

SOME COMPUTATIONS OF THE HOMOLOGY OF REAL GRASSMANNIAN MANIFOLDS

by

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Abstract

When computing the homology of Grassmannian manifolds, the first step is usually to look at the Schubert cell decomposition, and the chain complex associated with it. In the complex case and the real unoriented case with \mathbb{Z}_2 coefficients the additive structure is obtained immediately (i.e., generated by the homology classes represented by the Schubert cells) because the boundary map is trivial. In the real unoriented case (with \mathbb{Z}_2 coefficients) and the real oriented case, finding the additive structure is more complicated since the boundary map is nontrivial. In this paper, this boundary map is computed by cell orientation comparisons, using graph coordinates where the cells are linear, to simplify the comparisons. The integral homology groups for some low dimensional oriented and unoriented Grassmannians are determined directly from the chain complex (with the boundary map as computed).

The integral cohomology ring structure for complex Grassmannians has been completely determined mainly using Schubert cell intersections (what is known as Schubert Calculus). In this paper, a method using Schubert cell intersections to describe the \mathbb{Z}_2 cohomology ring structure of the real Grassmannians is sketched. The results are identical to those for the complex Grassmannians (with \mathbb{Z}_2 coefficients), but the notation used for the cohomology generators is not the usual one. It indicates that the products are to a certain degree independent of the Grassmannian.

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Introduction

A lot is known about the homology of the Grassmannian manifolds in general; e.g., from Characteristic Classes (see [1]) or using algebraic geometric methods (see [3] and [4]) and usually the Schubert cell decomposition is used. However, there do not seem to be readily available answers to such questions as:

- a) Given a finite dimensional oriented or unoriented real Grassmannian, what is the r -th homology group?
- b) Given two cocycles in such a Grassmannian, what is their cup product?

This paper is concerned with developing computational methods, using the geometry of the Schubert cell decomposition, by which explicit answers to the above can be determined.

In Part II, (a) is tackled by constructing two Universal chain complexes arising from the Schubert cell decomposition of the Universal oriented (real) Grassmannian and the Universal unoriented Grassmannian. (The main point is to compute the boundary maps.) From these complexes, the integral homology groups of some of the finite Grassmannians and, in low dimensions, for the infinite Grassmannians are calculated. Theoretically, it should be possible to determine all the homology groups for all the real Grassmannians (oriented and unoriented) from the formulas given for the boundary maps, but the amount of calculation required increases rapidly in the higher dimensions (above 6 for instance). However, by looking at the lower dimensions, it may be possible to detect patterns and make conjectures which could be proved by other means, e.g., characteristic classes. On the

other hand, comparing what is known about characteristic classes with what is obtained here may yield further information about the characteristic classes, e.g., which Schubert cells correspond to a given characteristic class. Homology for arbitrary Schubert varieties can be determined from the chain complexes also, and some examples are given.

Question (b) is completely solved in integral homology for the complex Grassmannians in [3] (pages 1072-1073), and the \mathbb{Z}_2 -cohomology ring for the infinite unoriented Grassmannians is known ([1] page 83 and [5] page 52) and some of the finite unoriented Grassmannians ([5] page 51). In Part III it is indicated, using intersection products, that the formulas in [3] for the complex case are valid also in \mathbb{Z}_2 cohomology for the real unoriented case. The \mathbb{Z}_2 cohomology products in the unoriented Schubert varieties can be determined also, using the induced map in cohomology of their embeddings in Grassmannians.

The intersection methods used are a first step in finding products in integral cohomology of oriented and unoriented Grassmannians, but this is a much more complicated problem (mainly because of signs) and will not be looked at in this paper.

PART I - DEFINITIONS AND NOTATION

Grassmannian Manifolds and Mappings Between Them

1.1 Definition: i) The real unoriented finite Grassmannian $G_{k,n}$ is the set of k dimensional planes through the origin (call them k -planes) in R^{k+n} , with topology given as follows:

Let $V_{k,n}$ be the set of ordered k -tuples of linearly independent vectors in R^{k+n} . $V_{k,n}$ is an open subset of $R^{k \times (n+k)}$ and thus inherits its topology. Define an equivalence relation \sim on $V_{k,n}$ as $A \sim B$ (where A and B are $k \times (k+n)$ matrices) if there is a linear transformation of the row space of A onto itself which maps the rows of A to the rows of B (i.e., $A \sim B$ if they have the same row space).

$G_{k,n}$ = the quotient of $V_{k,n}$ by \sim defines the topology for $G_{k,n}$.

ii) The real oriented finite Grassmannian $\tilde{G}_{k,n}$ is the set of oriented k -planes (through the origin) in R^{k+n} with topology given as follows.

Define an equivalence relation \approx on $V_{k,n}$ as above except that the linear transformation must have positive determinant.

$\tilde{G}_{k,n}$ = the quotient of $V_{k,n}$ by \approx defines the topology for $\tilde{G}_{k,n}$.

1.2 Notation: i) A representation for a k -plane P in R^{k+n} is a $k \times (k+n)$ matrix A having row space P .

ii) A representation for an oriented k -plane P in R^{k+n} is as above, with the additional condition that the orientation determined by the ordered row vectors of A coincides with the orientation of P .

1.3 Remarks: i) $\tilde{G}_{k,n}$ is always orientable, but $G_{k,n}$ is not in general (from [2] only if $k+n$ is even).

ii) There is an involution T on $\tilde{G}_{k,n}$ which takes an oriented plane P to the same plane with opposite orientation.

Notation: Call T the antipodal map, and if $Q = T(P)$ say that Q is antipodal to P . (In $\tilde{G}_{1,n} = S^n$, T is the usual antipodal map.)

iii) There is a double covering $\psi : \tilde{G}_{k,n} \rightarrow G_{k,n}$ which takes an oriented plane P to the same plane P with orientation ignored (ψ identifies antipodal points).

Note: (ii) and (iii) show that \tilde{G} is a Z_2 bundle over G .

1.4 Mappings between the Grassmannians:

The notation used here will be used throughout the paper.

i) For $i \leq p$, \bar{e}_i will denote the i -th standard basis vector of R^p .

ii) Define $\hat{j} : R^p \rightarrow R^q$ for $p \leq q$ by $\hat{j}(\bar{e}_i) = \bar{e}_i$, $1 \leq i \leq p$. For $p = k+n$ and $q = k+n'$ ($n' \geq n$), \hat{j} induces embeddings

$j : G_{k,n} \rightarrow G_{k,n'}$ by $j(P) =$ the k -plane $\hat{j}(P)$ in $R^{k+n'}$, and

$\tilde{j} : \tilde{G}_{k,n} \rightarrow \tilde{G}_{k,n'}$ by \tilde{j} (oriented plane P) = the plane $\hat{j}(P)$ in $\hat{j}(P)$ in $R^{k+n'}$ with orientation induced by the orientation of P .

iii) Define $\hat{l} : R^p \rightarrow R^q$, $p \geq q$, by $\hat{l}(\bar{e}_i) = \bar{e}_{q-p+i}$, $1 \leq i \leq p$. For $p = k+n$ and $q = k'+n$ ($k' \geq k$), \hat{l} induces embeddings

$l : G_{k,n} \rightarrow G_{k',n}$ by $l(P) =$ the k' -plane in $R^{k'+n}$ spanned by $\bar{e}_1, \dots, \bar{e}_{k'-k}$ and $\hat{l}(P)$

and

$\tilde{l} : \tilde{G}_{k,n} \rightarrow \tilde{G}_{k',n}$ by \tilde{l} (oriented plane P) = the k' -plane in $R^{k'+n}$ spanned by $\bar{e}_1, \dots, \bar{e}_{k'-k}$ and $\hat{l}(P)$, with orientation determined by the orientation of $\langle \bar{e}_1, \dots, \bar{e}_{k'-k} \rangle$ followed by the orientation of $\hat{l}(P)$ induced by the orientation of P .

Note: if $k' \geq k$ and $n' \geq n$ then the diagrams

$$\begin{array}{ccc} G_{k,n} & \xrightarrow{\quad} & G_{k,n'} \\ \downarrow l & \searrow j & \downarrow l \\ G_{k',n} & \xrightarrow{\quad} & G_{k',n'} \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{G}_{k,n} & \xrightarrow{\quad} & \tilde{G}_{k,n'} \\ \downarrow \tilde{l} & \searrow \tilde{j} & \downarrow \tilde{l} \\ \tilde{G}_{k',n} & \xrightarrow{\quad} & \tilde{G}_{k',n'} \end{array} \quad \text{commute.}$$

iv) For each k, n there are homeomorphisms

$\perp : G_{k,n} \rightarrow G_{n,k}$ taking a plane P to its orthogonal complement P^\perp in R^{k+n} ,

and

$\tilde{\perp} : \tilde{G}_{k,n} \rightarrow \tilde{G}_{n,k}$ taking an oriented plane P to its orthogonal complement P^\perp oriented so that the product orientation on $P \times P^\perp$ coincides with the standard orientation on R^{k+n} .

1.5 Definition: i) The infinite unoriented Grassmannian G_k is the union limit (via the embeddings $j : G_{k,n} \rightarrow G_{k,n'}$) as $n \rightarrow \infty$ of $G_{k,n}$.

ii) The infinite oriented Grassmannian \tilde{G}_k is the union limit (via the embeddings \tilde{j}) as $n \rightarrow \infty$ of $\tilde{G}_{k,n}$.

Note: These limits exist since for $n \leq n_1 \leq n_2$, the diagrams

$$\begin{array}{ccc}
 G_{k,n} & & \\
 \downarrow j & \searrow j & \\
 G_{k,n_1} & \xrightarrow{j} & G_{k,n_2}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \tilde{G}_{k,n} & & \\
 \downarrow \tilde{j} & \searrow \tilde{j} & \\
 \tilde{G}_{k,n_1} & \xrightarrow{\tilde{j}} & \tilde{G}_{k,n_2}
 \end{array}
 \quad \text{commute.}$$

By the note following 1.4 (iii) above, the embeddings i and \tilde{i} induce embeddings

$$i : G_k \rightarrow G_{k'}, \quad \text{and} \quad \tilde{i} : \tilde{G}_k \rightarrow \tilde{G}_{k'}, \quad \text{for } k \leq k'.$$

It is easy to see that here also the diagrams

$$\begin{array}{ccc}
 G_k & & \\
 \downarrow i & \searrow i & \\
 G_{k_1} & \xrightarrow{i} & G_{k_2}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 \tilde{G}_k & & \\
 \downarrow \tilde{i} & \searrow \tilde{i} & \\
 \tilde{G}_{k_1} & \xrightarrow{\tilde{i}} & \tilde{G}_{k_2}
 \end{array}
 \quad \text{commute for } k \leq k_1 \leq k_2.$$

Thus the following definitions are valid:

iii) The universal unoriented Grassmannian G is the union limit as $k \rightarrow \infty$ of G_k .

iv) The universal oriented Grassmannian \tilde{G} is the union limit as $k \rightarrow \infty$ of \tilde{G}_k .

Schubert Cells and Schubert Varieties

1.6 Definition: i) A Schubert symbol σ is a k -tuple of integers $(\sigma_1, \dots, \sigma_k)$ such that $0 \leq \sigma_1 \leq \dots \leq \sigma_k$. The "dimension" of σ is $|\sigma| = \sigma_1 + \dots + \sigma_k$.

ii) Given a Schubert symbol σ such that $\sigma_k \leq n$, define the Schubert "cell" e_σ in $G_{k,n}$ to be the set of k -planes P in \mathbb{R}^{k+n} satisfying the following conditions (called the Schubert conditions

associated with σ):

$$\text{dimension of } P \cap \hat{j}(R^{\sigma_i+i}) = i$$

and dimension of $P \cap \hat{j}(R^{\sigma_i+i-1}) = i - 1$ for $i = 1, \dots, k$.

Notation: P_σ is the k -plane $\langle e_{\sigma_1+1}, \dots, e_{\sigma_k+k} \rangle$ which lies in e_σ .

Remark: The validity of the terms "dimensions" and "cell" above will be shown below.

1.7 Theorem: Let $k > 0$ and $n > 0$ be given.

i) For any Schubert symbol $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$, the set $e_\sigma \subset G_{k,n}$ is an open cell of dimension $|\sigma|$.

ii) The collection of all such e_σ gives $G_{k,n}$ a cell complex structure.

pf: see [1] Section 6.

(i) is proved in 1.19.

1.8 Proposition: For e_σ a Schubert cell in $G_{k,n}$, $\psi^{-1}(e_\sigma)$ is a pair of antipodal open cells in $\tilde{G}_{k,n}$, each homeomorphic under ψ to e_σ .

pf: In general, if $f : X \rightarrow Y$ is a double covering and $A \subset Y$ is contractible, then $f^{-1}(A)$ is a pair of disjoint sets each homeomorphic under f to A . The proposition then follows from the fact that e_σ is an open cell and thus contractible, and that ψ identifies antipodal points.

Notation: Let e_σ^+ denote the half of $\psi^{-1}(e_\sigma)$ containing the plane P_σ with orientation $\langle \bar{e}_{\sigma_1+1}, \dots, \bar{e}_{\sigma_k+k} \rangle$ and e_σ^- denote the other half ($T(e_\sigma^+)$).

1.9 Corollary: Let $k > 0$ and $n > 0$ be given.

The collection of open cells e_σ^+ and e_σ^- where σ runs over

all Schubert symbols of the form $(\sigma_1, \dots, \sigma_k \leq n)$ gives $\tilde{G}_{k,n}$ a cell complex structure.

pf: By 1.8, the cell structure for $G_{k,n}$ given in 1.7 pulls back via ψ to a cell structure for $\tilde{G}_{k,n}$ made up of the cells e_σ^+, e_σ^- for all appropriate Schubert symbols.

1.10 Claim: With the above cell structures for $G_{k,n}$ and $\tilde{G}_{k,n}$, the maps $\underline{1}, j, l, \tilde{1}, \tilde{j}, \tilde{l}, T$ and ψ are cellular. Their actions on cells are

$$\begin{aligned} \underline{1}(e_\sigma) &= e_\sigma, & \text{where } \sigma_1' &= \text{the number of } j \text{ s.t. } \sigma_j \geq i \\ j(e_\sigma) &= e_\sigma & \text{considered as a cell in } G_{k,n}, \\ l(e_\sigma) &= e_\sigma, & \text{where } \sigma' = (\underbrace{0, 0, \dots, 0}_{k' - k}, \sigma_1, \sigma_2, \dots, \sigma_k) \\ \tilde{j}(e_\sigma^+) &= e_\sigma^+, & \tilde{l}(e_\sigma^+) &= e_\sigma^+, \\ \tilde{j}(e_\sigma^-) &= e_\sigma^- & \text{and } \tilde{l}(e_\sigma^-) &= e_\sigma^+, \text{ for } \sigma' \text{ as above} \\ T(e_\sigma^+) &= e_\sigma^-, & \psi(e_\sigma^+) &= \psi(e_\sigma^-) = e_\sigma \\ \tilde{1}(e_\sigma^+) &= e_\sigma^+, & \text{or } e_\sigma^-, & \text{from above, depending on } \sigma. \end{aligned}$$

The statements about $j, \tilde{j}, l, \tilde{l}, T$ and ψ are easily verified from the definitions. The statements about $\underline{1}$ and $\tilde{1}$ are not so easy to verify--one way is to go to graph coordinates--but since they are not used in any important way, they will not be proved here.

1.11 Remark: By 1.10, the cell structures for $G_{k,n}$ and $\tilde{G}_{k,n}$ yield in the union limit CW-complex structures for G_k and \tilde{G}_k which in turn yield CW-complex structures for G and \tilde{G} . (The CW properties are easy to check, e.g., [1] page 79.)

1.12 Definition: Let $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$, a Schubert symbol, be given.

i) The unoriented Schubert variety $\Omega(\sigma)$ in $G_{k,n}$ is the closure of e_σ in $G_{k,n}$.

ii) The oriented Schubert variety $\tilde{\Omega}(\sigma)$ in $\tilde{G}_{k,n}$ is $\psi^{-1}(\Omega(\sigma)) =$ the closure of $e_\sigma^+ \cup e_\sigma^-$ in $\tilde{G}_{k,n}$.

1.13 Remark: i) $\Omega(\sigma)$ in $G_{k,n}$ is the set of k -planes P in R^{k+n} satisfying the conditions $\dim P \cap \hat{j}(R^{\sigma_i+i}) \geq i \quad i = 1, \dots, k$ and $\tilde{\Omega}(\sigma)$ in $\tilde{G}_{k,n}$ is the set of oriented k -planes in R^{k+n} satisfying the same conditions.

ii) if $n' \geq n$ then $\Omega(n, n, \dots, n)$ in $G_{k,n'} = j(G_{k,n})$ and $\tilde{\Omega}(n, n, \dots, n)$ in $\tilde{G}_{k,n'} = \tilde{j}(\tilde{G}_{k,n})$.

iii) if $k' \geq k$ then $\Omega(0, 0, \dots, 0, \underbrace{n, n, \dots, n}_{k \text{ times}})$ in $G_{k',n}$ is $l(G_{k,n})$ and $\tilde{\Omega}(0, \dots, 0, \underbrace{n, n, \dots, n}_{k \text{ times}})$ in $\tilde{G}_{k',n}$ is $\tilde{l}(\tilde{G}_{k,n})$.

iv) Suppose $k' \geq k$ and $n' \geq n$.

then $j : G_{k,n} \rightarrow G_{k,n'}$ induces a homeomorphism between $\Omega(\sigma)$ in $G_{k,n}$ and $\Omega(\sigma)$ in $G_{k,n'}$ and

$l : G_{k,n} \rightarrow G_{k',n}$ induces a homeomorphism between $\Omega(\sigma)$ in $G_{k,n}$ and $\Omega(\sigma')$ in $G_{k',n}$ where $\sigma' = (\underbrace{0, 0, \dots, 0}_{k' - k}, \sigma_1, \sigma_2, \dots, \sigma_k)$. Similarly in the oriented case.

1.14 Claim: Let $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$ and $\sigma' = (\sigma_1', \dots, \sigma_k' \leq n)$ be given. Then $\Omega(\sigma) \subset \Omega(\sigma')$ in $G_{k,n} \Leftrightarrow \sigma_i \leq \sigma_i' \quad \forall i$ and $\tilde{\Omega}(\sigma) \subset \tilde{\Omega}(\sigma')$ in $\tilde{G}_{k,n} \Leftrightarrow \sigma_i \leq \sigma_i' \quad \forall i$.

pf: Using 1.13 (i):

Suppose σ and σ' are Schubert symbols as above and $\sigma_i \leq \sigma_i' \quad \forall i$. Then $P \in \Omega(\sigma)$

\Rightarrow dimension of $P \cap \hat{j}(R^{\sigma_i+i}) \geq i \quad \forall i$

\Rightarrow dimension of $P \cap j(R^{\sigma'_i + i}) \geq i \quad \forall i$, since $\hat{j}(R^{\sigma_i + i}) \subset \hat{j}(R^{\sigma'_i + i})$.

Thus $P \in \Omega(\sigma')$, i.e., for $\sigma_i \leq \sigma'_i \quad \forall i$, $P \in \Omega(\sigma)$

$\Rightarrow P \in \Omega(\sigma') \quad \text{or} \quad \Omega(\sigma) \subset \Omega(\sigma')$.

Suppose σ and σ' are Schubert symbols as in the claim, and $\sigma_{i_0} > \sigma'_{i_0}$

for some i_0 . Then $P \in e_\sigma \subset \Omega(\sigma)$

\Rightarrow dimension of $P \cap \hat{j}(R^{\sigma_{i_0} + i_0 - 1}) = i_0 - 1$

\Rightarrow dimension of $P \cap \hat{j}(R^{\sigma'_{i_0} + i_0}) \leq i_0 - 1$, since $\hat{j}(R^{\sigma_{i_0} + i_0 - 1}) \subset \hat{j}(R^{\sigma'_{i_0} + i_0})$.

Thus $P \notin \Omega(\sigma')$, i.e., $P \in e_\sigma \subset \Omega(\sigma)$

$\Rightarrow P \notin \Omega(\sigma')$, or $\Omega(\sigma) \not\subset \Omega(\sigma')$.

The same arguments work for the oriented case.

1.15 Remark: We can consider all the Schubert varieties as finite dimensional subcomplexes of the universal complexes \tilde{G} or \tilde{G} , since the inclusion maps j, \tilde{j}, l and \tilde{l} are homeomorphisms on any Schubert variety. The inclusions between the Schubert varieties can be shown by a diagram which is called the Hasse diagram. The diagram is valid for both oriented and unoriented Grassmannians.

Diagram 1.16 shows all Schubert varieties lying in $G_{4,4}$ up to dimension 8. Note the horizontal symmetry--it reflects the map \perp cellwise. There is also a vertical symmetry--the other half of the diagram for Schubert varieties in $G_{4,4}$ can be obtained by reflecting across dimension 8. This comes from Poincaré Duality.

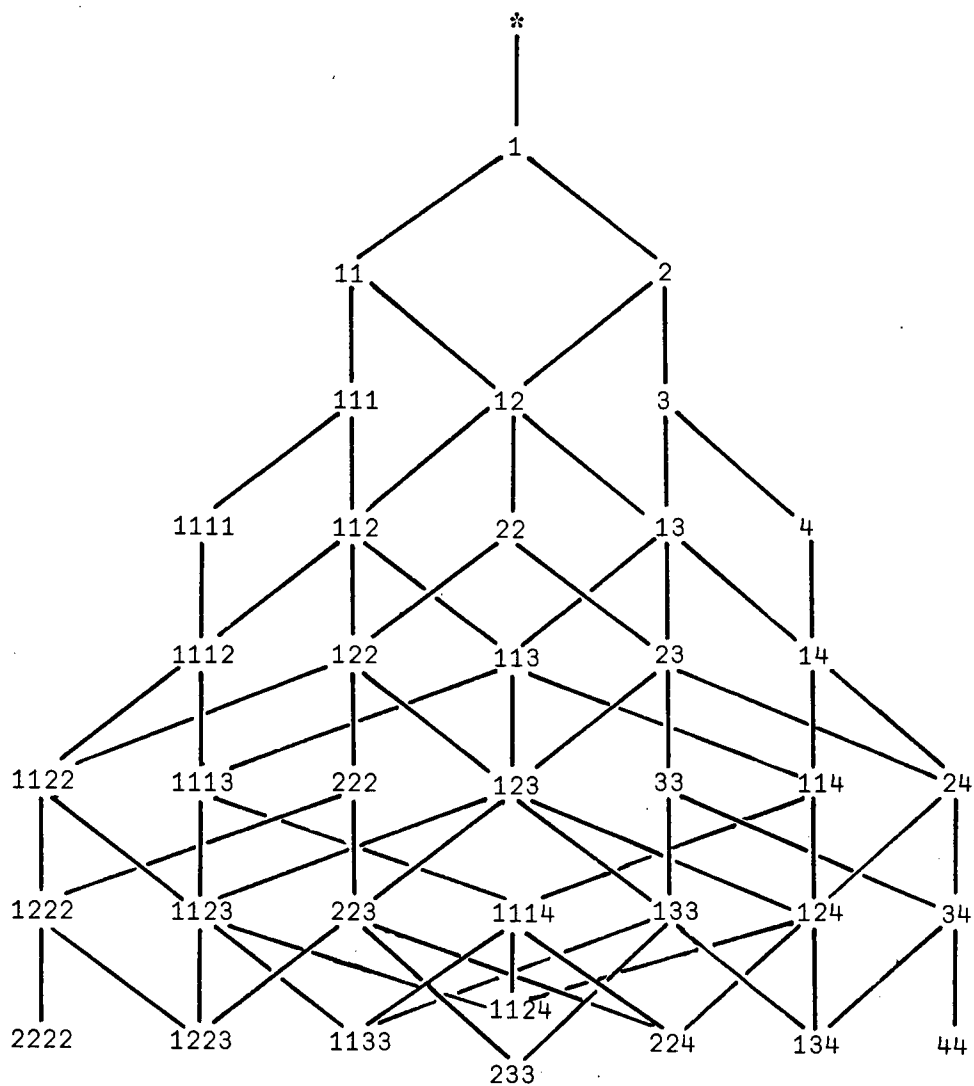
Graph Coordinates and Chain Complexes for the Grassmannians

1.17 Definition: Graph coordinates for $G_{k,n}$.

Fix the standard basis on $R^{k,n}$. Let P be a k -plane in $R^{k,n}$. Define the graph coordinates centred at P as follows:

let P^\perp be the orthogonal complement of P in R^{n+k}

1.16 The Hasse diagram (notation as in 2.13):



and

$h : P \times P^\perp \rightarrow R^{k+n}$ the isomorphism

$h(v, w) = v + w$ (vector addition)

Define

$\varphi_P : R^{k \times n} \approx \text{Hom}(P, P^\perp) \rightarrow G_{k,n}$ by

$\varphi_P(f : P \rightarrow P^\perp) = h(\text{graph}(f))$, a k -plane in R^{k+n}

i) This gives the graph coordinates centred at P for $G_{k,n}$.

ii) If P is an oriented plane, define

$\varphi_P : R^{k \times n} \rightarrow \tilde{G}_{k,n}$ as above, giving $\varphi_P(f : P \rightarrow P^\perp)$

the orientation induced by the orientation of P via the isomorphisms

$P \approx \text{graph}(f) \approx h(\text{graph}(f))$. This gives the graph coordinates centred at P for $\tilde{G}_{k,n}$.

1.18 Remark: In the graph coordinates centred at P_σ , there is a natural choice of isomorphism $\text{Hom}(P_\sigma, P_\sigma^\perp) \approx R^{k \times n}$ as follows. Give P_σ the ordered basis $\{\bar{e}_{\sigma_1+1}, \bar{e}_{\sigma_2+2}, \dots, \bar{e}_{\sigma_k+k}\}$ and give P_σ^\perp the ordered basis of the remaining basis vectors in R^{k+n} (in increasing order also).

Let $f : P_\sigma \rightarrow P_\sigma^\perp$ correspond to the matrix of f in the above bases.

Then with this correspondence, the map $\varphi_\sigma (= \varphi_{P_\sigma}) : R^{k \times n} \rightarrow G_{k,n}$ (or $\tilde{G}_{k,n}$)

is given by $A = (a_{ij}) \rightarrow$ the k -plane with representation (see 1.2)

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\sigma_1} & 1 & a_{1\sigma_1+1} & \dots & a_{1\sigma_2} & 0 & a_{1\sigma_2+1} & \dots & a_{1\sigma_k} & 0 & a_{1\sigma_k+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2\sigma_1} & 0 & a_{2\sigma_2+1} & \dots & a_{2\sigma_2} & 1 & a_{2\sigma_2+1} & \dots & a_{2\sigma_k} & 0 & a_{2\sigma_k+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & 0 & & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \dots & a_{k\sigma_k} & 0 & a_{k\sigma_k+1} & \dots & a_{k\sigma_2} & 0 & a_{k\sigma_2+1} & \dots & a_{k\sigma_k} & 1 & a_{k\sigma_k+1} & \dots & a_{kn} \end{pmatrix}$$

and $\varphi_{\sigma}(R^{k \times n})$ is the set of planes having such a representation. (The above is valid for $\tilde{G}_{k,n}$ also.)

Notation: Call $\varphi_{\sigma}(R^{k \times n})$ U_{σ} in $G_{k,n}$ and U_{σ}^{+} in $\tilde{G}_{k,n}$.

Give $R^{k \times n}$ the ordered basis $\{A_{11}, A_{12}, \dots, A_{1n}, A_{21}, \dots, A_{kn}\}$ where A_{ij} = matrix with 1 in ij^{th} position and zeros everywhere else.

1.19 Claim: Let $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$ be a Schubert symbol. Then

i) $e_{\sigma} \in U_{\sigma}$ in $G_{k,n}$ and $\varphi_{\sigma}^{-1}(e_{\sigma})$ is the plane

$$\langle A_{11}, A_{12}, \dots, A_{1\sigma_1}, A_{21}, A_{22}, \dots, A_{2\sigma_2}, A_{31}, \dots, A_{k1}, \dots, A_{k\sigma_k} \rangle$$

in $R^{k \times n}$. Call this plane L_{σ} .

ii) In $\tilde{G}_{k,n}$, $e_{\sigma}^{+} \in U_{\sigma}$ and $\varphi_{\sigma}^{-1}(e_{\sigma}^{+})$ is the same plane L_{σ} above.

pf: It is easy to see that a plane $P \in G_{k,n}$ is in

$e_{\sigma} \Leftrightarrow$ it has a representation of the form

$$\begin{pmatrix} * & * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 & * & \dots & * & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & * & \dots & 0 & \dots & \dots & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & * & \underbrace{0}_{\sigma_1+1} & * & \dots & * & \underbrace{0}_{\sigma_2+2} & * & \dots & * & \dots & * & 0 & * & \dots & * & \underbrace{1}_{\sigma_k+k} & 0 \end{pmatrix}$$

column column column

$\Leftrightarrow P = \varphi_{\sigma}(A)$ for some matrix $A = (a_{ij})$ such that

$$a_{ij} = 0 \text{ for } i \geq \sigma_i + i + 1.$$

This proves part (i). Part (ii) is proved the same way, noting that P_{σ} lies in e_{σ}^{+} .

1.20 Definition: Let $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$ be a Schubert symbol. Define the orientation of e_σ in $G_{k,n}$ and e_σ^+ in $\tilde{G}_{k,n}$ to be that induced by L_σ , where L_σ is given the orientation determined by the ordered basis vectors which span it. Carry over the orientation of e_σ^+ to e_σ^- via T .

1.21 Remark: i) We now have cell structures of oriented cells for $G_{k,n}$ and $\tilde{G}_{k,n}$.

ii) The maps j, \tilde{j}, l and \tilde{l} (also ψ and T) all preserve the cell orientations, so that these cell orientations induce cell orientations in G_k, \tilde{G}_k, G and \tilde{G} .

1.22 Definition: i) Define the graded group $C(G_{k,n})$ as $C_r(G_{k,n}) =$ the free abelian group generated by the Schubert cells e_σ in $G_{k,n}$ of dimension r . Define $C(G_k)$ and $C(G)$ similarly.

ii) Define the graded group $C(\tilde{G}_{k,n})$ as the free abelian group generated by all the Schubert cells e_σ^+ and e_σ^- in $\tilde{G}_{k,n}$ of dimension r . Define $C(\tilde{G}_k)$ and $C(\tilde{G})$ similarly.

Remark: These graded groups are the basis of chain complexes for the Grassmannians arising from the oriented cell decompositions.

PART II - ADDITIVE HOMOLOGY STRUCTURE

In this section, the additive structure of the integral homology of $G_{k,n}$ and $\tilde{G}_{k,n}$ will be studied by computing directly from cell orientations the boundary homomorphism d for the chain complex $(G_{k,n}; \mathbb{Z})$ arising from the Schubert cell decomposition.

The formula for d (Theorem 2.9) is the main result aimed for, and then some low dimensional homology groups for $G_{k,n}$ and $\tilde{G}_{k,n}$ are derived.

General Theory for Cell Complexes

In general, given a CW-complex K together with an orientation for each cell in K , there is a homomorphism $d : C^i(K; \mathbb{Z}) \rightarrow C^{i-1}(K; \mathbb{Z})$ making C into a chain complex so that the homology of $(C, d) \otimes G$ is $H_*(K, G)$ for any group G . (This can be done, for example, by triangulating K so that the closure of each cell is a finite subcomplex, and using simplicial methods to define d , see [6].)

2.1 Definition: Let K be a CW-complex with oriented cells, and d the resulting boundary homomorphism. For e_α and e_β cells of dimension r and $r - 1$ respectively, define the incidence number $[e_\alpha, e_\beta]$ to be the e_β -coefficient of de_α .

For the oriented Grassmannians, the only possibilities will be $[e_\alpha, e_\beta] = 0$ or ± 1 , which the following facts will take care of:

2.2 Let K be a CW-complex of dimension n and e_α, e_β oriented cells of dimension r and $r - 1$ respectively.

i) If $\bar{e}_\alpha \cap e_\beta = \emptyset$ then $[e_\alpha, e_\beta] = 0$

ii) If there is an open set U in K and a homeomorphism $\phi : (R^n, H^+, L) \rightarrow (U, e_\alpha \cap U, e_\beta \cap U)$ where H^+ is a linear r -half space and L a linear $r - 1$ space bounding H^+ , then

$$[e_\alpha, e_\beta] = \begin{cases} 1, & \text{if the orientation of } H^+ \text{ induced by } e_\alpha \\ & \text{coincides with the orientation of } L \text{ induced} \\ & \text{by } e_\beta \text{ followed by the normal of } L \text{ in } \bar{H}^+. \\ -1 & \text{otherwise.} \end{cases}$$

iii) Let e_γ also be an oriented cell of dimension r , and suppose there is a homeomorphism

$$\phi : (R^n, H^+, H^-, L) \rightarrow (U, e_\alpha \cap U, e_\gamma \cap U, e_\beta \cap U)$$

where L is a linear $r - 1$ space and $H = H^+ \cup L \cup H^-$ is a linear r space. Give H^+ and H^- orientations induced by e_α and e_γ respectively.

$$\text{Then } [e_\alpha, e_\beta] = \begin{cases} [e_\gamma, e_\beta] & \text{if there is a change of orientation across} \\ & L \text{ in } H \\ -[e_\gamma, e_\beta] & \text{otherwise.} \end{cases}$$

These facts will not be proved here, but can be checked by going to a simplicial definition of d (see, for example, 6).

2.3 If K and K' are CW-complexes with oriented cells, and

$f : K \rightarrow K'$ is a cellular continuous map taking r cells to r cells preserving orientations, then f induces a chain map

$$f_\# : C_i(K) \rightarrow C_i(K') \quad \forall i. \quad (\text{i.e., } f_\# \circ d = d \circ f_\#)$$

and if f is surjective or injective then so is $f_\#$.

This is a very weak form of the naturality of the chain complex $C_i(k)$.

Determining the Incidence Number for the Boundary Map for $C_i(\tilde{G}_{k,n})$

The manner in which the general theory is applied to $\tilde{G}_{k,n}$ is best explained by an example.

2.4 Example: The boundary map in $C_i(\tilde{G}_{2,2})$:

Rather than use the cell orientations given in 1.20, it is convenient to define the orientations as we proceed. Consider the graph coordinates centred at $P_{(0,0)}$, where

$$\begin{array}{c} \phi(0,0) \ a_{11} \ a_{12} \\ \ a_{21} \ a_{22} \end{array} = \text{the (oriented) row space of} \begin{pmatrix} 1 & 0 & a_{11} & a_{12} \\ 0 & 1 & a_{21} & a_{22} \end{pmatrix}.$$

It is easy to see that the cell $(0,1)^+$ corresponds to the linear half space $\langle A_{21} \rangle |_{a_{21} > 0}$ (recall A_{ij} from 1.18), and

$(0,1)^-$ corresponds to $\langle A_{21} \rangle |_{a_{21} < 0}$

$(0,2)^+$ corresponds to $\langle A_{21}, A_{22} \rangle |_{a_{22} > 0}$

$(1,1)^+$ corresponds to $\langle A_{11}, A_{21} \rangle |_{a_{11} < 0}$

$(1,1)^-$ corresponds to $\langle A_{11}, A_{21} \rangle |_{a_{11} > 0}$

$(0,2)^-$ corresponds to $\langle A_{21}, A_{22} \rangle |_{a_{22} < 0}$

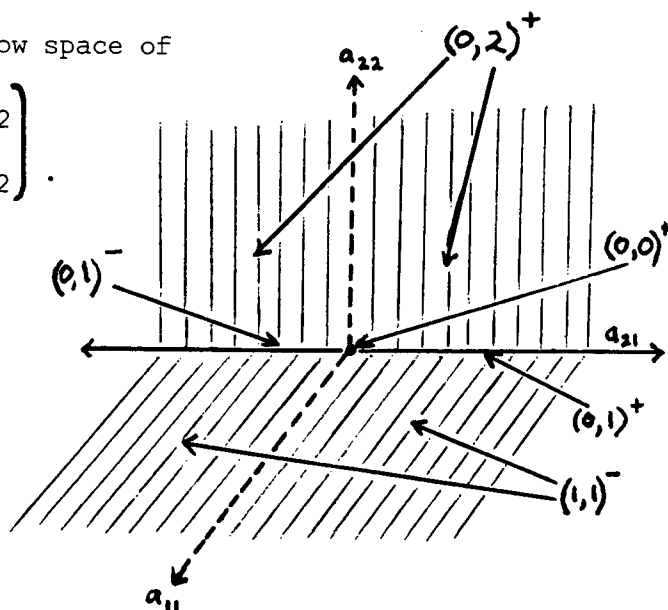


Fig. 1

Giving these linear half spaces orientations, it is easy to determine from them the incidence numbers $[(1,1)^+, (0,1)^+]$, $[(1,1)^+, (0,1)^-]$, etc. and thus obtain $d(1,1)^+$, $d(1,1)^-$, $d(0,2)^+$

and $d(0, 2)^-$. In order to determine $d(1, 2)^+$ and $d(1, 2)^-$, a more complicated procedure is needed since $(1, 2)^+$ is not a linear half space in these coordinates (it is the set $\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A : |A| = 0 \right\}$). We must go to new graph coordinates where the cell $(1, 2)^+$ is linear as well as $(1, 1)^+$ and $(0, 2)^+$ ($P_{(0,1)}$ will work), and keep track of the orientations induced by the cells $(1, 1)^+$, $(1, 1)^-$, $(0, 2)^+$ and $(0, 2)^-$ on their corresponding linear subspaces in the new coordinates. This is the main technicality in the proof of Proposition 2.8. Here, we can in a similar manner obtain $d(1, 2)^+$, $d(1, 2)^-$, $d(2, 2)^+$ and $d(2, 2)^-$ which together with the above will yield the homology of $\tilde{G}_{2,2}$ stated in Table II.

Note: The boundary map calculated above will not necessarily be the same as in 2.9 as the choice of cell orientations might be different.

2.5 Notation: For $\sigma = (\sigma_1, \dots, \sigma_k)$ a Schubert symbol, denote by $\sigma - \delta_s$ the symbol $(\sigma_1 - \delta_{1s}, \dots, \sigma_k - \delta_{ks})$ where δ_{js} is the Kronecker δ .

Note: $\sigma - \delta_s$ is a Schubert symbol $\Leftrightarrow \sigma_s \leq \sigma_s - 1$.

2.6 Lemma: Let $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$ and σ' be Schubert symbols such that $|\sigma| = |\sigma'| + 1$. Then in $\tilde{G}_{k,n}$,

$$[e_{\sigma}^+, e_{\sigma'}^+] = [e_{\sigma}^+, e_{\sigma'}^-] = 0 \text{ unless } \sigma' = \sigma - \delta_s \text{ for some } s.$$

pf: This follows from (the proof of) 1.14 and 2.2 (i) and the fact that if $\sigma' \neq \sigma - \delta_s$ for any s then $\sigma_{i_0} = \sigma'_{i_0} + 1$ for some i_0 .

2.7 Lemma: Let $\sigma = (\sigma_1, \dots, \sigma_k \leq n)$ and σ' be Schubert symbols such that $\sigma' = \sigma - \delta_s$ for some s . Then in $\tilde{G}_{k,n}$, (with cell orientations as defined in 1.20) we have

$$i) [e_{\sigma}^+, e_{\sigma'}^+] = (-1)^{\sigma_s + k - s} [e_{\sigma}^-, e_{\sigma'}^+]$$

$$ii) [e_{\sigma}^+, e_{\sigma'}^+] = (-1)^{1+k-s+\sigma_{s+1}+\sigma_{s+2}+\dots+\sigma_k}$$

pf: This is proved by going to graph coordinates where the lemma takes the following form.

2.8 Proposition: Suppose σ and σ' are Schubert symbols as above and $|\sigma'| = r$. Then in the graph coordinates centred at $P_{\sigma'}$ we have:

i) $e_{\sigma'}^+ \subset U_{\sigma'}$ and is a coordinate r -plane in $\mathbb{R}^{k \times n}$. As in 1.19 call this plane $L_{\sigma'}$.

ii) $e_{\sigma'}^+ \cap U_{\sigma'}$ and $e_{\sigma'}^- \cap U_{\sigma'}$ are coordinate $r+1$ half planes (call them H^+ and H^-) such that $H^+ \cup L_{\sigma'} \cup H^-$ is a coordinate $r+1$ plane.

iii) The orientations of H^+ and H^- induced by $e_{\sigma'}^+$ and $e_{\sigma'}^-$ are the same

$\Leftrightarrow \sigma_s + k - s$ is odd.

iv) The orientation of H^+ induced by $e_{\sigma'}^+$ coincides with the orientation of $L_{\sigma'}$ (as defined in 1.20) followed by the normal into H^+
 $\Leftrightarrow k - s + \sigma_{s+1} + \sigma_{s+2} + \dots + \sigma_k$ is odd.

Note: The above, together with 2.2 (ii) and (iii) immediately proves 2.7

pf: Using the above notation:

i) This is 1.19.

ii) $P \in e_{\sigma'}^+ \cap U_{\sigma'}$,

\Leftrightarrow it has representations (see 1.2 and 1.18) of the forms

$$a_{ij} = 0 \text{ for } j \geq \sigma_i + i + 1 \text{ and } a_{s, \sigma_s} > 0.$$

Conversely, for any such matrix A , $P = \varphi_{\sigma}(A)$ has representations of the forms X_1 and X_2 .

Similarly, $P \in e_{\sigma}^{-} \cap U_{\sigma} \Leftrightarrow T(P)$ has a representation of the form X_1 and P has a representation of the form X_2

\Leftrightarrow there is an orientation reversing linear transformation taking the X_1 representation to an X_2 representation

$\Leftrightarrow \otimes < 0 \Leftrightarrow P = \varphi_{\sigma}(A)$ where $A = (a_{ij})$ is a $k \times n$ matrix with $a_{ij} = 0$ for $j \geq \sigma_i + i + 1$ and $a_{s, \sigma_s} > 0$.

This proves (ii).

From (ii) we have the diagram shown. For (iii) and (iv) we must find the orientations of H^+ and H^- induced by e_{σ}^+ , which can be done by finding the Jacobian of the maps $\varphi_{\sigma}^{-1} \circ \varphi_{\sigma}$, and $\varphi_{\sigma}^{-1} \circ T \circ \varphi_{\sigma}$. (H^+ and H^- have natural orientations given by their ordered basis $\langle A_{11}, \dots, A_{1\sigma_1}, A_{21}, \dots, A_{k1}, \dots, A_{k\sigma_k} \rangle$. The orientation induced by e_{σ}^+ will be the same if $\varphi_{\sigma}^{-1} \circ \varphi_{\sigma}$ has Jacobian with positive

determinant, and the orientation induced by e_{σ}^- will be the same if $\varphi_{\sigma}^{-1} \circ T \circ \varphi_{\sigma}$ has Jacobian with positive determinant.) To write these maps coordinatewise, we must see how to go from one representation to another.

Given $P \in U_{\sigma} \cap U_{\sigma}$, let v_1, \dots, v_k be rows of the X_1 type representation for P . To obtain a representation of the form X_2 ,

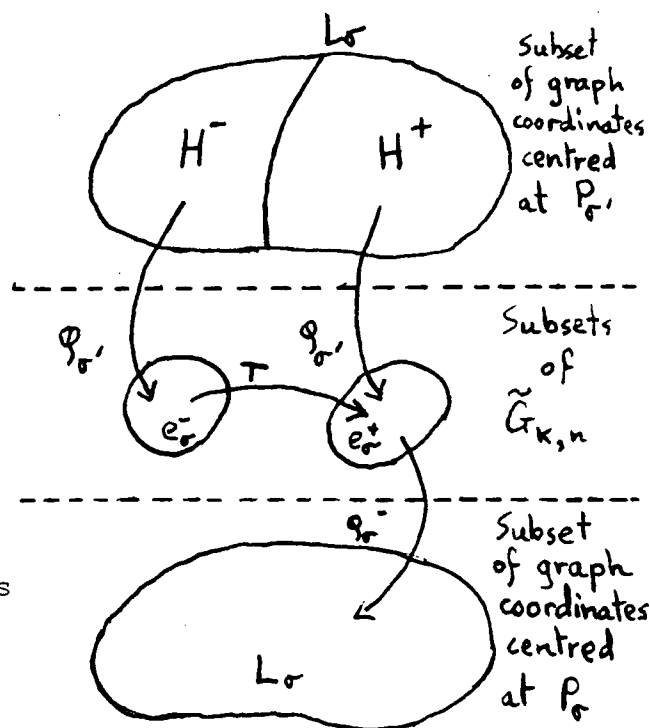


Fig. 2

use rows w_1, \dots, w_k where the w_i 's are obtained in the following way: write the $*$'s in X_1 as a_{ij} in the appropriate manner ($A = (a_{ij})$ will be $\phi_{\sigma'}^{-1}(P)$ and a_{s,σ_s} will be \otimes --see 1.18). Then $w_s = v_s/a_{s,\sigma_s}$ and $w_i = v_i - (a_{i\sigma_s}/a_{s\sigma_s})(v_s)$ $i \neq s$.

Note: the determinant of this transformation is $1/a_{s\sigma_s}$, and thus is orientation preserving for $a_{s\sigma_s} > 0$ and orientation reversing if $a_{s\sigma_s} < 0$.

Working out this linear transformation in coefficients, we get

$$\phi_{\sigma'}^{-1}(P) = (a_{lm}) \rightarrow (b_{ij}) = \phi_{\sigma}^{-1}(P) \text{ where}$$

$$b_{ij} = \begin{cases} \frac{1}{a_{s\sigma_s}} & \text{for } i = s, j = \sigma_s \\ \frac{-a_{i\sigma_s}}{a_{s\sigma_s}} & \text{for } i \neq s, j = \sigma_s \\ \frac{a_{sj}}{a_{s\sigma_s}} & \text{for } i = s, j \neq \sigma_s \\ a_{ij} - \frac{a_{sj} \cdot a_{i\sigma_s}}{a_{s\sigma_s}} & \text{for } i \neq s, j \neq \sigma_s \end{cases}$$

2.8a. For convenience call this map f . Then $f|_{a_{s\sigma_s} > 0}$ is $\phi_{\sigma'}^{-1} \circ \phi_{\sigma}$ and $f|_{a_{s\sigma_s} < 0}$ is $T \circ \phi_{\sigma'}^{-1} \circ \phi_{\sigma}$. We must now find the

determinant of the Jacobian when we restrict f to L_{σ} .

For $f_{ij}(a_{lm}) = b_{ij}$, the partial derivatives are

$$\frac{\partial f_{ij}}{\partial a_{lm}} = \delta_{ij} \delta_{jm} \text{ for } l \neq s, m \neq \sigma_s, i \neq s \text{ and } j \neq \sigma_s.$$

Thus restricting to these coefficients gives us the identity

matrix so for the determinant we need only worry about the

$k - s + \sigma_s \times k - s + \sigma_s$ matrix

$$\frac{\partial f_{ij}}{\partial a_{lm}} \quad \text{where } i = s \text{ or } j = \sigma_s, \quad l = s \text{ or } m = \sigma_s, \text{ and } j \leq \sigma_i \\ \text{and } m \leq \sigma_l.$$

This matrix is

$$\begin{array}{c} \partial f_{s1} \\ \partial f_{s2} \\ \vdots \\ \partial f_{s\sigma_s} \\ \partial f_{s+1\sigma_s} \\ \vdots \\ \partial f_{k\sigma_s} \end{array} \begin{bmatrix} \partial a_{s1} & \partial a_{s2} & \cdot & \cdot & \cdot & \partial a_{s\sigma_s} & \partial a_{s+1\sigma_s} & \cdot & \cdot & \partial a_{k\sigma_s} \\ (a_{s\sigma_s})^{-1} & & & & & * & & & & \\ & (a_{s\sigma_s})^{-1} & \bigcirc & & & * & & & & \\ & & \cdot & & & \vdots & & & & \\ & & \cdot & & & \vdots & & & & \\ & & \cdot & & (a_{s\sigma_s})^{-1} & * & & & & \\ & & & & & -(a_{s\sigma_s})^{-2} & & & & \\ & & & & & * & -(a_{s\sigma_s})^{-1} & & & \\ & & & & & \vdots & \cdot & & & \\ & & & & & * & \bigcirc & & & \\ & & & & & \underbrace{\quad}_{\text{the } \partial a_{s\sigma_s} \text{ column}} & & & & -(a_{s\sigma_s})^{-1} \end{bmatrix}$$

2.8b. The determinant of $Jf|_{L_\sigma}$ is thus $(-1)^{k-s+1}/(a_{s\sigma_s})^{\sigma_s+k-s+1}$.

2.8(iii). From 2.8b and 2.8a above we have that the orientations on H^+ and H^- agree

$\Leftrightarrow \sigma_s + k - s$ is odd.

2.8(iv). Comparing first the orientation

$\langle A_{11}, \dots, A_{1\sigma_1}, A_{21}, \dots, A_{2\sigma_2}, \dots, A_{k1}, \dots, A_{k\sigma_k} \rangle$ of H^+ with the orientation

$\langle A_{11}, \dots, A_{1\sigma_1}, A_{21}, \dots, A_{s1}, \dots, A_{s\sigma_s-1}, A_{s+1}, \dots, A_{k1}, \dots, A_{k\sigma_k} \rangle$ of L_σ , followed by $A_{s\sigma_s}$ (the normal into H^+) we have agreement

$\Leftrightarrow \sigma_{s+1} + \sigma_{s+2} + \dots + \sigma_k$ is even. Comparing the above orientation for

H^+ , with that induced by e_σ^+ , by 2.8b we have agreement
 $\Leftrightarrow (-1)^{k-s}$ is odd. Combining the two, we have agreement
 $\Leftrightarrow k - s + \sigma_{s+1} + \sigma_{s+2} + \dots + \sigma_k$ is odd. Q.E.D.

2.9 Theorem: The boundary map d in the chain complex for $\tilde{G}_{k,n}$ with cell orientations as in 1.20 is

$$d(e_\sigma^+) = \sum_{\substack{s \text{ s.t.} \\ \sigma_{s-1} \leq \sigma_s - 1}} (-1)^{1+k-s+\sigma_{s+1}+\dots+\sigma_k} (e_{\sigma-\delta_s}^+ + (-1)^{k-s+\sigma_s} e_{\sigma-\delta_s}^-)$$

and

$$d(e_\sigma^-) = Td(e_\sigma^+)$$

pf: This follows directly from 2.7 and 2.8 and the fact that T preserves cell orientations.

2.10 Corollary: The boundary map d in the chain complex for $G_{k,n}$ with cell orientations as in 1.20 is

$$d(e_\sigma) = \sum_{\substack{s \text{ s.t.} \\ \sigma_{s-1} \leq \sigma_s - 1}} (-1)^{1+k-s+\sigma_{s+1}+\dots+\sigma_k} (1 + (-1)^{k-s+\sigma_s}) e_{\sigma-\delta_s}.$$

pf: This follows from 2.9 and the fact that $\psi : \tilde{G}_{k,n} \rightarrow G_{k,n}$ maps e_σ^+ and e_σ^- to e_σ preserving orientation.

2.11 Remark: For j, l, \tilde{j} and \tilde{l} the embeddings in 1.4, the induced chain maps commute with d (from 1.21 and 2.3) so that the above formulas are valid in G_k and \tilde{G}_k , and also in G and \tilde{G} (if we think of each Schubert symbol σ as starting with a nonzero integer σ_1 to avoid the problem of having an infinite number of σ_i).

Some Low Dimensional Examples

Finding the homologies of the unoriented and oriented Grassmannians and Schubert varieties reduces via 2.9 and 2.10 to algebraic computation which will be carried out over \mathbb{Z} in some examples below. In general, homology over other groups can then be determined using the Universal Coefficients theorem, but in the following case it is easier to compute the \mathbb{Z}_2 homology directly from the chain complex:

2.12 Theorem: $H_r(G; \mathbb{Z}_2) = C_r(G; \mathbb{Z}_2)$ for all r , and the same is true for G_k and $G_{k,n}$ for all k and n .

pf: From 2.10, the boundary map d is $0 \bmod 2$ in all dimensions.

The method used in the examples is to find in dimension r a set of free generators for the group of cycles (denoted Z_r) and write out the boundaries $d(C_{r+1})$ (denoted B_r) in terms of these generators. The homology $H_r = Z_r/B_r$ is then the set of generators of Z_r together with relations given by setting the boundary elements to zero. The main difficulty is in looking for a set of free generators for Z_r , as it is not always clear whether or not a set of cycles spans the whole of Z_r (although to simplify things, linear independence in the examples given is obvious, and it is easy to determine what the rank of Z_r should be). In all the cases worked out, the above point has been settled by inspection, which in higher dimensions is not possible.

2.12a Note: In $\tilde{G}_{k,n}$, when writing boundary elements $d\sigma^+$ and $d\sigma^-$ (where $|\sigma| = r + 1$) in terms of generators for Z_r , if $d\sigma^+ = \pm d\sigma^-$ then we need only worry about $d\sigma^+$. Thus in writing the boundaries in terms

of generators of Z_r , some Schubert symbols yield two expressions and some only one.

In Table I and in the examples 2.15, a shortened notation will be used.

2.13 Notation:

i) Any Schubert symbol $\sigma = (\sigma_1, \dots, \sigma_k)$ will be written $\sigma_1\sigma_2 \dots \sigma_k$ (as $\sigma_k \leq 9$ in all cases, this will not give rise to confusion) and leading zeros will be omitted. The zero symbol will be denoted $*$.

ii) In \tilde{G} , the symbols $+\sigma$ (similarly $-\sigma$, $++\sigma$ and $--\sigma$) will refer to a linear combination of antipodal Schubert cells where the first sign refers to the coefficient of σ^+ and the second sign to that of σ^- . A Schubert symbol σ with one (or no) sign attached to it will refer to the positive cell σ^+ ;

e.g., $+ - 23$ refers to the chain element

$$(2, 3)^+ - (2, 3)^-$$

$23 + 14$ refers to the chain element

$$(2, 3)^+ + (1, 4)^+$$

TABLE I: THE BOUNDARY MAP IN $C(\tilde{G})$: (NOTATION FROM 2.13, " \rightarrow " REPRESENTS d).

	1	2	3	4	5	6	7	8	9	10
$G_{2,2}$	1 \rightarrow +*	2 \rightarrow -1 11 \rightarrow -1	12 \rightarrow -11 ++ 2							
$G_{2,3}$			3 \rightarrow -+ 2	13 \rightarrow -+ 12 -- 3	23 \rightarrow -+ 22 -+ 13	33 \rightarrow -- 23				
$G_{2,4}$				4 \rightarrow -- 3	14 \rightarrow -- 13 ++ 4	24 \rightarrow -- 23 +- 14	34 \rightarrow -- 33 ++ 24	44 \rightarrow +- 34		
$G_{2,5}$					5 \rightarrow -+ 4	15 \rightarrow -+ 14 -- 5	25 \rightarrow -+ 24 -+ 15	35 \rightarrow -+ 34 -- 25	45 \rightarrow -+ 44 -+ 35	55 \rightarrow -- 45
$G_{3,3}$			111 \rightarrow -+11	112 \rightarrow --111 +- 12	113 \rightarrow -+ 112 -+ 13 122 \rightarrow +- 112 -+ 22	123 \rightarrow -+ 122 -+ 113 +- 23 222 \rightarrow -- 122	223 \rightarrow -+ 222 ++123 133 \rightarrow -- 123 -+ 33	233 \rightarrow -- 223 -- 133	333 \rightarrow -+ 233	
$G_{3,4}$						114 \rightarrow -- 113 +- 14	124 \rightarrow -- 123 +- 114 -+ 24	134 \rightarrow -- 133 ++ 124 +- 34 224 \rightarrow -- 223 -- 124	144 \rightarrow +- 134 -+ 44 234 \rightarrow -- 233 ++ 224 ++ 134	244 \rightarrow +- 234 -- 144 334 \rightarrow -- 333 +- 234
$G_{3,5}$							115 \rightarrow -+ 114 -+ 15	125 \rightarrow -+ 124 -+ 115 +- 25	135 \rightarrow -+ 134 -- 125 -+ 35 225 \rightarrow -+ 224 ++ 125	145 \rightarrow -+ 144 -+ 135 +- 45 235 \rightarrow -+ 234 -- 225 -- 135
$G_{4,4}$				1111 \rightarrow --111	1112 \rightarrow --1111 ++ 112	1113 \rightarrow -+1112 -- 113 1122 \rightarrow +-1112 -- 122	1114 \rightarrow --1113 ++ 114 1123 \rightarrow -+1122 -+1113 ++ 123 1222 \rightarrow --1122 ++ 222	1124 \rightarrow --1123 +-1114 -- 124 1133 \rightarrow --1123 -- 133 1223 \rightarrow -+1222 ++1123 -- 223 2222 \rightarrow +-1222	1134 \rightarrow --1133 ++1124 ++ 134 1233 \rightarrow --1223 --1133 ++ 233 1224 \rightarrow --1223 --1124 ++ 224 2223 \rightarrow -+2222 -+1223	1144 \rightarrow +-1134 -- 144 1234 \rightarrow --1233 ++1224 ++1134 -- 234 1333 \rightarrow -+1233 -- 333 2233 \rightarrow --2223 +-1233 2224 \rightarrow --2223 +-1223

2.14 Theorem:

$$\begin{aligned} \text{i) } H_r(\tilde{G}_{1,n}; Z) &= \begin{cases} Z & \text{for } r = 0 \text{ and } n \\ 0 & \text{otherwise} \end{cases} \\ \text{ii) } H_r(G_{1,n}, Z) &= \begin{cases} Z & \text{for } r = 0 \text{ and } r = n \\ & \text{if } n \text{ is odd} \\ Z_2 & \text{for } r < n \text{ and odd} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

pf:

i) $Z_r(\tilde{G}_{1,n}, Z)$ are generated by the chain elements $(r)^+ + (r)^-$ for r odd, $r \leq n$, and $(r)^+ - (r)^-$ for $n \geq r > 0$ and even. For $r = 0$ Z_r is generated by $(0)^+$ and $(0)^-$. (This is easily seen from the Table I.) The boundary group B_r

(Image of $d : C_{r+1} \rightarrow C_r$) is generated by

$$\begin{aligned} (r)^+ + (r)^- & \text{ for } r \text{ odd } r < n \\ (r)^+ - (r)^- & \text{ for } 0 \leq r < n. \end{aligned}$$

Thus $H_r = Z_r/B_r$ is zero except for $r = 0$ and n where it is Z .

ii) From 2.10, the following can be verified

$$Z_r \text{ generated by } \begin{cases} (r) & r \text{ odd} \\ 0 & r > 0 \text{ even} \\ (0) & \text{(the 0-cell) for } r = 0 \end{cases}$$

$$B_r \text{ generated by } \begin{cases} 2(r) & r < n \text{ odd} \\ 0 & r \text{ even or } r = n \end{cases}$$

$$\text{Thus } Z_r/B_r \text{ is } \begin{cases} Z_2 & \text{for } r < n \text{ and odd} \\ Z & \text{for } r = 0 \text{ and } r = n \text{ if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

2.15 Examples: In these examples, Table I is used by inspection to find generators for Z_r . In labeling the cycles, no distinction is made between cycles of different dimensions (e.g., both $++1$ and $+-2$ are labeled "a"). As it will always be clear what dimension is being talked about, this should not cause any confusion. Notation is as described in 2.13.

i) $\tilde{G}_{2,3}$:

Dimension:	0	1	2	3	4	5	6
Generators	$a=*^+$	$a=++1$	$a=+-2$	$a=++3$	$a=++22$	$a=++23$	$a=+-33$
for Z_r :	$b=*^-$		$b=+-11$ $c=2-11$	$b=+-12$	$b=+-13$ $+ -22$		
B_r :	$a+b$	$d(2)=-a$	$d(3)=-a$ $d(12)=$ $2c-a-b$	$d(13)^+=-b-a$ $d(13)^-=b-a$ $d(22)=b$	$d(23)=-b$	$d(33)=-a$	

It is easy to see that $Z_r/B_r = \begin{cases} \mathbb{Z} & r = 0, 2, 4, 6 \\ 0 & \text{otherwise} \end{cases}$

(In dimension 2, Z_r/B_r has the relations $a \equiv 0$ $2c \equiv b$. Thus c generates Z_r/B_r and has order 0.)

In the next examples only the first homology groups are determined, as the rest can then be found using Universal Coefficients and Poincaré duality since $\tilde{G}_{k,n}$ is oriented for all k, n .

ii) $\tilde{G}_{3,3}$

Dimension (r):	0	1	2	3	4
$Z_r :$	$a=*^+$ $b=*^-$	$a=++1$	$a=+-2$ $b=+-11$ $c=2-11$	$a=++3$ $b=+-12$ $c=++111$	$a=++22$ $b=+-13$ $+-22$ $c=+-112$ $-+22$
$B_r :$	$a+b$	a	a $2c-a-b$ $111: b$	$b\pm a$ b $112: b\pm c$	b $113: b+c$ $122: c$

See note 2.8a.

$$Z_r/B_r = \begin{cases} \mathbb{Z} & \text{for } r = 0, 4 \\ \mathbb{Z}_2 & \text{for } r = 2 \\ 0 & \text{for } r = 1, 3 \end{cases}$$

iii) $\tilde{G}_{3,4}$:

Dimension (r):	3	4	5	6
Z_r :	a,b,c	a,b,c d=22+13-4 e=+-4	a=++23 b=++113 c=++122 d=122+113-23 e=+-14	a=+-33 b=+-222 c=++123 d=+-114-+24 e=++24--33
B_r :	4: a, b±a,b b±c	b,c,b+c 14: 2d-b+e-a	a, 24: a±e 114 b±e 222:±c 123: 2d-c-b+a	34: e 124: c±d 133: c±a 223: c±b

The unexplained elements in Z_r and B_r are from Examples (ii) and (i) above, using the same labels.

$$H_r = Z_r/B_r = \begin{cases} 0 & r = 3 \\ \mathbb{Z} \oplus \mathbb{Z} & r = 4 \\ \mathbb{Z}_2 & r = 5 \text{ and } 6. \end{cases}$$

The homology groups H_0 , H_1 and H_2 are the same as in $\tilde{G}_{3,3}$.

iv) $\tilde{\Omega}(1, 4)$:

Dimension (r):	1	2	3	4	5
C_r :	1^+	$11^+; 2^+$	$12^+; 3^+$	$13^+; 4^+$	14^+
	and antipodals				
Z_r :	$a=++1$	$a=+-2$ $b=+-11$ $c=2-11$	$a=++3$ $b=+-12$	$b'=++13--4$ $e=+-4$	$e=+-14$
B_r :	$11: a$ $2: a$	$12: 2c-a-b$ $3: a$	$13: b\pm a$ $4: a$	$14: b'$	
H_r :	0	Z	0	Z	Z

v) $\tilde{\Omega}(1, 2, 3)$:

Dimension (r):	1	2	3	4	5	6
C_r :	1^+	$11^+; 2^+$	$12^+; 3^+; 111^+$	$13^+; 22^+; 112^+$	$23^+; 122^+; 113^+$	123^+
	and antipodals					
Z_r :	a	a;b;c	a;b; $c=++111$	$a=++22$ $b=+-13+-22$ $c=+-112+-22$	$a=++23$ $b=++113$ $c=++122$ $d=122+113-23$	$c=++123$
B_r :	a	a;b;2c-a-b	$b\pm a; b; b\pm c$	b;b+c;c	2d-c-b+a	
H_r :	0	Z_2	0	Z	$Z\oplus Z\oplus Z$	Z

vi) $\tilde{G}_{4,4}$: Dimension 6

Cycles: a, b, c, d, e as in Example (iii) and in addition,

$$f = 1122 + 1113 - 114 - 222 + 24 - 33$$

$$g = ++1113--114$$

and $h = ++1122--222$.

Boundaries: $e, c \pm d, c \pm a, c \pm b$ as before, and in addition,

$$1114: g, \quad 1222: h$$

$$1123: 2f - h - g - e \pm c.$$

Thus in homology we have

$$e \equiv g \equiv h \equiv 0, \quad a \equiv b \equiv c \equiv d, \quad 2f \equiv c \quad \text{and} \quad 2c \equiv 0.$$

Thus f generates H_6 and $4f \equiv 0, \quad 2f \not\equiv 0$, so $H_6 = Z_4$.

Tables of Homology Groups of the Grassmannians

Table III tabulates the above results together with a few more that have been worked out in the above manner. By going to large enough Grassmannians, such results are valid for \tilde{G}_k and \tilde{G} as shown in Table III. Cohomology can be found using Universal Coefficients, and the results can be compared with those obtained using characteristic classes (see [1] pages 179 and 182). The copies of Z are generated by Pontrjagin classes and their products. Another method would be to use the cochain complex directly, where the incidence numbers defining δ (the coboundary map) would be $[e_\alpha, e_\beta] = [e_\beta, e_\alpha]$ from the boundary map. Going through the same procedure as in Example 2.15, explicit generators in terms of Schubert cell duals could be determined. In this way for instance it could be found which Schubert varieties represent the Pontrjagin classes.

Table II. Homology groups for $\tilde{G}_{k,n}$ where k and n are small:

	$\tilde{G}_{2,2}$	$\tilde{G}_{2,3}$	$\tilde{G}_{2,4}$	$\tilde{G}_{2,5}$	$\tilde{G}_{3,3}$	$\tilde{G}_{3,4}$	$\tilde{G}_{3,5}$	$\tilde{G}_{4,4}$
H_0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
H_1	0	0	0	0	0	0	0	0
H_2	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
H_3	0	0	0	0	0	0	0	0
H_4	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
H_5		0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
H_6		\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	0	\mathbb{Z}_4
H_7			0	0	0	0	\mathbb{Z}	0
H_8			\mathbb{Z}	\mathbb{Z}	0	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
H_9				0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_4
H_{10}				\mathbb{Z}		0	0	\mathbb{Z}_2
H_{11}						0	\mathbb{Z}	0
H_{12}						\mathbb{Z}	\mathbb{Z}_2	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
H_{13}							0	\mathbb{Z}_2
H_{14}							0	0
H_{15}							\mathbb{Z}	0
H_{16}								\mathbb{Z}

$$2.17 \text{ Assertion: } H_r(G_{2,n}) = \begin{cases} \mathbb{Z} & \text{for } r \text{ even and } r \neq n, \ r \leq 2n \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } r = n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

The method used for $G_{2,2}, \dots, G_{2,5}$ can be generalized easily.

Corollary: $H_r(G_2) = \mathbb{Z}$ for r even, 0 for r odd.

Table III. Low dimensional homology groups for \tilde{G}_k :

(Note: $H_r(\tilde{G}_k) = H_r(\tilde{G}_{k,r+1})$ since the embedding $\tilde{G}_{k,r+1} \rightarrow \tilde{G}_k$ covers all cells of \tilde{G}_k of dimension $r + 1$ or less. Also,

$H_r(\tilde{G}_{r+1}) = H_r(\tilde{G}_{r+2}) = \dots = H_r(\tilde{G})$ for the same reason.)

	H_0	H_1	H_2	H_3	H_4	H_5	H_6
\tilde{G}_1	\mathbb{Z}	0	0	0	0	0	0
\tilde{G}_2	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	0
\tilde{G}_3	\mathbb{Z}	0	\mathbb{Z}_2	0	\mathbb{Z}	\mathbb{Z}_2	0
\tilde{G}_4	\mathbb{Z}	0	\mathbb{Z}_2	0	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
\tilde{G}_5	\mathbb{Z}	0	\mathbb{Z}_2	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	\cdot	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
\tilde{G}_6	\mathbb{Z}	0	\mathbb{Z}_2	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	\cdot	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
\tilde{G}_7	\mathbb{Z}	0	\mathbb{Z}_2	\cdot	\cdot		$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
\cdot	\cdot	\cdot	\cdot	\cdot	\cdot		\cdot
\cdot	\cdot	\cdot	\cdot				\cdot
\cdot	\cdot	\cdot	\cdot				\cdot
\tilde{G}	\mathbb{Z}	0	\mathbb{Z}_2	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

In the unoriented Grassmannians and Schubert varieties, the computations are much simpler as there are only half as many cells to worry about and the boundary map is much simpler.

Table IV. Homology for the unoriented Grassmannians $G_{k,n}$ for small k and n :

	$G_{2,2}$	$G_{2,3}$	$G_{2,4}$	$G_{2,5}$	$G_{3,3}$	$G_{3,4}$
H_0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
H_1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
H_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2
H_3	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
H_4	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	\mathbb{Z}	$\mathbb{Z} \oplus \mathbb{Z}_2$
H_5		\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
H_6		0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
H_7			0	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
H_8			\mathbb{Z}	\mathbb{Z}	0	$\mathbb{Z} \oplus \mathbb{Z}_2$
H_9				\mathbb{Z}_2	\mathbb{Z}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
H_{10}				0		0
H_{11}						\mathbb{Z}_2
H_{12}						0

Note that there is Poincaré Duality in $G_{2,2}$, $G_{2,4}$ and $G_{3,3}$. This reflects the fact that $G_{k,n}$ is orientable whenever $k+n$ is even (see [2]).

Remark: The homology groups for $G_{k,n}$ have been determined in [7], but as this article was not available in Russian or English it was not possible to compare results.

PART III - HOMOLOGY AND COHOMOLOGY PRODUCTS

We now turn to the multiplicative structures. Only the Z_2 homology and cohomology products in the unoriented Grassmannians are studied, but the cohomology ring structure is determined entirely (3.16 and 3.17). The formulas describing the cup product are equivalent via Poincaré duality (described in terms of Schubert symbols in 3.7) to those describing products in Z cohomology of $G_{k,n}(C)$ (see [3]). In the form given, they can also be used to determine cup products in the unoriented Schubert varieties, and some examples are given.

The General Intersection Theory To Be Used

For M a manifold, there is a product theory for intersections of cycles in $H_*(M; Z_2)$ called the Lefschetz intersection

$$\cap_L : H_{n-a}(M; Z_2) \times H_{n-b}(M; Z_2) \rightarrow H_{n-a-b}(M; Z_2)$$

which is related to the cup product in cohomology in the following way.

3.1 Assertion: For $b = n - a$ above, the product \cap_L induces a map $D : H_{n-a}(M; Z_2) \times H_a(M; Z_2) \rightarrow Z \approx H_0(M; Z_2)$ which can be considered as a map

$$D : H_{n-a}(M; Z_2) \rightarrow \text{Hom}(H_a(M; Z_2) \supset Z_2) \approx H^a(M; Z_2)$$

by $\alpha \rightarrow \text{the map } f(\beta) = \alpha \cap_L \beta \in Z_2 \approx \sum_{\alpha \cap_L \beta = 1} \beta^*$

If $\alpha \cap_L \beta = \gamma$ in $H_*(M; Z_2)$ then $D(\alpha) \cup D(\beta) = D(\gamma)$ in $H^*(M; Z_2)$.

This is due to the Lefschetz intersection product being Poincaré dual to the cup product in the sense that the following diagram commutes:

$$\begin{array}{ccc}
 & \text{(with } \mathbb{Z}_2 \text{ coefficients)} & \\
 H^a(M) \times H^b(M) & \xrightarrow{\cup} & H^{a+b}(M) \\
 \uparrow D \cap [M] \quad \uparrow D \cap [M] & & \uparrow D \cap [M] \\
 H_{n-a}(M) \times H_{n-b}(M) & \xrightarrow{\cap_L} & H_{n-a-b}(M)
 \end{array}$$

where $\cap [M]$ is the cap product with the fundamental class of M (lying in $H_0(M; \mathbb{Z}_2)$) i.e. the Poincaré dual map.

3.2 Assertion: Let M be an n dimensional manifold with a cell complex structure, and e_α, e_β cells in the complex representing \mathbb{Z}_2 cycles α and β . Suppose there is continuous map $h : M \rightarrow M$ homotopic to the identity such that

for any cells $e_\alpha, \subset \overline{e_\alpha}$ and $e_\beta, \subset \overline{e_\beta}$, e_α is transverse to $h(e_\beta)$.

Then $\overline{e_\alpha} \cap h(\overline{e_\beta})$ is a \mathbb{Z}_2 cycle in M homologous to $\alpha \cap_L \beta$.

If $\overline{e_\alpha} \cap h(\overline{e_\beta}) = \emptyset$ then $\alpha \cap_L \beta' = 0$.

This is from general intersection theory (e.g., see [10]).

Simple Intersections in $G_{k,n}$ and the Poincaré Duality Map

A straightforward translation of 3.2 into Schubert cell terminology is given below (3.4) and, using it, the Poincaré duality map is described in terms of Schubert symbols (3.7) and some examples of explicit intersections are given.

3.3 Remark and Notation: From the chain complex $C_r(G_{k,n})$ associated with the Schubert cell decomposition (1.22) we obtain a mod 2 chain complex $C_r(G_{k,n}; \mathbb{Z}_2)$ and a mod 2 cochain complex

$$C^r(G_{k,n}; \mathbb{Z}_2) = \text{Hom}(C_r(G_{k,n}; \mathbb{Z}_2), \mathbb{Z}_2).$$

For $e_\sigma \in C_r(G_{k,n}; Z_2)$, σ a Schubert symbol, write the cochain element dual to e_σ as σ^* ;

i.e., $\sigma^* \in \text{Hom}(C^r(G_{k,n}; Z_2), Z_2)$ is the linear map

sending e_σ to 1 and

e_η to 0 for $\eta \neq \sigma$.

Since the cochain map δ is zero mod 2,

$$H^r(G_{k,n}; Z_2) = C^r(G_{k,n}; Z_2)$$

and $\{\sigma^* : |\sigma| = r\}$ is a basis for $H^r(G_{k,n}; Z_2)$ dual to the basis $\{e_\sigma : |\sigma| = r\}$ for $H_r(G_{k,n}; Z_2)$.

From here on, Z_2 homology and cohomology will be assumed unless otherwise stated.

3.4 Theorem: Let e_σ and e_η be cells in $G_{k,n}$ for σ and η Schubert symbols. Suppose there is an orthogonal linear transformation $\hat{\Phi} : R^{k+n} \rightarrow R^{k+n}$ inducing $\Phi : G_{k,n} \rightarrow G_{k,n}$ such that

i) For any $e_{\sigma'} \subset \bar{e}_\sigma$ and $e_{\eta'} \subset \bar{e}_\eta$, $e_{\sigma'}$ is transverse to $\Phi(e_{\eta'})$.

ii) $\bar{e}_\sigma \cap \Phi(\bar{e}_\eta) = \Phi_1 \bar{e}_{\sigma(1)} \cup \Phi_2 \bar{e}_{\sigma(2)} \cup \dots \cup \Phi_m \bar{e}_{\sigma(m)}$ for some orthogonal transformations Φ_1, \dots, Φ_m , where $\sigma(1) \dots \sigma(m)$ are distinct Schubert symbols of rank $|\sigma| + |\eta| - kn$.

Then in $H_*(G_{k,n})$, $e_\sigma \cap_L e_\eta$ (the Lefschetz intersection product) is $e_{\sigma(1)} + e_{\sigma(2)} + \dots + e_{\sigma(m)}$.

pf: This follows from 3.2 as $\Phi : G_{k,n} \rightarrow G_{k,n}$ is homotopic to the identity, and $\Phi_i \bar{e}_{\sigma(i)}$ is homologous to $\bar{e}_{\sigma(i)}$ (since Φ_i is also homotopic to the identity map).

3.5 Notation: Define $\hat{\Phi}^p : R^q \rightarrow R^q$, $p \leq q$ by $\hat{\Phi}^p(\bar{e}_i) = \begin{cases} \bar{e}_{p-i+1} & \text{for } i \leq p \\ \bar{e}_i & \text{for } i > p. \end{cases}$

If $q = k + n$, then $\Phi^P : G_{k,n} \rightarrow G_{k,n}$ is the induced homeomorphism.

3.6 Lemma: Let e_σ and e_η be Schubert cells in $G_{k,n}$ and $\Phi = \Phi^{k+n}$ as defined above. Then we have

- i) e_σ is transverse to Φe_η
- ii) $e_\sigma \cap e_\eta = \emptyset$ unless $\sigma_i + \eta_{k-i+1} \geq n \quad \forall i$.
- iii) if $\eta_i = n - \sigma_{k-i+1} \quad \forall i$ then $\bar{e}_\sigma \cap \Phi \bar{e}_\eta = \text{the point } \{P_\sigma\}$ in $G_{k,n}$. (Recall 1.6.)

pf: i) Suppose σ and η are Schubert symbols such that $\sigma_i + \eta_{k-i+1} \leq n - 1$ for some i .

If $P \in e_\sigma \cap \Phi e_\eta$, then

$$\text{dimension of } P \cap \hat{j}(R^{\sigma_i+i}) = i \quad (\hat{j} \text{ as in 1.4) and}$$

$$\text{dimension of } P \cap \hat{l}(R^{\eta_{k-i+1}+k-i+1}) = k - i + 1$$

$$\Rightarrow j(R^{\sigma_i+i}) \cap \hat{l}(R^{\eta_{k-i+1}+k-i+1}) \geq 1$$

$$\Rightarrow \sigma_i + i + \eta_{k-i+1} + k - i + 1 > n + k \quad \text{or} \quad \sigma_i + \eta_{k-i+1} > n - 1$$

which is a contradiction. Thus $e_\sigma \cap \Phi e_\eta = \emptyset$.

ii) Suppose σ and η are Schubert symbols such that $\sigma_i + \eta_{k-i+1} \geq n$ for all i .

Consider the graph coordinates centred at P_σ (see 1.17, 1.18 and 1.19). Since $e_\sigma \subset U_\sigma$, the intersection $e_\sigma \cap \Phi e_\eta$ lies entirely in U_σ , the domain of the graph coordinates. Recall that $L_\sigma \subset \mathbb{R}^{k \times n}$ is the linear subspace corresponding to e_σ .

Claim: $U_\sigma \cap \Phi e_\eta = (L_\sigma \cap \Phi e_\eta) \times L_\sigma \subset \mathbb{R}^{k \times n}$ in the above graph coordinates.

$$\text{pf: } L_\sigma = \{A = (a_{ij}) \text{ s.t. } a_{ij} = 0 \text{ for } j > \sigma_i\}$$

$$L_\sigma = \{A = (a_{ij}) \text{ s.t. } a_{ij} = 0 \text{ for } j \leq \sigma_i\}.$$

Suppose $A \in \Phi e_\eta \subset \mathbb{R}^{k \times n}$. The plane $\varphi_\sigma(A) = P$ is the row space of the matrix in 1.18 corresponding to A . Since P is in Φe_η , it must satisfy the Schubert conditions

$$\text{dimension of } P \cap \hat{l}(\mathbb{R}^{\eta_i + i}) = i \text{ and}$$

$$\text{dimension of } P \cap \hat{l}(\mathbb{R}^{\eta_i + i - 1}) = i - 1.$$

Looking at the matrix in 1.18, it can be seen that these conditions are independent of a_{ij} for $j \geq \sigma_i$ since $\eta_{k-i+1} + \sigma_i \geq n$.

$$\text{Thus } P \in \Phi e_\eta \cap U_\sigma$$

$$\Leftrightarrow P = \varphi_\sigma(A) \text{ for some } A = (a_{ij}) \text{ s.t. for } A' = (a'_{ij}) \text{ where}$$

$$a'_{ij} = \begin{cases} a_{ij} & \text{for } j \leq \sigma_i \\ 0 & \text{for } j > \sigma_i \end{cases}$$

$$\text{and } A' \in L_\sigma \cap \Phi e_\eta.$$

$$\text{Thus } \Phi e_\eta \cap U_\sigma = (L_\sigma \cap \Phi e_\eta) \times L_\sigma.$$

This claim proves 3.6(ii) since $L_\sigma \cap \Phi e_\eta$ has dimension $|\eta| - (kn - |\sigma|)$ and the intersection is transverse (since L_σ is orthogonal to L_σ and thus to $L_\sigma \cap \Phi e_\eta$).

iii) Suppose σ and η are Schubert symbols such that $\sigma_i + \eta_{k-i+1} = n \quad \forall i$.

$$P \in \Phi \bar{e}_\eta \cap \bar{e}_\sigma$$

$$\Leftrightarrow P \cap \hat{j}(\mathbb{R}^{\sigma_i + i}) \text{ in dimension } \geq i \text{ and}$$

$$P \cap \hat{l}(\mathbb{R}^{\eta_{k-i+1} + k - i + 1}) = \mathbb{R}^{n + k - \sigma_i - i + 1} \text{ in dimension } k - i \text{ for all } i$$

$$\Leftrightarrow P \supset \langle \bar{e}_{\sigma_i} \rangle \text{ for all } i$$

$$\Leftrightarrow P = P_\sigma.$$

Remark: Although 3.6 above shows that we can always make two cells \bar{e}_σ and \bar{e}_η in $G_{k,n}$ transverse (in the manner of 3.4(i)) using the orthogonal transformation Φ^{k+n} . However, although the intersection $\bar{e}_\sigma \cap \Phi \bar{e}_\eta$ must be a cycle homologous in $H_*(G_{k,n})$ to $e_\alpha \cap_L e_\beta$, it is not in general a union of orthogonal transformations of Schubert cells as required in 3.4(ii).

3.7 Theorem: In $G_{k,n}$, the intersection product \cap_L satisfies:

i) For Schubert symbols σ and η ,

$$e_\sigma \cap_L e_\eta = 0 \text{ unless } \sigma_i + \eta_{k-i+1} \geq n \quad \forall i.$$

ii) $\cap_L : H_{n-r} \times H_r \rightarrow H_0 \approx \mathbb{Z}_2$ is the map

$$(\sigma, \eta) \rightarrow \begin{cases} 1 & \text{if } \eta_i = n - \sigma_{k-i+1} \quad \forall i \\ 0 & \text{otherwise.} \end{cases}$$

iii) The Poincaré duality (inverse) map

$$D : H_{n-r} \rightarrow H^r \text{ is}$$

$$\sigma \rightarrow \eta^* \text{ where } \eta_i = n - \sigma_{k-i+1} \quad \forall i.$$

pf: (i) follows from 3.6(i) and 3.2.

(ii) follows from 3.6(ii), 3.2 and (i) above since if

$|\sigma| + |\eta| = n$ and $\eta_i = n - \sigma_{k-i+1}$ does not hold for all i then $\exists i_0$ such that $\eta_{i_0} + \sigma_{k-i_0+1} < n$.

(iii) is just another way of saying (ii).

This result is equivalent to Proposition, page 1072 in [3] which was first proved in [9].

3.8 Remark: The intersection product is unnatural in the sense that for $f : M \rightarrow M'$ a continuous (cellular) map, f_* does not preserve the intersection product. However, in cohomology, the induced map f^* does

preserve cup product, so that in the Grassmannians we have the following:

i) For $j : G_{k,n} \rightarrow G_{k,n'}$, $n' \geq n$, j as in 1.4,

$j^* : H^*(G_{k,n'}) \rightarrow H^*(G_{k,n})$ is the map

$$\sigma^* \rightarrow \sigma^*$$

(as can be seen from 3.3, 1.21 and 2.12) and

$$j^*(\sigma^* \cup \eta^*) = \sigma^* \cup \eta^*.$$

ii) For $l : G_{k,n} \rightarrow G_{k',n}$, $k' \geq k$, l as in 1.4,

$l^* : H^*(G_{k',n}) \rightarrow H^*(G_{k,n})$ is the map

$$\sigma^* \rightarrow (\sigma')^*$$

where $\sigma' = (\sigma_{n+k'-k+1}, \sigma_{n+k'-k+2}, \dots, \sigma_{n+k'})$

(as can be seen from 3.1, 1.21 and 2.12) and

$$l^*(\sigma^* \cup \eta^*) = (\sigma')^* \cup (\eta')^*$$

(η' defined from η as σ' is from σ).

We also have algebraic right inverses for the maps j^* and l^* defined as,

$$(j_*)^{-1} : H^*(G_{k,n}) \rightarrow H^*(G_{k,n'}) \quad n \leq n'$$

is the map $(\sigma^*) \rightarrow (j_*(\sigma))^*$ and

$$(l_*)^{-1} : H^*(G_{k,n}) \rightarrow H^*(G_{k',n}) \quad k \leq k'$$

is the map $\sigma^* \rightarrow (l_*(\sigma))^*$.

These maps are group homomorphisms (actually monomorphisms) but do not in general preserve cup product.

We now go to some specific examples of intersection product which use 3.4 directly. Checking transversality is in general more complicated to verify than in 3.6, so for the remainder of the paper we will assume the following.

3.9 Assumption: Let e_σ and e_η be Schubert cells in $G_{k,n}$, and suppose there is an orthogonal transformation $\hat{\Phi} : R^{k+n} \rightarrow R^{k+n}$ such that $\bar{e}_\sigma \cap \Phi \bar{e}_\eta$ (where $\Phi : G_{k,n} \rightarrow G_{k,n}$ is induced by $\hat{\Phi}$) is a (Z_2) cycle γ of dimension $|\sigma| + |\eta| - kn$. Then

i) e_σ is transverse to Φe_η

ii) there is an orthogonal transformation $\hat{\Phi}' : R^{k+n} \rightarrow R^{k+n}$ such that $\Phi \bar{e}_\eta = \Phi' \bar{e}_\eta$ and for any $e_{\sigma'} \subset \bar{e}_\sigma$ and $e_{\eta'} \subset \bar{e}_\eta$, $e_{\sigma'}$ is transverse to $\Phi' e_{\eta'}$.

iii) The cycle γ is $e_\sigma \cap_L e_\eta$.

Note: (ii) \Rightarrow (iii)

3.10 Examples: The same notation as in (2.16) will be used.

Remark about Schubert conditions: Recall (1.10) that for

$\sigma = (\sigma_1, \dots, \sigma_k)$ a Schubert symbol, the Schubert conditions associated with \bar{e}_σ in $G_{k,n}$ are $\dim P \cap \hat{j}(R^{\sigma_i+i}) \geq i \quad \forall i$.

If $\sigma_m = n$, then the above Schubert condition is redundant (since every k -plane in R^{k+n} intersects $\hat{j}(R^{n+i})$ in dimension i) and can be left out.

i) In $G_{2,2}$ look at $12 \cap_L 12$.

If we take the orthogonal transformation $\Phi^4 : G_{2,2} \rightarrow G_{2,2}$ (see 3.5), then 12 and $\Phi^4(12)$ satisfy 3.4(i) but not 3.4(ii), so we must use a different transformation. Take $\Phi^3 : G_{2,2} \rightarrow G_{2,2}$ (recall 3.5):

$$\begin{aligned} \overline{12} \cap \Phi^3(\overline{12}) = \{P \in G_{2,2} \text{ s.t. } \dim. P \cap \langle \bar{e}_1, \bar{e}_2 \rangle \geq 1 \\ \text{and } \dim. P \cap \langle \bar{e}_2, \bar{e}_3 \rangle \geq 1\} \end{aligned}$$

which is easily seen to be $X_1 \cup X_2$ where

$$X_1 = \{P \in G_{2,2} \text{ s.t. } \dim. P \cap \langle \bar{e}_2 \rangle = 1\}$$

$$X_2 = \{P \in G_{2,2} \text{ s.t. } P \subset \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle\}.$$

Since $X_2 = \overline{11}$ and $X_1 = \Phi(\overline{02})$ where $\hat{\Phi}$ is any orthogonal transformation taking \bar{e}_1 to \bar{e}_2 , by 3.9 we have $12 \cap_L 12 = 02 + 11$ in $G_{2,2}$.

ii) In $G_{2,3}$ look at $13 \cap_L 23$.

Here, we use $\Phi^4 : G_{2,3} \rightarrow G_{2,3}$:

$$\overline{13} \cap \Phi^4(\overline{23}) = \{P \in G_{2,3} \text{ s.t. } \dim. P \cap \langle \bar{e}_1, \bar{e}_2 \rangle \geq 1$$

$$\text{and } \dim. P \cap \langle \bar{e}_2, \bar{e}_3, \bar{e}_4 \rangle \geq 1\}$$

which is $X_1 \cup X_2$ where

$$X_1 = \{P \in G_{2,3} \text{ s.t. } \dim. P \cap \langle \bar{e}_2 \rangle = 1\}$$

$$X_2 = \{P \in G_{2,3} \text{ s.t. } \dim. P \cap \langle \bar{e}_1, \bar{e}_2 \rangle \geq 1$$

$$\text{and } P \subset \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \rangle\}$$

Since $X_2 = \overline{12}$ and $X_1 = \Phi(\overline{03})$ for Φ as in (i) above, we have

$$13 \cap_L 23 = 12 + 03 \text{ in } G_{2,3}.$$

iii) In $G_{3,3}$ look at $133 \cap_L 233$, using $\Phi^4 : G_{3,3} \rightarrow G_{3,3}$:

$$\overline{133} \cap \Phi^4(\overline{233}) = \{P \in G_{3,3} \text{ s.t. } \dim. P \cap \langle \bar{e}_1, \bar{e}_2 \rangle \geq 1$$

$$\text{and } \dim. P \cap \langle \bar{e}_2, \bar{e}_3, \bar{e}_4 \rangle \geq 1\}$$

which as in (ii) above is $X_1 \cup X_2$ where

$$X_1 = \{P \in G_{3,3} \text{ s.t. } P \cap \langle \bar{e}_2 \rangle = 1\}$$

$$X_2 = \{P \in G_{3,3} \text{ s.t. dim. } P \cap \langle \bar{e}_1, \bar{e}_2 \rangle \geq 1 \text{ and}$$

$$P \subset \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \rangle\}$$

However, here $X_2 = \overline{123}$ and $X_1 = \Phi(\overline{033})$ for Φ as above. Thus in $G_{3,3}$ we have

$$133 \cap_L 233 = 123 + 033.$$

iv) In $G_{3,3}$ look at $123 \cap_L 233$.

Let $\hat{\Phi}: R^6 \rightarrow R^6$ be an orthogonal transformation mapping $\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ to $\langle \bar{e}_2, \bar{e}_4, \bar{e}_5 \rangle$ and Φ the induced homeomorphism on $G_{3,3}$.

$$\overline{123} \cap \Phi(\overline{233}) = \{P \in G_{3,3} \text{ s.t. dim. } P \cap \langle \bar{e}_1, \bar{e}_2 \rangle \geq 1,$$

$$\text{dim. } P \cap \langle \bar{e}_2, \bar{e}_4, \bar{e}_5 \rangle \geq 1$$

$$\text{and dim. } P \cap \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \rangle \geq 2\}.$$

With a little difficulty, this can be seen to be $X_1 \cup X_2 \cup X_3$ where

$$X_1 = \{P \in G_{3,3} \text{ s.t. dim. } P \cap \langle \bar{e}_2 \rangle = 1$$

$$\text{and dim. } P \cap \langle \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \rangle \geq 2\}$$

$$X_2 = \{P \in G_{3,3} \text{ s.t. dim. } P \cap \langle e_1, e_2 \rangle \geq 1$$

$$\text{and dim. } P \cap \langle \bar{e}_1, \bar{e}_2, \bar{e}_4 \rangle \geq 2\}$$

$$X_3 = \{P \in G_{3,3} \text{ s.t. dim. } P \cap \langle e_1, e_2 \rangle \geq 1$$

$$\text{and } P \subset \langle \bar{e}_1, \dots, \bar{e}_5 \rangle\}.$$

Under suitable orthogonal transformations Φ_1 , Φ_2 , and Φ_3 ,

$$X_1 = \Phi_1(\overline{023}), \quad X_2 = \Phi_2(\overline{113}) \quad \text{and} \quad X_3 = \Phi_3(\overline{122}).$$

Thus $123 \cap_L 233 = 023 + 113 + 122$ in $H_*(G_{3,3})$.

v) In $G_{2,4}$ look at $34 \cap_L 24$, using $\Phi = \Phi^5$:

$$\overline{24} \cap \Phi \overline{34} = \{P \in G_{2,4} \text{ s.t. dim. } P \cap \hat{j}(R^3) \geq 1$$

$$\text{and dim. } P \cap \langle \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle \geq 1\}$$

which is $X \cup Y$ where

$$X = \{P \in G_{2,4} \text{ s.t. dim. } P \cap \langle \bar{e}_2, \bar{e}_3 \rangle \geq 1\} \quad \text{and}$$

$$Y = \{P \in G_{2,4} \text{ s.t. dim. } P \cap \hat{j}(R^3) \geq 1$$

$$\text{and } P \subset \hat{j}(R^5)\}.$$

$X = \Phi_1(\overline{14})$ and $Y = \overline{23}$ for some orthogonal transformation Φ_1 .

Thus $34 \cap_L 24 = 14 + 23$.

vi) In $G_{3,3}$ look at $222 \cap_L 033$.

By 3.7(i), $222 \cap_L 033 = 0$ in $G_{3,3}$, but let us try to make a nonempty intersection, using Φ^5 :

$$\overline{222} \cap \Phi^5(\overline{033}) = \{P \in G_{3,3} \text{ s.t. dim. } P \cap \langle \bar{e}_5 \rangle = 1$$

$$\text{and } P \subset \langle \bar{e}_1, \dots, \bar{e}_5 \rangle\}.$$

This is the cell $\Phi^5(\overline{022})$, but $|022| = 4$ whereas $|222| + |033| - 9 = 6 + 6 - 9 = 3$. Thus we cannot use 3.9, although 222 and $\Phi^5(\overline{033})$ (the open cells) are transverse, having empty intersection.

3.11 Remark: Using 3.7(iii) and 3.1 we can rewrite the above results as cup products in cohomology:

$$i) \text{ In } H^*(G_{2,2}), \quad 01^* \cup 01^* = 11^* + 02^*$$

ii) In $H^*(G_{2,3})$, $02^* \cup 01^* = 12^* + 03^*$

iii) In $H^*(G_{3,3})$ the same is true

iv) In $H^*(G_{3,3})$, $12^* \cup 01^* = 112^* + 22^* + 13^*$

v) In $H^*(G_{2,4})$, $02^* \cup 01^* = 12^* + 03^*$.

Note: $01^* \cup 01^* = 11^* + 02^*$ must hold in $H^*(G_{k,n})$ for all $k \geq 2$ and $n \geq 2$, since there are no other Schubert symbols of dimension 2.

Complicated Intersections and the General Formula

It is not always possible to intersect Schubert cells as in 3.10 so that 3.4 can be used--3.14 has such examples--and for the cases where it is not possible, a more complicated argument, such as the one developed below, is needed. The examples in 3.14 lead up to the main formula in 3.16.

3.12 Definition: For $k \leq k'$ define

$g : \{\text{subsets of } G_{k,n}\} \rightarrow \{\text{subsets of } G_{k',n}\}$
 as $X \subset G_{k,n} \rightarrow \{P \in G_{k',n} \text{ such that } P \text{ contains a } k\text{-plane } \hat{j}(P') \subset \hat{j}(\mathbb{R}^{k+n}) \text{ for some } P' \in X\}.$

3.13 Claim: If X is a cycle in $H_r(G_{k,n})$ homologous to $\sigma(1) + \dots + \sigma(m)$ where $\sigma(i)$ are Schubert symbols $i = 1, \dots, m$,

then $g(x)$ is a cycle in $H_{r+(k'-k)n}(G_{k',n})$ homologous to $\sigma'(1) + \dots + \sigma'(m)$ where

$$\sigma'(i) = (\sigma(i)_1, \sigma(i)_2, \dots, \sigma(i)_k, \underbrace{n, n, \dots, n}_{k' - k})$$

for $i = 1, \dots, m$.

This will not be proved, but its validity is suggested by the case $X = \Phi_1(\overline{\sigma(1)}) \cup \Phi_2(\overline{\sigma(2)}) \cup \dots \cup \Phi_m(\overline{\sigma(m)})$ for some orthogonal transformations Φ_1, \dots, Φ_m . Here, the claim is obviously true.

3.14 Examples: i) Look at $24 \cap_L 24$ in $G_{2,4}$, using $\Phi = \Phi^5$:

$$\overline{24} \cap \Phi(\overline{24}) = \{P \in G_{2,4} \text{ s.t. } \dim. P \cap \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle \geq 1$$

$$\text{and } \dim. P \cap \langle \bar{e}_3, \bar{e}_4, \bar{e}_5 \rangle \geq 1\}.$$

This is $X \cup Y$ where

$$X = \{P \in G_{2,4} \text{ s.t. } \dim. P \cap \langle \bar{e}_3 \rangle = 1\} \text{ and}$$

$$Y = \{P \in \overline{24} \cap \Phi(\overline{24}) \text{ s.t. } P \subset \langle \bar{e}_1, \dots, \bar{e}_5 \rangle\} = \overline{23} \cap \Phi(\overline{23})$$

Although $\overline{23}$ and $\Phi(\overline{23})$ do not satisfy the transversality conditions in 3.3 as cells in $G_{2,4}$, when considered as cells in $G_{2,3}$ they do (by 3.6). Thus $\overline{23} \cap \Phi(\overline{23})$ in $G_{2,3}$ is homologous to $23 \cap_L 23 = 13 + 22$ (by 3.11), so $Y = j(\overline{23} \cap \Phi(\overline{23}))$ in $G_{2,4}$ is homologous to $13 + 22$ also, since j preserves homology class.

X is $\Phi'(\overline{04})$ for some orthogonal transformation Φ' , so combining the homology classes determined by X and Y we obtain

$\overline{24} \cap \Phi \overline{24} = X \cup Y$ is a cycle in $G_{2,4}$ homologous to $13 + 22 + 04$. By 3.9 then,

$$24 \cap_L 24 = 13 + 22 + 04 \text{ in } H_*(G_{2,4}).$$

In cohomology, by 3.7(iii), this reads as

$$02^* \cup 02^* = 13^* + 22^* + 04^*.$$

ii) In $G_{4,4}$ look at $2334 \cap_L 2444$ using $\Phi = \Phi^7$:

$$\overline{2334} \cap \Phi \overline{2444} = \{P \in G_{4,4} \text{ s.t. } \dim. P \cap \hat{j}(R^3) \geq 1$$

$$\dim. P \cap \hat{j}(R^5) \geq 2$$

$$\dim. P \cap \hat{j}(R^6) \geq 3$$

$$\text{and } \dim. P \cap \langle e_5, e_6, e_7 \rangle \geq 1\}.$$

This is $X \cup Y$ where

$$X = \overline{2334} \cap \Phi \overline{2444} \cap j(G_{4,3}) = \overline{2333} \cap \Phi \overline{2333}, \text{ and}$$

$$Y = \{P \in \overline{2334} \cap \Phi \overline{2444} \text{ s.t. } \dim. P \cap \langle \bar{e}_5, \bar{e}_6 \rangle \geq 1\},$$

since for $P \in \overline{2334} \cap \Phi \overline{2444}$, if $P \cap \langle \bar{e}_5, \bar{e}_6 \rangle = 0$, then

$\dim. P \cap \hat{j}(R^7) = \dim. P \cap \hat{j}(R^6) + \dim. P \cap \langle \bar{e}_5, \bar{e}_6, \bar{e}_7 \rangle = 4$ so that P must be in X .

As in (i) above, $\overline{2333} \cap \Phi \overline{2333}$ in $G_{4,3}$ is homologous to $\overline{2333} \cap_L \overline{2333} = 1333 + 2233$ (by 3.11 and 3.7(iii)), so that in $G_{4,4}$ also $\overline{2333} \cap \Phi \overline{2333}$ is a cycle homologous to $1333 + 2233$.

$Y = g(\overline{233} \cap \Phi(\overline{133}) \subset G_{3,3})$ for g as in 3.12. (This can be easily checked.)

In $G_{3,3}$, $\overline{233}$ and $\Phi \overline{133}$ satisfy the transversality conditions in 3.3, thus $\overline{233} \cap \Phi(\overline{133})$ in $G_{3,3}$ is a cycle homologous to $123 + 033$ (by 3.10(iii)). Thus, by 3.13, $Y = g(\overline{233} \cap \Phi \overline{133} \subset G_{3,3})$ is a cycle in $G_{4,4}$ homologous to $1234 + 0334$.

Combining the homology classes of X and Y we have

$$\overline{2334} \cap_L \overline{2444} = 1333 + 2233 + 0334 + 1234 \text{ in } H_*(G_{4,4}).$$

In cohomology this reads as

$$112^* \cup 2^* = 1113^* + 1122^* + 114^* + 123^*.$$

The following two formulas (3.15 and 3.16) completely describe the cohomology ring structure in G as a ring generated by the Schubert cocycles $(0, 0, \dots, 0, a)^*$ over all integers $a \geq 1$.

3.15 Claim: Let $\sigma = (\sigma_1, \dots, \sigma_k)$ and $\eta = (0, 0, \dots, 0, \eta_k)$ be Schubert cycles in $G_{k,n}$.

i) For $j : G_{k,n} \rightarrow G_{k,n'}$, $n' \geq n$ and

$(j^*)^{-1}$ as in 3.8, we have

$$(j^*)^{-1}(\sigma^* \cup \eta^*) = \sigma^* \cup \eta^*.$$

ii) For $l : G_{k,n} \rightarrow G_{k',n}$, $k' \geq k$ and

$(l^*)^{-1}$ as in 3.8, we have

$$(l^*)^{-1}(\sigma^* \cup \eta^*) = (\sigma')^* \cup (\eta')^*$$

where $\sigma' = (0, 0, \dots, 0, \underbrace{\sigma_1, \dots, \sigma_k}_{k' - k})$ and $\eta = (0, 0, \dots, 0, \underbrace{\eta_k}_{n + k' - 1})$.

This can be proved by going to the intersection product via D (3.7) and generalizing

for (i), the way in which 3.10(ii) and (vi) give the same answer in cohomology (3.11) and

for (ii), the way in which 3.10(ii) and (iii) give the same answer in cohomology (3.11).

Note: In general it is not true that $(j^*)^{-1}$ and $(l^*)^{-1}$ preserve cup product--see 3.19(iv) and (v).

3.16 Claim: Let $\sigma = (\sigma_1 = 0, \sigma_2 > 0, \sigma_3, \dots, \sigma_k)$ and $\eta = (0, 0, \dots, 0, \eta_k)$ be Schubert symbols.

In $H^*(G)$ we have

$\sigma^* \cup \eta^* = \sum (\sigma')^*$, summed over all $\sigma' = (\sigma'_1, \dots, \sigma'_k)$, Schubert symbols of dimension $|\sigma'| = |\sigma| + \eta_k$ such that $\sigma_i \leq \sigma'_i \leq \sigma_{i+1}$ for $i = 1, \dots, k-1$ and $\sigma_k \leq \sigma'_k$.

Indication of proof: For σ and η as above, define $\sigma \cdot \eta$ as $\sum \sigma'$ for σ' as above.

Claim: $\sigma \cdot \eta$ splits into two sums $\sum_1 \sigma'$ and $\sum_2 \sigma'$ where \sum_1 is over σ' s.t. $\sigma' = (\sigma''_1, \sigma''_2 + 1, \sigma''_3 + 1, \dots, \sigma''_k + 1)$

for all σ'' in the sum for

$$(0, \sigma_2 - 1, \sigma_3 - 1, \dots, \sigma_k - 1) \cdot (0, 0, \dots, 0, \eta_k)$$

Σ_2 is over σ' s.t. $\sigma' = (\sigma_1'' + 1, \sigma_2'' + 1, \dots, \sigma_k'' + 1)$

for all σ'' in the sum for

$$(0, \sigma_2 - 1, \sigma_3 - 1, \dots, \sigma_k - 1) \cdot (0, 0, \dots, 0, \eta_k - 1).$$

pf: For σ' in the sum for $\sigma \cdot \eta$, either $\sigma_1' = \sigma_2$ in which case σ' is in Σ_2 , or $\sigma_1' < \sigma_2$ in which case σ' is in Σ_1 .

Conversely, for σ' in Σ_1 or Σ_2 , it is easy to see that σ' satisfies $\sigma_i \leq \sigma_i' \leq \sigma_{i+1}$ and $|\sigma'| = |\sigma| + \eta_k$.

By 3.15, the cup product $\sigma^* \cup \eta^*$ can be taken in $H^*(G_{k,n})$ without losing any terms. Go to the corresponding intersection product in $H_*(G_{k,n})$ via the Poincaré duality (3.7(iii)).

From here we can generalize the method and result in 3.14(ii), and we get

X is homologous to the dual (3.7(iii)) of Σ_2^* and Y is homologous to the dual (3.7(iii)) of Σ_1^* .

In this way, 3.16 can be proved by induction on $|\sigma|$ and η_k .

Note: The above formula in $H^*(G)$ holds in $H^*(G_{k,n}(\mathbb{C}); \mathbb{Z})$ where it is known as Pieri's formula (see [3]).

3.17 Claim: In $H^*(G)$ we have,

$$(\sigma_1, \sigma_2, \dots, \sigma_k)^* = \begin{vmatrix} \sigma(\sigma_k)^* & \sigma(\sigma_k + 1)^* & \dots & \sigma(\sigma_k + k - 1)^* \\ \sigma(\sigma_{k-1} - 1)^* & \sigma(\sigma_{k-1})^* & \dots & \sigma(\sigma_{k-1} + k - 2)^* \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(\sigma_1 - k + 1)^* & \sigma(\sigma_1 - k + 2)^* & \dots & \sigma(\sigma_1)^* \end{vmatrix}$$

the determinant, where the product is cup product, and

$\sigma(a) = (0, 0, \dots, 0, a)$ for $a \geq 0$ and 0 for $a < 0$.

Indication of proof: This follows algebraically from 3.16 by induction on the size of the matrix; e.g., for $k = 2$ we have

$$\begin{aligned}
 (\sigma_1, \sigma_2)^* &= \begin{vmatrix} \sigma(\sigma_2)^* & \sigma(\sigma_2 + 1)^* \\ \sigma(\sigma_1 - 1)^* & \sigma(\sigma_1)^* \end{vmatrix} \\
 &= \sigma(\sigma_2)^* \cup \sigma(\sigma_1)^* + \sigma(\sigma_1 - 1)^* \cup \sigma(\sigma_2 + 1)^* \\
 &= (\sigma_1, \sigma_2) + (\sigma_1 - 1, \sigma_2 + 1) + \dots + (0, \sigma_2 + \sigma_1) \\
 &\quad + (\sigma_1 - 1, \sigma_2 + 1) + (\sigma_1 - 2, \sigma_2 + 2) + \dots + (0, \sigma_2 + \sigma_1) \\
 &= (\sigma_1, \sigma_2)
 \end{aligned}$$

(The second line is from 3.16.)

Note: In $H^*(G_{k,n}(C), \mathbb{Z})$ this is called the determinantal formula (see [3]).

3.18 Remark: i) The results in 3.16 and 3.17 are also valid in $H^*(G_{k,n})$ if we use the projections

$$l^* : H^*(G) \rightarrow H^*(G_k) \quad \text{and}$$

$$j^* : H^*(G_k) \rightarrow H^*(G_{k,n}). \quad (\text{See 3.8.})$$

ii) They are also valid for $H^*(\Omega(\sigma))$ for any Schubert symbol σ , if we use

$$i^* : H^*(G_{k,n}) \rightarrow H^*(\Omega(\sigma)) \quad \text{where}$$

$$i : \Omega(\sigma) \rightarrow G_{k,n} \quad \text{is the embedding.}$$

3.19 Examples using 3.18 above: The shortened notation is used again, and for convenience we will drop the $*$'s, as everything is in cohomology.

ii) $H^*(\Omega(1, 3))$ cohomology products:

	1	11	2
1	11 + 2		
11	12	0	
2	12 + 3	13	13
12	13		
3	13		
13			

iii) $H^*(1, 1, 3)$ cohomology products:

	1	11	2
1	11 + 2		
11	111 + 12	0	
2	12 + 3	112 + 13	13
112	112	0	113
12	112 + 13	113	113
3	13	113	0
112	113		
13	113		
113			

The next two examples are of individual cup products in different Grassmannians.

iv) $124 \cup 2$:

In $H^*(G)$ it is

$$1224 + 1134 + 1125 + 234 + 225 + 144 + 135 + 126.$$

In $H^*(G_{3,4})$ it is $234 + 144$.

In $H^*(G_{3,5})$ it is $234 + 225 + 144 + 135$.

In $H^*(G_{4,4})$ it is $1224 + 1134 + 234 + 144$;

i.e., for $j : G_{3,4} \rightarrow G_{3,5}$ and $l : G_{3,4} \rightarrow G_{4,4}$

$(j^*)^{-1}$ and $(l^*)^{-1}$ do not preserve cup product.

v) $12 \cup 113 = (1 \cdot 2 + 3) \cup 113$:

In $H^*(G)$ it is

$11123 + 11114 + 1223 + 1133 + 1115 + 233 + 224 + 134 + 125$.

For $j : G_{3,3} \rightarrow G_{3,4}$ and $l : G_{3,3} \rightarrow G_{4,3}$

$(j^*)^{-1}$ and $(l^*)^{-1}$ do not preserve cup product.

Remark: In $G_{3,3}$, $12 \cup 113$ can be determined to be 233 (as above) by going to the Poincaré duals and using the intersection method as in 3.10.

3.20 Conclusion: The product structure in $H^*(G)$, $H^*(G_k)$ and $H^*(G_{k,n})$ is well known from characteristic classes (see [1]).

$H^*(G_k)$ is generated by the S-W classes $\omega_1, \dots, \omega_k$ of the tautological bundle, and

$H^*(G_{k,n})$ is generated by $\omega_1, \dots, \omega_k$ and $\tilde{\omega}_1, \dots, \tilde{\omega}_k$ under the conditions

$(1 + \omega_1 + \dots + \omega_k)(1 + \tilde{\omega}_1 + \dots + \tilde{\omega}_k) = 1$, where the $\tilde{\omega}_j$ are the S-W classes of the normal bundle.

It is known (see [2]) that $\tilde{\omega}_j$ is the cohomology class $\sigma(j)^*$ (from 3.17).

By the map $: G_{k,n} \rightarrow G_{n,k}$ which in cohomology must map ω_j to $\tilde{\omega}_j$, we can find the Schubert cocycle corresponding to ω_1 :

$(\sigma(j)) = (\underbrace{1, 1, \dots, 1}_{j \text{ times}})$ which we can call $\tau(j)$.

Thus $\tau(j)^* = \tilde{\omega}_j$, and $\tau(j)^*$ must generate $H^*(G_k)$. (This could be checked algebraically using 3.16 and 3.17.)

It can be determined algebraically from 3.16 and 3.17 that in G , (j) is obtained recursively from $\sigma(j)$ by

$$\begin{aligned} \tau(j)^* = & \sigma(j)^* + \sigma(j-1)^* \cup \tau(1)^* + \\ & + \sigma(j-2)^* \cup \tau(2)^* + \dots + \sigma(1)^* \tau(j-1)^*. \end{aligned}$$

This reflects the identity

$$(\omega_j + \omega_{j-1} + \dots + \omega_1)(\tilde{\omega}_j + \tilde{\omega}_{j-1} + \dots + \tilde{\omega}_1) = 1$$

in characteristic classes.

The above shows that for G , G_k and $G_{k,n}$, the cohomology ring has a simple description. However, in the cohomology of Schubert varieties, the product structure is more complicated, and the simplest method of description seems to be to give a table for cup product as in 3.19(ii) and (iii).

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