SKEIN MODULES AND CHARACTER VARIETIES

by

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B.Sc. The University of King's College, 2003

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE
in

THE FACULTY OF GRADUATE STUDIES

Mathematics

THE UNIVERSITY OF BRITISH COLUMBIA

September 2005

Abstract

We present a survey of the theory of skein modules of manifolds, and an introduction to skein algebras of groups. By applying a trick of Doug Bullock, we use $SL(2,\mathbb{C})$ character varieties to highlight some infinite linearly independent families of knots in the Kauffman Bracket skein module of a 3-manifold. These families are composed of a knot K, together with all $(1,n)$-cablings of K. We also exhibit a method of explicit computation based upon the work of Robert Riley, which can identify infinite linearly independent families in the skein algebras of 2-bridge knot groups.

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Introduction

1.1 Motivation

Skein modules are the foundation of a new algebraic approach to manifold theory, which has knot theory at its core [22], and this new approach is particularly applicable to manifolds of dimension three. Specifically, the skein module of a 3-manifold M is a module whose structure may allow one to distinguish M from many other 3-manifolds. As well, if one succeeds in finding a basis for the skein module of M, then this basis provides an entire family of invariants of links in M.

Previous invariants of manifolds, such as the homology or fundamental group of a 3-manifold, have been based upon equivalence relations between submanifolds of M that are relatively weak - namely the relations of being homologous or homotopic. Skein modules involve an importation of classical knot theory into arbitrary 3-manifolds, where one uses the relationship of "similarity of knots" to construct this new algebraic invariant, the skein module. As most geometric relationships between knots run deeper than (for example) either homology or homotopy, this new algebraic invariant will likely out-perform classical structures in many ways, as well as providing an arena for the study of knot theory in an arbitrary manifold.

At present, one of the main objects of study in the theory of skein modules is the Kauffman bracket skein module, which is a specialization based upon the well-known Jones polynomial of knots. However, as has happened in the past with new structures in algebraic topology, the problem of computing the Kauffman bracket skein module of a given manifold is initially proving to be quite difficult.

Doug Bullock observed a close relationship between the $SL(2,\mathbb{C})$ characters of the fundamental group of M (characters in the sense of representation theory) and the Kauffman bracket skein module [6]. This has provided a connection via which one may be able to translate intractable topological questions about the Kauffman bracket skein module into approachable problems in the world of algebra.

New results concerning the structure of this module will have direct impact on the theory of links in an arbitrary manifold, as well as the theory of 3-manifolds, both of which are interrelated quite closely. However, some results may have impact beyond even the scope of mathematics, as there is a very explicit relationship between the Kauffman bracket skein module and modern physics, via quantum invariants [1].

1.2 Introduction

1.2.1 Definition of Knots and Links and Equivalence

A **knot** in a 3-manifold M (typically $M = S^3$) is a piecewise-linearly embedded circle: $S^1 \hookrightarrow M$. Alternatively we can require that the embedding be smooth, these restrictions can be shown to be equivalent. A **link** L is a collection of disjoint circles, also piecewise-linearly embedded in M

$$L = \coprod_{i=1}^{n} S_i^1 \hookrightarrow M.$$

A knot or link is **oriented** if each copy of S^1 is assigned a preferred orientation. A knot or link is called **framed** if we think of each copy of S^1 as a skinny solid torus, together with a preferred longitude on its boundary. This longitude is sometimes referred to as the framing.

A better way of thinking of "preferred longitudes" is to think of the knot as an embedded ribbon, instead of a skinny torus with preferred longitude. The two ideas are equivalent, by taking the preferred longitude to be one side of the ribbon:

Two knots are considered equivalent if they are ambient isotopic. The formal definition of ambient isotopy is as follows:

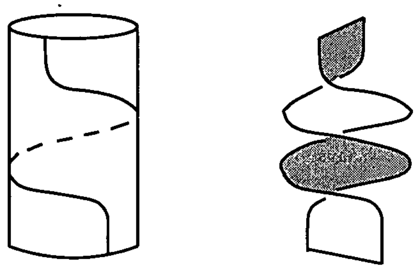


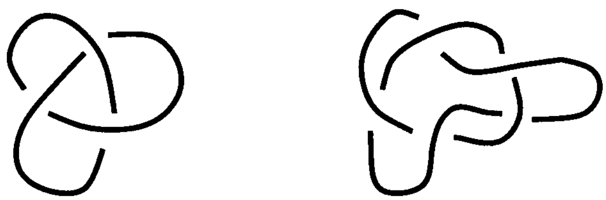
Figure 1.1: A torus with preferred longitude, and the corresponding ribbon

Definition 1.1. An **isotopy** between knots K_1 and K_2 in a 3-manifold M is a homotopy $H(x,t)$ between K_1 and K_2, such that for each fixed t, the map $H(-,t)$ is a piecewise-linear embedding. An **ambient isotopy** between K_1 and K_2 is an isotopy of M to itself that carries K_1 to K_2. This means that the ambient space is doing the deforming, and the knots simply "come along for the ride". For framed knots, the isotopy must carry the preferred longitude of one knot to the preferred longitude of the other, or equivalently one ribbon must be carried onto the other.

This formal definition is meant to model the commonplace idea of untangling/knotting a string. In plain English, two knots are equivalent if one can be manipulated (without the string breaking or passing through itself) so that it appears exactly as the other. This notion of equivalence generalizes to links in the obvious manner.

1.2.2 Diagrams and Local Diagrams

In the special case where our ambient space is $\mathbb{R}^3$ or S^3, there is a less cumbersome way of thinking of knots and links. Given a knot K we can create a **diagram** of K, which is essentially a picture of K obtained by projecting into a plane. To the image of the projection, we add information at doubly covered points to create **crossings**, allowing us to recover the original knot from a diagram. This gives us things as in figure 1.2.



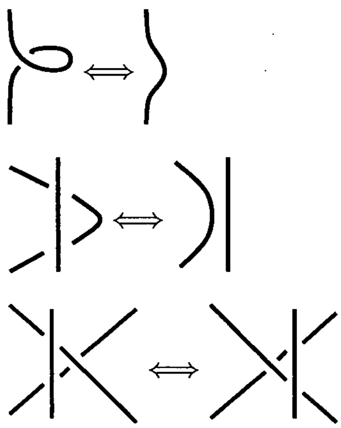
The trefoil The cinqfoil

Figure 1.2: Diagrams of the trefoil and cinqfoil

Here, the crossings are the regions appearing as , and the 'added information' is the break in one of the arcs; indicating that the broken arc appears below the unbroken arc in our original knot.

The equivalence relation of ambient isotopy can be carried into the world of diagrams. Two diagrams are equivalent if one can be reached from the other via a sequence of planar isotopies and **Reidemeister moves**, which are local diagram manipulations. There are three such moves:

1.
2.
3.



Theorem 1.2. ***(Reidemeister)*** *Two knots (in S^3) are ambiently isotopic if and only if their diagrams are related via a sequence of Reidemeister moves and planar isotopies.*

A more strict definition of the Reidemeister moves and a proof of this theorem can be found in [26].

If we are not working within S^3 or $\mathbb{R}^3$, then global projections are not possible, but we do have "local" diagrams. Suppose that we have a link L in M, and some 3-ball B in M such that $B \cap L \neq \emptyset$. Then we can produce a diagram of the part of L that lies inside B by projecting the contents of B onto some plane, and creating crossings as before. The intersection $B \cap L$ is called a **tangle**. The part of L which lies outside of B is often referred to as the **external wiring** of the **room** B. A tangle diagram appears as:

Figure 1.3: A tangle diagram

Often we restrict ourselves to tangles with $\partial B \cap L$ = four points, as in figure 1.4.

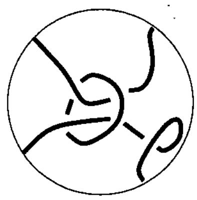
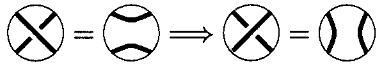


Figure 1.4: A tangle for which $\partial B \cap L =$ four points

Two tangles in B are equivalent if one can be obtained from the other via isotopies of B which fix the points in $\partial B \cap L$. Accordingly, we can rephrase this equivalence in terms of Reidemeister moves on tangle diagrams.

A common way of communicating the structure of a link L in some part of a manifold M is to say that L "looks like" a given diagram inside some 3-ball B. This colloquial language is just a way of communicating that there exists a series of ambient isotopies of the tangle $L \cap B$, and some plane of projection so that $L \cap B$ can be projected to the given diagram.

We also sometimes say that a certain equation involving diagrams is true "by tilting your head". This means that the desired equation results from the given equation by rotating all diagrams in the given equation in the same direction - usually by 90 degrees. For example, the implication



follows by tilting your head. We cannot say the same of equations that do not involve pictures. For example, even though

$$813 = 810 + 3$$

it is pure nonsense to say that

$$\frac{\infty}{\omega} = \frac{\infty}{0} + \omega.$$

Chapter 2
Skein Modules

2.1 Skein Modules by Example

We would like to build a setting for the study of knot theory in an arbitrary 3-manifold.

Let R be a commutative ring with 1, M an orientable 3-manifold, and $\mathcal{L}$ the set of all links in M considered up to ambient isotopy, including the empty link. If the links in $\mathcal{L}$ are oriented we add a subscript "o", and if they are framed we add a subscript "f". Thus, we have things like $\mathcal{L}_{o,f}$; the set of all oriented, framed links in M.

Let $R\mathcal{L}$ (resp. $R\mathcal{L}_o$, etc.) be the free R-module with basis $\mathcal{L}$ (resp $\mathcal{L}_o$, etc.). The idea of skein module theory is to start with this structure, and then cleverly select some family of relations between elements in $R\mathcal{L}$. The skein module of a 3-manifold M with coefficients in R is then an algebraic invariant of M:

$$\mathcal{S}(M;R) = R\mathcal{L} \Big/ relations$$

The relations that one chooses can vary greatly. If the relations are too weak, the resulting skein module may be intractible, and of no use. If the relations are too strong, the resulting module may contain no useful information. We illustrate this idea with a very geometric example.

We begin with a simple relation that allows us to eliminate all the crossings in any given knot. Suppose that L_1 and L_2 are two oriented knots in a 3-manifold M, and that they are identical everywhere in M, except in a 3-ball where they differ as in figure 2.1.

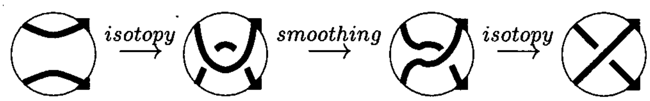
L_1 L_2

Figure 2.1: The appearance of the links L_1, L_2 in some 3-ball in M

In this case, we set $L_1 = L_2$ in the skein module, sometimes called 'smoothing a crossing'. We write:

$$\mathcal{S}_2(M) = {}^{R\mathcal{L}_o} \Big/$$  $\sim$ 

Remark 2.1. *As a consequence of this single relation, we can compute:*



so that we can smooth crossings of the opposite orientation.

With this choice of relations, we find $S_2(M) \cong RH_1(M;\mathbb{Z})$ via the homomorphism

$$\phi : R\mathcal{L}_o \to RH_1(M;\mathbb{Z}), \qquad \phi(L) = [L]$$

where we extend this definition linearly to all R-linear combinations of links. This fact is found in [16], we offer a proof here. Here $[L]$ is the equivalence class of L in the first homology of M.

First, to see that ϕ descends to an homomorphism

$$\Phi : S_2(M) \to RH_1(M;\mathbb{Z}),$$

suppose that we have links L_1 and L_2 in M that differ in some 3-ball B^3, as depicted in figure 2.1. In this case, we break L_1 and L_2 into arcs (think: 1-chains in singular homology) which lie inside B^3, and arcs which lie outside B^3. Label the arcs inside B^3 as in figure 2.2. Outside of B^3, L_1 and L_2 share

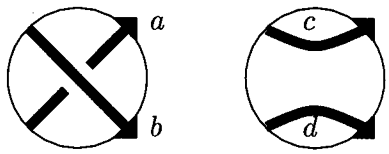


Figure 2.2: Labelling the arcs inside B^3

the same external wiring, and so denote this common external wiring by e. Then we compute:

$$\phi(L_1 - L_2) = [L_1] - [L_2] = [a + b + e] - [c + d + e],$$

where addition inside the square brackets corresponds to addition in the first homology group, and the subtraction is taking place inside the group ring $RH_1(M;\mathbb{Z})$. We can see that $[a + b + e] = [c + d + e]$ in $H_1(M;\mathbb{Z})$, as the difference:

$$(a + b + e) - (c + d + e) = a - c + b - d$$

is the boundary of a "twisted" disk, so that $[a + b + e] - [c + d + e] = 0$ in

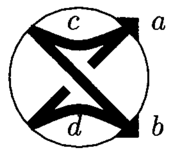


Figure 2.3: The twisted disk with boundary $a - c + b - d$

$RH_1(M;\mathbb{Z})$. Thus the map Φ is well-defined. It is evident that Φ is a surjection, because any element of the homology group $H_1(M;\mathbb{Z})$ can be represented by some link in $\mathcal{L}_o$.

To see that Φ is an monomorphism, we need a lemma:

Lemma 2.2. *Given any link L, we may choose a representative L' of the equivalence class of L in $S_2(M)$ such that L' single copy of S^1.*

Proof. It suffices to consider the case when $L = K_1 \sqcup K_2$ has only two components. We can isotope K_1 and K_2 so that they are very close to one another

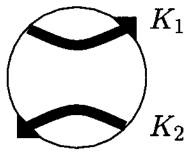
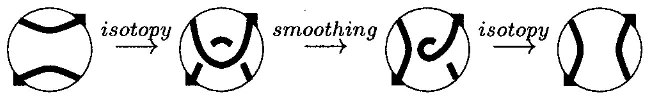


Figure 2.4: The appearance of K_1 and K_2 in some 3-ball

inside some 3-ball in M, and appear as in figure 2.4. Then in $\mathcal{S}_2(M)$, we find the equality:



so that we can apply this equation to figure 2.4 to find L', where L' is the connected sum $K_1 \# K_2$. □

Now suppose that we have two links L_1 and L_2 such that $\Phi(L_1) = \Phi(L_2)$, i.e. $\Phi(L_1 - L_2) = 0$. By the above lemma and by well-definedness of Φ, we may assume without loss of generality that $L_1 - L_2$ is a knot K, which is mapped to zero under Φ. This means that K is homologous to zero, so there exists a surface F such that $\partial F = K$. By the classification of surfaces, F must abstractly appear as in figure 2.5. However, from the calculation in our lemma,

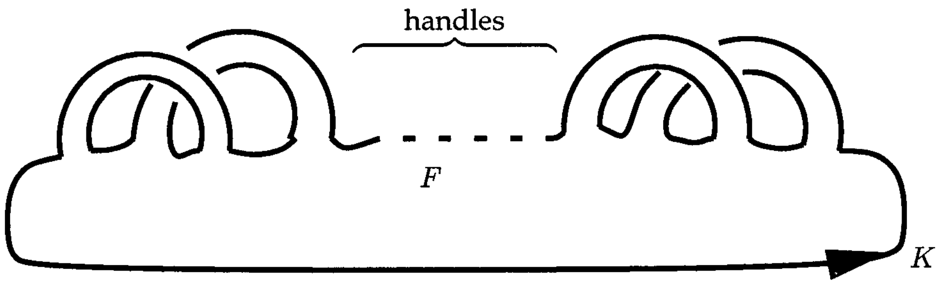
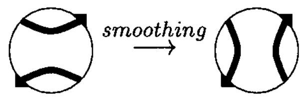


Figure 2.5: An abstract depiction of the surface F

we know that we can break each handle using the relation



to get a new representative K' of the equivalence class of K, as in figure 2.6.

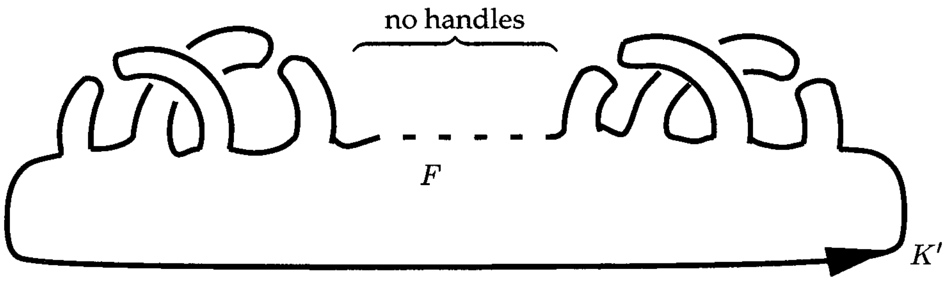


Figure 2.6: The surface F with broken handles

Evidently K' is the boundary of a disk, and so it is the unknot. This trivial representative of $L_1 - L_2$ shows that we must have $L_1 - L_2 = 0$ in $\mathcal{S}_2(M)$, so that Φ is an isomorphism

$$\Phi : \mathcal{S}_2(M) \to RH_1(M;\mathbb{Z})$$

as claimed, thus completing the proof.

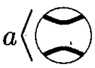
We will see shortly that by choosing different relations, we can recover a structure based upon the fundamental group of the manifold as well.

It is important to maintain the distinction between links and their diagrams when working through proofs of this nature. The relation $\sim$ is a relation between links in M, and not between diagrams of links. At present it is only known how to diagrammatically encode knots and links in S^3 or in a handlebody (virtual knots), there is no known way of creating diagrams of links in an arbitrary 3-manifold. For this reason, attempting such proofs in a diagrammatic manner can sometimes lead one astray.

2.2 The Kauffman Bracket Skein Module

In this section, we attempt to choose a more useful set of relations. To motivate this choice, be begin with some combinatorics.

Suppose that we are working in S^3, or $\mathbb{R}^3$, so that every link under consideration admits a diagram. One of the most powerful link invariants in this setting is the *Jones polynomial*. The Jones polynomial of a link L in S^3 can be computed from a diagram of L by using the *Kauffman bracket* of a link, denoted $\langle L \rangle$, which is a polynomial in $\mathbb{Z}[a, a^{-1}]$. The Kauffman bracket $\langle L \rangle$ is defined according to the following recursive (local) diagram manipulations [17]:

$\Big\langle$  $\Big\rangle = a\Big\langle$  $\Big\rangle + a^{-1}\Big\langle$  $\Big\rangle$ (2.1)

$$\langle L \sqcup \bigcirc \rangle = -(a^2 + a^{-2})\langle L \rangle \tag{2.2}$$

$$\langle \bigcirc \rangle = 1 \tag{2.3}$$

We are to interpret each of these equations as rules for mechanically decomposing a diagram into a union of disjoint circles, by eliminating crossings. One finds that since these rules produce only local changes in a knot diagram, crossings can be eliminated in any order, with no effect on the outcome of the calculation. Therefore, eliminating all crossings in a link diagram gives rise to a well-defined polynomial $\langle L \rangle$, for which there is an explicit combinatorial formula.

Next we assign an orientation to the diagram of L to create L_o, and compute the *writhe* of L_o (denoted $w(L_o)$). First, assign to each crossing point p in the diagram a value $\varepsilon(p)$, which is either $+1$ or -1, according to the convention:

$\varepsilon\Big($  $\Big) = +1 \quad \varepsilon\Big($  $\Big) = -1$

Then if X is the set of all crossing points in the diagram of L_o, we define:

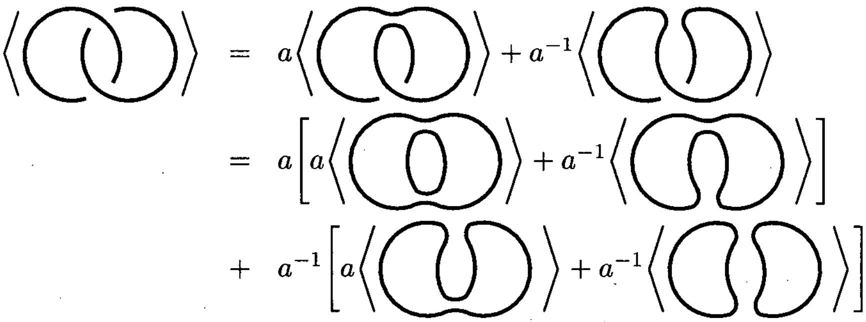
$$w(L_o) = \sum_{p \in X} \varepsilon(p)$$

We are now in a position to make the following definition:

Definition 2.3. The Jones polynomial of an oriented link L_o in S^3 is given by:

$$V_{L_o}(t) = -a^{-3w(L_o)}\langle L\rangle\big|_{a=t^{-\frac{1}{4}}}$$

Example 2.4. We compute the Jones polynomial of the Hopf link by first computing the Kauffman Bracket:

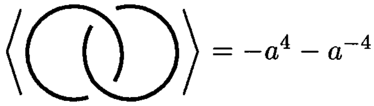


Since all the brackets now contain only circles, at this point we may use equation (2.2) to reduce the number of loops inside each bracket to one:

$$= -a^2(a^2+a^{-2})\langle \bigcirc \rangle + 2\langle \bigcirc \rangle - a^{-2}(a^2+a^{-2})\langle \bigcirc \rangle$$

$$= (-a^4 - a^{-4})\langle \bigcirc \rangle$$

Using rule (2.3) we get a final answer of:



We are only a step away from finding the Jones polynomial. Depending upon the orientation we choose at this point, we get two possible cases:

1. $Hopf_+$, satisfying $w(Hopf_+) = 2$
2. $Hopf_-$, satisfying $w(Hopf_-) = -2$.

This yields the two polynomials

$$V_{Hopf_+}(t) = -a^{-6}(-a^4 - a^{-4})\big|_{a=t^{-\frac{1}{4}}} = (a^{-10} + a^{-2})\big|_{a=t^{-\frac{1}{4}}} = t^{\frac{5}{2}} + t^{\frac{1}{2}}$$

and

$$V_{Hopf_-}(t) = -a^{6}(-a^4 - a^{-4})\big|_{a=t^{-\frac{1}{4}}} = (a^{10} + a^{2})\big|_{a=t^{-\frac{1}{4}}} = t^{-\frac{5}{2}} + t^{-\frac{1}{2}}$$

This illustrates a general
Fact: If L_1 and L_2 are identical, except for having opposite orientations, then

$$V_{L_2}(t) = V_{L_1}(t^{-1}).$$

That this construction defines an oriented link invariant follows from a check that the bracket is invariant under Reidemeister moves II and III, and that the factor $a^{-3w(L_o)}$ provides invariance under Reidemeister move I. More details can be found in the original expositions of this idea [18], [19].

This method of calculation using diagrams can be formally justified in the following way:

Let $\mathcal{D}$ be the set of all knot diagrams, considered up to Reidemeister moves, including the empty diagram. Thus two diagrams in $\mathcal{D}$ are considered "the same" if one can be obtained from the other by a sequence of Reidemeister moves. Then if we let $R = \mathbb{Z}[a^1, a^{-1}]$, the equations (2.1) and (2.2) can be interpreted as equivalences taking place in $R\mathcal{D}$. Let $\mathcal{I}$ be the smallest ideal in $R\mathcal{D}$ generated by these equivalences, i.e. the smallest ideal containing all expressions of the form:

$\langle$  $\rangle - a \langle$  $\rangle - a^{-1} \langle$  $\rangle$ (2.4)

and

$$\langle L \sqcup \bigcirc \rangle + (a^2 + a^{-2})\langle L \rangle, \tag{2.5}$$

define

$$S = R\mathcal{D}/\mathcal{I}.$$

Note that in equation 2.4 we have reinterpreted $\langle \cdot \rangle$, using the brackets to indicate that the diagrams in equation 2.4 differ only locally; with the differences appearing as indicated.

Within this formal framework, we can properly interpret the calculations in example 2.4 as computing a "nicer" representative of the equivalence class of the diagram  in S.

It is only natural to wish to extend this computational technique to embeddings $S^1 \hookrightarrow M$ for arbitrary 3-manifolds M. However, this clever machinery of Kauffman is impotent if we adhere to our diagrammatic interpretations, as links in an arbitrary 3-manifold do not admit diagrams. We therefore think in a more general setting.

Definition 2.5. (Przytycki, [22]) Let M be an oriented 3-manifold, R a commutative ring with identity, and $a \in R$ an invertible element. Let B^3 denote an arbitrary 3-ball in M. Suppose that L_+, L_0 and L_∞ are three links in M that differ from one another only inside B^3, where they can be projected to appear as depicted in figure 2.7.

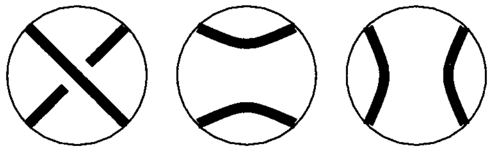


Figure 2.7: The appearance of the links L_+, L_0 and L_∞ in some 3-ball in M

When such a situation exists, the expression $L_+ - aL_0 - a^{-1}L_\infty$ is called the corresponding **skein expression.** Let $S_{2,\infty}$ be the smallest submodule of $R\mathcal{L}_f$ generated by:

1. All skein expressions
2. All expressions of the form $L \sqcup \bigcirc + (a^2 + a^{-2})L$, where L is any link in M and $\bigcirc$ is the unknot in M.

Define the **Kauffman bracket skein module**[1] to be the quotient:

$$\mathcal{S}_{2,\infty}(M; R, a) = R\mathcal{L}_f / S_{2,\infty}.$$

Hereafter, we suppress the subscript $2, \infty$ when the context is clear.

Example 2.6. It is clear that the definition of $\mathcal{S} = \mathbb{Z}[a, a^{-1}]\mathcal{D}/\mathcal{I}$ on the previous page and the definition of $\mathcal{S}(S^3, \mathbb{Z}[a, a^{-1}], a)$ are analagous in some sense, since Reidemeister moves are equivalent to ambient isotopy - the difference to be accounted for is framing. Consider the composition:

$$\mathcal{S}(S^3, \mathbb{Z}[a, a^{-1}], a) \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathbb{Z}[a, a^{-1}]$$

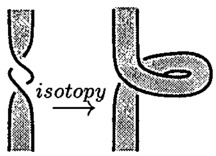
[1]The subscript $2, \infty$ in our notation arises from the paper [16], where these subscripts are meant to refer to certain 4-tangles.

where the arrows above have action

$$L \mapsto \textit{diagram of } L = \langle L \rangle \quad \langle \bigcirc \rangle \mapsto (-a^2 - a^{-2})\langle L \rangle,$$

each extended linearly to be maps of $\mathbb{Z}[a, a^{-1}]$ modules. We create our diagram via ϕ as follows:

Suppose that we plan on projecting a link L into some plane P to create our diagram. Before projecting, we isotope L so that the speficified meridian of each framed component is 'parallel' to P, so that an additional twist in the specified meridian contributes ± 1 to our writhe:



With this convention for creating diagrams, the composition

$$(\psi \circ \phi)(L) = (-a^2 - a^{-2})\langle L \rangle$$

is:

1. Well-defined, because our convention for creating diagrams via ϕ ensures that the different possible diagrams for a framed link all have equal writhe; and because the equivalence relations used in defining $\mathcal{S}(S^3, \mathbb{Z}[a, a^{-1}], a)$ are precisely the defining relations of the Kauffman bracket.

2. Surjective, as
$$(\psi \circ \phi)(p(a, a^{-1}) \cdot \emptyset) = p(a, a^{-1})$$
for any polynomial $p(a, a^{-1}) \in \mathbb{Z}[a, a^{-1}]$.

3. Injective, as any two links differing by a skein or framing relation will (by definition) have different Kauffman brackets.

We conclude that $\mathcal{S}(S^3, \mathbb{Z}[a, a^{-1}], a) \cong \mathbb{Z}[a, a^{-1}]$, in fact we can see that

$$\mathcal{S}(S^3, \mathbb{Z}[a, a^{-1}], a)$$

is free on the basis $\emptyset$.

For an arbitrary 3-manifold M, one may wonder if $\mathcal{S}(M, \mathbb{Z}[a, a^{-1}], a)$ admits such free bases – or at least an amenable set of generators. It is known that not all such modules are free, but it is not known which modules are free and which are not. Computing bases and sets of generators is an extremely difficult problem over which much ink has been spilled.

Remark 2.7. *Suppose that we can find a free basis for some $\mathcal{S}(M; R, a)$, say*

$$\{[L_1], [L_2], [L_3], ...\}.$$

Then if we are given any knot K in M, $[K] \in \mathcal{S}(M; R, a)$ has a unique representation relative to this basis as a finite sum:

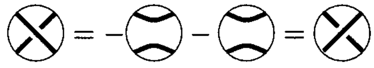
$$[K] = \sum_{i=1}^{n} r_i[L_i] \qquad \textit{for some } n.$$

In this case the ring elements $\{r_1, r_2, r_3, \cdots\}$ form a set of invariants of K in M, each analagous to the Jones polynomial in the case of S^3.

In this definition we have chosen our variable a arbitrarily, but a more deliberate choice can simplify the matter. If we choose our invertible ring element to be -1, then the skein module $\mathcal{S}(M; R, -1)$ enjoys additional structure.

Lemma 2.8. *$\mathcal{S}(M; R, -1)$ is a commutative algebra, with the product of two links given by taking their disjoint union, and identity $[\emptyset]$.*

Proof. Distributivity of this product follows immediately, since we extend our pair-wise definition of the product to all formal sums of links in precisely the way which conforms to the distributive law. Therefore, at issue is the commutativity and associativity of this product. Observe that with $a = -1$, skein relations become:



where the second equality follows from tilting one's head by 90 degrees. Thus, for any link L, $[L]$ is independent of crossing changes. In particular this means that $[L_1] \cdot [L_2] = [L_1 \sqcup L_2]$ is independent of the relative positioning of L_1 and L_2, so that

$$[L_1] \cdot [L_2] = [L_1 \sqcup L_2] = [L_2 \sqcup L_1] = [L_2] \cdot [L_1].$$

In the same manner we can argue that triple products are independent of our choice of bracketing, so that the product is associative. That the empty set is the identity is immediate from our definition. □

There is an easier way of thinking of elements in this specialized skein module. In this skein module, we allow crossing changes. It is a well-known fact (cited in [24]) that considering embedded graphs up to ambient isotopy and crossing changes is equivalent to considering embedded graphs up to homotopy. Therefore in the specialized skein module $\mathcal{S}(M; R, -1)$, two links are equivalent if and only if they are homotopic.

2.3 The Relationship Between $\pi_1(M)$ and $\mathcal{S}(M; R, a)$

We begin with the definition of a tensor algebra of a module. This is a standard algebraic object, but is not so commonly discussed in algebra classes.

Definition 2.9. Let R be a commutative ring, and M an R-module. Let

$$T^0(M) = R$$

$$T^1(M) = M$$

$$T^2(M) = M \otimes M$$

and in general

$$T^r(M) = M \otimes \cdots \otimes M$$

where the tensor above is taken r times. Define the *tensor algebra* over M to be

$$\mathbf{T}M = \bigoplus_{k=0}^{\infty} T^k(M)$$

In the langauge of category theory, $\mathbf{T}$ is a functor whose action on objects is given by the above equation, and whose action on maps is given by the formula:

$$\mathbf{T}(f)(m_1 \otimes \cdots \otimes m_r) = f(m_1) \otimes \cdots \otimes f(m_r).$$

The object $\mathbf{T}M$ arises naturally as the 'free' algebra over a module, in the sense that $\mathbf{T}$ is left adjoint to the forgetful functor U mapping from R-algebras to R-modules. Therefore in a sense, it is not a surprising or artificial structure to come across. Multiplication in the algebra $\mathbf{T}M$ is given by the tensor product of two elements, and the necessary distributive and associative properties of algebra multiplication follow immediately from the bilinearity and associativity of the tensor product.

We use tensor algebras in the following construction, due to Przytycki and Sikora in [24]:

Definition 2.10. Let G be a group with identity e, and R a comutative ring with 1. Denote the group ring over G with R coefficients by RG. Let $\mathcal{I}$ be the ideal of $\mathbf{T}RG$ generated by the expression $e - 2$ (here $2 = 1 + 1 \in R$), together with all expressions of the form:

1. $g \otimes h - h \otimes g$
2. $g \otimes h - gh - gh^{-1}$

where $g, h \in G$. Define the skein algebra of the group G with coefficients in R to be

$$\mathcal{S}(G; R) = \mathbf{T}RG/\mathcal{I}.$$

The elements of a skein algebra are therefore (equivalence classes of) formal sums of tensors of elements in G, weighted with coefficients from R. We use square brackets to denote the equivalence class of an element in $\mathcal{S}(G; R)$, for example $[g_1 \otimes g_2 \otimes g_3]$.

If we fix a ring R, then $\mathcal{S}(-; R)$ is a functor from the category of groups to the category of R-algebras. The action of $\mathcal{S}(-; R)$ on maps is to send a group homomorphism $\phi : G \to G'$ to the map denoted $\phi_* : \mathcal{S}(G; R) \to \mathcal{S}(G'; R)$, whose action is given completely by:

$$\phi_*([g]) = [\phi(g)]$$

for all $g \in G$.

The skein algebra of a group satisfies the following properties, whose proofs are largely computational [24].

1. For any $g \in G$, $[g] = [g^{-1}]$.
   Proof.
$$[g \otimes e] = [ge + ge^{-1}] = [g + g] = [g] + [g],$$

whereas we may also compute

$$[g \otimes e] = [e \otimes g] = [eg + eg^{-1}] = [g] + [g^{-1}]$$

so that $[g] + [g] = [g] + [g^{-1}]$, giving $[g] = [g^{-1}]$. □

2. For any pair $g, h \in G$, we have $[gh] = [hg]$, and consequently

$$[(hg)h^{-1}] = [h^{-1}(hg)] = [g].$$

Proof.

$$0 = [g \otimes h] - [h \otimes g] = [gh] + [gh^{-1}] - [hg] - [hg^{-1}]$$

$$= [gh] - [hg] - [hg^{-1}] + [(hg^{-1})^{-1}] = [gh] - [hg]$$

where in the last step, the cancellation of $[hg^{-1}]$ and $[(hg^{-1})^{-1}]$ follows from property (1). □

3. **The Universal Coefficient Theorem** If $\phi : R \hookrightarrow R'$ are rings, and we regard R' as an R-module with multiplication given by

$$r \cdot r' = \phi(r)r' \text{ for any } r \in R \text{ and } r' \in R',$$

then $\mathcal{S}(G; R') \cong \mathcal{S}(G; R) \otimes_R R'$.
Proof. The proof consists of showing that $u : \mathcal{S}(G; R) \otimes_R R' \to \mathcal{S}(G; R')$, defined by $u([g] \otimes_R r') = [r'g]$ is an isomorphism.

Let $\mathcal{I}(R)$ denote the ideal of $\mathbf{T}RG$ constructed in definition (2.10), and $\mathcal{I}(R')$ the analgous ideal in $\mathbf{T}R'G$. Then we have an exact sequence

$$\mathcal{I}(R) \hookrightarrow \mathbf{T}RG \to \mathcal{S}(G; R) \to 0,$$

and because the functor $\_\_ \otimes_R R'$ is right exact ([11], pp. 378-383), we obtain a second exact sequence

$$\mathcal{I}(R) \otimes_R R' \to \mathbf{T}RG \otimes_R R' \to \mathcal{S}(G; R) \otimes_R R' \to 0.$$

This second exact sequence fits into the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{I}(R) \otimes_R R' & \longrightarrow & \mathbf{T}RG \otimes_R R' & \longrightarrow & \mathcal{S}(G;R) \otimes_R R' & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle u} & & \downarrow \\
\mathcal{I}(R') & \longrightarrow & \mathbf{T}R'G & \longrightarrow & \mathcal{S}(G;R') & \longrightarrow & 0.
\end{array} \tag{2.6}$$

Here, f_2 is the isomorphism given by

$$f_2((r \cdot g_1 \otimes g_2 \otimes \cdots \otimes g_k) \otimes r') = (r'r) \cdot g_1 \otimes g_2 \otimes \cdots \otimes g_k.$$

To see that f_2 is an isomorphism, note that

$$\mathbf{T}RG \otimes_R R' = (R \oplus RG \oplus RG \otimes_R RG \oplus \cdots) \otimes_R R'$$

$$= (R \otimes_R R') \oplus (RG \otimes_R R') \oplus (RG \otimes_R RG \otimes_R R') \oplus \cdots$$

since tensors distribute over arbitrary direct sums. Therefore we can consider f_2 as a map

$$(R \otimes_R R') \oplus (RG \otimes_R R') \oplus (RG \otimes_R RG \otimes_R R') \oplus \cdots \to \mathbf{T}R'G$$

defined on each component of the direct sum by the same formula as before. We can then see that f_2 is an isomorphism component-wise on this direct sum, as the restriction of f_2 is the well known isomorphism

$$(RG \oplus \cdots \oplus RG) \otimes_R R' \cong R'G \oplus \cdots \oplus R'G$$

arising from *extension of scalars*.

We obtain f_1 by simply restricting f_2 to the subalgebra $\mathcal{I}(R) \otimes_R R'$, and so f_1 is clearly surjective.

These facts allow us to apply the five lemma to diagram 2.6 to conclude that u is an isomorphism. □

Remark 2.11. *Via an identical proof, we can show that the Universal Coefficient Theorem holds for topologically defined skein modules. Namely, if* $r : R \to R'$ *is a homomorphism of rings, then*

$$\mathcal{S}(M; R, a) \otimes R' \cong \mathcal{S}(M; R', r(a)).$$

(Recall that we have chosen to suppress the subscript $2, \infty$*.)*

At last the promised connection with the fundamental group of a manifold emerges, following a proof presented in [24].

Theorem 2.12. *If M is a 3-manifold and R is a commutative ring with 1, then*

$$\mathcal{S}(M; R, -1) \cong \mathcal{S}(\pi_1(M); R).$$

Proof. Define a function on framed links, $\psi : \mathcal{L}_f \to \mathcal{S}(\pi_1(M); R)$, in the following manner: Suppose that K is a knot in M, and that $\pi_1(M)$ is calculated with respect to the base point $x_0 \in M$. Then we can connect K to x_0 via a path α, yielding a representative element $\alpha K \alpha^{-1}$ of a conjugacy class in $\pi_1(M)$ (In doing this we have arbitrarily assigned an orientation to K). Let $\bar{K}$ denote this conjugacy class. Define ψ by the rules $\psi(K) = -[\bar{K}]$ and $\psi(\emptyset) = -1$.

First, note that ψ is well-defined, as property (1) of group skein modules shows that our choice of orientation for K does not affect $\psi(K)$, and property (2) of group skein modules shows that we may connect K to our base point using any path we please, so that our choice of α does not affect $\psi(K)$.

Suppose a link L in M has components $K_1, \cdots, K_n$. We define ψ on $L \in \mathcal{L}_f$ according to the rule

$$\psi(L) = \psi(K_1 \sqcup \cdots \sqcup K_n) = (-1)^n \psi(K_1) \otimes \cdots \otimes \psi(K_n)$$

i.e., we extend to all elements in $\mathcal{L}_f$ precisely the way that agrees with the algebra multiplication. After extending multiplicatively to all of $\mathcal{L}_f$, we then extend ψ linearly to all of $R\mathcal{L}_f$. Note that there is no problem regarding the ordering of $\psi(K_1), \cdots, \psi(K_n)$ in our product, because tensor products have been made abelian.

Next we check that ψ descends to an algebra homomorphism

$$\hat{\psi} : \mathcal{S}(M; R, -1) \to \mathcal{S}(\pi_1(M); R).$$

With our choice of $a = -1$, the relations defining $\mathcal{S}(M; R, -1)$ become

$$L \sqcup \bigcirc + 2L$$

and

$$L_+ + L_0 + L_\infty.$$

In this equation, $\bigcirc$ is a loop that is contained in some 3-ball, so we can directly compute

$$\psi(L \sqcup \bigcirc + 2L) = (-1)^2\psi(L) \otimes \psi(\bigcirc) - 2\psi(L) = [e - 2]\psi(L) = 0$$

by noting that $\bigcirc$ is the identity in $\pi_1(M)$, and so $[\bigcirc] = [e]$.

Dealing with the second relation is trickier. First, observe that we can write any skein relation as

$$L_1 \sqcup L_+ + L_1 \sqcup L_0 + L_1 \sqcup L_\infty = L_1 \sqcup (L_+ + L_0 + L_\infty)$$

where L_+ and L_0 are knots, and L_∞ is a two component link. We do this by absorbing into L_1 all components of our link, except for the component that intersects our 3-ball of interest. With this choice of L_1 we know that one of $\{L_0, L_\infty\}$ is a two-component link, and the other is a knot. Without loss of generality, we have chosen L_0 to be the knot, and L_∞ to be the two component link, for we can interchange L_0 and L_∞ in the skein relation by tilting our head.

By the above considerations, it suffices to show

$$\psi(L_+ + L_0 + L_\infty) = 0$$

in the case where L_+ and L_0 are knots, and $L_\infty = K_1 \sqcup K_2$ is a two-component link. In this case, choose a base point inside the 3-ball where L_+, L_0 and L_∞ differ. Then by connecting K_1 and K_2 to our base point and carefully choosing orientations, we get $\alpha, \beta \in \pi_1(M)$ such that

1. $\alpha \sim K_1$ and $\beta \sim K_2$
2. $\alpha\beta \sim L_+$
3. $\alpha\beta^{-1} \sim L_0$.

Now we may compute

$$\psi(L_+ + L_0 + L_\infty) = \psi(L_+) + \psi(L_0) + \psi(K_1 \cdot K_2) = -[\alpha\beta] - [\alpha\beta^{-1}] + [\alpha] \otimes [\beta] = 0$$

and hence ψ descends to $\hat{\psi} : \mathcal{S}(M; R, -1) \to \mathcal{S}(\pi_1(M); R)$.

To prove that ψ is an isomorphism, we define an inverse. Let

$$\phi : \mathbf{T}R\pi_1(M) \to \mathcal{S}(M; R, -1)$$

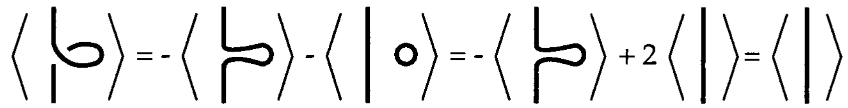
be the map defined by

$$\phi(\gamma) = -K_\gamma,$$

where $\gamma \in \pi_1(M)$ and K_γ is a knot representing γ, whose framing we choose arbitrarily. Note that if we are to have $\phi(r \cdot \alpha) = r \cdot \phi(\alpha)$, then we must also have $\phi(r) = r \cdot \emptyset$ for any pure ring element $r \in \mathbf{T}R\pi_1(M)$. This map is well-defined because:

1. Two homotopic knots will differ from one another by a sequence of ambient isotopies and crossing changes, and we can change crossings in $\mathcal{S}(M; R, -1)$.

2. Choosing $a = -1$ makes the knots independent of framing, so we may assign any framing to K_γ. We can convince ourselves of this pictorially by computing using the Kauffman bracket notation, and with $a = -1$:



Having convinced ourselves of well-definedness, we check that ϕ descends to a homomorphism

$$\hat{\phi} : \mathcal{S}(\pi_1(M); R) \to \mathcal{S}(M; R, -1).$$

1. Recalling that $\phi(r) = r \cdot \emptyset$,

$$\phi(e - 2) = -\bigcirc - 2 \cdot \emptyset = (a^2 + a^{-2}) \cdot \emptyset - 2 \cdot \emptyset = 0,$$

where the last equality follows when we take $a = -1$.

2. For any loops α, β, we compute

$$\phi(\alpha \otimes \beta - \beta \otimes \alpha) = K_\alpha \cdot K_\beta - K_\beta \cdot K_\alpha = 0$$

since our product in $\mathcal{S}(M; R, -1)$ is commutative.

3. For any loops α, β:

$$\phi(\alpha \otimes \beta - \alpha\beta - \alpha\beta^{-1}) = K_\alpha \cdot K_\beta + K_{\alpha\beta} + K_{\alpha\beta^{-1}} = L_\infty + L_+ + L_0 = 0.$$

The constructed homomorphism ϕ is the inverse of ψ, and so ψ is an isomorphism as claimed. $\square$

We construct an explicit connection with $\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a)$ via the following isomorphisms:

Corollary 2.13. *For any 3-manifold M, we have*

$$\mathcal{S}(\pi_1(M),\mathbb{C}) \cong \mathcal{S}(M;\mathbb{C},-1) \cong \mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a) \otimes_{\mathbb{Z}[a,a^{-1}]} \mathbb{C}.$$

Proof. The first isomorphism is the result of our theorem. The second arises from applying the Universal Coefficient Theorem to this special case, taking the map $r : R \to R'$ to be the map

$$\mathbb{Z}[a,a^{-1}] \to \mathbb{C}$$

defined by $a \mapsto -1$. $\square$

It is because of this isomorphism that the algebraic object $\mathcal{S}(G;R)$ is of interest to us. Knowledge of $\mathcal{S}(G;R)$ for some finitely generated group G can be translated into information about $\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a)$, whenever M satisfies $\pi_1(M) \cong G$. In particular, observe that any linear relationship between elements in $\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a)$ translates into a linear relationship between the same elements considered as elements of $\mathcal{S}(\pi_1(M),\mathbb{C})$, considered as a complex vector space. We can therefore state the following:

Fact: If a family of elements in $\mathcal{S}(\pi_1(M),\mathbb{C})$ are linearly independent, then the same elements considered in $\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a)$ are still linearly independent.

2.3.1 The skein module of an Abelian group

We start with some notation.

Definition 2.14. For G a group and R a ring, define $sym(RG)$to be the subalgebra of RG generated by elements of the form $g+g^{-1}$ for $g \in G$.

We first need the following fact in order to tackle the skein module of abelian group.

Theorem 2.15. *(Przytycki, Sikora, [24]) Let G be an abelian group. Then considered as an R-module, $sym(RG)$ is a free R-module with basis $\{e\} \cup \{g+g^{-1}\}_{g \in B}$, where*

1. $B = G - \{e\}$ *if* $2 \neq 0$ *in* R
2. $B = \{g \in G : g^2 \neq e\}$ *if* $2 = 0$ *in* R

Proof. First, note that though $sym(RG)$ is generated as an R-algebra by elements of the form $g+g^{-1}$, it is also generated as an *R-module* by such elements, as direct computation reveals:

$$(g+g^{-1})(h+h^{-1}) = gh + (gh)^{-1} + gh^{-1} + (gh^{-1})^{-1}.$$

The critical observation is that the right hand side is a sum of elements of the form $g+g^{-1}$.

To show that we indeed have a basis, suppose $0 \neq 2$ in R, and that

$$re + r_1(g_1+g_1^{-1}) + r_2(g_2+g_2^{-1}) + \cdots + r_n(g_n+g_n^{-1}) = 0$$

in RG. Then since the elements $g \in G$ form a basis for RG, we wish to rebracket this sum as an R-linear combination of elements in G so as to draw the desired conclusion - that all r_i are zero. The coefficients r_i distribute over the sums $g_i + g_i^{-1}$ giving a sum of $2n$ distinct elements in G, *provided* we do not have $g_k = g_k^{-1}$ for some g_k. In this case the term $2r_k g_k$ appears in our sum. Therefore in general, written as a sum of distinct elements in G, our rebracketing of the sum will have coefficients of the form r_i and $2r_i$. Since $2 \neq 0$, we can still use the fact that $g \in G$ is a basis of RG to conclude that all the r_i's are zero.

On the other hand, if $2 = 0$ in R, then any term of the form $g + g^{-1} = 2g$ in such a sum is zero, and hence we arrive at the generating set stipulated in (2). □

In the above proof, we have used the fact that $g \in G$ is a basis of RG in order to show that a certain generating set of $sym(RG)$ is a basis. We can now use this basis of $sym(RG)$ in a similar manner, to show that a certain generating set of $\mathcal{S}(G;R)$ is in fact a basis in the case that G is abelian.

Theorem 2.16. *(Przytycki, Sikora [24]) Let G be an abelian group, R a commutative ring with 1. Define $\phi : TRG \to RG$ by $\phi(g) = g + g^{-1}$ for all $g \in G$. Then provided* *either $2 \neq 0$ in R, or G has no elements of order 2, ϕ descends to an isomorphism of modules $\Phi : \mathcal{S}(G;R) \to sym(RG)$.*

Proof. The map Φ is well defined, as we compute:

1. $\phi(e-2) = \phi(e) - \phi(2) = e + e^{-1} - 2e = 0$

2. $\phi(g \otimes h - h \otimes g) = \phi(g)\phi(h) - \phi(h)\phi(g) = 0$, since G is abelian

3. Lastly,

$$\begin{aligned} \phi(g \otimes h - gh - gh^{-1}) &= \phi(g)\phi(h) - \phi(gh) - \phi(gh^{-1}) \\ &= (g+g^{-1})(h+h^{-1}) \\ &\quad -(gh+(gh)^{-1}) - (gh^{-1}+g^{-1}h) \\ &= 0, \end{aligned}$$

(upon expanding and collecting terms).

We now construct a generating set of $\mathcal{S}(G;R)$, as an R-module. We know that $\mathbf{T}RG$ is generated as an R-module by all finite tensors of elements of G, together with $1 \in R$. Therefore this set certainly generates $\mathcal{S}(G;R)$ as an R-module. However, we can reduce this set. Using the identity $g \otimes h = gh + gh^{-1}$, we represent any finite tensor as sum of elements in RG, so we can discard all tensors from the generating set. We have now reduced our generating set to $G \cup \{1\}$. However, further using the fact that $e = 2$ and that $g = g^{-1}$ in $\mathcal{S}(G;R)$, we can reduce this generating set to

$$X = \left(\left\{ {}^{G}\Big/_{g \sim g^{-1}} \right\} - \{e\} \right) \cup \{1\}$$

.

By definition of our map Φ, we have that $\Phi(\mathbf{T}RG) = sym(RG)$. By the previous theorem, the set

$$Y = \{g + g^{-1} : g \in G - \{e\}\} \cup \{e\}$$

is a basis for $sym(RG)$. As a map of R-modules, Φ carries the generating set X bijectively onto the basis Y, and hence Φ is an isomorphism of R-modules, and so is also an isomorphism of algebras.

$\square$

This gives us a complete description of $\mathcal{S}(G;R)$ in the event that G is abelian.

Chapter 3

Connections with $SL(2, \mathbb{C})$ Character Varieties

3.1 The $SL(2, \mathbb{C})$ Character Variety of a Group

Let G be a finitely generated group. A *representation* of G in $SL(2, \mathbb{C})$ is a homomorphism $\rho : G \to SL(2, \mathbb{C})$. Therefore we may think of elements $\rho(g)$ as invertible linear maps

$$\mathbb{C}^2 \xrightarrow{\rho(g)} \mathbb{C}^2.$$

There are two mutually exclusive types of representations: *irreducible* representations and *reducible* representations. A representation is called reducible if there exists a proper subspace V of $\mathbb{C}^2$ such that $\rho(g)$ fixes V for all $g \in G$. That is,

$$(\rho(g))(v) \in V \quad \text{for all } g \in G \text{ and for all } v \in V.$$

If a representation is not reducible, then it is irreducible.

The *character* of a representation ρ is the composition

$$\chi_\rho : G \xrightarrow{\rho} SL(2, \mathbb{C}) \xrightarrow{trace} \mathbb{C}.$$

Let $X(G)$ denote the set of all characters of the group G. For each $g \in G$, there is a map

$$\tau_g : X(G) \to \mathbb{C}$$

defined by $\tau_g(\chi_\rho) = \chi_\rho(g)$. The maps τ_g satisfy:

1. For any $g \in G$, $\tau_g = \tau_{g^{-1}}$. This follows from the identity $tr(A) = tr(A^{-1})$, which holds in $SL(2, \mathbb{C})$.

2. Since trace is invariant under conjugation, $\tau_g = \tau_h$ if g and h are conjugate elements in the group G.

Already we have a surprising result, which was proved independently by many people:

Theorem 3.1. *(Vogt, Fricke, Horowitz, Culler-Shalen [8]) There exists a finite set of elements*

$$\{g_1, ..., g_n\} \subset G$$

such that every τ_g is an element of the polynomial ring $\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}]$.

Proof. Suppose that G has generators $\{h_1, ..., h_m\}$. Let R be the ring generated by all the functions

$$\tau_{h_{i_1}...h_{i_r}}$$

where the $i_1, ..., i_r$ are distinct positive integers $\leq m$. The finite set of elements

$$\{h_{i_1}...h_{i_r} : i_1, ..., i_r \text{ are distinct positive integers } \leq m\}$$

will correspond to the finite set of elements $\{g_1, ..., g_n\}$ stipulated in the statement of the theorem. Under this correspondence, $\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] = R$, so we prove that $\tau_g \in R$ for every $g \in G$. The proof will be an induction, which relies heavily upon the following lemma.

Lemma 3.2. *For any $A, B \in SL(2,\mathbb{C})$,*

$$tr(A)tr(B) = tr(AB) + tr(AB^{-1}).$$

Proof. For any $A, B \in SL(2,\mathbb{C})$, we find that the characteristic polynomial of B is

$$\lambda^2 - tr(B)\lambda + 1$$

and so by the Cayley-Hamilton Theorem, B satisfies

$$B^2 - tr(B)B + I = 0.$$

We rearrange this expression to give

$$B^2 + I = tr(B)B$$

and multiply from the right by $B^{-1}A$ to find

$$BA + B^{-1}A = tr(B)A$$

which gives

$$tr(BA) + tr(B^{-1}A) = tr(B)tr(A)$$

upon taking the trace of both sides, which can equivalently be written as

$$tr(AB) + tr(AB^{-1}) = tr(B)tr(A)$$

by using $tr(AB) = tr(BA)$. □

Remark 3.3. *From this lemma we get*

$$\tau_g \tau_h = \tau_{gh} + \tau_{gh^{-1}}$$

for all $g, h \in G$.

We now start our first of two inductions. This induction will show that $\tau_g \in R$ whenever $g = h_{i_1}^{k_1} ... h_{i_r}^{k_r}$, where $i_1, ..., i_r$ are distinct integers between 1 and m, and $k_1, ..., k_r \in \mathbb{Z}$. We will handle the other elements of G with a second induction.

Our first induction is on the positive integer μ, defined by

$$\mu(g) = \sum_{j=1}^{m} \nu_j$$

where

$$\nu_j = \begin{cases} -k_j & \text{if } k_j \leq 0; \\ k_j - 1 & \text{if } k_j > 0. \end{cases}$$

Of course there may be many different ways of writing an element g as a product of generators, so that our definition of $\mu(g)$ above may not be well-defined. To remedy this, we take $\mu(g)$ to be the minimum arising from all possible ways of writing g as a product of generators.

As our base case, note that if $\mu(g) = 0$, then all the k_j are either 0 or 1, so that $\tau_g \in R$ by definition of R.

Claim: Without loss of generality, we can assume $k_1 \neq 0, 1$.
Proof of claim: Since $\mu(g) > 0$, there is some k_d that is not one or zero. Choose d to be the smallest integer for which $k_d \neq 1, 0$. In our argument we can replace the element g with the conjugate

$$g' = h_{i_{d-1}}^{-1} \dots h_{i_1}^{-1} g h_{i_1} \dots h_{i_{d-1}},$$

since $\tau_g = \tau_{g'}$, and because $\mu(g) = \mu(g')$, as a quick computation shows. The existence of g' proves the claim.

We can therefore consider two cases: $k_1 > 1$ and $k_1 < 0$.

1. If $k_1 < 0$, then we can use our remark to write

$$\tau_{h_{i_1}^{-1}} \tau_{h_{i_1} g} = \tau_g + \tau_{h_{i_1}^{-1} g^{-1} h_{i_1}^{-1}}$$

so that we find

$$\tau_g = -\tau_{h_{i_1}^{-1} g^{-1} h_{i_1}^{-1}} + \tau_{h_{i_1}^{-1}} \tau_{h_{i_1} g}.$$

Note that

$$\tau_{h_{i_1}^{-1} g^{-1} h_{i_1}^{-1}} = \tau_{h_{i_1} g h_{i_1}} = \tau_{h_{i_1}^2 g}$$

so we get

$$\tau_g = \tau_{h_{i_1}^{-1}} \tau_{h_{i_1} g} - \tau_{h_{i_1}^2 g}.$$

We show that the right hand side lies in R, completing the induction for the case $k_1 < 0$. Since $k_1 < 0$, we compute that $\mu(h_{i_1} g) = \mu(g) - 1$ and $\mu(h_{i_1}^2 g) < \mu(g) - 1$, so that $\tau_{h_{i_1}^2 g}, \tau_{h_{i_1} g} \in R$ by the induction hypothesis. By definition, $\tau_{h_{i_1}^{-1}} = \tau_{h_{i_1}} \in R$. So the right hand side lies in R.

2. If $k_1 > 0$, we can proceed in exact analogy with the first case. We write

$$\tau_g = \tau_{h_{i_1}} \tau_{h_{i_1}^{-1} g} - \tau_{h_{i_1} g^{-1} h_{i_1}}$$

and then note that

$$\tau_{h_{i_1} g^{-1} h_{i_1}} = \tau_{h_{i_1}^{-1} g h_{i_1}^{-1}} = \tau_{h_{i_1}^{-2} g}$$

so that we get

$$\tau_g = \tau_{h_{i_1}} \tau_{h_{i_1}^{-1} g} - \tau_{h_{i_1}^{-2} g}.$$

> Then both $\mu(h_{i_1}^{-2}g)$ and $\mu(h_{i_1}^{-1}g)$ are strictly less than $\mu(g)$, in analogy with before, so that $\tau_{h_{i_1}^{-2}g}, \tau_{h_{i_1}^{-1}g} \in R$ by the induction hypothesis. Noting that $\tau_{h_{i_1}} \in R$ (by definition) completes the induction in this second case.

Therefore, by induction, $\tau_g \in R$ whenever $g = h_{i_1}^{k_1}...h_{i_r}^{k_r}$, where $i_1, ..., i_r$ are distinct integers between 1 and m, and $k_1, ..., k_r \in \mathbb{Z}$.

We begin our second induction. Take any element $g \in G$, and we write $g = h_{i_1}^{k_1}...h_{i_r}^{k_r}$, where $i_1, ..., i_r$ are *not* necessarily distinct. We induct on r (again taken to be minimal over all ways of writing g as a product of generators) to reach our desired conclusion.

The cases $r = 1, 2$ are both covered by our first induction, so that the base case for this second induction holds.

If all $i_1, ..., i_r$ are distinct, this is a case we have already dealt with, so assume at least two of $i_1, ..., i_r$ are equal. Upon replacing g by a conjugate element with equal value r, we may assume without loss of generality that $i_s = i_r$ for some $s < r$. Then we split g into two pieces

$$X = h_{i_1}^{k_1}...h_{i_s}^{k_s} \qquad Y = h_{i_{s+1}}^{k_{s+1}}...h_{i_r}^{k_r}$$

and write

$$\tau_g = \tau_{XY} = \tau_X\tau_Y - \tau_{XY^{-1}}.$$

By the induction hypothesis, we clearly have $\tau_X, \tau_Y \in R$. Additionally, since $i_s = i_r$, the element XY^{-1} can be written as a "shorter" product of generators than g, namely:

$$XY^{-1} = h_{i_1}^{k_1}...h_{i_s}^{k_s-k_r}h_{i_{r-1}}^{-k_{r-1}}...h_{i_{s+1}}^{-k_{s+1}}$$

which has $r-1$ terms, so that $\tau_{XY^{-1}} \in R$ by the induction hypothesis as well. Therefore $\tau_g \in R$ for an arbitrary $g \in G$. □

This proof gives a flavour for the techniques involved in this subject, while setting the stage for an even more surprising result. Fix the set of generators of $R = \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}]$ provided by theorem 3.1. Then the map

$$t : X(G) \to \mathbb{C}^n \qquad t(\chi_\rho) = (\tau_{g_1}(\chi_\rho), ...\tau_{g_n}(\chi_\rho))$$

is an injection. To see this, suppose that $t(\chi_\rho) = t(\chi_{\rho'})$, in other words $\tau_{g_i}(\chi_\rho) = \tau_{g_i}(\chi_{\rho'})$ for $i = 1, ..., n$. Then given an arbitrary $g \in G$, by theorem

3.1 we can write τ_g as a polynomial in $\tau_{g_1}, ..., \tau_{g_n}$:

$$\tau_g = p(\tau_{g_1}, ..., \tau_{g_n}).$$

Now we check that $\chi_\rho = \chi_{\rho'}$ by verifying that they take the same value on this arbitrary $g \in G$. A quick computation shows

$$\begin{aligned}
\chi_\rho(g) &= \tau_g(\chi_\rho) \\
&= p(\tau_{g_1}(\chi_\rho), ..., \tau_{g_n}(\chi_\rho)) \\
&= p(\tau_{g_1}(\chi_{\rho'}), ..., \tau_{g_n}(\chi_{\rho'})) \\
&= \tau_g(\chi_{\rho'}) \\
&= \chi_{\rho'}(g)
\end{aligned}$$

so that t is an injection as claimed.

Recall that subset $V \subset \mathbb{C}^n$ is called an *algebraic set* if V is the set of common zeros of some set S of polynomials contained in $\mathbb{C}[x_1, ..., x_n]$.

Theorem 3.4. *(Culler, Shalen [8]) The set $t(X(G)) \subset \mathbb{C}^n$ is the zero set of an ideal in $\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}]$, and so is an algebraic set. For different choices of the elements $g_1, ..., g_n$, the resulting different parameterizations of $t(X(G))$ are equivalent via polynomial maps.*

This theorem has been the foundation for an entire branch of study. The proof is extremely difficult and can be found in [8].

Recall that the *coordinate ring* of an algebraic set $V \subset \mathbb{C}^n$ is defined as

$$\mathbb{C}[V] := \mathbb{C}[x_1, ..., x_n] \Big/ \mathcal{I}(V),$$

where $\mathcal{I}(V)$ is the unique largest ideal of polynomials that are identically zero on V:

$$\mathcal{I}(V) := \{f \in \mathbb{C}[x_1, ..., x_n] : f(a_1, ..., a_n) = 0 \text{ for all } (a_1, ..., a_n) \in V\}.$$

There is a more convenient way of thinking of the coordinate ring. The polynomials in $\mathbb{C}[x_1, ..., x_n]$ define functions on the algebraic set V simply by restriction. Two polynomials $f, g \in \mathbb{C}[x_1, ..., x_n]$ define the same function on V precisely when $f - g = 0$ on V, which means that $f - g \in \mathcal{I}(V)$. Therefore the cosets in $\mathbb{C}[V]$ can be thought of as polynomial functions restricted to V.

If two algebraic sets V, W are equivalent via a polynomial map $\phi : V \to W$, then $\tilde{\phi} : \mathbb{C}[W] \to \mathbb{C}[V]$ defined by $\tilde{\phi}(f) = f \circ \phi$ is an isomorphism of $\mathbb{C}$-algebras. From this algebraic fact, we choose not to study the object $t(X(G))$ of theorem 3.4, but instead the coordinate ring

$$\mathbb{C}[t(X(G))] = \mathbb{C}[\tau_{g_1}, \ldots, \tau g_n] \Big/ \mathcal{I}(t(X(G))).$$

We shorten the notation from $\mathbb{C}[t(X(G))]$ to $\mathbb{C}[X(G)]$, as the map t is immaterial - since t depends on the choice of coordinates, but our final object does not.

In this discussion, taking $G = \pi_1(M)$ for some 3-manifold M can already be used to yield powerful results. In this case, we often shorten $X(\pi_1(M))$ to $X(M)$. The following result was first proven by Thurston, then reworded in a much different language by Culler and Shalen.

Theorem 3.5. *(Thurston [1], Culler-Shalen [8]) Let M be a compact, orientable 3-manifold. Suppose M has s torus components in its boundary, $T_k \overset{i_k}{\hookrightarrow} \partial M$. Let*

$$\rho : \pi_1(M) \to SL(2,\mathbb{C})$$

be an irreducible representation such that

$$\rho(i_{k_*}(\pi_1(T_k))) \not\subseteq \{I, -I\} \qquad \text{for each } k.$$

Then any component of $X(M)$ containing χ_ρ has dimension (as a variety) of at least $s - 3\chi(M)$.

Corollary 3.6. *If M is an n-component link complement, and if $\pi_1(M)$ admits a representation ρ as stipulated in theorem 3.5, then $X(M)$ has a component whose dimension is at least n.*

Proof. Since M is an n-component link complement, we get immediately that ∂M has n torus components. There is a well known formula for odd-dimensional homology n-manifolds which says that [21], [14]:

$$\chi(M) = \frac{1}{2}\chi(\partial M),$$

[1] Culler and Shalen attribute this theorem to Thurston, although the source they give appears to never have been published.

which we can certainly apply in our case. Therefore, in the particular case that M is a link complement, we know that the component of $X(M)$ containing ρ has dimension at least

$$s - 3\chi(\partial M) = n - \frac{3}{2}\chi(\text{torus}) = n - 0 = n.$$

□

3.2 The Connection with Skein Algebras

It is reasonable to believe there could be a connection between $SL(2,\mathbb{C})$ character varieties and skein algebras, because of the striking similarities between the two defining identities:

$$[g] \otimes [h] = [gh] + [gh^{-1}]$$

and

$$tr(A)tr(B) = tr(AB) + tr(AB^{-1}).$$

This connection was provided by Bullock, in the form of the following theorem:

Theorem 3.7. *(Bullock, [6]) For any group G, there exists a surjective map of algebras*

$$\psi : \mathcal{S}(G;\mathbb{C}) \to \mathbb{C}[X(G)]$$

defined by $\psi([g]) = \tau_g$ for any $g \in G$. Furthermore,

$$ker(\psi) = \sqrt{0},$$

where $\sqrt{0}$ denotes the subalgebra of all nilpotent elements in $\mathcal{S}(G;\mathbb{C})$.

Proof. First, note that from its definition, ψ is clearly surjective. Next we show that ψ is well-defined, by showing that ψ maps the defining ideal of $\mathcal{S}(G;\mathbb{C})$ to zero. Applying the definition of ψ to the three types of elements in the ideal, we get:

1.
$$\psi([e-2]) = \psi([e]) - \psi([2]) = \tau_e - 2.$$
However, because $\rho(e) = Id \in SL(2,\mathbb{C})$ for any representation ρ, we find that:
$$\tau_e(\chi_\rho) = \chi_\rho(e) = trace(Id) = 2.$$
Therefore the function $\tau_e - 2$ is zero when evaluated on any character.

2.
$$\psi([g \otimes h - h \otimes g]) = \tau_g \tau_h - \tau_h \tau_g = 0,$$
where the last equality follows from the commutativity of $\mathbb{C}[X(G)]$.

3.
$$\psi([g \otimes h - gh - gh^{-1}]) = \tau_g \tau_h - \tau_{gh} - \tau_{gh^{-1}} = 0,$$
where the last equality is exactly remark 3.3.

We can readily observe that $\sqrt{0} \subset ker(\psi)$. Given $a \in \sqrt{0} \subset \mathcal{S}(G;\mathbb{C})$, if the image $\psi(a)$ is non-zero, it must be nilpotent in $\mathbb{C}[X(G)]$. However, from the definition of $\mathbb{C}[X(G)]$, we know that it cannot contain nilpotents (if the polynomial f is nonzero on some subset $A \subset \mathbb{C}$, no power f^n can be zero on A). Therefore $\psi(a) = 0$.

That $ker(\psi) \subset \sqrt{0}$ is much more difficult to prove, and it appears there are only two known proofs. One proof is algebraic, and the other is topological/combinatorial, both were created independently of one another.

The algebraic proof involves universal representation $\mathbb{C}$-algebras and the Brumfiel-Hilden algebra. The Brumfiel-Hilden algebra is a structure defined in [4] for the purposes of investigating $SL(2,\mathbb{C})$ representations of groups, and is denoted $TH_{\mathbb{C}}(G)$. In [23] it is shown to be isomorphic to $\mathcal{S}(G;\mathbb{C})$. The proof of our theorem appears in [4], and is done in the language of Brumfiel-Hilden algebras.

The topological and combinatorial proof is extremely lengthly, and deals with resolving trees, Young diagrams and Procesi identities. It is the subject of [6]. Both proofs involve a great deal of tangential material, and so are not presented here.

□

Also in [6] is a partial proof of the following fact:

Proposition 3.8. *The skein algebra $\mathcal{S}(G; \mathbb{C})$ is finite dimensional as a complex vector space if and only if $dim(X(G)) = 0$.*

Proof. First, we remark that a variety V has dimension zero if and only if the coordinate ring $\mathbb{C}[V]$ is finite dimensional as a vector space, the proof of which can be found in Appendix A.

Since $X(G)$ is a variety, it is the zero set of some ideal $I \subset \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}]$. Define a map

$$\phi : \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \to \mathcal{S}(G; \mathbb{C})$$

by $\phi(\tau_g) = [g]$. Since $\tau_g \in \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}]$ for every $g \in G$, the map ϕ is surjective. Theorem 10.2 in [6] tells us that

$$\sqrt{ker(\phi)} = \sqrt{I}.$$

Then any ideal contains some power of its radical (Chapter 15, Proposition 11 of [11]), so that we get

$$(\sqrt{I})^m \subset ker(\phi) \subset \sqrt{I}. \tag{3.1}$$

We have the following isomorphisms:

$$\mathbb{C}[X(G)] \cong \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \Big/ \mathcal{I}(\mathcal{Z}(I)) \cong \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \Big/ \sqrt{I}$$

where the last isomorphism follows from Hilbert's Nullstellensatz. This tells us that

$$\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \Big/ \sqrt{I}$$

is finite dimensional as a complex vector space, so that

$$\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \Big/ (\sqrt{I})^m$$

is also finite dimensional. However, recalling the inclusion of equation 3.1, we know that there is a quotient map

$$\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \Big/ (\sqrt{I})^m \longrightarrow \mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}] \Big/ ker(\phi) \cong \mathcal{S}(G; \mathbb{C})$$

so that $\mathcal{S}(G;\mathbb{C})$ is the image of a finite dimensional vector space.

Conversely, if $\mathcal{S}(G;\mathbb{C})$ is finite dimensional, then the surjection ψ from theorem 3.7 tells us that $\mathbb{C}[X(G)]$ is also finite dimensional, and so $\dim(X(G)) = 0$. □

Of course, we can restate this theorem as "$\dim\mathcal{S}(G;\mathbb{C}) = \infty$ if and only if $\dim(X(G)) \geq 1$". Both of these statements are useful to bear in mind.

Corollary 3.9. *Let M be a link complement. Suppose M has boundary components $T_k \overset{i_k}{\hookrightarrow} \partial M$. Let*

$$\rho : \pi_1(M) \to SL(2,\mathbb{C})$$

be an irreducible representation such that

$$\rho(i_{k_*}(\pi_1(T_k))) \not\subseteq \{I, -I\} \qquad \textit{for all } k.$$

Then $\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a)$ is infinitely generated as a module.

Proof. By applying corollary 3.6 and proposition 3.8 we get that $\mathcal{S}(\pi_1(M),\mathbb{C})$ is infinite dimensional, which gives the desired conclusion. □

Immediately we have some obvious questions: When $\dim\mathcal{S}(G;\mathbb{C}) < \infty$, what is the dimension? How does it relate to a manifold M in the event that we take $G = \pi_1(M)$? More obvious would be the question: In any event, what is a basis of $\mathcal{S}(G;\mathbb{C})$? We take some steps towards answering this last question by identifying some linearly independent families of elements in $\mathcal{S}(G;\mathbb{C})$.

3.3 An Application of Character Varieties

We present here an application, due to Doug Bullock in [5].

Fix a representation ρ of a finitely generated group G. For each fixed ρ, we get a map

$$ev_\rho : \mathbb{C}[X(G)] \to \mathbb{C},$$

the *evaluation map*, defined by

$$ev_\rho(\tau_g) = \tau_g(\chi_\rho) = \chi_\rho(g).$$

This evaluation "makes sense", as we recall that the elements of the coordinate ring $\mathbb{C}[X(G)]$ can be thought of as restrictions of polynomial maps to the set of all characters, $X(G)$.

Fix an element $g \in G$, and let k be an arbitrary integer. Recall that

$$\psi : \mathcal{S}(G;\mathbb{C}) \to \mathbb{C}[X(G)]$$

is defined by $\psi([g]) = \tau_g$, and consider the image of $[g^k] \in \mathcal{S}(G;\mathbb{C})$ under the composition $ev_\rho \circ \psi$. We first calculate that

$$ev_\rho \circ \psi([g^0]) = ev_\rho \circ \psi([e]) = ev_\rho(\tau_e) = \tau_e(\chi_\rho) = \chi_\rho(e) = 2,$$

and

$$ev_\rho \circ \psi([g]) = ev_\rho(\tau_g) = \tau_g(\chi_\rho) = \chi_\rho(g).$$

Denote the complex number $\chi_\rho(g)$ by z. Then using this notation,

$$\begin{aligned} ev_\rho \circ \psi([g^k]) &= ev_\rho \circ \psi([g^{k-1}] \otimes [g] - [g^{k-2}]) \qquad \text{as } g \otimes h = gh + gh^{-1} \\ &= ev_\rho \circ \psi([g^{k-1}])z - ev_\rho \circ \psi([g^{k-2}]). \end{aligned}$$

Therefore, we have a recursive formula for $ev_\rho \circ \psi([g^k])$ in terms of

$$ev_\rho \circ \psi([g^{k-1}]) \text{ and } ev_\rho \circ \psi([g^{k-2}]).$$

Defining polynomials $p_k(z)$ by the same recursion,

$$p_0(z) = ev_\rho \circ \psi([g^0]) = 2,$$

$$p_1(z) = ev_\rho \circ \psi([g]) = z,$$

and in general

$$p_k(z) = zp_{k-1}(z) - p_{k-2}(z),$$

we have that $ev_\rho \circ \psi([g^k]) = p_k(z)$. The first few polynomials defined by this recursion are:

$$\begin{aligned} p_0(z) &= 2 \\ p_1(z) &= z \\ p_2(z) &= z^2 - 2 \\ p_3(z) &= z^3 - 3z \\ p_4(z) &= z^4 - 4z^2 + 2 \\ p_5(z) &= z^5 - 5z^3 + 5z. \end{aligned}$$

Remark 3.10. *A quick way of computing the k^{th} such polynomial is by making use of the determinant identity*

$$p_k(z) = \begin{vmatrix} z & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & z & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & z & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & z & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & z \end{vmatrix},$$

where the above matrix is of size $k \times k$. This identity follows from remarking that cofactor expansion along the top row yields the same recursive relationship as the defining relationship of the p_k's. From this identity, or from a quick induction, one can see that $deg(p_k) = k$.

Define M_r to be the $r \times r$ matrix over $\mathbb{C}[z_1, ..., z_r]$ whose (i, j)-th entry is $p_i(z_j)$. Then

Lemma 3.11. *[6] The determinant $|M_r|$ is a degree $\frac{r(r+1)}{2}$ polynomial in the variables $z_1, ..., z_r$.*

Proof. The proof is by induction. First, the claim is certainly true for $r = 1$, as

$$M_1 = [p_1(z_1)] = [z].$$

Suppose that up to $r - 1$, the determinant $|M_{r-1}|$ is of the stipulated degree. Expanding M_r along the r-th row, we get:

$$M_r = \sum_{i=1}^{r} p_r(z_i)|C_i|, \tag{3.2}$$

where C_i is M_r with the r-th and i-th rows eliminated. But now the polynomials appearing in the columns of C_i still satisfy the defining recursion

$$p_0 = 2$$

$$p_1 = z$$

$$p_k = zp_{k-1} - p_{k-2},$$

but both the r-th row and the column containing polynomials in the variable z_i have been eliminated. Therefore, we may apply the inductive hypothesis to concluce that $|C_i|$ [2] is a degree $\frac{r(r-1)}{2}$ polynomial in the variables $z_1, ..., z_{i-1}, z_{i+1}, ..., z_r$. Therefore, each summand in equation 3.2 has degree:

$$deg(p_r(z_i)|C_i|) = deg(p_r(z_i)) + deg(|C_i|) = r + \frac{r(r-1)}{2} = \frac{r(r+1)}{2}.$$

The $\frac{r(r+1)}{2}$ degree term of the i-th summand is the only degree $\frac{r(r+1)}{2}$ term in equation 3.2 that contains z_i^r, so the degree $\frac{r(r+1)}{2}$ terms cannot cancel. □

Bullock applied this fact in a very clever way to $\mathcal{S}(G;\mathbb{C})$.

Theorem 3.12. *(Bullock [6]) If there exists $\tau_g \in \mathbb{C}[X(G)]$ such that the image $\tau_g(X(G))$ is open (or whose image contains an open set), then*

$$\{[g],[g^2],[g^3],...\}$$

form an infinite linearly independent set in $\mathcal{S}(G;\mathbb{C})$, when considered as a $\mathbb{C}$-vector space.

Proof. The claim follows by showing that

$$V_r := span\{[g],[g^2],...,[g^r]\} \cong \mathbb{C}^r$$

for every r. The polynomial $|M_r|$ is non-constant by lemma 3.11, and so cannot be identically zero on the open set $\tau_g(X(G))^r \subset \mathbb{C}^r$. Therefore, there must be a point

$$(\tau_g(\chi_{\rho_1}), \tau_g(\chi_{\rho_2}), ...\tau_g(\chi_{\rho_r})) \in \tau_g(X(G))^r$$

on which $|M_r|$ is non-zero, so that the matrix

$$[p_i(\tau_g(\chi_{\rho_j}))] = [ev_{\rho_j} \circ \psi([g^i])], \qquad 1 \leq i,j \leq r$$

is invertible.

Define

$$\Phi : \mathcal{S}(G;R) \to \mathbb{C}^r$$

[2] C_i is $(r-1) \times (r-1)$

by

$$\Phi = (ev_{\rho_1} \circ \psi, ev_{\rho_2} \circ \psi, ..., ev_{\rho_r} \circ \psi).$$

Then we compute

$$\begin{aligned}
\Phi(V_r) &= span\{\Phi([g]), \Phi([g^2]), ..., \Phi([g^r])\} \\
&= span\{(ev_{\rho_1} \circ \psi([g]), ..., ev_{\rho_r} \circ \psi([g])), ... \\
&\quad ..., (ev_{\rho_1} \circ \psi([g^r]), ..., ev_{\rho_r} \circ \psi([g^r]))\} \\
&= span\{\text{rows of } [ev_{\rho_j} \circ \psi([g^i])]\}
\end{aligned}$$

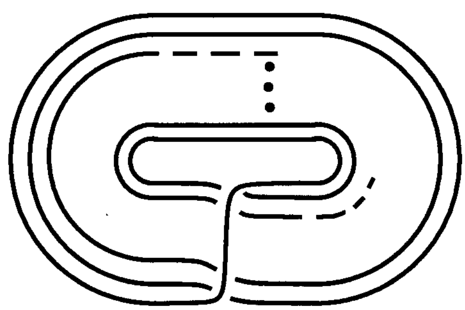
Therefore, by invertibility of the matrix $[ev_{\rho_j} \circ \psi([g^i])]$, the image of V_r under the map Φ is an r-dimensional subspace of $\mathbb{C}^r$—in other words, the map is surjective. Since V_r is *at most* r-dimensional, this forces

$$span\{[g], [g^2], ..., [g^r]\} \cong \mathbb{C}^r$$

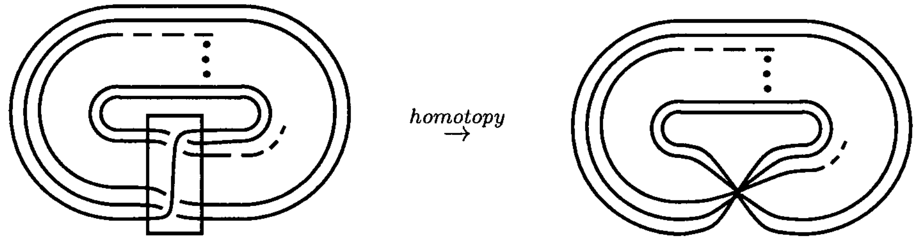
as claimed. □

This theorem admits a very nice topological interpretation, by using our isomorphism $\mathcal{S}(\pi_1(M);\mathbb{C}) \cong \mathcal{S}(M;\mathbb{C},-1)$ to translate the skein module elements $\{[g],[g^2],[g^3],...\}$ into knots.

Suppose that K is a knot in a 3-manifold M corresponding to an element $g \in \pi_1(M)$ that satisfies the hypotheses of theorem 3.12. Then corresponding to the elements $\{[g],[g^2],[g^3],...\}$ of the skein module $\mathcal{S}(\pi_1(M),\mathbb{C})$ are the knots $-[K_1], -[K_2], -[K_3], ... \in \mathcal{S}(M;\mathbb{C},-1)$, where K_i is an $(i,1)$-cabling of the original knot K. It should be noted that the negative signs arising from our isomorphism can be eschewed by using an alternate but isomorphic definition of $\mathcal{S}(G,\mathbb{C})$ (see [24]), but the correspondence used here introduces a negative sign: $[g] \iff -K_g$. The knots K_i look like:



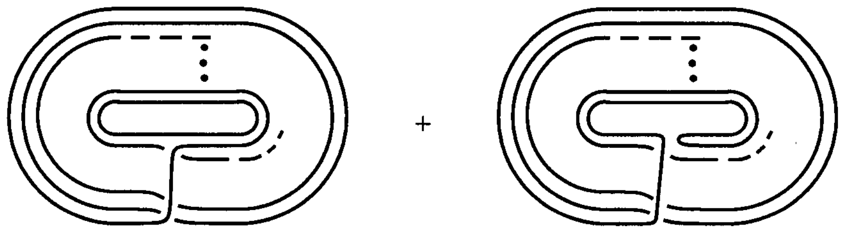
inside a tubular neighbourhood of K. Here, the vertical dots are meant to indicate i parallel strands. To see that this corresponds to the element g^i in the fundamental group, observe that we may homotope everything inside the indicated box to a single point:



which gives us the desired element of the fundamental group. The skein relation

$$[g^i] = [g^{i-1}] \otimes [g] - [g^{i-2}]$$

also has a nice topological interpretation. If we resolve the innermost crossing in $-K_i$ using the Kauffman bracket skein relation with $a = -1$, we get:



which, using our K_i notation, corresponds to $K_{i-1} \sqcup K_1 + K_{i-2}$, upon relaxing and homotoping some of the components into more agreeable positions. This agrees exactly with the right hand side of the skein relation

$$[g^i] = [g^{i-1}] \otimes [g] - [g^{i-2}],$$

under the image of our isomorphism, as one would hope.

In light of this theorem, we would like to know when there exists a map τ_g such that $\tau_g(X(G))$ contains an open set.

Theorem 3.13. *(Bullock [6]) If some Zariski component X_0 of $X(G)$ has dimension greater than zero, then there exists τ_g that is non-constant on X_0. Consequently, the image $\tau_g(X(G))$ contains an open set.*

Proof. Fix some system of coordinates $\mathbb{C}[\tau_{g_1}, ..., \tau_{g_n}]$, and choose a Zariski component X_0 of $X(G)$ that has dimension at least 1. Since X_0 is of dimension at least one, we can choose two distinct points χ_{ρ_1} and χ_{ρ_2} in X_0. Since the characters χ_{ρ_1} and χ_{ρ_2} are distinct, we can choose $g \in G$ on which they disagree. Then by this choice,

$$\tau_g(\chi_{\rho_1}) = \chi_{\rho_1}(g) \neq \chi_{\rho_2}(g) = \tau_g(\chi_{\rho_2})$$

so that τ_g is non-constant on X_0. In our chosen coordinates, τ_g is a polynomial map, and we have just shown that it is nonconstant on the variety X_0. Irreducible varieties (in general) admit a manifold structure on a dense open subset[3] [31], so that we can conclude the polynomial τ_g is nonconstant on some open neighbourhood U in X_0. In the chosen coordinates, τ_g is in fact a polynomial map, and so is holomorphic. Since non-constant holomorphic maps send open sets to open sets, τ_g sends the open neighbourhood U to an open set. □

Corollary 3.14. *Let M be a link complement, with boundary components included via the maps*

$$i_k : \partial M = T_k \hookrightarrow M.$$

Suppose the fundamental group of M admits an irreducible representation

$$\rho : \pi_1(M) \to SL(2,\mathbb{C})$$

such that

$$\rho(i_*(\pi_1(T_k))) \not\subseteq \{I, -I\}, \qquad \text{for each } k.$$

Then there is some knot K in M such that the cablings

$$\{K_1, K_2, K_3, \cdots\}$$

are linearly independent in $\mathcal{S}(M; R, -1)$.

[3] The nonsingular points in a variety admit neighbourhoods with manifold structure, and the nonsingular points form a dense open subset [31].

Proof. By applying corollary 3.6, we are able to conclude that $X(M)$ has a component X_0 of dimension at least 1. Hence, by theorem 3.13, there exists some $g \in \pi_1(M)$ such that the image $\tau_g(X(M))$ contains an open set. Theorem 3.12 then tells us that the infinite family of elements

$$\{[g],[g^2],[g^3],\ldots\}$$

is a linearly independent set in $\mathcal{S}(\pi_1(M),\mathbb{C})$. The isomorphism

$$\mathcal{S}(\pi_1(M);\mathbb{C}) \cong \mathcal{S}(M;\mathbb{C},-1)$$

gives us the desired conclusion. $\square$

We wonder: What knot and link complements admit representations as in corollary 3.14?

3.4 Hyperbolic Knots

We begin with a more approachable class of manifolds than a general knot complement, by considering hyperbolic knot complements. We recall some general facts about hyperbolic 3-manifolds [20]. If M is a complete hyperbolic 3-manifold, then there is a universal covering

$$\mathbb{H}^3 \xrightarrow{p} M.$$

From this, M can be realized as a quotient

$$M = \mathbb{H}^3 \big/ \Gamma$$

where Γ is the subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ that consists of all isometries γ satisfying $p \circ \gamma = p$. Since M arises as such a quotient, we know

$$\pi_1(M) \cong \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C})$$

so that there is a canonical inclusion $\pi_1(M) \hookrightarrow PSL(2,\mathbb{C})$.

Proposition 3.15. *(Thurston) Let M be a hyperbolic manifold. Then the canonical inclusion of $\pi_1(M)$ in $PSL(2,\mathbb{C})$ can be lifted to a representation in $SL(2,\mathbb{C})$.*

For a proof of this fact see [8]. We know immediately that the representation $\rho : \pi_1(M) \to SL(2,\mathbb{C})$ arising from this proposition is injective, as the canonical map $\pi_1(M) \hookrightarrow PSL(2,\mathbb{C})$ that we lifted is injective. If we further assume that M is a hyperbolic knot complement of finite volume, the representation ρ of $\pi_1(M)$ in $SL(2,\mathbb{C})$ provided by Thurston is necessarily irreducible [12], [8].

From this, we immediately get a fact found in much of the literature [9], [10], [2]:

Corollary 3.16. *If M is a hyperbolic 3-manifold of finite volume, and M is not closed, then some component of $X(M)$ has dimension at least 1,*[4] *and so M contains a knot whose $(1,i)$-cablings form an infinite linearly independent family in $\mathcal{S}(M;\mathbb{B},-1)$.*

Proof. We apply corollary 3.6, using the representation provided to us by proposition 3.15. □

Remark 3.17. *Alarmingly, this already appears to contradict the result of [6] if we choose M to be a small, hyperbolic knot complement of finite volume. In this case we have proven that $\mathcal{S}(M,\mathbb{C},-1)$ is infinite dimensional, whereas [6] asserts that*

$$dim\mathcal{S}(M,\mathbb{C},-1) < \infty$$

for all small 3-manifolds M. This is because the definition of "small" used in [6] is somewhat nonstandard, in that the incompressible surfaces inside the manifold M are not required to be closed. In this paper we take a small manifold to be a manifold that does not contain any closed, embedded, orientable surfaces that are both incompressible and nonboundary parallel.

Remark 3.18. *Through an entirely different approach, it is shown in proposition 2.4, [7], that in fact any knot complement M satisfies $dim(X(M)) \geq 1$. In fact, there can be no components of $X(M)$ having dimension zero! This much more powerful result tells us that $\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a)$ is infinitely generated as a module for any knot complement M.*

[4] This bound can be sharpened substantially to $X(M) = 1$ in the case that M is a small knot complement [7].

3.5 Comments

In light of these facts, we would liket to find elements g such that τ_g is non-constant on some 1-dimensional Zariski component of $X(G)$, for it is these elements g which will yield infinite linearly independent families

$$\{[g],[g^2],[g^3],\cdots\}$$

in $\mathcal{S}(G;\mathbb{C})$. In particular, if G is the fundamental group of some knot complement M, we know that such a linearly independent family must exist, by remark 3.18. In this case, such an element g will correspond to an infinite family of linearly independent knots

$$\{K_1,K_2,K_3,\cdots\}$$

in $\mathcal{S}_{2,\infty}(M;\mathbb{C},-1)$.

Chapter 4
Some Infinite Families of Linearly Independent Knots

4.1 Character Varieties of Hyperbolic Knot Complements

We focus on some results of Culler and Shalen, first published in [9]. Let M be a hyperbolic knot complement of finite volume, with torus boundary component

$$f : T \hookrightarrow M$$

and let

$$\rho_0 : \pi_1(M) \hookrightarrow SL(2,\mathbb{C})$$

be a lifting of the canonical embedding

$$\bar{\rho_0} : \pi_1(M) \hookrightarrow PSL(2,\mathbb{C}).$$

By comments in [15] and [9], the hypothesis that M has finite volume forces the induced map f_* between fundamental groups to be injective.

Let X_0 be the component of $X(M)$ containing the character χ_{ρ_0}, which we know (from the work of Thurston) has complex dimension 1. It is shown in [12] that the subgroup

$$\bar{\rho_0}(f_*(\pi_1(T))) \subset \bar{\rho_0}(\pi_1(M))$$

consists entirely of parabolic elements. This means that if $g \in \pi_1(M)$ lies in the image of the map f_*, then $\tau_g(\chi_{\rho_0}) = \chi_{\rho_0}(g) = \pm 2$. Therefore, if such maps

τ_g are to be constant on X_0, they must take on one of the two values ± 2 on the entire variety X_0.

Pick an arbitrary $g \in \pi_1(M)$ that lies in the image of f_*, and define a subvariety $Y \subset X_0$, that can be thought of as the union of the two level sets $\tau_g^{-1}(2)$ and $\tau_g^{-1}(-2)$. A more strict definition is as follows: for such an element g, define the algebraic set Y' to be the zero locus of the polynomial

$$\tau_g(\chi)^2 = 4.$$

Note that $\chi_{\rho_0} \in Y'$. The subvariety $Y \subset X_0$ that we are seeking is an irreducible component of $Y' \cap X_0$ that contains χ_{ρ_0}.

Theorem 4.1. *(Culler, Shalen, [9]). The variety Y has complex dimension 0, and so $Y = \{\chi_{\rho_0}\}$, since irreducible zero-dimensional varieties are singleton sets.*

From this theorem, we reason as follows: Given $g \in im(f_*)$, the corresponding map τ_g takes on the value $+2$ or -2 on the character $\chi_{\rho_0} \in X_0$. However, this is the *only* character in X_0 on which τ_g takes on the value ± 2, by our theorem. Therefore the map τ_g must be nonconstant on the one-dimensional component X_0. By the work of Bullock, this tells us that the infinite family

$$\{[g], [g^2], [g^3], \cdots\}$$

is linearly independent in $\mathcal{S}(G;\mathbb{C})$, and hence the corresponding knots are linearly independent in

$$\mathcal{S}(M;\mathbb{Z}[a,a^{-1}],a).$$

As before, the case of hyperbolic knot complements of finite volume is a simpler special case, whose results we can generalize.

4.2 Character Varieties of General Knot Complements

Let M be any knot complement, with boundary T included via the map f as before, and with f_* injective as before. The authors of [3], [2] have provided us with the following result. For any 1-dimensional component X_0 of $X(M)$, one of the following cases holds:

1. For every $g \in im(f_*)$, the map τ_g is constant on X_0.

2. For every $g \in im(f_*)$, the map τ_g is non-constant on X_0.

3. There is exactly one primitive[1] element $g \in im(f_*)$ such that

$$\tau_g, \tau_{g^2}, \tau_{g^3}, \cdots$$

are constant on X_0. All other maps τ_g are nonconstant.

Given our angle on this situation, we would like to know the circumstances under which cases (2) and (3) arise. We have the following partial answer.

Theorem 4.2. *(Boyer, Luft, Zhang, [2]) If M is small, then each one dimensional component X_0 of $X(M)$ satisfies either case (2) or case (3).*

From this, we follow a line of reasoning identical to before, and conclude that one of the following two cases hold:

1. Every $g \in im(f_*)$ gives rise to a linearly independent family of knots in

$$\mathcal{S}(M; \mathbb{Z}[a, a^{-1}], a).$$

2. There is exactly one primitive element $g \in im(f_*)$ whose powers

$$\{g, g^2, g^3, \cdots\}$$

may not give rise to linearly independent families of knots in

$$\mathcal{S}(M; \mathbb{Z}[a, a^{-1}], a).$$

All other elements in $im(f_*)$ give rise to infinite linearly independent families in

$$\mathcal{S}(M; \mathbb{Z}[a, a^{-1}], a).$$

[1]Here, we mean that the element g cannot be written as a power of any other element $h \in im(f_*)$

4.3 An Explicit Computation for 2-bridge knots

First, some general facts from the work of Riley in [27]. Define a *2-bridge group* of determinant $\alpha \geq 3$ ($\alpha \in \mathbb{Z}$ is odd) to be a group G with presentation

$$\langle x_1, x_2 \quad : \quad wx_1 = x_2 w \rangle,$$

where

$$w = x_1^{\epsilon_1} x_2^{\epsilon_2} x_1^{\epsilon_3} \cdots x_2^{\epsilon_{\alpha-1}}$$

and

$$\epsilon_j = \epsilon_{\alpha-j} = \pm 1 \qquad \text{for} \qquad j = 1 \cdots \alpha - 1.$$

We call these groups *2-bridge* groups, because this class of groups subsumes all 2-bridge knot groups [30]. Let C and D be the matrices:

$$C = \begin{bmatrix} t & 1 \\ 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} t & 0 \\ -tu & 1 \end{bmatrix}.$$

Given a 2-bridge group G, define $W \in GL(\mathbb{Z}[t, t^{-1}, u])$ by

$$W = C^{\epsilon_1} D^{\epsilon_2} \cdots D^{\epsilon_{\alpha-1}} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}.$$

Define a mapping

$$\rho_{t,u} : G \to GL(\mathbb{Z}[t, t^{-1}, u])$$

by $\rho_{t,u}(x_1) = C$, and $\rho_{t,u}(x_2) = D$, where subscript is to indicate that such a map depends on our choice of t and u, which we are to think of as complex numbers.

Lemma 4.3. *If the pair t, u satisfy the polynomial $w_{11} + (1-t)w_{12} = 0$, then $\rho_{t,u}$ defines a representation of G into $GL(\mathbb{Z}[t, t^{-1}, u])$.*

Proof. The assignment

$$\rho_{t,u}(x_1) = C, \qquad \rho_{t,u}(x_2) = D$$

defines a homomorphism precisely if $\rho_{t,u}$ preserves the single relation

$$wx_1 = x_2 w.$$

Under this assignment, the single relation becomes:

$$\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} t & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ -tu & 1 \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix},$$

or upon multiplying

$$\begin{bmatrix} tw_{11} & w_{11} + w_{12} \\ tw_{21} & w_{21} + w_{22} \end{bmatrix} = \begin{bmatrix} tw_{11} & tw_{12} \\ -tuw_{11} + w_{21} & -tuw_{12} + w_{22} \end{bmatrix}.$$

Equating entries and simplifying gives the four polynomial equations:

$$tw_{11} = tw_{11} \tag{4.1}$$

$$w_{11} + (1-t)w_{12} = 0 \tag{4.2}$$

$$(t-1)w_{21} + tuw_{11} = 0 \tag{4.3}$$

$$w_{21} + tuw_{12} = 0. \tag{4.4}$$

Equation 4.1 obviously does not concern us, as it is a simple identity. Equation 4.3 can be reduced to an identity if we employ equations 4.2 and 4.4:

$$\begin{aligned} 0 &= (t-1)w_{21} + tuw_{11} \\ &= (t-1)w_{21} + tu((t-1)w_{12}) \qquad \text{(by 4.2)} \\ &= (t-1)(-tuw_{12}) + tu((t-1)w_{12} \qquad \text{(by 4.4)} \\ &= 0. \end{aligned}$$

To complete the lemma we need only prove that 4.4 is an identity.

Define

$$V = \begin{bmatrix} \sqrt{-tu} & 0 \\ 0 & \frac{1}{\sqrt{-tu}} \end{bmatrix}.$$

Then a quick computation shows that

$$(C^{\epsilon_i})^T = VD^{\epsilon_i}V^{-1}, \qquad \text{and} \qquad (D^{\epsilon_i})T = VC^{\epsilon_i}V^{-1},$$

for $\epsilon_1 = \pm 1$. This allows the clever observation:

$$\begin{aligned}
W^T &= (C^{\epsilon_1} D^{\epsilon_2} \cdots D^{\epsilon_{\alpha-1}})^T \\
&= (D^{\epsilon_{\alpha-1}})^T (C^{\epsilon_{\alpha-2}})^T \cdots (C^{\epsilon_1})^T \\
&= (D^{\epsilon_1})^T (C^{\epsilon_2})^T \cdots (C^{\epsilon_{\alpha-1}})^T && \text{since the } \epsilon_i\text{'s are palindromic} \\
&= V C^{\epsilon_1} V^{-1} V D^{\epsilon_2} V^{-1} \cdots V D^{\epsilon_{\alpha-1}} V^{-1} && \text{by our choice of } V \\
&= V C^{\epsilon_1} D^{\epsilon_2} \cdots D^{\epsilon_{\alpha-1}} V^{-1} \\
&= V W V^{-1}.
\end{aligned}$$

In other words,

$$\begin{aligned}
\begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{bmatrix} &= \begin{bmatrix} \sqrt{-tu} & 0 \\ 0 & \frac{1}{\sqrt{-tu}} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{-tu}} & 0 \\ 0 & \sqrt{-tu} \end{bmatrix} \\
&= \begin{bmatrix} w_{11} & -tuw_{21} \\ \frac{w_{21}}{-tu} & w_{22} \end{bmatrix}
\end{aligned}$$

from which we can read off $w_{21} = -tuw_{12}$. □

In light of the importance of this polynomial, we define

$$\Phi(t,u) = w_{11} + (1-t)w_{12}.$$

The properties of this polynomial are the subject of [27]. As is standard, let $\Lambda = \mathbb{Z}[t, t^{-1}]$. Riley shows that $\Phi(t,u)$ always admits a factorization of the form

$$\Phi(t,u) = t^k \Phi_1(t,u)\Phi_2(t,u)\cdots\Phi_r(t,u)$$

where $k \in \mathbb{Z}$, and each $\Phi_j \in \Lambda[u]$ is irreducible and distinct, and has a leading monic term of the form u^n, $n \geq 1$. Additionally it is shown that though each factor lies in $\Lambda[u]$, no factor can lie in $\mathbb{Z}[u]$. This provides a decomposition of the zero locus of Φ into irreducible affine algebraic curves, and each point on such a curve corresponds to a representation $\rho_{t,u}$ of G.

Also of great importance is a remark in [27], proven by deRham in [25], that $\Phi(t,0)$ is a Λ-unit multiple of the Alexander polynomial $\Delta(t)$ of the group G. In the case that G is a 2-bridge knot group, there is a nice formula for the Alexander polynomial [13]:

$$\Delta(t) = 1 - t^{\epsilon_1} + t^{\epsilon_1+\epsilon_2} - t^{\epsilon_1+\epsilon_2+\epsilon_3} + \cdots - t^{\sum_{i=1}^{\alpha-1}\epsilon_i}.$$

We will make use of this information shortly.

There is also a partial "converse" to this correspondence between representations and pairs t, u, appearing as lemmas 7 and 8 in [28], and referenced in [27]. Suppose we are given a representation with nonabelian image,

$$\phi : G \to GL(2, \mathbb{C}),$$

defined by $\phi(x_1) = M_1$ and $\phi(x_2) = M_2$. Then there exists a matrix

$$U \in SL(2, \mathbb{C})$$

and some numbers t, u satisfying $w_{11} + (1-t)w_{12} = 0$ such that

$$UM_1U^{-1} = \sqrt{t^{-1}det(M_1)} \begin{bmatrix} t & 1 \\ 0 & 1 \end{bmatrix}, \quad UM_2U^{-1} = \sqrt{t^{-1}det(M_2)} \begin{bmatrix} t & 0 \\ -tu & 1 \end{bmatrix}.$$

Furthermore, the pair t, u is unique if M_1 and M_2 have a common eigenvector, otherwise the pair can only be replaced with t^{-1}, u. What this means is that each representation $\rho_{t,u}$ corresponds to at most two points in the zero locus $\Phi(t, u) = 0$.

We now wish to consider these results as they apply to representations into $SL(2, \mathbb{C})$. Note that if we instead had defined

$$\rho_{t,u}(x_1) = t^{-\frac{1}{2}}C = \begin{bmatrix} t^{\frac{1}{2}} & t^{-\frac{1}{2}} \\ 0 & t^{-\frac{1}{2}} \end{bmatrix}, \quad \rho_{t,u} = t^{-\frac{1}{2}}D = \begin{bmatrix} t^{\frac{1}{2}} & 0 \\ -t^{\frac{1}{2}}u & t^{-\frac{1}{2}} \end{bmatrix},$$

then the defining relation

$$wx_1 = x_2w$$

gets mapped to

$$(t^{-\frac{1}{2}})^{\sum \epsilon_i + 1}WC = (t^{-\frac{1}{2}})^{\sum \epsilon_i + 1}DW.$$

Since we can cancel the powers of t on both sides, this alternative assignment still defines a representation of G for each pair t, u in the zero locus $\Phi(t, u) = 0$, but whose image now lies in $SL(2, \mathbb{C})$. Therefore, though our original assignment of

$$\rho_{t,u}(x_1) = C, \qquad \rho_{t,u}(x_2) = D$$

is beneficial for illustrating the correspondence between points satisfying

$$\Phi(t, u) = 0$$

and representations of G, we shall henceforth take $\rho_{t,u}$ to be the new assignment

$$\rho_{t,u}(x_1) = t^{-\frac{1}{2}}C, \qquad \rho_{t,u}(x_2) = t^{-\frac{1}{2}}D.$$

We now consider the characters arising from this new assignment. To simplify notation, let

$$\chi_{t,u} = \chi_{\rho_{t,u}}.$$

For an arbitrary $g \in G$, we would like to know about the values $\tau_g(\chi_{t,u})$, which is now expressible as a polynomial in t and u. In general, $\tau_g(\chi_{t,u})$ is an unwieldy mess. However, we have the following lemma when we take $u = 0$:

Proposition 4.4. *Let g be an arbitrary element of the two-bridge group G, whose generators are x_1 and x_2. Then if*

$$g = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \cdots x_{j_s}^{\epsilon_s},$$

where $\epsilon_i = \pm 1$, we have that

$$\tau_g(\chi_{t,0}) = t^{\frac{\sigma(g)}{2}} + t^{-\frac{\sigma(g)}{2}}.$$

Here,

$$\sigma(g) = \sum_{i=1}^{s} \epsilon_i$$

is the exponent sum of g.

Proof. We set $u = 0$ in the matrices C, D, C^{-1}, D^{-1}, and get:

$$C_0 = \begin{bmatrix} t & 1 \\ 0 & 1 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_0^{-1} = \begin{bmatrix} t^{-1} & t^{-1} \\ 0 & 1 \end{bmatrix},$$

and

$$D_0^{-1} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

By induction on the length of the product, we will prove that an arbitrary product of these four matrices necessarily has the form:

$$\begin{bmatrix} t^{\sigma} & p(t) \\ 0 & 1 \end{bmatrix}.$$

where σ is the exponent sum of the matrix product, and $p(t)$ is some polynomial in t. Observe that for a product of length one, the claim holds, as we can see from inspection of C, D, C^{-1}, D^{-1}.

Assume that the claim holds for some matrix A that is a product of length s of C's and D's with exponent sum k, so that

$$A = \begin{bmatrix} t^{k} & p(t) \\ 0 & 1 \end{bmatrix}.$$

Then upon multiplying A on the left by C_0, C_0^{-1}, D_0, and D_0^{-1}, we get:

$$AC_0 = \begin{bmatrix} t^{k+1} & t^{k}p(t) \\ 0 & 1 \end{bmatrix}$$

$$AC_0^{-1} = \begin{bmatrix} t^{k-1} & t^{k-1}p(t) \\ 0 & 1 \end{bmatrix}$$

$$AD_0 = \begin{bmatrix} t^{k+1} & p(t) \\ 0 & 1 \end{bmatrix}$$

$$AC_0 = \begin{bmatrix} t^{k-1} & p(t) \\ 0 & 1 \end{bmatrix},$$

so that the claim holds true for an arbitrary product of length $s+1$. Therefore, with $g \in G$ arbitrary, we can compute

$$\tau_g(\chi_{t,0}) = tr(\rho_{t,u}(g)) = t^{-\frac{\sigma(g)}{2}} tr\left(\begin{bmatrix} t^{\sigma(g)} & p(t) \\ 0 & 1 \end{bmatrix}\right) = t^{\frac{\sigma(g)}{2}} + t^{-\frac{\sigma(g)}{2}}.$$

□

Suppose that we find two different points $(r_1, 0)$ and $(r_2, 0)$ in the zero locus of $\Phi(t, u)$, both of which lie in the same irreducible component of the curve

$$\Phi(t, u) = 0.$$

Since $\Phi(t,0) = \Delta(t)$ is the Alexander polynomial, finding r_1, r_2 amounts to finding roots of the Alexander polynomial, and somehow making an argument that the resulting points lie in the same irreducible component. Then if we find an element $g \in G$ such that

$$r_1^{\frac{\sigma(g)}{2}} + r_1^{-\frac{\sigma(g)}{2}} \neq r_2^{\frac{\sigma(g)}{2}} + r_1^{-\frac{\sigma(g)}{2}},$$

we will have

$$\tau_g(\chi_{r_1,0}) \neq \tau_g(\chi_{t,u}),$$

so that τ_g is a non-constant polynomial on the irreducible component of $\Phi(t,u) = 0$ that contains the points $(r_1, 0)$ and $(r_2, 0)$. This is sufficient for us to apply Doug Bullock's result, and conclude that the elements

$$[g], [g^2], [g^3], \cdots$$

are a linearly independent family in $\mathcal{S}(G, \mathbb{C})$.

4.3.1 Computational Applications to an Infinite Family of 2-Bridge Groups

First, we observe that

$$\begin{aligned} r_1^n + r_1^{-n} = r_2^n + r_2^{-n} \quad &\iff \quad \frac{r_1^{2n}+1}{r_1^n} - \frac{r_2^{2n}+1}{r_2^n} = 0 \\ &\iff \quad r_1^{2n} r_2^n + r_2^n - r_2^{2n} r_1^n - r_1^n = 0 \\ &\iff \quad (r_1^n r_2^n - 1)(r_1^n - r_2^n) = 0. \end{aligned}$$

If we restrict ourselves to real values, the last equation can only be satisfied if

$$r_1 = \frac{1}{r_2} \quad \text{or} \quad r_1 = r_2.$$

This is not to imply that considering complex roots of the Alexander polynomial cannot be fruitful. It is simply more to the point to restrict our attention to real roots for the exposition of these ideas.

For $p \equiv 0 \bmod 3$ an odd positive integer, we consider the family of 2-bridge knots $K_{\frac{p}{2p-1}}$. (For an explanation of this indexing, see [29].) From [13], the corresponding knot group has presentation

$$\langle x_1, x_2 : x_1 w = w x_2 \rangle,$$

where

$$w = x_1^{\epsilon_1} x_2^{\epsilon_2} x_1^{\epsilon_3} \cdots x_2^{\epsilon_{2p-2}}$$

and

$$\epsilon_j = \left\lfloor \frac{jp}{2p-1} \right\rfloor \qquad \text{for} \qquad j = 1 \cdots 2p-2.$$

The ϵ_j's obey they pattern:

$$\begin{aligned}
\epsilon_1 &= 1 \\
\epsilon_2 &= -1 \\
\epsilon_3 &= -1 \\
\epsilon_4 &= 1 \\
\epsilon_5 &= 1 \\
&\vdots \\
\epsilon_{2p-2} &= 1
\end{aligned}$$

here, the vertical dots indicate alternating pairs of -1 and $+1$. This is clearly a 2-bridge group, so we can compute the Alexander polynomial as:

$$\Delta(t) = 1 - t^1 + t^{1-1} - t^{1-1-1} + t^{1-1-1+1} - \cdots + t^{\sum \epsilon_i}$$

which gives

$$\Delta(t) = 1 - t + 1 - t^{-1} + 1 - t + 1 - t^{-1} + 1 - \cdots + 1$$

where there are $2p-1$ terms in the sum. Grouping together like terms, we get the simple formula

$$\Delta(t) = \left(\frac{p-1}{2}\right) t + pt + \left(\frac{p-1}{2}\right) t^{-1}.$$

To find the roots of this polynomial, we first multiply the Alexander polynomial by a factor of t and then use the quadratic formula on the resulting polynomial. This yields the roots:

$$r_1, r_2 = \frac{-p \pm \sqrt{2p-1}}{2p}.$$

Both roots of the Alexander polynomial are real, and distinct. The two roots are not inverses of one another, because

$$\frac{-p+\sqrt{2p-1}}{2p} = \frac{2p}{-p-\sqrt{2p-1}}$$

reduces to

$$-3p^2 - 2p - 1 = 0,$$

and considered modulo 3 this equation gives $2 \equiv 0$, since $p \equiv 0 \bmod 3$. Therefore the roots r_1 and r_2 satisfy

$$r_1^n + r_1^{-n} \neq r_2^n + r_2^{-n}$$

for all nonzero n. Thus we already have the following fact: If G is a knot group corresponding to one of the knots $K_{\frac{p}{2p-1}}$, and $g \in G$ has nonzero exponent sum, then

$$\tau_g(\chi_{r_1,0}) \neq \tau_g(\chi_{r_2,0}).$$

It remains to show that the points $(r_1, 0)$ and $(r_2, 0)$ lie ine the same component of the variety defined by $\Phi(t, u) = 0$.

Recalling that

$$\Phi(t,u) = t^k \Phi_1(t,u)\Phi_2(t,u)\cdots\Phi_r(t,u)$$

where $k \in \mathbb{Z}$, and each $\Phi_j \in \Lambda[u]$ is irreducible and distinct, we see that this must provide a factorization of $\Delta(t)$ over Λ of the form

$$\Delta(t) = \Phi(t,0) = t^k\Phi_1(t,0)\Phi_2(t,0)\cdots\Phi_r(t,0) = t^k c_1(t)\cdots c_r(t).$$

Here, $c_i(t)$ is the constant term of Φ_i, which lies in Λ. We consider the possible factorizations of

$$\Delta(t) = \left(\frac{p-1}{2}\right)t + pt + \left(\frac{p-1}{2}\right)t^{-1}$$

over Λ. Any such factorization can only have two nonzero roots, corresponding to the roots we have already computed. This means in our factorization $t^k c_1(t)\cdots c_r(t)$, one of two cases may occur:

1. Two of the c_i's are linear factors, say c_n and c_m, while all others must be units in Λ. Then the points $(r_1, 0)$ and $(r_2, 0)$ lie in zero loci of Φ_n and Φ_m, and so are in separate components of $\Phi(t, u) = 0$.

2. Only one of the c_i's is not a unit, and therefore has the same roots as the Alexander polynomial. Then the points $(r_1, 0)$ and $(r_2, 0)$ lie in the same irreducible component of $\Phi(t, u) = 0$.

For case (1) to occur, the polynomial

$$\left(\frac{p-1}{2}\right) t + pt + \left(\frac{p-1}{2}\right) t^{-1}$$

must factor over the integers into two linear factors. This is not possible, because this polynomial has descriminant $2p - 1 \equiv 2 \bmod 3$, and 2 is not a quadratic residue modulo 3. Therefore $(r_1, 0)$ and $(r_2, 0)$ lie in the same irreducible component of $\Phi(t, u) = 0$. This leads us to the following conclusion:

Proposition 4.5. *Suppose that G is the knot group of one of the knots $K_{\frac{p}{2p-1}}$, where $p \equiv 0 \, mod 3$ is odd, and that $g \in G$ has nonzero exponent sum. Then the elements*

$$[g], [g^2], [g^3], \cdots$$

are linearly independent in the skein module $\mathcal{S}(G, \mathbb{C})$.

We can use this to make a modest gain in our understanding of the nilradical of skein modules.

Proposition 4.6. *Suppose that G is the knot group of one of the knots $K_{\frac{p}{2p-1}}$,where $p \equiv 0 \, mod 3$ is odd, and that $g \in G$ has nonzero exponent sum. Then $[g]$ is not contained in the nilradical of $\mathcal{S}(G, \mathbb{C})$.*

Proof. This is an immediate consequence of the 'reduction formula'

$$[g] \otimes [h] = [gh] + [gh^{-1}].$$

Supposing that

$$[g] \otimes [g] \otimes \cdots \otimes [g] = 0,$$

we may apply our reduction formula to the left hand side repeatedly until we obtain a sum of elements of the form $[g^k]$. The equation we obtain in this way contradicts the linear independence of the elements

$$[g], [g^2], [g^3], \cdots .$$

□

4.4 Questions for Future Research

Computationally, skein modules are very difficult to tackle, any new computational techniques would be more than welcome in the field. The approach we have seen for finding infinite linearly independent families could be extended to yield new information by finding additional elements $g \in \pi_1(M)$ that yield functions τ_g which are non-constant on positive dimensional components of $X(M)$. This is certainly an appealing avenue for future research.

Based upon the explicit calculation for the knots $K_{\frac{p}{2p-1}}$ above, we would like to come up with better ways of finding pairs of points $(r_1, 0)$, $(r_2, 0)$ that lie in the same irreducible component of the curve $\Phi(t, u) = 0$, and use these pairs to create infinite linearly independent families. Alternatively, it seems that this approach of using the roots of the Alexander polynomial is inherently weak in some sense, since it only applies to elements of the group that have nonzero exponent sum. Perhaps there is some alternative approach based upon the calculations of Riley, which will provide more trenchant insights into the question

When is the map $\rho : \mathcal{S}(G; \mathbb{C}) \to \mathbb{C}[X(G)]$ injective?

specifically in the case that G is a 2-bridge group. This question was first posed by Przytycki in [23], and the answer is only known in a small number of cases.

Recent interest in the A-polynomial has also lead to questions as follows: Suppose that a positive-dimensional component $X_0 \in X(M)$ is defined by some set of polynomials

$$\{p_1, p_2, p_3, \cdots , p_n\},$$

and suppose that we find an element $g \in \pi_1(M)$ that is non-constant on the component X_0. It seems reasonable to expect that the linearly independent family of knots arising from this element would bear some connection to the defining polynomials $\{p_1, p_2, p_3, \cdots, p_n\}$. What would this relationship be?

This is of particular interest for the following reason. Suppose we have a knot K with complement M. In [7], the authors define the A-polynomial of a knot K as the defining polynomial of a certain one-dimensional algebraic subset of $X(M)$. From the work of Doug Bullock, we know that there must exist at least one function τ_g that is non-constant on this one-dimensional component. The obvious question is: Which elements $g \in \pi_1(M)$ give rise to functions that are non constant on the variety defined by the A-polynomial . Having found these elements, what relationship would they bear to the A-polynomial? An answer to this question could provide a connection between the skein module of a knot complement and the A-polynomial of the corresponding knot.

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