THE INFLUENCE OF TENSOR FORCES ON THE DIFFERENTIAL CROSS SECTION FOR THE SCATTERING OF POLARIZED NEUTRON BEAMS BY PROTONS

by

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ABSTRACT

Tensor forces between two particles involve a dependence upon the angle between the direction of spin quantization and the line joining the two particles. The effect of tensor forces upon the scattering of a polarized neutron beam has been investigated theoretically. An expression has been obtained for the differential scattering cross section of the triplet states as a function of the polarization of both neutrons and protons. In general, this cross section is also a function of the azimuthal angle to the direction of propagation of the neutron beam.

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Introduction

In order to explain the electric quadrupole moment of the deuteron, it has been necessary to introduce a tensor interaction potential of the form $S\,V(r)$, where

$$S = 3\frac{(\vec{\sigma}_n \cdot \vec{r})(\vec{\sigma}_p \cdot \vec{r})}{r^2} - \vec{\sigma}_n \cdot \vec{\sigma}_p$$

Rarita and Schwinger [1,2] have calculated the effects of this potential upon both the bound and the unbound states of the neutron-proton system. In particular, they have calculated the scattering cross section of a beam of neutrons by a proton target for neutrons of low energy. These calculations were extended by Ashkin and Wu [3], who used a more general phase shift analysis. Recently, Rohrlich and Eisenstein [4] have solved the same problem (and obtained identical results!) by means of a method which the authors find to be more satisfying theoretically.

In all of the above papers, the protons and neutrons were

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1. Rarita and Schwinger - Physical Review 59, 436, 1941.
2. Rarita and Schwinger - Physical Review 59, 556, 1941.
3. Ashkin and Wu - Physical Review 73, 973, 1948.
4. Rohrlich andEisenstein - Physical Review 75, 705, 1949.

considered to be completely unpolarized; i.e., the spins of the particles were assumed to have no preferential direction of alignment. The results of such a calculation* showed the scattering cross section to be dependent only upon the polar angle to the direction of propagation of the neutron beam. However, in order to answer a question raised by Dr. G.C. Laurence of Chalk River, it was decided to investigate whether a dependence upon azimuthal angle is introduced by certain polarization states of the neutron-proton system, and to determine how the polar dependence is modified by such states.

An expression has been obtained for the differential scattering cross section as a function of the azimuthal and polar angles and of parameters which are determined by the polarization of the neutrons and protons.

Calculation of the Differential Cross Section

Because the tensor interaction operator yields zero when applied to any singlet spin function, only the contribtuion of the triplet scattering to the total cross section will be considered. In the centre of mass co-ordinate system, the initial incident wave is represented by the expression

$$\psi_{inc} = e^{i\bar{k}\cdot\bar{r}} \sum_{m_s=-1}^{+1} a_{m_s} \chi^{m_s} \tag{1}$$

where $\bar{r}$ is the vector from proton to neutron, $\hbar\bar{k}$ is the momentum ~~of~~

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* - See Reference 3. This paper is the starting point for the calculations which follow.

of the incident neutron in the centre of mass co-ordinate system, χ^{m_s} $(m_s = 1, 0, -1)$ are the three triplet spin functions, defined for convenience with respect to $\bar{k}$, and a_{m_s} $(m_s = 1, 0, -1)$ are constants which depend upon the polarization states of neutron and proton.

As a result of the tensor interaction, the asymptotic form of the scattered wave is

$$\psi_{sc} \sim \frac{e^{ikr}}{r} \sum_{m_s' = -1}^{+1} \sum_{m_s = -1}^{+1} \chi^{m_s'} S_{m_s' m_s}(\theta, \varphi)\, a_{m_s} \quad (2)$$

The matrix elements $S_{m_s' m_s}$* depend upon both θ and φ, the polar and azimuthal angles respectively; the dependence upon φ is a direct result of the asymmetry of the tensor force.

To obtain the triplet scattering cross section per unit solid angle, one calculates the square modulus of the coefficient of $\frac{e^{ikr}}{r}$ in (2), which yields (using the orthonormality of the triplet spin functions):

$$\sum_{m_s} \sum_{m_s'} \sum_{m_s''} S^*_{m_s' m_s} S_{m_s' m_s''}\, a^*_{m_s} a_{m_s''} \quad (3)$$

$$= \sum_{m_s''} \sum_{m_s'} |a_{m_s''}|^2 \, |S_{m_s' m_s''}|^2 + \sum_{m_s''} \sum_{m_s'} \sum_{\substack{m_s \\ (m_s \neq m_s'')}} a^*_{m_s} a_{m_s''} S^*_{m_s' m_s} S_{m_s' m_s''} \quad (3a)$$

An examination of the specific form of the matrix elements $S_{m_s' m_s}$ (displayed as Equations 10 and 11 in this paper) leads one immediately to the conclusion that φ-dependence disappears completely from the first term in (3a), and may enter the final expression only by virtue

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*- See Reference 3, Page 981. The symbol $S_{m_s' m_s}$ in this paper is identical in meaning to the corresponding symbol in Ashkin and Wu. These elements have been written on the assumption that the z-axis is the direction of propagation of the incident beam of neutrons.

of the cross-product terms. If both neutrons and protons are completely unpolarized, the cross-products disappear in averaging over the phases of the amplitudes a_{m_s}: a simple calculation shows that $a^*_{m_s} a_{m_s''}$ averages to zero if $m_s \neq m_s''$, and to $1/4$ if $m_s = m_s''$. Therefore,

$$\sigma_{trip} = 1/4 \sum_{m_s'} \sum_{m_s''} \left| S_{m_s' m_s''} \right|^2 \tag{4}$$

As will be seen, if there is a specific polarization, these cross-products do not disappear except in special, symmetric cases.

It is necessary to calculate the coefficients a_{m_s} for an arbitrary but specific polarization of the neutrons and protons. In the following, it will be assumed that the polar axis is both the direction of propagation and the direction of reference of the spin functions.

The following physical situation is considered: in the direction θ_i, φ_i (i = n,p and refers thus to neutron or proton), the ratio of the number of particles in a state with spin angular momentum $\frac{\hbar}{2}$ in the positive sense to the number having a spin of $\frac{\hbar}{2}$ in the negative sense is $\gamma_i / 1-\gamma_i$. To obtain the differential cross section appropriate to this physical situation, we may first compute the cross sections for four specific spin orientations of the neutron-proton system (protons and neutrons parallel and anti-parallel to the directions θ_i, φ_i), and then compound the results by means of a statistical argument. However, it may be shown that it is also possible to proceed by introducing at the outset a wave function which refers to an assembly of neutrons and protons in the state described above:

$$\psi = \left(\sqrt{\gamma_n}\, \alpha_n' + e^{i\lambda_n}\sqrt{1-\gamma_n}\, \beta_n'\right)\left(\sqrt{\gamma_p}\, \alpha_p' + e^{i\lambda_p}\sqrt{1-\gamma_p}\, \beta_p'\right) \tag{5}$$

In the above expression, the λ_i are the random statistical phase factors which must be averaged out in arriving at a final result.

~~and the phases $e^{i\lambda_i}$ appear as a result of the physical conditions imposed.~~ α_i', β_i' represents the states of positive and negative spin in the usual way. These states may be expressed in terms of the spin functions α_i, β_i referred to the z-axis by the following transformations:

$$\begin{aligned} \alpha_i' &= \cos\theta_i/2 \, e^{-\frac{i\varphi_i}{2}}\alpha + \sin\theta_i/2 \, e^{\frac{i\varphi_i}{2}}\beta \\ \beta_i' &= -\sin\theta_i/2 \, e^{-\frac{i\varphi_i}{2}}\alpha + \cos\theta_i/2 \, e^{\frac{i\varphi_i}{2}}\beta \end{aligned} \tag{6}$$

Thus the total spin function (including both singlet and triplet states) is:

$$\psi_{sp} = (a_n\alpha_n + b_n\beta_n)(a_p\alpha_p + b_p\beta_p)$$

where:

$$\begin{aligned} a_i &= e^{-\frac{i\varphi_i}{2}}\left(\sqrt{f_i}\cos\theta_i/2 - e^{i\lambda_i}\sqrt{1-f_i}\sin\theta_i/2\right) \\ b_i &= e^{\frac{i\varphi_i}{2}}\left(\sqrt{f_i}\sin\theta_i/2 + e^{i\lambda_i}\sqrt{1-f_i}\cos\theta_i/2\right) \end{aligned} \tag{7}$$

Now, we have written in (1) only the triplet part of ψ_{sp} which is

$$\psi_{trip\,sp.} = a_1\chi^1 + a_{-1}\chi^{-1} + a_0\chi^0$$

with χ^1, χ^{-1}, and χ^0 so defined that

$$\begin{aligned} a_1 &= a_n a_p \qquad a_{-1} = b_n b_p \\ a_0 &= 1/\sqrt{2}\,(a_n b_p + a_p b_n) \end{aligned} \tag{8}$$

Averaging the products of the amplitudes $a^*_{m_s} a_{m_s}$ over the phases, λ_i, one obtains:

$$\begin{aligned} a_1^* a_1 &= \left(f_n\cos^2\frac{\theta_n}{2} + (1-f_n)\sin^2\frac{\theta_n}{2}\right)\left(f_p\cos^2\frac{\theta_p}{2} + (1-f_p)\sin^2\frac{\theta_p}{2}\right) \\ a_{-1}^* a_{-1} &= \left(f_n\sin^2\frac{\theta_n}{2} + (1-f_n)\cos^2\frac{\theta_n}{2}\right)\left(f_p\sin^2\frac{\theta_p}{2} + (1-f_p)\cos^2\frac{\theta_p}{2}\right) \\ a_0^* a_0 &= \frac{1}{2}\Big\{\left(f_n\cos^2\frac{\theta_n}{2} + (1-f_n)\sin^2\frac{\theta_n}{2}\right)\left(f_p\sin^2\frac{\theta_p}{2} + (1-f_p)\cos^2\frac{\theta_p}{2}\right) \\ &\quad + \left(f_p\cos^2\frac{\theta_p}{2} + (1-f_p)\sin^2\frac{\theta_p}{2}\right)\left(f_n\sin^2\frac{\theta_n}{2} + (1-f_n)\cos^2\frac{\theta_n}{2}\right) \\ &\quad + \frac{1}{2}\sin\theta_p\sin\theta_n\cos(\varphi_n - \varphi_p)(2f_n - 1)(2f_p - 1)\Big\} \end{aligned} \tag{9}$$

$$a_1^* a_0 = \tfrac{1}{2\sqrt{2}}\, e^{i\varphi_p} \sin\theta_p\,(2f_p-1)\left(f_n \cos^2\tfrac{\theta_n}{2} + (1-f_n)\sin^2\tfrac{\theta_n}{2}\right)$$
$$\quad + \tfrac{1}{2\sqrt{2}}\, e^{i\varphi_n} \sin\theta_n\,(2f_n-1)\left(f_p \cos^2\tfrac{\theta_p}{2} + (1-f_p)\sin^2\tfrac{\theta_p}{2}\right)$$
$$a_0^* a_1 = \tfrac{1}{2\sqrt{2}}\, e^{-i\varphi_p} \sin\theta_p\,(2f_p-1)\left(f_n \cos^2\tfrac{\theta_n}{2} + (1-f_n)\sin^2\tfrac{\theta_n}{2}\right)$$
$$\quad + \tfrac{1}{2\sqrt{2}}\, e^{-i\varphi_n} \sin\theta_n\,(2f_n-1)\left(f_p \cos^2\tfrac{\theta_p}{2} + (1-f_p)\sin^2\tfrac{\theta_p}{2}\right)$$
$$a_{-1}^* a_0 = \tfrac{1}{2\sqrt{2}}\, e^{-i\varphi_p} \sin\theta_p\,(2f_p-1)\left(f_n \sin^2\tfrac{\theta_n}{2} + (1-f_n)\cos^2\tfrac{\theta_n}{2}\right)$$
$$\quad + \tfrac{1}{2\sqrt{2}}\, e^{-i\varphi_n} \sin\theta_n\,(2f_n-1)\left(f_p \sin^2\tfrac{\theta_p}{2} + (1-f_p)\cos^2\tfrac{\theta_p}{2}\right)$$
$$a_0^* a_{-1} = \tfrac{1}{2\sqrt{2}}\, e^{i\varphi_p} \sin\theta_p\,(2f_p-1)\left(f_n \sin^2\tfrac{\theta_n}{2} + (1-f_n)\cos^2\tfrac{\theta_n}{2}\right)$$
$$\quad + \tfrac{1}{2\sqrt{2}}\, e^{i\varphi_n} \sin\theta_n\,(2f_n-1)\left(f_p \sin^2\tfrac{\theta_p}{2} + (1-f_p)\cos^2\tfrac{\theta_p}{2}\right)$$
$$a_1^* a_{-1} = \tfrac{1}{4}\, e^{i(\varphi_p+\varphi_n)} \sin\theta_n \sin\theta_p\,(2f_n-1)(2f_p-1)$$
$$a_{-1}^* a_1 = \tfrac{1}{4}\, e^{-i(\varphi_p+\varphi_n)} \sin\theta_n \sin\theta_p\,(2f_n-1)(2f_p-1)$$

The above expressions represent the coefficients in (3); it is now necessary to calculate the terms $S^*_{m_s' m_s} S_{m_s' m_s''}$. In order to make the φ-dependence of these terms explicit, we reproduce the Ashkin and Wu matrix as follows:

$$\begin{array}{ll} S_{1,1} = S_{-1,-1} = A & S_{0,0} = E \\ S_{0,1} = e^{i\varphi} B & S_{0,-1} = e^{-i\varphi} B \\ S_{1,0} = e^{-i\varphi} D & S_{-1,0} = e^{i\varphi} D \\ S_{-1,1} = e^{2i\varphi} C & S_{1,-1} = e^{-2i\varphi} C \end{array} \qquad (10)$$

WHERE

$$A = \frac{1}{2ik} \sum_{L=0}^{\infty} [2(2L+1)]^{1/2} P_L(\theta) \left\{ \frac{L-1}{2(2L+1)}\left(e^{2i\delta_L^{L-1,1}} - 1\right) + \frac{\left(e^{2i\delta_L^{L,1}} - 1\right)}{2} + \frac{L+2}{2(2L+1)}\left(e^{2i\delta_L^{L+1,1}} - 1\right) \right\}$$

$$B = \frac{1}{2ik}\sum_{l=1}^{\infty}[2(2l+1)]^{1/2}\frac{P_l^1(\theta)}{[2l(l+1)]^{1/2}}\left\{-\frac{(l^2-1)(e^{2i\delta_l^{l-1,1}}-1)}{2l+1}-(e^{2i\delta_l^{l,1}}-1)+\frac{l(l+2)(e^{2i\delta_l^{l+1,1}}-1)}{2l+1}\right\}$$

$$C = \frac{1}{2ik}\sum_{l=2}^{\infty}[2(2l+1)]^{1/2}\frac{P_l^2(\theta)}{[(l-1)l(l+1)(l+2)]^{1/2}}\left\{\frac{(e^{2i\delta_l^{l-1,1}}-1)}{2l(2l+1)}-\frac{(e^{2i\delta_l^{l,1}}-1)}{2l(l+1)}+\frac{(e^{2i\delta_l^{l+1,1}}-1)}{(2l+1)(2l+2)}\right\}$$

$$D = \frac{1}{2ik}\sum_{l=1}^{\infty}[2(2l+1)]^{1/2}P_l^1(\theta)\left[\frac{l(l+1)}{2}\right]^{1/2}\left\{-\frac{(e^{2i\delta_l^{l-1,0}}-1)}{2l+1}+\frac{(e^{2i\delta_l^{l+1,0}}-1)}{2l+1}\right\} \qquad (11)$$

$$E = \frac{1}{2ik}\sum_{l=0}^{\infty}[2(2l+1)]^{1/2}P_l(\theta)\left\{\frac{l(e^{2i\delta_l^{l-1,0}}-1)}{2l+1}+\frac{(l+1)(e^{2i\delta_l^{l+1,0}}-1)}{2l+1}\right\}$$

In the above expressions, $P_l^{|m|}(\theta)$ are the normalized associated Legendre polynomials, and $\delta_l^{J,m}$ are the phase shifts of the scattered wave. (It is to be remarked that each of A, B, and C is also equal to a similar expression with $\delta_l^{J,1}$ replaced by $\delta_l^{J,-1}$.)

We now define $T_{m_s m_s''}$ to be $\sum_{m_s'} S^*_{m_s' m_s} S_{m_s' m_s''}$: then

$$\begin{aligned}
T_{1,1} &= T_{-1,-1} = |A|^2+|B|^2+|C|^2 = M \\
T_{0,0} &= 2|D|^2+|E|^2 = N \\
T_{0,1} &= Re^{i(\varphi+\rho)} & T_{1,0} &= Re^{-i(\varphi+\rho)} \qquad (12)\\
T_{0,-1} &= Re^{-i(\varphi-\rho)} & T_{-1,0} &= Re^{i(\varphi-\rho)} \\
T_{-1,1} &= Qe^{2i\varphi} & T_{1,-1} &= Qe^{-2i\varphi}
\end{aligned}$$

where R and ρ are defined by: $Re^{i\rho} = [AD^* + BE^* + CD^*]$
and $Q = [C^*A + A^*C + |B|^2]$.

If expression (3) is now expanded as indicated, the result is:

$$
\begin{aligned}
\sigma_{trip} = {} & \Big\{\Big(f_n \cos^2\tfrac{\theta_n}{2} + (1-f_n)\sin^2\tfrac{\theta_n}{2}\Big)\Big(f_p \cos^2\tfrac{\theta_p}{2} + (1-f_p)\sin^2\tfrac{\theta_p}{2}\Big) \\
& + \Big(f_n \sin^2\tfrac{\theta_n}{2} + (1-f_n)\cos^2\tfrac{\theta_n}{2}\Big)\Big(f_p \sin^2\tfrac{\theta_p}{2} + (1-f_p)\cos^2\tfrac{\theta_p}{2}\Big)\Big\} M \\
+ {} & \Big\{\Big(f_n \cos^2\tfrac{\theta_n}{2} + (1-f_n)\sin^2\tfrac{\theta_n}{2}\Big)\Big(f_p \sin^2\tfrac{\theta_p}{2} + (1-f_p)\cos^2\tfrac{\theta_p}{2}\Big) \\
& + \Big(f_n \sin^2\tfrac{\theta_n}{2} + (1-f_n)\cos^2\tfrac{\theta_n}{2}\Big)\Big(f_p \cos^2\tfrac{\theta_p}{2} + (1-f_p)\sin^2\tfrac{\theta_p}{2}\Big) \\
& + \tfrac{1}{2}\cos(\varphi_n-\varphi_p)\sin\theta_p \sin\theta_n (2f_n-1)(2f_p-1)\Big\} N/2 \\
+ {} & \Big\{ \sin\theta_p (2f_p-1)\Big(f_n \cos^2\tfrac{\theta_n}{2} + (1-f_n)\sin^2\tfrac{\theta_n}{2}\Big)\cos(\varphi+\rho-\varphi_p) \\
& + \sin\theta_n (2f_n-1)\Big(f_p \cos^2\tfrac{\theta_p}{2} + (1-f_p)\sin^2\tfrac{\theta_p}{2}\Big)\cos(\varphi+\rho-\varphi_n) \\
& + \sin\theta_p (2f_p-1)\Big(f_n \sin^2\tfrac{\theta_n}{2} + (1-f_n)\cos^2\tfrac{\theta_n}{2}\Big)\cos(\varphi-\rho-\varphi_p) \\
& + \sin\theta_n (2f_n-1)\Big(f_p \sin^2\tfrac{\theta_p}{2} + (1-f_p)\cos^2\tfrac{\theta_p}{2}\Big)\cos(\varphi-\rho-\varphi_n)\Big\} P/2 \\
+ {} & \Big\{ \sin\theta_n \sin\theta_p (2f_n-1)(2f_p-1)\cos(2\varphi-\varphi_p-\varphi_n)\Big\} Q/2
\end{aligned}
\tag{13}
$$

<u>Conclusions</u>

It may be concluded after an examination of the expression for the triplet scattering cross section that there are two conditions which are separately sufficient for azimuthal symmetry:

1. $\theta_n = \theta_p = 0 \text{ or } \pi$, that is, for any polarization state in which both neutrons and protons are aligned parallel to the direction of the neutrons, and

2. $f_p = f_n = \tfrac{1}{2}$, that is, for complete non-polarization.

For other cases there is a dependence upon the azimuthal angle which becomes stronger as the polarization becomes more complete.

It is of considerable experimental interest to note that a partially polarized beam of neutrons impinging upon an unpolarized proton target would, as a result of the tensor interaction, be expected to show an azimuthally asymmetric cross section. The magnitude of this asymmetry and the conditions under which the asymmetry will be a maximum are to be calculated shortly.

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