A CRITICAL ASSESSMENT OF A
CONSTITUTIVE THEORY FOR SOILS

by

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A constitutive theory for soil proposed by J.H. Prevost is explained in detail and examined for its limitations, appropriateness of assumptions, and capabilities. Three models using this theory are appraised: an undrained total stress formulation, an effective stress model for cohesionless soil, and a general effective stress model for any type of soil. The necessary equations are derived and the consequences of the implicit assumptions discussed. Methods for determining the parameters are presented and comparisons of the model predictions with actual test data are also made.

It was found that the undrained total stress model is remarkably accurate for predicting the behavior of a kaolinite clay under a complex monotonic stress path. However, caution must be exercised when applying the model to heavily overconsolidated clays. Problems may also be encountered when applying the model to strain softening soils subjected to complex load paths.

The effective stress models do not give good comparisons between predictions and test data. Some of the problems include a restriction of the models to monotonic loading and a mathematical inconsistency in formulating the general effective stress model. In their present form, the effective stress models are not suitable for predicting the effective stress behavior of soils.
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To my family, friends and fellow students in Vancouver who have tolerated my non-technical discussions and bad jokes, you are not forgotten. And to the reader who takes it upon him/herself to wade through this dissertation, be forewarned that the text contains no intentional humour and a pillow might come in handy.
NOMENCLATURE

A - a material parameter associated with shear induced volume change
B - elastic bulk modulus
B₁ - elastic bulk modulus at p=p₁
B' - a parameter associated with the plastic bulk modulus
c - a constant in the effective stress yield function
dμ - a scalar parameter associated with μ to describe the kinematic hardening of the yield surfaces
eₐ - total deviatoric strain tensor
eₐₐ - plastic deviatoric strain tensor
f - yield surface or yield function
F - loading function
g - plastic potential function
G - elastic shear modulus
G₁ - elastic shear modulus at p=p₁
h' - plastic shear modulus
H - elastoplastic modulus
H' - plastic modulus
k - a measure of the size of the yield surface
k₀ - original size of the largest yield surface
N - number of cycles applied beyond initial loading curve plus 1
p=σₘ - mean normal or hydrostatic stress
p₁ - a reference mean normal stress
pₐ - normal to the plastic potential surface g
pₐₐ' - projection of the normal to the plastic potential surface onto the deviatoric plane
\( P' \) - projection of the normal to the plastic potential surface onto the hydrostatic axis (a scalar)

\( q \) - stress difference: \( q_y - q_x \)

\( Q_{ij} \) - normal to the yield surface

\( Q'_{ij} \) - projection of the normal to the yield surface onto the deviatoric plane

\( Q''_{ij} \) - projection of the normal to the yield surface onto the hydrostatic axis (a scalar)

\( |Q| \) - scalar magnitude of \( Q_{ij} \)

\( s_{ij} \) - deviatoric stress

\( S_i \) - stress in the 1,2,3 coordinate system

\( a_{ij} \) - stress coordinates of the center of the yield surface in the deviatoric subspace

\( \beta \) - stress coordinate of the center of the yield surface along the hydrostatic axis (a scalar)

\( \delta_{ij} \) - Kronecker delta; the hydrostatic axis

\( e_{ij} \) - strain tensor

\( e_v \) - volumetric strain (a scalar)

\( e \) - strain difference: \( e_y - e_x \)

\( \gamma \) - shear strain = \( 2e_{xy} \)

\( \lambda \) - a measure of the total accumulated plastic shear strain

\( \xi_{ij} \) - stress coordinates of the center of the yield surface in stress space

\( \mu_{ij} \) - a tensor in the direction of a line joining the stress point on \( f_a \) and a point on \( f_m \), with the same outward normal

\( \Omega \) - Lame's constant (elasticity)

\( \rho \) - \( d(ln \, p)/d\epsilon_v \) in a consolidation test

\( \tau \) - shear stress

\( \Theta_{ij} \) - \( \sigma_{ij} - \xi_{ij} \)
INTRODUCTION

It is a well known fact that the particulate nature of soils causes its stress-strain behavior to be non-linear, hysteretic, and stress path dependent. Cycling the stresses may cause the strength and stiffness to increase or decrease. In addition, the mode of deposition and stress history can cause the soil to behave differently for different orientations of a set of stresses, resulting in pronounced anisotropy.

Another peculiarity of soil is the phenomenon of coupling between the different components of stress or strain. For example, under drained conditions, dense sands dilate when subjected to shear strains. This shear-induced volume change is an important consideration when analysing the behavior of soil.

Any realistic soil model should include the above behavioral characteristics. However, most present day models are able to describe only certain aspects with reasonable accuracy.

Recently, J.H. Prevost (18,19) has introduced a total stress model to describe the non-linear, hysteretic, stress path dependent behavior of anisotropic, undrained clays subjected to monotonic or quasi-static cyclic loading conditions. He has also extended the concept to describe the effective stress behavior of soils (20-22). Published comparisons with experimental data have shown the undrained total stress model to be remarkably accurate and its future in
geotechnical applications appears to be very promising. However, little work has been done on the model by people other than Prevost.

To examine the limitations, assumptions, and capabilities of the Prevost model, a study was undertaken by the author and the results are the subject of this thesis. The first chapter reviews the concepts of stress and strain tensors so that the presentation can be self-contained. The second deals with the fundamental theoretical considerations which form the basis of the model. Chapters 3 to 7 and 8 to 11 present an undrained total stress and effective stress formulation, respectively. Methods for determining the necessary parameters and a comparative study of predictions and experimental results are also presented.
CHAPTER 1

THE STRESS AND STRAIN TENSORS

Tensor Notation

Tensor notation is used to define stresses, strains, and yield criteria discussed in this thesis. Therefore, a brief description of tensor notation, and in particular the stress and strain tensors, will be presented here. A tensor is a set of numbers or functions which can undergo coordinate transformations according to certain mathematical laws. Only the notation of tensors, and not their transformation laws, will be discussed here.

If we wish to write the vector \( \{x_1, x_2, x_3\} \) in tensor notation, it would appear as \( x_i \) where the subscript \( i \) (a latin or other non-numeric symbol) can take on the values 1, 2, and 3. \( x_i \) is a tensor of order one since it has only one subscript. Stresses however, require two subscripts and are second order tensors: one to describe the direction of the associated force and the other to specify the direction of the normal to the face of the material element it acts upon. There are then nine components of stress \( \sigma_{ij} \) which are shown in eq.1.1:
\[
\begin{bmatrix}
\sigma_x & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z
\end{bmatrix}
= \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
= \sigma_{ij} = \sigma (1.1)
\]

Just as \(i\) took on values of 1, 2, and 3 in the first example, both \(i\) and \(j\) take on these values here. More generally, when a non-numeric subscript appears only once in a term, the subscript takes on the values 1, 2, and 3. In practical terms, the indices may be thought of as representing the \(x\), \(y\), and \(z\) axes, respectively.

Similarly for the strains:

\[
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
= \varepsilon_{ij} = \varepsilon (1.2)
\]

where:

\[
e_{ij} = \frac{\gamma_{ij}}{2} \quad i \neq j
\]

If a subscript is repeated in a term, the range of the subscript is then summed:

\[
\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}
\]

\[
\sigma_{ij} d\varepsilon_{ij} = \sigma_{11} d\varepsilon_{11} + \sigma_{12} d\varepsilon_{12} + \ldots + \sigma_{21} d\varepsilon_{21} + \ldots + \sigma_{33} d\varepsilon_{33}
\]
No subscript may be repeated more than once.

If the subscripts are interchangeable without altering the components, then the system is said to be symmetric. Except in very special cases, the stress and strain tensors are symmetric:

\[ \sigma_{ij} = \sigma_{ji}, \]
\[ \varepsilon_{ij} = \varepsilon_{ji}. \]

A special second order tensor, called the Kronecker delta \( \delta_{ij} \), is defined as:

\[
\delta_{ij} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{cases}
1 & \text{if } i=j \\
0 & \text{if } i\neq j
\end{cases}
\]

(1.3)

Some special properties of this tensor are:

\[ \sigma_{ij} \delta_{ik} = \sigma_{jk}, \]
\[ \sigma_{ij} \delta_{ij} = \sigma_{ii}, \]
\[ \delta_{ij} \delta_{ij} = 3. \]
Deviatoric Stresses and Strains

It is often convenient in plasticity theory and soil mechanics to separate the stress tensor into two parts. One is called the spherical stress tensor and is defined as:

\[ p \delta_{ij} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \] (1.4)

where:

\[ p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}a_{kk} = a_m \] (1.5)

The quantity \( p \) is also called the mean normal stress or hydrostatic stress. It is the same for all possible orientations of axes and is thus an invariant.

The second part of the stress tensor is called the deviatoric stress tensor \( s_{ij} \) and is defined as:

\[ s_{ij} = a_{ij} - a_m \delta_{ij} \] (1.6)

Obviously, the subtraction of the hydrostatic stress from the stress tensor does not change the direction of the principal deviator stresses.

The three invariants of the stress deviator tensor are:

\[ J_1 = 0 \]
The second invariant $J_2$ is especially significant in plasticity because it forms part of the Von Mises yield criterion, which will be discussed later.

Strains can be treated in much the same way as stresses. The spherical strain tensor is given by:

$$1 \delta_{ij} = \frac{1}{3} \begin{bmatrix} \epsilon_v & 0 & 0 \\ 0 & \epsilon_v & 0 \\ 0 & 0 & \epsilon_v \end{bmatrix}$$

where:

$$\epsilon_v = \epsilon_{ii} = (\epsilon_x + \epsilon_y + \epsilon_z)$$

= volumetric strain

The deviatoric strain tensor is defined as:

$$e_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon_v \delta_{ij}$$

The deviatoric strain invariants are:

$$J_1' = 0$$

$$J_2' = \frac{1}{2} e_{ij} e_{ij}$$

$$J_3' = \frac{1}{3} e_{ij} e_{jk} e_{ki}$$
CHAPTER 2

GENERAL THEORY

The theory of plasticity provides a powerful basis upon which to construct models to describe the stress-strain behavior of soils. The simplest formulation would be to assume ideal or perfect plasticity. A sample with this property subjected to a uniaxial load would deform plastically at constant load when a certain stress, called the yield stress, is reached. However, soils often exhibit hardening; that is, they have load-deformation curves which monotonically increase during plastic deformation. The constitutive model described herein uses plasticity theory to model this phenomenon. A general formulation will now be presented. Particularizations to describe the undrained and drained behavior of soils will be presented in later chapters.

Components of Strain

The incremental theory of plasticity is based on the assumption that a given strain increment can be separated into an elastic and a plastic component:

\[ \text{d} \varepsilon_{ij} = \text{d} \varepsilon_{ij}^e + \text{d} \varepsilon_{ij}^p \]

Where:

- \( \text{d} \varepsilon_{ij}^e \) = elastic strain increment tensor
- \( \text{d} \varepsilon_{ij}^p \) = plastic strain increment tensor
For convenience, the elasticity of the soil is assumed to be isotropic. The elastic strain increments are then related to the stress increments by generalized Hooke's law expressed in terms of the shear modulus $G$ and the bulk modulus $B$. These moduli may be assumed to be constant or functions of the effective stress and/or void ratio.

To describe the plasticity of the soil, a yield criterion, flow rule, and hardening rule are required.

The yield criterion or yield function specifies the stress states for which plastic deformation may take place. It is represented by a scalar function $f(\sigma_{ij}, e^p_{ij}, k)$ which forms a closed surface in stress space. As such, $f$ is also called a yield surface. Stress states on $f$ may cause plastic deformations whereas only elastic behavior occurs for states within the surface. Mathematically, it is defined as:

$$f = F(\sigma_{ij}) - k(e^p_{ij}) = 0 \quad (2.1)$$

where:

$F$ describes the shape of the yield surface and is a function of only the stresses

$k$ is a measure of the size of the yield surface and may be a function of the plastic strains and history of loading.

If $f<0$, i.e. $F<k$, then the stress point lies inside the
yield surface and only elastic deformations occur. Plastic deformations may occur if \( f=0 \). Now suppose \( f>0 \); then \( F>k \) and the stress point would be outside the yield surface. Since loading from a plastic state must lead to another plastic state, \( f=0 \) at all times during plastic deformation. This requirement is known as the consistency condition (16) and demands that the stress point always be on the current yield surface during loading. \( f>0 \) is therefore inadmissible and has no meaning.

To determine whether plastic deformation occurs for a perfectly plastic or hardening material when the stress point is on the yield surface \( f=0 \), the scalar (dot) product between the outward normal to the yield surface \( \frac{\partial f}{\partial \sigma_{ij}} \) and the stress increment \( d\sigma_{ij} \) is evaluated:

a) For loading, \( \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \) and hardening occurs.

b) For neutral loading \( \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \). Plastic deformation may occur for perfectly plastic materials, but no hardening may take place.

c) For unloading \( \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \) and only elastic strains occur.

Notice that \( \frac{\partial F}{\partial \sigma_j} = \frac{\partial f}{\partial \sigma_{ij}} \) since \( k \) is not a function of \( \sigma_{ij} \). \( F \) is often called the "loading function" because it is only a function of the stresses and defines the loading condition.
The quantity \( \frac{\partial f}{\partial \sigma_j} \, d\sigma_j \) can be thought of as being a measure of the components of the stress increment tensor in the direction of the outward normal. Thus for loading to occur, the stress "vector" must be directed outwards from the yield surface.

The flow rule relates the plastic strain increment tensor to the stress tensor. The most common rule is the normality rule, which assumes that the strain increments are normal to a surface in stress space called the plastic potential surface, \( g \). Now let us assume that the plastic strain increment and stress axes coincide in stress space. If \( f = g \), the yield surface also plays the role of the potential surface and the flow rule is known as an associated flow rule (i.e. the flow rule is associated with the yield function \( f \)). A non-associated flow rule results if \( f \neq g \), and the strain increment tensor is not related to the yield surface.

Hardening rules are a consequence of the consistency condition. Figure 2.1 shows how plastic hardening is achieved in a uniaxial test. Let \( \sigma_A \) be the original yield stress. Now suppose a stress increment \( d\sigma \) is applied causing plastic deformation. The consistency condition requires that the yield point be shifted to point B. Similarly, the yield surface in stress space must assume a new position consistent with the new stress state. The modification of the yield surface's position

---

1 Although stress is a tensor, it can be visualized as being a "vector" in stress space.
Fig. 2.1. Stress-strain curve of a uniaxial system and stress space representation of stress state. a) Initial; b) After load increment has been applied.
in stress space as plastic flow occurs is described by the hardening rule.

Isotropic and kinematic hardening are two fundamental types of hardening. Isotropic hardening assumes that the yield surface expands proportionately outwards while the center of the yield surface remains stationary. Figure 2.2 gives two examples of this.

Fig. 2.2. Isotropic hardening. The dashed lines show the yield surface's size before plastic deformation.

Now suppose that the material in Fig. 2.3 is loaded uniaxially and has its yield surface centered at the origin ($\sigma_j = 0$). Once the applied stress exceeds the yield stress $\sigma_y$, the material hardens. If the load is reversed from $\sigma'$, isotropic hardening would predict that plastic deformation will not occur until a stress of $\sigma''_e$, where $|\sigma''_e| = |\sigma'_e|$, is reached.
Fig. 2.3. Effect of hardening type on the stress-strain curve of a uniaxial system. Dashed yield surface lines show yield surface's position after hardening.
Fig. 2.4. Kinematic hardening.

in extension. Soils, however, often exhibit the Bauschinger effect; that is, plastic deformations occur well before the stress has been fully reversed. Thus isotropic hardening, which is a useful concept for monotonic loading situations, is not convenient to use when loading reversal occurs.

Kinematic hardening (17) provides a simple means of modelling the Bauschinger effect. It assumes that the yield surface's size and orientation is retained but its center is allowed to translate in stress space (Fig.2.4). If we use kinematic hardening on the material in Fig.2.3, it can be seen that $\sigma_y$ in extension occurs well before $\sigma'_e$. 
General Plastic Stress-Strain Relation

A general form of the plastic stress-strain relation for hardening materials can be developed using three assumptions:

1) The stress state lies on the yield surface $f$ and further plastic deformation occurs only for:

$$\frac{\partial f}{\partial \sigma_j} d\sigma_j > 0 \quad (2.2)$$

2) There exists a surface $g$ in stress space such that the normal to $g$ gives the direction of the plastic strain increments.

3) The relation between the infinitesimals of stress and strain is linear:

$$d\varepsilon_{ij}^p = C_{ijkl} d\sigma_{kl} \quad (2.3)$$

where $C_{ijkl}$ may be a function of stress, strain, or history of loading but is independent of $d\varepsilon_{ij}^p$ or $d\sigma_{kl}$.

Using assumption 2, $C_{ijkl}$ can be written:
\[ C_{ijkl} = \frac{\partial g}{\partial \sigma_{ij}} \Gamma_{kl} \]  

(2.3a)

where: \[ \frac{\partial g}{\partial \sigma_{ij}} \] outward normal to the plastic potential; this tensor is co-directional with \( \varepsilon_{ij} \).

\( \Gamma_{kl} \) = an as yet undefined tensor with subscripts "kl"

The tensor \( \Gamma_{kl} \) can be obtained by noting that assumption 3 allows the principle of superposition to apply. Thus:

\[ d\sigma_{kl} = d\sigma'_{kl} + d\sigma''_{kl} \]  

(2.4)

where \( d\sigma'_{kl} \) produces no plastic deformation and \( d\sigma''_{kl} \) is proportional to the gradient of \( f \) (Fig.2.5):

\[ d\sigma''_{kl} = r \frac{\partial f}{\partial \sigma_{kl}} \]  

(2.5)

\( r > 0 \)

Eq.2.4 into eq.2.2 gives:

\[ \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} = \frac{\partial f}{\partial \sigma_{kl}} (d\sigma'_{kl} + d\sigma''_{kl}) > 0 \]  

(2.6)
Fig. 2.5. Decomposition of the stress increment \( d\sigma_{kl} \)

\( d\sigma_{kl}'' \) is perpendicular and \( d\sigma_{kl}' \) is tangent to the yield surface at the stress point.

But \( d\sigma_{kl}' \), by definition, produces no plastic flow; therefore:

\[
\frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl}' = 0 \tag{2.7}
\]

Combining equations 2.5, 2.6, and 2.7 gives:

\[
\frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl} = \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl}'' = \frac{\partial f}{\partial \sigma_{kl}} r \frac{\partial f}{\partial \sigma_{kl}}
\]

or:
Comparing eq. 2.8 with eq. 2.3a, it can be concluded that:

\[
\Gamma_{kl} = \frac{1}{H'} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{mn}}
\] (2.9)

where \(1/H'\) is a factor of proportionality and \(H'\) is called the plastic modulus.

The most general form of the plastic stress-strain relation for hardening materials can therefore be written:

\[
d\epsilon_{ij}^p = \frac{1}{H'} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{mn}}
\] (2.10)

This can be expressed more succinctly as:
\[ d\varepsilon_{ij}^p = \frac{\langle \mathbf{L} \rangle}{|Q|^2} \hat{P} \quad (2.11) \]

where:

\[ \hat{P} = p_{ij} = \frac{\partial g}{\partial \sigma_{ij}} \]

= normal to the plastic potential surface (i.e. in the direction of the plastic strain increments)

\[ \hat{Q} = Q_{ij} = \frac{\partial f}{\partial \sigma_{ij}} \]

= normal to the yield surface

\[ |Q| = \left| \frac{\partial f}{\partial \sigma_{nn}} \frac{\partial f}{\partial \sigma_{mn}} \right|^{1/2} \]

\[ L = \frac{1}{H'} \hat{Q} \cdot d\hat{\sigma} = \frac{1}{H'} \frac{\partial f}{\partial \sigma_{kl}} \, d\sigma_{kl} \]

\[ \langle \mathbf{L} \rangle = \begin{cases} L & \text{if } L \geq 0 \\ 0 & \text{if } L < 0 \end{cases} \]

Drucker (3) further restricts the form of the stress-
strain relation by defining that a work hardening material be in equilibrium when a self-equilibrating set of stresses is added to the material and further that:

a) positive work is done by the application of a set of self-equilibrating forces
b) the net work performed over a complete cycle of application and removal is zero or positive.

Consider a volume of material under a homogeneous state of stress $\sigma_{ij}$ and strain $\epsilon_{ij}$. If a self-equilibrating set of forces applies small surface tractions resulting in a stress change at each point of $d\sigma_{ij}$, the resulting strain increments are $d\epsilon_{ij}$. Now if these small surface tractions are removed, the elastic strain increments will be released. Thus condition "a" becomes:

$$d\sigma_{ij} \ d\epsilon_{ij} > 0 \quad (2.12)$$

Condition "b" results in:

$$d\sigma_{ij} \ (d\epsilon_{ij} - d\epsilon_{ij}^e) \geq 0$$

$$d\sigma_{ij} \ d\epsilon_{ij}^e \geq 0 \quad (2.13)$$
where: \( d\sigma_{ij} \, d\varepsilon_{ij}^p = 0 \) if \( d\varepsilon_{ij}^p = 0 \)

Equations 2.12 and 2.13 are a mathematical definition of Drucker's work hardening hypothesis. Recalling from Fig. 2.5 that the stress increment can be decomposed, eq. 2.13 may be written as:

\[
d\sigma_{ij} \, d\varepsilon_{ij}^p = (d\sigma_{ij}' + d\sigma_{ij}'') \, d\varepsilon_{ij}^p \geq 0 \quad (2.14)
\]

But \( d\sigma_{ij}' \) produces no plastic flow and therefore the stress increment \( d\sigma_{ij} = C \, d\sigma_{ij}' + d\sigma_{ij}'' \) will produce the same plastic strain increment \( d\varepsilon_{ij}^p \) regardless of the value of \( C \). However, \( d\sigma_{ij}' \, d\varepsilon_{ij}^p \) must vanish, otherwise \( C \) could be chosen to be a large negative number such that eq. 2.14 is violated. Therefore:

\[
d\sigma_{ij} \, d\varepsilon_{ij}^p = 0
\]

or from eq. 2.11:

\[
d\sigma_{ij}' \frac{\partial g}{\partial \sigma_{ij}} \frac{\langle L \rangle}{|Q|^2} = 0 \quad (2.15)
\]

Plastic loading requires that \( \langle L \rangle > 0 \). Eq. 2.15 therefore implies that \( \partial g/\partial \sigma_{ij} \) is perpendicular to \( d\sigma_{ij}' \) and thus also to
f at the stress point during yielding. This indicates that \( f = g \) and therefore equation 2.10 becomes:

\[
\frac{\delta f}{\partial \sigma_{kl}} = \frac{\delta f}{\partial \sigma_{ij}} H' \frac{\delta f}{\delta \sigma_{mn}} d^{ij}_{kl} \quad (2.16)
\]

This proves that a work hardening material, in the Drucker sense, must have an associated flow rule.

Drucker (3) has pointed out that the definition of work hardening defined by equations 2.12 and 2.13 implies stability of the system; that is, no deformations may occur without an input of work. However, the behavior of soils under drained conditions (i.e. when the behavior is a function of the mean normal stress) does not always satisfy Drucker's definition of work hardening. For example, consider a specimen of dry sand at the critical void ratio, subjected to a shear stress \( \tau \) and a mean normal stress \( p \) (Fig.2.6). If the stress point for the system lies on the yield surface, then a cycle of adding and then removing a hydrostatic stress decrement \( dp \) would cause an increment in plastic shear strain. Assuming that no volume change occurs, the work done by the added stress \( dp \) would be zero. This behavior is in violation of eq.2.13 and therefore soils under drained conditions are not, in general, work hardening materials. Materials which may show the characteristics of hardening (i.e. a monotonically increasing
stress-strain curve) but do not necessarily satisfy Drucker's requirement for stability are called hardening materials and the flow rule is given by eq. 2.10. It should be recognized that work hardening materials are a subset of hardening materials.

**General Yield Function**

It will be assumed herein that the yield surface can be represented by an expression of the form:

\[ f = F(\sigma_j - \xi_j) - k^n = 0 \]  

(2.17)
where: $\xi_{ij} =$ offset of the yield surface from the origin in stress space.

$k(\varepsilon_{ij}^p) =$ a measure of the size of the yield surface.

$n =$ an integer exponent; $n \geq 1$

For soils, $F$ is conveniently chosen to be a homogeneous function of degree $n$ in $(\sigma_{ij} - \xi_{ij})$ (see Appendix A). Thus, for quadratic $F$, $n=2$.

For convenience (19) the size of the yield surface $k$ may be a constant or a function of:

$$
\varepsilon_{ij}^p = \int d\varepsilon_{ij}^p \
\lambda = \frac{1}{2} \sqrt{\frac{1}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p}
$$

where:

$\varepsilon_{ij}^p =$ plastic volumetric strain

$\varepsilon_{ij}^p = \varepsilon_{ij}^p - \frac{1}{2} \varepsilon_{kk} \delta_{ij}$

= plastic deviatoric strain tensor

To allow for the anisotropic hardening of a material subjected to any path in stress space, the concept of a field of hardening moduli surfaces, $f_1, f_2, \ldots, f_p$, with sizes $k^{(1)} < k^{(2)} < \ldots < k^{(p)}$, respectively, was proposed by Iwan (8) and Mroz (12) and adapted for modelling soil behavior by Prevost (18-22). These surfaces represent stress loci at which the plastic modulus $H'$ (or other plastic parameters) acquires a new value. For example, when the stress point $M$ in Fig.2.7 touches $f_m$, a new plastic modulus (or other plastic parameters)
is used in the stress-strain relation.

![Diagram of hardening moduli surfaces in stress space.](image)

**Fig. 2.7. Field of hardening moduli surfaces in stress space.**

Although the functions describing each hardening surface need not be the same, $f_2, f_3, \ldots, f_p$ are chosen to be of an identical form as the yield function $f_1$ for simplicity.

$$f_r = F_r (\sigma_{ij} - f_{ij}^{(r)}) - (k^{(r)})^n = 0 \quad (2.18)$$

for all $r$

When the stress point reaches the hardening surface $f_m$ during loading, $f_m$ acts as a yield surface and equation 2.10 becomes:
Prevost calls the hardening surfaces "yield surfaces". This is not entirely correct if yielding is defined as the "limit of elasticity under any possible combination of stresses" (7). For example, if \( f \) is a true yield surface in Fig.2.7 (i.e. purely elastic behavior within it) then \( f_m \) cannot also be a yield surface since plastic deformations can occur within it. Nevertheless, when the stress point lies on the hardening surface \( f_m \), this function acts like a yield surface. In particular, if a stress increment is applied tangent to \( f_m \):

\[
\frac{\partial f}{\partial \sigma_{kl}} \ d\sigma_{kl} = 0
\]

and no plastic strain occurs. If stress increments are continuously applied in this manner, the hardening surface may theoretically be circumscribed by the stress point without causing plastic strains. This is a property of a yield surface. To simplify the terminology, Prevost's convention of calling hardening surfaces "yield surfaces" will be used herein.

Now let us qualitatively examine one of the advantages of
Fig. 2.8. Example showing how multiple yield surfaces can be used to simulate the different behavior under compression and extension, and also anisotropy.

a) Yield surfaces in stress space.

b) Corresponding stress-strain curve.
a multiple yield surface approach using a uniaxial system. Let $f_1$, $f_2$, and $f_3$ in Fig.2.8a be yield surfaces having constant plastic moduli of $H_1$, $H_2$, $H_3$ and constant sizes $k_1<k_2<k_3$, respectively. Assuming the elastic strains are zero, the corresponding stress-strain curves are illustrated in Fig.2.8b. Notice the different behavior in compression and extension if the yield surfaces are offset with respect to the origin 0. The multiple yield surface approach would thus be a convenient method of simulating the different behavior under compression and extension, and also anisotropy.

One of the most well known yield criterion in plasticity is the Von Mises yield criterion:

$$J_2 = \frac{1}{2} s_{ij} s^{ij} = k$$

where: $k$ = a measure of the size of the yield surface, which may vary due to strain hardening or softening. If the above relation is satisfied, then plastic deformation may occur.

Kinematic Hardening Rule

The translation and change in size of a yield surface due to plastic deformation is described by the kinematic and isotropic hardening rules respectively. Isotropic hardening is the simplest of the two and is basically a correlation between the observed behavior of a material and the changes in the sizes of the yield surfaces (i.e. $dk$). An example of an
isotropic hardening rule is given in Chapter 6. However, kinematic hardening can be quite complex for a multi-yield surface model because the yield surfaces interact with one another. The mathematics which describes these movements will now be explained.

If a material hardens, then the consistency condition must be satisfied. Applying the consistency condition to:

\[ F(\sigma_{ij} - \xi_{ij} - \xi_{ij} - \xi_{ij}) = F(\Theta_{ij} + \Xi_{ij}) = h(k) = k^n \] (2.20)

requires that:

\[ F(\sigma_{ij} + d\sigma_{ij} - \xi_{ij} - d\xi_{ij}) = F(\Theta_{ij} + d\Theta_{ij}) = h(k + dk) = (k + dk)^n \] (2.21)

Expanding \( F(\Theta_{ij} + d\Theta_{ij}) \) into a Taylor series we get:

\[ F(\Theta_{ij} + d\Theta_{ij}) = F(\Theta_{ij}) + \frac{\partial F}{\partial \Theta_{ij}} d\Theta_{ij} + \frac{1}{2!} \frac{\partial^2 F}{\partial \Theta_{ij}^2} (d\Theta_{ij})^2 + \ldots \ldots \ldots \] (2.22)

Similarly:
\[ h(k + \Delta k) = h(k) + \frac{\partial h}{\partial k} \Delta k + \frac{1}{2!} \frac{\partial^2 h}{\partial k^2} (\Delta k)^2 + \ldots \]

(2.23)

Substituting eq. 2.22 and eq. 2.23 into eq. 2.21 and since \[ F(\Theta_{ij}) = h(k) \]:

\[ \frac{\partial F}{\partial \Theta_{ij}} \frac{\partial \Theta_{ij}}{\partial k} + \frac{1}{2!} \frac{\partial^2 F}{\partial \Theta_{ij}^2} (\partial \Theta_{kl})^2 + \ldots = \frac{\partial h}{\partial k} \Delta k + \frac{1}{2!} \frac{\partial^2 h}{\partial k^2} (\Delta k)^2 + \ldots \]

(2.24)

From the definition of \( \Theta_{ij} \):

\[ \frac{\partial F}{\partial \Theta_{ij}} = \frac{\partial F}{\partial \sigma_j} = Q_{ij} \]

Equation 2.24 then becomes:

\[ Q_{ij} (d\sigma_j - d\xi_j) + \frac{1}{2!} \frac{\partial Q_{ij}}{\partial \sigma_j} (d\sigma_{kl} - d\xi_{kl})^2 + \ldots \]

\[ = \frac{\partial h}{\partial k} \Delta k + \frac{1}{2!} \frac{\partial^2 h}{\partial k^2} (\Delta k)^2 + \ldots \]

(2.25)
or:

\[ Q_{ij} (d\sigma_{ij} - d\xi_{ij}) + \frac{1}{2!} \frac{\partial Q_{ij}}{\partial \sigma_{ij}} (d\sigma_{kl} - d\xi_{kl})^2 + \ldots \]

\[ = n k^{n-1} \Delta k + \frac{n (n-1) k^{n-2}}{2!} (\Delta k)^2 + \ldots \]

(2.26)

This is a mathematical statement of the consistency condition.

If \((d\sigma_{ij} - d\xi_{ij})\) and \(\Delta k\) are small, then the second order terms can be neglected:

\[ Q_{ij} (d\sigma_{ij} - d\xi_{ij}) = n k^{n-1} \Delta k \]

(2.27)

This simpler form of the consistency condition will be used in subsequent derivations unless specified otherwise.

Since \(F\) is a homogeneous function, Euler's theorem (see Appendix A) can be used on eq.2.27 to give:

\[(\sigma_{ij} - \xi_{ij}) Q_{ij} = n F = n k^n \]

(2.28)

or:
It is imperative that the yield surfaces never overlap. A physical reason for this can be seen in Fig. 2.9.

\[ n k^{n-1} \frac{dk}{k} = \frac{d}{\partial \sigma_j} Q_j (\sigma_j - \delta_{ij}) \] (2.29)

Fig. 2.9. Intersection of two yield surfaces gives two different outward normals at A and thus the strain increment would not be unique.

If the stress point reaches point A, two different outward normals, corresponding to \( f_1 \) and \( f_2 \), would exist. From eq. 2.9, it can be seen that the strain increment would not be unique at this point and thus overlappings cannot be permitted. A point where \( \partial f/\partial \sigma_j \) is not uniquely defined is called a singular point.

To prevent the yield surfaces which are smaller than \( f_m \) from intersecting one another, Mroz (12) proposed that \( f_1, f_2, \ldots, f_{m-1} \) be made tangent to \( f_m \) at the stress point M. Since \( f_1, f_2, \ldots, f_{m-1} \) differ only in size but not form:
The geometric meaning of this relation is that the yield surfaces are tangent at the stress point and have their centers lying on the "vector" \((\sigma_{ij} - \xi_{ij}^{(m)})\) as shown in Fig.2.7. It should be noted that eq.2.30 is not a hardening rule since plastic deformation need not occur for it to be true.

When the stress point is on \(f_m\) and when plastic hardening occurs, the yield surfaces \(f_1, f_2, \ldots, f_m\) are translated together and remain tangent at the stress point as it moves along its stress path towards \(f_{m+1}\). Thus the stress point must never be outside any yield surface.

To avoid the overlapping of \(f_m\) and \(f_{m+1}\), these yield surfaces may touch only at points with the same outward normal. Let \(M\) be the stress point lying on \(f_m\) and let \(R\) be the point on \(f_{m+1}\) which has an outward normal in the same direction as at \(M\) in Fig.2.10. For illustrative purposes, the yield surfaces are shown as "circles" in stress space. We shall now demand that points \(M\) and \(R\) move towards each other during kinematic and isotropic hardening of the yield surfaces. Points \(A\) and \(B\) denote the centers of the yield surfaces \(f_m\) and \(f_{m+1}\), respectively. Since similar triangles are formed by \(\triangle EBM\) and \(\triangle RMB\), \(|\overrightarrow{BM}| = |\overrightarrow{MB}|\) and \(|\overrightarrow{MB}| = -(\sigma - \xi^{(m+1)})\). Also, \(|\overrightarrow{BR}| = k^{(m+1)}\) and \(|\overrightarrow{AM}| = k^{(m)}\). But from the similarity of triangles, \(|\overrightarrow{EM}| = |\overrightarrow{BR}| = k^{(m+1)}\). Therefore \(\overrightarrow{MR}\) is given by:

\[
\frac{\sigma_{ij} - \xi_{ij}^{(1)}}{k^{(1)}} = \frac{\sigma_{ij} - \xi_{ij}^{(2)}}{k^{(2)}} = \ldots = \frac{\sigma_{ij} - \xi_{ij}^{(m)}}{k^{(m)}}
\]
Fig. 2.10. Translation of $f_m$ due to hardening.
\[
\hat{\mu} = \hat{M} \mathbf{R} = \hat{E} \mathbf{B} \\
= \hat{E} \hat{M} + \hat{M} \mathbf{B} \\
= \frac{k^{(m+1)}}{k^{(m)}} \hat{A} \mathbf{M} + \hat{M} \mathbf{B}
\]

\[
\hat{\mu} = \frac{k^{(m+1)}}{k^{(m)}} (\hat{\sigma} - \hat{\xi}^{(m)}) - (\hat{\sigma} - \hat{\xi}^{(m+1)})
\]

(2.31)

This result was first obtained by Mroz (12).

Following an analysis by Prevost (20), we shall now find the movements of \( f_m \) and \( f_{m+1} \) with respect to each other such that no overlapping may occur (i.e., we seek an expression for \( \delta \xi^{(m)} - \delta \xi^{(m+1)} \)).

Let \( \delta \hat{\sigma} \) and \( \delta \hat{\sigma}' \) be the movements of \( M \) and \( R \) respectively due to kinematic (\( d\xi^{(m)} \) and \( d\xi^{(m+1)} \)) and isotropic (\( dk^{(m)} \) and \( dk^{(m+1)} \)) hardening. The stress increment \( \delta \hat{\sigma} \) should not be confused with \( \delta \hat{\sigma}' \). Its purpose is to track the change in position of \( M \) due to hardening. The new position of \( M \) after hardening will not necessarily coincide with the position of the stress point.

The consistency condition for point \( M \) can be written:

\[
Q_{ij}^{(m)} (\delta \sigma_{ij} - d\xi_{ij}^{(m)}) = n k^{(m)^{n-1}} d\kappa^{(m)}
\]

But \( (\delta \sigma_{ij} - d\xi_{ij}^{(m)}) \) is in the same direction as \( Q_{ij}^{(m)} \) since this
quantity only contains isotropic hardening. This can be mathematically stated as:

\[ Q_{ij}^{(m)} = \eta (\delta \sigma_{ij} - d\xi_{ij}^{(m)}) \]  \hspace{1cm} (2.32)

where: \( \eta = \) a scalar constant

To find \( \eta \):

\[ |Q_{ij}^{(m)}| |\delta \sigma_{ij} - d\xi_{ij}^{(m)}| = n (k^{(m)})^{n-1} dk^{(m)} \]

\[ |Q_{ij}^{(m)}| |Q_{ij}^{(m)}|_{\eta} = n (k^{(m)})^{n-1} dk^{(m)} \]

Therefore:

\[ \eta = \frac{|Q_{ij}^{(m)}|^2}{n (k^{(m)})^{n-1} dk^{(m)}} \]  \hspace{1cm} (2.33)

Equation 2.33 into 2.32 gives:

\[ \delta \sigma_{ij} = d\xi_{ij}^{(m)} + \frac{n (k^{(m)})^{n-1} dk^{(m)}}{|Q^{(m)}|^2} Q_{ij}^{(m)} \]  \hspace{1cm} (2.34)

For point R, \( (\delta \sigma_{ij}' - d\xi_{ij}^{(m+1)}) \) will be in the same direction as \( Q_{ij}^{(m)} \) since the outward normals at M and R are in the same direction.
\( Q^{(m)}_{ij} = \psi(\delta \sigma_{ij}^* - d\hat{s}_{ij}^{(m+1)}) \)

Where \( \psi \) is a scalar constant.

To get an expression for \( \psi \):

\[
|Q^{(m+1)}_{ij}||\delta \sigma_{ij}^* - d\hat{s}_{ij}^{(m+1)}| = n (k^{(m+1)})^{-\frac{1}{n}} \, dk^{(m+1)}
\]

\[
|Q^{(m+1)}_{ij}||Q^{(m)}_{ij}| = n (k^{(m+1)})^{-\frac{1}{n}} \, dk^{(m+1)}
\]

\[
\psi = \frac{|Q^{(m+1)}_{ij}| |Q^{(m)}_{ij}|}{n (k^{(m+1)})^{-\frac{1}{n}} \, dk^{(m+1)}}
\]

Therefore:

\[
\delta \sigma_{ij}^* = d\hat{s}_{ij}^{(m+1)} + \frac{n (k^{(m+1)})^{-\frac{1}{n}} \, dk^{(m+1)}}{|Q^{(m)}_{ij}| |Q^{(m+1)}_{ij}|} Q^{(m)}_{ij}
\] (2.35)

Note that although \( Q_{ij}^{(m)} \) and \( Q_{ij}^{(m+1)} \) are in the same direction, their magnitudes are different.

In accordance with our demand that M and R move towards each other:

\[
\delta \vec{\sigma} - \delta \vec{\sigma}' = d\mu \vec{\mu}
\] (2.36)

The scalar parameter \( d\mu \) can be obtained from eq.2.36:

\[
\vec{Q}^{(m)} \cdot \delta \vec{\sigma} - \vec{Q}^{(m)} \cdot \delta \vec{\sigma}' = d\mu \vec{Q}^{(m)} \cdot \vec{\mu}
\]
\[ d\mu = \frac{\hat{Q}^{(m)} \cdot (\delta \hat{\sigma} - \delta \hat{\sigma}')}{\hat{Q}^{(m)} \cdot \hat{\mu}} \] (2.37)

Since the consistency condition specifies that point \( M \) and the stress point must both remain on the yield surface after the stress increment \( d\sigma \) has been applied, for point \( M \):

\[ F(\hat{\sigma} + \delta \hat{\sigma} - \check{\xi}^{(m)} - d\check{\xi}^{(m)}) = 0 \]

\[ = k^{(m)} + \frac{\partial F}{\partial \hat{\sigma}} (\delta \hat{\sigma} - d\check{\xi}^{(m)}) \]

\[ = k^{(m)} + \hat{Q} \cdot (\delta \hat{\sigma} - d\check{\xi}^{(m)}) \] (2.38)

For the stress point:

\[ F(\hat{\sigma} + d\hat{\sigma} - \check{\xi}^{(m)} - d\check{\xi}^{(m)}) = 0 \]

\[ = k^{(m)} + \hat{Q} \cdot (d\hat{\sigma} - d\check{\xi}^{(m)}) \] (2.39)

Equating eq.2.38 and eq.2.39 gives:

\[ \hat{Q}^{(m)} \cdot \delta \hat{\sigma} = \hat{Q}^{(m)} \cdot d\hat{\sigma} \] (2.40)

Letting \( \hat{\sigma}' \) be the stress coordinates for \( R \), the consistency condition requires:
\[ F(\hat{\sigma}^t + \delta \hat{\sigma}^t - \hat{\sigma}'^{(m+1)} - d\hat{\sigma}'^{(m+1)}) = 0 \]

\[ = k^{(m+1)} + n k^{(m+1)^{-1}} dk^{(m+1)} \]

\[ \hat{Q}^{(m+1)} \cdot \delta \hat{\sigma}^t - \hat{Q}^{(m+1)} \cdot d\hat{\sigma}'^{(m+1)} - n (k^{(m+1)^{-1}} dk^{(m+1)} = 0 \]

\[ (\hat{Q}^{m} \cdot \delta \hat{\sigma}^t) \frac{|Q^{(m+1)}|}{|Q^{(m)}|} - (\hat{Q}^{m} \cdot d\hat{\sigma}'^{(m+1)}) \frac{|Q^{(m+1)}|}{|Q^{(m)}|} \]

\[ - n (k^{(m+1)^{-1}} dk^{(m+1)} = 0 \]

\[ \hat{Q}^{(m)} \cdot \delta \hat{\sigma}^t = \hat{Q}^{(m)} \cdot d\hat{\sigma}'^{(m+1)} + n (k^{(m+1)^{-1}} dk^{(m+1)} \frac{|Q^{(m)}|}{|Q^{(m+1)}|} \]

(2.41)

Substituting eq.2.40 and eq.2.41 into eq.2.37 yields:

\[ \hat{d}\mu = \frac{\hat{Q}^{(m)} \cdot \hat{d}\hat{\sigma} - \hat{Q}^{(m)} \cdot d\hat{\sigma}'^{(m+1)} - n k^{(m+1)^{-1}} dk^{(m+1)} |Q^{(m)}|}{|Q^{(m+1)}|} \]

(2.42)

Combining equations 2.34, 2.35, and 2.36 gives the relation between \( d\hat{\sigma}'^{(m)} \) and \( d\hat{\sigma}'^{(m+1)} \):
\[ d\tilde{\xi}^{(m)} - d\tilde{\xi}^{(m+1)} = \]
\[
\frac{n}{|Q^{(m)}|} \left[ \frac{k^{(m+1)} n^{-1} dk^{(m+1)}}{|Q^{(m+1)}|} - \frac{k^{(m)} n^{-1} dk^{(m)}}{|Q^{(m)}|} \right] \dot{Q}^{(m)} + \dot{d}\mu \dot{\mu}
\]

(2.43)

Note that \( \dot{d}\mu \) is also a function of \( d\tilde{\xi}^{(m+1)} \).

The yield surfaces \( f_m \) and \( f_{m+1} \) translate in stress space in a manner which prevents their intersection. But eq.2.43 does not describe the kinematic movement of \( f_m \) or \( f_{m+1} \); it only specifies their motions relative to each other. Thus no restrictions are placed on how any one of the yield surfaces translates.

In the following, several kinematic hardening rules for \( f_m \) (i.e. expressions for \( d\tilde{\xi}^{(m)} \)) are considered. \( d\tilde{\xi}^{(m+1)} \) can then be found from eq.2.43.

Prager's Kinematic Rule

According to (17), \( d\tilde{\xi}^{(m)} = d\nu \tilde{Q}^{(m)} \) and the instantaneous translation of \( f_m \) occurs in the direction of the local outward normal. The analogy of a frictionless ring (representing the yield surface) being dragged by a stick (the stress point) is often used. From the simplified consistency condition (eq.2.27):

\[
\tilde{Q}^{(m)} \cdot d\tilde{\sigma} - n k^{(m)} n^{-1} dk^{(m)} = \tilde{Q}^{(m)} \cdot d\tilde{\xi}^{(m)} = \tilde{Q}^{(m)} \cdot \dot{\tilde{Q}}^{(m)} d\nu
\]
and:

\[ d\tilde{\xi}^{(m)} = \left[ \frac{\hat{Q}^{(m)} \cdot d\hat{\sigma} - n k^{(m)^{n-1}} dk^{(m)}}{|Q^{(m)}|^2} \right] \hat{Q}^{(m)} \]  \hspace{1cm} (2.44)

Ziegler's Kinematic Rule

According to (32), \( d\tilde{\xi}^{(m)} = d\nu (\hat{\sigma} - \tilde{\xi}^{(m)}) \) and the instantaneous translation of \( f_m \) occurs in the direction of the 'vector' connecting the center of the yield surface to the stress point. From the consistency condition (eq.2.27):

\[ \hat{Q}^{(m)} \cdot d\hat{\sigma} - n k^{(m)^{n-1}} dk^{(m)} = \hat{Q}^{(m)} \cdot d\tilde{\xi}^{(m)} \]

\[ = \hat{Q}^{(m)} \cdot (\hat{\sigma} - \tilde{\xi}^{(m)}) \, d\nu \]

and:

\[ d\tilde{\xi}^{(m)} = \left[ \frac{\hat{Q}^{(m)} \cdot d\hat{\sigma} - n k^{(m)^{n-1}} dk^{(m)}}{\hat{Q}^{(m)} \cdot (\hat{\sigma} - \tilde{\xi}^{(m)})} \right] (\hat{\sigma} - \tilde{\xi}^{(m)}) \]  \hspace{1cm} (2.45)

Using equation 2.28, eq.2.45 becomes

\[ d\tilde{\xi}^{(m)} = \left[ \frac{\hat{Q}^{(m)} \cdot d\hat{\sigma} - n k^{(m)^{n-1}} dk^{(m)}}{n k^{(m)^n}} \right] (\hat{\sigma} - \tilde{\xi}^{(m)}) \]  \hspace{1cm} (2.46)
Phillip's Kinematic Rule

From (15), \(\tilde{d}\\xi^{(m)} = \nu d\sigma\) and the instantaneous translation is in the direction of the stress 'vector' at the stress point. From the consistency condition (eq.2.27):

\[
\tilde{Q}^{(m)} \cdot d\sigma - n k^{(m)\cdot n^{-1}} \, dk^{(m)} = \tilde{Q}^{(m)} \cdot \tilde{d}\\xi^{(m)} = \tilde{Q}^{(m)} \cdot d\sigma \, \nu
\]

and:

\[
\tilde{d}\\xi^{(m)} = \left[ 1 - \frac{n k^{(m)\cdot n^{-1}} \, dk^{(m)}}{\tilde{Q}^{(m)} \cdot d\sigma} \right] d\sigma \tag{2.47}
\]

Mroz's Kinematic Rule

According to (12), \(\tilde{d}\\xi^{(m)} = d\nu \tilde{\mu}\). The consistency condition (eq.2.27) gives:

\[
\tilde{Q}^{(m)} \cdot d\sigma - n k^{(m)\cdot n^{-1}} \, dk^{(m)} = \tilde{Q}^{(m)} \cdot d\nu \tilde{\mu} \, d\nu
\]

\[
\tilde{d}\\xi^{(m)} = \left[ \tilde{Q}^{(m)} \cdot d\sigma - n k^{(m)\cdot n^{-1}} \, dk^{(m)} \right] \tilde{\mu} \tag{2.48}
\]

A New Simplified Kinematic Rule

The tensors \(\tilde{d}\\xi^{(m)}\), given by equations 2.44 through 2.48 would all result in a translation of \(f_{m+1}\) unless some
additional restrictions are placed on $d\xi^{(m)}$ and/or $d\xi^{(m+1)}$. However, eq.2.43 is quite complex to solve since $d\xi^{(m+1)}$ also occurs in a scalar product with $\hat{\Phi}^{(m)}$. In addition, any movement of $f_{m+1}$ would require a similar analysis to prevent its intersection with $f_{m+2}$. This procedure must be done for all $f_r$ in which $m \leq r \leq p$. The resulting mass of equations would be quite cumbersome to handle unless some simplifying assumptions were made.

In order to retain generality without having to compute the translation of successive yield surfaces, $d\xi^{(m+1)}$ may be set equal to zero. If it is also assumed that $d\kappa^{(r)} \leq d\kappa^{(r+1)}$ for $r \geq m+1$, $d\xi^{(r)}$ can be computed from eq.2.43:

$$d\xi^{(m+1)} = 0$$

$$d\xi^{(m)} =$$

$$\frac{n}{|\Phi^{(m)}|} \left[ \frac{k^{(m+1)^{n-1}}}{|\Phi^{(m+1)}|} \frac{d\kappa^{(m+1)}}{d\kappa^{(m)}} - \frac{k^{(m)^{n-1}}}{|\Phi^{(m)}|} \frac{d\kappa^{(m)}}{d\kappa^{(m)}} \right] \hat{\Phi}^{(m)} +$$

$$\left[ \frac{\hat{\Phi}^{(m)} \cdot \hat{\sigma} - n k^{(m+1)^{n-1}} \frac{d\kappa^{(m+1)}}{d\kappa^{(m)}} |\Phi^{(m)}|}{|\Phi^{(m+1)}|} \right] \hat{\mu}$$

The conditions that $d\xi^{(r)} = 0$ and $d\kappa^{(r)} \leq d\kappa^{(r+1)}$ for all $r \geq m+1$ assures that no overlappings will occur for $r \geq m+1$. 

Practical Considerations in Applying Kinematic Hardening

In general, the kinematic rules given by equations 2.44, 2.46, 2.47, 2.48, and 2.49 only describe the instantaneous translation of the yield surfaces $f_m$ and $f_{m+}$ when the stress point is at $M$ in Fig.2.10. This means that only small and theoretically infinitesimal stress increments may be applied when following a stress path. The reason for this can be demonstrated using Fig.2.11. For clarity, it is assumed that Mroz's kinematic rule is applicable, $d\xi^{(m+1)}=0$, and no isotropic hardening occurs (i.e. $dk^{(r)}=0$ for all $r$).

If a large stress increment $\Delta \sigma$ is applied to the stress point $M$ such that the new stress state is at point $A$, then Mroz's kinematic rule would demand that point $M$ translate in the direction of $MR$. This raises the problem of how $f_m$ can translate along $MR$ and yet still have point $A$ on $f_m$, as the consistency condition demands, after $\Delta \sigma$ has been applied. It is evident that the choice of a large value of $\Delta \sigma$ is responsible for this difficulty.

If a small stress increment $d\sigma$ is applied instead such that the new stress state is at point $B$, then the situation becomes manageable. $f_m$ translates towards $f_{m+}$ in the direction of $MR$ but only for a short distance because of the small value of $d\sigma$. Point $B$ can therefore remain on $f_m$ while simultaneously satisfying the kinematic hardening rule. Fig.2.12 illustrates the positions of the yield surfaces after $d\sigma$ has been applied. Note that the corresponding outward normal to point $B$ (on $f_m$) on yield surface $f_{m+}$ is point $R'$ and this point lies between $R$
Fig. 2.11. Translation of $f_m$ requires that many small stress increments be used, such as the initial one to point B, instead of one large one, such as a single one to point A.
Fig. 2.12. Position of $f_m$ after a small stress increment has been applied.
and the intersection of the stress path with \( f_{m+1} \) (point G). When another stress increment is applied along the indicated stress path, \( f_m \) translates in the direction of \( BR' \) which is different, in general, to \( MR \). Clearly, the application of many more small stress increments would cause the corresponding outward normal on \( f_{m+1} \) to shift along the surface of \( f_{m+1} \) towards point G, with \( f_m \) ultimately being tangent to \( f_{m+1} \) at this point. It is evident that only "small" stress increments can give consistent results.

Of course, this raises the problem of defining what a small stress increment is. Unfortunately, there is no single answer and the maximum stress increment which can be used depends on the desired accuracy of the computations and the positions of the yield surfaces with respect to one another. For example, many stress increments would be required for the yield surfaces shown in Fig.2.11 for reasons discussed earlier. In contrast, a single stress increment between yield surfaces would be adequate if the centers of \( f_m \) and \( f_{m+1} \) lie along an extension to a linear stress path, as illustrated in Fig.2.13, since \( \mu_{ij} \) would have the same value at intermediate points (i.e. using small stress increments) as the initial value.

It should be noted that the specification of small \((d\sigma_{ij} - d\Sigma_{ij})\) and \(dk\) in the shortened form of the consistency condition (eq.2.27) is not related to the problem just discussed. The consistency condition, whether it be eq.2.26 or 2.27 only influences the magnitude of the kinematic movement and not the direction. The direction is specified by the particular
Fig.2.13. A large stress increment may be applied when the centers of $f_m$ and $f_{m+1}$ lie along an extension to a linear stress path because $\mu_h$ would not change even if small stress increments were used.
hardening rule used.
CHAPTER 3

THEORY FOR THE UNDRAINED TOTAL STRESS ANALYSIS OF CLAY

Total stress type analyses have been widely used in the past for determining the stability of slopes and for stress-deformation problems. Although it is widely acknowledged that effective stresses govern the behavior of soils, the total stress approach is still widely used because of its simplicity and its successful application to many geotechnical problems in the past. This is particularly true for saturated clays subjected to loadings of short duration. Under these conditions, no volume change occurs and any increase in the total mean normal stress has no effect on the effective mean normal stress. The behavior of the soil is therefore independent of the total mean normal stress and only the deviatoric stresses need to be considered in any constitutive relation. Changes in the stress-strain characteristics due to pore pressure generation during deformation must be indirectly accounted for by varying the yield stress and strength of the soil.

The behavior of plastically deformed, work hardening metals is also considered to be independent of the mean normal stress. Mathematical models for describing the hysteresis and Bauschinger effect during plastic deformation of metals were proposed by Iwan (8) and Mroz (12) in 1967. But it was not until Prevost published his work in 1977 (18) that these ideas
were utilized to construct a practical model for clays. This chapter particularizes the theory of Chapter 2 to model the undrained total stress behavior of clay.

To describe the stress-strain behavior of a soil subjected to undrained loading, the strains are separated into elastic and plastic deformations. The elasticity is assumed to be constant, linear, and isotropic. The relation between the stress and elastic strain is then given by:

\[ \sigma_{ij} = 2G\epsilon_{ij}^e + \delta_{ij} \Omega \epsilon_v^e \]  \hspace{1cm} (3.1)

where:

\[ \Omega = \text{Lame's constant} \]
\[ \epsilon_{ij}^e = \text{elastic strain} \]
\[ \epsilon_v^e = \text{elastic volumetric strain} \]
\[ \epsilon_d^e = \text{elastic deviatoric strain} \]
\[ \sigma_m = \text{mean normal stress} \]

Multiplying eq.3.1 by \( \delta_{ij} \) gives:

\[ \sigma_j \delta_{ij} = 3\sigma_m = 2G\epsilon_{ij}^e \delta_{ij} + \delta_{ij} \delta_{ij} \Omega \epsilon_v^e \]

\[ \sigma_m = \frac{3}{2} G \epsilon_v^e + \Omega \epsilon_v^e \]

Thus:
Non-linearity is assumed to be a result of the plasticity. The yield criterion for undrained saturated clays, as mentioned previously, is usually considered to be independent of the total mean normal stress. The yield function then requires only deviatoric stresses for its arguments. For mathematical simplicity, a Von Mises type yield surface is assumed. Equation 2.18 then becomes:

\[ f_m = 0 = \frac{2}{3}(s_{ij} - a^{(m)\cdot}))(s_{ij} - a^{(m)\cdot}) - (k^{(m)})^2 \]  

The absence of the total mean normal stress from eq.3.3 indicates that there is no plastic volume change. This is in contrast to the common assumption in constitutive theories for soil which assumes that the total volume change is zero with the plastic volume change equal and opposite to the elastic volume change. Although it is possible to calculate an elastic
volume change, this quantity is ordinarily set to zero in this model for consistency with the assumption of zero total volume change in clays during undrained loading.

The $a_j^{(m)}$ are a subset of the $\xi_j$ defined in eq.2.17 and represent the coordinates of the center of the yield surface, $f_m$, in the deviatoric stress subspace. The $k^{(m)}$ are a measure of the size of the yield surfaces.

To conceptualize eq.3.3, let us assume that there are only two deviatoric stress dimensions: 12 (one-two) and 13 (one-three), with the remaining deviatoric stresses and offsets ($a_{ij}$) being zero. Expanding eq.3.3 we have:

$$(s_{12}-a_{12})^2 + (s_{13}-a_{13})^2 = \frac{2}{3}k^2 \quad (3.4)$$

This is the equation of a circle in $s_{12}-s_{13}$ space with offsets of $a_{12}$ and $a_{13}$ from the origin along the 12 and 13 axes respectively. Thus, yield surfaces can be thought of as being 2 dimensional circles having radii of $\sqrt{\frac{2}{3}}k$ in a 2 dimensional deviatoric stress space. In a three dimensional deviatoric stress space, the yield surfaces would appear as spheres.

To obtain the plastic strains, a flow rule is required. For mathematical simplicity, an associated flow rule is adopted. Although some researchers have questioned its validity (4,27), the associated flow rule has enjoyed some success in other soil plasticity models (24) and it is widely used. Equation 2.19 then becomes:
\[ \text{de}^p_{ij} = \frac{1}{H'_m} \frac{(Q^{(m)}' \cdot ds_{kl}) Q^{(m)}'}{|Q^{(m)}'|^2} \]  \quad (3.5) \\

where:

\[ Q^{(m)}'_{ij} = \frac{\partial f_m}{\partial s_{ij}} = \frac{\partial f_m}{\partial s_{ij}} \]

= outward normal to yield surface \( f_m \)

\[ |Q^{(m)}'| = (Q^{(m)}'_k Q^{(m)}'_l) \]

\( ds_{ij} \) = deviatoric stress increment

\( \text{de}^p_{ij} \) = plastic deviatoric strain increment

To obtain a practical stress-strain relation, eq.3.5 is simplified by recalling that:

\[ Q^{(m)}'_{ij} = \frac{\partial f_m}{\partial s_{ij}} = 3(s_{ij} - a^{(m)}_{ij}) \]  \quad (3.6) \\

\[ |Q^{(m)}'| = \sqrt{6k^{(m)}} \]  \quad (3.7) \\

Therefore:

\[ \text{de}^p_{ij} = \frac{1}{H'_m} \frac{[3(s_{ij} - a^{(m)}_{ij})] 3(s_{kl} - a^{(m)}_{kl}) ds_{kl}}{6k^{(m)}^2} \]  \quad (3.8) \\

The complete constitutive relation then becomes:
\[ \text{de}_{ij} = \frac{ds_{ij}}{2G} + \frac{3}{2H_m^{'}} \frac{(s_{ij} - a_{ij}^{(m)})}{k^{(m)}_l} (s_{kl} - a_{kl}^{(m)}) ds_{kl} \] \hspace{1cm} (3.9)

This can be inverted (Appendix B) to give the stress increment in terms of the strain increment:

\[ ds_{ij} = 2G \text{de}_{ij} - \frac{3}{2} \frac{(2G - H_m)(s_{ij} - a_{ij}^{(m)})}{k^{(m)}_l} (s_{kl} - a_{kl}^{(m)}) \text{de}_{kl} \] \hspace{1cm} (3.10)

where:

\[ H_m = \frac{1}{H_m^{'}} + \frac{1}{2G} \] \hspace{1cm} (3.11)

Now let us examine the form of \( H_m^{'}. \) By forming a scalar product of both sides of eq.3.5 with itself we have:

\[ \text{de}^p_{i} \text{de}^p_{j} = \frac{1}{H_m^{'}} \left[ \frac{(Q_{ij}^{(m)'} \ ds_{ij})}{|Q^{(m)'}|^2} \right]^2 Q_{ki}^{(m)'} Q_{kl}^{(m)'} \]

The square root of this becomes:

\[ H_m^' = \frac{(Q_{ij}^{(m)'} \ ds_{ij})}{|Q^{(m)'}| \ |\text{de}^p|} \]

where:

\[ |\text{de}^p| = (\text{de}_{kl}^p \text{de}_{kl}^p)^{\frac{1}{2}} \]
The stress increment $\mathbf{d}s_j$ can be split into two portions:

$$\mathbf{d}s_j = \mathbf{d}s_j^i + \mathbf{d}s_j^\prime$$

where:

$\mathbf{d}s_j^\prime$ is in the direction of the outward normal to the yield surface;

$\mathbf{d}s_j^i$ is tangent to the yield surface and produces no plastic flow.

Therefore:

$$Q_j^{(m)'} \mathbf{d}s_j = Q_j^{(m)'} \mathbf{d}s_j^\prime$$

$$= |Q^{(m)'}| |\mathbf{d}s^\prime|$$

and:

$$H'_m = \frac{|\mathbf{d}s^\prime|}{|\mathbf{d}e^p|} \quad (3.12)$$

This shows that the plastic modulus is not a function of $Q_j^i$ and thus $H'_m$ may assume a constant value along the yield surface.

Both isotropic and kinematic hardening are incorporated in this model. Kinematic hardening is adopted to model the
Bauschinger effect while isotropic hardening is used for incorporating the effects of strength and stiffness loss during cyclic loading.

The kinematic hardening of the yield surfaces is described by Mroz's kinematic rule (12):

$$da_{ij}^{(m)} = d\mu \mu_{ij} \quad (3.13)$$

in which (eq.2.31):

$$\mu_{ij} = \frac{k^{(m+1)}}{k^{(m)}} (s_{ij} - a_{ij}^{(m)}) - (s_{ij} - a_{ij}^{(m+1)}) \quad (3.14)$$

where $s_{ij}$ may be substituted for $a_{ij}$ since only deviatoric stresses are considered. The scalar $d\mu$ can be found from eq.2.48 (assuming small $ds_{ij} - da_{ij}$ and $dk$):

$$d\mu = \frac{Q_{ij}^{(m)} ds_{ij} - 2k^{(m)} dk^{(m)}}{Q_{kl}^{(m)} \mu_{kl}}$$

$$= \frac{3(s_{ij} - a_{ij}^{(m)}) ds_{ij} - 2k^{(m)} dk^{(m)}}{3(s_{kl} - a_{kl}^{(m)}) \mu_{kl}} \quad (3.15)$$

If it is further assumed that $dk^{(r)} = dk^{(r+1)}$ for all $r$ and $Q_{ij}^{(m)} da_{ij}^{(m+1)} = 0 \quad (20)$, then eq.2.43 gives:
and the yield surfaces larger than the one which the stress point is currently on do not translate (i.e. the unengaged surfaces do not move). This resulting computational simplicity is the primary reason for using Mroz's kinematic rule over Prager's, Ziegler's, or Phillip's rule.

Alternatively, eq.3.15 could have been derived by using the hardening rule proposed in eq.2.49 with $dk^{(r)} = dk^{(r+1)}$ for all $r$.

A more exact expression for $d\mu$ without assuming small $(ds_j - da_j)$ and $dk$ can be derived by first assuming $dk^{(r)} = dk^{(r+1)}$ for all $r$ and $da_j^{(m+1)} = 0$ (20). Eq.2.26 for $n=2$ becomes:

$$Q_{ij}(ds_{ij} - da_{ij}) + \frac{1}{2} \frac{\partial Q_{ij}}{\partial s_{ij}} (ds_{kl} - da_{kl})^2 = 2kdk + dk^2$$

$$da_j^2 \left( \frac{1}{2} \frac{\partial Q_{kl}}{\partial s_{kl}} \right)^2 + da_j \left( - \frac{\partial Q_{kl}}{\partial s_{kl}} ds_{ij} - Q_{ij} \right) + \left( \frac{1}{2} \frac{\partial Q_{kl}}{\partial s_{kl}} ds_{ij}^2 + Q_{ij} ds_{ij} - 2kdk - dk^2 \right) = 0$$

Letting:

$$Q_{ij} = \frac{\partial f}{\partial s_{ij}} = 3(s_j - a_j)$$

then:
\[
d a_{ij}^2 \left( \frac{3}{2} \right) + da_{ij} \left[ -3ds_{ij} - 3(s_{ij} - a_{ij}) \right] + \left[ \frac{3}{2} ds_{ij}^2 + 3(s_{ij} - a_{ij}) ds_{ij} - 2kd - dk^2 \right] = 0
\]

But \( da_{ij} = d\mu \mu_{ij} \).

Therefore:

\[
d\mu = \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\]

(3.16)

where:

\[
A = \frac{3}{2} \mu_{ij} \mu_{ij}
\]

\[
B = -[3\mu_{ij} ds_{ij} + 3(s_{ij} - a_{ij}) \mu_{ij}]
\]

\[
C = \left[ \frac{3}{2} ds_{ij}^2 + 3(s_{ij} - a_{ij}) ds_{ij} - 2kd - dk^2 \right]
\]

For isotropic hardening/softening, the size of the yield surface \( k \), is conveniently taken to be a function of (19):

\[
\lambda = \int \left( \frac{3}{2} \delta e^p_{ij} \delta e^p_{ij} \right)^{\frac{1}{4}}
\]

(3.17)

The parameter \( \lambda \) then becomes a measure of the total plastic strain which has taken place. For simplicity, the surfaces may retain their original size until the first load reversal occurs.

By varying the size of the yield surfaces, the behavior of
a soil subjected to a small number of load cycles can be adequately modelled. For a large number of cycles, both \( k \) and \( H' \) will vary for each yield surface. This will be explained further in the chapter dealing with cyclic loading.

As mentioned previously, the \( a_i^{(r)} \) represent the coordinates of the center of the yield surface \( f_r \), in the deviatoric stress subspace. The significance of the \( a_i^{(r)} \) can best be examined by first assuming that they are all zero (Fig.3.1). Any circle \( A \) with its center at \( O \) would represent an invariant measure of the deviatoric stress state of the system.

From eq.3.12, it can be seen that \( |\text{de}^p| \) would be the same for any monotonic stress path which starts at the origin \( O \) and terminates on circle \( A \). Since \( |\text{de}^p| \) is an invariant measure of the plastic strain and the elasticity is assumed to be isotropic, the soil would behave isotropically. Thus if \( a_i^{(r)} = 0 \), the material is isotropic.

If, on the other hand, we offset yield surface \( B \) as in Fig.3.2, the \( |\text{ds}''| \) (stress increment component perpendicular to the yield surface) to circle \( A \) would be different and hence \( |\text{de}^p| \) would depend on the loading path. The soil would then behave anisotropically.

In general, there are six deviatoric axes (of which five are independent) and thus six components of \( a \) must be evaluated for each material type. However, if the physical coordinate axes coincide with the principal axes of material anisotropy, \( a_{xy}^{(r)} \), \( a_{yz}^{(r)} \) and \( a_{zx}^{(r)} \) are then initially all equal to zero for
Fig. 3.1. Yield surfaces in stress space all centered about the origin. Isotropic behavior would result.

Fig. 3.2. Yield surfaces in stress space with yield surface B offset from origin. Anisotropic behavior would result.
all \( r \). Further, if \( a_x(r) = a_y(r) \) for all \( r \), then the material initially exhibits cross-anisotropy about the \( y \) axis (i.e. the behavior is the same on a given plane which is perpendicular to the \( y \) axis). This is particularly relevant to geotechnical engineering because fine-grained soils are often considered to be deposited in horizontal layers.
CHAPTER 4

APPLICATION OF THE TOTAL STRESS MODEL
TO UNDRAINED CLAYS

The application of the model requires the determination of:

1. The elastic shear modulus $G$
2. The initial positions $(a_0)$ and sizes $(k_0)$ of the yield surfaces and their associated plastic moduli $(H')$
3. The changes in the plastic moduli or size of the yield surfaces as loading occurs.

To describe the undrained behavior of clays which maintain their stiffness and do not soften requires only the first two sets of parameters. Under these conditions, $H'$ and $k$ are constant. This section will be confined to the application of the model to such soils. Clays which soften when subjected to cyclic loads require that the changes in $H'$ and $k$ be determined. A simple example illustrating how the functional change in $k$ (and $H'$) can be determined will be described in Chapter 6.

Although the model is capable of analyzing a soil which is completely anisotropic, only a procedure to determine the parameters for the case of cross-anisotropy (or rotational symmetry) about the vertical $y$ axis will be outlined here. In
this case, both the experimental triaxial \((\sigma_x=\sigma_z)\) compression and extension stress-strain curves are required. Theoretically, it is immaterial whether the triaxial compression test is conducted by holding \(\sigma_x\) and \(\sigma_z\) constant and increasing \(\sigma_y\), or by holding \(\sigma_y\) constant and decreasing the confining pressure since both loading patterns would have identical stress paths in the deviatoric plane. Similarly, the method of imposing the stress difference \((\sigma_y-\sigma_z)\) is immaterial for triaxial extension. Test data to support this finding will be presented further on.

The triaxial compression and extension tests described above are not true tests for anisotropy. This is demonstrated in Fig.4.1 where the equal and opposite stresses necessary for equilibrium have been removed for clarity. However, if the material is cross-anisotropic about the vertical axis, an associated flow rule is assumed, and a Von Mises type yield surface is adopted, then the behavior of the soil under triaxial compression and extension tests is also a reflection of its anisotropy.

To illustrate this, stress paths corresponding to the stress states I, II, and III in Fig.4.1 have been shown in Fig.4.2 in the deviatoric stress space. Cross-anisotropy about the y axis and an associated flow rule require that the center of each yield surface lie on the (positive or negative) \(s_y\) axis. If a yield point is selected along the positive \(s_y\) axis, say at point A, then only one other corresponding yield point in the deviatoric plane, call it B, is needed to completely
a) Triaxial Tests

Compression

Extension

b) True Tests for Anisotropy

I

II

III

Triaxial Compression

Fig. 4.1. a) The triaxial compression and extension tests subject the soil to a different set of stresses in each case whereas b) the same set of stresses at different orientations is required for a true test for anisotropy.
describe the position of that yield surface. Thus if a test corresponding to stress state II or III (which is truly indicative of anisotropy) is performed, the corresponding yield point in triaxial extension can be interpolated from the yield function. Similarly, if $B$ were chosen from a triaxial extension test, the corresponding yield points for II and III may be determined. Therefore the extension test is a reflection of the soil's anisotropy in this particular case.

If the axes of material anisotropy coincide with the principal axes of loading in the test apparatus, then for rotational symmetry about the y axis:
\[ a_{xy} = a_{yz} = a_{zx} = 0 \] \hspace{1cm} (4.1)

\[ a_x = a_z \]

Since the \( a_j \) are deviatoric stresses, then \( a_n = 0 \) and:

\[ a_x = -\frac{a_y}{2} \] \hspace{1cm} (4.2)

A procedure to find \( G \) and the initial values of \( H', k, \) and \( a \) is as follows:

1. Choose a piecewise linear approximation of the stress difference \( (a_y - a_x) \) versus axial strain \( (\varepsilon_y) \) curve in compression as shown in Fig.4.3a. A corresponding representation in stress space is shown with it for explanatory purposes in Fig.4.3b. Appendix D derives the scale factor \( \sqrt{3/2} \) which is necessary to establish equivalence between the stress axes of Fig.4.3a and Fig.4.3b.

2. Points where the piecewise linear curve changes slope are referred to here as yield points. An elasto-plastic modulus \( H_f \) is associated with the linear segment above yield point \( r \):
Fig. 4.3. a) Piecewise linear approximation of the stress-strain curve for an axisymmetric triaxial compression and extension test. b) Corresponding representation of yield surfaces in stress space.
\[
\frac{3}{H_r} = \frac{3}{\frac{2}{H_r^2} - \frac{1}{G}} = \text{slope of segment on stress curve vs. axial strain difference}
\]

This is derived in Appendix C.

3. Find corresponding yield points on the extension side.
   As mentioned previously, the plastic shear modulus \( H_r \) is a constant for each yield surface \( m \). Thus the corresponding yield stresses in extension represent points where the slopes of the idealized, piecewise linear, stress-strain curves are the same as in compression.

4. Calculate \( k^{(m)} \) for each \( m \):

\[
2k^{(m)} = \text{difference between the yield stress in compression and extension of surface } m.
\]

\[
= (\sigma_y - \sigma_x)^{(m)}_{\text{comp}} - (\sigma_y - \sigma_x)^{(m)}_{\text{ext}}
\]

5. Calculate \( a^{(m)} \) for each \( m \) where:

\[
a^{(m)} = \text{offset of center of yield surface } m \text{ from origin.}
\]
\[
\begin{align*}
\frac{3}{2} a_y^{(m)} &= -3 a_x^{(m)} = -3 a_i^{(m)} \\
= (\sigma_y - \sigma_x)_{\text{comp}}^{(m)} - k^{(m)}
\end{align*}
\]

These parameters together with equations 3.9 and 3.13-3.15 give the complete set of equations necessary to describe the stress-strain behavior of non-softening soils. It should be noted that the parameters obtained by the above method are a function of the selection of the piecewise linear approximation of the stress-strain curve in compression. Thus, there are an infinite number of possible parameters for a given stress-strain curve.

For many common problems, some components of \( a_{ij} \) and the stress tensor vanish. For example, in situations for which either axisymmetric or plane strain loading conditions occur (i.e. two dimensional analysis), \( \tau_{yz} = \tau_{xz} = 0 \). The following stress coordinates become convenient for this case(19):

\[
\begin{align*}
S_1 &= \frac{3}{2} s_y \\
S_2 &= \sqrt{3}(a_z - a_x) = \sqrt{3}(s_z - s_x) \\
S_3 &= \sqrt{3} \tau_{xy}
\end{align*}
\]

These stress coordinates and their associated strains will be referred to as 1, 2, 3 coordinates. The distance from the origin to the stress point in the orthogonal stress space
\[ \sqrt{S_1^2 + S_2^2 + S_3^2} = \sqrt{3J_2} = \sqrt{\frac{3}{2}S_{ij}S_{ij}} \]

By analogy with eq. 3.3, we have:

\[ \left( \Sigma(S_i - a_i^{(m)})^2 \right) - k^{(m)^2} = 0 \]

\[ = \frac{3}{2}(s_{ij} - a_{ij}^{(m)})(s_{ij} - a_{ij}^{(m)}) - k^{(m)^2} \]

\[ (4.5) \]

in which the coordinates of the center of the yield surface \( m \) are:

\[ a_i^{(m)} = \frac{3}{2}a_y^{(m)} \]

\[ a_z^{(m)} = \frac{\sqrt{3}}{2}(a_z^{(m)} - a_x^{(m)}) \]

\[ a_i^{(m)} = \sqrt{3}a_{xy}^{(m)} \]

The strains \( E_1, E_2, \) and \( E_3 \) associated with the stresses \( S_1, S_2, \) and \( S_3 \) respectively are defined so that an increment of work would be the same in either \( x, y, z \) or \( 1, 2, 3 \) coordinates:

\[ dW = s_{ij} \, de_{ij} = \sum_i S_i \, dE_i \]

\[ (4.6) \]

Combining equations 4.4 and 4.6 gives:
To get an expression for $E_3$, note that the last term on the left hand side of eq. 4.7 corresponds to the last term of the right hand side since these are the only terms involving $s_{xy}$. Therefore:

$$2s_{xy}e_{xy} = \sqrt{3}s_{xy}dE_3$$

Integrating the left hand side between 0 and $E_3$ and the right hand side between 0 and $\gamma_{xy}$ gives:

$$E_3 = \frac{1}{\sqrt{3}} \gamma_{xy}$$

$E_1$ can be found from the terms in eq. 4.7 which do not have any $xy$ terms:

$$s_xde_x + s_yde_y + s_zde_z = \frac{3}{2}s_ydE_1 + \frac{\sqrt{3}}{2}(s_z-s_x)dE_2 + \sqrt{3}s_{xy}dE_3$$

Recall that:

$$s_x + s_y + s_z = 0$$
and if Poisson's ratio=.5:

\[ e_x + e_y + e_z = 0 \]

then:

\[ 2s_x \, de_x + 2s_y \, de_y + s_y \, de_x + s_x \, de_y = \frac{3}{2} s_y \, dE_1 - \sqrt{\frac{3}{2}} (s_y + 2s_x) \, dE_2 \]  
\((4.11)\)

\[ 2s_x \, (de_x + \frac{1}{2}de_y) + 2s_y \, de_y + s_y \, de_x = \frac{3}{2} s_y \, dE_1 - \sqrt{\frac{3}{2}} (s_y + 2s_x) \, dE_2 \]  
\((4.12)\)

Taking terms containing only \(s_x\):

\[ -\sqrt{3} s_x \, dE_2 = 2s_x \, (de_x + \frac{1}{2}de_y) \]

\[ = 2s_x \, (de_x - \frac{1}{2}de_x - \frac{1}{2}de_z) \]

\[ = s_x \, (de_x - de_z) \]

Therefore:

\[ dE_2 = \frac{1}{\sqrt{3}} (de_z - de_x) \]

Integrating, we get:

\[ E_2 = \frac{1}{\sqrt{3}} (e_z - e_x) \]  
\((4.13)\)

Finally \(E_1\) can be determined by substituting the expression for \(E_2\) into eq.4.12 and integrating the resulting total differential to give:
\[ E_1 = e_y \] (4.14)

The stress-strain relation is obtained by substituting eq. 4.5 into eq. 3.5 giving:

\[
\text{d}E_i = \frac{\text{d}S_i}{3G} + \frac{2}{3H_m} \left( \frac{S_i - a_i^{(m)}}{(k^{(m)})^2} \right) \sum_j (S_j - a_j^{(m)}) \text{d}S_j \quad (4.15)
\]

in which the inverse is:

\[
\text{d}S_i = 3G\text{d}E_i - \left( 3G - \frac{3}{2} H_m \right) \frac{S_i - a_i^{(m)}}{(k^{(m)})^2} \sum_j (S_j - a_j^{(m)}) \text{d}S_j \quad (4.16)
\]

The kinematic hardening rule follows from equations 2.31 and 2.48:

\[
da_i = d\mu \left[ \frac{k^{(m+1)}}{k^{(m)}} \left( S_i - a_i^{(m)} \right) - \left( S_i - a_i^{(m+1)} \right) \right] \quad (4.17)
\]

in which:

\[
d\mu = \frac{-B - \sqrt{B^2 - 4AC}}{2A}
\]

\[
A = \mu_i \mu_i
\]

\[
B = -2\mu_i (S_i + \text{d}S_i - a_i^{(m)})
\]
\[ C = 2(S_i - a_i^{(m)})dS_i + dS_i dS_i - 2k(m)dk(m) - (dk(m))^2 \]

or for small \((dS_i - da_i)\) and \(dk\):

\[ d\mu = \frac{\Sigma(S_i - a_i^{(m)})dS_i - k(m)dk(m)}{k(m+1)k(m) - \Sigma_j(S_j - a_j^{(m+1)})(S_j - a_j^{(m)})} \]  

(4.18)

The \(S_1, S_2, S_3\) stress space is particularly useful when interpreting triaxial, simple shear, or plane strain loading conditions.

During axisymmetric triaxial tests, \(\sigma_x = \sigma_z\) and \(\tau_{xy} = 0\). Thus \(S_2 = S_3 = 0\) and the stress point moves along the \(S_1\) axis only.

Simple shear soil tests demand that \(d\varepsilon_x = d\varepsilon_y = d\varepsilon_z = 0\). For a soil initially subjected to equal horizontal normal stresses: \(E_2 = 0\), \(S_2 = 0\), and \(\sigma_2 = 0\). Applying equations 4.16 and 4.17 gives \(dS_2 = 0\) and \(da_2 = 0\) respectively. The stress point therefore remains in the \(S_1, S_3\) plane at all times. If the elastic strains are assumed to be zero, \(d\varepsilon_y = 0\) and \(dE_1 = 0\). The stress point then follows a stress path such that the outward normal to the yield surface which it is on, has no component in the \(S_1\) direction (Fig.4.4).

Plane strain tests require \(\tau_{xy} = \tau_{yz} = \tau_{zx} = 0\) and thus \(S_3 = 0\) at all times. Conventional plane strain tests constrain \(d\varepsilon_z = 0\) and thus:

\[ d\varepsilon_x + d\varepsilon_y + d\varepsilon_z = 0 \]
Fig. 4.4. a) Stress path for simple shear test in $S_1-S_3$ plane. Resulting b) shear stress vs. shear strain and c) stress difference vs. shear strain curves. [from Prevost (18)]

\[
\begin{align*}
\text{Insitu pressuremeter tests, which are a type of plane strain test, ideally have } & \text{ } \\
\text{they have } d\varepsilon_y = 0 \text{ and thus } dE_1 = 0. \text{ If the elastic strains are negligible, then the stress point lies in the } & \text{ } \\
S_1-S_2 \text{ plane and follows a path such that the normal to the yield } & \text{ }
\end{align*}
\]

\[
\begin{align*}
d\varepsilon_x &= -d\varepsilon_y \\
-\sqrt{3}dE_2 &= -dE_1 \\
dE_2 &= \frac{1}{\sqrt{3}}dE_1
\end{align*}
\]
surface which it is on has no component in the $S_1$ direction.
CHAPTER 5

PREDICTIONS USING THE UNDRAINED TOTAL STRESS MODEL

To evaluate the accuracy of the model proposed by Prevost, the stress-strain behavior of a clay subjected to four different undrained, monotonic stress paths was predicted from the results of undrained triaxial compression and extension tests. The predictions were subsequently compared with stress-strain curves obtained from actual test data.

Computer Program Check

In order to verify that the computer program developed for this study was able to duplicate the calculation procedures as outlined in Prevost's papers (18,19), a prediction of the simple shear behavior of Drammen clay $K_0$ overconsolidated to a ratio of 4, was made using the model parameters published in (18) and repeated here in Table 1. Figure 5.1 shows the stress-strain curves as calculated by the computer program. The curves match those published in (18) very well, thus verifying the accuracy of the computer program used.

Test Data

The tests were conducted as part of the May 1980 NSF/NSERC North American Workshop on Plasticity Theory and Generalized Stress-Strain Modelling of Soils (29). Full details of the
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<tr>
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<th>$\frac{a}{\sigma'}$</th>
<th>$\frac{H}{\sigma'}$</th>
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<tr>
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<td>0.150</td>
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<td>0.600</td>
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<td>0.400</td>
<td>73.33</td>
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<tr>
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<td>0.775</td>
<td>0.475</td>
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<td>0.875</td>
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<td>0.575</td>
<td>24.33</td>
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<td>0.600</td>
<td>17.33</td>
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</table>

$G = 200\sigma'$

testing procedure are available in the Proceedings of the above conference and only a summary will be given here.

A laboratory prepared kaolinite clay with a liquid limit of 62.5%, a plastic limit of 39.0%, and an average undrained water content of 38.8% was used for the tests. The samples
were typically $K_0$ consolidated with an effective confining pressure of 40 psi and an additional axial load sufficient for a $K_0=0.48$ condition to be maintained. After consolidation was completed, the additional ($K_0$) axial load was removed and the sample was allowed to rebound under an isotropic effective stress of 40 psi. The samples were then subjected to stress-controlled tests in an undrained condition.

Four basic tests were provided upon which model parameters were to be derived:

**Test 1** - A compression test in which the axial stress was increased while the confining pressure was held constant.
**Test 4** - A compression test in which the axial stress was increased and the lateral stress decreased such that the total mean normal stress remained constant.

**Test 10** - An extension test in which the axial stress is decreased while the confining pressure was held constant.

**Test 13** - An extension test in which the axial stress was decreased and the lateral stress increased such that the total mean normal stress remained constant.

The stress difference versus axial strain curves are plotted in Fig.5.2. Note that tests 1 and 4 in compression have almost identical behavior. This is to be expected since these tests have identical stress paths in the deviatoric plane and the changes in the total mean normal stress do not affect the effective mean normal stress (because of undrained conditions). A similar argument holds for tests 10 and 13 in extension.

Table 2 lists a set of parameters for this soil using the procedure outlined in Chapter 4 while Tables 3, 4, 5, and 6 tabulate the stress paths for prediction. Note that tests 2 and 5 have identical stress paths in the deviatoric plane. Fig.5.3 shows how the angle $\omega$ is defined for the stresses paths to be predicted. A representation of the yield surfaces in the $S_1, S_3$ plane is given in Fig.5.4 together with the stress paths
Fig. 5.2. Triaxial stress-strain curve for Kaolinite. Note the almost identical behavior of the conventional triaxial test and the constant total mean normal stress test. [data from (29)]
TABLE 2

Model Parameters for Kaolinite

<table>
<thead>
<tr>
<th>m</th>
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<th>a</th>
<th>H</th>
</tr>
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<tbody>
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<td>psi</td>
<td>psi</td>
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<tr>
<td>8</td>
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</table>

G = 3030. psi

Fig. 5.3. Definition of $\omega$. 
to be predicted.

Predictions

Figures 5.5, 5.6, 5.7 and 5.8 compare the predicted stress-strain curves for tests 2, 3, 5, and 7 respectively with actual test data. While tests 2, 3, and 7 agree very well, the results of test 5 appear to be somewhat disappointing. Because the stress paths in the deviatoric plane of samples 2 and 5 are the same, one would expect the correspondence between the predicted and experimental stress-strain curves to be the same for both tests. One explanation for this discrepancy could be that test 5 had a water content almost 1% higher than the average of the other tests. In effect, this sample was less overconsolidated than the others and thus one would expect its strength to be lower than that predicted from parameters obtained from specimens with lower water contents.

With these considerations in mind, it appears that the model was able to predict the behavior of this clay very well. These comparisons further demonstrate the applicability of the model to the undrained, monotonic loading of clays as previously shown by Prevost (19).
TABLE 3

Test 2

Total Principal Stresses (psi)

<table>
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<th>$\sigma_1$</th>
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water content = 38.8%

$\omega = 15^\circ$
<table>
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water content = 38.8%

$\omega = 37.5^\circ$
### TABLE 5

**Test 5**

Total Principal Stresses (psi)

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Water content = 39.7%

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water content = 38.8%

$\omega = 45^\circ$
Fig. 5.4. Kaolinite yield surfaces and stress paths for prediction in $S_1-S_3$ plane.
Fig. 5.5. Stress-strain comparisons for Kaolinite – Test 2. [data form (29)]
Fig. 5.6. Stress-strain comparisons for Kaolinite - Test 3. [data from (29)]
Fig. 5.7. Stress-strain comparisons for Kaolinite - Test 5. [data from (29)]
Fig. 5.8. Stress-strain comparisons for Kaolinite - Test 7. [data from (29)]
CHAPTER 6

APPLICABILITY OF THE MODEL TO CYCLIC LOADING

One of the most attractive features of this model is its ability to simulate some aspects of cyclic loading. Foremost amongst these is the ability to exhibit Masing behavior.

Figures 6.1 and 6.2 illustrate how the model generates hysteresis loops in a simple shear test. The yield surfaces and stress paths are plotted in $S_1, S_3$ space whereby $S_1 = \sqrt{3} \sigma_y/2$ (eq.4.4a) and $S_3 = \sqrt{3} \tau_{xy}$ (eq.4.4c). Let us first assume that the soil is cross-anisotropic about the vertical axis and that the elastic strains are negligible. Then the $\sqrt{3} \tau_{xy}$ required to reach any yield surface $m$ upon initial loading is equal to $k^{(m)}$ since the stress path goes through the apex of $f_m$ (Fig.6.1).

If no softening occurs, then the change in $\sqrt{3} \tau_{xy}$ to reach yield surface $f_m$ upon unloading is $2k^{(m)}$ as shown in Fig.6.2. This is precisely the behavior predicted by applying Masing's rules.

Soils which soften when subjected to cyclic loads require that the yield surfaces decrease in size. A simplistic method of doing this would be to utilize an isotropic softening rule whereby all the yield surfaces contract in size simultaneously and by the same amount (i.e. $dk^{(r)} = dk^{(r+1)}$ for all $r$) without changing their positions $(a_{ij}^{r+1})$. While this may be appropriate if only a few load cycles are applied, the remolding effect of large cumulative plastic straining eventually causes a loss of
Fig. 6.1. a) Yield surfaces before loading and initial monotonic stress path to point N. b) Shear stress – strain curve resulting from monotonic stress path. [from Prevost (18)]

Fig. 6.2. a) Yield surfaces upon reaching point N and loading reversal stress path. b) Hysteretic loop formed by stress cycle. [from Prevost (18)]
the soil's inherent anisotropy. To account for this, Prevost (18) suggests that, in addition to contracting the size of all the yield surfaces by the same amount, the offsets along the $S_1$ axis be made to decrease (i.e. a kinematic change) such that $a_i = a_i(\lambda)$ until $a_i(r) = 0$ for all yield surfaces, $f_r$, which have not yet been translated. Inspection of the experimental cyclic load test data for loss of inherent anisotropy (i.e. a tendency towards isotropic behavior) can determine whether this additional sophistication is warranted.

To demonstrate the effect of decreasing the sizes of the yield surfaces, the isotropic softening rule will be applied to model the behavior of Santa Barbara silt when subjected to a strain controlled, cyclic simple shear test with an amplitude of $\gamma_{x'y_{max}} = 1\%$. Fig.6.3 shows the results of a monotonic triaxial compression and an extension test as reported in reference 23 and Table 7 lists a set of parameters derived from this data. The predicted simple shear stress-strain curve together with the experimental test results are compared in Fig.6.4. Good correspondence between the predicted curve and experimental points was achieved for the $\tau_{xy}$ versus $\gamma_{xy}$ plot.

To model the softening behavior during cyclic loading, the change in the resisting stress when the soil is subjected to a constant cyclic strain amplitude must be found experimentally. Fig.6.5 plots this change versus the number of cycles for a strain amplitude of 1% as reported by Prevost (23). The quantity $N$ is defined to be the number of cycles applied beyond the initial loading curve plus one. Thus the end of the
Fig. 6.3. Triaxial test results used for determining the parameters for Santa Barbara silt. [data from (23)]
TABLE 7
Santa Barbara Silt

<table>
<thead>
<tr>
<th>m</th>
<th>( \frac{k}{\sigma'} )</th>
<th>( \frac{a}{\sigma'} )</th>
<th>( \frac{H}{\sigma'} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.150</td>
<td>0.500</td>
<td>52.630</td>
</tr>
<tr>
<td>2</td>
<td>0.225</td>
<td>0.425</td>
<td>27.780</td>
</tr>
<tr>
<td>3</td>
<td>0.298</td>
<td>0.403</td>
<td>16.667</td>
</tr>
<tr>
<td>4</td>
<td>0.340</td>
<td>0.390</td>
<td>13.333</td>
</tr>
<tr>
<td>5</td>
<td>0.375</td>
<td>0.375</td>
<td>8.888</td>
</tr>
<tr>
<td>6</td>
<td>0.405</td>
<td>0.365</td>
<td>4.444</td>
</tr>
<tr>
<td>7</td>
<td>0.450</td>
<td>0.350</td>
<td>3.030</td>
</tr>
<tr>
<td>8</td>
<td>0.550</td>
<td>0.300</td>
<td>1.587</td>
</tr>
<tr>
<td>9</td>
<td>0.615</td>
<td>0.335</td>
<td>0.001</td>
</tr>
</tbody>
</table>

G = 83.33\( \sigma' \)

monotonic stress-strain curve would be cycle 1 and the completion thereafter of a hysteresis loop would be cycle 2. Neglecting elastic strains (and therefore the cumulative plastic strain invariant \( \lambda \) also equals the invariant measure of the total cumulative strain) the increment of plastic strain is equal to \( d\lambda \):
Using the above definition for $N$, the total cumulative plastic strain for a cyclic strain controlled simple shear test for one cycle would be:
Fig. 6.5. Decrease of stress ratio during cycling in a simple shear strain-controlled test at a shear strain amplitude of 1%. [data from (23)]

\[
\lambda = \frac{\gamma_{xy}}{\sqrt{3}}
\]

or after 2 cycles:

\[
\lambda = \frac{\gamma_{xy}}{\sqrt{3}} [1 + 4]
\]

or after \( N \) cycles:

\[
\lambda = \frac{\gamma_{xy}}{\sqrt{3}} [1 + 4(N-1)] \quad N \geq 1 \quad (6.1)
\]
For simplicity in presentation, only the stress-strain traces during five cycles with $\gamma_{xy_{\text{max}}} = 1\%$ will be demonstrated here. From the experimentally obtained curves in Fig. 6.5, the relation between the resisting stress $\tau_{xy}$ and the number of cycles $N$ for a strain amplitude of $1\%$ can be approximated to be (for $N \leq 10$):

$$\tau_h = -0.0767 \log(N) + 0.230 \quad (6.2)$$

where:

$$-0.0767 = \frac{\Delta \tau_h}{\Delta \log N}$$

Equation 6.1 may be rewritten:

$$N = \frac{1}{4} \left[ \frac{\sqrt{3} \lambda}{\gamma_{xy_{\text{max}}}} + 3 \right] \quad N \geq 1$$

Substituting this expression into eq. 6.2 for $\gamma_{xy_{\text{max}}} = 1\%$ gives:

$$\sqrt{3} \tau_h = k(r) = \sqrt{3} \left[ -0.0767 \log \left( \frac{\sqrt{3} \lambda}{4} + 0.75 \right) + 0.230 \right]$$

$$\quad (6.3)$$
for all \( r \), where \( \lambda \) is in percent. Eq.6.3 describes \( k \) as a function of \( \lambda \) for a cyclic strain-controlled simple shear test with a strain amplitude of 1\% and \( 10^N > 1 \).

Fig.6.6 shows the effect of the decrease in size of the yield surfaces on its stress-strain behavior. A reduction in the shear resistance at a given strain and softening of the soil are a result of this degradation. The dotted line shows the hysteresis loop generated by cycling at a constant strain amplitude of 1\% without changing the sizes of the yield surfaces.

The above analysis is a very simplistic one and is meant only to demonstrate the mechanics of the model. A more sophisticated approach for describing the changes in \( k \) is required if the model is to describe the behavior under a general cyclic stress path. Prevost has developed a set of equations for completely describing the degradation of Drammen clay at an overconsolidation ratio of 4 (18). These equations are essentially a curve fitting approach but are quite complex. A simpler method for describing the degradation of the yield surfaces would be desirable if possible.

If a soil strain softens enough, it is apparent that the smaller yield surfaces would disappear as isotropic softening occurs. This could lead to unrealistically "flat" stress-strain curves as only the larger, and consequently softer, yield surfaces remain. To overcome this, Prevost (18) suggests a minimum size \( k'_r \) for each yield surface "\( r \)" so that:
Fig. 6.6. Softening of stress-strain curve with cycling at a strain amplitude of 1%. Dotted line shows the hysteretic loop formed if no softening occurred.
\[ k_l(r) = \chi k_0(r) \]  

where:

\[ \chi = \frac{k'(P)}{k_0'(P)} \]

\[ k'(P) = \text{size of largest yield surface (i.e. the failure surface) after very large cumulative plastic straining or remolding.} \]

\[ k_0'(P) = \text{initial size of largest yield surface.} \]

When the surfaces reach their limiting sizes (usually individually) their associated plastic shear moduli \( H'_r \) then start to vary. \( k'(r) \) then remains constant while \( H'_r \) becomes a function of \( \lambda \). A procedure similar to finding \( k(\lambda) \) must then be used for \( H'(\lambda) \) (18). The change in slope of the stress-strain curve (elastoplastic modulus \( H_r \)) can be plotted versus the number of load cycles in a manner similar to Fig.6.5 at the various stress levels corresponding to the individual yield surfaces. Using eq.3.11, a plot of \( H'_r \) with the number of load cycles can be computed. A function can then be found equating \( H'_r \) with \( N \), much like eq.6.2 for the resisting stress versus \( N \). Equation 6.1 can be used to eliminate \( N \) such that \( H'_r \) becomes a function of only \( \lambda \). Unlike the expression for the change in size of the yield surfaces, which assumes that they all change by the same amount, \( H'_r(\lambda) \) is likely to be different for each yield surface \( r \) (18). As such, individual expressions of \( H'_r(\lambda) \)
are required for each yield surface.

It should be noted that \( H' \) is a constant for each yield surface \( r \) until the yield surfaces have reached their limiting sizes. Only after yield surface \( r \) has reached its limiting size \( k_L(r) \) does \( H' \) become a function of \( \lambda \).
CHAPTER 7

SOME LIMITATIONS OF THE UNDRAINED TOTAL STRESS MODEL

Although the comparisons between the test data and the model predictions given in Chapter 5 together with those reported by Prevost (19) show that the model can be quite accurate, many more such comparisons with a wide variety of clays must be made before the model can be used with greater confidence. The results of a study of this nature would actually be an assessment of the validity of the assumptions made in formulating the model. To achieve a workable model, simplifying assumptions must necessarily be made. However, some of these assumptions need to be reassessed to allow the model to describe a wider range of behavior. In the following, some of the limitations of the model in its present form will be discussed from a speculative point of view.

The assumption of a Von Mises type yield surface may not always be valid. Yong and McKyes (30) showed that a Von Mises yield surface existed in a clay they tested provided the applied stress was less than one half the peak stress. Beyond this limit, the yield surface gradually changed in shape until a Mohr Coulomb surface (31) developed at failure. This change was due to the development of a localized region of high disturbance which ultimately developed into a failure plane. Thus, a Mohr Coulomb type yield surface may be more appropriate for soils once a localized slip zone develops. Heavily
overconsolidated clays are particularly prone to failing along a slip plane and therefore a Von Mises type yield surface may not give very good results for such soils.

If the stress-strain curve of a material decreases beyond the peak stress with continued straining and without rupturing, the material is said to be strain softening in this portion of the curve. The application of the model to such soils subjected to a complicated loading path presents the problem of defining when plastic deformations occur. Since the condition \( \frac{\partial f}{\partial \sigma_j} \delta \sigma_j > 0 \) is not applicable in this case, only two conditions exist:

A) \( \frac{\partial f}{\partial \sigma_j} \delta \sigma_j = 0 \). No plastic flow occurs, neutral loading.

B) \( \frac{\partial f}{\partial \sigma_j} \delta \sigma_j < 0 \). Plastic flow may occur.

If the stress point lies on the yield surface, elastic behavior is also possible and \( \frac{\partial f}{\partial \sigma_j} \delta \sigma_j < 0 \) in this case as well. By multiplying the outward normal to \( f \) with the stress increment, it is not possible to distinguish between plastic and elastic loading states. Therefore the model may not be suitable for describing the behavior of strain softening soils subjected to a complicated loading path.

Hill (7) suggested that eq.3.17 is applicable to initially isotropic metals. This expression may not be entirely suitable
for anisotropic soils. For example, consider the monotonic, triaxial, stress-strain curve for $K_0$ normally consolidated, resedimented, Boston Blue clay (19) illustrated in Fig. 7.1. If elastic strains are neglected, the horizontal axis $|\varepsilon_y|$, could also be denoted as $\lambda$ since:

$$\sqrt{\frac{2}{3} \text{de}_{ij} \text{de}_{ij}} = \sqrt{\frac{2}{3} [\varepsilon_y^2 + 2 \varepsilon_y^2]}$$

$$= \sqrt{\frac{2}{3} [\varepsilon_y^2 + 2(-\frac{1}{2} \varepsilon_y)^2]}$$

$$= \sqrt{\varepsilon_y^2}$$

$$= |\varepsilon_y|$$

Notice that at a compressive strain of $|\varepsilon_y| = 0.25\%$, the soil strain-softens and therefore the yield surfaces contract in size beyond this point. In extension however, hardening continues to occur up to a strain of 3.0%. Thus the hardening-softening behavior of the soil is different.

One approach to solving this problem would be to pseudo-harden the extension curve so that when the function $k(\lambda)$ in compression is applied to it, the resulting softening would give the actual stress-strain curve in extension. Fig. 7.2 illustrates this concept.

This procedure is workable for monotonic loading conditions but its application to cyclic loads is questionable. To illustrate this, consider a strain controlled, cyclic...
triaxial test of the soil in Fig. 7.1. If the strain is cycled between +0.5% and 0.0% in compression, one would expect a relatively rapid reduction in compressive strength since the applied strain exceeds the strain at peak strength (+0.25%) and the original structure is destroyed. However, if the strain is cycled between -0.5% and 0.0% in extension, the soil would still be in the strain hardening portion of the curve and the rate of degradation in strength would be significantly less. Nevertheless, the pseudo-hardening approach would incorrectly predict identical rates of degradation for both loading conditions.
A more consistent approach would be to use a method which incorporates the anisotropic strain softening rates directly without having to modify the experimentally obtained curves. To the author's knowledge, such a method has yet to be developed. Clearly, further research in this area must be undertaken.

The simplification that the plastic shear moduli vary only after the yield surfaces reach their limiting sizes during cyclic loading imposes a restriction on the type of behavior which the model is capable of handling. Soils which undergo little loss in peak strength during cycling but experience a
significant reduction in stiffness cannot be modelled using the existing procedure. Figure 7.3 illustrates this problem for a hypothetical clay.

If the isotropic softening rule is applied to a yield surface $m$ corresponding to point B on the initial loading curve, then after, say 10 cycles, the model would predict the dotted curve in Figure 7.3 with the slope at point B being the same as at point A. Since the remolded strength is less than the strength after 10 cycles, yield surface $m$ would not have reached its limiting size (although the smaller yield surfaces may have). As such, the plastic modulus for $f_m$ cannot vary and thus a softer response than the dotted curve in Fig.7.3 cannot be modelled. It is therefore not possible to obtain the "actual" stress-strain curve after 10 cycles if the plastic moduli are not permitted to vary until the limiting yield surface's size is reached.
Fig. 7.3. Behavior predicted by model as compared to "actual" behavior for a hypothetical clay.
CHAPTER 8

THEORY FOR THE EFFECTIVE STRESS ANALYSIS
OF SOILS

The behavior of soils is governed to a large extent by its void ratio and the imposed effective mean normal stress. Other peculiarities of soil behavior include plastic volume changes and a coupling between volume change and shear strain under drained loading conditions. Consequently, it is desirable to have the effects of these variables in a constitutive relation. Using the general theory of Chapter 2, Prevost has extended his isotropic/kinematic plasticity model to describe the behavior of soil in terms of effective stresses under undrained or drained conditions. The theoretical considerations of the model he has proposed will be discussed in this chapter. All stresses are effective stresses unless stated otherwise.

Theory

Elasticity

If the elasticity is assumed to be isotropic, two parameters are required to describe the elastic behavior: the shear modulus $G$ and the bulk modulus $B$. The deviatoric elastic strain is then given by:
and the volumetric elastic strain is defined to be:

\[ \text{d} \varepsilon^e = \frac{\text{d}s_{ij}}{2G} \]  \hspace{1cm} (8.1)

For simplicity, the bulk modulus may be taken to be a function of the mean normal effective stress while the shear modulus is held constant (21). Janbu (9) has proposed the following relation for \( B \):

\[ B = B_1 \left( \frac{p}{p_1} \right)^n \]  \hspace{1cm} (8.3)

where \( B_1 \) = bulk modulus at \( p = p_1 \),
\( n \) = a constant depending on the soil type
\( p_1 \) = a normalizing (reference) stress

A more sophisticated approach would be to also make \( G \) a function of the mean normal effective stress. The shear modulus could then assume a form similar to \( B \):
Hardin and Drenovich (6) have suggested that $n=0.5$ in equation 8.4 and show that $G$ should also be a function of the void ratio and the overconsolidation ratio. The latter variable however is only meaningful for undisturbed clays since straining destroys the original fabric of the soil.

Plasticity

The description of the behavior of soils in terms of effective stresses requires that the yield criterion include some measure of the effective mean normal stress. In addition, if hysteresis during hydrostatic loading and unloading is to be modelled, then a parameter to track the kinematic change of the yield surface along the hydrostatic axis should be incorporated. To satisfy these requirements, Prevost (20) has proposed the following yield criterion:

$$f_m = 0 = \frac{3}{2}(s_{ij} - \alpha_{ij}^{(m)})(s_{ij} - \alpha_{ij}^{(m)}) + c^2(p - \beta^{(m)})^2 - (k^{(m)})^2$$

(8.5)

where:

$$p = \text{the effective mean normal stress} = \frac{1}{3}q_u$$
\[ \beta = \text{coordinate along the p axis of the center of the yield surface} \]

\[ c = \text{a constant} \]

Let:

\[ \overline{Q} = Q' + Q''\hat{\delta} \]

\( Q' = \text{projection of } \overline{Q} \text{ onto the deviatoric plane} \)

\[ = \frac{\partial F}{\partial s_{ij}} = \frac{\partial f}{\partial s_{ij}} \]

\( 3Q'' = \text{projection of } \overline{Q} \text{ along hydrostatic axis p} \)

\[ = \overline{Q} \cdot \hat{\delta} \]

\[ = (\overline{Q}' \cdot \hat{\delta} + Q'' \hat{\delta} \cdot \hat{\delta}) \]

\[ = 3 \frac{\partial f}{\partial \sigma_i} = \frac{\partial f}{\partial p} \]

Then:

\[ |Q'(m)|^2 = \overline{Q}(m) \cdot \overline{Q}(m) \]

\[ = \overline{Q}' \cdot \overline{Q}' + (Q'')^2 \hat{\delta} \cdot \hat{\delta} \]

\[ = 9(s_{ij} - a_{ij})(s_{ij} - a_{ij}) + 3(\frac{3}{4}c^4(p - \beta)^2) \]

\[ = 6[\frac{3}{2}(s_{ij} - a_{ij})(s_{ij} - a_{ij}) + \frac{3}{4}c^4(p - \beta)^2] \]

For similarity with eq. 3.7, Prevost has set \(|Q|^2 = 6k^2\), which gives \( c = 3/\sqrt{2} \). Other values of \( c \) may be used to give a better fit with experimental data, but it is not mathematically
convenient to do so since $|Q|$ must then be computed from $s_{ij}$, $a$, $p$, and $\beta$. The influence of $c$ can be seen by rewriting eq.8.5 as:

$$\left[\frac{3}{2}(s_{ij} - a_{ij})(s_{ij} - a_{ij})\right] + [c(p - \beta)]^2 = (k^{'m})^2 \quad (8.5a)$$

The first term contains only deviatoric stresses while the second term involves only the mean normal stress. The yield surfaces can then be thought of as being circles in the $\sqrt{\frac{3}{2}s_{ij}s_{ij}}$ versus $cp$ stress space. Alternatively, the yield surfaces would appear as ellipsoidal surfaces if plotted in the $\sqrt{\frac{3}{2}s_{ij}s_{ij}}$ versus $p$ space with $c$ being the ratio between the major and minor axes. A value of $c$ greater than 1 would mean that the yield surface would be elongated in the $p$ direction.

If the physical axes of anisotropy coincide with the principle axes of applied stress, then $a^{(r)}=0$ for $i \neq j$ and for all $r$. Further, if the soil is cross-anisotropic about the $y$ axis (i.e. rotational symmetry) then $a_x = a_z$. These assumptions will be implicit in all the following analyses.

To conceptualize eq.8.5a, let us examine the special case of the axisymmetric triaxial test whereby $a_x = a_z$. Recalling that $\frac{3}{2}s_{yy} = (\sigma_y - \sigma_x)$ in this case, then eq.8.5 gives:

$$[q-a_1]^2 + c^2(p-\beta)^2 = k^2 \quad (8.6)$$
where:

\[ a_1 = \frac{2}{3} a_y \quad (8.7) \]
\[ p = \frac{1}{3} (a_y + 2a_x) \quad (8.8) \]
\[ q = (a_y - a_x) \quad (8.9) \]

Figure 8.1 shows how the yield surfaces appear on the triaxial plane in stress space. The line OC is called the Critical State line for compression and represents the locus of stress points for which large shear deformations occur at constant stress and sample volume. It does not necessarily indicate the stress conditions at which the peak strength is reached. Cohesion or dilation of the soil could allow the stress point to cross above this line temporarily. However, if large deformations occur so that these effects are overcome, then the stress point will ultimately lie on the Critical State line. Thus, it represents the upperbound for the stress point after large deformations occur. The slope \( M \) can be determined from the stress conditions in the triaxial compression test when the Critical State has been reached:

\[ M = \frac{q}{p} = \frac{3 (\sigma_1 - \sigma_3)}{2\sigma_1 + \sigma_3} \quad (8.10) \]

Using the Mohr Coulomb failure criterion, for an axisymmetric state of stress, to represent the Critical State line (24):
\[ \frac{\sigma_1}{\sigma_3} = \frac{1 + \sin\phi'}{1 - \sin\phi'} \]  

(8.11)

where: \( \phi' \) = friction angle at the critical state

An expression for \( M_c \) can be found in terms of \( \phi' \):

\[ M_c = \frac{6 \sin\phi'}{3 - \sin\phi'} \]  

(8.12)

Similarly, the Critical State line can be defined for extension \( (\sigma_y < \sigma_z) \):

\[ M_e = \frac{-6 \sin\phi'}{3 + \sin\phi'} \]  

(8.13)

The plastic strains are obtained from the flow rule. As Palmer et. al. (13) have pointed out, frictional materials do not exhibit normality. So, for generality, a non-associated flow rule (eq.2.11) is adopted. \( \bar{P} \) can be decomposed into:

\[ \bar{P} = \bar{P}' + P''\delta \]  

(8.14)

where:

\( \bar{P} \) = normal to the plastic potential surface \( g \)
\( \bar{P}' \) = projection of the plastic potential surface onto the deviatoric plane

\[ = \frac{\partial g}{\partial S_{ij}} \]

\( 3P'' \) = projection of the plastic potential surface onto
the hydrostatic axis. Note that this is a scalar quantity in which \( \dot{\delta} \) is the associated tensor.

\[
\mathbf{p} \cdot \dot{\delta} = \frac{\partial q}{\partial p}
\]

Then:

\[
d\varepsilon_{ij}^p = d\varepsilon_{ij}^p + \frac{1}{3} d\varepsilon_{ij}^\varphi \delta_{ij}
\]  
(8.15)

and:

\[
q
\]

\[
M_c/3
\]

\[
f_{m+1}
\]

\[
f_m
\]

\[
f_p
\]

\[
E
\]

\[
M_{c}/3
\]

\[
M_{E}/3
\]

Fig. 8.1. Yield surfaces in triaxial plane. [from Prevost(20)]
\[ \text{de}_{ij}^p = \frac{\langle L \rangle}{|Q|^2} P_{ij} \]  
\[ = \text{plastic deviatoric strain increment tensor} \]  
\[ \text{de}_{ij}^\epsilon = \text{de}_{ij}^p \delta_{ij} = \frac{3}{|Q|} \langle L \rangle P'' \]  
\[ = \text{plastic volumetric strain increment (a scalar)} \]

where:

\[ \langle L \rangle = 1 \frac{\bar{Q} \cdot d\bar{\sigma}}{H'} = \frac{1}{H'} [\bar{Q}' \cdot d\hat{\sigma} + 3Q''dp] \]
\[ |Q| = |\bar{Q} \cdot \bar{Q}|^{\frac{1}{4}} \]

Prevost has chosen a plastic potential \( q_m \) associated with yield surface \( f_m \) such that the plastic deviatoric strain increment vector is normal to the projection of \( f_m \) onto the deviatoric subspace:

\[ \frac{\delta q}{\delta s_{ij}} = \frac{\delta f}{\delta s_{ij}} \]
\[ P_{ij}' = Q_{ij}' \]

However, \( P'' \neq Q'' \) in general. Instead \( P'' \) is related to \( Q'' \) by the following relation:

\[ P'' = Q'' + A_m |Q'| \]  
\[ (8.18) \]
where:

\[ A_m = \text{a material parameter to be determined from soil tests.} \]

Equations 8.16 and 8.17 then become:

\[
de_{\text{ef}} = \frac{1}{H'} \left( \frac{\hat{Q}' \cdot \hat{d}\hat{s} + 3\hat{Q}'' \hat{d}p}{|\hat{Q}|^2} \right) \hat{Q}' \tag{8.19}
\]

\[
de_{\text{ce}} = \frac{1}{H'} \left( \frac{3(\hat{Q}' \cdot \hat{d}\hat{s} + 3\hat{Q}'' \hat{d}p)}{|\hat{Q}|^2} \right) \left[ \hat{Q}'' + A_m |\hat{Q}'| \right] \tag{8.20}
\]

The motivating factor for choosing \( \hat{P}' = \hat{Q}' \) was the good predictive capability of the undrained total stress model. Since the modelling capabilities were reasonably good, it follows that the assumptions of a Von Mises type yield function given by eq.3.3 and the use of an associated flow rule were acceptable for undrained, total stress loading. The latter is especially important since it implies that the relative magnitudes of the different plastic deviatoric strain increment components (i.e. the direction of the plastic strain increment vector) were compatible with experimental observations. Thus a logical extension to these findings would be to also have \( \hat{P}' = \hat{Q}' \) for the effective stress model. However, only experimental comparisons with the model predictions can determine the true validity of this assumption.

The plastic modulus \( H' \) serves as a factor of proportionality for the plastic strain increment. One special
case for $H'$ is when $\tilde{Q}'=0$ (i.e. the yield surface has no component of its outward normal at the stress point in the deviatoric plane). Equation 8.20 then becomes:

$$\frac{d\varepsilon'}{d\tilde{Q}} = \frac{1}{H'} \frac{3\tilde{Q}'dp\tilde{Q}''}{|\tilde{Q}|^2}$$

Recalling that $\tilde{Q} = \tilde{Q}' + \tilde{Q}'' \delta$ and rearranging:

$$H' = \frac{dp}{d\varepsilon'}$$

(8.21)

$H'$ then becomes the plastic bulk modulus. If $\tilde{Q}''=0$, the result is given by eq.3.12 in which case $H'$ becomes the plastic shear modulus.

The kinematic hardening rule describes the changes in the yield surface(s) as plastic deformation occurs. For drained loading, Mroz's kinematic rule may be used: $d\tilde{\xi}^{(m)} = d\nu\tilde{\mu}$ where $\tilde{\mu}$ is given by eq.2.31 and $d\tilde{\xi} = da + \delta d\tilde{\xi}$. If $(d\tilde{\sigma} - d\tilde{\xi})$ and $dk$ are small, the scalar $d\mu$ can be found from eq.2.42:

$$d\mu = \frac{\tilde{Q}^{(m)} \cdot d\tilde{\sigma} - n(k^{(m)})^{n-1} \cdot dk^{(m)}}{\tilde{Q}^{(m)} \cdot \tilde{\mu}}$$

(8.22)

If it is further assumed that $dk^{(r)} = dk^{(r+1)}$ for all $r$ and $\tilde{Q}^{(m)} \cdot d\tilde{\xi}^{(m+1)} = 0$ then eq.2.43 gives $d\tilde{\xi}^{(m+1)} = 0$. With these assumptions, Mroz's kinematic rule simplifies to a special case
of the hardening rule proposed in eq. 2.49.
The theory presented in Chapter 8 is not usable by itself. Before it can be put to use, a method for obtaining the necessary parameters must be formulated. Additional simplifying assumptions may also be required in order that the model be practical.

The first and simplest model proposed by Prevost (21) requires only the results of an isotropic compression/rebound test and an axisymmetric triaxial test but is applicable only to cohesionless soils. The primary assumptions used were:

1. "For cohesionless soil specimens prepared in the laboratory: \( a_{i}(m) = 0 \) and \( \beta(m) = \sqrt{2}k(m)/3 \) initially for all \( m \) and \( \sigma_{ij} = 0 \) for all \( i,j \). The yield surfaces are thus initially all centered along the hydrostatic axis and tangent to each other at the origin". Note that this implies isotropic material behavior and no initially imposed stresses. The expression for \( \beta \) comes from the yield function, eq.8.6, with \( \sigma_{y} = \sigma_{x} = \sigma_{z} = p = 0 \).

2. \( k(r) \) is a different constant for each yield surface \( r \).

3. The shear modulus \( G \) is a constant.

4. The plastic modulus is given by:
\[ H'_m = \frac{2B'_m}{(1-t) \left[ \frac{2B'_m}{h'_m} (1+t) - t \right]} \]  

(9.1)

where:

\[ t = \frac{3 (p-\beta)}{\sqrt{2} k^{(m)}} \quad -1 \leq t \leq 1 \]

Note the following limiting conditions:

a) \( t = 1 \) and \( \dot{Q}'=0 \); i.e. the soil is being isotropically compressed; and \( H' = \infty \). The strains are then purely elastic.

b) \( t = 0 \); i.e. \( p=\beta \). The stress path then goes through the apex of the yield surface and \( Q''=0 \). Then the result is identical to eq.3.12: \( H'_m = \) the plastic shear modulus = \( h'_m \).

c) \( t = -1 \); i.e. the isotropic stress is being released. Since \( \dot{Q}'=0 \) for this case, then it follows from eq.8.21 that \( H'_m = \) plastic bulk modulus = \( B'_m \).

Cases "a" and "c" appear to be an anomaly since the usual convention is to have the plastic strains occur during compression (loading). However, the choice of whether to record the plastic strains during compression or rebound during computations is largely a matter of bookkeeping. What is
significant is the permanent (plastic) deformation during a cycle of loading and unloading. Whether this quantity is recorded during compression or during rebound is of minor importance in the following.

As a result of assumption 4 and equations 8.2 and 8.20, the volumetric strain increment during isotropic compression is:

\[ d\varepsilon_v = d\varepsilon_v^e = \frac{dp}{B} \quad (9.2) \]

with B given by eq.8.3. When the compressive stress reaches a maximum value of \( p_c \) and lies on yield surface \( c, \beta^{(m)} + \frac{\sqrt{2}}{3} k^{(m)} = p_c \) for \( m \leq c \) since all the yield surfaces which have been translated are tangent to each other at \( p_c \) (Fig.9.1).

During isotropic rebound, equations 8.2 and 8.20 give:

\[ B_m = \frac{3}{d\varepsilon_v - \frac{1}{dp} B} \quad (9.3) \]

The value for \( \beta^{(m)} \) can be found by noting from Figure 9.1 that:

\[ \beta^{(m)} = \frac{1}{2} (p_c + p_m) \quad (9.4) \]
Fig. 9.1. Yield surfaces during isotropic compression to $p=p_c$. 
The size of the yield surface can be determined similarly:

\[ k^{(m)} = \frac{3}{12} \left( \frac{p_e - p_m}{2} \right) \]  \hfill (9.5)

Thus \( B_1, B'_1, \beta^{(r)}, \) and \( k^{(r)} \) for yield surfaces \( f_r, r \leq c \) can be determined from an isotropic compression/rebound test. The other parameters, such as \( h', A_r \) and \( G \) cannot be determined from this test. It should be noted that \( p_e \) during testing should be at least as large as the \( p \) expected to be attained in the problem at hand since only \( \beta^{(r)} \) and \( k^{(r)} \) (but not \( B'_1 \)) for \( r > c \) can be determined from a triaxial test.

If a cohesionless soil is isotropically compressed to a confining pressure of \( p_1 \), and then a conventional axisymmetric triaxial compression test is performed, then:

\[ dq = 3dp \]

where:

\[ q = (\sigma_y - \sigma_x) \]
\[ p = \frac{1}{3}(\sigma_y + 2\sigma_x) \]

When the stress point initially reaches \( f_m \), equations 8.1, 8.6, and 8.19 and 8.2, 8.6, and 8.20 give:

\[ \frac{d(\varepsilon_y - \varepsilon_x)}{dq} = \frac{1}{2G} + \frac{3q}{2H'} + \frac{3(p - \beta^{(m)})}{(k^{(m)})^2} \frac{dp/dq}{dq} = \frac{d\bar{\varepsilon}}{dq} \]  \hfill (9.6)
\[
\frac{d\varepsilon_v}{dp} = \frac{1}{B} + \frac{3}{2H_m} \frac{9(p - \beta^{(m)}) + \sqrt{6} A_m q}{(k^{(m)})^2} \left( \frac{3q + 3(p - \beta^{(m)})}{dp/dq} \right)
\]

(9.7)

in which:

\[
\beta^{(m)} = \frac{\sqrt{2}}{3} k^{(m)}
\]

(9.8)

\[m \geq c\]

since the yield surfaces are tangent at \(p=0, q=0\). Therefore from eq.8.6:

\[
k^{(m)} = \frac{3}{9} \frac{q^2 + q^2}{\frac{1}{2} 9p}
\]

(9.9)

The shear modulus \(G\) is obtained from the steepest slope of the stress-strain curve:

\[
G = \frac{1}{2} \left( \frac{dq}{d(\varepsilon_v - \varepsilon_x)} \right)_{\text{steepest}}
\]

(9.10)

\(H_m\) can then be found from eq.9.6:
\[ H'_m = \frac{q (q + \frac{3}{2}(p-\beta))}{(k'(m))^2 \left[ d(\epsilon_y - \epsilon_x) - \frac{1}{dq} \right] - \frac{1}{2G}} \]  

(9.11)

while \( A_m \) is obtained from eq.9.7:

\[ \sqrt{6}A = \frac{d\epsilon_v}{dp} \left( \frac{1}{B} \right) - \frac{p(p-\beta)}{q} \]  

(9.12)

Lastly, the plastic shear modulus \( h'_m \) can be determined from:

\[ h'_m = \frac{2B'_m(1+t)}{2B'_m/H'_m(1-t) + t} \]  

(9.13)

It should be noted that while \( G, H'_m, \beta'(m), k'(m), \) and \( A_m \) are obtainable solely from a triaxial test, where \( m \) is greater than the largest yield surface reached in isotropic compression under \( p_1 \) (i.e. \( f_r \)), \( B'_m \) cannot be determined from this test.

These equations will now be used to determine the parameters for Cook's Bayou Sand from the data given in Figures 9.2 and 9.3.
Fig. 9.2. Experimental, isotropic compression/rebound curve for Cook's Bayou sand. [from Prevost (21)]
Fig. 9.3. Experimental, drained triaxial compression curves for Cook's Bayou sand. [from Prevost (21)]
Upon initial compression, the strains are purely elastic; thus:

\[ B = \frac{dp}{d\varepsilon_v} \quad (9.14) \]

Applying eq.8.3 to eq.9.14 with \( n=0.5 \) for sands gives:

\[ d\varepsilon_v = \frac{dp}{B \left( \frac{p}{\sigma_{vc}} \right)^{n/2}} = \frac{dp}{B \left( \frac{p}{\sigma_{vc}} \right)^{5/2}} \quad (9.15) \]

where \( \sigma_{vc} = p_1 \) = a normalizing (reference) stress.

Letting \( p_0 \) and \( \varepsilon_{vo} \) be the initial stress and strain states respectively:

\[ \int_{\varepsilon_{vo}}^{\varepsilon_v} d\varepsilon_v = \frac{\sqrt{\sigma_{vc}}}{B_1} \int_{p_0}^{p} p^{-\frac{1}{4}} dp \]

Therefore during compression:

\[ \varepsilon_v - \varepsilon_{vo} = \frac{2}{B_1} \frac{\sqrt{p}}{\sqrt{\sigma_{vc}}} - \frac{\sqrt{p_0}}{\sqrt{\sigma_{vc}}} \quad (9.16) \]
Using the data in Figure 9.2, $B_1/o_{vc} = 472$. Upon isotropic rebound, values for $p$ are selected to correspond to intervals at which the yield surfaces are desired. Values for $k^{(m)}$ and $B'_m$ are then calculated from eq.9.5 and eq.9.3 respectively.

\begin{table}[h]
\centering
\caption{Model Parameters For Cook's Bayou Sand}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$m$ & $k/o_{vc}$ & $\beta/o_{vc}$ & $B'_m/o_{vc}$ & $h'/o_{vc}$ & $\lambda$ \\
\hline
1 & 1.061 & 0.500 & ----- & ----- & ----- \\
2 & 1.432 & 0.675 & -8123. & 338.1 & -2.328 \\
3 & 1.945 & 0.917 & -6860. & 205.1 & -1.508 \\
4 & 2.663 & 1.255 & -6222. & 125.1 & -1.004 \\
5 & 3.182 & 1.500 & -4996. & 77.6 & -0.814 \\
6 & 3.450 & 1.626 & -4657. & 56.4 & -0.773 \\
7 & 3.722 & 1.755 & -4581. & 24.8 & -0.637 \\
8 & 4.561 & 2.150 & ----- & ----- & ----- \\
\hline
\end{tabular}
\end{table}

$G = 400 \sigma_{vc}$

$B_1 = 472 \sigma_{vc}$

$\sigma_{vc} = 50$ psi

Turning to the triaxial test on a new but identical sample which has been isotropically compressed to a value of $p/\sigma_{vc} = 1$,
eq. 9.8 gives values for $\beta^{(m)}$, $m \geq 2$. Note that for $m=1$, $f_1$ was translated during isotropic compression of the triaxial sample and thus $\beta^{(1)}$ is given by eq. 9.4.

Values for $p$ and $q$ corresponding to yield surfaces $m \geq 2$ can be determined by recalling that $3dp=dq$ or $p=1+\frac{1}{3}q$, $q \geq 0$ in the case of the triaxial compression test. Equation 9.9 then becomes:

$$k^{(m)} = \frac{3}{\sqrt{2}} \frac{\left[\frac{q}{2}(1 + \frac{1}{3}q_m)\right]^2 + q_m^2}{9(1 + \frac{1}{3}q_m)} \quad (9.17)$$

and the yield stresses $q_m$ (and $p_m=1+\frac{1}{3}q_m$) associated with yield surface $f_m$ are easily determined. The parameters $G$, $H'_m$, $A_m$, and $h'_m$ for $m \geq 2$ can then be found from equations 9.10, 9.11, 9.12, and 9.13 respectively. It must be emphasized that $H_1$, $A_1$, and $h_1$ are indeterminate since the triaxial test was not conducted in its stress range (i.e. the stress point had $q=0$ when it first reached $f_1$). Table 8 tabulates a complete set of parameters for the soil based on a piecewise linear approximation of the stress-strain curve and Figure 9.4 shows the yield surfaces in stress space. Other values for the parameters are possible depending upon the choice for the yield stresses.

The parameters obtained from the above procedure are only appropriate for the instant the stress point first touches the yield surface. From eq. 9.6 and 9.7, it can be seen that any
Fig. 9.4. Yield surfaces in the triaxial plane for Cook's Bayou sand.
change in the stress point which causes a change in $p-\beta$ and $q-\alpha$ would alter the value of $d\bar{e}/dq$ and $d\varepsilon_v/dp$, assuming $G$, $h'$, $B'$, and $A$ remain constant. To prevent any changes in $d\bar{e}/dq$ and $d\varepsilon_v/dp$ during a change in the stress point, Prevost computes these values when the stress point first touches $f_m$ and these remain constant until the next largest yield surface is reached.

In comparison, the kinematic movement of the yield surfaces does not impose any such restriction on the undrained total stress model for vertically cross-anisotropic soils. In this case, $d\varepsilon_{ij}/ds_{kl}$ is constant during the triaxial tests to determine the parameters, by virtue of the yield surfaces being centered along the $\sqrt{3}/2 \alpha_y$ axis (Fig. 4.3b). Since $(s_{ij}-a_{ij})$ remains constant during any stress increment in the triaxial test, $d\varepsilon_{ij}/ds_{kl}$ also remains constant. Thus the $d\varepsilon_{ij}/ds_{kl}$ determined when the stress point first touches yield surface $f_m$ remains the same until $f_{m,i}$ is reached.

It should be noted that the parameters obtained from equations 9.11, 9.12, and 9.13 do not necessarily imply that a piecewise linear approach to matching the experimental stress-strain curve must be used. These equations are also applicable for instantaneous values of the parameters at given values of $q$, $p$, $\alpha$, and $\beta$, whereby $d\bar{e}/dq$ and $d\varepsilon_v/dp$ are the tangents on the respective stress-strain curves at the stress point of interest.

Using the parameters given in Table 8, a comparison between actual test data, the model fit assuming constant $d\bar{e}/dq$
and $d\varepsilon_v/dp$ between yield surfaces, and the model fit using the same parameters but permitting $d\varepsilon/dq$ and $d\varepsilon_v/dp$ to vary as the kinematic movement of the yield surfaces demands is shown in Figure 9.5. Clearly, the parameters of Table 8 obtained from the initial ("original") positions of the yield surfaces, using the previously described method, are not compatible with kinematic hardening in this case. If a larger number of yield surfaces had been used, the model fit for the non-constant $d\varepsilon/dq$ and $d\varepsilon_v/dp$ curves would have been better because $q-\alpha$ and $p-\beta$ would not change as much during kinematic hardening. As an example, Fig.9.6 compares the model fit using 8 and 19 yield surfaces. It is evident that, although using 19 yield surfaces gives a relatively better fit, many more yield surfaces are required if the test data is to be matched more closely.

To evaluate the predictive capabilities of the model the parameters for dry Ottawa sand at a relative density of 87% and an initial, isotropic confining pressure of 5 psi were determined from data given in Reference 29. The stress-strain curves were then predicted for a triaxial compression and an extension test in which the mean normal stress was held constant at 5 psi.

Table 9 gives a set of parameters for this sand. To give a realistic ultimate strength for the sand, the Mohr Coulomb failure criterion was used. A comparison between the predictions and the actual test data is shown in Fig.9.7. Except for the shear stress-strain curve at low strains, the performance of the model is not good.
Fig. 9.5. Cook's Bayou sand. Comparison of model fit and test data.
Movement of yield surfaces not considered.

8 yield surfaces with kinematic movement considered.

19 yield surfaces with kinematic movement considered.

Fig. 9.6. Comparison of model fit using 8 and 19 yield surfaces using "original" parameters and considering the kinematic movement of the yield surfaces.
Table 9

Ottawa Sand Parameters

<table>
<thead>
<tr>
<th>m</th>
<th>(\frac{k}{\sigma_0})</th>
<th>(\frac{\beta}{\sigma_0})</th>
<th>(\frac{B'}{\sigma_0})</th>
<th>(\frac{h'}{\sigma_0})</th>
<th>A'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.061</td>
<td>0.500</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>2</td>
<td>1.140</td>
<td>0.537</td>
<td>-18362</td>
<td>15.00</td>
<td>-7.036</td>
</tr>
<tr>
<td>3</td>
<td>1.591</td>
<td>0.750</td>
<td>-16906</td>
<td>249.72</td>
<td>-1.276</td>
</tr>
<tr>
<td>4</td>
<td>1.871</td>
<td>0.882</td>
<td>-21591</td>
<td>260.00</td>
<td>-1.354</td>
</tr>
<tr>
<td>5</td>
<td>2.334</td>
<td>1.100</td>
<td>-21654</td>
<td>209.08</td>
<td>-1.130</td>
</tr>
<tr>
<td>6</td>
<td>2.834</td>
<td>1.336</td>
<td>-19726</td>
<td>165.58</td>
<td>-1.044</td>
</tr>
<tr>
<td>7</td>
<td>3.182</td>
<td>1.500</td>
<td>-17296</td>
<td>145.53</td>
<td>-0.993</td>
</tr>
<tr>
<td>8</td>
<td>3.540</td>
<td>1.669</td>
<td>-16645</td>
<td>93.79</td>
<td>-0.998</td>
</tr>
<tr>
<td>9</td>
<td>4.091</td>
<td>1.929</td>
<td>-11466</td>
<td>54.10</td>
<td>-0.929</td>
</tr>
<tr>
<td>10</td>
<td>4.846</td>
<td>2.285</td>
<td>-5281</td>
<td>19.46</td>
<td>-0.909</td>
</tr>
<tr>
<td>11</td>
<td>5.619</td>
<td>2.649</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
</tbody>
</table>

\(G = 770\sigma_0\)

\(B_1 = 1724\sigma_0\)

\(\sigma_0 = 5 \text{ psi}\)
Fig. 9.7. Predictions. Ottawa sand. [data from (29)]
CHAPTER 10

A MORE GENERAL FORMULATION FOR SOILS

The formulation given in Chapter 9 was restricted to a particular class of soils, namely isotropic, cohesionless soils. Further simplifying assumptions include a constant shear modulus and a constant yield surface size $k$ during initial loading. In actuality, the shear modulus is known to be a function of the effective mean normal stress $(\sigma)$ and $k^{(m)}$ is related to the volumetric strain as will be demonstrated later.

In light of these deficiencies, Prevost proposed a model to characterize the behavior of any given soil under monotonic loads (22) in which the parameters can be obtained solely from a triaxial compression and an extension test. However, the following presentation will be concerned mainly with modelling the behavior of cohesive soils. The applicability of the model to cohesionless soils will be discussed in Chapter 11. The primary assumptions of this method are that:

\begin{align}
B &= B_1 \left( \frac{\sigma}{P_1} \right)^n \quad (10.1a) \\
G &= G_1 \left( \frac{\sigma}{P_1} \right)^n \quad (10.1b) \\
h^{(m)} &= (h^{(m)})_1 \left( \frac{\sigma'}{P_1} \right)^n \quad (10.1c)
\end{align}
\[ B_m' = (B_m')_i \left( \frac{p'}{P_1} \right)^n \]  
\[ k(m) = k_i(m) e^{\rho \varepsilon} \]  
\[ a(m) = a_i(m) e^{\rho \varepsilon} \]  
\[ \beta(m) = \beta_i(m) e^{\rho \varepsilon} \]  
\[ H_m' = h_m' + \frac{\sqrt{3}}{|Q(m)|} B_m' \]  

where \( \rho = \frac{d(\ln p)}{d\varepsilon_v} \) in the normally consolidated range of the \( \varepsilon_v - \ln(p) \) isotropic consolidation curve. Since the slope of the normal consolidation line for one-dimensional compression is parallel to the slope for isotropic compression (1), either test may be used to obtain \( \rho \).

While equations 10.1a and 10.1b, as previously demonstrated, are justifiable from a soil mechanics point of view, evidence to prove the applicability of 10.1c and 10.1d is scarce at the present time. The prime motivation for having the latter relations appears to be the mathematical simplicity it affords in subsequent derivations.

The basis for equation 10.2a lies in the experimental observation that for a given void ratio, there exists a unique value for \( p \) and \( q \) at which the Critical State line is reached. Thus there is a unique relationship between the size of the largest yield surface at large strains and the volumetric strain \( \varepsilon_v \). Further, it has been shown by Roscoe and Burland (24) that the projection of the Critical State line onto the
volume-$\ln(p')$ plane is parallel to the normal consolidation line (Figure 10.1). In light of these observations, the relation $k'(p') = k_1(p')e^{\rho_e}$ is acceptable and a logical extension would be eq.10.2a.

![Diagram](image)

**Fig. 10.1.** The Critical State line in $v - \ln(p')$ space. [from Atkinson and Bransby (1)]

It is an experimental observation that for normally consolidated clays, the stress-strain curve for samples consolidated under different consolidation pressures would appear to be the same if the stress difference is normalized by the consolidation pressure $p_e$ (Fig.10.2). Analogously, the yield surfaces for a normally consolidated clay should also be the same if the stresses are normalized with respect to the consolidation pressure. It then follows that $\alpha$ and $\beta$ should have the same form as $k$; thus giving eq.10.2b and c. Note that
Fig. 10.2.  

a) Relationship between stress difference $q$ and axial strain $\varepsilon_a$ in undrained triaxial tests on samples normally consolidated to $p_c = a, 2a, 3a$. 

b) Relationship between normalized stress difference $q/p_c$ and axial strain $\varepsilon_a$. 

[from Atkinson and Bransby (1)]
eq.10.2 is a "quasi-hardening" rule since elastic volume changes can modify the yield surfaces. True hardening rules, as defined in plasticity theory, are caused by plastic strains only.

The relation for the plastic modulus (eq.10.3) was proposed by Prevost for its simplicity. Note the three limiting values for $H_m'$:

a) $Q'' = 0$ ; $H_m' = h_m' =$ plastic shear modulus

b) $Q' = 0$ , $Q'' = |Q|$ ;

$H_m' = h_m' + B_m' =$ plastic bulk modulus in loading

c) $Q' = 0$ , $Q'' = -|Q|$ ;

$H_m' = h_m' - B_m' =$ plastic bulk modulus in unloading

Let us now consider a sample, which is cross-anisotropic about the vertical y axis, subjected to a stress path in the axisymmetric triaxial plane (i.e. q-p plane). Equation 8.6 is then applicable. Defining $\theta$ to be as in Figure 10.3:

\[
\sin \theta^{(m)} = \frac{q - a^{(m)}}{k^{(m)}} \tag{10.4}
\]

\[
\cos \theta^{(m)} = \frac{c(p - \beta^{(m)})}{k^{(m)}} \tag{10.5}
\]

Eq.8.19 then becomes:
For this formulation, Prevost (22) has modified the expression for $P''$ such that:

$$P'' = Q'' + A |Q'| \frac{Q''}{|Q''|}$$

Substituting this expression into eq.8.17 gives:

$$\frac{d\bar{\varepsilon}}{dq} = \frac{1}{2G} + \frac{1}{H_m} \sin\theta (\sin\theta + cT\cos\theta)$$

(10.6)

$$\frac{d\varepsilon}{dp} = \frac{1}{B} + \frac{1}{H_m} \frac{(2c \cos\theta + A_m \sqrt{6} \cos\theta |\tan\theta|)^{-1} (\sin\theta + cT\cos\theta)}{3T}$$

(10.7)
in which: \[ \epsilon_v = \epsilon_y + 2 \epsilon_z \]
\[ \overline{\epsilon} = \epsilon_y - \epsilon_z \]
\[ T = \frac{dp}{dq} \]

Let \( \theta_{c}^{(m)} \) and \( \theta_{e}^{(m)} \) be the values of \( \theta \) when the stress point first reaches yield surface \( f_m \) in triaxial compression and extension respectively under drained conditions. Dividing the plastic component of the volumetric strain (eq. 10.7) by the deviatoric component (eq. 10.6):

\[
\frac{d\epsilon_v^p}{d\overline{\epsilon}^p} = \frac{dp}{H_m^p} \frac{\left(2c \cos \theta + A_m \sqrt{6} \cos \theta |\tan \theta|\right)}{3T} \frac{1}{(sin \theta + cT \cos \theta)}
\]

\[
= \frac{1}{3} \left(\frac{2c}{\tan \theta} + A_m \sqrt{6} \frac{|\tan \theta|}{\tan \theta}\right) \tag{10.8}
\]

Subtracting \( \frac{d\epsilon_v^p}{d\overline{\epsilon}^p} \) of the compression side from the extension side:
\[
\frac{d\varepsilon_{vc}^f}{d\varepsilon_c^f} - \frac{d\varepsilon_{ve}^f}{d\varepsilon_e^f} = \frac{1}{3} \left[ \left( \frac{2c}{\tan \theta_c} + \lambda_m \sqrt{c} \frac{|\tan \theta_c|}{\tan \theta_e} \right) - \left( \frac{2c}{\tan \theta_e} + \lambda_m \sqrt{c} \frac{|\tan \theta_e|}{\tan \theta_e} \right) \right]
\]  

(10.9)

\[
\frac{2c}{3} \left( \frac{1}{\tan \theta_c} \pm \frac{1}{\tan \theta_e} \right) = \frac{d\varepsilon_{vc}^f}{d\varepsilon_e^f} \pm \frac{d\varepsilon_{ve}^f}{d\varepsilon_c^f}
\]

if:

\[
\tan \theta_c \tan \theta_e < 0 \quad +
\]

\[
\tan \theta_c \tan \theta_e > 0 \quad -
\]

But:

\[
d\varepsilon_v^f = d\varepsilon_v - d\varepsilon_e^e
\]

\[
= dp \frac{d\varepsilon_v}{dp} - \frac{dp}{B}
\]

Similarly for \(d\varepsilon^f\):

\[
d\varepsilon_c^f = dq \frac{d\varepsilon_c}{dq} - \frac{dq}{2G}
\]

Therefore:
\[
\frac{d\epsilon_p}{d\tilde{\epsilon}^p} = \frac{dp}{dq} \left( \frac{d\epsilon_v}{dp} - \frac{1}{B} \right) \left( \frac{p^n}{p_1^n} \right)
\]

\[
= \frac{dp}{dq} \left[ \left( \frac{p^n}{p_1^n} \right) \frac{d\epsilon_v}{dp} - \left( \frac{p^n}{p_1^n} \right) \frac{1}{B} \right]
\]

\[
= \frac{dp}{dq} \left[ \left( \frac{p^n}{p_1^n} \right) \frac{d\epsilon_v}{dq} - \left( \frac{p^n}{p_1^n} \right) \frac{1}{2G_1} \right]
\]

\[
= \frac{T^x}{y}
\]

\[\text{(10.10)}\]

Therefore:

\[
T_c \frac{x_c}{y_c} \pm T_\epsilon \frac{x_\epsilon}{y_\epsilon} = \frac{2c}{3} \left( \frac{1}{\tan \theta_c} \pm \frac{1}{\tan \theta_\epsilon} \right)
\]

\[\text{(10.11)}\]

where:

\[
T = \frac{dp}{dq}
\]

\[\text{(10.12)}\]
where the subscripts \( c \) and \( \varepsilon \) refer to compression and extension loading, respectively.

To get another relation involving \( \theta_c \) and \( \theta_\varepsilon \), recall that:

\[
k_1 = k e^{-\rho \varepsilon_\varepsilon}
\]

(10.15)

\[
a_1 = a e^{-\rho \varepsilon_\varepsilon}
\]

(10.16)

\[
\beta_1 = \beta e^{-\rho \varepsilon_\varepsilon}
\]

(10.17)

Therefore:

\[
a_1^c = (q_c - k \sin \theta_c) e^{-\rho \varepsilon_\varepsilon} = q_c e^{-\rho \varepsilon_\varepsilon} - k_1 \sin \theta_c
\]

\[
a_1^\varepsilon = (q_\varepsilon - k \sin \theta_\varepsilon) e^{-\rho \varepsilon_\varepsilon} = q_\varepsilon e^{-\rho \varepsilon_\varepsilon} - k_1 \sin \theta_\varepsilon
\]

\[
\beta_1^c = (p_c - k \cos \theta_c) e^{-\rho \varepsilon_\varepsilon} = p_c e^{-\rho \varepsilon_\varepsilon} - k_1 \cos \theta_c
\]

\[
\beta_1^\varepsilon = (p_\varepsilon - k \cos \theta_\varepsilon) e^{-\rho \varepsilon_\varepsilon} = p_\varepsilon e^{-\rho \varepsilon_\varepsilon} - k_1 \cos \theta_\varepsilon
\]

Then:
\[
\frac{\cos \theta_c - \cos \theta_e}{\sin \theta_c - \sin \theta_e} = \frac{c(p^c_1 - \rho^c_1) - c(p^e_1 - \rho^e_1)}{(q^c_1 - \alpha^c_1) - (q^e_1 - \alpha^e_1)} \\
= \frac{c(p^c_1 - p^e_1)}{(q^c_1 - q^e_1)} \\
= \frac{c(p^c \epsilon^c \epsilon^c - p^e \epsilon^e \epsilon^e)}{(q^c \epsilon^c \epsilon^c - q^e \epsilon^e \epsilon^e)} \\
= \frac{c p^c - p^e e^{\rho (\epsilon^c - \epsilon^e)}}{q^c - q^e e^{\rho (\epsilon^c - \epsilon^e)}} = R \quad (10.18)
\]

Using algebraic manipulations, eq. 10.18 can be rewritten:

\[
\left(\frac{1}{\tan \theta_c} + \frac{1}{\tan \theta_e}\right) + \frac{2R}{1 - R^2} \left(\frac{1}{\tan \theta_c} \frac{1}{\tan \theta_e} - 1\right) = 0 \quad (10.19)
\]

The smooth stress-strain curves are approximated by piecewise linear segments along which the tangent modulus is a constant. To find the yield surfaces, yield points are chosen along the compression stress-strain curve. The corresponding yield point in extension is determined by specifying that the slope \(dq/d\epsilon\) is to be the same in both compression and extension once the stress point has first reached that particular yield surface. Values for \(\theta_c\) and \(\theta_e\) for each yield surface can then
be determined from the two simultaneous equations eq.10.11 and 10.19. The parameters associated with each yield surface can be found by rearranging eq.10.6 and 10.7:

\[
B'_m = \frac{x_c \sin \theta_c \, TC - x_e \sin \theta_e \, TE}{\cos \theta_c - \cos \theta_e} \tag{10.20}
\]

\[
h'_m = x_c \sin \theta_c \, TC - B'_m \cos \theta_c \tag{10.21}
\]

\[
A_m \sqrt{\delta} = \frac{1}{|\tan \theta_c|} \left[ 3T_c \frac{x_c}{y_c} \tan \theta_c - 2c \right] \tag{10.22}
\]

\[
k_1^{(m)} = \frac{q_c \, e^{-\rho \varepsilon'} - q_e \, e^{-\rho \varepsilon_e}}{\sin \theta_c - \sin \theta_e} \tag{10.23}
\]

\[
a_1^{(m)} = q_c \, e^{-(\rho \varepsilon_c)} - k_1^{(m)} \sin \theta_c \tag{10.24}
\]

\[
\beta_1^{(m)} = p_c \, e^{-(\rho \varepsilon_e)} - k_1^{(m)} \cos \theta_c \tag{10.25}
\]

where:

\[
TC = \sin \theta_c + cT_c \cos \theta_c \tag{10.26}
\]

\[
TE = \sin \theta_e + cT_e \cos \theta_e \tag{10.27}
\]

The specification that the smooth stress-strain curves be approximated by linear segments demands that \(d \varepsilon/dq\) and \(d \varepsilon_e/dp\) be constant. For this to be true, the value of \((d \varepsilon/dq)\) and \((d \varepsilon_e/dp)\) must be calculated when the stress point first reaches \(f_m\) and cannot change, despite the translation and
expansion of the yield surfaces according to eq. 10.2, until $f_{m,1}$ is reached. As with the model for cohesionless soil, kinematic and isotropic hardening does not influence the strain increments.

Let us now use this model to predict the drained behavior of two normally consolidated clays, one natural and the other remolded, subjected to two different axisymmetric triaxial stress paths: active compression (AC) whereby $\sigma_\alpha = \sigma_\beta$ is decreased while $\sigma_\gamma$ is held constant, and passive extension (PE) whereby the confining stress is increased while the vertical stress is held constant. The two stress paths required for the determination of the model parameters are: passive compression (PC) whereby the vertical stress is increased with the confining stress constant, and active extension (AE) in which $\sigma_\gamma$ is decreased with $\sigma_\alpha = \sigma_\beta$ held constant.

The natural clay used for the study is known as Haney clay. It has a liquid limit of 46%, a plastic limit of 26%, and a natural water content of 41-44%. Its sensitivity is of the order of 6-10. A complete description of the soil properties and testing program can be found in Reference 28.

The samples were $K_0$ consolidated to a vertical effective stress of 5.95 kg/cm$^2$ and a horizontal effective stress of 3.30 kg/cm$^2$ giving a $K_0$ value of 0.56. The specimens were then loaded at a constant rate of strain in a drained condition. Figures 10.4 and 10.5 show stress difference versus axial strain and volumetric strain versus axial strain curves respectively for the passive and active extension tests. Also
shown are the predictions which the model made using the parameters in Table 10. The good correspondence is to be expected since the parameters were derived from this set of data. The stress paths of these two tests as well as the two for prediction are shown along with the initial sizes and positions of the yield surfaces in Figure 10.6.

A comparison between the predicted stress-strain curves and the actual test data for AC and PE are shown in Figures 10.7 and 10.8. The predicted behavior bears no resemblance to the actual data. Obviously, the model was of little value in predicting the effective stress behavior of this soil.

Remolded Weald clay was the second soil used in this evaluation. It has a liquid limit of 20% and a plastic limit of 30%. The samples were isotropically consolidated to an effective mean normal stress of 30 psi and then sheared in a drained condition. Details of the test program are given in Reference 14.

The test data for stress paths PC and AE used to derive the parameters in Table 11 is shown in Figures 10.9 and 10.10. The comparison between the predicted stress-strain curves and the actual test data for AC and PE is shown in Fig.10.11. The predictions for this clay are not very good either.

The Prevost effective stress model can be used to predict pore pressures under undrained conditions by setting \( d \varepsilon_v = 0 \) and rearranging eq.10.7 to express \( dp \) as a function of \( dq \). Although the total volume change is zero, the elastic and plastic volume changes are non-zero, and equal and opposite.
Fig. 10.4. Haney clay. Stress difference versus axial strain curves. [from Vaid(28)]
Table 10

Parameters For Haney Clay

\[
\begin{array}{ccccccc}
 m & k_1 & a_1 & \beta_1 & h' & B' & A/6 \\
1 & 0.6723 & 2.1316 & 3.9129 & 192.094 & -54.9757 & -2.2646 \\
2 & 2.1338 & 1.4605 & 3.5315 & 41.569 & -14.7826 & -2.1709 \\
3 & 2.4957 & 1.1329 & 3.4464 & 15.609 & -5.1436 & -2.4666 \\
5 & 3.0040 & 1.2339 & 3.2061 & 5.280 & -2.0401 & -1.9167 \\
6 & 3.1094 & 1.0528 & 3.2363 & 0.001 & -0.001 & 0.0000 \\
\end{array}
\]

\[G_1 = 400 \text{ kg/cm}^2\]
\[B_1 = 201 \text{ kg/cm}^2\]
\[\rho = 3.612\]
\[n = 0.5\]

Note: \(k, a, \beta, h',\) and \(B'\) all have units of kg/cm\(^2\).

\(A/6\) is dimensionless.

Prevost (22) has used this method, to predict the pore pressures of a clay during undrained shear, with only mixed success. Fig.10.12 shows two comparisons between predicted and measured pore pressures as reported in reference 22 for the Kaolinite clay discussed in Chapter 5.
Fig. 10.5. Haney clay. Volumetric strain versus axial strain curves. [from Vaid (28)]
Fig. 10.6. Haney Clay. Original positions of yield surfaces and stress paths used for study.
Fig. 10.7. Haney clay. Stress difference $q$ versus axial strain $\varepsilon_y$ predictions. [data from (29)]
Fig. 10.8. Haney clay. Volumetric strain $\varepsilon_v$ versus axial strain $\varepsilon$, predictions. [data from (29)]
Table 11

Parameters For Weald Clay

<table>
<thead>
<tr>
<th>m</th>
<th>( k_i )</th>
<th>( a_i )</th>
<th>( \beta_i )</th>
<th>( h' )</th>
<th>( B' )</th>
<th>( \Lambda \sqrt{\delta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.574</td>
<td>-1.0591</td>
<td>29.4514</td>
<td>1123.71</td>
<td>-309.95</td>
<td>-2.4616</td>
</tr>
<tr>
<td>2</td>
<td>6.945</td>
<td>-0.6027</td>
<td>28.8374</td>
<td>571.44</td>
<td>-120.37</td>
<td>-2.2525</td>
</tr>
<tr>
<td>3</td>
<td>12.254</td>
<td>2.5129</td>
<td>28.8502</td>
<td>264.30</td>
<td>-65.40</td>
<td>-1.9343</td>
</tr>
<tr>
<td>4</td>
<td>16.411</td>
<td>3.1065</td>
<td>28.0681</td>
<td>159.32</td>
<td>-43.54</td>
<td>-1.8564</td>
</tr>
<tr>
<td>5</td>
<td>19.759</td>
<td>5.5773</td>
<td>27.8338</td>
<td>81.74</td>
<td>-27.78</td>
<td>-1.5626</td>
</tr>
<tr>
<td>6</td>
<td>21.727</td>
<td>6.0652</td>
<td>27.4839</td>
<td>28.82</td>
<td>-10.96</td>
<td>-1.5154</td>
</tr>
<tr>
<td>7</td>
<td>22.783</td>
<td>5.5821</td>
<td>27.0053</td>
<td>.01</td>
<td>-.01</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

\( G = 833.33 \text{ psi} \)
\( B = 8333.3 \text{ psi} \)
\( n = 0.5 \)
\( \rho = 7.443 \)

Note: \( k, a, \beta, h', \text{ and } B' \) all have units of psi.
\( \Lambda \sqrt{\delta} \) is dimensionless.
Fig. 10.9. Normally consolidated, remolded Weald clay. Drained triaxial data used to get the parameters. Stress difference $q$ versus strain difference $\bar{\varepsilon} = \epsilon_y - \epsilon_x$.

[data from (14)]
Fig. 10.10. Normally consolidated, remolded Weald clay. Drained triaxial data used to get the parameters. Volumetric strain $\varepsilon_v$ versus strain difference $\bar{\varepsilon}$. [data from (14)]
Fig. 10.11. Normally consolidated, remolded Weald clay. Predictions. [data from (14)]
Fig. 10.12. Pore water pressure predictions for Kaolinite. [data from (22,29)]
A DISCUSSION OF THE EFFECTIVE STRESS MODEL

The comparisons shown in Figures 9.7, 10.7, 10.8 and 10.11 demonstrate quite clearly that the effective stress formulation of the theory of Chapter 2 is not very good for predicting the behavior of soil under drained conditions. This implies that some of the assumptions used to formulate the model are not valid. An assessment of the propriety of each assumption would be very difficult to make. For example, the value for "c" in the yield criterion (eq.8.5a), the normal to the plastic potential component for volumetric strain (eq.8.18), and the expression for H' (eq.9.1 and eq.10.3) were chosen largely for mathematical convenience. It could then be said that one, two, or possibly all three are not good assumptions; but to isolate which combination is responsible for the poor predictions would be extremely difficult.

In some cases however, criticism can be made on a more rational basis. The assumption of initial isotropy for cohesionless soils prepared in the laboratory, which the model in Chapter 9 makes, is one of these. Ladd et. al. (10) have shown that anisotropy can be quite pronounced in laboratory samples prepared using standard methods. The \( a_{ij} \neq 0 \) in this case. Verification of anisotropy or isotropy requires at least one other test, under the same stress path, which shears the sample in a different direction. Thus the two tests prescribed
for obtaining the parameters for the cohesionless soil model are not adequate.

Fixing the value of $d\bar{e}/dq$ and $d\epsilon_p/dp$ from when the stress point first touches $f_m$ until $f_{m+1}$ is reached may only be suitable for monotonic loading unless a very large number of yield surfaces are used. Under more complex stress paths, the appropriate value of $d\bar{e}/dq$ and $d\epsilon_p/dp$ would be difficult to determine.

To illustrate this point, consider the monotonic stress path of Fig. 11.1 from point A to point B. If we now follow a new stress path, say to point C, the problem of obtaining an appropriate value of $d\bar{e}/dq$ and $d\epsilon_p/dp$ becomes apparent. Obviously, using the initial values of $d\bar{e}/dq$ computed under stress path AB would be incorrect. From eq.9.6, $d\bar{e}/dq=1/2G$ in this case since $q=0$ initially for all the yield surfaces. Thus no plastic deviatoric strains would be predicted, which is incorrect.

It could be argued that the appropriate values for $d\bar{e}/dq$ and $d\epsilon_p/dp$ should be computed from the new orientation of the yield surfaces. However, an inconsistency can arise whereby the plastic strains can be different under the same plastic stress path (i.e. That portion of the stress path which causes plastic strains). For example, let $f_1$ enclose an entirely elastic region and let $f_2$ be the yield surface which the stress point has just reached (Fig. 11.2). $d\bar{e}/dq$ and $d\epsilon_p/dp$ can then be computed and these values remain constant until $f_3$ is reached.

By moving the stress point to $B$, $(p-\beta^{(m)})$ and $(q-a^{(m)})$
Fig. 11.1. Position of yield surfaces upon translation from A to B.
Fig. 11.2. When stress point first reaches $f_2$.

Fig. 11.3. When stress point is on $f_2$ after unloading and reloading elastically.
change due to the kinematic movement of the yield surfaces.

We shall now unload elastically to point C and then reload elastically to B again (Fig. 11.3). A new value for \( \frac{d\bar{e}}{dq} \) and \( \frac{d\varepsilon}{dp} \) must now be computed if plastic loading occurs; but these quantities would be different from those computed earlier.

If loading is now continued along the same stress path to point D, the total deviatoric plastic strain which occurs from A to D would be:

\[
\bar{\varepsilon}_{A\overline{B}D}^p = \bar{\varepsilon}_{A\overline{B}}^p + \bar{\varepsilon}_{\overline{B}D}^p
\]

\[
= \left( \frac{d\bar{e}}{dq} \right)_{AB} (q_A - q_B) + \left( \frac{d\varepsilon}{dp} \right)_{BD} (q_B - q_D) \quad (11.1)
\]

If the loading from A to D increased monotonically instead, the total plastic deviatoric strain would be:

\[
\bar{\varepsilon}_{A\overline{D}}^p = \left( \frac{d\bar{e}}{dq} \right)_{AB} (q_A - q_B) \quad (11.2)
\]

But \( \bar{\varepsilon}_{A\overline{D}}^p \neq \bar{\varepsilon}_{A\overline{B}D}^p \) when, in fact, they should be equal since the unloading and reloading to point C was purely elastic. This shows that fixing the value of \( \frac{d\bar{e}}{dq} \) and \( \frac{d\varepsilon}{dp} \) is only suitable for monotonic loading.

Of course, special rules can always be created to deal with these situations. But such rules would complicate the
computational process. The simplest means for modelling the behavior under any stress path would be to calculate a set of parameters which would be compatible with changing values of $\frac{d\sigma}{dq}$ and $d\epsilon_v/dp$ during each stress increment.

One method for doing this would be to calculate $h'_m$, $H'_m$ and $A_m$ using an average value for $(p - \beta_m)$ and $(q - \alpha_m)$ obtained in the triaxial test. Let $(p_0 - \beta_0(m))$ and $(q_0 - \alpha_0(m))$ denote original values when the stress point first touches $f_m$. However, as the yield surface translates, $(p-\beta)$ and $(q-\alpha)$ change and, according to eq. 2.30, reach values of:

$$
(p - \beta_m)_{f} = \frac{k(m)}{k(m+1)} (p - \beta_{(m+1)})_0
$$

$$
(q - \alpha_m)_{f} = \frac{k(m)}{k(m+1)} (q - \alpha_{(m+1)})_0
$$

when the next largest yield surface is reached (subscript "f" denotes their final value upon reaching $f_{m+1}$). Average values can then be found from:

$$
(q - \alpha_m)_{avg} = \frac{(q - \alpha_m)_0 + (q - \alpha_m)_f}{2}
$$

$$
(p - \beta_m)_{avg} = \frac{(p - \beta_m)_0 + (p - \beta_m)_f}{2}
$$
and these can be used in equations 9.11, 9.12, and 9.13. The computation can then proceed in the same manner as for the undrained total stress model, whereby updating of $\frac{\partial e}{\partial s}$ and $\frac{\partial e}{\partial p}$ can be done after each stress increment.

Figure 11.4 compares the response predicted for the triaxial compression test on Cook's Bayou sand by a) using parameters determined by this averaging method together with kinematically moving yield surfaces, and b) using Prevost's method whereby "original" parameters are used without considering the kinematic movement of the yield surfaces. Clearly, the averaging method can give a very good model fit while permitting full consideration of the kinematic hardening which has occurred.

The assertion that the model discussed in Chapter 10 is applicable to any given soil is questionable. For example, eq.10.2a is justifiable if the Critical State line also defines the state at which the peak stress is achieved. This coincidence is true for normally and lightly overconsolidated clays and for very loose sands. However, heavily overconsolidated clays and medium to very dense sands have peak strengths considerably above the Critical State. After reaching the peak stress, these soils strain soften towards the Critical State condition. Unlike the Critical State condition, this peak strength cannot be uniquely determined for a given void ratio. Thus the size of the largest yield surface, as given by eq.10.2a in which $k$ is solely a function of the volumetric strain, cannot also define failure at peak stress.
Fig. 11.4. Model fit using Prevost's method and the averaging method.
for these soils.

As explained earlier, equations 10.2b and 10.2c are justifiable if the "strength" of a normally consolidated soil can be normalized by the consolidation pressure. While this is true for clays, it is unreasonable in general for sands. Even the loosest sands will show a peak stress and then subsequent strain softening behavior if the confining stress is low enough. On the other hand, if this loose sand were subjected to a very high confining pressure, "normally consolidated" behavior would occur. Clearly, a normalization procedure based on a consolidation pressure cannot be used for sands in general.

These arguments show that the model discussed in Chapter 10, while not being applicable to certain types of soils, uses assumptions which are valid for normally consolidated clays. A closer examination of eq.10.7 shows a possible reason why the model does not give good predictions for this soil either. Recall that:

\[ Q'_{ij} = \frac{\partial f}{\partial s_{ij}} = 3(s_{ij} - a_{ij}) \]  \hspace{1cm} (11.7)

\[ Q_{ij} \cdot ds_{ij} = \frac{\partial f}{\partial s_{ij}} \cdot ds_{ij} = 3(s_{ij} - a_{ij}) \cdot ds_{ij} \]

\[ = 3[(q-a_1)\frac{4}{3}dq + 2(-\frac{1}{3}(q-a_1))(-\frac{1}{3}dq)] \]
\[ = 2(q-a_1)dq \]  \hspace{1cm} (11.8)

\[ Q'' = \frac{\partial f}{\partial p} = \frac{2}{3}c^2(p-\beta) = 3(p-\beta) \]  \hspace{1cm} (11.9)
\[ |Q'| = \sqrt{3 \cdot 3 (s_{ij} - a_{ij}) (s_{ij} - a_{ij})} \]
\[ = 3 \sqrt{\left( \frac{3}{2} (q-a_i) \right)^2 + 2 \left( -\frac{1}{2} (q-a_i) \right)^2} \]
\[ = |\sqrt{6} (q-a_i)| \]  

(11.10)

Thus equation 8.20 becomes:

\[
\frac{d\tilde{\varepsilon}}{dq} = \frac{1}{H_m^*} \frac{(2(q-a_i) dq + 2c^2(p-\beta) dp) \left[ 2c^2(p-\beta) + 3 \sqrt{6} A_m |q-a_i| \right]}{6k^2} 
\]

(11.11)

\[
\frac{d\varepsilon}{dp} = \frac{1}{3H_m^*} \frac{[2c \cos \theta + 3 \sqrt{6} A_m |\sin \theta|]}{(q-a_i) + c \text{Tc}(p-\beta))} \]

(11.12)

Substituting equations 10.4 and 10.5 into the above gives:

\[
\frac{d\varepsilon}{dp} = \frac{1}{H_m^*} \frac{[2c \cos \theta + 3 \sqrt{6} A_m |\sin \theta|]}{(\sin \theta + c \text{Tcos}\theta)} \]

(11.13)

Notice that the second term in square brackets has a numerical factor of $3 \sqrt{6}$ compared with $\sqrt{6}$ for eq.10.7. The difference lies in the definition of $A_m$. Equation 10.7 includes the factor 3 in it whereas eq.11.13 separates it from $A_m$. Thus the $A_m$ of equation 10.7 is three times larger than the $A_m$ defined by eq.8.20 or 11.13. To avoid confusion, only eq.10.7 will be used herein.

Also significant is that the second term in square brackets...
brackets has $\cos \theta |\tan \theta|$ for eq.10.7 whereas eq.11.13 has $|\sin \theta|$. Note that:

$$\cos \theta |\tan \theta| = \frac{\cos \theta}{|\cos \theta|} |\sin \theta| \neq |\sin \theta| \quad (11.14)$$

If eq.11.13 is used instead of eq.10.7 to generate a second equation, instead of eq.10.11, to be used together with eq.10.19 to solve for $\theta_c'(m)$ and $\theta_e'(m)$, no solution can be obtained for either Weald clay or Haney clay. This implies that a solution for $\theta_c'(m)$ and $\theta_e'(m)$ using eq.11.13 is not, in general, obtainable and thus an alternative equation (eq.10.7) is necessary.

Although eq.10.7 allows a solution to be obtained, it has certain implications as to the type of behavior the model will describe. Figure 11.5 and the following text describes one of these.

Assume that the stress point lies infinitesimally to the right of $f_m$ and that the stress increment points outward to $f_m$ (vector $A$ in Fig.11.5). Thus:

$$\cos \theta = \text{an infinitesimally small positive number}$$
$$\sin \theta = 1$$
$$H'_m = h'_m = \text{a constant}$$
$$\Lambda_m = \text{a constant}$$
$$d\epsilon_v = \frac{dp}{B} + \frac{1}{H'_m} \left(0+A_m\sqrt{6}\right) \frac{1}{3} dq \quad (11.15)$$
Fig. 11.5. Discontinuity of volumetric strain when \( \theta=90^\circ \).

Let us now examine what happens if the stress point lies infinitesimally to the left of \( f_m \) and the stress increment is in the same direction as in the case above (vector B in Figure 11.5):

\[
\cos \theta = \text{an infinitesimally small negative number} \\
\sin \theta = 1 \\
H'_m = h'_m = \text{a constant} \\
A_m = \text{a constant} \\
d\epsilon_v = \frac{dp}{B} + \frac{1}{H'_m} (0-A \sqrt{6}) \frac{1}{3} (dq) \quad (11.16)
\]

For clarity, let us assume that \( dp=0 \). Stress vector A gives:
\[ \frac{d\varepsilon}{dq} = \frac{A_m \sqrt{6}}{3H_m} \] i.e. compression \hspace{1cm} (11.17)

while stress vector B gives:

\[ \frac{d\varepsilon}{dq} = -\frac{A_m \sqrt{6}}{3H_m} \] i.e. expansion \hspace{1cm} (11.18)

Thus two stress points which are infinitesimally apart can give radically different volume changes. This is not consistent with the observed behavior of soils in which infinitesimal changes in stress give only infinitesimal changes in strain (i.e. continuity) at stress states below failure. Given the above inconsistency, one would not expect this model to give good predictions.
From the results of the thesis, the following conclusions can be drawn:

1. Plasticity theory utilizing multiple yield surfaces and kinematic hardening provides a convenient means of modelling the nonlinear, hysteretic, stress path dependent behavior of anisotropic soils.

2. The total stress model proposed by Prevost appears to be quite accurate for predicting the monotonic behavior of undrained clays under complex stress paths. However, many more such comparisons between the stress-strain curves predicted by the model and actual test data must be made before the model can be used with confidence.

3. The reduction in strength and stiffness of a soil during cyclic loading can be simulated by incorporating isotropic hardening/softening into the model. However, simpler expressions for describing the degradation of the yield surfaces than what have been proposed would be desirable.

4. Neither of the effective stress models is capable of accurately predicting the stress-strain behavior of the soils examined.

5. The general formulation for soils has a mathematical inconsistency in the derivation of the equations for the parameters. This is one of the reasons why this effective stress model does not give good predictions.
6. Unless a very large number of yield surfaces are used, the practice of maintaining $\frac{d\epsilon}{ds}$ and $\frac{d\epsilon}{dp}$ constant in the effective stress model between yield surfaces implies that only monotonic loads can be considered. To handle the more general stress paths, it is proposed that a set of parameters calculated from the "average" position of each yield surface during translation towards the next largest yield surface, be used.
REFERENCES


APPENDIX A

Homogeneous Functions

A function \( f(x_1, x_2, \ldots, x_r) \) is called a homogeneous function of degree \( n \) if:

\[
f(vx_1, vx_2, \ldots, vx_r) = v^n f(x_1, x_2, \ldots, x_r)
\]

where \( v \) is an arbitrary parameter.

For example, the invariants of the stress deviation, \( J_2 \) and \( J_3 \), are homogeneous functions of degree 2 and 3 respectively. The following illustrates this for \( J_2 \):

\[
J_2 = \frac{1}{2} s_{ij} s_{ij}
\]

\[
\frac{1}{2} (vs_{ij})(vs_{ij}) = \frac{1}{2} v^2 s_{ij} s_{ij}
\]

Therefore \( J_2 \) is a homogeneous function of degree 2.

Euler's theorem on homogeneous functions states that if \( f(x_1, x_2, \ldots, x_r) \) is homogeneous of degree \( n \), then:

\[
x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_r \frac{\partial f}{\partial x_r} = nf
\]
or:

\[ x_i \frac{\partial f}{\partial x_i} = nf \]
APPENDIX B

Inversion of Equation 3.9

Let \( \delta e_{ij}^e = v(s_{ij} - a_{ij}) \); where:

\[
v = \frac{3}{2H'} \frac{(s_{ij} - a_{ij})}{k^2} \delta s_{ij} = \text{a scalar}
\]

Recalling that \( \delta s_{ij} = 2G \delta e_{ij}^e \), equation 3.9 can be rewritten:

\[
\delta e_{ij} = \delta e_{ij}^e + \frac{3}{2H'} \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) 2G(\delta e_{kl} - \delta e_{kl}^e)
\]

\[
= \delta e_{ij}^e + \frac{3G}{H'} \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) \delta e_{kl} - \frac{3G}{H'} \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) (s_{kl} - a_{kl}) v
\]

\[
= \delta e_{ij}^e + \frac{3G}{H'} \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) \delta e_{kl} - \frac{2G}{H'} (s_{ij} - a_{ij}) v
\]

\[
= \delta e_{ij}^e + \delta e_{ij}^f
\]

Therefore:
\[ \text{de}_{ij}^\rho = v(s_{ij} - a_{ij}) \]

\[ = \frac{3G}{H'} \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) \text{de}_{kl} - \frac{2G}{H'} (s_{ij} - a_{ij}) v \]

Since:

\[ \frac{1}{H} = \frac{1}{H'} + \frac{1}{2G} \]

then:

\[ v(s_{ij} - a_{ij}) \left(1 + \frac{2G}{H'}\right) = \frac{3G}{H'} \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) \text{de}_{kl} \]

\[ = v(s_{ij} - a_{ij}) \left(1 + \frac{2G}{H} - 1\right) \] (B1)

Equation B1 becomes:

\[ \frac{2v(s_{ij} - a_{ij})}{H} = 3 \left(1 - \frac{H}{2G}\right) \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) \text{de}_{kl} \]

\[ \text{de}_{ij}^\rho = v(s_{ij} - a_{ij}) \]

\[ = \frac{3}{2} \left(1 - \frac{H}{2G}\right) \frac{s_{ij} - a_{ij}}{k^2} (s_{kl} - a_{kl}) \text{de}_{kl} \]
\[ d_{ij} = d_{ij} - \frac{d_{ij}^2}{2G} \]

Therefore:

\[ \frac{ds_{ij}}{2G} = d_{ij} - \frac{3}{2} \left( 1 - \frac{H}{2G} \right) \frac{(s_{ij} - a_{ij})}{k^2} \frac{(s_{kl} - a_{kl})d_{kl}}{k^2} \]

\[ ds_{ij} = 2Gd_{ij} - (2G - H) \frac{3}{2} \frac{(s_{ij} - a_{ij})}{k^2} \frac{(s_{kl} - a_{kl})d_{kl}}{k^2} \]
APPENDIX C

Derivation of Expression for Elastoplastic Modulus $H$

For a triaxial test in which $a_x = a_z$, equation 3.9 gives:

$$\text{de}_y = \frac{ds_y}{2H'} + \frac{3}{2H'} \frac{(s_y - a_y)}{k^2} \left[ (s_y - a_y) ds_y + 2(s_x - a_x) ds_x \right]$$

(C1)

But:

$$s_x = -\frac{1}{2}s_y$$

$$a_x = -\frac{1}{2}a_y$$

$$ds_x = -\frac{1}{2}ds_y$$

Therefore equation C1 becomes:

$$\text{de}_y = \frac{ds_y}{2G} + \frac{3}{2H'} \frac{(s_y - a_y)^2}{k^2} (ds_y - ds_x)$$

(C2)

Recall that:

$$\frac{3}{2} (s_{ij} - a_{ij})(s_{ij} - a_{ij}) - k^2 = 0$$
which for a triaxial test becomes:

\[
\frac{3}{2}[(s_y-a_y)^2 + 2(s_x-a_x)^2] - k^2 = 0
\]

or:

\[
\frac{9}{4}(s_y-a_y)^2 = k^2
\]

Equation C2 is then:

\[
\frac{\partial e_y}{2G} = \frac{2}{3H'} (\frac{\partial s_y}{\partial s_x})
\]

\[
= \frac{\partial s_y}{2G} + \frac{2}{3H'} (\sigma_y - \sigma_x)
\]

But \(e_y = e_y\) since \(e_v = 0\) and \(s_y = \frac{2}{3}(\sigma_y - \sigma_x)\). Therefore:

\[
\frac{\partial e_y}{\partial e_y} = \frac{\partial (\sigma_y - \sigma_x)}{3G} + \frac{2}{3H'} (\sigma_y - \sigma_x)
\]

\[
= d\epsilon_y^e + d\epsilon_y^f
\]

And:

\[
\frac{\partial (\sigma_y - \sigma_x)}{d\epsilon_y} = \frac{1}{\frac{1}{3G} + \frac{2}{3H'}}
\]

\[
= \frac{3}{-H}
\]
APPENDIX D

The Scale Factor $\sqrt{3/2}$ of Figure 4.3

A stress space representation of Fig.4.3 from a different perspective is shown in Fig.D1. Note that $\sqrt{3/2}\sigma_y$ is the

![Diagram](https://via.placeholder.com/150)

**Fig.D1.** a) Oblique view of axisymmetric triaxial plane (shaded) in stress space. b) View perpendicular to triaxial plane.

projection of $\sigma_y$ onto a plane perpendicular to the space diagonal. This vector lies in the deviatoric plane and is equal to $s_y$; $\sqrt{3/2}\sigma_y = s_y$. But since we are comparing Figure 4.3a with Figure 4.3b and recalling that $\sigma_y - \sigma_z = \frac{3}{2}s_y$ for a triaxial test, the axis $\sqrt{3/2}\sigma_y$ must be multiplied by 3/2 to achieve correspondence:
\[ \frac{3}{2} \sqrt{\frac{2}{3}} \sigma_\gamma = \sqrt{\frac{3}{2}} \sigma_\gamma \]