Caterpillars, Ribbons, and The Chromatic Symmetric Function

by

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Abstract

For every $n$-vertex tree $T$, it is known that the chromatic polynomial $\chi(T, k)$ is equal to $k(k - 1)^{n-1}$. It is known that the function in noncommuting variables, $Y_G(x)$, distinguishes all simple graphs. In the midground, the question of whether or not the chromatic symmetric function $X_G(x)$ distinguishes nonisomorphic trees is still open.

We look at Stanley's expansion of $X_G(x)$ in terms of the power sum symmetric basis $\{p_\lambda(x) | \lambda \vdash n\}$ of $\Lambda^n$, and identify properties of our trees in various coefficients of the $p_\lambda$ in this expansion for $X_G(x)$. By restricting to the case when our tree is a caterpillar $C$, we shall use a correspondence between ribbons and caterpillars to look at the coefficients of the $p_\lambda(x)$ in the expansion of $X_C(x)$ using ribbon classes. Among these ribbon classes we will have special interest in those which are symmetric. We show that the chromatic symmetric function distinguishes these symmetric classes from all other caterpillars.
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<td>The number of stable partitions of $G$ of type $\lambda$.</td>
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<td>$b$</td>
<td>A box of a diagram.</td>
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<td>$c$</td>
<td>A corner of a ribbon.</td>
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<td>$c_j$</td>
<td>The $j$-th column of a diagram, or the number of boxes in the $j$-th column of a diagram.</td>
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<td>$c_{\alpha}$</td>
<td>The number of corners of the ribbon $\rho_{\alpha}$.</td>
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<td>$c_{\rho}$</td>
<td>The number of corners of the ribbon $\rho$.</td>
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<td>$\deg(v)$</td>
<td>The degree of the vertex $v$.</td>
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<td>$e$</td>
<td>An edge of the graph, tree, or caterpillar.</td>
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<td>The incomparability graph of the poset $P$.</td>
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<td>$l(\lambda)$</td>
<td>The number of parts of $\lambda$.</td>
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<td>$m$</td>
<td>The number of vertices.</td>
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<td>$m_{\lambda}(x)'$</td>
<td>The augmented monomial symmetric function corresponding to $\lambda$.</td>
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<td>$n$</td>
<td>The number of boxes in a ribbon. (Except in §2.3.)</td>
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<td>$o$</td>
<td>The number of rows of a diagram.</td>
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<td>$p_{\lambda}(x)$</td>
<td>The power symmetric function corresponding to $\lambda$.</td>
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<td>$[p_{\lambda}]_{X_G}$</td>
<td>The coefficient of $p_{\lambda}$ in $X_G$.</td>
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<td>$q$</td>
<td>The number of columns of a diagram.</td>
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<td>$r_i$</td>
<td>The $i$-th row of $\rho$, or the number of boxes in the $i$-th row of $\rho$.</td>
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<td>$r(\alpha_i)$</td>
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<td>The number of vertices in the spine $S$.</td>
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<td>The number of boxes strictly south and west of $b$.</td>
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<td>$u, v, w$</td>
<td>Vertices of a graph.</td>
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<td>A</td>
<td>A subset of {2, 3, \ldots, n - 1}.</td>
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<td>A_ρ</td>
<td>The corner set of the ribbon (ρ).</td>
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<td>A'</td>
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<td>C</td>
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<td>C(ρ)</td>
<td>The conjugate of the ribbon (ρ).</td>
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<td>D</td>
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<td>A subset of (E).</td>
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<td>S</td>
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<td>S_n</td>
<td>The symmetric group on (n) letters.</td>
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<td>T</td>
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<td>V</td>
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<td>x</td>
<td>The sequence of indeterminants ((x_1, x_2, x_3, \ldots)).</td>
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<td>x^r</td>
<td>The monomial corresponding to the coloring (K).</td>
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<td>The monomial corresponding to the tableau (T).</td>
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<td>N, E, S,</td>
<td>A step in the north, east, south, or west direction respectively.</td>
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<tr>
<td>or W</td>
<td>A sequence of (t) steps north.</td>
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<tr>
<td>N^t</td>
<td>The set of caterpillars.</td>
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<td>G(n + 4)</td>
<td>The number of ((n + 4))-vertex caterpillars.</td>
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<td>C(n, c)</td>
<td>The number of (C)-symmetric (n)-box ribbon classes with (c) corners.</td>
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<td>F_{T,\lambda}</td>
<td>The family of sets (F \subseteq E(T)) with (\lambda(F) = \lambda).</td>
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<td>G</td>
<td>The group (G = {id, R, C, I}).</td>
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<td>N_{\rho}(n + 4)</td>
<td>The number of nonsymmetric ((n + 4))-vertex caterpillars.</td>
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<td>N(n, c)</td>
<td>The number of nonsymmetric (n)-box ribbon classes with (c) corners.</td>
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<td>The number of (R)-symmetric (n)-box ribbon classes with (c) corners.</td>
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<td>The set of symmetric ribbons.</td>
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<td>The number of symmetric (n)-box ribbon classes with (c) corners.</td>
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<td>The number of symmetric ((n + 4))-vertex caterpillars.</td>
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<td>T</td>
<td>A tableau.</td>
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<td>The (i, j)-th entry of the tableau (T).</td>
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<td>The (i)-th part of (\alpha).</td>
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<td>(\alpha^c)</td>
<td>The conjugate of (\alpha).</td>
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<td>(\alpha_\rho)</td>
<td>The composition corresponding to (\rho).</td>
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<td>([\alpha])</td>
<td>The equivalence class of (\alpha).</td>
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<td>(\chi(G,k))</td>
<td>The chromatic polynomial of (G).</td>
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<tr>
<td>(\kappa)</td>
<td>A coloring of a graph or diagram.</td>
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<td>(\lambda, \mu)</td>
<td>Partitions of (n) (or (m)).</td>
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<td>(\lambda/\mu)</td>
<td>The skew diagram, (\lambda) skew (\mu).</td>
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<td>(\lambda_i)</td>
<td>The (i)-th part of (\lambda).</td>
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<td>(\lambda(c))</td>
<td>The partition obtained by deleting (c) from (\rho).</td>
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<td>(\lambda(F))</td>
<td>The partition induced by the set (F \subseteq E).</td>
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<td>(\Lambda^n)</td>
<td>The set of homogeneous symmetric functions in (x) of degree (n).</td>
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<td>(\rho, \sigma)</td>
<td>Ribbons.</td>
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<td>The ribbon corresponding to (\alpha).</td>
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<td>(\rho_A)</td>
<td>The ribbon associated to the set (A).</td>
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<td>([\rho])</td>
<td>The equivalence class of (\rho).</td>
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<td>The equivalence class of (\rho) (modulo (I)).</td>
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<td>(2_T)</td>
<td>The set of partitions of (T) into two parts.</td>
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<td>(2_\rho)</td>
<td>The set of partitions of (G(\rho)) into two parts.</td>
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<td>(2_\alpha)</td>
<td>The set of partitions of (G(\rho_{\alpha})) into two parts.</td>
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<td>(\zeta(\alpha))</td>
<td>The corners of (\alpha) at the end of the rows.</td>
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<td>(j^t)</td>
<td>The element (j), repeated (t) times.</td>
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<tr>
<td>((r_1, \ldots, r_n))</td>
<td>The partition with (r_i) parts of size (i).</td>
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<tr>
<td>(&quot;)</td>
<td>&quot;is a partition of&quot;</td>
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<td>(\mid)</td>
<td>&quot;is a composition of&quot;</td>
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<td>(\cong)</td>
<td>&quot;is isomorphic to&quot;</td>
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Chapter 1

Introduction

We first recall the definitions relevant for the following discourse.

We shall be concerned with the topic of graphs, in particular trees, and their colorings. We shall also find use in looking at partitions and compositions of $n$, ribbon diagrams and the Ferrers graphs thereof. Additionally, we need to mention some well-known bases for the space $\Lambda^n$ of homogeneous symmetric functions of degree $n$. We shall remind the reader of the general definitions here, recalling the results that we will require.

1.1 Partitions, Compositions, Tableaux, and Ribbons

A partition $\lambda$ of a positive integer $n$, written $\lambda \vdash n$, is a sequence of weakly decreasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\sum_{i=1}^k \lambda_i = n$. Given a partition $\lambda$, we can represent it via the diagram of left justified rows of boxes whose $i$-th row contains $\lambda_i$ boxes. The diagrams of these type are called Ferrers diagrams. We shall use the symbol $\lambda$ when referring to both the partition and its Ferrers diagram. We may sometimes write $\lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$ for the partition which has $r_1$ parts of size one, $r_2$ parts of size two, ..., and $r_k$ parts of size $k$.

We say $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition of $n$ and write $\alpha \vdash n$ if each $\alpha_i$ is a positive integer and $\sum_{i=1}^k \alpha_i = n$. If we relax the conditions to consider sums of nonnegative integers, that is, allowing some of the $\alpha_i$ to be zero, we obtain the concept of a weak composition of $n$. If $\alpha \vdash n$ we obtain a partition of $n$ by reordering the $\alpha_i$ into weakly decreasing order. We shall sometimes abuse notation and refer to a composition of $n$ as a partition of $n$ when we really mean the associated partition.

If $\lambda \vdash n$, we can fill the boxes of the Ferrers diagram $\lambda$ with the numbers $1, 2, \ldots, n$, obtaining a tableau $T$ of shape $\lambda$. We denote the entry in the $i$-th row and $j$-th column of $\lambda$ by $T_{i,j}$.

A generalized Young tableau of shape $\lambda$ is the array $T$ obtained by filling the boxes of the Ferrers diagram of $\lambda$ with positive integers, repetition allowed. The content of a generalized tableau $T$ is the weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_j)$, where $j$ is the largest integer appearing in $T$, and for each $1 \leq i \leq j$, $\alpha_i$ is the number of times $i$ occurs as an entry of $T$. A generalized tableau is said to
be semistandard if each row gives a weakly increasing sequence of integers and each column gives a strictly increasing sequence of integers.

Example Here we consider a semistandard Young tableau with shape $\lambda = (4, 4, 2)$ and content $\mu = (2, 2, 3, 2, 1)$.

\[
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 3 & 5 \\
3 & 4 \\
\end{array}
\]

Suppose partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash m$, with $m \leq n$, $k \leq j$, and $\mu_i \leq \lambda_i$ for each $i = 1, 2, \ldots, k$ are given. Then diagrammatically, $\lambda$ contains a copy of $\mu$ in its top left corner. We can form the skew diagram $\lambda / \mu$ by removing that copy of $\mu$ from the top left corner of $\lambda$.

Given any diagram or skew diagram $D$, the conjugate diagram $D'$ is obtained by simply transposing the array of boxes along the main diagonal.

Example Here we consider the partitions $\lambda = (4, 4, 2)$ and $\mu = (3, 1)$, and form the skew diagram $\lambda / \mu$.

The conjugates, $\lambda' = (3, 3, 2, 2), \mu' = (2, 1, 1)$ and $(\lambda / \mu)' = \lambda' / \mu'$ are shown below.

A ribbon is a skew diagram that has a connected interior and contains no $2 \times 2$ array of boxes. The skew diagrams $\lambda / \mu$ and $(\lambda / \mu)' = \lambda' / \mu'$ shown in the above example are ribbons. In fact whenever $\rho$ is a ribbon, then so is its conjugate $\rho'$. For ribbons $\rho$, we shall typically use $C(\rho)$ to denote the conjugate diagram $\rho'$. We shall use $n$ for the number of boxes in the ribbon under consideration.

An equivalent definition of ribbons consists of taking all walks formed by a finite sequence of north and east steps (or equivalently, south and west steps). We shall denote a step in the north, east, south, and west direction by $N$, $E$, $S$, and $W$ respectively, and denote a walk by its sequence of steps in these four
1.1. Partitions, Compositions, Tableaux, and Ribbons

directions. We shall write $N^t$ for $t$ successive steps in the north direction, and similarly for the other directions.

Another way of obtaining ribbons is by compositions. Given the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$, we can relate the $n$-box ribbon $\rho_\alpha$ with $\alpha_{k+1-j}$ boxes in the $j$-th row. We construct $\rho_\alpha$ so that each row shares exactly one column with the row before it. Additionally, for each ribbon $\rho$, we can obtain a composition $\alpha_\rho$ of $n$ by listing, from bottom to top, the number of boxes in each row of $\rho$. This gives a bijection.

We shall use this bijection to pass back and forth between ribbon and composition representatives as we see fit. Both ribbons and compositions will later serve as representatives for certain classes of graphs known as caterpillars, which will be defined in §1.3.

**Example** Below we see a ribbon $\rho$ with 11 boxes. We may write $\rho$ in terms of a walk as either $E^5 N^2 E N E$ or $W S W S^2 W^5$.

![Diagram of a ribbon](image)

Counting the number of boxes per row from bottom to top we obtain $\alpha_\rho = (6, 1, 2, 2) \models 11$.

In a ribbon, a box $b$ may fall into one of two categories: either the ribbon bends at that box or it does not. More precisely, given a box $c$ in a ribbon $\rho$, we say that $c$ is a **corner** of $\rho$ if either:

1. There is a box of $\rho$ immediately west of $c$ and a box of $\rho$ immediately north of $c$, or
2. There is a box of $\rho$ immediately east of $c$ and a box of $\rho$ immediately south of $c$.

Otherwise, we say that the box is a **noncorner** of $\rho$.

**Example** Looking at the ribbon from the previous example, $\rho = E^5 N^2 E N E = W S W S^2 W^5$, we see that $\rho$ has 4 corners.
1.2 Symmetric Functions

In this section we consider an infinite set of variables $x = (x_1, x_2, \ldots)$ and the ring of formal power series $\mathbb{C}[x]$. By $S_n$, we mean the group of permutations of the letters $\{1, 2, \ldots, n\}$.

We can let each $\pi \in S_n$ act on elements $f(x_1, x_2, \ldots) \in \mathbb{C}[x]$ by defining

$$\pi f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{n+1}, x_{n+2}, \ldots). \quad (1.1)$$

We are interested in certain functions $f(x_1, x_2, \ldots) \in \mathbb{C}[x]$ which are fixed by each $\pi \in S_n$, for every $n \in \mathbb{N}$. For instance, the $n$-th power sum symmetric function, $p_n(x)$, is such a function, as is the $n$-th elementary symmetric function, $e_n(x)$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ by the power sum (resp. elementary) symmetric function corresponding to $\lambda$ we shall mean

$$p_\lambda(x) = \prod_{i=1}^k p_{\lambda_i}(x) \quad (e_\lambda(x) = \prod_{i=1}^k e_{\lambda_i}(x) \text{ resp.}).$$

Additionally, by the monomial symmetric function corresponding to $\lambda$ we mean

$$m_\lambda(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},$$

where the above sum is taken over all distinct monomial terms of the form $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}$. We note that $m_{(n)}(x) = p_n(x)$ and $m_{(1^n)}(x) = e_n(x)$.

Each of the $p_\lambda$, $e_\lambda$, and $m_\lambda$, are invariant under the action described in Equation 1.1. Further, it turns out that each of the sets $\{p_\lambda(x) | \lambda \vdash n\}$, $\{e_\lambda(x) | \lambda \vdash n\}$, and $\{m_\lambda(x) | \lambda \vdash n\}$ are independent and span the same space of functions; that is, they are all bases of a common space. This space, denoted $\Lambda^n$, is called the set of homogeneous symmetric functions of degree $n$ and is usually defined as the span of the $m_\lambda(x)$ for $\lambda \vdash n$.

For any tableau $\mathcal{T}$ of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we have the weighting

$$x^\mathcal{T} = \prod_{(i,j) \in \lambda} x_{T_{i,j}}^{\lambda_i} = \prod_{i=1}^k \prod_{j=1}^{\lambda_i} x_{T_{i,j}}.$$

With this, we can now define the last family of symmetric functions we shall have interest in mentioning. The Schur symmetric function corresponding to $\lambda$ is defined to be

$$s_\lambda(x) = \sum_T x^\mathcal{T},$$
where the sum is taken over all semistandard tableaux $T$ of shape $\lambda$.

If $\lambda$ is a partition of $n$, then $s_\lambda(x)$ is a homogeneous symmetric function of degree $n$. As with the other symmetric functions, the set \{s_\lambda(x) \mid \lambda \vdash n\} is a basis of $\Lambda^n$. Further details can be found in [Sagan 2001]. We shall henceforth drop reference to the variables $x$ and just write $s_\lambda$ in place of $s_\lambda(x)$, and similarly with the other symmetric functions.

1.3 Graphs and the Chromatic Symmetric Function

By a graph $G$ we mean a pair $(V, E)$, where $V$ is a finite set of elements, called the vertices of $G$, and $E$ is a subset of $V \times V$ which we consider as unordered pairs of vertices, and call the set of edges of $G$. Often times one is interested in the cases when $G$ can contain the same edge $\{u, v\}$ more than once, which is called a multiple edge, or when $G$ can have edges of the form $\{v, v\}$, which are called loops. All the graphs that we shall work with are simple graphs, that is, those graphs with no loops or multiple edges.

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we say that $G_1$ and $G_2$ are isomorphic, written $G_1 \cong G_2$, if there is a one-to-one map $\Phi$ from $V_1$ onto $V_2$ which preserves edges. That is, a bijection $\Phi : V_1 \rightarrow V_2$ such that $\{u, v\} \in E_1$ if, and only if, $\{\Phi(u), \Phi(v)\} \in E_2$ (with correct multiplicity in the case of multiple edges).

If an edge $e$ has $e = \{v_1, v_2\}$, we shall say that the vertices $v_1$ and $v_2$ are endpoints of $e$, that the vertices $v_1$ and $v_2$ are incident to $e$, and that the vertices $v_1$ and $v_2$ are adjacent. We shall say that two edges are incident when they share a common vertex among their endpoints. For each vertex $v$, the degree of the vertex $v$, written $\text{deg}(v)$, is the number of vertices adjacent to $v$.

A walk in $G$ is a sequence of vertices $u = W_0, W_1, W_2, \ldots, W_k = v$ such that $W_i$ is adjacent to $W_{i+1}$ for each $i = 0, 1, 2, \ldots, k-1$. In the case when all the vertices are distinct, we call the walk a path in $G$ and we call $u$ and $v$ the endpoints. We say that $u$ and $v$ are connected by a path if there is a path with endpoints $u$ and $v$. Path-connectedness gives an equivalence relation on $V$ whose equivalence classes are called the connected components of $G$.

A graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ if both $V' \subseteq V$ and $E' \subseteq E$. The subgraph $G'$ is called an induced subgraph of $G$ if $E'$ is the set of all edges in $E$ whose endpoints lie in $V'$. The subgraph $G'$ is a spanning subgraph if $V' = V$.

A subset of vertices is said to be stable if the subgraph induced by the set contains no edges. A graph in which the vertex set can be partitioned into two stable subsets is called a bipartite graph.

Trees are graphs which are minimally connected. That is, those graphs $G$ which have a single connected component and have the property that removing any edge of $G$ increases the number of connected components. As such, trees are
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acyclic, that is, there are no walks of the form \( w_0, w_1, w_2, \ldots, w_k \) with \( w_0 = w_k \).

If \( T \) is an \( m \)-vertex tree one can show that \( T \) has \( m - 1 \) edges. Conversely, any connected \( m \)-vertex graph with \( m - 1 \) edges is a tree. For the details of these and other results on trees, see [West].

The vertices of \( T \) which have degree equal to one are called leaves, while the other vertices are called internal vertices. If both endpoints of an edge are internal vertices, it is called an internal edge. If \( v \) is a leaf, the only paths containing \( v \) have \( v \) as one of their endpoints. Hence, deleting \( v \) from \( G \) does not disconnect any pair of vertices from this graph. Therefore the induced subgraph corresponding to the vertex set \( V - \{v\} \) is a connected graph, and hence, another tree. Similarly, deleting all the leaves of \( T \), by considering the subgraph induced by the set of internal vertices, we obtain another tree \( T' \).

A caterpillar \( C \) is a tree for which this derived tree \( T' \) is a path. Equivalently, a caterpillar is a tree in which there is a path consisting of internal vertices of \( T \), such that every vertex of \( T \) is either on the path, or adjacent to a vertex of the path. This path is called the spine of the caterpillar and is denoted \( S \). We shall use \( \mathcal{C} \) to denote the set of all caterpillars.

**Example** The tree we see below is a caterpillar; its spine has been highlighted.

![Caterpillar Diagram]

A \( k \)-coloring of a graph \( G \), is a map \( \kappa \) from the vertex set \( V \) into a \( k \)-set of "colors", typically the set \( \{1, 2, 3, \ldots, k\} \). A coloring is said to be proper if there are no monochromatic edges. That is, \( \kappa \) is proper when \( \kappa(u) \neq \kappa(v) \) for every edge \( \{u, v\} \in E \).

For a coloring \( \kappa \), the color class of the color \( j \) is just the set of vertices \( v \) with \( \kappa(v) = j \). The color classes partition the vertex set, with their cardinalities giving a partition \( \lambda \vdash |V| \). Further, for a proper coloring the color classes are stable subsets of \( V \).

The function \( \chi(G, k) \) is defined to be the number of proper \( k \)-colorings of \( G \). It is called the chromatic polynomial of the graph \( G \), and, as its name implies, it is a polynomial function in \( k \). Details can once again be found in [West].

For every \( m \)-vertex tree \( T \) we have

\[
\chi(T, k) = k(k - 1)^{m-1}.
\] (1.2)

There are \( k \) choices for the color of the first vertex of \( T \) we choose to label. Now, if there are uncolored vertices left, then there is one which is adjacent to exactly one of the colored vertices, and there are \( k - 1 \) choices for its color. Therefore Equation 1.2 follows.
1.3. Graphs and the Chromatic Symmetric Function

We turn to the case when our set of colors is \( N = \{1, 2, 3, \ldots\} \). Given a coloring \( \kappa \) of an \( m \)-vertex graph \( G \) we shall write \( x^\kappa \) for the monomial term of degree \( m \) defined by

\[
x^\kappa = \prod_{v \in V} x_{\kappa(v)}.
\]

For a graph \( G \), the chromatic symmetric function \( X_G(x) \) is defined by

\[
X_G(x) = \sum_{\kappa} x^\kappa,
\]

where the sum is over all proper colorings \( \kappa \). We note that \( X_G(1^k) = \chi(G, k) \), where \( 1^k \) denotes setting \( x_1 = x_2 = \ldots = x_k = 1 \) and \( x_{k+1} = x_{k+2} = \ldots = 0 \), since then a monomial survives if, and only if, it comes from a proper coloring using the colors \( \{1, 2, \ldots, k\} \), in which case the contribution to the sum is 1.

Permuting the color classes of a proper coloring of \( G \) yields another proper coloring of \( G \), so the series \( X_G(x) \) is indeed a symmetric function in the indeterminants \( (x_1, x_2, x_3, \ldots) \). We shall again drop reference to the variables and write \( X_G \) in place of \( X_G(x) \).

In this thesis we take interest in the following question of Stanley.

**Problem 1.3.1** Does the chromatic symmetric function distinguish every pair of nonisomorphic trees? That is, given nonisomorphic trees \( T_1 \) and \( T_2 \), do we have \( X_{T_1} \neq X_{T_2} \)?

To this end we look at expanding \( X_G \) in terms of the bases for \( \Lambda^m \) introduced in §1.2. For the power sum symmetric basis \( \{p_\lambda \mid \lambda \vdash m\} \) we have the following expansion for \( X_G \).

**Theorem 1.3.2** [Stanley, 1995, Theorem 2.5]

For an \( m \)-vertex graph \( G \)

\[
X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(F)},
\]

where \( E \) is the edge set of \( G \) and given \( F \subseteq E \), \( \lambda(F) \) is the partition of \( m \) whose parts correspond to the sizes of the connected components of the spanning subgraph of \( G \) with edge set \( F \).

In this thesis we shall use Theorem 1.3.2 to attack Problem 1.3.1. Results on \( X_G \) for general graphs trees are collected at the beginning of Chapter 3. Chapter 2 yields, among other results, a correspondence between caterpillars and certain equivalence classes of ribbons, which we shall use in the latter portion of Chapter 3 to look at the analogue of Problem 1.3.1 restricted to the set of caterpillars.

Finally in Chapter 4, we shall look at the chromatic symmetric function for some special classes of ribbons. We shall end by showing that there are collections \( Q_m \) of \( m \)-vertex graphs such that:
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1. \( \lim_{m \to \infty} |Q_m| = \infty \),

2. All graphs in \( Q_m \) have the same chromatic polynomial, and

3. No two graphs from the \( Q_m \) share the same chromatic symmetric function.

Although we shall only use the expansion in Theorem 1.3.2 for our work with \( X_G \), it is worth mentioning that there are expansions of \( X_G \) into the other bases we mentioned for which the coefficients of \( X_G \) have graph theoretic interpretations. We now turn to defining the concepts needed to state the relevant theorems. As the remainder of this section is not used or referred to later on, it may be skipped by the busy reader.

Instead of stating the result for the monomial symmetric basis, it is easier to deal with the augmented monomial symmetric basis \( m'_\lambda \) defined by \( m'_\lambda = r_1!r_2! \cdots r_k! m_\lambda \), where \( \lambda \) is the partition \( (1^{r_1}, 2^{r_2}, \ldots, k^{r_k}) \). Since \( \{m_\lambda | \lambda \vdash m \} \) is a basis for \( \Lambda^m \), the set \( \{m'_\lambda | \lambda \vdash m \} \) is also a basis for \( \Lambda^m \).

A stable partition of \( G \) is a partition \( \pi \) of the vertex set \( V \) in which every block of the partition is a stable subset—that is, in any given block of \( \pi \), no two vertices are adjacent. The type of \( \pi \) is defined to be the partition of \( n \) whose parts are just the sizes of the blocks of \( \pi \). From these definitions we find the following result.

**Theorem 1.3.3** [Stanley, 1995, Proposition 2.4]

Let \( G \) be a graph and \( a_\lambda \) be the number of stable partitions of \( G \) with type \( \lambda \). Then

\[
X_G = \sum_{\lambda \vdash m} a_\lambda m'_\lambda.
\]

Given a graph \( G \), we can obtain an oriented graph by giving each edge a direction, that is, by choosing one of the ordered edges \((u, v)\) or \((v, u)\) for each unordered edge \( \{u, v\} \in E \). An oriented graph \( G \) is acyclic if there are no sequences of ordered edges of the form \((w_1, w_2), (w_2, w_3), \ldots, (w_{k-1}, w_k)\), where \( w_k = w_1 \). A vertex \( u \) of an oriented graph is called a sink if there are no edges of the form \((u, v)\), for any vertex \( v \).

For the elementary symmetric functions, Stanley proves the following result.

**Theorem 1.3.4** [Stanley, 1995, Theorem 3.3]

If

\[
X_G = \sum_{\lambda \vdash m} c_\lambda e_\lambda,
\]

then the number of acyclic orientations of \( G \) with \( j \) sinks is

\[
\sum_{\lambda \vdash m} c_\lambda, \quad l(\lambda) = j
\]

where \( l(\lambda) \) is the number of parts of \( \lambda \).
A relation $R$ on a set $Z$ is a subset of $Z \times Z$, that is, ordered pairs of elements of $Z$. A poset (or partially-ordered set) $P$ is a set with a relation $R$ on it, that is, a subset of $P \times P$, which is reflexive, antisymmetric, and transitive. More precisely,

1. $(p, p) \in R$ for all $p \in P$,
2. $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, and
3. $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

Given a poset $P$, the incomparability graph, $\text{inc}(P)$, is the graph obtained by using the elements of $P$ as vertices and having two vertices $u$ and $v$ adjacent if, and only if, they are incomparable in the poset, that is, neither $(u, v)$ nor $(v, u)$ lies in $R$. A poset is said to be $(3+1)$-free if it contains no induced subposet isomorphic to the disjoint union of a three-element chain and a one-element chain.

Finally, we let $f^\lambda(P)$ be the number of $P$-tableaux of shape $\lambda$, where a $P$-tableau is a generalization of the Standard Young Tableau of shape $\lambda$.

With the preceding definitions we may now state the following result.

**Theorem 1.3.5** [Gasharov, 1994]

*Let $P$ be a $(3+1)$-free poset and $G = \text{inc}(P)$. Then*

$$X_G = \sum_{\lambda \vdash m} f^\lambda(P)s_\lambda.$$  

A graph $G$ is said to be claw-free if it has no induced subgraph isomorphic to $K_{1,3}$, the complete bipartite graph on partite sets of size 1 and 3 respectively.

Unfortunately, $P$ is $(3+1)$-free if, and only if, $\text{inc}(P)$ is claw-free, so Theorem 1.3.5 only expresses the claw-free graphs $G$ in terms of $s_\lambda$. As the only trees which are claw-free are the paths, this expansion holds little use for Problem 1.3.1 unless it is shown to hold for graphs with claws as well. Failing that, perhaps one can obtain a different expansion for $X_G$ in terms of the $s_\lambda$ in the case when $G$ is not claw-free.

The noncommutative version of the chromatic symmetric function, $Y_G$, is defined in exactly the same way as $X_G$. We take

$$x^V = x_{\kappa(v_1)}x_{\kappa(v_2)}\cdots x_{\kappa(v_m)},$$

where $V = \{v_1, v_2, \ldots, v_m\}$ and define

$$Y_G = \sum_{\kappa} x^V.$$  

The only difference is now the variables $(x_1, x_2, \ldots)$ no longer commute.

This noncommutative function $Y_G$ is certainly more robust than $X_G$ in the sense that it distinguishes more graphs.

**Theorem 1.3.6** [Gebhard/Sagan, 1999, Proposition 8.2]

*The noncommutative symmetric function $Y_G$ distinguishes all graphs $G$ with no loops or multiple edges.*
1.4 Ferrers Graphs

Considering a Ferrers diagram $D$ with $o$ rows $r_1, r_2, \ldots, r_o$ and $q$ columns $c_1, c_2, \ldots, c_q$, one has the corresponding bipartite Ferrers graph $G(D)$ obtained by letting the vertex set be $\{r_1, r_2, \ldots, r_o\} \cup \{c_1, c_2, \ldots, c_q\}$ and having vertex $r_i$ adjacent to vertex $c_j$ whenever there is a box in the $i$-th row and $j$-th column of $D$.

**Example** Here we see the Ferrers diagram corresponding to the partition $\lambda = (4, 4, 2)$ and its associated Ferrers graph.

Values intrinsic to the graph can be obtained from the diagram that gave rise to it. The number of edges is the number of boxes. The number of vertices is the number of rows plus the number of columns. The degree of the vertex $v$ is the number of boxes in the row (or column resp.) that is labelled $v$. Two edges are incident if, and only if, the corresponding boxes share a row or column. Thus, given an edge $e$ with associated box $b$, the number of edges incident to $e$ is the number of boxes in the row and column containing $b$, not counting the box $b$ itself.

For a Ferrers diagram $D$ with rows $r_1, r_2, \ldots, r_o$ and columns $c_1, c_2, \ldots, c_q$, we say that a coloring of $D$ is a map

$$\kappa : \{r_1, r_2, \ldots, r_o, c_1, c_2, \ldots, c_q\} \to \mathbb{N}.$$  

We say that the coloring $\kappa$ is proper if $\kappa(r_i) \neq \kappa(c_j)$ whenever there is a box in the $i, j$-th position of $D$.

Given a coloring $\kappa$ of a diagram $D$ with rows $r_1, r_2, \ldots, r_o$ and columns $c_1, c_2, \ldots, c_q$, we set

$$x^\kappa = \left(\prod_{i=1}^{o} x_{\kappa(r_i)}\right)\left(\prod_{j=1}^{q} x_{\kappa(c_j)}\right),$$

and define

$$X_D = \sum_{\kappa} x^\kappa,$$

where the sum is taken over all proper colorings of $D$.

The proper colorings of $G(D)$ are exactly the proper colorings of $D$, so we have $X_D = X_{G(D)}$ for every Ferrers diagram $D$. We shall use this fact to avoid having to mention the Ferrers graph of the diagram, even when we are computing the chromatic symmetric function for that Ferrers graph. This way we can stay solely within the context of diagrams when we find it convenient.
We will see in the next chapter, that the set of Ferrers graphs of ribbon diagrams is exactly the set of caterpillars. We will then use the graph $G$ and ribbon diagram $\rho$ to look at $X_G$ ($X_\rho$ resp.), in the power sum symmetric basis for $\Lambda^m$. By looking at various coefficients of the chromatic symmetric function in terms of properties of the graph (ribbon, composition resp.), we shall see that the chromatic symmetric function does distinguish certain classes of caterpillars.
Chapter 2

Ribbons

In this chapter we will see that the class of caterpillars arises naturally from the set of ribbon diagrams. We then proceed to use these ribbon representatives to count certain symmetric caterpillars.

2.1 Ribbon-Caterpillar Correspondence

We begin by finding which diagrams $D$ give rise to trees.

**Proposition 2.1.1** A skew diagram is a ribbon if, and only if, its Ferrers graph is a tree.

**Proof** For the forward direction, let $\rho$ be a ribbon.

Looking at $\rho$ as a northeast walk, $\rho$ consists of a finite number of north and east steps, say $j$ north and $k$ east, the order of the steps being immaterial. We shall show $G(\rho)$ is a tree by showing that

1. $G(\rho)$ has one fewer edges than vertices,
2. $G(\rho)$ is connected.

The number of edges in the graph $G(\rho)$ is the number of boxes in the diagram $\rho$, which is one more than the number of steps in the walk, as an extra box is needed for the starting box of the walk. Hence $G(\rho)$ has $j + k + 1$ edges. The $j$ steps north imply $j + 1$ rows in $\rho$, and the $k$ steps east imply $k + 1$ columns in $\rho$, so $G(\rho)$ has $(j + 1) + (k + 1) = j + k + 2$ vertices. Thus $G(\rho)$ has the correct number of edges to be a tree.

Since $\rho$ is a ribbon, the diagram $\rho$ is connected. Since adjacent boxes of the diagram correspond to incident edges in the graph, $G(\rho)$ will also be connected. Thus every ribbon gives rise to a tree.

Conversely, suppose $\xi$ is a skew diagram but not a ribbon. Then either $\xi$ is not connected, or it contains a $2 \times 2$ subdiagram of boxes. In the latter case we have a 4-cycle in $G(\xi)$, so it cannot be a tree.

Thus, we are left considering the case when $\xi$ is not connected. Taking $\psi$ to be a maximal connected subdiagram of $\xi$, then $\psi$ is also a skew diagram and since it cannot be all of $\xi$, there is some box $b$ of $\xi$ not in $\psi$. This box cannot be in the same row or column as any box in $\psi$, since $\psi$ was chosen as maximal. This is because in any skew diagram, if two boxes share a row (column resp.),
then the diagram contains all the boxes between those two in that row (column resp.). Hence $G(\xi)$ is not connected, as there is no path from any vertex in $G(\psi)$ to either vertex corresponding to the row and column containing of $b$.

Therefore $G(\xi)$ is not a tree. □

Having seen that ribbons are the only skew diagrams giving rise to trees, we now identify which trees appear.

**Proposition 2.1.2** A graph $G$ is a caterpillar if, and only if, $G \cong G(\rho)$ for some ribbon $\rho$.

**Proof** Recall that the caterpillars are those trees $T$ that contain a path $S$, called the spine of the caterpillar, for which every vertex of $T$ is either in $S$, or adjacent to a vertex in $S$. Since every edge has some pair of vertices as its endpoints, every edge of $T$ is either on the spine (when both endpoints are internal vertices), or incident to an edge of the spine (when an endpoint is a leaf). For the duration of the proof we shall think of the spine in terms of its set of edges.

We start by proving the converse direction.

If $\rho$ is a ribbon, we consider the set $S$ of edges corresponding to the corners of $\rho$. It is clear that all boxes of $\rho$ are in the same row (or column) as one of the corners, so every edge of $G(\rho)$ is either associated to corner, or incident to an edge associated to a corner. Hence, to show $G(\rho)$ is a caterpillar, we need only check that the edges in $S$ form a path. Let $\text{Cor}(\rho)$ be the set of corners of $\rho$ and $G(\text{Cor}(\rho))$ be the induced subgraph which we wish to show is a path.

Traversing the ribbon as a walk from southwest to northeast we find that each corner shares a row or a column with the next corner. Hence, the edge associated with a given corner is incident to the edge associated with the next corner in the ribbon. Hence, $G(\text{Cor}(\rho))$ is connected.

Since there are no more than two corners in any given row or column, every vertex of $G(\text{Cor}(\rho))$ has degree $\leq 2$. Since additionally $G(\text{Cor}(\rho))$ is connected and acyclic, it is a path.

Now consider the forward direction.

Given a caterpillar $C$, let $S$ be its spine. As $C$ is a tree, there is only one bipartition of the vertex set of $C$ (up to swapping the partite sets). Say there are partite sets $Z_1$ and $Z_2$, of sizes $o$ and $q$ respectively.

We take an endpoint of $S$, which we may assume lies in $Z_1$, and label it $r_1$. Now starting at $r_1$, and following the path $S$, we iterate:

1. If vertex $x$ is labelled $r_j$ for some $j$, then we label the leaves adjacent to $x$ with the unused labels among $c_1, c_2, \ldots, c_q$ with the largest indices, then label the next element along the path $S$, if there is one, with the largest indexed label remaining unused among $c_1, c_2, \ldots, c_q$. 
2. If vertex \( x \) is labelled \( c_j \) for some \( j \) then we label the leaves adjacent to \( x \) with the unused labels among \( r_1, r_2, \ldots, r_o \) with the smallest indices, then label the next element along the path \( S \), if there is one, with the smallest indexed label remaining unused among \( r_1, r_2, \ldots, r_o \).

Once we finish traversing \( S \), we will have labelled all of \( Z_1 = \{ r_1, r_2, \ldots, r_o \} \) and \( Z_2 = \{ c_1, c_2, \ldots, c_q \} \). By labelling the rows of a grid, top to bottom, by \( r_1, r_2, \ldots, r_o \), the columns, left to right, by \( c_1, c_2, \ldots, c_q \), and by placing a box in the \( r_i, c_j \)-th position whenever the vertex labelled \( r_i \) is adjacent to the vertex labelled \( c_j \) in the caterpillar \( C \), we obtain a diagram \( \rho_C \) with \( G(\rho_C) \cong C \). We now wish to show that the diagram \( \rho_C \) is a ribbon.

If \( w_1, w_2, w_3, \ldots, w_s \) is the spine of \( C \) written as a path of vertices, with \( w_1 = r_1 \), we let \( \delta = (\delta_1, \delta_2, \ldots, \delta_s) \) be the sequence of degrees of \( S \), that is \( \delta_i = \text{deg}(u_i) \) for each \( 1 \leq i \leq s \).

Then \( \rho_C \) is the ribbon obtained by the walk \( W^{\delta_1 - 1} S^{\delta_2 - 1} W^{\delta_3 - 1} \ldots \), since, starting with a box at coordinates \( (r_1, c_q) \), taking \( \delta_1 - 1 \) steps west corresponds to the \( \delta_1 - 1 \) edges adjacent to \( w_1 \) besides \( \{ r_1, c_q \} \) (first the leaf edges, then the edge \( \{ w_1, w_2 \} \)), then \( \delta_2 - 1 \) steps south for the \( \delta_2 - 1 \) edges adjacent to \( w_2 \) besides \( \{ w_1, w_2 \} \) (first the leaf edges, then the edge \( \{ w_2, w_3 \} \)), and so forth.

**Example** Here we see the process of going from caterpillars to ribbons with a caterpillar with eight vertices. On the left the spine is highlighted and an end of the spine has been labelled \( r_1 \). On the right we have finished labelling the vertices by the above procedure. Here, \( \alpha = \gamma = 4 \), and \( s = 3 \).

Since the spine \( S = w_1, w_2, w_3 = r_1, c_2, r_4 \) we obtain the degree sequence \( \delta = (3, 4, 2) \) and predict the ribbon \( \rho = WWSSSW \).

Starting with the edge \( \{ r_1, c_4 \} \) of \( C \) we place a box in the position \( (r_1, c_4) \) of the \( 4 \times 4 \) grid whose rows are labelled top to bottom with coordinates \( r_1, r_2, r_3, r_4 \) and whose columns are labelled left to right with coordinates \( c_1, c_2, c_3, c_4 \).
Now for each edge \(\{r_1, c_i\}\) incident to \(r_1\) besides \(\{r_1, c_4\}\), we place a box in position \((1, i)\). Since these edges are incident to \(r_1\) these boxes are all in the row labelled \(r_1\), and by our method of labelling the the labels decrease one by one, so the boxes are placed one next to the other.

There are \(\text{deg}(r_1) - 1\) such edges, so we have \(\text{deg}(r_1) - 1 = \delta_1 - 1 = 2\) westwards steps from \((r_1, c_4)\). By the choice of how we labelled the adjacent leaves and then we labelled the next vertex in the spine, the last box we add will correspond to an edge of the spine: in this case, the edge \(\{r_1, c_2\}\).

We proceed in the same way creating boxes for the vertices adjacent to the next vertex in the spine, \(c_2\) in this case. As the edge \(\{r_1, c_2\}\) has already been used, we obtain \(\text{deg}(c_2) - 1 = \delta_2 - 1 = 3\) more boxes, all placed in the column of \(c_2\). This adds \(\delta_2 - 1 = 3\) steps south to the walk.

This continues until we have reached the last vertex in our spine \(r_4\). Of the \(\text{deg}(r_4)\) edges incident to \(r_4\), only the one spine edge incident to it has been counted so far, and the other edges give \(\text{deg}(r_4) - 1 = 1\) boxes to add to the row of \(r_4\), giving \(\text{deg}(r_4) - 1 = 1\) west step to finish our walk.
Had we started from the other end of the spine we would have obtained the following labelling of $C$, and ribbon:

Hence, starting at opposite ends of the spine, we can potentially obtain two different ribbons from each caterpillar $C$. We shall write $p_C$ and $q_C$, to denote these two ribbons.

### 2.2 Ribbon Symmetries and Classes

For any ribbon $\rho$ there are a few simple operations on the diagram that preserve the structure of the underlying caterpillar $G(\rho)$. For instance rotating $\rho$ by $180^\circ$ gives another ribbon $R(\rho)$ whose Ferrers graph is isomorphic to $G(\rho)$. Rotating the diagram only serves to reverse the ordering of the rows and the columns, hence only reverses the indices of the vertex labels.

If we think of $\rho$ as a skew diagram, then the conjugate skew diagram $C(\rho)$ is also a ribbon whose Ferrers graph is isomorphic to $G(\rho)$. This action only swaps the row labels with the column labels.

We shall call these two operations rotation and conjugation respectively. Composing these two operations acts to both swap the sets of row and column labels and reverse the orders of the labellings, and it is clear that the order in which these two actions are done is immaterial. We let $I(\rho) = CR(\rho) = RC(\rho)$, and call $I$ inversion.

Looking at a ribbon as a sequence of steps, $I$ acts to invert the direction of each step. For instance, inverting the northeast walk $N^1E^2N^3\ldots N^t$ gives the northeast walk $E^tN^tE^3\ldots E^1$ and inverting the southwest walk $W^1S^2W^3\ldots S^t$ gives the southwest walk $S^tW^tS^3\ldots W^1$. 
2.2. Ribbon Symmetries and Classes

Example Here we see the ribbon $\rho = \text{ENNNEEEE}$, its rotation $R(\rho)$, its conjugate $C(\rho)$, its inversion $I(\rho)$, and their Ferrer graph $G(\rho)$.

Let $id$ be the identity operation on the set of ribbons, that is for every ribbon $\rho$, $id(\rho) = \rho$. Then we claim the set $G = \{id, R, C, I\}$ forms a group under composition. Each of the operations $f \in G$ has $f \circ f = id$, the identity operation on the set of ribbons, so every element is its own inverse. Since $RC = I = CR$, $RI = C = IR$, and $CI = R = IC$, this set is closed under composition. As function composition is always associative, $G$ is a group as claimed.

This group acts on the set of ribbons, so the orbits of the ribbons under this action partition the set of ribbons into equivalence classes of the form $[\rho] = \{\rho, R(\rho), C(\rho), I(\rho)\}$.

We shall call any such set a ribbon class.

From the comments at the beginning of this section the operations in $G$ preserve the underlying Ferrers graph. That is, members of a ribbon class share the same Ferrers graph.

By the equivalence class of a ribbon $\rho$ modulo $I$, denoted $\rho$ (modulo $I$), we mean the set $\{\rho, I(\rho)\}$. Since $I(\rho) \neq \rho$ for every ribbon $\rho$, each equivalence class $\rho$ (modulo $I$) has exactly two elements. Since $IR = CRR = C$, we have
\[ \{R(\rho)\}_I = \{R(\rho), C(\rho)\} \]. Thus for each ribbon class \([\rho]\) we have the decomposition

\[ [\rho] = [\rho]_I \cup \{R(\rho)\}_I. \]

For both \(f = R\) and \(f = C\) we ask for which \(\rho\) do we have \(f(\rho) = \rho\), that is, which ribbons are \(f\)-symmetric. We shall call a ribbon symmetric if it is either \(R\)- or \(C\)-symmetric. We note that no ribbon with more than one box could be \(I\)-symmetric, since looking at \(\rho\) as northeast walk, \(\rho\) begins with a step north if, and only if, \(I(\rho)\) begins with a step east.

Recall from the example and comments after the proof of Proposition 2.1.2 that from a caterpillar \(C\) we can obtain the pair of ribbons \(\rho_C, \sigma_C\) with Ferrers graph \(C\).

**Lemma 2.2.1** If \(C\) is a caterpillar, the only ribbons (modulo \(I\)) that have \(G(\rho) \cong C\) are \(\rho_C\) and \(\sigma_C\).

**Proof** Given a ribbon \(\rho\) with \(G(\rho) \cong C\), numbering the \(k - 1\) corners in the northeast to southwest direction \(1, 2, \ldots, k - 1\) will form a traversal of the spine.

We label the first box of \(\rho\) with a 0, and call it the zeroeth corner of \(\rho\). Additionally, we label the last box of \(\rho\) with a \(k\) and call it the \(k\)-th corner of \(\rho\). Up to inversion we assume \(\rho\) begins with a step west. For each \(j\) from 1 to \(k\) we label the column or row (depending on the parity of \(j\)) that contains both the \((j - 1)\)-th and \(j\)-th corners with a \(w_j\). We label the column when \(j\) is even, and the row when \(j\) is odd.

This acts to label every row and column that contains a corner with a \(w_i\).

That is, it labels every vertex which is part of the spine.

**Example** Given the ribbon \(\rho = W^2S^3W\), we label the \(i\)-th corner of \(\rho\) with an \(i\), and label the rows and columns that contain a corner with the various \(w_i\) as described in the proof of Lemma 2.2.1.

On the right we show the Ferrers graph \(G(\rho)\), with the vertices of the spine labelled with the row/column labels inherited from the ribbon \(\rho\).

Now, back to proving the lemma. If, as a walk, \(\rho = W^{t_1}S^{t_2}W^{t_3} \ldots\), then

\[ t_j = \text{the number of steps from the } (j - 1)\text{-th corner to the } j\text{-th corner} \]
2.2. Ribbon Symmetries and Classes

\[ \text{[the number of boxes in the row (or column resp.) of } w_j] - 1 \]
\[ = \text{deg}(w_j) - 1. \]

In the proof of Proposition 2.1.2, we created the ribbon \( \rho_C \) by using the walk \( W^{\delta_1-1}S^{\delta_2-1}W^{\delta_3-1} \ldots \) where \( \delta = (\delta_1, \delta_2, \ldots) \) was the degree sequence of the caterpillar's spine. With this labelling we have

\[ \rho_C = W^{\text{deg}(w_1)-1}S^{\text{deg}(w_2)-1}W^{\text{deg}(w_3)-1} \ldots \]
\[ = W^{\delta_1}S^{\delta_2}W^{\delta_3} \ldots \]
\[ = \rho, \]

as desired.

For each caterpillar \( C \), Lemma 2.2.1 shows the only ribbons with Ferrers graph \( C \) are \( \rho_C \) and \( \sigma_C \). In particular, if the degree sequence of the spine is \((\delta_1, \delta_1, \ldots, \delta_k)\), the proof translated it into a southwest walk, giving a ribbon

\[ \rho_C = W^{\delta_1-1}S^{\delta_2-1}W^{\delta_3-1} \ldots \]

Starting from the opposite end of the spine, we create another ribbon

\[ \sigma_C = W^{\delta_k-1}S^{\delta_{k-1}-1}W^{\delta_{k-2}-1} \ldots. \]

**Lemma 2.2.2** For any caterpillar \( C \) we have \( \sigma_C = R(\rho_C) \) (modulo 1).

**Proof** From \( \rho_C = W^{\delta_1-1}S^{\delta_2-1}W^{\delta_3-1} \ldots \) we obtain

\[ R(\rho_C) = E^{\delta_1-1}N^{\delta_2-1}E^{\delta_3-1} \ldots. \]

If \( k \) is even, then

\[ R(\rho_C) = E^{\delta_1-1}N^{\delta_2-1} \ldots E^{\delta_{k-1}-1}N^{\delta_k-1} \]
\[ = S^{\delta_k-1}W^{\delta_{k-1}-1} \ldots S^{\delta_2-1}W^{\delta_1-1}, \]

and this latter walk is equivalent to \( \sigma_C = W^{\delta_k-1}S^{\delta_{k-1}-1} \ldots W^{\delta_2-1}S^{\delta_1-1} \) under the action of \( I \), since \( I \) inverts the south and west steps in a southwest walk.

If \( k \) is odd, then

\[ R(\rho_C) = E^{\delta_1-1}N^{\delta_2-1} \ldots N^{\delta_{k-1}-1}E^{\delta_k-1} \]
\[ = W^{\delta_k-1}S^{\delta_{k-1}-1} \ldots S^{\delta_2-1}W^{\delta_1-1}, \]

and this latter walk is \( \sigma_C \).
We summarise Lemma 2.2.1 and Lemma 2.2.2 as follows.

**Corollary 2.2.3** Given a caterpillar \( C \) and a ribbon \( \rho \), \( C \cong G(\rho) \) if, and only if, \( \rho \in [\rho_C] \). That is, the map taking \( C \) to \([\rho_C]\) gives a bijection between the set of caterpillars and the set of ribbon classes.

We now turn to look at the number of \( R \)- and \( C \)-symmetric ribbons.

**Lemma 2.2.4** Let \( n \) be a positive integer.

1. If \( n \) is even there are \( 2^\frac{n}{2} \) \( R \)-symmetric \( n \)-box ribbons, and no \( C \)-symmetric \( n \)-box ribbons.

2. If \( n \) is odd there are the same number of \( R \)-symmetric \( n \)-box ribbons as there are \( C \)-symmetric \( n \)-box ribbons, namely \( 2^{\frac{n+1}{2}} \) of each.

**Proof** In the case where \( n \) is even we construct an \( \frac{n}{2} \)-box ribbon of half the desired length by choosing \( \frac{n}{2} - 1 \) steps north or east, starting from the initial box. From this we obtain \( 2^\frac{n}{2} - 1 \) ribbons with \( \frac{n}{2} \) boxes. We can now obtain two \( R \)-symmetric \( n \)-box ribbons from each of these by rotating 180° around the midpoint of either the north or east face of the top right box.

These are the only two possibilities of an even boxed \( \rho \) stemming from the given half ribbon with \( R(\rho) = \rho \), since once the direction from the last box (either north or east) is specified the rest of the desired ribbon is fixed from the ribbon's rotational symmetry.

All \( R \)-symmetric ribbons \( \rho \) arise in this way, since taking the first half of \( \rho \), the only ways of completing the half ribbon to be \( R \)-symmetric are those above. Hence we find \( 2 \times 2^\frac{n}{2} - 1 = 2^\frac{n+1}{2} \) \( R \)-symmetric \( n \)-box ribbons.

If \( \rho \) is \( C \)-symmetric, then we claim it has an odd number of boxes. To prove this, we note that the main diagonal along which we transpose the array of boxes must cross the diagonal of exactly one box of \( \rho \). This single box is fixed under the action of \( C \), whereas all other boxes in the ribbon are paired up by the action of \( C \). Hence the total number of boxes is odd.

In the case when \( n \) is odd we construct ribbons of approximately half the length; namely, we create \( 2^\frac{n+1}{2} \) ribbons of length \( \frac{n+1}{2} \) by choosing \( \frac{n+1}{2} \) steps. As in the previous case, each of these half ribbons can be extended to two different symmetric \( n \)-box ribbons. This time one is \( R \)-symmetric while the other is \( C \)-symmetric.

Rotating by 180° through the center of the top right box gives an \( R \)-symmetric ribbon, and it is the only symmetric ribbon extending the half ribbon whose next step after the initial ribbon was in the same direction as the last step of the half ribbon.

A \( C \)-symmetric ribbon can be obtained by reflecting the half ribbon across the line \( y = -x \) with the origin taken as being in the center of the top right box. This is the only symmetric ribbon extending the half ribbon whose first
step after the half ribbon was in the orthogonal direction to the last step of the half ribbon.

Again, the two ribbons we obtain are distinct because they have a different number of corners, and all symmetric ribbons with an odd number of boxes arise in these ways. Since one of these was $R$-symmetric and one was $C$-symmetric, we find that there are $2^{n-1}$ $R$-symmetric and $2^{n-1}$ $C$-symmetric $n$-boxed ribbons.

**Example** Suppose we wish to create two $R$-symmetric ribbons with $n = 8$ boxes stemming from the half ribbon $ENN$ with four boxes. If we chose the next step in our ribbon to be north (resp. east), then the point that the completed ribbon will be rotated around will be the midpoint of the northern (resp. eastern) face of the top right box of our initial ribbon. Rotating the initial ribbon $180^\circ$ through that point completes the $R$-symmetric ribbon.

The ribbons we obtain from these two rotations are certainly distinct from one another as they have a different number of corners.

Any $R$-symmetric ribbon $\rho$ of length eight can similarly be constructed. The first four boxes of $\rho$ determine a ribbon of half the desired length, and rotating this half ribbon about the midpoint of either the north or east face of the top right box will give $\rho$.

**Example** Here we look at the case $n = 11$. We are looking to create symmetric 11-box ribbons. After the first 6 boxes of the ribbon are determined, choosing the direction of the next step determines the type of symmetry we can obtain.

If the direction of the next step is the same as the direction of the last step, then one can create an $R$-symmetric ribbon by rotating the half ribbon $180^\circ$ around the center of the top right box.

Here we use the half ribbon given by $EENEE$, and choose the direction of the next step to be east:
If the direction of our next step is in a different direction than the last step of our half ribbon, then one obtains a $C$-symmetric ribbon by reflecting across the line $y = -x$.

Hence, starting with the same half ribbon EEENEE, and choosing the next step to be north we obtain the following $C$-symmetric ribbon:

We shall say that a ribbon class $[\rho]$ is $R$-symmetric when the members of its class are $R$-symmetric, and that it is $C$-symmetric when the member of its class are $C$-symmetric. We shall say that a ribbon class is symmetric when it is either $R$- or $C$-symmetric.

If $[\rho]$ is $R$-symmetric ($C$-symmetric resp.), then, since $\rho$ is a member of $[\rho]$, $\rho$ must be $R$-symmetric ($C$-symmetric resp.). As we shall see the converse also holds.

Lemma 2.2.5 Let $\rho$ be a ribbon. The ribbon class $[\rho]$ is $R$-symmetric ($C$-symmetric resp.) if, and only if, $\rho$ is $R$-symmetric ($C$-symmetric resp.).

Proof First we shall treat the $R$-symmetric case. As the forward direction was proved directly before the statement of the lemma, we need only consider the converse direction.

Let $\rho$ be $R$-symmetric, that is $R(\rho) = \rho$. We intend to show that each other member of $[\rho] = \{\rho, R(\rho), C(\rho), I(\rho)\}$ is $R$-symmetric.

Since $R(R(\rho)) = R(\rho)$, $R(\rho)$ is $R$-symmetric. Using $CR = I = RC$ we find that

$$R(C(\rho)) = C(R(\rho)) = C(\rho)$$

so $C(\rho)$ is $R$-symmetric, and lastly

$$R(I(\rho)) = R(C(R(\rho))) = R(C(\rho)) = I(\rho)$$
2.3 Counting Ribbon Classes / Caterpillars

It is a known result that

**Theorem 2.3.1 [Harary/ Schwenk, 1973]**

The number of nonisomorphic caterpillars with \( n + 4 \) vertices is \( 2^n + 2^{\lfloor n/2 \rfloor} \).

We shall prove the result here using ribbon diagrams.

**Proof (of Theorem 2.3.1)**

For the duration of the proof we shall say that a caterpillar \( C \) is symmetric or nonsymmetric if its associated ribbon \( \rho_C \) is symmetric or nonsymmetric respectively. We note that Harary and Schwenk used an equivalent notion of symmetric caterpillars in their proof.

As \( G(\rho) = G(I(\rho)) \), we shall consider ribbon diagrams (modulo \( I \)). Each ribbon comes from some walk consisting of north and east steps. We will use \( I \) to always choose the representative (modulo \( I \)) that begins with an east step.

Having \( n + 4 \) vertices in a caterpillar requires \( n + 3 \) edges, that is, a ribbon with \( n + 3 \) boxes, or a northeast walk with \( n + 2 \) steps. We have two choices (east or north) for all but the first step in the walk, so there are \( 2^{n+1} (n+3) \)-box ribbons (modulo \( I \)).

By Lemma 2.2.1 and Lemma 2.2.2, for each caterpillar \( C \) we obtain the pair of ribbons \( (\rho_C, R(\rho_C)) \), that have \( C \) as their Ferrers graph. We have that \( \rho_C = R(\rho_C) \) if, and only if, \( \rho_C \) is symmetric. So nonsymmetric caterpillars receive pairs of distinct ribbons. Applying \( G \) to either member of the pair yields \( C \), so distinct caterpillars will receive distinct ordered pairs. Every pair of the form \( (\rho, R(\rho)) \) will show up—namely, from the caterpillar \( C = G(\rho) \) provided in the proof of Proposition 2.1.2. Further, by Lemma 2.2.1 and Lemma 2.2.2, \( \rho_C \) and \( R(\rho_C) \) are the only ribbons (modulo \( I \)) that have \( C \) as their Ferrers graph.

Hence, in computing \( G(\rho) \) for each of the \( 2^{n+1} (n+3) \)-box ribbons \( \rho \) (modulo \( I \)), we will obtain each of the nonsymmetric \((n+4)\)-vertex caterpillars twice (for both \( \rho_C \) and \( \sigma_C \)) and the symmetric \((n+4)\)-vertex caterpillars only once, since they have \( \rho_C = \sigma_C \) by Lemma 2.2.2.

Thus, if \( S_\varphi(n + 4) \) is the number of symmetric \((n + 4)\)-vertex caterpillars, \( N_\varphi(n+4) \) is the number of nonsymmetric \((n+4)\)-vertex caterpillars, and \( \varphi(n+4) \) is the number of nonisomorphic \((n + 4)\)-vertex caterpillars, then

\[
2^{n+1} = 2N_\varphi(n + 4) + S_\varphi(n + 4).
\]
Since \( \mathcal{C}(n+4) = \mathcal{N}_\varphi(n+4) + \mathcal{S}_\varphi(n+4) \), this becomes

\[ 2^{n+1} = 2 \mathcal{C}(n+4) - \mathcal{S}_\varphi(n+4). \]

Hence \( \mathcal{C}(n+4) = 2^n + \frac{\mathcal{S}_\varphi(n+4)}{2} \), and it only remains to count \( \mathcal{S}_\varphi(n+4) \).

By Lemma 2.2.4, we find that there are \( 2^{\frac{n+3}{2}} \) \( R \)-symmetric \((n+3)\)-box ribbons when \( n \) is odd (i.e., \( n+3 \) is even), and both \( 2^{\frac{n+1}{2}} \) \( R \)-symmetric \((n+3)\)-box ribbons and \( 2^{\frac{n+1}{2}} \) \( C \)-symmetric \((n+3)\)-box ribbons when \( n \) is even (i.e. \( n+3 \) is odd).

Since \( \rho \neq I(\rho) \) for every \( \rho \) we find

1. If \( n \) is odd there are \( \frac{2^{\frac{n+3}{2}}}{2} = 2^{\frac{n+1}{2}} = 2^{\frac{n+1}{2}+1} \) symmetric \((n+3)\)-box ribbons (modulo \( I \)), and

2. If \( n \) is even there are \( \frac{2^{\frac{n+1}{2}}+2^{\frac{n+1}{2}+1}}{2} = 2^{\frac{n+1}{2}+1} = 2^{\frac{n+1}{2}+1} \) symmetric \((n+3)\)-box ribbons (modulo \( I \)).

Hence in each case we obtain \( |\mathcal{S}_\varphi(n+4)| = 2^{\frac{n+1}{2}+1} \).

Thus \( \mathcal{C}(n+4) = 2^n + 2^{\frac{n+1}{2}} \) as desired.

2.4 More with Corners

Given an \( n \)-box ribbon \( \rho \), we shall enumerate its boxes from southwest to northeast with the numbers \( 1, 2, 3, \ldots, n \). The corners of \( \rho \) give rise to a set \( A_\rho \subseteq \{2, 3, \ldots, n-1\} \) called the corner set of \( \rho \).

Conversely, given a set \( A \subseteq \{2, 3, \ldots, n-1\} \), we associate the \( n \)-box ribbon \( \rho_A \) obtained by taking the northeast walk starting with an step east that changes direction at the \( i \)-th box if, and only if, \( i \in A \). In this way, we are guaranteed that \( A_{\rho(A)} = A \) for each \( A \subseteq \{2, 3, \ldots, n-1\} \).

Example Consider the ribbon class (modulo \( I \)),

\[ [\rho]_I = \{ENNEEE, NEENNN\} \]

shown below. We have labelled the boxes of each representative of \( [\rho]_I \) from southwest to northeast with the numbers \( 1, 2, \ldots, 7 \) and obtained the corner set \( \{2, 4\} \) in each case. Further, the members of \( [\rho]_I \) are the only seven-box ribbons that have corner set \( \{2, 4\} \).
Lemma 2.4.1 Let \( \rho \) and \( \sigma \) be ribbons. Then \( A_\rho = A_\sigma \) if, and only if, \( \sigma \in [\rho]_I \).

Proof We shall think of \( \rho \) and \( \sigma \) as northeast walks. Both ribbons \( \rho \) and \( I(\rho) \) as walks change directions after the same number of steps, hence their corner sets will be identical.

Conversely, if \( \rho \) and \( \sigma \) have the same corner set they change directions after the same number of steps. Hence, if the first step of \( \rho \) and \( \sigma \) are in the same direction, then \( \rho = \sigma \), while if the first step of \( \rho \) and \( \sigma \) are in different directions, then \( \rho = I(\sigma) \).

Given a set \( A \subseteq \{2, 3, \ldots, n-1\} \), we call the set \( A' = \{n+1-i \mid i \in A\} \) the reflection of \( A \). If the set \( A \) satisfies \( A = A' \) we shall call \( A \) a symmetric subset.

Lemma 2.4.2 Let \( \rho \) be a ribbon. Then \( A_{R(\rho)} = A'_\rho = A_C(\rho) \).

Proof In computing \( A_\rho \) and \( A_{R(\rho)} \) (or \( A_\rho \) and \( A_{C(\rho)} \) resp.) we are enumerating the boxes from opposite ends of the ribbon. Since the the \( i \)-th box from one end of \( \rho \) is the \( (n+1-i) \)-th box from the other end, the result follows.

Lemma 2.4.3 An \( n \)-box ribbon \( \rho \) is symmetric if, and only if, its corner set \( A_\rho \) is a symmetric subset of \( \{2, 3, \ldots, n-1\} \). Moreover, if \( \rho \) is symmetric, then \( \rho \) is \( C \)-symmetric if, and only if, \( \frac{n+1}{2} \in A_\rho \).

Proof If \( \rho \) is symmetric, then by Lemma 2.4.2 we find that \( A_\rho \) is symmetric.

Now suppose \( A_\rho \) is symmetric, that is \( A_\rho = A'_\rho \). We take the sets \( A_1 = A_\rho \cap \{2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \}, \ A_2 = A'_\rho \), and construct \( n \)-box ribbons \( \rho_i = \rho_{A_i} \) for \( i = 1, 2 \). Since we can assume both \( \rho \) and \( \rho_1 \) begin with a step east, \( \rho_1 \) shares its first half with \( \rho \), while one of \( \rho_2 \) or \( I(\rho_2) \) (call it \( \rho_3 \) for convenience) shares its second half with \( \rho \).

Since \( A_\rho \) is symmetric, \( A_\rho = A_1 \cup A_2 \). Given \( A_2 = A'_\rho \), from Lemma 2.4.2 we have that \( A_{R(\rho_1)} = A_2 \). Then by using Lemma 2.4.1 we find that \( \rho_3 \in [R(\rho_1)]_I = \{R(\rho_1), C(\rho_1)\} \). When \( \rho_3 = R(\rho_1) \) the ribbon \( \rho \) is \( R \)-symmetric, while if \( \rho_3 = C(\rho_1) \) the ribbon \( \rho \) is \( C \)-symmetric.

From Lemma 2.2.4 a symmetric ribbon \( \rho \) can be \( C \)-symmetric if, and only if, \( n \) is odd. Further, the proof showed \( \rho \) was \( C \)-symmetric if, and only if, the \( \frac{n+1}{2} \)-th box was made to be a corner.

We will let \( R(n, c) \), \( C(n, c) \), and \( S(n, c) \) denote the number of distinct \( n \)-box ribbon classes with \( c \) corners which are \( R \)-symmetric, \( C \)-symmetric, and symmetric, respectively. Hence

\[
S(n, c) = R(n, c) + C(n, c).
\]

Further, we have

Proposition 2.4.4 Let \( c \) be a nonnegative integer.

1. If \( c \) is even, then for each \( n \in \mathbb{N} \)
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(a) \( R(n, c) = \left( \frac{n-2}{2} \right) \) and

(b) \( C(n, c) = 0 \).

2. If \( c \) is odd, then

(a) \( C(n, c) = \left( \frac{n-3}{2} \right) \) when \( n \) is odd,

(b) \( C(n, c) = 0 \) when \( n \) is even, and

(c) \( R(n, c) = 0 \) for each \( n \in \mathbb{N} \).

Proof We shall use the notation \( A_1 = \{2, 3, \ldots, \left\lceil \frac{n+1}{2} \right\rceil \} \cap A_\rho \) and \( A_2 = A_\rho' \) for the duration of the proof. Since \( A_2 = A_\rho' \), we have \( |A_1| = |A_2| \).

Suppose that \( \rho \) is an \( R \)-symmetric ribbon. Since \( \rho \) is symmetric, we must have \( A_1 \cup A_2 = A_\rho \). If \( n \) is even then we find \( A_1 \subseteq \{2, 3, \ldots, \frac{n}{2}\} \) and \( A_2 \subseteq \{\frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n-1\} \) so clearly \( A_1 \cap A_2 = \emptyset \). In the case when \( n \) is odd have \( A_1 \subseteq \{2, 3, \ldots, \frac{n+1}{2}\} \) and \( A_2 \subseteq \{\frac{n+1}{2}, \frac{n+1}{2} + 1, \ldots, n-1\} \) and using Lemma 2.4.3 to note that \( \frac{n+1}{2} \not\in A_\rho \), we have \( A_1 \cap A_2 = \emptyset \). In either case

\[ c = |A_\rho| = |A_1| + |A_2| = 2|A_1| \]

is even. Thus we find \( R(n, c) = 0 \) for odd values of \( c \).

When \( n \) is even, then from Lemma 2.2.4 there are no \( C \)-symmetric \( n \)-box ribbons. That is, \( C(n, c) = 0 \) whenever \( n \) is even. We have so far proved parts (b) and (c) of 2.

For (b) we use Lemma 2.4.3. If \( \rho \) is \( C \)-symmetric then, exactly as above, \( A_1 \cup A_2 = A_\rho \). Further, from Lemma 2.4.3 we have \( \frac{n+1}{2} \in A_\rho \). Hence

\[ c = |A_\rho| = |A_1| + |A_2| - 1 = 2 \times |A_1| - 1 \]

is odd, contradicting our assumption that the number of corners was even. Therefore there are no \( C \)-symmetric even-cornered ribbons.

Now when both \( n \) and \( c \) are even, we create a map from the \( R \)-symmetric classes \( [\rho] \) to the subsets of \( \{2, 3, \ldots, \frac{n}{2}\} \) of cardinality \( \frac{c}{2} \) by sending \( [\rho] \) to \( A_1 \). Since \( \rho \) is symmetric, Lemma 2.4.1 and Lemma 2.4.2 show this map is well-defined.

To construct an inverse, given a set \( A \subseteq \{2, 3, \ldots, \frac{n}{2}\} \) with \( |A| = \frac{c}{2} \), we first take the ribbon \( \hat{\rho} = \rho(A) \) with \( \frac{n}{2} \) boxes. We then rotate \( \hat{\rho} \) around the midpoint of either the north or east face of its top right box to complete the desired \( n \)-box ribbon. Both of the rotations of \( \hat{\rho} \) will form an \( R \)-symmetric ribbon, one of these having its \( \frac{n}{2} \)-th box a corner, while the other does not. We choose one or the other depending on whether or not \( \frac{n}{2} \in A \). The resulting \( R \)-symmetric ribbon \( \rho \) clearly has \( A_1 = A_\rho \cap \{2, 3, \ldots, \frac{n}{2}\} = A \).

Since we have constructed an inverse for our map, it is a bijection. Hence, if both \( n \) and \( c \) are even \( R(n, c) = \left( \frac{n-2}{2} \right) \), proving half of (a).
For the case when \( n \) is odd, we have a similar correspondence.

Let \( \rho \) be a symmetric \( n \)-box ribbon, then \( A_1 \subseteq \{2, 3, \ldots, \frac{n+1}{2}\} \) implies \( A_2 \subseteq \{\frac{n+1}{2}, \ldots, n-2, n-1\} \) and thus \( A_1 \cap A_2 \subseteq \{\frac{n+1}{2}\} \). Since \( \rho \) is symmetric, Lemma 2.4.3 shows that \( A_\rho \) is also symmetric; this gives \( A_1 \cup A_2 = A_\rho \).

If \( \frac{n+1}{2} \notin A_1 \) then \( c = 2 \times |A_1| \) is even. If \( \frac{n+1}{2} \in A_1 \), then \( \frac{n+1}{2} \in A_2 \) as well, and thus \( c = 2 \times |A_1| - 1 \) is odd. These two cases correspond to \( R \)-symmetric and \( C \)-symmetric ribbon classes respectively.

We create a map from the symmetric \( n \)-box ribbon classes to the subsets of \( \{2, 3, \ldots, \frac{n+1}{2}\} \) of cardinality \( \lfloor \frac{c+1}{2} \rfloor \) by sending \( [\rho] \) to \( A_1 \). Again, this is well-defined by Lemma 2.4.1 and Lemma 2.4.2.

To construct an inverse, given a set \( A \subseteq \{2, 3, \ldots, \frac{n+1}{2}\} \) we first take the half ribbon \( \hat{\rho} = \rho(A) \). We then rotate this ribbon around the center of its top right box or reflect across that top right box to complete the desired \( n \)-box ribbon. Both of the ribbons obtained from \( \hat{\rho} \) will be symmetric; the ribbon obtained by reflecting has its \( \frac{n+1}{2} \)-th box a corner and hence is \( C \)-symmetric, while the ribbon obtained by rotating does not have its \( \frac{n+1}{2} \)-th box a corner and is therefore \( R \)-symmetric. We shall choose one or the other depending on whether or not \( \frac{n+1}{2} \in A \). The resulting symmetric ribbon \( \rho \) has \( A_1 = A_\rho \cap \{2, 3, \ldots, \frac{n+1}{2}\} = A_{\hat{\rho}} = A \), as desired.

Since we have constructed an inverse for our map, we have a bijection. Hence, for odd \( n \):

1. If \( c \) is even, then \( \mathcal{R}(n, c) = \binom{n-3}{\frac{c+1}{2}} \), and

2. If \( c \) is odd then \( \mathcal{C}(n, c) = \binom{n-3}{\frac{c+1}{2}} \).

This completes the proof. \( \square \)

**Example** Directly after the proof of Lemma 2.2.4 we looked at an example where \( n = 8 \), so \( \frac{n}{2} = 4 \). Labelling the boxes of the diagrams of that example from southwest to northeast we see the two cornered \( R \)-symmetric class generated by \( \{2\} \subseteq \{2, 3, 4\} \), and the four cornered \( R \)-symmetric class generated by \( \{2, 4\} \subseteq \{2, 3, 4\} \).
Both the ribbons obtained are $R$-symmetric.

**Example** Directly after the proof of Lemma 2.2.4 we looked at an example where $n = 11$. Labelling the boxes from southwest to northeast we saw the two cornered $R$-symmetric ribbon generated by $\{3, 4\} \subseteq \{2, 3, 4, 5, 6\}$ and the $C$-symmetric ribbon generated by $\{3, 4, 6\} \subseteq \{2, 3, 4, 5, 6\}$.

The set with $\frac{n+1}{2} = 6 \notin A$ gives rise to the $R$-symmetric ribbon, while the set with $\frac{n+1}{2} = 6 \in A$ gives rise to the $C$-symmetric ribbon.

We let $\mathcal{N}(n, c)$ denote the number of nonsymmetric, $n$-box ribbon classes with $c$ corners. Then there are $\mathcal{S}(n, c) + \mathcal{N}(n, c)$ $n$-box ribbon classes with $c$ corners, and so by Corollary 2.2.3 and Theorem 2.3.1 we obtain

$$\sum_c (\mathcal{S}(n, c) + \mathcal{N}(n, c)) = 2^{n-3} + 2^\lceil \frac{n-3}{2} \rceil.$$ 

In computing $\mathcal{N}(n, c)$ we find

**Proposition 2.4.5** $\mathcal{N}(n, c) = \frac{1}{2} \binom{n-2}{c} - \frac{1}{2} \mathcal{S}(n, c)$.

**Proof** Given a set of size $c$, say $A = \{a_1, a_2, \ldots, a_c\} \subseteq \{2, 3, \ldots, n-1\}$ where $a_1 < a_2 < \ldots < a_c$, we can obtain the ribbon $\rho_A$ by taking the northeast walk $E^{a_1-1}N^{a_2-a_1}E^{a_3-a_2} \ldots (N$ or $E)^{n-a_c}$. The last step being north when $c$ is odd, and east when $c$ is even.

From Lemma 2.4.3, $\rho$ is symmetric if, and only if, $A_\rho$ is a symmetric subset. Hence, looking at $[\rho_A]$ for each $A \subseteq \{2, 3, \ldots, n-1\}$, the nonsymmetric classes $[\rho]$ will appear exactly twice, for both $A_\rho$ and $A'_\rho$, while the symmetric ribbon classes will appear only once.

Hence $2\mathcal{N}(n, c) + \mathcal{S}(n, c) = \binom{n-2}{c}$, giving the desired result.  


Chapter 3

Coefficients of $p_\lambda$ in $X_G$

3.1 Graphs

In this chapter we begin looking at the coefficients of $X_G$. From Theorem 1.3.2 we have the following expansion for the chromatic symmetric function of a $m$-vertex graph:

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(F)},$$

(3.1)

where $E$ is the edge set of $G$ and given $F \subseteq E$, $\lambda(F)$ is the partition of $m$ whose parts correspond to the size of the connected components of the spanning subgraph of $G$ with edge set $F$.

It is clear that $X_G$ is homogeneous of degree $m$. Hence graphs with different number of vertices have different chromatic symmetric functions.

We use the notation $[p_\lambda]X_G$ to denote the coefficient of $p_\lambda$ in the expansion of $X_G$ in terms of the basis $\{p_\lambda \mid \lambda \vdash m\}$ of $\Lambda^m$. We have

$$[p_\lambda]X_G = \sum_{F \subseteq E} (-1)^{|F|}.$$

(3.2)

The only way to obtain a partition of type $(1^m)$ would be to include no edges of $G$, so the only contribution to the coefficient of $p_{(1^m)}$ comes from $F = \emptyset$. Hence, for every graph $G$

$$[p_{(1^m)}]X_G = 1.$$

(3.3)

The only way to obtain a partition of type $(2,1^{m-2})$ is to include only one edge of $G$. Hence the only contributions to the coefficient of $p_{(2,1^{m-2})}$ comes from the $\lambda(F)$ with $|F| = 1$. From Equation 3.2 we obtain

$$[p_{(2,1^{m-2})}]X_G = -|E|,$$

(3.4)

for every graph $G$.

Similarly we see

$$[p_{(2^k,1^{m-2k})}]X_G = (-1)^k \mu_k(G),$$

(3.5)

where $\mu_k(G)$ is the number of ways of selecting $k$ vertex-disjoint edges in $G$, that is, the number of matchings in $G$ of size $k$.

Further if $m$ is even, then the number of matchings which use every vertex as an endpoint exactly once, that is the number of perfect matchings $\mu(G)$, has

$$[p_{(\frac{m}{2})}]X_G = (-1)^{\frac{m}{2}} \mu(G).$$

(3.6)
3.2 Trees

We now consider the case where our graph $G$ is an $m$-vertex tree $T$, for some $m \geq 3$. As trees are connected graphs with minimal edge sets, the number of connected components increases by one for each edge that is removed from the tree. Hence to obtain a partition with $j$ parts, we would need to remove exactly $j - 1$ of the $m - 1$ edges in $E$. Hence we are looking at sets $F \subseteq E$ with $|F| = (m - 1) - (j - 1) = m - j$. Thus we find

$$X_T = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(F)}$$

$$= \sum_{j=1}^{m} \sum_{F \subseteq E, \lambda(F) = j} (-1)^{m-j} p_{\lambda(F)}$$

where the inner sum is over all $F \subseteq E$ whose induced partition $\lambda(F)$ has $j$ parts.

Collecting terms that share the same partition type $\lambda$ gives

$$[p_{\lambda}]X_T = \sum_{F \subseteq E, \lambda(F) = \lambda} (-1)^{m-l(\lambda)}. \quad (3.7)$$

If we let $\mathcal{F}_{T,\lambda} = \{ F \subseteq E : \lambda(F) = \lambda \}$ be the set of all edge subsets with induced partition of type $\lambda$ we obtain

$$[p_{\lambda}]X_T = |\mathcal{F}_{T,\lambda}|(-1)^{m-l(\lambda)}. \quad (3.8)$$

Thus we can write

$$X_T = \sum_{\lambda=m}^{m} [\mathcal{F}_{T,\lambda}](-1)^{m-l(\lambda)}p_{\lambda}. \quad (3.9)$$

**Corollary 3.2.1** For $m$-vertex trees $T_1$ and $T_2$, $X_{T_1} = X_{T_2}$ if, and only if, $|\mathcal{F}_{T_1,\lambda}| = |\mathcal{F}_{T_2,\lambda}|$ for every $\lambda \vdash m$.

For $\lambda = (1^m)$ we get $\mathcal{F}_{T,\lambda} = \{\emptyset\}$, so as we already saw for general graphs in Equation 3.3, $[p_{(1^m)}]X_T = 1$. Since removing any edge disconnects the tree, for $\lambda = (m)$ we find that $\mathcal{F}_{T,\lambda} = \{E\}$. Hence

$$[p_{(m)}]X_T = (-1)^{m-1}. \quad (3.10)$$

In considering the $\lambda(F)$ with exactly two parts, we are looking at those partitions obtained by removing a single edge from $T$. We use $2_T$ to denote this set of partitions, that is

$$2_T = \{\lambda(E - e) : e \in E\}, \quad (3.11)$$
and shall call it the collection of two part partitions of $T$. We note that $2_T$ is a multiset. We shall use the notation $j^t$ to denote $t$ copies of $j$ so that, for example, $\{1^3, 3^2, 4\}$ denotes the multiset $\{1, 1, 1, 3, 3, 4\}$.

We have

**Corollary 3.2.2** If $T_1$ and $T_2$ are trees with $2_{T_1} \neq 2_{T_2}$, then $X_{T_1} \neq X_{T_2}$.

**Proof** If $X_{T_1} = X_{T_2}$, then $|\mathcal{F}_{T_1, \lambda}| = |\mathcal{F}_{T_2, \lambda}|$ for each $\lambda$, in particular, for each $\lambda$ with two parts. Hence $2_{T_1} = 2_{T_2}$.

Each leaf of $T$ is the endpoint of some edge, and since $m \geq 3$ no edge of $T$ has both its endpoints being leaves. Thus we have the same number of leaves in $T$ as the number of edges in $T$ which have a leaf as an endpoint. Removing a single edge with a leaf endpoint from $T$ gives the partition $(m-1, 1)$, and such a removal is the only way to obtain this partition.

Hence if $L(T)$ is the number of leaves of $T$, then we have

$$[p_{(m-1,1)}]X_T = (-1)^{m-2}L(T). \tag{3.12}$$

Thus we obtain

**Corollary 3.2.3** The chromatic symmetric function distinguishes trees with different number of leaves.

If $\lambda = (k, 1^{m-k})$, then the sets $F \subseteq E$ with $\lambda(F) = \lambda$ are those edge sets that give graphs with a single connected component of size $k$, and have all other connected components of size 1. The component of size $k$, being a connected subgraph of a tree, must also be a tree. Further, every $k$-vertex subtree shows up in this way. Hence we find

$$[p_{(k,1^{m-k})}]X_T = (-1)^{k-1}T_k, \tag{3.13}$$

where $T_k$ is the number of $k$-vertex subtrees of $T$. From this we obtain the following.

**Corollary 3.2.4** Let $k$ be a positive integer. The chromatic symmetric function $X_G$ distinguishes $m$-vertex trees which contain a different number of $k$-vertex subtrees.

Given a $\lambda \vdash m$, we let $T_\lambda$ be the number of partitions of $T$ into disjoint subtrees of size $\lambda_1, \lambda_2, \ldots, \lambda_j$, where $j = l(\lambda)$ is the number of parts of $\lambda$. Generalizing Corollary 3.2.4 we obtain

**Theorem 3.2.5** If $T$ is a tree and $\lambda$ is a partition of $m$, then

$$[p_\lambda]X_T = (-1)^{m-l(\lambda)}T_\lambda.$$
Given two edges of a tree, they are either incident or disjoint. Hence we have

\[ [p_{(2^2,1^{m-4})}]X_T + [p_{(3,1^{m-3})}]X_T = \binom{|E|}{2}. \quad (3.14) \]

From this we see that knowing one of the coefficients of \( p_{(2^2,1^{m-4})} \) and \( p_{(3,1^{m-3})} \) is equivalent to knowing the other. Consequently, when checking which coefficients of \( X_T \) and \( X_T' \) are the same, we need only check one of the coefficients of \( p_{(2^2,1^{m-4})} \) or \( p_{(3,1^{m-3})} \).

Now consider looking at the sets \( F \subseteq E \) with \( |F| = k \). Since these are edge subsets of a tree, each additional edge in \( F \) decreases the number of connected components of the spanning subgraph with edge set \( F \) by one. Hence, we have \( l(\lambda(F)) = m - k \). Conversely, if we assume that \( F \subseteq E \) is such that \( l(\lambda(F)) = m - k \), then we must have \( |F| = k \). Thus, extending Equation 3.14 we find

**Proposition 3.2.6** Let \( T \) be an \( m \)-vertex tree and \( 0 \leq k \leq m - 1 \), then

\[ \sum_{\lambda \vdash m \atop l(\lambda) = m - k} [p_{\lambda}]X_T = (-1)^k \binom{|E|}{k} = (-1)^k \binom{m - 1}{k}. \]

**Proof** By the comments preceding the proposition, we find

\[ \sum_{\lambda \vdash m \atop l(\lambda) = m - k} [p_{\lambda}]X_T = \sum_{F \subseteq E \atop |F| = k} [p_{\lambda}]X_T \]

and by Equation 3.1 we have

\[ \sum_{F \subseteq E \atop |F| = k} [p_{\lambda}]X_T = \sum_{F \subseteq E \atop |F| = k} (-1)^{|F|} = (-1)^k \binom{|E|}{k} \]

where \( |E| = m - 1 \), since \( T \) is a tree.

### 3.3 Caterpillar Coefficients via Ribbon Diagrams

We take \( m = n + 1 \) and specialize to the case when the tree \( T \) is an \((n+1)\)-vertex caterpillar. By Proposition 2.1.2 we have an \( n \)-box ribbon \( \rho \) with \( T = G(\rho) \). By Corollary 2.2.3, \( [\rho] = \{\rho, R(\rho), C(\rho), I(\rho)\} \) is the set of all ribbons with \( T = G(\rho) \).

From the proof of Proposition 2.1.2 we saw that given a ribbon \( \rho \), the number of leaves in \( G(\rho) \) is precisely the number of boxes of \( \rho \) which are noncorners. Thus, the number of leaves of \( G(\rho) \) determines the number of corners of \( \rho \), and vice-versa. Hence from Corollary 3.2.3 we obtain
Corollary 3.3.1 The chromatic symmetric function distinguishes ribbons with different numbers of corners.

We define $c_\rho = |A_\rho|$ to be the number of corners of $\rho$. For each corner $c \in A_\rho$, deleting the box $c$ from the diagram $\rho$ would leave two ribbons with $c - 1$ and $n - c$ boxes respectively. Hence removing the corresponding edge from $G(\rho)$ induces the partition

$$\lambda(c) = (c, n - c + 1). \quad (3.15)$$

Thus, from Equation 3.11 we obtain

**Proposition 3.3.2** For an $n$-box ribbon $\rho$ with set of corners $A_\rho$, we find that the collection of two part partitions of the tree $G(\rho)$ is given by

$$2_{G(\rho)} = \{\lambda(c) | c \in A_\rho\} \cup \{(n, 1)^{n-c}\}.$$ 

For the sake of brevity, we shall henceforth write $2_\rho$ in place of $2_{G(\rho)}$.

**Example** For the seven-box ribbon $\rho$ shown below, we use Equation 3.15 and Proposition 3.3.2 to calculate $2_\rho$. We have five boxes in $\rho$ which are noncorners, so we obtain five copies of $(7, 1) \vdash 8$ in $2_\rho$. For the corners of $\rho$ we use Equation 3.15 with $A_\rho = \{2, 5\}$ and $A'_\rho = \{3, 6\}$ to obtain the partitions $(6, 2)$ and $(5, 3)$ respectively.

Hence $2_\rho = \{(7, 1)^5, (6, 2), (5, 3)\}$.

Whenever $\lambda$ is a partition with two parts and $T$ is a caterpillar there is a trivial bound on the coefficients of $p_\lambda$, namely

**Proposition 3.3.3** Let $T$ be an $(n + 1)$-vertex caterpillar and $\lambda$ have two parts. Then either

1. $(-1)^{n-1}[p_\lambda]X_T = L(T)$, if $\lambda = (n, 1)$, or
2. $0 \leq (-1)^{n-1}[p_\lambda]X_T \leq 2$ otherwise.
Proof From Equation 3.12, we have \([p_\lambda X_T = (-1)^{n-1}L(T)\) in the case \(\lambda = (n, 1)\. Any other partition \(\lambda\) with two parts may arise as \(\lambda(c)\) for some corner \(c\) of \(\rho\). We shall show that any such \(\lambda\) can arise at most twice.

Consider computing \(\lambda(c) = (c, n + 1 - c)\) for each corner \(c\) by starting from the bottom left corner and moving along the ribbon in a northeast direction. The first component of the pair \((c, n + 1 - c)\) strictly increases and the second strictly decreases as we travel from corner to corner in this direction, hence any partition \((c, n + 1 - c)\) could only appear at most twice, that is if both \(c\) and \(n + 1 - c \in A_\rho\)

Recall from the comments at the end of §1.4 we can use the notation \(X_\rho\) in place of \(X_{G(\rho)}\). From the proof of Proposition 3.3.3 we see that

**Corollary 3.3.4** If \(\lambda = (k, n + 1 - k)\), \(1 < k < n\), is a partition of \(n + 1\) into two parts and \(\rho\) is a \(n\)-box ribbon, then \([p_\lambda X_\rho = (-1)^{n-1}\{(k, n-k) \cap A_\rho\}\).

We now return to Equation 3.14, and inspect the coefficients of both \(p_{(2,1^n-3)}\) and \(p_{(3,1^n-2)}\) in terms of the ribbon diagram. First, looking at the case of disjoint pairs of edges in \(T\), we find

**Proposition 3.3.5** We have \([p_{(2,1^n-3)} X_\rho = \sum_{b \in \rho} ssw(b) = \sum_{b \in \rho} sne(b)\), where both sums are over all boxes \(b\) in the ribbon \(\rho\) and \(ssw(b)\) (\(sne(b)\) respectively) denotes the number of boxes both strictly south and strictly west (north and east respectively) of \(b\).

**Proof** The coefficient of \(p_{(2,1^n-3)}\) is added to for each \(F \subseteq E\) with \(|F| = 2\) and \(\lambda(F) = (2, 1^{n-3})\). This occurs exactly when \(F\) consists of a pair of disjoint edges. Hence we are interested in how many pairs of boxes in our diagram do not share a row or a column.

For each \(b\), all \(ssw(b)\) boxes strictly south and strictly west of \(b\) are edges that are disjoint from the edge corresponding to \(b\). Conversely, all pairs of disjoint edges correspond to boxes in different rows and columns, and so one is strictly south and west of the other. Hence, each pair of edges will be counted exactly once. This shows \([p_{(2,1^n-3)} X_\rho = \sum_{b \in \rho} ssw(b)\). The fact that \([p_{(2,1^n-3)} X_\rho = \sum_{b \in \rho} sne(b)\), is proved in exactly the same manner.

We now turn to the coefficient of \(p_{(3,1^n-2)}\) in \(X_\rho\).

**Proposition 3.3.6** Let \(\rho\) have \(o\) rows and \(q\) columns, and let \(r_j\) (\(c_k\) resp.) denote the number of boxes in the \(j\)-th row (\(k\)-th column resp.) of \(\rho\). Then

\[
[p_{(3,1^n-2)} X_\rho = \sum_{j=1}^{o} \binom{r_j}{2} + \sum_{k=1}^{q} \binom{c_k}{2}\]
3.4 Composition Classes

Proof A pair of edges in \( T = G(\rho) \) are incident if, and only if, the corresponding pair of boxes in \( \rho \) share a row or column in the diagram of \( \rho \). ■

Example Consider the ribbon \( \rho \) generated by the walk ENNE. For each box \( b \) in \( \rho \), we highlight the boxes of \( \rho \) which are both strictly north and strictly east of \( b \).

![Diagram of ENNE ribbon]

Proposition 3.3.5 shows that \( |\rho_{(2,1^{n-3})}|X_{\rho} = 3 + 1 + 1 + 0 + 0 = 5 \). From Proposition 3.3.6 we have \( |\rho_{(3,1^{n-2})}|X_{\rho} = 1 + 0 + 1 + 0 + 3 + 0 = 5 \). Additionally, we can check that

\[
|\rho_{(2,1^{n-3})}|X_{\rho} + |\rho_{(3,1^{n-2})}|X_{\rho} = 5 + 5 = 10 = \binom{5}{2},
\]

as predicted in Equation 3.14.

3.4 Composition Classes

Recall that given a composition of \( n \), say \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) \models n \), we may associate an \( n \)-box ribbon, \( \rho_\alpha \), with \( k \) rows by letting the \((k + 1 - i)\)-th row of the diagram have \( \alpha_i \) boxes. Conversely, starting with a ribbon diagram \( \rho \) with \( n \) boxes, we get a composition of \( n \), \( \alpha_\rho \) by counting the number of boxes in a given row, for each row from bottom to top. This gives a bijection between the set of compositions of \( n \) and the set of \( n \)-box ribbon diagrams. Hence we shall sometimes use \( \alpha \) in referring to both the composition and its corresponding ribbon. Thus for a composition \( \alpha \) with corresponding ribbon \( \rho_\alpha \), we shall abbreviate the terms \( c_{\rho_\alpha}, X_{\rho_\alpha}, 2_{\rho_\alpha}, \) and \( A_{\rho_\alpha} \), by \( c_\alpha, X_\alpha, 2_\alpha, \) and \( A_\alpha \) respectively.

Since we are only interested in the underlying Ferrers graph, we only care about the various ribbon classes \( [\rho] = \{\rho, R(\rho), C(\rho), I(\rho)\} \). We use the notation \( \alpha^r = R(\alpha), \alpha^c = C(\alpha), \) and \( \alpha^i = I(\alpha) \) when we are dealing with compositions as ribbons. Thus we have the corresponding composition classes \( [\alpha] = \{\alpha, \alpha^r, \alpha^c, \alpha^i\} \). We shall call \( \alpha^c \) the conjugate of \( \alpha \), \( \alpha^i \) the inversion of \( \alpha \), and \( \alpha^r \) the reversal of \( \alpha \). The choice of these names will be made clear in Proposition 3.4.1.

Since the ribbon classes partition the set of ribbons, the set of compositions are partitioned by the classes \( [\alpha] = \{\alpha, \alpha^r, \alpha^c, \alpha^i\} \). Each composition class represents one of the \( n \)-box ribbon classes, and hence one of the caterpillars with \((n + 1)\) vertices.
Example Let $\alpha = (2, 1, 1, 5)$, then

$$[\alpha] = \{(2, 1, 1, 5), (5, 1, 1, 2), (1, 1, 1, 1, 4, 1), (1, 4, 1, 1, 1, 1)\}.$$ 

Let $\beta = (1, 2, 1, 2, 1, 2, 1)$, which is palindromic, that is, it is its own reversal and

$$[\beta] = \{(1, 2, 1, 2, 1, 2, 1), (2, 3, 3, 2)\},$$ 

Let $\gamma = (2, 3, 2, 2, 1, 2, 1)$, which is its own conjugate and

$$[\gamma] = \{(2, 3, 2, 2, 1, 2, 1), (1, 2, 1, 2, 2, 3, 2)\}.$$ 

To calculate the members of the composition classes, we have the following.
Proposition 3.4.1 If \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) \) is a composition of \( n \) into \( k \) parts, then \( \alpha^r = (\alpha_k, \ldots, \alpha_3, \alpha_2, \alpha_1) \).

Now let \( \alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \ldots, \alpha_{i_j} \) be the subsequence of those parts of \( \alpha \) greater than 1.

1. If \( j = 0 \), (i.e. \( \alpha = (1,1,\ldots,1) \)), then \( \alpha^c = (n) \).
2. If \( 1 < i_1 \) and \( i_j < k \) (i.e. \( \alpha_1 = \alpha_k = 1 \)) then
   \[
   \alpha^c = (k - i_j + 1, 1^{\alpha_j-2}, \ldots, 1^{\alpha_k+1-2}, i_{h+1} - i_h + 1, 1^{\alpha_h-2}, \ldots, 1^{\alpha_1-2}, i_1).
   \]
3. If \( 1 = i_1 \) (i.e. \( \alpha_1 > 1 \)) then \( \alpha^c \) is the same as the case when \( 1 < i_1 \) and \( i_k < n \), except it ends with
   \[
   (\ldots, 1^{\alpha_2-2}, i_2 - i_1 + 1, 1^{\alpha_1-1}).
   \]
4. If \( i_j = k \) (i.e. \( \alpha_k > 1 \)) then \( \alpha^c \) is the same as the case when \( 1 < i_1 \) and \( i_k < n \), except it starts with
   \[
   (1^{\alpha_j-1}, i_j - i_{j-1} + 1, 1^{\alpha_{j-1}-2} \ldots).
   \]

Example If \( \alpha = (4,3,3,1,2) \), then \( i_1 = 1, i_2 = 2, i_3 = 3 \), and \( i_4 = 5 \). We can use the third and fourth cases of the above proposition to obtain

\[
\alpha^c = (1^{4-1}, 5 - 3 + 1, 1^{3-2}, 3 - 2 + 1, 1^{2-2}, 2 - 1 + 1, 1^{1-1})
= (1, 3, 1, 2, 1, 2, 1, 1, 1).
\]

Suppose \( \beta = (1,3,1,2,1,5,1) \). We have \( i_1 = 2, i_2 = 4, i_3 = 6 \), and \( k = l(\beta) = 7 \). This falls into the second case, and we obtain

\[
\beta^c = (k - 6 + 1, 1^{\beta_6-2}, 6 - 4 + 1, 1^{\beta_4-2}, 4 - 2 + 1, 1^{\beta_2-2}, 2)
= (7 - 6 + 1, 1^{5-2}, 6 - 4 + 1, 1^{3-2}, 4 - 2 + 1, 1^{3-2}, 2)
= (2, 1, 1, 1, 3, 3, 1, 2).
\]

Proof (of Proposition 3.4.1)

It is easy to see that rotating a ribbon reverses the order of the rows, so \( \alpha^r = (\alpha_k, \ldots, \alpha_3, \alpha_2, \alpha_1) \).

Part 1 merely states that conjugating a column with \( n \) boxes gives a row with \( n \) boxes. We now look to proving part 2. In this case, \( \alpha \) both begins and ends with a 1, say

\[
\alpha = (1^{i_1-1}, \alpha_{i_1}, \ldots, \alpha_{i_h}, 1^{i_h+1-i_h-1}, \alpha_{i_{h+1}}, \ldots, \alpha_{i_j}, 1^{k-i_j}).
\]
Chapter 3. Coefficients of \( p_\lambda \) in \( X_\lambda \)

For convenience, we shall let \( r(\alpha_i) \) denote the row corresponding to \( \alpha_i \).
Starting from the single box in \( r(\alpha_1) \), the ribbon takes \( i_1 - 1 \) north steps to arrive in the first box of \( r(\alpha_{i_1}) \).

For each \( h, 1 \leq h \leq j \), starting at the first box in \( r(\alpha_{i_h}) \), the ribbon takes \( \alpha_{i_h} - 1 \) steps east to traverse all the boxes in this row.
Then, for each \( 1 \leq h \leq j - 1 \), starting from the last box in \( r(\alpha_{i_h}) \), it takes \( i_{h+1} - i_h - 1 \) steps north to traverse the \( i_{h+1} - i_h - 1 \) rows of length one between \( r(\alpha_{i_h}) \) and \( r(\alpha_{i_{h+1}}) \), and then one more step to arrive in the first box of \( r(\alpha_{i_{h+1}}) \).
Thus there are \( i_{h+1} - i_h \) steps north between the steps east from the rows \( r(\alpha_{i_h}) \) and \( r(\alpha_{i_{h+1}}) \) in the diagram of \( \alpha \).
Finally, from the last box of \( r(\alpha_{i_j}) \) the ribbon takes \( k - i_j \) steps to traverse the \( k - i_j \) rows of length one that lie above \( r(\alpha_{i_j}) \) in the diagram of \( \alpha \).

From this we find the associated ribbon
\[
\rho_\alpha = N^{i_1} E^{\alpha_{i_1} - 1} \ldots E^{\alpha_{i_h} - 1} N^{i_{h+1} - i_h} E^{\alpha_{i_{h+1}} - 1} \ldots E^{\alpha_j - 1} N^{k-i_j}.
\]
Since conjugation switches north with west, and south with east, we have
\[
C(\rho_\alpha) = W^j S^{\alpha_{i_1} - 1} \ldots S^{\alpha_{i_h} - 1} W^{i_{h+1} - i_h} S^{\alpha_{i_{h+1}} - 1} \ldots S^{\alpha_j - 1} W^{k-i_j} = E^{k-i_j} N^{\alpha_{i_1} - 1} \ldots N^{\alpha_{i_{h+1}} - 1} E^{i_{h+1} - i_h} N^{\alpha_{i_h} - 1} \ldots N^{\alpha_j - 1} E^{i_j - 1}.
\]
Rewriting this as a composition by reversing the above procedure, we find
\[
\alpha^c = (k - i_j + 1, 1^{\alpha_{i_j} - 2}, \ldots, 1^{\alpha_{i_{h+1}} - 2}, i_{h+1} - i_h + 1, 1^{\alpha_{i_h} - 2}, \ldots, 1^{\alpha_2 - 2}, i_1)
\]
as desired.

We now turn to proving part 3. In this case \( \alpha \) has the form
\[
\alpha = (\alpha_{i_1}, 1^{i_2-i_1 - 1}, \alpha_{i_2}, \ldots, \alpha_{i_h}, 1^{i_{h+1} - i_h - 1}, \alpha_{i_{h+1}}, \ldots).
\]
In this case we obtain the corresponding ribbon
\[
\rho_\alpha = E^{\alpha_{i_1} - 1} N^{i_2 - i_1} E^{\alpha_{i_2} - 1} \ldots E^{\alpha_{i_h} - 1} N^{i_{h+1} - i_h} E^{\alpha_{i_{h+1}} - 1} \ldots
\]
and obtain
\[
C(\rho_\alpha) = S^{\alpha_{i_1} - 1} W^{i_2 - i_1} S^{\alpha_{i_2} - 1} \ldots S^{\alpha_{i_h} - 1} W^{i_{h+1} - i_h} S^{\alpha_{i_{h+1}} - 1} \ldots = \ldots N^{\alpha_{i_{h+1}} - 1} E^{i_{h+1} - i_h} N^{\alpha_{i_h} - 1} \ldots N^{\alpha_{i_2} - 1} E^{i_2 - i_1} N^{\alpha_{i_1} - 1}
\]
from which we compute the corresponding composition \( \alpha^c \) to be
\[
\alpha^c = (\ldots, 1^{\alpha_{i_{h+1}} - 2}, i_{h+1} - i_h + 1, 1^{\alpha_{i_h} - 2}, \ldots, 1^{\alpha_2 - 2}, i_2 - i_1 + 1, 1^{\alpha_1 - 1})
\]
as desired.
Now looking at part 4, \( \alpha \) has the form
\[
\alpha = (\ldots, \alpha_{ih}, 1^{i_{h+1} - i_h - 1}, \alpha_{ih+1}, \ldots, \alpha_{ij}, 1^{i_j - i_{j-1} - 1}, \alpha_{ij}).
\]
In this case we obtain the corresponding ribbon
\[
\rho_\alpha = \ldots E^{\alpha_h - 1} N^{i_{h+1} - i_h} E^{\alpha_{ih+1} - 1} \ldots E^{\alpha_{ij} - 1} N^{i_j - i_{j-1}} E^{\alpha_{ij}} - 1
\]
and obtain
\[
C(\rho_\alpha) = \ldots S^{\alpha_h - 1} W^{i_{h+1} - i_h} S^{\alpha_{ih+1} - 1} \ldots S^{\alpha_{ij} - 1} W^{i_j - i_{j-1}} S^{\alpha_{ij}} - 1
\]
\[
= N^{\alpha_j - 1} E^{i_j - i_{j-1}} N^{\alpha_{ij} - 1} \ldots N^{\alpha_{ih+1} - 1} E^{i_{h+1} - i_h} N^{\alpha_{ih} - 1} \ldots
\]
from which we compute the corresponding composition \( \alpha^c \) to be
\[
\alpha^c = (1^{\alpha_j - 1}, i_j - i_{j-1} + 1, 1^{\alpha_{j-1} - 2}, \ldots, 1^{\alpha_{i+1} - 1}, i_{i+1} - i_h + 1, 1^{\alpha_{i} - 2}, \ldots)
\]
as desired.

We obtain a few straightforward observations from this. For instance, \( \alpha \) is palindromic, that is \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) = (\alpha_k, \ldots, \alpha_3, \alpha_2, \alpha_1), \) if, and only if, the ribbon \( \rho_\alpha \) is \( R \)-symmetric. Since \( CR = RC, \) we see that \( (\alpha^c)^c = (\alpha^c)^R. \)

Also, since the number of rows plus the number of columns of a \( n \)-box ribbon gives the number of vertices in \( G(\rho), \) we have
\[
n + 1 = o + q,
\]
where \( o \) and \( q \) are the number of rows and columns of \( \rho_\alpha. \) Further, the number of rows (columns resp.) of \( \rho_\alpha \) is the number of parts of \( \alpha \) (\( \alpha^c \) resp.), hence
\[
n + 1 = l(\alpha) + l(\alpha^c).
\]

Also, as one may have already guessed, we have the following.

**Proposition 3.4.2** Let \( \alpha \) be a composition of \( n, \) where \( n \geq 2. \) Then \( ||\alpha|| = 2 \) or 4.

**Proof** We begin by showing we cannot have both \( \alpha = \alpha^r \) and \( \alpha = \alpha^c. \) Firstly, if a composition \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k) \) has \( \alpha = \alpha^r \) then from Proposition 3.4.1
\[
\alpha_1 = \alpha_k.
\]
Now, if \( \alpha = \alpha^c, \) from the Proposition 3.4.1 \( \alpha_1 > 1 \) if, and only if, \( \alpha_k = 1 \) and \( \alpha_1 = 1 \) if, and only if, \( \alpha_k > 1. \) That is, \( \alpha = \alpha^c \) gives
\[
\alpha_1 \neq \alpha_k.
\]
Therefore we cannot have both $\alpha = \alpha^r$ and $\alpha = \alpha^c$.

If all four members are distinct, we are done.

If $\alpha$ is palindromic, then $\alpha^r \neq \alpha^c$, lest both be $\alpha$, and since $(\alpha^c)^r = (\alpha^r)^c = \alpha^c$, $\alpha^c$ is also palindromic. Hence $[\alpha] = \{\alpha, \alpha^c\}$. Similarly, if $\alpha$ is its own conjugate, then $\alpha^c \neq \alpha^r$, lest both be $\alpha$, and since $(\alpha^r)^c = (\alpha^c)^r = \alpha^r$, $\alpha^r$ is also its own conjugate, giving $[\alpha] = \{\alpha, \alpha^r\}$.

The cases $\alpha^r = \alpha^t$ and $\alpha^c = \alpha^t$ can be reduced to one of the above cases by rotating or conjugating. The case $\alpha = \alpha^t$, and equivalently the case $\alpha^r = \alpha^c$, cannot occur since there are no $I$-symmetric ribbons for $n \geq 2$.

Hence the size of the composition classes is always 2 or 4, as desired.

Now, in interest of letting our corner work for ribbons carry over to compositions, we look to see, given $\alpha$, how many corners $c_\alpha$ does the ribbon $p_\alpha$ have?

**Proposition 3.4.3** If $\alpha$ is a composition with $k$ parts greater than one, then

1. $c_\alpha = 2k$ if $\alpha$ both begins and ends in a 1,

2. $c_\alpha = 2k - 2$ if neither the first nor last part of $\alpha$ is a 1, or

3. $c_\alpha = 2k - 1$ otherwise.

**Proof** Every row of $p_\alpha$ which has more than one box, besides the first and last, has exactly two corners in it. The first and last row have one corner if they have more than a single box, and none otherwise. Since the numbers of boxes in the rows of $p_\alpha$ are just the parts of $\alpha$, we are done.

From this we obtain the following.

**Corollary 3.4.4** If we have compositions $\alpha$ and $\beta$, one with exactly one 1 among its first and last parts, while the other has either zero or two, then the corresponding ribbons have a different number of corners.

**Proof** From Proposition 3.4.3, the number of corners of the two ribbons have different parity, and so cannot be equal.

**Example** Let $\alpha = (1, 3, 2, 1, 3, 1)$ and $\gamma = (2, 3, 2, 2, 1, 2, 1)$, then by Corollary 3.4.3, $p_\alpha$ has 6 corners while $p_\gamma$ has 9.

By merely looking at the entries $\alpha_1, \alpha_6, \gamma_1$, and $\gamma_7$, Corollary 3.4.4 shows that $c_\alpha \neq c_\gamma$.

**Corollary 3.4.5** Let $\alpha$ and $\beta$ be compositions with $k$ and $j$ parts greater than one respectively. If $|k - j| > 1$, then $c_\alpha \neq c_\beta$.
Proof If $\gamma$ is a composition with $i$ parts greater than one, then by Proposition 3.4.3 $c_\gamma \in \{2i, 2i - 1, 2i - 2\}$.

For $|k - j| \geq 2$, the sets $\{2k, 2k - 1, 2k - 2\}$ and $\{2j, 2j - 1, 2j - 2\}$ are disjoint as the largest element of one set is strictly smaller than the smallest of the other. Hence $c_\alpha \neq c_\beta$, as desired.

Example Let $\alpha = (1, 3, 2, 1, 3, 1)$ and $\gamma = (2, 3, 2, 1, 2, 3)$. Using Corollary 3.4.5, since $5 - 3 = 2 > 1$, we find that $\rho_\alpha$ and $\rho_\gamma$ have a different number of corners. In fact by Corollary 3.4.3 we find that $c_\alpha = 6$ while $c_\gamma = 8$.

If two compositions do have ribbons with the same number of corners, we can still look at what partitions we obtain when we remove those corners.

To this end, for $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k)$ with parts $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k > 1$ (not including $\alpha_k$ even if $\alpha_k > 1$), we define

$$\zeta(\alpha) = \{\sum_{i \leq i_l} \alpha_i : l = 1, 2, \ldots, j\}.$$

Observe $c \in \zeta(\alpha)$ if, and only if, $c$ is a corner of $\alpha$ which appears at the end of a row.

Proposition 3.4.6 If $\alpha$ is a composition of $n$ then

$$2\alpha = \{(c, n + 1 - c) : c \in \zeta(\alpha) \cup \zeta(\alpha^r)\} \cup \{(n, 1)^{n-c}\}.$$

Proof The set $\zeta(\alpha)$ contains the corners of $A_\alpha$ which occur at the end of a row of $\alpha$ with more than one box, excepting the top row. Similarly the set $\zeta(\alpha^r)$ contains the corners of $A_\alpha$ which occur at the beginning of a row of $\alpha$ with more than one box, excepting the bottom row.

Applying Proposition 3.3.2, we get the desired result.

Example Consider the composition $\alpha = (2, 3, 1, 1, 4, 1) \models 12$. From Proposition 3.4.3 we compute $c_\alpha = 5$.

Since only $\alpha_1$, $\alpha_2$, and $\alpha_5 > 1$, we have $i_1 = 1$, $i_2 = 2$, and $i_3 = 5$, and we compute

$$\zeta(\alpha) = \{2, 2 + 3, 2 + 3 + 1 + 1 + 4\} = \{2, 5, 11\}.$$

For $\alpha^r = (1, 4, 1, 1, 3, 2)$, we have $i_1 = 2$, and $i_2 = 5$, (as $\zeta$ ignores the final part of $\alpha$) so we obtain

$$\zeta(\alpha^r) = \{1 + 4, 1 + 4 + 1 + 1 + 3\} = \{5, 10\}.$$

Hence

$$2\alpha = \{(11, 2), (8, 5), (11, 2)\} \cup \{(8, 5), (10, 3)\} \cup \{(12, 1)^7\}$$

$$= \{(8, 5)^2, (10, 3), (11, 2)^2, (12, 1)^7\}.$$
Thus, looking at various \([p_\lambda]X_\alpha\) for \(l(\lambda) = 2\), we have

\[
[p_{(11,2)}]X_\alpha = -2, \ [p_{(10,3)}]X_\alpha = -1, \ [p_{(9,4)}]X_\alpha = 0, \ [p_{(8,5)}]X_\alpha = -2,
\]

\[
[p_{(7,6)}]X_\alpha = 0, \text{ and } [p_{(12,11)}]X_\alpha = -7.
\]

When \(|\alpha_1| \neq |\alpha_2|\) but \(2\alpha_1 = 2\alpha_2\), we look to other coefficients to distinguish the caterpillars. We have

**Proposition 3.4.7** If \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k)\) is a composition of \(n\) into \(k\) parts with conjugate \(\alpha^c = (\beta_1, \beta_2, \ldots, \beta_{n-k+1})\), then

\[
[p_{(3\ldots n-2)}]X_\alpha = \sum_{i=1}^{k} \binom{\alpha_i}{2} + \binom{n-k+1}{\beta_i}.
\]

**Proof** The lengths of the rows and columns of \(\alpha\) viewed as a ribbon are the parts of \(\alpha\) and \(\alpha^c\) viewed as a composition. The result follows from Proposition 3.3.6.

**Example** Consider the two compositions of 9, \(\alpha = (1,2,2,1,1,1,1)\) and \(\beta = (1,2,1,2,1,1,1)\).

We discover \([\alpha] = \{ (1,2,2,1,1,1,1), (1,1,1,2,2,1), (5,2,2), (2,2,5) \}\), and \([\beta] = \{ (1,2,1,2,1,1,1), (1,1,1,2,1,2,1), (4,3,2), (2,3,4) \}\), so \([\alpha] \neq [\beta]\).

If we compute the two part partitions we find \(2_\alpha = 2_\beta\). However, from Proposition 3.4.7 we find

\[
[p_{(3\ldots 7)}]X_\alpha = \left( \binom{1}{2} + \binom{2}{2} + \binom{2}{2} + \binom{1}{2} + \binom{1}{2} \right) + \left( \binom{1}{2} + \binom{5}{2} + \binom{2}{2} + \binom{2}{2} \right) + (0 + 1 + 1 + 0 + 0 + 0) + (0 + 10 + 1 + 1) = 14.
\]

Similarly, we compute

\[
[p_{(3\ldots 7)}]X_\beta = \left( \binom{1}{2} + \binom{2}{2} + \binom{1}{2} + \binom{2}{2} + \binom{1}{2} + \binom{1}{2} \right) + \left( \binom{1}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} \right) + (0 + 1 + 0 + 1 + 0 + 0) + (0 + 6 + 3 + 1) = 12,
\]

so that \(X_\alpha \neq X_\beta\).
Chapter 4

Distinguishing Certain Ribbon Classes

As before we fix $m$ to be the number of vertices in the tree.

Since we are only interested in the Ferrers graph of the ribbons $\rho$, we shall work with the ribbon classes $[\rho] = \{\rho, R(\rho), C(\rho), I(\rho)\}$.

4.1 Hooks

In this section we treat the case of the ribbons with a single corner. That is those ribbon diagrams which are a hook diagrams. We fix a total of $m$ vertices and label the rows from top to bottom $r_1, r_2, \ldots, r_o$, and the columns from left to right $c_1, c_2, \ldots, c_q$, so that $m = o + q$.

Due to our operations $R$ and $C$, if $\rho$ has only one corner we may, up to our operations, assume that the corner is in the position $(r_i, c_j)$ and that $o \leq q$. Hence, for two hooks $\rho_1$ and $\rho_2$ to be distinct, we may assume both $o_1 < q_1$ and $o_2 < q_2$, where $o_1 \neq o_2$. Hence as partitions, $(o_1, q_1) \neq (o_2, q_2)$.

Here we see the hook $\rho$ with its boxes numbered from southwest to northeast.

Looking at our corner set we have $A_\rho = \{o\}$. Since $2_\rho$ is determined by the values $\lambda(o) = (o, m - o) = (o, q)$ and $(o_1, q_1) \neq (o_2, q_2)$, we have $2_{\rho_1} \neq 2_{\rho_2}$. Hence $X_{\rho_1} \neq X_{\rho_2}$, that is

**Proposition 4.1.1** Let $\rho_1$ and $\rho_2$ be hook diagrams, then $X_{\rho_1} = X_{\rho_2}$ if, and only if, $[\rho_1] = [\rho_2]$. 
Moreover, we have the explicit expression for the chromatic symmetric function of a hook diagram:

**Proposition 4.1.2** If \( \rho \) is the hook with \( o \) rows and \( q \) columns, where \( o \leq q \), then

\[
X_{\rho} = p_{(1^o+q)} + \sum_{i=1}^{o-1} (-1)^{i}(\begin{pmatrix} o + q - 2 \\ i - 1 \end{pmatrix} + \begin{pmatrix} q - 1 \\ i \end{pmatrix})p_{(i+1,1^o+q-i-1)}
\]

\[
+ \sum_{i=0}^{q-1} (-1)^{i}(\begin{pmatrix} o + q - 2 \\ i - 1 \end{pmatrix} + \begin{pmatrix} q - 1 \\ i \end{pmatrix})p_{(i+1,1^o+q-i-1)}
\]

\[
+ \sum_{i=0}^{o+q-1} (-1)^{i}(\begin{pmatrix} o + q - 2 \\ i - 1 \end{pmatrix} + \begin{pmatrix} q - 1 \\ i \end{pmatrix})p_{(i+1,1^o+q-i-1)} + \sum_{t=1}^{o-1} c_{it}p_{((i+1)^2,1^o+q-2t-2)}
\]

\[
+ \sum_{1 \leq j < k < o-1} (-1)^{j+k}(c_{jk} + c_{kj})p_{(j+1,k+1,1^o+q-j-k-2)}
\]

\[
+ \sum_{j=1}^{o-1} \sum_{k=0}^{q-1} (-1)^{j+k}c_{jk}p_{(j+1,k+1,1^o+q-j-k-2)},
\]

where \( c_{jk} = \begin{pmatrix} o - 1 \\ j \\ k \end{pmatrix} \).

**Proof** As before we can assume the rows are labelled from top to bottom \( r_1, r_2, \ldots, r_o \) and the columns are labelled from left to right \( c_1, c_2, \ldots, c_q \) with the corner box in position \( r_1c_1 \).

We have \( E = \{r_1c_1\} \cup \{r_ic_1|1 < i \leq o\} \cup \{r_1c_j|1 < j \leq q\} \).

Let \( E_1 = \{r_ic_1|1 < i \leq o\} \), \( E_2 = \{r_1c_j|1 < j \leq q\} \), and \( e = \{r_1, c_1\} \), so that \( E = \{e\} \cup E_1 \cup E_2 \). Then \( e \) is incident with every edge of \( G(\rho) \), so for each \( F \subseteq E \) containing \( e \), the edge set \( F \) induces a connected component with \( |F| + 1 \) vertices, that is \( \lambda(F) = (|F| + 1, 1^o+q-|F|-1) \).

Hence

\[
X_{\rho} = \sum_{F \subseteq E} (-1)^{|F|}p_{\lambda(F)}
\]

\[
= \sum_{e \in F \subseteq E} (-1)^{|F|}p_{\lambda(F)} + \sum_{F \subseteq E_1 \cup E_2} (-1)^{|F|}p_{\lambda(F)}
\]

\[
= \sum_{i=1}^{o+q-1} (-1)^{i}p_{(i+1,1^o+q-i-1)}
\]

\[
+ \sum_{F_1 \subseteq E_1} \sum_{F_2 \subseteq E_2} (-1)^{|F_1 \cup F_2|}p_{\lambda(F_1 \cup F_2)}
\]
Thus we find that

\[ X_p = \sum_{i=1}^{o+q-1} (-1)^i \binom{0+q-2}{i-1} p_{i+1,1^o+q-i-1} + \sum_{F_1 \subseteq E_1} \sum_{F_2 \subseteq E_2} (-1)^{|F_1|+|F_2|} p_{\lambda(F_1 \cup F_2)}. \quad (4.1) \]

Now, every edge of \( E_1 \) is incident with every other edge in \( E_1 \). Thus for each \( F_1 \subseteq E_1 \), the only nontrivial component in the subgraph induced by the set of edges in \( F_1 \) has \( |F_1| + 1 \) vertices. Similarly with \( E_2 \). Further, these components are disjoint in the subgraph induced by the edge set \( F_1 \cup F_2 \), so \( \lambda(F_1 \cup F_2) = (|F_1| + 1, |F_2| + 1, 1^{o+q-|F_1|-|F_2|-2}) \).

Therefore

\[
\sum_{F_1 \subseteq E_1} \sum_{F_2 \subseteq E_2} (-1)^{|F_1|+|F_2|} p_{\lambda(F_1 \cup F_2)} = \\
= \sum_{j=0}^{o-1} \sum_{F_1 \subseteq E_1} \sum_{k=0}^{q-1} \sum_{F_2 \subseteq E_2} (-1)^{|F_1|+|F_2|} p_{\lambda(F_1 \cup F_2)} \\
= \sum_{j=0}^{o-1} \sum_{k=0}^{q-1} \sum_{F_1 \subseteq E_1} \sum_{F_2 \subseteq E_2} (-1)^{j+k} p_{\lambda(j+1,k+1,1^{o+q-j-k-2})} \\
= \sum_{j=0}^{o-1} \sum_{k=0}^{q-1} (-1)^{j+k} \binom{o-1}{j} \binom{q-1}{k} \ p_{\lambda(j+1,k+1,1^{o+q-j-k-2})}.
\]

Rearranging this last sum while writing \( c_{jk} = \binom{o-1}{j} \binom{q-1}{k} \) gives

\[
p_{(1^{o+q})} + \sum_{j=1}^{o-1} (-1)^j \binom{o-1}{j} p_{\lambda(j+1,1^{o+q-j-1})} + \sum_{k=1}^{q-1} (-1)^k \binom{q-1}{k} p_{\lambda(k+1,1^{o+q-k-1})} + \sum_{j=1}^{o-1} \sum_{k=1}^{q-1} (-1)^{j+k} c_{jk} p_{\lambda(j+1,k+1,1^{o+q-j-k-2})}
\]
Chapter 4. Distinguishing Certain Ribbon Classes

\[ = p_{(1+q)} + \sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} p_{\lambda(i+1,1^{q-i-1})} \]
\[ + \sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} p_{\lambda(i+1,1^{q-i-1})} \]
\[ + \sum_{i=1}^{q-1} c_{ii} p_{\lambda(i+1,1^{q-2q-2i-2})} \]
\[ + \sum_{1 \leq j < k \leq q-1} (-1)^{j+k}(c_{jk} + c_{kj}) p_{(j+1,k+1,1^{q-j-k-2})} \]
\[ + \sum_{j=1}^{q-1} \sum_{k=0}^{q-1} (-1)^{j+k} c_{jk} p_{(j+1,k+1,1^{q-j-k-2})}. \]

Substituting this into Equation 4.1 gives the desired result. 

4.2 Two Row Ribbons

In this section we consider those ribbon diagrams which have only two rows. More precisely, we consider those ribbon classes \( [p] \) which contain a member with two rows.

Fixing \( m \) as \( q + 2 \), and labelling the rows \( r_1 \) and \( r_2 \) from top to bottom and the columns \( c_1, c_2, \ldots, c_q \) from left to right, then the only possible ribbons with two rows are the \( p_j \), \( 1 \leq j \leq q \), shown below.

\[
\begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
  & & & \\
\end{array}
\]

Moreover, we need only consider \( 1 \leq j \leq \left\lfloor \frac{q+1}{2} \right\rfloor \) since \( R(p_j) = p_{q+1-j} \) gives \( [p_j] = [p_{q+1-j}] \) for each \( 1 \leq j \leq q \). Of these ribbons, \( p_j \) has two corners except when \( j = 1 \), in which case it has only one corner.

By Corollary 3.3.1 we have \( X_{p_i} \neq X_{p_j} \) for \( 2 \leq j \leq \left\lfloor \frac{q+1}{2} \right\rfloor \).

Now consider \( j \) and \( k \) with \( 1 < j < k \leq \left\lfloor \frac{q+1}{2} \right\rfloor \). From Proposition 3.3.2, we get at most three distinct partitions of \( m = q + 2 \) into two parts—each obtained by removing a single edge. We obtain a copy of \( (q+1,1) \) from each of the noncorners and one more partition from each of the two corners. In the case of \( p_j \) (\( p_k \) resp.) we shall obtain the partitions \( (j, q-j+2) \) and \( (j+1, q-j+1) \) (\( k \) resp.) from the corners and the copies of \( (q+1,1) \) from the noncorners.
If $X_{p_1} = X_{p_2}$ then by Corollary 3.2.2, $2T_j = 2T_k$. Therefore the partition $(j, q - j + 2)$ must be equal to either $(k, q - k + 2)$ or $(k + 1, q - k + 1)$. Clearly

$$j < k < k + 1$$

and this, together with $k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, gives

$$j < k \leq 2k - k \leq q + 1 - k < q - k + 2.$$

Hence $j$, being strictly less than each of $k, k + 1, q - k + 1,$ and $q - k + 2$ makes it impossible for the partition $(j, q - j + 2)$ to be either $(k, q - k + 2)$ or $(k + 1, q - k + 1)$.

We have proven the following.

**Proposition 4.2.1** The chromatic symmetric function distinguishes the two row ribbon classes.

### 4.3 Symmetric and Near-Symmetric Ribbons

We saw in Lemma 2.4.3 that a given $n$-box ribbon $\rho$ was symmetric if, and only if, $A_\rho$ is symmetric, that is if $n + 1 - k \in A_\rho$ whenever $k \in A_\rho$. Further if $\rho$ is symmetric, then $\rho$ is $C$-symmetric if, and only if, $\frac{n+1}{2} \in A_\rho$.

Given an $R$-symmetric $n$-box ribbon $\rho$, we let $A_\rho \cap \{2, 3, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \} = \{c_1, c_2, \ldots, c_j\}$, where $1 < c_1 < c_2 < \ldots < c_j < \frac{n+1}{2}$. Then

$$A_\rho = \{c_1, c_2, \ldots, c_j\} \cup \{n + 1 - c_1, \ldots, n + 1 - c_j\}$$

which gives

$$2_\rho = \{(c_1, n+1-c_1)^2, (c_2, n+1-c_2)^2, \ldots, (c_j, n+1-c_j)^2\} \cup \{(n, 1)^{n-2j}\}. \quad (4.2)$$

Conversely by Corollary 3.3.4, if a ribbon $\rho$ satisfies Equation 4.2, then both $c_i$ and $n + 1 - c_i \in A_\rho$ for each $i = 1, 2, \ldots, j$. Hence

$$A_\rho = \{c_1, c_2, \ldots, c_j\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_j\},$$

which shows that $\rho$ is $R$-symmetric. Thus the set $2_\rho$ distinguishes $R$-symmetric ribbons from the ribbons which are not $R$-symmetric.
Further, since the two part partitions of any symmetric ribbon determines the ribbons corner set we have

\[ 2p_1 = 2p_2 \Leftrightarrow A_{p_1} = A_{p_2} \Leftrightarrow |p_1|_I = |p_2|_I, \]

and hence \( X_p \) does distinguish among those ribbon classes (modulo \( I \)) that are \( R \)-symmetric.

Now if \( \rho \) is a \( C \)-symmetric \( n \)-box ribbon, then \( n \) is odd by Lemma 2.2.4 and \( \frac{n+1}{2} \in A_\rho \) by Lemma 2.4.3. Let \( A_\rho \cap \{2, 3, \ldots, \frac{n+1}{2}\} = \{c_1, c_2, \ldots, c_i, \frac{n+1}{2}\} \), where \( 1 < c_1 < c_2 < \ldots < c_i < \frac{n+1}{2} \). Then \( A_\rho = \{c_1, c_2, \ldots, c_i\} \cup \{\frac{n+1}{2}\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_i\} \), which gives

\[ 2^p = \{(\frac{n+1}{2}, \frac{n+1}{2})\} \cup \{(c_1, n + 1 - c_1)^2, \ldots, (c_i, n + 1 - c_i)^2\} \cup \{(n, 1)^{n-1-2^i}\}. \]

Conversely, using Corollary 3.3.4, if a ribbon \( \rho \) satisfies Equation 4.3 with \( 1 < c_i < \frac{n+1}{2} \), then \( \frac{n+1}{2} \in A_\rho \) and both \( c_i \) and \( n + 1 - c_i \in A_\rho \) for each \( i = 1, 2, \ldots, l \). Hence

\[ A_\rho = \{c_1, c_2, \ldots, c_i\} \cup \{\frac{n+1}{2}\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_i\}, \]

showing that \( \rho \) is \( C \)-symmetric.

Hence \( 2p \) distinguishes \( C \)-symmetric ribbons from the ribbons which are not \( C \)-symmetric. Further, it distinguishes among those ribbon classes (modulo \( I \)) that are \( C \)-symmetric since

\[ 2p_1 = 2p_2 \Leftrightarrow A_{p_1} = A_{p_2} \Leftrightarrow |p_1|_I = |p_2|_I. \]

Since \( |p_1|_I = |p_2|_I \) implies \( |p_1| = |p_2| \), we have shown the following.

**Theorem 4.3.1** The chromatic symmetric function distinguishes the symmetric ribbon classes from the nonsymmetric ribbon classes. Further, it distinguishes among the symmetric ribbon classes.

The caterpillars whose associated ribbon classes are symmetric can be visualized in an easy way. They are those caterpillars for which there is a line of symmetry that splits the spine in half.

**Example** Here we see the Ferrers graph of two symmetric ribbons.
By collecting various results, we find that we have proved the following.

**Proposition 4.3.2** There are collections $Q_m$ of $m$-vertex graphs such that:

1. $\lim_{m \to \infty} |Q_m| = \infty$,
2. $\chi(G_1, k) = \chi(G_2, k)$ for every pair of graphs $G_1$ and $G_2 \in Q_m$, and
3. If $G_1$ and $G_2 \in Q_m$ and $X_{G_1} = X_{G_2}$, then $G_1 = G_2$.

**Proof** We look at the collection $Q_m$ of caterpillars corresponding to symmetric ribbons with $m - 1$ boxes. We have property 1 by Lemma 2.2.4. Since all the caterpillars in $Q_m$ have the same number of vertices, we obtain property 2 via Equation 1.2. Finally Theorem 4.3.1 gives property 3. 

Unfortunately, the method by which we proved Theorem 4.3.1 fails to distinguish ribbons that are nonsymmetric, as the implication $2_{\rho_1} = 2_{\rho_2} \Rightarrow A_{\rho_1} = A_{\rho_2}$ fails to hold in this setting.

**Example** The nonsymmetric ribbons $\rho_1$ and $\rho_2$ shown below both have $$2_{\rho} = \{(5, 2), (4, 3), (6, 1)\}^4,$$

but clearly $[\rho_1] \neq [\rho_2]$.

---

We shall call a nonsymmetric ribbon $\rho$ near-symmetric if $A_\rho \cup \{i\}$ is a symmetric subset for some number $i \in \{2, 3, \ldots, n-1\}$. That is, $\rho$ is near-symmetric if it is nonsymmetric, yet by changing one of its noncorners into a corner we obtain a symmetric ribbon.

**Example** The ribbon $\rho$ with ten boxes whose corner set is $A_\rho = \{3, 4, 8\}$ is near-symmetric, as $\{3, 4, 7, 8\}$ is a symmetric subset of $\{2, 3, \ldots, 9\}$.

We can transform $\rho$ into the symmetric ribbon it is near, by making the seventh box into a corner. This can be done by performing an inversion on the subdiagram of $\rho$ consisting of the seventh box onward.
If \( \rho \) is near-symmetric, then for some \( i \) the multiset \( 2_\rho \) is given by either
\[
2_\rho = \{(c_1, n + 1 - c_1)^2, (c_2, n + 1 - c_2)^2, \ldots, (c_j, n + 1 - c_j)^2\}
\cup \{(i, n + 1 - i)\} \cup \{(n, 1)^n - 2j - 1\}
\]
when it is near to an \( R \)-symmetric ribbon, or
\[
2_\rho = \{(c_1, n + 1 - c_1)^2, (c_2, n + 1 - c_2)^2, \ldots, (c_j, n + 1 - c_j)^2\}
\cup \{((\frac{n + 1}{2}, \frac{n + 1}{2})\} \cup \{(i, n + 1 - i)\} \cup \{(n, 1)^n - 2j - 2\}
\]
when it is near to a \( C \)-symmetric ribbon.

The first of these two cases has only the two possible corner sets
\[
\{c_1, c_2, \ldots, c_j\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_j\} \cup \{i\},
\]
and
\[
\{c_1, c_2, \ldots, c_j\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_j\} \cup \{n + 1 - i\},
\]
which are reflections of one another. Conversely, given the above two sets, Lemma 2.4.2 shows they belong to the same ribbon class, which is clearly a near-symmetric ribbon class.

The second of these two cases has only the two possible corner sets
\[
\{c_1, c_2, \ldots, c_j\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_j\} \cup \{i, \frac{n + 1}{2}\},
\]
and
\[
\{c_1, c_2, \ldots, c_j\} \cup \{n + 1 - c_1, n + 1 - c_2, \ldots, n + 1 - c_j\} \cup \{n + 1 - i, \frac{n + 1}{2}\},
\]
which again are reflections of one another. Again, given these sets, Lemma 2.4.2 shows they belong to the same ribbon class, which again is near-symmetric.

Thus we find:

**Theorem 4.3.3** The chromatic symmetric function distinguishes the near-symmetric ribbon classes from those ribbon classes which are not near-symmetric. Further, it distinguishes among the near-symmetric ribbon classes.

Hence if \( C \) is the set of caterpillars, \( S \) is the set of caterpillars whose ribbons are symmetric, and \( S' \) is the set of caterpillars whose ribbons are near-symmetric, and if \( C_1 \in C \) and \( C_2 \in S \cup S' \), we find that \( X_{C_1} = X_{C_2} \) if, and only if, \( C_1 \cong C_2 \).
Chapter 5

Conclusions

We had set out to study the chromatic symmetric function $X_G$ in the case of $G = T$ a tree, seeing how the chromatic symmetric function relates to the tree in question. The problem inspiring this thesis asked whether the chromatic symmetric function determines the tree it came from. We have been unable to obtain such a result at the present, but have had considerable success in the analogous problem restricted to caterpillars.

We summarise our main results.

Theorem 5.0.4

1. Let $T_1$ and $T_2$ be trees with $X_{T_1} = X_{T_2}$, then
   (a) $T_1$ and $T_2$ have the same number of edges and the same number of vertices.
   (b) $T_1$ and $T_2$ have the same number of leaves.
   (c) For any $k$, $T_1$ and $T_2$ have the same number of $k$-vertex subtrees.
   (d) For every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash m$, $T_1$ and $T_2$ have the same number of partitions into disjoint subtrees of sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$.

2. The map taking $[\rho]$ to $G(\rho)$ is a bijection from the set of ribbon classes to the set of caterpillars. This map sends each ribbon class with $n$ boxes and $k$ corners to a caterpillar with $n + 1$ vertices and $k$ internal edges.

3. For ribbon classes (or caterpillars equivalently),
   (a) If $\rho_1$ and $\rho_2$ are ribbon classes with $X_{\rho_1} = X_{\rho_2}$, then $\rho_1$ and $\rho_2$ have the same number of boxes and the same number of corners.
   (b) $X_\rho$ distinguishes among the hooks, that is, among those ribbon classes which contain a single corner.
   (c) $X_\rho$ distinguishes among those ribbon classes that have two rows.
   (d) $X_\rho$ distinguishes the symmetric and near-symmetric ribbon classes from all other ribbon classes, as well as distinguishes among the symmetric and near-symmetric ribbon classes.
   (e) The composition classes $[\alpha]$ of $n$ generate the $n$-box ribbon classes, so we can use these representations to:
      i. Enumerate the nonisomorphic caterpillars by generating the composition classes.
ii. Explicitly calculate \( [p_{\lambda}]X_{\alpha} \) for \( \lambda = (2, 1^{n-1}), (3, 1^{n-2}), (2^2, 1^{n-3}) \), and every partition \( \lambda \) with two parts without resorting to diagrams of any sort.

4. Additionally, we have used our ribbon representatives to

   (a) Calculate the number of \( n \)-box ribbon classes with \( c \) corners.

   (b) Calculate the number of \( n \)-box ribbons that are R- and C-symmetric respectively, and the number of \( c \)-cornered \( n \)-box ribbons that are R- and C-symmetric respectively.

   (c) Calculate the number of nonisomorphic caterpillars with \( n \) vertices.

   It has been checked that \( X_G \) distinguishes all nonisomorphic \( n \)-vertex trees for values of \( n \) up to 9. Using the composition and ribbon representations of caterpillars, I have checked that that \( X_G \) distinguishes all \( n \)-vertex caterpillars, for \( n = 10 \).

   It is still unknown whether \( X_G \) distinguishes all nonisomorphic trees. We invite the reader to help investigate this and other open problems related to the chromatic symmetric function.
Bibliography


