O-MINIMAL EXPANSIONS OF THE REALS

by

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Abstract

We survey recent results on o-minimal theories, and in particular o-minimal expansions of real closed fields. The recent work in the classification of reducts of the field of real numbers, largely the work of Peterzil, is presented, as is the basic groundwork of o-minimality. It is shown that if $X \subseteq \mathbb{R}^n$ is semialgebraic, but not semilinear, then multiplication on $\mathbb{R}$ may be defined locally in terms of X, modulo the vector space properties of the reals. If $X \setminus K$ is not semilinear for any compact K, then the condition of locality can be removed.

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Introduction

The aim of this thesis is to explore the subject of o-minimal structures, and in particular survey the use of o-minimal structure theory to classify reducts of the field of real numbers. The thesis itself is written with the assumption that those reading it will have a basic understanding of model theory, but we will lay out here the requisite fundamentals informally.

Initially we should attempt to define model theory. Wilfrid Hodges[1] describes model theory as 'algebraic geometry minus fields.' This is a rather good description both because it gives a fairly good idea of what the subject is, and because it makes model theory seem very cutting-edge. Indeed, model theory was recently used to solve a significant problem in algebraic geometry [Bou98]. One wanting a description more thorough (and less glib) might like this: Model theory is the subject in mathematics interested in determining what mathematical structures exist, and what mathematical structures are worth studying, primarily by examining the subsets definable in structures by first-order sentences. In the classical sense, a 'structure' is a non-empty set with certain distinguished relations, functions, and elements. For example, a group, $(G, \cdot, e)$ is a structure. A definable set in a structure is the set of tuples from the structure satisfying some first order formula, with possible additional constants from the structure. If any additional constants used come from the set A, the set is said to be A-definable. The center of G is a definable subset of G, namely the set of all x satisfying

$$\forall y(xy = yx).$$

In fact, it is $\emptyset$-definable. For those unfamiliar with the concept of a first-order formula, it suffices to consider any coherent string of symbols either from the 'language' of the structure (the distinguished relations, functions, and constants) or from the language of logic: $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\neg$ (not), $\forall$ (for all), $\exists$ (there exists), and variables. The variables which are not bound by quantifiers are 'free variables' and we can view a first order formula as a condition on elements (or tuples of elements) in the structure. If φ is such a formula, and $\mathscr{M}$ is a structure, $\varphi(\mathscr{M})$ is exactly that set of elements (or tuples)

[1] [Hod93]

from the structure of which φ is true. We use a great deal the fact that first order formulas are of finite length.

We use $\equiv$ to denote equivalence of first order formulas to avoid confusion with the uses of $=$ in the formula. This symbol also has another important usage, however: we say that two structures $\mathscr{M}$ and $\mathscr{N}$ are elementarily equivalent, written $\mathscr{M} \equiv \mathscr{N}$, if they have the same language, and the same first order sentences (formulas without free variables) are true in $\mathscr{M}$ and $\mathscr{N}$. An embedding of $\mathscr{M}$ into $\mathscr{N}$ is an injective function from $\mathscr{M}$ to $\mathscr{N}$ which sends the various constants of $\mathscr{M}$ to the appropriate constants in $\mathscr{N}$, commutes with all of the distinguished functions, and preserves the distinguished relations (i.e., a tuple $\bar{a} \in \mathscr{M}^n$ is related by R if and only if $(f(a_0), ..., f(a_n)) \in \mathscr{N}^n$ is related by R). An elementary embedding is an injective function which preserves the relations defined by *any* first order formulas. We say that $\mathscr{M}$ is a substructure of $\mathscr{N}$ (resp. elementary substructure), $\mathscr{M} \leqslant \mathscr{N}$ (resp. $\mathscr{M} \preccurlyeq \mathscr{N}$), if it is a subset and the identity map is an embedding (resp. elementary embedding). If $\mathscr{M}$ is a substructure of $\mathscr{N}$, $\mathscr{N}$ is an extension of $\mathscr{M}$. An isomorphism is a surjective embedding. A first order theory T is a set of first order sentences (formulas without free variables), and a model of a theory is a structure $\mathscr{M}$ which makes each of these sentences true, denoted $\mathscr{M} \models T$.

It is common, however, in modern model theory to view a structure somewhat differently. We typically consider two structures on the same underlying set to be 'the same' if the same sets are $\emptyset$-definable (or even, for some purposes, simply definable) in both. Thus $(\mathbb{Q}, +)$ and $(\mathbb{Q}, -)$ are more or less the same structure. A strong reduct of a structure $\mathscr{M}$ is another structure $\mathscr{N}$ on the same set with every $\emptyset$-definable set in $\mathscr{N}$ being $\emptyset$-definable in $\mathscr{M}$. A structure $\mathscr{N}$ is simply a reduct of $\mathscr{M}$, $\mathscr{N} \ll \mathscr{M}$, if every definable set in $\mathscr{N}$ is definable in $\mathscr{M}$. If $\mathscr{N}$ is a reduct of $\mathscr{M}$, $\mathscr{M}$ is an expansion of $\mathscr{N}$. As a more general sense of reduct, the structure $\mathscr{N}$ is said to be 'definable' in $\mathscr{M}$ if $\mathscr{N} \subseteq \mathscr{M}^n$ is definable, and each definable subset of $\mathscr{N}^m$ is a definable subset of $\mathscr{M}^{mn}$. For example, the group of invertible 2×2 matrices over some field F (viewed as a subset of F^4) is definable in F.

We also make use of a sort of infinite conjunction: an n-type over $A \subseteq \mathscr{M}$ is an ultrafilter of A-definable subsets of $\mathscr{M}^n$, the set of all such being denoted $S_n(A)$. The realizations of p (the intersection of all of the definable sets) is $p(\mathscr{M})$. In a given structure, and for a given type, this may or may not be empty, but the compactness theorem ensures that every type is realized in some elementary extension. A type p is atomic if it is generated (as an ultrafilter) by a single formula, or definable set. Atomic types are always realized. The type of a tuple $\bar{a}$ over a set A is denoted $\mathrm{tp}(\bar{a}/A)$.

A great deal of work has been completed in an attempt to separate those structures which are easy study in some sense or other, and those that are not. One particular class of structure which has been studied is that of minimal structures: a structure $\mathscr{M}$ is minimal if for any definable $X \subseteq \mathscr{M}$, either X or $\mathscr{M} \setminus X$ is finite. Note that in any structure $\mathscr{M}$ the set $\{a_0, ..., a_n\}$ is defined by the formula

$$x = a_0 \vee ... \vee x = a_n,$$

and $\mathscr{M} \setminus \{a_0, ..., a_n\}$ is defined by the negation of this formula, so a minimal structure is a structure in which only those subsets of $\mathscr{M}$ which must be definable are definable. However, this does not mean that there are no interesting definable subsets of $\mathscr{M}^n$ for $n > 1$. It is not true that any structure elementarily equivalent to a minimal structure is again minimal, but structures with that property form an even more interesting class, called strongly minimal structures.

The archetypical example of a strongly minimal structure is an algebraically closed field. It is easy to show (using, say, Theorem 1.2.2) that if T_p is the set of axioms for an algebraically closed field of characteristic p, then T_p is complete (that is, any two models of T_p are elementarily equivalent) and has elimination of quantifiers (any definable set is definable using a formula without $\forall$ or $\exists$). In particular, the definable subsets of $k \models T_p$ are simply boolean combinations of zero sets of polynomials, and so are finite or cofinite. And it is rather easy to study models of T_p as there is exactly one, up to isomorphism, for each cardinal number κ, namely the algebraically closed field with transcendence degree κ over the prime field of characteristic p.

More generally, interest has recently been focussed on structures which show strong geometric properties and independence relations. Given a set $A \subseteq \mathscr{M}$ (definable or otherwise) we define the algebraic closure of A to be the union of all finite A-definable sets. In algebraically closed fields this corresponds to algebraic closure in the usual sense[2]; in vector spaces, span. Several

[2]Precisely: the 'algebraic closure' of a set is the algebraic closure of the field generated

of the familiar properties from these settings transfer to more general settings:

- if $A \subseteq \operatorname{acl}(B)$ then $\operatorname{acl}(A) \subseteq \operatorname{acl}(B)$
- for all A, $A \subseteq \operatorname{acl}(A)$
- if $x \in \operatorname{acl}(A)$ then $x \in \operatorname{acl}(A_0)$ for some finite[3] $A_0 \subseteq A$

In certain structures, however, we have another useful property. We say that a is independent from b over A, denoted $a \downarrow_A b$, if $a \notin \operatorname{acl}(\{b\} \cup A) \setminus \operatorname{acl}(A)$. If this relation is symmetric (for any A) in a given structure, we say that this structure satisfies the exchange law. In structures with this property we can define a reasonable sense of algebraic dimension. The dimension of a tuple $\bar{a} \in \mathscr{M}^n$ over A, $\dim(\bar{a}/A)$, is the number of elements in any algebraically independent (over A) $\bar{a}' \subseteq \bar{a}$ such that $\operatorname{acl}(\bar{a} \cup A) = \operatorname{acl}(\bar{a}' \cup A)$. If $X \subseteq \mathscr{M}^n$ is A-definable, then $\dim(X) = \max\{\dim(\bar{a}/A) : \bar{a} \in X\}$ (which turns out to be independent of A).

Another related tool used to study structures is Morley dimension or rank. If $\mathscr{M}$ is a structure we may define a relation R between nonempty definable subsets of $\mathscr{M}$ and ordinals by transfinite induction, using

- $R(X, 0)$ for all definable X
- $R(X, \lambda)$ if and only if $R(X, \alpha)$ for all $\alpha < \lambda$, when λ is a limit ordinal
- $R(X, \alpha + 1)$ if and only if there is a countable family $X_0, X_1, \ldots$ of definable disjoint subsets of X such that $R(X_n, \alpha)$ for all n.

It is a simple observation that if $R(X, \alpha)$ fails for some ordinal then there is a greatest ordinal for which it is true, and this ordinal is the Morley rank of X, $\operatorname{Mrank}(X)$. If $R(X, \alpha)$ is true for all α then we put $\operatorname{Mrank}(X) = \infty$. Thus $R(X, \alpha)$ is equivalent to $\alpha \leqslant \operatorname{Mrank}(X)$, and $\operatorname{Mrank}(X) = 0$ if and only if X is finite. Structures of finite Morley rank have been studied at great length. Note that if $\mathscr{M}$ is strongly minimal (and infinite), then $\operatorname{Mrank}(\mathscr{M}) = 1$. We know that $R(\mathscr{M}, 1)$ because $\mathscr{M}$ is infinite, but if we have infinitely many disjoint definable subsets of $\mathscr{M}$, at most one may be infinite (as that one will be cofinite)

by that set.

[3]This is because x is in some finite set defined using constants from A, and the defining formula is of finite length.

and so it is false that $R(\mathscr{M}, 2)$. It is currently conjectured (Cherlin's conjecture) that every simple group of finite rank is an algebraic group. As it is also true that any structure definable in a structure of finite Morely rank is again of finite Morely rank, the truth of this conjecture would imply that any simple group definable in a strongly minimal structure is an algebraic group. Many other conjectures have been put forward regarding strongly minimal structures. In particular, it was conjectured that the only non-degenerate examples of strongly minimal structures arose from vector spaces and algebraically closed fields. This turned out to be true given additional assumptions.

In this thesis we will study o-minimal structures. An ordered structure is, for us, one in which one may define a dense linear ordering without endpoints. Such a structure cannot be strongly minimal as one may define open intervals, which are infinite but not cofinite. But here again we introduce a subfamily of 'minimal' structures, where only the bare minimum of sets may be defined. In the case of ordered structures, at minimum we may define boolean combinations of singletons and open intervals, and so a structure in which *only* sets of this form may be defined is called order-minimal, or o-minimal. Again, this does not prohibit interesting behaviour in higher dimensions. Although the study of o-minimal structures is much younger than the study of strongly minimal structures, progress has been rapid. A version of the conjecture above, called the trichotomy theorem, has been proven. Any o-minimal structure is, locally, either trivial, a vector space, or an expansion of a real-closed field. It is also known that the only simple groups definable in o-minimal structures are algebraic groups, either over real closed fields or algebraically closed fields. Vaught's conjecture and several other well known model theoretic claims have been demonstrated in the o-minimal case. O-minimal structures provide a natural context for studying structures over $\mathbb{R}$, and so are a natural class to consider. By a 'structure over $\mathbb{R}$', here, we mean an expansions of $(\mathbb{R}, <)$. Indeed, as we shall see, many familiar structures over $\mathbb{R}$ are o-minimal. As we are allowing (in fact insisting on) consideration of the order on $\mathbb{R}$, we may also study some topological properties of structures over $\mathbb{R}$. Recently some work has been done on the fundamental groups of sets definable in o-minimal structures over $\mathbb{R}$.

Another way of viewing the study of o-minimal structures is as a generalization of semialgebraic geometry (geometry in the field[4] $(\mathbb{R},+,\cdot)$). Any subset of $\mathbb{R}^n$ constructible in this field turns out to have a very simple form. A cell is defined recursively to be a singleton or open interval, or the graph of a continuous rational function on a cell, or the region above or below the graph of some continuous rational function on a cell, or the region between two continuous, nonintersecting rational functions on a cell. Every semialgebraic set then turns out to be a finite union of cells. A similar statement is true in the study of subanalytic geometry. If one makes the same definition of a cell in an o-minimal structure, replacing 'rational' with 'definable', the analogous theorem can be derived (Theorem 1.1.2). Sets and functions definable in o-minimal structures over $\mathbb{R}$ can then be shown to have various nice topological and analytic properties.

Recent work has used strong minimality to classify, to some extent, the reducts of $(\mathbb{C},+,\cdot)$. It was shown by Marker and Pillay in [MP90] that for algebraic sets $X \subseteq \mathbb{C}^n$, $(\mathbb{C},+,X)$ is either locally modular[5] or defines all algebraic sets[6]. Of course, it suffices to show that one can recover multiplication from X. One of the more concrete reasons to study o-minimal structures is that it allows us to make great strides towards classifying the reducts of the field of real numbers. This may seem, as we are working within $(\mathbb{R},+,\cdot)$, to be a question of semialgebraic geometry, but a significant amount of model theory is required as well. It was conjectured by van den Dries that there was no nontrivial reduct of the field of reals which properly expanded the vector space of real numbers. The motivation for this is obvious: given many simple rational functions it is possible to recover multiplication using only vector space algebra. For example, if $f(x) = x^2$, $xy = \frac{1}{2}(f(x+y) - f(x) - f(y))$; if $g(x) = 1/x$, $x^2 = g(g(x) - g(x+1)) - x$, and so on. The conjecture, however, turns out not to be true. The structure $\mathscr{B} = (\mathbb{R},+,*,<)$, where $*$ is the restriction of multiplication to $[0,1]^2$, is one such proper reduct, but this is the only counterexample. We can then show, using similar methods, that the

[4]Although not explicitly an ordered structure, the field of real numbers defines the usual ordering on the reals, by $x < y \leftrightarrow \exists z(z^2 = y - x)$.

[5]See [Bou98]. A structure is modular if for any sets A and B, $\dim(A) + \dim(B) = \dim(A \cap B) + \dim(A \cup B)$. A structure is locally modular if it is modular in terms of the closure operator $\mathrm{cl}_Y(A) = \mathrm{acl}(A \cup Y)$ for some large set Y.

[6]I.e., is the same structure as $(\mathbb{C},+,\cdot)$.

only proper reduct of $(\mathbb{R}, +, \cdot)$ which properly expands $(\mathbb{R}, \cdot, <)$ is $(\mathbb{R}, \oplus, \cdot, <)$, where $\oplus$ is the restriction of $+$ to $[1,2]^2$. These facts can be expressed in terms of growth rates of functions, as can a similar fact about exponentiation.

The limits of space and time restrict our ability to prove certain results. For the most part, any result which is not model theoretic is deferred. Well-known results of model theory which do not require knowledge of o-minimality are also assumed without proof. We will also omit the proof that $(\mathbb{R}, +, \cdot, x \mapsto e^x)$ is o-minimal, a result shown by Wilkie. The main substance of this rather long proof is the proof of model completeness. This tells us that every definable set is the projection of some quantifier-free definable set. As we do demonstrate the o-minimality of $(\mathbb{R}, +, \cdot)$ we can take this result to be a corollary of Speissegger's theorem on Pfaffian differential equations ([Spe99, LS00]): if $\mathscr{R}$ is an o-minimal expansion of $(\mathbb{R}, +, \cdot)$, U is an open, connected, $\mathscr{R}$-definable subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is a C^1 function satisfying a system of differential equations

$$\frac{\partial f}{\partial x_i}(\bar{x}) = F_i(f(\bar{x}), \bar{x})$$

with each $F_i : \mathbb{R} \times U \to \mathbb{R}$ definable in $\mathscr{R}$ and C^1 then $(\mathscr{R}, f)$ is o-minimal. Applying this to $f(x) = e^x, U = \mathbb{R}, F_1(y, x) = y$ we have our result. The proof of Speisseger's theorem requires a great deal of heavy differential geometry, and not a great deal of model theory.

Current research in o-minimal structures is too broad to survey succinctly, but there are a few large questions which remain unanswered. Possibly the greatest of these is whether or not there are any examples of o-minimal structures not arising from the real numbers. So far all known examples are generated in a simple fashion[7]: one constructs some o-minimal structure on $\mathbb{R}$, builds an elementary extension, adds constants from the new structure, and then restricts attention to an elementary substructure. There is no proof, however, that this generates all o-minimal structures. There are also some questions regarding proper o-minimal expansions of the prime models[8] of certain theories. It is, for example, widely conjectured that there is no proper o-minimal expansion of $(\mathbb{Q}, +, <)$. In [LS95] the question is widened to the

[7] With the possibility of some steps being trivial.

[8] A (the) prime model of a theory is a model which elementarily embeds into every other model.

field of algebraic numbers and the prime model of the theory of the exponential field of real numbers. The general statement, however, cannot be true as the ordered vector space $(\mathbb{R}, +, <, \lambda_a : a \in \mathbb{R})$, where $\lambda_a(x) = ax$, is prime and o-minimal, but properly expanded by $(\mathbb{R}, +, \cdot)$.

Chapter 1
O-minimal structures

1.1 Introduction

A structure is o-minimal if it defines a linear ordering $<$ and, modulo this fact, has the smallest possible class of definable subsets. Stated formally:

Definition 1.1.1. Let $\mathscr{M}$ be an ordered structure, and $X \subseteq \mathscr{M}$ be definable. We say that X is *of finite type* if X is a boolean combination of sets of the form

$$
\begin{aligned}
(a,b) &= \{x : a < x < b\} \\
(-\infty, a) &= \{x : x < a\} \\
(b, \infty) &= \{x : b < x\}
\end{aligned}
$$

and $\{a\}$, with $a, b \in \mathscr{M}$. $\mathscr{M}$ is said to be *o-minimal* if every definable $X \subseteq \mathscr{M}$ is of finite type.

We will consider only o-minimal structures whose underlying order is a dense linear order with no endpoints. It was shown by Pillay and Steinhorn that any o-minimal structure can be definably 'split' into a trivial part, and a part with the above order. Note also that every ordered structure $\mathscr{M}$ comes equipped with a natural topology, generated by the open intervals

$$(a, b) = \{x : a < x \wedge x < b\},$$

where $a, b \in \mathscr{M} \cup \{\pm\infty\}$, and where $\pm\infty$ are imaginary elements added above and below the structure. Similarly, we form a topology on $\mathscr{M}^n$ in the usual way, by taking products. Unless noted, all topological references will be to these topologies. One should also note that while an o-minimal structure may not be complete, it must be *definably complete*. That is, for every definable set $X \subseteq \mathscr{M}$, both $\inf(X)$ and $\sup(X)$ must exist in $\mathscr{M} \cup \{\pm\infty\}$.

Lemma 1.1.1. *Let* $\mathrm{int}(X)$, $\mathrm{cl}(X)$, *and* $\mathrm{bd}(X)$ *denote the interior, closure, and boundary of a set* X*, in the usual topological sense. Then if* X *is a definable subset of* $\mathscr{M}^n$*, for some o-minimal structure* $\mathscr{M}$*,* $\mathrm{int}(X)$, $\mathrm{cl}(X)$, *and* $\mathrm{bd}(X)$ *are definable.*

Proof. Let $\varphi(x)$ define $X \subseteq \mathscr{M}$. Then the formula

$$\exists y \exists z (y < x < z \wedge \forall w (y < w < z \rightarrow \varphi(w)))$$

defines the interior of X. Then, $\mathrm{cl}(X)$ is the complement of the interior of the complement of X, and $\mathrm{bd}(X)$ is the closure of X less the interior of X. Now, if $X \subseteq \mathscr{M}^n$ is defined by $\varphi(x_1, ..., x_n)$, it is enough to point out that $(a_1, ..., a_n) \in \mathrm{int}(X)$ if and only if $a_i \in \mathrm{int}(\varphi(a_1, ..., a_{i-1}, \mathscr{M}, a_{i+1}, ..., a_n))$ for each i, simply by the definition of the product topology. ❑

Notice, in fact, that all three sets are definable using only the parameters used to define X and that they are all 'uniformly' definable, in that each operation is given by a scheme in the formula defining the set.

The groundwork for the study of o-minimal structures was set down in [PS86, KPS86, PS88], with perhaps the most key theorem being the cell decomposition theorem. This theorem allows us to perform many useful inductions on the definable sets in an o-minimal structure, by building these definable sets in a simple, recursive fashion. The theorems and lemmas in this section are all contained in the three papers listed above.

Definition 1.1.2. Let $\mathscr{M}$ be an ordered structure. Any singleton subset of $\mathscr{M}$ will be called a 0-dimensional cell, and any open interval a 1-dimensional cell. If $C \subseteq \mathscr{M}^n$ is a k-dimensional cell, and $f, g : C \to \mathscr{M}$ are continuous, definable functions with $f < g$ then $f \subseteq \mathscr{M}^{n+1}$ (viewed as a set of tuples) is a k-dimensional cell, and

$$(f, g) = \{(\bar{x}, y) : f(\bar{x}) < y < g(\bar{x})\} \subseteq \mathscr{M}^{n+1}$$

is a $k+1$-dimensional cell. In the latter we allow $f = -\infty$ and/or $g = \infty$.

Theorem 1.1.2. *Let $\mathscr{M}$ be an o-minimal structure, and let $X \subseteq \mathscr{M}^n$ be a definable set. Then there are cells $C_1, ..., C_k \subseteq \mathscr{M}^n$ such that $X = C_1 \cup ... \cup C_k$. Furthermore, if $f : X \to \mathscr{M}$ is definable, the C_i may be chosen such that $f \restriction C_i$ is continuous for each i.*

This theorem cannot be proven right away. Its proof is a rather complex induction, intertwined with the following, very useful, result.

Theorem 1.1.3 (Uniform bounds). *Suppose $\mathscr{M}$ is an o-minimal structure, and that $X \subseteq \mathscr{M}^{n+1}$ has the property that each fibre*

$$X_{\bar{a}} = \{x : (x, \bar{a}) \in X\},$$

$\bar{a} \in \mathscr{M}^n$, is finite. Then there is a uniform bound on the sizes of the $X_{\bar{a}}$.

Note that the theorem extends easily to $X \subseteq \mathscr{M}^{(n+k)}$. It is well known that not every minimal structure is strongly minimal. As a witness to this, $(\omega, <)$ is minimal, but is not strongly minimal as

$$(\omega, <) \equiv (\omega \times \{0\}) \cup (\mathbb{Z} \times \{1\}),$$

ordered reverse lexicographically, and $x < (0, 1)$ defines a set in the latter which is neither finite nor cofinite. Assuming the above theorem we may easily prove that such an example cannot be constructed in the o-minimal case. 'Strongly o-minimal' is equivalent to 'o-minimal.'

Theorem 1.1.4. *Let $\mathscr{M}$ be o-minimal, and $\mathscr{N} \equiv \mathscr{M}$. Then $\mathscr{N}$ is o-minimal.*

Proof. We will assume Theorem 1.1.3. Suppose that $\varphi(x, \bar{y})$ is a formula in the language of $\mathscr{M}$. Let $\mathscr{M} \models \psi_\varphi(a, \bar{b})$ if and only if a is a boundary point of $\varphi(\mathscr{M}, \bar{b})$, and let

$$\sigma_\varphi^n \equiv \forall \bar{y} \exists^{\leqslant n} x(\psi_\varphi(x, \bar{y})).$$

As $\mathscr{M}$ is o-minimal, $\psi_\varphi(\mathscr{M}, \bar{b})$ is finite for all $\bar{b}$, and so, by the theorem on finite bounds, there is some n for which $\mathscr{M} \models \sigma_\varphi^n$. But then $\mathscr{N} \models \sigma_\varphi^n$, and so each fibre of φ in $\mathscr{N}$ is of finite type. As φ was arbitrary, $\mathscr{N}$ is o-minimal. ❑

Theorem 1.1.2 also allows us to define a reasonable dimension for definable sets: if $\mathscr{M}$ is o-minimal and $X \subseteq \mathscr{M}^n$ is a definable set, then

$$\dim(X) = \max\{\dim(C) : C \subseteq X,\ C \text{ is a cell}\}.$$

It is easy to verify that this is defined for all definable sets X, in particular because $C \subseteq X$ implies $\dim(C) \leqslant n$. By a curve in $\mathscr{M}$ we mean a 1-dimensional subset of $\mathscr{M}^2$.

We will prove Theorems 1.1.2 and 1.1.3 in Section 1.3. First, however, we will establish a technical lemma needed in the result. This lemma is clearly the base case for the induction to prove Theorem 1.1.2.

Lemma 1.1.5. *Assume that $\mathscr{M}$ is o-minimal, and that $f : (a, b) \to \mathscr{M}$ is definable, where $a, b \in \mathscr{M} \cup \{\pm\infty\}$. Then there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that for all i, $f \restriction (a_i, a_{i+1})$ is either constant, or a bijection which preserves or reverses order.*

Proof. To simplify the proof we introduce a claim. Also, for the remainder of this proof, a function which is either constant or strictly monotone on a given interval is said to be *basic* on that interval.

Claim. *Let f and $\mathscr{M}$ be as in the lemma. Then there is an interval $I \subseteq (a, b)$ such that f is basic on I^*.*

Proof of claim. Let

$$\begin{aligned}
\varphi_0(x) &\equiv a < x < b \wedge \exists y(y < x \wedge \forall u(y < u < x \to f(u) = f(x))) \\
&\quad \vee \exists y(y > x \wedge \forall u(x < u < y \to f(u) = f(x))) \\
\varphi_1(x) &\equiv a < x < b \wedge \exists y, z(y < x < z \wedge \forall u(y \leqslant u < x \to f(u) < f(x)) \\
&\quad \wedge \forall u(x < u \leqslant z \to f(x) < f(u))) \\
\varphi_2(x) &\equiv a < x < b \wedge \exists y, z(y < x < z \wedge \forall u(y \leqslant u < x \to f(u) > f(x)) \\
&\quad \wedge \forall u(x < u \leqslant z \to f(x) > f(u))) \\
\varphi_3(x) &\equiv a < x < b \\
&\quad \wedge \exists y, z(y < x < z \wedge \forall u(y < u < z \to (u = x \vee f(u) > f(x)))) \\
\varphi_4(x) &\equiv a < x < b \\
&\quad \wedge \exists y, z(y < x < z \wedge \forall u(y < u < z \to (u = x \vee f(u) < f(x))))
\end{aligned}$$

We first need to show that $(a, b) = \varphi_0(\mathscr{M}) \cup \ldots \cup \varphi_4(\mathscr{M})$. Suppose that $x \in (a, b) \setminus \varphi_0(\mathscr{M})$. Let

$$X^+ = \{y \in (a, b) : f(y) > f(x)\},$$
$$X^- = \{y \in (a, b) : f(y) < f(x)\}.$$

Then for all $y \in (a, b)$, $y < x$, either X^+ or X^- must intersect (y, x), or else that y witnesses $\varphi_0(x)$. By o-minimality, then, either X^+ or X^- must contain an interval of the form (α, x), with $\alpha < x$. Similarly, either X^+ or X^- must contain an interval of the form (x, β), with $\beta > x$. The four cases, where $(\alpha, x) \in X^{\pm}$ and $(x, \beta) \in X^{\pm}$, correspond to the four cases, $\varphi_1(x), \ldots, \varphi_4(x)$. Thus one of the sets $\varphi_i(\mathscr{M})$ must contain an interval I.

Suppose that $I \subseteq \varphi_0(\mathscr{M})$ (indeed, suppose $\varphi_0(\mathscr{M}) \neq \emptyset$). If $x \in \varphi_0(\mathscr{M})$ then there is an interval to the left or right of x on which f is constant, and so basic. We have our I.

Suppose that $I \subseteq \varphi_1(\mathscr{M})$, let $x \in I$ be arbitrary, and let

$$X = \{y : y \in I' \wedge y > x \wedge f(x) \geqslant f(y)\}.$$

If there is some $y > x$ with $f(y) \leqslant f(x)$ then $X \neq \emptyset$. If $y \in X$ then, as $y \in \varphi_1(\mathscr{M})$, there is an interval to the left of y which is also contained in X. As X is definable, then, it must be a finite union of intervals. Also note that there is, by $\varphi_1(x)$, some $z > x$ such that X is disjoint from $(-\infty, z) \cap I$. So let $c = \inf(X)$. Of course, $c \notin X$, as any element of X has an interval to its left contained in X. Thus $f(x) < f(c)$. But because $\varphi_1(c)$, there is some interval J to the right of c such that $j \in J \to f(c) < f(j)$. Clearly J must be disjoint from X, contradicting the assumption that c was the left endpoint of an interval in X. So X is empty and, as x was arbitrary, f is increasing (whence basic) on I.

Suppose that $I \subseteq \varphi_2(\mathscr{M})$. Repeating the proof in the above paragraph with the order reversed, we see that f is decreasing on I.

Suppose that $I \subseteq \varphi_3(\mathscr{M})$. Let

$$X = \{x \in I : \forall y \in I(x < y \to f(x) < f(y))\}.$$

If X contains an interval then f is clearly increasing on this interval, and we are done. As X is definable, if it does not contain an interval it is finite, so let $I' = \{x \in I : x > \max(X)\}$. Then I' is a subinterval of I, and

$$\mathscr{M} \models \forall x \in I' \exists y \in I'(y > x \wedge f(y) \leqslant f(x)).$$

Suppose that for some $x_0 \in I'$ there is no y such that the above statement holds with strict inequality. Then there is some $x_1 \in I'$, $x_1 > x_0$, with $f(x_1) = f(x_0)$. There can be no y for x_1 for which the inequality is strict (or this y would also serve for x_0), so there is an $x_2 \in I'$, $x_2 > x_1$, with $f(x_2) = f(x_1) = f(x_0)$. Continuing this way we have an infinite set, and so an interval, on which f is constant. So we may assume that

$$\mathscr{M} \models \forall x \in I' \exists y \in I' (y > x \wedge f(y) < f(x)). \tag{1.1}$$

By a similar argument, and another refinement of the interval, to I'', we may also assume that

$$\mathscr{M} \models \forall x \in I' \exists y \in I' (y < x \wedge f(y) > f(x)). \tag{1.2}$$

Now fix some $x \in I''$, and let

$$\begin{aligned} X^+ &= \{y \in I'' : f(x) < f(y)\} \\ X^0 &= \{y \in I'' : f(x) = f(y)\} \\ X^- &= \{y \in I'' : f(x) > f(y)\}. \end{aligned}$$

Clearly these three sets are definable and partition I'', and clearly X^0 is finite, or we are done. Thus either X^+ or X^- must contain an interval (c, d), where d is the right endpoint of I''. Take c to be the least such point, and suppose $(c, d) \subseteq X^+$. We cannot have $f(c) > f(x)$, as $\varphi_3(c)$ implies that there is some $u < c$ with $u < v < c \to f(v) > f(c) > f(x)$, and so (u, d), a proper superset of (c, d), would be contained in X^+. So we must have $f(c) \leqslant f(x)$. By 1.1 there is some $y > c$ (in I'') with $f(c) > f(y)$. But then $f(y) < f(x)$, and so

$y \in X^-$, contradicting $(c,d) \cap X^- = \emptyset$. If we assume that $(c,d) \subseteq X^-$, for some $c < d$, then the same argument (reversing inequalities and replacing 1.1 with 1.2) we also get a contradiction.

Finally, if $I \subseteq \varphi_4(\mathscr{M})$, we can modify the proof above in the obvious way to see that the claim is true. ❑

Note that any subinterval of (a,b) meets the criteria in the lemma, and so will contain a subinterval on which f is basic. Now let $\psi(x)$ be a formula saying that x is the left endpoint of some interval $I \subseteq (a,b)$ such that f is basic on I and for no $I' \supset I$ is f basic on I'. Suppose $x \in \psi(\mathscr{M})$. Then there is some $y > x$ such that f is basic on (x,y). If $z \in (x,y)$, then, we cannot have $\psi(z)$, because if I is the interval witnessing $\psi(z)$, I is properly expanded by $I \cup (x,y)$ which leads to a contadiction[1]. So $\psi(\mathscr{M})$ contains no interval. Let $\psi(\mathscr{M}) = \{a_0, a_1, ..., a_n\}$ where $a_i < a_{i+1}$. We claim that f is basic on each (a_i, a_{i+1}). If not, let (α, β) be a maximal subinterval of (a_i, a_{i+1}) on which f is basic. We know by the claim that there must be one such interval, say I', and we know that there must be a maximal such as it is defined by

$$\chi(x) \equiv \exists u, v((u,v) \supseteq I' \wedge x \in (u,v) \wedge f \text{is basic on}(u,v)).^2$$

If $\alpha > a_i$ then, since $\alpha < a_{i+1}$, we have missed a member of $\psi(\mathscr{M})$ in our list. If $\alpha = a_i$ then $\beta < a_{i+1}$, and so there must be a maximal subinterval of (β, a_{i+1}) on which f is basic, and again we have missed a point on our list.

[1]It is clear that if f is basic on I_1 and I_2, and $I_1 \cap I_2 \neq \emptyset$, then f is basic on $I_1 \cup I_2$
[2]We can take the union of all intervals extending I' on which f is basic, but it is not clear a priori that this set has endpoints in $\mathscr{M}$.

Similarly, if $a_0 > a$ then we are missing a point on our list, and so $a = a_0$ and $b = a_n$. Thus f is basic on each (a_i, a_{i+1}).

Now, finally, suppose that f is strictly increasing on (a_i, a_{i+1}). Then

$$Y = \{f(x) : a_i < x < a_{i+1}\},$$

being definable, is a finite union of points and open intervals. Let $\{\beta_1, ..., \beta_m\}$ be the points, and let $f(\alpha_j) = \beta_j$, $\alpha_0 = a_i$, and $\alpha_{m+1} = a_{i+1}$. Then the image of f over each (α_j, α_{j+1}) is an interval, and f is strictly increasing on each (α_j, α_{j+1}), so f must be an order preserving bijection on each of these intervals. By performing a similar refinement on the intervals of decrease for f, we can expand our list $\{a_0, ..., a_n\}$ to the list described in the lemma. ❑

This lemma, at first glance, may make o-minimal structures seem like a rather trivial class, but we shall see that many important structures fall into this family. The class of o-minimal structures can also be shown, from this lemma, to have a very important geometric property.

Definition 1.1.3. In a structure (ordered or otherwise) $\mathscr{M}$, we say that an element a is *algebraic over* a set (not necessarily definable) $A \subseteq \mathscr{M}$ if there is a formula with constants from A which a and only finitely many other elements of $\mathscr{M}$ satisfy. The set of all elements algebraic over A is $\mathrm{acl}(A)$.

The element a is further said to be *definable* over A if there is some formula with constants from A which a, and only a, satisfies. The set of elements definable over A is $\mathrm{dcl}(A)$.

Note that both acl and dcl are closure operators in the algebraic sense. It is clear, for example, that $\mathrm{acl}(A)$ and $\mathrm{dcl}(A)$ both contain A, and that $A \subseteq \mathrm{acl}(B)$

implies $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$ (similarly for dcl). It is also true that these closure operators are *finitary* (i.e., if $a \in \mathrm{acl}(A)$ then $a \in \mathrm{acl}(A_0)$ for some finite $A_0 \subseteq A$), as any formula may contain only finitely many symbols. In o-minimal structures, as in strongly minimal structures, we have another important property:

Theorem 1.1.6 (The exchange law). *Let $\mathscr{M}$ be an o-minimal structure, $A \subseteq \mathscr{M}$ (possibly not definable), and $a, b \in \mathscr{M}$. Then if $a \in \mathrm{acl}(\{b\} \cup A) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(\{a\} \cup A)$.*

What this theorem says is that if there is a dependence of a on b over the set A that is more than simply a dependence of a on A, then there is a symmetric dependence of b on a (over the same set).

Proof. The first step in the proof of this theorem is to note that in an o-minimal structure (indeed, any linearly ordered structure), $\mathrm{acl}(\cdot) = \mathrm{dcl}(\cdot)$. If a is one of only n realizations in M of $\varphi(x)$, where the constants in φ come from A, then a is the sole realization of one of the formulae "a is the first realization of φ," "a is the second realization of φ," et cetera.

Now we wish to show that a is definable over $A \cup \{b\}$ if and only if there is an A-definable interval or singleton X, and an A-definable function $f : X \to \mathscr{M}$ with $f(b) = a$. One direction is clear, so suppose $\varphi(\mathscr{M}, \bar{a}, b) = \{a\}$, where $\bar{a} \in A$. Let

$$f(x) = \inf\{y : \varphi(y, \bar{a}, x)\}.$$

Then f is clearly A-definable, as is $\mathrm{dom}(f)$, and $f(b) = a$. As shown above, the boundary points of $\mathrm{dom}(f)$ are A-definable, and so $\mathrm{dom}(f)$ is a union of points and intervals, each of which is A-definable. Let X be the point or interval containing b.

So suppose X is an A-definable point or interval, $f : X \to \mathscr{M}$ an A-definable function, and $f(b) = a$. Because $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$, we cannot have $b \in \mathrm{dcl}(A)$ (or $\mathrm{dcl}(\{b\} \cup A) \setminus \mathrm{dcl}(A) = \emptyset$), and so X is an interval. Let $a_0, \ldots, a_n$ be as in Lemma 1.1.5. A close examination of the proof of the lemma shows that these elements are also definable over A, and so we cannot have $b = a_i$ for any i. Suppose $b \in (a_i, a_{i+1})$. If f is constant on this interval, then a is the sole realization of $\exists y \in (a_i, a_{i+1})(f(y) = x)$, and so is in $\mathrm{dcl}(A)$. So f must be a bijection of (a_i, a_{i+1}) onto some open interval containing a which either preserves or reverses order, and thus is invertible. This inverse is definable over A by "$f(y) = x$," and shows that $b \in \mathrm{dcl}(\{a\} \cup A)$. ❑

1.2 The Tarski-Seidenberg theorem

Before continuing, we wish to give at least one example of an o-minimal structure. Of course, quantifier elimination tells us that $(\mathbb{Q}, <)$ is o-minimal, but the following example is somewhat less trivial (and somewhat more interesting).

Theorem 1.2.1. *Let $X \subseteq \mathbb{R}^n$ be definable in the field $\mathbb{R}$. Then X is a boolean combination of sets of the form*

$$\{\bar{x} : f(\bar{x}) > 0\}$$

$$\{\bar{x} : f(\bar{x}) = 0\},$$

as f runs through $\mathbb{R}[\bar{x}]$. In particular, the field of real numbers is o-minimal.

Sets definable in this structure are called *semi-algebraic*. To prove this theorem, we will quote the following from [Hod93].

Theorem 1.2.2. *If T is a theory satisfying*

a. *if $\mathscr{M}, \mathscr{N} \models T$, $\mathscr{M} \leqslant \mathscr{N}$, and $\varphi(x)$ is a quantifier free formula with parameters from $\mathscr{M}$ then $\mathscr{N} \models \exists x \varphi(x)$ implies $\mathscr{M} \models \exists x \varphi(x)$*

b. *if $\mathscr{N} \models T$ and $\mathscr{M} \leqslant \mathscr{N}$ then there is an $\bar{\mathscr{M}} \models T$ such that $\mathscr{M} \leqslant \bar{\mathscr{M}} \leqslant \mathscr{N}$ and if $\mathscr{N}' \models T$ contains $\mathscr{M}$ then $\bar{\mathscr{M}}$ may be embedded into $\mathscr{N}'$ over $\mathscr{M}$*

then T has elimination of quantifiers. If in addition T has a model which can be embedded into all others (an algebraically prime *model) then T is complete.*

For what follows, we take a real closed field to be a structure

$$\mathscr{M} = (\mathscr{M}, +, \cdot, 0, 1, <)$$

such that

a. $(\mathscr{M}, +, \cdot, 0, 1)$ is a field

b. $<$ is a linear ordering on $\mathscr{M}$ which preserves the field structure, in the sense that if $x < y$, then $x + z < y + z$ for all z and $xz < yz$ or $yz < xz$ if $z > 0$ or $z < 0$ respectively

c. for all $f \in \mathscr{M}[x]$ and $a < b \in \mathscr{M}$, if $f(a)f(b) < 0$ then for some $a < c < b$, $f(c) = 0$.

Clearly we can give a(n infinite) set of first order axioms equivalent to the above. One can reformulate these axioms in an entirely field-theoretic way, but this is the most convenient context for our purposes.

Proof of Theorem 1.2.1. The proof here follows the proof in [Hod93]. Let $\mathscr{M}$ and $\mathscr{N}$ be as in part b of Theorem 1.2.2, with T the theory of real closed fields (described above). Let $\bar{\mathscr{M}}$ be the set of elements from $\mathscr{N}$ which are algebraic over $\mathscr{M}$ (in the usual, field-theoretic, sense). It is known (see, for example, [Fra93]) that this is a field. As it is a substructure of $\mathscr{N}$, all of the universal axioms of $\mathscr{N}$ hold in $\bar{\mathscr{M}}$. It is also clear that $<$ induces a dense linear order without endpoints on $\bar{\mathscr{M}}$, as

$$x < y \rightarrow x < \frac{x+y}{2} < y.$$

Now suppose $f \in \bar{\mathscr{M}}[x]$, and that $f(a)f(b) < 0$. Then there is some $c \in \mathscr{N}$ with $a < c < b$ and $f(c) = 0$. But then c is algebraic over $\bar{\mathscr{M}}$, and so over $\mathscr{M}$ (again, see [Fra93]). Thus $c \in \bar{\mathscr{M}}$.

To show a, let $\varphi(x)$ be a quantifier-free formula with parameters from $\mathscr{M}$, and suppose $\mathscr{N} \models \exists x \varphi(x)$. Using basic logic, we may rearrange $\exists x \varphi(x)$ into the form

$$\exists x \bigvee_{j=0}^{n} \bigwedge_{i=0}^{m_i} \varphi_{ji}(x) \equiv \bigvee_{j=0}^{n} \exists x \bigwedge_{i=0}^{m_i} \varphi_{ji}(x),$$

where each $\varphi_{ji}(x)$ is either an atomic formula, or the negation of one. In this structure each negated atomic formula is a disjunction of atomic formulae, and so by rearranging each φ_{ji} (and possibly factoring out disjunctions in the negated case) we see that each φ_{ji} may be taken to be $p_{ji}(x) = 0$ or $p_{ji}(x) > 0$ for some $p_{ji} \in \mathscr{M}[x]$. We clearly need only show the result when $n = 0$. So let (writing m for m_0 and i for $(0, i)$)

$$\varphi(x) \equiv \bigwedge_{i=0}^{m} \varphi_i(x)$$

where each φ_i is atomic or negated atomic. If, for some i, we have $\varphi_i(x) \equiv$ '$p_i(x) = 0$' then, as $\mathscr{M}$ is real closed, any realization of φ is already in $\mathscr{M}$. Thus we may suppose that for each i, $\varphi_i(x)$ is $p_i(x) > 0$. Let $c_0 < c_1 < ... < c_k$ be a list of all roots of the p_i. If $b \in \mathscr{N}$ realizes φ then b is on one of the intervals $(-\infty, c_0), (c_0, c_1), ..., (c_k, \infty)$. By the reasoning above, $c_j \in \mathscr{M}$ for all j. Now simply notice that any c in the same interval as b must also satisfy φ, by the intermediate value property. So we choose one such $c \in \mathscr{M}$ and we are done. ❑

Notice also that the field of real algebraic numbers is an algebraically prime model of the theory of real-closed fields, and so this theory is complete, and any real closed field is o-minimal.

1.3 The proof

In this section we will complete the proofs of the Cell Decomposition Theorem and the Theorem on Uniform Bounds. In what follows, $\Gamma(f)$ will denote the graph of a function f.

Definition 1.3.1. Let $X \subseteq \mathscr{M}^n$ be an open cell, and let $\varphi(x, \bar{y})$ be a formula (with parameters from $\mathscr{M}$) such that for each $\bar{a} \in \mathscr{M}^m$, $\varphi(\mathscr{M}, \bar{a})$ is finite. We will say that $\bar{a} \in X$ is *good for* $\varphi(\bar{x}, y)$ if for all $b \in \mathscr{M}$, there is an open box (i.e., a product of intervals) $B \subseteq X$ containing $\bar{a}$ and an interval I containing b such that

(i) $\mathscr{M} \models \varphi(\bar{a}, b)$ implies that $\varphi(\mathscr{M}) \cap (B \times I)$ is the graph of a continuous function from B to I and

(ii) $\mathscr{M} \not\models \varphi(\bar{a}, b)$ implies that $\varphi(\mathscr{M}) \cap (B \times I) = \emptyset$.

The proof is a simultaneous induction of the following claims:

I$_n$ Given any cell $X \subseteq \mathscr{M}^n$, and finite collection $\{X_i\}$ of definable subsets of X, there is a (finite) partition of X into cells which partitions each X_i.

II$_n$ If $X \subseteq \mathscr{M}^n$ is definable and $f : X \to \mathscr{M}$ a definable function, there is a partition of X into cells such that the restriction of f to each cell in the partition is continuous.

III$_n$ Let $X \subseteq \mathscr{M}^n$ be definable, and $\varphi(\bar{x}, y)$ a formula where $\bar{x}$ is an n-tuple. Then if for each $\bar{a} \in \mathscr{M}^n$, $\varphi(\bar{a}, \mathscr{M})$ is finite, the set $\{|\varphi(\bar{a}, \mathscr{M})| : \bar{a} \in \mathscr{M}^n\}$ is bounded.

IV$_n$ Let $X \subseteq \mathscr{M}^n$ be an open cell, and $\varphi(\bar{x}, y)$ a formula where $\bar{x}$ is an n-tuple. Then if for each $\bar{a} \in \mathscr{M}^n$, $\varphi(\bar{a}, \mathscr{M})$ is finite, if the mappings

$$\bar{x} \mapsto \min\{y : \varphi(\bar{x}, y)\}$$

and

$$\bar{x} \mapsto \max\{y : \varphi(\bar{x}, y)\}$$

are continuous, and if $\bar{a}$ is good for φ, for all $\bar{a} \in X$, then $|\varphi(\bar{a}_1, \mathscr{M})| = |\varphi(\bar{a}_2, \mathscr{M})|$ for all $\bar{a}_1, \bar{a}_2 \in \mathscr{M}$.

The Base Case. First note that $\mathbf{I}_1$ follows from the definition of an o-minimal structure, and $\mathbf{II}_1$ follows from Lemma 1.1.5.

Let us now show $\mathbf{IV}_1$. Assume that $X = (a, b)$, each $c \in X$ is good for φ, each $\varphi(c, \mathscr{M})$ is finite, and that

$$f_F(x) = \min\{y : \varphi(x, y)\}$$

and

$$f_L(x) = \max\{y : \varphi(x, y)\}$$

are continuous. If the statement is false, then there is some $k \in \omega$ for which the set $X_1 = \{c : \exists^{=k} y \varphi(c, y)\}$ is non-empty, but not all of X. Let $c \in X$ be a boundary point of X_1. Our aim is to show that c is not good for φ, which contradicts an hypothesis. Let $\varphi(c, \mathscr{M}) = \{d_0, ..., d_N\}$, where $d_i < d_{i+1}$. We may inductively choose disjoint intervals $J_0, ..., J_N$, such that $d_i \in J_i$ for all i, and an interval I containing c, such that for each i, $\varphi(\mathscr{M}) \cap (I \times J_i)$ is the graph of the continuous function $g_i : I \to J_i$. So for all $d \in I$, $|\varphi(d, \mathscr{M})| \geqslant N$. If there were an interval $I' \subseteq I$ containing c such that for all $d \in I'$, $|\varphi(d, \mathscr{M})| = N$, then we would have $I' \cap X_1 \neq \emptyset$, whence $N = k$, whence c is interior to X_1, not a boundary point. As $\{d : |\varphi(d, \mathscr{M})| = N\}$ is definable, there must be an interval to the left or right of c which is not contained in it. Suppose, without loss of generality, that there is an interval I' on the right of c with

$$\forall d \in I', |\varphi(d, \mathscr{M})| > N. \qquad (*)$$

Set

$$g(x) = \min\{y : \varphi(x, y) \wedge \bigwedge_{i=0}^{N} y \neq g_i(x)\}.$$

By $*$ and by the fact that $\varphi(d, \mathscr{M})$ is finite for all $d \in X$, g is defined (and definable) on I'. As $f_F < g < f_L$ on I', $\lim_{x \to c^+} g(x)$ exists, say d. If $\neg\varphi(c, d)$, then c is not good for φ as any box containing (c, d) also contains part of the graph of g. If $\varphi(c, d)$ then $d = d_i$ for some i. But $g \neq g_i$, and $\lim_{x \to c^+} g_i(x) = \lim_{x \to c^+} g(x)$, so no box about (c, d) intersects $\varphi(\mathscr{M})$ as the graph of a continuous function.

Finally, we will prove $\mathbf{III}_1$. Let X, φ be as in the statement. Note that we can get away with proving the statement for each of the cells in some finite decomposition of X, as the maximum of the uniform bounds on the sizes of fibres on each cell in the decomposition will serve as a bound on the sizes of the fibres over all of X. If X is a singleton the result is trivial, so let $X = (a, b)$, where we may have $a, b = \pm\infty$. We may assume, by $\mathbf{I}_1$ and the above remark, that $X = \mathrm{dom}(\varphi)$, and so f_F and f_L are defined. By $\mathbf{II}_1$ we may also assume that these two functions are continuous. Let Y be the (definable) set of points which are not good for φ. If $Y = \{a_0, ..., a_n\}$ then on each interval (a_i, a_{i+1}), where $a_{-1} = a$ and $a_{n+1} = b$, the size of the fibres of φ is a constant by $\mathbf{IV}_1$, and so we are done. We shall prove, by contradiction, that Y is indeed finite.

Suppose that $(a_1, b_1) \subseteq Y$. We say that $(c, d) \in \mathscr{M}^2$ is *type I nasty* for φ if $\varphi(c, d)$ and for all boxes B about (c, d), $\varphi(\mathscr{M}) \cap B$ is not the graph of a continuous function. We say that (c, d) is *type II nasty* for φ if $\neg\varphi(c, d)$ and for all boxes B about (c, d), $B \cap \varphi(\mathscr{M}) \neq \emptyset$. Thus for each $c \in Y$ there is at least one d such that (c, d) is nasty for φ (type I or II).

Claim. *For all $c \in (a_1, b_1)$ there is a least $d \in \mathscr{M}$ such that (c, d) is nasty for φ.*

Proof of the claim. As we are assuming that φ has finite fibres on X, φ may only have finitely many type I nasty points for each c, thus it remains only to show that if (d_1, d_2) is an interval such that for all $d \in (d_1, d_2)$, (c, d) is type II nasty for φ then (c, d_1) is nasty (type I or II) as well. First we establish that d_1 is, in fact, finite. If not, let $a_0 > a_1 > \ldots$ be elements of $(-\infty, d_2)$, decreasing without bound. We may inductively construct boxes B_i about (c, a_i) with $B_i \cap B_j = \emptyset$ whenever $i \neq j$ and, as $\varphi(\mathscr{M}) \cap B_i \neq \emptyset$ for all i, conclude that $\varphi(c, \mathscr{M}) \supseteq (-\infty, \alpha)$ for some α. But then $\varphi(c, \mathscr{M})$ is not finite.

If $\neg\varphi(c, d_1)$ then let B be a box containing (c, d_1). Of course, B contains (c, d) for some $d \in (d_1, d_2)$, and so, by hypothesis, $\varphi(\mathscr{M}) \cap B \neq \emptyset$. On the other hand, if $\varphi(c, d_1)$ then take I and J to be intervals containing c and d_1 respectively. If $(I \times J) \cap \varphi(\mathscr{M}) = \Gamma(f)$ for some continuous $f : I \to J$, let $d' \in (d_1, d_2)$. For any box $B \subseteq (I \times J)$ containing (c, d'), $\Gamma(f) \cap B \neq \emptyset$. Thus $\lim_{x \to c} f(x) = d'$. But d' was arbitrary, contradicting the uniqueness of limits. Thus $(I \times J) \cap \varphi(\mathscr{M})$ is not the graph of a continuous function, and again (c, d) is nasty for φ. ❑

Let

$$\begin{aligned} g(x) &= \min\{d : (x,d) \text{ is nasty for } \varphi\} \\ g_1(x) &= \max\{y : \varphi(x,y) \wedge y < g(x)\} \\ g_2(x) &= \min\{y : \varphi(x,y) \wedge y > g(x)\}, \end{aligned}$$

allowing g_1 and g_2 to take the values $-\infty, \infty$ when undefined. By applying $\mathbf{II}_1$, we may refine our interval to one on which g, g_1, and g_2 are continuous (or $\pm\infty$, in the latter two cases). Suppose that there is a sub-interval I of this interval such that for all $c \in I$, $(c, g(c))$ is type I nasty. Fix some $c \in I$, and choose d_1, d_2 such that $g_1(c) < d_1 < g(c) < d_2 < g_2(c)$. By the continuity of these functions on this interval, we may choose a sub-interval I' such that for all $x \in I'$, $g_1(x) < d_1 < g(x) < d_2 < g_2(x)$. But then $(I' \times (d_1, d_2)) \cap \varphi(\mathscr{M}) = \Gamma(g)$, which is a contradiction. So suppose that there is a sub-interval I such that for all $c \in I$, $(c, g(c))$ is type II nasty[3]. Construct c, d_1, d_2, and I' as above. Then $(I' \times (d_1, d_2)) \cap \varphi(\mathscr{M}) = \emptyset$. ❑

To prove the inductive step we require the following, useful, lemma:

Lemma 1.3.1. *Let $C \subseteq \mathscr{M}^n$ be a cell, $k = \dim(C) \geqslant 1$. Then there is a cell $C' \subseteq \mathscr{M}^k$ definably homeomorphic to C.*

Proof. If $n = 1$ the result is trivial. Suppose that $C_1 \subseteq \mathscr{M}^n$ is a cell, suppose that $f : C_1 \to \mathscr{M}$ is continuous and definable, and let $C^* = \Gamma(f)$. Let $h : C_1 \to C^*$ be given by $h(\bar{x}) = (\bar{x}, f(\bar{x}))$. Then clearly h is a bijection. Also, if B is an open box about some $\bar{x} \in \mathscr{M}^n$, and $I \subseteq \mathscr{M}$ is an open interval containing

[3]There must be one or the other, as nastiness is definable.

$f(\bar{x})$, then $h^{-1}(B \times I) = B \cap f^{-1}(I)$ is open. Conversely, h^{-1} is continuous as $h(B) = (B \times \mathcal{M}) \cap \Gamma(f)$.

Now suppose that $C^* = (f, g)$, where $f, g : C_1 \to \mathcal{M}$ are continuous and definable (C_1 a cell), and suppose that $h : C_1 \to C_2$ is a definable homeomorphism. Then define $h^* : (f, g) \to \mathcal{M}^{n+1}$ by $(\bar{x}, y) \mapsto (h(\bar{x}), y)$. Again, this is clearly a bijection from (f, g) onto $(f \circ h^{-1}, g \circ h^{-1})$. If $B \times I \subseteq (f, g)$, then $h^*(B \times I) = h(B) \times I$, and *vice versa*, so h^* is a homeomorphism. The lemma follows. ❑

The Inductive Case. In light of Lemma 1.3.1, and the observation that cell decompositions will carry through a definable homeomorphism, we can assume that the statement $\mathbf{I}_n$ holds for any cell X with dimension less than n. So suppose $X = (f, g)$, where $f, g : X^* \to \mathcal{M}$ are continuous and definable, X^* a cell in $\mathcal{M}^{n-1}$. For a given $X_i \subseteq X$, let $X_i^* \subseteq X^*$ be the projection of X_i. We may choose some partition $\{C_j^*\}$ of X^* which partitions each X_i^*. Let $C_j = (f \restriction C_j^*, g \restriction C_j^*)$ for each j. Fix some $C = C_j$ and let $I' = \{i : X_i^* \cap C_j^* \neq \emptyset\}$.

Claim. *If C is open then it admits a partition partitioning each X_i, $i \in I'$.*

Proof of the claim. Let $(X)_{\bar{a}} = \{b \in \mathcal{M} : (\bar{a}, b) \in X\}$ for all definable sets X. By the induction hypothesis, the boundary of $(X_i)_{\bar{a}}$ is uniformly finite. By using cell decomposition for $\mathcal{M}$, we can clearly assume that each $(X_i)_{\bar{a}}$ (as $\bar{a}$ ranges through $\mathcal{M}^{n-1}$) is of the same *type*[4]. Let $f_i^1(\bar{a}), \ldots, f_i^{k_i}(\bar{a})$ be the boundary points of $(X_i)_{\bar{a}}$ and, again using cell decomposition at lower levels, assume

[4]Here *type* refers to the set's representation as a union of points and intervals.

that the set of functions $\{f_i^j : 1 \leqslant i \leqslant |I'|, 1 \leqslant j \leqslant k_i\}$ can be rewritten as a sequence $g_1 < g_2 < \ldots < g_k$. Note that each of these functions is definable. One more application of the induction hypothesis allows us to assume that each g_i is continuous on its domain, and then

$$\bigcup_{i=0}^{k}(g_i \upharpoonright C, g_{i+1} \upharpoonright C) \cup \bigcup_{i=1}^{k} \Gamma(g_i),$$

where $g_0 = f$ and $g_{k+1} = g$, is a cell decomposition of C partitioning each X_i, $i \in I'$. ❑

The result follows by the fact that the non-open cells are of dimension at most $n-1$.

In order to prove $\mathbf{II}_n$, suppose $X \subseteq \mathscr{M}^n$ is a cell and $f : X \to \mathscr{M}$ a definable function. Again, if X has dimension less that n a definable homeomorphism with a cell in $\mathscr{M}^{n-1}$ shows the result. So assume X is open, and set

$$\begin{aligned} X_1 &= \{(\bar{a}, b) \in X : f(x_1, ..., x_{n-1}, b) \text{ is cts on some box} \\ &\quad\ B \ni \bar{a} \text{ with } B \times \{b\} \subseteq X\} \\ X_2 &= \{(\bar{a}, b) \in X : f(\bar{a}, x_n) \text{ is constant or a monotone bijection on some} \\ &\quad\ \text{interval } I \ni b \text{ with } \{\bar{a}\} \times I \subseteq X\}. \end{aligned}$$

Let $\mathscr{P}$ be a cell decomposition of X which partitions both X_1 and X_2, and let $C \in \mathscr{P}$ be open. We intend to prove that $f \upharpoonright C$ is continuous. Suppose $C = (f_1, f_2)$ for some $f_1, f_2 : C^* \to \mathscr{M}$. If $(\bar{a}, b) \in C$ then $f(x_1, ..., x_{n-1}, b)$ is defined on some open subset of C^*. By the induction hypothesis, there is an open cell $D \subseteq C^*$ such that $f(\bar{x}, b)$ is continuous of D. Now, if $\bar{a}' \in D$,

$(\bar{a}', b) \in X_1$. Since $X_1 \cap C \neq \emptyset$, we have $C \subseteq X_1$. A similar proof shows that $C \subseteq X_2$.

Now, if $f(\bar{a}, x)$ is not constant or a monotone bijection on $(f_1(\bar{a}), f_2(\bar{a}))$, the let $f_1(\bar{a}) < b_1 < \ldots < b_m < f_2(\bar{a})$ be the points shown to exist in the monotonicity lemma, with m minimal. Then there can be no $I \ni b_1$ with $f(\bar{a}, x)$ constant or a monotone bijection on I, so $(\bar{a}, b_1) \notin X_2$. But $C \subseteq X_2$, so this is impossible.

Let $(\bar{a}, b) \in C$, and let $J \ni f(\bar{a}, b)$. We want to find an open box about $(\bar{a}, b)$ which f maps into J. We can find a closed interval $I = [b_1, b_2] \subset (f_1(\bar{a}), f_2(\bar{a}))$ such that $f(\bar{a}, I) \subseteq J$ by continuity. As $(\bar{a}, b_1), (\bar{a}, b_2) \in X_1$ we can find open boxes B_1 and B_2 with $f(B_1, b_1) \subseteq J$ and $f(B_2, b_2) \subseteq J$. Let $B' = B_1 \cap B_2$. Then $B' \times \operatorname{int}(I) \subseteq C$. Also, if $(\bar{a}', b') \in B' \times I$, we have $f(\bar{a}', b') \in J$ as $f(\bar{a}', b_1), f(\bar{a}', b_2) \in J$ and $f(\bar{a}', x)$ is constant or monotone on I.

Thus f is continuous on every open cell in $\mathscr{P}$, and is continuous on every other cell as well (by the induction hypothesis).

To show $\mathbf{III}_n$, let $\varphi(x_1, \ldots, x_n, y)$ be a formula (with parameters from $\mathscr{M}$) and $X \subseteq \mathscr{M}^n$ an open cell. Suppose φ is finite on X, and let

$$\begin{aligned} X_1 &= \{(\bar{a}, b) \in X : \bar{a} \text{ is good for } \varphi(x_1, \ldots, x_{n-1}, b, y)\} \\ X_2 &= \{(\bar{a}, b) \in X : b \text{ is good for } \varphi(\bar{a}, x_n, y)\}. \end{aligned}$$

Again let $\mathscr{P}$ be a cell decomposition of X partitioning both X_1 and X_2, and let $C \in \mathscr{P}$ be open. We shall show that $C \subseteq X_1 \cap X_2$.

Let $B \subseteq C$ be an open box, $(\bar{a}, b) \in B$, and $B^* \subseteq \mathscr{M}^{n-1}$ be its projection. Then $\varphi(x_1, \ldots, x_{n-1}, b, y)$ is uniformly finite on B^*. By cell decomposition, pick some open cell $D \subseteq B^*$ such that for all $\bar{a}' \in D$, $|\varphi(\bar{a}', b, \mathscr{M})| = k$ and such that the first, second, ..., kth points functions are continuous. Clearly if $\bar{a}' \in D$, then $\bar{a}'$ is good for $\varphi(\bar{x}, b, y)$, so $(\bar{a}', b) \in X_1$. But then $C \subseteq X_1$. The proof is the same to show that $C \subseteq X_2$.

The following claim proves $\mathbf{III}_n$. It also finishes the theorem, as the assumptions of $\mathbf{IV}_n$ imply $X = X_1 = X_2$.

Claim. *If $C \subseteq X$ is an open cell and $C \subseteq X_1 \cap X_2$ then for all $\bar{c}_1, \bar{c}_2 \in C$, $|\varphi(\bar{c}_1, \mathscr{M})| = |\varphi(\bar{c}_2, \mathscr{M})|$.*

Proof of the claim. Suppose that for some $k < \omega$, $C_k = \{\bar{c} \in Y : |\varphi(\bar{c}, \mathscr{M})| = k\}$ is a nonempty, proper subset of C. Then there is a boundary point $\bar{c}$ of C_k in C. Let B be an open box in C containing $\bar{c}$, choose $(\bar{a}_1, b_1), (\bar{a}_2, b_2) \in B$, and let B^* be the projection of B (in $\mathscr{M}^{n-1}$). By hypothesis every $\bar{a} \in B^*$ is good for $\varphi(\bar{x}, b_1, y)$, and so

$$|\varphi(\bar{a}_1, b_1, \mathscr{M})| = |\varphi(\bar{a}_2, b_1, \mathscr{M})|,$$

by $\mathbf{IV}_{n-1}$. Similarly, by $\mathbf{IV}_1$,

$$|\varphi(\bar{a}_2, b_1, \mathscr{M})| = |\varphi(\bar{a}_2, b_2, \mathscr{M})|.$$

But then B cannot contain both points in C_k and points not in C_k. ❑

❑

1.4 Algebraic structures

In Section 1.2 we saw that the field of real numbers, and hence the ordered group of real numbers, is o-minimal. Here we will see that these are, from the view point of first order logic, the only o-minimal ordered rings and groups. For the remainder of this section we will lift the restriction that ordered structures have a dense underlying order. The structures below are only assumed to be ordered linearly.

Theorem 1.4.1. *Let $\mathscr{G}$ be an o-minimal ordered group. Then $\mathscr{G}$ is abelian and divisible. In particular, $\mathscr{G} \equiv (\mathbb{R}, <, +)$.*

Proof. We will prove that $\mathscr{G}$ must be abelian and divisible, and let folklore take care of the completeness of these axioms (see, e.g., [Hod93]). The key is that if $H \subseteq \mathscr{G}$ is a definable subgroup, then H is trivial. If $H \neq \{0\}$, let $h > 0$ be in H. If there is some $g \in (0, h) \setminus H$ then $nh \in H$ for all n, and $g + nh \notin H$ for all n, but

$$\ldots < nh < g + nh < (n+1)h < g + (n+1)h < \ldots,$$

so H must have infinitely many connected components, which is impossible. Thus H is connected. Now let $h = \sup(H)$, and suppose that $h < \infty$. Clearly we cannot have $h \in H$, or $2h \leqslant h$ as $2h \in H$, whence $h \leqslant 0$. So $h \notin H$. Let $0 < g < h$, so that $g \in H$. Then $0 < h - g < h$, and so $h - g \in H$. But this would mean that $(h - g) + g = h \in H$, which is a contradiction. So $h = \infty$, and consequently $H = \mathscr{G}$.

We can now easily show that $\mathscr{G}$ is abelian and divisible simply by noting that the centralizer of any element of $\mathscr{G}$ is a nontrivial definable subgroup of $\mathscr{G}$, as is $n\mathscr{G}$, for each n. ❑

Theorem 1.4.2. *Let $\mathscr{R}$ be an o-minimal ordered ring. Then $\mathscr{R}$ is a real closed field (and hence $\mathscr{R} \equiv (\mathbb{R}, +, \cdot)$).*

To prove this we will need a version of the intermediate value property that will hold in all o-minimal structures.

Definition 1.4.1. Let $\mathscr{M}$ be a structure, and $X \subseteq \mathscr{M}^n$ be a definable set. X is said to be *definably connected* if there do not exist two open, definable sets U and V not disjoint from X such that $X \subseteq U \cup V$, but $X \cap U \cap V = \emptyset$.

It should be noted right away that open and closed intervals are definably connected, even though they may not be connected. The interval $(0, 1)$ in $(\mathbb{Q}, <)$ is diconnected by the open sets $(0, 1/\pi) \cap \mathbb{Q}$ and $(1/\pi, 1) \cap \mathbb{Q}$, but any definable pair of open subsets of $(0, 1)$ would themselves have to be finite unions of open intervals, and so could not disjointly cover $(0, 1)$.

Lemma 1.4.3. *If $X \subseteq \mathscr{M}^n$ is definable and definably connected, $Y \subseteq \mathscr{M}^m$ is definable, and $f : X \to Y$ is definable, continuous, and surjective, then Y is definably connected.*

Proof. The standard topological proof works, just noting that the inverse image of a definable set by a definable map is definable. ❑

Proof of Theorem 1.4.2. It was shown above that $(\mathscr{R}, <, +)$ must be an abelian group. Also, if $x \in \mathscr{R}$, $x\mathscr{R}$ is a subgroup of the additive group of $\mathscr{R}$, and so is the entire group. This implies that $\mathscr{R}$ has inverses and an indentity. Now consider the group $\mathscr{G}$ whose underlying set is $\{x \in \mathscr{R} : x > 0\}$, and whose operation is the multiplication operation in $\mathscr{R}$. The sets definable in this group are clearly all definable in the ring $\mathscr{R}$, so $\mathscr{G}$ is an o-minimal ordered group. Thus $\mathscr{R}$ is a field. Now all that needs to be shown is that every polynomial assuming both positive and negative values has a root. But it is clear that any

$f \in \mathscr{R}[x]$ is definable and continuous. If $f(a) < 0$ and $f(b) > 0$ suppose, w.l.o.g., that $a < b$. Then, as (a, b) is definably connected, the image under f of (a, b) must also be, and so f has a root. ❑

1.5 Prime models

For the remainder of the section we fix some o-minimal L-theory T.

Definition 1.5.1. A model $\mathscr{M} \models T$ is prime over the set $A \subseteq \mathscr{M}$ if whenever there is an elementary map of A into $\mathscr{N} \models T$, the map can be extended to an elementary embedding of $\mathscr{M}$ into $\mathscr{N}$.

The existence and uniqueness of prime models over arbitrary sets, demonstrated in [PS86], shows a strong similarity between o-minimal structures and strongly minimal structures. The proof presented here is much simpler and, we feel, more intuitive than the original, but requires an extra hypothesis.

Theorem 1.5.1. *If $\mathscr{M} \models T$ and there are $\emptyset$-definable functions $f : \mathscr{M}^2 \to \mathscr{M}$ and $g : \mathscr{M} \to \mathscr{M}$ satisfying*

$$\forall x \forall y (x < y \to x < f(x, y) < y)$$

$$\forall x (g(x) > x)$$

then there is, for any nonempty $A \subseteq \mathscr{M}$, a unique prime model over A.

The hypothesis above requires that we have Skolem functions demonstrating the order type of the structure. The added hypothesis does not weigh us down too much, however, as any group satisfies this requirement (using $f(x, y) = (x + y)/2$, $g(x) = x + 1$).

Proof. Let T have the special property above. For the remainder of this proof, an o-minimal theory will be called *regular* if every $\emptyset$-definable n-ary partial function definable in T is already represented by a function symbol in L. Of course, if T is *not* regular, we may expand the language to a new one L^* which contains a function symbol for every $\emptyset$-definable function, and extend T and its models in a unique way to L^* structures. It is clear that the concepts of elementary embeddings, isomorphisms, et cetera are not changed by doing this. The concepts of embedding and substructure, however, are.

Claim. *Let $\mathscr{M} \models T$ and let $\mathscr{N} \leqslant \mathscr{M}$ (not necessarily a model of T). Then if T is regular, $\mathscr{N} \preccurlyeq \mathscr{M}$.*

Proof of the claim. By the Tarski-Vaught criterion, we must check that if $\varphi(\bar{x}, y)$ is an L-formula (without parameters) and $\bar{a} \in \mathscr{N}$ with $\mathscr{M} \models \exists y \varphi(\bar{a}, y)$ then there is a $d \in \mathscr{N}$ with $\mathscr{M} \models \varphi(\bar{a}, d)$. Suppose $\mathscr{M} \models \exists y \varphi(\bar{a}, y)$. By o-minimality, $X = \varphi(\bar{a}, \mathscr{M})$ is a finite union of points and intervals. Suppose that $\inf(X)$ is in X, and let $\psi(\bar{x}, y) \equiv$ "$y = \inf\{z : \varphi(\bar{x}, z)\}$". Clearly ψ defines a function (where it is defined), and so there is an n-ary function symbol f_ψ such that $f_\psi(\bar{x}) = y \leftrightarrow \psi(\bar{x}, y)$. As $\mathscr{N} \leqslant \mathscr{M}$, $f_\psi(\bar{a}) \in \mathscr{N}$. But $\mathscr{M} \models \varphi(\bar{a}, f_\psi(\bar{a}))$. Now suppose that $\inf(X)$ is not in X. Then set $\psi(\bar{x}, y)$ as above, and $\chi(\bar{x}, y) \equiv$ "$y = \sup\{z : (f_\psi(\bar{x}), z) \subseteq X)\}$". Then both $b = f_\psi(\bar{a})$ and $c = f_\chi(\bar{a})$ are in $\mathscr{N}$. As $\mathscr{N}$ is a dense order (by our added hypothesis), let $d \in (b, c) \cap \mathscr{N}$. Then $d \in (b, c) = (f_\psi(\bar{a}), f_\chi(\bar{a})) \subseteq X$, and $\mathscr{M} \models \varphi(\bar{a}, d)$. ❑

Now assume T is regular, let $A \subseteq \mathscr{M} \models T$, let $\mathscr{A}$ be the substructure of $\mathscr{M}$ generated by A, and let $\mathscr{N}$ be a structure into which A may be elementarily mapped. Without loss of generality, we can assume that $A \subseteq \mathscr{N}$. Then $\mathscr{A} \leqslant \mathscr{N}$, and so $\mathscr{A} \preccurlyeq \mathscr{N}$. For uniqueness, suppose $\mathscr{B}$ is another structure prime over A, $A \subseteq \mathscr{B}$. Then, again, $\mathscr{A} \preccurlyeq \mathscr{B}$. Let $f : \mathscr{B} \to \mathscr{A}$ be an elementary embedding fixing A. Then f must also fix the structure generated by A, namely $\mathscr{A}$. So we have $\mathscr{A} = \mathscr{B}$. ❑

Note that the claim in the proof is not true if the special assumption is dropped. In particular, if $f(x) = x$, the theory of $(\mathbb{Q}, <, f)$ is regular.

We may use similar ideas to prove the following useful lemma which will be needed in Chapter 2:

Lemma 1.5.2. *Let $\mathscr{M}$ be an o-minimal expansion of a group, and let $\sim$ be a definable equivalence relation on some definable $X \subseteq \mathscr{M}^n$. Then there is a definable transversal for X, that is, a function $f : X \to X$ such that $x \sim y \leftrightarrow f(x) = f(y)$.*

Proof. We will deal first with the case where $n = 1$, $X = \mathscr{M}$. For each x, let $S_x = \{y : x \sim y\}$, $a_x = \inf S_x$, and $b_x = \sup\{y : (a_x, y) \subseteq S_x\}$, and let $c > 0$ be an arbitrary point. We will define $f(x)$ as follows if $a_x \neq -\infty$: if $a_x \sim x$ then $f(x) = a_x$; if $a_x \not\sim x$ and $b_x < \infty$, $f(x) = (a_x + b_x)/2$; if $b_x = \infty$, $f(x) = a_x + c$. If a_x is $-\infty$, we define $f(x)$ by: if $b_x \neq \infty$, $f(x) = b_x - c$; if $b_x = \infty$, $f(x) = c$. It has been engineered such that $f(x) \sim x$ for all x. It is also clear that $f(x) = f(y) \leftrightarrow x \sim y$, as $x \sim y \leftrightarrow S_x = S_y$.

If X is a proper subset of $\mathscr{M}$ we may extend $\sim$ to the rest of $\mathscr{M}$ by setting $S_x = \{x\}$ for $x \in \mathscr{M} \setminus X$. If $n > 1$ we may use the same definition of $f(x)$ using the lexicographical ordering. ❑

Chapter 2
The real numbers

One of the goals of this chapter is to examine reducts of the field of real numbers. In particular, we will outline the work leading to the classification of all reducts of this structure. We will also look at some important o-minimal expansions of the reals. We begin, however, with a justification of our focus on the real numbers.

2.1 Archimedean ordered groups

Recall that an ordered group $\mathscr{G}$ is *Archimedean* if for any $g_1, g_2 \in \mathscr{G}$ where $g_1 > 0$, there is a natural number n such that $ng_1 > g_2$. It is well known that any abelian Archimedean ordered group may be embedded into the group of real numbers. Laskowski and Steinhorn [LS95] extended this result using the strong structure of o-minimal ordered groups.

Theorem 2.1.1. *Let $\mathscr{M}$ be an o-minimal expansion of an Archimedean ordered group. Then $\mathscr{M}$ can be elementarily embedded in an o-minimal expansion of the group* $(\mathbb{R}, <, +, 0)$.

In fact, if $c \in \mathscr{M}$ is positive, we will construct a unique elementary embedding of $(\mathscr{M}, c)$ into a unique expansion of $(\mathbb{R}, <, +, 0, 1)$. This has the important consequence that, to study o-minimal expansions of Archimedean ordered groups, we need only study o-minimal expansions of the real group.

Definition 2.1.1. For any structure $\mathscr{M}$, let $\mathrm{qf}(\mathscr{M})$ be the set of elements of $\mathscr{M}$ definable over $\emptyset$ by quantifier free formulas. An ordered structure $\mathscr{M}$ is *standard* if $\mathrm{qf}(\mathscr{M})$ is dense in $\mathscr{M}$.

Lemma 2.1.2. *Let $\mathscr{G} = (G, <, +, 0, 1)$ be o-minimal. Then $\mathscr{G}$ is Archimedean if and only if it is standard.*

Proof. If we embed the rational numbers into $\mathscr{G}$ in the canonical way, note that $\mathrm{qf}(\mathscr{G}) = \mathbb{Q}$. Suppose $\mathbb{Q}$ is dense in $\mathscr{G}$, and let $g_1, g_2 \in \mathscr{G}$, $g_1 > 0$. If there is no positive integer m with $g_2 < m$, then $(g_2, \infty) \cap \mathbb{Q} = \emptyset$, contradicting the hypothesis, so let $g_2 < m$. By the density of the rationals, let $0 < \frac{p}{q} < g_1$ (where p and q are positive integers). From the Archimedean property of the integers, there is an n with $qm < pn$. Thus $\frac{m}{n} < \frac{p}{q} < g_1$, and so $g_2 < m < ng_1$.

Now suppose that $\mathscr{G}$ is Archimedean, and let $a < b$ be elements of $\mathscr{G}$. Also suppose, without loss of generality, that $a > 0$. Then there is some natural number n such that $n(b-a) > 1$, and thus $1+na < nb$. Also, $na > 0$, so we may choose a natural number m with $m-1 \leqslant na < m$. Then $na < m \leqslant na+1 < nb$, so $a < \frac{m}{n} < b$, as desired.

❑

Notice that if $\mathscr{M}$ and $\mathscr{N}$ are elementarily equivalent, there is a unique elementary map $f_0 : \mathrm{qf}(\mathscr{M}) \to \mathrm{qf}(\mathscr{N})$.

Lemma 2.1.3. *If $\mathscr{M}$ and $\mathscr{N}$ are elementarily equivalent and o-minimal, f_0 is the map above, and $\mathscr{N}$ is standard, then an order preserving map $f : \mathscr{M} \to \mathscr{N}$ extending f_0 is an elementary embedding.*

Proof. Note that f would be trivially injective, and unique by the density of $\mathrm{qf}(\mathscr{N})$. All that remains to be shown is that if $\varphi(\bar{x})$ is any formula (without parameters), and $\bar{a} \in \mathscr{M}$, then $\mathscr{M} \models \varphi(\bar{a})$ implies $\mathscr{N} \models \varphi(f(\bar{a}))$. We will proceed by induction on n, the number of free variables in φ. If $n = 0$, the claim reduces to $\mathscr{M} \equiv \mathscr{N}$.

Suppose the remark is true for all formulae with at most n free variables, and let $\varphi(x, \bar{y})$ have $n + 1$. As a first case, suppose that $\varphi(\mathscr{M}, \bar{b})$ is a singleton, $\{a\}$. Then, by the induction hypothesis on $\exists^{=1}x\varphi(x, \bar{y})$, $\varphi(\mathscr{N}, f(\bar{b}))$ is a singleton. Suppose $c \in \mathscr{N}$, and $c < f(a)$. Then, by the density of $\mathrm{qf}(\mathscr{N})$, let $q \in \mathrm{qf}(\mathscr{N}) \cap (c, f(a))$. As $q < f(a)$, $q^{\mathscr{M}} < a$, where by $q^{\mathscr{M}}$ we mean the interpretation of q in $\mathscr{M}$. If $\psi(x)$ is a quantifier (and parameter) free formula defining q, then, $\mathscr{M} \models \neg\exists x\exists y(x < y \wedge \psi(y) \wedge \varphi(x, \bar{b}))$. But then $\mathscr{N} \models \neg\exists x\exists y(x < y \wedge \psi(y) \wedge \varphi(x, f(\bar{b})))$ by the induction hypothesis, and so $\mathscr{N} \models \neg\varphi(c, f(\bar{b}))$. Similarly, if $f(a) < c$, $\mathscr{N} \models \neg\varphi(c, f(\bar{b}))$, and so we must have $\mathscr{N} \models \varphi(f(a), f(\bar{b}))$.

Now suppose that $\varphi(\mathscr{M}, \bar{b})$ defines an interval (a_1, a_2) in $\mathscr{M}$. Let $\psi_1(x, \bar{b})$ say that x is the infimum of realizations of $\varphi(\mathscr{M}, \bar{b})$, and $\psi_2(x, \bar{b})$ that x is the supremum. By the above, applied to ψ_1 and ψ_2, $f(a_1)$ is the infimum of $\varphi(\mathscr{N}, f(\bar{b}))$, and $f(a_2)$ the supremum. But

$$\mathscr{M} \models \forall x\forall y\forall z((x < y < z \wedge \varphi(x, \bar{b}) \wedge \varphi(z, \bar{b})) \to \varphi(y, \bar{b}))$$

so by the induction hypothesis

$$\mathcal{N} \models \forall x \forall y \forall z((x < y < z \wedge \varphi(x, f(\bar{b})) \wedge \varphi(z, f(\bar{b}))) \rightarrow \varphi(y, f(\bar{b}))),$$

whence $\varphi(\mathcal{N}, f(\bar{b}))$ is convex, and equal to $(f(a_1), f(a_2))$. By the order preserving property, then, $\mathcal{M} \models \varphi(a, \bar{b})$ implies $\mathcal{N} \models \varphi(f(a), f(\bar{b}))$.

Finally, let $\varphi(\mathcal{M}, \bar{b})$ be any finite union of points and intervals, defined by $\psi_1(x, \bar{b}), \psi_2(x, \bar{b}), \ldots, \psi_k(x, \bar{b})$. By the induction hypothesis, $\varphi(\mathcal{N}, f(\bar{b}))$ is the union of the sets defined by $\psi_1(x, f(\bar{b})), \ldots, \psi_k(x, f(\bar{b}))$, and by the above work, $\mathcal{M} \models \psi_i(a, \bar{b})$ if and only if $\mathcal{N} \models \psi_i(f(a), f(\bar{b}))$, for each i. So we are done. ❑

Definition 2.1.2. A type $p \in S_1(\mathcal{M})$ is an *irrational cut* if

$$C = \{c : (c < x) \in p(x)\}$$

is a nonprincipal Dedekind cut, that is, if C has no least upper bound. Any other nonprincipal type in $S_1(\mathcal{M})$ is a *noncut*. A type $p \in S_1(\mathcal{M})$ is *uniquely realizable* if for any $\mathcal{N} \succcurlyeq \mathcal{M}$ and any $a \in p(\mathcal{N})$, p is realized by only a in some (the, up to isomorphism) prime model over $\mathcal{M} \cup \{a\}$.

The following lemma is from [Mar86].

Lemma 2.1.4. *Suppose $\mathcal{M}$ is o-minimal, and $p, q \in S_1(\mathcal{M})$ (not necessarily distinct). If $\mathcal{M}' \succcurlyeq \mathcal{M}$ is a model prime over a realization a of p and $b \in q(\mathcal{M}')$ is not a then there is an $\mathcal{M}$-definable function f such that $f(a) \neq a$ realizes q.*

Proof. The Omitting Types Theorem (see [Hod93]) shows that b must realize an atomic type[1] over $\mathcal{M} \cup \{a\}$, and so (by o-minimality) we must have either

[1]A type in which one formula implies all others.

$b \in \mathrm{dcl}(\mathscr{M} \cup \{a\})$ or there is some $\mathscr{M} \cup \{a\}$-atomic interval (β_1, β_2) containing b. In the latter case, note that β_1, β_2, and b must all realize the same type over $\mathscr{M}$. Otherwise there is some $\mathscr{M}$-definable interval $I \subseteq (\beta_1, \beta_2)$, contradicting either the atomicity of (β_1, β_2) in $\mathscr{N}$ or the non-principality of q. Thus there is some realization of q in $\mathrm{dcl}(\mathscr{M} \cup \{a\})$. But we cannot have either β_1 or β_2 definable over $\mathscr{M}$, or $\beta_i \in \mathscr{M}$ and q is not a type. So without loss of generality, assume $b \in \mathrm{dcl}(\mathscr{M} \cup \{a\}) \setminus \mathrm{dcl}(\mathscr{M})$. By the proof of the exchange law, we see that there is an $\mathscr{M}$-definable interval I and a function $f : I \to \mathscr{M}'$ such that $f(a) = b$. ❑

In the above we can, of course, assume that f is a bijection preserving or reversing order by the monotonicity lemma.

Lemma 2.1.5. *Let $\mathscr{M}$ be an o-minimal structure on a dense subset of $\mathbb{R}$ (with the usual ordering), and suppose that every irrational cut in $\mathscr{M}$ is uniquely realizable. Then $\mathscr{M}$ has a unique extension to $\mathbb{R}$.*

Proof. We use Zorn's lemma. Let $\mathscr{M} \preccurlyeq \mathscr{N} \subset \mathbb{R}$, and let $a \in \mathbb{R} \setminus \mathscr{N}$. We claim that the prime model $\mathscr{N}'$ over $\mathscr{N} \cup \{a\}$ may be (order) embedded in $\mathbb{R}$ over $\mathscr{N}$. If this is true, then any maximal elementary extension of $\mathscr{M}$ on a subset of $\mathbb{R}$ must have all of $\mathbb{R}$ as its underlying set. The uniqueness is then not hard to prove.

Claim. *Let $\mathscr{N}$ and $\mathscr{N}'$ be as above. Then $\mathscr{N}'$ realizes no non-cut q over $\mathscr{N}$.*

Proof of the claim. Suppose $\mathscr{N}'$ realizes some non-cut $q \in S_1(\mathscr{N})$. By Lemma 2.1.4 there is an $\mathscr{N}$-definable function $f : I \to \mathscr{N}'$ such that $f(a)$ realizes q.

Case 1: $q = \{``n < x < r_q" : m < r_q, n \in \mathcal{N}\}$ for some $r_q \in \mathcal{N}$. Without loss of generality, assume that f is increasing, and let $I = (\alpha, \beta)$ (where $\alpha, \beta \in \mathcal{N}$). As f is $\mathcal{N}$-definable and $r_q \in \mathcal{N}$, let $c \in \mathcal{N}$ satisfy $a < c < f^{-1}(r_q)$. Then $\mathcal{N}$, and so $\mathcal{N}'$, models $\forall x(\alpha < x < c \rightarrow f(x) < f(c) < q)$. But $a < c$, and so $f(a) < f(c) < q$, and so it is false that $q(f(a))$.

Case 2: $q = \{``n < x" : n \in \mathcal{N}\}$. Again assume that f is increasing, and let $I = (\alpha, \beta)$. Again, as p is a cut, pick some $c \in (a, \beta) \cap \mathcal{N}$. Because f is increasing, $\mathcal{N}$ (and hence $\mathcal{N}'$) models $\forall x(\alpha < x < c \rightarrow f(x) < f(c))$. But then $f(a)$ cannot realize q.

The other two cases (for the other two types of noncut) are the same. ❑

The following is assumed without mention in [LS95], but we feel that it is not trivial, and warrants proof.

Claim. *Suppose $\mathcal{M} \preccurlyeq \mathcal{N} \subseteq \mathbb{R}$. Then every cut in $\mathcal{N}$ is uniquely realizable.*

Proof of the claim. First we prove that if $\mathcal{N}_1$ has this property, then a model $\mathcal{N}_2$ prime over a realization of some cut $p \in S_1(\mathcal{N}_1)$ has the property as well.

Let $a \in p(\mathcal{N}_2)$, and let $b \in \mathcal{N}_2 \setminus \mathcal{N}_1$. Then, using Lemma 2.1.4 we can find a function f such that f is a bijection between realizations of p and realization of the type of b over $\mathcal{N}_1$. In particular, as a is the only realization of p, b is the only realization of its type, $f(a) = b$, and $b \in \mathrm{dcl}(\mathcal{N}_1 \cup \{a\})$. Now let $\mathcal{N}_3$ be the prime model over $\mathcal{N}_2 \cup \{b\}$, and let there be another realization in $\mathcal{N}_3$ of the type of b over $\mathcal{N}_2$. Then we can find a function f_2 such that $f_2(b) \neq b$ realizes this type. As f_2 is definable over $\mathcal{N}_2 = \mathrm{dcl}(\mathcal{N}_1 \cup \{a\})$ we can find a function $g(x, y)$ such that $g(a, y) = f_2(y)$. Assume (by refining if necessary) that $g(x, y) \neq y$. Then set $h(x) = f^{-1} \circ g(x, f(x))$. We can see that h is $\mathscr{N}_1 \cup \{a\}$-definable, and that $h(a) \neq f(a) = b$ are two distinct realizations of $\mathrm{tp}(b/\mathscr{N}_1)$. This is a contradiction.

By induction, if $A \subseteq \mathbb{R}$ is a finite set then the prime model over $\mathscr{M} \cup A$ has the property in question.

Now, suppose $\mathscr{M} \preccurlyeq \mathscr{N} \subseteq \mathbb{R}$ has a cut p which is not uniquely realizable. Again, there is an $\mathscr{N}$-definable function f such that in any $\mathscr{N}' \succcurlyeq \mathscr{N}$, if $p(a)$ then $p(f(a))$ ($f(a) \neq a$). Let $A \subseteq \mathbb{R}$ be a minimal set of parameters used to define f. As A is finite, the prime model over $\mathscr{M} \cup A$ has uniquely realizable cuts. But f contradicts this, by the density of $\mathscr{M}$ in $\mathbb{R}$. ❑

So now we are essentially done. If $\mathscr{M} \preccurlyeq \mathscr{N} \subseteq \mathbb{R}$ then we have shown that the prime model over $\mathscr{N}$ realizes only cuts, and realizes them uniquely. Each cut p over $\mathscr{N}$ can be realized by a unique real r, specifically the supremum of $\{c : c < x \in p(x)\}$.

Now let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two elementary expansions of $\mathscr{M}$ to $\mathbb{R}$, and let $\mathscr{R}_i' = (\mathscr{R}_i, a : a \in \mathscr{M})$. Then clearly $\mathscr{R}_1' \equiv \mathscr{R}_2'$ and both are standard. By Lemma 2.1.3, applied to the identity map from $\mathscr{M}$ to $\mathscr{M}$, we have that $\mathscr{R}_1 = \mathscr{R}_2$. ❑

Lemma 2.1.6. *Let $\mathscr{M}$ be an o-minimal expansion of $\mathscr{G} = (G, <, +, 0, 1)$. Then $\mathscr{G}$ is Archimedean if and only if every irrational cut in $\mathscr{M}$ is uniquely realizable.*

Proof. First we will assume that $\mathscr{G}$ is Archimedean, and suppose that the cut $p \in S_1(\mathscr{M})$ is not uniquely realizable. Let $f : I \to I$ be the function described in Lemma 2.1.4 and assume, without loss of generality, that f is increasing.

Set $g(x) = f(x) - x$. We can assume again, without loss of generality, that $g(x) > 0$ on I. Let $C = \{c \in \mathscr{M} : c < x \in p(x)\}$.

Claim. *For all $\varepsilon > 0$ there is a $c \in C$ such that $x > c$ implies $g(x) < \varepsilon$.*

This claim clearly implies that for all $x \notin C$, $g(x) \leqslant 0$, which is a contradiction.

Proof of the claim. Let $\varepsilon > 0$ and let $c_0 \in C$ be arbitrary. As $\mathscr{G}$ is Archimedean, let $n \in \omega$ be the least n such that $c_0 + (n+1)\varepsilon \notin C$. Since we must have $f(C) \subseteq C$, $x > c_0 + n\varepsilon \to g(x) < \varepsilon$, or else $f(x) - x = g(x) \geqslant \varepsilon$, and so $f(x) \geqslant x + \varepsilon \notin C$. ❑

Now suppose that $\mathscr{G}$ is not Archimedean, and select $a > 0$ and $b > 0$ such that $b < na$ for no $n \in \omega$. Now let $C = \{na : n \in \omega\}$, and let p be the cut determined by C, i.e., the unique type extending $\{c < x : c \in C\}$. We can see that p is a cut, and not a non-cut, as if $x < d \in p(x)$, $x < d - a$ is as well. But if p is realized by d in some extension of $\mathscr{M}$, p is also realized by $a + d$, and so p is not a uniquely realizable type. ❑

This proves the Theorem 2.1.1. If $\mathscr{M} = (\mathscr{M}, +, <, 0, 1, ...)$ is as in the theorem, and $\mathscr{N} = (\mathscr{M}, +, <, 0, 1)$ is the reduct of $\mathscr{M}$ to the language of groups, then $\mathscr{N} \equiv (\mathbb{R}, +, <, 0, 1)$ as both are divisible, abelian ordered groups on dense linear orderings. By quantifier elimination, $\mathrm{qf}(\mathscr{N}) = \{q \cdot 1 : q \in \mathbb{Q}\}$, and so the unique order-preserving extension of $f_0(q \cdot 1) = q$ is an elementary embedding f of $\mathscr{N}$ into $(\mathbb{R}, +, <, 0, 1)$. If $\mathscr{M}'$ is the expansion of the image

$f(\mathscr{N})$ which is isomorphic to $\mathscr{M}$, Lemma 2.1.6 and Lemma 2.1.5 give us an elementary extension of $\mathscr{M}'$ to $\mathbb{R}$.

2.2 Semilinear and semibounded sets

We will be viewing the real numbers in several different ways: as a group, a vector space, a field, or a few other, unnamed, structures. We will always, however, include the order on $\mathbb{R}$, and use properties of o-minimality.

Let $\mathscr{L}$ be the structure $(\mathbb{R}, <, +, 0, 1, \lambda_a : a \in \mathbb{R})$, where $\lambda_a(x) = ax$. This is the set of real numbers viewed as an ordered vector space over the field of real numbers. One can look at the additive group of real numbers as the vector space of real numbers over the field of rationals.

Definition 2.2.1. A set $X \subseteq \mathbb{R}^n$ is *semilinear* if it is definable in $\mathscr{L}$.

Theorem 2.2.1. *$\mathscr{L}$ is exactly the structure generated on $\mathbb{R}$ by the sets*

$$\{\bar{x} \in \mathbb{R}^n : \sum a_i x_i = b\},$$

as b and the a_i run through $\mathbb{R}$.

Proof. One direction is trivial. The other direction requires a quantifier elimination result.

For an arbitrary subfield[2] $F \subseteq \mathbb{R}$, let T_F be the axioms for the group $(\mathbb{R}, +, 0, 1, <)$ plus the universal closures of the following axioms:

V1 $\lambda_a(x + y) = \lambda_a(x) + \lambda_a(y)$

[2]We are proving a more general result which will be used later.

V2 $\lambda_{a+b}(x) = \lambda_a(x) + \lambda_b(x)$

V3 $\lambda_{ab}(x) = \lambda_a(\lambda_b(x))$

for all $a, b \in F$.

We now suppose we have some $\mathscr{N} \models T_F$ and some $\mathscr{M} \leqslant \mathscr{N}$. As the additional axioms here are universal, $\mathscr{M} \models T_F$. Now suppose $\mathscr{M} \leqslant \mathscr{N}$, $\mathscr{M}, \mathscr{N} \models T_F$, and $\varphi(x)$ is a quantifier free formula with parameters from $\mathscr{M}$ with $\varphi(\mathscr{N}) \neq \emptyset$. As in the proof in Section 1.2, we will assume that $\varphi(x)$ is a conjunction of atomic formulas of the form $\lambda_a(x) + b = 0$ or $\lambda_a(x) + b > 0$. Of course, if there are any formulas of the first form then $\varphi(\mathscr{N}) = \varphi(\mathscr{M})$ as, if $x \in \varphi(\mathscr{N})$, $x = \lambda_{-1/a}(b) \in \mathscr{M}$. But the set of solutions to a system of equations of the second form will be an interval with endpoints in $\mathscr{M}$, and so again we have $\varphi(\mathscr{M}) \neq \emptyset$. So T_F is complete. Also, F, as a vector space over itself, must certainly embed into each model of T_F, so the theory admits elimination of quantifiers as well. Taking $F = \mathbb{R}$ we have the result we require. ❑

It was conjectured by van den Dries that there is no structure properly in between (in the sense of reduction) $\mathscr{L}$ and $\mathscr{F}$. This question was answered negatively by Pillay, Steinhorn, and Scowcroft.

Definition 2.2.2. Suppose X is definable in the structure $(\mathbb{R}, +, <, B_i : i \in I)$, where $\{B_i : i \in I\}$ is the collection of all bounded subsets of $\mathbb{R}^n$ for any n. Then we say that X is semibounded. We denote by $\mathscr{B}$ the expansion of $\mathscr{L}$ generated by all bounded semialgebraic sets (that is, the structure whose definable sets are exactly the semibounded semialgebraic sets)[3].

[3]Perhaps not a priori true, but true.

As we will see in subsequent sections, it is possible to characterize $\mathscr{B}$ as the structure $(\mathbb{R}, +, *, <)$ where $*$ is the restriction of multiplication to $[0,1]^2$. In this language one can prove elimination of quantifiers, although we will not do so here.

It is clear that $\mathscr{B}$ is a proper expansion of $\mathscr{L}$ (by the quantifier elimination result above), but what is not clear is that $\mathscr{B}$ is a proper reduct of $\mathscr{F}$. This was shown by Pillay, Steinhorn, and Scowcroft in [PSS87]. The following result gives a simple proof. Once one has shown elimination of quantifiers (also in [PSS87]) it is simple to show directly that any curve (in $\mathbb{R}^2$) definable in this structure is semilinear outside of some bounded set, which gives another proof that multiplication is not definable.

Theorem 2.2.2. *Let $\mathscr{M} \preccurlyeq \mathscr{N}$ be two ω-saturated o-minimal structures, and φ be a formula with parameters from $\mathscr{M}$ such that $\varphi(\mathscr{M}) = \varphi(\mathscr{N})$. Then if $X \subseteq \varphi(\mathscr{M})$, $(\mathscr{M}, X) \preccurlyeq (\mathscr{N}, X)$.*

Notice that the above theorem does not assume that X has any nice properties of its own. A lemma is needed:

Lemma 2.2.3. *Let $\mathscr{M}$ be an o-minimal structure, X an $\emptyset$-definable subset of $\mathscr{M}$, and $\bar{a} \in \mathscr{M}^n$. Then there is some finite $A \subseteq X$ such that whenever $\bar{a}$ and $\bar{b}$ share the same type over A, they share the same type over X.*

Proof. We will first assume that $\bar{a}$ is algebraically independent over X and show that if $\mathrm{tp}(\bar{a}/\emptyset) = \mathrm{tp}(\bar{b}/\emptyset)$ then $\mathrm{tp}(\bar{a}/X) = \mathrm{tp}(\bar{b}/X)$. We will derive the result from this.

Let $\ell(\bar{a})$ denote the length of the tuple $\bar{a}$. We will proceed by induction on

$\ell(\bar{a})$. If $\ell(\bar{a}) = 1$ then $\bar{a}$ is a single element a. Assuming that $\mathrm{tp}(a/\emptyset) = \mathrm{tp}(b/\emptyset)$ we will show, now by induction on n, that for any formula φ with $n+1$ free variables and no parameters,

$$\varphi(a, \bar{c}) \leftrightarrow \varphi(b, \bar{c}), \ \forall \bar{c} \in X^n.$$

Starting with $n = 1$, let $C = \{c \in X : \varphi(a, c)\}$, and let $C^* = \mathrm{cl}(C) \setminus \mathrm{int}(C)$. Note that as X is $\emptyset$-definable, both C and C^* are $\{a\}$-definable. Suppose C^* is not $\emptyset$-definable. Then there is some $d \in C^*$ such that $d \in \mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$. Note that d cannot be in $X^* = \mathrm{cl}(X) \setminus \mathrm{int}(X)$, as this set is $\emptyset$-definable and finite, so $d \in X$. By the exchange law, $a \in \mathrm{acl}(d)$. But this contradicts our assumption of algebraic independence as $d \in X$. Thus C^*, and so C, is $\emptyset$-definable, and the statement holds for $n = 1$. Now suppose the statement holds for n, and let $\bar{c} \in X^{n+1}$, and let φ have $n+2$ free variables. Again we assume that $\mathrm{tp}(a/\emptyset) = \mathrm{tp}(b/\emptyset)$. Let $C = \{c' \in X : \varphi(a, \bar{c}, c')\}$, and C^* be as before. The points in C^* are now $\{a, \bar{c}\}$-definable. Suppose that there is some $d \in C^*$ which is not $\{\bar{c}\}$-definable. Then $d \in \mathrm{acl}(a, \bar{c}) \setminus \mathrm{acl}(\bar{c})$. The exchange law says that $a \in \mathrm{acl}(d, \bar{c}) \setminus \mathrm{acl}(\bar{c}) \subseteq \mathrm{acl}(X)$. Again this is a contradiction. So C is $\bar{c}$-definable and, by the induction hypothesis, $\varphi(a, \bar{c}, c') \leftrightarrow \varphi(b, \bar{c}, c')$.

Now suppose that the result holds when $\ell(\bar{a}) = n$, and let $\bar{a}, a' \in \mathscr{M}$, algebraically independent over X. If $\mathrm{tp}(\bar{a}, a'/\emptyset) = \mathrm{tp}(\bar{b}, b'/\emptyset)$ then the induction hypothesis tells us that $\mathrm{tp}(\bar{a}/X) = \mathrm{tp}(\bar{b}/X)$. Then there is some elementary extension, say $\mathscr{N}$, of $\mathscr{M}$ and an automorphism g of $\mathscr{N}$ such that $g(\bar{b}) = \bar{a}$ (see [Hod93]). So we need only prove that $\mathrm{tp}(\bar{a}, a'/X) = \mathrm{tp}(\bar{a}, g(b')/X)$. But, adding new symbols to the language for $\bar{a}$, this reduces to the previous case (where $\ell(\bar{a}) = 1$).

We will now show that, given any $\bar{a} \in \mathscr{M}$ and any $\emptyset$-definable $X \subseteq \mathscr{M}$ there is some finite (possibly empty) $A \subseteq \mathscr{M}$ such that $\bar{a}$ is algebraically independent over X in $(\mathscr{M}, A)$. The lemma will then follow as the type of $\bar{a}$ over A in $\mathscr{M}$ is the type of $\bar{a}$ over $\emptyset$ in $(\mathscr{M}, A)$. Again we can proceed by induction on $\ell(\bar{a})$. If $\ell(\bar{a}) = 1$ and $\bar{a} = a$ is algebraically dependent over X, $a \in \mathrm{acl}(X)$. By the properties of algebraic closure there is some finite $A \subseteq X$ such that $a \in \mathrm{acl}(A)$. Now suppose the result is true for $\ell(\bar{a}) = n$, and let $\bar{a}, a' \in \mathscr{M}$. By the induction hypothesis there is some finite $A_0 \subseteq X$ such that $\bar{a}$ is independent over X in $(\mathscr{M}, A_0)$. If $\bar{a}, a'$ is independent over X in this structure then we are done. Otherwise, $a' \in \mathrm{acl}(\bar{a} \cup X)$. But then there is some finite $A_1 \subseteq (\bar{a} \cup X)$ such that $a' \in \mathrm{acl}(A_1)$. Let $A = A_0 \cup (A_1 \cap X)$. ❑

Proof of Theorem 2.2.2. Let $\mathbb{F}$ be the set of functions from $\varphi(\mathscr{M}) \cup \{a_0, ..., a_k\} \subseteq \mathscr{M}$ to $\varphi(\mathscr{N}) \cup \{b_0, ..., b_k\} \subseteq \mathscr{N}$ such that

a. f is the identity on $\varphi(\mathscr{M})$

b. $\mathrm{tp}(\bar{a}/\varphi(\mathscr{M})) = \mathrm{tp}(\bar{b}/\varphi(\mathscr{N}))$

c. $\forall i, f(a_i) = b_i$.

We will show that, for every $f \in \mathbb{F}$ and $a' \in \mathscr{M}$ there is an extension $g \in \mathbb{F}$ of f such that $a' \in \mathrm{dom}(g)$. We will also show that for every $b' \in \mathscr{N}$ there is an extension $h \in \mathbb{F}$ of f such that $b' \in \mathrm{rng}(h)$. As the identity map on $\varphi(\mathscr{M})$ demonstrates that $\mathbb{F} \neq \emptyset$, $\mathbb{F}$ forms a back and forth system[4] which fixes $X \subseteq \varphi(M)$, and consequently $(\mathscr{M}, X)$ is elementarily embedded in $(\mathscr{N}, X)$.

[4] See [Hod93]

If $a' \in \mathscr{M}$, and $f \in \mathbb{F}$, we need to find a $b' \in \mathscr{N}$ such that $\mathrm{tp}(\bar{a}, a'/\varphi(\mathscr{M})) = \mathrm{tp}(\bar{b}, b'/\varphi(\mathscr{N}))$. By the lemma above, there is a finite $A \subseteq \varphi(\mathscr{M}) = \varphi(\mathscr{N})$ such that if $tp(\bar{a}, a'/A) = \mathrm{tp}(\bar{c}, c'/A)$, $tp(\bar{a}, a'/\varphi(\mathscr{N})) = \mathrm{tp}(\bar{c}, c'/\varphi(\mathscr{N}))$. As $\mathscr{N}$ is ω-saturated, we can find some $b' \in \mathscr{N}$ such that $\mathrm{tp}(b'/A \cup \{b_0, ..., b_k\}) = \mathrm{tp}(a'/A \cup \{a_0, ..., a_k\})$, and we may extend f by adding the point (a', b'). The construction of h is identical (resting on the ω-saturation of $\mathscr{M}$). ❑

By a pole in a structure $\mathscr{R}$ we mean a bijection between a bounded interval (an interval with endpoints in $\mathscr{R}$) and an unbounded interval (an interval with an endpoint $= \pm\infty$). If a structure $\mathscr{R}$ on $\mathbb{R}$ defines multiplication then it must define a pole, namely $x \mapsto 1/x$ (from $(0,1)$ to $(1,\infty)$).

Corollary 2.2.4. *If $X \subseteq \mathbb{R}^n$ is bounded then $(\mathbb{R}, +, 1, <, X)$ defines no pole.*

Proof. Let $\mathscr{M}$ be an ω-saturated extension of $(\mathbb{R}, +, 1, <)$ and $\mathscr{N} = \mathscr{M} \times \mathscr{M}$ equipped with the product group operation and lexicographical order. If we identify $\mathscr{M}$ and $\mathscr{M} \times \{0\}$, it is true that $\mathscr{M} \preccurlyeq \mathscr{N}$ (as both are divisible, abelian ordered groups and so eliminate quantifiers). It is simple to show that $\mathscr{N}$ is ω-saturated, in particular because it is definable in $\mathscr{M}$. Now, if X is a bounded set, $X \subseteq [-c, c]^n$ for some $c \in \mathscr{M}$. As $\mathscr{N}$ is an end-extension of $\mathscr{M}$, the interval $[-c, c]$ is the same in both structures, so $(\mathscr{N}, X)$ is elementarily equivalent to $(\mathscr{M}, X)$. But if $(\mathscr{M}, X)$ defines a pole, say $f(x)$, $(\mathscr{N}, X)$ must as well. By the cell decomposition theorem we can assume that f is a decreasing bijection, and by scaling we may assume that $f : (0, \varepsilon) \to (a, \infty)$, $\varepsilon < c$. But then if $a \in \mathscr{N}$ is larger than any element of $\mathscr{M} \subseteq \mathscr{N}$, $f^{-1}(a)$ must be smaller than any positive element of $\mathscr{M}$ but still positive. This contradicts the fact that

$$\{x \in \mathscr{M} : 0 < x < \varepsilon\} = \{x \in \mathscr{N} : 0 < x < \varepsilon\}.$$

❑

Strictly speaking, this does not prove our result. But building a similar end-extension of $\mathscr{L}$ and using the same proof we get our result.

2.3 Defining fields

The goal of the next two sections is to prove that the only nontrivial reduct of $\mathscr{F}$ which properly expands $\mathscr{L}$ is $\mathscr{B}$.

In what follows we will need to know something about the smoothness of functions definable in o-minimal expansions of $(\mathbb{R}, +, <)$.

Theorem 2.3.1 (Laskowski, Steinhorn [LS95]). *If $f : (a, b) \to \mathbb{R}$ is definable in some o-minimal expansion of $(\mathbb{R}, +, <)$ then f is piecewise C^n, for all n.*

Proof. We define a sequence of functions by

$$\begin{aligned} \Delta_h^0 f(x) &= f(x) \\ \Delta_h^{k+1} f(x) &= \Delta_h^k f(x+h) - \Delta_h^k f(x). \end{aligned}$$

Intuitively, $\Delta_h^k f(x) \approx h^k f^{(k)}(x)$. We also define the formula $\Phi_{f,k,a,b}$ to be

$$\forall x \forall h(a < x < x + kh < b \rightarrow \Delta_h^k f(x) \geqslant 0) \\ \vee \forall x \forall h(a < x < x + kh < b \rightarrow \Delta_h^k f(x) \leqslant 0).$$

It is a theorem of Boas and Widder [BW40] that if $\Phi_{f,k,a,b}$ holds, $f^{(k-2)}$ exists and is continuous on (a, b). We will first show that there is a subinterval (c, d) of (a, b) such that $\Phi_{f,k,c,d}$. Let

$$X = \{(x, h) : x \in (a, b), h \in (0, (b - x)/(n + 2))\},$$

and let

$$X_+ = \{(x, h) \in X : \Delta_h^{n+2} f(x) \geqslant 0\}$$
$$X_- = \{(x, h) \in X : \Delta_h^{n+2} f(x) < 0\}.$$

By cell decomposition, there is a partition of X into cells which partitions both X_+ and X_-. We can clearly find a cell which contains a subset of the form

$$\{(x, h) : x \in (c, d), h \in (0, c - x/(n + 2))\}$$

for some (c, d). So then $\Phi_{f,n+2,c,d}$. So now we know that the set on which f is not n-times differentiable is finite, or it would contain an interval and the above argument would show a contradiction. ❑

We will also need to refer to the following theorem of semialgebraic geometry, by Pillay and Nesin [NP91]. By a one-dimensional semialgebraic field, we mean a field definable in $\mathscr{F}$ whose underlying set is one-dimensional.

Theorem 2.3.2. *Let F be a one-dimensional semialgebraic field. Then there is a semialgebraic isomorphism $f : \mathbb{R} \to F$.*

Note that the theorem implies that a one-dimensional semialgebraic field is pure. If $X \subseteq F^n$ is semialgebraic then $f^{-1}(X) \subseteq \mathbb{R}^n$ is semialgebraic (as f is). But then $X = f(f^{-1}(X))$ is definable in F, by f being an isomorphism.

In this section we will show that if a reduct of $\mathscr{F}$ is not a reduct of $\mathscr{L}$, one can use this structure define on an interval of $\mathbb{R}$ a real closed field (with the usual ordering). This field will allow us to construct all semibounded sets. We first need a reduction lemma.

Lemma 2.3.3 (Laskowski, Steinhorn [LS95]). *Suppose an o-minimal expansion $\mathscr{R}$ of $\mathscr{L}$ defines no function $f : \mathbb{R} \to \mathbb{R}$ which is not semilinear. Then it defines no semialgebraic subsets of $\mathbb{R}^n$, for any n, which are not semilinear.*

Proof. By cell decomposition we need to show that every cell definable in $\mathscr{R}$ is semilinear. It suffices to show that for every semilinear cell C and continuous, definable function $f : C \to \mathbb{R}$, the graph of f is semilinear, and we will prove this by induction on the dimension of the cell. For zero-dimensional cells (singletons), the result is trivial. We saw in Chapter 1 that if $C \subseteq \mathbb{R}^n$ is a semilinear cell then there is a semilinear cell in $\mathbb{R}^{\dim(C)}$ to which C is semilinearly homeomorphic, so we need only prove our claim for open cells in $\mathbb{R}^n$. Assume the claim holds for all cells of dimension less than n, and let $C \subseteq \mathbb{R}^n$ be an open cell, $f : C \to \mathbb{R}$ definable and continuous.

Let $B \subseteq \mathbb{R}^{n-1}$ be an open box, and $a, b \in \mathbb{R}$ such that $B \times (a, b) \subseteq C$. Write f as $f(\bar{x}, y)$, and for each $\bar{c} \in B$ let $g_{\bar{c}} = f(\bar{c}, y)$. By the induction hypothesis, each $g_{\bar{c}}$ is semilinear, or piecewise linear, on (a, b). Let $h(\bar{x})$ be the least $d \in (a, b)$ such that $g_{\bar{x}}$ is linear on (a, d). Again, h is semilinear by the induction hypothesis. So there is a $B_0 \subseteq B$ and a $d \in (a, b)$ such that each $g_{\bar{c}}$, $\bar{c} \in B_0$, is linear on (a, d). Let $m(\bar{c})$ be the slope of $g_{\bar{c}}$. This is definable as $\frac{1}{r}(g_{\bar{c}}(\alpha + r) - g_{\bar{c}}(\alpha))$ for some fixed $\alpha \in (a, d)$ and sufficiently small rational r. Again, m is semilinear. We reduce to a smaller box, $B_1 \subseteq B_0$ on which m is linear,

$$m(\bar{x}) = \sum \lambda_{a_i}(x_i) + d.$$

As $g_{\bar{c}}$ is linear on (a, d), we can find a definable function e such that

$$g_{\bar{x}}(y) = m(\bar{x})y + e(\bar{x}).$$

Consider the function $k(\bar{x}, y) = f(\bar{x}, y) - \lambda_d(y) - e(\bar{x})$. This function, on $B_1 \times (a, d)$ is definable and equal to $(\sum \lambda_{a_i} x_i)y$. If $a_i \neq 0$ for some i, then $k(0, ..., 0, \lambda_{1/a_i} x, 0, ..., 0, x)$ defines $x \mapsto x^2$ on some interval, contradicting the induction hypothesis. So m is constant on B_1, and $f(\bar{x}, y) = \lambda_m(t) + e(\bar{x})$, and e is semilinear by the induction hypothesis, therefore f is semilinear.

The above shows that given any open subset U of C there is an open subset of U on which f is semilinear. Using cell decomposition, then, there is a partition of C into cells such that on all open cells f is semilinear. On non-open cells the induction hypothesis applies. ❑

Theorem 2.3.4 (Marker, Pillay, Peterzil [MPP92]). *If $X \subseteq \mathbb{R}^n$ is not semilinear then there is a real-closed field on some subinterval of $\mathbb{R}$ whose ordering is the usual ordering and which is definable in $(\mathscr{L}, X)$.*

To prove this theorem we will also need the following lemma on analytic functions:

Lemma 2.3.5. *Suppose f and g are analytic on $(-\alpha, \alpha)$ and $f(0) = g(0) = 0$, $f'(0) = g'(0)$ and $f''(0) \neq 0$. Then there is a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an $x \in (-\delta, \delta)$ such that*

$$f(x + \varepsilon) = g(x) + f(\varepsilon).$$

Proof of Theorem 2.3.4. We will assume that X is a curve, and in particular the null set of a nonlinear function $F(x, y)$. We write $F'(p)$, where $p \in \mathbb{R}^2$, for the slope of F at p. Because F is a non-linear function, we can find an interval in the set of $F'(p)$, and a rational number α such that $F'(p) = \alpha$ and $F''(p) \neq 0$. We may then translate F linearly that $F(0,0) = 0$, $F'(0,0) = 0$, $F''(0,0) \neq 0$, and the range of F' contains an interval around 0. Using the implicit function theorem[5] we may shrink to an interval $(-a, a)$ on which X is the graph of some analytic function, g. By shrinking again if required, we can also assume that g' is an injection from $(-a, a)$ into $(-b, b)$ for some $0 < b < 1$, $g'' \neq 0$ on $(-a, a)$.

Claim. *There is some $c \in (0, a)$ and two definable functions, A and M, such that for all $x, y \in (-c, c)$,*

$$g'(A(x,y)) = g'(x) + g'(y)$$

$$g'(M(x,y)) = g'(x)g'(y).$$

Given the claim we will finish the result by defining a field of fractions using A and M. Let $Z = \{(a, b) \in (-c, c)^2 : b \neq 0\}$, and define $(a_0, b_0) \sim (a_1, b_1)$ if and only if $M(a_0, b_1) = M(a_1, b_0)$. Let $Y = Z/\sim$, let $\langle a, b\rangle$ denote the residue of (a, b) in Y, and define two operations on Y by

$$\langle a_0, b_0\rangle \oplus \langle a_1, b_1\rangle = \langle \text{half}(A(M(a_0, b_1), M(a_1, b_0)), \text{half}(M(b_0, b_1))\rangle,$$

$$\langle a_0, b_0\rangle \otimes \langle a_1, b_1\rangle = \langle M(a_0, a_1), M(b_0, b_1)\rangle,$$

[5] See [Che01].

where $\mathrm{half}(x)$ is the unique y such that $A(y,y) = x$. Then $(Y, \oplus, \otimes)$ is a real closed field. To show this we will simply prove that

$$\sigma(\langle a,b\rangle) = g'(a)/g'(b)$$

is an isomorphism from $(Y, \oplus, \otimes)$ to $(\mathbb{R}, +, \cdot)$. From this we can then use Lemma 1.5.2 to find a definable transversal for $\sim$ in $\mathbb{R}^2$ making $(Y, \oplus, \otimes)$ definable in $(\mathbb{R}, +, <, X)$.

First we should note that

$$\begin{aligned} \langle a,b\rangle = \langle c,d\rangle &\leftrightarrow M(a,d) = M(c,b) \\ &\leftrightarrow g'(M(a,d)) = g'(M(c,d)) \\ &\leftrightarrow g'(a)g'(d) = g'(c)g'(d) \\ &\leftrightarrow \sigma(\langle a,b\rangle) = \sigma(\langle c,d\rangle). \end{aligned}$$

So σ is well defined and injective. If we fix some $\beta \in (0,c)$, and $0 < x < 1$, pick α such that $g'(\alpha) = xg'(\beta)$, and then $\sigma(\langle \alpha, \beta\rangle) = x$. If $g'(\beta) < x$, choose α so that $g'(\alpha) = \frac{1}{x}g'(\beta)$, and $\sigma(\langle \beta, \alpha\rangle) = x$. Thus, multiplying by -1 if required, σ is onto. Notice that $x = \mathrm{half}(y)$ iff $A(x,x) = y$ iff $g'(x) = \frac{1}{2}g'(y)$. Now, finally,

$$\begin{aligned}
\sigma(\langle a_0, b_0\rangle \otimes \langle a_1, b_1\rangle) &= \sigma(\langle M(a_0, a_1), M(b_0, b_1)\rangle) \\
&= g'(M(a_0, a_1))/g'(M(b_0, b_1)) \\
&= (g'(a_0)g'(a_1))/(g'(b_0)g'(b_1)) \\
&= \sigma(\langle a_0, b_0\rangle)\sigma(\langle a_1, b_1\rangle), \\
\sigma(\langle a_0, b_0\rangle \oplus \langle a_1, b_1\rangle) &= (\frac{1}{2}g'(A(M(a_0, b_1), M(a_1, b_0))))/(\frac{1}{2}M(b_0, b_1)) \\
&= \frac{g'(a_0)g'(b_1) + g'(a_1)g'(b_0)}{g'(b_0)g'(b_1)} \\
&= \sigma(\langle a_0, b_0\rangle) + \sigma(\langle a_1, b_1\rangle).
\end{aligned}$$

Now all that remains is to check the claim.

Proof of the claim. Let $c \in (0, a)$ be small enough that for any $u, v \in (-c, c)$ there are $y, z \in (-a, a)$ such that

$$g'(u) + g'(v) = g'(y)$$
$$g'(u)g'(v) = g'(z).$$

For each $\alpha \in (-c, c)$ let

$$g_\alpha = g(x + \alpha) - g(\alpha).$$

Then we want $A(u, v)$ to be the unique w such that for all sufficiently small δ and all sufficiently small ε, $g_u + g_v$ and $g_{w+\varepsilon}$ have at least two points of intersection with x-coordinate in $(-\delta, \delta)$. So $A(u, v) = w$ will be equivalent to the formula

$$\exists\delta_0 > 0\forall\delta \in (0, \delta_0)\exists\varepsilon_0 > 0\forall\varepsilon \in (0, \varepsilon_0)$$
$$((g_u + g_v) \cap g_{w+\varepsilon} \cap ((-\delta, \delta) \times \mathbb{R}) \text{ contains} \geqslant 2 \text{ points}),$$

taking the functions to be set of ordered pairs.

Our claim is, of course, that this w is precisely the w for which $g'(u) + g'(v) = g'(w)$. Note that this is the unique w for which $g_u + g_v$ and g_w are tangential at 0. For any ε,

$$g_w(x+\varepsilon) - g_w(\varepsilon) = g(x+w+\varepsilon) - g(w) - g(w+\varepsilon) + g(w) = g_{w+\varepsilon}(x).$$

So Lemma 2.3.5 guarantees that w will satisfy the formula above, taking $f = g_w$, $g = g_u + g_v$.

Now suppose z is some number other than w above. Then $g_u + g_v$ is not tangential to g_z, and so we can see that for small enough ε, $g_u + g_v$ and $g_{z+\varepsilon}$ have only one point of intersection near 0.

We can define $M(u, v)$ in the same way, using $g_v \circ g_u$. ❑

❑

Note in particular that if X is semialgebraic but not semilinear, we may define in $(\mathscr{L}, X)$ a real-closed field on some interval $I \subseteq \mathbb{R}$. Every $\mathbb{R}$-semialgebraic subset of I must also be I-semialgebraic as I is a pure field. By linear translations we may construct in $(\mathscr{L}, X)$ any bounded semialgebraic set.

2.4 Nonlinear structures

Our next theorem towards this classification of reducts was shown by Peterzil [Pet92].

Theorem 2.4.1. *Let $\mathscr{C} \subseteq \mathbb{R}^2$ be a curve definable in some o-minimal expansion of $\mathscr{F}$. Then if $\mathscr{C} \setminus I$ is not semilinear for any bounded interval $I \subseteq \mathbb{R}$, a bijection between a bounded and an unbounded interval is definable in $(\mathbb{R}, +, <, \mathscr{C})$.*

To prove this we will need a few lemmas about growth rates of real functions. We will say that f grows like g towards ∞, denoted $f \overset{\infty}{\sim} g$, if

$$\lim_{x \to \infty} f(x)/g(x) = c \neq 0.$$

We will say that f grows less than g towards ∞, $f \overset{\infty}{\ll} g$, if

$$\lim_{x \to \infty} f(x)/g(x) = 0.$$

Note that $\overset{\infty}{\sim}$ is an equivalence relation and $\overset{\infty}{\ll}$ a partial ordering. Moreover, if f and g are definable in o-minimal structures, the monotonicity theorem gives us the trichotomy $f \overset{\infty}{\ll} g$, $g \overset{\infty}{\ll} f$, or $f \overset{\infty}{\sim} g$.

Lemma 2.4.2. *If $l : (a, \infty) \to \mathbb{R}$ is definable in an o-minimal expansion of $\mathscr{F}$ and $\lim_{x \to \infty} l(x) = c$ then $\lim_{x \to \infty} xl'(x) = 0$ and $\lim_{x \to \infty} x^2 l''(x) = 0$.*

Proof. Of course we are done if $l'(x) = 0$ eventually. We can assume, by the monotonicity theorem, that l and l' are eventually strictly monotone (l' is also definable). By the mean value theorem, there is some $y_x \in (x, 2x)$ (for any x) such that $l(2x) - l(x) = xl'(y_x)$. Thus $\lim_{x \to \infty} xl'(y_x) = 0$. By the eventual monotonicity of l' we have either

$$xl'(x) < xl'(y_x) < xl'(2x)$$

or

$$xl'(x) > xl'(y_x) > xl'(2x)$$

for large enough x. But $\lim_{x \to \infty} xl'(x) = 2 \lim_{x \to \infty} xl'(2x)$, so if the first limit is negative, so is $\lim_{x \to \infty} xl'(y_x)$, by the squeeze theorem. Similarly, the first limit cannot be positive. Thus $\lim_{x \to \infty} xl'(x) = 0$.

Now we apply the above result to $xl'(x)$. This tells us that

$$\begin{aligned} 0 &= \lim_{x\to\infty} x(xl'(x))' \\ &= \lim_{x\to\infty} x(l'(x)+xl''(x)) \\ &= \lim_{x\to\infty} xl'(x) + \lim_{x\to\infty} x^2 l''(x), \end{aligned}$$

and so we are done. ❑

Lemma 2.4.3. *Let $f : (a,\infty) \to \mathbb{R}$ be definable in some o-minimal expansion of $\mathscr{F}$ and let $\alpha \in \mathbb{Q}$. Then*

- *if $\alpha \neq 0$ and $f \overset{\infty}{\approx} x^\alpha$, then $f(x+1) - f(x) \overset{\infty}{\approx} x^{\alpha-1}$*
- *if $f \overset{\infty}{\ll} x^\alpha$, then $f(x+1) - f(x) \overset{\infty}{\ll} x^{\alpha-1}$*
- *if $\lim_{x\to\infty} f(x) = \infty$, $\alpha > 0$, and $x^\alpha \ll f$, then $f^{-1} \ll x^{1/\alpha}$.*

Proof. Let $l(x) = f(x)/x^\alpha$. By the mean value theorem we can find, for each x, a $y_x \in (x, x+1)$ such that $f(x+1) - f(x) = f'(y_x)$. Thus

$$\begin{aligned} \frac{f(x+1)-f(x)}{x^{\alpha-1}} &= \frac{f'(y_x)}{x^{\alpha-1}} \\ &= \frac{\alpha y_x^{\alpha-1} l(y_x) + y_x^\alpha l'(y_x)}{x^{\alpha-1}} \\ &= \alpha l(y_x)\frac{y_x^{\alpha-1}}{x^{\alpha-1}} + y_x l'(y_x)\frac{y_x^{\alpha-1}}{x^{\alpha-1}} \end{aligned}$$

which tends to the same thing as $\alpha l(y_x)$ by the previous lemma and because $y_x/x \to 1$. But in the first case $\alpha l(y_x) \to \alpha c \neq 0$ and in the second case $\alpha l(y_x) \to 0$, so the first two statements are true.

Now notice that, in the third claim, as f is increasing, so is f^{-1}. Thus we have (under the hypothesis)

$$\begin{aligned} 0 &= \lim_{x\to\infty} \frac{x^\alpha}{f(x)} \\ &= \lim_{x\to\infty} \frac{(f^{-1}(x))^\alpha}{f(f^{-1}(x))} \\ &= \left(\lim_{x\to\infty} \frac{f^{-1}(x)}{x^{1/\alpha}}\right)^\alpha, \end{aligned}$$

and so we are done. ❑

We may now complete the proof of the theorem.

Proof of Theorem 2.4.1. Let $\mathscr{C}$ be non-linear outside of every bounded rectangle, as in the theorem. We may assume, without loss of generality, that $\mathscr{C}$ is the graph of some eventually non-linear function f. We may also assume that f has no vertical or horizontal asymptotes, or the monotonicity theorem would allow us to find a bounded interval on which either f or f^{-1} would be a pole. As f is definable in an o-minimal expansion of the reals, we know that either $f(x) \overset{\infty}{\approx} x$, $f(x) \overset{\infty}{\ll} x$, or $x \overset{\infty}{\ll} f(x)$. In the final case Lemma 2.4.3 allows us to conclude that $f^{-1}(x) \overset{\infty}{\ll} x$, so without loss of generality we may assume that one of the first two cases occurs.

If $f(x) \overset{\infty}{\approx} x$, then Lemma 2.4.3 tells us that $f(x+1) - f(x) \overset{\infty}{\approx} x^0 = 1$. Thus $\lim_{x\to\infty} f(x+1) - f(x) = c \neq 0$. Suppose $f(x+1) - f(x)$ is eventually constant. Then, fixing some x_0 after which $f(x+1) - f(x)$ is constant, we know that there is, by the mean value theorem, some $y_{x_0} \in (x_0, x_0+1)$ such that $f'(y_{x_0}) = f(x_0+1) - f(x_0) = c$. There is also some $y_{x_0+1} \in (x_0+1, x_0+2)$ such that $f'(y_{x_0+1}) = c$, and so on. As $\{x : f'(x) = c\}$ is definable, and contains arbitrarily large reals, f' is eventually constant. But this contradicts the non-linearity of f. The $f(x+1)-f(x)$ is non-constant, and $\lim_{x\to\infty} f(x+1)-f(x) = c$, and so we have a horizontal asymptote.

Now, if $f(x) \overset{\infty}{\ll} x$ we have $f(x+1)-f(x) \overset{\infty}{\ll} 1$, so $\lim_{x\to\infty} f(x+1)-f(x) = 0$. As above, $f(x+1)-f(x)$ is not eventually constant, and so we are done. ❑

With one further lemma this theorem will allow us to conclude that there is no proper reduct of $\mathscr{F}$ which properly expands $\mathscr{B}$. This lemma gives us insight into the structure of semibounded sets definable in a certain class of o-minimal structures. We say that an o-minimal structure $\mathscr{R}$ satisfies the partition condition if for any $\mathscr{R}$-definable set $X \subseteq \mathbb{R}^n$ there are disjoint analytic sets (perhaps not definable) $X_1, ..., X_n \subseteq X$ such that $X \setminus (\bigcup X_i)$ has no interior in X. It is well known that $\mathscr{F}$ and $(\mathbb{R}, +, \cdot, <, x \mapsto e^x)$ satisfy the partition condition.

Lemma 2.4.4 (Peterzil, [Pet92]). *Let X be a definable in an o-minimal expansion of the reals satisfying the partition condition and suppose that every curve (in $\mathbb{R}^2$) definable in $(\mathbb{R}, +, <, X)$ is semilinear outside of some rectangle. Then there are bounded sets $B_1, ..., B_k$ and scalings $\lambda_{a_1}, ..., \lambda_{a_k}$ definable in $(\mathbb{R}, +, <, X)$ such that X is definable in $(\mathbb{R}, +, <, \lambda_{a_1}, ..., \lambda_{a_j}, B_1, ..., B_k)$. In particular, X is semibounded.*

We know already that X being semibounded implies that every curve definable in $(\mathbb{R}, +, <, X)$ is semilinear outside of some rectangle. Otherwise, the lemma above would allow us to construct a pole from X, which is in turn constructed from a bounded set (or several bounded sets, but the difference is superficial) which would contradict an earlier result. So this lemma proves, for example, that the structure of semibounded semialgebraic sets is the same as the structure generated over $\mathscr{L}$ by the bounded semialgebraic sets: if X is semialgebraic and semibounded then $(\mathbb{R}, +, <, X)$ is a reduct of $\mathscr{F}$, and the bounded sets needed to define X are definable in this reduct of $\mathscr{F}$. It also proves that if X is not semibounded, then a curve such as that needed in the hypothesis of the lemma above is definable, offering a converse to the statement made above.

Proof. In this proof we will treat tuples of reals as vectors to ease notation. In particular, if $\bar{a}, \bar{b}$ are n-tuples, $\langle \bar{a}, \bar{b} \rangle = \sum_{i=1}^{n} a_i b_i$. A set X definable in an o-minimal expansion $\mathscr{R}$ of $(\mathbb{R}, +, <)$ is said to be almost linear if there is a bounded $\mathscr{R}$-cell C and vectors $\bar{v_1}, ..., \bar{v_k} \in \mathbb{R}^n$ such that

$$\begin{aligned} X &= \left\{ \bar{c} + \sum_i t_i \bar{v_i} : t_i > 0 \forall i, \bar{c} \in C \right\} \\ &= C + \text{span}^+\{\bar{v_1}, ..., \bar{v_k}\}. \end{aligned} \tag{2.1}$$

If the vectors are linearly independent and the scalars representing each point are unique we say that X is in normal form. If X is as above, we let $\hat{X}$ be the set defined as in 2.1 but with the scalars possibly 0. If $f : X \to \mathbb{R}$ is a continuous function (and X is almost linear), we say that (X, f) is almost linear if f can be extended to $\hat{f} : \hat{X} \to \mathbb{R}$ in such a way that $\hat{f}$ is bounded on C and there are scalars $\bar{a}$ with

$$\hat{f}\left(\bar{c} + \sum_i t_i \bar{v_i} \right) = \hat{f}(\bar{c}) + \langle \bar{t}, \bar{a} \rangle.$$

Note that if (X, f) is almost linear, then the graph of f over X is almost linear, as exhibited by the cell C', the graph of $\hat{f}$ over C, and the vectors $\bar{v_1}', ..., \bar{v_k}'$

which are obtained by concatenating $\bar{v}_i$ and a_i for each i, denoted $(\bar{v}_i, a_i)$. Also, if Y is an almost linear subset of X then (Y, f) is also almost linear.

Fix the structure $\mathscr{R}$ meeting the conditions of the theorem. We will show that, if X is a cell, X can be written as a finite disjoint union of almost linear sets in normal form. This will, of course, cover the case of any definable set. We will prove this by induction on the n such that $X \subseteq \mathbb{R}^n$. If X is a bounded cell, then the statement is trivial. In particular, if $X \subseteq \mathbb{R}$ we need only consider the case where $X = (a, \infty)$ (or $(-\infty, a)$) in which case $X = \{a\} + \text{span}^+\{1\}$ (or $\{a\} + \text{span}^+\{-1\}$). The induction step will be demonstrated modulo the following claim, which is the real content of the result.

Claim. *Suppose Y is almost linear in normal form, and $f : Y \to \mathbb{R}$ is definable and continuous. Then there is a partition $Y_1, ..., Y_k$ of Y such that (Y_i, f) is almost linear[6] for each i.*

Proceeding with the induction, suppose that the claim is true for all subsets of $\mathbb{R}^n$, and let $X \subseteq \mathbb{R}^{n+1}$. There are two cases for $\dim(X)$:

If $\dim(X) < n + 1$ then X is (up to permutation of co-ordinates) the graph of a continuous, definable function g on some cell $Y \subseteq \mathbb{R}^n$. By the induction hypothesis and the claim, we may partition Y into almost linear sets in normal form, and then partition those sets further until we have X represented as the union of the graphs of g over some Y_i where each pair (Y_i, g) is almost linear.

If $\dim(X) = n+1$ then X is the region between the graphs of two functions

[6]We will be sloppy with domains of functions in this proof.

$g, h : Y \to \mathbb{R}$, where $Y \subseteq \mathbb{R}^n$ is an open cell. By the induction hypothesis and the claim, we may partition $Y = Y_1 \cup ... \cup Y_k$ such that (Y_i, g) is almost linear for each i. We may then partition each $Y_i = Y_{i,1} \cup ... \cup Y_{i,l}$ so that $(Y_{i,j}, h)$ is almost linear for all i, j. By the comment at the end of the first paragraph of this proof, $(Y_{i,j}, g)$ is still almost linear. So assume without loss of generality that

$$Y = C + \text{span}^+\{\bar{v}_1, ..., \bar{v}_k\}$$
$$g\left(\bar{c} + \sum_i t_i\bar{v}_i\right) = g(\bar{c}) + \langle \bar{a}, \bar{t}\rangle$$
$$h\left(\bar{c} + \sum_i t_i\bar{v}_i\right) = h(\bar{c}) + \langle \bar{b}, \bar{t}\rangle$$

for some tuples $\bar{a}, \bar{b}$. Note that we are dropping the distinction between g and $\hat{g}$. As $g < h$ on Y, continuity tells us that $g \leqslant h$ on C. In fact, $g < h$ on C. This implies that $a_i \leqslant b_i$ for all i. If $a_i = b_i$ for all i then

$$X = C^* + \text{span}^+\{(\bar{v}_1, a_1), ..., (\bar{v}_k, a_k)\},$$

where C^* is the cell between the graph of g and the graph of h over C. The new vectors are clearly linearly independent, as their projections to a lower dimension are. If $(\bar{c}, d) + \sum t_i\bar{v}_i{}^* = (\bar{c}', d') + \sum t_i'\bar{v}_i{}^*$ for some $\bar{c}$, d, etc, then, because Y is in normal form, $\bar{c} = \bar{c}'$ and $t_i = t_i'$ for all i. But then $d + \langle \bar{t}, \bar{a}\rangle = d' + \langle \bar{t}', \bar{a}\rangle$, and so $d = d'$. Thus points in X have unique representation.

If $a_i < b_i$ for some i then we will re-arrange indices such that $a_i < b_i$ for all $i \leqslant l$ and $a_i = b_i$ for all $i > l$. By definition,

$$X = \left\{(\bar{c} + \sum_{i=1}^{k} t_i\bar{v}_i, y) : \bar{c} \in C, t_i > 0\forall i, y \in (g(\bar{c}) + \langle \bar{a}, \bar{t}\rangle, h(\bar{c}) + \langle \bar{b}, \bar{t}\rangle)\right\}.$$

We will definably split X into three pieces: X_1, X_2, X_3. In each the defining condition will remain the same except for the range of ys.

$$X_1 = \left\{ (\bar{c} + \sum_{i=1}^{k} t_i \bar{v}_i, y) : \bar{c} \in C, t_i > 0 \forall i, y \in (h(\bar{c}) + \langle \bar{a}, \bar{t} \rangle, h(\bar{c}) + \langle \bar{b}, \bar{t} \rangle) \right\}$$

$$X_2 = \left\{ (\bar{c} + \sum_{i=1}^{k} t_i \bar{v}_i, y) : \bar{c} \in C, t_i > 0 \forall i, y = h(\bar{c}) + \langle \bar{b}, \bar{t} \rangle \right\}$$

$$X_3 = \left\{ (\bar{c} + \sum_{i=1}^{k} t_i \bar{v}_i, y) : \bar{c} \in C, t_i > 0 \forall i, y \in (g(\bar{c}) + \langle \bar{a}, \bar{t} \rangle, h(\bar{c}) + \langle \bar{a}, \bar{t} \rangle) \right\}.$$

X_2 is the graph of an almost linear function over Y and so is almost linear in normal form. Also, X_3 is the case above. All that remains to be shown is that X_1 can be written as an almost linear set in normal form.

Let

$$Z = \left\{ (\sum t_i \bar{v}_i, y) : y \in (\langle \bar{a}, \bar{t} \rangle, \langle \bar{b}, \bar{t} \rangle), t_i > 0 \right\}.$$

Then X_1 is the graph of h over C shifted by Z. In particular, if Z is almost linear in normal form, so is X_1. Let $\bar{\alpha_1} = (a_1, a_2, ..., a_k) = (a_1, ..., a_{l-1}, b_l, ..., b_k)$, $\bar{\alpha_2} = (a_1, ..., a_{l-2}, b_{l-1}, ..., b_k)$ and so on. Then by construction,

$$\langle \bar{\alpha_1}, \bar{t} \rangle < \langle \bar{\alpha_2}, \bar{t} \rangle < ... < \langle \bar{\alpha_l}, \bar{t} \rangle,$$

for $\bar{t} \in (\mathbb{R}^+)^k$. If we set

$$A_{\bar{\beta}}^{\bar{\alpha}} = \left\{ (\sum t_i \bar{v}_i, y) : y \in (\langle \bar{\alpha}, \bar{t} \rangle, \langle \bar{\beta}, \bar{t} \rangle) \right\}$$

then

$$Z = \bigcup_{i=1}^{l-1} A_{\bar{\alpha}_{i+1}}^{\bar{\alpha}_i} \cup \bigcup_{i=1}^{l-1} \left\{ (\sum t_i \bar{v}_i, \langle \bar{\alpha}_i, \bar{t} \rangle) \right\}.$$

The second type of set is clearly almost linear in normal form. So we need only show that the $A^{\bar{\alpha_i}}_{\alpha_{\bar{i}+1}}$ are as well. Suppose $y \in (\langle \bar{\alpha_i}, \bar{t}\rangle, \langle \alpha_{\bar{i}+1}, \bar{t}\rangle)$. Then

$$a_1t_1 + ... + a_{l-i}t_{l-i} + b_{l-i+1}t_{l-i+1} + ... + b_kt_k$$
$$< y < a_1t_1 + ... + a_{l-i-1}t_{l-i-1} + b_{l-i}t_{l-i} + ... + b_kt_k.$$

Writing $y_2 = y - (a_1t_1 + ... + a_{l-i-1}t_{l-i-1} + b_{l-i+1}t_{l-i+1} + ... + b_kt_k)$ we have $a_{l-i}t_{l-i} < y_2 < b_{l-i}t_{l-i}$. Choose $0 < t' < 1$ such that $y_2 = a_{l-i}t_{l-i}t' + b_{l-i}t_{l-i}(1-t')$. Now

$$(\sum t_i\bar{v}_i, y)$$
$$-(t_1(\bar{v_1}, a_1) + ... + t_{l-i-1}(v_{l-\bar{i}-1}, a_{l-i-1}) + t_{l-i+1}(v_{l-\bar{i}+1}, b_{l-i+1}) + ... + t_k(\bar{v_k}, a_k))$$
$$= t_{l-i}(v_{\bar{l}-i}, y_2)$$
$$= t't_{l-i}(v_{\bar{l}-i}, a_{l-i}) + (1-t')t_{l-i}(v_{\bar{l}-i}, b_{l-i}).$$

So $A^{\bar{\alpha_i}}_{\alpha_{\bar{i}+1}}$ is the positive span of $(\bar{v}_j, a_j)$, for $0 \leqslant j \leqslant l-i$, and $(\bar{v}_j, b_j)$, for $l-i \leqslant j \leqslant k$.

Proof of the claim. Again we proceed by induction, this time on $\dim(Y)$ and the n such that $Y \subseteq \mathbb{R}^n$, and again we assume that Y is a cell. If $\dim(Y) < n$, then Y is the graph of some function g on a cell $Z \subseteq \mathbb{R}^{n-1}$. Let $h : Z \to \mathbb{R}$ be given by $h(x_1, ..., x_{n-1}) = f(x_1, .., x_{n-1}, g(x_1, ..., x_{n-1}))$. By the induction hypothesis we can partition Z into Z_i such that (Z_i, g) and (Z_i, h) are almost linear. Fixing an index, let

$$Z_j = \left\{ \bar{c} + \sum t_i \bar{v}_i : \bar{c} \in C, t_i > 0 \right\}$$

$$g(\bar{c} + \sum t_i \bar{v}_i) = g(\bar{c}) + \langle \bar{a}, \bar{t} \rangle$$

$$h(\bar{c} + \sum t_i \bar{v}_i) = h(\bar{c}) + \langle \bar{b}, \bar{t} \rangle.$$

Of course, $\bar{a}$, $\bar{b}$, and the $\bar{v}_i$ depend on j. Let

$$Y_j = \{(\bar{c}, g(\bar{c})) + \sum t_i(\bar{v}_1, a_i) : \bar{c} \in C, t_i > 0\}.$$

By construction, on Y_j we have

$$\begin{aligned} f((\bar{c}, g(\bar{c})) + \sum t_i(\bar{v}_1, a_i)) &= f(\bar{c} + \sum t_i \bar{v}_i, g(\bar{c}) + \langle \bar{t}, \bar{a} \rangle) \\ &= f(\bar{c} + \sum t_i \bar{v}_i, g(\bar{c} + \sum t_i \bar{v}_i)) \\ &= h(\bar{c} + \sum t_i \bar{v}_i) \\ &= h(\bar{c}) + \langle \bar{b}, \bar{t} \rangle \\ &= f(\bar{c}) + \langle \bar{b}, \bar{t} \rangle. \end{aligned}$$

Also, as the Z_i partition Z, the Y_i partition Y.

Now suppose that $\dim(Y) = n$. By the partition condition there are open, connected (not necessarily definable) sets $U_1, ..., U_r$ and analytic functions $f_1, ..., f_r$ defined on the U_i such that $f_i = f_j$ on $U_i \cap U_j$, for all i and j, and $Y \setminus \bigcup U_i$ is contained in a definable set of dimension less than n. We will abbreviate this by saying $\dim(Y \setminus \bigcup U_i) < n$. We will prove the result by induction on $r(Y, f)$, the minimum number of U_i required. If $r(Y, f) = 0$ then $\dim(Y) < n$, so this case has been dealt with. Now let $Y = C + \text{span}^+\{\bar{v}_1, ..., \bar{v}_k\}$. For each $\bar{t} \in (\mathbb{R}^+)^{k-1}$ and $\bar{c} \in C$, set

$$F_{(\bar{c},\bar{t})}(t) = f(\bar{c} + \sum_{i=1}^{k} t_i\bar{v}_i + t\bar{v}_k).$$

It is our aim to use these functions to show that f is essentially linear in the $\bar{v}_k$ direction. We will then apply induction to complete the proof. By hypothesis, $F_{\bar{c},\bar{t}}$ (for fixed $\bar{c}$ and $\bar{t}$) is eventually linear. Let $l(\bar{c},\bar{t})$ be the least t such that for all $s > 0$ and $t_1, t_2 > t$,

$$F_{(\bar{c},\bar{t})}(t_1 + s) - F_{(\bar{c},\bar{t})}(t_1) = F_{(\bar{c},\bar{t})}(t_2 + s) - F_{(\bar{c},\bar{t})}(t_2).$$

This is the (or a) point after which $F_{(\bar{c},\bar{t})}$ is linear, and the function is definable. Finally, let $s(\bar{c},\bar{t})$ be the eventual slope of $F_{(\bar{c},\bar{t})}$. This is definable as

$$F_{(\bar{c},\bar{t})}(t+1) - F_{(\bar{c},\bar{t})}(t),$$

for some $t > l(\bar{c},\bar{t})$.

We claim that the range of s is finite. If not, then it contains some interval. As $\dim(Y) = \dim(C) + k$, we will assume, to simplify matters, that C is an open subcell of $\mathbb{R}^{n-k}$. If it is not we can find a definable homeomorphism between C and such a cell, and examine the images of s, etc through this homeomorphism. So s is defined on $C \times (\mathbb{R}^+)^{k-1}$. By cell decomposition we can find an open, connected cell V on which s is continuous and has infinite range. Suppose that for all i and $a_1, a_2, ..., a_{n-1} \in C \times (\mathbb{R}^+)^{k-1}$, $s^*(x) = s(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_{n-1})$ has finite range. Then, by continuity and by following paths along axes, we can show that $s(\bar{a}) = s(\bar{a}')$ for any $\bar{a}, \bar{a}' \in V$. This is a contradiction, so for some $a_1, ...,$ $s^*(x)$ has infinite range. Choose, by the monotonicity lemma, some $[a,b] \subseteq \mathbb{R}$ on which s^* is strictly monotone, and let $M = \sup\{l(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_{n-1}) : x \in [a, b]\}$. Because there are no poles definable in the structure, M is finite. For any $x \in [a, b]$, $F_{(a_1,\ldots,x,\ldots,a_{n-1})}(t + M)$ is defined on $\mathbb{R}^+$ and is linear (in t) with slope $s^*(x)$. Let

$$g_0(x, y) = F_{(a_1,\ldots,x,\ldots,a_{n-1})}(y + M) = s^*(x)y + g_0(x, 0).$$

Now let $g_1(x, y) = g_0(x, y) - g_0(x, 0) = s^*(x)y$. Now let

$$g_2(x, y) = g_1(s^{*-1}(x), y) = xy.$$

If we reduce $[a, b]$ to a subinterval with rational endpoints, we can now define, using only the group structure,

$$g_3 : [0, b - a] \times \mathbb{R}^+ \to \mathbb{R} \text{ by } g_3(x, y) = g_2(x + a, y) - ay,$$

as a and b are rational. Again, $g_3(x, y) = xy$, now defined on $[0, b - a]$. We can define $g_4(x)$ on $[1/(b - a), \infty)$ to be the (unique) y such that $g_3(x, y) = 1$. This is a pole (namely $x \mapsto 1/x$).

So s has finite range, say $\{m_1, ..., m_p\}$. Let m be the least m_i such that $s^{-1}(m_i)$ is $n - 1$-dimensional, and set

$$G = \{\bar{c} + \sum_{i=1}^{k-1} t_i\bar{v}_i + t\bar{v}_k : \bar{c} \in C, t_i, t > 0, F_{(\bar{c},\bar{t})} \text{ is linear with slope } m$$
$$\text{on some interval containing } t\}.$$

Then G is, as shown above, definable. We will show that either $Y = G$ or $r(G, f)$ and $r(Y \setminus G, f)$ are both less than $r(Y, f)$.

Let

$$Y_{\bar{c},\bar{t}} = \{\bar{c} + \sum_{i=1}^{k-1} t_i\bar{v}_i + t\bar{v}_k : t > 0\}.$$

If $(\bar{c},\bar{t}) \neq (\bar{c}',\bar{t}')$ then, by unique representation, $Y_{\bar{c},\bar{t}} \cap Y_{\bar{c}',\bar{t}'} = \emptyset$. Let

$$Z = \{\bar{c} + \sum_{i=1}^{k-1} t_i\bar{v}_i\},$$

and

$$Z_m = \{\bar{c} + \sum_{i=1}^{k-1} t_i\bar{v}_i \in Z : s(\bar{c},\bar{t}) = m\}.$$

If $\bar{c}+\sum_{i=1}^{k-1} t_i\bar{v}_i \in Z_m$ then $Y_{\bar{c},\bar{t}}\cap G$ is infinite. So the $Y_{\bar{c},\bar{t}}\cap G$ with $\bar{c}+\sum_{i=1}^{k-1} t_i\bar{v}_i \in Z_m$ is a family of infinite subsets, and since $\dim(Z_m) = n-1$, the dimension of G must be[7] n.

Suppose that $\dim(Y \setminus G) = n$. Let $U_1, .., U_r$, $f_1, ..., f_r$ be witnesses to $r(Y,f) = r$. As $\dim(G) = n$, there is some i_0 with $\dim(G \cap U_{i_0}) = n$ (that is, this set has non-empty interior). Similarly, there must be some j_0 with $\dim((Y \setminus G) \cap U_{j_0}) = n$. But if, for some given i, $U_i \cap G$ contains an open set then, f_i must be $\bar{v}_k$-linear (with slope m) on $U_i \cap G$ by the defining condition of G. But f_i is analytic on U_i, so the same is true on U_i. In particular, $U_i \cap (Y \setminus G) = \emptyset$, and $i_0 \neq j_0$. Re-arranging the indices, we may assume that, for some s, $i \leqslant s$ implies $U_i \cap Y \subseteq G$, and $s < i$ implies $\dim(U_i \cap G) < n$. Thus

$$\dim((Y \setminus G) \setminus \bigcup_{i=s+1}^{r} U_i) < n$$

$$\dim(G \setminus \bigcup_{i=1}^{s} U_i) < n,$$

[7] See [Pil88].

demonstrating $r(Y \setminus G, f) \leqslant r - s < r$ and $r(G, f) \leqslant s < r$ as desired.

Now suppose that $\dim(Y \setminus G) < n$. In this case we will have $Y = G$, and the result will follow. If, for some $\bar{c}, \bar{t}$ we have $Y_{\bar{c},\bar{t}} \setminus G$ finite then, as $F_{\bar{c},\bar{t}}$ is continuous and linear of slope m on G, $Y_{\bar{c},\bar{t}} \subseteq G$. So we can define the set

$$\begin{aligned} Z_1 &= \left\{ \bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i \in Z : Y_{\bar{c},\bar{t}} \setminus G \text{ is finite} \right\} \\ &= \left\{ \bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i \in Z : Y_{\bar{c},\bar{t}} \not\subseteq G \right\}. \end{aligned}$$

Note that the set of $Y_{\bar{c},\bar{t}} \setminus G$ with $\bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i \in Z_1$ partitions a subset of $Y \setminus G$ which, by hypothesis, has dimension $< n$, so $\dim(Z_1) < n - 1$. Now let

$$Y_1 = \left\{ \bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i + t\bar{v}_k : \bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i \in Z_1, t > 0 \right\}.$$

So $\dim(Y_1) = \dim(Z_1) + 1 < n$. Now if $Y_2 = Y \setminus Y_1$, $Y_2 \subseteq G$ and if $\bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i + t\bar{v}_k \in Y_2$,

$$f(\bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i + t\bar{v}_k) = f(\bar{c} + \sum_{i=1}^{k-1} t_i \bar{v}_i) + tm.$$

But as $\dim(Y_2 \setminus Y) < n - 1$, Y_2 is dense in Y, so the above equation holds throughout Y (as both sides are continuous). By the induction hypothesis, we can assume that (Z, f) is almost linear, and so we are done. ❑

❑

We can now prove that $\mathscr{B}$ is the only structure 'between' $\mathscr{L}$ and $\mathscr{F}$, as observed in [Pet93].

Theorem 2.4.5. *$\mathscr{B}$ is the only proper reduct of $\mathscr{F}$ which properly expands $\mathscr{L}$.*

Proof. Let $\mathscr{R}$ be a reduct of $\mathscr{F}$ which properly expands $\mathscr{L}$. Because $\mathscr{R}$ properly expands $\mathscr{L}$, we know from Theorem 2.3.4 there there is an interval I such that every semialgebraic subset of I^n is $\mathscr{R}$-definable. If X is any bounded semialgebraic subset of $\mathbb{R}^n$ we can linearly translate X into I^n, so X is $\mathscr{R}$-definable as well. Thus $\mathscr{B} \ll \mathscr{R}$. If $\mathscr{R}$ is not $\mathscr{B}$ then the theorems of this section tell us that there is an $\mathscr{R}$-definable bijection from a bounded interval to an unbounded interval. Linearly scaling and pasting as needed, we may assume that there is an $\mathscr{R}$-definable bijection $f : I \to \mathbb{R}$. Now, if X is any semialgebraic subset of $\mathbb{R}^n$, $f^{-1}(X)$ is a semialgebraic subset of I^n, and so is $\mathscr{R}$-definable. So X is $\mathscr{R}$-definable, and $\mathscr{R}$ is $\mathscr{F}$. ❑

2.5 Multiplicative reducts

A similar result to those above can be had when considering reducts of $\mathscr{F}$ which expand $\mathscr{P} = (\mathbb{R}, \cdot, <)$, as in [Pet93]. The only reduct of $\mathscr{F}$ which properly expands $\mathscr{P}$ is $(\mathbb{R}, +^*, \cdot, <)$, where $+^*$ is the restriction of $+$ to the interval $[1, 2]$. In order to prove the following results we will need to assume that the structure $(\mathbb{R}, +, \cdot, x \mapsto e^x)$ is o-minimal. This was shown by Wilkie in [Wil96] by demonstrating that every definable set is the projection of some quantifier-free definable set (model completeness). It also follows that this structure satisfies the partition condition. We will work over $\mathscr{P}^+ = (\mathbb{R}^+, \cdot, <)$ for the most part, but this is for ease of notation. The results transfer readily to $\mathscr{P}$.

Definition 2.5.1. A set $X \subseteq (\mathbb{R}^+)^n$ will be called p-bounded if $X \subseteq [a, b]^n$ for some $0 < a < b$. We call $X \subseteq \mathbb{R}^n$ p-semibounded if it is definable in a structure $(\mathbb{R}, \cdot, <, B)$ where the B is a p-bounded set. We will let $\mathscr{B}_\text{p}$ denote the structure generated on $\mathscr{P}$ by the p-semibounded sets.

Lemma 2.5.1. *Let X be a p-bounded set. Then in $(\mathbb{R}, \cdot, <, X)$ one cannot define a bijection between a p-bounded interval and a p-unbounded interval.*

Proof. Let $\sigma(x) = \ln(x)$. Note that σ provides a natural isomorphism between $\mathscr{P}^+$ and $(\mathbb{R}, +, <)$. Note that $X \subseteq \mathscr{P}^+$ is p-bounded if and only if $\sigma(X)$ is bounded. Consequently, by the isomorphism, X is p-semibounded if and only if $\sigma(X)$ is semibounded. So if there is a bijection f between a p-bounded interval and a p-unbounded interval definable in $(\mathscr{P}^+, X)$ then there is a bijection between a bounded interval and an unbounded interval definable in $(\mathbb{R}, +, <, \sigma(X))$. By Theorem 2.2.2, then, $\sigma(X)$ is not semibounded, whence X is not p-semibounded. ❑

Lemma 2.5.2. *Suppose $X \subseteq (\mathbb{R}^+)^n$ is semialgebraic but not $\mathscr{P}^+$-definable. Then $\sigma(X)$ is not $\mathscr{L}$ definable.*

Proof. Suppose that $\sigma(X)$ is $\mathscr{L}$-definable. Then there are certain $\alpha_1, ..., \alpha_k$ such that $\sigma(X)$ is definable in $(\mathbb{R}, +, \lambda_{\alpha_1}, ..., \lambda_{\alpha_k}, <)$. Notice that σ is a natural isomorphism to this structure from $\mathscr{R} = (\mathbb{R}^+, \cdot, \mu_{\alpha_1}, ..., \mu_{\alpha_k}, <)$, where $\mu_\beta(x) = x^\beta$. By Lemma 2.4.4, the λ_{α_i} are all $(\mathbb{R}, +, <, \sigma(X))$-definable, and so the μ_{α_i} are $(\mathscr{P}^+, X)$-definable. Now, if X is semi-algebraic, the μ_{α_i} need all be semi-algebraic, and so the α_i are all rational. But then the μ_{α_i} are $\mathscr{P}^+$-definable. ❑

Lemma 2.5.3. *Let $X \subseteq (\mathbb{R}^+)^n$ be $\mathscr{F}$-definable. Then if $\sigma(X)$ is semibounded, X is p-semibounded.*

Proof. As X is $\mathscr{F}$-definable, $\sigma(X)$ is $(\mathbb{R},+,\cdot,x\mapsto e^x)$-definable. So the structure $(\mathbb{R},+,<,\sigma(X))$ is o-minimal and satisfies the partition condition. If $\sigma(X)$ is semibounded then there are $\lambda_{\alpha_1},...,\lambda_{\alpha_k}$ and bounded sets $B_1,...,B_l$ definable in $(\mathbb{R},+,<,\sigma(X))$ such that $\sigma(X)$ is definable in

$$(\mathbb{R},+,<,\lambda_{\alpha_1},...,\lambda_{\alpha_k},B_1,...,B_l).$$

If we let $C_i=\sigma^{-1}(B_i)$ for each i, we see as above that X is

$$(\mathbb{R},\cdot,\mu_{\alpha_1},...,\mu_{\alpha_k},C_1,...,C_l)$$

-definable. As above, the C_i are all p-bounded and the μ_{α_i} must all be $\mathscr{P}^+$ definable, and so X is p-semibounded. ❑

Theorem 2.5.4. *The only proper expansion of $\mathscr{P}$ which is a proper reduct of $\mathscr{F}$ is $\mathscr{B}_{\mathrm{p}}$.*

Proof. We let $\mathscr{R}$ be some proper reduct of $\mathscr{F}$ properly expanding $\mathscr{P}$ and demonstrate that $\mathscr{R}=\mathscr{B}_{\mathrm{p}}$. First suppose that $X\subseteq\mathbb{R}^n$ is $\mathscr{R}$-definable but not $\mathscr{P}$-definable. Without loss of generality, we can assume that $X\subseteq(\mathbb{R}^+)^n$. Then $(\mathbb{R},+,<,\sigma(X))$ is o-minimal (it is a reduct of the exponential field) but not semilinear. In particular, we can define a real closed field on some subinterval of $(\mathbb{R},+,<,\sigma(X))$. Let $I=(a,b)$ be the σ^{-1}-image of this interval. By Theorem 2.3.2 every semialgebraic subset of I^n is $(\mathscr{P},X)$-definable. If (γ,δ) is any other interval in $\mathbb{R}^+$, and $\alpha\in(a,b)$ then there is a rational number q sufficiently small such that if $\beta=\alpha(\delta/\gamma)^q$, $(\alpha,\beta)\subseteq(a,b)$. Then

$$y=\gamma\left(\frac{x}{\alpha}\right)^{\frac{\ln(\delta/\gamma)}{\ln(\beta/\alpha)}}$$

is a $\mathscr{P}$-definable bijection from (α, β) to (γ, δ), and so every semialgebraic subset of (γ, δ) is definable. In particular, $\mathscr{B}_{\mathrm{p}} \ll \mathscr{R}$.

Now suppose that some non-p-semibounded X is $\mathscr{R}$-definable. Again we may assume that $X \subseteq (\mathbb{R}^+)^n$, and again we may apply previous theorems to $(\mathbb{R}, +, <, \sigma(X))$. As $\sigma(X)$ is not semibounded we may, in $(\mathbb{R}, +, <, \sigma(X))$, define a bijection between a bounded interval and an unbounded interval. This translates in $(\mathscr{P}, X)$ to a bijection f between a p-bounded interval and a non-p-bounded interval. But any non-p-bounded interval is either of the form (a, ∞), or of the form $(-\infty, -a)$ or contains 0 in its closure. Using $x \mapsto 1/x$ if required, we can assume that $\operatorname{rng}(f) = (a, \infty)$. We may then also construct[8] bijections with range $(0, 1/a)$, $(-1/a, 0)$, and $(-\infty, -a)$. Now any semialgebraic set $Y \subseteq \mathbb{R}^m$ may be divided into $Y \cap (a, \infty)^m$, $Y \cap [1/a, a]^m$, $Y \cap (0, 1/a)^m$, et cetera, each of which is $(\mathscr{P}, X)$-definable. So $\mathscr{R} = \mathscr{F}$. ❑

The lemmas above show that both of these reductions are strict.

2.6 Polynomially unbounded structures

As in the previous section, the natural isomorphism $(\mathbb{R}, +, <) \to (\mathbb{R}^+, \cdot, <)$ yields useful results. We will here prove a result due to Miller [Mil94], but in a form presented by Poston [Pos95].

Theorem 2.6.1. *Let $\mathscr{R}$ be an o-minimal expansion of $\mathscr{F}$ in which some definable function is not bounded by any polynomial. Then $x \mapsto e^x$ is definable in $\mathscr{R}$.*

[8] The function $x \mapsto -x$ is definable as for all $x \neq 0$, $-x$ is the unique y such that $x^2 = y^2$ and $xy < 0$.

Proof. We will prove that $(x, y) \mapsto x^{\ln(y)}$ is definable in $\mathscr{R}$. Then, if $g(x) = x^{\ln x} = e^{(\ln(x))^2}$, $g'(x) = 2\ln(x)e^{(\ln(x))^2}/x$ is definable in $\mathscr{R}$, and then so is $\frac{1}{2}xg'(x)/g(x) = \ln(x)$.

Let $\mathscr{P} = (\mathbb{R}^+, \cdot, <)$, and $\mathscr{G} = (\mathbb{R}, +, <)$. Then $\mathscr{P} \cong \mathscr{G}$ by $x \mapsto \ln(x)$. Now, if f is definable in an o-minimal expansion of $\mathscr{F}$, $(\mathscr{P}^+, f)$ is also o-minimal. Let g be the image of f in $\mathscr{G}$, so that $(\mathscr{P}, f) \cong (\mathscr{G}, g)$. If $g(x) < nx$ for any n, we have $f(x) < x^n$, so g is not linearly bounded. Thus multiplication is definable in $(\mathscr{G}, g)$. Lifting through the isomorphism, $(x, y) \mapsto x^{\ln(y)}$ is definable in $(\mathscr{P}, f)$. ❑

2.7 O-minimal expansions of $(\mathbb{Q}, +, <)$

Though it is widely believed that there is no proper o-minimal expansion of $(\mathbb{Q}, +, <)$, it is not known. The evidence mounting towards this conjecture is quite convincing, though, and the above sections allow us to put some restrictions on possible counterexamples to the conjecture.

Theorem 2.7.1 (Laskowski, Steinhorn [LS95]). *There is no proper semialgebraic o-minimal expansion of* $(\mathbb{Q}, +, <)$.

By a semialgebraic expansion we mean an expansion by a set $X \cap \mathbb{Q}^n$, where $X \subseteq \mathbb{R}^n$ is semialgebraic.

Proof. Suppose X is a semialgebraic set, let $X_{\mathbb{Q}} = X \cap \mathbb{Q}^n$, and suppose that $\mathscr{Q} = (\mathbb{Q}, +, <, X_{\mathbb{Q}})$ is o-minimal. As shown at the beginning of this chapter, $\mathscr{Q}$ can be elementarily embedded in a unique o-minimal expansion $\mathscr{R}$ of $\mathbb{R}$ (unique assuming $1 \mapsto 1$, in which case the embedding is the identity map).

Claim. *For every semialgebraic set $Y \subseteq \mathbb{R}^k$, if $Y_{\mathbb{Q}} = Y \cap \mathbb{Q}^k$ is definable in $\mathscr{Q}$ then the interpretation of $Y_{\mathbb{Q}}$ in $\mathscr{R}$ (denoted $Y_{\mathbb{Q}}^{\mathscr{R}}$) is simply Y.*

Proof. We proceed by induction on the dimension of Y, and restrict our attention to cells. As seen before, we need only demonstrate the case where Y is the graph of a continuous, definable $f : C \to \mathbb{R}$, for some $\dim(Y) - 1$-dimensional cell C. Given the induction hypothesis (which is trivial for 0-dimensional cells), $C_{\mathbb{Q}}^{\mathscr{R}} = C$. Also, $Y_{\mathbb{Q}}$ is the graph of f restricted to $C_{\mathbb{Q}}$. Let $g : C \to \mathbb{R}$ be the function whose graph $Y_{\mathbb{Q}}^{\mathscr{R}}$ is. Then $f = g$ on $C_{\mathbb{Q}}$, a dense subset of C. This tells us that $f = g$ on C (see, for example, [Kel55]). ❑

If X is not semilinear, then multiplication is definable on $[-2, 2]$ in $\mathscr{R}$, and hence in $\mathscr{Q}$. But this is impossible, as $\mathscr{Q} \equiv \mathscr{R}$ would imply that $\mathbb{Q}$ contains a square root for 2. Thus $\mathscr{R}$ is a reduct of $\mathscr{L}$. Again, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then λ_α cannot be $\mathscr{R}$-definable, as $\lambda_\alpha(q)$ is irrational for any rational q. But if α is rational, λ_α is already definable in $(\mathbb{Q}, +, <)$. Thus, if $\mathscr{Q}$ is o-minimal, $\mathscr{Q} = (\mathbb{Q}, +, <)$. ❑

In fact, in light of the theorems above, if $\mathscr{Q}$ is an o-minimal expansion of $(\mathbb{Q}, +, <)$ then $\mathscr{Q} \preccurlyeq \mathscr{R}$, for some o-minimal $\mathscr{R}$ on $\mathbb{R}$. If $\mathscr{Q}$ is proper, $\mathscr{R}$ cannot be semilinear, by the argument above. So there must be be some $\oplus$ and $\otimes$ definable in $\mathscr{Q}$ which make some interval of $\mathscr{Q}$ into a real closed field. Although it seems unlikely, it has not yet been shown that this cannot happen.

2.8 End extensions

We conclude with a simple observation on the nature of non-Archimedean o-minimal groups. Theorem 2.2.2 allowed us to conclude that in any o-minimal structure for which we may construct end-extensions, multiplication cannot be definable. We can give a sort of converse to this in light of the theorems above.

Definition 2.8.1. Let $\mathscr{M}$ be an o-minimal structure. We define two types in $S_1(\mathscr{M})$: $p_\infty(x)$ is the type generated by the formulae $x > a$ for all $a \in \mathscr{M}$, and $p_\varepsilon(x)$ is the type generated by $0 < x < a$ for all $a \in \mathscr{M}$, $a > 0$. We say that $\mathscr{M}$ admits end-extensions if there is an elementary extension of $\mathscr{M}$ in which p_∞ is realized but p_ε is not.

Lemma 2.8.1. *Let $\mathscr{M}$ be an o-minimal expansion of a group. Then if $\mathscr{M}$ does not admit end-extension, there is a definable pole in $\mathscr{M}$.*

Proof. Suppose $\mathscr{M}$ does not admit end-extension. By Lemma 2.1.4 there is a definable function $f : \mathscr{M} \to \mathscr{M}$ such that in any elementary extension of $\mathscr{M}$, $p_\infty(a)$ implies $p_\varepsilon(f(a))$. By cell decomposition there are $-\infty = a_0, ..., a_n = \infty$ such that the restriction of f to (a_i, a_{i+1}) for any i is either constant or a bijection between intervals. If $I = (a_{n-1}, a_n)$ we cannot have f constant on I, as in elementary extensions of $\mathscr{M}$, $f(I)$ contains realizations of p_ε (and in particular elements outside $\mathscr{M}$). So f is a bijection. Similarly, the closure of $f(I)$ must contain 0. Suppose f is increasing, and let $c \in f(I) \cap (0, \infty)$. Then there is a $b \in I$ such that $x > b \to f(x) > c$. But if a realizes p_∞ in an elementary extension of $\mathscr{M}$, $a > b$ and $f(a) < c$. So f is decreasing. Finally, suppose that $\lim_{x \to \infty} f(x) = L < 0$. Then there is some $c \in (L, 0)$ and some $b \in I$ such that $x > b \to f(x) < c$. Again this is a contradiction, and so $\lim_{x \to \infty} f(x) = 0$. Thus f restricted to some subinterval of I is a pole. ❑

Applying the theorems above, if $\mathcal{M}$ is Archimedean, $\mathcal{M}$ is an elementary substructure of some o-minimal expansion of $(\mathbb{R}, +, <)$. Our pole above will allow us to define multiplication in this structure if the corresponding structure on $\mathbb{R}$ is semialgebraic, or at least a real-closed field on some subinterval otherwise. Note that Lemma 2.1.4 also tells us that if $\mathcal{M}$ is end-extendible, there must also be an elementary extension of $\mathcal{M}$ realizing p_ε and not p_∞. Also, every noncut over $\mathcal{M}$ is a translation of p_ε or a reflection across 0 of such a translation, and so an end-extension may realize no non-cuts.

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