{Ric}}_{00} \\
&\quad + \frac{1}{\Delta^3}\Bigg(\ \Big\{ (n-1)\,\Delta^2\,(1-\Delta)\,K \,+\, \Delta^2\,\breve{W}^s{}_{:t}\breve{W}^t{}_{:s} \\
&\qquad\quad - (n+1)\,\Delta\,W^s W^t \breve{W}^r{}_{:s}\breve{W}_{r:t} \\
&\qquad\quad + \left[\tfrac{1}{4}(n+1) \,-\, \tfrac{1}{16}(7n-3)\,\Delta \,-\, \tfrac{1}{16}(n-1)\,\Delta^2\right]\sigma^2\,\Delta\Big\}\,\breve{\alpha}^2 \\
&\qquad + \Big\{2\,(n-1)\,\Delta\,K \,-\, 2(n+1)\,W^s W^t \breve{W}^r{}_{:s}\breve{W}_{r:t} \\
&\qquad\quad + \Delta\,\breve{W}^s{}_{:t}\breve{W}^t{}_{:s} \,+\, \left[\tfrac{1}{2}(n+1) \,-\, \tfrac{1}{8}\,\Delta\,(5n+1)\right]\sigma^2\Big\}\,\breve{W}_0{}^2 \\
&\qquad - (n+1)\,\sigma\,\Delta\,\breve{W}_0\,W^s\breve{W}_{s:0} \,-\, (n+1)\,\Delta\,\left(W^s\breve{W}_{s:0}\right)^2 \\
&\qquad - 2(n+1)\,\Delta\,\breve{W}_0\,W^s\breve{W}^t{}_{:s}\breve{W}_{t:0} \,-\, (n+1)\,\Delta^2\,\breve{W}^s{}_{:0}\breve{W}_{s:0} \\
&\qquad - (n-1)\,\Delta^2\,W^s\breve{W}_{s:0:0}\Bigg)\,.
\end{aligned}$$

Taken together, the four expressions computed above allow us to miraculously rewrite the the Curvature Equation as

$$\breve{\mathrm{Ric}}_{00} = (n-1)\left(K + \tfrac{1}{16}\sigma^2\right)\breve{\alpha}^2\,.$$

Call this the **Einstein Equation**, because it says the Riemannian metric $\breve{a}$ is Einstein, with Ricci scalar $(n-1)(K + \frac{1}{16}\sigma^2)$.

Let's take a moment to review. We assumed the Lie_W and $\breve{\mathrm{E}}_{23}$ Equations hold, and then algebraically derived the Einstein Equation from the Curvature Equation. We may reverse the process and resurrect the Curvature Equation from Einstein Equation. In particular, given the Lie_W and $\breve{\mathrm{E}}_{23}$ Equations, the Curvature and Einstein Equations are equivalent.

At the end of §3.4 we remarked that the Lie_W and $\breve{\mathrm{E}}_{23}$ Equations hold if and only if the Basic and E_{23} Equations do. Therefore, the Lie_W, $\breve{\mathrm{E}}(23)$ and Einstein Equations are equivalent to the characterizing Basic, E_{23} and Curvature Equations. The characterization theorem (Theorem 2.3) implies

> The Randers metric given by perturbing $\breve{a}$ by W is Einstein with Ricci scalar $Ric = (n-1)K$ is if and only if the Lie_W, $\breve{\mathrm{E}}(23)$ and Einstein Equations are satisfied.

6. The theorem

We have just seen that a Randers metric with underlying $\breve{a}$ and W is Einstein if and only if the Lie_W, $\breve{\mathrm{E}}(23)$ and Einstein Equations hold. The $\breve{\mathrm{E}}_{23}$ Equation is redundant. (This is not a surprise. I remarked, in §2.6, that the Basic and Curvature Equations imply the E_{23} Equation.)

LEMMA 3.1. *The Lie_W and Einstein Equations imply the $\breve{E}_{23}$ Equation.*

PROOF. The implication may be seen in the following way. With the Ricci identities for W, and the Lie_W Equation identity $\breve{W}_{i:j:k} + \breve{W}_{j:i:k} = 0$, we have the following 5 equations

$$\begin{aligned}
\breve{W}_{i:j:k} - \breve{W}_{i:k:j} &= W^s \breve{R}_{isjk} \\
\breve{W}_{i:k:j} + \breve{W}_{k:i:j} &= 0 \\
-\breve{W}_{k:i:j} + \breve{W}_{k:j:i} &= -W^s \breve{R}_{ksij} \\
-\breve{W}_{k:j:i} - \breve{W}_{j:k:i} &= 0 \\
\breve{W}_{j:k:i} - \breve{W}_{j:i:k} &= W^s \breve{R}_{jski}\,,
\end{aligned}$$

which sum to produce

$$\breve{W}_{i:j:k} - \breve{W}_{j:i:k} = -2W^s \breve{R}_{ksij}$$

with a little help from the first Bianchi identity. Trace on (i,k) and apply the Einstein Equation to derive

$$\begin{aligned}
\breve{W}^s{}_{:j:s} - \breve{W}_{j:}{}^s{}_{:s} &= 2W^s \breve{\mathrm{Ric}}_{sj} \\
&= W^s\,(n-1)(2K + \tfrac{1}{8}\sigma^2)\breve{a}_{sj} \\
&= (n-1)(2K + \tfrac{1}{8}\sigma^2)\breve{W}_j\,.
\end{aligned}$$

This is the indexed form of the $\breve{\mathrm{E}}_{23}$ Equation. Contracting with y^j produces the $\breve{\mathrm{E}}_{23}$ Equation. □

We are now ready to state the navigation description for Einstein metrics of Randers type. Before I do so, recall the perturbing vector field W must satisfy $\breve{a}(W) < 1$, if the resulting Randers metric is to be strongly convex.

THEOREM 3.2 (Einstein navigation description). *Suppose the Randers metric F solves Zermelo's problem of navigation on the Riemannian manifold $(M, \breve{a})$ under the external force W, $\breve{a}(W) < 1$. Then (M, F) is Einstein with Ricci scalar $Ric =: (n-1)K$ if and only if*

- *The Einstein Equation*

$$\breve{Ric}_{00} = (n-1)(K + \tfrac{1}{16}\sigma^2)\breve{\alpha}^2$$

holds. That is, the Riemannian metric $\breve{a}$ is Einstein with Ricci scalar $(n-1)(K+\frac{1}{16}\sigma^2)$.
- *The vector field W is an infinitesimal homothety of $\breve{a}$. To be precise, the Lie_W Equation*

$$\breve{W}_{i:j} + \breve{W}_{j:i} = -\sigma\breve{a}_{ij}$$

is satisfied.

I refer to this theorem as a 'description' and distinguish it from the 'characterization' theorem of §2.6.2 because it provides a concise description of Einstein metrics of Randers type, in which the underlying geometry is explicit.

One consequence of the theorem is that the Ricci scalars of the Einstein F and $\breve{a}$ agree when W is Killing. Matsumoto's Identity of §3.8 describes in greater detail the values σ may take, and allows us to refine the relationship between the Ricci scalars.

Constant flag curvature and 3-D rigidity. We say a Finsler metric is *of constant flag curvature* if the flag curvature $K(x,y,w) = K$ is constant (§1.4). Since the Ricci scalar is the sum of $n-1$ flag curvatures, we see that constant flag curvature metrics are Einstein with Ricci scalar $(n-1)K$. These metrics are the Finslerian analogs of Riemannian metrics of constant sectional curvature.

Recollect that a three-dimensional Riemannian metric is Einstein if and only if it is of constant sectional curvature. It is not known if this rigidity holds for arbitrary Finsler metrics.

However, the rigidity does hold for Randers metrics. The proof rests on a navigation description for Randers metrics of constant flag curvature similar to the Einstein description above [**BRS03**].

THEOREM 3.3 (Constant flag curvature navigation description). *Suppose the Randers metric F solves Zermelo's problem of navigation on the Riemannian manifold $(M, \breve{a})$ under the external force W. Then (M, F) is of constant flag curvature K if and only if*

- *The Riemannian metric $\breve{a}$ is of constant sectional curvature $(K + \frac{1}{16}\sigma^2)$. That is,*

$$\breve{R}_{hijk} = \left(K + \tfrac{1}{16}\sigma^2\right)\left(\breve{a}_{ij}\breve{a}_{hk} - \breve{a}_{ik}\breve{a}_{hj}\right).$$

- *The vector field W is an infinitesimal homothety of $\breve{a}$. To be precise, the Lie_W Equation*

$$\breve{W}_{i:j} + \breve{W}_{j:i} = -\sigma\breve{a}_{ij}$$

is satisfied.

As in the Einstein setting, the derivation begins with a characterization of Randers metrics of constant flag curvature in terms of $\tilde{a}$ and $\tilde{b}$. The description is then produced by re-writing the characterizing equations within the context of Zermelo navigation. This is joint work with David Bao. Z. Shen has solved the Lie_W Equation and obtained a list of W for each of the three standard Riemannian space forms $\breve{a}$; see also §4.2. The detailed classification of constant flag curvature Randers metrics is derived in [**BRS03**].

The three dimensional rigidity is immediate. A Randers metric is Einstein if and only if the Lie_W Equation holds, and $\breve{a}$ is Einstein with Ricci scalar $(n-1)(K + \frac{1}{16}\sigma^2)$. In three dimensions, all Riemannian Einstein metrics are of constant sectional curvature. Hence $\breve{a}$ is of constant sectional curvature $K + \frac{1}{16}\sigma^2$, and the conditions of Theorem 3.3 are met.

Interestingly, the two navigation descriptions also tell us any Einstein Randers metric arising as a solution to Zermelo's problem of navigation on a Riemannian space form is necessarily of constant flag curvature.

7. A Schur lemma

It is my goal in this section to prove a Schur lemma for the Ricci scalar of a Randers metric. Essentially, the lemma constrains the geometry of Einstein metrics in dimension greater than two, insisting that the Ricci scalar be constant.

Historically, this is the second Schur lemma in (non-Riemannian) Finsler geometry. In 1973 del Riego [**dR73**] proved a Schur lemma for the flag curvature of Finsler metrics. Matsumoto [**Mat86**] presented a second proof in 1986.

In two dimensions, the Ricci scalar *Ric* of a Riemannian metric is the Gaussian curvature of the surface. That is, all Riemannian surfaces are Einstein. Since the Gaussian curvature is not constant in general, we see the Schur lemma fails for Riemannian, and therefore Randers, metrics in two dimensions.

At this point it is natural to ask if the Schur lemma must also fail for non-Riemannian ($\tilde{b} \neq 0$) Randers surfaces. The answer is 'yes'. Section 4.1 develops a class of non-Riemannian Randers metrics with Ricci scalar a non-constant function of x alone. In particular, these non-Riemannian metrics are Einstein, but fail the Schur lemma.

LEMMA 3.4 (Schur). *The Ricci scalar $Ric(x)$ of any Einstein Randers metric in dimension greater than two is necessarily constant.*

PROOF. The lemma follows immediately from the Riemannian Schur lemma. To see this, suppose F is an Einstein metric of Randers type, solving Zermelo's problem on the Riemannian manifold $(M, \breve{a})$. The Einstein Equation says $\breve{a}$ is Einstein with Ricci scalar $(n-1)(K + \frac{1}{16}\sigma^2)$. By the Riemannian Schur lemma (see Appendix B), $K + \frac{1}{16}\sigma^2$ is constant when $n > 2$. Since σ is constant (see §2.4), we see $K = \frac{1}{n-1} Ric$ must be as well. ☐

Discussion. One proof of the Riemannian Schur lemma is based on the second Bianchi Identity. Unfortunately, the Finslerian second Bianchi Identity does not yield itself to a similar argument. This has led geometers to expect the Schur lemma to fail in general. However, the establishement here of a Randers Schur lemma permits some optimism for arbitrary Finsler metrics.

The interested reader may find a discussion of these issues in the second appendix. There I include

- a proof of the Riemannian Schur lemma,
- a discussion of the obstacle presented by the Finslerian second Bianchi identity, and
- a second proof of the Randers Schur lemma based on the characterization theorem of §2.6.2.

8. The Matsumoto Identity

Matsumoto has shown [**Mat01**] that any Randers metric of *constant flag curvature* K satisfies

$$\sigma\,\left(K + \tfrac{1}{16}\,\sigma^2\right) = 0\,.$$

The identity may be extended to Einstein metrics. (Recall, constant flag curvature metrics are Einstein, with Ricci scalar $Ric = (n-1)K$.)

I will show any Einstein metric of Randers type, with Ricci scalar $Ric =: (n-1)K$, satisfies the **Matsumoto Identity for Einstein Randers metrics**

$$\begin{aligned} \sigma(K + \tfrac{1}{16}\sigma^2) &= W^i K_{:i}\,, &\quad& \text{when } n = 2 \\ \sigma(K + \tfrac{1}{16}\sigma^2) &= 0 && \text{when } n > 2\,. \end{aligned}$$

The proof rests on the Ricci Identity for $\breve{W}_{i:j} - \breve{W}_{j:i}$

$$\begin{aligned} &\left(\breve{W}_{i:j} - \breve{W}_{j:i}\right)_{k:h} - \left(\breve{W}_{i:j} - \breve{W}_{j:i}\right)_{h:k} \\ &\qquad = \left(\breve{W}_{s:j} - \breve{W}_{j:s}\right) \breve{R}_i{}^s{}_{kh} + \left(\breve{W}_{i:s} - \breve{W}_{s:i}\right) \breve{R}_j{}^s{}_{kh}\,, \end{aligned}$$

where $\breve{R}_h{}^i{}_{jk}$ is the curvature tensor of $\breve{a}$. Trace the Ricci Identity on (i,k) and (h,j) to obtain

$$\left(\breve{W}^{i:j}{}_{:i} - \breve{W}^{j:i}{}_{:i}\right)_{:j} = \left(\breve{W}^{i:j} - \breve{W}^{j:i}\right) \breve{Ric}_{ij}\,.$$

The Ricci tensor $\breve{Ric}_{ij}$ of $\breve{a}$ is symmetric. Since $\breve{W}^{i:j} - \breve{W}^{j:i}$ is skew-symmetric the right hand side must vanish. With the help of the $\breve{\mathrm{E}}_{23}$ Equation this implies

$$\begin{aligned} 0 &= \left(\breve{W}^{i:j}{}_{:i} - \breve{W}^{j:i}{}_{:i}\right)_{:j} \\ &= \left\{(n-1)(2K + \tfrac{1}{8}\sigma^2)W^j\right\}_{:j} \\ &= 2(n-1)\left\{W^j K_{:j} + (K + \tfrac{1}{16}\sigma^2)\breve{W}^j{}_{:j}\right\} \\ &= 2(n-1)\left\{W^j K_{:j} - \frac{n}{2}\sigma(K + \tfrac{1}{16}\sigma^2)\right\}. \end{aligned}$$

The identity now follows from the Schur lemma.

A second proof of the Matsumoto Identity, based on the Einstein characterization theorem of §2.6.2, may be found in Appendix A.

8.1. The refined description for constant *Ric*. The navigation description tells us that every Einstein Randers metric arises as the solution to Zermelo's problem of navigation on an Einstein Riemannian manifold $(M, \breve{a})$ under an infinitesimal homothety W. Matsumoto's identity further refines this description *when Ric is constant.* To see how this is so, suppose the Ricci scalar $Ric = (n-1)K$ of F is constant. The Matsumoto Identity reads

$$\sigma\,(K + \tfrac{1}{16}\sigma^2) = 0\,.$$

Recollecting that $(n-1)\,(K + \frac{1}{16}\sigma^2)$ is the Ricci scalar of the Einstein $\breve{a}$, consider the following three cases:

(+) If $Ric > 0$, then $\sigma = 0$. In particular, the Ricci scalar of $\breve{a}$ is Ric, and W is Killing. (Equivalently, W is an infinitesimal isometry.)

(0) If $Ric = 0$, then $\sigma = 0$, and F solves Zermelo's problem of navigation on a Ricci-flat Riemannian metric under an infinitesimal isometry.

(-) When $Ric < 0$, either $\sigma = 0$ or $\sigma = \pm 4\sqrt{|K|}$.
- If $\sigma = 0$, the Ricci scalar of $\breve{a}$ is Ric and W is an infinitesimal isometry.
- If $\sigma = \pm 4\sqrt{|K|}$, then $\breve{a}$ is Ricci-flat, and W is *not* Killing. Alternatively, any solution of Zermelo's problem of navigation on a Ricci-flat Riemannian manifold under a infinitesimal homothety with $\sigma \neq 0$, is an Einstein Randers metric with *negative* Ricci scalar $Ric = -\frac{1}{16}(n-1)\sigma^2$.

CHAPTER 4

Case Studies

In this chapter I apply the Einstein navigation description of Theorem 3.2 to examine some special classes of Einstein metrics of Randers type.

I begin in dimension two, constructing non-trivial Randers metrics with Ricci scalar a non-constant function of position x alone. The metrics are Einstein, and provide non-Riemannian counter-examples to the Schur lemma in two dimensions.

Next, in Section 4.2, I discuss a recent classification of constant flag curvature Randers spaces. The result is analogous to the Hopf classification of Riemannian geometry.

Last, Section 4.3 investigates closed Einstein manifolds. I will demonstrate that any Einstein Randers metric of negative Ricci scalar on a closed manifold is necessarily Riemannian. Also, any Ricci-flat Einstein Randers metric on a closed manifold must be Berwald.

1. Randers metrics with $K = K(x)$ in dimension 2

The first case study is of surfaces of rotation in $\mathbb{R}^3$. We shall see that solutions to Zermelo's problem of navigation under infinitesimal rotations are Einstein with a non-constant Ricci scalar $Ric = Ric(x)$. These metrics, constructed jointly with D. Bao [**BR03a**], are non-Riemannian counter examples to Shur's lemma in dimension 2.

To begin take any surface of revolution M, obtained by revolving a profile curve

$$\varphi \mapsto (\,0,\ f(\varphi),\ g(\varphi)\,)$$

in the right half of the yz-plane around the z axis. The ambient Euclidean space induces a Riemannian metric $\breve{a}$ on M. Parameterize M as follows:

$$(\theta, \varphi) \ \mapsto\ (\,f(\varphi)\cos(\theta),\ f(\varphi)\sin(\theta),\ g(\varphi)\,)\,.$$

Now consider an external force acting on M represented by the infinitesimal isometry $W := \epsilon\,\partial_\theta$. Here ϵ is a constant to be specified momentarily. By limiting the size of our profile curve if necessary, there is no loss of generality in assuming that f is bounded. Choose ϵ so that $\epsilon|f| < 1$ for all φ. Then the solution to Zermelo's problem is the following Randers metric on M:

$$\alpha = \frac{\sqrt{u^2\,f^2 + v^2\,(1-\epsilon^2\,f^2)\,(\dot{f}^2+\dot{g}^2)}}{1-\epsilon^2\,f^2}\,, \qquad \beta = \frac{-\epsilon\,u\,f^2}{1-\epsilon^2\,f^2}\,.$$

Here, $u\,\partial_\theta + v\,\partial_\varphi$ represents an arbitrary tangent vector on M, and $\dot f$, $\dot g$ abbreviate the derivative of f, g with respect to φ.

Because the vector field W is Killing σ vanishes, and the Einstein navigation description says the Ricci curvature K of F is identical to the Gaussian curvature of the Riemannian metric $\breve{a}$

$$K = \frac{\dot g\,(\dot f\,\ddot g - \ddot f\,\dot g)}{f\,(\dot f^2 + \dot g^2)^2}\,,$$

where the dots indicate derivatives by φ.

A Randers metric on the elliptic paraboloid. Specialize to the surface of revolution $z = x^2 + y^2$ in R^3, and set the multiple ϵ in W to be 1. The resulting Randers metric lives on the $x^2 + y^2 < 1$ portion of the elliptic paraboloid, and has Ricci curvature $4/(1 + 4x^2 + 4y^2)^2$. It reads:

$$\alpha = \frac{\sqrt{(-yu + xv)^2 + \{(1+4x^2)\,u^2 + 8xy\,uv + (1+4y^2)\,v^2\}\,\mathcal{D}}}{\mathcal{D}}\,,$$
$$\beta = \frac{yu - xv}{\mathcal{D}}\,,\quad \|\tilde b\|^2 = x^2 + y^2\,,\quad \text{where }\ \mathcal{D} := 1 - x^2 - y^2\,.$$

A Randers metric on the torus of revolution. Specialize to a torus of revolution with parameterization

$$(\vartheta, \varphi) \mapsto (\ [2 + \cos(\varphi)]\cos(\vartheta),\ [2 + \cos(\varphi)]\sin(\vartheta),\ \sin(\varphi)\ )\,.$$

Set the multiple ϵ in W to be $\frac{1}{4}$. The resulting Randers metric on the torus has Ricci curvature $\cos(\varphi)/[2 + \cos(\varphi)]$. It is given by:

$$\alpha = \frac{4\,\sqrt{16\,[2 + \cos(\varphi)]^2\,u^2 + \{16 - [2+\cos(\varphi)]^2\}\,v^2}}{16 - [2 + \cos(\varphi)]^2}\,,$$
$$\beta = \frac{-4\,[2 + \cos(\varphi)]^2\,u}{16 - [2 + \cos(\varphi)]^2}\,,\qquad \text{with }\ \|\tilde b\|^2 = \tfrac{1}{16}\,[2 + \cos(\varphi)]^2\,.$$

2. Spaces of constant flag curvature

Constant flag curvature spaces form a distinguished subclass of Einstein spaces. As indicated in Section 3.6, all Randers metrics of constant flag curvature arise as solutions to Zermelo's problem on an Riemannian space of constant curvature under an infinitesimal homothety. This result, along with the Hopf classification of constant curvature Riemannian spaces, allows a complete classification of constant flag curvature Randers metrics. The work is joint with D. Bao and Z. Shen; the details and additional results may be found in [**BRS03**]. My primary goal in presenting the classification theorem here is to provide the reader with an explicit recipe for the construction of these special Einstein Randers metrics. First, some notation

- $Q = (Q_{ij})$ is a skew-symmetric matrix and $C = (C^i)$ a vector, both constant;
- Qx denotes $(Q^i{}_j\,x^j)$, and $x := (x^i)$;

- all indices on Q, C, x are manipulated by the Kronecker deltas δ_{ij} and δ^{ij};
- "$\cdot$" is the standard Euclidean dot product.

THEOREM 4.1 (Classification [**BRS03**]). *Let $F(x,y)$ be a Randers metric on a smooth manifold M of dimension $n \geq 2$, solving Zermelo's problem of navigation for the Riemannian metric $\breve{a}$ and vector field W. Then F is of constant flag curvature K if and only if $\breve{a}$ is of constant curvature, and W is an infinitesimal homothety. Moreover, up to local isometry, the Riemannian space form $\breve{a}$ and the vector field W must belong to one of the following four families.*

(+) *When $K > 0$: $\breve{a}$ is the standard metric on the n-sphere of radius $\frac{1}{\sqrt{K}}$ and $W = Qx + C + (x \cdot C)\, x$, with*

$$\frac{1}{1+(x\cdot x)}\{(Qx+C)\cdot(Qx+C)+(x\cdot C)^2\} \;<\; K\,.$$

(Here W is given with respect to the projective coordinates parameterizing a hemisphere. See [**BRS03**].*)*

(0) *When $K = 0$: $\breve{a}$ is the Euclidean metric on $\mathbb{R}^n$ and $W = Qx + C$, with*

$$(Qx+C)\cdot(Qx+C) \;<\; 1\,.$$

(−) *When $K < 0$:*

$(-)_e$ *either $\breve{a}$ is the Euclidean metric on $\mathbb{R}^n$, and $W = -\frac{1}{2}\sigma x + Qx + C$ satisfies*

$$(Qx+C)\cdot(Qx+C)+\sigma x\cdot(\tfrac{1}{4}\sigma x - C) \;<\; 1$$

with $\sigma = \pm 4\sqrt{|K|}$;

$(-)_k$ *or $\breve{a}$ is the Klein model of sectional curvature K on the unit ball in $\mathbb{R}^n$, and $W = Qx + C - (x\cdot C)x$ satisfies*

$$\frac{1}{1-(x\cdot x)}\{(Qx+C)\cdot(Qx+C)-(x\cdot C)^2\} \;<\; |K|\,.$$

Furthermore, if M is simply-connected and $\breve{a}$ is complete, then the said local isometry is in fact a global isometry.

The perhaps mysterious inequalities for Q and C are determined by the restraint that $\|W\| < 1$. Often times W will satisfy this restraint only on an open subset U of M. In these cases the Randers metric solving Zermelo's problem is defined only on U. In fact, the Klein model admits no non-trivial globally defined W satisfying the necessary inequality. It does however admit many locally defined Killing fields of norm less than one. In constrast, both Euclidean space and the sphere admit non-trival global infinitesimal isometries with $\|W\| < 1$ ([**BRS03**]).

In addition to describing the Randers metrics of constant flag curvature, the classification may be used to construct Einstein metrics of non-constant flag curvature. To do so, I recall the familiar fact that the product of two

Einstein Riemannian spaces with a common Ricci scalar λ is itself Einstein with Ricci scalar λ. Fix λ. Let M_i be an n_i-dimensional Riemannian manifold of constant curvature $\frac{\lambda}{n_i}$, $i = 1, 2$. Then M_i is Einstein with Ricci scalar λ. Let W_i be an infinitesimal homothety on M_i. Modulo local isometry, W_i must correspond to one of the four families listed in the classification theorem.

The product space $M = M_1 \times M_2$ is Einstein, with infinitesimal homothety $W = (W_1, W_2)$. Theorem 3.2 tells us the Randers metric F on M solving Zermelo's problem of navigation under W is Einstein.

Note that when λ is nonzero, the Riemannian metric on M is not of constant sectional curvature. In particular F, though Einstein, is not of constant flag curvature (Theorem 3.3).

3. Closed Einstein spaces

In the final case study, I restrict attention to closed (compact and boundaryless) manifolds. Assume throughout the present section that

> **F is an Einstein Randers metric, with Ricci scalar Ric, on a connected, closed manifold M.**

In particular, F solves Zermelo's problem of navigation for a Einstein Riemannian metric $\breve{a}$ on M under an infinitesimal homothety W. It is my goal in this discussion to determine the constraints placed on W, and therefore F, by the hypothesis that M is closed.

I begin by observing that W must be an infinitesimal isometry. (Equivalently, W is a Killing vector field.) This is easily seen with the following divergence lemma.

LEMMA 4.2 ([**BCS00**]). *Let V be any globally defined vector field on a closed Riemannian manifold (M, g). Let ∇ denote the Riemannian connection. Then*

$$\int_M \nabla_i V^i dV_g = 0\,.$$

Setting $g = \breve{a}$ and $V = W$ establishes the following lemma.

LEMMA 4.3. *The infinitesimal homothety W is Killing. Equivalently, σ vanishes.*

PROOF. By the divergence lemma we have

$$0 = \int_M W^i{}_{:i}\, dV = \int_M -\tfrac{1}{2} n\, \sigma\, dV\,.$$

Hence σ is zero, and W is Killing. □

> It follows now, from the Einstein navigation description, that the Ricci scalar of $\breve{a}$ is Ric.

The following result of Bochner addresses the case $Ric < 0$.

THEOREM 4.4 (Bochner [**Boc46, KN96**]). *Let M be a connected Riemannian manifold whose Ricci tensor is negative definite everywhere on M. If the length of an infinitesimal isometry V attains a relative maximum at some point of M, then V vanishes identically on M.*

Because $\breve{a}$ is Einstein, the Ricci tensor of $\breve{a}$ is negative definite when $Ric < 0$. By Lemma 4.3 W is an infinitesimal isometry. The norm $\|\breve{W}\|$ must attain a maximum as M is compact. Hence $W = 0$, and $F = \breve{\alpha}$ is Riemannian. I have proven the following

PROPOSITION 4.5. *Let F be an Einstein Randers metric with negative Ricci scalar on a closed manifold M. Then F is Riemannian.*

Let's now turn our attention to the 1-form $\omega = \breve{W}_i dx^i$ dual to W. The Laplacian of ω is given by the Weitzenböck formula [**BCS00**]

$$\Delta\,\omega = \left(\breve{\mathrm{Ric}}_i{}^j\,\breve{W}_j\; - \;\breve{W}_{i:}{}^j{}_{:j}\right)dx^i\,.$$

By the Einstein navigation description

$$\breve{\mathrm{Ric}}_i{}^j = Ric\,\delta_i{}^j\,.$$

Similarly, the Lie_W and $\breve{\mathrm{E}}(23)$ equations of Sections 3.3 and 3.4 imply

$$\breve{W}_{i:}{}^j{}_{:j} = -Ric\,\breve{W}_i\,.$$

Whence we compute

$$\Delta\,\omega = -2\,Ric\,\omega\,.$$

(I have made use of the fact that $\sigma = 0$ in these computations.)

Assume $Ric = 0$. Then the Ricci tensor of $\breve{a}$ vanishes and ω is harmonic. Another well-known result of Bochner implies ω is parallel.

THEOREM 4.6 (Bochner [**BCS00**]). *Let θ be a globally defined 1-form on a closed Riemannian manifold (M, g). Suppose the Ricci tensor of g is non-negative. Then θ is harmonic if and only if it is parallel, that is, $\nabla\theta = 0$.*

The 1-form ω is parallel precisely when W is. In particular, W is parallel whenever $Ric = 0$. The converse holds as well, so long as F is non-Riemannian (ie. $W \neq 0$). (To see why we need to assume F is non-Riemannian, notice that F is Riemannian precisely when $W = 0$. In which case W is certainly parallel, but the Ricci tensor of $F = \breve{a}$ may assume any value.)

To prove the converse assume that W is parallel, and not identically zero. Since σ vanishes the $\breve{\mathrm{E}}(23)$ Equation reads

$$0 = 2\,Ric\,W\,.$$

Because W is not identically zero on M, the Ricci scalar Ric must be zero. We have established the following

PROPOSITION 4.7. *Assume F is a non-Riemannian Einstein Randers metric on a closed manifold M. Then $Ric = 0$ if and only if W is parallel.*

It now follows as a corollary that a non-Riemannian Einstein Randers metric is Berwald if and only if $Ric = 0$. Assume $(\mathcal{M}, \mathcal{F})$ is an arbitrary Finsler space. Let $\mathcal{G}^i$ denote the geodesic spray coefficients of $\mathcal{F}$ (cf. §1.4). Then $(\mathcal{M}, \mathcal{F})$ is said to be a *Berwald space* if the quantities $(\mathcal{G}^i)_{y^j y^k}$ do not depend on y [**BCS00**].

For the moment, let us consider an arbitrary Randers metric $F = \alpha + \beta$, solving Zermelo's problem for $(\breve{a}, W)$. In particular, M need not be closed, nor F Einstein. It is known that F is Berwald if and only if b is parallel [**BCS00**]. Look back at the expression for $\tilde{b}_{i|j}$ in §3.3. Recollect our computation of this formula involves no assumptions on F. Notice that $\breve{W}_{i:j} = 0$ implies $\tilde{b}_{i|j} = 0$. Conversely, if we assume $\tilde{b}$ is parallel, and successively contract the right-hand side of the expression for $\tilde{b}_{i|j}$ in §3.3 with $\breve{W}^i \breve{W}^j$, $\breve{W}^i$, and $\breve{W}^j$, we may deduce that $\breve{W}_{i:j} = 0$. In particular,

> Let F be *any* Randers metric. Then $\tilde{b}$ is parallel with respect to $\tilde{a}$ if and only if W is parallel with respect to $\breve{a}$.

Now return to the realm of Einstein Randers metrics on closed manifolds. Proposition 4.7 and the discussion above imply

COROLLARY 4.8. *Assume F is a non-Riemannian Einstein Randers metric on a closed manifold M. Then $Ric = 0$ if and only if F is Berwald.*

Finally, let me close this section by observing that the rigidity of Proposition 4.5 does not hold when $Ric = 0$.

PROPOSITION 4.9. *The flat torus admits non-Riemannian Randers metrics of constant flag curvature $K = 0$.*

PROOF. A flat torus is simply Euclidean $\mathbb{R}^n$ modulo n linearly independent translations. The parallel vector fields on $\mathbb{R}^n$ are the constant vector fields. A constant vector field on $\mathbb{R}^n$ is certainly invariant under translation, and therefore defines a global, parallel vector field W on the torus. In particular, Zermelo navigation on the torus under W defines a non-trivial Randers metric. □

CHAPTER 5

Open Questions

It is natural to wonder which of the results established here, among Randers metrics, may be extended to Finsler spaces in general. To that end I have selected a few problems to discuss.

1. A Schur lemma

Does the Einstein Schur lemma hold for arbitrary Finsler metrics? One proof of the lemma for Riemannian spaces rests on the second Bianchi identity. As I mention in §3.7 (and discuss in Appendix B), the Finslerian second Bianchi identity does not lend itself to an analogous proof. The intractable nature of the identity has led geometers to doubt the existence of a Schur lemma for Finsler metrics. However, my success in the Randers setting encourages a sanguine speculation.

2. Chern's conjecture

S.S. Chern has conjectured that

Every manifold admits a Finslerian Einstein metric.

It is known that topological obstructions prevent some manifolds from admitting Einstein Riemannian metrics [**Hit74, LeB99**]. According to the navigation description (Theorem 3.2) any manifold that admits an Einstein Randers metric must also admit a Einstein Riemannian metric. Hence the same topological obstructions restrict Einstein metrics of Randers type.

To understand why the conjecture fails among Randers metrics, yet may still be expected to hold in the larger class of Finsler metrics, it is helpful to consider the indicatrix. The *indicatrix* of a Finsler manifold $(\mathcal{M}, \mathcal{F})$ is the set of points in the tangent space $T_x\mathcal{M}$ of norm 1, $S_x(\mathcal{F}) := \{y \in T_x\mathcal{M} : \mathcal{F}(y) = 1\}$. Given the Randers metric F solving Zermelo's problem of navigation for $(\breve{a}, W)$ the indicatrices are related by $S_x(F) = S_x(\breve{\alpha}) + W_x$. In particular, the Randers indicatrix is simply an ellipse centered at W_x.

In this context, it seems asking a metric to be both Einstein and elliptical (ie. Randers) restricts the topology of M. In pursuing Chern's conjecture we are asking if relaxing the elliptical assumption on the indicatrix to strict convexity (this is, moving from Randers to Finsler metrics) is sufficient to lift the topological restraints.

3. Einstein rigidity

Consider closed Einstein manifolds and, in particular, Proposition 4.5 and Corollary 4.8 of §4.3. Together the results imply the following rigidity theorem.

THEOREM 5.1 (Ricci rigidity). *Suppose (M, F) is a connected compact boundaryless Einstein Randers manifold with Ricci scalar Ric.*

- *If $Ric < 0$, then (M, F) is Riemannian.*
- *If $Ric = 0$, then (M, F) is Berwald.*

This statement generalizes a theorem of Akbar-Zadeh's for Finsler metrics of constant flag curvature [**AZ88**].

THEOREM 5.2 (Akbar-Zadeh). *Suppose (M, F) is a connected compact boundaryless Finsler manifold of constant flag curvature λ.*

- *If $\lambda < 0$, then (M, F) is Riemannian.*
- *If $\lambda = 0$, then (M, F) is locally Minkowski.*

The Ricci rigidity theorem is a straightforward extension of Akbar-Zadeh's result when $Ric < 0$. To appreciate the generalization when $Ric = 0$, it is helpful to note that locally Minkowski spaces are the Berwald spaces with constant flag curvature $K = 0$ [**BCS00**].

Notice though, that Akbar-Zadeh's theorem holds for arbitrary Finsler metrics, while the Ricci rigidity theorem above is restricted to the Randers setting. So, towards a complete generalization of Akbar-Zadeh's result: may we replace 'Randers' with 'Finsler' in the Ricci rigidity theorem?

4. A generalized navigation problem

The Zermelo navigation structure of a Randers space is the *sine qua non* of this thesis. As such I am indebted to Z. Shen for realizing Randers metrics as solutions to Zermelo's problem of navigation [**She02a, She02b**].

The pivotal role of the navigation structure is made clear by the Einstein navigation description of §3.6. The concise, geometrically transparent nature of Theorem 3.2 is in stark contrast with the unintuitive characterization of Theorem 2.3. The sleek format of the navigation description is ideally suited to the theory of Einstein metrics. This is nowhere more evident than in the Schur lemma, in which an unexpected result is realized as a corollary to Riemannian geometry.

It is desirable then, to extend Zermelo's problem of navigation as a means of parameterizing Finsler metrics in general. Given the integral role of the navigation structure in the study of Randers metrics, we may reasonably hope that such a parameterization will provide an arena to approach the questions outlined above.

APPENDIX A

The Matsumoto Identity: Preliminary Form

Here I present a second proof of Matsumoto's identity. The argument is based on the characterization of §2.6.2. Unlike the proof of the Matsumoto Identity in §3.8 the following computation is considerably more involved. It is my hope that the juxtaposition of the two proofs, one straightforward, the second abstruse, illuminates the merit of the navigation description over the characterization of §2.6.2.

The final form of the Matsumoto Identity requires the Schur lemma. For now, I show any Randers metric, for which the Basic Equation (§2.5.1) and the E_{23} Equation (§2.5.3) hold, with $n \geq 2$, satisfies the following preliminary identity:

$$0 = n\,\sigma\{1 - \|\tilde{b}\|^2\}\left(K + \tfrac{1}{16}\sigma^2\right) + 2\tilde{b}^i K_{|i}$$

Here $K = \frac{1}{n-1}Ric$, and Ric is given by the E_{23} Equation.

In Appendix B, I will show, via the Einstein characterization theorem of §2.6.2 and the preliminary identity above, that the Ricci scalar $Ric(x)$ must be constant when $n > 2$. Once we see Ric is constant the identity is updated to its final form,

$$\begin{aligned} \sigma\{1 - \|\tilde{b}\|^2\}\left(K + \tfrac{1}{16}\sigma^2\right) + \tilde{b}^i K_{|i} &= 0 \qquad \text{when } n = 2, \\ \sigma\left(K + \tfrac{1}{16}\sigma^2\right) &= 0 \qquad \text{when } n > 2. \end{aligned}$$

To obtain the preliminary version I begin with the Ricci identity for curl_{ij},

$$\mathrm{curl}_{ij|k|h} - \mathrm{curl}_{ij|h|k} = \mathrm{curl}_{sj}\,\tilde{R}_i{}^s{}_{kh} + \mathrm{curl}_{is}\,\tilde{R}_j{}^s{}_{kh}\,.$$

Contract this expression with $\tilde{a}^{ik}\tilde{a}^{hj}$ and apply the skew-symmetry of curl_{ij} to generate

$$\mathrm{curl}^{ij}{}_{|i|j} = -\mathrm{curl}^{ij}\widetilde{\mathrm{Ric}}_{ij}\,.$$

Since curl^{ij} is skew-symmetric, and $\widetilde{\mathrm{Ric}}_{ij}$ is symmetric, the right-hand side of this equation is zero. Hence,

$$\mathrm{curl}^{ij}{}_{|i|j} = 0\,.$$

Let us compute $\mathrm{curl}^{ij}{}_{|i|j}$. Virtue of the E_{23} Equation we have

$$\begin{aligned} \mathrm{curl}^{ij}{}_{|i|j} \;=\; & 2\tilde{b}^i Ric_{|i} + \{2Ric + \tfrac{1}{8}(n-1)\,\sigma^2\}\tilde{b}^i{}_{|i} \\ & + \tfrac{1}{2}(n-1)\{\mathrm{curl}^{ij}\Theta_{i|j} + \mathrm{curl}^{ij}{}_{|j}\Theta_i + \sigma\Theta^i{}_{|i}\}\,. \end{aligned}$$

We need to address the terms

$$\tilde{b}^i{}_{|i}\,,\quad \mathrm{curl}^{ij}{}_{|j}\Theta_i\,,\quad \Theta^i{}_{|i}\quad\text{and}\quad \mathrm{curl}^{ij}\Theta_{i|j}$$

above.

Note that $\tilde{b}^i{}_{|i} = \frac{1}{2}\mathrm{lie}^i{}_i$, and the Basic Equation implies

$$\tilde{b}^i{}_{|i} = \tfrac{1}{2}\,\sigma\,(n - \|\tilde{b}\|^2)\,.$$

Next, we compute the terms $\mathrm{curl}^{ij}{}_{|j}\Theta_i$ and $\Theta^i{}_{|i}$ with the E_{23} Equation. In these computations I make frequent use of the facts $\tilde{b}_{i|j} = \frac{1}{2}(\mathrm{lie}_{ij} + \mathrm{curl}_{ij})$ and $\tilde{b}_{i|j}\mathrm{curl}^{ij} = \frac{1}{2}\mathrm{curl}_{ij}\mathrm{curl}^{ij}$. The two terms are given by

- $\mathrm{curl}^{ij}{}_{|j}\Theta_i = -\,\frac{1}{2}(n-1)\,\sigma\,\Theta_i\,\Theta^i$, and
- $\Theta^i{}_{|i} = \frac{1}{2}\mathrm{curl}_{ij}\,\mathrm{curl}^{ij} \;-\; \{2Ric + \frac{1}{8}(n-1)\,\sigma^2\}\,\|\tilde{b}\|^2 + \frac{1}{2}(n-1)\,\Theta_i\,\Theta^i$.

The calculation of the fourth term, $\mathrm{curl}^{ij}\Theta_{i|j}$, is more involved. Notice, by the skew-symmetry of curl^{ij}, that

$$\mathrm{curl}^{ij}\Theta_{i|j} = \tfrac{1}{2}\mathrm{curl}^{ij}\,\left(\Theta_{i|j} - \Theta_{j|i}\right)\,.$$

While it is difficult to compute $\Theta_{i|j}$, the computation of $\Theta_{i|j} - \Theta_{j|i}$ is relatively straightforward. The calculation proceeds as follows. In §2.4 I presented a formula for $\mathrm{curl}_{ij|k}$

$$\mathrm{curl}_{ij|k} = -2\tilde{b}^s\tilde{R}_{ksij} + \mathrm{lie}_{ik|j} - \mathrm{lie}_{kj|i}\,.$$

With this expression we have

$$\begin{aligned}
\Theta_{i|j} - \Theta_{j|i} &= (\tilde{b}^h\mathrm{curl}_{hi})_{|j} - (\tilde{b}^h\mathrm{curl}_{hj})_{|i}\\
&= (\tilde{b}^h{}_{|j}\mathrm{curl}_{hi} + \tilde{b}^h\mathrm{curl}_{hi|j}) \;-\; (\tilde{b}^h{}_{|i}\mathrm{curl}_{hj} + \tilde{b}^h\mathrm{curl}_{hj|i})\\
&= \tfrac{1}{2}(\mathrm{lie}^h{}_j + \mathrm{curl}^h{}_j)\mathrm{curl}_{hi} \;-\; \tfrac{1}{2}(\mathrm{lie}^h{}_i + \mathrm{curl}^h{}_i)\mathrm{curl}_{hj}\\
&\quad\ \tilde{b}^h\Big\{ -2b^s(\tilde{R}_{jshi} - \tilde{R}_{ishj})\\
&\qquad\qquad + (\mathrm{lie}_{hj|i} - \mathrm{lie}_{ji|h}) \;-\; (\mathrm{lie}_{hi|j} - \mathrm{lie}_{ij|h})\Big\}\\
&= -\sigma\mathrm{curl}_{ij}\,.
\end{aligned}$$

The third equality follows from the expression for $\mathrm{curl}_{ij|k}$ above. It can be shown, via the symmetries of the Riemann curvature tensor, that $\tilde{b}^h\tilde{b}^s(\tilde{R}_{jshi} - \tilde{R}_{ishj}) = 0$. So the curvature tensors to not contribute to the computation. The last equality is then a result of the Basic Equation.

It follows now that

$$\mathrm{curl}^{ij}\Theta_{i|j} = \tfrac{1}{2}\mathrm{curl}^{ij}\,\left(\Theta_{i|j} - \Theta_{j|i}\right) = -\,\tfrac{1}{2}\,\sigma\,\mathrm{curl}_{ij}\,\mathrm{curl}^{ij}\,.$$

Recollect $\mathrm{curl}^{ij}\Theta_{i|j}$ is the fourth, and final expression computed for $\mathrm{curl}^{ij}{}_{|i|j}$. We may now obtain the identity by substituting the derived formulas for $\tilde{b}^i{}_{|i}$, $\mathrm{curl}^{ij}{}_{|j}\Theta_i$, $\Theta^i{}_{|i}$ and $\mathrm{curl}^{ij}\Theta_{i|j}$ into $\mathrm{curl}^{ij}{}_{|i|j} = 0$.

APPENDIX B

A Schur Lemma

In §3.7 the proof of the Schur lemma is based on the Einstein navigation description of §3.6. This appendix contains a proof of the Schur lemma based on the Einstein characterization of §2.6.2. Two purposes are served by the appendix. First, to confirm the first proof of §3.7. As in the case of the Matsumoto Identity in Appendix A, the reader will find this proof considerably more complex than the one based on the navigation description. The dichotomy serves the second purpose – to illustrate the strength of the navigation description over the characterization.

1. The Riemannian Schur Lemma

Before addressing the Randers case, it is helpful to review the proof of the Riemannian Schur lemma.

LEMMA B.1. *Schur Suppose $n \geq 3$ and that the Ricci scalar Ric of α is a function of x alone. Then Ric is constant.*

PROOF. Since *Ric* is independent of x, the Riemannian metric α is Einstein. In particular, $\widetilde{\mathrm{Ric}}_{ij} = Ric(x)\,\tilde{a}_{ij}$. Tracing on (i,j) we have $\tilde{S} := \widetilde{\mathrm{Ric}}{}^i{}_i = Ric(x)\,n$. Now rewrite the Einstein condition as

$$\widetilde{\mathrm{Ric}}_{ij} = \tfrac{1}{n}\,\tilde{S}\,\tilde{a}_{ij}\,.$$

Recollect the second Bianchi identity for α,

$$0 = \tilde{R}_h{}^i{}_{jk|l} + \tilde{R}_h{}^i{}_{kl|j} + \tilde{R}_h{}^i{}_{lj|k}\,.$$

Contract the identity with $\delta^j{}_i\,\tilde{a}^{hl}$ to obtain

$$\begin{aligned} 0 &= 2\widetilde{\mathrm{Ric}}{}^i{}_{k|i} - \tilde{S}_{|k} \\ &= 2\left(\tfrac{1}{n}\tilde{S}\,\delta^i{}_k\right)_{|i} - \tilde{S}_{|k} \\ &= \frac{2}{n}\tilde{S}_{|k} - \tilde{S}_{|k} \\ \Longrightarrow\quad 0 &= (n-2)\,\tilde{S}_{|k}\,. \end{aligned}$$

Hence, when $n > 2$, $\tilde{S}_{|k} = n\,\tilde{Ric}_{|k} = 0$. Since *Ric* is a scalar, $Ric_{|k} = Ric_{x^k}$, and the partial derivatives of *Ric* vanish. Hence, *Ric* is constant. □

2. A Finslerian obstacle

Above we saw that the Riemannian Schur lemma relies on the second Bianchi identity. It is reasonable then to consider the Finslerian second Bianchi identity

$$R_h{}^i{}_{jk:l} + R_h{}^i{}_{lj:k} + R_h{}^i{}_{kl:j} = P_h{}^i{}_{js}R^s{}_{kl} + P_h{}^i{}_{ks}R^s{}_{lj} + P_h{}^i{}_{ls}R^s{}_{jk}\,,$$

where $R^s{}_{jk} := \frac{1}{F}y^i R_i{}^s{}_{jk}$. Here R and P are respectively the *hh*- and *hv*- curvature tensors of the Chern connection for F; and the colon ':' denotes the horizontal covariant derivative generated by the Chern connection. Details may be found in [**BCS00**].

When F is Riemannian, the identity reduces to the Riemannian second Bianchi identity. For general Finsler metrics however, the right hand side is non-zero. In particular, we cannot mimic the Riemannian proof. We shall see in the following section that the Einstein characterization allows us a way around this obstacle.

3. The Randers Schur Lemma

Given the intractable nature of the Finslerian second Bianchi identity we can not hope to mimic the Riemannian proof with the Finsler Ricci tensor Ric_{ij}. Instead we turn to the Curvature Equation of the Einstein characterization (§2.6.2), and apply the *Riemannian* second Bianchi identity to the given expression for $\widetilde{\mathrm{Ric}}_{ij}$.

In our discussion of the Riemannian Schur lemma we saw, via the second Bianchi identity for α, $0 = \widetilde{\mathrm{Ric}}^i{}_{i|k} - 2\widetilde{\mathrm{Ric}}^i{}_{k|i}$. With the Curvature Equation (§2.5.2) we have

$$\begin{aligned} 0 &= \widetilde{\mathrm{Ric}}^i{}_{i|k} - 2\widetilde{\mathrm{Ric}}^i{}_{k|i} \\ &= (2\,\Theta_k - \mathrm{lie}^i{}_i\,\tilde b_k)\,Ric \,+\, (n + \|\tilde b\|^2 - 2)\,Ric_{|k} - 2\,\tilde b^i Ric_{|i}\,\tilde b_k \\ &\quad -\mathrm{curl}^{ij}{}_{|i}\mathrm{curl}_{jk} - \tfrac{1}{2}\,n\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k} + \mathrm{curl}^{ij}\mathrm{curl}_{ik|j} \\ &\quad +(n-1)\,\Big\{ \tfrac{1}{16}\,\sigma^2\,(2\,\Theta_k - \mathrm{lie}^i{}_i\,\tilde b_k) \,+\, \tfrac{1}{2}\Theta^i{}_{|i}\Theta_k + \\ &\qquad\qquad \tfrac{1}{2}\Theta^i(\Theta_{k|i} - \Theta_{i|k}) \,+\, \tfrac{1}{2}(\Theta^i{}_{|k|i} + \Theta_k{}^{|i}{}_{|i} - \Theta^i{}_{|i|k})\Big\} \end{aligned}$$

We shall refer to this as the **second Bianchi equation**. Many of the terms above we may directly evaluate with the Einstein characterization or the (preliminary) Matsumoto Identity (§A). A few of them, however, require special attention:

- $-\frac{1}{2}\,n\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k} + \mathrm{curl}^{ij}\mathrm{curl}_{ik|j}$,
- $\Theta^i(\Theta_{k|i} - \Theta_{i|k})$, and
- $\frac{1}{2}(\Theta^i{}_{|k|i} + \Theta_k{}^{|i}{}_{|i} - \Theta^i{}_{|i|k})$.

The first term. In the computation of the first term we will make use of the formula

$$\mathrm{curl}_{ij|k} = -2\tilde{b}^s \tilde{R}_{ksij} + \mathrm{lie}_{ik|j} - \mathrm{lie}_{kj|i},$$

initially presented in §2.4. With this expression for $\mathrm{curl}_{ij|k}$ and the skew-symmetry of curl_{ij} it is straightforward to check that

$$\mathrm{curl}^{ij}\mathrm{curl}_{ik|j} = \tfrac{1}{2}\mathrm{curl}^{ij}\mathrm{curl}_{ij|k}.$$

Whence we may rewrite the first term as

$$-\tfrac{1}{2}\, n\, \mathrm{curl}^{ij}\mathrm{curl}_{ij|k} + \mathrm{curl}^{ij}\mathrm{curl}_{ik|j} = -\tfrac{1}{2}\,(n-1)\,\mathrm{curl}^{ij}\mathrm{curl}_{ij|k}.$$

The second term. In Appendix A we computed

$$\Theta_{k|i} - \Theta_{i|k} = -\sigma \mathrm{curl}_{ki}.$$

It is now easy to see that the second term is given by

$$\Theta^i(\Theta_{k|i} - \Theta_{i|k}) = \sigma\, \Theta^i\, \mathrm{curl}_{ik}.$$

The third term. We begin with two observations.

- First, the Ricci identity for Θ implies ${\Theta^i}_{|k|i} = {\Theta^i}_{|i|k} + \Theta^i\widetilde{\mathrm{Ric}}_{ik}$.
- The second observation, ${\Theta_k}^{|i}{}_{|i} = {\Theta^i}_{|i|k} + \Theta^i\widetilde{\mathrm{Ric}}_{ik} + \sigma{\mathrm{curl}^i}_{k|i}$, follows from the first as a consequence of $\Theta_{i|k} - \Theta_{k|i} = -\sigma\,\mathrm{curl}_{ik}$.

Now we see the last term may be re-expressed as

$$\tfrac{1}{2}\left({\Theta^i}_{|k|i} + {\Theta_k}^{|i}{}_{|i} - {\Theta^i}_{|i|k}\right) = \tfrac{1}{2}\,{\Theta^i}_{|i|k} + \Theta^i\widetilde{\mathrm{Ric}}_{ik} + \tfrac{1}{2}\sigma{\mathrm{curl}^i}_{k|i}.$$

Notice we have a good understanding of the right-hand side of this equation. We computed ${\Theta^i}_{|i}$ in Appendix A. The computation of ${\Theta^i}_{|i|k}$ is then straightforward. The quantity $\Theta^i\widetilde{\mathrm{Ric}}_{ik}$ may be computed with the Curvature Equation. Similarly, ${\mathrm{curl}^i}_{k|i}$ is given by the E_{23} Equation.

The finale. We have reached the final step in the proof of the Randers Schur lemma. Substitute our expressions for the three terms computed above into the second Bianchi equation. Simplify the result with

- the Basic, Curvature and E_{23} Equations of the Einstein characterization (§2.5),
- the preliminary form of the Matsumoto Identity in Appendix A,
- $b_{i|j} = \tfrac{1}{2}(\mathrm{lie}_{ij} + \mathrm{curl}_{ij})$, a tautology, and
- $\Theta_{i|k} - \Theta_{k|i} = -\sigma\mathrm{curl}_{ik}$, computed in §A.

The result is

$$\begin{aligned} 0 &= \tilde{S}_{|k} - 2{\widetilde{\mathrm{Ric}}^i}_{k|i} \\ &= (n-2)(1 - \|\tilde{b}\|^2)\, Ric_{|k}. \end{aligned}$$

We have proven the following

LEMMA B.2 (Schur). *The Ricci scalar $Ric(x)$ of any Einstein Randers metric in dimension greater than two is necessarily constant.*

4. The Matsumoto Identity: Final Form

In light of this result, we may update the Matsumoto Identity, as computed in Appendix A, for Einstein Randers metrics to

$$\begin{aligned} \sigma\{1-\|\tilde{b}\|^2\}\left(K+\tfrac{1}{16}\sigma^2\right)+\tilde{b}^i K_{|i} &= 0 \qquad \text{when } n=2, \\ \sigma(K+\tfrac{1}{16}\sigma^2) &= 0 \qquad \text{when } n>2. \end{aligned}$$

The identity now agrees with the navigation version computed in §3.8.

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