ASYNCHRONOUS HOPPING AND CODE DIVERSITY IN FREQUENCY HOPPED CODE DIVISION MULTIPLE ACCESS SYSTEMS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

DEPARTMENT OF ELECTRICAL ENGINEERING

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

September 1997

Department of Electrical Engineering.

The University of British Columbia
Vancouver, Canada

Date 29 Sept 97.

Abstract

This thesis is about Frequency Hopped-Code Division Multiple Access (FH-CDMA) systems. More specifically, it studies the packet error rates and system performance of FH-CDMA systems with guard times. In the past, a number of simplifying assumptions have been made in the studies of these systems. This work investigates the effects of these simplifying assumptions by deriving exact expressions, which do not make these assumptions, and comparing the results using both methods. By considering edge effects due to the inclusion of guard times, it is shown that the probability of codeword error can be significantly lower. Furthermore, it is found that the independence assumption, where frequency hits within a packet is assumed are independent, leads to larger probability of codeword error. On the other hand, system performance measures such as normalized maximum local traffic and throughput are not significantly altered by these simplifying assumptions.

In addition, this work also proposed a new diversity scheme in FH-CDMA systems, which we refer to as code diversity. In such a diversity scheme, the transmitters are allowed to transmit in more than one frequency bin simultaneously. A variety of decoding schemes, including some with optimal bit error rate (BER) performance, for the code diversity system are proposed and studied. It is shown that the code diversity scheme can have a lower BER than conventional FH-CDMA systems and that such a method of transmitting can be used to establish priority classes among the users in the system. It is further shown that error control coding can be included in the code diversity transmission scheme to further improve the BER performance. The performance of several code diversity schemes is studied in a Rayleigh fading and additive white Gaussian noise environment. From the analysis of the various schemes it is found that code diversity can improve the BER.

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Glossary

Acronyms

BER	Bit Error Rate
CDMA	Code Division Multiple Access
TDMA	Time Division Multiple Access
FDMA	Frequency Division Multiple Access
DS	Direct Sequence
FH	Frequency Hopped
FFH	Fast Frequency Hopped
AWGN	Additive White Gaussian Noise
SIR	Signal to Interference Ratio
SNR	Signal to Noise Ratio
FSK	Frequency Shift Keying
BFSK	Binary Frequency Shift Keying
MFSK	M-ary Frequency Shift Keying
CSMA	Carrier Sense Multiple Access

Notations

P_h	Probability of a frequency hit by another transmitter
q	Number of frequency bins
N_s	Number of symbols transmitted per hop interval
G_t	Length of guard time in terms of hop durations
J	Number of active transmitters
K	Number of active interferers
$L^*(P_E)$	Maximum normalized local load
$W^*(P_E)$	Normalized local throughput
W_{max}	Maximum normalized local thoughput
L	Diversity degree
$\overline{\rho}$	Average signal to noise ratio
β	Detection threshold normalized with respect to noise

p_D	Probability of deletion
p_F	Probability of false alarm

Acknowledgment

I would like to express my sincere thanks and gratitude to my thesis advisor, Professor Cyril Leung, for his guidance and encouragement. Through the many discussions I had with him, he has provided me with many useful insights and ideas which were essential to the completion of this work. His timely and critical review of this work is also very much appreciated.

Financial support from the Natural Sciences and Engineering Research Council (NSERC), Science Council of British Columbia, University of British Columbia and British Columbia Telephone Company in the form of various scholarships is gratefully acknowledged. Additional financial support from NSERC Grant OGP0001731 is also gratefully acknowledged. I would also like to thank Dr. Andrew Wright, Mr. Mike Walker and Mr. Greg Acres from MPR Teltech who have acted as my industrial collaborators for the Science Council of British Columbia's GREAT award.

Friends and fellow students at UBC have certainly made my studies here a memorable and interesting one. I would especially like to thank fellow students from the communications and high performance computing and networks labs.

Last but not least, my family has been a constant source of love and encouragement. My wife, son and daughter have made my life as a student an unusual but enriching one. They, too, have played a significant role in the completion of this work.

Chapter 1

Introduction

Spread spectrum techniques have long been used in telecommunication systems [1] for their anti-jamming and low probability of interception features. In recent years, these techniques have been proposed and studied for multiple access systems. Such systems are known as code division multiple access (CDMA) or spread spectrum multiple access (SSMA) systems. A number of researchers [2]–[5] have found CDMA to have better spectral utilization characteristics and be able to support more users with a given bandwidth allocation than other more traditional multiple access schemes. This has made CDMA an active and important topic of research in the telecommunication field.

This thesis is concerned with frequency hopped CDMA (FH-CDMA) systems. The first part of the work deals with time slotted FH-CDMA systems with asynchronous hopping. It extends some previous work by Hegde and Stark [6] and Pursley [7]. The exact packet error probabilities and system throughput are derived and investigated for time slotted FH-CDMA systems with an integer number of hop intervals as guard times.

The second part of this work deals with a new diversity technique, called *code diversity*, in FH-CDMA. Here, we study a number of possible ways in which this technique can be used to improve the system performance in a multiple access situation. Several decoding schemes are proposed and the performances of these schemes are evaluated initially for the case of no coding and no background noise. The effects of background noise and coding are then taken into consideration.

This thesis is organized as follows:

A brief introduction to CDMA systems and the motivation for this work is given in the latter part of this chapter. The various types of CDMA systems are described and a summary of the advantages of using CDMA over other types of multiple-access systems is given.

In Chapter 2, some of the related works found in the literature is reviewed. Various aspects of FH systems such as asynchronous and synchronous hopping, slotted and unslotted systems are briefly described. Models and assumptions commonly used in the study of FH-CDMA system are also introduced. This chapter also contains a description of the various forms of diversity in the FH-CDMA systems.

An analysis of the time slotted asynchronous hopping system can be found in Chapter 3. The concept of code diversity is introduced in Chapter 4. A number of code diversity decoding schemes are proposed and analyzed for an environment where the background noise is negligible. The effects of error control coding, using a random coding approach, is investigated in the latter part of this chapter. In Chapter 5, the performance of code diversity systems is studied in an environment where there is noise and fading. A summary of the main results of this work and some suggestions for possible future work appear in Chapter 6.

1.1 Multiple Access Schemes

In communication systems with a large number of users, where it is highly improbable that all users are transmitting at any given time, it is often desirable for the users to share a common channel; this results in a multiple-access channel. The main objective of a multiple-access system is to improve efficiency of channel utilization. While it is desirable to allow many transmissions over the multiple-access channel at any given time, this must be done at a manageable risk of transmission corruption due to collisions between transmissions which will result in re-transmissions. In addition, it is also desirable to reduce the amount of overhead associated with managing the transmissions such as guard times. There are two extremes among the many strategies that have been developed (see Figure 1.1). One is the random access approach in which all users use the same channel and the users send new packets immediately (or transmit in the next time slot in the case of slotted systems), hoping for no interference from the other users. The other extreme is the "perfectly scheduled" approach in which each user is allocated channel resources (i.e. time slot and/or bandwidth) in some orderly manner, as defined by the scheme, for transmission of its packet.

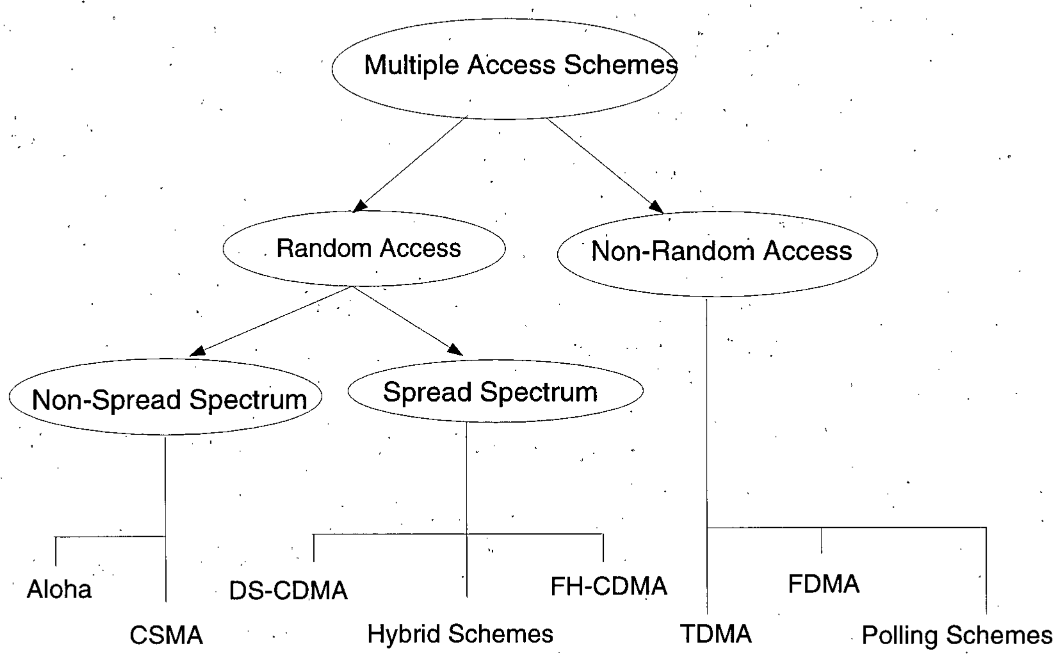

Figure 1.1 Multiple Access Schemes

ALOHA, slotted ALOHA and CSMA [8] are examples of random access schemes. Compared to the "perfectly scheduled" schemes, random access schemes have good delay characteristics under low offered traffic conditions and since each user can potentially use the whole channel, they are especially effective in handling bursty type traffic.

Examples of "rigidly scheduled" schemes include polling schemes, frequency division multiple access (FDMA) and time division multiple access (TDMA) [8], [9]. These schemes perform well under heavy traffic conditions especially when all the transmitters are transmitting regularly. Under such conditions, channel resources such as time slots in the case of TDMA and frequency channels in FDMA are heavily utilized and there is little wastage. The main drawback in most of these schemes is the longer delay compared to random access schemes under light traffic situations.

1.2 Code Division Multiple Access (CDMA)

CDMA which applies spread-spectrum waveform technology is a relatively new approach to the multiple access communications problem. It is recognized as a viable alternative to the schemes mentioned above. Pickholtz *et al* [1], Pursley [10] and Sklar [9] provide some good background on the theory behind CDMA. There are primarily three basic spread-spectrum techniques used in CDMA. These are

- direct sequence (DS) spread spectrum.
- frequency hopping (FH) spread spectrum which can be further classified into
    a. fast hopping in which there is more than one hop per data symbol.
    b. slow hopping in which one or more data symbols is transmitted per hop.
- hybrid schemes which combine both DS and FH spread spectrum features.

The idea behind these techniques is to take the energy that is to be transmitted and spread it over a very wide bandwidth so that the energy per unit bandwidth is small. (We shall see later in Section 1.3 that there are several advantages for transmitting using this technique.) This is achieved via a pseudonoise (PN) sequence that is unique to each user in the system. Unlike schemes such as FDMA or TDMA where the signals from each transmitter is separated in either frequency or time, all transmitters in a CDMA system can potentially be using the same transmission bandwidth simultaneously.

In DS systems, as shown in Figure 1.2, spreading is accomplished by multiplying the data by a PN sequence with a rate that is much faster than the data rate, prior to modulation. In the frequency domain, this is equivalent to convolving the original signal with another signal with a much larger bandwidth. Hence the transmitted signal occupies a much larger bandwidth. At the receiver end, an estimate of the data sent can be recovered by despreading using the same PN sequence with the appropriate time delay prior to demodulation. Since all active transmitters use the same bandwidth simultaneously, there is mutual interference among these transmitters. This interference is commonly called multiple-access interference. To minimize multiple-access interference, it is desirable that the PN sequences used have low cross-correlation properties. Various sequences such as Gold sequences and Reed-Solomon codes [11] are known to have long periods and low crosscorrelation characteristics, making them suitable for CDMA applications.

In FH systems, the total channel bandwidth is divided into a number of frequency bins. Unlike the DS system, each user in a FH system does not occupy the entire bandwidth at any given time. Instead, as shown in Figure 1.3, a hopping pattern determined by the PN sequence is used to specify which frequency bin is used for each hop interval. The same hopping pattern is generated at the receiver end and is used to reconstruct the message transmitted. When two or more users transmit using the same frequency bin simultaneously, they may interfere with each other's transmission. This is referred to as a frequency hit. A collection of hopping patterns are said to be orthogonal if there are no frequency hits throughout the length of the hopping patterns [12]. In such a case, no two transmitters will be using the same frequency bin simultaneously throughout the duration of the hopping patterns. In reality, the hopping patterns are normally quasi-orthogonal and hence frequency hits do occur. This is especially true in systems where the length of the hopping patterns is long, the number of transmitters is large (i.e. large number of hopping patterns) and there is a small number of frequency bins.

The PN sequence rate, also known as the chip rate, of a DS-CDMA system is normally much faster than the hopping rate of a FH-CDMA system. This is mainly due to the technology limitation of the frequency synthesizer [5], [13]. In a multipath fading environment where the delay spread is larger than the chip interval, a multipath-combining receiver such as the RAKE receiver [10], may be used to combine the signal received from the various paths in a DS-CDMA system. Such a receiver offers a form of diversity and can improve the performance of a DS-CDMA system. Hence DS-CDMA systems generally perform better than FH-CDMA systems in a frequency selective multipath environment [10], [14]. However, depending on the transmission protocol and nature of the channel, FH-CDMA can have better capture, multiple access and narrow-band interference rejection characteristics than DS-CDMA [10]. Hybrid schemes which combine both FH-CDMA and DS-CDMA features have been suggested [14]–[16] as yet another alternative CDMA system. Such schemes can lead to improved performance but at the cost of increased transmitter and receiver complexity. In hybrid systems, two PN sequences are used. One to spread the bandwidth of the message signal and the other for frequency hopping. Unlike a true DS system, the spreading of the message signal is not over the entire system bandwidth but rather over a fraction of it. Such fractions of the entire system bandwidth will be used for transmission depending on the hopping pattern. The reverse process of reconstructing the hopped signal and despreading is performed at the receiver. Such a system is illustrated in Figure 1.4.

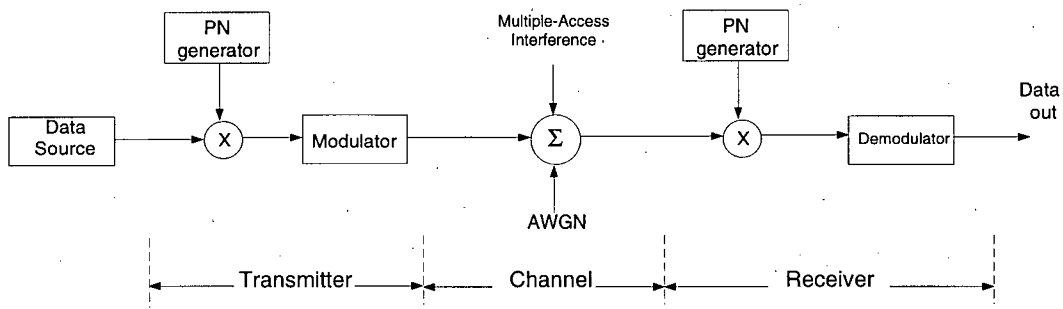

Figure 1.2 A Direct Sequence CDMA system

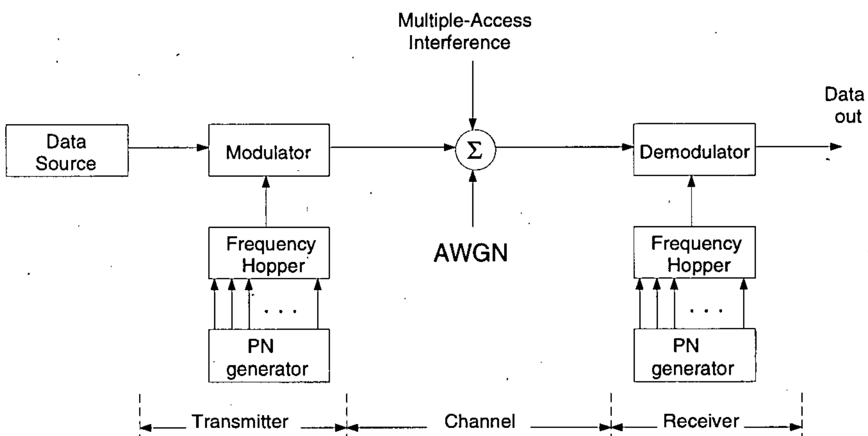

Figure 1.3 A Frequency Hopped CDMA system

1.3 Advantages of CDMA Scheme

The two most common multiple access techniques in the area of wireless communications are FDMA and TDMA. In FDMA, all users may transmit simultaneously, and use disjoint frequency bands with some guard bands between adjacent frequency bands. In TDMA, all users occupy the same frequency bandwidth, but transmit sequentially in time. CDMA is essentially a random access scheme. However, unlike conventional random access schemes

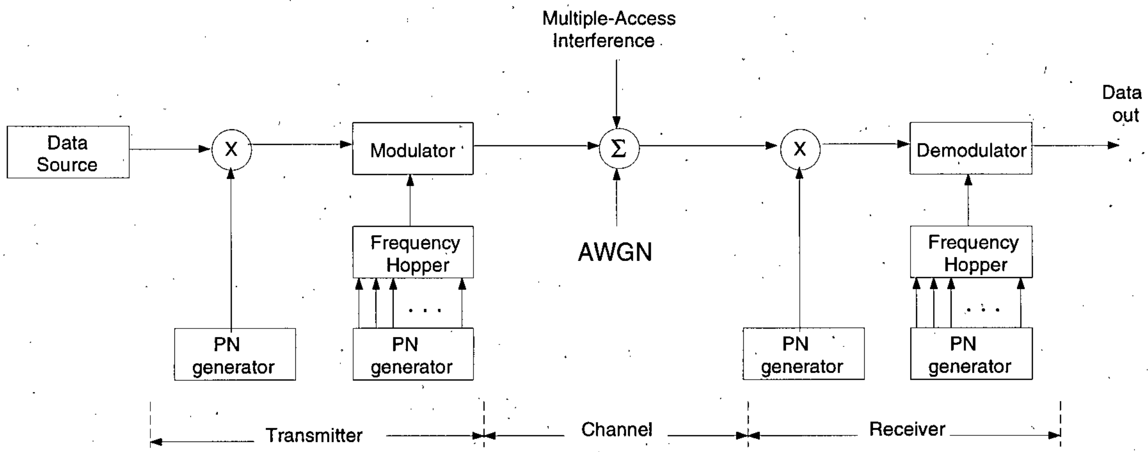

Figure 1.4 A Hybrid CDMA system

such as ALOHA or CSMA, because each user has a unique spreading sequence, simultaneous transmissions from several users in the system do not necessarily result in packet errors. Listed below are some of the advantages of CDMA over TDMA and FDMA that have been suggested.

- Increased capacity. Lee [5] and Gilhousen *et al* [4] have suggested that the number of users that can be supported for a fixed amount of bandwidth of a cellular CDMA system is about 20 and 4 times that of an equivalent analog FM/FDMA and TDMA cellular system respectively. Johannsen [3] showed that CDMA is a superior system in that it is able to support more users for a given bandwidth than FDMA in a mobile satellite communication environment.
- No hard handoff in cellular systems is required. Since every cell uses the same CDMA frequency spectrum, a mobile does not have to switch to another frequency channel when it crosses over to another cell, unlike FDMA and TDMA. In a CDMA system, the mobile can receive transmission from the adjacent cell sites at the cell boundary and switch over to a particular cell only when the received power from that particular cell is

significantly higher. Furthermore since each cell uses the same frequency spectrum, no frequency management or assignment is required in CDMA [5].

- Some guard time is often needed in TDMA systems between time slots to allow for possible synchronization problems between the users. This guard time represents an overhead which reduces the capacity of the system. Since CDMA is usually a random access system, such guard time is normally not required [5].
- Less prone to fading. CDMA uses wideband transmission which is effective in combatting frequency selective multipath fading [10]. An equalizer to combat such fading in FDMA and TDMA is not necessary in a CDMA system. This reduces the complexity of the receiver.
- Soft capacity. In CDMA, a new user can be added at the expense of a slight degradation in quality [5].
- Co-existence with narrowband system. Pickholtz *et al* [17] have shown that a carefully designed CDMA system and an existing narrowband system can share the same bandwidth without much adverse effect.
- Relatively higher level of privacy and security. Each transmitter-receiver pair uses a unique spreading sequence to generate and despread the spread-spectrum waveform which make it relatively more difficult for a casual listener to eavesdrop.

It should be noted that the issue of channel capacity has been the subject of an on-going debate. In most of the literature, the calculation of CDMA capacity has been based on theoretical models; Gilhousen *et al* [18] do provide some field results. The capacity issue is discussed further in [4], [18]–[21].

Chapter 2

Review of Related Work

CDMA is a wide research topic and numerous aspects of CDMA have been discussed in the literature. These include DS-CDMA, fast hopping FH-CDMA, slow hopping FH-CDMA, coding, scheduling, stability, capacity, synchronization of clocks, interference and fading channels [22]–[30]. In this chapter, we will review previous work that is related to this thesis. In particular we will focus the review on slow hopping FH-CDMA and diversity techniques, which form the main focus of this thesis.

2.1 Frequency Hopped CDMA System Model

The entire frequency spectrum of a FH-CDMA system is divided into a number, q, of frequency bins. Within each frequency bin is a number of frequency tones, M, which the transmitters can select to transmit their symbols. In slow FH systems, the number of symbols transmitted, N_s, per hop interval is at least one and the time duration to transmit a packet is normally assumed fixed. The system may be slotted or unslotted. In slotted systems, all transmissions which begin in a slot must be completed within the same slot and there is no overlap between any two time slots. In addition, it is commonly assumed that a transmitter can transmit at most a single packet in a time slot [7]. Hence the number of packets which may potentially interfere with a particular packet is fixed within the time slot. In unslotted systems, there is no restriction on the start of transmission time of the users and transmitters can begin transmitting at any time.

In slotted systems, the slot duration is larger than the time taken to transmit a packet. This is necessary to maintain synchronization at the packet (or slot) level in the system as there may be different time delays associated with the users in the system. There are delay compensation schemes which can be employed to improve synchronization, thereby reducing the slot duration. For example, through the use of pilot tones transmitted by the base station, mobiles can estimate their distance from the base station and make the appropriate timing adjustments to compensate for their relative delay in their reverse link transmission [4]. There is another level of synchronization, that is the hopping times of the users. The hopping times of the users can be synchronous i.e. identical or asynchronous i.e. random. In synchronous hopping, the users synchronize their hopping time, therefore the number of potential interferers is fixed for the hop duration. In asynchronous hopping, there is no restriction imposed on the hopping times of the users other than transmission of a packet must begin and end in the same time slot. Figure 2.1 illustrate how FH-CDMA systems may be classified. For unslotted systems, since the transmitters can transmit at any time, there would be no synchronization at the hopping times. For such systems, asynchronous hopping is used.

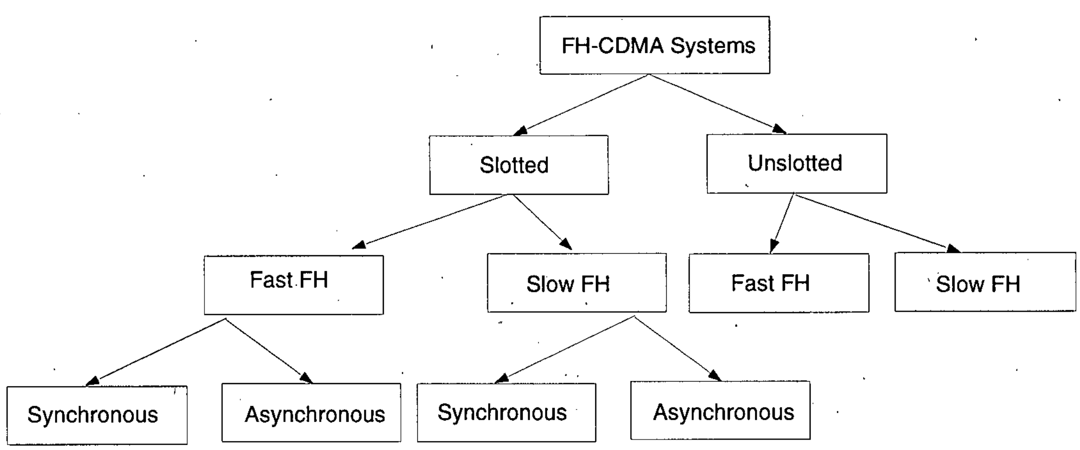

Figure 2.1 Classification of FH-CDMA systems

The multiple-access capability of a system is largely determined by its hopping patterns which determine which of the q frequency bins are used in a hop interval. Since these patterns are usually quasi-orthogonal rather than truly orthogonal, two or more users in the system may transmit in the same frequency bin simultaneously. Such an event is called a *frequency hit* and can result in loss of data even in the absence of noise or fading. Random hopping patterns are often used in the literature to model the extremely complex hopping patterns [12], [31]–[33]. These random hopping patterns can be either Markovian, in which the transmitter hops to a frequency bin other than the currently used frequency bin with equal probability, or memoryless, where the next frequency bin can be any frequency bin with equal probability [31]. In a synchronous frequency hopping system, where the hopping times of all users are synchronized, the probability of a frequency hit by another user is

$$P_h = 1/q \tag{2.1}$$

for both memoryless and Markovian random hopping pattern models. Users in an asynchronous frequency hopping system make no attempt to synchronize their hopping times. The probability of a hit in such a system, where N_s is the number of symbols transmitted per hop interval, is given by [31]

$$P_h = \frac{1}{q}\left(1 + \frac{1}{N_s}\right) \tag{2.2}$$

and

$$P_h = \frac{1}{q}\left[1 + \frac{1}{N_s}\left(1 - \frac{1}{q}\right)\right] \tag{2.3}$$

for Markovian and memoryless hopping patterns respectively. For asynchronous frequency hopping systems with a large number of frequency bins (i.e. large q) the probability of hit for both memoryless and Markovian random hopping patterns are essentially equal. For simplicity of analysis, it is commonly assumed that all frequency hits result in symbol errors [31], [33], [6], [12], regardless of the time duration of the hit or the symbols transmitted by the users. This is a pessimistic assumption and the actual symbol error rate should be lower than that obtained using this assumption.

Reception of the transmitted signal at the receiver may be coherent if the phase of the symbol signal is known. However, it is more realistic to assume noncoherent reception, especially for fast frequency hopping (FFH) situations [16]. For such cases, the phase of the received signal is normally assumed to be uniformly distributed over $[0, 2\pi)$. The received signal is composed of the signal from the transmitter, background additive white Gaussian noise (AWGN) and multiple-access interference. Rayleigh or Rician fading is commonly used to model mobile communication channels [16], [31], [34]. Multiple-access interference, due to frequency hits from other users, is commonly modelled as a Gaussian random variable at the receiver's demodulator to simplify the analysis [16], [20], [31], [35]. This approximation, motivated by the central limit theorem, is called the Gaussian approximation. By dividing the power level, relative to a particular user, into a number of groups, Geraniotis [36] derived an exact expression for the bit error probability and by comparison showed that the Gaussian approximation technique is reasonably accurate for systems using binary frequency shift keying (BFSK). For MFSK, the Gaussian approximation method is accurate only for low signal-to-inteference ratio i.e. when the transmission power of the interferers is large. For higher signal-to-inteference ratio, it gives very optimistic results. By assuming uniformly distributed random phases, independent data bits and a finite number of power levels, Cheun and Stark [32] also derived an exact expression for the error rate. It is shown [32] that results obtained using this expression give a very good fit to simulated results while the Gaussian approximation method gives results that are optimistic.

2.2 Slotted and Unslotted Systems

Transmission of data packets may be time-slotted with synchronous or asynchronous frequency hopping or unslotted. Fixed-rate hopping, so that all hop intervals (also known as dwell intervals) are of the same length T_h, is commonly assumed. Hopping times of the transmitters are the same in a synchronous FH-CDMA system. In certain applications where the propagation delay is small, synchronous frequency hopping may be accomplished by including a small guard interval at the ends of each dwell interval [37]. In time-slotted asynchronous FH-CDMA systems, we do not require that the transmission and the hopping times be completely synchronized. However, each packet transmission must be completed within a time slot which is normally slightly larger than the time required to transmit a packet. The time-slotted synchronous FH-CDMA scheme may be viewed as a special case of a time-slotted asynchronous FH-CDMA system where the time slot is exactly equal to the packet duration. In an unslotted FH-CDMA, no attempt is made to synchronize the hopping times or transmission times. As a result, unlike the time-slotted cases, the number of interferers during the transmission of a particular packet can vary.

In a time-slotted asynchronous FH-CDMA communication system [7], some guard time between packets is required to maintain slotting of the network. This situation is especially true in systems where the variations in propagation delays can be large. Frank and Pursley [38] mentioned that due to this guard time, symbols at the beginning and at the end of a particular packet may be subjected to less multiple access interference than the other symbols within the packet. However, no quantitative study of this "edge" effect was reported.

It is known that if the number J, of transmitters in an asynchronous FH-CDMA system is greater than 2, the frequency hits within a packet (which for simplicity consist of a single codeword [39], [33]) for a particular transmitter are not independent [39]. In calculating the probability of a packet error, it is quite commonly assumed that the hits within a packet are independent even when J is greater than 2 [7], [31]. We shall refer to this assumption as the *independence assumption*. Pursley [7] used the independence assumption and $P_h = 2/q$, derived from equation (2.2) for $N_s = 1$, in his calculation of codeword error rate. Hegde and Stark [6] and Georgiopoulos [40] calculated the codeword error rate without making the independence assumption but did not take the edge effect into account. The results in [6] indicate that the independence assumption can yield a good approximation to the actual codeword error probability if the edge effects are neglected. The difference is typically less than 1%.

2.4 Diversity Techniques in Spread Spectrum Systems

A number of spread spectrum diversity schemes used in different channel models can be found in the literature [34], [35], [41]–[44]. Together with the use of error control coding, diversity schemes can improve system performance significantly.

An early paper using FFH as a form of diversity in a multiple-access system is by Cooper and Nettleton [2]. The system uses phase-shift-keying (PSK) modulation and maximum likelihood decoding. Each data symbol is divided into L subsymbols and each subsymbol is transmitted in a hop interval. It is suggested that the number of users that can be supported by this scheme may exceed those using narrow-band schemes by a factor of five for a given BER. This work was extended by Goodman *et al* [45] who examined a scheme which uses using frequency-shift-keying modulation and majority logic decoding with a simple frequency hopping encoder/decoder structure. It is shown that this system can further improve the capacity of the system in [2] by a factor of almost three.

Atkin and Blake [46] studied a diversity system which uses FFH and multiple tone transmission per hop in the presence of jamming. For each hop interval, instead of sending a single tone as in the case of conventional FFH, a number of tones are transmitted simultaneously. It is shown that significant performance improvements can be achieved by such a multitone system over one that uses only FFH.

Solaiman *et al* [34] studied the performance of FFH CDMA with binary frequency shift keying (BFSK) in a frequency selective Rayleigh fading environment. A scheme using L antennas at the receiver with equal gain was proposed and analyzed. It is shown that for bit energy to noise ratios, E_b/N_o, less than 20 dB and diversity degrees, L of 3 or higher, the use of diversity can significantly improve the BER.

A FFH BFSK system with self-normalization combining in a partial band interference and Rician fading environment was studied by Robertson and Ha [47]. Each data symbol is divided into L subsymbols and transmitted over a fading channel with partial band interference, using a different frequency bin for each subsymbol. At the receiver, the output of each matched filter is squared and the sum is used to normalize the output of each detector before the L subsymbols receptions are combined. The sum of the square of the match filter outputs is directly proportional to the interference level detected by the receiver. As a result, subsymbols of hops which contain a large amount of interference will have less influence on the decision statistics than subsymbols of hops with less interference. It is shown that diversity can reduce receiver performance degradation due to partial band interference and fading when the signal-to-interference ratio is about 13 dB or more, especially when the signal does not contain a strong direct component.

A number of studies [48], [44] have modelled diversity as L independent receptions at the receiver without specifying the way in which such receptions can be achieved. These studies have proposed a variety of diversity combining schemes, some requiring side-information which gives a measure of the reliability of the received symbols. Schemes that do not require side-information include ratio-threshold test (RTT) [48], ratio-statistic combining [49] and clipped diversity combining [43]. The basic idea of the RTT and ratio-statistic combining schemes is to discard unreliable diversity receptions by comparing each reception against a derived threshold limit based on the received signal levels. By doing so, only diversity receptions that are considered reliable enter into the decision device. Clipped diversity combining [43] limits the influence of partial-band interference by clipping the output of each envelope detector at the receiver prior to combining. Diversity combining schemes that require side information have been studied in [35], [41] under different conditions. In these schemes, Reed-Solomon codes are used and the side information provides a measure of the reliability of the received symbols. Unreliable symbols are decoded as erasures rather than to one of the valid symbols. In [35], each code symbol is transmitted L times. Assuming Reed-Solomon coding scheme, frequency nonselective Rician channel, pulsed Gaussian interference and perfect side information, an expression for the symbol error probability as a function of the interference duty cycle of the pulsed Gaussian interference was derived. In [41], it is shown that with a proper combination of frequency hopping, diversity transmission and side information, a FH spread spectrum system can render a narrow band jammer harmless. As an example, numerical results indicate that employing a (256, 200) Reed-Solomon code with a diversity degree of 5 and 16–ary orthogonal signalling requires that over 68 percent of the total bandwidth be jammed and a signal-to-noise ratio below 6.81 dB in order for the symbol error probability to be above 10^{-4}.

Chapter 3

Asynchronous Hopping Slotted Systems

In this chapter, we present a method for determining the exact codeword error probability in a time-slotted asynchronous FH-CDMA communication system employing G_t hop intervals as guard time [50], [51]. In particular, we consider the cases where the guard time is 1 and 2 hop intervals long. It is shown that the codeword error probabilities when the edge effect is taken into consideration can be significantly different from those obtained in [6], [40]. It is also found that the independence assumption leads to larger probability of codeword error, for $G_t = 2$. Using the codeword error probability obtained using the new method, other performance measures such as maximum local traffic and throughput as defined by Pursley [7] are also evaluated. It is shown that the normalized values of these quantities are not significantly altered by using the independence assumption.

3.1 System Model

The hopping pattern for each transmitter T_i, $i = 1, 2, ..., J$ is assumed to be memoryless [6], [31], [40]. This means that the frequency bin used by T_i at each hop is chosen independently and uniformly from an available set of q frequency bins, numbered $1, 2, ..., q$, and independently of the bins used by the other transmitters. For simplicity, we shall assume that a packet consists of a single codeword and that hop rate is one per symbol. The same method of analysis can be used for calculating the probability of codeword error for system with hop intervals that are greater than one symbol but with the added complication of determining where the interferers hop epochs are relative to the symbols within a hop interval. Let $F_{i,j}$, $i = 1, 2, ..., J$, $j = 1, 2, ..., n$ denote the frequency bin used by T_i for the j^{th} symbol of its codeword. The j^{th} symbol of T_i's packet is said to have suffered a hit if $F_{i,j}$ is also simultaneously used by another transmitter.

From Figure 3.1, we observe that the symbol of a particular transmitter (hereafter referred to as **T**) can overlap (in time) with one or two symbols from each of the other transmitters due to the guard time interval and the asynchronous nature of the hopping times. As in [6], [40] we assume that **T**'s symbol is received in error (or erased in the case of an erasure demodulator) whenever a frequency hit occurs, even if the frequency hit occurs for only a small fraction of the hop interval. A symbol error can result from frequency hits by the other $K = J - 1$ transmitters either from the left or the right or both. We define the following three binary valued random variables:

$$H_j^L = \begin{cases} 1, & \text{if } \mathbf{T}'\text{s } j^{th} \text{ frequency bin is hit from the left} \\ 0, & \text{otherwise} \end{cases}$$
$$H_j^R = \begin{cases} 1, & \text{if } \mathbf{T}'\text{s } j^{th} \text{ frequency bin is hit from the right} \\ 0, & \text{otherwise} \end{cases}$$
$$H_j = \begin{cases} 1, & \text{if at least one of } H_j^L \text{ or } H_j^R \text{ is } 1 \\ 0, & \text{otherwise} \end{cases} . \tag{3.1}$$

Note that **T**'s j^{th} symbol is assumed to be demodulated correctly if and only if $H_j = 0$.

Since each packet must be transmitted within a slot, the start of packet transmission for each packet must be within the first G_t hop intervals of each slot. This start of packet transmission time is assumed to be uniformly distributed over the range $[0, G_t]$.

3.2 Codeword Error Probability

We will explicitly examine two cases, $G_t = 1$ and $G_t = 2$. The techniques used to analyze these cases can also be used for larger values of G_t, even though the computational

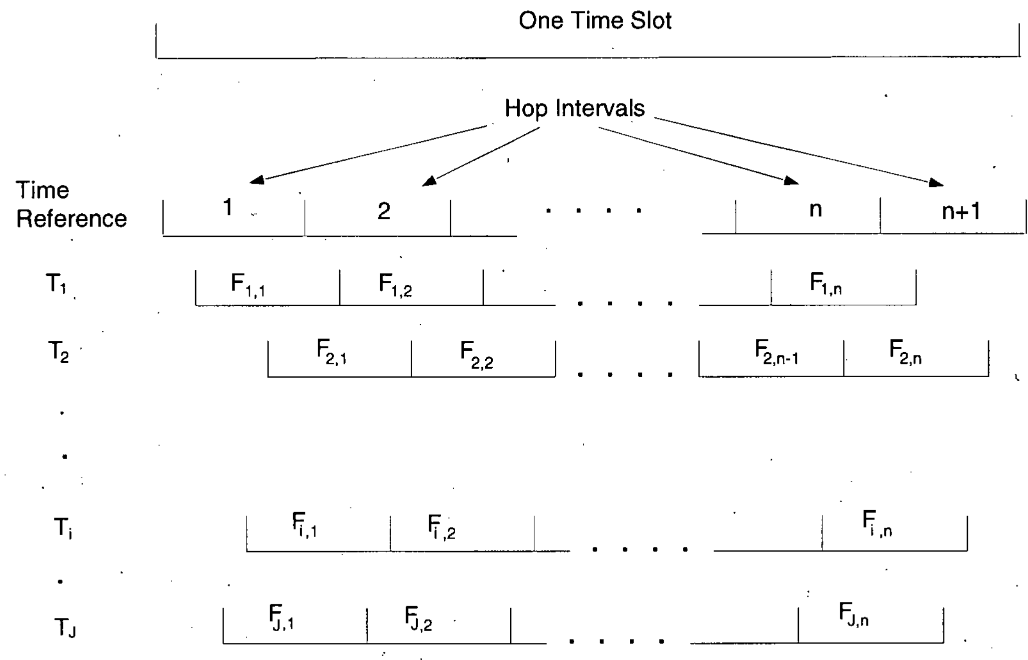

Figure 3.1 Time Slotted Asynchronous Hopping FH-CDMA System

complexity grows rapidly. This is because the number of possible ways by which **T** can begin transmitting and the number of possible ways by which the other transmitters can cause the edge effects in **T**'s transmission increases with G_t. Furthermore, to obtain the codeword error probability, an averaging over the total number of possible cases has to be performed.

3.2.1 Case for $G_t = 1$

Let A_1 and A_2 denote the two sets of interferers (of cardinality K_1 and K_2) that start transmission before and after **T** respectively. The first hop interval of **T**'s packet can be hit from the left only by interferers in A_1. Similarly, the last hop interval of **T**'s packet can be hit from the right only by interferers in A_2. The first and last hop intervals of **T**'s packet have to be considered separately from the other hop intervals within the packet. Let

$$\alpha_1 = (1 - 1/q)^{K_1}$$

$$\alpha_2 = (1 - 1/q)^{K_2}$$

$$
\begin{aligned}
\beta_1 &= (1 - 2/q)^{K_1} \\
\beta_2 &= (1 - 2/q)^{K_2} \\
\alpha &= (1 - 1/q)^{K} \\
\beta &= (1 - 2/q)^{K}.
\end{aligned}
\tag{3.2}
$$

Let the four possible values of H_j^L, H_j^R for **T**'s j^{th} hop interval be represented by states

- 0 : where **T**'s first frequency bin is not hit by any interferer either from the left or right.
- 1 : where **T**'s first frequency bin is not hit by any interferer from the left but is hit by at least one interferer from right.
- 2 : where **T**'s first frequency bin is hit by at least one interferer from the left but is not hit by any interferer from right.
- 3 : where **T**'s first frequency bin is hit by at least one interferer from the left and at least one interferer from right.

Since **T**'s first hop interval can be hit from the left only by the interferers in A_1, the probabilities associated with the states in **T**'s first hop interval are given by

$$
\begin{aligned}
P_0 &= \Pr\Big(H_1^L = 0, H_1^R = 0\Big) = \alpha_1.\alpha \\
P_1 &= \Pr\Big(H_1^L = 0, H_1^R = 1\Big) = \alpha_1(1 - \alpha) \\
P_2 &= \Pr\Big(H_1^L = 1, H_1^R = 0\Big) = (1 - \alpha_1)\alpha \\
P_3 &= \Pr\Big(H_1^L = 1, H_1^R = 1\Big) = (1 - \alpha_1)(1 - \alpha).
\end{aligned}
\tag{3.3}
$$

It has been shown [29] that $\Big(H_j^L, H_j^R\Big)$, $j = 1, 2, \ldots n$ form a Markov chain, i.e., $\Pr\Big(H_j^L, H_j^R | H_{j-1}^L, H_{j-1}^R, H_{j-2}^L, H_{j-2}^R, \ldots\Big) = \Pr\Big(H_j^L, H_j^R | H_{j-1}^L, H_{j-1}^R\Big)$. Furthermore, it is also shown [29] that

$$\Pr\left(H_j^L, H_j^R | H_{j-1}^L, H_{j-1}^R\right) = \Pr\left(H_j^L, H_j^R | H_{j-1}^R\right). \tag{3.4}$$

For the purpose of determining **T**'s packet error probability, there is no need to distinguish between the interferers after all interferers in A_2 have begun transmission. This is true until **T**'s last hop. The evolution of **T**'s hop intervals can be described by a 4–state Markov chain as shown in Figure 3.2. From (3.4), we can see that the transition probabilities of the Markov chain is only dependent on H^R of the previous state. Therefore the transition probabilities from states 0 and 2 to any other state are equal. Likewise the transition probabilities from state 1 and 3 to any other state are equal. . The transition probabilities for hop intervals between **T**'s first and last hop intervals are thus given by [6]

$$\begin{aligned}
P_{00} = P_{20} &= \frac{1}{q}\alpha + \left(1 - \frac{1}{q}\right)\beta \\
P_{01} = P_{21} &= \left[\frac{1}{q}\alpha + \left(1 - \frac{1}{q}\right)\beta\right]\left(\frac{1-\alpha}{\alpha}\right) \\
P_{02} = P_{22} &= \left(1 - \frac{1}{q}\right)(\alpha - \beta) \\
P_{03} = P_{23} &= \left(1 - \frac{1}{q}\right)(\alpha - \beta)\left(\frac{1-\alpha}{\alpha}\right) \\
P_{10} = P_{30} &= \left(1 - \frac{1}{q}\right)(\alpha - \beta)\left(\frac{\alpha}{1-\alpha}\right) \\
P_{11} = P_{31} &= \left(1 - \frac{1}{q}\right)(\alpha - \beta) \\
P_{12} = P_{32} &= \left[\frac{1}{q}(1-\alpha) + \left(1 - \frac{1}{q}\right)(2(1-\alpha) - (1-\beta))\right]\left(\frac{\alpha}{1-\alpha}\right) \\
P_{13} = P_{33} &= \left[\frac{1}{q}(1-\alpha) + \left(1 - \frac{1}{q}\right)(2(1-\alpha) - (1-\beta))\right].
\end{aligned} \tag{3.5}$$

The detailed derivation of these transition probabilities is shown in Appendix A.1.

Due to possible edge effects, the transition probabilities for the first and last hop must be considered separately. Denote $P_{ij}^{(k)}$ as the transition probability for the i^{th} state to the j^{th} state in the k^{th} hop. From (3.4) and the fact that all transmitters must begin transmitting within the first hop interval of each time slot, the transition probabilities from the first to the second hop interval are given by

$$\begin{aligned}
P_{00}^{(1)} &= P_{20}^{(1)} = P_{00} \\
P_{01}^{(1)} &= P_{21}^{(1)} = P_{01} \\
P_{02}^{(1)} &= P_{22}^{(1)} = P_{02} \\
P_{03}^{(1)} &= P_{23}^{(1)} = P_{03} \\
P_{10}^{(1)} &= P_{30}^{(1)} = P_{10} \\
P_{11}^{(1)} &= P_{31}^{(1)} = P_{11} \\
P_{12}^{(1)} &= P_{32}^{(1)} = P_{12} \\
P_{13}^{(1)} &= P_{33}^{(1)} = P_{13}.
\end{aligned} \tag{3.5}$$

In **T**'s last hop interval, the K_1 interferers which started their transmissions earlier than **T** can only hit the from the left. The other K_2 interferers can hit the last hop interval from both the left and right. Using (3.4), the transition probabilities from the second last to the last hop interval of **T** can be obtained as

$$\begin{aligned}
P_{00}^{(n-1)} &= P_{20}^{(n-1)} = \frac{1}{q}\alpha_2 + (1-1/q)\frac{\beta\alpha_2}{\alpha} \\
P_{01}^{(n-1)} &= P_{21}^{(n-1)} = \frac{1}{q}(1-\alpha_2) + (1-1/q)\frac{\beta(1-\alpha_2)}{\alpha} \\
P_{02}^{(n-1)} &= P_{22}^{(n-1)} = (1-1/q)\left(1-\frac{\beta}{\alpha}\right)\alpha_2 \\
P_{03}^{(n-1)} &= P_{23}^{(n-1)} = (1-1/q)\left(1-\frac{\beta}{\alpha}\right)(1-\alpha_2) \\
P_{10}^{(n-1)} &= P_{30}^{(n-1)} = (1-1/q)\left(\frac{\alpha-\beta}{1-\alpha}\right)\alpha_2
\end{aligned}$$

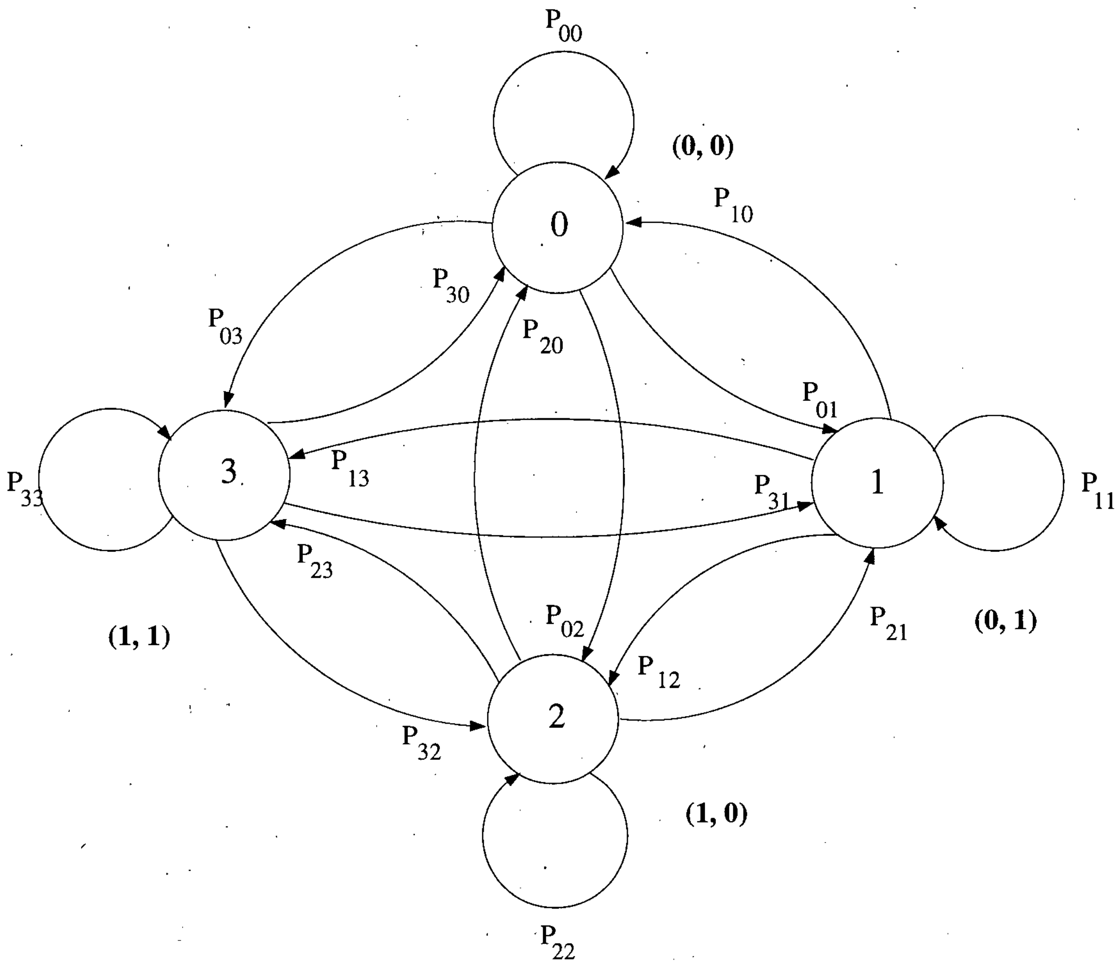

Figure 3.2 Four State Markov Chain

$$
\begin{aligned}
P_{11}^{(n-1)} &= P_{31}^{(n-1)} = (1-1/q)\left(\frac{\alpha-\beta}{1-\alpha}\right)(1-\alpha_2) \\
P_{12}^{(n-1)} &= P_{32}^{(n-1)} = \frac{1}{q}\alpha_2 + (1-1/q)[2(1-\alpha)-(1-\beta)]\frac{\alpha_2}{(1-\alpha)} \\
P_{13}^{(n-1)} &= P_{33}^{(n-1)} = \frac{1}{q}(1-\alpha_2) \\
&\qquad + (1-1/q)[2(1-\alpha)-(1-\beta)]\frac{(1-\alpha_2)}{(1-\alpha)}.
\end{aligned}
\tag{3.6}
$$

These transition probabilities are derived in Appendix A.2.

Following an approach similar to that in [6], let $r_{i,j}^{m}(L)$ be the probability that **T**'s m^{th} hop interval is in state j and exactly L of the m corresponding symbols are not hit, given that **T**'s first hop interval is in state i. We can write

$$r_{i,j}^{m+1}(L) = \sum_{l \in \{0,1,2,3\}} r_{i,l}^{m}(L) P_{lj}, \qquad j \neq 0 \tag{3.7}$$

when the Markov chain goes into one of the 3 states where **T**'s frequency bin is hit in the next transition and

$$r_{i,j}^{m+1}(L) = \sum_{l \in \{0,1,2,3\}} r_{i,l}^{m}(L-1) P_{lj}, \qquad j = 0 \tag{3.8}$$

when **T**'s frequency bin is not hit from either the left or right in the next transition. The initial conditions are

$$\begin{aligned}
r_{i,j}^{2}(0) &= P_{ij}, && i \neq 0, j \neq 0 \\
r_{i,j}^{2}(1) &= P_{i0}, && i \neq 0, j = 0 \\
r_{i,j}^{2}(1) &= P_{0j}, && i = 0, j \neq 0 \\
r_{i,j}^{2}(2) &= P_{00}, && i = 0, j = 0.
\end{aligned} \tag{3.9}$$

Assuming that every frequency hit results in a symbol error, the probability of exactly c correct symbols in a packet of n symbols given K_1 and K_2 interferers start packet transmission before and after **T** respectively is given by

$$\Pr(C = c | K_1, K_2) = \sum_{i \in \{0,1,2,3\}} \sum_{j \in \{0,1,2,3\}} \sum_{k \in \{0,1,2,3\}} P_i r_{i,j}^{n-1}(L) P_{jk}^{(n-1)},$$
$$L = \begin{cases} c-1, & \text{for } k = 0 \\ c, & \text{for } k \neq 0. \end{cases} \tag{3.10}$$

From (3.10), we can calculate the probability that a received packet is successfully decoded, given K_1 and K_2, as

$$\Pr\{\text{success}|K_1, K_2\} = \sum_{c=n-t}^{n} \Pr(C = c|K_1, K_2), \tag{3.11}$$

where t is the error correcting capability of the code.

The start of transmission times for the users are assumed to be independent, identically and uniformly distributed random variables with outcomes in the range $[0, 1]$. The probability of having K_1 interferers starting before **T**, given that there is a total of K interferers, is given by

$$\begin{aligned}\Pr(K_1, K - K_1) &= \int_0^1 \binom{K}{K_1} x^{K_1}(1-x)^{K-K_1}\, dx \\ &= \binom{K}{K_1} \mathrm{B}(K - K_1 + 1, K_1 + 1),\end{aligned} \tag{3.12}$$

where $\mathrm{B}(.,.)$ is the beta function [52]. Using (3.11) and (3.12), we can write the probability of packet error after decoding as

$$\begin{aligned}P_{e,1} &= 1 - \sum_{K_1=0}^{K} \Pr(K_1, K - K_1) \Pr\{\text{success}|K_1, K - K_1\} \\ &= 1 - \sum_{K_1=0}^{K} \binom{K}{K_1} \mathrm{B}(K - K_1 + 1, K_1 + 1) \sum_{c=n-t}^{n} \Pr(C = c|K_1, K_2).\end{aligned} \tag{3.13}$$

3.2.2 Case for $G_t = 2$

In this case, it is convenient to divide the interferers into 4 groups.

Group 1: Interferers in this group begin transmission within 1 hop interval earlier than **T**.

Group 2: Interferers in this group begin transmission within 1 hop interval later than **T**.

Group 3: Interferers in this group begin transmission more than 1 hop interval later than **T**. Note that this set of interferers is non-empty only if **T** starts transmission within the first hop interval of the time slot.

Group 4: Interferers in this group begin transmission more than 1 hop interval earlier than **T**. This set of interferers is non-empty only if **T** starts transmission within the second hop interval of the time slot.

In situation i, $i = 1, 2$, **T** begins transmission in the i^{th} hop interval of the time slot. By symmetry, the packet error probability is the same regardless of whether **T** begins transmission within the first or second hop interval of the time slot. The various groups of interferers are depicted in the two situations in Figure 3.3. We shall consider the situation where **T** starts its transmission in the first hop interval. The number of interferers in group 4 is zero in this case and the probability associated with the distribution of interferers in the other groups is given by

$$\begin{aligned}
&\Pr(K_1, K_2, K_3 = K - K_1 - K_2) \\
&= \int_0^1 \frac{K!}{K_1! K_2! \,(K - K_1 - K_2)!} \left(\frac{x}{2}\right)^{K_1} \left(\frac{1}{2}\right)^{K_2} \left(\frac{1-x}{2}\right)^{(K-K_1-K_2)} dx \\
&= \frac{K!}{2^K \, K_1! \, K_2! \, (K - K_1 - K_2)!} \int_0^1 x^{K_1} (1-x)^{(K-K_1-K_2)} dx \\
&= \frac{K!}{2^K \, K_1! \, K_2! \, (K - K_1 - K_2)!} \mathrm{B}(K_1 + 1, K - K_1 - K_2 + 1).
\end{aligned} \tag{3.14}$$

The only interferers that can hit **T**'s first hop interval are those in Group 1 and 2. The interferers in Group 1 can hit from left or right and the interferers in Group 2 can only hit

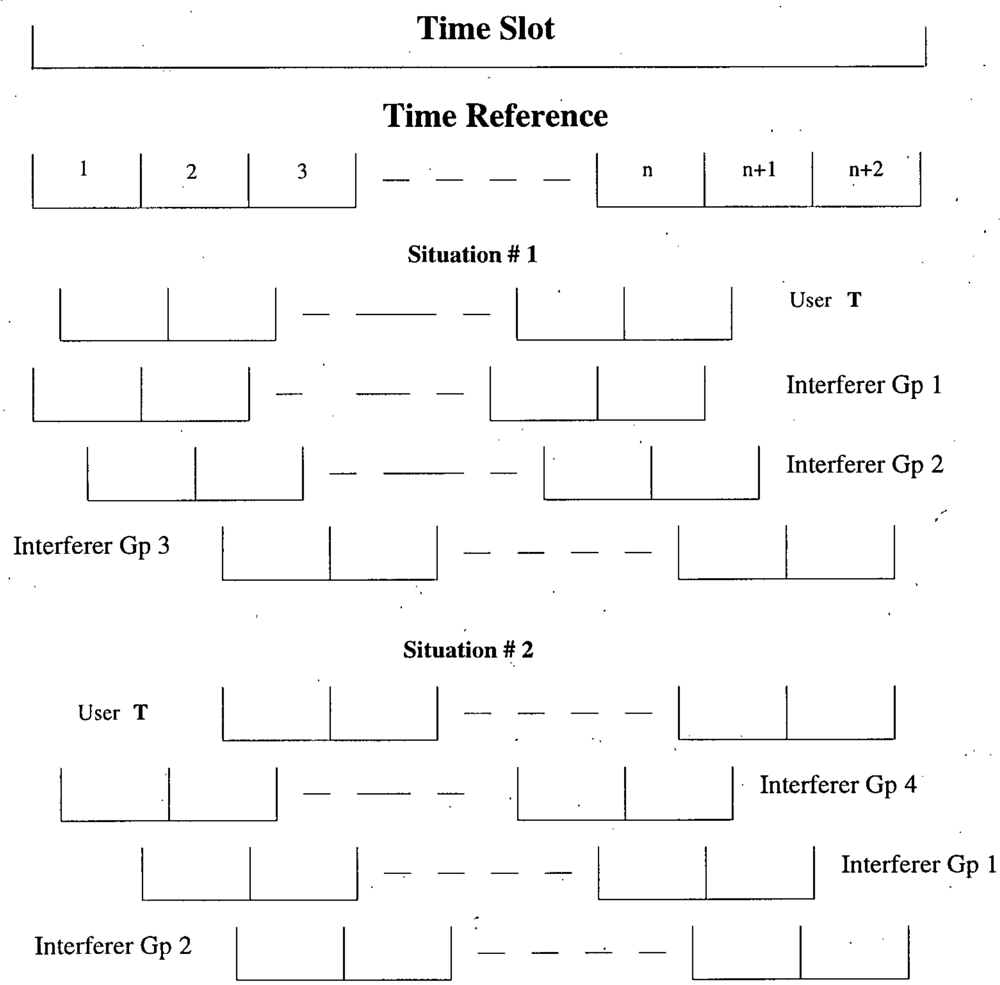

Figure 3.3 Possible Situations in the Case Where $G_t = 2$

from the right. Hence the starting state probabilities can be written as

$$\begin{aligned}
P_0 &= P\Big(H_1^L = 0, H_1^R = 0\Big) = \alpha_1^2\alpha_2 \\
P_1 &= P\Big(H_1^L = 0, H_1^R = 1\Big) = \alpha_1(1 - \alpha_1\alpha_2) \\
P_2 &= P\Big(H_1^L = 1, H_1^R = 0\Big) = (1 - \alpha_1)\alpha_1\alpha_2 \\
P_3 &= P\Big(H_1^L = 1, H_1^R = 1\Big) = (1 - \alpha_1)(1 - \alpha_1\alpha_2).
\end{aligned} \tag{3.15}$$

Using a similar method to those in Appendix A.1 and A.2, the transition probabilities from the first to the second hop interval of **T** is given by

$$\begin{aligned}
P_{00}^{(1)} = P_{20}^{(1)} &= \frac{1}{q}\alpha + \left(1 - \frac{1}{q}\right)\frac{\beta_1\beta_2}{\alpha_1\alpha_2}\alpha \\
P_{01}^{(1)} = P_{21}^{(1)} &= \frac{1}{q}(1-\alpha) + \left(1 - \frac{1}{q}\right)\frac{\beta_1\beta_2}{\alpha_1\alpha_2}(1-\alpha) \\
P_{02}^{(1)} = P_{22}^{(1)} &= \left(1 - \frac{1}{q}\right)\left(1 - \frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)\alpha \\
P_{03}^{(1)} = P_{23}^{(1)} &= \left(1 - \frac{1}{q}\right)\left(1 - \frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)(1-\alpha) \\
P_{10}^{(1)} = P_{30}^{(1)} &= \left(1 - \frac{1}{q}\right)\left(\frac{\alpha_1\alpha_2 - \beta_1\beta_2}{1-\alpha_1\alpha_2}\right)\alpha \\
P_{11}^{(1)} = P_{31}^{(1)} &= \left(1 - \frac{1}{q}\right)\left(\frac{\alpha_1\alpha_2 - \beta_1\beta_2}{1-\alpha_1\alpha_2}\right)(1-\alpha) \\
P_{12}^{(1)} = P_{32}^{(1)} &= \frac{1}{q}\alpha + \left(1 - \frac{1}{q}\right)[2(1-\alpha_1\alpha_2) - (1-\beta_1\beta_2)]\frac{\alpha}{(1-\alpha_1\alpha_2)} \\
P_{13}^{(1)} = P_{33}^{(1)} &= \frac{1}{q}(1-\alpha) + \\
&\qquad \left(1 - \frac{1}{q}\right)[2(1-\alpha_1\alpha_2) - (1-\beta_1\beta_2)]\frac{(1-\alpha)}{(1-\alpha_1\alpha_2)}.
\end{aligned} \tag{3.16}$$

The transition probability from the second last to the last hop interval of **T**'s packet can be written as

$$\begin{aligned}
P_{00}^{(n-1)} = P_{20}^{(n-1)} &= \frac{1}{q}\alpha_2\alpha_3 + \left(1 - \frac{1}{q}\right)\frac{\beta\alpha_2\alpha_3}{\alpha} \\
P_{01}^{(n-1)} = P_{21}^{(n-1)} &= \frac{1}{q}(1-\alpha_2\alpha_3) + \left(1 - \frac{1}{q}\right)\frac{\beta(1-\alpha_2\alpha_3)}{\alpha} \\
P_{02}^{(n-1)} = P_{22}^{(n-1)} &= \left(1 - \frac{1}{q}\right)\left(1 - \frac{\beta}{\alpha}\right)\alpha_2\alpha_3 \\
P_{03}^{(n-1)} = P_{23}^{(n-1)} &= \left(1 - \frac{1}{q}\right)\left(1 - \frac{\beta}{\alpha}\right)(1-\alpha_2\alpha_3) \\
P_{10}^{(n-1)} = P_{30}^{(n-1)} &= \left(1 - \frac{1}{q}\right)\left(\frac{\alpha-\beta}{1-\alpha}\right)\alpha_2\alpha_3 \\
P_{11}^{(n-1)} = P_{31}^{(n-1)} &= \left(1 - \frac{1}{q}\right)\left(\frac{\alpha-\beta}{1-\alpha}\right)(1-\alpha_2\alpha_3)
\end{aligned}$$

$$\begin{aligned}
P_{12}^{(n-1)} = P_{32}^{(n-1)} &= \frac{1}{q}\alpha_2\alpha_3 + \left(1-\frac{1}{q}\right)[2(1-\alpha)-(1-\beta)]\frac{\alpha_2\alpha_3}{(1-\alpha)} \\
P_{13}^{(n-1)} = P_{33}^{(n-1)} &= \frac{1}{q}(1-\alpha_2\alpha_3)+ \\
&\qquad \left(1-\frac{1}{q}\right)[2(1-\alpha)-(1-\beta)]\frac{(1-\alpha_2\alpha_3)}{(1-\alpha)} \qquad (3.17)
\end{aligned}$$

where $\alpha_3 = (1-1/q)^{K_3}$. Using (3.7), (3.8), (3.15), (3.16) and (3.17), the probability of having exactly c correct symbols in a packet of n symbols given that there are K_1, K_2 and K_3 interferers in Groups 1, 2 and 3 respectively, is given by

$$\Pr(C=c|K_1,K_2,K_3) = \sum_{i\in\{0,1,2,3\}}\sum_{l\in\{0,1,2,3\}}\sum_{j\in\{0,1,2,3\}}\sum_{k\in\{0,1,2,3\}} P_i P_{il}^{(1)} r_{l,j}^{n-1}(L) P_{jk}^{(n-1)},$$
$$L = \begin{cases} c-1, & \text{for } k=0 \\ c, & \text{for } k \neq 0. \end{cases} \qquad (3.18)$$

The initial conditions for $r_{lj}^{n-1}(L)$ are as listed in (3.9) except that they are determined by the modified first transition probabilities, $P_{il}^{(1)}$. The probability of successfully decoding a codeword with an error correcting capability of t given K_1, K_2 and K_3 can be written as

$$\Pr\{\text{success}|K_1,K_2,K_3\} = \sum_{c=n-t}^{n} \Pr(C=c|K_1,K_2,K_3). \qquad (3.19)$$

Using (3.14) and (3.19), the probability of codeword error with a guard time interval of 2 is given by

$$\begin{aligned}
P_{e,2} &= 1 - \sum_{K_1=0}^{K} \Pr(K_1,K_2,K_3 = K-K_1-K_2) \\
&\qquad\qquad \times \Pr\{\text{success}|K_1,K_2,K_3 = K-K_1-K_2\} \\
&= 1 - \sum_{K_1=0}^{K}\sum_{K_2=0}^{K-K_1} \frac{K!}{2^K K_1! K_2! (K-K_1-K_2)!} \mathrm{B}(K_1+1, K-K_1-K_2+1) \\
&\qquad\qquad \times \sum_{c=n-t}^{n} \Pr(C=c|K_1,K_2,K_3). \qquad (3.20)
\end{aligned}$$

3.2.3 Numerical Results

Tables 1—6 list the numerical results obtained using (3.13) and (3.20) for $P_{e,1}$ and $P_{e,2}$ respectively. The results are obtained for various Reed-Solomon codes which are commonly used in data communication systems [53]. Tables 1–4 are for the case of an erasure demodulator, whereas Tables 5 and 6 are for the case in which the demodulator makes a symbol error whenever there is a frequency hit. The codeword error probabilities, $P_{e,0}$ obtained by Hegde and Stark [6] are included in these tables for comparison. The percentage difference between $P_{e,0}$ and $P_{e,1}$ can be as large as 30% and the percentage difference between $P_{e,0}$ and $P_{e,2}$ can be as large as 40%. The codeword error probabilities, $P_{i,1}$ and $P_{i,2}$, calculated using the independence assumption and taking the edge effect into consideration for a guard time of 1 and 2 hop intervals respectively, are also shown in these tables. In calculating $P_{i,1}$ and $P_{i,2}$, the interferers are divided into groups depending on the time that they begin transmission relative to **T**. This approach is identical the one used in calculating $P_{e,1}$ and $P_{e,2}$. However, the probability of codeword error is calculated using the steady state probability for state 0 of the Markov chain, assuming that the probability of being in this state in the next hop interval is independent of the state of the previous and present hop interval. The percentage difference between $P_{e,1}$ and $P_{i,1}$ is comparable to that between $P_{e,0}$ and $P_{i,0}$ as reported in [6]. The percentage difference between $P_{e,2}$ and $P_{i,2}$ is larger and can be as large as 10%.

J	$P_{e,0}$	$P_{e,1}$	$P_{e,2}$	$P_{i,1}$	$P_{i,2}$	
3	2.444267	1.882547	1.640901	1.865608	1.533955	$(\times 10^{-11})$
4	9.854872	7.657452	6.712215	7.576007	6.264310	$(\times 10^{-9})$
5	5.337109	4.214235	3.718516	4.169784	3.476572	$(\times 10^{-7})$
6	9.696240	7.774854	6.908226	7.698326	6.478714	$(\times 10^{-6})$
7	8.849878	7.202879	6.445627	7.138844	6.067116	$(\times 10^{-5})$
8	5.034209	4.157422	3.747022	4.124798	3.540926	$(\times 10^{-4})$
9	2.030218	1.700660	1.543748	1.689109	1.464772	$(\times 10^{-3})$
10	6.308742	5.358689	4.898808	5.327774	4.667201	$(\times 10^{-3})$

Table 1 Probability of Codeword Error for (32,16)
Reed-Solomon Code with Erasure Correction ($e = 16$), $q = 50$

J	$P_{e,0}$	$P_{e,1}$	$P_{e,2}$	$P_{i,1}$	$P_{i,2}$	
3	1.731250	1.196324	0.973509	1.148531	0.867806	$(\times 10^{-4})$
4	5.531554	4.077022	3.400610	3.926648	3.041058	$(\times 10^{-3})$
5	4.060710	3.166278	2.712301	3.080231	2.465057	$(\times 10^{-2})$
6	1.382431	1.133423	0.996744	1.113716	0.922752	$(\times 10^{-1})$
7	2.987969	2.560861	2.308538	2.536849	2.174660	$(\times 10^{-1})$
8	4.854427	4.322844	3.985808	4.307251	3.811838	$(\times 10^{-1})$
9	6.561842	6.032867	5.673601	6.033098	5.493900	$(\times 10^{-1})$
10	7.8777426	7.429979	7.104640	7.444487	6.947213	$(\times 10^{-1})$

Table 2 Probability of Codeword Error for (16,4)
Reed-Solomon Code with Erasure Correction ($e = 12$), $q = 10$

J	$P_{e,0}$	$P_{e,1}$	$P_{e,2}$	$P_{i,1}$	$P_{i,2}$	
3	3.339100	2.430601	2.064979	2.417253	1.920713	$(\times 10^{-7})$
4	8.670030	6.420760	5.478466	6.378077	5.081422	$(\times 10^{-6})$
5	7.812909	5.875259	5.041124	5.835487	4.678273	$(\times 10^{-5})$
6	3.948459	3.012379	2.600416	2.992773	2.417683	$(\times 10^{-4})$
7	1.385786	1.072042	0.931246	1.065524	0.867938	$(\times 10^{-3})$
8	3.785735	2.968415	2.594962	2.951859	2.425295	$(\times 10^{-3})$
9	8.617319	6.846324	6.023150	6.811779	5.646031	$(\times 10^{-3})$
10	1.707893	1.374423	1.216847	1.368225	1.144146	$(\times 10^{-2})$

Table 3 Probability of Codeword Error for (15,7)
Reed-Solomon Code with Erasure Correction ($e = 8$), $q = 50$

J	$P_{e,0}$	$P_{e,1}$	$P_{e,2}$	$P_{i,1}$	$P_{i,2}$	
3	9.256974	6.641580	5.611106	6.621845	5.225990	$(\times 10^{-10})$
4	2.923396	2.119269	1.791927	2.111252	1.662394	$(\times 10^{-8})$
5	3.198852	2.339003	1.981957	2.329500	1.836645	$(\times 10^{-7})$
6	1.959040	1.443791	1.226652	1.437820	1.136733	$(\times 10^{-6})$
7	8.313498	6.173097	5.259966	6.147744	4.877187	$(\times 10^{-6})$
8	2.739603	2.049098	1.751318	2.040853	1.625317	$(\times 10^{-5})$
9	7.503881	5.652559	4.846238	5.630467	4.502449	$(\times 10^{-5})$
10	1.784956	1.353993	1.164540	1.348884	1.083239	$(\times 10^{-4})$

Table 4 Probability of Codeword Error for (15,7)
Reed-Solomon Code with Erasure Correction ($e = 8$), $q = 100$

J	$P_{e,0}$	$P_{e,1}$	$P_{e,2}$	$P_{i,1}$	$P_{i,2}$	
3	4.263326	3.772204	3.542540	3.765302	3.430818	$(\times 10^{-4})$
4	6.336397	5.696838	5.389806	5.686733	5.236310	$(\times 10^{-3})$
5	3.313866	3.024533	2.882686	3.020391	2.811121	$(\times 10^{-2})$
6	9.873741	9.140107	8.773584	9.131715	8.588309	$(\times 10^{-2})$
7	2.080458	1.951427	1.885804	1.950422	1.852692	$(\times 10^{-1})$
8	3.483900	3.307527	3.216267	3.306883	3.170408	$(\times 10^{-1})$
9	4.977498	4.776962	4.671439	4.777137	4.618713	$(\times 10^{-1})$
10	6.357719	6.159570	6.053561	6.160697	6.000956	$(\times 10^{-1})$

Table 5 Probability of Codeword Error for (31,15) Reed-Solomon with Error Correction ($e = 8$), $q = 50$

J	$P_{e,0}$	$P_{e,1}$	$P_{e,2}$	$P_{i,1}$	$P_{i,2}$	
3	2.095480	1.824375e	1.700443	1.822202	1.642756	$(\times 10^{-6})$
4	4.979229	4.372417	4.089914	4.366336	3.955115	$(\times 10^{-5})$
5	4.111137	3.639668	3.417290	3.634726	3.309877	$(\times 10^{-4})$
6	1.905278	1.700154	1.602336	1.698039	1.554739	$(\times 10^{-3})$
7	6.137291	5.518855	5.220994	5.512786	5.075341	$(\times 10^{-3})$
8	1.540237	1.395468	1.325087	1.394147	1.290548	$(\times 10^{-2})$
9	3.224276	2.942668	2.804527	2.940328	2.736552	$(\times 10^{-2})$
10	5.884108	5.408527	5.173190	5.405009	5.057140	$(\times 10^{-2})$

Table 6 Probability of Codeword Error for (31,15) Reed-Solomon with Error Correction ($e = 8$), $q = 100$

3.3 System Performance

In a conventional slotted ALOHA system where multiple-access interference is the only source of interference, the codeword error probability, $P_e(\mu)$ is 0 when there are no interfering packets in the slot and $P_e(\mu) = 1$ otherwise.* This is because packets in the same time slot are transmitted using the same frequency channel and when there are two or more packets using the same time slot, it is assumed that the packets interfere with each other and none of them can be successfully received by their respective intended receiver. Thus in the analysis of the slotted ALOHA system [8], it is assumed that the codewords are uncoded. In FH-CDMA system however, since symbols from the same packet may be transmitted using different frequency bins, $P_e(\mu)$ can be less than one even when there are several packets being transmitted in a time slot. In the case where some of the symbols in a packet are corrupted due to interference from other packets, the packet may still be successfully received, depending on the type of error correcting code used. Furthermore, if a packet in a particular time slot cannot be decoded successfully, it does not necessarily mean that all other packets transmitted during that time slot cannot be decoded successfully. This suggests that the performance of FH-CDMA systems cannot be analyzed using the same techniques as the slotted ALOHA case. Pursley [7] suggested some new measures of system performance and these are based on the constraint that the average probability of codeword error, $P_e(\mu)$, for the expected number of interfering packets, μ, is not to exceed a pre-determined error rate, P_E. To allow valid comparison between FH-CDMA and slotted ALOHA systems, it is necessary to normalize these new measures with the code rate, r, and the number of frequency bins, q.

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* This is the usual assumption that is made in the analysis of the slotted ALOHA system although some authors [54] have developed capture models wherein due to the difference in received powers of the contending packets, it may be possible to successfully decode a packet even though it has been collided with.

In [7] the maximum normalized local load, $L^*(P_E)$, normalized local throughput, $W^*(P_E)$, and maximum normalized local throughput, W_{max}, were used as performance measures for a FH-CDMA system. These measures were calculated using the independence assumption with the probability of a frequency hit by another user equal to $2/q$. The traffic was modelled as a Poisson distribution with an average number, μ, of interferers per time slot.

The normalized local load is defined by $L(P_E) \triangleq \frac{r}{q}\mu(P_E)$ where $\mu(P_E)$ is the number of interferers which can be accommodated if $P_e(\mu)$, is not to exceed P_E, and r is the code rate. The maximum normalized local load, $L^*(P_E)$, can be calculated using

$$L^*(P_E) = r\mu^*(P_E)/q, \tag{3.21}$$

where $\mu^*(P_E) = \max\{\mu : \; P_e(\mu) \leq P_E\}$ is the maximum expected number of arrivals in a time slot that is allowed subjected to the constraint that the average packet error remains below or equal to P_E.

The local throughput is defined as the number of successfully decoded packets per slot and is given by

$$s(\mu) = \mu P_c(\mu), \tag{3.22}$$

where $P_c(\mu) = 1 - P_e(\mu)$. When the expected number of arrivals, μ, increases the number of interfering packets increases and $P_c(\mu)$ decreases. Therefore the local throughput is a product of an increasing function of μ and a decreasing function of μ and there exist a maximum value. This maximum value can be determined by

$$S_{\max} = \max\{\mu P_c(\mu) : \; \mu \geq 0\}. \tag{3.23}$$

Since the local throughput is a product of an increasing function of μ and a decreasing function of μ, $P_c(\mu)$ may be unacceptably low when the local throughput is maximum.

As an alternatively, we may be interested in determining the maximum local throughput subjected to a constraint on P_E. This maximum value is given by

$$S^*(P_E) = \max\{s(\mu) : \mu \leq \mu^*(P_E)\}. \tag{3.24}$$

To allow for valid comparison against the slotted ALOHA system, we normalize the local throughput and the maximum local throughput. The normalized local throughput is defined as

$$w(\mu) = rs(\mu)/q, \tag{3.25}$$

and we can write the maximum normalized local throughput for a given P_E as

$$W^*(P_E) = rS^*(P_E)/q \tag{3.26}$$

and the unconstrained maximum normalized local throughput as

$$W_{\max} = \max\left\{\frac{r}{q}\mu P_c(\mu) : \mu \geq 0\right\}. \tag{3.27}$$

It is shown by Pursley [7] that in applications in which the probability of packet error is required to be low, for example $P_E \doteq 10^{-2}$, the maximum normalized local throughput for FH-CDMA is higher than that of slotted ALOHA.

3.3.1 Numerical Results

The Poisson distribution, $f(j)$ which is used in the performance measures is an infinite population model i.e.

$$f(j) = \mu^j e^{-\mu}/j! \qquad\qquad j \in \{0, 1, ...\}. \tag{3.28}$$

To obtain numerical results for $L^*(P_E)$, $W^*(P_E)$ and W_{max} a truncated Poisson series as in [7] was used. The maximum number of users, j_{max}, considered in a time slot is selected such that

$$1 - \sum_{j=0}^{j_{max}} f(j) \leq 10^{-10}. \tag{3.29}$$

This ensures that the accuracy of the results obtained is at least 10^{-10}. The code rate used to normalize these performance measures is modified, taking guard time into consideration. The modified code rate is given by

$$r = k/(n + G_t). \tag{3.30}$$

The results obtained, using (3.21), (3.26) and (3.27), for guard times of 1 and 2 hop intervals are shown in Tables 7 — 11. Subscript i is used to denote a guard time of i hop intervals. Results from [7], denoted by the subscript "0", are also included in these tables for comparison. The results indicate that the three performance measures are not sensitive to the inclusion of guard time into the time slots. The inclusion of a larger guard time will reduce the packet error probability due to edge effects. This will increase the throughput and local load. However, the inclusion of a larger guard time decrease the modified code rate which will reduce the normalized local load and throughput. These two factors appear to offset each other making the performance measures quite insensitive to the inclusion of guard time or the amount of guard time. It should be noted from the results, however, that these three performance measures do vary as a function of the code rate. For example, for $P_E = 10^{-2}$, $n = 64$ and $q = 25$, the maximum normalized local load and normalized maximum local throughput occurs at $k = 16$.

k	$L_0^*(10^{-2})$	$L_1^*(10^{-2})$	$L_2^*(10^{-2})$	$L_0^*(10^{-6})$	$L_1^*(10^{-6})$	$L_2^*(10^{-6})$
24	0.0165	0.0171	0.0168	4.17E-5	5.19E-5	5.33E-5
20	0.0348	0.0358	0.0353	0.0013	0.0014	0.0014
16	0.0531	0.0544	0.0535	0.0054	0.0058	0.0057
12	0.0672	0.0686	0.0676	0.0117	0.0123	0.0122
10	0.0712	0.0727	0.0716	0.0150	0.0157	0.0156
8	0.0723	0.0737	0.0726	0.0178	0.0186	0.0184
6	0.0692	0.0705	0.0695	0.0195	0.0202	0.0200
5	0.0656	0.0668	0.0659	0.0196	0.0203	0.0201
4	0.0602	0.0613	0.0605	0.0190	0.0196	0.0194

Table 7 Maximum Normalized Local Load for (32, k)
Reed-Solomon with Erasure Correction, $q = 25$

k	$L_0^*(10^{-2})$	$L_1^*(10^{-2})$	$L_2^*(10^{-2})$	$L_0^*(10^{-6})$	$L_1^*(10^{-6})$	$L_2^*(10^{-6})$
32	0.0592	0.0608	0.0603	0.0097	0.0103	0.0102
24	0.0742	0.0761	0.0755	0.0183	0.0192	0.0191
16	0.0796	0.0815	0.0806	0.0258	0.0268	0.0267
12	0.0762	0.0780	0.0774	0.0276	0.0286	0.0285
10	0.0723	0.0739	0.0734	0.0275	0.0285	0.0283

Table 8 Maximum Normalized Local Load for (64, k)
Reed-Solomon with Erasure Correction, $q = 25$

k	$L_0^*(10^{-2})$	$L_1^*(10^{-2})$	$L_2^*(10^{-2})$	$L_0^*(10^{-6})$	$L_1^*(10^{-6})$	$L_2^*(10^{-6})$
32	0.0926	0.0927	0.0918	0.0372	0.0373	0.0370
24	0.1067	0.1068	0.1058	0.0503	0.0505	0.0501
20	0.1089	0.1089	0.1079	0.0545	0.0546	0.0542
16	0.1067	0.1067	0.1058	0.0561	0.0562	0.0558
12	0.0988	0.1001	0.0981	0.0541	0.0545	0.0538

Table 9 Maximum Normalized Local Load for (64, k)
Reed-Solomon with Erasure Correction, $q = 100$

k	$W_0^*(10^{-2})$	$W_1^*(10^{-2})$	$W_2^*(10^{-2})$	$W_{0,\max}$	$W_{1,\max}$	$W_{2,\max}$
16	0.0526	0.0538	0.0530	0.0996	0.1007	0.0988
12	0.0665	0.0679	0.0676	0.1085	0.1098	0.1078
10	0.0705	0.0720	0.0709	0.1088	0.1101	0.1082
8	0.0716	0.0729	0.0719	0.1053	0.1067	0.1049
6	0.0685	0.0698	0.0688	0.0968	0.0981	0.0967

Table 10 Normalized Maximum Local Throughput for
$(32,\ k)$ Reed-Solomon Codes with Erasure Decoding, $q = 25$

k	$W_0^*(10^{-2})$	$W_1^*(10^{-2})$	$W_2^*(10^{-2})$	$W_{0,\max}$	$W_{1,\max}$	$W_{2,\max}$
40	0.0390	0.0403	0.0399	0.0818	0.0832	0.0823
32	0.0586	0.0602	0.0597	0.0996	0.1013	0.1003
24	0.0735	0.0753	0.0748	0.1094	0.1113	0.1103
20	0.0777	0.0796	0.0790	0.1100	0.1120	0.1110
16	0.0788	0.0806	0.0801	0.1068	0.1087	0.1078
15	0.0784	0.0803	0.0797	0.1053	0.1072	0.1063
12	0.0754	0.0772	0.0766	0.0985	0.1004	0.0996
10	0.0723	0.0732	0.0727	0.0919	0.0937	0.0930

Table 11 Normalized Maximum Local Throughput for $(64,\ k)$ Reed-Solomon Codes, $q = 25$

Chapter 4

Code Diversity Schemes

Consider a spread-spectrum multiple-access data communication system [1], [10], [55] with q frequency bins available for hopping. The transmitters use M-ary frequency-shift-keyed (FSK) modulation and slow frequency-hopping, i.e. the symbol (baud) rate is a multiple of the hopping rate. Suppose that there are J active transmitters. In a conventional FH-CDMA system, at each hop each active transmitter sends its data in a single frequency bin. Here, we study a code diversity system [56], [57] in which at each hop, a transmitter uses L distinct frequency bins. As described in Section 2.4, a similar technique has been previously proposed [46] to reduce the probability of "confusion" in systems which are subject to jamming.

The symbol error probabilities $P_e(J,q,L)$ for several different receiver decoding schemes are derived. It is shown that depending on the values of J and q, a large reduction in symbol error rates may be achieved using code diversity. Furthermore, the code diversity scheme can be used to establish priority classes among the users in the system by giving higher values of L to users in the higher priority groups. The optimal diversity degree, L^*, which minimizes $P_e(J,q,L)$ for each decoding scheme is also examined.

We consider a synchronous FH-CDMA system in which the hop times of the transmitters are synchronized. That is, the system is synchronous. Analysis for asynchronous hopping code diversity systems would be more complex and is not considered here. This complexity is mainly due to the fact that each user selects a set of $L > 1$ distinct frequency bins for transmission, so that the probability of a particular user choosing a particular frequency bin is dependent on the frequency bins that the user has already selected. Because of this dependency between the bins chosen by a user, the techniques used in Chapter 3 for the analysis of conventional asynchronous hopping FH-CDMA systems cannot be directly applied here. The probability of a frequency hit on one of L frequency bins is not only dependent on the probability of a hit from the right for the previous symbol but also dependent on the probability of a frequency hit on the remaining $L-1$ frequency bins.

4.1 Analysis of Symbol Error Probability for a Noiseless System

In this section we will consider a system where the main source of errors is interference from other users so that thermal noise can be neglected. The case where the effects of noise and fading are not negligible is considered in Chapter 5. Four receiver decoding schemes are studied. In Scheme 1, the receiver knows which symbol frequencies (in the transmitter's L frequency bins) are being sent, but does not know *how many* transmitters are transmitting a given symbol frequency tone. As long as there is only one particular M-ary symbol $m \in \{0, 1, ..., M-1\}$ that is present in *all* the L frequency bins used by transmitter, the receiver can correctly decode the transmitted symbol. In Scheme 2, the receiver is assumed to have knowledge of the number $n_{m,l}$ of transmitters sending symbol "m" in the l^{th} bin of the transmitter's L frequency bins. As in Scheme 1, the receiver is able to decode correctly if there is only one particular M-ary symbol present in all the L frequency bins used by transmitter. If this is not the case, the receiver in Scheme 2 sums up the number of transmitters in the L bins for each of the M symbols i.e. it forms $h_m \triangleq \sum_{l=1}^{L} n_{m,l}$; the receiver then chooses symbol m^* such that $h_{m^*} \geq h_m$, for $m \in \{0, 1, ..., M-1\}$. The decoding process in Scheme 3 is similar to that in Scheme 2 except that the receiver uses the maximum a posteriori (MAP) decision rule [58] based on $\{h_0, h_1, ..., h_m\}$ to choose the symbol which was transmitted.

Instead of using the MAP decision rule based on $\{h_0, h_1, ..., h_m\}$, Scheme 4 bases the MAP decision rule on $\{n_{0,1}, n_{1,1}, n_{0,2}, n_{1,2}, ..., n_{0,L}, n_{1,L}\}$. The decoding process used in schemes 3 and 4 are optimal in the sense that they achieve the lowest BER based on the information available to them.

4.1.1 Scheme 1

We first derive an expression for the symbol error probability $P_e(J, q, L)$ for Scheme 1. The following assumptions are used in this derivation:

- The data symbols for each transmitter are statistically independent and take on one of M possible values $\{0, 1, \ldots, (M-1)\}$ with equal probability.
- For each hop, each transmitter independently selects its L distinct frequency bins from the available q bins. Each of the $\binom{q}{L}$ possible sets is chosen with equal probability. A similar assumption has been commonly used in the study of conventional FH-CDMA systems [6], [31].
- If a transmitter, **T**, uses a certain set of L frequency bins to transmit a symbol $m \in \{0, 1, \ldots, (M-1)\}$, and no other symbol is simultaneously present in all the L frequency bins used by **T**, then **T**'s symbol can be correctly detected. This assumption is valid in an environment where the predominant cause of errors is interference from other users.
- Let the set of K interferers (for a marked transmitter **T**) be partitioned into M symbol groups $G_0, G_1, ..., G_{M-1}$. Each interferer in group G_i transmits symbol i, $i = 0, 1, ..., M-1$ and we denote the number of such interferers by K_i. A *hit* refers to a symbol transmission in a given frequency bin by any transmitter (including **T**). A *complete hit* on symbol s is said to occur if *all* the L bins used by **T** are hit by at least

one interferer from symbol group G_s. Note that if a receiver for **T** finds that there is a complete hit on only one symbol s^*, then the receiver knows that s^* must be the symbol transmitted by **T**. In the event of a complete hit on l symbols, the receiver randomly declares one of these l symbols as having been sent. Note that this definition of a hit is slightly different from that used in Section 3 where a hit refers to a frequency hit which is the event where two transmitters simultaneously use the same frequency bin to transmit. Here, we account the various possible symbols that can be transmitted within a frequency bin and we do not make the pessimistic assumption that all frequency hits results in decoding ambiguity or error.

Let $Q_1(i|k_s)$ be the probability of having exactly i of **T**'s L frequency bins hit by k_s interferers transmitting symbol "s". $Q_1(i|k_s)$ can be calculated using the following recursive equation :

$$\begin{aligned}
Q_1(i|k_s) &= Q_1(i|k_s-1)\frac{\binom{L-i}{0}\binom{q-L+i}{L}}{\binom{q}{L}} \\
&+ Q_1(i-1|k_s-1)\frac{\binom{L-i+1}{1}\binom{q-L+i-1}{L-1}}{\binom{q}{L}} \\
&+ Q_1(i-2|k_s-1)\frac{\binom{L-i+2}{2}\binom{q-L+i-2}{L-2}}{\binom{q}{L}} + \ldots \\
&+ Q_1(i-j|k_s-1)\frac{\binom{L-i+j}{j}\binom{q-L+i-j}{L-j}}{\binom{q}{L}} + \ldots \\
&+ Q_1(1|k_s-1)\frac{\binom{L-1}{i-1}\binom{q-L+1}{L-i+1}}{\binom{q}{L}} \\
&+ Q_1(0|k_s-1)\frac{\binom{L}{i}\binom{q-L}{L-i}}{\binom{q}{L}} \\
&= \sum_{j=0}^{i} Q_1(i-j|k_s-1)\frac{\binom{L-i+j}{j}\binom{q-L+i-j}{L-j}}{\binom{q}{L}} \qquad (4.1)
\end{aligned}$$

with initial conditions

$$Q_1(i|0) = \begin{cases} 1, & \text{for } i = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

By symmetry, the symbol error probability is the same regardless of the symbol transmitted by **T**. For convenience, we shall assume that **T** transmits a "0". Let P_i, $i \in \{1, 2, ..., M-1\}$ denote the probability that a complete hit is caused by the k_i interferers transmitting symbol "i". Then

$$P_i = Q_1(L|k_i), \qquad i \in \{1, 2, ..., M-1\}. \tag{4.3}$$

The probability of symbol error given the distribution of the K interferers can be written as

$$\begin{aligned} P_e(K+1, q, L|K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1}) = & \frac{1}{2} \sum_{i=1}^{M-1} P_i \prod_{j \neq i} (1 - P_j) \\ & + \frac{2}{3} \sum_{1 \leq i < j \leq M-1} P_i P_j \prod_{k \neq i,j} (1 - P_k) \\ & + ... \\ & + \frac{M-1}{M} \prod_{i=1}^{M-1} P_i. \end{aligned} \tag{4.4}$$

The probability associated with a particular distribution of interferers is given by the multinomial distribution [59]

$$\Pr\{K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1}\} = \frac{K!}{k_0!\, k_1! \dots k_{M-1}!} \left(\frac{1}{M}\right)^K. \tag{4.5}$$

Since

$$\begin{aligned} P_e(K+1, q, L) = & \sum_{k_0 + k_1 + ... + k_{M-1} = K} \Pr\{K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1}\} \\ & \times P_e(K+1, q, L|K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1}), \end{aligned} \tag{4.6}$$

the symbol error rate, $P_e(K+1, q, L)$, can be obtained using (4.1), (4.3), (4.4) and (4.5).

4.1.1.1 Numerical Results

Equation (4.6) was used to calculate the symbol error probability $P_e(J, q, L)$ when there are J active transmitters for $M = 2$, $q = 200$ and different diversity degrees. The results are plotted in Figure 4.1. It can be seen that depending on the value of J, a large decrease in the probability of bit error can be obtained through the use of diversity. As an example, for $J = 20$, $P_e = 2.32 \times 10^{-2}$ with no diversity whereas $P_e = 2.69 \times 10^{-4}$ with $L = 5$. Figure 4.2 shows the results obtained for $q = 200$ and $M = 3$ and Figure 4.3 show the plots for $q = 100$ for the binary case. It can be seen from these figures that the general behavior of the symbol error rate is similar in all three cases.

For $M = 2$ and $q = 200$, the BER $P_e(J, q, L)$ is plotted as a function of the diversity degree L for different values of J in Figure 4.4. It can be seen that $P_e(J, q, L)$ is moderately sensitive to L.

To illustrate the dependence of the optimal diversity degree L^* on J and q, we have plotted L^* as a function of q/J for different values of q in Figure 4.5. As would be expected, for a fixed value of q, L^* increases with q/J (i.e. decreases with J). For a fixed value of q/J, L^* tends to increase slightly with q. For $q = 100$, the optimal diversity degree can be approximated by

$$L^* = \max\left\{1, \left\lceil 3.84 \left(\frac{q}{J}\right)^{1/2} - 3.05 \right\rceil\right\} \tag{4.7}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. It was determined numerically that the use of L^* as given by (4.7) when q is in fact 50 or 200 does not result in a serious BER degradation. The percentage difference in BER's, for $q = 50$ is largest at $J = 6$ and is 11%. For $q = 200$, the largest percentage difference is 4% and occurs at $J = 25$.

It can be noted from Figure 4.1 that for some ranges of BER's, code diversity does increase the capacity (number of active transmitters which can be supported) of the system. For example, when the BER is 10^{-2}, the number of active transmitters that can be supported in the system is about 9 in the case where there is no code diversity. For this same BER, the capacity can be increased to 48 active transmitters by using a diversity degree of 5. The system capacity increase generally applies to all the code diversity schemes considered in this thesis when operating at lower BER's.

4.1.2 Scheme 2

In this section, we derive an expression for the symbol error probability $P_e(J, q, L)$ for Scheme 2 under the following assumptions:

- The hopping patterns, frequency bin and data symbol selection procedures are as described for Scheme 1.
- A receiver, **R**, which attempts to decode **T**'s transmission has knowledge of the number of hits (including **T**'s) on each of the symbols within the L frequency bins used by **T**. The decoding process proceeds as follows :
    - Step 1: If **R** finds that there is a complete hit on only one symbol, then **R** knows that this must be the symbol transmitted by **T**, and the decoding is complete.
    - Step 2: If there is a complete hit on $l\ (\geq 2)$ symbols, **R** sums the number of hits in the L frequency bins for each of these l symbols and chooses the symbol with the largest number of hits as the symbol sent by **T**. In the event that two or more

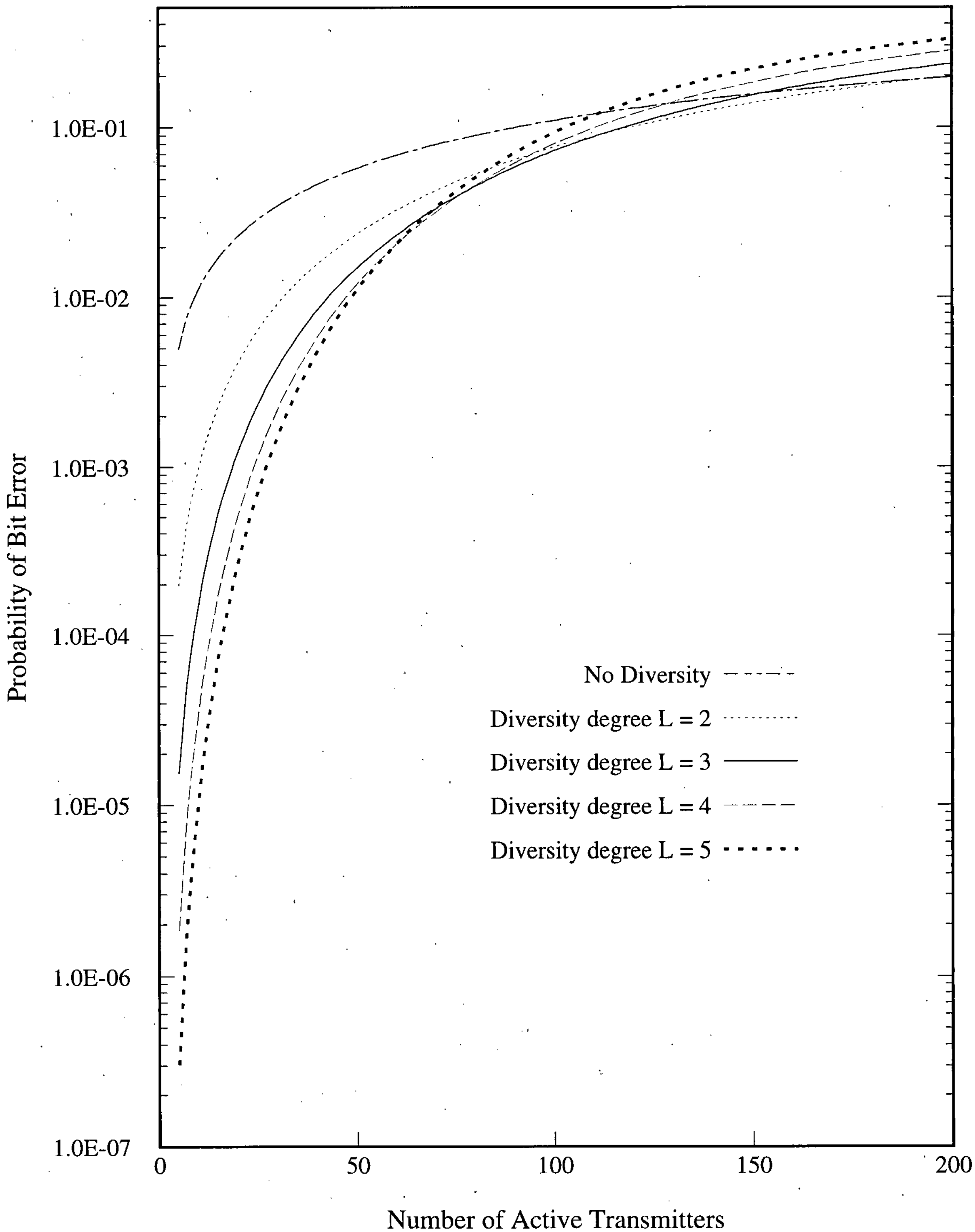

Figure 4.1 Probability of Bit Error Vs Number of Active Users in System, Scheme 1, $q = 200$.

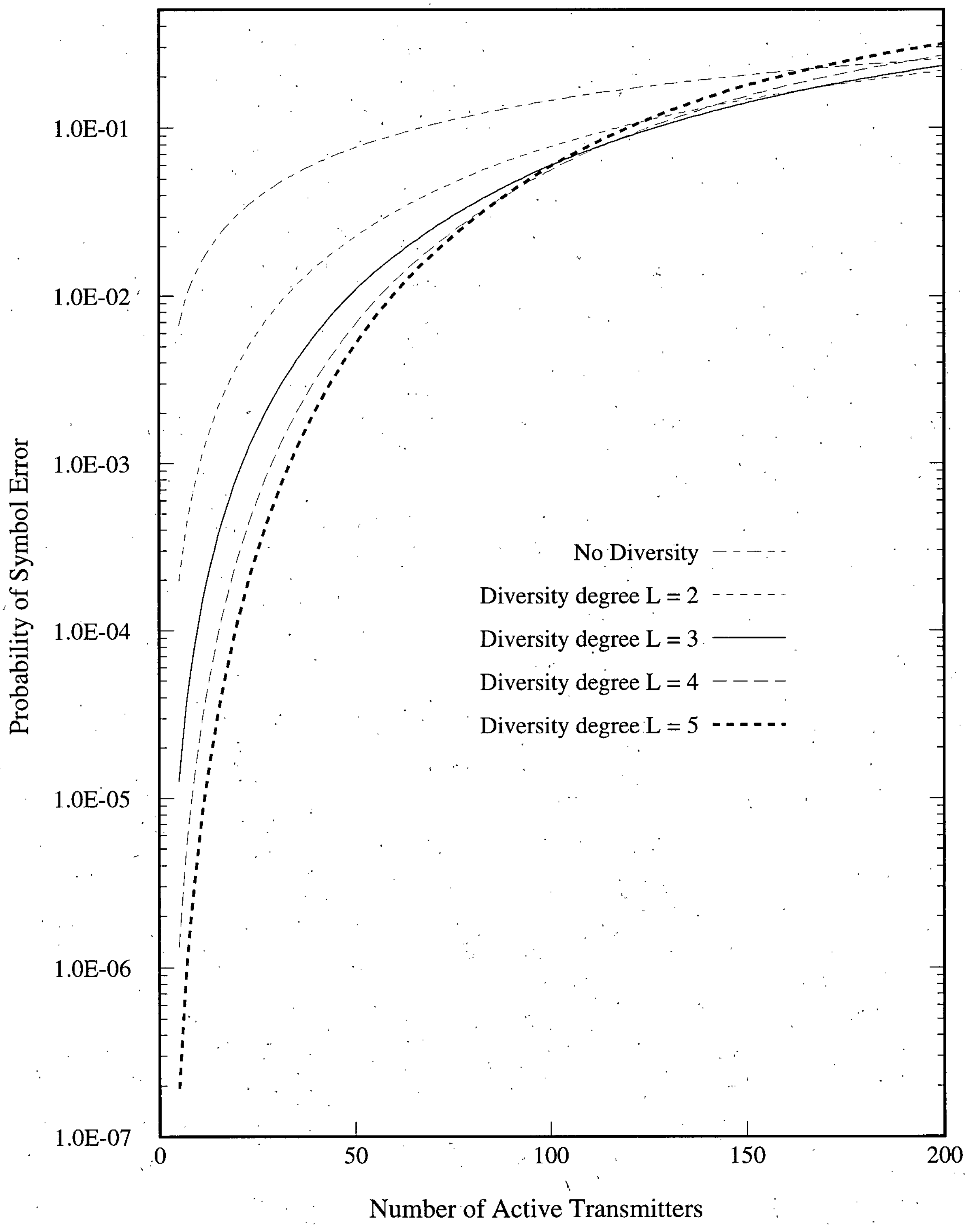

Figure 4.2 Probability of Symbol Error Vs Number of Active Users in System, Scheme 1, $q = 200$, $M = 3$.

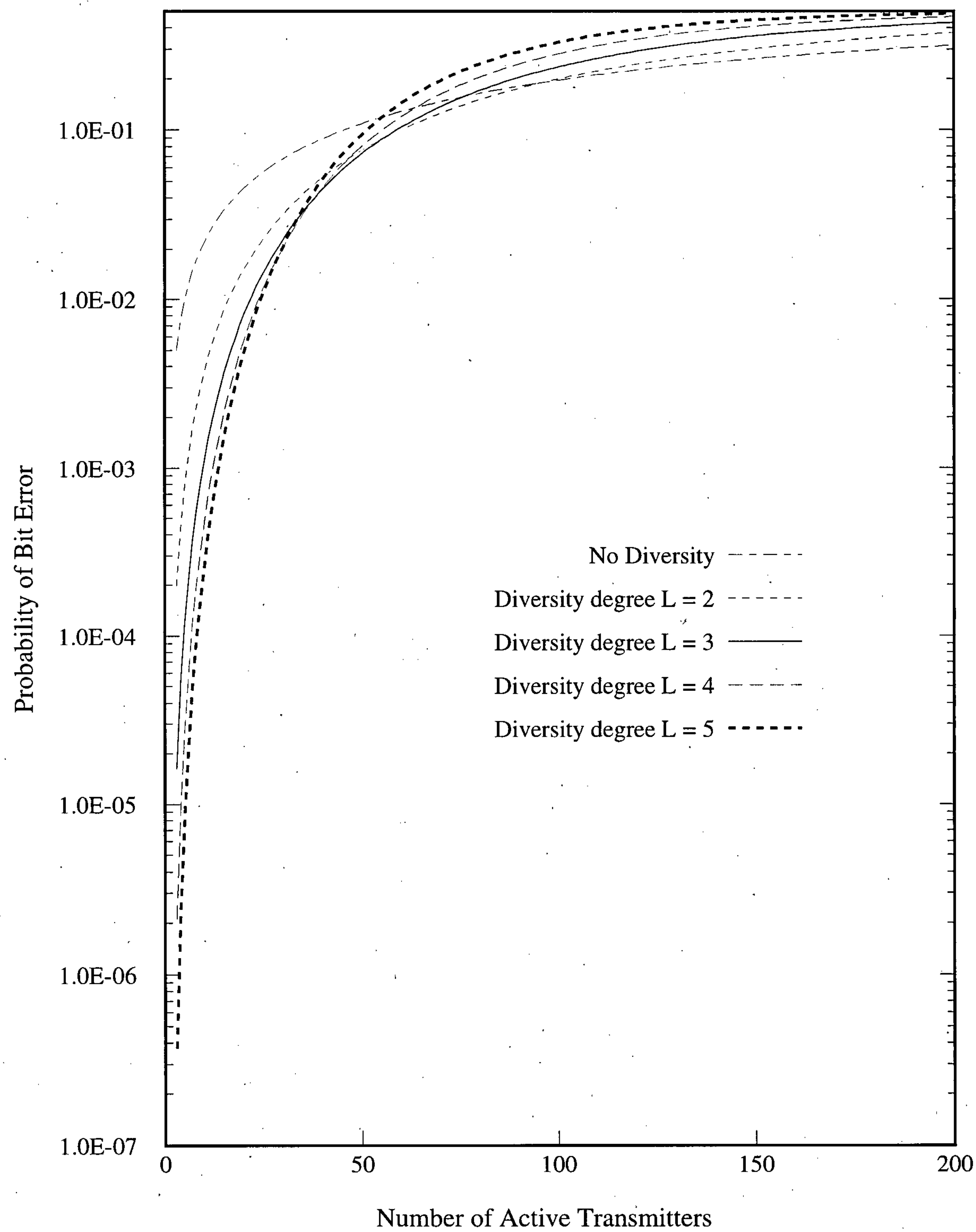

Figure 4.3 Probability of Bit Error Vs Number of Active Users in System, Scheme 1, $q = 100$.

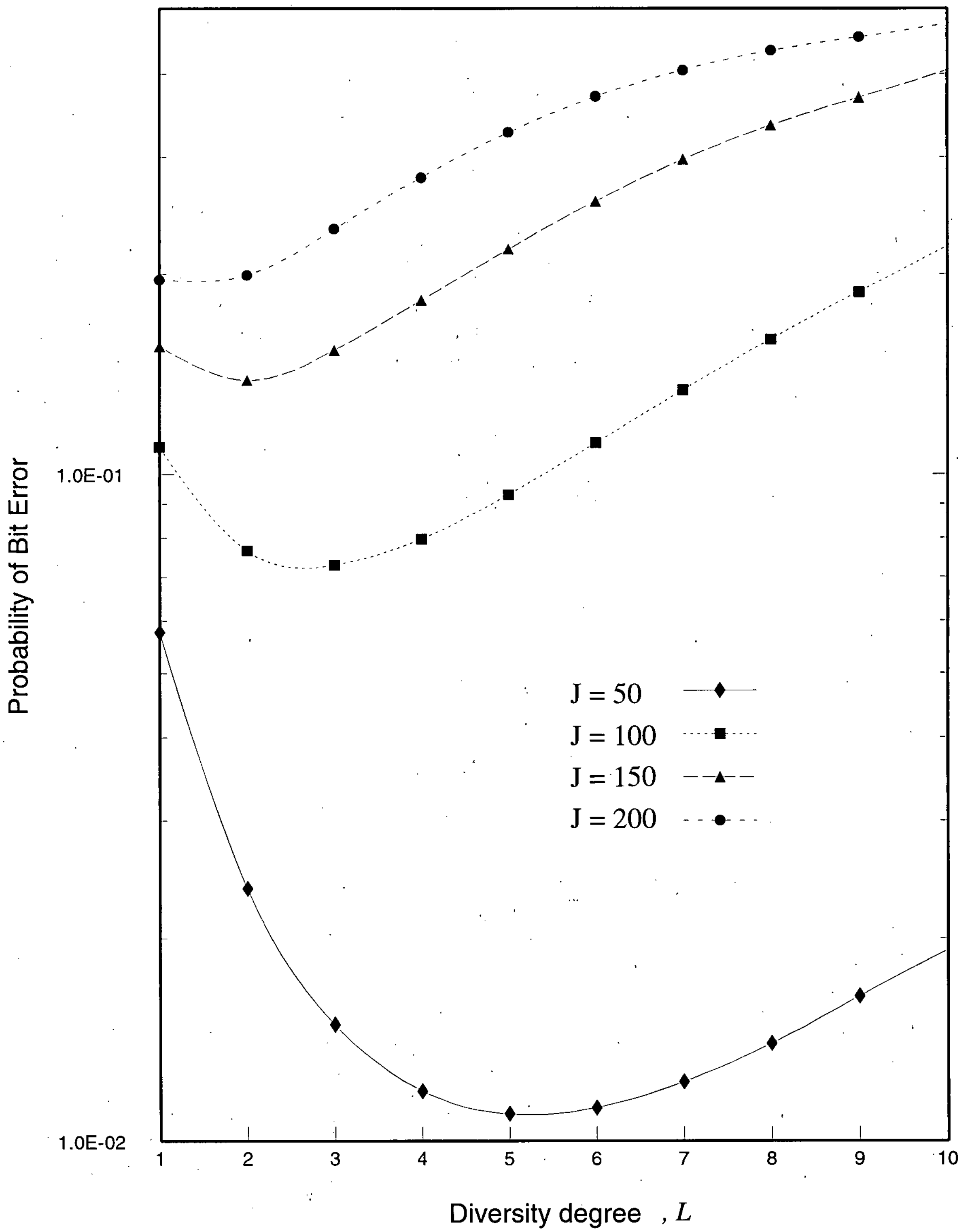

Figure 4.4 Probability of Bit Error Vs Diversity Degree, Scheme 1, $q = 200$.

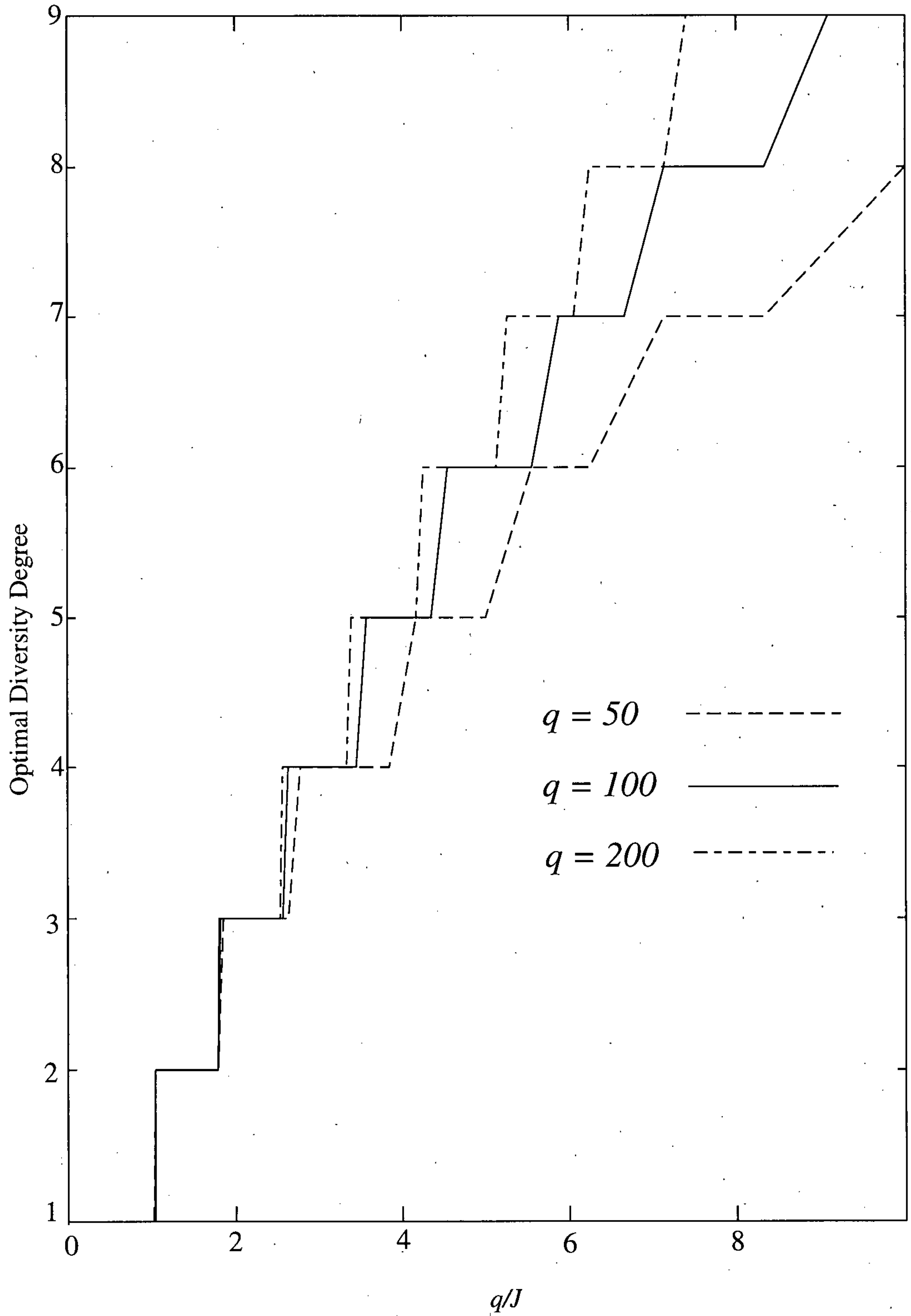

Figure 4.5 Optimal Diversity Level Vs q/J, Scheme 1.

symbols have the same number of hits, **R** randomly declares one of these symbols as having been sent.

For simplicity, we shall consider the binary symbol case, i.e. $M = 2$, first. The analysis is then extended to the general case in Section 4.1.2.2.

4.1.2.1 Binary Symbol Case

It is convenient to divide the interferers into two groups, K_0 interferers transmitting the symbol "0" and the remaining $K_1 = (K - K_0)$ interferers transmitting the symbol "1". Since each symbol is chosen independently and with equal probability by the interferers, the probability that there are exactly k_s interferers transmitting symbol "s" is given by

$$\Pr(K_s = k_s) = \binom{K}{k_s}\left(\frac{1}{2}\right)^K, \qquad s = 0 \text{ or } 1. \tag{4.8}$$

Let $Q_2(z_s, i|k_s)$ be the probability of having exactly i of **T**'s L frequency bins hit and a total number of z_s hits by the k_s interferers transmitting symbol "s". $Q_2(z_s, i|k_s)$ can be calculated using the following recursive equation

$$\begin{aligned}
Q_2(z_s, i|k_s) = {} & Q_2(z_s, i|k_s - 1)\frac{\binom{q-L}{L}}{\binom{q}{L}} + Q_2(z_s - 1, i|k_s - 1)\frac{\binom{i}{1}\binom{q-L}{L-1}}{\binom{q}{L}} + \ldots \\
& + Q_2\left(z_s - j, i|k_s - 1\right)\frac{\binom{i}{j}\binom{q-L}{L-j}}{\binom{q}{L}} + \ldots + Q_2(z_s - i, i|k_s - 1)\frac{\binom{i}{i}\binom{q-L}{L-i}}{\binom{q}{L}} \\
& + Q_2(z_s - 1, i - 1|k_s - 1)\frac{\binom{L-i+1}{1}\binom{q-L}{L-1}}{\binom{q}{L}} \\
& + Q_2(z_s - 2, i - 1|k_s - 1)\frac{\binom{i-1}{1}\binom{L-i+1}{1}\binom{q-L}{L-2}}{\binom{q}{L}} + \ldots \\
& + Q_2(z_s - j, i - 1|k_s - 1)\frac{\binom{i-1}{j-1}\binom{L-i+1}{1}\binom{q-L}{L-j}}{\binom{q}{L}} + \ldots \\
& + Q_2(z_s - i, i - 1|k_s - 1)\frac{\binom{i-1}{i-1}\binom{L-i+1}{1}}{\binom{q}{L}} + \ldots
\end{aligned}$$

$$+ Q_2(z_s - l, i - l | k_s - 1) \frac{\binom{L-i+l}{l}\binom{q-L}{L-l}}{\binom{q}{L}}$$

$$+ Q_2(z_s - l - 1, i - l | k_s - 1) \frac{\binom{i-l}{1}\binom{L-i+l}{l}\binom{q-L}{L-l-1}}{\binom{q}{L}} + \ldots$$

$$+ Q_2(z_s - l - j, i - l | k_s - 1) \frac{\binom{i-l}{j}\binom{L-i+l}{l}\binom{q-L}{L-l-j}}{\binom{q}{L}} + \ldots$$

$$+ Q_2(z_s - i, i - l | k_s - 1) \frac{\binom{i-l}{i-l}\binom{L-i+l}{l}}{\binom{q}{L}} + \ldots$$

$$+ Q_2(z_s - i, 0 | k_s - 1) \frac{\binom{L}{i}\binom{q-L}{L-i}}{\binom{q}{L}}$$

$$= \sum_{l=0}^{i} \sum_{j=0}^{i-l} Q_2(z_s - l - j, i - l | k_s - 1) \frac{\binom{i-l}{j}\binom{L-i+l}{l}\binom{q-L}{L-l-j}}{\binom{q}{L}} \tag{4.9}$$

with initial conditions

$$Q_2(z_s, i|0) = \begin{cases} 1, & \text{for } z_s = i = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

By symmetry, the symbol error probability is the same regardless of whether **T** transmits a "0" or a "1". Assuming that a "0" was sent, the probability of having a total of exactly z_0 hits given $K_1 = k_1$ can be written as

$$\Pr\{Z_0 = z_0 | K_0 = K - k_1\} = \sum_{i=0}^{L} Q_2(z_0, i | K_0 = K - k_1). \tag{4.11}$$

Using (4.9) and (4.11), the probability of symbol error given $K_1 = k_1$ can be written as

$$P_e(K+1, q, L|k_1) = \sum_{z_1=L}^{k_1 L} Q_2(z_1, L|k_1) \Big(\frac{1}{2} \Pr\{Z_0 = z_1 - L | K_0 = K - k_1\} + \sum_{z_0=L}^{z_1-L-1} \Pr\{Z_0 = z_0 | K_0 = K - k_1\} \Big). \tag{4.12}$$

Since

$$P_e(K+1,q,L) = \sum_{k_1=1}^{K} \Pr\{K_1 = k_1\}\, P_e(K+1,q,L|k_1), \tag{4.13}$$

the BER, $P_e(K+1,q,L)$ can be obtained using (4.8), (4.12) and (4.13).

4.1.2.2 Extension to Non-binary Symbol Case

The probability distribution of the interferers in the M symbol groups is given by (4.5). As before, we shall assume that the marked transmitter **T** sends symbol "0". For a given distribution of interferers $\{k_0, k_1, ..., k_{M-1}\}$, let

- A_i, $i \in \{1, 2, ..., M-1\}$ be the event that all L frequency bins chosen by **T** are hit by at least one of the k_i interferers and $z_i > z_0 + L$.
- B_i, $i \in \{1, 2, ..., M-1\}$ be the event that all L frequency bins chosen by **T** are hit by at least one of the k_i interferers and $z_i = z_0 + L$.

From the definition of $Q_2(.,.|.)$, we can write

$$\begin{aligned}\Pr\{A_i\} &= \sum_{z_i=L+1}^{k_i L} Q_2(z_i, L|k_0, k_1, ..., k_{M-1}) \sum_{z_0=0}^{z_i-L-1} \Pr\{Z_0 = z_0|k_0, k_1, ..., k_{M-1}\} \\ &= \sum_{z_i=L+1}^{k_i L} Q_2(z_i, L|k_i) \sum_{z_0=0}^{z_i-L-1} \Pr\{Z_0 = z_0|k_0\}, \qquad i = 1, 2, ..., M-1\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\Pr\{B_i\} &= \sum_{z_i=L}^{k_i L} Q_2(z_i, L|k_0, k_1, ..., k_{M-1}) \Pr\{Z_0 = z_i - L|k_0, k_1, ..., k_{M-1}\} \\ &= \sum_{z_i=L}^{k_i L} Q_2(z_i, L|k_i) \Pr\{Z_0 = z_i - L|k_0\}, \qquad i = 1, 2, ..., M-1.\end{aligned} \tag{4.15}$$

From equations (4.14) and (4.15), we can calculate $\Pr\{A_i\}$ and $\Pr\{B_i\}$ using (4.9) and (4.11). The probability of symbol error given a distribution of interferers $\{k_0, k_1, ..., k_{M-1}\}$ can be written as

$$
\begin{aligned}
P_e(K+1, q, L | K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1}) = \\
\Pr\{A_1 \cup A_2 \cup \ldots \cup A_{M-1}\} + \frac{1}{2} \sum_{i=1}^{M-1} \Pr\{B_i\} \prod_{j \neq i} (1 - \Pr\{B_j\} - \Pr\{A_j\}) \\
+ \frac{2}{3} \sum_{1 \leq i < j \leq M-1} \Pr\{B_i\} \Pr\{B_j\} \prod_{k \neq i,j} (1 - \Pr\{B_k\} - \Pr\{A_k\}) + \ldots \\
+ \frac{(M-1)}{M} \prod_{i=1}^{M-1} \Pr\{B_i\}.
\end{aligned}
\tag{4.16}
$$

The first term of (4.16) can be written as [59]

$$
\begin{aligned}
\Pr\{A_1 \cup A_2 \cup \ldots \cup A_{M-1}\} = \sum_{i=1}^{M-1} \Pr\{A_i\} - \sum_{1 \leq i < j \leq M-1} \Pr\{A_i\} \Pr\{A_j\} + \\
\sum_{1 \leq i < j < k \leq M-1} \Pr\{A_i\} \Pr\{A_j\} \Pr\{A_k\} + \ldots + (-1)^M \\
\times \prod_{1 \leq i \leq M-1} \Pr\{A_i\}.
\end{aligned}
\tag{4.17}
$$

Since

$$
\begin{aligned}
P_e(K+1, q, L) = \sum_{k_0 + k_1 + \ldots + k_{M-1} = K} \Pr\{K_0 = k_0, K_1 = k_1, \ldots, K_{M-1} = k_{M-1}\} \\
P_e(K+1, q, L | K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1})
\end{aligned}
\tag{4.18}
$$

we can obtain $P_e(K+1, q, L)$ using (4.5), (4.14), (4.15), (4.16) and (4.17).

4.1.2.3 Numerical Results

Equation (4.13) was used to calculate the symbol error probability $P_e(J, q, L)$ given J active transmitters for $M = 2$, $q = 200$ and different diversity degrees. The results are plotted in Figure 4.6. In general, for the same value of L, Scheme 2 yields a lower BER than Scheme 1, especially for $J \geq 10$. However, there are instances e.g. $q = 200$, $L = 4$, $J \in \{2, 3, 4, 5, 6\}$ in which Scheme 1 yields a lower BER, even though the difference is less than 3%. This somewhat surprising result can be attributed to the fact that when most of the interferers are transmitting a symbol that is different from **T**'s, Scheme 2 is more likely to make a decoding error than Scheme 1. Results for $q = 200, M = 3$ and $q = 100, M = 2$ are shown in Figure 4.7 and 4.8 respectively. The optimal diversity degree for a given number of transmitter is at least equal to or higher in Figure 4.7 than in Figure 4.6. This is because, for a same number of interferers, the probability of a complete hit is less when $M = 3$. The reverse situation is true when comparing Figure 4.6 with Figure 4.8. With a reduced number of frequency bins, the probability of a complete hit is increased for a given number of interferers. Hence, for a given number of transmitters, the optimal diversity degree in Figure 4.8 tends to be lower than in Figure 4.6.

For $M = 2$ and $q = 200$, the BER $P_e(J, q, L)$ is plotted as a function of the diversity degree L for different values of J in Figure 4.9. It can be seen that $P_e(J, q, L)$ is less sensitive to L than in Scheme 1.

To illustrate the dependence of the optimal diversity degree L^* on J and q, we have plotted L^* as a function of q/J for different values of q in Figure 4.10. For $q = 100$, the optimal diversity degree can be approximated by

$$L^* = \max\left\{1, \left\lceil 2.803\left(\frac{q}{J}\right)^{1/2} - 0.734 \right\rceil \right\}. \tag{4.19}$$

As in Scheme 1, it was determined numerically that the use of L^* as given by (4.19) when q is in fact 50 or 200 does not result in a serious bit error degradation. The percentage difference in BER's for both cases is largest at $J = 200$ and is about 10%.

4.1.2.4 Upperbound on Scheme 2

A simplified version of Scheme 2 in which the receiver does not perform step 1 of Scheme 2 is considered in this section. The BER of the resulting scheme, which we will be referred to as Scheme 2A, is an upperbound on the BER of Scheme 2. Its performance relative to that of Scheme 2 gives the loss in omitting step 1.

Using (4.9) and the law of total probability, the probability that the total number of hits on symbol "s" given k_s interferers can be written as

$$\Pr\{Z_s = z_s | k_s\} = \sum_{i=0}^{L} Q_2(z_s, i | k_s). \tag{4.20}$$

For a given distribution of interferers $\{k_0, k_1, ..., k_{M-1}\}$, let

- D_i, $i \in \{1, 2, ..., M-1\}$ be the event that $z_i > z_0 + L$.
- E_i, $i \in \{1, 2, ..., M-1\}$ be the event that $z_i = z_0 + L$.

We can write

$$\Pr\{D_i\} = \sum_{z_i=L+1}^{k_i L} \Pr\{Z_i = z_i | k_i\} \sum_{z_0=0}^{z_i-L-1} \Pr\{Z_0 = z_0 | k_0\}, \qquad i = 1, 2, ..., M-1 \tag{4.21}$$

and

$$\Pr\{E_i\} = \sum_{z_i=L}^{k_i L} \Pr\{Z_i = z_i | k_i\} \Pr\{Z_0 = z_i - L | k_0\}, \qquad i = 1, 2, ..., M-1. \tag{4.22}$$

The probability of symbol error given a distribution of interferers $\{k_0, k_1, ..., k_{M-1}\}$, $P_e(K+1, q, L | K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1})$, is again given by (4.16), except with A_i and B_i replaced by D_i and E_i respectively.

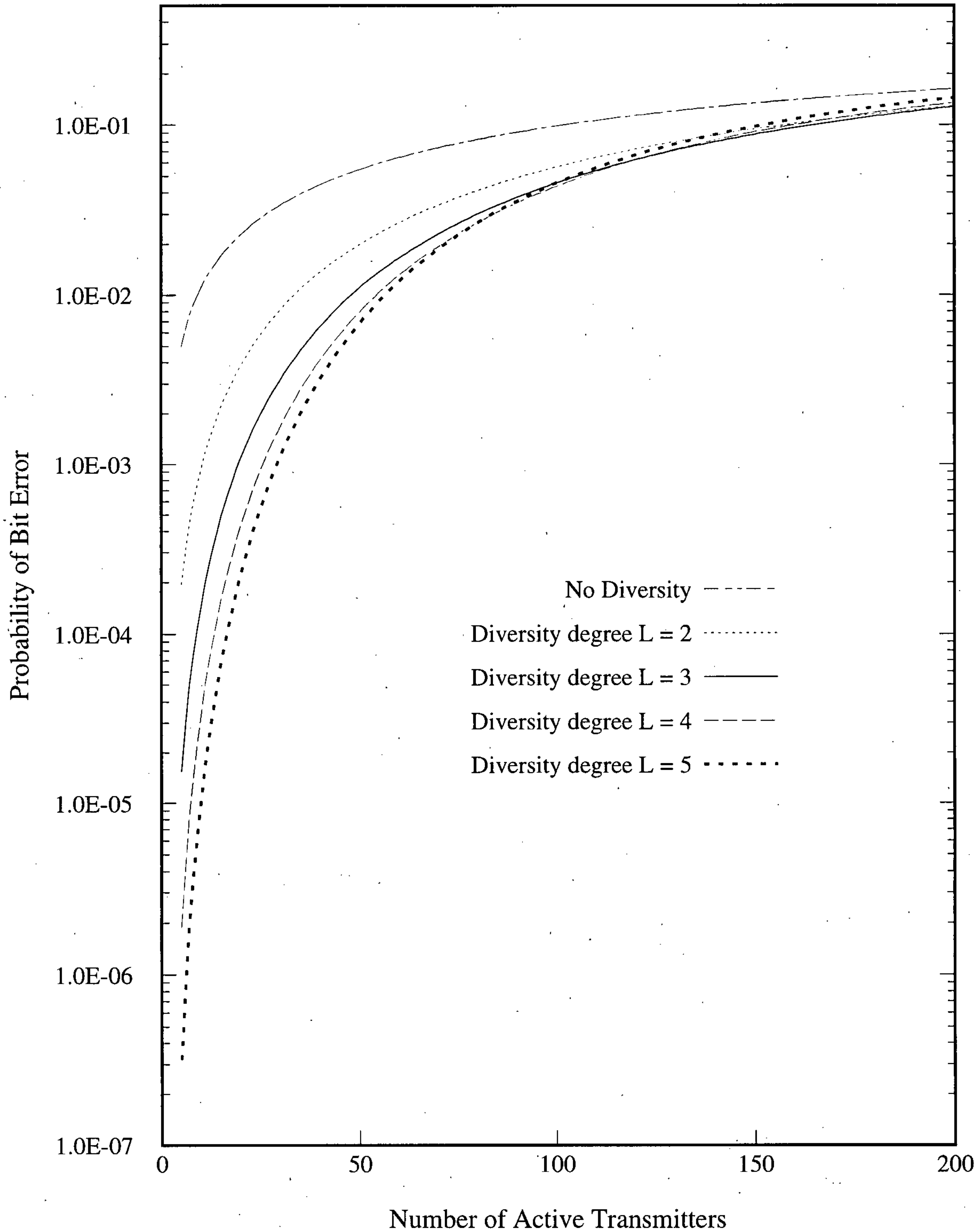

Figure 4.6 Probability of Bit Error Vs Number of Active Users, Scheme 2, $q = 200$.

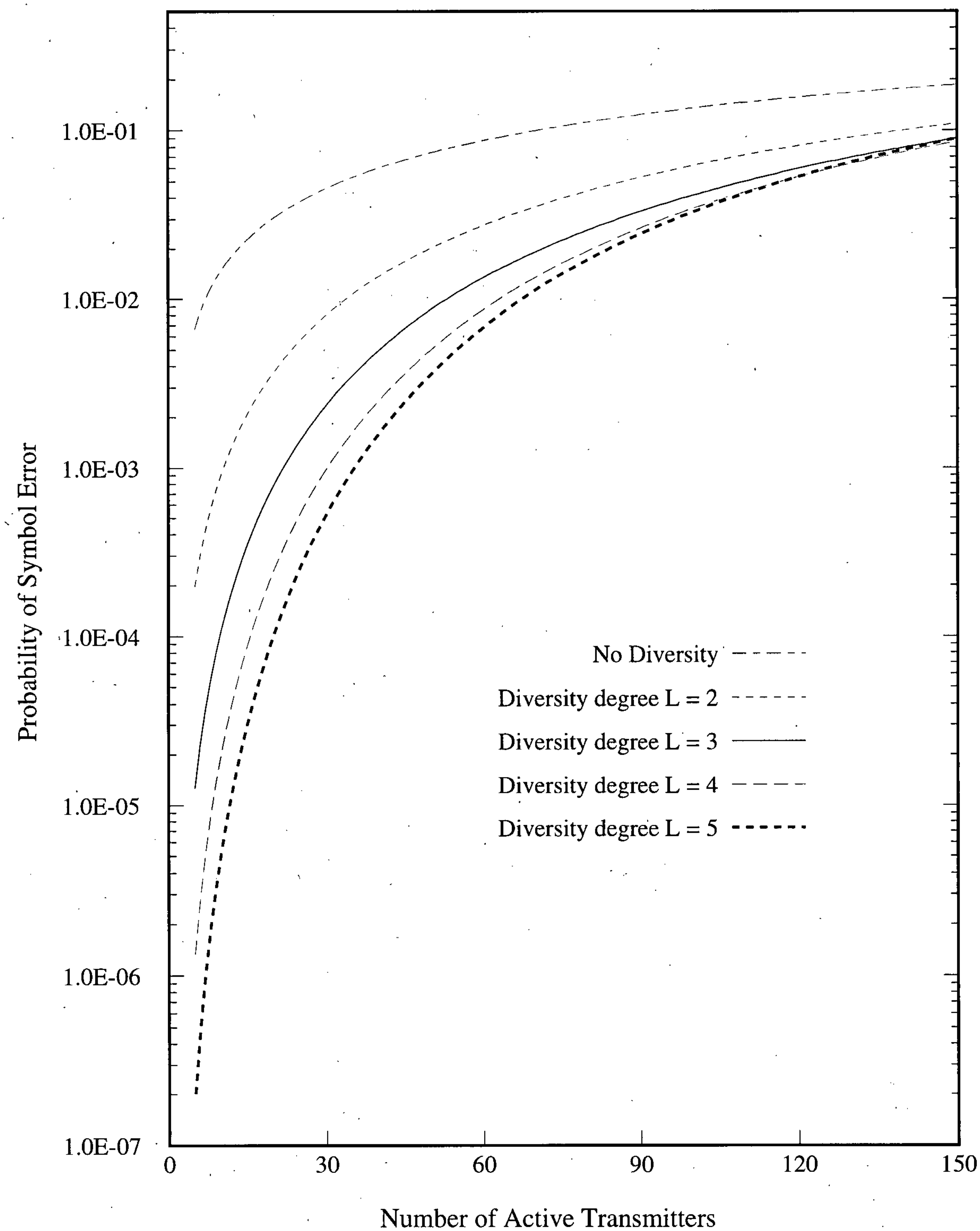

Figure 4.7 Probability of Symbol Error Vs Number of Active Users in System, Scheme 2, $q = 200$, $M = 3$.

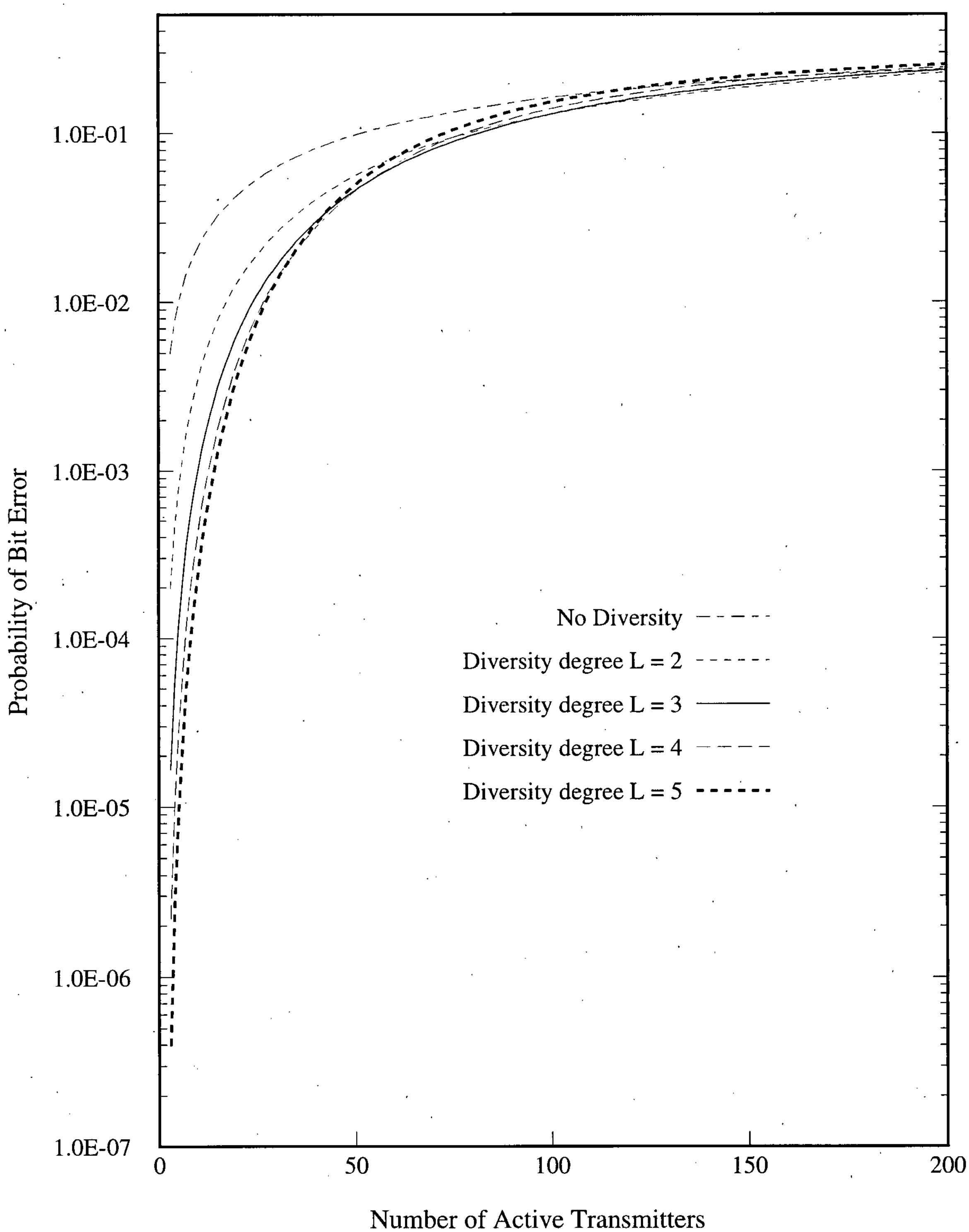

Figure 4.8 Probability of Bit Error Vs Number of Active Users, Scheme 2, $q = 100$.

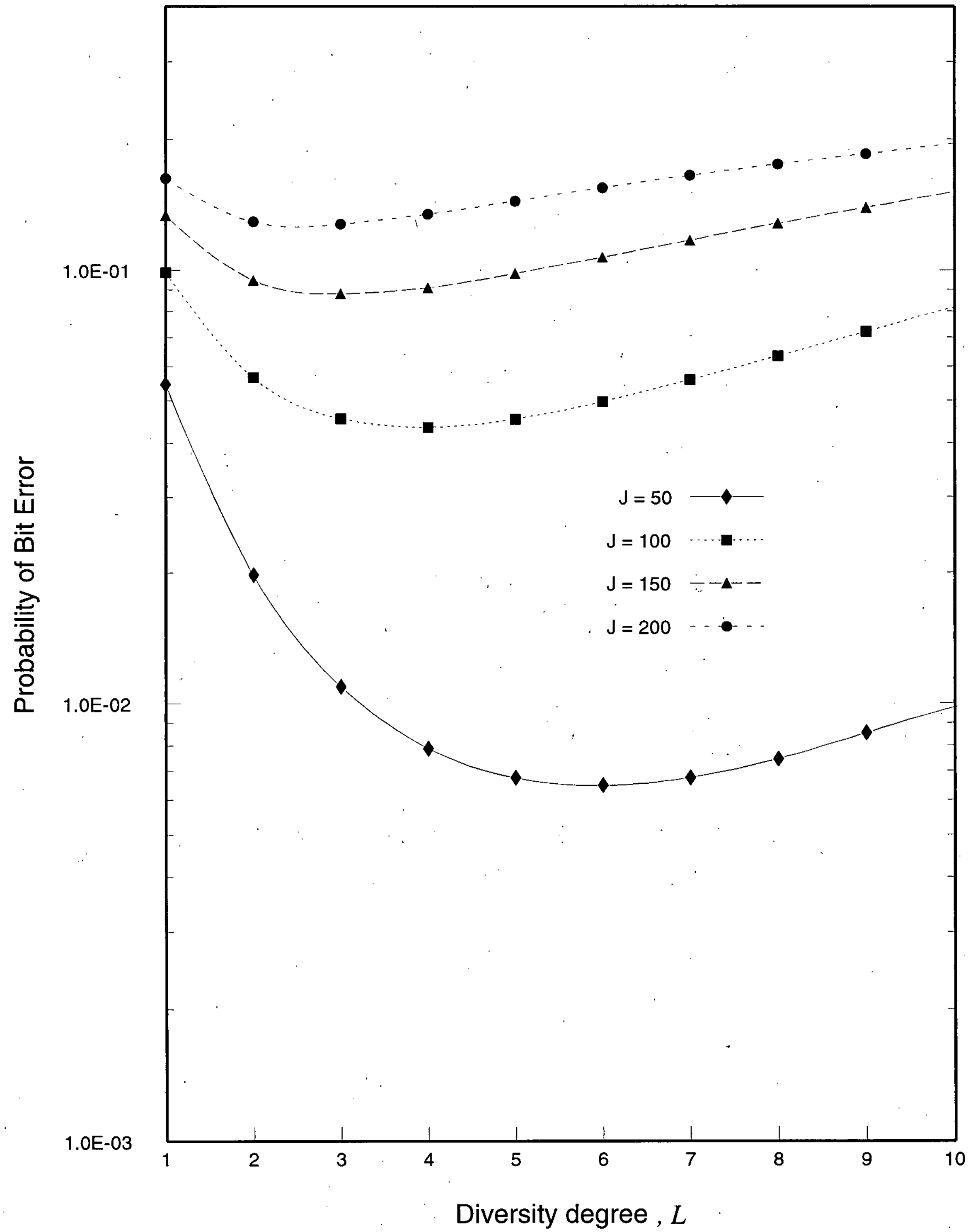

Figure 4.9 Probability of Bit Error Vs Diversity Degree, Scheme 2, $q = 200$.

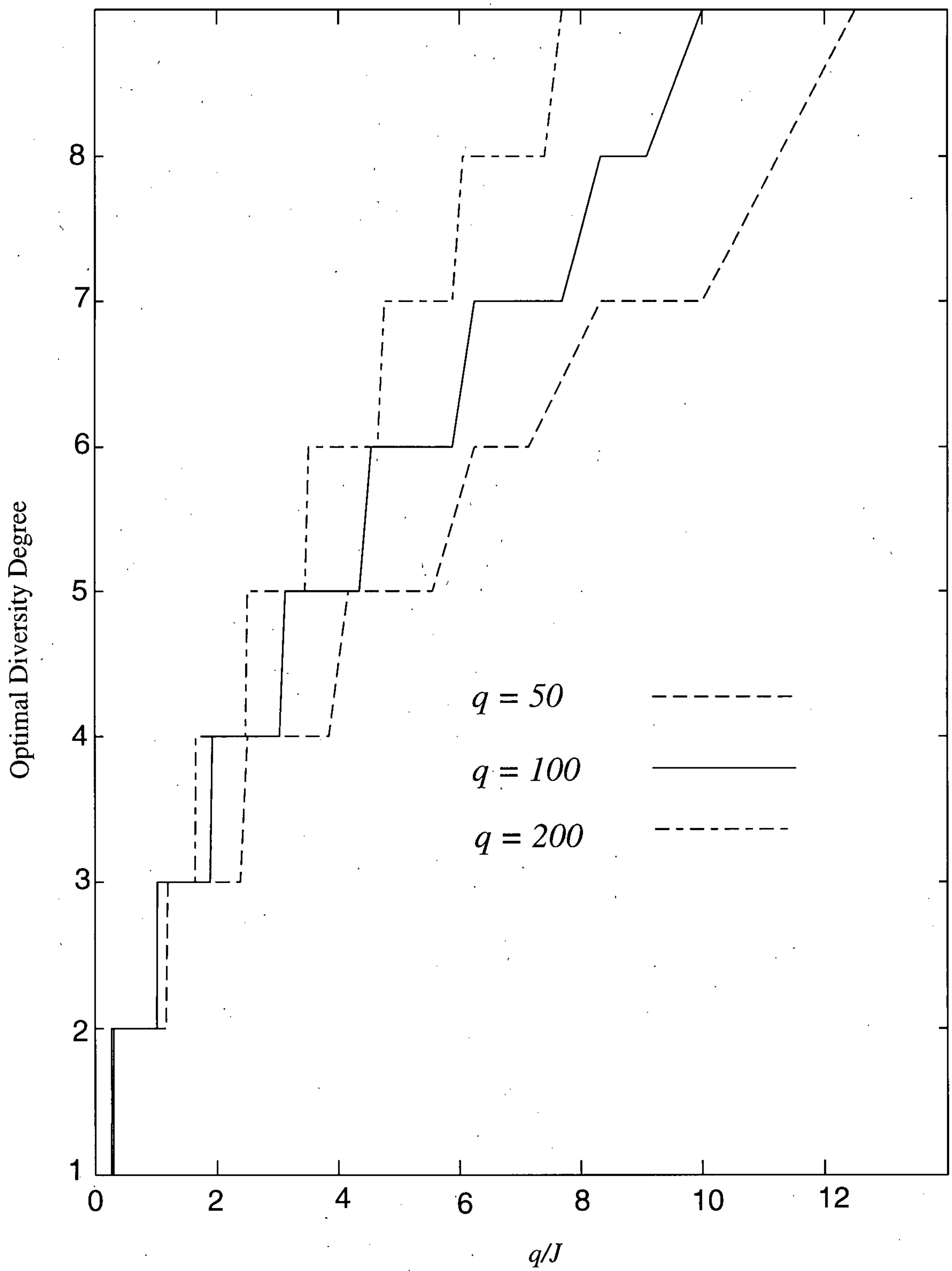

Figure 4.10 Optimal Diversity Level Vs q/J, Scheme 2.

The probability of symbol error, $P_e(K+1,q,L)$, can be obtained using the same method as used for Scheme 2, i.e. (4.17) with A_i replaced by D_i and (4.18).

For the binary case, i.e. $M = 2$, the BER is given by

$$\begin{aligned} P_e(K+1,q,L) &= \sum_{k_0=0}^{K} \binom{K}{k_0} \left(\frac{1}{2}\right)^K \sum_{z_1=L+1}^{k_1 L} \left(\Pr\{D_1\} + \frac{1}{2}\Pr\{E_1\}\right) \\ &= \sum_{k_0=0}^{K} \binom{K}{k_0} \left(\frac{1}{2}\right)^K \sum_{z_0=0}^{k_0 L} P(z_0|k_0) \Big\{ P(Z_1 > z_0 + L | K_1 = K - k_0) \\ &\quad + \frac{1}{2} P(Z_1 = z_0 + L | K_1 = K - k_0) \Big\}. \end{aligned} \tag{4.23}$$

For $L = 1$, Schemes 2 and 2A have the same symbol error probabilities. Figure 4.11 shows the BER's for Scheme 1, Scheme 2 and Scheme 2A for $L = 3$. It can be seen that a significant degradation in the BER may be incurred if step 1 of Scheme 2 is not performed. For $J = 20$, the BER's are 1.09×10^{-3} for Scheme 2 and 3.71×10^{-3} for Scheme 2A. The BER for Scheme 1 is lower than that for Scheme 2A for small values of J. However, for higher values of J, Scheme 2A has a lower BER than Scheme 1.

4.1.3 Scheme 3

In this section, we determine $P_e(J,q,L)$ for Scheme 3. The decoding process in Scheme 3 is identical to that in Scheme 2 except that when there is a complete hit on more than one symbol, the receiver uses the MAP decision rule based on $\{h_0, h_1, ..., h_{M-1}\}$ instead of randomly choosing one of the symbols whose h_m is largest as the symbol sent by **T** [60]. We consider the binary case here; similar techniques can be used to extend the results to the M-ary case. Note that $h_m, m \in \{0,1,...,M-1\}$ is the sum of the number of hits (including those caused by **T**) on the m^{th} symbol of **T**'s L frequency bins. This is different

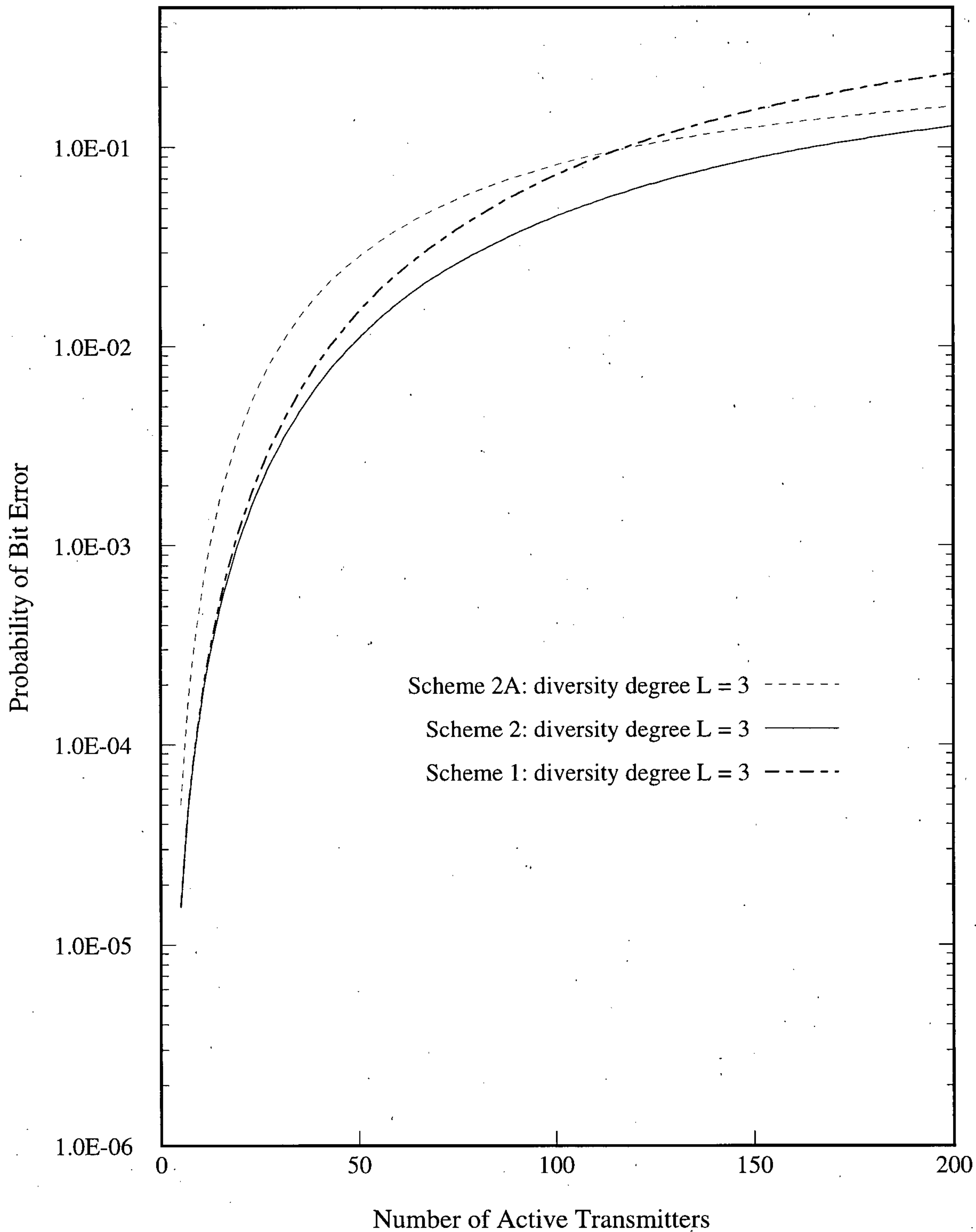

Figure 4.11 Probability of Bit Error Vs Number of Active Users, Scheme 1, Scheme 2 and Scheme 2A for $q = 200, L = 3$.

from $z_i, i \in \{0, 1, ..., M-1\}$ which is the sum of the number of hits in the i^{th} symbol of **T**'s L frequency bin caused by the interferers. Because the MAP decision rule is used in Scheme 3, using the sum of the hits caused by all the transmitters (including **T**) is seen as a better approach in the derivation.

Let F be the event that a complete hit on more than one symbol has occurred. Furthermore, let $\Pr\{\mathbf{T} \text{ sent } 0|h_0, h_1, J, F\}$ be the probability that **T** transmitted symbol "0" given that event F occurred and there is a total of h_0 and h_1 hits by the J transmitters in **T**'s L frequency bins. Based on the given information, the receiver in Scheme 3 chooses the most probable symbol that was sent by **T**. It might be noted from symmetry that,

$$\Pr\{\mathbf{T} \text{ sent } 0|H_0 = h_0, H_1 = h_1, J, F\} = \Pr\{\mathbf{T} \text{ sent } 1|H_0 = h_1, H_1 = h_0, J, F\}, \quad (4.24)$$

$$\Pr\{H_0 = h_0, H_1 = h_1|\mathbf{T} \text{ sent } 0, J\} = \Pr\{H_0 = h_1, H_1 = h_0|\mathbf{T} \text{ sent } 1, J\}, \quad (4.25)$$

and

$$\Pr\{F|\mathbf{T} \text{ sent } 0, J\} = \Pr\{F|\mathbf{T} \text{ sent } 1, J\}. \quad (4.26)$$

Furthermore

$$\begin{aligned}\Pr\{F|J\} &= \Pr\{\mathbf{T} \text{ sent } 0|J\}\Pr\{F|\mathbf{T} \text{ sent } 0, J\} + \Pr\{\mathbf{T} \text{ sent } 1|J\}\Pr\{F|\mathbf{T} \text{ sent } 1, J\} \\ &= \Pr\{F|\mathbf{T} \text{ sent } 0, J\}. \end{aligned} \quad (4.27)$$

Using Bayes' rule, we can write

$$\Pr\{\mathbf{T} \text{ sent } 0|h_0, h_1, J, F\} = \frac{\Pr\{h_0, h_1|\mathbf{T} \text{ sent } 0, J, F\}\Pr\{\mathbf{T} \text{ sent } 0|J, F\}}{\Pr\{h_0, h_1|J, F\}} \quad (4.28)$$

where

$$\Pr\{h_0, h_1|\mathbf{T} \text{ sent } 0, J, F\} = \frac{\Pr\{h_0, h_1, F|\mathbf{T} \text{ sent } 0, J\}}{\Pr\{F|\mathbf{T} \text{ sent } 0, J\}} \quad (4.29)$$

and

$$\Pr\{\mathbf{T}\text{ sent }0|J,F\} = \frac{1}{2}. \tag{4.30}$$

Using (4.28), (4.29) and (4.30), we can write the probability of symbol error for Scheme 3 given $H_0 = h_0$ and $H_1 = h_1$ as

$$\begin{aligned}
\Pr\{\text{error}|h_0,h_1,J\} &= \Pr\{F|h_0,h_1,J\} \times \min(\Pr\{\mathbf{T}\text{ sent }0|h_0,h_1,J,F\}, \\
&\qquad \Pr\{\mathbf{T}\text{ sent }1|h_0,h_1,J,F\}) \\
&= \frac{\Pr\{F|h_0,h_1,J\}}{2\Pr\{h_0,h_1|J,F\}} \\
&\quad \times \min\left(\frac{\Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }0,J\}}{\Pr\{F|\mathbf{T}\text{ sent }0,J\}}, \frac{\Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }1,J\}}{\Pr\{F|\mathbf{T}\text{ sent }1,J\}}\right) \\
&= \frac{\Pr\{F|J\}}{2\Pr\{h_0,h_1|J\}} \\
&\quad \times \min\left(\frac{\Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }0,J\}}{\Pr\{F|\mathbf{T}\text{ sent }0,J\}}, \frac{\Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }1,J\}}{\Pr\{F|\mathbf{T}\text{ sent }1,J\}}\right).
\end{aligned} \tag{4.31}$$

Using (4.26), (4.27) and (4.31),

$$\begin{aligned}
\Pr\{\text{error}|h_0,h_1,J\} &= \frac{1}{2\Pr\{h_0,h_1|J\}} \\
&\quad \times \min\left(\Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }0,J\}, \Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }1,J\}\right).
\end{aligned} \tag{4.32}$$

Hence

$$\begin{aligned}
P_e(J,q,L) &= \sum_{h_0=L}^{(J-1)L} \sum_{h_1=L}^{(JL-h_0)} \Pr\{h_0,h_1|J\}\Pr\{\text{error}|h_0,h_1,J\} \\
&= \frac{1}{2} \sum_{h_0=L}^{(J-1)L} \sum_{h_1=L}^{(JL-h_0)} \min(\Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }0,J\}, \\
&\qquad \Pr\{h_0,h_1,F|\mathbf{T}\text{ sent }1,J\}).
\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}\Pr\{h_0, h_1, F|\mathbf{T}\text{ sent } m, J\} &= \sum_{k_1=1}^{K} \Pr\{k_1 \text{ interferers sent } \overline{m}|\mathbf{T}\text{ sent } m, J\} \\ &\qquad \times \Pr\{h_{\overline{m}}, F|\mathbf{T}\text{ sent } m, J, k_{\overline{m}}\} \\ &\qquad \times \Pr\{h_m|\mathbf{T}\text{ sent } m, J, k_{\overline{m}}, h_{\overline{m}}, F\} \\ &= \sum_{k_{\overline{m}}=0}^{K} \binom{K}{k_{\overline{m}}} \left(\frac{1}{2}\right)^K Q_2(h_{\overline{m}}, L|k_{\overline{m}}) \\ &\qquad \times \sum_{i=0}^{L} Q_2(h_m - L, i|K - k_{\overline{m}}). \end{aligned} \tag{4.34}$$

4.1.3.1 Numerical Results

Numerical results obtained using (4.33) and (4.34) indicate that for $q = 200$ and $J > 2(L-1)$, there is almost no difference between the BER's for Scheme 2 and Scheme 3. For $1 < J \leq 2(L-1)$, the maximum percentage difference for $L = 2, 3, 4$ and 5 is about 10%. The difference between the BER's of Schemes 2 and 3 indicates that there are instances in which the receiver should choose symbol "m" even though $h_m < h_{m^*}$. To explain this somewhat counter-intuitive statement, consider the case where $q = 8,\ L = 3,\ J = 3,\ h_0 = 3$ and $h_1 = 4$. A sufficient condition to show that $\mathbf{T}$ is more likely to have transmitted symbol "0" given $h_0 = 3$ and $h_1 = 4$, is

$$\Pr\{h_0 = 3, h_1 = 4, F|\mathbf{T}\text{ sent } 0\} > \Pr\{h_0 = 3, h_1 = 4, F|\mathbf{T}\text{ sent } 1\}. \tag{4.35}$$

It can be verified that $\Pr\{h_0 = 3, h_1 = 4, F|\mathbf{T}\text{ sent } 0\} = 0.01674$ whereas $\Pr\{h_0 = 3, h_1 = 4, F|\mathbf{T}\text{ sent } 1\} = 0.004783$.

The results obtained for $L = 1, 2, 3, 4$ and $q = 200$ are plotted in Figure 4.12. Due to the computational complexity involved, the results shown are limited to 50 active transmitters.

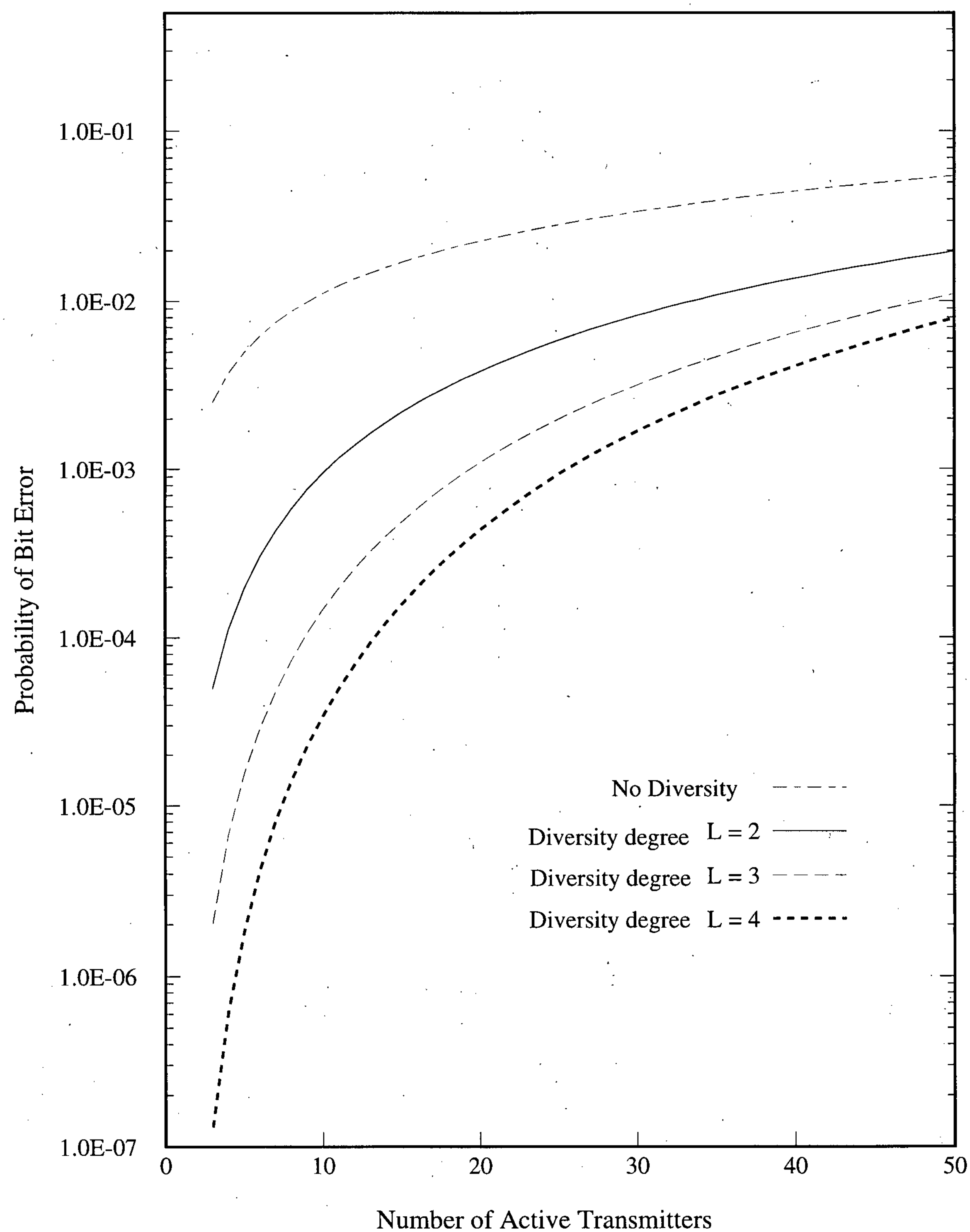

Figure 4.12 Probability of Bit Error Vs Number of Active Users, Scheme 3, $q = 200$.

4.1.4 Scheme 4

The assumptions made in the derivation of $P_e(J,q,L)$ for Scheme 4 are identical to those for Scheme 3 except that Scheme 4 uses the MAP decision rule based on $\{n_{0,1}, n_{1,1}, n_{0,2}, n_{1,2}, ...n_{0,L}, n_{1,L}\}$ where $n_{m,l}$ is the total number of hits on the m^{th} symbol of **T**'s l^{th} frequency bin [60].

Using Bayes' rule, we can write

$$\begin{aligned}
&\Pr\{\mathbf{T}\text{ sent }0|n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}, J, F\} \\
&= \frac{\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}|\mathbf{T}\text{ sent }0, J, F\}}{\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}|J, F\}} \\
&\quad \times \Pr\{\mathbf{T}\text{ sent }0|J, F\}
\end{aligned} \tag{4.36}$$

where

$$\begin{aligned}
&\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}|\mathbf{T}\text{ sent }0, J, F\}. \\
&= \frac{\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }0, J\}}{\Pr\{F|\mathbf{T}\text{ sent }0, J\}}
\end{aligned} \tag{4.37}$$

Using (4.30), (4.36), and (4.37) we can write the probability of symbol error given $\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}\}$ as

$$\begin{aligned}
&\Pr\{\text{error}|n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, J\} \\
&= \Pr\{F|n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, J\} \\
&\quad \times \min\left(\Pr\{\mathbf{T}\text{ sent }0|n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, J, F\},\right. \\
&\qquad \left.\Pr\{\mathbf{T}\text{ sent }1|n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, J, F\}\right) \\
&= \frac{\Pr\{F|n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, J\}}{2\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}|J, F\}}
\end{aligned}$$

$$\times \min\left(\frac{\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }0, J\}}{\Pr\{F|\mathbf{T}\text{ sent }0, J\}},\right.$$
$$\left.\frac{\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }1, J\}}{\Pr\{F|\mathbf{T}\text{ sent }1, J\}}\right)$$
$$= \frac{\Pr\{F|J\}}{2\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}|J\}}$$
$$\times \min\left(\frac{\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }0, J\}}{\Pr\{F|\mathbf{T}\text{ sent }0, J\}},\right.$$
$$\left.\frac{\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }1, J\}}{\Pr\{F|\mathbf{T}\text{ sent }1, J\}}\right). \tag{4.38}$$

Using (4.26), (4.27) and (4.38),

$$\Pr\{\text{error}|n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, J\}$$
$$= \frac{1}{2\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}|J\}}$$
$$\times \min(\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }0, J\},$$
$$\Pr\{n_{0,1}, n_{1,1}, ...n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }1, J\}). \tag{4.39}$$

Hence

$$P_e(J, q, L)$$
$$= \sum_{n_{0,1}=0}^{J-1} \sum_{n_{1,1}=0}^{J-1} \cdots \sum_{n_{0,L}=0}^{J-1} \sum_{n_{1,L}=0}^{J-1} \Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}|J\}$$
$$\times \Pr\{\text{error}|n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}, J\}$$
$$= \frac{1}{2} \sum_{n_{0,1}=0}^{J-1} \sum_{n_{1,1}=0}^{J-1} \cdots \sum_{n_{0,L}=0}^{J-1} \sum_{n_{1,L}=0}^{J-1}$$
$$\min(\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }0, J\},$$
$$\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}, F|\mathbf{T}\text{ sent }1, J\})$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{n_{0,1}=1}^{J-1} \sum_{n_{1,1}=1}^{J-1} \cdots \sum_{n_{0,L}=1}^{J-1} \sum_{n_{1,L}=1}^{J-1} \\
&\min(\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L} | \mathbf{T} \text{ sent } 0, J\}, \\
&\quad \Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L} | \mathbf{T} \text{ sent } 1, J\}).
\end{aligned} \tag{4.40}
$$

where

$$
\begin{aligned}
&\Pr\{n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L} | \mathbf{T} \text{ sent } m, J\} \\
&= \sum_{k_{\overline{m}}=1}^{K} \Pr\{k_{\overline{m}} \text{ interferers sent } \overline{m} | \mathbf{T} \text{ sent } m, J\} \\
&\times \Pr\{n_{\overline{m},1}, ..., n_{\overline{m},L} | \mathbf{T} \text{ sent } m, J, k_{\overline{m}}\} \\
&\times \Pr\{n_{m,1}, ..., n_{m,L} | \mathbf{T} \text{ sent } m, J, k_{\overline{m}}, n_{1,1}, ..., n_{1,L}\} \\
&= \sum_{k_{\overline{m}}=1}^{K} \binom{K}{k_{\overline{m}}} \left(\frac{1}{2}\right)^{K} Q_4(n_{\overline{m},1}, ..., n_{\overline{m},L} | k_{\overline{m}}) \\
&\quad \times Q_4(n_{m,1}-1, ..., n_{m,L}-1 | K - k_{\overline{m}})
\end{aligned} \tag{4.41}
$$

and $Q_4(n_{s,1}, n_{s,2}, ..., n_{s,L} | k_s)$ is the probability of having exactly $n_{s,1}, n_{s,2}, ..., n_{s,L}$ hits on $\mathbf{T}$'s $1^{\text{st}}, 2^{\text{nd}}, ..., L^{\text{th}}$ frequency bins respectively by the k_s interferers transmitting symbol "s". Denoting

$$
P_l = \frac{\binom{L}{l}\binom{q-L}{L-l}}{\binom{q}{L}}, \tag{4.42}
$$

$Q_4(., ., ..., ., .|.)$ can be calculated recursively as

$$
\begin{aligned}
&Q_4(n_{m,1}, n_{m,2}, ..., n_{m,L} | k_m) \\
&= P_0 Q_4(n_{m,1}, n_{m,2}, ..., n_{m,L} | k_m - 1) \\
&+ P_1 \frac{1}{\binom{L}{1}} \Big[Q_4(n_{m,1}-1, n_{m,2}, ..., n_{m,L} | k_m - 1) \\
&\quad + Q_4(n_{m,1}, n_{m,2}-1, ..., n_{m,L} | k_m - 1) + ...
\end{aligned}
$$

$$
\begin{aligned}
&\quad + Q_4\big(n_{m,1}, n_{m,2}, ..., n_{m,L} - 1 | k_m - 1\big)\Big] \\
&+ P_2 \frac{1}{\binom{L}{2}} \Big[Q_4\big(n_{m,1} - 1, n_{m,2} - 1, ..., n_{m,L} | k_m - 1\big) \\
&\quad + Q_4\big(n_{m,1} - 1, n_{m,2}, n_{m,3} - 1, ..., n_{m,L} | k_m - 1\big) \\
&\quad + ... \\
&\quad + Q_4\big(n_{m,1} - 1, n_{m,2}, ..., n_{m,L} - 1 | k_m - 1\big) + \\
&\quad + Q_4\big(n_{m,1}, n_{m,2} - 1, n_{m,3} - 1, ..., n_{m,L} | k_m - 1\big) \\
&\quad + ... \\
&\quad + Q_4\big(n_{m,1}, ..., n_{m,L-1} - 1, n_{m,L} - 1 | k_m - 1\big)\Big] \\
&+ ... \\
&+ P_L Q_4\big(n_{m,1} - 1, n_{m,2} - 1, ..., n_{m,L} - 1 | k_m - 1\big) \qquad (4.43)
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
&Q_4\big(n_{m,1}, n_{m,2}, ..., n_{m,L} | 0\big) \\
&\quad = \begin{cases} 1 & n_{m,1} = n_{m,2} = ... = n_{m,L} = 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (4.44)
\end{aligned}
$$

The BER for Scheme 4 was evaluated using (4.40) for $q = 200$, $L = 1, 2, 3$ and 4. A comparison with the BER for Scheme 3 indicated that there is very little difference between the performances of the two schemes. This difference is approximately 0.01%. However, the computational task for evaluating the performance of Scheme 4 is L fold more than Scheme 3. Hence the technique developed for Scheme 3 may be used to obtain a tight upperbound on the BER of Scheme 4.

4.2 Error Control Coding

All the code diversity schemes considered in previous sections used a repetition code as a means of error control coding. The same symbol is transmitted by a particular by a particular transmitter in all its diversity branches. In this section, an error control scheme using the random code approach is studied. The only source of interference is multiple access interference caused by other transmitters and except for the error control coding scheme, the system model used is the same as the one in the previous section. It will be shown that the BER with random coding is lower than the repetition code BER.

4.2.1 Random Coding Scheme

In this scheme, each transmitter/receiver pair uses a strategy where L distinct frequency bins are selected for transmission for each symbol. However, unlike the repetitive coding scheme, the tone transmitted in each of the L frequency bins is randomly selected. For example, it might be decided a priori that when a particular transmitter wants to transmit symbol "0", the actual transmission will involve transmitting symbols "0" and "1" in the first and second frequency bin respectively. Any receiver wishing to decode transmission from this particular transmitter must know a priori the encoding scheme to decode the actual symbol.

For simplicity we shall assume, without loss of generality, that the marked transmitter **T** transmits the same symbol in all its L frequency bins. A receiver which decodes the transmission from **T** will not be able to decode **T**'s transmitted symbol without ambiguity only when the L frequency bins contains more than 1 set of identical symbols which have been hit. In such cases, the receiver randomly select any one of these symbols which have been detected in all L frequency bins as the decoded symbol.

To evaluate $P_e(J,q,L)$, we need to evaluate the joint probability of the number of interferers transmitting in **T**'s L frequency bins, $\Pr(b_1,b_2,...,b_L|K)$, where b_l is the total number of interferers transmitting in the l^{th} frequency bin. This probability can be easily evaluated using the following equation

$$\begin{aligned}\Pr(b_1,...,b_L|K) &= \Pr(b_1,...,b_L|K-1)\frac{\binom{q-L}{L}}{\binom{q}{L}} \\ &+ \frac{\binom{q-L}{L-1}}{\binom{q}{L}}\{\Pr(b_1-1,b_2,...,b_L|K-1)+\ldots \\ &\qquad + \Pr(b_1,b_2,...,b_L-1|K-1)\}+... \\ &+ \frac{\binom{q-L}{L-2}}{\binom{q}{L}}\{\Pr(b_1-1,b_2-1,...,b_L|K-1)+... \\ &\qquad + \Pr(b_1,...,b_{L-1}-1,b_L-1|K-1)\}+... \\ &+ \frac{1}{\binom{q}{L}}\{\Pr(b_1-1,b_2-1,...,b_L-1|K-1)\}\end{aligned} \tag{4.45}$$

with initial conditions

$$\Pr(b_1,b_2,...,b_L|0) = \begin{cases} 1 & b_1=b_2=...=b_L=0 \\ 0 & \text{otherwise}\end{cases}. \tag{4.46}$$

As in the previous cases, we shall assume that **T** transmitted symbol "0". Given the number of interferers transmitting in a particular frequency bin, the distribution of hits on the symbols of that particular frequency bin is a multinomial distribution which can be written as

$$\begin{aligned}\Pr\left(n_{0,l},n_{1,l},...,n_{M-1,l}|b_l\right) &= \frac{b_l!}{\left(n_{0,l}-1\right)!\,n_{1,l}!\ldots n_{M-1,l}!}\left(\frac{1}{M}\right)^{b_l}, \\ & n_{0,l}-1+\sum_{m=1}^{M-1}n_{m,l}=b_l\end{aligned} \tag{4.47}$$

where $n_{m,l}$ is as defined in Section 4.1.4. Furthermore conditioned on the number of interferers transmitting in each frequency bin, the distribution of hits on the symbols for each bin are independent. Hence we can write the conditional joint distribution of hits on the symbols for the L frequency bins as

$$\Pr\left(n_{0,1}, n_{1,1}, ..., n_{M-1,1}, n_{1,1}, ..., n_{M-1,1}, ..., n_{M-1,L} | b_1, b_2, ..., b_L\right) = \prod_{l=1}^{L} \Pr\left(n_{0,l}, n_{1,l}, ..., n_{M-1,l} | b_l\right). \tag{4.48}$$

Using equations (4.47) and (4.48), we can determine the number of symbols besides "0" that is detected in all the L frequency bins. Therefore using equations (4.45), (4.47) and (4.48), we can write

$$\begin{aligned} P(K+1, q, L) = \sum_{b_1=1}^{K} \sum_{b_2=1}^{K} ... \sum_{b_L=1}^{K} \Pr\left(b_1, b_2, ..., b_L | K\right) \times \\ \Bigg[\frac{1}{2}\Big\{\Pr(\text{only symbol 1 detected}|b_1, b_2, ..., b_L) \\ + \Pr(\text{only symbol 2 detected}|b_1, b_2, ..., b_L) + \; ... \\ + \Pr(\text{only symbol } M-1 \text{ detected}|b_1, b_2, ..., b_L)\Big\} + \\ \frac{2}{3}\Big\{\Pr(\text{only 2 other symbols detected}|b_1, b_2, ..., b_L)\Big\} + \; ... \\ \frac{M-1}{M}\Big\{\Pr(\text{all other symbols detected}|b_1, b_2, ..., b_L)\Big\}\Bigg]. \end{aligned} \tag{4.49}$$

For the binary case, a much simpler form of the BER expression can be derived. Conditioned on the number of interferers transmitting in each of **T**'s L frequency bins, the probability that symbol "1" is detected in all L frequency bins is given by

$$\begin{aligned} \Pr\left(n_{1,1} > 0, n_{1,2} > 0, ..., n_{1,L} > 0 | b_1, b_2, ..., b_L\right) &= \prod_{l=1}^{L}\Big(1 - \Pr\left(n_{0,l} = b_l | b_l\right)\Big) \\ &= \prod_{l=1}^{L}\left(1 - \left(\frac{1}{2}\right)^{b_l}\right). \end{aligned} \tag{4.50}$$

Therefore

$$P_e(K+1,q,L) = \frac{1}{2} \sum_{b_1=1}^{K} \sum_{b_2=1}^{K} \cdots \sum_{b_L=1}^{K} \Pr(b_1, b_2, \ldots b_L | K) \prod_{l=1}^{L} \left(1 - \left(\frac{1}{2}\right)^{b_l}\right). \tag{4.51}$$

4.2.2 Numerical Results

Equations (4.45) and (4.51) were used to calculate the BER for $q = 200,\ 100$ and $L = 1,\ 2$ and 3. The results showing the BER versus J are plotted in Figures 4.13 and 4.14. Unlike the other schemes, the interferers for this scheme cannot be partitioned into symbol groups. This results in fairly extensive computational memory usage and the BER's for values of J beyond 50 transmitters were not computed.

It can be seen that the general behavior of the BER is quite similar to those shown for repetition code schemes. However, from the numerical results, the BER's using the random coding scheme are lower than those obtained using scheme 1. For example, when $J = 20$ and $L = 3$, $P_e(J,q,L)$ is 1.130×10^{-3} and 1.236×10^{-3} for the random coding scheme and scheme 1 respectively. The percentage difference is larger for smaller values of J. This is because for large values of J in scheme 1, the number of possible ways of hitting a particular symbol in one of $\mathbf{T}$'s L frequency bins increases. This tends to make the interference random-like in nature. Hence the difference between the two schemes diminishes for a fixed value of L as J increases. From this reasoning and the numerical results obtained, we conjecture that the BER of the random code scheme is a lower bound to that of scheme 1.

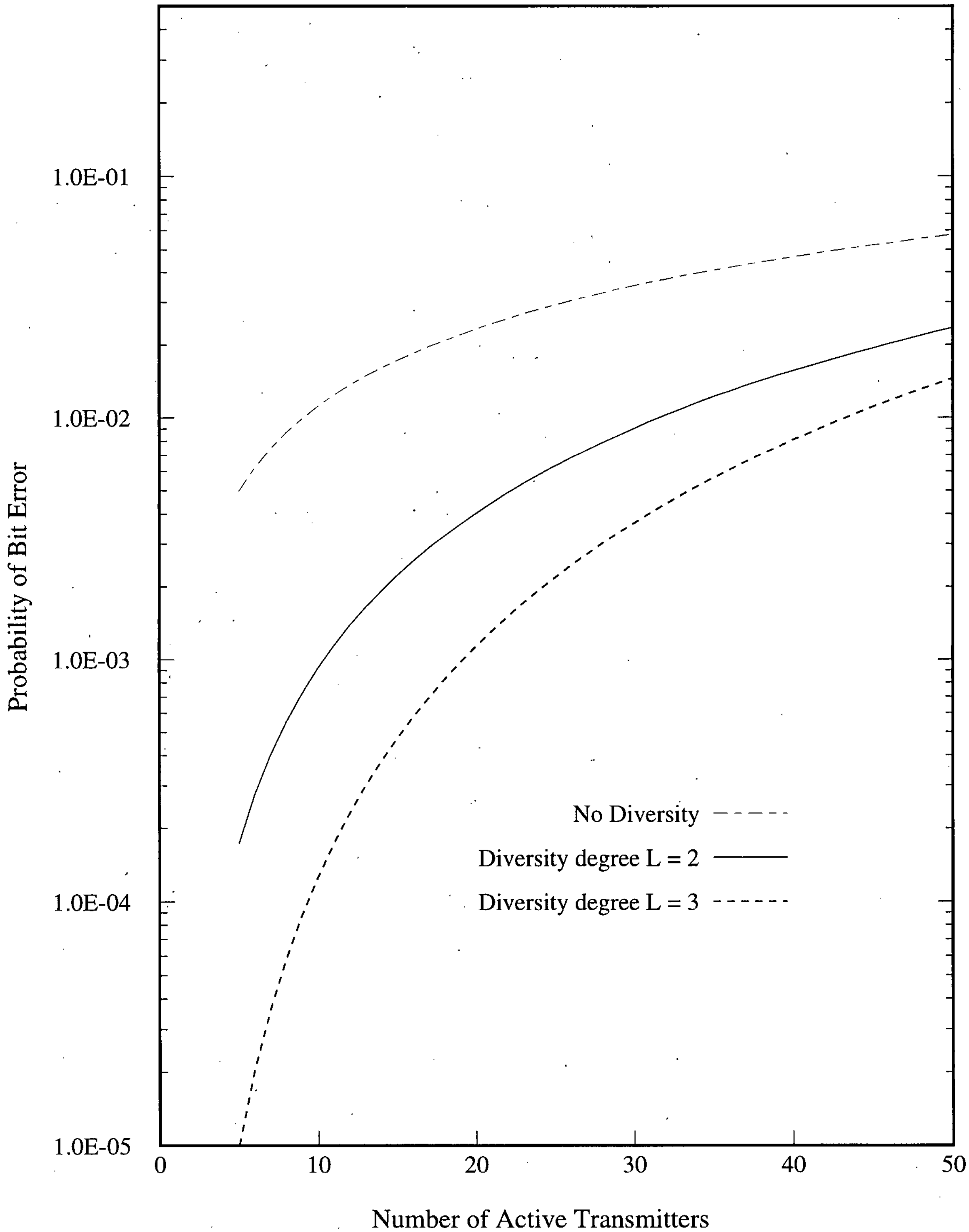

Figure 4.13 Probability of Bit Error Versus J, Random Code Scheme with $q = 200$.

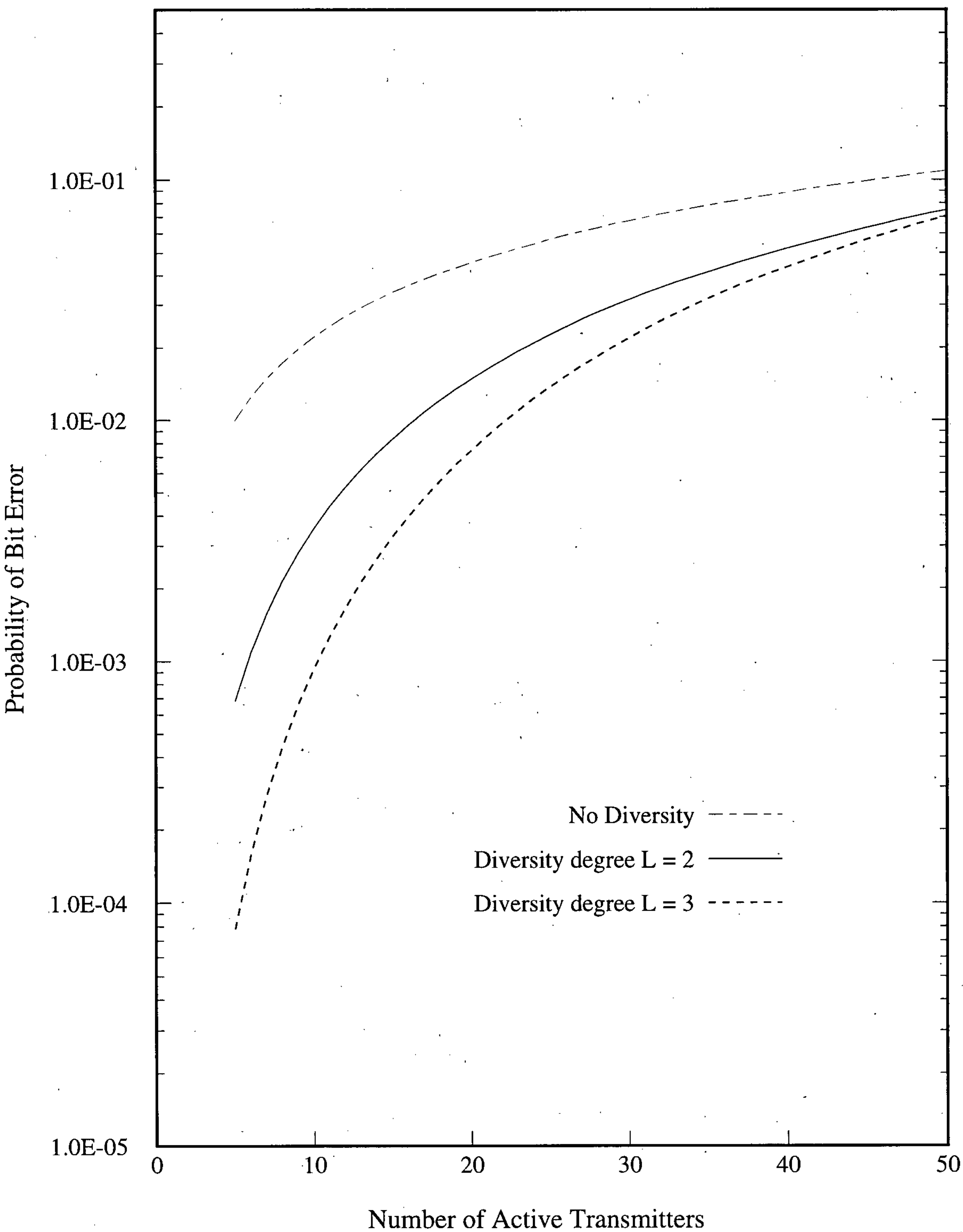

Figure 4.14 Probability of Bit Error Versus J, Random Code Scheme with $q = 100$.

Chapter 5
Code Diversity Schemes in the Presence of Noise and Rayleigh Fading

In this chapter, we consider the BER performance of code diversity schemes in the presence of noise and fading. The channel considered is a Rayleigh fading channel with AWGN. Two different models are considered here.

In the first model, the signal strength detected at the receiver for a given tone is the same, whether one or more transmitters send that tone. This simplifying assumption is valid in cases where there is usually only a small number of transmitters transmitting a particular tone simultaneously and hence the probability of a hit is low. A detection threshold is used by the receiver to determine if a tone is detected and a majority voting rule is used to determine the symbol transmitted. In the second model, the signal strength of a given tone is dependent on the number of transmitters transmitting that tone. The signals from the transmitters sending a given tone are assumed to be independently faded and are vector added to determine the resultant signal for that tone. The simplifying assumption made in the first model is thus removed. Detection of the tone is based on the resultant signal strength.

5.1 Majority Vote Decoding Scheme with Threshold Detection

In this section, we derive an expression for the symbol error probability $P_e(J, q, L)$ for a system in Rayleigh fading and AWGN [61]. Due to the fading, the receiver may not detect a symbol tone which was transmitted; this event is referred to as a deletion. The probability of such a deletion is given by [62]

$$p_D = 1 - \exp\left(-\frac{\beta^2}{2(1+\overline{\rho})}\right). \tag{5.1}$$

where β is the detection threshold normalized with respect to the noise and $\overline{\rho}$ is the average signal-to-noise ratio.

Due to the noise, it is possible that a receiver detects a certain tone even though that tone was not sent by any of the transmitters. This is known as a false alarm and its probability can be expressed as [62]

$$p_F = \exp\left(-\frac{\beta^2}{2}\right). \tag{5.2}$$

The events, deletion and false alarm, are mutually exclusive. A deletion on a particular tone can occur only if at least one transmitter sent that particular tone and a false alarm on a particular tone can occur only if no transmitter sent that particular tone. Furthermore, deletion and false alarm events are assumed independent across all the tones in the system. For example, if there are two different tones transmitted in a particular frequency bin of a $5-ary$ system, possible deletions at the two symbol tones occur independently and possible false alarms in the other three symbol tones occur independently.

A tone is said to be detected, if the receiver detection threshold is exceeded (either because the tone was sent by one of the transmitters or a false alarm has occurred). To decode the symbol sent by a particular transmitter **T**, the receiver sums up the number of times each symbol has been detected in **T**'s L frequency bins and selects the symbol detected the largest number of times as the symbol sent by **T**. In the case of a tie, the receiver randomly selects one of these symbols as the symbol sent by **T**.

As in previous sections, it is convenient to partition the total number of interferers, $K = J - 1$, into the M symbol groups where $\sum_{s=0}^{M-1} k_s = K$ and k_s is the number of interferers transmitting symbol "s".

In addition to the assumptions made in Section 4.1.1, the following assumptions are made in the derivation:

- as long as there is one or more hits caused by the users on a particular tone, p_D remains unchanged.
- the receiver uses a majority vote decoding rule and chooses the symbol that has been hit most frequently in the L frequency bins. (Note that because p_D in general is non-zero, the receiver may not be able to find at least one set of tones corresponding to a particular symbol in all of **T**'s L frequency bins.) If there are two or more symbols with the largest number of hits, the receiver declares randomly one of these symbols as the symbol decoded.

Let $X(d, f|l, k_s)$ be the probability that there are exactly $d \in \{0, 1, ..., l\}$ deletions and $f \in \{0, 1, ..., L-l\}$ false alarms in **T**'s L frequency bins given that there are exactly l of these L bins hit by the k_s interferers. This probability can be calculated using

$$X(d, f|l, k_s) = \binom{l}{d} p_D^d (1 - p_D)^{l-d} \binom{L-l}{f} p_F^f (1 - p_F)^{L-l-f}. \tag{5.3}$$

Furthermore let $R(i|k_s)$ be the probability of having exactly i of the symbol $s \in \{1, 2, ..., M-1\}$ tones within **T**'s L frequency bins detected by the receiver. Using (4.1) and (5.3), $R(i|k_s)$ can be calculated as

$$\begin{aligned} R(i|k_s) &= Q_1(0|k_s)\binom{L}{i} p_F^i \\ &+ Q_1(1|k_s)\left[\sum_{d=\max(1-i,0)}^{1} \binom{1}{j} p_D^d (1-p_D)^{1-d} \binom{L-1}{i+d-1} p_F^{i+d-1} (1-p_F)^{L-i}\right] \\ &+ \ldots \\ &+ Q_1(t|k_s)\left[\sum_{d=\max(t-i,0)}^{t} \binom{t}{d} p_D^d (1-p_D)^{t-d} \binom{L-t}{i+d-t} p_F^{i+d-t} (1-p_F)^{L-i}\right] \end{aligned}$$

$$+ \ldots$$
$$+ Q_1(L|k_s)\Big[\sum_{d=L-i}^{L} \binom{L}{d} p_D^d (1-p_D)^{L-d}\Big]$$
$$= \sum_{t=0}^{L} Q_1(t|k_s) \sum_{d=\max(t-i,0)}^{t} X(d, i+d-t|t, k_s) \tag{5.4}$$

where $Q_1(.|.)$ is as defined in (4.1).

As in the previous sections, assuming that **T** transmitted the symbol "0", the number of frequency bin hits detected at the receiver on symbol "0" at **T**'s chosen frequency bins can be calculated as

$$S(i) = \binom{L}{L-i} p_D^{L-i}(1-p_D)^i. \tag{5.5}$$

Let F_s be the event that the number of frequency bins (of **T**'s L bins) with tones detected on the s^{th} symbol exceeds the corresponding number on the "0" symbol and E_s be the event that the number of frequency bins with tones detected on the symbol s^{th} equals the corresponding number on the "0" symbol. Using (5.4) and (5.5), the conditional probabilities associated with these event can be written as

$$\Pr\{F_s|k_s\} = S(i) \sum_{j=i+1}^{L} R(j|k_s) \tag{5.6}$$

and

$$\Pr\{E_s|k_s\} = S(i)R(i|k_s). \tag{5.7}$$

The probability of symbol error given a distribution of interferers $\{k_0, k_1, ..., k_{M-1}\}$ can be written as

$$P_e(K+1, q, L|K_0 = k_0, K_1 = k_1, ..., K_{M-1} = k_{M-1}) =$$

$$
\Pr\{F_1 \cup F_2 \cup \ldots \cup F_{M-1}\} + \frac{1}{2}\sum_{i=1}^{M-1} \Pr\{E_i\} \prod_{j \neq i} (1 - \Pr\{E_j\} - \Pr\{F_j\})
$$
$$
+ \frac{2}{3} \sum_{1 \leq i < j \leq M-1} \Pr\{E_i\} \Pr\{E_j\} \prod_{k \neq i,j} (1 - \Pr\{E_k\} - \Pr\{F_k\}) + \ldots
$$
$$
+ \frac{(M-1)}{M} \prod_{i=1}^{M-1} \Pr\{E_i\}. \tag{5.8}
$$

The first term of (5.8) can be written as [59]

$$
\Pr\{F_1 \cup F_2 \cup \ldots \cup F_{M-1}\} = \sum_{i=1}^{M-1} \Pr\{F_i\} - \sum_{1 \leq i < j \leq M-1} \Pr\{F_i\} \Pr\{F_j\} +
$$
$$
\sum_{1 \leq i < j < k \leq M-1} \Pr\{F_i\} \Pr\{F_j\} \Pr\{F_k\} + \ldots
$$
$$
+ (-1)^M \prod_{1 \leq i \leq M-1} \Pr\{F_i\}. \tag{5.9}
$$

Since

$$
P_e(K+1, q, L) = \sum_{k_0 + k_1 + \ldots + k_{M-1} = K} \Pr\{K_0 = k_0, K_1 = k_1, \ldots, K_{M-1} = k_{M-1}\}
$$
$$
P_e(K+1, q, L | K_0 = k_0, K_1 = k_1, \ldots, K_{M-1} = k_{M-1}) \tag{5.10}
$$

we can obtain $P_e(K+1, q, L)$ using (4.5), (5.4), (5.5), (5.6), (5.7), (5.8) and (5.9).

5.1.1 Numerical Results

For given values of the average signal to noise ratio, $\overline{\rho}$, and detection threshold, β, equations (5.1) and (5.2) allow us to determine the corresponding values of p_D and p_F. These values are used to obtain the symbol error rate according to the procedure described above. In the results presented below, binary signalling (i.e. $M = 2$) and $q = 200$ is assumed.

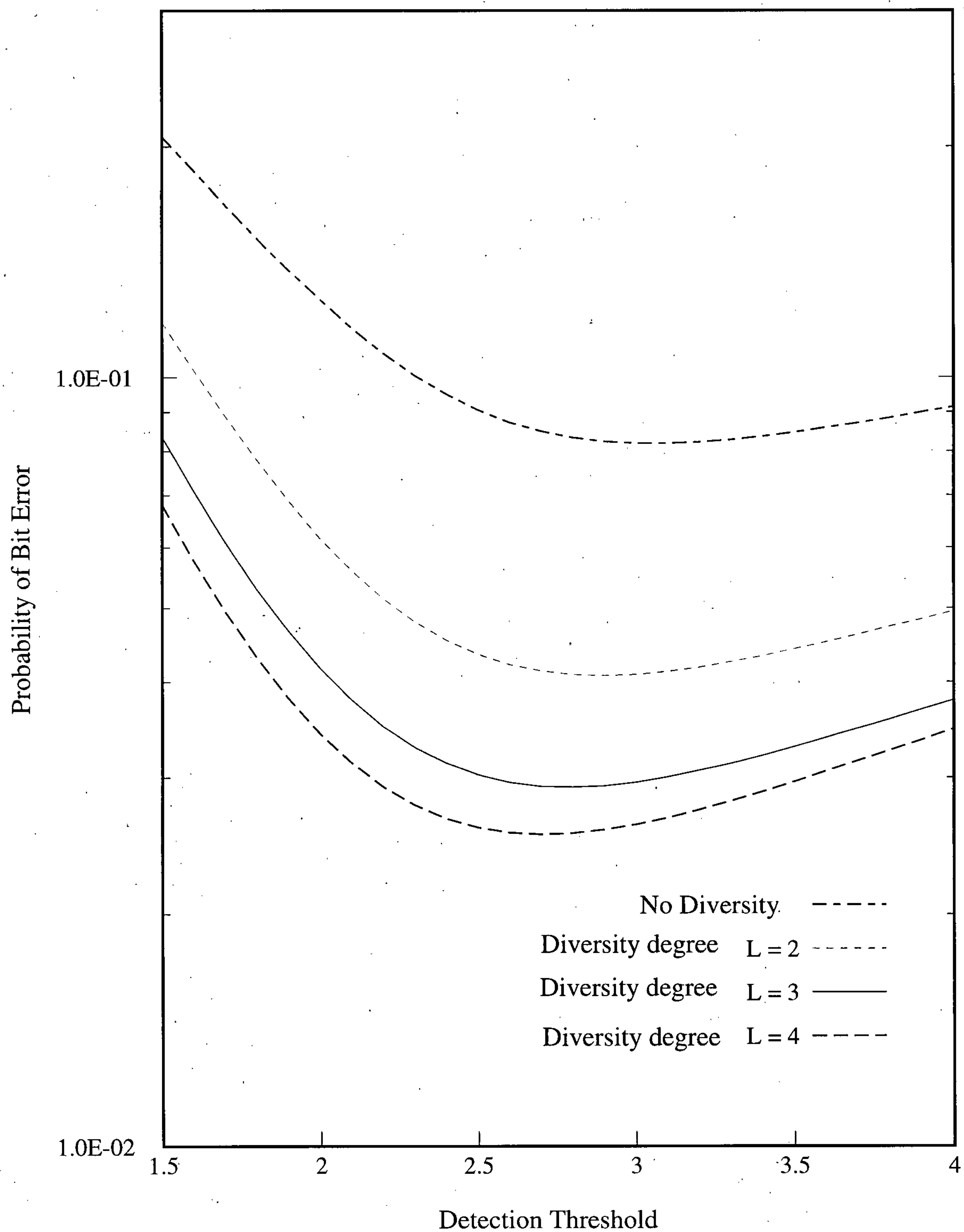

Figure 5.1 Probability of Bit Error Versus Detection Threshold, $\overline{\rho} = 20$ dB and $J = 50$.

Plots of bit error rate, p_b, as a function of the detection threshold, β for $L = 1, 2, 3$ and 4, $\overline{\rho} = 20$ dB, and $J = 50$ are shown in Figure 5.1. It can be seen that the optimum value, β_{opt}, of the detection threshold which minimizes p_b is quite insensitive to changes in L. The value of β_{opt} is about 2.6. The curves also show that p_b is not very sensitive to changes in β around β_{opt}. Corresponding plots for $J = 150$ are shown in Figure 5.2. The observations drawn from Figure 5.1 also hold for Figure 5.2.

Curves of p_b versus β for $J = 25, 50, 75$ and 100, $\overline{\rho} = 20$ dB and $L = 2$ appear in Figure 5.3. The value of β_{opt} is about 2.9 for the various values of J. In Figure 5.5 where $L = 4$, β_{opt} is again about 2.9 for the various values of J. Figure 5.4 illustrates how p_b changes with β for $\overline{\rho} = 10$, 15, 20 and 25 dB with $J = 50$ and $L = 2$. Similar plots for $L = 4$ is shown in Figure 5.6. The value of β_{opt} tends to increase with $\overline{\rho}$ in both cases.

The p_b versus J curves for $L = 1$, 2, 3 and 4, $\overline{\rho} = 20$ dB and β = 2.6 (a value which is close to the optimum) are shown in Figure 5.7. It can be seen that code diversity generally yields a lower BER. Corresponding curves for $\overline{\rho} = 15$ dB and β = 2.6 appear in Figure 5.8.

In the numerical results above, it is assumed that the transmitters for each diversity group is transmitting at the same power for each diversity branch as the transmitter with no diversity. This assumption is valid if there is no constraint on the total power transmitted and there is a limit on the power emission level at the frequency bin level. Because of the power emission limit at the frequency bin level, the transmitters with lower diversity degrees are not able to transmit at maximum power. In some situations, these assumptions may not be valid and it may be more appropriate to fix the overall transmitted power by each transmitter. In Figure 5.9, we have plotted the curves where the SNR for each diversity branch is $\overline{\rho}/L$ for $\overline{\rho} = 20$ dB and $\beta = 2.6$. It can be seen that code diversity still yields a lower BER.

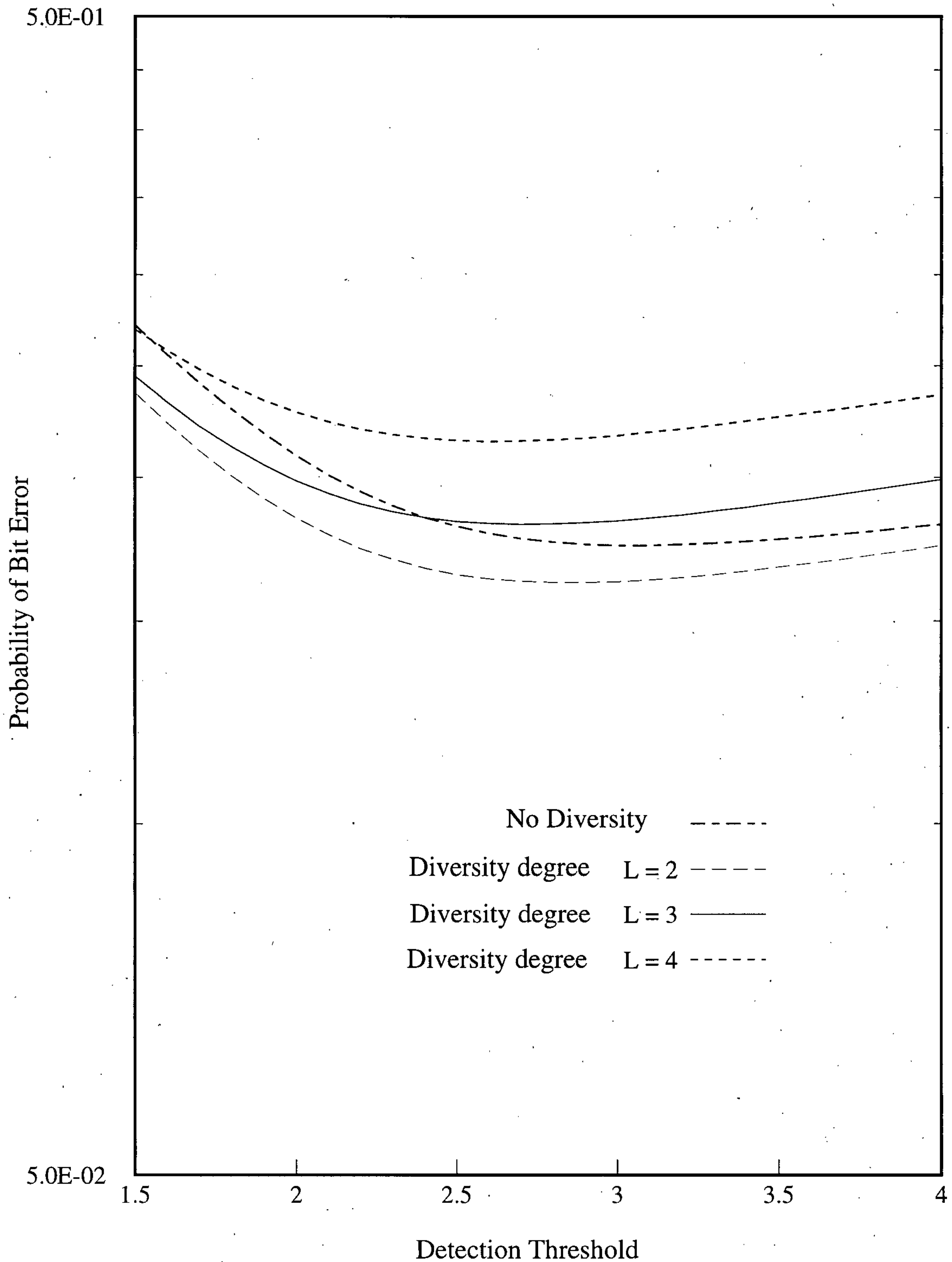

Figure 5.2 Probability of Bit Error Versus Detection Threshold, $\overline{\rho} = 20$ dB and $J = 150$.

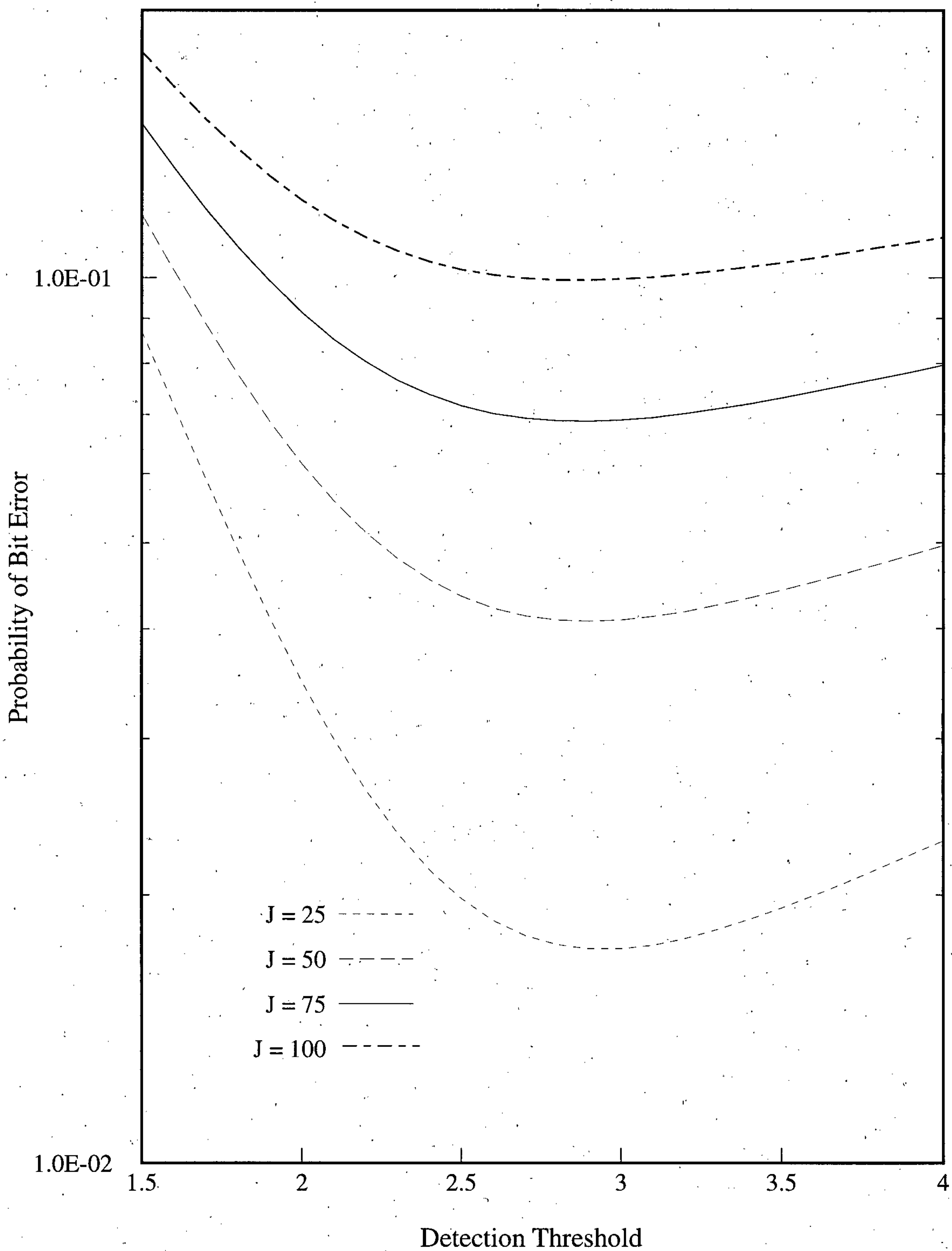

Figure 5.3 Probability of Bit Error Versus Detection Threshold, $\overline{\rho} = 20$ dB and $L = 2$.

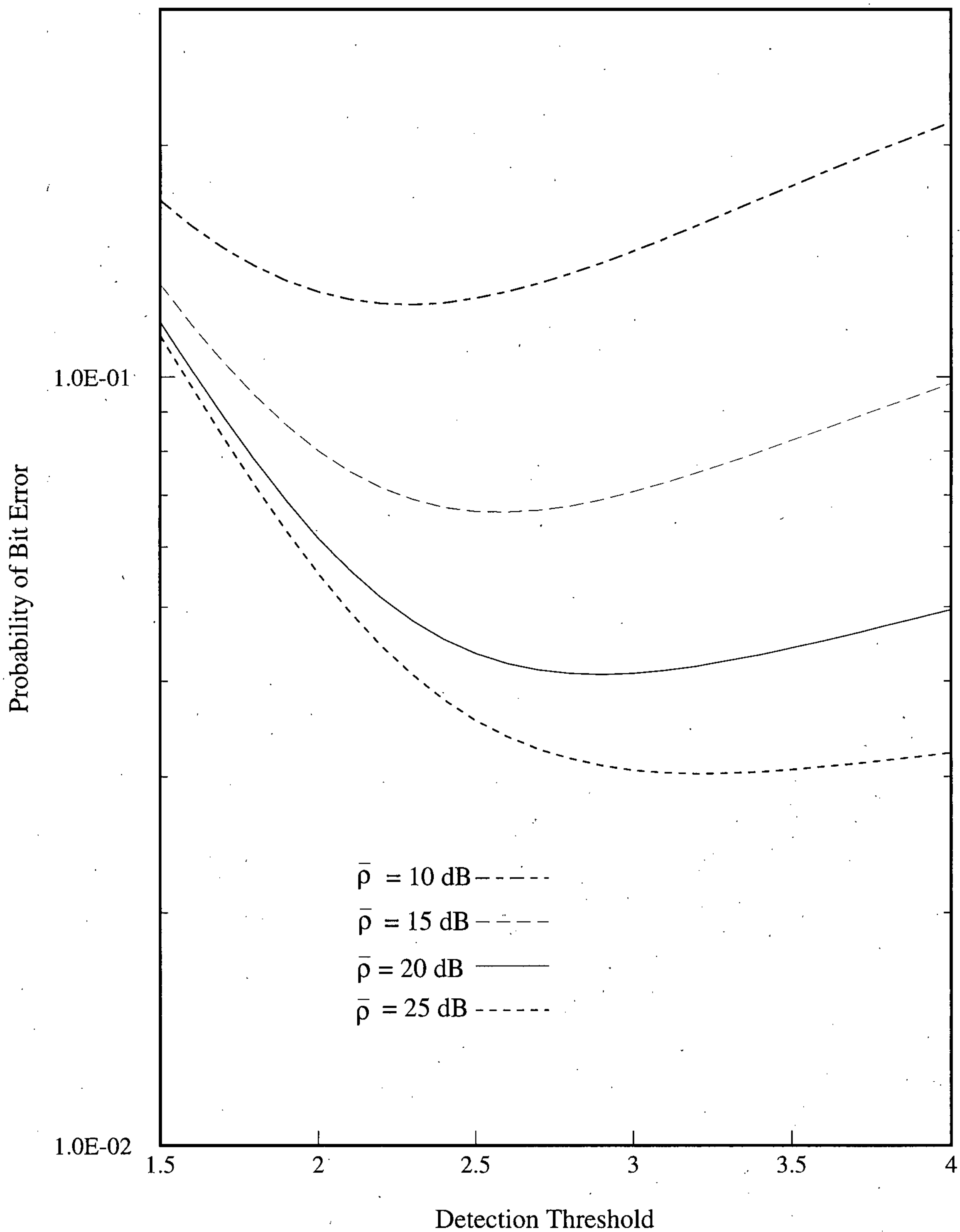

Figure 5.4 Probability of Bit Error Versus Detection Threshold, $J = 50$ and $L = 2$.

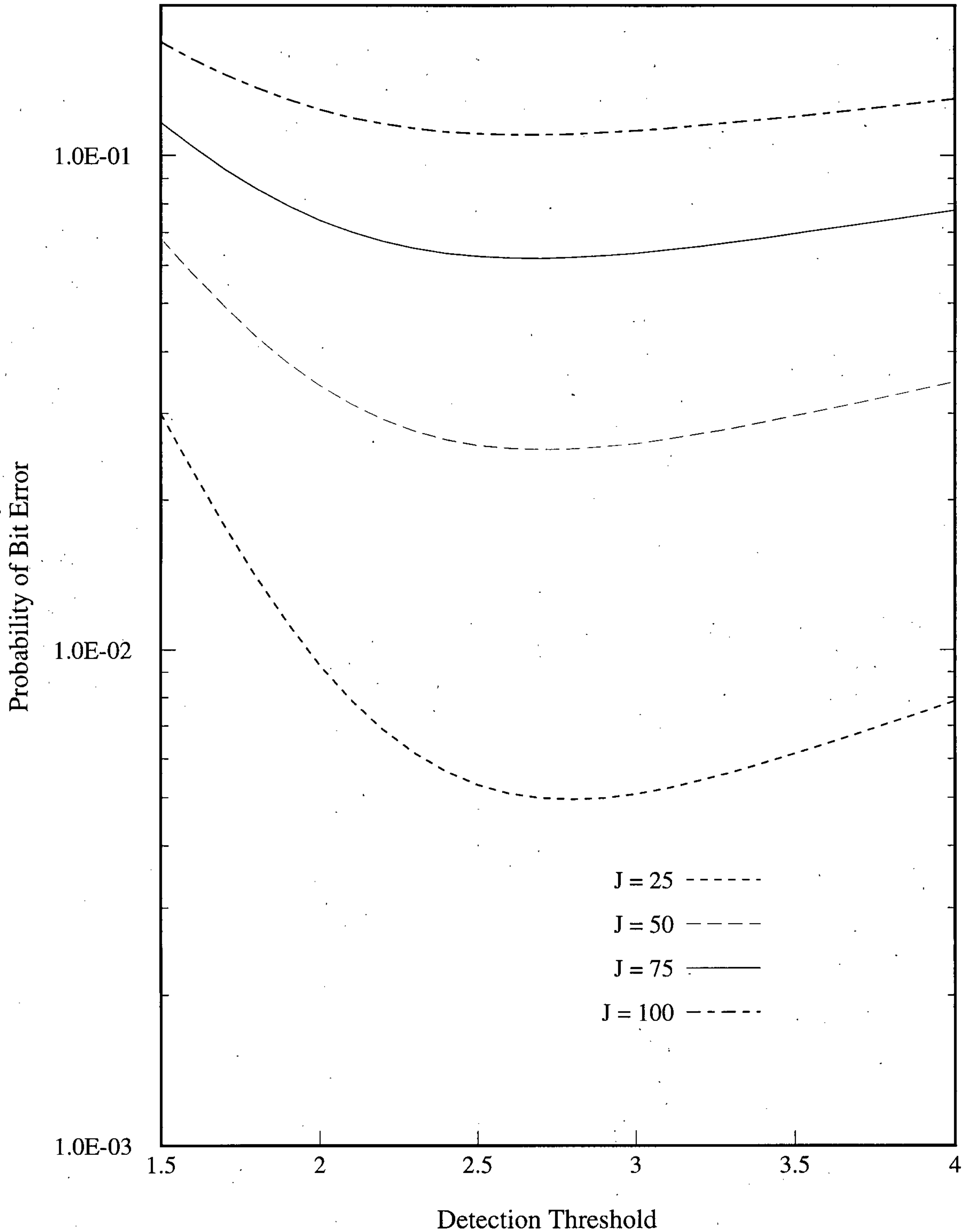

Figure 5.5 Probability of Bit Error Versus Detection Threshold, $\overline{\rho} = 20$ dB and $L = 4$.

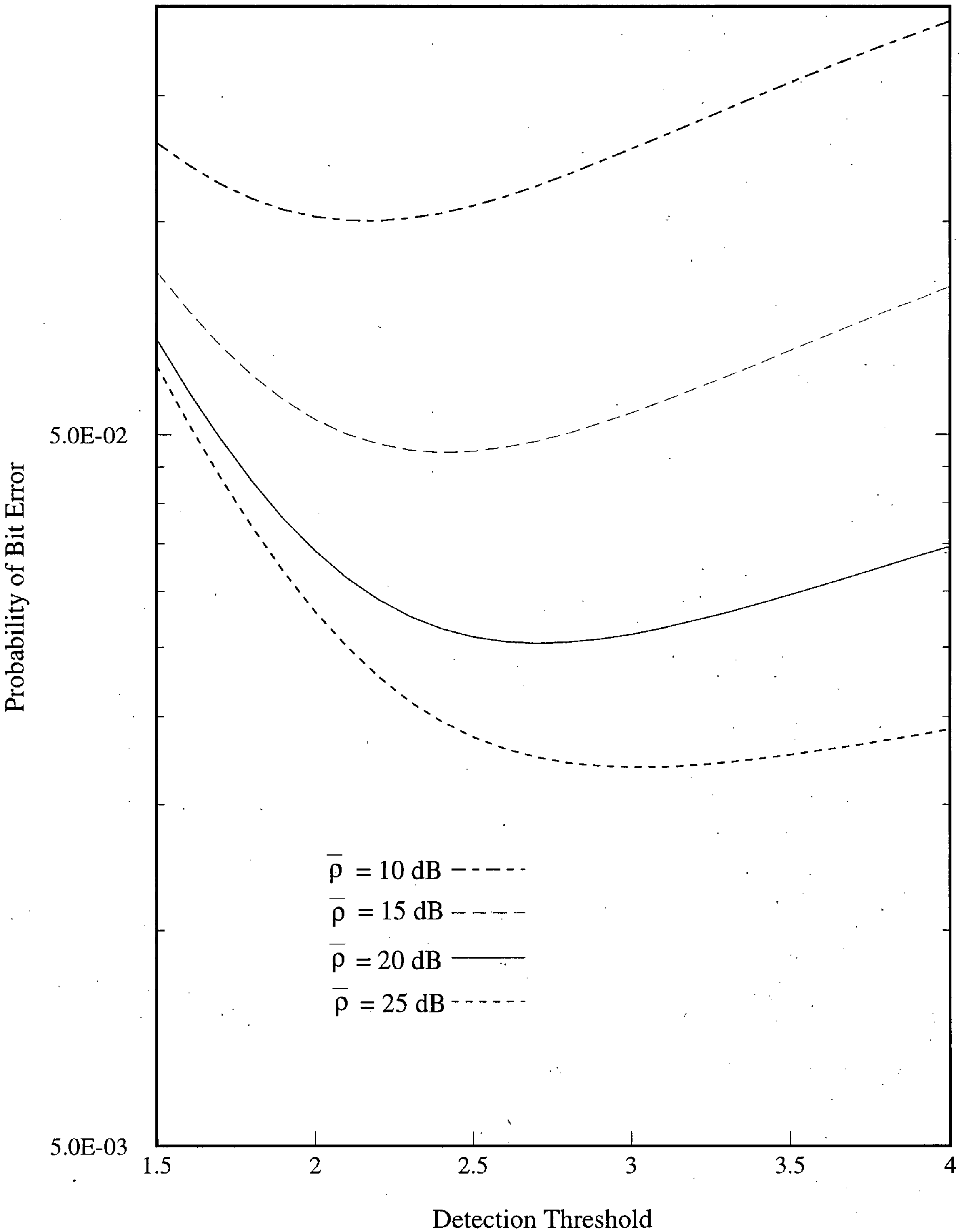

Figure 5.6 Probability of Bit Error Versus Detection Threshold, $J = 50$ and $L = 4$.

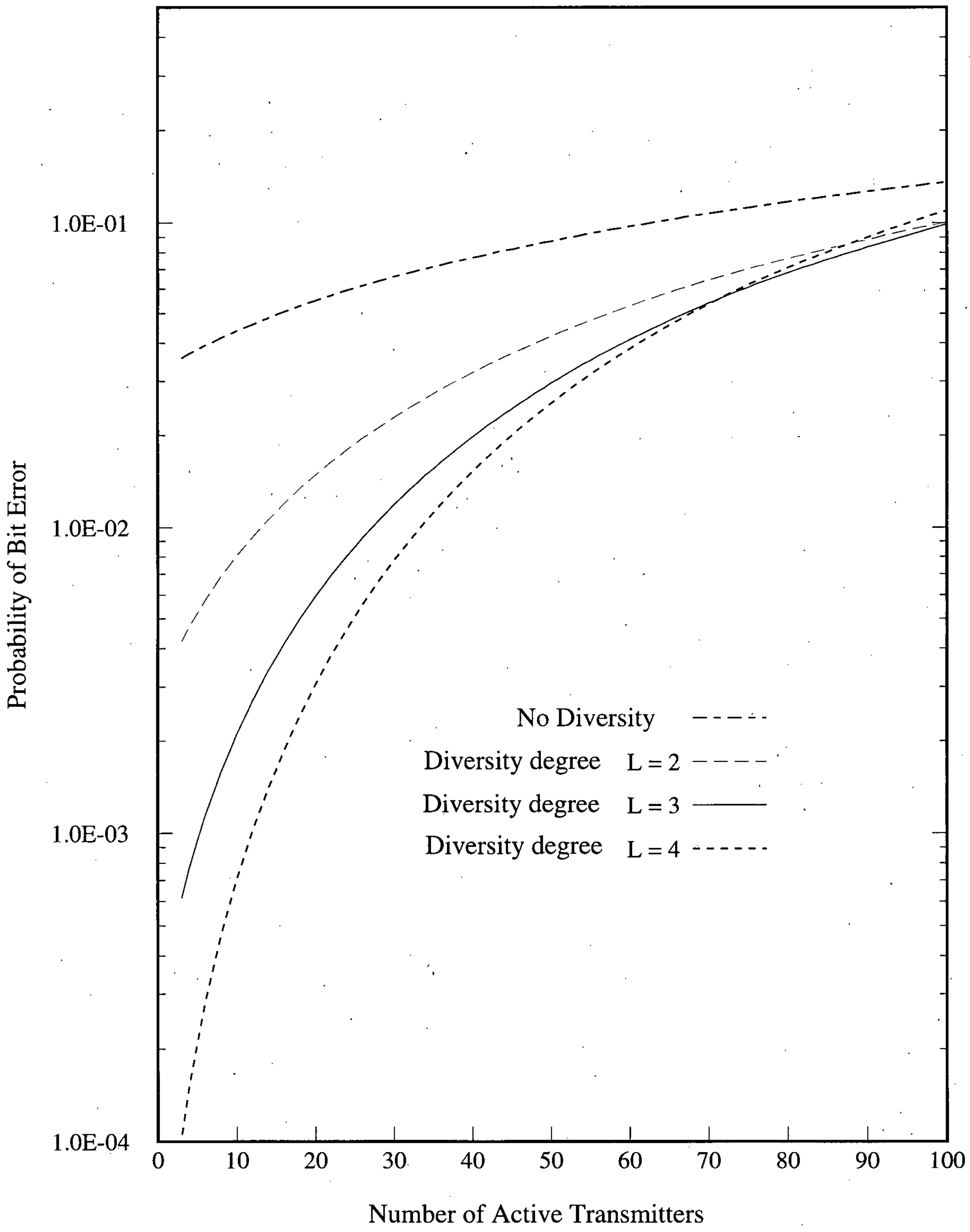

Figure 5.7 Probability of Bit Error Versus J, $\overline{\rho} = 20$ dB and $\beta = 2.6$.

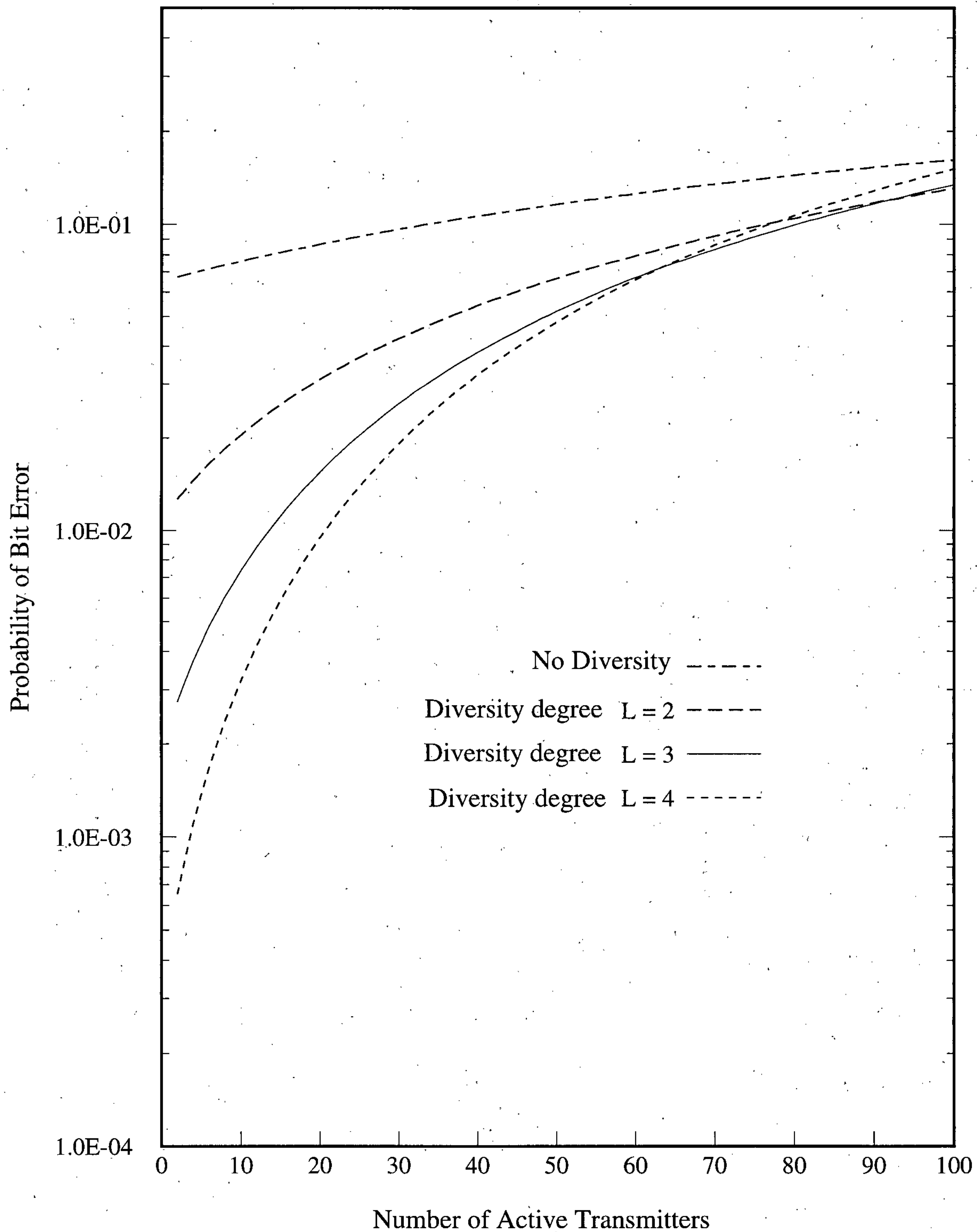

Figure 5.8 Probability of Bit Error Versus J, $\overline{\rho} = 15$ dB and $\beta = 2.6$.

Corresponding curves for $\overline{\rho} = 15$ dB and $\beta = 2.6$ appear in Figure 5.10.

5.2 Majority Vote Decoding Without Threshold Detection

In this section, we derive an expression for the symbol error probability $P_e(J, q, L)$ for a system model in which the resultant signal for each symbol tone at the receiver is a vector addition of each individual signal transmitted by the transmitters for that particular symbol tone. The signal transmitted by each individual transmitter is assumed to undergo independent fading and the signal strength of each individual signal is assumed to be statistically identical. In other words, the individual signals are assumed to have independent and identically distributed (i.i.d) Rayleigh distributed amplitudes and uniformly distributed phase over $(0,\ 2\pi]$ and the variance of the in-phase and quadrature phase is σ^2 if the signals from all transmitters are received at the same average power. The noise is assumed to be complex Gaussian with variance σ_n^2. These signals (including the noise) are added vectorially to form a resultant signal. As an example, in Figure 5.11, three individual signals and the noise are vector added to produce the resultant signal.

The modulation scheme considered here is Non-coherent Binary Frequency Shift Keying (BFSK) and in the decoding process, the receiver selects the tone with the largest signal strength as the tone detected in each of **T**'s L frequency bins. This signal strength at the sampling instance of the filter output is Rayleigh distributed according to the magnitude of the resultant signal of that particular tone. Majority vote decoding is then used to determine the symbol transmitted by **T**. In the event of a tie between the two symbols, the receiver randomly chooses any of them. This decoding process is illustrated in Figure 5.12 for the case of $L = 5$.

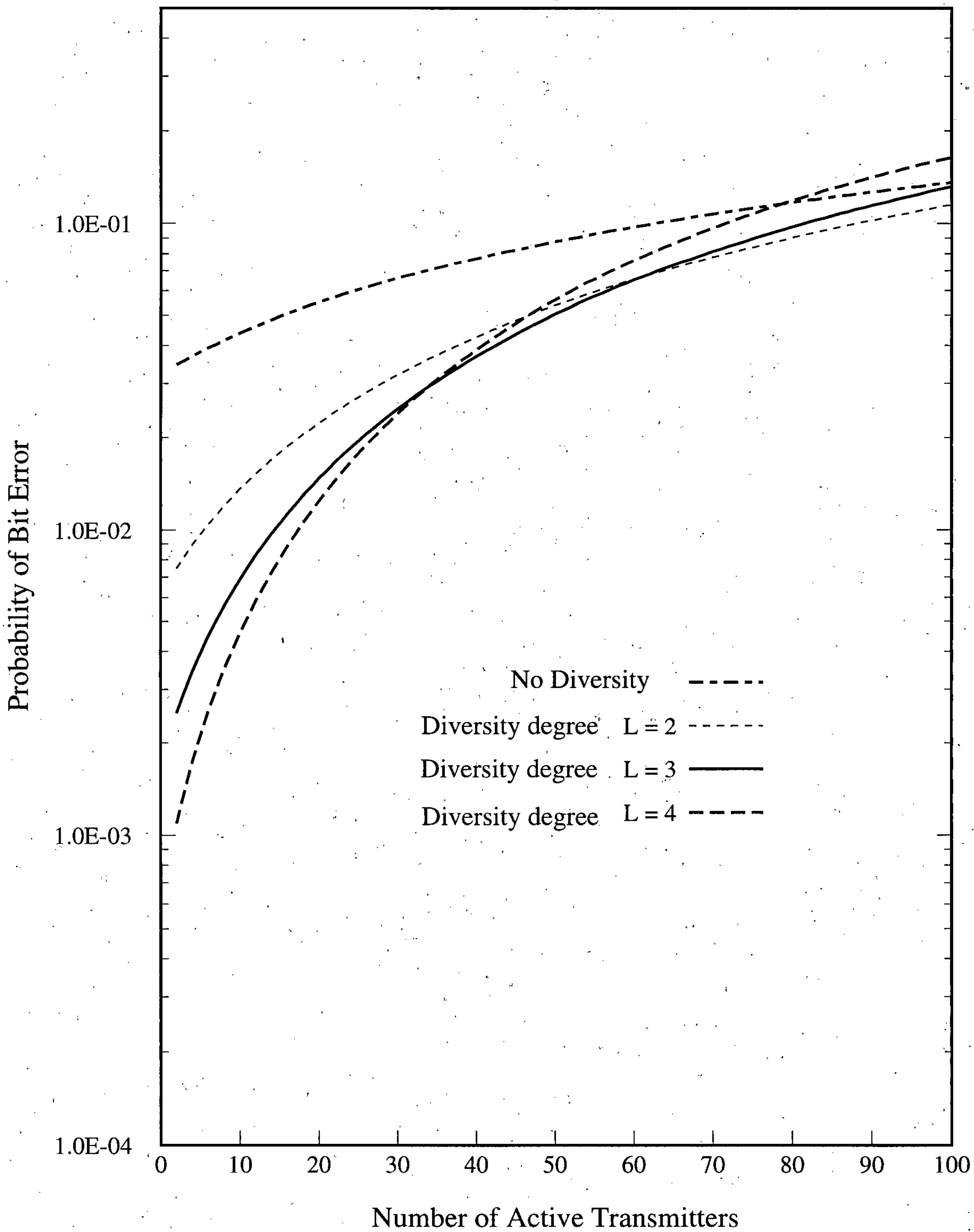

Figure 5.9 Probability of Bit Error Versus J, $\overline{\rho} = 20$ dB and $\beta = 2.6$; the total power per transmitter is constant.

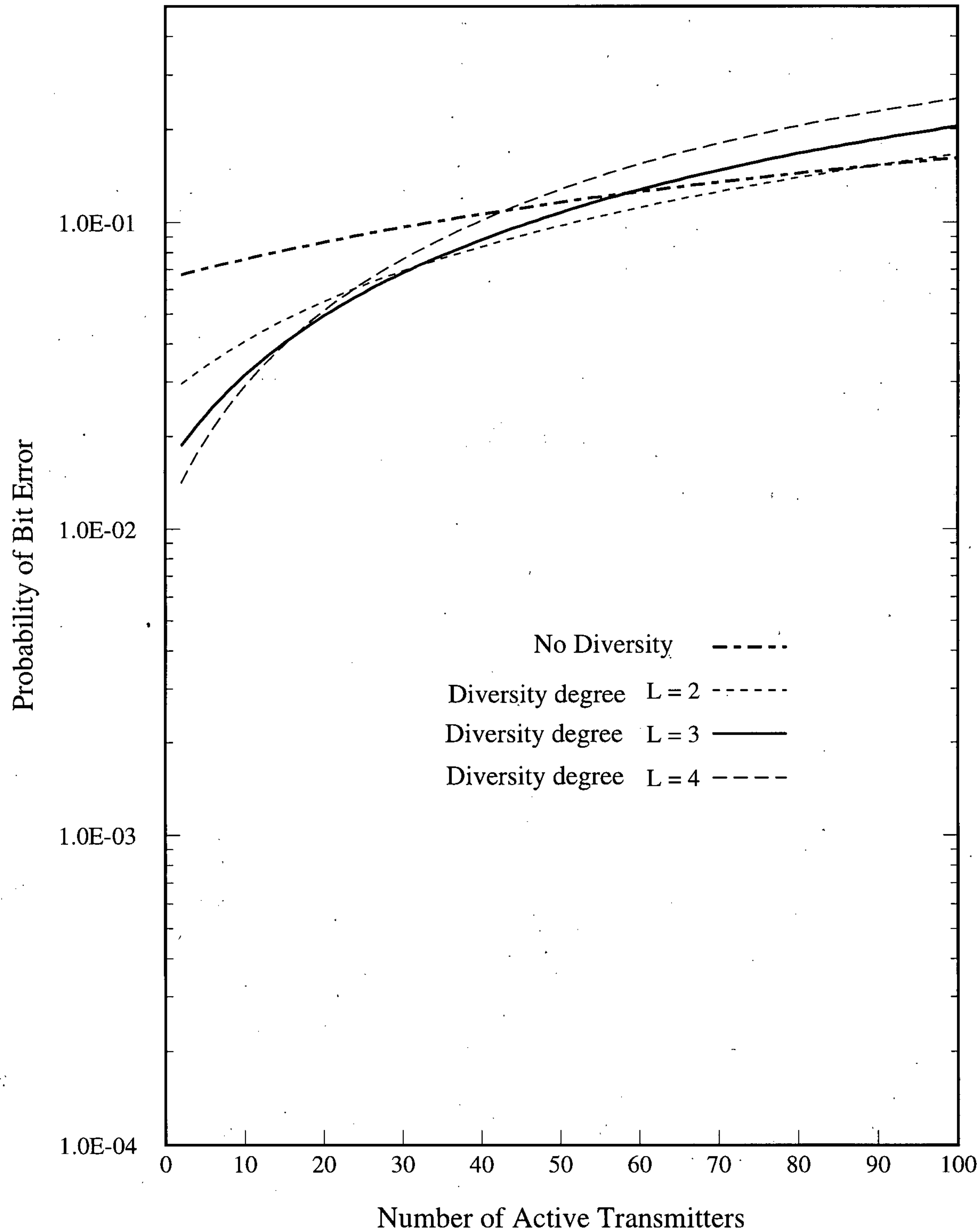

Figure 5.10 Probability of Bit Error Versus J, $\overline{\rho} = 15$ dB and $\beta = 2.6$; the total power per transmitter is constant.

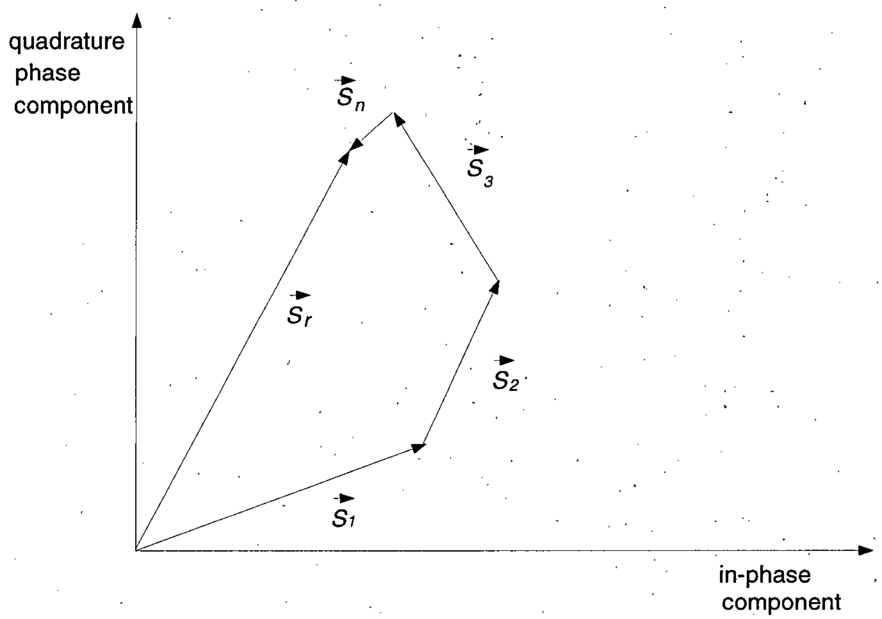

Figure 5.11 Vector Addition of Signals.

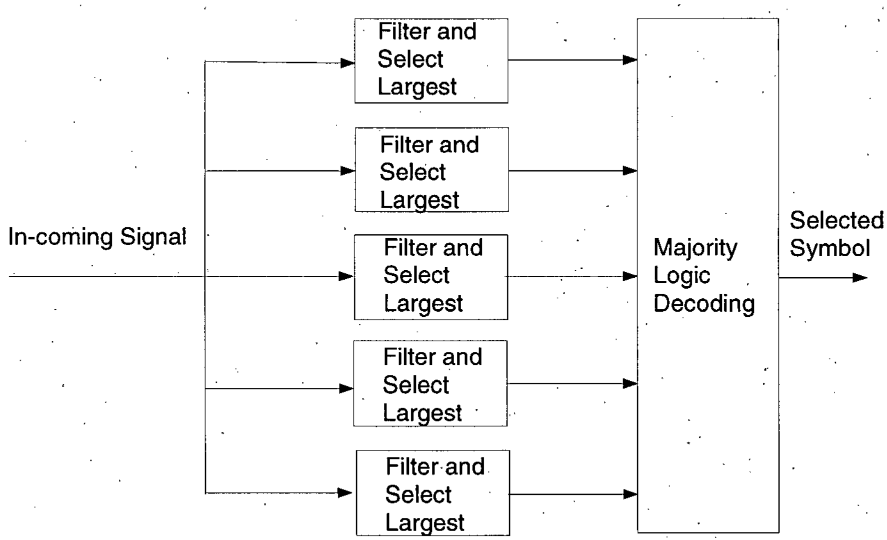

Figure 5.12 Majority Vote Decoding Without Detection Threshold, $L = 5$.

Let $\vec{S}_{m,l}$ be the resultant vector of the m^{th}, $m \in \{0, 1\}$, symbol of **T**'s l^{th} frequency bin. It is shown in Appendix B that $\vec{S}_{m,l}$ is Rayleigh distributed with uniform phase over $(0,\ 2\pi]$. If $n_{m,l}$ is the total number of hits on the m^{th} symbol of **T**'s l^{th} frequency bin, as defined in Section 4.1.4, the probability density function of the magnitude, $S_{m,l}$, of $\vec{S}_{m,l}$

can be written as

$$f_{S_{m,l}}(s_{m,l}) = \begin{cases} \frac{s_{m,l}}{n_{m,l}\sigma_i^2+\sigma_n^2} e^{-s_{m,l}^2/2\left(n_{m,l}\sigma_i^2+\sigma_n^2\right)} & \text{if } \mathbf{T} \text{ did not transmit } m \\ \frac{s_{m,l}}{(n_{m,l}-1)\sigma_i^2+\sigma^2+\sigma_n^2} e^{-s_{m,l}^2/2\left((n_{m,l}-1)\sigma_i^2+\sigma^2+\sigma_n^2\right)} & \text{if } \mathbf{T} \text{ transmitted } m \end{cases}, \tag{5.11}$$

i.e. a Rayleigh distribution with mean and variance of the following form $\sqrt{\pi(\sigma_c^2+\sigma_n^2)/2}$ and $(2-\pi/2)(\sigma_c^2+\sigma_n^2)$ respectively where

$$\sigma_c^2 = \begin{cases} n_{m,l}\sigma_i^2 & \text{if } \mathbf{T} \text{ did not transmit } m \\ (n_{m,l}-1)\sigma_i^2+\sigma^2+\sigma_n^2 & \text{if } \mathbf{T} \text{ transmitted } m \end{cases} \tag{5.12}$$

and the average signal to interferer power ratio (SIR), $\overline{\rho_i} \equiv \sigma^2/\sigma_i^2$. Hence the conditional probability of the receiver detecting "0" and "1" in the l^{th} frequency bin given that **T** transmitted "0" can be written as

$$\begin{aligned} P_{0,l|\mathbf{T}_0} &= \Pr\{S_{0,l} > S_{1,l}|n_{0,l}, n_{1,l}, \mathbf{T} \text{ transmitted } 0\} \\ &= \int_0^\infty \Big[\int_{S_{1,l}}^\infty f_{S_{0,l}}(s_{0,l}|n_{0,l}, n_{1,l}, \mathbf{T} \text{ transmitted } 0)\, ds_{0,l}\Big] f_{S_{1,l}} \\ &\qquad (s_{1,l}(s_{0,l}|n_{0,l}, n_{1,l}, \mathbf{T} \text{ transmitted } 0)\, ds_{1,l}) \\ &= \frac{(n_{0,l}-1)\sigma_i^2+\sigma^2+\sigma_n^2}{(n_{0,l}+n_{1,l}-1)\sigma_i^2+\sigma^2+2\sigma_n^2} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} P_{1,l|\mathbf{T}_0} &= 1 - P_{0,l|\mathbf{T}_0} \\ &= \frac{n_{1,l}\sigma_i^2+\sigma_n^2}{(n_{0,l}+n_{1,l}-1)\sigma_i^2+\sigma^2+2\sigma_n^2} \end{aligned} \tag{5.14}$$

respectively. Similar conditional probabilities can be obtained in the case where **T** transmitted "1" by symmetry. Using the law of total probability, we can write

$$P_{0,l} = \Pr\{T \text{ transmitted } 0\}\, P_{0,l|\mathbf{T}_0} + \Pr\{T \text{ transmitted } 1\} P_{0,l|\mathbf{T}_1}$$

$$= \frac{(2n_{0,l}-1)\sigma_i^2+\sigma^2+2\sigma_n^2}{2\big((n_{0,l}+n_{1,l}-1)\sigma_i^2+\sigma^2+2\sigma_n^2\big)} \tag{5.15}$$

and

$$\begin{aligned} P_{1,l} &= 1-P_{0,l} \\ &= \frac{(2n_{1,l}-1)\sigma_i^2+\sigma^2+2\sigma_n^2}{2\big((n_{0,l}+n_{1,l}-1)\sigma_i^2+\sigma^2+2\sigma_n^2\big)}. \end{aligned} \tag{5.16}$$

Note that when the SIR is one and the average signal to noise ratio, $\bar{\rho} \equiv \sigma^2/\sigma_n^2$, is large $P_{0,l}$ and $P_{1,l}$ is approximately $n_{0,l}/(n_{0,l}+n_{1,l})$ and $n_{1,l}/(n_{0,l}+n_{1,l})$ respectively. In such situations, assuming that the tone detected at each frequency bin is dependent solely on the relative number of transmitters transmitting that particular tone as compared to the other tone in the frequency bin is a good approximation, just as in Section 4.1.1. Conversely, when $\bar{\rho}$ is small, $P_{0,l}$ and $P_{1,l}$ tends toward 0.5 as expected.

Let X_m be the number of frequency bins among **T**'s L frequency bins that detected the symbol "m". The probability of error, given **T** transmitted "0" (the dependency on $\{n_{0,1}, n_{1,1}, n_{0,2}, n_{1,2}, ..., n_{0,L}, n_{1,L}\}$ is not explicitly stated) can be written in terms of X_m as

$$P_e(X_0, X_1|\mathbf{T}\text{ transmitted } 0) = \begin{cases} \Pr\{X_1 > L/2\} & L \text{ odd} \\ \Pr\{X_1 > L/2\} + \frac{1}{2}\Pr\{X_1 = L/2\} & L \text{ even.} \end{cases} \tag{5.17}$$

Given $\{n_{0,1}, n_{1,1}, n_{0,2}, n_{1,2}, ..., n_{0,L}, n_{1,L}\}$, this probability can be written in terms of $P_{m,l}$. For example in the case of $L = 3$,

$$P_e(X_0, X_1|\mathbf{T}\text{ transmitted } 0) = P_{0,1}P_{1,2}P_{1,3} + P_{1,1}P_{0,2}P_{1,3} + P_{1,1}P_{1,2}P_{0,3}. \tag{5.18}$$

The probability of having exactly $n_{m,1}$, $n_{m,2}$, ..., $n_{m,L}$ hits on **T**'s 1^{st}, 2^{nd}, ..., L^{th} frequency bins respectively by k_m interferers transmitting symbol "m" can be calculated using equation (4.41) and (4.43). Hence the average probability of error given the number of transmitters, J, can be written as

$$\begin{aligned}
P_e(J,q,L) &= \Pr\{\text{error}|J,q,L,\mathbf{T}\text{ sent }0\} \\
&= \sum_{n_{0,1},n_{1,1},\ldots,n_{0,L},n_{1,L}} \Pr\{n_{0,1},n_{1,1},\ldots,n_{0,L},n_{1,L}|J,\mathbf{T}\text{ sent }0\} \\
&\qquad \times \Pr\{X_1 > X_0|n_{0,1},n_{1,1},\ldots,n_{0,L},n_{1,L}\} \\
&= \sum_{k_1=0}^{K}\binom{K}{k_1}\left(\frac{1}{2}\right)^K \sum_{n_{0,1}=1}^{k_0}\sum_{n_{1,1}=0}^{k_1}\cdots\sum_{n_{0,L}=1}^{k_0}\sum_{n_{1,L}=0}^{k_1} Q_4(n_{1,1},\ldots,n_{1,L}|k_1) \\
&\qquad \times Q_4(n_{0,1}-1,\ldots,n_{0,L}-1|k_0) \\
&\qquad \times \Pr\{X_1 > L/2\} \qquad (5.19)
\end{aligned}$$

for odd values of L and

$$\begin{aligned}
P_e(J,q,L) &= \Pr\{\text{error}|J,q,L,\mathbf{T}\text{ sent }0\} \\
&= \sum_{k_1=0}^{K}\binom{K}{k_1}\left(\frac{1}{2}\right)^K \sum_{n_{0,1}=1}^{k_0}\sum_{n_{1,1}=0}^{k_1}\cdots\sum_{n_{0,L}=1}^{k_0}\sum_{n_{1,L}=0}^{k_1} Q_4(n_{1,1},\ldots,n_{1,L}|k_1) \\
&\qquad \times Q_4(n_{0,1}-1,\ldots,n_{0,L}-1|k_0) \\
&\qquad \times [\Pr\{X_1 > L/2\} + \frac{1}{2}\Pr\{X_1 > L/2\}]. \qquad (5.20)
\end{aligned}$$

for even values of L.

5.2.1 Numerical Results

Using equations (5.15), (5.16), (5.17), (5.19) and (5.20), $P_e(J,q,L)$ for $L=1$ and 2 can be calculated. Figures 5.13 and 5.14 show the plots for SNR of 20 dB and 10 dB respectively with the SIR of one and Figure 5.15 shows the plots for SNR and SIR of 20 dB. It can be seen that the BER for $L=1$ is lower even for small values of J and the percentage difference is greater in the case of higher SNR values. For example, for $J = 6$, the percentage BER difference is about 5% and 27% for SNR values of 10 dB and 20 dB respectively. The explanation for this somewhat surprising result is due to the fact that for higher SNR values, multiple access interference is the predominant cause of error. The scheme which makes a hard decision on the symbol transmitted at the frequency bin level is especially susceptible to multiple access interference especially for larger diversity levels where the interferers have a higher chance of transmitting in the frequency bins of the marked transmitter. This is shown analytically for the case of $J = 2$ in Appendix C.

It should also be noted that at high SNR values, this scheme is similar to scheme 1 of Chapter 4 which shows that $P_e(J, q, L)$ is lower for higher values of L in cases where the number of interferers is reasonably low. There is however an important difference — scheme 1 of Chapter 4 does a majority vote only when there is a complete hit on more than 1 symbol. This suggests that threshold detection to determine if there is more than one set of complete hit is useful to lower the BER in cases where the SNR is large.

5.3 Soft Decision Decoding Scheme

The scheme considered in this section is a soft decision decoding scheme. The receiver for such a scheme with $L = 3$ is shown in Figure 5.16. All other aspects of the system model, apart from the decoding process, are as described in Section 5.2.

The incoming signal is filtered at each tone in all of **T**'s L frequency bins, resulting in a Rayleigh distributed random variable at the output of each filter. Instead of making a decision on the symbol detected at each frequency bin as in the previous scheme, the outputs of these filters for the same symbol are added together, resulting in M scalar quantities for a

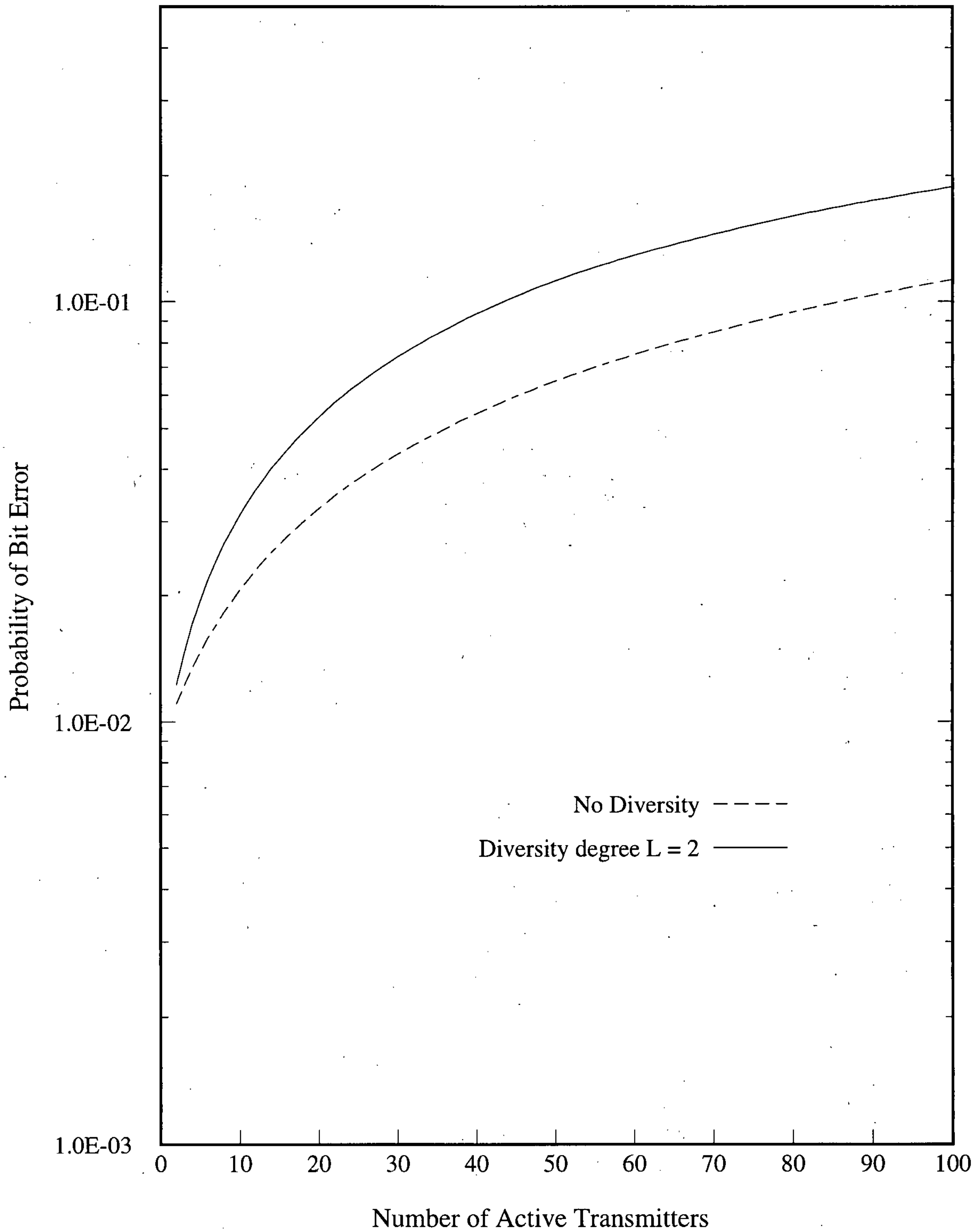

Figure 5.13 Probability of Bit Error Versus J, SNR = 20 dB and $q = 200$.

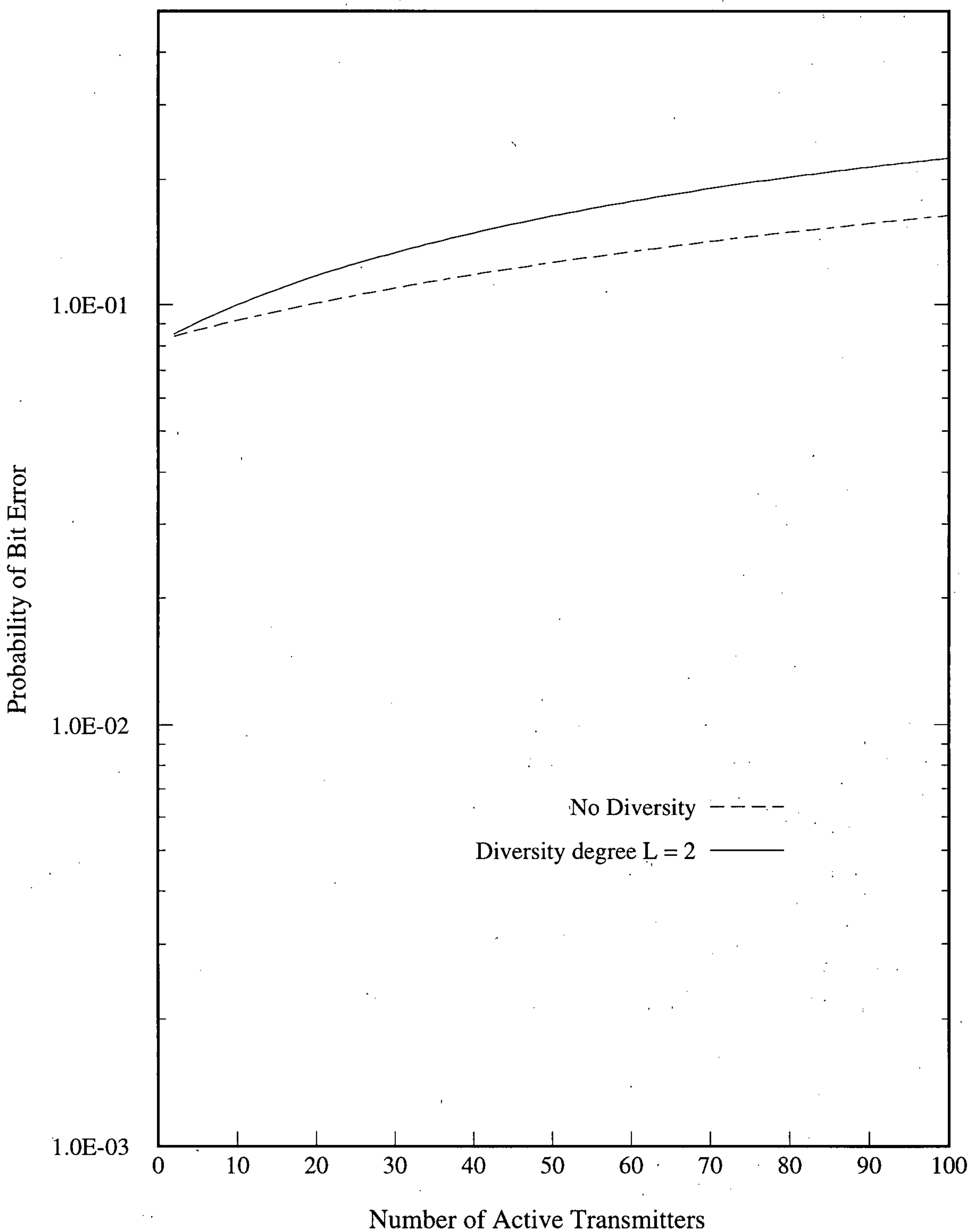

Figure 5.14 Probability of Bit Error Versus J, SNR = 10 dB and q = 200.

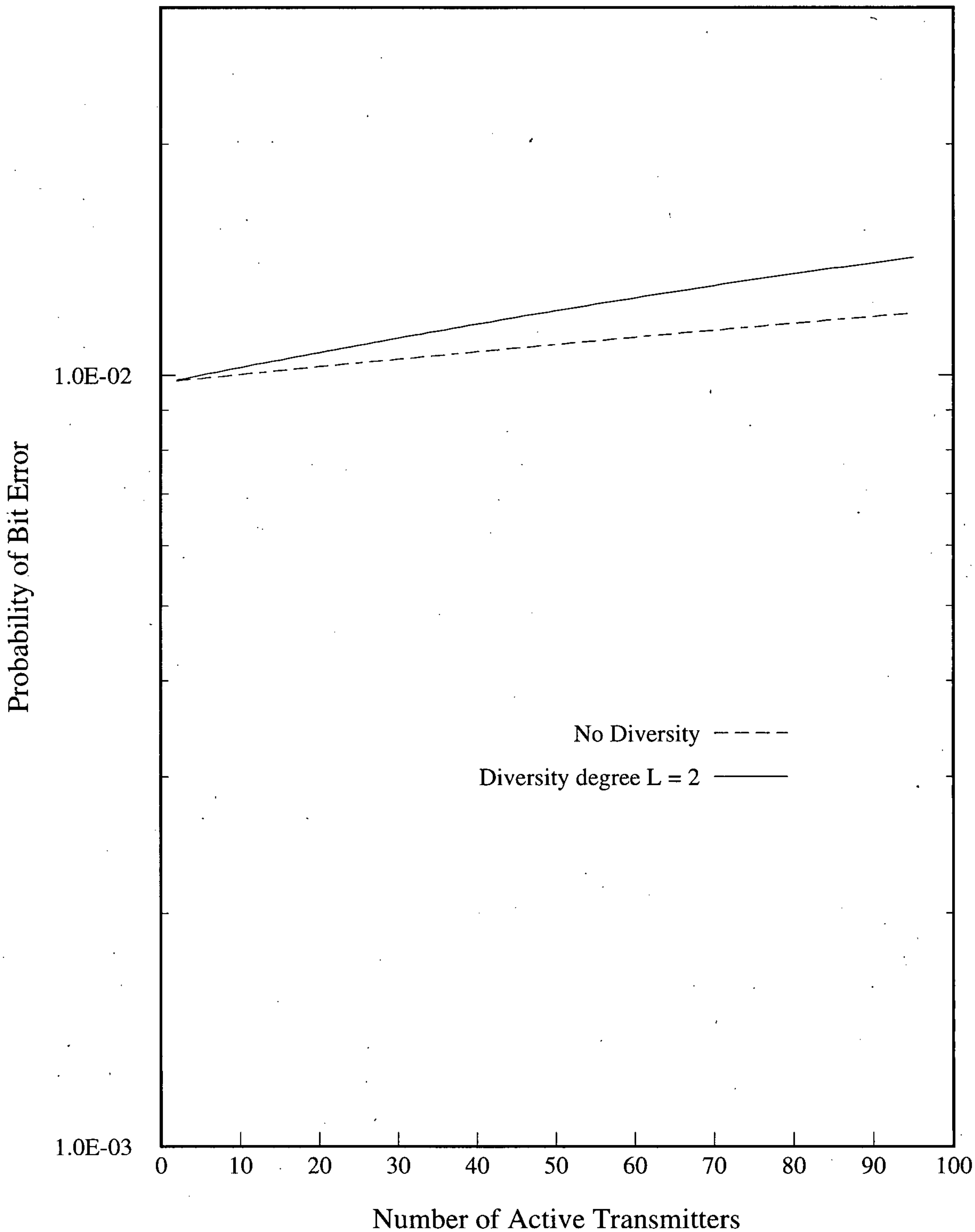

Figure 5.15 Probability of Bit Error Versus J, SNR = 20 dB, SIR = 20 dB and $q = 200$.

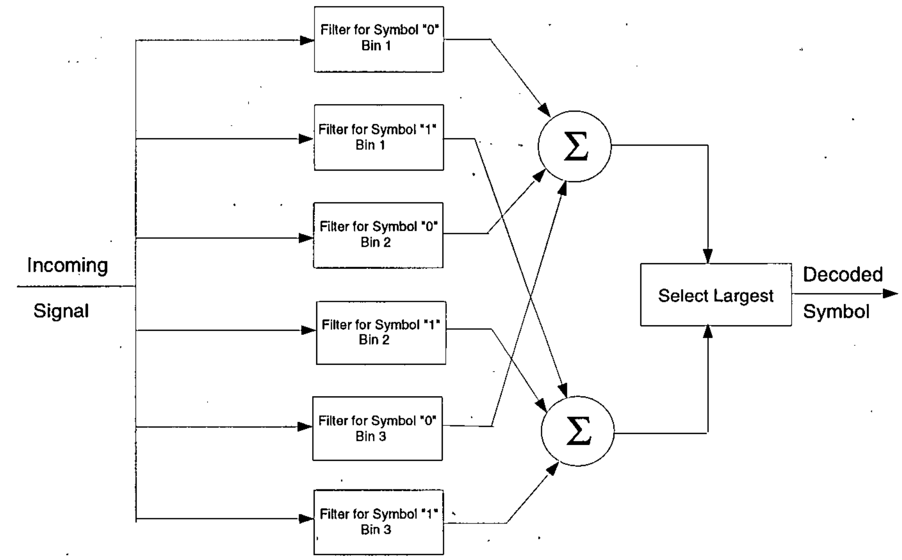

Figure 5.16 Soft Decision Decoding

M-ary system. The decision rule used by the decoder is to declare the symbol corresponding to the largest of these M values as the decoded symbol.

Conditioned on the number of transmitters transmitting in each tone of **T**'s L frequency bins, $n_{0,1}$, $n_{1,1}$, ..., $n_{0,L}$, $n_{1,L}$, the magnitude of the filter output corresponding to symbol m and the l^{th} frequency bin, $S_{m,l}$, is only dependent on $n_{m,l}$. Hence we can write

$$f_{S_{m,l}}(s_{m,l}|n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}) = f_{S_{m,l}}(s_{m,l}|n_{m,l}) \tag{5.21}$$

and $f_{S_{m,l}}(s_{m,l}|n_{m,l})$ is Rayleigh distribution as given by equation (5.11). Since $S_{m,l}$ is only dependent on $n_{m,l}$, the probability density of the composite signal of symbol "m" can be obtained by

$$\begin{aligned} f_{S_m}(s_m|n_{0,1}, n_{1,1}, ..., n_{0,L}, n_{1,L}) &= f_{S_m}(s_m|n_{m,1}, ..., n_{m,L}) \\ &= f_{S_{m,1}}(s_{m,1}|n_{m,1}) \otimes f_{S_{m,2}}(s_{m,2}|n_{m,2}) \ldots \otimes f_{S_{m,L}}(s_{m,L}|n_{m,L}) \end{aligned} \tag{5.22}$$

where $\otimes$ is the convolution operation symbol.

Given that $\mathbf{T}$ transmits symbol "0", the conditional probability of error is given by

$$P_e(J,q,L|n_{0,1},n_{1,1},\ldots,n_{0,L},n_{1,L},\mathbf{T}\text{ sent }0) = \int_0^\infty f_{S_0}(s_0|n_{0,1},\ldots,n_{0,L}) \cdot \int_{s_0}^\infty f_{S_1}(s_1|n_{1,1},\ldots,n_{1,L})\, ds_1 ds_0. \tag{5.23}$$

Using equations (4.41), (4.43) and (5.23), the average probability of bit error can be written as

$$\begin{aligned} P_e(J,q,L) &= \Pr\{\text{error}|J,q,L,\mathbf{T}\text{ sent }0\} \\ &= \sum_{k_1=0}^{K}\binom{K}{k_1}\left(\frac{1}{2}\right)^K \sum_{n_{0,1}=1}^{k_0}\sum_{n_{1,1}=0}^{k_1}\cdots\sum_{n_{0,L}=1}^{k_0}\sum_{n_{1,L}=0}^{k_1} Q_4(n_{1,1},\ldots,n_{1,L}|k_1) \\ &\quad \times Q_4(n_{0,1}-1,\ldots,n_{0,L}-1|k_0) \\ &\quad \times \int_{s_0}^\infty f_{S_1}(s_1|n_{1,1},\ldots,n_{1,L})\, ds_1 \cdot \int_0^\infty f_{S_0}(s_0|n_{0,1},\ldots,n_{0,L})\, ds_0. \end{aligned} \tag{5.24}$$

5.3.1 Numerical Results

By using equations (4.41), (4.43), (5.23) and (5.24), the probability of error can be determined. Unfortunately the computation complexity of this calculation quickly increases as the number of interferers and diversity level increase. Instead, Monte Carlo simulation was used to obtain numerical results. The probability of bit error versus number of transmitters for SNR of 20 dB and 10 dB are shown in Figure 5.18 and 5.19 respectively. The results were obtained for cases where the number of interferers is 5, 10, 20, 50 and 100 for diversity degrees of 2, 3, 4 and 5. Along with the mean BER's, the 99% confidence levels of the simulation results are also shown. In the case where there is no diversity, this scheme is identical to the majority vote scheme described in Section 5.2. Therefore the exact BER's, calculated using equations (5.15), (5.16), (5.17) and (5.19) are shown in the figures.

Simulation results for cases where the total transmitter power is divided equally among the diversity branches are also obtained. Figures 5.20 and 5.21 shows the curves for cases where the SNR for transmitters with no diversity is 20 dB and 10 dB respectively. It can be seen that code diversity can yield a better BER when the SNR is relatively high as in the 20 dB case. However, in cases where the SNR is low, dividing the signal power by the number of diversity branches may not offer much improvement in the BER. Noise in each of the diversity branches becomes an important factor and will reduce the performance of the system. Apart from the number of active users and frequency bins in the system, the optimal diversity degree is also dependent on the total power per transmitter and noise power ratio.

With this receiver implementation, it can be seen that there is some advantage to using diversity in cases where the number of active transmitters is low. For example when $J = 21$, at an SNR of 20 dB, the BER is 46% less when $L = 4$ compared to the case of no diversity. Simulation results were also obtained using the optimal receiver structure for FSK signalling with diversity over a Rayleigh AWGN channel [63]. An example of such a receiver structure is shown in Figure 5.17. This receiver performs an additional operation by squaring the output of each filter prior to the summing operation. It was found that the BER for this receiver structure is not necessarily less than the soft decision receiver structure which uses the sum of the absolute values. In fact, using just the sum of the absolute values tends to achieve a better BER for parameter values in Figures 5.18 and 5.19.

The reason why the optimal receiver structure for FSK signalling over a Rayleigh AWGN channel is no longer optimal in code diversity schemes is the distribution of hits on the symbol tones by the interferers. Unlike normal FSK signalling where the only source of interference is noise, code diversity schemes have to contend with multiple access interference. Unlike

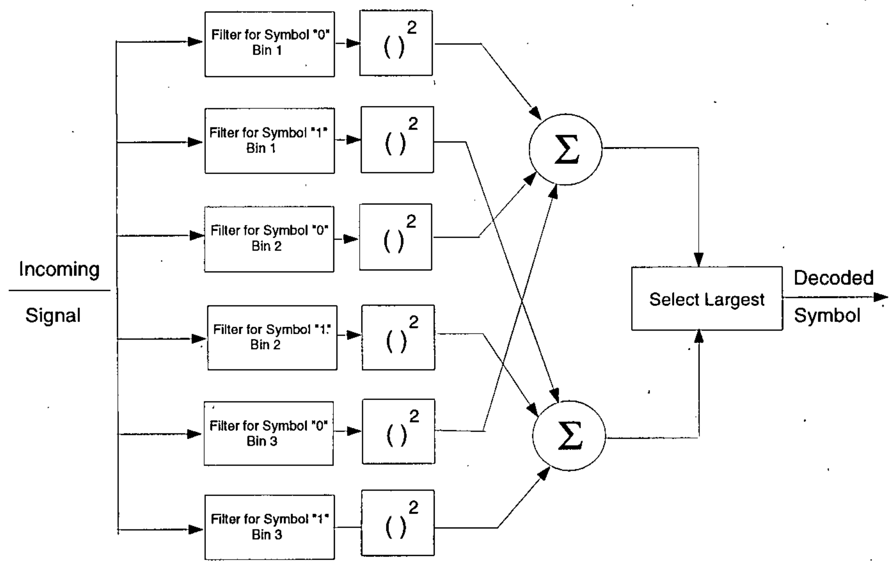

Figure 5.17 Optimal Receiver Structure for FSK Signalling over Rayleigh Channel with Diversity Degree of Three

noise which is present and identically distributed in all diversity branches, the statistics of the multiple access interference is dependent on the number of interferers transmitting in that particular diversity branch and can be asymmetric across all diversity branches.

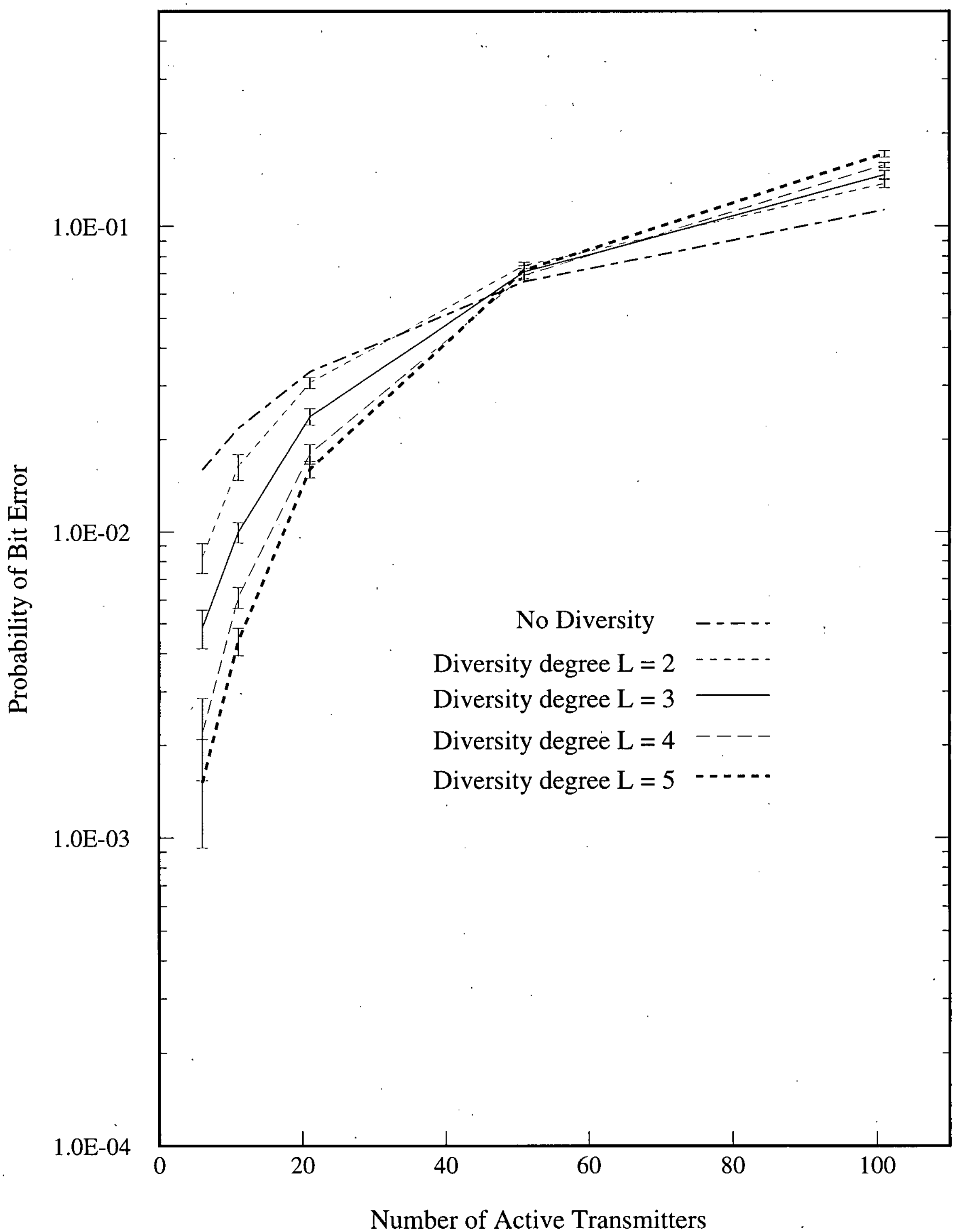

Figure 5.18 Probability of Bit Error Versus J, SNR = 20 dB and q = 200.

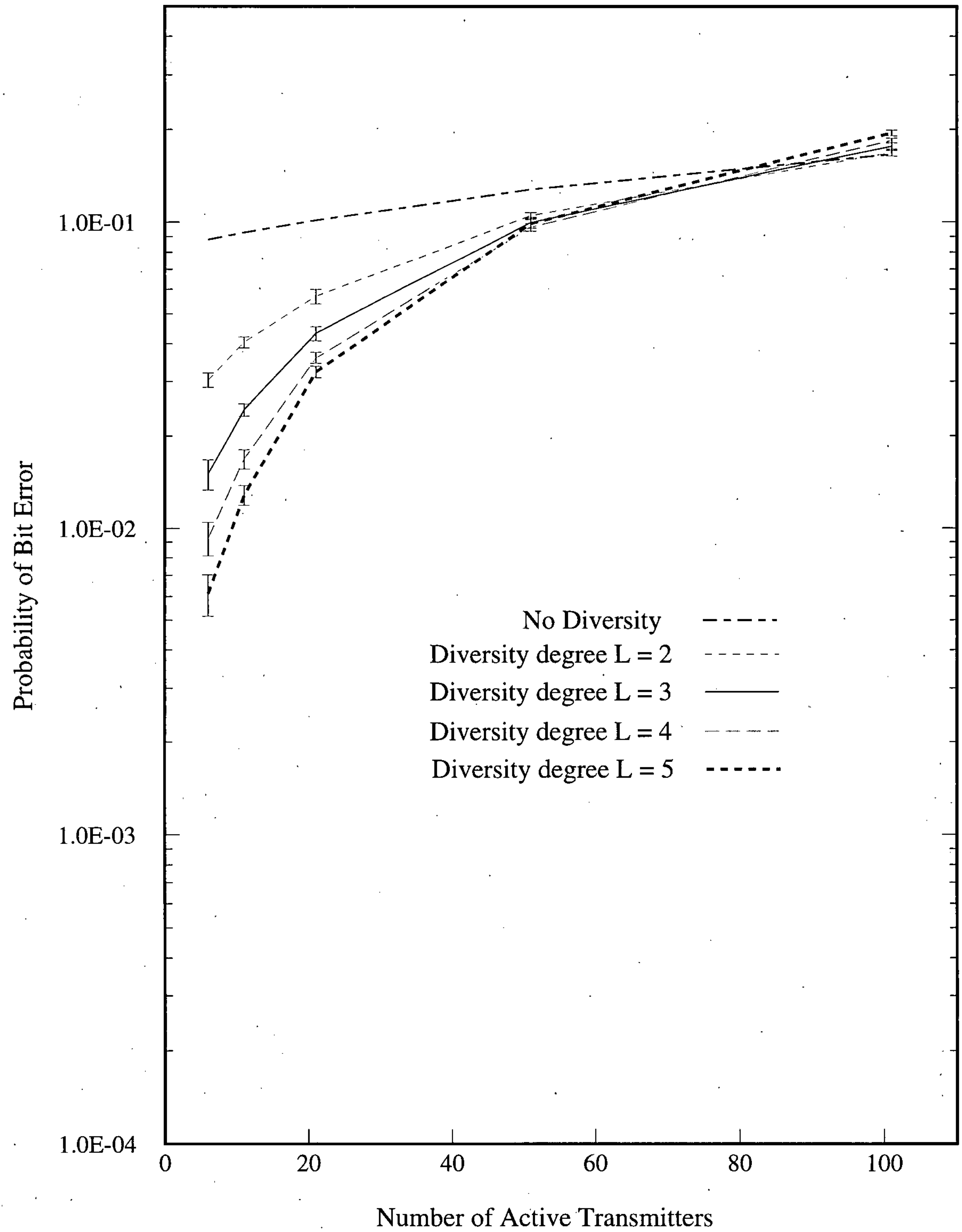

Figure 5.19 Probability of Bit Error Versus J, SNR = 10 dB and q = 200.

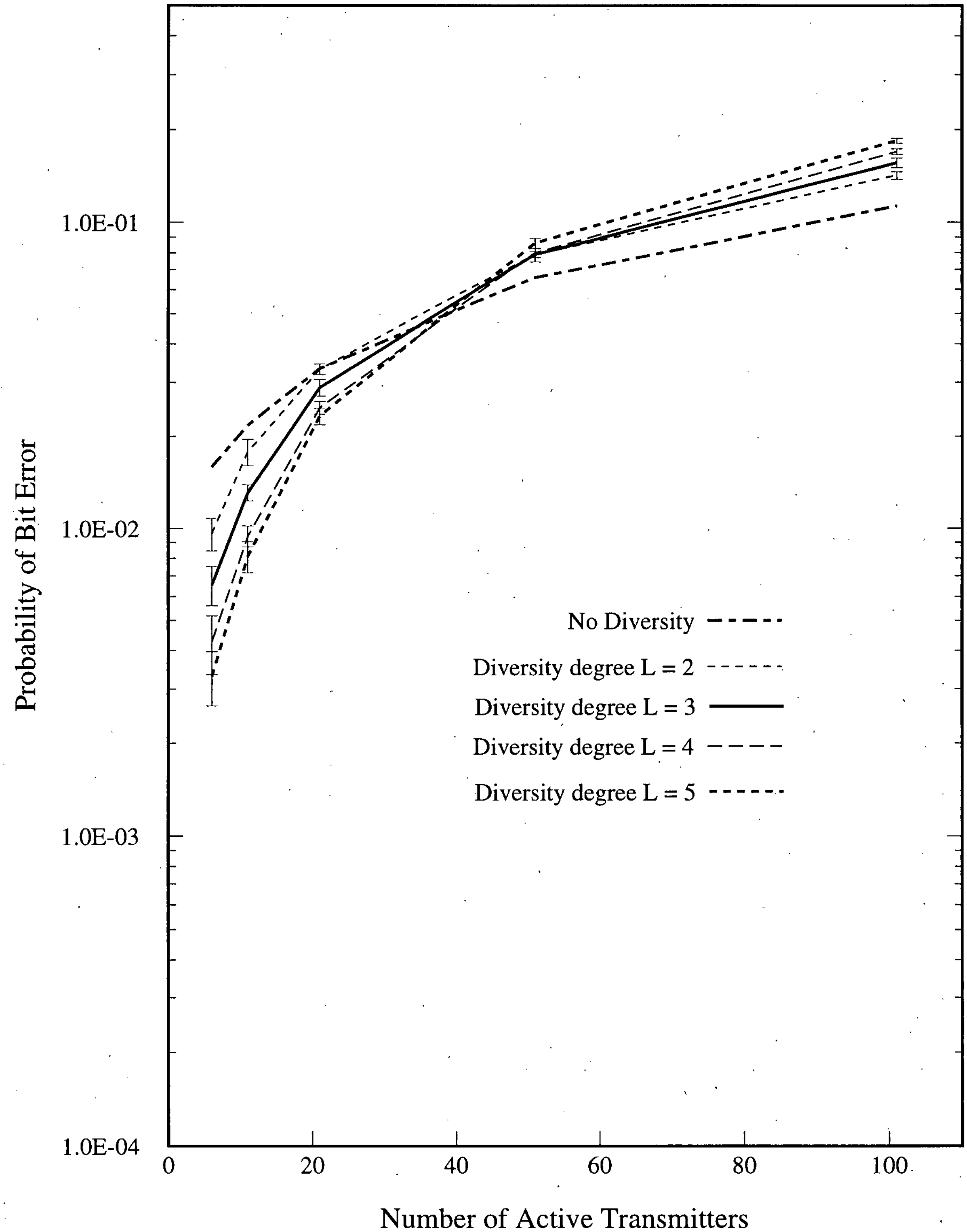

Figure 5.20 Probability of Bit Error Versus J, SNR = 20 dB for Transmitters with no Diversity and $q = 200$, equal total transmitted power for all transmitters.

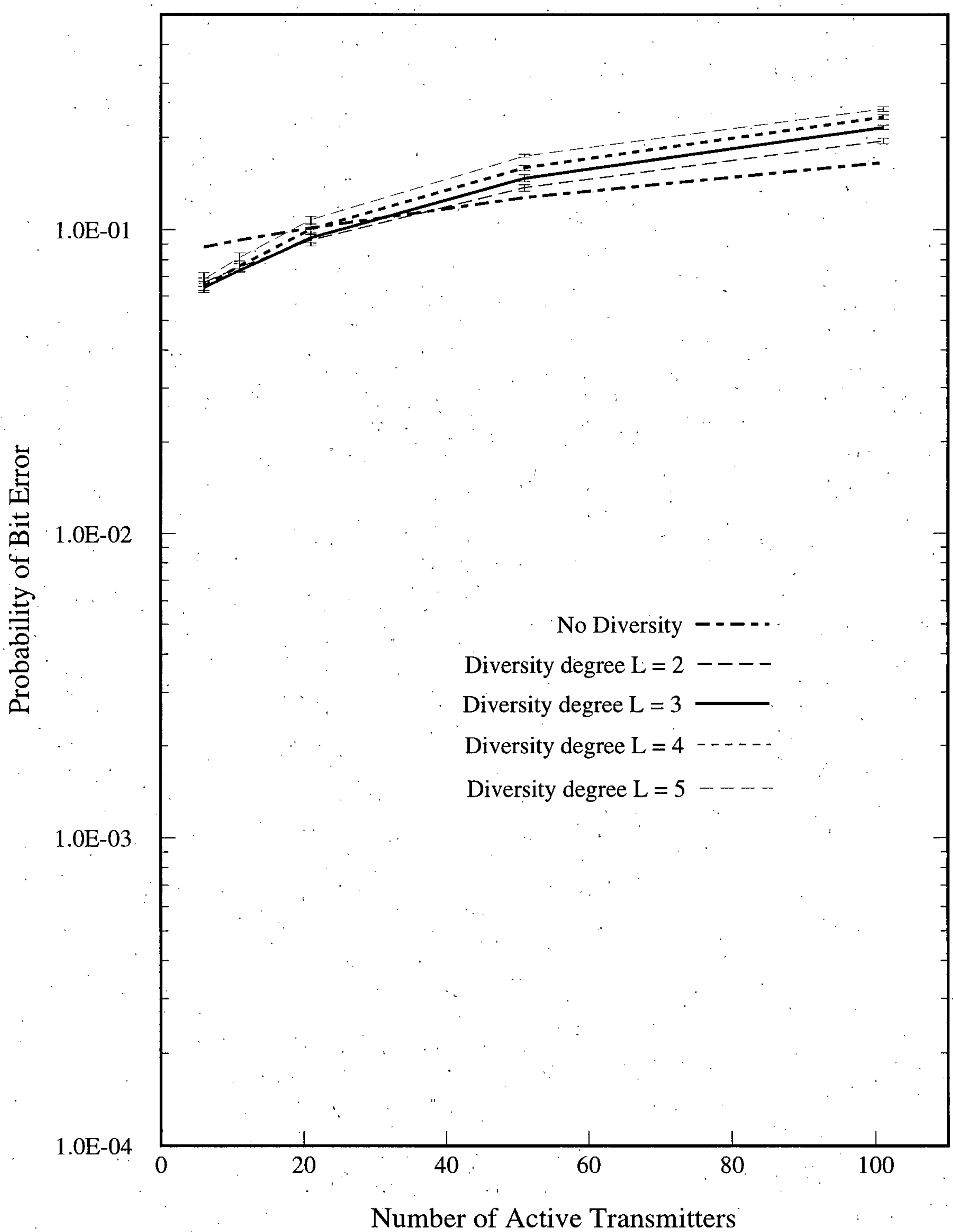

Figure 5.21 Probability of Bit Error Versus J, SNR = 10 dB for Transmitters with no Diversity and $q = 200$, equal total transmitted power for all transmitters.

Chapter 6

Conclusions

Several aspects of FH-CDMA systems have been examined in this thesis. The effects of guard times in an asynchronous hopping slotted FH-CDMA system have been studied and a code diversity FH-CDMA system where the transmitters can transmit using L frequency bins for each symbol has been proposed. A number of decoding schemes for the code diversity system have been studied.

In the asynchronous hopping systems, a method for calculating the packet error probability which takes into account the guard time has been proposed. This method does not use the independence assumption which is normally assumed in the literature. It is found that when the guard time is considered, the packet error rate can be as much as 40% lower. From the numerical results obtained, it is also found that the independence assumption is not as good when the guard time is 2 hop intervals long. System performance measures such as the maximum normalized local throughput and unconstrained maximum normalized local throughput have also been evaluated for systems with guard time using the exact expression of codeword error probability. It was found that these system performance measures are not very sensitive to the inclusion of guard time into the time slots. The reduction in packet error probability due to edge effects is offset by the reduction in the normalized code rate.

A code diversity scheme for FH-CDMA system has been proposed. In general, the code diversity scheme gives better BER than a conventional FH-CDMA system. Various code diversity decoding schemes, including two optimal receiver schemes, were studied. The exact symbol error probability expressions were derived for these schemes. The optimal diversity degree as a function of the number of active transmitters was also investigated. Instead of using repetitive coding in which the same tone is transmitted in each diversity branch, it was shown that some improvement in performance can be achieved by using a random coding scheme. We have also studied the effects of noise and fading on code diversity FH-CDMA systems. It was again found that code diversity schemes can have better performance than the conventional FH-CDMA system in most situations. It can also be seen that the code diversity scheme can be used to establish priority classes among the users in the system, with the high priority classes have a higher diversity level.

6.1 Future Work

The work described in this thesis can be extended in several directions.

- In the asynchronous hopping slotted system, it was assumed that a symbol is received in error if there exist at least one other transmitter transmitting in that frequency bin at any time during the transmission of that symbol. This model can be improved especially if we were to consider the modulation scheme aspect of the model. One way of improving this model may be to assume that the symbol is in error only if there is at least one other transmitter transmitting in that frequency bin for at least 50% of that symbol's transmission time interval. A further refinement of the model could take into account the symbol transmitted by the other transmitters. For example, if the other transmitter is transmitting the same symbol as the marked transmitters, the probability of detection of this particular symbol may increase. These are more realistic but more complex models. The derivation of the exact codeword probability can probably be derived using a more complex Markov chain model which also keeps track of the amount of transmission time overlap between the transmitters. And the transmitters may have to be partitioned into symbol groups as described in the code diversity scheme.

- For this more complex asynchronous hopping slotted system model, once the codeword error has been evaluated, the same techniques used in this thesis can be used to evaluate the maximum normalized local load and normalized local throughput. It would be interesting to see if the general conclusion is the same.
- We have assumed synchronous hopping in the code diversity schemes studied in this thesis. This work can be extended to the case of asynchronous hopping.
- The issue of error control coding in the code diversity scheme could be studied further to include codes other than the repetition and random code used in this thesis. In particular, some of the more commonly used codes such as Hamming and BCH codes can be studied. One possibility is to encode the k information bits into a codeword of n bits. These n bits can be transmitted using the code diversity scheme using blocks of $\lfloor n/L \rfloor$ transmissions.
- For the code diversity scheme over a Rayleigh fading channel, the simulation results indicate that for the range of parameters considered, a receiver structure based on the sum of the absolute values of outputs of all filters across the diversity branches is superior to one which uses the sum of the squared values. Some further work could be done to derive an analytical result which confirms this. This could also be extended to study the optimal receiver structure for the code diversity scheme over a Rayleigh fading channel.

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Appendix A

Derivation of Transition Probabilities

A.1 Transition Probabilities without Edge Effects

Here we derive the transition probabilities associated with the Markov chain in Figure 3.2. Let A be the event that **T** hops onto the same frequency bin and $\overline{A}$ be its' complementary event. Since we are using the memoryless hopping pattern model, the probability associated with these events is $1/q$ and $1 - 1/q$ for A and $\overline{A}$ respectively. Using (3.2),

$$\begin{aligned}
P_{00} = P_{20} &= \Pr(A) \times \Pr\Big(H_j^L = 0, H_j^R = 0 | A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0, H_j^R = 0 | \overline{A}, H_{j-1}^R = 0\Big) \\
&= \frac{1}{q}(1 - 1/q)^K + \Big(1 - \frac{1}{q}\Big)(1 - 2/q)^K \\
&= \frac{1}{q}\alpha + \Big(1 - \frac{1}{q}\Big)\beta
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
P_{01} = P_{21} &= \Pr(A) \times \Pr\Big(H_j^L = 0, H_j^R = 1 | A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0, H_j^R = 1 | \overline{A}, H_{j-1}^R = 0\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 0 | A, H_{j-1}^R = 0\Big) \times \\
&\quad \Pr\Big(H_j^R = 1 | A, H_{j-1}^R = 0, H_j^L = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0 | \overline{A}, H_{j-1}^R = 0\Big) \times \\
&\quad \Pr\Big(H_j^R = 1 | \overline{A}, H_{j-1}^R = 0, H_j^L = 0\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 0 | A, H_{j-1}^R = 0\Big) \times \\
&\quad \Pr\Big(H_j^R = 1 | A, H_{j-1}^R = 0, H_j^L = 0\Big) +
\end{aligned}$$

$$
\begin{aligned}
&\Pr(\overline{A}) \times \Pr\Big(H_j^L = 0, H_{j-1}^R = 0|\overline{A}\Big)/\Pr\Big(H_{j-1}^R = 0|\overline{A}\Big)\times \\
&\quad \Pr\Big(H_j^R = 1|\overline{A}, H_{j-1}^R = 0, H_j^L = 0\Big) \\
&= \frac{1}{q}\Big(1-(1-1/q)^K\Big) + \Big(1-\frac{1}{q}\Big)\frac{(1-2/q)^K}{(1-1/q)^K}\Big(1-(1-1/q)^K\Big) \\
&= \frac{1}{q}(1-\alpha) + \Big(1-\frac{1}{q}\Big)\beta\frac{(1-\alpha)}{\alpha} \\
&= \left[\frac{1}{q}\alpha + \Big(1-\frac{1}{q}\Big)\beta\right]\frac{(1-\alpha)}{\alpha} \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
P_{02} = P_{22} &= \Pr(A) \times \Pr\Big(H_j^L = 1, H_j^R = 0|A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_j^R = 0|\overline{A}, H_{j-1}^R = 0\Big) \\
&= \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1|\overline{A}, H_{j-1}^R = 0\Big) \times \Pr\Big(H_j^R = 0|\overline{A}, H_{j-1}^R = 0, H_j^L = 1\Big) \\
&= \Big(1-\frac{1}{q}\Big)\Big(1-(q-2/q-1)^K\Big)(1-1/q)^K \\
&= \Big(1-\frac{1}{q}\Big)(\alpha-\beta) \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
P_{03} = P_{23} &= \Pr(A) \times \Pr\Big(H_j^L = 1, H_j^R = 1|A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_j^R = 1|\overline{A}, H_{j-1}^R = 0\Big) \\
&= \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1|\overline{A}, H_{j-1}^R = 0\Big) \times \Pr\Big(H_j^R = 1|\overline{A}, H_{j-1}^R = 0, H_j^L = 1\Big) \\
&= \Big(1-\frac{1}{q}\Big)\Big(1-(q-2/q-1)^K\Big)\Big(1-(1-1/q)^K\Big) \\
&= \Big(1-\frac{1}{q}\Big)\Big(1-\frac{\beta}{\alpha}\Big)(1-\alpha) \\
&= \Big(1-\frac{1}{q}\Big)(\alpha-\beta)\Big(\frac{1-\alpha}{\alpha}\Big) \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
P_{10} = P_{30} &= \Pr(A) \times \Pr\left(H_j^L = 0, H_j^R = 0 | A, H_{j-1}^R = 1\right) + \\
&\qquad \Pr(\overline{A}) \times \Pr\left(H_j^L = 0, H_j^R = 0 | \overline{A}, H_{j-1}^R = 1\right) \\
&= \Pr(\overline{A}) \times \Pr\left(H_j^L = 0 | \overline{A}, H_{j-1}^R = 1\right) \times \Pr\left(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 0\right) \\
&= \Pr(\overline{A}) \times \Pr\left(H_j^L = 0, H_{j-1}^R = 1 | \overline{A}\right) / \Pr\left(H_{j-1}^R = 1 | \overline{A}\right) \\
&\qquad \times \Pr\left(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 0\right) \\
&= \left(1 - \frac{1}{q}\right) \frac{(1 - 1/q)^K \left(1 - (q - 2/q - 1)^K\right)}{1 - (1 - 1/q)^K} (1 - 1/q)^K \\
&= \left(1 - \frac{1}{q}\right) (\alpha - \beta) \left(\frac{\alpha}{1 - \alpha}\right) \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
P_{11} = P_{31} &= \Pr(A) \times \Pr\left(H_j^L = 0, H_j^R = 1 | A, H_{j-1}^R = 1\right) + \\
&\qquad \Pr(\overline{A}) \times \Pr\left(H_j^L = 0, H_j^R = 1 | \overline{A}, H_{j-1}^R = 1\right) \\
&= \Pr(\overline{A}) \times \Pr\left(H_j^L = 0 | \overline{A}, H_{j-1}^R = 1\right) \times \Pr\left(H_j^R = 1 | \overline{A}, H_{j-1}^R = 1, H_j^L = 0\right) \\
&= \left(1 - \frac{1}{q}\right) \frac{\left(1 - (q - 2/q - 1)^K\right)(1 - 1/q)^K}{1 - (1 - 1/q)^K} \left(1 - (1 - 1/q)^K\right) \\
&= \left(1 - \frac{1}{q}\right) (\alpha - \beta) \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
P_{12} = P_{32} &= \Pr(A) \times \Pr\left(H_j^L = 1, H_j^R = 0 | A, H_{j-1}^R = 1\right) + \\
&\qquad \Pr(\overline{A}) \times \Pr\left(H_j^L = 1, H_j^R = 0 | \overline{A}, H_{j-1}^R = 1\right) \\
&= \Pr(A) \times \Pr\left(H_j^L = 1 | A, H_{j-1}^R = 1\right) \times \Pr\left(H_j^R = 0 | A, H_{j-1}^R = 1, H_j^L = 1\right) + \\
&\qquad \Pr(\overline{A}) \times \Pr\left(H_j^L = 1 | \overline{A}, H_{j-1}^R = 1\right) \times \Pr\left(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 1\right) \\
&= \Pr(A) \times \Pr\left(H_j^L = 1 | A, H_{j-1}^R = 1\right) \times \Pr\left(H_j^R = 0 | A, H_{j-1}^R = 1, H_j^L = 1\right) +
\end{aligned}
$$

$$
\begin{aligned}
&\Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_{j-1}^R = 1|\overline{A}\Big)/\Pr\Big(H_{j-1}^R = 1|\overline{A}\Big)\times \\
&\quad \Pr\Big(H_j^R = 0|\overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \frac{1}{q}(1-1/q)^K + \left(1-\frac{1}{q}\right)\frac{1-2(1-1/q)^K+(1-2/q)^K}{\Big(1-(1-1/q)^K\Big)}(1-1/q)^K \\
&= \frac{1}{q}\alpha + \left(1-\frac{1}{q}\right)\left(\frac{1-2\alpha+\beta}{1-\alpha}\right)\alpha \\
&= \left[\frac{1}{q}(1-\alpha) + \left(1-\frac{1}{q}\right)(2(1-\alpha)-(1-\beta))\right]\left(\frac{\alpha}{1-\alpha}\right)
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
P_{13} = P_{33} &= \Pr(A) \times \Pr\Big(H_j^L = 1, H_j^R = 1|A, H_{j-1}^R = 1\Big)+ \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_j^R = 1|\overline{A}, H_{j-1}^R = 1\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 1|A, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 1|A, H_{j-1}^R = 1, H_j^L = 1\Big)+ \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1|\overline{A}, H_{j-1}^R = 1\Big)\times \\
&\quad \Pr\Big(H_j^R = 1|\overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 1|A, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 1|A, H_{j-1}^R = 1, H_j^L = 1\Big)+ \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_{j-1}^R = 1|\overline{A}\Big)/\Pr\Big(H_{j-1}^R = 1|\overline{A}\Big)\times \\
&\quad \Pr\Big(H_j^R = 1|\overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \frac{1}{q}\Big(1-(1-1/q)^K\Big)+ \\
&\quad \left(1-\frac{1}{q}\right)\frac{1-2(1-1/q)^K+(1-2/q)^K}{\Big(1-(1-1/q)^K\Big)}\Big(1-(1-1/q)^K\Big) \\
&= \frac{1}{q}(1-\alpha) + \left(1-\frac{1}{q}\right)(1-2\alpha+\beta) \\
&= \left[\frac{1}{q}(1-\alpha) + \left(1-\frac{1}{q}\right)(2(1-\alpha)-(1-\beta))\right]
\end{aligned} \tag{A.8}
$$

A.2 Transition Probabilities for the Last Hop, $G_t = 1$

In **T**'s last hop interval, K_1 interferers can hit from the left only and the remaining K_2 interferers can hit from both the left and right. Using the same notation for A and $\overline{A}$ as in A.1, the transition probabilities are given by

$$\begin{aligned}
P_{00}^{(n-1)} = P_{20}^{(n-1)} &= \Pr(A) \times \Pr\Big(H_j^L = 0, H_j^R = 0 | A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0, H_j^R = 0 | \overline{A}, H_{j-1}^R = 0\Big) \\
&= \frac{1}{q}(1 - 1/q)^{K_2} + \left(1 - \frac{1}{q}\right)((q-2)/(q-1))^K (1 - 1/q)^{K_2} \\
&= \frac{1}{q}\alpha_2 + \left(1 - \frac{1}{q}\right)\frac{\beta\alpha_2}{\alpha}
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
P_{01}^{(n-1)} = P_{21}^{(n-1)} &= \Pr(A) \times \Pr\Big(H_j^L = 0, H_j^R = 1 | A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0, H_j^R = 1 | \overline{A}, H_{j-1}^R = 0\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 0 | A, H_{j-1}^R = 0\Big) \times \Pr\Big(H_j^R = 1 | A, H_{j-1}^R = 0, H_j^L = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0 | \overline{A}, H_{j-1}^R = 0\Big) \times \Pr\Big(H_j^R = 1 | \overline{A}, H_{j-1}^R = 0, H_j^L = 0\Big) \\
&= \frac{1}{q}\Big(1 - (1 - 1/q)^{K_2}\Big) + \left(1 - \frac{1}{q}\right)(1 - 2/q)^K \frac{\Big(1 - (1 - 1/q)^{K_2}\Big)}{(1 - 1/q)^K} \\
&= \frac{1}{q}(1 - \alpha_2) + \left(1 - \frac{1}{q}\right)\beta\frac{(1 - \alpha_2)}{\alpha}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
P_{02}^{(n-1)} = P_{22}^{(n-1)} &= \Pr(A) \times \Pr\Big(H_j^L = 1, H_j^R = 0 | A, H_{j-1}^R = 0\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_j^R = 0 | \overline{A}, H_{j-1}^R = 0\Big) \\
&= \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1 | \overline{A}, H_{j-1}^R = 0\Big) \times \Pr\Big(H_j^R = 0 | \overline{A}, H_{j-1}^R = 0, H_j^L = 1\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{1}{q}\right)\left(1 - (q - 2/q - 1)^{K}\right)(1 - 1/q)^{K_2} \\
&= \left(1 - \frac{1}{q}\right)\left(1 - \frac{\beta}{\alpha}\right)\alpha_2 \qquad \text{(A.11)}
\end{aligned}
$$

$$
\begin{aligned}
P_{03}^{(n-1)} = P_{23}^{(n-1)} &= \Pr(A) \times \Pr\left(H_j^L = 1, H_j^R = 1 | A, H_{j-1}^R = 0\right) + \\
&\quad \Pr(\overline{A}) \times \Pr\left(H_j^L = 1, H_j^R = 1 | \overline{A}, H_{j-1}^R = 0\right) \\
&= \Pr(\overline{A}) \times \Pr\left(H_j^L = 1 | \overline{A}, H_{j-1}^R = 0\right) \times \Pr\left(H_j^R = 1 | \overline{A}, H_{j-1}^R = 0, H_j^L = 1\right) \\
&= \left(1 - \frac{1}{q}\right)\left(1 - (q - 2/q - 1)^{K}\right)\left(1 - (1 - 1/q)^{K_2}\right) \\
&= \left(1 - \frac{1}{q}\right)\left(1 - \frac{\beta}{\alpha}\right)(1 - \alpha_2) \qquad \text{(A.12)}
\end{aligned}
$$

$$
\begin{aligned}
P_{10}^{(n-1)} = P_{30}^{(n-1)} &= \Pr(A) \times \Pr\left(H_j^L = 0, H_j^R = 0 | A, H_{j-1}^R = 1\right) + \\
&\quad \Pr(\overline{A}) \times \Pr\left(H_j^L = 0, H_j^R = 0 | \overline{A}, H_{j-1}^R = 1\right) \\
&= \Pr(\overline{A}) \times \Pr\left(H_j^L = 0 | \overline{A}, H_{j-1}^R = 1\right) \times \Pr\left(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 0\right) \\
&= \Pr(\overline{A}) \times \Pr\left(H_j^L = 0, H_{j-1}^R = 1 | \overline{A}\right) / \Pr\left(H_{j-1}^R = 1 | \overline{A}\right) \\
&\qquad \times \Pr\left(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 0\right) \\
&= \left(1 - \frac{1}{q}\right) \frac{(1 - 1/q)^{K}\left(1 - (q - 2/q - 1)^{K}\right)}{1 - \left(1 - \frac{1}{q}\right)^{K}} (1 - 1/q)^{K_2} \\
&= \left(1 - \frac{1}{q}\right)\left(\frac{\alpha - \beta}{1 - \alpha}\right)\alpha_2 \qquad \text{(A.13)}
\end{aligned}
$$

$$
P_{11}^{(n-1)} = P_{31}^{(n-1)} = \Pr(A) \times \Pr\left(H_j^L = 0, H_j^R = 1 | A, H_{j-1}^R = 1\right) +
$$

$$\begin{aligned}
&\Pr(\overline{A}) \times \Pr\Big(H_j^L = 0, H_j^R = 1 | \overline{A}, H_{j-1}^R = 1\Big) \\
&= \Pr(\overline{A}) \times \Pr\Big(H_j^L = 0 | \overline{A}, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 1 | \overline{A}, H_{j-1}^R = 1, H_j^L = 0\Big) \\
&= \left(1 - \frac{1}{q}\right) \frac{\Big(1 - (q - 2/q - 1)^K\Big)(1 - 1/q)^K}{1 - (1 - 1/q)^K} \Big(1 - (1 - 1/q)^{K_2}\Big) \\
&= \left(1 - \frac{1}{q}\right) \left(\frac{\alpha - \beta}{1 - \alpha}\right)(1 - \alpha_2) \qquad \text{(A.14)}
\end{aligned}$$

$$\begin{aligned}
P_{12}^{(n-1)} = P_{32}^{(n-1)} &= \Pr(A) \times \Pr\Big(H_j^L = 1, H_j^R = 0 | A, H_{j-1}^R = 1\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_j^R = 0 | \overline{A}, H_{j-1}^R = 1\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 1 | A, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 0 | A, H_{j-1}^R = 1, H_j^L = 1\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1 | \overline{A}, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 1 | A, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 0 | A, H_{j-1}^R = 1, H_j^L = 1\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_{j-1}^R = 1 | \overline{A}\Big) / \Pr\Big(H_{j-1}^R = 1 | \overline{A}\Big) \times \\
&\quad \Pr\Big(H_j^R = 0 | \overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \frac{1}{q}(1 - 1/q)^{K_2} + \left(1 - \frac{1}{q}\right) \frac{1 - 2(1 - 1/q)^K + (1 - 2/q)^K}{\Big(1 - (1 - 1/q)^K\Big)} (1 - 1/q)^{K_2} \\
&= \frac{1}{q}\alpha_2 + \left(1 - \frac{1}{q}\right)\left(\frac{1 - 2\alpha + \beta}{1 - \alpha}\right)\alpha_2 \\
&= \frac{1}{q}\alpha_2 + \left(1 - \frac{1}{q}\right)[2(1 - \alpha) - (1 - \beta)] \frac{\alpha_2}{(1 - \alpha)} \qquad \text{(A.15)}
\end{aligned}$$

$$\begin{aligned}
P_{13}^{(n-1)} = P_{33}^{(n-1)} &= \Pr(A) \times \Pr\Big(H_j^L = 1, H_j^R = 1 | A, H_{j-1}^R = 1\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_j^R = 1 | \overline{A}, H_{j-1}^R = 1\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 1 | A, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 1 | A, H_{j-1}^R = 1, H_j^L = 1\Big) +
\end{aligned}$$

$$
\begin{aligned}
&\Pr(\overline{A}) \times \Pr\Big(H_j^L = 1|\overline{A}, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 1|\overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \Pr(A) \times \Pr\Big(H_j^L = 1|A, H_{j-1}^R = 1\Big) \times \Pr\Big(H_j^R = 1|A, H_{j-1}^R = 1, H_j^L = 1\Big) + \\
&\quad \Pr(\overline{A}) \times \Pr\Big(H_j^L = 1, H_{j-1}^R = 1|\overline{A}\Big)/\Pr\Big(H_{j-1}^R = 1|\overline{A}\Big) \times \\
&\quad \Pr\Big(H_j^R = 1|\overline{A}, H_{j-1}^R = 1, H_j^L = 1\Big) \\
&= \frac{1}{q}\Big(1 - (1 - 1/q)^{K_2}\Big) + \\
&\qquad \left(1 - \frac{1}{q}\right) \frac{1 - 2(1 - 1/q)^K + (1 - 2/q)^K}{\Big(1 - (1 - 1/q)^K\Big)} \Big(1 - (1 - 1/q)^{K_2}\Big) \\
&= \frac{1}{q}(1 - \alpha_2) + \left(1 - \frac{1}{q}\right)(1 - 2\alpha + \beta)\frac{(1 - \alpha_2)}{(1 - \alpha)} \\
&= \frac{1}{q}(1 - \alpha_2) + \left(1 - \frac{1}{q}\right)[2(1 - \alpha) - (1 - \beta)]\frac{(1 - \alpha_2)}{(1 - \alpha)}.
\end{aligned}
\tag{A.16}
$$

Appendix B
Envelope of Resultant Signal is Rayleigh Distributed

In this part of the appendix, it is shown that if the envelope of the individual signal is Rayleigh distributed and the phase is uniformly distributed over $(0,\ 2\pi]$, the envelope of the resultant signal obtained by the vector addition of these individual signals is also Rayleigh distributed.

Let n be the number of i.i.d Rayleigh distributed individual signals. Furthermore let S_{pi} and S_{qi} be the in-phase and quadrature components of $\vec{S}_i$, the i^{th}, $i \in \{1, ..., n\}$ signal, respectively. Since $\vec{S}_i$, is Rayleigh distributed, S_{pi} and S_{qi} are independent Gaussian random variables. We shall denoted the mean and variance of these Gaussian random variable by μ and σ^2 respectively. The in-phase component of the resultant vector, S_{pr}, is obtained by adding the in-phase component of each individual signal and is given by

$$S_{pr} = \sum_{i=1}^{n} S_{pi}. \tag{B.1}$$

The characteristic function of the in-phase component is

$$\phi_{S_{pi}}(\omega) = e^{j\omega\mu - \omega^2\sigma^2/2}. \tag{B.2}$$

Since S_{pr} is the sum of i.i.d random variables, the characteristic function of S_{pr} can be written as

$$\begin{aligned} \phi_{S_{pr}}(\omega) &= \prod_{i=1}^{n} \phi_{Spi}(\omega) \\ &= \prod_{i=1}^{n} e^{j\omega\mu - \omega^2\sigma^2/2} \\ &= e^{j\omega(n\mu) - \omega^2(n\sigma^2)/2} \end{aligned} \tag{B.3}$$

which is the characteristic function of a Gaussian random variable with mean $n\mu$ and variance $n\sigma^2$. This shows that in-phase component of the resultant signal is a Gaussian random variable.

Since the phase of each individual signal is uniformly distributed over $(0,\ 2\pi]$, by symmetry the quadrature component of the of the resultant vector is Gaussian distribution identical to the in-phase component. Hence the envelope of the resultant signal must be Rayleigh distributed.

Appendix C
Majority Vote Decoding without Detection Threshold, $J = 2$

We shall show here that in the majority vote without detection threshold scheme, described in Section 5.2, the BER is lower for the case when there is no diversity than the case where $L = 2$ for the case of $J = 2$. We shall assume that the marked transmitter, **T**,, transmitted the symbol "0" in both cases and determine the probability that the symbol decoded is "1". By symmetry, the unconditional BER is same as this probability.

Consider first the case where the diversity degree is two. Let $P_{n_{01}n_{11}n_{02}n_{12}}$ be the joint probability distribution of the number of hits in the various tones in **T**'s selected frequency bins. For example, P_{0101} is the probability that there is none and one transmitter transmitting symbol "0" and "1" respectively in both frequency bins. For $q = 200$, the non-zero probabilities of such distribution of hits is as follow

$$P_{1010} = \frac{\binom{198}{2}}{\binom{200}{2}} = \frac{19503}{19900}$$

$$P_{1110} = P_{1011} = P_{2010} = P_{1020} = \frac{1}{2}.\frac{\binom{2}{1}\binom{198}{1}}{\binom{200}{2}} = \frac{99}{9950}$$

$$P_{1111} = P_{2020} = \frac{1}{2}.\frac{1}{\binom{200}{2}} = \frac{1}{39800}. \tag{C.1}$$

It is convenient to denote

$$\phi_1 = \left(\frac{\sigma_n^2}{\sigma^2 + 2\sigma_n^2}\right)$$
$$\phi_2 = \left(\frac{\sigma^2 + \sigma_n^2}{\sigma^2 + 2\sigma_n^2}\right)$$

$$\phi_3 = \left(\frac{\sigma_n^2}{2\sigma^2 + 2\sigma_n^2}\right)$$
$$\phi_4 = \left(\frac{2\sigma^2 + \sigma_n^2}{2\sigma^2 + 2\sigma_n^2}\right)$$
$$\phi_5 = \left(\frac{\sigma^2 + \sigma_n^2}{2\sigma^2 + 2\sigma_n^2}\right). \tag{C.2}$$

These equations were derived using (5.15) and (5.16) and they are the conditional probabilities of decoding a particular symbol at frequency bin given the number of hits on the tones in that frequency bin. For example, ϕ_1 is the conditional probability of decoding symbol "1" at a particular frequency bin given the number of hits on symbol "0" and "1" is one and none respectively. (Note that $\phi_5 = 1/2$, this is the probability of decoding either symbol given that there is an equal number of hits on both symbols.) Hence the probability of error can be written as

$$\begin{aligned} P_e(J = 2, q = 200, L = 2) = {} & P_{1010}(\phi_1^2 + \phi_1\phi_2) + (P_{2010} + P_{1020})\left(\phi_1\phi_3 + \frac{1}{2}\phi_2\phi_3 + \frac{1}{2}\phi_1\phi_4\right) \\ & + (P_{1110} + P_{1011})\left(\phi_1\phi_5 + \frac{1}{2}\phi_2\phi_5 + \frac{1}{2}\phi_1\phi_5\right) + P_{1111}(2\phi_5^2) \\ & + P_{2020}\left(\phi_3^2 + \frac{1}{2}\phi_3\phi_4\right). \end{aligned} \tag{C.3}$$

After simplification

$$P_e(J = 2, q = 200, L = 2) = \frac{400\sigma_n^4 + 400\sigma^2\sigma_n^2 + \sigma^4}{400(\sigma^2 + \sigma_n^2)(\sigma^2 + 2\sigma_n^2)}. \tag{C.4}$$

Using a similar approach for $L = 1$, the probability of bit error can be written as

$$P_e(J = 2, q = 200, L = 1) = \frac{800\sigma_n^4 + 800\sigma^2\sigma_n^2 + \sigma^4}{800(\sigma^2 + \sigma_n^2)(\sigma^2 + 2\sigma_n^2)}. \tag{C.5}$$

Since σ^2 and σ_n^2 are positive values, it is clear from (C.4) and (C.5) that $P_e(J = 2, q = 200, L = 1) < P_e(J = 2, q = 200, L = 2)$. Note that when noise is negligible, $P_e(J = 2, q = 200, L = 2) \approx 2P_e(J = 2, q = 200, L = 1)$.