APPLICATION OF GEOMETRIC PROGRAMMING TO PID CONTROLLER TUNING WITH STATE CONSTRAINTS

by

L. James Carver

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ABSTRACT

In the thesis, geometric programming is considered as a numerical optimization technique. The problem of minimizing the integral square error of a system characterized by a second order plant with proportional-integral-derivative (PID) controller is investigated. Constraints are imposed upon the state of the system in order to obtain feasible solutions and conditions that are amenable to the geometric programming technique.

The application of geometric programming requires the use of approximation procedures to eliminate untenable conditions in the objective and constraint functions. The techniques utilized render solutions that are easily obtainable, usually amounting to solving a set of linear equations and requiring no differentiation of terms. In addition, there is rapid convergence to an optimum. The accuracy of the results is dependent upon the validity of the approximations.

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SYMBOLS

Some common mathematical symbols used include:

$\approx$	approximately equal to
$>$	greater than
$\geq$	equal to or greater than
$<$	less than
$\leq$	equal to or less than
$\triangleq$	defined as
$\sum_{m}^{n} i$	summation of terms with index i ranging from m to n
$\prod_{i=m}^{n}$	product of terms with index i ranging from m to n
L[ ]	Laplace transform
$\int_{-j\infty}^{j\infty} ds$	integral over the same dimension as the vector

The principal symbols used are defined below:

a_i	variable associated with error transfer function
a_{ij}	exponent of variable in primal program
α_i	variable associated with error transfer function
$\overline{\alpha}_i$	initial value of primal variable
α_i^*	optimum value of primal variable
b_i	coefficient of constraint equation
β_i	variable associated with error transfer function for the time delay system
c_i	variable associated with error transfer function for time delay system

c_j	coefficient of a posynomial term
$C(s)$	Laplace transform of system output response
$c(t)$	system output response in time domain
δ_i	variable vector in dual program
δ_i^*	optimum value of dual variable
Δ_i	exponent of dual variable
$E(s)$	Laplace transform of error transfer function
$e(t)$	system error transfer function
η_i	dual exponent
$G(s)$	system transfer function
$G_c(s)$	PID controller transfer function
$G_p(s)$	plant transfer function
$g_k(t)$	$k = 1 \ldots n$, posynomial constraints in primal program
$g_o(t)$	posynomial, primal objective function
γ	plant parameter
ISE	system objective function: Integral Square Error
K_c	PID controller gain
λ	Lagrange multiplier
$\lambda_k(\delta)$	dual variable associated with a forced constraint in the primal program
m	number of primal variables
N_p	number of terms in the primal program
ω_n	plant parameter
$P_j(t)$	a posynomial term
$\psi(\delta)$	dual objective function
$R(s)$	Laplace transform of system input
$r(t)$	system input in time domain

T	system time delay
t_i	variable vector in primal program
T_s	settling time
τ_d	PID derivative time constants
τ_i	PID integral time constants
ϕ	cost or objective function
ϕ_a	augmented cost function
x_i	primal variable
ζ	plant parameter

LIST OF ILLUSTRATIONS

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I. INTRODUCTION

1.1 Basic Concept of Optimization

The optimal design and control of systems and industrial processes has long been of concern to the applied scientist and engineer, and, indeed might be taken as the definition of the function and goal of engineering. Usually in design problems a system is characterized in mathematical terms by defining a desired objective function or performance index which consists of fixed parameters and decision variables. The objective of an optimal design is to choose those variables which yield an optimum (usually maximum or minimum) value of the performance criterion. Traditional methods of achieving this are by differential calculus and/or some kind of a search over the design variables, which is often slow and exhausting. A recently developed theory called geometric programming offers a more efficient method of solution in as much as it is computationally more convenient and in addition rapidly approaches the optimum of the objective function.

1.2 Fundamentals of Geometric Programming

Geometric programming was first formulated by Zener (1961) as a method for optimizing engineering design. He considered the problem of minimizing the sum of component costs of a unit, when each cost depends on products of the design variables, each variable raised to an arbitrary but known power. In addition, it was required that the number of components in the unit exceed the number of design variables by one. Zener showed that the solution of such a problem can always be obtained simply by solving a square system of linear algebraic equations.

Zener's method was subsequently extended (Duffin, 1962a;

Duffin, 1962b; Duffin and Peterson, 1964) to optimization problems of a more general nature, including those with inequality constraints. The most comprehensive reference on geometric programming is the book of Duffin, Peterson and Zener (1967) where the mathematical derivations and some engineering applications are described in detail. The mathematical treatment relies on a generalization of the inequality and equality relationships between arithmetic and geometric means, as well as other geometric concepts, such as orthogonality of vectors. Geometric programming derives its name from its intimate connection with these geometric concepts. The underlying premise is based on the fact that the minimum of the arithmetic mean is equal to the maximum of the geometric mean.

The basic mathematical concepts of geometric programming deal with real valued functions called posynomials. A posynominal is given by

$$g(t) = \sum_{j=1}^{n} P_j(t) \tag{1.1}$$

where

$$P_j(t) = c_j \prod_{i=1}^{m} t_i^{a_{ij}} \tag{1.2}$$

Each coefficient c_j is positive, while the exponents a_{ij} are arbitrary real numbers and the variables t_i are restricted to take on positive values, i.e. the domain of the posynomial is the positive orthant.

Geometric programming deals with minimizing posynomials subject to a certain type of posynomial constraint. This minimization problem is called <u>the primal problem of geometric programming</u> or simply, <u>the primal program</u>.

In the primal program, the objective is to find a vector t* that minimizes the function $g_o(t)$ subject to the constraints

$$t_1 > 0,\ t_2 > 0 \ldots,\ t_m > 0 \tag{1.3}$$

and

$$g_1(t) \le 1,\ g_2(t) \le 1,\ \ldots,\ g_p(t) \le 1 \tag{1.4}$$

Here

$$g_k(t) = \sum_{j=M_k}^{N_k} P_j(t),\ k = 0,1,\ldots,p \tag{1.5}$$

where

$$P_j(t) = c_j \prod_{i=1}^{m} t_i^{a_{ij}} \tag{1.6}$$

and

$$M_o = 1,\ M_k = N_{k-1} + 1,\ k = 1,\ldots,p \tag{1.7}$$

The exponents a_{ij} are real numbers and the coefficients c_j are positive. Thus the functions $g_k(t)$ are posynomials.

An important feature of geometric programming is the central role played by the terms P_j in the posynomials g_k. Instead of focusing on determining the optimal t*, the approach is to concentrate on evaluating the minimum of $g_o(t)$ and the relative contribution of the terms $P_j(t)$ to this minimum. Only after they are determined is the optimal t* found. Another important feature of geometric programming is the concept of a duality theory which associates with each primal program a "dual" programming problem which is usually easier to solve than the primal program.

The objective of the dual program is to find a vector δ* that maximizes the product function

$$\psi(\delta) = \left[\prod_{j=1}^{N_p} (c_j/\delta_j)^{\delta_j}\right] \prod_{k=1}^{p} \lambda_k(\delta)^{\lambda_k(\delta)} \tag{1.8}$$

where

$$\lambda_k(\delta) = \sum_{j=M_k}^{N_k} \delta_j,\ k = 1,\ldots,p \tag{1.9}$$

subject to the constraints

$$\delta_1 \geq 0,\ \delta_2 \geq 0,\ \ldots,\ \delta_{N_p} \geq 0 \qquad (1.10)$$

$$\sum_{j=1}^{N_o} \delta_j = 1 \qquad (1.11)$$

and

$$\sum_{j=1}^{N_o} a_{ij}\delta_j = 0, \quad i = 1,\ldots,m \qquad (1.12)$$

Here a_{ij}, c_j, M_k, N_k are the same as in the primal program.

The dual program is obtained from the primal program. The constants c_j appearing in the dual function $\psi(\delta)$ are the coefficients of the posynominals g_k appearing in the primal objective function and constraints. The variable δ_j is associated with the j^{th} term, P_j, of g_k, $k = 0,1,\ldots,p$, so that each δ_j is associated with one and only one posynomial term P_j. Moreover, each $\lambda_k(\delta)$ is associated with the k^{th} primal constraint $g_k(t) \leq 1$. The λ_k, therefore, are similar to Lagrange multipliers. The normality condition (1.11) is imposed on the dual variables associated with the primal objective function only, while the orthogonality conditions (1.12) apply to all dual variables. It should be noted, also, that the i^{th} constraint in (1.12) is associated with the i^{th} primal variable t_i through the exponents a_{ij}.

A primal or dual program is consistent if there exists at least one vector that satisfies its constraints. A primal geometric program is superconsistent if there exists at least one vector $\hat{t} > 0$ such that

$$g_k(\hat{t}) < 1, \quad k = 1,\ldots,p \qquad (1.13)$$

A vector t is called primal feasible if it satisfies the primal constraints (1.3) and (1.4) while a vector δ is called dual feasible if it satisfies the dual constraints (1.10), (1.11), and (1.12).

A primal program is said to be solvable if there is a feasible vector, $\bar{t}$, such that $g_o(\bar{t}) \leq g_o(t)$ for all feasible t. Similarly, a dual program is said to be solvable if there exists a feasible vector $\bar{\delta}$ such that $\psi(\bar{\delta}) \geq \psi(\delta)$ for all feasible δ.

In terms of the preceding concepts, the following duality theorem of geometric programming can be stated:

Theorem 1. If a primal geometric program is superconsistent and solvable, then:

(i) The corresponding dual program is solvable.

(ii) The constrainted minimum of the primal program is equal to the constrained maximum of the dual program, i.e.

$$g_o(t^*) = \psi(\delta^*) \tag{1.14}$$

(iii) The relations between optimal primal and dual variables are given by:

$$\delta_j^* = \begin{cases} P_j(t^*)/g_o(t^*), & j = 1,\ldots,N_o \\ \lambda_k(\delta^*)\ P_j(t^*), & j = M_k,\ldots,N_k \quad k = 1,\ldots,p \end{cases} \tag{1.15}$$

and

$$\lambda_k(\delta^*)\ [1 - g_k(t^*)] = 0 \qquad k = 1,\ldots,p \tag{1.16}$$

Thus, if a given primal geometric program is superconsistent and solvable then Theorem 1 implies that instead of solving the primal geometric program, the corresponding dual program may be solved and by (1.14) the maximum of the dual is equal to the minimum of the primal. In addition, the upper and lower bounds on the solution of the primal and dual programs can be obtained by evaluating $g_o(t)$ and $\psi(\delta)$ for feasible vectors t and δ, respectively. For such t and δ, then $g_o(t) \geq g_o(t^*) = \psi(\delta^*) \geq \psi(\delta)$.

The duality theorem enables the minimum value of the primal objective function to be found without actually solving the primal program, but relation (1.15) also gives a method to find the minimizing vector t^* from the knowledge of a maximizing vector δ^*. From (1.15) it follows that:

$$c_j \prod_{i=1}^{m} t_i^{*a_{ij}} = \delta_j^* \psi(\delta^*), \quad j = 1,\dots,N_o \tag{1.17}$$

and for $k = 1,\dots,p$, such that $\lambda_k(\delta^*) > 0$

$$c_j \prod_{i=1}^{m} t_i^{*a_{ij}} = \delta_j^*/\lambda_k(\delta^*), \quad j = M_k,\dots,N_k \tag{1.18}$$

Taking the logarithm of both sides of each equation in (1.17) and (1.18) and rearranging them yields

$$\sum_{i=1}^{m} a_{ij} \log t_i^* = \log(\delta_j^* \psi(\delta^*)/c_j), \quad j = 1,\dots,N_o \tag{1.19}$$

and

$$\sum_{i=1}^{m} a_{ij} \log t_i^* = \log[\delta_j/(c_j\ \lambda_k(\delta^*)], \quad j = M_k,\dots, N_k \tag{1.20}$$

The optimal primal variables t_i^* are thus found by solving the above system of linear equations in the variables $\log t_i^*$.

Knowing that each term P_j in the primal objective function g_o is associated with one of the dual variables, it can be seen from (1.15) that each optimal dual variable δ_j^*, $j = 1,\dots,N_o$, represents the weight or relative contribution of the term P_j to the minimum of $g_o(t)$. Thus by solving the dual program, one obtains first the minimum of the primal objective function and then the relative contribution of each term P_j to the optimal solution.

Since the feasible t are restricted to be positive, t^* is also positive; it follows then that each $P_j(t^*)$ in the objective function is positive. By (1.15), therefore, those dual variables which correspond to terms in the objective function are positive, i.e. $\delta_j^* > 0$ for $j = 1,\ldots,N_o$. The remaining δ_j, i.e. those δ_j corresponding to terms in the constraints, are zero or positive, according as the particular constraint is loose ($g_k(t^*) < 1$) or tight ($g_k(t^*) = 1$) at primal optimum. More precisely relations (1.9), (1.10) and (1.16) imply:

(i) whenever $g_k(t^*) < 1$, the optimal dual variables δ_j^*, $j = M_k,\ldots N_k$ vanish;

(ii) if $\delta_j^* > 0$, for some $j = M_k,\ldots,N_k$, then $g_k(t^*) = 1$.

Moreover, by the second equation of (1.15) and the positivity of t^*, it can be concluded that $\lambda_k(\delta^*) > 0$ for some k, implies $\delta_j^* > 0$ for $j = M_k,\ldots,N_k$.

1.3 Some Properties of Geometric Programming

The dual problem of geometric programming consists of maximizing a given function subject to linear equations and nonnegativity constraints on the variables. A unique and easily obtained solution for the linear equation arises when the number of equations in the dual constraint is the same as the number of dual variables. This case occurs when the total number of terms, N_p, in the primal objective function and constraints exceeds the number of primal variables, m, by one, i.e.

$$N_p = m + 1 \tag{1.21}$$

If the number of variables in the dual exceeds the number of equations by one (i.e. the number of terms in the primal program is two greater than the number of variables) then the dual constraints may be solved explicitly in terms of only one variable, thus reducing the problem to a maximization over a single variable. The next case, in which the number of primal terms exceeds the number of variables by three, would lead to a dual program in which a two-variable maximization needs to be carried out. Each case is succeedingly more difficult than the previous; the 'degree of difficulty' of a geometric program may be defined by

$$\text{Degree of difficulty} = N_p - m - 1 \tag{1.22}$$

Generally, well-formulated geometric programs can have arbitrary non-negative degrees of difficulty.

The linear dual constraints (1.10) - (1.12) have the important property that they are independent of the primal coefficients c_j. Hence the dual optimal solution for a problem with zero degree of difficulty is invariant in the sense that no matter what the numerical values of the coefficients c_j are, the optimal dual variables are the same, since they are uniquely determined by solving the dual constraints. When the primal objective function represents a cost to be minimized, the optimal dual variables measure the relative contribution of the various cost items to the minimum cost. In the case of zero degrees of difficulty each term in the primal objective function at the optimum has an invariant weight represented by the unique solution of the linear dual constraints, thus providing insight into the engineering or economic structure of the problem. An analysis to find the relative importance of the terms in the primal objective function can be made, in the zero degree of difficulty case, without prior knowledge of the numerical values of the coefficients. From a computational point of view, evaluation of the optimal primal variables from (1.19) and (1.20) for positive values of c_j can be achieved without resolving the programming problem. Thus the optimal primal variables are easily adjusted for any change in the coefficients (a condition that could reflect altered design parameters or market fluctuations).

Complete solution of a geometric programming problem requires either the minimization of a posynomial, subject to posynomial inequality constraints (primal program) or the maximization of a nonlinear product function, subject to linear inequality and equality constraints (dual program). If the degree of difficulty is low i.e. zero or one, solving the dual is preferable since the optimization procedure is fairly easy. For higher degrees of difficulty, the question is not as clearly decidable although the dual offers some advantages as has been noted by Duffin, Peterson and Zener (1967).

1.4 Introductory Examples

As a preliminary, the general idea of geometric programming will be illustrated by some examples that are indicative of the kind of reasoning and results that pertain to geometric programming.

Consider the cost of producing a product to be made up of several factors: the cost of raw materials is $1000n^{.8}$ \$/year, the cost of operating one machine in the production process is $\frac{4000}{nr^{.2}}$ \$/year, while the cost of operation of a second machine is 1000r \$/year. The objective is to find values for n and r which minimize the total annual cost, given by

$$\phi = 1000n^{.8} + \frac{4000}{nr^{.2}} + 1000r \qquad (1.23)$$

In this unconstrained example, the optimal value of the parameters n* and r* may be found by setting the first partial derivatives equal to zero:

$$(\frac{\partial\phi}{\partial n}) = \frac{(.8)(1000)}{n^{.2}} - \frac{4000}{n^{*2}r^{*.2}} = \frac{1}{n^*}((.8)(1000n^{*.8} - \frac{4000}{n^*r^{*.2}}) = 0 \qquad (1.24)$$

$$\left(\frac{\partial\phi}{\partial r}\right) = -\frac{4000(.2)}{n^*r^{*1.2}} + 1000 = \frac{1}{r^*}\left(-\frac{4000(.2)}{n^*r^{*.2}} + 1000r^*\right) = 0 \qquad (1.25)$$

The values of n^* and r^* can be found by solving the nonlinear equations (1.24) and (1.25). However, the method of geometric programming does not require the solution of nonlinear equations. Since the minimum cost ϕ^* is made up of three cost terms, there is a unique distribution among the terms that contribute to the cost. Let δ_1, δ_2, and δ_3 be respectively, the fractions of the total minimum cost represented by raw materials, operation of machine one and operation of machine two. Then

$$\delta_1 = \frac{1000n^{*.8}}{\phi^*} \qquad (1.26)$$

$$\delta_2 = \frac{4000}{\phi^* n^* r^{*.2}} \qquad (1.27)$$

$$\delta_3 = \frac{1000r^*}{\phi^*} \qquad (1.28)$$

Since the weights must sum to unity

$$\delta_1 + \delta_2 + \delta_3 = 1 \qquad (1.29)$$

Substituting (1.26), (1.27), and (1.28) into (1.24) and (1.25) gives

$$.8\delta_1 - \delta_2 = 0 \qquad (1.30)$$

$$-.2\delta_2 + \delta_3 = 0 \qquad (1.31)$$

Equations (1.29), (1.30) and (1.31) represent a system of three linear equations in three unknowns which has the unique solution

$$\delta_1 = 25/49, \ \delta_2 = 20/49, \ \delta_3 = 4/49 \qquad (1.32)$$

Hence in the optimal design, raw materials contribute 25/49 of the total cost, machine one contributes 20/49, while machine two contributes the remainder or 4/49. Note that to find this optimal cost distribution, the optimal values of the variables n and r did not have to be found.

Since the sum of the weights is equal to unity then the minimum cost can be written as

$$\phi^* = \phi^{*(\delta_1 + \delta_2 + \delta_3)} = \phi^{*\delta_1} \cdot \phi^{*\delta_2} \cdot \phi^{*\delta_3} \tag{1.33}$$

Substituting (1.26), (1.27) and (1.28) into (1.33) gives

$$\phi^* = \left(\frac{1000n^{*.8}}{\delta_1}\right)^{\delta_1} \left(\frac{4000}{n^*r^{*.2}\delta_2}\right)^{\delta_2} \left(\frac{1000r^*}{\delta_3}\right)^{\delta_3} \tag{1.34}$$

Rearranging,

$$\phi^* = \left(\frac{1000}{\delta_1}\right)^{\delta_1} \left(\frac{4000}{\delta_2}\right)^{\delta_2} \left(\frac{1000}{\delta_3}\right)^{\delta_3} n^{*(.8\delta_1 - \delta_2)} \; r^{*(-.2\delta_2 + \delta_3)} \tag{1.35}$$

But by (1.30) and (1.31) the exponents of n* and r* vanish so that

$$\phi^* = \left(\frac{1000}{\delta_1}\right)^{\delta_1} \left(\frac{4000}{\delta_2}\right)^{\delta_2} \left(\frac{1000}{\delta_3}\right)^{\delta_3} \tag{1.36}$$

$$\text{or } \phi^* = \left(\frac{1000}{25/49}\right)^{25/49} \left(\frac{4000}{20/49}\right)^{20/49} \left(\frac{1000}{4/49}\right)^{4/49} = 4359.61 \tag{1.37}$$

i.e. the minimum cost is \$4359.61.

The optimum design variables can now be found. From (1.28)

$$\frac{4}{49} = \frac{1000r^*}{4359.61}$$

$$\text{or } r^* = 0.36 \tag{1.38}$$

Similarly from (1.27)

$$\frac{20}{49} = \frac{4000}{(4359.61)n^*(.36)^{.2}}$$

$$\text{or } n^* = 2.74 \tag{1.39}$$

These are the identical values to be obtained from a solution of the nonlinear equations (1.24) and (1.25). It is clear that the geometric programming approach was advantageous in that it required only minor computation of linear equations to obtain a solution.

Consider a second example, similar to one appearing in Eveleigh, 1967. For a system that is described by the transfer function

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s^3 + x_2 s^2 + x_1 s + 1} \tag{1.40}$$

the objective is to find values for the system parameters, x_1, and x_2, which will minimize the Integral Square Error (ISE) for the system. This is given by:

$$ISE = \frac{x_2^2 + x_1^2 x_2 - x_1}{2(x_1 x_2 - 1)} = \frac{x_1}{2} + \frac{x_2^2}{2(x_1 x_2 - 1)} \tag{1.41}$$

As before, the optimal variables x_1^* and x_2^* may be obtained by setting the first partial derivatives equal to zero

$$\frac{\partial ISE}{\partial x_1} = \frac{1}{2} - \frac{2x_2^3}{(2x_1 x_2 - 2)^2} = 0 \tag{1.42}$$

$$\frac{\partial ISE}{\partial x_2} = \frac{2x_2}{(2x_1 x_2 - 2)} - \frac{2x_1 x_2^2}{(2x_1 x_2 - 2)^2} = 0 \tag{1.43}$$

and solving the nonlinear equations.

Alternately, let

$$x_o \le 2x_1 x_2 - 2 \tag{1.44}$$

and substitute into (1.41) to obtain

$$ISE = \frac{x_1}{2} + \frac{x_2^2}{x_o} \tag{1.45}$$

Thus a related problem is to solve the posynomial objective function (1.45), subject to the constraint

$$\frac{x_o}{2x_1 x_2} + \frac{2}{2x_1 x_2} \le 1 \tag{1.46}$$

To show the concepts and development of geometric programming as they apply to this problem, consider the introduction of the Lagrangian multiplier λ_1 to form the augmented equation.

$$\phi_a = \frac{x_1}{2} + \frac{x_2^{\,2}}{x_o} + \lambda_1\left(\frac{x_o}{2x_1x_2} + \frac{2}{2x_1x_2}\right) \tag{1.47}$$

Now let δ_1 and δ_2 be respectively the fractions of the total ISE represented by the first and second terms in the ISE and let δ_3 and δ_4 be the respective fractions of the restraining equation.

$$\delta_1 = \frac{x_1}{2\phi_a} \tag{1.48}$$

$$\delta_2 = \frac{x_2^{\,2}}{x_o\phi_a} \tag{1.49}$$

$$\delta_3 = \frac{\lambda_1 x_o}{2x_1x_2\phi_a} \tag{1.50}$$

$$\delta_4 = \frac{2\lambda_1}{2x_1x_2\phi_a} \tag{1.51}$$

The optimum of the augmented equation, (1.47), can be found by the traditional method of taking partial derivatives and equating to zero. Thus,

$$\frac{\partial\phi_a}{\partial x_1} = \frac{1}{2} - \frac{\lambda_1 x_o}{2x_1^{\,2}x_2} - \frac{2\lambda_1}{2x_1^{\,2}x_2} = \frac{1}{x_1}\left(\frac{x_1}{2} - \frac{\lambda_1 x_o}{2x_1x_2} - \frac{2\lambda_1}{2x_1x_2}\right) = 0 \tag{1.52}$$

$$\frac{\partial\phi_a}{\partial x_2} = \frac{2x_2}{x_o} - \frac{\lambda_1 x_o}{2x_1x_2^{\,2}} - \frac{2\lambda_1}{2x_1x_2^{\,2}} = \frac{1}{x_2}\left(\frac{2x_2^{\,2}}{x_o} - \frac{\lambda_1 x_o}{2x_1x_2} - \frac{2\lambda_1}{2x_1x_2}\right) = 0 \tag{1.53}$$

$$\frac{\partial\phi_a}{\partial x_o} = -\frac{x_2^{\,2}}{x_o^{\,2}} + \frac{\lambda_1}{2x_1x_2} = \frac{1}{x_o}\left(-\frac{x_2^{\,2}}{x_o} + \frac{\lambda_1 x_o}{2x_1x_2}\right) = 0 \tag{1.54}$$

Substituting (1.48) - (1.51) into (1.52), (1.53) and (1.54) gives

$$\delta_1 - \delta_3 - \delta_4 = 0 \tag{1.55}$$

$$2\delta_2 - \delta_3 - \delta_4 = 0 \tag{1.56}$$

$$-\delta_2 + \delta_3 = 0 \tag{1.57}$$

Equations (1.55), (1.56) and (1.57) are termed the orthogonality conditions. The normality condition, applicable only to the primal objective function is formed by noting that the fractions, δ_1 and δ_2, of (1.45) must sum to unity. Therefore,

$$\delta_1 + \delta_2 = 1 \tag{1.58}$$

Equations (1.55) through (1.58) represent a system of four linear equations in four unknowns which has the unique solution

$$\delta_1{*} = 2/3, \ \delta_2{*} = 1/3, \ \delta_3{*} = 1/3, \ \delta_4{*} = 1/3 \tag{1.59}$$

The general relationship of the arithmetic mean to the geometric mean for a series of weighted terms states that the weighted arithmetic mean is equal to or greater than the corresponding weighted geometric mean. From (1.47) this relationship can be expressed as:

$$\phi_a = \delta_1\left(\frac{x_1}{2\delta_1}\right) + \delta_2\left(\frac{x_2^{\,2}}{x_o\delta_2}\right) + \delta_3\left(\frac{\lambda_1 x_o}{2x_1x_2\delta_3}\right) + \delta_4\left(\frac{2\lambda_1}{2x_1x_2\delta_4}\right) \geq$$

$$\left(\frac{x_1}{2\delta_1}\right)^{\delta_1}\left(\frac{x_2^{\,2}}{x_o\delta_2}\right)^{\delta_2}\left(\frac{\lambda_1 x_o}{2x_1x_2\delta_3}\right)^{\delta_3}\left(\frac{2\lambda_1}{2x_1x_2\delta_4}\right)^{\delta_4} \tag{1.60}$$

Equation (1.60) indicates the transition from the primal problem (minimizing the arithmetic series) to the associated dual problem (maximizing the geometric series). At the extremes there is equality between the expressions. Hence

$$\phi_a{*} = \left(\frac{1}{2\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{2\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4} x_1{*}^{\delta_1-\delta_3-\delta_4} x_2{*}^{2\delta_2-\delta_3-\delta_4} x_o{*}^{-\delta_2+\delta_3} \cdot \lambda_1^{\delta_3+\delta_4} \tag{1.61}$$

But by (1.55), (1.56) and (1.57) the exponents of $x_1{*}$, $x_2{*}$, and $x_o{*}$ vanish so that

$$\phi_a{*} = \left(\frac{1}{2\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{2\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4} \lambda_1^{\delta_3+\delta_4} \tag{1.62}$$

and from (1.50), (1.54) and (1.55)

$$\frac{1}{\lambda_1}(\delta_3+\delta_4) = 1 \qquad \text{or} \qquad \delta_3+\delta_4 = \lambda_1 \tag{1.63}$$

Thus,

$$\phi_a^* = \left(\frac{1}{2\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{2\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{(\delta_3+\delta_4)} \tag{1.64}$$

or

$$\phi_a^* = \left(\frac{1}{2(2/3)}\right)^{2/3}\left(\frac{1}{(1/3)}\right)^{1/3}\left(\frac{1}{2(1/3)}\right)^{1/3}\left(\frac{1}{(1/3)}\right)^{1/3}(1/3+1/3)^{(1/3+1/3)} = 3/2 \tag{1.65}$$

The theory of geometric programming states that the maximum of the dual problem is equal to the minimum of the primal problem. Hence the minimum of the ISE is 3/2. It should be noted that this was obtained without first obtaining the optimum of the design variables, x_1 and x_2. The relationship between the optimal primal and dual variables is given by (1.17) and (1.18). Substituting in values obtained from (1.59) and (1.65) yields

$$\frac{x_1^*}{2} = \delta_1^*\phi_a(\delta^*) = 1 \tag{1.66}$$

$$\frac{x_2^{*2}}{x_o^*} = \delta_2^*\phi_a(\delta^*) = \tfrac{1}{2} \tag{1.67}$$

$$\frac{x_o^*}{2x_1^*x_2^*} = \frac{\delta_3^*}{\delta_3^* + \delta_4^*} = \tfrac{1}{2} \tag{1.68}$$

$$\frac{1}{x_1^*x_2^*} = \frac{\delta_4^*}{\delta_3^* + \delta_4^*} = \tfrac{1}{2} \tag{1.69}$$

Solution of these equations yields the optimal parameters $x_1^* = 2$, $x_2^* = 1$, which may be verified by substituting into Equations (1.46) and (1.43).

II. APPROXIMATION TECHNIQUES

2.1 Standard Approximation Procedures

The direct application of geometric programming as an optimization technique is restricted by two main limitations - for problems with high 'degree of difficulty', the solution could require a multi-variable maximization procedure to be employed in addition to the geometric programming procedures; and the requirement that each term in the primal objective and constraining equations be positive. Recently several techniques (Avriel, Dembo and Passy, 1975; Avriel and Williams, 1970; Avriel and Williams, 1971; Duffin, 1970) have been developed to overcome the limitations of nonpositive terms. In general, these methods require the introduction of new variables in any objective function which contains negative coefficients, or a rearrangement of the inequality constraints if these contain negative terms. Unfortunately, the resultant functions may still contain many complicated and inconvenient terms. The method of Duffin, Peterson and Zener, 1967, in their expository book does not require the introduction of additional terms and yet can still handle some nonposynomial functions. Hence it may be potentially more useful in the solution of many engineering problems.

In this method, a function $f(t_1,t_2,\ldots,t_m)$ that is not a posynomial might be <u>approximated</u> by a posynomial. There is no unique method of doing this but the following is typical. Suppose that

$$f(t_1,t_2,\ldots,t_m) = g(t_1,t_2,\ldots,t_m) + h(t_1,t_2,\ldots,t_m) \tag{2.1}$$

where g is a posynomial and h is not. To approximate h by a single term posynomial a rough estimate of the range of variability of each variable t is made. Let $\overline{t}_j$ be the geometric mean of this range. Then

$(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m)$ is termed the operating point. Now, if $u(t)$ is a single-term posynomial such that

$$u(t) = ct_1^{a_1} t_2^{a_2} \ldots t_m^{a_m} \tag{2.2}$$

then

$$u(t) = u(\bar{t}) \left(\frac{t_1}{\bar{t}_1}\right)^{a_1} \left(\frac{t_2}{\bar{t}_2}\right)^{a_2} \ldots \left(\frac{t_m}{\bar{t}_m}\right)^{a_m} \tag{2.3}$$

and

$$\frac{t_j}{u} \frac{\partial u}{\partial t_j} \triangleq a_j, \quad j = 1,2,\ldots,m \tag{2.4}$$

thus, h is approximated as

$$h(t) \approx h(\bar{t}) \left(\frac{t_1}{\bar{t}_1}\right)^{a_1} \left(\frac{t_2}{\bar{t}_2}\right)^{a_2} \ldots \left(\frac{t_m}{\bar{t}_m}\right)^{a_m} \tag{2.5}$$

where

$$a_j = \left[\left(\frac{t_j}{h}\right) \frac{\partial h}{\partial t_j}\right]_{t=\bar{t}}, \quad j = 1,2,\ldots,m \tag{2.6}$$

This approximation is equivalent to expanding log h in a power series in terms of the variables $z_j = \log(t_j/\bar{t}_j)$ and neglecting all but the linear terms. If $h(\bar{t})$ is positive, f is approximated by a posynomial The approximation is such that f and the posynomial have the same value and the same first partial derivatives at the operating point $\bar{t}$.

2.2 Approximation Using a Non-derivative Technique

It is interesting to note that similar approximation results can be obtained without the use of differentiation. This result is derived from the basic arithmetic - geometric relationships and the resultant equality at the extremes.

Consider the expression

$$g(t) = \sum_i c_i t_1^{a_{i1}} \ldots t_m^{a_{im}} \tag{2.7}$$

where c_i can be either positive or negative. At the operating point $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m$, (2.7) takes on the value $\bar{g}(t)$ and each term contributes a fraction of this total equal to its value divided by the total. That is

$$\Delta_i = \frac{c_i \bar{t}_1^{\,a_{i1}} \ldots \bar{t}_m^{\,a_{im}}}{\bar{g}\,(t)} \tag{2.8}$$

Now the arithmetic - geometric relationship is given by:

$$\sum_i c_i t_i^{\,a_{i1}} \ldots t_m^{\,a_{im}} \approx \prod_i \left(\frac{c_i t_i^{\,a_{i1}} \ldots t_m^{\,a_{im}}}{\Delta_i}\right)^{\Delta_i} \tag{2.9}$$

and at the extremes, equality holds. Hence

$$g(t) \approx \prod \left(\frac{c_i t_i^{\,a_{i1}} \ldots t_m^{\,a_{im}}}{c_i \bar{t}^{\,a_{i1}} \ldots \bar{t}_m^{\,a_{im}}} \cdot \bar{g}(t)\right)^{\Delta_i} \tag{2.10}$$

Since the sum of the weights, Δ_i, must add to unity, then

$$g(t) \approx \bar{g}(t) \prod \left(\frac{t_i^{\,a_{i1}} \ldots t_m^{\,a_{im}}}{\bar{t}_i^{\,a_{i1}} \ldots \bar{t}_m^{\,a_{im}}}\right)^{\Delta_i} \tag{2.11}$$

where g(t) is now a posynomial.

An additional and important benefit to be realized by this approximation and the one of Section 2.1, is that although the original expression may contain several terms, the approximation can condense them into a single term. Hence, techniques of this type may be used as a means of reducing the degree of difficulty of the original problem since they can reduce the difference between the number of terms and the number of variables in the primal problem.

With initially appearing to be universally applicable, these methods must be thoroughly investigated before being applied to engineering problems. However, the techniques of approximation, together with the features of geometric programming when judiciously applied, yield considerable benefits in the analysis of optimization problems.

III. GEOMETRIC PROGRAMMING APPLICATION TO A PID CONTROL SYSTEM

3.1 Introduction

Many industrial processes can be adequately defined by a relatively simple mathematical model consisting of a second order lag plus dead time. The design of a control system for this type of process requires the choice of a controller and adjustment of the system parameters to produce an "optimum" result, i.e. usually obtaining a minimum or maximum value of a performance index or cost function.

Geometric programming can be applied to this problem with optimum or near optimum solutions obtained with a minimum of computational intricacies. That is, from an arbitrary starting condition, the optimum is approached in a single computational step by basically solving only linear equations.

3.2 PID Controller and System Definitions

A type of controller that is commonly used in industrial processes is the proportional - integral - derivative (PID) controller. The transfer function for this is

$$G_c(s) = K_c(1 + \frac{1}{\tau_i s} + \tau_d s) \tag{3.1}$$

and the control problem requires selection of the parameters K_c, τ_i, and τ_d so as to optimize a selected performance index such as the integral square error for the closed loop system to which the controller is applied.

Initially let the time delay be zero and consider a plant with the open-loop transfer function

$$G_p(s) = \frac{\omega_n^2 \gamma}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{3.2}$$

Letting

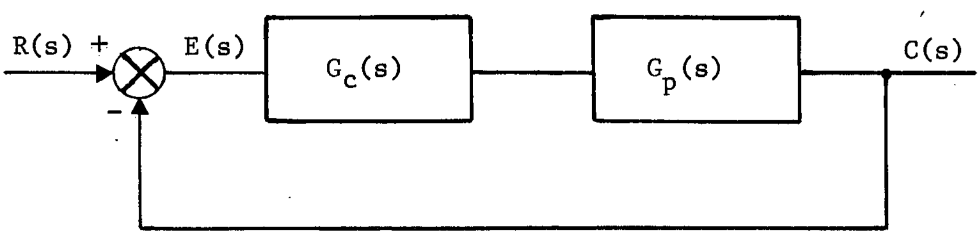
$$a_1 = 2\zeta\omega_n \tag{3.3}$$

$$a_2 = \omega_n^2 \tag{3.4}$$

$$a_3 = \omega_n^2\gamma$$

yields

$$G_p(s) = \frac{a_3}{s^2 + a_1 s + a_2} \tag{3.6}$$



$$G_p(s) \triangleq \text{plant} = \frac{\omega_n^2\gamma}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G_c(s) \triangleq \text{PID controller} = K_c(1 + \frac{1}{\tau_i s} + \tau_d s)$$

Fig. 1 System Configuration

For a closed-loop system of the form shown in Figure 1, with the cascade compensation network, G_c, given by (3.1) then

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G_c(s)G_p(s)} \tag{3.7}$$

Substituting (3.1) and (3.6) into (3.7) gives

$$\frac{E(s)}{R(s)} = \frac{s(s^2+a_1s+a_2)}{s(s^2+a_1s+a_2) + \frac{K_c a_3}{\tau_i}(1+\tau_i s+\tau_i\tau_d s^2)}$$

$$= \frac{s(s^2+a_1s+a_2)}{s^3+s^2(a_1+a_3K_c\tau_d) + s(a_2+a_3K_c) + a_3K_c/\tau_i} \tag{3.8}$$

Letting

$$\alpha_o = \frac{a_3K_c}{\tau_i} \tag{3.9}$$

$$\alpha_1 = a_2 + a_3K_c \tag{3.10}$$

$$\alpha_2 = a_1 + a_3K_c\tau_d \tag{3.11}$$

then,

$$\frac{E(s)}{R(s)} = \frac{s(s^2+a_1s+a_2)}{s^3+\alpha_2s^2+\alpha_1s+\alpha_o} \tag{3.12}$$

If the input to the system is a unit step, then $R(s) = 1/s$, and consequently

$$E(s) = \frac{s^2+a_1s+a_2}{s^3+\alpha_2s^2+\alpha_1s+\alpha_o} \tag{3.13}$$

Equation (3.13) is the error transfer function for the system of Fig. 1. It has been shown (Newton, Gould, Kaiser, 1964) that the integral square error, ISE, can be expressed in terms of functions like (3.13) by means of Parseval's theorem in the form

$$\text{ISE} = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} ds\frac{c(s)c(-s)}{d(s)d(-s)}$$

From tabulated values of the integral, the closed-loop system objective function is given by:

$$\text{ISE} = \frac{\alpha_o\alpha_1 + (a_1^2-2a_2)\alpha_o + a_2^2\alpha_2}{2\alpha_o\alpha_1\alpha_2(1 - \frac{\alpha_o}{\alpha_1\alpha_2})} \tag{3.14}$$

The optimization problem is to determine values of α_o, α_1, and α_2 that minimize Equation (3.14).

3.3 Constraint Derivation

As many aspects of design theory apply to mathematical models, the usefulness of the theory is determined by how closely the model agrees with the physical problem under consideration. For example, optimization could yield a mathematically realizable system in theory but one that physically results in unrealistic responses. That is, the act of optimization could drive signals in parts of the model corresponding to the fixed elements of the control system to such high peak values that the model is no longer a valid approximation. Thus the design theory is restricted in usefulness. One method of avoiding this condition is to impose constraints upon the system to insure that the resultant optimum parameter choice yields a system that is practical. For the PID controller problem, a constraint condition that could be imposed upon the system is to force the output response and consequently the error, e(t), to be a specified value at some specified time, t. In this manner, the response can be controlled so as to avoid a condition of over-driven signals within the system.

From (3.13)

$$E(s) = \frac{s^2+a_1 s+a_2}{s^3+\alpha_2 s^2+\alpha_1 s+\alpha_o} \tag{3.15}$$

Therefore

$$E(s) = \frac{s^2(1+a_1/s+a_2/s^2)}{s^3(1+\alpha_2/s+\alpha_1/s^2+\alpha_o/s^3)} \tag{3.16}$$

Dividing the numerator by the denominator and disregarding terms greater than s^4 gives

$$E(s) \approx s^{-1}\left(1 + \frac{a_1-\alpha_2}{s} + \frac{a_2-\alpha_1-a_1\alpha_2+\alpha_2^{\,2}}{s^2} + \frac{2\alpha_1\alpha_2+\alpha_2^{\,2}-a_1\alpha_1-a_2\alpha_2-\alpha_2^{\,3}-\alpha_o}{s^3}\right) \tag{3.17}$$

which yields in the time domain,

$$e(t) = 1 + (a_1-\alpha_2)t + (a_2-\alpha_1-a_1\alpha_2+\alpha_2^2)\frac{t^2}{2} + (2\alpha_1\alpha_2+\alpha_2^2 a_1-\alpha_1 a_1-a_2\alpha_2-\alpha_2^3-\alpha_o)t^3/6 \tag{3.18}$$

By rearranging terms, (3.18) can be written as

$$\alpha_1(t^2/2+a_1t^3/6) + \alpha_2(t+a_1t^2/2+a_2t^3/6) + \alpha_2^3t^3/6 - 2(t^3/6)\alpha_1\alpha_2 -$$

$$(a_1t^3/6+t^2/2)\alpha_2^2 = 1 - e(t) + a_1t + a_2t^2/2 - \alpha_o t^3/6 \tag{3.19}$$

Now,

$$c(t) = r(t) - e(t) \tag{3.20}$$

and for a unit step input,

$$r(t) = 1 \tag{3.21}$$

Therefore

$$c(t) = 1 - e(t) \tag{3.22}$$

and

$$c(t) + a_1t + a_2t^2/2 - \alpha_o t^3/6 = \alpha_1(t^2/2+a_1t^3/6) + \alpha_2(t+a_1t^2/2+a_2t^3/6) +$$

$$\alpha_2^3t^3/6 - 2(t^3/6)\alpha_1\alpha_2 - (t^2/2+a_1t^3/6)\alpha_2^2 \tag{3.23}$$

In general $\alpha_o << 1$ and the term $\alpha_o t^3/6$ can be disregarded. Thus by specifying a value of c(t) at a time t, the L.H.S. of Equation (3.23) is a constant, say Y. Therefore, the constraint may be written as

$$Y = b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_2^3 - 2b_3\alpha_1\alpha_2 - b_1\alpha_2^2 \tag{3.24}$$

where

$$Y \triangleq c(t) + a_1t + a_2t^2/2 \tag{3.25}$$

$$b_1 \triangleq t^2/2 + a_1t^3/6 \tag{3.26}$$

$$b_2 \triangleq t + a_1t^2/2 + a_2t^3/6 \tag{3.27}$$

$$b_3 \triangleq t^3/6 \tag{3.28}$$

The optimization problem is now to minimize

$$ISE = \frac{\alpha_o\alpha_1 + (a_1^2-2a_2)\alpha_o + a_2^2\alpha_2}{2\alpha_o\alpha_1\alpha_2(1-\alpha_o/\alpha_1\alpha_2)} \tag{3.29}$$

subject to the constraint

$$Y \geq b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_2{}^3 - 2b_3\alpha_1\alpha_2 - b_1\alpha_2{}^2 \qquad (3.30)$$

Inspection of (3.29) and (3.30) indicate that the equations contain both positive and negative terms, and as such, are not solvable by ordinary geometric programming procedures. In addition, the number of terms compared to the number of variables, indicates a high degree of difficulty. To apply geometric programming to the optimization problem then requires some degree of approximation of the two functions in order to reduce the degree of difficulty and eliminate the negative terms. The following Sections contain various ways of approximation and a comparison of the results achieved by the different methods.

3.3.1 Approximation of the Constraint by a Straight Line

Consider the plant of Fig. 1 to have the following parameters.

$$a_1 = 2\zeta\omega_n = 0.25 \qquad (3.31)$$

$$a_2 = \omega_n{}^2 = 0.01 \qquad (3.32)$$

$$a_3 = \omega_n{}^2\gamma = 0.03 \qquad (3.33)$$

In addition, it is desired to make the output c(t) have the value 0.5 at time $\tau = 0.25$ seconds. Scaling the time axis by a factor of 10, and substituting the above values into the constraint equation, (3.30), gives:

$$Y = 1.15625 = 3.776\alpha_1 + 3.307\alpha_2 + 2.604\alpha_2{}^3 - 5.208\alpha_1\alpha_2 - 3.776\alpha_2{}^2 \qquad (3.34)$$

A plot of this equation is shown in Fig. 2. Since a large portion of the curve is nearly linear a simple method of approximating the constraint is by a straight line of the form

$$c_1\alpha_1 + c_2\alpha_2 = 1 \qquad (3.35)$$

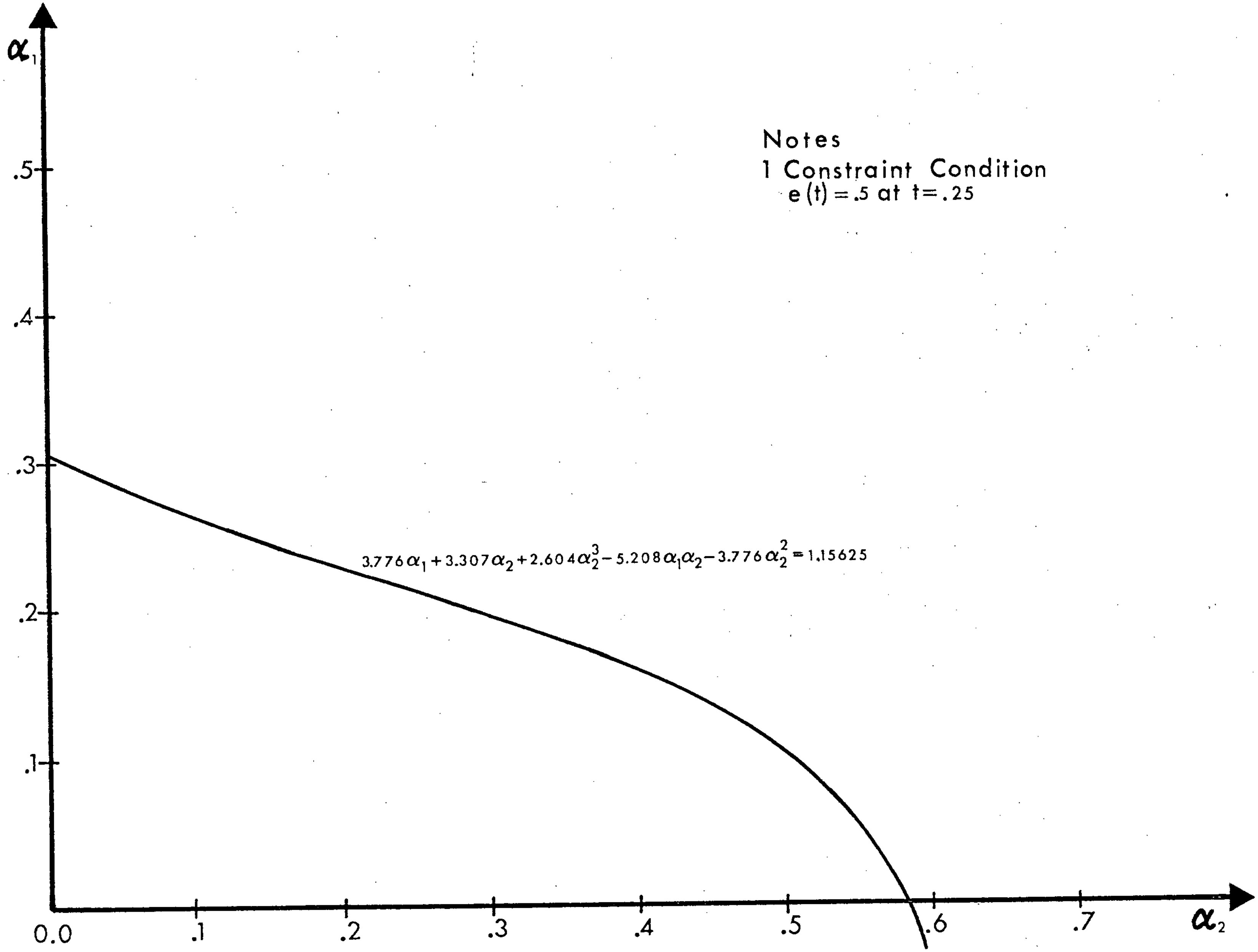


Figure 2 Constraint Equation

To determine the values of the parameters c_1 and c_2, a technique that matches (3.34) and (3.35) at one point plus equating their slopes is used.

(1) at the point $\alpha_2 = 0$

$$\alpha_1 = \frac{1}{c_1} = .3062 \tag{3.36}$$

Therefore,
$$c_1 = \frac{1}{\alpha_1} = 3.2669 \tag{3.37}$$

(2) at the point $\alpha_2 = 0$, $\alpha_1 = 0.3062$ the slope of (3.34) is

$$\frac{d\alpha_2}{d\alpha_1} = -\frac{3.776}{3.307 - 5.208(.3062)} = -2.2052 \tag{3.38}$$

and from (3.35)

$$\frac{d\alpha_2}{d\alpha_1} = -\frac{c_1}{c_2} \tag{3.39}$$

From (3.37), (3.38), and (3.39)

$$c_2 = 1.4815 \tag{3.40}$$

Therefore the original constraint equation has been approximated by the straight line,

$$3.2269\alpha_1 + 1.4815\alpha_2 = 1 \tag{3.41}$$

A plot of this function and Equation (3.34) are shown in Fig. 3.

The optimization problem is now to minimize (3.29) subject to the constraint of (3.41). Inspection of the equations indicate that only the denominator of (3.29) contains a negative coefficient (since, from (3.31) and (3.32), $a_1^2 > 2a_2$) and, as such, the expression is not a posynomial. However, for α_o small,

$$\left(1 - \frac{\alpha_o}{\alpha_1\alpha_2}\right)^{-1} \approx 1 + \frac{\alpha_o}{\alpha_1\alpha_2} \tag{3.42}$$

and

$$\text{ISE} \approx \frac{\alpha_o\alpha_1 + (a_1^2 - 2a_2)\alpha_o + a_2^2\alpha_2}{2\alpha_o\alpha_1\alpha_2}\left(1 + \frac{\alpha_o}{\alpha_1\alpha_2}\right) \tag{3.43}$$

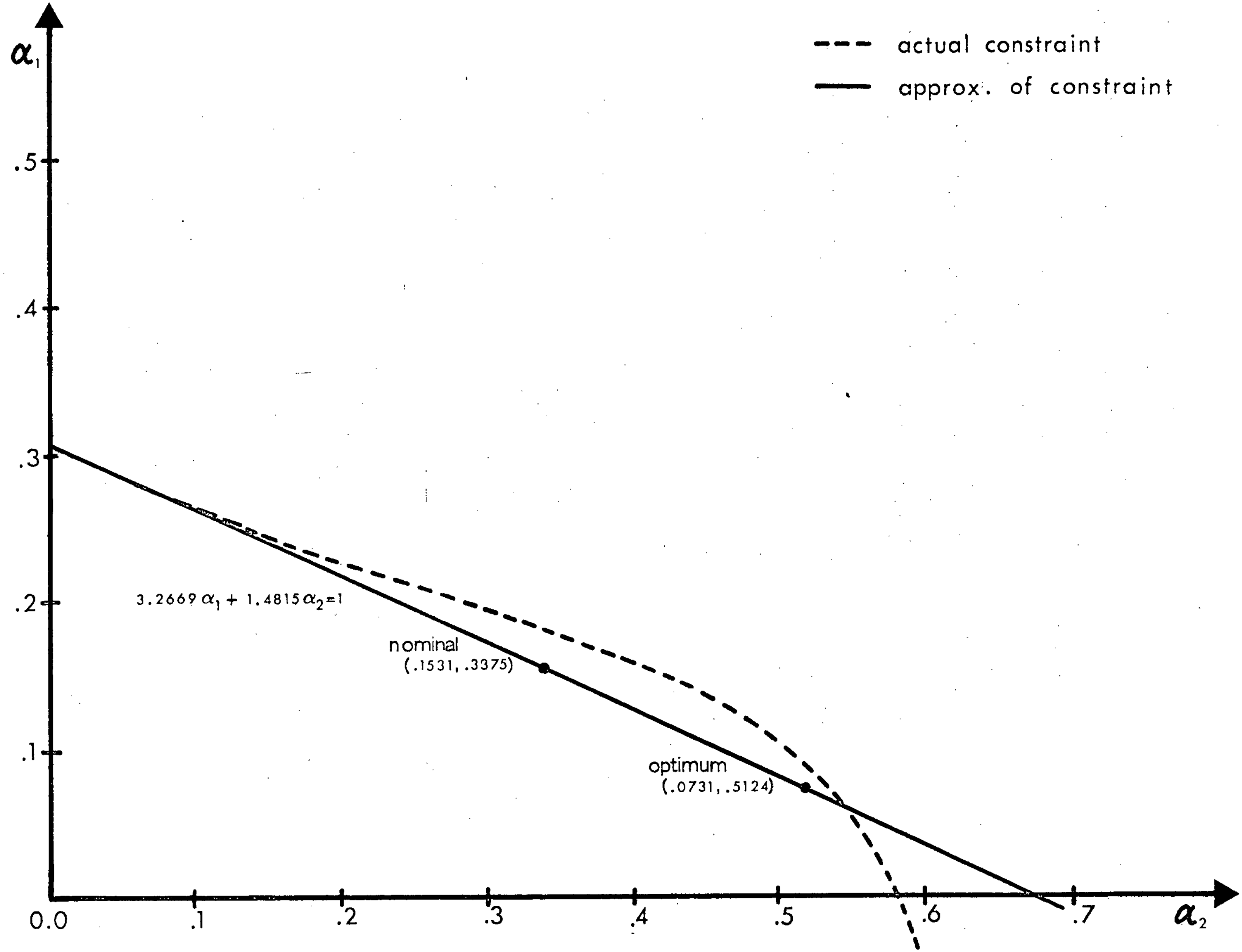


Figure 3 Linear Approximation of Constraint

Both (3.43) and (3.41) are posynomials, containing a total of eight terms. Thus,

$$N_p = 8 \tag{3.44}$$

and, since there are three primal variables,

$$m = 3 \tag{3.45}$$

From (1.22)

$$\text{degree of difficulty} = N_p - m - 1 = 4 \tag{3.46}$$

which indicates that a unique solution would be difficult to obtain by geometric programming. To overcome this, the number of terms in Equation (3.43) can be reduced to ultimately yield a zero degree of difficulty problem. Since (3.41) contains two terms, this requires (3.43) to be condensed to a total of two terms also.

Take as a nominal starting point

$$\bar{\alpha}_1 = .1531, \quad \bar{\alpha}_2 = .3375 \tag{3.47}$$

which lies at the approximate mid-point of (3.41). The initial value α_o, $\bar{\alpha}_o$, is chosen as that value which will minimize (3.43). That is,

$$\frac{\partial ISE}{\partial \alpha_o} = (1 + \frac{\alpha_o}{\alpha_1\alpha_2})(-\frac{a_2^{\,2}}{2\alpha_o^{\,2}\alpha_1}) + \frac{1}{\alpha_1\alpha_2}(\frac{1}{2\alpha_2} + \frac{a_1^{\,2}-2a_2}{2\alpha_1\alpha_2} + \frac{a_2^{\,2}}{2\alpha_o\alpha_1}) = 0 \tag{3.48}$$

Solving (3.48) gives

$$\alpha_o = \sqrt{\frac{a_2^{\,2}\alpha_2^{\,2}\alpha_1}{\alpha_1 + a_1^2 - 2a_2}} \tag{3.49}$$

Substituting, (3.31), (3.32), and (3.47) into (3.49) yields

$$\bar{\alpha}_o = 0.0031 \tag{3.50}$$

Condensation of the ISE into two terms is done by the method outlined in Section 2.1 .

Letting,

$$\phi_1 \triangleq 1 + \frac{\alpha_o}{\alpha_1\alpha_2} \tag{3.51}$$

then for

$$\bar{\alpha}_o = .0031, \quad \bar{\alpha}_1 = .1531, \quad \bar{\alpha}_2 = .3375 \tag{3.52}$$

$$\bar{\phi}_1 = 1 + \frac{\bar{\alpha}_o}{\bar{\alpha}_1\bar{\alpha}_2} = 1.609 \tag{3.53}$$

and

$$\left.\frac{\partial\phi_1}{\partial\alpha_o}\right|_{\bar{\alpha}_i} = \frac{1}{\bar{\alpha}_1\bar{\alpha}_2} = 19.3531 \tag{3.54}$$

$$\left.\frac{\partial\phi_1}{\partial\alpha_1}\right|_{\bar{\alpha}_i} = -\frac{\bar{\alpha}_o}{\bar{\alpha}_1^2\bar{\alpha}_2} = -.3919 \tag{3.55}$$

$$\left.\frac{\partial\phi}{\partial\alpha_2}\right|_{\bar{\alpha}_i} = -\frac{\bar{\alpha}_o}{\bar{\alpha}_1\bar{\alpha}_2^2} = -.1778 \tag{3.56}$$

Now

$$b_o = \bar{\alpha}_o \frac{\partial\phi_1}{\partial\alpha_o} \cdot \bar{\phi}_1^{-1} = .0566 \tag{3.57}$$

$$b_1 = \bar{\alpha}_1 \frac{\partial\phi_1}{\partial\alpha_1} \cdot \bar{\phi}_1^{-1} = -.0566 \tag{3.58}$$

$$b_2 = \bar{\alpha}_2 \frac{\partial\phi_1}{\partial\alpha_2} \cdot \bar{\phi}_1^{-1} = -.0566 \tag{3.59}$$

and,

$$\phi_1 \approx \bar{\phi}_1\left(\frac{\alpha_o}{\bar{\alpha}_o}\right)^{b_o}\left(\frac{\alpha_1}{\bar{\alpha}_1}\right)^{b_1}\left(\frac{\alpha_2}{\bar{\alpha}_2}\right)^{b_2} \tag{3.60}$$

Substituting (3.52), (3.53), and (3.57) through (3.59) into (3.60) gives as an approximation to (3.51), the single term expression

$$\phi_1 \approx 1.2440\left(\frac{\alpha_o}{\alpha_1\alpha_2}\right)^{.0566} \tag{3.61}$$

Similarily, letting

$$\phi_2 \triangleq \frac{\alpha_o\alpha_1 + (a_1{}^2-2a_2)\alpha_o + a_2{}^2\alpha_2}{2\alpha_o\alpha_1\alpha_2} \tag{3.62}$$

then, by rearranging terms

$$\phi_2 = \frac{1}{2\alpha_2}\left(1 + \frac{(a_1{}^2-2a_2)}{\alpha_1}\right) + \frac{a_2{}^2}{2\alpha_o\alpha_1} \tag{3.63}$$

If

$$k \triangleq 1 + \frac{a_1{}^2-2a_2}{\alpha_1} \tag{3.64}$$

then

$$\bar{k} = 1 + \frac{a_1{}^2-2a_2}{\bar{\alpha}_1} = 1.2776 \tag{3.65}$$

and

$$\left.\frac{\partial k}{\partial \alpha_1}\right|_{\bar{\alpha}_1} = -\frac{a_1{}^2-2a_2}{\alpha_1{}^2} = -1.8131 \tag{3.66}$$

$$b_o = \bar{\alpha}_1 \frac{\partial k}{\partial \alpha_1} \cdot \bar{k}_1{}^{-1} = -.2173 \tag{3.67}$$

Hence,

$$k \approx \bar{k}\left(\frac{\alpha_1}{\bar{\alpha}_1}\right)^{b_o} = .8497\alpha_1{}^{-.2173} \tag{3.68}$$

Now from (3.63),

$$\phi_2 = .4249\alpha_1{}^{-.2173}\alpha_2{}^{-1} + .00005\alpha_o{}^{-1}\alpha_1{}^{-1} \tag{3.69}$$

and since,

$$\text{ISE} = \phi_1 \cdot \phi_2 \tag{3.70}$$

therefore,

$$\text{ISE} \approx .5286\alpha_o{}^{.0566}\alpha_1{}^{-.2739}\alpha_2{}^{-1.0566} + .000062\alpha_o{}^{-.9434}\alpha_1{}^{-1.0566}\alpha_2{}^{-.0566} \tag{3.71}$$

Eq. (3.71) is an approximation of (3.43) in which the number of terms has been reduced to two. The optimization problem is now:

Minimize

$$\text{ISE} \approx .5286\alpha_o{}^{.0566}\alpha_1{}^{-.2739}\alpha_2{}^{-1.0566} + .000062\alpha_o{}^{-.9434}\alpha_1{}^{-1.0566}\alpha_2{}^{-.0566} \tag{3.72}$$

subject to

$$3.2669\alpha_1 + 1.4815\alpha_2 \leq 1 \tag{3.73}$$

Applying geometric programming to the above problem results in the formulation of the dual problem

Maximize

$$\psi(\delta) = \left(\frac{.5286}{\delta_1}\right)^{\delta_1}\left(\frac{.000062}{\delta_2}\right)^{\delta_2}\left(\frac{3.2669}{\delta_3}\right)^{\delta_3}\left(\frac{1.4815}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{\delta_3+\delta_4} \tag{3.74}$$

subject to the normality and orthogonality conditions given by (1.11) and (1.12)

Therefore

$$\delta_1 + \delta_2 = 1 \tag{3.75}$$

$$.0566\delta_1 - .9434\delta_2 = 0 \tag{3.76}$$

$$-.2739\delta_1 - 1.0566\delta_2 + \delta_3 = 0 \tag{3.77}$$

$$-1.0566\delta_1 - .0566\delta_2 + \delta_4 = 0 \tag{3.78}$$

Solution of Equation (3.75) through (3.78) results in

$$\delta_1{}^* = 0.9434 \tag{3.79}$$

$$\delta_2{}^* = 0.0566 \tag{3.80}$$

$$\delta_3{}^* = 0.3174 \tag{3.81}$$

$$\delta_4{}^* = 1 \tag{3.82}$$

and substitution of these values into (3.74) gives

$$\psi(\delta^*) = 1.7471 \tag{3.83}$$

From (1.14) the maximum of the dual function is equal to the minimum of the primal objective function

Hence

$$\min \text{ISE} = \max \psi(\delta^*) = 1.7471 \tag{3.84}$$

By Equations (1.17) and (1.18) the primal variables that give this result are determined from

(1) $0.5286\alpha_o^{.0566}\alpha_1^{-.2739}\alpha_2^{-1.0566} = \delta_1^*\psi(\delta^*) = 1.6483$ (3.85)

(2) $0.000062\alpha_o^{-.9434}\alpha_1^{-1.0566}\alpha_2^{-.0566} = \delta_2^*\psi(\delta^*) = 0.0989$ (3.86)

(3) $3.2669\alpha_1 = \delta_3^*/(\delta_3^* + \delta_4^*) = 0.2409$ (3.87)

(4) $1.4815\alpha_2 = \delta_4^*/(\delta_3^* + \delta_4^*) = 0.7591$ (3.88)

The values of α_o, α_1, and α_2 that simultaneously satisfy (3.85) through (3.88) are

$$\alpha_o^* = 0.0078 \qquad (3.89)$$

$$\alpha_1^* = 0.0737 \qquad (3.90)$$

$$\alpha_2^* = 0.5124 \qquad (3.91)$$

From Equations (3.9), (3.10), and (3.11), the system parameters are given as

$$K_c \doteq \frac{\alpha_1^* - a_2}{a_3} = 2.12 \qquad (3.92)$$

$$\tau_i = \frac{a_3K_c}{{}_o^*} = 8.17 \qquad (3.93)$$

$$\tau_d = \frac{\alpha_2^* - a_1}{a_3K_c} = 4.13 \qquad (3.94)$$

A plot of the output response of the system for a unit step displacement input is shown in Fig. 4. From the plot it can be seen that the constraining condition has been met since the output has an approximate value of 0.5 at a scaled time of t = 2.5 seconds. Additionally, a measure of the largest error between input and output during the transient state is called overshoot and can be defined by

$$\text{Per cent overshoot} = \frac{\text{Maximum overshoot}}{\text{Final desired value}} \times 100 \qquad (3.95)$$

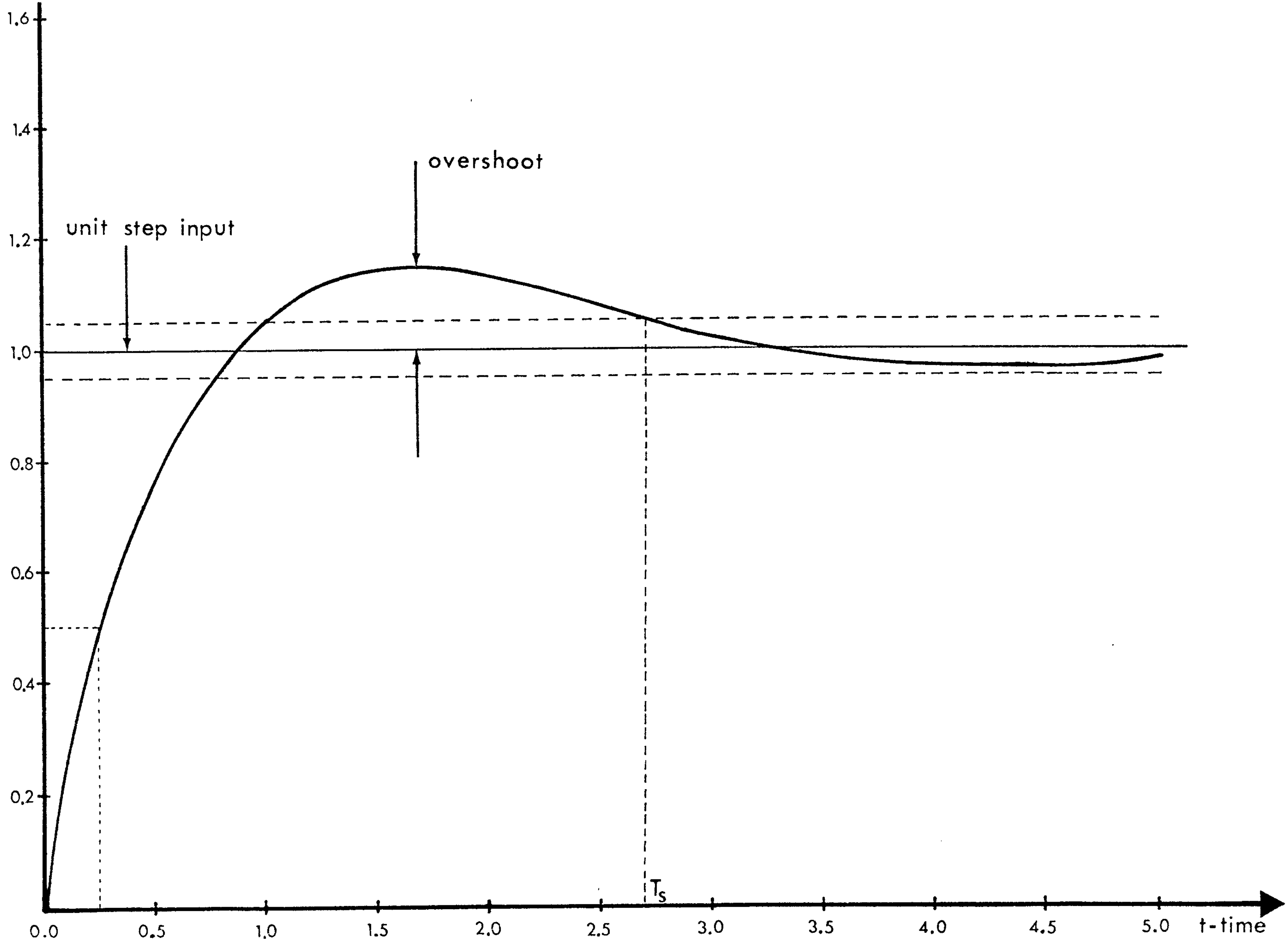


Figure 4 Output Response- Linear Constraint

From Fig. 4

$$\text{Per cent overshoot} = \frac{0.15}{1.00} \times 100 = 15\% \qquad (3.96)$$

The settling time can be defined as the time required for the response to decrease to and stay within a specified percentage, say 5 per cent of its final value. From Fig. 4

$$T_s \triangleq \text{settling time} = 2.7 \text{ seconds} \qquad (3.97)$$

Finally a measure of the degree of conformity of the approximations of (3.72) and (3.73) to the actual value of the integral square error and constraint can be obtained by comparing a computer calculated value of ISE = 2.0479 with the value of 1.7471 calculated using geometric programming. From the above, it can be seen that the methods and procedures utilized can produce valid results with a minimum of computational complexity.

3.3.2 Approximation of the Constraint by a Curved and Straight Line

Although the optimized value of the ISE obtained by making the approximations of Section 3.3.1 is close to the actual value calculated with no approximations, it is apparent that the agreement between the two can be improved. Furthermore it is noted, from Fig. 3, that while the straight line given by (3.41) generally follows the original constraint, there are regions of large deviation. An enhancement in this approximation can be obtained if Equation (3.41) is replaced by two curves - a straight line for that portion of the curve that appears to be fairly linear ($\alpha_1 > 0.16$) and a curved line for the non-linear portion of the constraint ($\alpha_1 < 0.16$).

With values previously given, from Equations (3.31), (3.32), and (3.33) the constraint equation is

$$1.15625 = 3.776\alpha_1 + 3.307\alpha_2 + 2.604\alpha_2^3 - 5.208\alpha_1\alpha_2 - 3.776\alpha_2^2 \quad (3.98)$$

For the region $\alpha_1 < 0.16$, (3.98) can be approximated by an equation of the form

$$c_1\alpha_1 + c_2\alpha_2^{1+\sigma} = 1 \quad (3.99)$$

A method that matches (3.98) and (3.99) at two different points, plus making their slopes equal at one of the points can be utilized to determine values for c_1, c_2, and σ.

(1) Substituting $\alpha_1 = 0$ into (3.98) and solving for α_2 yields

$$\alpha_2 = .5800 \quad (3.100)$$

(2) Similarly at the point $\alpha_1 = 0.16$

$$\alpha_2 = .3998 \quad (3.101)$$

(3) The slope of (3.98) is obtained by taking the derivative of α_2 with respect to α_1.

Therefore $$\frac{\partial\alpha_2}{\partial\alpha_1} = \frac{5.208\alpha_1 - 3.776}{3.307 + 7.81\alpha_2^2 - 5.208\alpha_1 - 7.552\alpha_2} \quad (3.102)$$

At $\alpha_1 = 0.16$, $\alpha_2 = 0.3998$,

$$\frac{\partial\alpha_2}{\partial\alpha_1} = -2.4090 \quad (3.103)$$

Now the slope of (3.99) is

$$\frac{\partial\alpha_2}{\partial\alpha_1} = -\frac{c_1}{c_2(1+\sigma)\alpha_2^{\sigma}} \quad (3.104)$$

and substitution of the points (0.0, 0.58) and (0.16, 0.3998) into (3.99) and equating (3.103) to (3.104) gives the following

$$c_2(.58)^{1+\sigma} = 1 \quad (3.105)$$

$$c_1(.16) + c_2(.3998)^{1+\sigma} = 1 \quad (3.106)$$

$$-\frac{c_1}{c_2(1+\sigma)(.3998)^{\sigma}} = -2.4090 \quad (3.107)$$

which can be solved to yield

$$c_1 = 5.0785 \tag{3.108}$$

$$c_2 = 11.6031 \tag{3.109}$$

$$\sigma = 3.5 \tag{3.110}$$

Thus the original constraint can be partially approximated by the equation

$$5.0785\alpha_1 + 11.631\alpha_2^{4.5} = 1 \tag{3.111}$$

A plot of (3.111) and (3.98) is shown in Fig. 5. From this, it can be seen that there is good agreement between the two equations for the region $\alpha_1 < .16$ but for $\alpha_1 > .16$, the approximation is no longer valid. However, in this region, the original constraint appears to be almost linear, and it can be approximated by a straight line of the form

$$c_3\alpha_1 + c_4\alpha_2 = 1 \tag{3.112}$$

Choosing two points on the original constraint, say (0.16, 0.3998) and (0.306, 0) and substituting into (3.112) gives

$$c_3 = 3.2680 \tag{3.113}$$

$$c_4 = 1.1934 \tag{3.114}$$

Thus, the additional approximation of the constraint curve is

$$3.2680\alpha_1 + 1.1934\alpha_2 = 1 \tag{3.115}$$

for the region $\alpha_1 > 0.16$. A plot of both approximations plus the original constraint curve is shown in Fig. 6. It is apparent that the agreement is good.

The optimization problem is now given by:

$$\text{Minimize ISE} = \frac{\alpha_o\alpha_1 + (a_1{}^2 - 2a_o)\alpha_o + a_2{}^2\alpha_2}{2\alpha_o\alpha_1\alpha_2(1 - \frac{\alpha_o}{\alpha_1\alpha_2})} \tag{3.116}$$

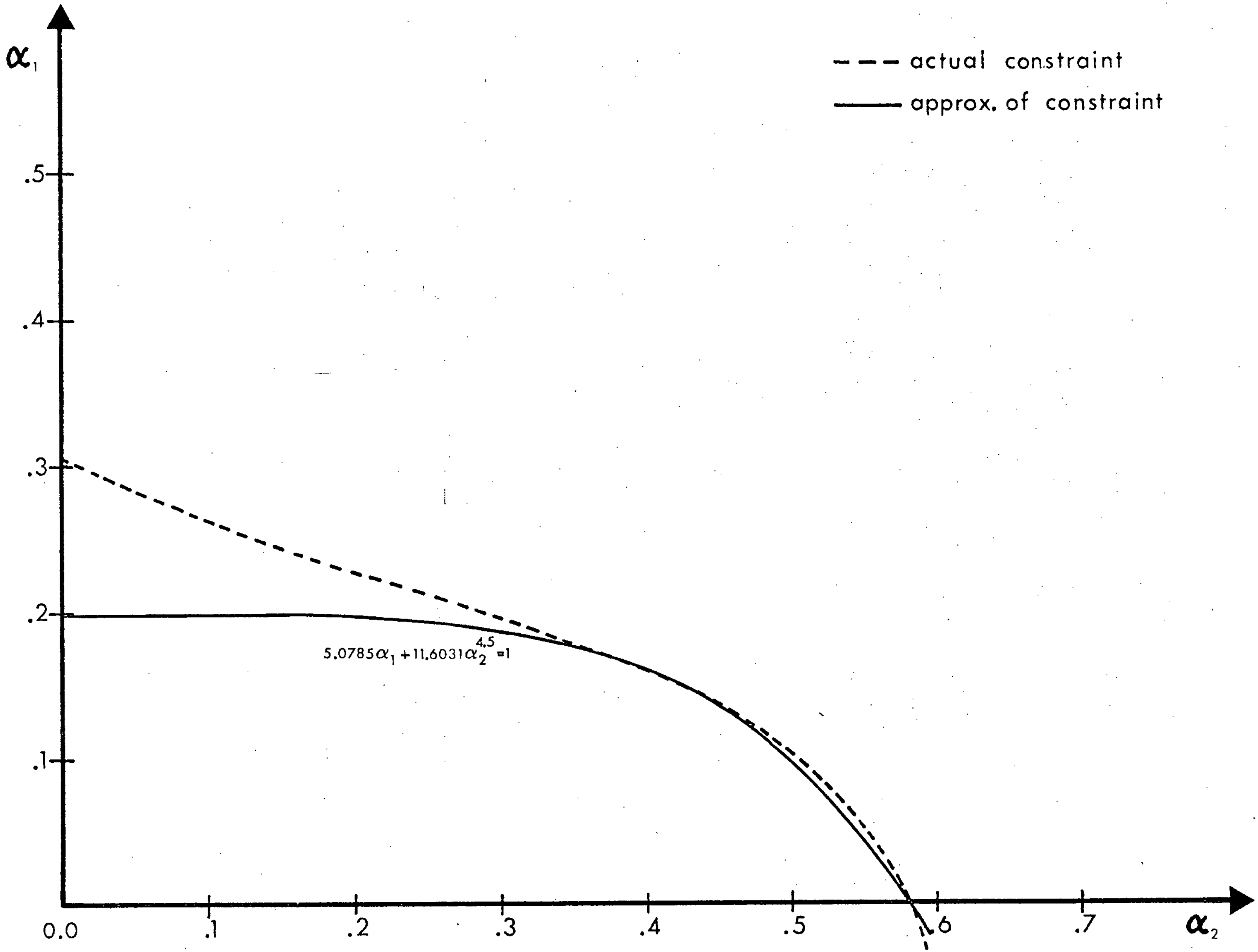


Figure 5 Partial Approximation of Constraint

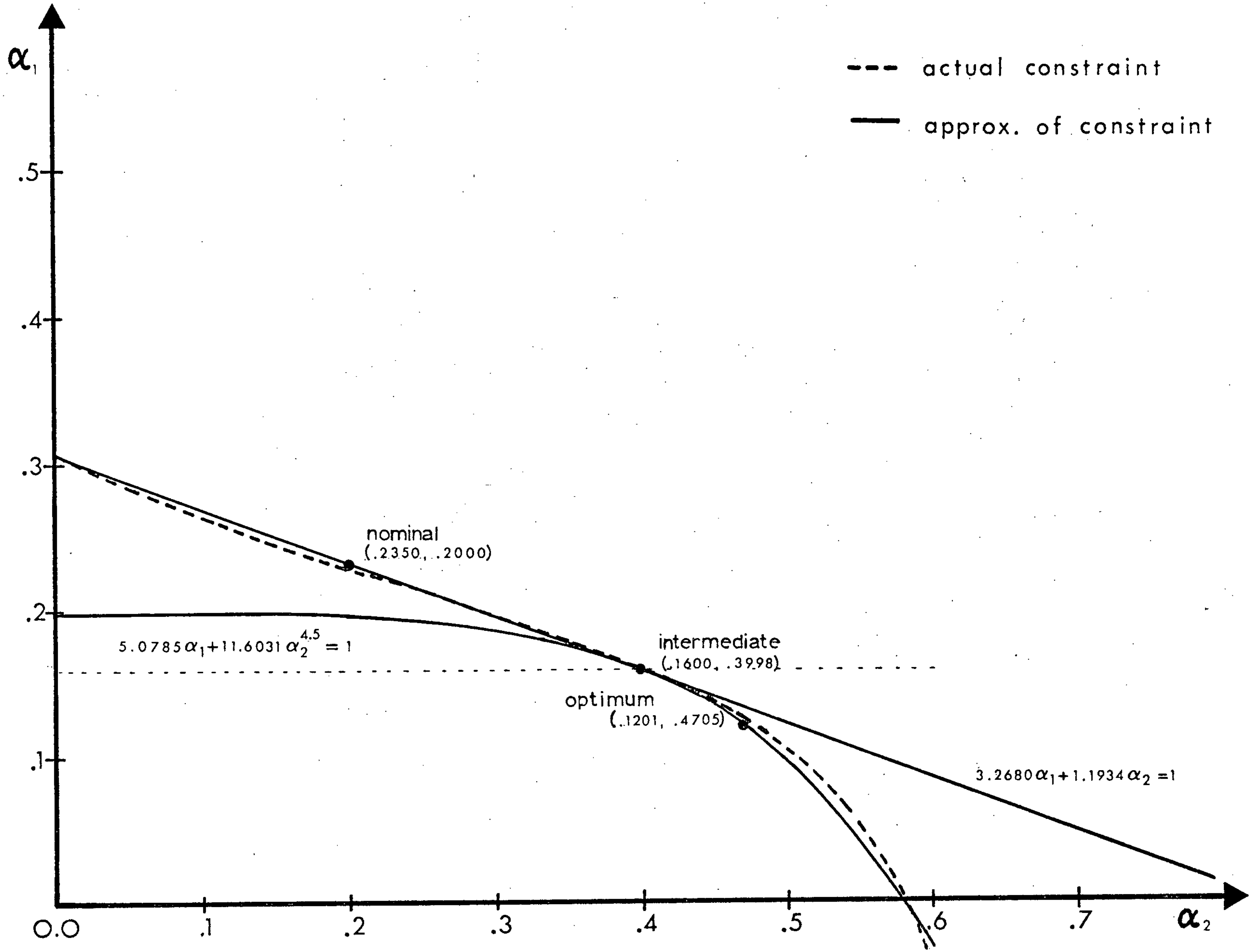


Figure 6 Two Line Approximation of Constraint

Subject to

$$5.0785\alpha_1 + 11.6031\alpha_2^{4.5} \leq 1 \tag{3.117}$$

and

$$3.2680\alpha_1 + 1.1934\alpha_2 \leq 1 \tag{3.118}$$

To facilitate a solution to this problem it is necessary to make the same type approximations and condensation of (3.116) as used in Section 3.3.1. In addition, since there are two constraint equations involved, computational complexity can be avoided if the problem is solved in a piecewise fashion. That is, if each constraint is considered individually it is possible to determine if only one is valid and at the same time, the degree of difficulty associated with the problem will remain low.

First consider (3.118) to be the effective constraint and take as initial conditions, the point $\bar{\alpha}_1 = 0.235$, $\bar{\alpha}_2 = 0.2$. In a manner similar to that of Section 3.3.1, (3.116) can be approximated by the two term expression

$$ISE \approx .5535\alpha_o^{.0364}\alpha_1^{-.1896}\alpha_2^{-1.0364} + .000059\alpha_o^{-.9636}\alpha_1^{-1.0364}\alpha_2^{-.0364} \tag{3.119}$$

Now consider a bound on α_1, that is $\alpha_1 \geq a$. Thus, the optimization problem is to minimize (3.119) subject to (3.118) and

$$a_1\alpha_1^{-1} \leq 1 \tag{3.120}$$

Applying the concept of duality to the problem results in the formation of the alternate problem of maximizing the dual objective function subject to the constraints of normality and orthogonality. This is given as

$$\text{Max. } \psi(\delta) = \left(\frac{.5535}{\delta_1}\right)^{\delta_1}\left(\frac{.000059}{\delta_2}\right)^{\delta_2}\left(\frac{3.2680}{\delta_3}\right)^{\delta_3}\left(\frac{1.1934}{\delta_4}\right)^{\delta_4}\left(\frac{a}{\delta_5}\right)^{\delta_5}(\delta_3+\delta_4)^{\delta_3+\delta_4}\delta_5^{\delta_5} \tag{3.121}$$

subject to

$$\delta_1 + \delta_2 = 1 \tag{3.122}$$

$$.0364\delta_1 - .9636\delta_2 = 0 \tag{3.123}$$

$$-.1896\delta_1 - 1.0364\delta_2+\delta_3 - \delta_5 = 0 \tag{3.124}$$

$$-1.0364\delta_1 - .0364\delta_2 + \delta_4 = 0 \tag{3.125}$$

Since (3.122) through (3.125) represent a set of four linear equations in five unknowns, there is no unique solution. However, solving in terms of the variable δ_3 yields

$$\delta_1{}^* = .9636 \tag{3.126}$$

$$\delta_2{}^* = .0364 \tag{3.127}$$

$$\delta_3{}^* = \delta_3{}^* \tag{3.128}$$

$$\delta_4{}^* = 1 \tag{3.129}$$

$$\delta_5{}^* = \delta_3{}^* - .2204 \tag{3.130}$$

Substituting into (3.121) now gives

$$\psi(\delta^*) = \left(\frac{.5535}{.9636}\right)^{.9636}\left(\frac{.000059}{.0364}\right)^{.0364}\left(\frac{3.2680}{\delta_3{}^*}\right)^{\delta_3{}^*}\left(\frac{1.1934}{1}\right)^{1}\cdot$$
$$(\delta_3{}^*+1)^{\delta_3{}^*+1}\left(\frac{a}{\delta_3{}^*-.2204}\right)^{\delta_3{}^*-.2204}(\delta_3{}^*-.2204)^{\delta_3{}^*-.2204} \tag{3.131}$$

Hence

$$\psi(\delta^*) = .5415\left(\frac{3.2680}{\delta_3{}^*}\right)^{\delta_3}\ a^{\delta_3{}^*-.2204}(\delta_3{}^*+1)^{\delta_3{}^*+1} \tag{3.132}$$

It has been shown (Duffin, Peterson, Zener, 1967) that the functions $\psi(\delta)$ and $\ln\,\psi(\delta)$ have the same maximizing points. Consequently the maximum value of (3.132) can be obtained by taking the natural logarithm of both sides of the equation, taking the derivative with respect to $\delta_3{}^*$ and then equating to zero. This gives

$$\ell n\ \psi(\delta^*) = -.6134+\delta_3^*[\ell n\ 3.2680-\ell n\delta_3^*] + (\delta_3^*-.2204)\ell n\ a + (\delta_3^*+1)\ell n(\delta_3^*+1) \tag{3.133}$$

and

$$\frac{\partial \ell n\ \psi(\delta^*)}{\partial \delta_3^*} = \ell n\ 3.2680 - \ell n\delta_3^* - \frac{\delta_3^*}{\delta_3^*} + \ell n\ a + \ell n(\delta_3^*+1) + (\frac{\delta_3^*+1}{\delta_3^*+1}) = 0 \tag{3.134}$$

Therefore

$$\ell n\ 3.2680a = \ell n[\delta_3^*/(\delta_3^*+1)] \tag{3.135}$$

and

$$3.2680a = \frac{\delta_3^*}{\delta_3^*+1} \tag{3.136}$$

Note

$$\frac{\partial^2 \ell n\psi}{\partial \delta_3^{*2}} = -\frac{1}{\delta_3^*} + \frac{1}{\delta_3^*+1} = -\frac{1}{\delta_3^*(\delta_3^*+1)} < 0 \text{ for } \delta_3^* > 0$$

Equation (3.136) indicates the conditions for a maximum value of the dual objective function. In addition, from Equation (1.18) the following holds

$$3.2680\alpha_1 = \frac{\delta_3^*}{\delta_3^*+1} \tag{3.137}$$

and

$$a\alpha_1^{-1} = 1 \tag{3.138}$$

Substituting $a = \alpha_1$ into (3.137) yields

$$3.2680a = \frac{\delta_3^*}{\delta_3^*+1} \tag{3.139}$$

which is identical to that condition necessary for a maximum of the dual function as shown in Equation (3.136). This implies that the optimum solution lies on the line $\alpha_1 = a$. Consequently, the region for $\alpha_1 > a$ can be disregarded in the search for a global optimum. That is, if $a = 0.16$, then α_1^* has a value of 0.16 or less and the constraint given by

$$3.2680\alpha_1 + 1.1934\alpha_2 = 1$$

should no longer be considered applicable since the optimum lies in the region $\alpha_1 < 0.16$. The problem now consists of minimizing (3.116) subject to the single constraint given by (3.117).

Taking a = 0.16 and substituting into (3.136) yields

$$\delta_3{}^* = 1.0960 \tag{3.140}$$

Therefore, from (3.132)

$$\psi(\delta^*) = .5415(\frac{3.2680}{1.0960})^{1.0960}(.16)^{.8756}(2.0960)^{2.0960} = 1.6997 \tag{3.141}$$

Equation (3.141) gives a sub-optimal value to the problem that is attainable with the constraints imposed by (3.117) and (3.120). Under these conditions the following also holds true:

$$.5535\alpha_o{}^{*.0364}\alpha_1{}^{*-.1896}\alpha_2{}^{*-1.0364} = \delta_1{}^*\psi(\delta^*) = 1.6375 \tag{3.142}$$

$$.000059\alpha_o{}^{*-.9636}\alpha_1{}^{*-1.0364}\alpha_2{}^{*-.0364} = \delta_2{}^*\psi(\delta^*) = .0619 \tag{3.143}$$

$$3.2680\alpha_1{}^* = \delta_3{}^*/(\delta_3{}^* + \delta_4{}^*) = .5229 \tag{3.144}$$

$$1.1934\alpha_2{}^* = \delta_4{}^*/(\delta_3{}^* + \delta_4{}^*) = .4771 \tag{3.145}$$

$$.16\alpha_1{}^{*-1} = 1 \tag{3.146}$$

A solution to the above set of equations gives

$$\alpha_o{}^* = 0.0053 \tag{3.147}$$

$$\alpha_1{}^* = 0.16 \tag{3.148}$$

$$\alpha_2{}^* = 0.3998 \tag{3.149}$$

With these values,(3.116) is re-condensed in a manner similar to before to give

$$\text{ISE} \approx .5649\alpha_o^{.0768}\alpha_1^{-.2867}\alpha_2^{-1.0768} + .000066\alpha_o^{-.9232}\alpha_1^{-1.0768}\alpha_2^{-.0768} \tag{3.150}$$

which is now to be minimized subject to the constraint

$$5.0785\alpha_1 + 11.6031\alpha_2{}^{4.5} \leq 1 \tag{3.151}$$

Applying the concept of duality results in maximizing

$$\psi(\delta) = \left(\frac{.5649}{\delta_1}\right)^{\delta_1}\left(\frac{.000066}{\delta_2}\right)^{\delta_2}\left(\frac{5.0785}{\delta_3}\right)^{\delta_3}\left(\frac{11.6031}{\delta_4}\right)^{\delta_4}(\delta_3+\delta_4)^{\delta_3+\delta_4} \tag{3.152}$$

subject to

$$\delta_1 + \delta_2 = 1 \tag{3.153}$$

$$.0768\delta_1 - .9232\delta_2 = 0 \tag{3.154}$$

$$-.2867\delta_1 - 1.0768\delta_2 + \delta_3 = 0 \tag{3.155}$$

$$-1.0768\delta_1 - .0768\delta_2 + 4.5\delta_4 = 0 \tag{3.156}$$

Equations (3.153) through (3.156) can be solved uniquely to give

$$\delta_1' = 0.9232 \tag{3.157}$$

$$\delta_2' = 0.0768 \tag{3.158}$$

$$\delta_3' = 0.3474 \tag{3.159}$$

$$\delta_4' = 0.2222 \tag{3.160}$$

Therefore,

$$\psi(\delta') = \left(\frac{.5649}{.9232}\right)^{.9232}\left(\frac{.000066}{.0768}\right)^{.0768}\left(\frac{5.0785}{.3474}\right)^{.3474}\left(\frac{11.6031}{.2222}\right)^{.2222}(.5696)^{.5696}$$
$$= 1.6397 \tag{3.161}$$

which is the optimum value of the integral square error. From (1.17) and (1.18) the value of the system parameters that give this optimum are determined from

$$.5649\alpha_o^{.0768}\alpha_1^{-.2867}\alpha_2^{-1.0768} = \delta_1'\psi(\delta') = 1.5138 \tag{3.162}$$

$$.000066\alpha_o^{-.9232}\alpha_1^{-1.0768}\alpha_2^{-.0768} = \delta_2'\psi(\delta') = .1259 \tag{3.163}$$

$$5.0785\alpha_1 = \delta_3'/(\delta_3' + \delta_4') = .6099 \tag{3.164}$$

$$11.6031\alpha_2^{4.5} = \delta_4'/(\delta_3' + \delta_4') = .3901 \tag{3.165}$$

which yield

$$\alpha_o' = .0035 \tag{3.166}$$

$$\alpha_1' = .1201 \tag{3.167}$$

$$\alpha_2' = .4705 \tag{3.168}$$

Substitution of (3.166), (3.167) and (3.168) into (3.9), (3.10) and (3.11) gives the following optimum controller parameters

$$K_c = 3.67 \tag{3.169}$$

$$\tau_i = 31.46 \tag{3.170}$$

$$\tau_d = 2.00 \tag{3.171}$$

Fig. 7 is a plot of the output response of the system with the above controller parameters. From the plot it is seen that

$$\text{Per cent overshoot} = \frac{.06}{1.00} \times 100 = 6\%$$

and

$$T_s = 1.15 \text{ seconds}$$

It is also evident that the constraint condition ($e(t) = .5$ at $t = 2.5$) is met. The computer calculated value of the integral square error is 1.6612 which compares favorably with the value of 1.6397 determined by geometric programming. This indicates that the method of representing the constraint equation by two less complex expressions and condensing the objective function into a form convenient for geometric programming can yield very good results.

3.3.3 Approximation of the Constraint by the Method of Duffin, Peterson and Zener

The Duffin, et al, approximation procedure of Section 2.1 is

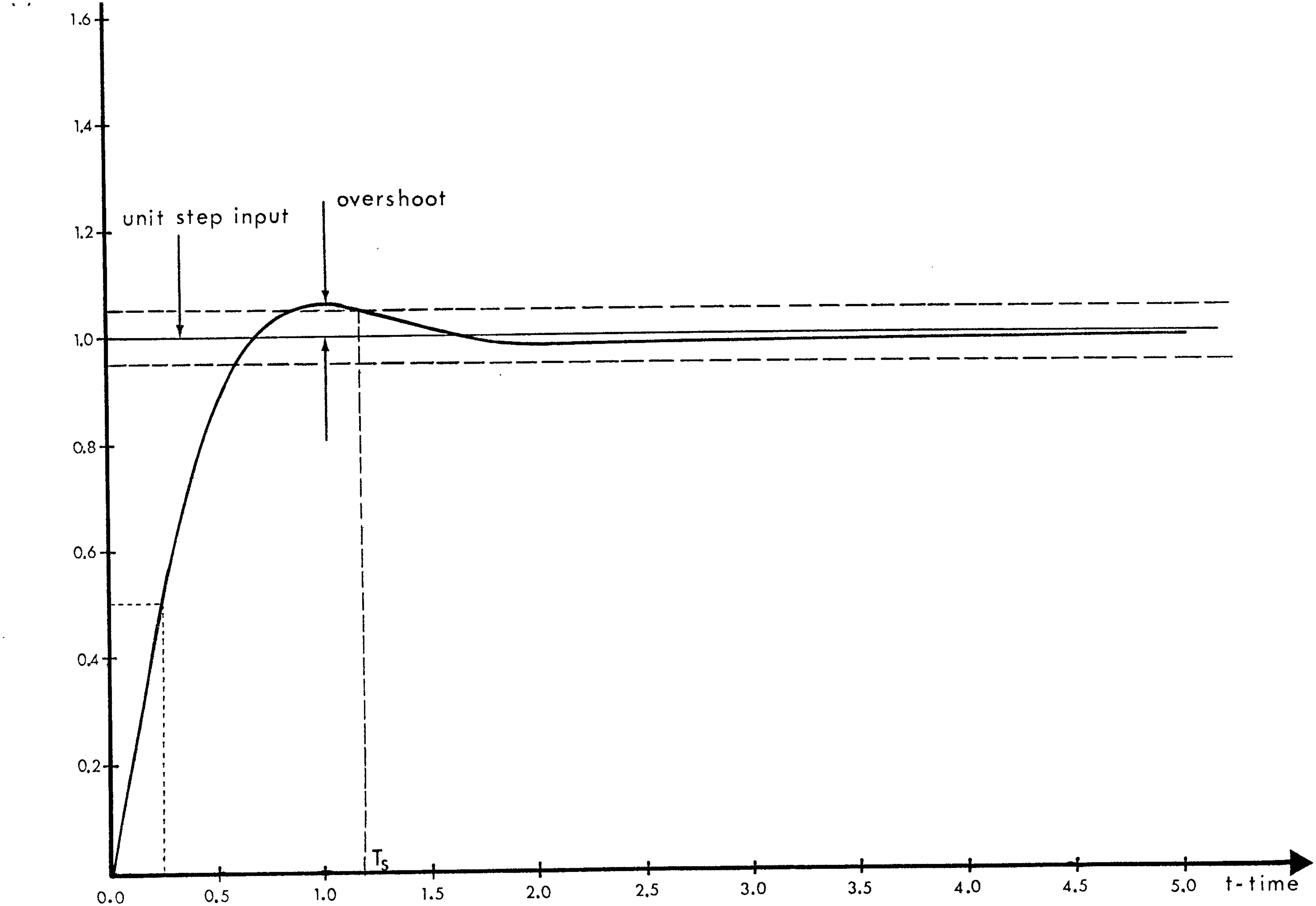


Figure 7 Output Response – Two Line Approx. of Constraint

very powerful as it allows the problems of non-posynomial terms and high degrees of difficulty to be overcome. The method does have a drawback as the approximation is often a very poor representation of a function throughout its entire domain. However, if the condensation can be made in a region that is close to the optimum, then it is possible to obtain valid results.

By re-arranging (3.18) an expression for the error can be written as

$$e(t) = 1 + a_1 t + (a_2+\alpha_2^2)t^2/2 + (2\alpha_1\alpha_2 + a_1\alpha_2^2)t^3/6 - [\alpha_2 t + (\alpha_1+a_1\alpha_2)t^2/2 + (\alpha_o+a_1\alpha_1+a_2\alpha_2+\alpha_2^3)t^3/6 \quad (3.172)$$

Since

$$c(t) = 1 - e(t) \quad (3.173)$$

Therefore

$$-c(t) = a_1 t + (a_2+\alpha_2^2)t^2/2 + (2\alpha_1\alpha_2 + a_1\alpha_2^2)t^3/6 - [\alpha_2 t + (\alpha_1+a_1\alpha_2)t^2/2 + (\alpha_o+a_1\alpha_1+a_2\alpha_2+\alpha_2^3)t^3/6 \quad (3.174)$$

Let

$$g_1 \triangleq \alpha_o t^3/6 + \alpha_1(t^2/2+a_1 t^3/6) + \alpha_2(t+a_1 t^2/2+a_2 t^3/6) + \alpha_2^3 t^3/6 \quad (3.175)$$

and

$$g_2 \triangleq c(t) + a_1 t + a_2 t^2/2 + 2\alpha_1\alpha_2 t^3/6 + \alpha_2^2(t^2/2+a_1 t^3/6) \quad (3.176)$$

For the same plant parameters and constraint condition, i.e. $e(t) = .5$ at $t = 2.5$ as in Section 3.3.1, Equations (3.175) and (3.176) give:

$$g_1 = 2.604\alpha_o + 3.776\alpha_1 + 3.307\alpha_2 + 2.604\alpha_2^2 \quad (3.177)$$

$$g_2 = 1.156 + 5.208\alpha_1\alpha_2 + 3.776\alpha_2^2 \quad (3.178)$$

and for the constraint condition

$$g_1 = g_2 \quad (3.179)$$

then

$$2.604\alpha_o + 3.776\alpha_1 + 3.307\alpha_2 + 2.604\alpha_2^2 = 1.156 + 5.208\alpha_1\alpha_2 + 3.776\alpha_2^2 \tag{3.180}$$

To determine a feasible point around which to approximate the constraint condition consider the following. From (3.174), the slope of the output response is

$$-\frac{dc}{dt} = a_1 + (a_2+\alpha_2^2)t + (2\alpha_1\alpha_2 + a_1\alpha_2^2)t^2/2 - [\alpha_2 + (\alpha_1+a_1\alpha_2)t + (\alpha_o+a_1\alpha_1+a_2\alpha_2+\alpha_2^3)t^2/2] \tag{3.181}$$

at $t = 0$,

$$-\frac{dc}{dt} = a_1 - \alpha_2 \tag{3.182}$$

Since the desired output response has a slope of 0.2, i.e. $\Delta c/\Delta t = .5/2.5$, for small values of t, then substituting this into Equation (3.182) gives

$$\bar{\alpha}_2 = .2 + a_1 = .2 + .25 = .45 \tag{3.183}$$

Now assume that $\alpha_o = 0$ and since $g_1 = g_2$,

$$3.776\alpha_1 + 3.307(.45) + 2.604(.45)^3 = 1.156 + 5.208(.45)\alpha_1 + 3.776(.45)^2 \tag{3.184}$$

from which

$$\alpha_1 = 0.137 \tag{3.185}$$

It has been shown by Equation (3.44) that a nominal $\bar{\alpha}_o$ can be obtained once values of α_1 and α_2 are known. From (3.183) and (3.185) then

$$\bar{\alpha}_o = \sqrt{\frac{a_2{}^2\alpha^2\alpha_1}{\alpha_1 + a_1{}^2 - 2a_2}} = .00393 \tag{3.186}$$

Substituting (3.186) and (3.183) into (3.184) now gives

$$\bar{\alpha}_1 = .1297 \tag{3.187}$$

Using (3.183), (3.186) and (3.187) a condensation of the ISE, Equation (3.14), can be made in a manner similar to that previously calculated. This gives:

$$ISE \approx .5076\alpha_o^{.0633}\alpha_1^{-.3108}\alpha_2^{-1.0633} + .000063\alpha_o^{-.9367}\alpha_1^{-1.0633}\alpha_2^{-.0633} \tag{3.188}$$

From (3.177) let

$$g_1 = K_1 + 2.604\alpha_2{}^3 \tag{3.189}$$

where

$$K_1 \triangleq 2.604\alpha_o + 3.776\alpha_1 + 3.307\alpha_2 \tag{3.190}$$

Now condense K_1 into a single posynomial term around $\bar{\alpha}_o$, $\bar{\alpha}_1$, $\bar{\alpha}_2$ by the method of Duffin:

$$\bar{K}_1 = 2.604\bar{\alpha}_o + 3.776\bar{\alpha}_1 + 3.307\bar{\alpha}_2 = 1.9881 \tag{3.191}$$

and

$$\left.\frac{\partial K_1}{\partial \alpha_o}\right|_{\bar{\alpha}_i} = 2.604 \tag{3.192}$$

$$\left.\frac{\partial K_1}{\partial \alpha_1}\right|_{\bar{\alpha}_i} = 3.776 \tag{3.193}$$

$$\left.\frac{\partial K_1}{\partial \alpha_2}\right|_{\bar{\alpha}_i} = 3.307 \tag{3.194}$$

From which

$$b_o = \bar{\alpha}_o \left.\frac{\partial K_1}{\partial \alpha_o}\right|_{\bar{\alpha}_i} \cdot \bar{K}_1^{-1} = .0051 \tag{3.195}$$

$$b_1 = \bar{\alpha}_1 \left.\frac{\partial K_1}{\partial \alpha_1}\right|_{\bar{\alpha}_i} \cdot \bar{K}_1^{-1} = .2463 \tag{3.196}$$

$$b_2 = \bar{\alpha}_2 \left.\frac{\partial K_1}{\partial \alpha_2}\right|_{\bar{\alpha}_i} \cdot \bar{K}_1^{-1} = .7485 \tag{3.197}$$

and

$$K_1 \approx \bar{K}_1\left(\frac{\alpha_o}{\bar{\alpha}_o}\right)^{b_o}\left(\frac{\alpha_1}{\bar{\alpha}_1}\right)^{b_1}\left(\frac{\alpha_2}{\bar{\alpha}_2}\right)^{b_2} = 6.1484\alpha_o^{.0051}\alpha_1^{.2463}\alpha_2^{.7485} \tag{3.198}$$

Therefore

$$g_1 \approx 6.1484\alpha_o^{.0051}\alpha_1^{.2463}\alpha_2^{.7485} + 2.604\alpha_2^{3} \tag{3.199}$$

In a similar fashion g_2 can be condensed to give

$$g_2 \approx 5.6783\alpha_1^{.1366}\alpha_2^{.8289} \tag{3.200}$$

Substituting (3.199) and (3.200) into (3.179),

$$6.1484\alpha_o^{.0051}\alpha_1^{.2463}\alpha_2^{.7485} + 2.604\alpha_2^{3} = 5.6783\alpha_1^{.1366}\alpha_2^{.8289} \tag{3.202}$$

which can be re-written as

$$\frac{6.1484\alpha_o^{.0051}\alpha_1^{.2463}\alpha_2^{.7485} + 2.604\alpha_2^{3}}{5.6783\alpha_1^{.1366}\alpha_2^{.8289}} = 1 \tag{3.202}$$

Therefore

$$1.0828\alpha_o^{.0051}\alpha_1^{.1097}\alpha_2^{-.0754} + .4586\alpha_1^{-.1366}\alpha_2^{2.1761} = 1 \tag{3.203}$$

To satisfy the requirements of geometric programming this is written as an inequality constraint and the problem in an approximated form is:

Minimize

$$ISE = .5076\alpha_o^{.0638}\alpha_1^{-.3208}\alpha_2^{-1.0633} + .000063\alpha_o^{.9367}\alpha_1^{-1.0633}\alpha_2^{-.0633} \tag{3.204}$$

subject to

$$1.0828\alpha_o^{.0051}\alpha_1^{.1097}\alpha_2^{-.0754} + .4586\alpha_1^{-.1366}\alpha_2^{2.1761} \leq 1 \tag{3.205}$$

A plot of the approximate form of the constraint equation is shown in Fig. 8. It can be seen that over a small region the approximation holds to the original constraint but diverges rapidly beyond the region of interest.

Solving (3.204) and (3.205) in the standard geometric programming manner gives

$$\text{Max } \psi(\delta^*) = \text{min ISE} = 1.66226 \tag{3.206}$$

and

$$\alpha_o^* = 0.0033 \tag{3.207}$$

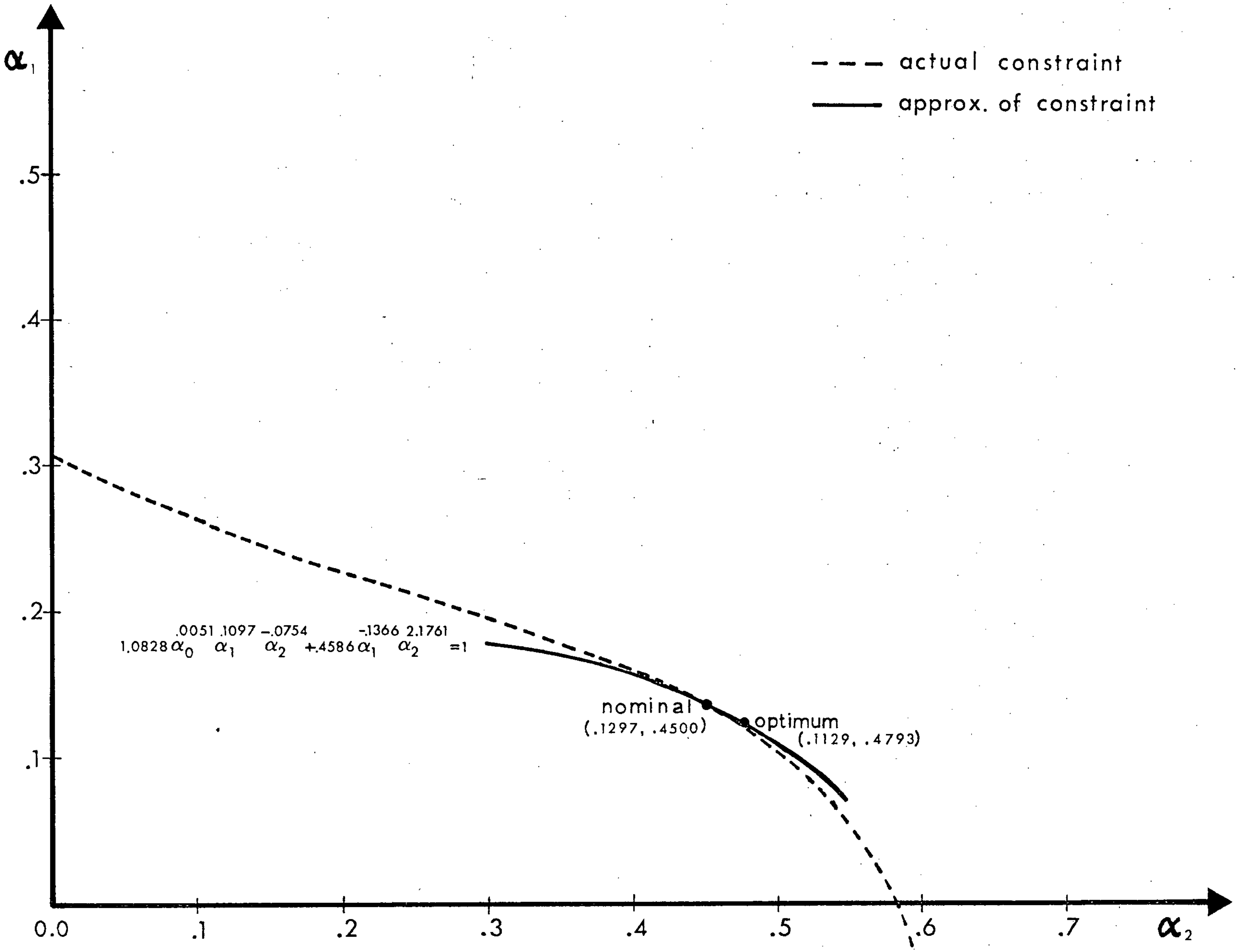


Figure 8 Duffin Approximation of Constraint

$$\alpha_1* = 0.1129 \tag{3.208}$$

$$\alpha_2* = 0.4793 \tag{3.209}$$

from which, by Equations (3.9), (3.10) and (3.11)

$$K_c = 3.43 \tag{3.210}$$

$$\tau_i = 30.89 \tag{3.211}$$

$$\tau_d = 2.23 \tag{3.212}$$

A plot of the output response is shown in Fig. 9. It can be seen that the constraint condition has been met and

$$\text{Per cent overshoot} = \frac{.04}{1.00} \times 100 = 4\% \tag{3.213}$$

In addition, the response settles right into the 5 per cent limit. The computer calculated value of the integral square error is 1.6707 which compares favorably with the approximate value of 1.6623. It is interesting to note that although the response of Fig. 9 appears better than that of Section 3.3.2 the value of the performance index is higher (1.66226 versus 1.66106). This is due to the fact that the response of Fig. 9 rises slower than that of Fig. 7 resulting in a slightly larger error.

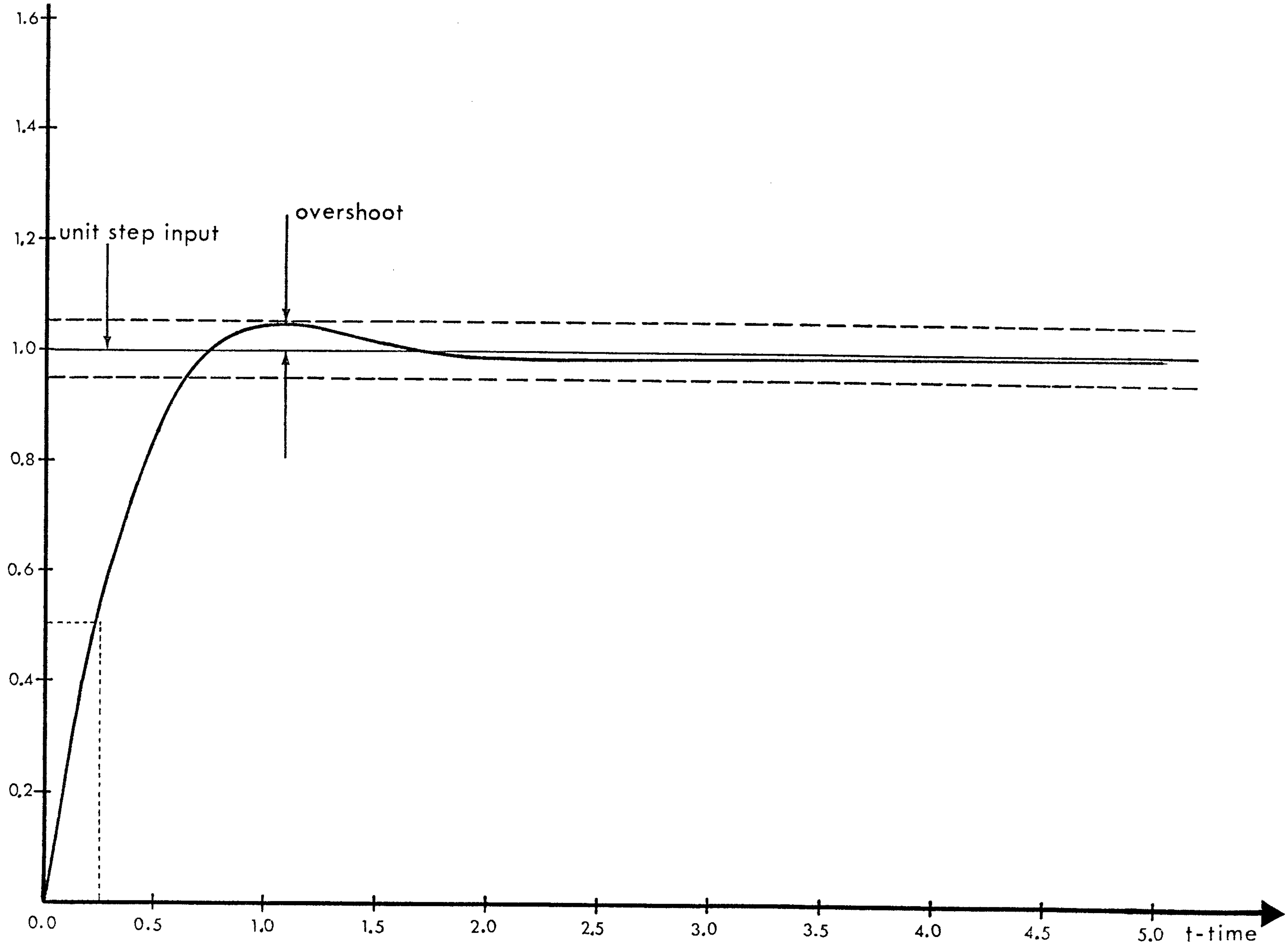
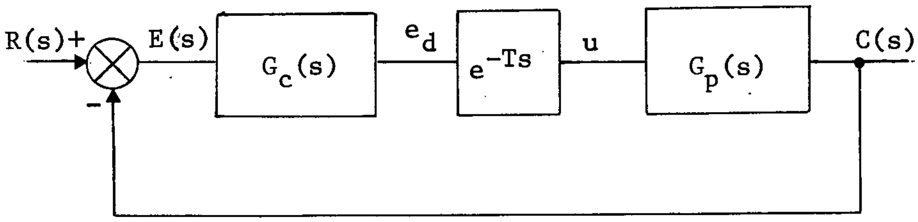


Figure 9 Output Response- Duffin Approx. of Constraint

IV. GEOMETRIC PROGRAMMING APPLICATION TO A PID CONTROL SYSTEM WITH TIME DELAY

4.1 System Definitions and Constraint Derivation

A large number of control systems are characterized by the fact that the output responds to a transient input only after a given time interval. Due to the time delay effect, the transfer function of these systems are no longer quotients of polynomials, but usually contain the term e^{-Ts} where T denotes the time delay or transportation lag. Figure 10 depicts the PID controller system with time delay.



$$G_p(s) \triangleq \text{plant} = \frac{\omega_n^2 \gamma}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G_c(s) \triangleq \text{controller} = K_c(1 + \frac{1}{\tau_i s} + \tau_d s)$$

$$e^{-Ts} \triangleq \text{time delay}$$

Fig. 10 PID Controller with Time Delay

The over-all system transfer function is

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)e^{-Ts}}{1 + G_c(s)G_p(s)e^{-Ts}} \tag{4.1}$$

As a result of the numerator exponential, there is a direct lag of T seconds between input and output. In addition, the closed loop performance of the system is affected by the delay because of the factor e^{-Ts} in the denominator. For example, the stability of the system is modified by the presence of this factor.

In any analytical analysis, the transcendental transfer function has classically been considered by approximating the exponential by a rational algebraic function, such as the first few terms of the Maclaurin series. Thus

$$e^{-Ts} \approx 1 - Ts + \frac{T^2s^2}{2!} - \frac{T^3s^3}{3!} + \ldots \tag{4.2}$$

Considering only small values of T, this can be further simplified to

$$e^{-Ts} \approx 1 - Ts \tag{4.3}$$

Let the system of Fig. 10 be defined by the following state equations.

$$x_1 = c(t) \tag{4.4}$$

$$x_2 = \dot{x}_1 \tag{4.5}$$

$$x_3 = \int e\,dt \tag{4.6}$$

Therefore, from Fig. 10

$$X_1(s) = \frac{\omega_n^{\,2}\gamma U}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}} \tag{4.7}$$

or

$$s^2X_1 + 2\zeta\omega_n sX_1 + \omega_n^{\,2}X_1 = \omega_n^{\,2}\gamma U \tag{4.8}$$

which can be written as

$$\ddot{x}_1 + 2\zeta\omega_n\dot{x}_1 + \omega_n^{\,2}x_1 = \omega_n^{\,2}\gamma u \tag{4.9}$$

Since $$e(t) = r(t) - x_1(t) \tag{4.10}$$

and for $$r(t) = 1 \tag{4.11}$$

then $$x_1 = 1 - e \tag{4.12}$$

$$\dot{x}_1 = \delta(t) - \dot{e} \tag{4.13}$$

where $\delta(t)$ is the impulse function and can be disregarded for $t > 0$.

Therefore

$$\ddot{x}_1 = -\ddot{e} \tag{4.14}$$

and Equation (4.9) can be re-written as

$$-\ddot{e} = -\omega_n^2(1 - e) + 2\zeta\omega_n\dot{e} + \omega_n^2\gamma u \tag{4.15}$$

Now

$$U(s) = E_d e^{-Ts} \approx E_d(1 - Ts) \tag{4.16}$$

or,

$$u(t) = e_d - T\dot{e}_d \tag{4.17}$$

But, from Fig. 10

$$e_d(s) = K_c E(s) + \frac{K_c E(s)}{\tau_i s} + K_c \tau_d E(s)s \tag{4.18}$$

which becomes in the time domain

$$e_d(t) = K_c e + \frac{K_c}{\tau_i}\int e dt + K_c \tau_d \dot{e} \tag{4.19}$$

and

$$\dot{e}_d = K_c \dot{e} + \frac{K_c e}{\tau_i} + K_c \tau_d \ddot{e} \tag{4.20}$$

Therefore, from (4.17)

$$u(t) = e_d - T\,(K_c\dot{e} + \frac{K_c e}{\tau_i} + K_c\tau_d\ddot{e}) \tag{4.21}$$

Substituting (4.18) and (4.21) into (4.15) gives

$$-\ddot{e} = -\omega_n^2(1-e) + 2\zeta\omega_n\dot{e} + \omega_n^2\gamma[K_c e + \frac{K_c}{\tau_i}\int e dt + K_c\tau_d\dot{e} - TK_c\dot{e} - \frac{TK_c e}{\tau_i} - TK_c\tau_d\ddot{e}] \tag{4.22}$$

Rearranging,

$$\ddot{e}(1-\omega_n^2\gamma K_c\tau_d T) + \dot{e}(2\zeta\omega_n - \omega_n^2\gamma K_c T + \omega_n^2\gamma K_c\tau_d) +$$
$$e(\omega_n^2 + \omega_n^2\gamma K_c - \frac{\omega_n^2\gamma K_c T}{\tau_i}) + \frac{\omega_n^2\gamma K_c}{\tau_i}\int e dt = \omega_n^2 \tag{4.23}$$

As before, let

$$a_1 = 2\zeta\omega_n \tag{4.24}$$

$$a_2 = \omega_n^{\ 2} \tag{4.25}$$

$$a_3 = \omega_n^{\ 2}\gamma \tag{4.26}$$

and define

$$c_1 \triangleq \omega_n^{\ 2}\gamma K_c \tag{4.27}$$

$$c_2 \triangleq \omega_n^{\ 2}\gamma K_c \tau_d \tag{4.28}$$

$$c_3 \triangleq \frac{\omega_n^{\ 2}\gamma K_c}{\tau_i} \tag{4.29}$$

Substituting into (4.23) gives

$$\ddot{e}(1-c_2T) + \dot{e}(a_1+c_2-c_1T) + e(a_2+c_1-c_3T) + c_3\int e\,dt = \omega_n^{\ 2} \tag{4.30}$$

When taking Laplace Transforms of the error the following initial conditions hold

$$L[\dot{e}] = sE(s) - e(0^+) = sE(s) - 1 \tag{4.31}$$

$$L[\ddot{e}] = s^2E(s) - se(0^+) - \dot{e}(0^+) = s^2E(s)-s-\dot{e}(0^+) \tag{4.32}$$

where $\dot{e}(0^+)$ is the initial rate of the error for the system.
However, from (4.13)

$$-\dot{e} = \dot{x}_1 = x_2 \tag{4.33}$$

and from (4.9)

$$\ddot{x}_1 = \dot{x}_2 = \omega_n^{\ 2}\gamma u - 2\zeta\omega_n\dot{x}_1 - \omega_n^{\ 2}x_1 \tag{4.34}$$

Integrating gives

$$-\dot{e} = x_2 = \omega_n^{\ 2}\gamma\int u\,dt - 2\zeta\omega_n x_1 - \omega_n^{\ 2}\int x_1\,dt \tag{4.35}$$

Due to the time delay in the system, there is no output response to any input for the time period $0 \le t \le T+0^+$. Consequently, $x_1 = 0$ and (4.35) can be written as

$$-\dot{e} = x_2 = \omega_n^{\ 2}\gamma\int u dt \tag{4.36}$$

Additionally, since there is no feedback occurring within the system for this time period

$$e_d = K_c r(t) + \frac{K_c}{\tau_i}\int r(t)dt + K_c\tau_d\dot{r}(t) \tag{4.37}$$

For a unit step displacement input

$$r(t) = 1, \qquad \dot{r}(t) = \delta(t) \tag{4.38}$$

and since $\tau_i >> T$

$$e_d \approx K_c + K_c\tau_d\delta(t) \tag{4.39}$$

Now

$$u(t) = e_d(t-T) \approx K_c\tau_d\delta(t-T) \tag{4.40}$$

since

$$r(t-T) = 0 \quad \text{for} \quad 0 \leq t \leq T+0^+$$

Hence Equation (4.36) becomes

$$-\dot{e} = x_2(T+0^+) = \omega_n^{\ 2}\gamma K_c\tau_d \tag{4.41}$$

and so (4.32) can be written as

$$L[\ddot{e}] = s^2E(s) - s + \omega_n^{\ 2}\gamma K_c\tau_d \tag{4.42}$$

Taking the Laplace transform of (4.30) with initial conditions of (4.31) and (4.42) yields

$$E(s)\ [(1-c_2T)s^2 + (a_1 + c_2 - c_1T)s + a_2 + c_1(1 - T/\tau_i) + c_3/s] =$$
$$(1-c_2T)s + a_1 + c_2^{\ 2}T - c_1T + a_2/s \tag{4.43}$$

or

$$E(s) = \frac{s^2 + \beta_1 s + \beta_o}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_o} \tag{4.44}$$

where

$$\alpha_o = \frac{c_3}{1 - c_2T} \tag{4.45}$$

$$\alpha_1 = \frac{a_2 + c_1 - c_3 T}{1 - c_2 T} \tag{4.46}$$

$$\alpha_2 = \frac{a_1 + c_2 - c_1 T}{1 - c_2 T} \tag{4.47}$$

$$\beta_o = \frac{a_2}{1 - c_2 T} \tag{4.48}$$

$$\beta_1 = \frac{a_1 + c_2{}^2 T - c_1 T}{1 - c_2 T} \tag{4.49}$$

In the analysis of Section 3, it was convenient to impose a constraint upon the output response of the system. For the system with time delay, it is advantageous to consider an alternate constraint - that of making the initial rate of response of the output to be equal to or less than a fixed value. That is,

$$x_2(0^+) = c_2 = \omega_n{}^2 \gamma K_c \tau_d \leq \text{constant} \tag{4.50}$$

From (4.47)

$$\alpha_2 + \frac{c_1 T}{1 - c_2 T} = \frac{a_1 + c_2}{1 - c_2 T} \tag{4.51}$$

and from (4.46)

$$\alpha_1 + \frac{c_3 T}{1 - c_2 T} - \frac{a_2}{1 - c_2 T} = \frac{c_1}{1 - c_2 T} \tag{4.52}$$

Substituting (4.45) into (4.52) gives

$$\alpha_1 + \alpha_o T - \frac{a_2}{1 - c_2 T} = \frac{c_1}{1 - c_2 T} \tag{4.53}$$

which when substituted into (4.51) gives

$$\alpha_2 + \alpha_1 T + \alpha_o T^2 = \frac{a_1 + c_2 + a_2 T}{1 - c_2 T} = \text{constant} \tag{4.54}$$

From this, it can be seen that the system variables α_o, α_1 , α_2 are constrained.

4.2 Examination of the Performance Index

The integral square error for the system of Fig. 10 and Equation (4.44) can be determined from tables (Newton, Gould and Kaiser, 1964) to be

$$ISE = \frac{\alpha_o\alpha_1 + (\beta_1^2 - 2\beta_o)\alpha_o + \beta_o^2\alpha_2}{2\alpha_o\alpha_1\alpha_2 - 2\alpha_o^2} \tag{4.55}$$

In the above equation, α_o, α_1, α_2 and β_1 are function of the system variables K_c, τ_i, and τ_d and are thus variables themselves, while β_o is a constant.

However, from Equation (4.49)

$$\beta_1 = \frac{a_1 + c_2^2T - c_1T}{1 - c_2T} = \frac{a_1 - c_1T}{1 - c_2T} + \frac{c_2^2T}{1 - c_2T} \tag{4.56}$$

and from Equation (4.47)

$$\frac{a_1 - c_1T}{1 - c_2T} = \alpha_2 - \frac{c_2}{1 - c_2T} \tag{4.57}$$

Therefore,

$$\beta_1 = \alpha_2 + \frac{c_2(-1 + c_2T)}{(1 - c_2T)} = \alpha_2 - c_2 \tag{4.58}$$

In addition, it has been shown in Section 3.3.1, that by differentiating the expression for the integral square error with respect to α_o and equating to zero, then α_o can be expressed as a function of the other variables in the equation. That is

$$\alpha_o = \alpha_2\beta_o\sqrt{\frac{\alpha_1}{\alpha_1 + \beta_1^2 - 2\beta_o}} \tag{4.59}$$

Thus it can be seen that the number of independent variables in the performance index can be reduced to two.

When applying the approximation procedures of Section 2 to an expression, it is convenient to first plot that expression in order to determine a feasible region over which the approximation can be made. Fig. 11 is a plot of Equation (4.55) showing a family of constant cost (ISE) curves. It is obvious, that there is a region of discontinuity of the function and that minimization is possible only for those values of α_2 beyond the discontinuity.

Consider now a condensation of Equation (4.55) around the operating point $\bar{\alpha}_1$, $\bar{\alpha}_2$ into a single term posynomial by the method of Section 2.2. From (4.59), and (4.58), let

$$\phi_1 = \alpha_1 + \beta_1^2 - 2\beta_o = \alpha_1 + \alpha_2^2 - 2\alpha_2 c_2 + c_2^2 - 2\beta_o \tag{4.60}$$

which can be approximated by

$$\phi_1 \approx \bar{\phi}_1 \left(\frac{\alpha_1}{\bar{\alpha}_1}\right)^{\Delta_1} \left(\frac{\alpha_2^2}{\bar{\alpha}_2^2}\right)^{\Delta_2} \left(\frac{\alpha_2}{\bar{\alpha}_2}\right)^{\Delta_3} \tag{4.61}$$

where

$$\bar{\phi}_1 = \bar{\alpha}_1 + \bar{\alpha}_2^2 - 2\bar{\alpha}_2 c_2 + c_2^2 - 2\beta_o \tag{4.62}$$

$$\Delta_1 = \frac{\bar{\alpha}_1}{\bar{\phi}_1} \tag{4.63}$$

$$\Delta_2 = \frac{\bar{\alpha}_2^2}{\bar{\phi}_1} \tag{4.64}$$

$$\Delta_3 = -\frac{2\bar{\alpha}_2 c_2}{\bar{\phi}_1} \tag{4.65}$$

Thus, (4.59) can be expressed as

$$\alpha_o \approx \Phi_1 \alpha_1^{(1/2-\Delta_1/2)} \alpha_2^{(1-\Delta_2-\Delta_3/2)} \tag{4.66}$$

where

$$\Phi_1 = [\bar{\phi}_1 \bar{\alpha}_1^{-\Delta_1} \bar{\alpha}_2^{-2\Delta_2-\Delta_3}]^{-1/2} \beta_o \tag{4.67}$$

Now from (4.55) let

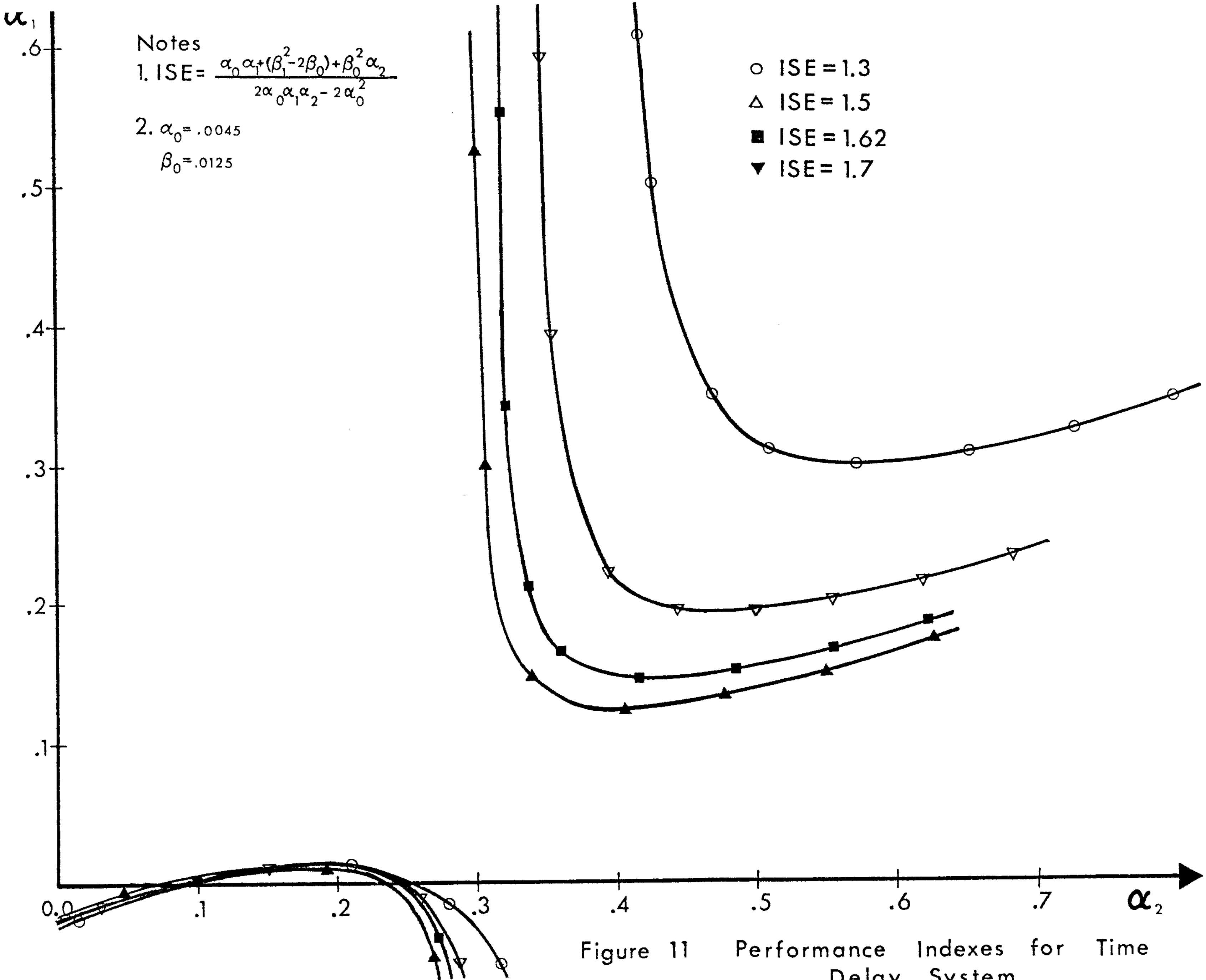


Figure 11 Performance Indexes for Time Delay System

$$\phi_2 = 2\alpha_o\alpha_1\alpha_2 - 2\alpha_o^2 \tag{4.68}$$

Substituting (4.66) into (4.68) yields

$$\phi_2 = 2\Phi_1\alpha_1^{3/2-\Delta_1/2}\alpha_2^{2-\Delta_2-\Delta_3/2} - 2\Phi_1^2\alpha_1^{1-\Delta_1}\alpha_2^{2-2\Delta_2-\Delta_3} \tag{4.69}$$

which can be approximated by

$$\phi_2 \approx \overline{\phi}_2\left(\left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{3/2-\Delta_1/2}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{2-\Delta_2-\Delta_3/2}\right)^{\Delta_4}\left(\left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{1-\Delta_1}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{2-2\Delta_2-\Delta_3}\right)^{\Delta_5} \tag{4.70}$$

where

$$\overline{\phi}_2 = 2\Phi_1\overline{\alpha}_1^{3/2-\Delta_1/2}\overline{\alpha}_2^{2-\Delta_2-\Delta_3/2} - 2\Phi_1^2\overline{\alpha}_1^{1-\Delta_1}\overline{\alpha}_2^{2-2\Delta_2-\Delta_3} \tag{4.71}$$

$$\Delta_4 = \frac{2\Phi_1\overline{\alpha}_1^{3/2-\Delta_1/2}\overline{\alpha}_2^{2-\Delta_2-\Delta_3/2}}{\overline{\phi}_2} \tag{4.72}$$

$$\Delta_5 = \frac{2\Phi_1^2\overline{\alpha}_1^{1-\Delta_1}\overline{\alpha}_2^{2-2\Delta_2-\Delta_3}}{\overline{\phi}_2} \tag{4.73}$$

Similarly, let

$$\phi_3 = \alpha_o\alpha_1 + (\alpha_2^2-2\alpha_2c_2+c_2^2-2\beta_o)\alpha_o + \beta_o^2\alpha_2 \tag{4.74}$$

which becomes,

$$\phi_3 = (\alpha_1+\alpha_2^2-2\alpha_2c_2+c_2^2-2\beta_o)\Phi_1\alpha_1^{1/2-\Delta_1/2}\alpha_2^{1-\Delta_2-\Delta_3/2} + \beta_o^2\alpha_2 \tag{4.75}$$

Equation (4.75) can be approximated by

$$\phi_3 \approx \overline{\phi}_3\left(\left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{3/2-\Delta_1/2}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{1-\Delta_2-\Delta_3/2}\right)^{\Delta_6}\left(\left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{1/2-\Delta_1/2}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{3-\Delta_2-\Delta_3/2}\right)^{\Delta_7}$$

$$\cdot\left(\left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{1/2-\Delta_1/2}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{2-\Delta_2-\Delta_3/2}\right)^{\Delta_8}\left(\frac{\alpha_1}{\overline{\alpha}_1}\right)^{1/2-\Delta_1/2}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{1-\Delta_2-\Delta_3/2)\Delta_9}\left(\frac{\alpha_2}{\overline{\alpha}_2}\right)^{\Delta_{10}} \tag{4.76}$$

where

$$\overline{\phi}_3 = (\overline{\alpha}_1+\overline{\alpha}_2^2-2\overline{\alpha}_2c_2+c_2^2-2\beta_o)\Phi_1\overline{\alpha}_1^{1/2-\Delta_3/2}\overline{\alpha}_2^{1-\Delta_2-\Delta_3/2} + \beta_o^2\overline{\alpha}_2 \tag{4.77}$$

$$\Delta_6 = \frac{\overline{\alpha}_1^{\,3/2-\Delta_1/2}\,\overline{\alpha}_2^{\,1-\Delta_2-\Delta_3/2}}{\overline{\phi}_3} \cdot \Phi_1 \tag{4.78}$$

$$\Delta_7 = \frac{\overline{\alpha}_1^{\,1/2-\Delta_1/2}\,\overline{\alpha}_2^{\,3-\Delta_2-\Delta_3/2}}{\overline{\phi}_3} \cdot \Phi_1 \tag{4.79}$$

$$\Delta_8 = \frac{\overline{\alpha}_1^{\,1/2-\Delta_1/2}\,\overline{\alpha}_2^{\,2-\Delta_2-\Delta_3/2}}{\overline{\phi}_3} \cdot \Phi_1(-2c_2) \tag{4.80}$$

$$\Delta_9 = \frac{\overline{\alpha}_1^{\,1/2-\Delta_1/2}\,\overline{\alpha}_2^{\,1-\Delta_2-\Delta_3/2}}{\overline{\phi}_3} \cdot \Phi_1(c_2^{\,2}-2\beta_o) \tag{4.81}$$

$$\Delta_{10} = \frac{\beta_o^{\,2}\overline{\alpha}_2}{\overline{\phi}_3} \tag{4.82}$$

and finally the integral square error can be approximated by

$$\mathrm{ISE} \approx \frac{\phi_3}{\phi_2} \tag{4.83}$$

To examine this method and the resultant posynomial expression consider several examples. From Fig. 11, take the point $\overline{\alpha}_1 = .170$, $\overline{\alpha}_2 = .36$ which lies on the curve ISE = 1.62. In addition, assume

$$T = \text{time delay} = .1 \text{ or } 1 \text{ scaled} \tag{4.84}$$

$$c_2 = x_2(0^+) = \text{initial rate} = \omega_n^{\,2}\gamma K_c\tau_d = 0.2 \tag{4.85}$$

From (4.48)

$$\beta_o = \frac{a_2}{1 - c_2} = 0.0125 \tag{4.86}$$

Substituting the above values into the derived expressions yields

$$\overline{\phi}_1 = .1706 \tag{4.87}$$

$$\Delta_1 = .9965 \tag{4.88}$$

$$\Delta_2 = .7597 \tag{4.89}$$

$$\Delta_3 = -.8441 \tag{4.90}$$

$$\Phi_1 = .0089 \tag{4.91}$$

Therefore

$$\alpha_o \approx .0089\alpha_1^{.0018}\alpha_2^{.4256} \tag{4.92}$$

Similarly,

$$\overline{\phi}_2 = .0005 \tag{4.93}$$

$$\Delta_4 = 1.1026 \tag{4.94}$$

$$\Delta_5 = -.1026 \tag{4.95}$$

Therefore

$$\phi_2 \approx \left[\left(\frac{\alpha_1}{.170}\right)^{1.0018}\left(\frac{\alpha_2}{.36}\right)^{1.6642}\right]^{1.1026}\left[\left(\frac{\alpha_1}{.170}\right)^{.0035}\left(\frac{\alpha_2}{.36}\right)^{1.3247}\right]^{-.1026} \tag{4.96}$$

$$= .0204\alpha_1^{1.1042}\alpha_2^{1.1970} \tag{4.97}$$

and

$$\overline{\phi}_3 = .0008 \tag{4.98}$$

$$\Delta_6 = .9283 \tag{4.99}$$

$$\Delta_7 = .7068 \tag{4.100}$$

$$\Delta_8 = -.7855 \tag{4.101}$$

$$\Delta_9 = .0823 \tag{4.102}$$

$$\Delta_{10} = .0678 \tag{4.103}$$

$$\phi_3 \approx .0008\left[\left(\frac{\alpha_1}{.170}\right)^{1.0018}\left(\frac{\alpha_2}{.36}\right)^{.6624}\right)^{.9283}\left(\left(\frac{\alpha_1}{.170}\right)^{.0018}\left(\frac{\alpha_2}{.36}\right)^{2.6624}\right)^{.7068}$$

$$\cdot\left(\left(\frac{\alpha_1}{.170}\right)^{.0018}\left(\frac{\alpha_2}{.36}\right)^{1.6624}\right)^{-.7855}\left(\left(\frac{\alpha_1}{.170}\right)^{.0018}\left(\frac{\alpha_2}{.36}\right)^{.6624}\right)^{.0823}\left(\frac{\alpha_2}{.36}\right)^{.0678} \tag{4.104}$$

$$= .0164\alpha_1^{.9300}\alpha_2^{1.3132} \tag{4.105}$$

Equation (4.83) gives

$$ISE \approx \frac{\phi_3}{\phi_2} = \frac{.0164\alpha_1^{.9300}\alpha_2^{1.3132}}{.0204\alpha_1^{1.1042}\alpha_2^{1.6970}} = .8021\alpha_1^{-.1742}\alpha_2^{-.3838} \qquad (4.106)$$

Now take the point $\overline{\alpha}_1 = .152$, $\overline{\alpha}_2 = .375$ which also lies on the curve ISE = 1.62. With the same time delay and initial rate of response as given in Equations (4.84) and (4.85), the ISE can be approximated in a similar manner to give

$$ISE = .8939\alpha_1^{-.1957}\alpha_2^{-.2348} \qquad (4.107)$$

The approximations to the ISE given by Equations (4.107) and (4.106) are compared to the actual ISE Equation (4.55) in Fig. 12. It can be seen that the choice of an operating point or point of approximation has a great influence on the shape of the condensed curve. That is, for a comparatively small shift of the operating point, the resultant shape and slope of the condensed curve are very different. This is due to the unique shape of Equation (4.55), making the approximation valid only over a local region, and indicates that whenever approximations are to be made, it is important to thoroughly examine the function when choosing an operating point in order to obtain feasible results.

4.3 Optimization Procedure

In general, it has been shown in Section 4.2 that the cost (ISE) equation for the time delay system can be condensed into a single term posynomial of the form

$$ISE \approx \Phi\, \alpha_1^{\eta_1}\alpha_2^{\eta_2} \qquad (4.108)$$

where

$$\Phi = \frac{\overline{\phi}_3}{\overline{\phi}_2}\, \overline{\alpha}_1^{-\eta_1}\overline{\alpha}_2^{-\eta_2} \qquad (4.109)$$

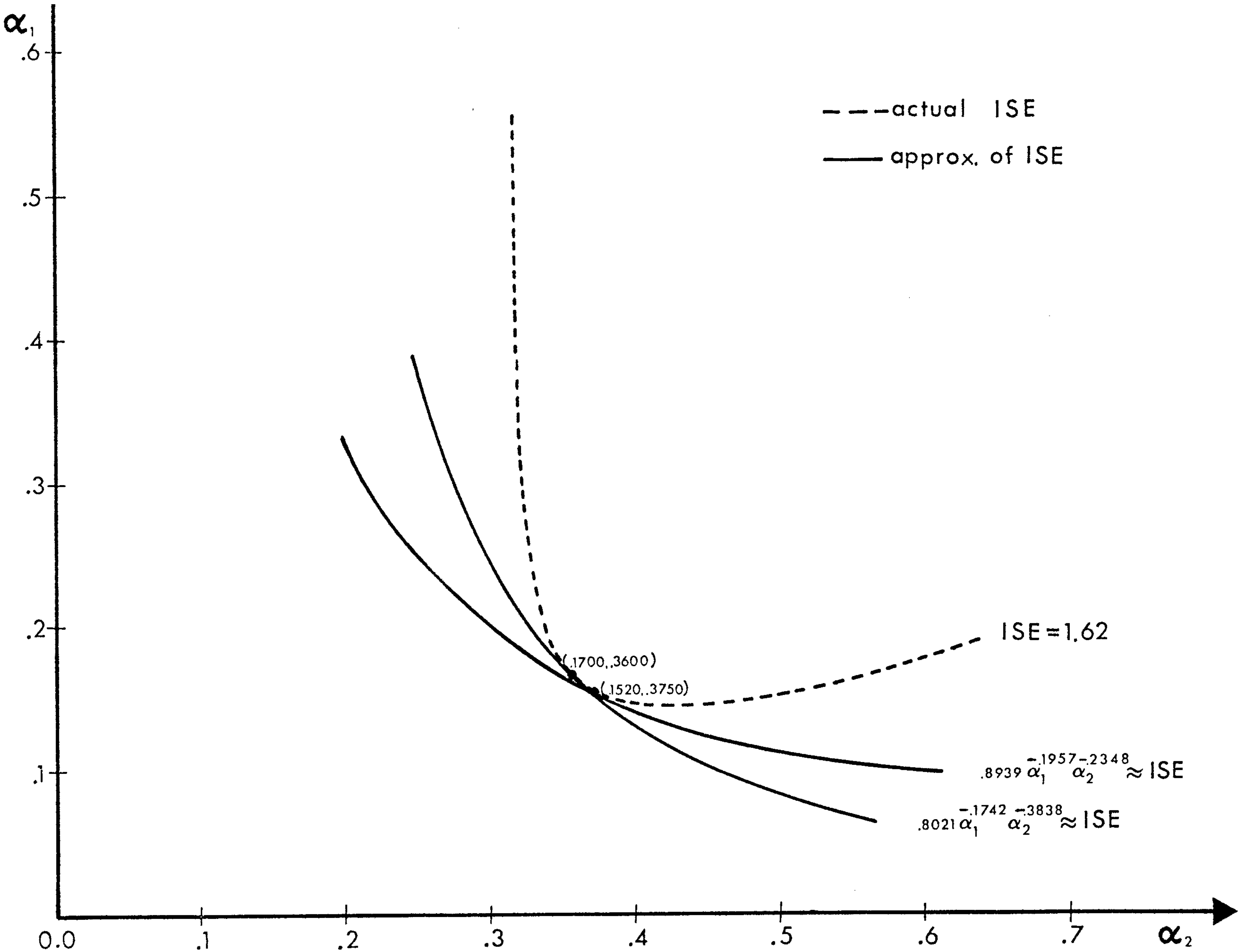


Figure 12 Approximations of Performance Index

$$\eta_1 = (3/2-\Delta_1/2)(\Delta_6-\Delta_4) + (1/2-\Delta_1/2)(\Delta_7+\Delta_8+\Delta_9-2\Delta_5) \tag{4.110}$$

$$\eta_2 = (1-\Delta_2-\Delta_3/2)(\Delta_6+\Delta_9-2\Delta_5) + (2-\Delta_2-\Delta_3/2)(\Delta_8-\Delta_4) + (3-\Delta_2-\Delta_3/2)\Delta_7+\Delta_{10} \tag{4.111}$$

In addition, Section 4.1 derived a constraint for the system in the form of Equation (4.54). For small α_0, this can be approximated by the linear equation

$$\alpha_1 T + \alpha_2 \leq K_1 \tag{4.112}$$

where

$$K_1 \triangleq \text{constant} = \frac{a_1 + c_2 + a_2 T}{1 - c_2 T} \tag{4.113}$$

or

$$1/K_1(\alpha_1 T+\alpha_2) \leq 1 \tag{4.114}$$

Equations (4.108) and (4.114) are both posynomials in which the total number of terms (three) is one greater than the number of variables (α_1 and α_2). Hence, the degree of difficulty is zero and a solution by geometric programming is readily accomplished. First the dual is formed.

$$\psi(\delta) = \left(\frac{\Phi}{\delta_1}\right)^{\delta_1}\left(\frac{T(1/K_1)}{\delta_2}\right)^{\delta_2}\left(\frac{1/K_1}{\delta_3}\right)^{\delta_3}(\delta_2+\delta_3)^{\delta_2+\delta_3} \tag{4.115}$$

Equation (4.115) is to be maximized subject to the normality and orthogonality conditions of (1.11) and (1.12), which for this system are:

$$\delta_1 = 1 \tag{4.116}$$

$$\eta_1\delta_1 + \delta_2 = 0 \tag{4.117}$$

$$\eta_2\delta_1 + \delta_3 = 0 \tag{4.118}$$

These have the unique solution

$$\delta_1{}^* = 1 \tag{4.119}$$

$$\delta_2{}^* = -\eta_1 \tag{4.120}$$

$$\delta_3^* = -\eta_2 \qquad (4.121)$$

which when substituted into (4.115) yields the maximum value of $\psi(\delta)$, and simultaneously the minimum value of ISE. At an optimum, the following conditions from Equations (1.17) and (1.18) are satisfied:

$$\Phi\alpha_1^{\eta_1}\alpha_2^{\eta_2} = \delta_1^*\psi(\delta^*) \qquad (4.122)$$

$$(T/K_1)\alpha_1 = \delta_2^*/(\delta_2^* + \delta_3^*) \qquad (4.123)$$

$$(1/K_1)\alpha_2 = \delta_3^*/(\delta_2^* + \delta_3^*) \qquad (4.124)$$

The optimum value of the parameters α_1 and α_2 are then found from simultaneously satisfying (4.122), (4.123) and (4.124). In addition, from (4.59)

$$\alpha_o^* = \alpha_2^*\beta_o\sqrt{\frac{\alpha_1^*}{\alpha_1^* + \beta_1^2 - 2\beta_o}} \qquad (4.125)$$

Substitution of α_o^*, α_1^*, and α_2^* into Equations (4.45), (4.46) and (4.47) will yield solutions for the parameter c_1, c_2, and c_3 which then can be used in Equations (4.27), (4.28) and (4.29) to find values of the controller parameters, K_c, τ_i, and τ_d which yield the minimum ISE.

It is interesting to note that from Equation (1.10) it is required that all δ_i be equal to or greater than zero for a feasible solution to the optimization problem. From Equations (4.120) and (4.121) this is possible only if

$$-\eta_1 = (3/2-\Delta_1/2)(\Delta_4-\Delta_6)+(1/2-\Delta_1/2)(2\Delta_5-\Delta_7-\Delta_8-\Delta_9) \geq 0 \qquad (4.126)$$

$$-\eta_2 = (1-\Delta_2-\Delta_3/2)(2\Delta_5-\Delta_6-\Delta_9)+(2-\Delta_2-\Delta_3/2)(\Delta_4-\Delta_8)-\Delta_7(3-\Delta_2-\Delta_3/2) - \Delta_{10} \geq 0 \qquad (4.127)$$

Equation (4.126) and (4.127) can be used as a means of determining if an approximation around a particular point will yield a feasible solution.

Consider an example of the optimization procedure. Previously it has been shown that an approximation of Equation (4.55) around the point $\overline{\alpha}_1 = .152$, $\overline{\alpha}_2 = .375$ gives

$$\text{ISE} \approx .8939\alpha_1^{-.1957}\alpha_2^{-.2348} \tag{4.128}$$

For a_1, a_2, a_3 as given in Section 3 and T = 1 then

$$K_1 = \frac{a_1 + c_2 + a_2T}{1 - c_2T} = .575 \tag{4.129}$$

and from Equation (4.114) the constraint on the system becomes

$$1.7391\ (\alpha_1 + \alpha_2) \leq 1 \tag{4.130}$$

Substituting values from (4.128) and (4.130) into (4.115) gives

$$\psi(\delta) = \left(\frac{.8939}{\delta_1}\right)^{\delta_1}\left(\frac{1.7391}{\delta_2}\right)^{\delta_2}\left(\frac{1.7391}{\delta_3}\right)^{\delta_3}(\delta_2+\delta_3)^{\delta_2+\delta_3} \tag{4.131}$$

from which the normality and orthogonality conditions are

$$\delta_1 = 1 \tag{4.132}$$

$$-.1957\delta_1 + \delta_2 = 0 \tag{4.133}$$

$$-.2348\delta_1 + \delta_3 = 0 \tag{4.134}$$

These have the unique solution

$$\delta_1{}^* = 1, \quad \delta_2{}^* = .1957, \quad \delta_3{}^* = .2348 \tag{4.135}$$

Substitution of (4.135) into (4.131) gives the maximum of the dual (equal to the minimum of the primal objective function) as:

$$\psi(\delta^*) = \left(\frac{.8939}{1}\right)^{1}\left(\frac{1.7391}{.1957}\right)^{.1957}\left(\frac{1.7391}{.2348}\right)(.1957+.2348)^{.1957+.2348} = 1.5261 \tag{4.136}$$

At an optimum Equation (4.122), (4.123) and (4.124) hold.

Therefore:

$$.8939\alpha_1^{-.1957}\alpha_2^{-.2348} = \delta_1{*}\psi(\delta{*}) = 1.5261 \qquad (4.137)$$

$$1.7391\alpha_1 = \delta_2{*}/(\delta_2{*} + \delta_3{*}) = .4546 \qquad (4.138)$$

$$1.7391\alpha_2 = \delta_3{*}/(\delta_2{*} + \delta_3{*}) = .5454 \qquad (4.139)$$

from which

$$\alpha_1{*} = .2614 \qquad (4.140)$$

$$\alpha_2{*} = .3136 \qquad (4.141)$$

and from Equation (4.125)

$$\alpha_o{*} = 0.0040 \qquad (4.142)$$

Equations (4.45), (4.46) and (4.47) can be solved to give

$$c_1 = .2017 \qquad (4.143)$$

$$c_2 = .202 \qquad (4.144)$$

$$c_3 = .0032 \qquad (4.145)$$

The controller parameters are

$$K_c = c_1/a_3 = 6.73 \qquad (4.146)$$

$$\tau_i = c_1/c_3 = 63.03 \qquad (4.147)$$

$$\tau_d = c_2/c_1 = 1.00 \qquad (4.148)$$

A pictorial representation of the procedure is shown in Fig. 13. The approximate performance index is shown in relationship to the constraint equation and it is seen that the point of condensation $\overline{\alpha}_1 = .152$, $\overline{\alpha}_2 = .375$ does not initially satisfy the constraint condition. At an optimum, the parameters α_1 and α_2 have been shifted to lie on the constraint curve and there is tangency with the performance

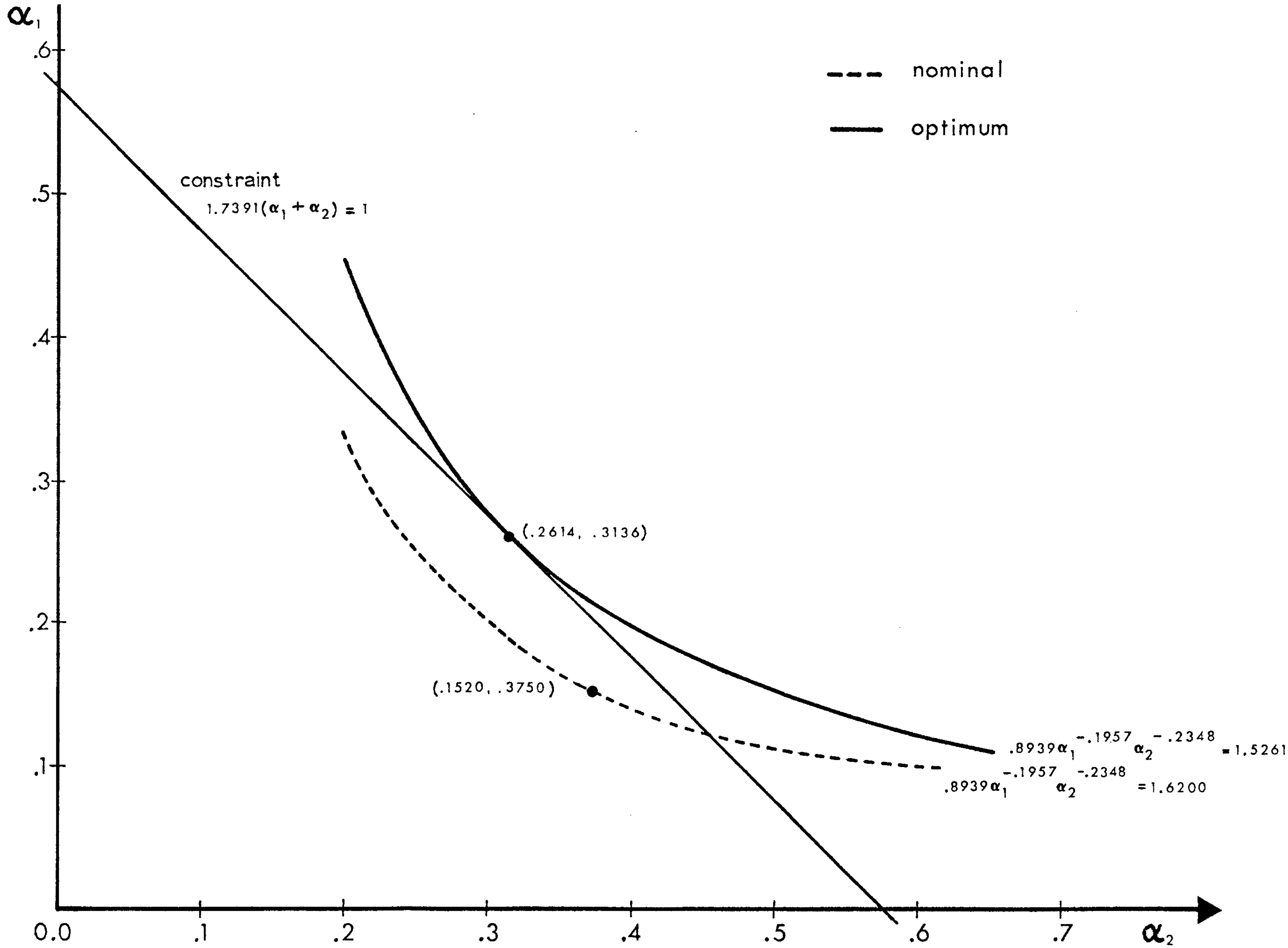


Figure 13 Curve Shift Due to Optimization Procedure

index curve.

A plot of the output response for the system with controller parameters given by Equations (4.146), and (4.147) and (4.148) is shown in Fig. 14. From this

$$\text{Per cent overshoot} = \frac{.45}{1.00} \times 100 = 45\% \tag{4.149}$$

and the settling time is

$$T_s = 3.0 \text{ seconds} \tag{4.150}$$

These values indicate a somewhat oscillatory nature for the system that could be undesirable. To determine why an "optimization" procedure yielded these results requires an examination of the approximate and actual forms of the performance index.

As seen from Fig. 13, the optimization procedure has resulted in a shift of the curve of Equation (4.128) to a new position tangent with the constraint equation at the point $\alpha_1^* = .2614$, $\alpha_2^* = .3136$. At this new position, the value of (4.128) is found from (4.136) to be 1.5261. The system response, Fig. 14, has been determined from controller parameter values that satisfy (4.136). However, when these values are substituted in the actual form of the performance index its value is

$$\text{ISE} = \frac{\alpha_o^*\alpha_1^* + (\beta_1^{*2} - 2\beta_o)\alpha_o^* + \beta_o^2\alpha_2^*}{2\alpha_o^*\alpha_1^*\alpha_2^* - 2\alpha_o^{*2}} \doteq 1.6772 \tag{4.151}$$

This is higher than the value (ISE = 1.6200) at the initial conditions, ($\overline{\alpha}_1 = .152$, $\overline{\alpha}_2 = .375$) and since the response of Fig. 14 is derived from equations that contain no approximations, explains why the output response is non-optimum. In addition from Fig. 15 it is evident that at an "optimum" (i.e. $\alpha_1^* = .2614$, $\alpha_2^* = .3136$), the agreement between the

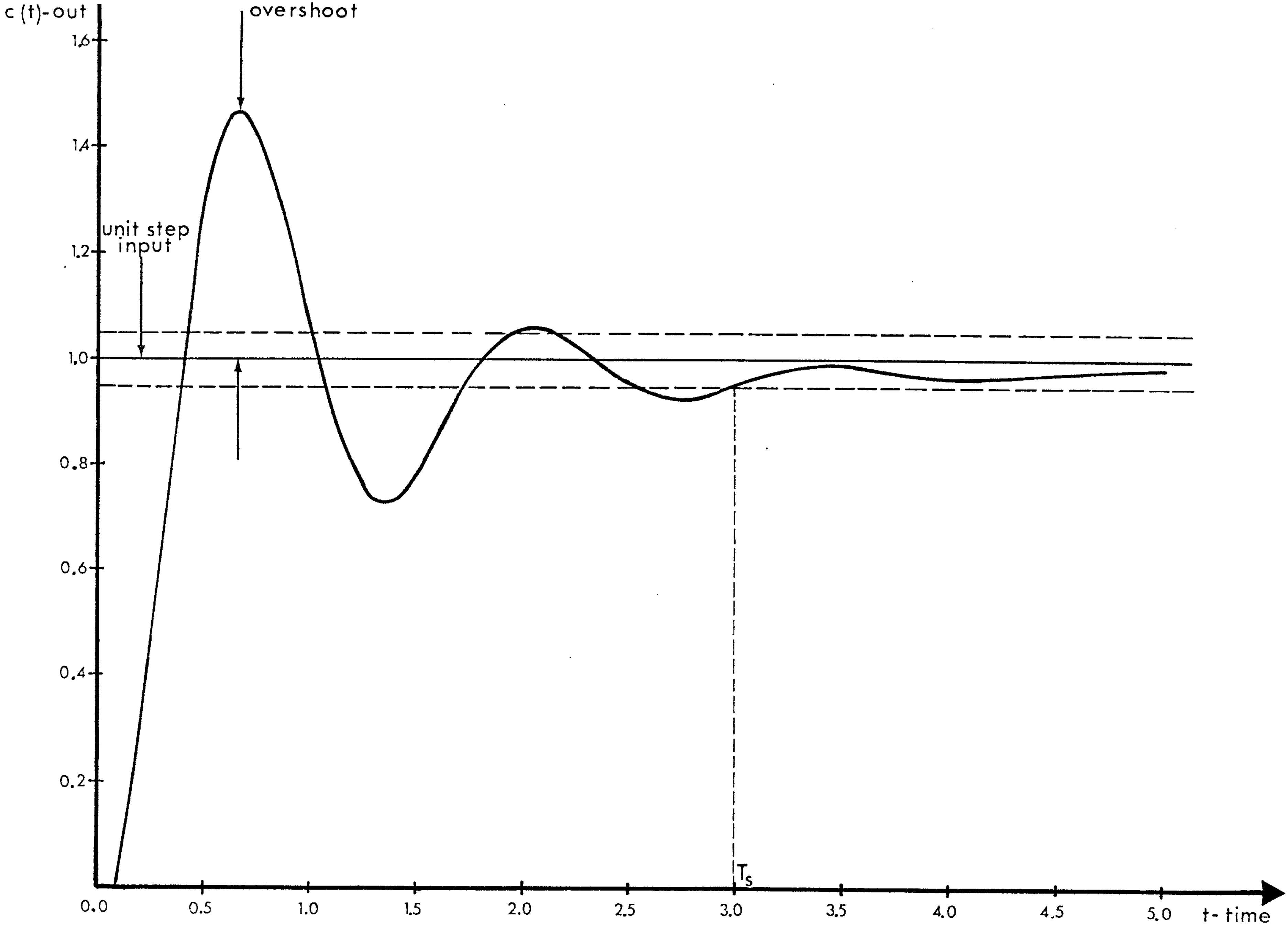


Figure 14 Output Response– Time Delay System [Non-optimum Case]

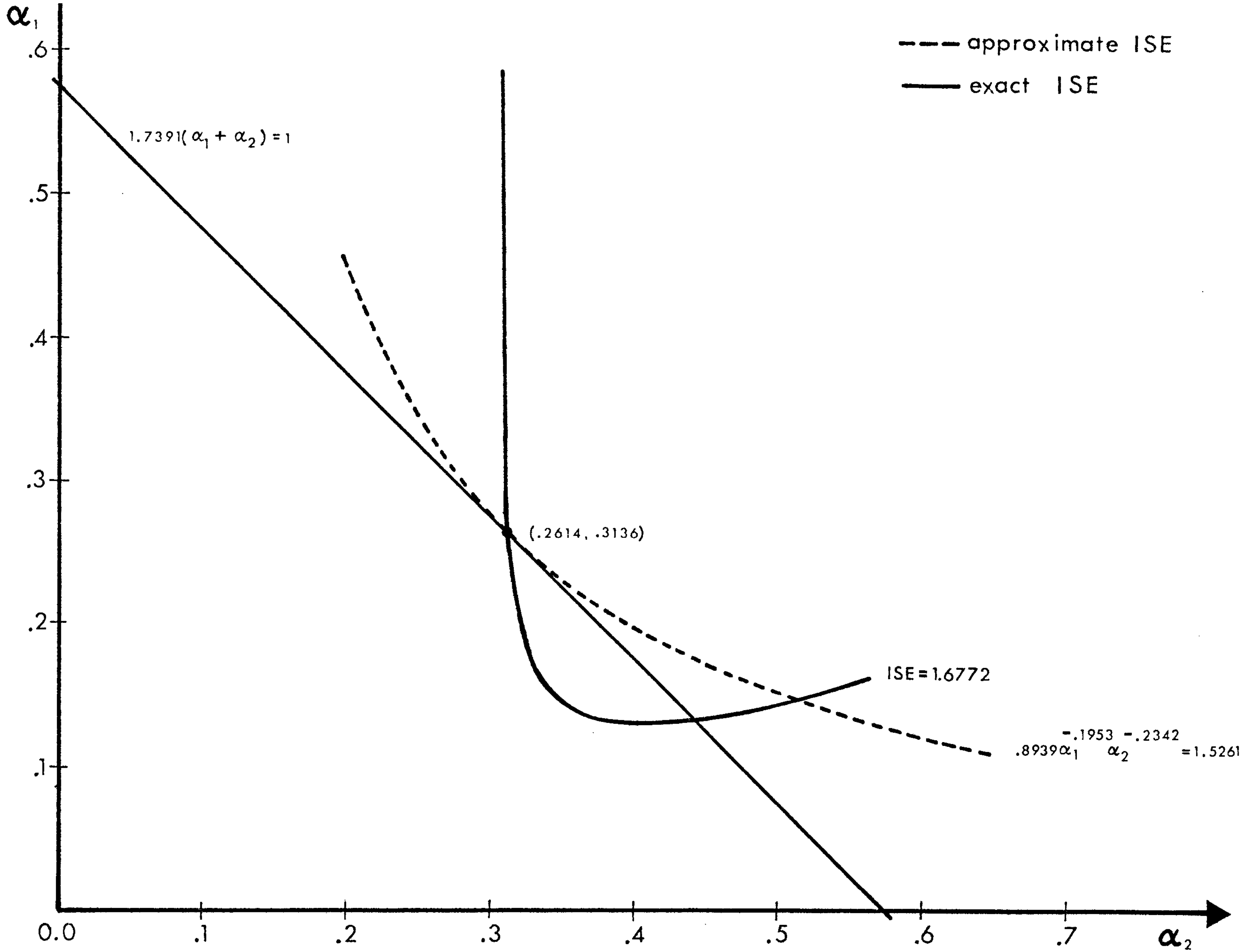


Figure 15 Non-optimum Case– Comparison of Exact and Approximate ISE

approximate and actual forms of the integral square error is not good. This is evident by the dissimilar shapes of the curves and the fact that only the point α_1*, α_2* is common to both.

As previously seen, an approximation of (4.55) at the point $\overline{\alpha}_1 = .170$, $\overline{\alpha}_2 = .36$ gives

$$ISE \approx .8021\alpha_1^{-.1742}\alpha_2^{-.3838} \tag{4.152}$$

Minimization of this equation, subject to the constraint condition of Equation (4.130) is as follows:

$$\psi(\delta) = \left(\frac{.8021}{\delta_1}\right)^{\delta_1}\left(\frac{1.7391}{\delta_2}\right)^{\delta_2}\left(\frac{1.7391}{\delta_3}\right)^{\delta_3}(\delta_2+\delta_3)^{\delta_2+\delta_3} \tag{4.153}$$

$$\delta_1 = 1 \tag{4.154}$$

$$-.1742\delta_1 + \delta_2 = 0 \tag{4.155}$$

$$-.3838\delta_1 + \delta_3 = 0 \tag{4.156}$$

Equations (4.154), (4.155) and (4.156) have the unique solution

$$\delta_1^* = 1, \quad \delta_2^* = .1742, \quad \delta_3^* = .3838 \tag{4.157}$$

Substitution of (4.157) into (4.153) gives the maximum of the dual and minimum of (4.152) to be

$$\psi(\delta^*) = \left(\frac{.8021}{1}\right)\left(\frac{1.7391}{.1742}\right)^{.1742}\left(\frac{1.7391}{.3838}\right)^{.3838}(.5580)^{.5580} = 1.5445 \tag{4.158}$$

In addition

$$.8021\alpha_1^{-.1702}\alpha_2^{-.3838} = \delta_1^*\psi(\delta^*) = 1.5445 \tag{4.159}$$

$$1.7391\alpha_1 = \delta_2^*/(\delta_2^* + \delta_3^*) = .3122 \tag{4.160}$$

$$1.7391\alpha_2 = \delta_3^*/(\delta_2^* + \delta_3^*) = .6878 \tag{4.161}$$

from which

$$\alpha_1^* = .1795 \tag{4.162}$$

$$\alpha_2^* = .3955 \qquad (4.163)$$

and from Equation (4.125)

$$\alpha_o^* = .0048 \qquad (4.164)$$

Using these values the controller parameters are found to be

$$K_c = 4.58 \qquad (4.165)$$

$$\tau_i = 36.16 \qquad (4.166)$$

$$\tau_d = 1.46 \qquad (4.167)$$

Substitution of (4.162), (4.163) and (4.164) into Equation (4.55) gives the actual value of the performance index to be

$$ISE = \frac{\alpha_o^*\alpha_1^* + (\beta_1^{*2}-2\beta_o)\alpha_o^* + \beta_o^2\alpha_2^*}{2\alpha_o^*\alpha_1^*\alpha_2^* - 2\alpha_o^{*2}} = 1.5530 \qquad (4.167)$$

The agreement between Eqn. (4.158) and (4.167) is good, indicating that the output response with parameter values from (4.165) through (4.167) should be better than that of Fig. 14. Fig. 16 confirms that this is so. It is seen that

$$\text{Per cent overshoot} = \frac{.27}{1.00} \times 100 = 27\% \qquad (4.168)$$

and

$$T_s = 2.0 \text{ seconds} \qquad (4.169)$$

In addition Fig. 17 shows a plot of Equations (4.158) and (4.167). The shape of the curves confirms the close agreement between the two expressions at the point of interest.

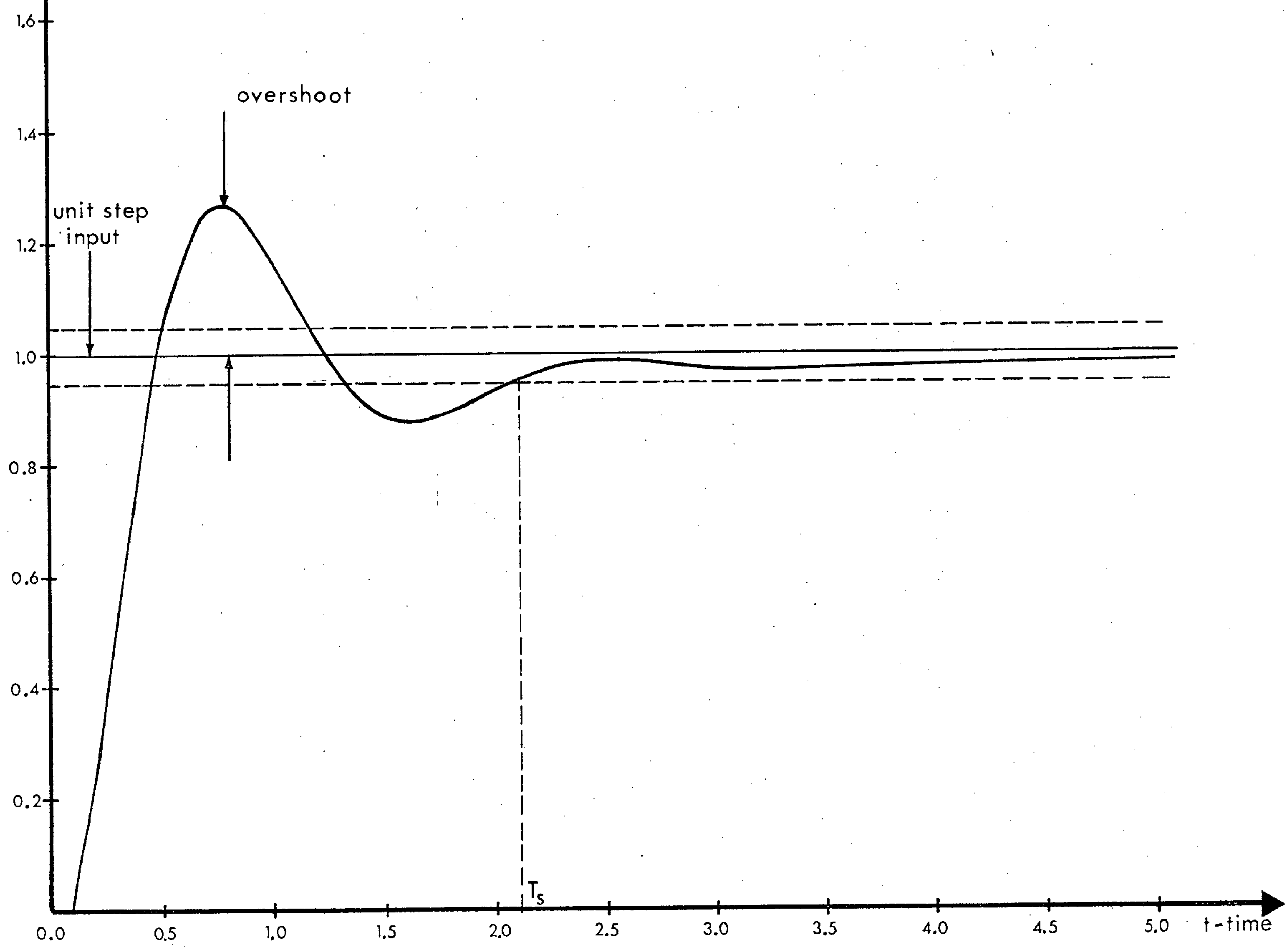


Figure 16 Output Response - Time Delay System [Optimum Case]

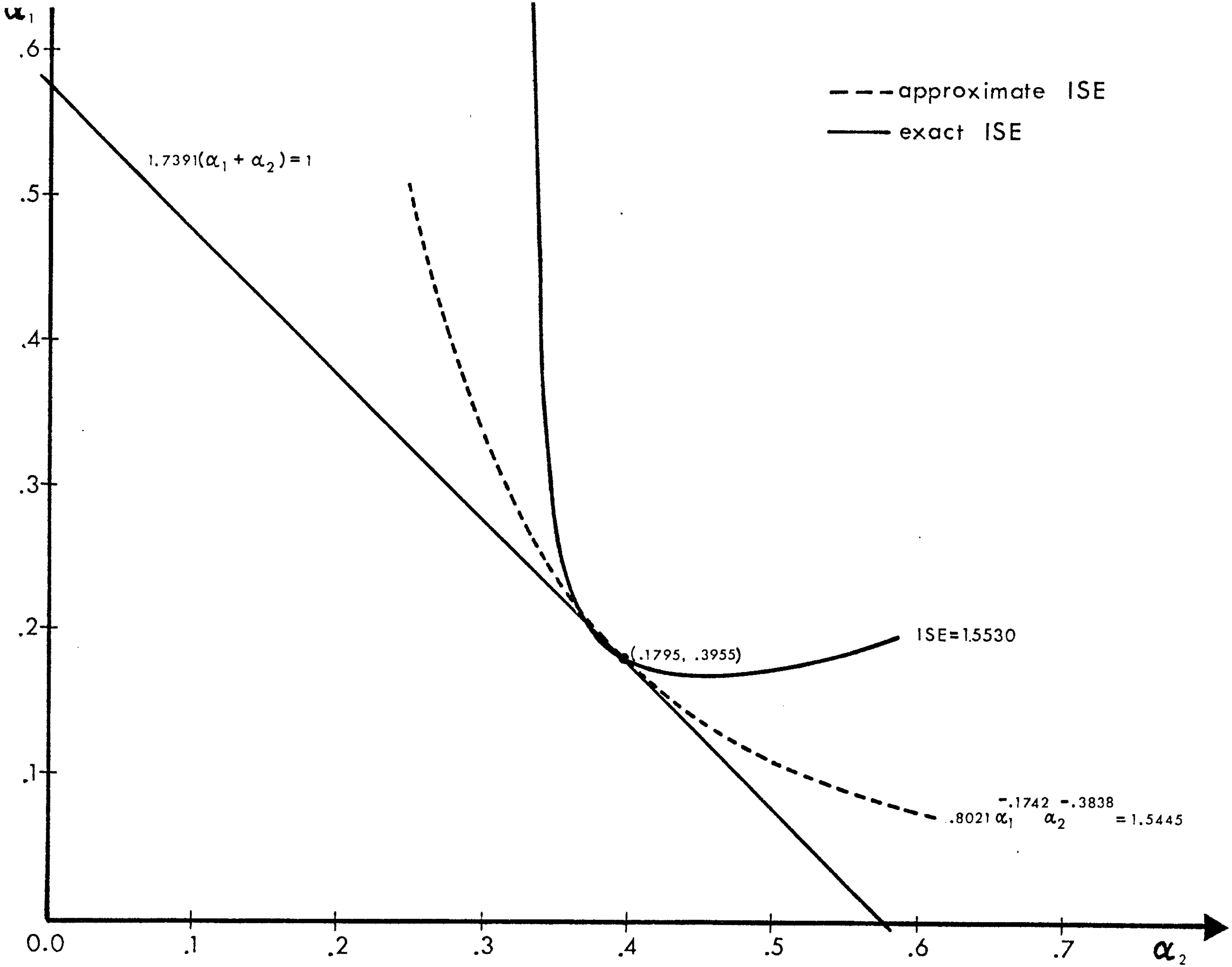


Figure 17 Optimum Case– Comparison of Exact and Approximate ISE

V. SUMMARY AND CONCLUSIONS

A problem expressed as

$$\begin{aligned} &\text{minimize} \quad g_o(t) \\ &\text{subject to} \quad g_k(t) \leq G_k \quad k = 1, \ldots \end{aligned} \tag{5.1}$$

where the g's are posynomials, can be expressed as a geometric programming problem. It is possible to solve (5.1) by a direct search in the $\bar{t}$ variables. However, the presence of nonlinear constraints poses particular difficulties, and it has been demonstrated that it is more feasible to maximinze a dual function $\psi(\bar{\delta})$ where the constraints that $\bar{\delta}$ must satisfy are linear. A knowledge of the maximizing vector $\bar{\delta}$ provides a complete solution to the problem.

Generally, engineering problems like (5.1) contain more terms than variables. This constitutes a high degree of difficulty and an explicit solution is not readily available unless access to extensive computer programs (Zener, 1971; Avriel, Dembo, and Passy, 1975) is possible. In addition, not all terms in the equation are likely to be positive. These difficulties can be overcome by using techniques of approximation, in which non-posynomial equations containing many terms can be condensed to positive expressions that contain fewer terms. Successively applying this technique can eventually lead to a zero degree of difficulty problem. An approximation technique was developed from the arithmetic-geometric relationships that required no differentiation of terms making it particularly easy to apply to complicated expressions.

Although the techniques of approximation can be utilized to obtain problems that are readily solvable the method must be applied judiciously. An examination of the intended expression should be made in order that a feasible operating point be chosen around which to condense the equation. Failure to do this could result in an optimization of the approximate equation but a final value of the actual expression that is greater than that for the original operating point.

Formulation of a control system problem requires the choice of an objective function, such as the integral square error, plus the imposition of constraints upon the system performance. For a second order system with proportional-integral-derivative controller, the constraints can be developed to not only yield a physically plausible system, but also in a manner to make a solution by geometric programming more viable. That is for a given objective function, a choice of constraints is made that mathematically restricts the system variables to feasible values, and at the same time lends itself to the geometric programming concepts.

Once a problem has been formulated and expressed in a suitable fashion, a solution by geometric programming is easily accomplished, usually amounting to solving a set of linear equations. The minimum value of the objective function is obtained without first determining parameter values for achieving this minimum. However, the power of the method becomes apparent when it is seen that the optimum is approached in a single computational step. This, together with the ease of the mathematical operations involved, make geometric programming an attractive optimization technique.

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