A NON LINEAR INCREMENTAL
FINITE ELEMENT PROGRAM FOR THE ANALYSIS OF
SHAFTS AND TUNNELS IN OILSANDS
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ABSTRACT

A method for analysing the deformation behaviour of oilsand adjacent to shafts and tunnels is presented.

Oilsand is comprised of a dense sand matrix with its pore spaces filled with bitumen, water and free or dissolved gases. The Engineering behaviour of oilsand is governed by the stresses in the sand matrix. The bitumen does not contribute directly to the strength of the sand. However, indirectly the presence of bitumen may greatly affect its behaviour. This is because the presence of bitumen reduces the effective permeability of the oilsand and very often undrained conditions occur. Then the pressure of the pore gases remain high reducing the effective stresses for unloading conditions.

A nonlinear incremental finite element model is used to analyse the oilsand skeleton behaviour. Dilation or shear induced volume change is an important characteristic of a dense sand and this is included in the analysis using a modified form of Rowe's stress dilatancy theory.

The unloading condition at the face of a tunnel or shaft can lead to a violation of the failure criterion and this condition is rectified by a stress redistribution technique.

The compressibilities of the oil and water phases are neglected in comparison with that of the gas phase and pore pressure changes are predicted by the ideal gas laws. Under undrained conditions the pore pressure is coupled into the skeleton stresses by maintaining volumetric strain compatibility between the skeleton and pore fluid phases.

The results have been checked with drained and undrained closed form solutions.

The solution for the unloading of a tunnel in oilsand is presented and it shows that the limiting support pressures can be reduced by venting elements to a reasonable distance from the tunnel. It is also found that the effects of shear dilation are significant only when the limiting support pressure is approached.
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CHAPTER 1

INTRODUCTION

1.1 Background

Most of the methods that have been established for the recovery of heavy oil from deep lying oilsand deposits, involve construction of shafts and tunnels in oilsands itself. Thus the design of a suitable support system for them turns out to be a chief concern in recovery processes.

In order for the supports to be designed, the limit state loads to which they will be subjected, will have to be estimated. In the conventional design of support systems reasonable estimates of loads on tunnel support are made by reviewing previous tunnelling experience in materials with similar properties.

Oilsand is comprised of a dense sand matrix with its pore spaces filled with bitumen, water and free or dissolved gases. Unloading associated with the excavations for tunnels and shafts in oilsand, causes these gases to evolve. The presence of bitumen reduces the effective permeability of the oilsand and very often undrained conditions occur. Then the effective stresses will drop leading to instability at the face of the excavation. Thus oilsand exhibits unusual behaviour. Moreover, tunnelling experience in oilsand, is also scarce.

Therefore the empirical design approach may not be reliable in this particular situation.

In view of this, an analytical method for computing stresses and deformations in oilsands adjacent to tunnel openings is invaluable. As the pore gases play a dominant role in the behaviour of oilsands, such a method should contain a suitable pore pressure model.

Once the convergence curve (the inward movement of the tunnel face plotted against the support pressure) is obtained, the convergence-confinement method as described in Smith and Byrne (1980) could be used to design the confinement system.

Harris and Sobkowicz (1977) presented a method of predicting the gas porosity changes in oilsands, due to the changes in total stress or temperature, based on ideal gas laws. The sand skeleton was considered to be linear elastic, and by maintaining the volumetric compatibility between the pore fluid and the skeleton, pore pressure changes were extracted.

In 1979, Thurber consultants, in collaboration with Byrne extended this to a more realistic case of, a non linear elastic sand skeleton. Byrne and Grigg (1980) further modified the model by introducing plastic volumetric strains, (due to shear dilation), into the dense sand skeleton. In their finite element
formulation, an iterative process is used to achieve volumetric strain compatibility. The constitutive relations used, are of the equivalent elastic type, where the moduli are computed from the secants of the non-linear stress-strain curves. In the region where the soil strength is fully mobilized, which very often is the case of tunnel openings, this solution is stable. The load is applied in a single step and the total load equilibrium is satisfied.

A solution method, whereby equilibrium is satisfied for each load increment up to the final load, using moduli obtained from the tangents of the non-linear stress-strain curves at any stage, is preferred to the above one. The simulation of the construction procedure (which essentially is a step by step excavation), and the possibility of incorporating the influence of the support system, are considered to be advantages of this incremental technique.

1.2 Purpose and Scope

The purpose of this thesis is to develop a nonlinear incremental finite element method for the analysis of the deformation behaviour of oil sand in shafts and tunnels.

The engineering behaviour of oil sand is governed by the stresses in the sand skeleton. This is to be modelled by an effective stress model which incorporates the dilatant properties of the dense oil sand matrix.

The Non Linear Soil Structure Interaction Program (NLSSIP) developed by Byrne and Duncan (1979), adopts an incremental technique, and is capable of analysing static soil-structure interaction problems such as flexible retaining walls and tunnel linings. It is possible to achieve the above purpose by introducing the following modifications to the NLSSIP.

The constitutive parameters used are inadequate to cope up with the mobilization of the full soil strength, and the yield criterion (Mohr-Coulomb) is violated by the results under the unloading conditions. Furthermore, the volumetric strains due to shear dilation are not considered even in the case of a dense sand.

First phase of the thesis is to overcome the violation of the yield criterion as mentioned above, by employing a stress redistribution procedure, and to improve the volumetric strains of the sand skeleton by adding a shear dilation component, when sands with dilatant properties are analysed. Rowe's stress - dilatancy theory is used to assess the appropriate dilation and this is introduced into the finite element formulation by using a temperature analogy.

This version of the NLSSIP program, could be utilized in the analysis of any plane strain problem of a sand, exhibiting dilatant behaviour.
Secondly the NLSSIP will be modified to incorporate the gas laws representing the behaviour of oilsand porefluid. This would enable the program to predict pore pressure changes in each load step under undrained conditions. The final solution is obtained by maintaining the volumetric strain compatibility between the sand skeleton and the pore fluid.

In essence, the work presented herein consists of modification of an incremental effective stress model, and a method of coupling it with a pore pressure model for soils containing free or dissolved gases in their pore fluids.
Fig. 1 Possible Intersitial Conditions of Oilsand. (After Dusseault, 1979)
CHAPTER 2

OILSAND BEHAVIOUR

Before attempting to predict the deformation behaviour of oilsand, on unloading, it is important to visualize the structure of oilsand that leads to its characteristic behaviour.

Generally oilsands are very dense sands with high strength and low compressibility. The interstitial spaces may contain any combination of fluids of the following. (Fig. 1).

a. Water only
b. Water + oil (bitumen)
c. Water + free gas
d. Water + oil + free gas

Another important property of the bitumen rich oilsand is its very low effective permeability.

With this picture in mind it is easy to perceive the oilsand behaviour on undrained unloading.

In the field the unusual properties of oilsand are manifested in a number of ways. Swelling ranging between 5-15% of the original volume has been observed, when oilsand samples are brought to the surface from deposits and left in an unconfined state. The properties such as the irrecoverable reduction in strength and density associated with such swelling have also been noticed.

Unloading associated with the excavations for tunnels and shafts in oilsand, causes the dissolved gases to evolve and if the rate of loading is high enough, undrained conditions will result due to the low permeability of the pore fluids. As explained below the pore pressure will then remain high allowing the effective stresses to drop. This leads to instability at the face of the excavation.

Surficial slabbing of oilsands is another consequence of the instability reached under undrained unloading. This happens to be a very natural phenomenon in the excavations for open pit mines, shafts and tunnels in deep oilsand deposits. At some stage the slabs break off and fall.

On the other hand, natural unloading of oilsand deposits, such as the erosion of oilsands that take place over years, leave competent sandstone with no or little visual slabbing. This proves how the prolonged unloading, facilitates the gas drainage, thereby preventing the undrained conditions.

When the total all round stress acting on an oilsand specimen with a pore state defined by Fig. 1d, is reduced the
Fig. 2 Unloading of Oilsand
(After Harnis and Sobokowicz, 1977)
following things would happen.

1. The overall volume will increase, essentially due to an increase in the pore fluid volume.

2. This expansion will pull apart the sand grains thereby reducing the effective stress, which is propagated only through the grain contacts. Consequently the density of the sand matrix as well as its strength is reduced.

3. The increase in pore volume will be accompanied by a decrease in pore pressure, which in turn will cause evolution of the dissolved gases, and further expansion of the free gases.

4. The lower pore pressure and the effective stresses as well as the larger volumes, will result in a higher pore fluid and skeleton compressibility, as opposed to its relatively low compressibility in the initial state.

All these will lead to an unstable structure on unloading, as observed in the field.

Yet there are limitations to the above phenomena, depending on the location of the sample and the duration of unloading. For example, in shallow deposits, the pore pressure are not high enough for the existence of pressure dissolved gases.

Furthermore, despite the low permeability, if ample time is allowed on unloading for the evolved gases to vent, the undrained conditions cannot be maintained.

Therefore the situation will be most critical, if deep lying oilsand deposits, which are rich in bitumen (with very low permeability), and pressure dissolved gases, are unloaded in a short duration.

Fig. 2 shows how the structure is deformed, and venting paths are formed on unloading of oilsands.

In the ensuing analysis the oilsand skeleton and the pore fluid are treated independently.

A nonlinear incremental effective stress model described in the following Chapter is used to represent the skeleton behaviour.

The above described pore fluid behaviour is represented by a pore pressure model based on ideal gas laws, which is presented in Chapter 5. Under undrained conditions the skeleton and the pore fluid are finally coupled by the volumetric strain compatibility condition.
CHAPTER 3

EFFECTIVE STRESS MODEL

The oilsand skeleton is treated as a continuum, in the effective stress model. Therefore continuum mechanics is applied in the stress strain analysis.

3.1 Stresses

Terzaghi defined the effective stress in the following manner.

\[ \sigma' = \sigma - u \]

where \( \sigma' \) - effective normal stress on any plane
\( \sigma \) - total normal stress on that plane
\( u \) - pore pressure

Recently Skempton(1957), introduced 3 theories on how, effective stress depends on the grain contacts and the internal friction, in saturated porous media. But under most practical situations Terzaghi's theory was validated.

Further since water carries no shear stresses:

\[ \tau' = \tau \]

\( \tau' \) and \( \tau \) being the effective and the total shear stresses on the plane.

3.2 Deformations and Strains

If \( u, v, \) and \( w \) are the displacements at a point in a continuum, along the \( x, y, \) and \( z \) directions, in a cartesian coordinate system, the six strains can be defined as,

\[ \varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \varepsilon_z = \frac{\partial \omega}{\partial z} \]

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y} \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x} \]

where only the linear terms of the displacements are concerned.

3.3 Constitutive Relations

Various forms of elastic and plastic constitutive relations are used in Soil mechanics. Most soils exhibit elastic behaviour under low stress levels. The simplest elastic model is the linear elastic one.
\( \sigma_1 - \sigma_3 \quad (= \sigma_d) \)

![Nonlinear Elastic Curve](image)

Fig. 3 Nonlinear Elastic Curve

\( \sigma_1 - \sigma_3 \)

\( (\sigma_1 - \sigma_3)_{ult} \)

![Hyperbolic Curve](image)

Fig. 4 Hyperbolic Curve
3.3.1 Linear Elasticity

Linear elastic constitutive relations would be in the following form.

\[ \sigma = [D] \varepsilon \]  \hspace{1cm} (4)

where

- \( \sigma \) - stress vector.
- \( \varepsilon \) - strain vector.
- \([D]\) - constitutive matrix with 36 coefficients.

By limiting our interest to isotropic materials and assuming symmetry of \([D]\), we can reduce these 36 material constants to any two of the following defined.

Young's modulus \( (E) \) = \( \frac{\sigma_x}{\varepsilon_x} \)  \hspace{1cm} (5)

Poisson's ratio \( (\mu) \) = \( -\frac{\varepsilon_x}{\varepsilon_y} \)  \hspace{1cm} (6)

Bulk modulus \( (B) \) = \( \frac{\sigma_m}{\varepsilon_y} \)  \hspace{1cm} (7)

where

\[ \sigma_m = \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \] - mean normal stress  \hspace{1cm} (8)

\[ \varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z \] - volumetric strain.  \hspace{1cm} (9)

Shear modulus \( (G) \) = \( \frac{\tau_{xy}}{\gamma_{xy}} \)  \hspace{1cm} (10)

Unlike for most common materials, very often the behaviour of soil cannot be represented by linear elastic models. The stress-strain parameters are highly dependent on the magnitudes of the stresses in the soil.

For the prediction of realistic results, the constitutive relations used in the soil deformation analysis should be reasonable for that soil. The non-linear elastic approach encompass two significant characteristics of the stress-strain behaviour of soils, namely, non-linearity and stress dependency.

In this analysis, nonlinear elastic constitutive relations are used. Further the plastic volume changes are also incorporated by using a shear volume coupling concept.

3.3.2 Nonlinear Elasticity

The use of hyperbolic stress-strain curves (Fig. 3) is common in nonlinear elasticity. Two distinct models can be identified depending on the method of tracing these curves.
3.3.2.1 Equivalent Elastic or Secant Method

The moduli are defined by the initial and the anticipated final state.

\[ E_s = \frac{\sigma_d}{\varepsilon_i} \]  \hspace{1cm} (11)

This method works well when the failure is approached, where the curve is flatter.

3.3.2.2 Incremental or Tangent Method

The curve is more closely traced, since loading is done in a series of increments, the moduli for each one is defined by the tangent to the curve, at the average stress state.

This method resembles the usual construction sequence and as such is advantageous over the previous one, although it may be a little unstable in the failure region, which aspect is discussed in detail later.

This analysis involves an incremental approach and from here onwards it will be discussed in more detail.

Incremental Young's modulus \( (E_t) = \frac{d\sigma_i}{d\varepsilon_i} \)  \hspace{1cm} (12)

Incremental Bulk modulus \( (B_t) = \frac{d\sigma_m}{d\varepsilon_v} \)  \hspace{1cm} (13)

where \( \sigma_i \) and \( \sigma_m \) - axial and lateral principal stresses.

\( \varepsilon_i, \varepsilon_v \) - axial strain (in triaxial tests).

\( \sigma_m, \varepsilon_v \) - as defined in Chapter 3.3.1

Using linear elastic principles for each increment, it is shown in Appendix A.1 that the

Incremental Poisson's ratio \( (\mu_t) = \frac{1}{2} \left( 1 - \frac{E_t}{3B_t} \right) \)  \hspace{1cm} (14)

3.3.3 Hyperbolic Relation

Kondner and Zelasko (1963), have shown that the stress-strain curves for many soils can be approximated by hyperbolas of the form,

\[ (\sigma_i - \sigma_3) = \frac{\varepsilon}{E_i^t} + \frac{\varepsilon}{(\sigma_i - \sigma_3)_{ult}} \]  \hspace{1cm} (15)
One advantage of using this, is that the equation parameters, $E_i$ and $(\sigma_1-\sigma_3)_{ult}$ have physical significance, being the initial tangent modulus and the asymptotic value of deviator stress, respectively, as shown in Fig. 4.

It should be noted here that, in the triaxial test, since $\sigma_3$ is usually kept constant, $E_i$ (defined by $d\sigma_d/d\varepsilon$) becomes a tangent Young's modulus. ($\sigma_d$ is defined by $d\sigma_d/d\varepsilon$).

### 3.3.4 Dependency of $E_i$ on confining pressure

It has been observed that for soils (except for those tested under unconsolidated undrained conditions), an increase in confining pressure will result in a steeper stress-strain curve and a higher ultimate strength. Consequently $E_i$ and $(\sigma_1-\sigma_3)_{ult}$ should increase with increasing confining pressure.

Empirical formulae have been presented to represent this variation. Janbu presented the following equation for $E_i$:

$$E_i = K_E P_a \left(\frac{\sigma_3}{P_a}\right)^n$$  \hspace{1cm} (16)

where the dimensionless parameters, $K_E$ and $n$ are referred to as the modulus number and the modulus exponent respectively. $P_a$ - the atmospheric pressure has been introduced to make the equation non-dimensional. Units of $\sigma_3$, $P_a$ and $E_i$ should be consistent.

### 3.3.5 Failure Ratio

In order to determine $(\sigma_1-\sigma_3)_{ult}$, it is empirically related to $(\sigma_1-\sigma_3)_f$ (the shear strength), by the failure ratio as follows:

$$(\sigma_1-\sigma_3)_{ult} = \frac{(\sigma_1-\sigma_3)_f}{R_f}$$  \hspace{1cm} (17)

$R_f$ - Failure ratio

Since $(\sigma_1-\sigma_3)_f$ is always smaller than $(\sigma_1-\sigma_3)_{ult}$, $R_f$ is less than unity, and varies from 0.5 to 0.9 for most soils.

### 3.3.6 Yield Criterion

The yield criterion will determine the maximum shear strength that could be developed by the soil. Herein, the Coulomb criterion is used.

$$\tau_f = c + \sigma' tan \phi$$  \hspace{1cm} (18)
Fig. 5 Variation of $\phi$ with $\sigma_3$. 
Fig. 6. Unloading-reloading Curve
\( T_f \) - strength developed on the failure plane
\( \sigma' \) - effective normal stress on the failure plane.
\( \phi \) - maximum apparent angle of friction
\( c \) - apparent cohesion.

Experimental results have shown that the value of \( \phi \) decreases considerably with the increase in confining pressure, as shown in Fig. 5.

This is accommodated in the analysis by using the following equation.

\[
\phi = \phi_1 - \Delta \phi \log \left( \frac{\sigma_3}{P_a} \right)
\]  

\( \phi, \phi_1 \) - Friction angles at conf. pressures of \( \sigma_3 \) and \( P_a \) resp.

\( \Delta \phi \) - Reduction in \( \phi \), for a tenfold increase in \( \sigma_3 \).

3.3.7 Relation between \( E_t \) and stresses

The tangent modulus can be obtained by differentiating Eqn. 15 w.r.t. \( \varepsilon \), and by the substitution of \( E_t \) and \( (\sigma_1-\sigma_3)_{ult} \), from Eqn. 16 and 17, together with the yield criterion (represented by Eqn. 18).

The expression, as derived in the Appendix A.2 is as follows,

\[
E_t = K_E \, P_a \left( \frac{\sigma_3}{P_a} \right)^n \left[ 1 - \frac{R_f(1-\sin \phi)(\sigma_1-\sigma_3)}{2c \cos \phi + 2\sigma_3 \sin \phi} \right]^2
\]  

If the soil parameters given by \( K_E, n, c, \phi, \) and \( R_f \) are known, this expression can be used to obtain the instantaneous elastic modulus, at any deviator stress and any confining pressure.

3.3.8 Other Non Linear Relations

3.3.8.1 Unloading Reloading Modulus

Duncan, Byrne, Wong and Mabry (1978) describe a way of representing the inelastic behaviour of soil by using different moduli values for loading and unloading.

If a sample is unloaded during the progress of a triaxial test, and reloaded the stress-strain curve in Fig. 6 is obtained.
Fig. 7 Volume Change Curves
The unloading (and reloading) curve is steeper than the primary loading curve thereby leaving an irrecoverable strain ($\varepsilon_p$) on unloading. By assuming that unloading and reloading is linear elastic and that there are no hysteresis effects, they present the following expression for $E_{ur}$.

$$E_{ur} = K_{ur} P_a \left( \frac{\sigma_3}{P_a} \right)^n$$  \hspace{1cm} (21)

From Fig. 6.

$$\varepsilon_r = \frac{(\sigma_1 - \sigma_3)}{E_{ur}}$$  \hspace{1cm} (22)

$$\varepsilon_p = \varepsilon - \varepsilon_r$$  \hspace{1cm} (23)

$\varepsilon_r =$ elastic (or recoverable) axial strain.

$\varepsilon_p =$ plastic (or irrecoverable) axial strain.

It has been found out that the ratio $K_{ur}/K$ decreases with the increasing soil density. Therefore $E_{ur}/E$ ratio will be smaller for the dense sands. This implies that at very low stress levels, $\varepsilon_r/\varepsilon_p$ ratio will be high for dense sands.

3.3.8.2 Elastic Volume change

The volume change characteristics obtained when triaxial compression tests are performed under different confining pressures are shown in Fig. 7.

It is seen that the volumetric compression is highly dependent on $\sigma_3$. It can be assumed that the elastic volume changes are independent of the deviator stress, and depend only on $\sigma_3$.

Thus, in non-linear elasticity, the bulk modulus is defined as,

$$B = K_B P_a \left( \frac{\sigma_3}{P_a} \right)^m$$  \hspace{1cm} (24)

where $K_B$ = Bulk modulus number.

$m$ = Bulk modulus exponent.

$P_a$ = Atmospheric pressure.

$K_B$ and $m$, just as $K_\varepsilon$ and $n$, are dimensionless.

Duncan et al (1978), say that usually $m$ ranges between 0.0 and 1.0, except in clays compacted to dry of optimum, tested under undrained conditions, whose bulk modulus decrease with the
Fig. 8 Stress Strain Curves

Fig. 9 Violation of the Failure Criterion
increase in $\sigma_3$.

### 3.3.9 Constitutive relations in the failure region.

Once failure is reached, the shearing will be entirely governed by the Mohr-Coulomb failure criterion.

In the failure region

$$\sigma_d = \frac{2c\cos\phi + 2\sigma_3 \sin\phi}{(1 - \sin\phi)}$$

(25)

The volumetric strains are still predicted by Eqns. 24.

The NLSSIP uses the parameters obtained from above equations (i.e. $E_t$, $B_t$ etc), in forming its constitutive matrix as discussed in Chapter 4. In the failure region, inspection of Eqn 25 will reveal that $E_t$, (which is $\frac{d\sigma_d}{dE}$), will be zero, whereas $B_t$ will have a finite value, from Eqn. 24.

$$\mu_t = \frac{1}{2} \left(1 - \frac{E_t}{3B_t}\right)$$

(26)

From this it follows that,

$$\mu_f = 0.5$$

Thus the tangential shear modulus ($G_t = \frac{E_t}{2(1+\mu_t)}$ from Appendix D), will also become zero at failure. (since $E_t = 0$).

If this condition is used as it is, Eqn. 67 will produce a singular constitutive matrix. This could be prevented by avoiding the limiting value of $\mu_t$ of 0.5 and use 0.495 instead. Then if the $B$ value from Eqn. 24 is believed in the failure region, $E_t$ will get a very low value compared to $B_t$, from Eqn. A3;

i.e $E_t = 0.03 B_t$

and $G_t = 0.01 B_t$

Fig. 8 shows the final stress-strain curve used in the present program.

The inaccuracy of these unstable conditions is reflected by the violation of the failure criterion, by the results obtained in problems associated with unloading of a soil. The problem of unloading of a tunnel under plane strain conditions, which will be analysed later is a good example. Initially the program predicted a stress path travelling above the Mohr-Coulomb failure line (along QR), when the mean normal stress of the soil adjacent to the cavity is reduced due to unloading, as shown in Fig. 9, whereas it should have travelled along QS. Excess shear
strains were predicted for a particular mean normal stress. This shear, in excess of the shear strength was distributed to the surrounding elements by applying an equivalent set of nodal forces. The technique used will be discussed along with the finite element formulation.

3.4 Comment on the Non linear Elastic parameters

The parameters have been found useful for many reasons.

1. They can be determined from triaxial compression test results using simple plotting techniques.

2. They could be applied to effective stress (the parameters being obtained from drained tests), and total stress (by using unconsolidated undrained test results), analyses.

3. Since data is available for a wide variety of soils, approximate values for a problem in hand are easily accessible, which is of immense help under difficult testing situations.

On the other hand, the volume changes (or deformations) given by these relations are incomplete as the plastic volume changes or the shear induced dilation has not been introduced. In this analysis they have been considered along the guidelines of Rowe's dilatancy theory.

3.5 Determination of Non Linear Parameters

Determination of $K_E$, $n$, $\phi$, $\Delta \phi$, and $R_f$ is described in detail in Duncan et al (1978). Since the volume changes are modified, herein a new method of evaluating $K_E$ and $m$ is suggested.

3.5.1 Evaluation of $K_E$ and $m$

The method adopted by Duncan et al (1978) could be outlined as follows.

Using the volume change curves shown in Fig.7, the Bulk modulus for each $\sigma_3$ is calculated.

\[
B = \frac{d\sigma_m}{d\Delta \nu} \quad \text{(27)}
\]

we get
\[
B = \frac{\Delta (\sigma_1 + \sigma_2 + \sigma_3)}{3 \Delta \nu} \quad \text{(28)}
\]

For triaxial tests with constant confining pressure,
\[
B = \frac{\Delta \sigma_1}{3 \Delta \nu} \quad \text{(29)}
\]
Fig. 10 Isotropic Consolidation

Fig. 11 Determination of $K_B$. 

$\sigma_3^3 > \sigma_3^2 > \sigma_3^1$
If $B$ is evaluated at several values of axial strain, it is noticed that the values obtained would be different from each other. If the assumption, that $B$ is independent of the deviator stress, is reasonable (for elastic volume changes), this only points out the fact that inelastic volume changes (depending on the deviator stress), also contribute to the volume changes observed in Fig. 7.

Since we take the shear volume coupling into account in our model, we refrain from using Eqn. 11 to calculate $B$—the modulus related to the elastic volume changes.

Duncan and others, circumvent this problem by specifying the stress level at which $B$ is to be evaluated, as follows,

1. If the volume change curve does not reach a horizontal tangent prior to the stage at which 70% of the strength is mobilized, use the points on the stress-strain and volume change curves corresponding to a stress level of 70%.

2. If the volume change curve reaches a horizontal tangent prior to the stage at which 70% of the strength is mobilized, in which case the points selected falls on the dilating part, the point corresponding to the inflexion point in the $\varepsilon_v$ vs. $\varepsilon$ curve, is selected so as to avoid the plastic volume change. This situation usually occurs for highly dilatant sands.

The method suggested herein, is based on isotropic consolidation in the triaxial apparatus. If a constant all round pressure is applied to a soil sample, the volume change characteristics in Fig. 10 are obtained for different pressures. Under the isotropic conditions, in the absence of shear stresses the shear dilatancy will not be exhibited. Thus it is reasonable to assume, that the maximum $\varepsilon_v$ obtained is the total elastic volume change due to the corresponding confining pressure.

$$B = \frac{\sigma_3}{\varepsilon_v}$$ (31)

Infact this could be done as the first stage of all previous measurements, since the sample has to be consolidated to $\sigma_3$, in any CU or CD test before shearing is introduced.

Once $B$ values for a number of values (in the working range), are calculated, the normalised $B$ (i.e. $B/P_0$) can be plotted against normalised $\sigma_3$ (i.e. $\sigma_3/P_0$) on a logarithmic scale.
Fig. 12 Shear Dilation
From Eqn. 24

$$B = K_B \rho_a \left( \frac{\sigma_3}{\rho_a} \right)^m$$

$$\therefore \frac{B}{\rho_a} = K_B \left( \frac{\sigma_3}{\rho_a} \right)^m$$

By taking logarithms,

$$\log \left( \frac{B}{\rho_a} \right) = \log K_B + m \log \left( \frac{\sigma_3}{\rho_a} \right)$$

The plot as shown in Fig. 11 would be linear, with the intercept and the gradient (properly converted), indicating the $K_B$ and $m$ values.

At this stage, it should be mentioned that hyperbolic stress-strain relations and the parameters have been widely applied in Soil Dynamics problems (such as liquifaction problems) as well. In this class of applications it is common to relate the moduli parameters to the relative density of the soil, which is justified as relative density ($D_r$) could possibly be the dominant factor in the moduli parameters. Such empirical relationships between the shear modulus parameter and $D_r$ have been presented by Hardin and Drnevich (1972).

### 3.6 Shear Volume Coupling

A characteristic feature of any linear or a non-linear elastic theory is the shear-volume decoupling, i.e., the assumption that volume changes can only be caused by a change in mean normal stress, and not by shearing. Most structural materials like metals, timber, and concrete behave in this way, with the volume changes and shearing taking place independently.

On the other hand, shear-induced volume changes are a salient feature in sands and clays. For instance, if a simple shear test is performed on a sample of dense or medium dense sand, volume change characteristics as shown in Fig. 12 are obtained.

With the mobilization of the shear strength, initially a reduction in volume occurs. This part diminishes if the sand is made denser. Later, under higher shear strains, a marked increase in volume takes place, even though the external normal loads are kept constant. This phenomenon is termed the shear-induced dilation in Soil mechanics, and is predominant in dense and medium dense sands.

The maximum dilation rate usually occurs in the failure region, where the full shear strength of the sand is mobilized. When the shear strains reach very high values (20–30%), dilation ceases.

When triaxial compression tests are performed on medium dense sands, the volume of the sample starts increasing rapidly
after the initial decrease, inspite of the increasing(compressive) mean normal stress. This again provides a classic example for the demonstration of shear dilation in sands. A significant feature of this dilation is that unlike the bulk of the volume change due to any change in mean normal stress, it is totally irrecoverable. It is thought to be due to plastic deformations in the sand skeleton, such as slip between grains, to assume a new arrangement.

In the cavity expansion problems in dense sands (as that taking place in pressuremeter tests), the dilation has an enormous effect on the boundary deflections. On the other hand in the undrained cyclic loading of sands, a shear induced contraction will give rise to rapid increases in pore pressure, causing liquifaction problems. Therefore in the analysis of medium dense or dense sands, the assumption of shear volume decoupling appears to be a poor one.

The fundamental relationship in shear volume coupling, as used in this model, could be expressed as,

$$\Delta \varepsilon_v = \Delta \varepsilon_{v}^{\text{elastic}} + \Delta \varepsilon_{v}^{\text{plastic}}$$

(35)

Using the definition of $B$, we can simplify this to give,

$$\Delta \varepsilon_v = \frac{\Delta \sigma_m}{B} + f(\Delta \gamma)$$

(36)

where $\Delta \gamma =$ shear strain increment

and $f =$ function of the density, the maximum shear strength, of the soil and the stress level.

Different theories that account for this dilatant behaviour of sand, have been put forward. The revised modified Cam-Clay model with kinematic hardening, and Rowe's stress-dilatancy theory are two of them. The physical formulation of the latter is more attractive and logical, while the predicted results seem to be in good agreement with the observed ones. Rowe's principles included in this analysis will be outlined next.

3.7 Basic Concepts of Rowe's Stress-Dilatancy Theory

In contrast to the popular approach to stress-strain relations for soils by assuming elastic behaviour (by Duncan et al), or by applying concepts of plasticity theory (Roscoe, Prevost et al), a study of particulate mechanics can also be used to explain the behaviour of soils, especially in the case of dense and medium dense sand assemblies.

Rowe has attempted to understand the mechanics governing deformation characteristics by treating soil as discrete
particulate matter. Stress-Dilatancy theory is a result of his progress in this direction.

Rowe first introduced this theory in 1963, where energy ratio criterion is used to determine the critical angle of sliding in a random assembly of particles. This original form was subjected to criticism, in the hands of Gibson and Morgenstern (1963), Roscoe and Schofield (1964), and other workers.

Later Horne (1965), Barden (1966), reanalysed the theory behind, and established it on firmer scientific grounds. In 1971 Rowe presented the modified theory, giving substantial experimental evidence for dense and medium dense sands, in support of it.

The following assumptions form the framework of the theory,

a. The number of rolling contacts in a particle assembly is negligible compared to the number of sliding contacts, as sliding was shown to be a more stable mechanism of deformation than rolling.

b. At any instant sliding is confined to some preferred angle.

c. Deformation is a result of relative motion between temporary rigid group of particles, which can continuously reform by division and coalescence.

d. The individual particles are rigid, and could not contribute an elastic component towards the deformation.

As shown in Appendix B, by basic mechanics and the application of minimum energy principle (i.e., the sliding angle is chosen such that the energy absorbed for a given energy input is a minimum), results in the following relationship,

$$\frac{\sigma_1'}{\sigma_3'} = \left(1 - \frac{dv}{d\varepsilon_1}\right) \tan^2\left(45 + \frac{\phi_m}{2}\right)$$  (37)

where $\sigma_1'$ and $\sigma_3'$ have their usual meaning.

$dv$ and $d\varepsilon_1$ are the increments in volumetric strain, and the major principal strain resp.

$\phi_m$ is the angle of interparticle friction.

It should be mentioned here that Eqn. 37 is valid under the conditions bounded by the above assumptions. But in practice rolling and rotation occur in addition to slip, thereby lowering the energy ratio below the one specified in the Appendix B. Moreover, at high dilatancy rates, the groups slide at various angles in contrast to the assumption b.

Consideration of these, and observation of triaxial test
results, have lead to the presentation of the stress-dilatancy relation as:

\[
\frac{\sigma'_i}{\sigma'_3} = \left(1 - \frac{dv}{d\varepsilon_i}\right) \tan^2 \left(45 + \frac{\phi_f}{2} \right) \tag{38}
\]

for cohesionless soils.

At the critical state, where the soil deforms under constant volume,

\[
dv = 0 \\
\Rightarrow \frac{\sigma'_i}{\sigma'_3} = \tan^2 \left(45 + \frac{\phi_f}{2} \right) \tag{39}
\]

But we also know that at very large strains,

\[
\frac{\sigma'_i}{\sigma'_3} = \tan^2 \left(45 + \frac{\phi_{cv}}{2} \right) \tag{40}
\]

From Eqns 39 and 40, it follows that at the critical state (where all the dilation has ceased) \(\phi_f = \phi_{cv}\), where \(\phi_{cv}\) = angle of internal friction measured under constant volume deformation.

Rowe specified the range of values of \(\phi_{cv}\) as follows.

\[
\phi_{\mu} < \phi_f < \phi_{cv} \tag{41}
\]

It can be concluded that, a lower bound for \(\phi_f\) is provided by the interparticle friction angle (effective under ideal conditions specified by the assumptions), whereas the angle of friction at constant volume deformation, provides the upper bound.

Very often the relationship is expressed as,

\[
R = K D \\
D = 1 - \frac{dv}{d\varepsilon_i} \tag{43} \\
K = K_{pf} = \tan^2 \left(45 + \frac{\phi_f}{2} \right) \tag{44}
\]
Fig. 13 Plastic Potential
Experimental observations have shown that, for dense sand,

at pre-peak strains \( \phi_f = \phi_{\mu} \)
at large strains \( \phi_f = \phi_{CV} \)
for loose sands,

at all strains \( \phi_f = \phi_{CV} \)
Under plane strain conditions \( \phi_f = \phi_{CV} \)

Triaxial test results for sand, seem to be in good agreement with the theory, according to Rowe (1971).

3.7.1 Plastic Potentials

A significant feature of Eqn 38, is that it can be used as a flow rule, i.e., a relationship governing the direction of the plastic strain increment (such as the normality condition used in theories of plasticity).

\[
\text{dv} = \text{d}\varepsilon_1 + \text{d}\varepsilon_3 \quad \text{(for plane strain conditions)} \quad (45)
\]

Substituting in Eqn. 38,

\[
\frac{\sigma'_1}{\sigma'_3} = - \frac{\text{d}\varepsilon_3}{\text{d}\varepsilon_1} \cdot K_{pf} \quad (46)
\]

rewriting

\[
\frac{\text{d}\varepsilon_1}{\text{d}\varepsilon_3} = - \frac{\sigma'_3}{\sigma'_1} K_{pf} \quad (47)
\]
or

\[
\frac{\text{d}\varepsilon_1}{\text{d}\varepsilon_3} = f (\sigma'_1, \sigma'_3) \quad (48)
\]

which is the form of a flow rule in theories of plasticity. The presence of a flow rule gives rise to the concept of plastic potential which is defined as the surface to which the plastic strain increment is normal at the current stress point.

In Fig. 13, PS is the plastic potential and PQ is the strain increment vector.

3.8 Stress-Dilatancy Relationship for Plane Strain Conditions

As discussed previously, the Eqn. 38 could be rewritten for the special case of plane strain deformation as,

\[
\frac{\sigma'_1}{\sigma'_3} = \tan^2 \left( 45 + \frac{\phi_{CV}}{2} \right) \left( 1 - \frac{\text{dv}}{\text{d}\varepsilon_1} \right) \quad (49)
\]

\[
\text{dv} = \text{d}\varepsilon_1 + \text{d}\varepsilon_3 \quad (50)
\]
Fig. 14 Mohr Circle for Strains

\[ \frac{1}{2} \delta \]

\[ d \varepsilon_3 \rightarrow d \varepsilon_1 \]

Fig. 15 Mohr Circle for Stresses

\[ OQ = \frac{1}{2} (\sigma'_1 + \sigma'_3) \]

\[ OP = \frac{1}{2} (\sigma'_1 - \sigma'_3) \]
Fig. 16 Dilation Angle
\(d\varepsilon_1, d\varepsilon_3\) are the major and minor principal strain increments in the x-y plane. (The third component \(d\varepsilon_z\) being zero).

If \(d\gamma\) is the maximum shear strain increment, as shown in Fig. 14,

\[
d\gamma = d\varepsilon_1 - d\varepsilon_3
\]

(51)

From Eqns. 50 and 51,

\[
dv + d\gamma = 2d\varepsilon_1
\]

(52)

Substitution in Eqn. 49 gives,

\[
\frac{\sigma'_1}{\sigma'_3} = \tan^2 \left(45 + \frac{\phi_v}{2}\right) \left(\frac{d\gamma - dv}{dv + dv}\right)
\]

(53)

or

\[
\frac{\sigma'_1}{\sigma'_3} = \tan^2 \left(45 + \frac{\phi_v}{2}\right) \left(\frac{1 - \frac{dv}{d\gamma}}{1 + \frac{dv}{d\gamma}}\right)
\]

(54)

A Mohr circle consideration, as in Fig. 15, will show that, for cohesionless soil,

\[
\frac{\sigma'_1}{\sigma'_3} = \frac{1 + \sin \phi_d}{1 - \sin \phi_d} = \tan^2 \left(45 + \frac{\phi_d}{2}\right)
\]

(55)

\(\phi_d\) being the developed angle of friction.

3.8.1 Dilation Angle

From the volume change characteristics obtained from a simple shear test, the dilation angle \(\nu\) is defined for a particular stress state, under constant confining pressure, as follows,

\[
\sin \nu = -\frac{dv}{d\gamma}
\]

(56)

It is a convenient measure of the rate of dilation at any stress level. (Fig. 16)

Eqns. 55 and 56, could be substituted in Eqn. 54, to yield,

\[
\tan^2 \left(45 + \frac{\phi_d}{2}\right) = \tan^2 \left(45 + \frac{\phi_v}{2}\right) \tan^2 (45 + \nu/2)
\]

(57)
Fig. 17  Comparison of Dense and Loose Sand Behaviour in Simple Shear.

Fig. 18  Unloading-reloading in Shear
If a sand is sheared starting under isotropic (or $K_0$) conditions, the developed angle of friction ($\phi_d$) is small compared to $\phi_{cv}$ initially, with the result that the Eqn. 57 will predict a negative dilation angle.

Typical simple shear test results obtained for loose and medium dense sand, shows a compression at the initial stages as shown in Fig. 12. After reaches value, becomes positive and keeps on increasing till the maximum friction is mobilized, which again is observed experimentally.

Even in the case of a dense sand, Eqn. 57, predicts an initial decrease in volume which is not usually observed in the laboratory. In dense sands the initial gradient is sharper, and thus $\phi_{cv}$ (the starting point of dilation) is mobilized sooner. Since this occurs over a small range of shearing, the measurement of this initial compression would be difficult, which may be one of the reasons for the above discrepancy. Moreover, if the density of the sand is such that the attainment of $\phi_{cv}$ is almost instantaneous, only a volume increase is predicted by Eqn. 57. Fig. 17 depicts this situation, and the resulting stress-volume change-strain curves are not very far from the test results for very dense sands.

3.8.2 Unloading-reloading shear modulus and plastic shear strains

The ideas of an unloading-reloading elastic modulus discussed in Chapter 3.3.8.1 can be extended to form its shear counterpart, as shown in Fig. 18.

$G_{ur}$ can be evaluated at any confining pressure ($\sigma_3$), by a procedure similar to the determination of $E_{ur}$ (Duncan et al. 1978). The elastic and the plastic components of the shear strains can be decoupled as follows.

$$\gamma_r = \frac{\epsilon}{G_{ur}}$$

$$\gamma_p = \gamma - \gamma_r$$

Then the plastic shear strain increments can be used in Eqns. 36 and 56, instead of the total shear strain increment.

If the same argument that $\epsilon_r/\epsilon_p$ at low stress levels is high for dense sands is applied to the ratio of $\gamma_r/\gamma_p$ as well, it would mean that at low stress levels in dense sands, a considerable portion of the shear strain is elastic, which does not produce any volumetric strain. This further explains the excess volumetric contraction predicted for dense sands by Eqn.
Fig. 19 Variation of $\nu$ with $\sigma_{vo}$ (After Vaid et al. 1980)

Fig. 20 Simple Shear Test Results. (After Stroud)
Experimentally, for triaxial tests, in the post peak region there is a rapid reduction in the dilation angle, before all the dilation ceases at the ultimate state (or critical state), reached under large strains. The maximum dilation angle, for dense sand occurs at the peak strength. This behaviour is also modelled very well by the theory, as when \( \phi_d \) starts to reduce from \( \phi_{\text{max}} \), the Eqn. 57 will predict lower values for \( \nu \). Once the residual strength is mobilized (at the critical state), a constant volume deformation is correctly predicted, by zero \( \nu \), since \( \phi_{\text{CV}} \) is developed under such conditions for any sand.

For very loose sands on the other hand, the residual strength is the peak strength. Thus an apparent friction angle, higher than \( \phi_{\text{CV}} \) (the residual strength of any sand), is never developed, with the result that the dilatancy theory will not predict a volume increase. Under excessive strains, when \( \phi_{\text{CV}} \) is developed, a constant volume deformation, as in the case of a dense or a medium dense sand is predicted, as shown in Fig. 17. This situation is not observed in practice, due to the difficulty in preparing very loose samples. According to laboratory investigations, even a sand of relative density of 27%, dilate a little after the initial compression.

The foregoing discussion enables one to understand how Rowe's Dilatancy theory explains the different rates of dilation, of initially dense and loose samples. Under the same confining pressure, dense samples will dilate more than the initially looser ones, to attain a more or less the same voids ratio (and therefore the same ultimate strength), at large strains.

Apart from the initial density, the variations in the confining pressure also has a marked effect on the dilation rate.

### 3.8.3 Dilation angle and confining pressure

Vaid, Byrne, and Hughes (1980), investigated the dependency of the dilation rates on the confining pressure (\( \sigma_v \)), and they concluded that, an increase in \( \sigma_v \) will result in a decrease in the dilation rate and also the ultimate voids ratio, for a sand of a particular relative density.

When a dilation angle at a specified strain (i.e. 10%) is selected, it was shown that, this relationship between \( \nu \) and confining pressure, is linear as shown in Fig. 19.

This phenomenon can be explained satisfactorily using the dilatancy theory in the following manner.

Shear tests on sand has revealed that, the maximum strength mobilized (\( \phi_{\text{max}} \)), decreases with the increase in effective confining pressure, (Seed et al 1966), as depicted in Fig. 3. Thus
under higher confining pressures, if $\phi_{\text{max}}$ is reduced, the Eqn. 57 will predict lower values of $\psi$, which implies that the rate of dilation, throughout the shearing, will be lower for a particular relative density.

The NLSSIP can incorporate this behaviour, as it can accommodate the parameter $\Delta \phi$ (Chapter 3.3.6), which is the reduction in $\phi$ for a tenfold increase in the effective confining pressure. Along with this, the theory will predict lower $\psi$ at higher confining pressures for a sand of any initial density.

3.9 Modified NLSSIP and the shear dilation

The existing NLSSIP uses non-linear elastic incremental constitutive relations, and predicts only the deformations due to elastic strains. It has been modified to take into account the contribution from shear induced dilation towards the volumetric strains. This is done by an analogy with the Temperature-Stress problem.

The mechanics and the finite element techniques adopted, will be outlined in detail in Chapter 4.

The user of the program can select between two options in introducing shear dilation.

1. Starting from a certain specified shear strain, a constant dilation angle is used throughout the shearing process. (which is the maximum dilation angle). Here the dilation angle and the starting shear strain have to be entered.

2. Rowe's dilatancy equation for plane strain conditions (Eqn. 26), is used, and the program calculates the rate of dilation to be used at any shear strain, for any finite element, depending on the stress state. The value of $\psi_{\text{cy}}$ has to be specified by the user.

The option 1, an alternative to the Rowe's approach, follows from the notable behaviour of sands under plane strain conditions. As first observed by Stroud (1971), from simple shear tests on Leighton Buzzard sand (of different initial relative densities), the deformation occurs at a constant principal stress ratio, after the peak stress ratio is reached, over a substantial range of strain. Consequently, the dilation angle also remains unchanged so long as this happens, as seen by Figs. 20a and 20b.

Hughes et al (1977) made use of this to determine an elegant solution to the cavity expansion problem in sand due to a pressuremeter, which will be presented in Appendix G.

Byrne et al (1980) used the same technique to introduce dilation to a problem of unloading of a tunnel in oil sand. Their volume change curves were similar to the one shown in Fig. 20b except for the compression part.
Fig. 21  Idealized Volume Change Curve
The constant dilation angle they used is the $\gamma_{max}$ as it occurred at the peak stress-ratio (Fig. 21). The former group defined $\gamma_o$ as the strain at which failure is first reached, whereas the latter group of workers selected $\gamma_o$ as $2\gamma_{cv}$, where $\gamma_{cv}$ is the shear strain at which the friction angle developed is $\phi_{cv}$. It can be concluded that although both options seem to be equally good for sand, the option 2 is advantageous over the other in that it predicts the initial compression of the medium dense and loose sands.
The soil is modelled by 2D isoparametric quadrilateral or triangular elements. As the name implies the same interpolation functions are used to define the element shapes as well as the displacements within the elements.

4.1 Constitutive matrix

Most of the engineering problems encountered in practice can be identified as falling into one of the following classes.

1. Plane Strain - 2 dimensional
2. Plane Stress - 2 dimensional
3. Axisymmetric - 3 dimensional

Plane strain problems are characterised by the following two properties.

a. There is no deflection in the z direction
b. First derivatives of the other deflections w.r.t. z are zero.

With these two conditions Eqn. 3 will produce

\[ \varepsilon_z = 0 \quad \gamma_{yz} = 0 \quad \gamma_{xz} = 0 \quad (60) \]

Thus the plane strain vector will be

\[ \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (61) \]

From the generalized Hooke's law

\[ \tau_{xz} = G \gamma_{xz} = 0 \quad \tau_{yz} = G \gamma_{yz} = 0 \quad (62) \]

but as shown in the Appendix D.1

\[ \sigma_z = \mu (\sigma_y + \sigma_z) \quad (63) \]

Therefore we use

\[ \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (64) \]
but calculate \( \sigma_z \) based on Eqn. 63, if required.

In this case, the \((D)\) matrix will be a \(3 \times 3\) one. Based on the assumption that generalized Hooke's law can be applied to incremental elasticity in its original form, the \((D)\) matrix has been shaped out in Appendix D.1 as,

\[
[D] = \frac{E_t}{(1+\mu_t)(1-2\mu_t)} \begin{pmatrix}
1-\mu_t & \mu_t & 0 \\
\mu_t & 1-\mu_t & 0 \\
0 & 0 & \frac{1-2\mu_t}{2} \\
\end{pmatrix}
\] (65)

It is also shown that if, equivalent bulk and shear moduli are defined for the plane strain condition, such that,

\[
B_t' = \frac{3B_t}{2(1+\mu_t)} \quad \text{and} \quad G_t' = \frac{E_t}{2(1+\mu_t)}
\] (66)

the constitutive relationships reduce to the following form, as directly used in the program.

\[
\begin{pmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \tau_{xy}
\end{pmatrix} = \begin{pmatrix}
B_t' + G_t' & B_t' - G_t' & 0 \\
B_t' - G_t' & B_t' + G_t' & 0 \\
0 & 0 & G_t'
\end{pmatrix} \begin{pmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \gamma_{xy}
\end{pmatrix}
\] (67)

4.2 Stiffness matrix

The nodal force increment and the nodal displacement vectors will be \(8 \times 1\) in general.

\[
\Delta f = \begin{pmatrix}
\Delta f_1 \\
\Delta f_2 \\
\Delta f_3
\end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_3
\end{pmatrix}
\] (68)

The displacement field can be expressed as

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{bmatrix} A_1 \end{bmatrix} \boldsymbol{\alpha}
\] (69)

where \(\boldsymbol{\alpha}\) is a set of generalized coordinates \([A_1]\) is a matrix containing coefficients which are functions of
the coordinates of the point considered.

Applying Eqn. 69 to the nodal points gives,
\[
\delta = \begin{bmatrix} A_2 \end{bmatrix} \alpha \tag{70}
\]
where \([A_2]\) is a matrix containing coefficients which are dependent on the nodal coordinates.

By differentiating Eqn. 69 and combining it with Eqn. 3 gives,
\[
\Delta \epsilon = [C] \alpha \tag{71}
\]
where \(\Delta \epsilon\) is the plane strain increment vector.

From Eqn. 70
\[
\alpha = [A_2]^{-1} \delta \tag{72}
\]
\[
\Delta \epsilon = [C] [A_2]^{-1} \delta \tag{73}
\]

The following effective stress vector \(\Delta \sigma'\) has been obtained using the ideas about effective stresses in Chapter 3.1
\[
\Delta \sigma' = \begin{bmatrix} \Delta \sigma_x \\ \Delta \sigma_y \\ \Delta \tau_{xy} \end{bmatrix} - \Delta u \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{74}
\]

It is shown in Appendix D.2 that the force - displacement relation for a general finite element can be written as,
\[
\Delta \mathbf{f} = [k^*] \Delta \mathbf{u} = [k] \delta \tag{75}
\]
where \([k]\) - Element stiffness matrix dependent on the geometry of the element and \((D)\) and \([k^*]\) - A matrix dependent on the geometry of the element only.

Further it is pointed out in Appendix D.2 how all the element relationships could be built up to form the nodal force increment - displacement relationship for the whole structure, which is
\[
\Delta \mathbf{F} = [k] \Delta \tag{76}
\]
where \((K)\) -global stiffness matrix and \((F)\) -Total external force increment vector modified by the pore pressure forces.

4.3 Solution

Gauss' elimination technique is used to obtain the solution to Eqn. 76.

By knowing the nodal displacements \(\delta\) of each element

\[
\Delta \varepsilon = [B] \Delta \delta
\]

(77)

and

\[
\Delta \sigma' = [D] \Delta \varepsilon
\]

(78)

are used to evaluate the average strain and stress increments in each element.

4.4. Shear dilation

It was mentioned in Chapter 3.9, that the additional volume changes due to shear are introduced by means of temperature analogy.

The two equations for volumetric strain in each case, presented below, will clarify this analogy further.

Temperature-Stress problem

\[
\Delta \varepsilon_v = \frac{\Delta \sigma_m}{B} - \alpha \Delta T
\]

(79)

Shear dilation

\[
\Delta \varepsilon_v = \frac{\Delta \sigma_m}{B} - \sin \psi \cdot \Delta \gamma
\]

(78)

where

\(\Delta \varepsilon_v\) = Total volumetric strain

\(\frac{\Delta \sigma_m}{B}\) = elastic strain

\(\alpha\) = Temperature coefficient of the material

\(\Delta T\) = change in temperature. (increase)

Eqn. 79 shows how the volumetric strain in a temperature stress problem is made up of the strain associated with the mean normal stress change, as well as that due to a temperature change. (\(\Delta T\)). In the shear dilation problem the change in shear can be thought of as equivalent to a temperature change as far as the volumetric strain is concerned, and it's contribution is added to the total volumetric strain, alongside that due to the
changes in the mean normal stress (the elastic component of the volumetric strain), yielding Eqn. 80. Comparison of these two equations will show the analogy between the rate of expansion ($\alpha$) and the rate of dilation. ($\sin \gamma$). In the finite element method, effects due to a temperature change are incorporated by a set of equivalent nodal forces. The procedure for obtaining them in the case of shear dilation is described below.

Suppose $\Delta \xi^d$ is the additional strain (due to shear dilation), required in one element and this is to be obtained by applying a force vector of $\Delta f^d$ on its nodes.

If $\Delta \gamma$ is the maximum shear strain in the element, from Eqn. 56, the plastic volume change is

$$\Delta \xi^p = - \sin \gamma \cdot \Delta \gamma$$

(81)

From this we construct the $\Delta \xi^d$ as follows,

$$\Delta \xi^d = \begin{bmatrix} \Delta \xi_x^d \\ \Delta \xi_y^d \\ \Delta \gamma_{xy}^d \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \sin \gamma \cdot \Delta \gamma \\ -\frac{1}{2} \sin \gamma \cdot \Delta \gamma \\ 0 \end{bmatrix}$$

$$\gamma$$ - average dilation angle during the load increment.

Now if $\delta^d$ is the set of deflections produced by the application of $\Delta f^d$

For equilibrium we can write the virtual work equation, as in Appendix D.2,

$$(\delta^d)^T \Delta f^d = \int (\Delta \xi^d)^T (\Delta \sigma)^d dV$$

(83)

But from Eqn. 77,

$$\Delta \xi^d = [B] \delta^d$$

(84)

Substitution from Eqns. 78 and 84 in Eqn. 83 yields

$$(\delta^d)^T \Delta f^d = (\delta^d)^T \int [B]^T [D] \Delta \xi^d dV$$

(85)

$$\Delta f^d = [B]^T [D] \Delta \xi^d V_e$$

(86)

since (B) and (D), consists of coefficients which are constants for the element. (Appendix D.2).
Considering a unit thickness,
\[
\Delta f^d = [B]^T [D] \Delta \xi^d \Lambda_e \tag{87}
\]

Equations 82 and 87 can be used to assess the required nodal forces for each element, and the global force vector \(\Delta f^d\) is applied in addition to the pore pressure modified load increment vector.

Evaluation of the results is done as described in Chapter 4.3, except that Eqn. 78 has to be modified, as done in the temperature problem, by subtracting \(\Delta \xi^d\) from the final \(\Delta \xi\), since \(\Delta \xi^d\) does not create additional stresses (similar to strains caused by temperature changes). The modified Eqn. 78 is then,
\[
\Delta \sigma' = [P][ \Delta \xi - \Delta \xi^d ] \tag{38}
\]

It will be realised that, the \(\Delta \sigma'\) value from \(\Delta \xi\), would differ from the previous \(\Delta \gamma\) (used in Eqn. 82), when a mesh of elements is analysed. Thus an iterative procedure has to be adopted, for a convergent solution.

In constructing \(\Delta \xi^d\) (Eq. 82), it has been assumed that the shear dilation is isotropic, i.e. \(\Delta \xi^d\) is made up of equal contributions from \(\Delta \xi_x^d\) and \(\Delta \xi_y^d\).

This is not strictly true in the light of the plastic flow rule in Chapter 3.7.1 (Eqn. 48).

\[
\frac{d \xi_i}{d \xi} = \frac{\sigma'_i}{\sigma'_j} K_{pf} \tag{39}
\]

Although it is more complicated, Eqn. 82 can be built up by Eqns. 90 and 91, in more concise work.

\[
\frac{\Delta \xi_x^d}{\Delta \xi_y^d} = f (\sigma'_3, \sigma'_1) \tag{90}
\]

(from Eqn. 48)

\[
\Delta \xi_x^d + \Delta \xi_y^d = - \sin \nu \cdot \Delta \gamma \tag{91}
\]

(from Eqn. 56).

4.5 Stress redistribution technique

It was mentioned in Chapter 3.3.9, how the stresses can be corrected in the elements, that violated the failure criterion, by applying an additional set of nodal forces. Herein the method of obtaining these nodal forces is described.
Fig. 22 Stress Redistribution

OP' - Failure line in modified Mohr plot
Suppose $\Delta \sigma^l$ is the stress correction to be applied for a single element, and this is to be achieved by applying a set of force $\Delta f^l$ to the nodes of that element.

If $\delta^l$ is the additional set of deflections, due to the application of $\Delta f^l$,

By the principle of virtual work as in Appendix D.2,

$$[\delta^l]^T \Delta f^l = \int \left( [\Delta \varepsilon^l]^T \Delta \sigma^l \right) dV \quad (92)$$

$\Delta \varepsilon^l$ - additional strain vector

From Eqn. 77

$$\Delta \varepsilon^l = [B] \delta^l \quad (93)$$

Substitution in Eqn. 92 gives

$$[\delta^l]^T \Delta f^l = [\delta^l]^T \int \left[ [B] \Delta \sigma^l \right] dV \quad (94)$$

which on simplification as described in the previous article, results in

$$\Delta f^l = [B]^T \Delta \sigma^l \lambda \varepsilon \quad (95)$$

$\Delta \sigma^l$ is constructed in the following manner.

Fig 22 shows the Mohr circle for a typical element that violates the failure criterion. If it is to abide by the Mohr-Coulomb criterion,

$$QP \leq QP'$$

The stress state of the element is given by $A(\sigma_x, \tau_{xy})$ and $B(\sigma_y, -\tau_{xy})$ respectively.

At this stage an assumption is made that the mean normal stress remains constant. Consequently, the corrected circle will acquire the position of the dotted circle, with the stress points A and B landing on A' and B' respectively.

$$QP' = \sigma_m \tan \alpha = \sigma_m \sin \phi_{\text{max}} \quad (96)$$

$$QP = \tau_{\text{max}} \quad (97)$$

$$PP' = \tau_{\text{max}} - \sigma_m \sin \phi_{\text{max}} \quad (98)$$

$$= AA' = BB'$$

= the reduction in the radius of the Mohr circle
Fig. 23  Inclusion of Springs
Now from the \( \Delta A A'R \)

\[
- \Delta T_{xy} = AA' \sin \theta \\
- \Delta \sigma_x = AA' \cos \theta
\]  

(99)

Similarly

\[
\Delta \sigma_y = BB' \cos \theta = - \Delta \sigma_x \]  

(100)

\[
\Delta \sigma_x = -(T_{\text{max}} - \sigma_m \sin \phi_{\text{max}}) \cos \theta \]  

(101)

\[
\Delta \sigma_y = (T_{\text{max}} - \sigma_m \sin \phi_{\text{max}}) \cos \theta \]  

(102)

\[
\Delta T_{xy} = -(T_{\text{max}} - \sigma_m \sin \phi_{\text{max}}) \sin \theta \]  

(103)

\[
\Delta \sigma^* = \begin{bmatrix} \Delta \sigma_x \\ \Delta \sigma_y \\ \Delta T_{xy} \end{bmatrix} \]  

(105)

Then Eqn. 95 generates the nodal force increment vector, and as previously mentioned, the global force vector (for all such elements), is applied to give the desired stress conditions, after an iterative procedure.

If an element that first starts yielding at \( Q \) (Fig. 9), is assigned the constitutive parameters described in Chapter 3.3.9 for failure region,

\[
\mu = 0.01 B_t
\]

the value of \( \mu_t \) will be very low compared to \( B_t \) (from Eqn. 24), with the result that from then onwards, the stress path will be along QR on further unloading. (with a very little shear stress change compared to the change in mean normal stress).

Now on the redistribution of stresses, if we retain the same moduli values, a reduction in the excess shear stress under constant mean normal stress, is impossible to be achieved, due to the above reasons. Thus on the stress-redistribution, \( \mu_t \) has to be upgraded w.r.t \( B_t \), since our aim is to bring the stress point along RS. This would amount to lowering of \( \mu_t \), from 0.495 (at R). In keeping with the practical limits, \( \mu_t \) was increased to such an extent that \( \mu_t \) becomes zero in the process of redistribution.

4.6 Inclusion of elastic springs

Fig. 23a shows the finite element domain around the shaft, for the particular problem of analysing the unloading of a shaft in oil sand.

A problem of an infinite space has been converted to one of a finite space, by the displacement boundary conditions at \( A, B, C \).
and D. The solution is not affected if the boundary is set at a distance which is large in comparison to the diameter of the shaft. This involves extending the mesh to areas which have a little influence on the shaft deformation. This problem was overcome by fixing radial elastic springs at A, B, C and D, to account for the influence of the soil mass from the boundary to infinity. The advantage in this modification is that the problem remains that of an infinite soil mass, with the opportunity of using a finer mesh closer to the shaft without generating an enormous mesh. (Fig. 23b).

With the inclusion of the springs (with stiffness \( k \)), the force-displacement relationship in Eqn. 38 was modified by,

a. Assigning usual degrees of freedom at the spring boundary \((j, j+1 \text{ etc})\).

b. Specifying zero forces at them.

c. Modifying the \((K)\) by adding the spring stiffness to the diagonal terms, corresponding to the above degrees of freedom \((j, j+1 \text{ etc})\).

\[
\begin{pmatrix}
\Delta F_1 \\
\Delta F_2 \\
0 \\
0 \\
\Delta F_n
\end{pmatrix} =
\begin{pmatrix}
K_{11} \\
K_{22} \\
K_{jj} + k_{j,j+1} \\
K_{jj+1} + k_{j,j+1} \\
K_{nn} \\
\end{pmatrix}
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_j \\
\delta_{j+1} \\
\delta_n
\end{pmatrix}
\]  \( (106) \)

The method of obtaining \( \lambda \) has been described in the Appendix E.

It should be emphasized here that the above outlined method is valid only for the axisymmetric case, where the deflections are radial at any point.
Fig. 24  Time Effect in Pore Pressure Changes
CHAPTER 5

PORE PRESSURE MODEL

5.1 Idealization of the pore-fluid behaviour

One of the most important assumptions made regarding the pore-pressure distribution is that, the pressure is the same in all three fluid phases. This is reasonable as the surface tension at the interfaces, which makes the difference is negligible. Dusseault (1979), says that the gas bubbles are likely to be in contact with bitumen only, and hence only the surface tension at the gas-bitumen and water-bitumen interfaces should be considered, and that they can be neglected, since their magnitudes are insignificant.

The second assumption is that, the compressibility of water and bitumen is neglected, when compared to that of gas bubbles. It should be borne in mind that this is strictly valid under fairly high initial gas porosities. Consequent to this assumption, the volume changes of water and bitumen on unloading shall not be taken into account.

Finally the time taken for gas evolution \( \tau \) is neglected. (As shown in Fig. 24, since the gas evolution is not instantaneous, a certain time elapses before the equilibrium pore pressure is attained, on the reduction of external stress).

5.2 Free gas behaviour

The volume changes occurring in the free gas under pressure and temperature changes, are predicted by the ideal gas equation of state.

\[
P V = n R T
\]

(107)

For real gases like the organic gaseous compounds, present in the oilsands, this equation gives accurate results only under high pressures and low temperatures.

5.3 Pressure solubility

There is a marked increase in the solubility of gases in liquids, with increased pressure. Conversely, as previously mentioned, occlusion of the gas from solution will result, on removal of the excess pressure.

This was quantitatively expressed by Harris and Sobkowicz (1977), by the following equation, which we would be using in our program due to its simplicity.

\[
V_g = -H V \Delta u
\]

(108)

where \( V_g \) is the gas going into solution (indicated by the -ive sign), when the pressure increases by \( \Delta u \), at constant temperature. (This volume is referred to a set of standard
The pressure solubility of gases, in dilute solutions is governed by Henry's law, which says that, at constant temperature, the partial pressure of a gas in contact with a solution is proportional to the mole fraction of that gas in solution.

Starting from Henry's law, an equation similar to Eqn. 108, has been derived in Appendix C.2.a and it is also shown that the coefficient $H$ is highly dependent on the temperature.

5.4 Temperature solubility

An increase in temperature will decrease the solubility of gases, and therefore changes in gas porosities due to temperature changes associated with excavation work in oilsands, could be expected.

Harris and Sobokowicz (1977), use the following relationship to take this into account.

$$\nu_g' = \beta V \Delta T$$  \hspace{1cm} (109)

where $\nu_g'$ is the gas coming out of the solution (referred to the same set of standard conditions), when the temperature is increased by $\Delta T$, at constant pressure.

$\nu$ - volume of the solution
$\beta$ - a coefficient measured at the standard conditions.

It could also be shown that, the coefficient $\beta$ turns out to be a function of the pressure and the temperature at which the gas is evolved.

5.5 Volumetric strain compatibility (for undrained conditions)

From the assumptions made regarding the compressibility of oilsand phases, the volume change of the gas phase for a unit volume of oilsand, should be the same as the volumetric strain of the soil skeleton obtained from the effective stress considerations, under undrained conditions.

This compatibility condition, along with Eqns. 107, 108, and 109 were combined by Byrne et al. (1980), as in Appendix C.1, to obtain the expression for the pore pressure increment, under pressure and temperature changes, as,

$$\Delta u = \frac{u_0 \Delta \varepsilon_v + \frac{T_i}{T_a} u_0 (\beta_\omega n_\omega + \beta_\rho n_\rho) \Delta T + n_g \frac{u_0}{T_a} \Delta T}{n_g - \Delta \varepsilon_v + \frac{u_0 T_i}{T_a} (H_\rho n_\rho + H_\omega n_\omega)}$$  \hspace{1cm} (110)
where
\( n_w \) - Porosity of the water phase (dimensionless)
\( n_o \) - Porosity of the oil phase (dimensionless)
\( n_g \) - Porosity of the gas phase (dimensionless)
\( \beta_w \) - Temperature solubility coefficient of water (°K\(^{-1}\))
\( \beta_o \) - Temperature solubility coefficient of oil (°K\(^{-1}\))
\( H_w \) - Pressure solubility coeff. of water (pressure unit)
\( H_o \) - Pressure solubility coeff. of oil (pressure unit)
\( U_a \) - Reference (atmospheric) pressure (pressure unit)
\( T_a \) - Reference (room) temperature (283°K)
\( T_0 \) - Initial temperature (°K)
\( T_f \) - Final temperature (°K)
\( \Delta T \) - Change in temperature (°K)
\( U_p \) - Initial absolute pressure (pressure unit)
\( \Delta u \) - Change in pore pressure (pressure unit)
\( \Delta \varepsilon \) - Volumetric strain increment (dimensionless)

5.6 Comment on the use of Eqn. 110

In addition to being subjected to the validity of the assumptions made in deriving it, this equation has other limitations as well. Whether the behaviour of the gases present in oilsand obeys the ideal gas or Henry's law, and if so under what conditions they do so without much deviation, still lies in question, due to the lack of test data. Possible chemical activity between the gases and the bitumen or gases and sand grains may also prevent them from behaving ideally.

It is advisable to work with a \( H \) value obtained at the working temperature range, and a \( \beta \) value valid for the working pressure and temperature range, since the Henry's law refutes the fact that \( H \) and \( \beta \) are constants. Yet this could only be done if data is available from tests done under a wide range of pressures and temperatures.

Further the significance of the time effects, (such as the time taken for gas evolution) has to be checked by a laboratory investigation, if one is to rely on Eqn. 110 which ignores them.
CHAPTER 6

SOLUTION METHOD

6.1 Drainage conditions

Depending on the drainage conditions two types of analysis can be done. The drained analysis (when sufficient time is allowed for the pore fluid to escape on loading) and the undrained analysis (due to restricted or no drainage paths, or due to rapid loading when the pore fluid is prevented from escaping).

The drained conditions present no problems since the pore pressures are known. The undrained conditions can be handled in two ways.

An effective stress analysis can be performed if the pore pressure changes are evaluated. In numerical analysis, changes in pressure ($\Delta u$) are obtained as fractions of the total stress changes, using Skempton's pore pressure parameters $A$ and $B$. Instead, in this program the pore pressures are predicted by Eqn. 110, and calculation of porosities is done based on Eqns. H4 to H6.

The trouble of evaluating $\Delta u$ can be avoided by adopting a total stress analysis, where total stress constitutive parameters are used.

The program can be used to analyse problems involving deformation of sand in both drained and undrained modes. In the undrained case, the program predicts the pore pressure changes as a part of the iterative procedure. If the user has some idea of the changes in pore pressure taking place in a subsequent load case, the number of iterations can be cut down by initially assigning them. Otherwise the program assumes $\Delta u$ to be zero in all elements, in its initial trial.

6.2 Iterative procedures

This analysis is comprised of a number of load increments. Each increment involves the following steps.

6.2.1 Drained Analysis

1. The external load increment is applied to the mesh at the nodal points ($\Delta F$).

2. The global stiffness matrix is built up by using the constitutive parameters $(E, B)$, evaluated at the initial condition.

3. The corresponding deflections are obtained by solving Eqn. 76.

4. Using Eqns. 78 and 77 stress and strain increments are
predicted, and $E_\tau$ and $B_\tau$ are updated to the average stress condition during the increment.

5. If shear dilation is to be introduced, the corresponding equivalent set of nodal forces ($\Delta F^d$), is calculated based on the above shear strain increment, using Eqn. 87.

6. Steps 1, 2, 3, and 4 are repeated with the following changes.
   a. $\Delta F + \Delta F^d$ is applied instead of $\Delta F$.
   b. Updated $E_\tau$ and $B_\tau$ are used, in step 2.
   c. Eqn. 88 is used, instead of Eqn. 78 to calculate the stress increments.

7. Then it is checked whether all the elements have dilated in accordance with their current shear strain increment, and if so the process of iterating terminates. Otherwise the steps 1-5 are retraced over and over, each time updating $\Delta F^d$, $E_\tau$, and $B_\tau$ until convergence is reached.

8. Finally it is checked whether any of the elements violate the failure criterion. If they do so, an additional set of nodal forces ($\Delta F^d$) is calculated from Eqn. 95 and applied separately along steps 1-4, until stress redistribution is done so that all the elements conform to the yield criterion.

By making the external load increment ($\Delta F$) as small as possible, more accurate results can be predicted using a low number of iterations.

6.2.2 Undrained analysis

1. External load increment is applied to the mesh at the nodal points ($\Delta F$)

2. The stiffness matrix $(K)$, (in Eqn. 76), is formed in a manner identical to that for the drained case, since an effective stress analysis is done.

3. The set of additional nodal forces, prescribed by the vector $(K'\Delta u)$ (as discussed in Chapter 4.2), is calculated. Initially the program assumes $\Delta u = 0$ for all elements, if values are not provided by the user.

4. Eqn. 76 is solved to obtain the nodal deflections.

5. Using Eqns. 77 and 78, effective stress and strain increments are predicted and the incremental moduli ($E_\tau$ and $B_\tau$) are updated to the average effective stress condition during the increment.

6. If shear dilation is introduced $\Delta F^d$ is calculated as in the previous case.

7. $\Delta u$ values are estimated for each element based on the
volumetric strain increments and Eqn. 110. Then the current \( \Delta u \)

is compared with the guessed \( \Delta u_0 \) in step 3, and if they differ by more than a specified tolerance, a new pore pressure increment \( \Delta u_2 \) is prepared for the next iteration in the following way.

i. For elements in the pre-failure region, where the conditions are stable, \( \Delta u_2 = \Delta u_1 \).

ii. For elements in the failure region, even a small reduction of the external load, will cause a sensational drop in the mean normal stress, followed by a similar expansion giving rise to an excessive \( \Delta u \), which if applied in the next iteration might even overbalance the external load effect, thereby leading to a divergent solution. On the other hand, if \( \Delta u_0 \) was a reasonable guess (or the correct guess), \( \Delta u \) would have been approximately equal to \( \Delta u_0 \). With this in mind it is logical to assume that, the correct \( \Delta u \) lies in between \( \Delta u_0 \) and \( \Delta u_2 \) if they differ from each other. Thus

\[
\Delta u_2 = \frac{1}{2}(\Delta u_0 + \Delta u_1)
\]

8. In the next trial, steps 1, 2, 3, 4, and 5 are cycled through once again with the following modifications.

a. \( \Delta F + \Delta F^d \) is applied instead of \( \Delta F \).

b. \((K)\) is built up with updated moduli.

c. \( \Delta u_2 \) is used to evaluate the pore pressure force vector.

d. Eqn. 88 is used to extract the effective stress changes.

9. It is checked whether a dilation, appropriate to the present shear strain increment, has been introduced into each element. If not \( \Delta F^d \) is updated.

10. Pore pressure increments for the next iteration are prepared \( (\Delta u_4) \). For elements in the pre-failure region, the same procedure as described in step 7 is followed. For the other elements, the next guess is based on the fact that \( \Delta u \) lies between

i. \( \Delta u_0 \) and \( \Delta u_1 \) (from the previous trial)

ii. \( \Delta u_2 \) and \( \Delta u_3 \) (from the current trial).

\( (\Delta u_3 \) is the one predicted by Eqn. 110, for the volumetric strain increment in the current iteration, using a guessed \( \Delta u_4 \)).

If for any iteration, the guessed \( \Delta u \) agrees with the \( \Delta u \) evaluated from Eqn. 110, within the specified tolerance, for all elements, the iterative procedure terminates.

11. Finally stress-redistribution is done, if needed, as described in step 8 of the drained case.
Fig. 25  Sign Convention for Loads and Displacements

Fig. 26  Sign Convention for Stresses and Strains
6.2.3 Undrained analysis followed by Venting

Due to the high compressibility of the system at any stage, or even otherwise the user may decide to vent some of the critical elements. (In a certain region). This sudden reduction in pore pressure is handled by the program as another load case at constant total load. The equilibrium values of the stress, strain and pore pressures (in other elements), are obtained by tracing steps 1-11 in the undrained case with the following exceptions:

i. \( \Delta F \) in step 1 is a zero vector (constant total load).

ii. \( \Delta u \) for vented elements are not predicted by Eqn. 110, but instead are assigned the sudden fall in pore pressure in them, for all iterations.

In the subsequent analysis, the vented elements are set to undergo the drained iterative procedure, with no changes in pore pressure, while the other elements are cycled through the steps of the undrained analysis.

6.3 Limitations

1. The mean normal stress used to calculate the elastic and bulk moduli are limited to minimum of 0.1 times the atmospheric pressure.

2. The incremental Poisson's ratio (\( \mu_t \)) is in the range of 0 to 0.495.

3. Dilation may not be introduced to the elements which have developed tension.

4. Gas laws cannot be used to predict pore pressure changes where positive volume changes exceed the gas porosity which would imply negative gas porosities. i.e. an unrealistic condition.

5. Once a region reaches failure, subsequent load increments should be an order of magnitude smaller than the previous ones.

6. On venting, the pore pressure in vented elements will drop to zero instantaneously, and after that they will never behave in an undrained manner.

6.4 Sign conventions

The following sign conventions are used in evaluating the results (stresses, strains, and displacements).

6.4.1 Loads and displacements

Positive directions of the nodal loads and displacements are indicated in Fig. 25.
6.4.2 Stresses and strains

Fig. 26 indicates the positive directions of the stresses and strains.

The same conventions are used in the input of stresses and strains for the pre-existing soil elements.

Instructions regarding the input of external forces is provided in the user's manual in Appendix I.
Fig. 27  Some Plane Strain Situations
CHAPTER 7

COMPARISON WITH EXISTING SOLUTIONS

Under both drained and undrained conditions the program predictions were checked with available closed form solutions and other results.

7.1 Elastic closed form solutions

- Deformation behaviour of
  1. Thick walled cylinder, (Fig. 27a)
  2. Opening in an infinite space, (Fig. 27b)

are two plane strain problems, to which complete, linear elastic closed form solutions are available.

If \( a \) and \( b \) are the internal and external radii of a thick walled cylinder as shown, subject to pressures \( P_i \) and \( P_o \) from the inside and outside respectively. The stress and the displacement conditions at any radius \( r \) are given by the following expressions.

\[
\sigma_r = - \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} + \frac{(P_i - P_o) a^2 b^3}{(b^2 - a^2) r^2}
\]

\[
\sigma_\theta = - \frac{a^2 P_i - b^2 P_o}{b^2 - a^2} - \frac{(P_i - P_o) a^2 b^3}{(b^2 - a^2) r^2}
\]

\[
\tau_{r\theta} = \frac{(-2 \mu)(1 + \mu)(a^2 P_i - b^2 P_o) r + (1 + \mu)(P_i - P_o) a^2 b^3}{E (b^2 - a^2) r^3}
\]

( \( \tau_{r\theta} = 0 \) due to axisymmetry)

Stresses and the displacements in such a cylinder (\( a=10 \) ft., \( b=30 \) ft.), at various locations when subjected to an internal suction of 500 psf, are plotted in Fig. 30, along with the results obtained from the modified NLSSIP, and the results are in good agreement.

If the pressure on the inner face of a shaft (of radius \( a \)) in an infinite medium is changed from \( P_o \) to \( P_i \), the final stress state at any point (at a radius \( r \)) is given by

\[
\sigma_r = P_o + (P_i - P_o) \frac{a^4}{r^4}
\]

\[
\sigma_\theta = P_o - (P_i - P_o) \frac{a^2}{r^2}
\]

\[
\tau_{r\theta} = 0
\]
and the radial deflection resulting from the change is

\[ \delta = \frac{(1 + \mu)}{E} \frac{a^2}{r} (P_i - P_o) \quad (117) \]

(The derivation of the eqns. 111 and 117, from principles of mechanics is presented in Appendix F).

The stresses and displacements in such a situation, where the shaft radius is 10 ft., is plotted (when the internal pressure has been reduced to 1000 psf from 2000 psf), in Fig. 31. Again the results are in remarkably good agreement with those from the modified NLSSIP.

7.2 Elasto-plastic closed form solution

The conventional pressuremeter tests in sand, where relatively high strains occur, is a good plane strain problem to consider. Many non-elastic solutions have been presented to interpret the pressuremeter test data, out of which the one given by Hughes et al. (1977) appears to be adequate for the particular situation.

As stated in Chapter 3.9, they assume the failure of sand at a constant effective stress ratio (based on the simple shear test data), and their solution is superior to the others in that shear dilation of sand at failure is taken into account.

The theory and the assumptions behind it are reviewed in Appendix G, and the following relationships generated by this solution for the plastic zone are combined with the previous linear elastic ones in formulating a complete solution to the problem of a cylindrical opening in an infinite sand mass.

\[ \frac{\sigma'_\theta}{\sigma_r} = N = \tan^2 \left( 45 + \frac{\phi_{\text{max}}}{2} \right) \quad (113) \]

\[ \sigma'_R = P_0 (1 - \sin \phi_{\text{max}}) \quad (119) \]

\[ r = a \left( \frac{P_o}{P_i} (1 - \sin \phi_{\text{max}}) \right)^{\frac{1 - \sin \phi_{\text{max}}}{2 \sin \phi_{\text{max}}}} \quad (120) \]

where \( a, P_o, P_i \), are the same quantities defined in the previous article, and \( \sigma'_R \) is the radial stress at the elastic-plastic interface occurring at a radius \( R \).

\[ \log \frac{\sigma'_R}{\sigma'_R} = (1 - N) \log \frac{R}{r} \quad (12.1) \]

and finally the radial deflections in the failure region is
Non linear elastic-plastic at $R_f = COC1$

Linear elastic plastic

Fig. 28  Linear Elasticity in NLSSIP
expressed by

\[ U = r \left( \frac{R}{r} \right)^{D+1} \frac{u_R}{\tau} \] (122)

\( U \) -Deflection at the interface

where \( D = \tan^2 \left( \frac{45 + \nu}{2} \right) \) (from the stress-dilatancy theory)

When the internal pressure of a shaft of radius 10 ft is reduced from 2000 psf to 800 psf failure sets in, and the closed form predictions (using Eqns. 118-122) are plotted in Fig. 32, along with the modified NLSSIP results. The same situation predicted using the linear elastic relations is also plotted.

It can be seen that NLSSIP results lie between the elastic and elasto-plastic curves. This deviation could be reviewed in the context of the following approximations made.

1. The Poisson's ratio was assumed to be 0.495, in the failure region, to avoid instability, (Chapter 3.3.9), whereas the actual \( \mu_\perp \) should be 0.5.

2. The coarseness of the finite element mesh used.

3. In the stress-redistribution procedure, additional loads coming on the shaft boundary, disturbs the stress boundary conditions to some extent.

In the above predictions a low value of \( R_f (=0.001) \) was used, as under such conditions, the hyperbolic stress-strain curve, resembles a typical linear elastic plastic curve, as indicated in Fig. 28.

The constitutive relations are set up in NLSSIP in such a way that, the parameters \( E_\perp \) and \( B_\perp \) do not depend on the stress level at very low \( R_f \) values.

7.3 One dimensional unloading of oilsand

Up to now it was shown how well the drained behaviour of sand was predicted in a few cases by the modified NLSSIP, which appears to be a satisfactory check on the effective stress model used.

The pore pressure model for oilsand, described in Chapter 3 will now be checked against some existing predictions.

When the oilsand core samples are recovered in drill holes and brought up in steel containers, they swell by 5-15% of the original volume. If the core liners are rigid enough it could be assumed that, this expansion is totally axial. But as stated in Byrne et al (1980), some radial deformation could also occur, since the cut diameter of the core is slightly less than that of
Fig. 29 One Dimensional Unloading of Oilsand

Initial effective stress 7 MPa

pore pressure 3

Oilsand
the liner. Thus if we ignore this effect, the situation could be simulated by confined axial unloading of a cylindrical oil sand sample, initially under a very high effective stress and a pore pressure.

While the system shown in Fig. 29 is unloaded, its behaviour is plotted in Fig. 33 with the predictions of the OILSTRESS program.

From this it is seen that the change in total stress is apportioned in such a way between the effective stress and the total stress, that the effective stress drops rapidly to zero, while the pore pressure reduces slightly. When such a stage is approached, the volume is increased at such a high rate that the gas may be allowed to vent. On this assumption the pore pressure was dropped to atmospheric pressure instantaneously, whereby the effective stress shoots up causing a contraction in the skeleton. Further unloading takes place under drained conditions.

Venting usually occurs when the swelling is about 5-15% of the original volume. The corresponding gas porosity is 2-5%.

In Fig. 33, under the condition of zero effective stress, the gas porosity

\[ = \text{initial } n^g + \text{volumetric strain} \]

\[ = 0.015 + 0.012 \]

\[ = 2.7\% \]

This figure is believed to lie in the usual range of venting porosity.

As in the OILSTRESS program, here the effects due to shear dilation were not considered.

7.4 Dusseault's solution

Dusseault (1979) derived a rigorous one-dimensional equation of state for an undrained soil, based on certain assumptions. It expresses \( B \) (Skempton's pore pressure parameter) as,

\[ B = \frac{du}{d\sigma} = \frac{1}{1 + f(u, \sigma)} \]  \( \text{(24)} \)

where \( f(u, \sigma) \) constitutes a relationship between the skeleton compressibility, pore pressure, porosities of each phase and temperature and pressure solubility constants.

The assumptions made regarding the pore fluid behaviour are very similar to those indicated in Chapter 5, but the skeleton compressibility was expressed as a logarithmic relationship, (as
Fig. 30.a. Elastic Stresses in a Thick Walled Cylinder

- \( \sigma_\theta \)
- \( \sigma_r \)

- \( E = 1058100 \) psi
- \( \mu = 0.333 \)
- \( a = 10 \) ft
- \( b = 30 \) ft
- \( p_0 = 0 \)
- \( p_1 = -500 \) psi

- \( \sigma_\theta \) vs. \( \sigma_r \)
- \( \sigma_\theta \) and \( \sigma_r \) vs. radial distance (ft.)
- \( \sigma_\theta \) and \( \sigma_r \) vs. closed form
Fig. 30.b. Displacements in a Thick Walled Cylinder

\[ E = 1058100 \text{ psf} \]
\[ \mu = 0.333 \]
\[ a = 10 \text{ ft} \]
\[ b = 30 \text{ ft} \]
\[ p_0 = 0 \]
\[ p_1 = -500 \text{ psf} \]
Fig. 31.a. Elastic Stresses in Around a Shaft
Fig. 31.b. Elastic Displacements Around a Shaft
Elasto-Plastic Stresses Around a Shaft

Fig. 32.a. Elasto-Plastic Stresses Around a Shaft
UNLOADING OF A CYLINDRICAL SHAFT

\[ E = 1058100 \text{ psf} \]
\[ \mu = 0.333 \]
\[ a = 10 \text{ ft} \]
\[ P_0 = 2000 \text{ psf} \]
\[ P_1 = 800 \text{ psf} \]
\[ \phi = 30^\circ \]
\[ \nu = 10^\circ \]

Fig. 32.b. Elasto-Plastic Displacements Around a Shaft
Fig. 33  One Dimensional Unloading of Oilsand
Fig. 34 Comparison With Dusseault's Results
Fig. 35 Unloading of a Cylindrical Shaft in Oilsand
compared to our exponential form), as shown below,

\[ e = A - C \ln \sigma' \]  

(125)

where \( \sigma' \) - vertical effective stress
\( e \) - voids ratio
and \( A \) and \( C \) are soil parameters.

Since this was the only analytical solution available, the predictions of our program for the one-dimensional unloading was compared with Dusseault's results, as shown in Fig. 34.

First and foremost, the constitutive parameters were selected so as to give approximately the same system compliance as Eqn. 125 (for \( A=0.46 \) and \( C=0.005 \)). The soil was unloaded from an initial total stress of 308.9 kpa (corresponding to a selected depth) and a pore pressure of 147.1 kpa.

The variation of \( u \) and \( B \) with the total stress, compared well with the corresponding curves obtained from OILSTRESS program (Byrne and Grigg-1980) and Dusseault (1979).

7.5 Unloading of a cylindrical shaft in oilsand

Finally, the neighbourhood of a cylindrical opening in oilsand was modelled by a finite element domain and a number of springs which represented the effect of the soil mass up to infinity from the boundary of the domain.

The soil properties as indicated Fig. 35 were assigned to the elements. Unloading at the inner surface of the opening was done starting with an initial overall effective stress of 1.0 MPa and a pore pressure of 1.5 MPa.

First a totally undrained analysis was performed, and the convergence of the shaft was plotted as the total stress was reduced. As seen from Fig. 35, as a limiting support pressure (the total external stress below which the shaft becomes unstable) of approximately 1.2 MPa was reached, the deflections grow rapidly, and these results are in remarkably good agreement with those of the OILSTRESS.

Next, the analysis was repeated, by venting the elements up to a radius of 5m, by reducing their pore pressure to zero and maintaining drained conditions thereupon. This way the limiting support pressure could be reduced to approximately 0.15 MPa. Once again the NLSSIP and the OILSTRESS predictions were close to each other.

A characteristic feature of Fig. 35 is that the convergence of the shaft has been markedly boosted up by the shear dilation, only in the vicinity of the limiting support pressures.
CHAPTER 8

SUMMARY AND CONCLUSIONS

A finite element method for analysing the deformation behaviour of oilsand adjacent to shafts and tunnels has been developed.

An existing, nonlinear elastic incremental effective stress model is modified in two aspects, to represent the dense oilsand skeleton behaviour.

Shear dilation is incorporated in it using a temperature analogy. Rowe's stress dilatancy theory forms the basis of this.

The stresses predicted using the existing program, violated the failure criterion at the face of the tunnel, under heavy unloading conditions. A stress redistribution technique is used to remedy this.

A pore pressure model formulated on the basis of ideal gas laws is used to represent the oilsand pore fluid behaviour. Under undrained conditions the influence of the pore fluid is coupled into the oilsand skeleton by achieving volumetric compatibility between them, through an iterative process.

The present program is capable of predicting the limiting support pressures, required by underground tunnels in oilsand.

The predictions were checked against available closed form solutions and other results. They are in good agreement with the linear elastic and linear elasto-plastic closed form solutions, for the problem of a circular opening in an infinite space, deforming under plane strain conditions.

Furthermore, the program yielded results that agreed well with those obtained for the one-dimensional undrained unloading of oilsand by Dusseault (1979), and for the unloading of a circular shaft in oilsand by Byrne and Grigg (1980). The solution for the unloading of a tunnel in oilsand shows that the limiting support pressures can be reduced by venting elements to a reasonable distance from the tunnel. It is also found that the effects of shear dilation are significant only when the limiting support pressure is approached.
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APPENDIX A

A.1 Proof of $\mu_t = \frac{1}{2(1 - \frac{E_t}{3B_t})}$

If for any stress increment $\Delta \sigma = [\Delta \tau_{xz}]$, the corresponding stress increment is $\Delta \varepsilon = [\Delta \gamma_{xz}]$ then using, Generalized Hooke's law for the increment:

$$\Delta \varepsilon_x = \frac{1}{E_t} [\Delta \sigma_x - \mu_t (\Delta \sigma_y + \Delta \sigma_z)] \quad (A0)$$

$$\Delta \varepsilon_y = \frac{1}{E_t} [\Delta \sigma_y - \mu_t (\Delta \sigma_x + \Delta \sigma_z)] \quad (A1)$$

$$\Delta \varepsilon_z = \frac{1}{E_t} [\Delta \sigma_z - \mu_t (\Delta \sigma_y + \Delta \sigma_x)] \quad (A2)$$

By A0 + A1 + A2,

$$\Delta \varepsilon_x + \Delta \varepsilon_y + \Delta \varepsilon_z = \frac{1}{E_t} (1 - 2 \mu_t) (\Delta \sigma_x + \Delta \sigma_y + \Delta \sigma_z)$$

$$\Delta \varepsilon_y = \frac{3}{E_t} (1 - 2 \mu_t) \Delta \sigma_m$$

$$B_t = \frac{E_t}{3(1 - 2 \mu_t)}$$

$$\mu_t = \frac{1}{2(1 - \frac{E_t}{3B_t})} \quad (A3)$$

A.2 Derivation of the expression for $E_t$

Kondner's hyperbolic stress-strain relation from eqn. 15

$$\frac{(\sigma - \sigma)}{1 - 3} = \frac{\varepsilon}{E_t + \frac{E}{(\sigma_1 - \sigma_3)_{ult}}} \quad (A4)$$

$$(\sigma_1 - \sigma_3)\left[\frac{1}{E_i} + \frac{\varepsilon}{(\sigma_1 - \sigma_3)_{ult}}\right] = \varepsilon$$
Differentiating w.r.t. $\varepsilon$

$$(\sigma_1 - \sigma_3)\left[\frac{1}{(\sigma_1 - \sigma_3)_{ult}}\right] + \frac{d}{d\varepsilon}(\sigma_1 - \sigma_3)\left[\frac{1}{E_i + (\sigma_1 - \sigma_3)_{ult}}\right] = 1$$

$$\frac{d}{d\varepsilon}(\sigma_1 - \sigma_3) = \frac{1}{\left[\frac{1}{E_i + (\sigma_1 - \sigma_3)_{ult}}\right]} \frac{(\sigma_1 - \sigma_3)}{[1 - (\sigma_1 - \sigma_3)_{ult}]}$$

From eqn. A4,

$$\varepsilon = \frac{(\sigma_1 - \sigma_3)}{\frac{d\varepsilon}{d\varepsilon}/[1 - (\sigma_1 - \sigma_3)_{ult}]}$$

$$\frac{d}{d\varepsilon}(\sigma_1 - \sigma_3) = \frac{1}{E_i + (\sigma_1 - \sigma_3)_{ult}} \frac{1}{[\frac{1}{E_i + (\sigma_1 - \sigma_3)_{ult}}]} \frac{1}{E_i [1 - (\sigma_1 - \sigma_3)_{ult}]}$$

$$\frac{d}{d\varepsilon}(\sigma_1 - \sigma_3) = E_i [1 - (\sigma_1 - \sigma_3)_{ult}]^2$$

For triaxial tests $\frac{d}{d\varepsilon}(\sigma_1 - \sigma_3) = E_t$

$$\eta E_t = E_i [1 - (\sigma_1 - \sigma_3)_{ult}]^2$$

Using eqns. 16 and 17

$$E_t = K_E P_a \left(\frac{\sigma_3}{P_a}\right)^n \left[1 - \frac{R_f (\sigma_1 - \sigma_3)}{(\sigma_1 - \sigma_3)_f}\right]^2 \quad (A5)$$
Fig. 36. Mohr Circle at Failure
An expression for $(\sigma_1 - \sigma_3)_f$ could be derived from the Mohr-Coulomb plot in terms of $C$, $\phi$, and $\sigma_3$.

In the Fig. 36,

$$\begin{align*}
BQ &= \frac{1}{2}(\sigma_1 - \sigma_3)_f \sin \phi \\
AB &= \frac{1}{2}(\sigma_1 - \sigma_3)_f \cos \phi \\
OB &= OQ - BQ \\
&= \frac{(\sigma_1 - \sigma_3) \sigma_3}{2} - \frac{(\sigma_1 - \sigma_3)_f}{2} \sin \phi \\
&= \frac{(\sigma_1 - \sigma_3)_f}{2} + \sigma_3 - \frac{(\sigma_1 - \sigma_3)_f}{2} \sin \phi \\
&= \frac{(\sigma_1 - \sigma_3)_f}{2} [1 - \sin \phi] + \sigma_3 \\
\tan \phi &= \frac{AP}{CP} = \frac{AB - C}{OB} \\
&= \frac{1/2(\sigma_1 - \sigma_3)_f \cos \phi - C}{1/2(\sigma_1 - \sigma_3)_f + [1 - \sin \phi] + \sigma_3} \\
&= \frac{1/2(\sigma_1 - \sigma_3)_f \tan \phi - 1/2(\sigma_1 - \sigma_3)_f}{\cos \phi} \frac{\sin^2 \phi}{\sigma_3 \tan \phi} \\
&= 1/2(\sigma_1 - \sigma_3)_f \cos \phi - C \\
(\sigma_1 - \sigma_3)_f &= \frac{2C \cos \phi + 2\sigma_3 \sin \phi}{(1 - \sin \phi)} \quad (A6)
\end{align*}$$

By substituting in eqn. A5

$$E_t = K E \frac{\sigma_3}{P_e} a (\frac{P}{P_e})^n \left[1 - \frac{R_f (\sigma_1 - \sigma_3) (1 - \sin \phi)}{2C \cos \phi + 2\sigma_3 \sin \phi}\right]^2 \quad (A7)$$
Fig. 37. Particle Slip
B.1 Derivation of stress-dilatancy equation

The actual picture will be made simple in Fig. 37 by a 2-D consideration with the slip orientation, along the plane of the paper.

\[
\tan \beta = \frac{dx_3}{dx_1}
\]

From interparticle friction

\[
\tan(\phi + \beta) = \frac{L_1}{L_3}
\]

Energy ratio, E is defined as,

\[
E = \frac{\text{increment of energy applied to the contact by } L_1}{\text{Increment of work done against the external } L_3}
\]

\[
E = \frac{-L_1 \, dx_1}{L_3 \, dx_3}
\]  
(B1)

Absolute energy absorbed

\[
= L_1 \, dx_1 + L_3 \, dx_3
\]

\[
= L_1 \, dx_1 \left[1 - \frac{1}{E}\right]
\]  
(B2)

For a given energy input \((L, dx)\), absolute energy absorbed is a minimum, when \(E\) is a minimum.

From eqn. B1,

\[
E = \frac{\tan(\phi + \beta)}{\tan \beta}
\]  
(B3)
\[
\frac{dE}{d\beta} = \frac{\tan\beta \sec^2(\phi + \beta) - \tan(\phi + \beta) \sec^2\beta}{\tan^2\beta}
\]

For minimum \(E\),
\[
\tan\beta \sec^2(\phi + \beta) = \tan(\phi + \beta) \sec^2\beta
\]
\[
\sin 2\beta = \sin^2(\phi + \beta)
\]
\[
2\beta = \pi - 2\phi - 2\beta
\]
\[
\beta = \frac{\pi}{4} - \frac{\phi}{2}
\]

Substituting in eqn. B3,
\[
E_{\text{min}} = \frac{\tan(\pi/4 + \phi/2)}{\tan(\pi/4 - \phi/2)}
\]
\[
E_{\text{min}} = \tan^2(\pi/4 + \phi/2) = K_p \mu
\]  \hspace{1cm} (B4)

If all the particles satisfy the minimum energy condition in slipping, combining equations B1 and B4 gives,
\[
\frac{n}{\Sigma L_1 dx_1} = K_p \mu
\]  \hspace{1cm} (B5)

where \(n\) refers to the no. of slipping particles.

Under triaxial conditions (axi-symmetric)
\[
\sigma_a d\varepsilon_a = \Sigma L_1 dx_1
\]

and
\[
2\sigma_r d\varepsilon_r = \Sigma L_3 dx_3
\]

From eqn. B5,
\[
\frac{\sigma_a d\varepsilon_a}{2\sigma_r d\varepsilon_r} = K_p \mu
\]
If \( dv \) is the volumetric strain

\[
dv = d\varepsilon_a + 2d\varepsilon_r
\]

\[
\frac{dv}{d\varepsilon_a} = 1 + \frac{2d\varepsilon_r}{d\varepsilon_a}
\]

\[
\frac{\sigma_a}{\sigma_r} = (1 - \frac{dv}{d\varepsilon_a}) K_\mu
\]

or

\[
\frac{\sigma_a}{\sigma_r} = (1 - \frac{dv}{d\varepsilon_a}) \tan^2(45 + \phi/2)
\]

(B6)
Fig. 38. Effect of Pressure and Temperature on Oilsand
APPENDIX C

C.1 Proof of the expression for the change in pore pressure in undrained oilsand.

Consider a unit volume of oilsand under a pressure and temperature of \( U_0 \) and \( T_0 \) respectively. (Fig. 38).

The final state \( U_1, T_1 \) as indicated is attained by the changes \( \Delta u \) and \( \Delta T \) occurring simultaneously, or one after the other.

The initial porosities \( n_g, n_w \) and \( n_o \) become the initial volumes of the three phases as, a unit total volume is considered.

\( \Delta V \) - the volume of gas coming out of solution due to the increase in pressure \( \Delta u \), measured at standard conditions of \( U_a \) and \( T_a \), can be obtained by using equation 108.

\[
V_g = -H V \Delta u \\
\Delta V_1 = -(H_o n_o + H_w n_w) \Delta u
\]  
(C1)

Similarly if \( \Delta V_2 \) is the volume of gas coming out of the solution due to the increase in temperature of \( \Delta T \), (measured at \( U_a \) and \( T_a \)). From equation 109,

\[
V_g = \beta V \Delta T \\
\Delta V_2 = (\beta_o n_o + \beta_w n_w) \Delta T
\]  
(C2)

where \( H_o, H_w, \beta_o, \beta_w \) are the pressure and temperature solubility constants for oil and water phases.

From eqns. C1 and C2 the total volume of gas coming out at STP,
\[ \Delta V_1 + \Delta V_2 = (\beta_o n_o + \beta_w n_w) \Delta T \]
\[ - (H_o n_o + H_w n_w) \Delta u \]

From \[ PV = nRT \]
the volume of the gas coming out at the final state is,

\[ \Delta V_3 = \left[ (\beta_o n_o + \beta_w n_w) \Delta T - (H_o n_o + H_w n_w) \Delta u \right] \frac{U_a}{T_a} \frac{T_1}{T_1} \]  

(C3)

The expansion of the free gas also takes place under changes of \( \Delta u \) and \( \Delta T \); and from eqn. 107 if \( \Delta V_4 \) is the increase in volume of the free gas,

\[ \Delta V_4 = \frac{u_o}{u_1} \frac{T_1}{T_o} - 1 \]  

(C4)

\[ \Delta V_g = \Delta V_3 + \Delta V_4 \]
\[ = n_g \left[ \frac{u_o}{u_1} \frac{T_1}{T_o} - 1 \right] + \frac{u_a}{u_1} \frac{T_1}{T_a} \left[ (\beta_o n_o + \beta_w n_w) \Delta T \right] \]
\[ - (H_o n_o + H_w n_w) \Delta u \]  

= the net increase in the gas volume.

Now if \( \Delta \varepsilon_v \) is the compressive strain of the sand skeleton,

The increase in volume of the sand skeleton

\[ = -\Delta \varepsilon_v \times 1 \]
\[ = -\Delta \varepsilon_v \]

Since the conditions are undrained, and following the assumptions regarding the compressibility of oil water and sand particles,

\[ -\Delta \varepsilon_v = n_g \left[ \frac{u_o}{u_1} \frac{T_1}{T_o} - 1 \right] + \frac{u_a}{u_1} \frac{T_1}{T_a} \left[ (\beta_o n_o + \beta_w n_w) \Delta T \right] \]
\[ - (H_o n_o + H_w n_w) \Delta u \]
which simplifies to

\[
\Delta u = \frac{u_o \Delta \varepsilon_v + \frac{T_1}{T_a} u_o (\beta_o n_o + \beta_w n_w) \Delta T + \frac{n_o u_o}{T_o} \Delta T}{n_g \Delta \varepsilon_v + \frac{u_o T_1}{T_a} (H_o n_o + H_w n_w)}
\]

C.2 Review of constant \( H \)

Henry's law for the solubility of gases in liquids mentioned in Chapter 5, could be stated as follows.

The concentration of a solute in a solution is proportional to the partial pressure of the solute at the interface; at constant temperature or mathematically

\[
P_{\text{solute}} = k \frac{N_g}{N_g + N_s}
\]

\( N_g \) - no. of moles of the gas in solution

\( N_s \) - " " " " " " " " solvent.

\( K \) - Henry's law constant, specified for the particular temperature.

Since Henry's law could be accurately applied for dilute solutions

\[ N_g + N_s = N_s \]

Further if \( V_s \) is the volume of the solvent and \( \rho \) is its density at that temperature,

\[ N_s = \frac{\rho V_s}{M} \]
M = molecular wt. of the solvent.

Substitution in eqn. C5 yields,

\[ P_{\text{solute}} = \frac{kM}{\rho V} N_g \]

Since it was assumed that the partial pressure all of phases in oilsand interstices is the same,

\[ P_{\text{solute}} = u \]

\[ u = \frac{kM}{\rho V} N_g \]  \hspace{1cm} (C6)

C.2.a Changes in pressure under constant temperature (T)

The quantities, \( k, M, \rho \) and \( V_s \) are unaffected by the pressure changes, and by differentiating eqn. C6 we get

\[ \Delta N_g = \frac{\rho V}{kM} \Delta u \]  \hspace{1cm} (C7)

which gives the no. of moles sent into solution, on increasing the pressure, by \( \Delta u \).

If \( \Delta V_g \) is the volume going into solution (measured at the present pressure \( u_o + \Delta u \) and the constant temperature \( T \));

using the ideal gas equation

\[ pV = nRT \]

\[ (u_o + \Delta u) \Delta V_g = \Delta N_g RT \]

Substituting in eqn. C7,

\[ \frac{(u_o + \Delta u)}{RT} \Delta V_g = \frac{\rho V}{kM} \Delta u \]

\[ \Delta V_g = \left( \frac{\rho RT}{kM} \right) V \left( \frac{\Delta u}{u_o + \Delta u} \right) \]  \hspace{1cm} (C8)

If we refer this volume to the STP of \( U_a \) and \( T_a \) we get,

\[ \Delta V_g u_o, T_a = \left( \frac{\rho RT}{kM} \right) V \frac{\Delta u}{(u_o + \Delta u)} \left( \frac{u_o + \Delta u}{T} \right) \frac{T_a}{u_o} \]

$\Delta V_g \Delta u = \frac{\rho RT}{kM_u a} V_L \Delta u \tag{C9}$

$\Delta V_g \Delta u = H V_L \Delta u$

By comparison of eqns. C9 and 108,

$$H = \frac{\rho RT}{kM_u a} \tag{C10}$$

If we ignore the temperature dependence of $\rho$ - the density of the liquid, and make sure that all the measurements are made at $u_a$ and $T_a$ always, then $M$, $\rho$, $R$, $T_a$ and $u_a$ will be constants; but not $k$.

It should be remembered here that, although the volume of gas is referred to the STP of $u_a$ and $T_a$ for simplicity, the gas could be evolved at any other temperature - $T$. Thus, from the previous definition of Henry's constant ($k$) in eqn. C5, $k$ is specified for $T$.

$\Delta k = f(T)$

From this result and eqn. C10 we can conclude that the coefficient $H$ is highly dependent on the temperature at which the gas is evolved.
D.1 Constitutive matrix

The generalized Hooke's law for incremental elasticity, can be stated as,

\[ \Delta \varepsilon_x = \frac{1}{E_t} [\Delta \sigma_x - \mu_t (\Delta \sigma_y + \Delta \sigma_z)] \]  \hspace{1cm} (A0)

\[ \Delta \varepsilon_y = \frac{1}{E_t} [\Delta \sigma_y - \mu_t (\Delta \sigma_x + \Delta \sigma_z)] \]  \hspace{1cm} (A1)

\[ \Delta \varepsilon_z = \frac{1}{E_t} [\Delta \sigma_z - \mu_t (\Delta \sigma_y + \Delta \sigma_x)] \]  \hspace{1cm} (A2)

\[ \Delta \gamma_{xy} = \frac{\Delta \tau_{xy}}{G_t} \]  \hspace{1cm} (D1)

\[ \Delta \gamma_{xz} = \frac{\Delta \tau_{xz}}{G_t} \]  \hspace{1cm} (D2)

\[ \Delta \gamma_{yz} = \frac{\Delta \tau_{yz}}{G_t} \]  \hspace{1cm} (D3)

where \( G_t = \frac{E_t}{2(1+\mu_t)} \)

From the discussion in Chapter 4.1, (for plane strain)

\[ \Delta \varepsilon_z = \Delta \gamma_{xz} = \Delta \gamma_{yz} = 0 \]

Substitution in eqns. D1 - D3 gives

\[ \Delta \tau_{xy} = \frac{E_t}{2(1+\mu_t)} \]  \hspace{1cm} (D4)

\[ \Delta \tau_{xz} = \Delta \tau_{yz} = 0 \]

Substitution in eqn. A2 gives

\[ \Delta \sigma_z = \mu_t (\Delta \sigma_y + \Delta \sigma_x) \]  \hspace{1cm} (D5)

which is equivalent to

\[ \sigma_z = \mu_t (\sigma_y + \sigma_x) \]
Eqn. D5 along with eqns. A0 and A1 yield

\[
\Delta \varepsilon_x = \frac{1}{E_t} [(1-\mu_t^2)\Delta \sigma_x - \mu_t(1+\mu_t)\Delta \sigma_y]
\]

\[
\Delta \varepsilon_y = \frac{1}{E_t} [(1-\mu_t^2)\Delta \sigma_y - \mu_t(1+\mu_t)\Delta \sigma_x]
\]

In the matrix form

\[
\begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y
\end{bmatrix} = \frac{1}{E_t} \begin{bmatrix}
(1-\mu_t^2) & -\mu_t(1+\mu_t) \\
-\mu_t(1-\mu_t) & (1-\mu_t^2)
\end{bmatrix} \begin{bmatrix}
\Delta \sigma_x \\
\Delta \sigma_y
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y
\end{bmatrix} = \begin{bmatrix}
(1+\mu_t) \\
E_t
\end{bmatrix} \begin{bmatrix}
(1-\mu_t^2) & -\mu_t(1+\mu_t) \\
-\mu_t(1-\mu_t) & (1-\mu_t^2)
\end{bmatrix} \begin{bmatrix}
\Delta \sigma_x \\
\Delta \sigma_y
\end{bmatrix}
\]

on inversion

\[
\begin{bmatrix}
\Delta \sigma_x \\
\Delta \sigma_y
\end{bmatrix} = \frac{E_t}{(1+\mu_t)(1-2\mu_t)} \begin{bmatrix}
1-\mu_t & \mu_t \\
\mu_t & 1-\mu_t
\end{bmatrix} \begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y
\end{bmatrix}
\]

(E6)

Equations D4 and D6 could be combined to give,

\[
\begin{bmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \tau_{xy}
\end{bmatrix} = \frac{E_t}{(1+\mu_t)(1-2\mu_t)} \begin{bmatrix}
1-\mu_t & \mu_t & 0 \\
\mu_t & 1-\mu_t & 0 \\
0 & 0 & (1-2\mu_t)
\end{bmatrix} \begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \delta_{xy}
\end{bmatrix}
\]

(D7)

Now, if we define

\[
B'_t = \frac{3B_t}{2(1+\mu_t)} \quad \text{and} \quad G'_t = \frac{E_t}{2(1+\mu_t)} = G_t
\]

but

\[
B'_t = \frac{E_t}{3(1-2\mu_t)}
\]

(A3)
Further $B'_t + G'_t = \frac{E_t}{2(1+\mu_t)(1-2\mu_t)} + \frac{E_t}{2(1+\mu_t)}$

Substituting these values in the coefficients of the $[D]$ matrix in the eqn. D7,

$$
\begin{bmatrix}
\Delta \sigma_x \\
\Delta \sigma_y \\
\Delta \tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
B'_t + G'_t & B'_t - G'_t & 0 \\
B'_t - G'_t & B'_t + G'_t & 0 \\
0 & 0 & G'_t
\end{bmatrix}
\begin{bmatrix}
\Delta \varepsilon_x \\
\Delta \varepsilon_y \\
\Delta \gamma_{xy}
\end{bmatrix}
$$

D.2 Force-Displacement relationship

As mentioned in Chapter 4, the same function is used to define the shape of the element as well as the displacements within it. This function is expressed in natural co-ordinates $(\xi$ and $\eta)$ - a system of co-ordinates intrinsic to the element.
Fig. 39. Isoparametric Element
Thus since the displacements are to be related to the system co-ordinates n and y, first of all a relationship between these two co-ordinates has to be found, i.e., the shape of the element has to be defined by natural co-ordinates; as follows.

\[
\begin{bmatrix}
X \\
Y
\end{bmatrix} =
\begin{bmatrix}
N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}
\begin{bmatrix}
X_1 \\
Y_1 \\
Y_4
\end{bmatrix}
\]  
(D8)

where \(N_1, N_2, N_3, \) and \(N_4\) are called **shape functions** whose values are given by,

\[
N_1 = \frac{(1-\xi)(1-n)}{4} \quad N_2 = \frac{(1-\xi)(1-n)}{4}
\]
\[
N_3 = \frac{(1+\xi)(1+n)}{4} \quad and \quad N_4 = \frac{(1-\xi)(1+n)}{4}
\]

This transformation, in fact maps the quadrilateral into a square as shown by Fig. 39.

In the finite element procedure, the next step is to assume a compatible set of displacements.

\[
\begin{bmatrix}
u \\
v
\end{bmatrix} = [N]
\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_8
\end{bmatrix} = [N] \delta
\]  
(D9)
u and \( v \) are the displacements in \( x, y \) directions at any point \((x, y)\) or \((\xi, \eta)\); whereas the \( \delta_i \) values are the nodal values of the \( u \) and \( v \). In keeping with the isoparametric element properties, \([N]\) happens to be the same matrix introduced in eqn. D8.

The function selected in eqn. D9, defines the assumed displacement field, and in order for the solution obtained to converge at the correct solution, on subdivision into smaller elements, this assumed field should satisfy two properties, namely

a. admissibility

b. completeness, which will not be discussed herein.

Eqns. D8 and D9 could be combined with

\[
\begin{align*}
\epsilon_x &= \frac{\partial v}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{align*}
\]

to obtain,

\[
\Delta \tilde{\xi} = [B] \tilde{\delta}
\]  

(D10)

where \([B]\) - the strain-displacement matrix - a function of the nodal co-ordinates.

Now from eqn. D7,

\[
\Delta \tilde{\xi}' = [D] \Delta \tilde{\xi}
\]  

(D7)

the effective stress has been introduced here as it is the stress responsible for the deformation.

\[
\Delta \tilde{\xi}' = \Delta \tilde{\xi} - \Delta u
\]

\[
\frac{1}{2} \Delta \tilde{\xi} - \Delta u[1] = [D] \Delta \tilde{\xi}
\]

0

combining with eqn. D10

\[
\frac{1}{2} \Delta \tilde{\xi} - \Delta u[1] = [D][B] \tilde{\delta}
\]  

(D11)
External vertical work done by nodal displacements riding through the nodal force increments
\[ \delta^T \Delta f \]

Internal vertical work done by the compatible strain increments riding through the stress increments
\[ = \int \Delta \varepsilon^T \Delta \sigma \, dv \]

From the principle of vertical work (since the force-stress equilibrium exists)

Ext. virtual work = Int. virtual work
\[ \delta^T \Delta f = \int \Delta \varepsilon^T \Delta \sigma \, dv \]

Using eqn. D11
\[ = \int \Delta \varepsilon^T ([D][B] \dot{\xi} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Delta u) \, dv \]

From eqn. D10, \( \Delta \varepsilon^T = \dot{\xi}^T [B]^T \)
\[ \dot{\xi}^T \Delta f = \int \dot{\xi}^T [B]^T ([D][B] \dot{\xi} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Delta u) \, dv \]
\[ = \dot{\xi}^T \left( \int [B]^T [D][B] \, dv \right) \dot{\xi} + \int [B]^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} dv \Delta u \]
\[ = \int [B]^T [D][B] \, dv \dot{\xi} + \int [B]^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} dv \Delta u \]

If we define \( \int [B]^T [D][B] \, dv = [K] = \) element stiffness matrix
and \( \int [B] \begin{bmatrix} 1 \\ 0 \end{bmatrix} dv = [K^*] \)

the force-displacement relationship results
\[ \Delta f = [K] \dot{\xi} + [K^*] \Delta u \]

If \([K^*] \Delta u\) is treated as an additional set of nodal deflections in modified nodal force increment vector \( \Delta f' \) is expressed as
\[ \Delta f' = [K] \delta \]  

If \( n \) is the total no. of degrees of freedom for the system, eqn. D12 can be arranged, in such a way that the generalized coordinates of this first element, tally with their corresponding degree of freedom (if d.o.f. is non-zero) then

\[ \Delta F_1 = [K_1] \Delta_1 \]

\[ \sum \Delta F_1 = \sum [K_1] \Delta_1 \] (summed to all elements)

yields

\[ \Delta F = [K] \Delta \]

Since the coefficients of \([B]\) are functions of the nodal coordinates and those of \([D]\) are only stress-strain parameters

\[ [K] = [B]^T[D][B] V_e \]

\( V_e \) - Vol. of the element

or if a unit thickness is considered and \( A_e \) - area of the element,

\[ [K] = [B]^T[D][B] A_e \]  

(D13)
Fig. 40. Approximation of an Infinite Problem
Spring Stiffness $\lambda$

With reference to Fig. 40, if the uniforming distributed load in the spring boundary is $p$, and its equivalent point load on each spring is $P$

$$2\pi a p = NP$$  \hspace{1cm} (E1)

$N$ - no. of springs.

Radial deformation (elastic) resulting at the inside boundary of an infinite mass, under an inside pressure of $p$ is

$$\delta = \frac{1+\mu}{E} a p$$  \hspace{1cm} (E2)

(This is further discussed in Appendix F)

Combining eqns. E1 and E2,

$$\frac{2\pi a E \delta}{a (1+\mu)} = NP$$

$$P = \frac{2\pi}{N} \frac{E}{(1+\mu)} \delta$$  \hspace{1cm} (E3)

By comparison with $P = \lambda \delta$

$$\lambda = \frac{2\pi}{N} \frac{E}{(1+\mu)}$$

For non linear incremental elasticity

$$\lambda = \frac{2\pi}{N} \frac{E_t}{(1+\mu_t)}$$

$$= \frac{4\pi}{N} \frac{G_t}{E_t}$$

Since $G_t = \frac{E_t}{2(1+\mu_t)}$

$$\therefore \lambda = \frac{4\pi}{N} G_t$$
Appendix F

Elastic Closed Form Solutions

The elastic stresses for axisymmetric, plane strain problems, (using an Airy stress function), can be shown to be the form

\[\sigma_r = \frac{A}{r^2} + B(1+2 \log r) + 2C \quad \text{(F1)}\]
\[\sigma_\theta = \frac{A}{r^2} + B(3+2 \log r) + 2C \quad \text{(F2)}\]
\[\tau_{r\theta} = 0 \quad \text{(F3)}\]

(Timoshenko, 1941)

For continuous regions, as shown in Fig. 27 a full circle solution exists and the term B - which gives rise to tangential displacements, vanishes.

\[\sigma_r = \frac{A}{r^2} + 2C \quad \text{(F4)}\]
\[\sigma_\theta = \frac{A}{r^2} + 2C \quad \text{(F5)}\]

At \( r = a \), \( \sigma_r = P_i \) (comp. stress curve)

and at \( r = b \), \( \sigma_T = P_o \)

Substituting in eqn. F4

\[P_i = \frac{A}{a^2} + 2C\]
\[P_o = \frac{A}{b^2} + 2C\]

which gives \( A = (P_i - P_o) \frac{a^2 b^2}{(b^2-a^2)} \)

and \( 2C = -\frac{(a^2 P_i - b^2 P_o)}{(b^2-a^2)} \)

\[\sigma_r = \frac{-(a^2 P_i - b^2 P_o)}{b^2-a^2} + \frac{(P_i - P_o) a^2 b^2}{(b^2-a^2) r^2}\]
\[ \sigma_\theta = \frac{-(a^2 p_i - b^2 p_0)}{(b^2-a^2)} - \frac{(p_i-P_0)a^2 b^2}{(b^2-a^2)r^2} \]

\[ \tau_{r\theta} = 0 \]

Using generalized Hooke's laws,

\[ \varepsilon_\theta = \frac{1}{E} \left[ \sigma_\theta - \mu (\sigma_r + \sigma_z) \right] \]  

(F6)

\[ \sigma_\theta = \mu (\sigma_r + \sigma_\theta) \]

(for plane strain condition)

\[ \varepsilon_\theta = \frac{(1+\mu)}{E} \left[ (1+\mu) \sigma_\theta - \mu \sigma_r \right] \]  

(F7)

Substituting eqns. F4 and F5 in eqn. F7

\[ \varepsilon_\theta = \frac{(1+\mu)}{E} \left[ (1-2\mu)(2C-A/r^2) - \mu(2C+A/r^2) \right] \]

\[ = \frac{(1+\mu)}{E} \left[ 2C - \frac{A}{r^2} - 4C \mu \right] \]

\[ = \frac{(1-2\mu)(1+\mu)2C}{E} - \frac{(1+\mu)A}{E} \frac{1}{r^2} \]

Under axisymmetric conditions,

\[ \varepsilon_\theta = \frac{\delta}{r} \]

where \( \delta \) is the radial displacement

\[ \delta = \frac{1(1-2\mu)(1+\mu)2C}{E} \frac{1}{r} + \frac{(1+\mu)A}{E} \frac{1}{r} \]

Subs. for 2C and A

\[ \delta = \frac{(1-2\mu)(1+\mu)(a^2 p_i - b^2 p_0)}{E(b^2-a^2)} \frac{1}{r} + \frac{(1+\mu)(P_i-P_0)a^2 b^2}{E(b^2-a^2)r} \]
For the case of a shaft in an infinite medium, the solution could be deduced, from the limiting case of \( b \to \infty \),

From eqns. 111 and 112

\[
\sigma_r = P_0 + \frac{(P_i - P_0)a^2}{r^2}
\]

and

\[
\sigma_\theta = P_0 - \frac{(P_i - P_0)a^2}{r^2}
\]

\( \tau_{r\theta} = 0 \)

Here \( P_0 \) could be interpreted as the initial stress of the whole soil mass.

From eqn. 113, the initial deflection due to a constant pressure \( P_0 \) is

\[
\delta_i = \frac{-(1-2\nu)(1+\nu) r P_0}{E}
\]

(Since \( P_i = P_0 \))

Similarly the total deflection, when \( P_0 \) and \( P_i \) are applied is

\[
\delta_f = \frac{(1-2\nu)(1+\nu) P_0}{E} + \frac{(1+\nu)(P_i - P_0)a^2}{E r}
\]

\[
\delta_i + \delta
\]

The additional deflection due to the change of the inner surface pressure from \( P_0 \) to \( P_i \) is \( \delta \)

\[
\delta = \frac{(1+\nu)}{E} \frac{a^2}{r} (P_i - P_0)
\]
Fig. 41. Elasto-Plastic Solution
APPENDIX G

Elasto-plastic solution

(Based on Hughes et al (1977), solution for an expanding cylindrical cavity in sand).

When shearing is introduced into the soil mass by either increasing or decreasing the pressure at the inside face, the principal stress ratio, at any point will change in a manner shown in Fig. G.b. The immediate vicinity of the cavity will reach failure first, and on further shearing the failure zone will be propagating away as seen in Fig. 41a.

The positions of the points P, Q and R in the stress-strain and volume change curves are marked in the figures. For analytical convenience these curves are idealized in each case, such that states of all points lie on ST. It has already been discussed in Chapter 3.9, how this idealization is justified by simple shear test results.

For the plastic zone defined by (a < r < R)

\[ \frac{\sigma'}{\sigma} = N = \frac{1 + \sin \phi_{\text{max}}}{1 - \sin \phi_{\text{max}}} = \tan^2 \left(45^\circ + \frac{\phi_{\text{max}}}{2}\right) \]

Equilibrium equation in polar co-ordinates,

\[ \frac{d\sigma'}{dr} + \frac{\sigma' - \sigma}{r} = 0 \]  

\[ \frac{d\sigma'}{dr} = (N-1) \log \frac{R}{r} \]

R also lies on the inner boundary of the elastic zone and on applying eqn. 115,
\[ \sigma'_{R} = P_{o} - (\sigma'_{R} - P_{o}) \frac{R^2}{R^2} \]
\[ = 2P_{o} - \sigma'_{R} \]

But from eqn. 118, \( \sigma'_{R} = N \sigma'_{R} \)
\[ \sigma'_{R} = 2P_{o} - \sigma'_{R} \]
\[ \sigma'_{R} = \frac{2P_{o}}{1+N} \]
\[ = \frac{2P_{o}}{1-N} (1-\sin \phi_{max}) \]
\[ \sigma'_{R} = P_{o} (1-\sin \phi_{max}) \]

Since eqn. 121 satisfies the inner b.c.,
(i.e. \( \sigma'_{R} = P_{i} \) at \( r = a \))

\[ \log \frac{P_{i}}{\sigma'_{R}} = (1-N) \log \frac{R}{a} \]
\[ \frac{P_{i}}{\sigma'_{R}} = (\frac{R}{a})^{1-N} \quad (G3) \]

Substituting \( \sigma'_{R} \) from eqn. 119 in eqn. G3,

\[ \frac{P_{i}}{P_{o}(1-\sin \phi_{max})} = (\frac{R}{a})^{1-N} \]
\[ \frac{R}{a} = \left[ \frac{P_{i}}{P_{o}} \frac{1}{(1-\sin \phi_{max})} \right]^{(1-N)} \]
\[ = \left[ \frac{P_{i}}{P_{o}} \frac{1}{(1-\sin \phi_{max})} \right]^{(1-N)} \]
\[ R = a \left[ \frac{P_{o}}{P_{i}} (1-\sin \phi_{max}) \right]^{(1-N)} \]

For the shear dilation from eqn. 56

\[ \sin v = - \frac{dv}{dy} \]
Since \( v \) is constant along ST
\[
v = -(\sin v) \gamma + C
\]
The constant C results from the initial compression that occurs mainly in loose sands. For dense sand C could be ignored without loss of accuracy
\[
\gamma = -(\sin v) y
\]
\[\text{For plane strain conditions,} \]
\[
\gamma = \epsilon_y + \epsilon_\theta \quad (v \epsilon_z = 0)
\]
\[
\gamma = \epsilon_\theta - \epsilon_r \quad \text{(since } \epsilon_y \text{ and } \epsilon_\theta \text{ are principle strains with } \gamma_{\theta\theta} = 0 \text{ due to axisymmetry)}
\]
\[
\epsilon_r = \text{radial compressive strain}
\]
\[
\epsilon_\theta = \text{circumferential comp. strain}
\]
From eqn. G4
\[
\epsilon_r + \epsilon_\theta = -(\sin v) (\epsilon_\theta - \epsilon_r)
\]
\[
\epsilon_r = \frac{(1+\sin v)}{(1+\sin v)} \epsilon_\theta
\]
But \[
\frac{1+\sin v}{1-\sin v} = \tan^2(45+v/2) \quad D
\]
where D is the dilation factor in the stress-dilatancy relation
\[
\epsilon_r = -D \epsilon_\theta
\]
For axisymmetric polar co-ordinate system
\[
\epsilon_r = \frac{du}{dr}
\]
\[
\epsilon_\theta = \frac{u}{r}
\]
u - radial deflection
Substituting in eqn. G5,
\[
\frac{d u}{d r} = -D \frac{u}{r}
\]

(G6)

Imposing the outer b.c of \( u = u_R \) at \( r = R \) and integrating

\[
\log \frac{u}{u_R} = D \log \frac{R}{r} \\
\Rightarrow \frac{u}{u_R} = \left(\frac{R}{r}\right)^D
\]

or

\[
u = r \left(\frac{R}{r}\right)^{D+1} \frac{u_R}{R}
\]

\( u_R \) can be obtained from applying eqn. 117 for the outer boundary of the elastic zone

\[
u_R = \frac{1+\mu}{E} \frac{R^2}{R} (\sigma'_R - P_O)
\]

But \( \sigma'_R = P_O (1 - \sin \phi_{max}) \)

\[
\therefore \nu_R = - \frac{(1+\mu)}{E} R P_O \sin \phi_{max}
\]
If a unit volume of soil is considered, the volumes of each phase become their respective porosities, as shown in Fig. C.a.

After a volumetric strain of $\varepsilon_v$ (increase of volume) the situation is depicted by Fig. C.b, in accordance with the assumption that soil grains, water and bitumen are incompressible.

Thus if $n'_{g}$, $n'_{o}$ and $n'_{w}$ are the new porosities,

$$n'_{g} = \frac{n_{g} + \varepsilon_v}{1 + \varepsilon_v} \quad (H1)$$

$$n'_{o} = \frac{n_{o}}{1 + \varepsilon_v} \quad (H2)$$

$$n'_{w} = \frac{n_{w}}{1 + \varepsilon_v} \quad (H3)$$

Following the sign convention that compressive strains are positive

$$n'_{g} = \frac{n_{g} - \varepsilon_v}{1 - \varepsilon_v} \quad (H4)$$

$$n'_{o} = \frac{n_{o}}{1 - \varepsilon_v} \quad (H5)$$

$$n'_{w} = \frac{n_{w}}{1 - \varepsilon_v} \quad (H6)$$
APPENDIX I

PROGRAM ORGANISATION

USERS' MANUAL

I.1 Variable Description

The following additional variables have been used in the modified NLSSIP, where the stress redistribution technique and the shear dilation are found.

AREA - Area of any element at the beginning of any load case.

BULK - Bulk modulus of any element, used in the present iteration.

BULKS - Bulk modulus of any element, used in the previous iteration.

DELEPS - Additional strain to be introduced in the x and y directions for any dilating element.

DELV - Total volumetric strain in the element.

DILFAC = 1 for dilating elements.
        = 0 for other elements.

DSIG - Twice the change in mean normal stress for any element.

EPSIN - The initial shear strain(%) at which the dilating starts for any material (if constant dilation rate option is used).

ETA - Constant dilation angle. (if constant dilation angle option is used).

IT = 1 for trials.
    = 2 for the final iteration, where tape recording is done and results are printed.

ITR - Counter of the no. of iterations.

MMM - Counter of the no. of elements without a satisfactory dilation.

NOPT = 1 if constant dilation angle option is used.
       = 2 if stress-dilatancy theory is used.

NITR - Maximum no. of iterations required by the user.

NMI1 - Counter of the no. of elements still violating the yield criterion.
PHYCV  - The constant volume friction angle for any material.

PRS3  - The array containing the maximum shear strain for any element.

PRS5  - The array containing the principal stress ratio for any element.

STAR  = 0 for the first run in a load case.
    = 1 for the iterations.

Apart from the above described, the following variables have been used in programming the pore pressure model.

CAPPA  - Elastic spring stiffness.

BSAVE  - Array saving the load vector for the iterations.

DELU  - Pore pressure change for any element.

DOFAS  - Degree of freedom connected with the first spring. (located by the program itself).

FAC  = 1 for the elements in the failure region.
    = 0 for others.

ITR1  - Counter of the elements without an accurate pore pressure change.

IVENT  = 1 for the elements to be vented.
       = 0 for others.

MOON  - Counter of the load case number.

NAVE  = 1 For elements for which pore pressure averaging is done.
       = 0 for others.

NEL1  - The element no. of the element to be vented.

NREAD  = 1 if pore pressures are to be read.
        = 0 if the program assigns zero pore pressure changes initially.

NSPRIN  - No. of elastic springs.

NVEL  - Total no. of elements to be vented.

NVENT  = 1 if venting is to be done for a certain load case.
        = 0 if venting is not to be done.

PS1,PS2  - Arrays containing the previously located limits of the accurate pore pressure change.

PWI  - Array containing the pore pressure increment used in
the previous iteration for any element.

PWO - Pore pressure at the inception of loading, of any element. (entered by the user.)

PWW - Pore pressure change (guessed), for any element to be used in the next iteration.

SUN = 0 at the first entry into the subroutine ITRAN.
     = 1 for subsequent entries.
1.2 Description of subroutines

The subroutines of the original NLSSIP affected by the modifications will be described herein.

LAYOUT - Reads and prints the soil input data. (Including dilation data.) In the program by which oilsand can be analysed it reads the original pore pressures in each element, the initial porosities, and the solubility constants for each material and prints them. Finally it computes and prints the initial stresses and the initial moduli values for the soil elements.

FVECT - Calculates the nodal point forces due to weights of added elements, reads concentrated load data and/or boundary pressure data, prints the nodal point forces, sets up the force vector and stores it in array BSAVE, for the use in the future iterations. Further it modifies this vector in every iteration depending on the additional nodal forces due to dilation. Later, when stress redistribution is done it forms a separate force vector for each iteration. For the subsequent use in the current run the force vector is stored in tape 10.

POREF - Reads element incremental pore water pressures in the first run in a particular load case, computes equivalent nodal forces, stores current areas of all elements in the AREA array, and modifies tape 10, by adding the force vector due to Δu. If oilsand is being analysed, it checks whether any element is to be vented, after reading the venting data. (The modified Δu values for any element are used in calculating the nodal forces.)

ELAW - Calculates the moduli values for the soil elements in accordance with the magnitudes of the stresses. When redistributing the stresses it uses μ of 0 for the elements in the failure region.

ISQUAD - Formulates the constitutive equations, forms the element stiffness matrix for each element, and writes it on tape 2. If elastic springs are used to analyse an infinite domain, it calculates the required spring stiffness based on the elastic modulii, and modifies the diagonal terms of the stiffness matrices of the elements concerned. It also forms the strain displacement matrix for each element, and writes it on tape 11.
ISRS LT - Calculates stress/strain increments, average stresses and cumulative strains for the trial iterations and evaluates the modulii for each element after each iteration. In the final iteration it calculates the incremental and cumulative stresses and strains, for the elements and accordingly updates the modulii values to be used in the next load case. The internal forces in the structural elements are also computed and printed. Further it prints out, the displacements, strains and stresses for the elements, before winding up the iterative procedure.

DILAT - Calculates the change in maximum shear strain for each element in a load case, and by using the average stress condition within the load case (if stress-dilatancy theory is used), determines the dilation angle to be used, in computing the dilation for each element. It also evaluates the additional nodal forces to be introduced. Moreover, here a check is imposed on each element to ensure that a plastic volume change, within a reasonable tolerance of the exact value has been introduced into each element.

ITRAN - Calculates the volumetric strain that has occurred so far, and by using the incremental volumetric strain, gets subroutine PORVOL to predict the $\Delta u$ for each non-venting element. In addition to this it prepares $\Delta u$ values to be used in the next iteration, employing an averaging technique if necessary. After ascertaining that the correct $\Delta u$ is predicted for all elements (within a reasonable limit), it also updates the pore pressure of each element.

PORVOL - Updates the porosities of each element, and using the solubility coefficients and the incremental volumetric strain, calculates $\Delta u$.

LSHED - Selects the elements violating the Mohr-Coulomb failure criterion, and calculates an incremental set of nodal forces to be applied to such elements.
I.3 Instructions on data input

Input of data is done according to NLSSIP manual (Byrne and Duncan, 1979), except for the below items.

Modified NLSSIP

1. Control cards
After the control card 1(b) enter,

1 - 5 NCHECK  
= 1 if the maximum no. of iterations is specified.
= 2 if the maximum no. of iterations is set to 3.

Next card
1 - 5 NITR  - Maximum no. of iterations required, if NCHECK=1

2. Material property cards
After the card 4(b) enter,
1 - 5 NOPT  
= 1 if constant dilation angle option is used.
= 2 if stress - dilatancy theory is used.

Next card
If NOPT=1 enter,
1 - 5 L
5 - 15 ETA  - Constant dilation angle.
16 - 25 EPSIN  - The shear at which dilation starts. (%)

If NOPT=2 enter,
1 - 5 L
6 - 15 PHYCV  - Constant volume friction angle.

Modified NLSSIP (for analysing oilsand)

1. Control cards
In the card 1(b), enter NITR instead of IPW
66 - 70 NITR  - Maximum no. of iterations required.
  Enter 5 if NCHECK=2.

after card 1(b) enter,
1 - 5 NREAD  - 1 if pore pressure increments are to read for all load cases.
  0 if pore pressure increments are not to be read.
6 - 10 NSPRIN  - No. of elastic springs used.

Next card
1 - 5 NCHECK  = 1 (As previously mentioned.)
= 2 if max. no. of iterations is set to 5.

2. Material property cards

Dilation data - enter dilation data as specified above, after the card 4(b).

Pore fluid data - enter after the dilation data.

1 - 5 KL - Material number.
6 - 15 PN(KL,1) - Water porosity.
16 - 25 PN(KL,2) - Oil porosity.
26 - 35 PN(KL,3) - Gas porosity.
36 - 45 BTT(KL,1) - Temperature solubility Coefficient for water.
46 - 55 BTT(KL,2) - Temperature solubility Coefficient for oil.
56 - 65 HP(KL,1) - Pressure solubility Coefficient for water
1 - 10 HP(KL,1) - Pressure solubility Coefficient for oil.

3. Initial Pore pressure data - enter after card 5

1 - 5 MN - Element number.
6 - 15 PWO(MN) - Pore pressure.

Use NUMELT cards.

4. Pore pressure increments and Venting data

Enter after each load case
If NREAD=1 enter the pore pressure increments as specified in the NLSSIP write-up (item 13)

Next card
1 - 5 NVENT = 0 if venting is not to be done.
= 1 if venting is to be done during this load case.

Next card
If NVENT=1 enter

1 - 5 NVEL - No. of elements to be vented.

Next card
1 - 5 NELL(1) - First element to be vented.

6 - 10 NELL(2) - Second element to be vented.
Upto NELL(NVEL) ,upto a maximum of 5 elements per card.