A NON-LINEAR DYNAMIC FINITE ELEMENT ANALYSIS

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ABSTRACT

A two-dimensional finite element method of analysis for predicting the stress and permanent displacements of earth structures to seismic loading is presented. The inelastic behavior of the soil is modelled by an incremental linear approach in which the tangent shear modulus is varied with the level of both the shear strain and the mean normal stress. The shear modulus strain dependency is based on hyperbolic relationships governing initial loading and unloading behavior, leading to a hysteretic type energy dissipation. The tangent bulk modulus is varied with the level of the mean normal stress only, and hysteretic effects are not considered.

The incremental linear equations of motion of the structure are solved using the Newmark step-by-step integration procedure in the time domain allowing the stresses and displacement to be computed. After each time step the tangent shear and bulk modulus are re-evaluated. Hysteretic damping as a result of the hyperbolic shear stress-strain law is inherent in the model. Viscous damping may also be included.

The analysis is applied to a number of dams and slopes and the earthquake induced displacements are compared with those predicted by a simpler Newmark single degree of freedom rigid plastic analysis. As well, a comparison is made with Makdisi's prediction of deformation of embankments. For a clay slope structure, the overall displacements are of similar order. For a clay dam structure the non-linear finite element results
indicate that the Newmark type methods are overly conservative. The more rigorous multi-degree of freedom analysis allows the distribution of displacements within and on the surface of the embankment to be obtained.
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CHAPTER 1

INTRODUCTION

1.1 Preliminary Remarks

The frequent occurrence of destructive earthquakes during the past few decades has caused billions of dollars in property damage and loss of thousands of lives. It is therefore, important that earth structures located in an active seismic region be designed to withstand safely the expected earthquake motion for that region. In February 1971, an earthquake measuring 6.6 on the Richter scale occurred in California. Fifty-eight people were killed, over two thousand injured, and 1500 buildings were damaged beyond safe occupational levels. This earthquake, said to be the most severe to occur in California in the past 80 years, caused an estimated five billion dollars in damage.

Of particular geotechnical importance was the partial failure and near total collapse of the Lower San Fernando Dam, and downstream movement of the Upper San Fernando Dam. Prior to the earthquake the water level in the reservoir was 35 feet below the crest of the dam. The slide which occurred in the upstream shell of the Lower San Fernando Dam following the earthquake resulted in a free board of only 4 to 5 feet of cracked material. At the same time, the Upper dam which forms part of the same reservoir complex, displaced some 6 feet downstream. Fortunately this movement did not cause a release of water from the reservoir of the Upper Dam. If it had, the
resulting overtopping of what remained of the Lower Dam might have caused considerable damage and loss of life (Seed 1979).

Five years previous, based on extensive state of the art procedures, a consulting board, a design agency, and review board had deemed the design of the Lower San Fernando Dam to be adequate against any earthquake to which it could be subjected. But the margin by which the total collapse of the Lower San Fernando Dam was avoided was uncomfortably small. Because of this and other recent events with associated disastrous consequences, (Osaki 1966, Seed et. al. 1966) there was a need for a rethinking of the earthquake resistant design philosophy of earth and rock fill dams and the development of alternate analysis procedures.

Up to this time, the standard method for 40 years or more for evaluating the safety of earth dams or embankment slopes to earthquake forces has been the so-called pseudo-static method analysis. In this method the effect of the earthquake on a potential slide mass is represented by an equivalent static horizontal force, determined by the product of a seismic coefficient and the total weight of the slide mass. Assuming the earthquake force acts permanently on the slope material in one direction only, and thru the centroid the mass, conventional slope stability analysis are applied to evaluate the factor of safety against movement. A factor of safety of less than unity implies movement, but because of the transient and randomly oscillating nature of earthquake motions, a collapse (in terms
of an excessive displacement criteria) may not occur. In this sense, a factor of safety cannot determine adequate seismic performance.

It is now generally accepted that the magnitude of relative displacements is a more logical and better criterion than the factor of safety for assessing the dynamic performance of earth structures. The allowable displacements may vary from a few inches to a few yards depending on the functional aspects of the structures concerned. Newmark (1965) first proposed a procedure for predicting permanent deformations in earth embankments during earthquakes using a simplifying rigid plastic single degree of freedom approach. More recently, dynamic response analyses of soil structures (Seed et al., 1973) have been included in a more elaborate and encompassing procedure in predicting displacements. These procedures will be discussed in Chapter 2.

Dynamic response analyses as practised today had its origin in the pioneering attempts of Seed and his co-workers at the University of California at Berkeley to predict the accelerations in horizontal soil deposits by bedrock earthquake motions. The Seed Approach (Dynamic stress path Method) was used to evaluate the seismic performance of structures founded on soil deposits. For such structures, the performance of the soil deposit to act as an adequate foundation base when subjected to earthquake motions and the modification of the bedrock motion as it propagates through the soil are important
considerations. It has been identified that amplification or de-amplification (depending on the characteristics of the soil deposit) occurs when seismic waves travel through soil. In addition, there is a tendency for the resultant motion at the top of the deposit to be affected by the natural frequency of the deposit. The predominant period of the motion at the top of a deposit is generally longer than the predominant period of the bedrock motion.

The prediction of seismic response of horizontally layered deposits has been well researched (Schnabel et al, 1972; Streeter et al, 1974; Finn et al, 1977). These methods assume that soil properties vary in the vertical direction and remain uniform in the horizontal direction. These methods vary in the simplifying assumptions that are made, the modelling of the non-linearity and possible strength loss of soil during the dynamic loading, and in the methods used to integrate the equation of motion.

Often soil structures cannot be modelled adequately as horizontally layered deposits, and it becomes important to consider two or even three dimensions. For example, in zoned dams, where the cross-sectional shape as well as the spatial variability of the soil properties necessitate a second dimension. Where the third dimension is much larger than the other two and properties in this third dimension are uniform, two dimensional analyses are adequate. The effects of the third dimension have been considered in two dimensional analyses by
Makdisi (1976). The predominant period of the two dimensional cross section was altered according to a length of structure to height relationship. Dynamic response analyses were generalized to two dimensions with the development of the computer programs QUAD4, LUSH and FLUSH. Their method of analysis is based on an equivalent linear elastic approach. Linear elastic methods do not allow computation of displacements directly, but are used in conjunction with laboratory and addition analytical procedures to determine displacements. These procedures will be discussed in greater detail in Chapter 2.

Streeter proposed the first true non-linear analysis where a Ramberg-Osgood representation of the stress-strain behavior of soil was used. The analysis can only be used in one dimensional total stress applications. The development of a two dimensional non-linear dynamic analysis is the next logical step. The work herein, is presented with this aim in mind.

1.2 Scope of the Thesis

The thesis presents a non-linear finite element dynamic response analysis to predict deformations during earthquake loadings.

The following basic assumptions are made while formulating the model:

1) Theory of linear incremental elasticity is applicable.
2) Soil behaves isotropically.
3) Plane strain conditions prevail.
4) The earthquake ground motion is identical at all points along the base of the structure.

The stress-strain behavior of soil under shear loading is assumed to be a hyperbolic stress-strain relationship similar to that used by Martin, Finn and Seed (1975). Volume change behavior as proposed by Duncan et. al., (1980) is followed. The dynamic analysis is carried out using an incremental time step approach where for each element the stress-strain curve is followed in an incremental manner. With this approach, permanent deformations can be evaluated directly.

The seismic excitation is assumed to be vertically travelling shear waves which are defined by a specified acceleration time history at the rigid base of the finite element model.

In Chapter 2, a description of current methods in assessing seismic instability and predicting permanent deformations is given. Their main assumptions, analysis procedures and limitations are critically reviewed.

Chapter 3 and 4 deals with the formulation of the non-linear dynamic finite element analysis. Assumptions, constitutive behavior, stiffness and damping properties of the soil materials, and the method of solution of the equations of motion are presented.

The method is used to evaluate the seismic behavior of some
typical earth structures. These results are presented in Chapter 5. Displacement predictions are compared against results obtained from procedures developed by earlier investigators. A brief summary, conclusion, suggestions for further research are presented in Chapter 6.
CHAPTER 2
CRITICAL REVIEW OF CURRENT METHODS IN THE
ASSESSMENT OF SEISMIC PERFORMANCE
EARTH STRUCTURES

The seismic performance of earth embankments and dams have received considerable attention in recent years. There are currently several methods being used in engineering practice for assessing the seismic stability and predicting deformations of earth structures. By way of introduction to later chapters in this thesis, these methods will be examined herein. The assumptions made in the formulation, analysis procedures, and limitations of each method will be critically reviewed.

2.1 Pseudo-static Method

In this method the randomly oscillating inertia forces caused by an earthquake is represented by an equivalent static horizontal force and a conventional slope stability analysis is applied to evaluate the factor of safety against collapse. The equivalent static horizontal force is determined by multiplying the design seismic coefficient by the total weight of the potential slide mass. The method is illustrated in Figure 2-1.

The effectiveness of the method depends upon, amongst other things, the selection of a design seismic coefficient value. Values for the seismic coefficient in the design of earth dams worldwide have varied from .05 to .20. The standard North American practice is to use seismic coefficient values of 0.05,
FIG. 2-1 PSEUDO-STATIC METHOD

\[ F.S. = \frac{S \cdot R}{E W + F \cdot K W} \]
0.10, 0.15, in areas of low, medium and high seismicity. The selection of a seismic coefficient value seems largely based on past experience. Prior to 1971, very few dams designed in accordance with these principles, have been subjected to very strong earthquake shaking. For this reason, there had been no real field experience in which to base the adequacy of the pseudo-static method for assessing seismic stability.

The pseudo-static approach failed to predict the slope failures of the Lower and Upper San Fernando Dam (Seed, 1979). The primary cause of failure was the build-up of pore water pressures in the embankment and the loss of strength resulting from these pore pressures. The seismic instability which can occur in loose cohesionless soils depends upon; the peak ground acceleration and frequency content of the earthquake motion, initial stress state of the soil, residual pore water pressures. These factors cannot be represented in any rational way by an equivalent static horizontal force. Seed (1979) suggests that the pseudo-static method can only be used with any assurance for soil materials that do not suffer significant strength or stiffness loss during dynamic loading. Under these conditions, factors of safety against movement in the range of 1.0 to 1.2 are considered adequate for seismic stability.

2.2 Newmark Analysis Procedure

Newmark (1965), and Seed (1966) have both criticized the concept of factor of safety as a means of assessing the probable
performance of an earth dam during an earthquake. A factor of safety of less than unity in a static analysis is not acceptable as it implies catastrophic displacements. However, factors of safety of less than unity are acceptable in dynamic analyses, since the earthquake forces acts for a short time and alternate in direction, only small displacements may occur and these may be quite acceptable. The pseudo-static method does not provide a prediction on the magnitude of displacements. Therefore it cannot be viewed as an adequate method for evaluating seismic performance.

In the Rankine Lecture of 1965, Newmark first outlined the basic elements of a procedure for predicting the potential deformations of an embankment slope due to earthquake forces. It was assumed that slope failure would be initiated and outward movement would begin to occur if the inertia forces on a potential slide mass were large enough to overcome the yield resistance and that the movement would stop when the inertia forces were reversed. Newmark proposed that the movement of slide mass along its failure surface could be adequately modelled by the movement of a rigid block on an inclined plane (Figure 2-2). By computing an acceleration at which the inertia forces become sufficiently high to cause sliding to begin and integrating the effective acceleration on the rigid block in excess of this yield acceleration as a function of time for the duration of the earthquake motion (Figure 2-3), velocities and ultimately displacements of the rigid block could be determined. Newmark (1965) presented a chart for computing such
FIG. 2-2 FORCES ON SLIDING BLOCK
FIG. 2-3 INTEGRATION OF EFFECTIVE ACCELERATION TIME HISTORY
displacements. In the development of the chart, the displacements were computed using four earthquake motions normalized to a maximum acceleration of .5g and a maximum velocity of 30 in/sec. The narrow scatter in the data indicates that the earthquake motions used have essentially the same number of significant pulses. This may not be necessarily true for other earthquakes. For a conservative estimate of the permanent deformation of an embankment slope, the equations proposed by Newmark (and shown on the chart to give an upper bound fit to the data) may be used. The maximum velocity and acceleration values are the values appropriate to the design earthquake motion. The maximum resistance coefficient is defined as the value of the horizontal seismic coefficient which will give a factor of safety equal to unity in a pseudo-static analysis of the embankment. The resistance coefficient can be considered the yield acceleration of the slide mass.

Evidence had been presented that slip in dense sands, when subjected to uniform accelerations, occurs along a thin surface zone. The sliding mass may be considered analogous to a block resting on a inclined plane. Shaking tests performed by Goodman and Seed (1966) on small scale embankments of dry dense sands, demonstrated the validity of the fundamental principles of the Newmark approach.

Refinements to allow for the variations in acceleration throughout the embankment and slide mass were proposed by Seed and Martin (1966), Ambraseys and Sarma (1967), and Seed and
Makdisi (1978). On the basis of two dimensional response analysis on embankments subjected to a given earthquake acceleration time history, average induced acceleration time histories for a number of potential sliding masses were computed by Seed and Makdisi. Deformations are determined using the average induced accelerations time histories and Newmark's procedure. Design curves presented by Seed and Makdisi show good agreement with Ambraseys and Sarma results.

The determination of displacement values using a Newmark approach is straightforward enough. A complex multi-degree of freedom system is presented by a simple single degree of freedom model. But in doing so, there arises the problem of interpretation of results. Where does this displacement occur? It can be seen either as, the horizontal movement or the sliding movement along a slip surface of the failure mass. In any case, the Newmark approach gives some idea of the average displacement. Ideally, maximum displacement as well as a distribution of displacements within an embankment is desired.

2.3 Seed Lee Idriss Analysis Procedure

Seed and his co-workers at the University of California, Berkeley have developed a comprehensive dynamic analysis procedure for predicting deformations in earth structures. The method initially proposed by Seed (1966) and later has undergone refinements (Seed et. al.,1973), endeavours to account for dynamic forces induced by the earthquake and the
effect of stiffness loss due to dynamic cycling. The method has been used with reasonable success in back calculating the response of a number of earth dams (Seed et. al., 1975). Commonly referred to as the Seed Dynamic stress path Approach, the procedure is summarized by the following steps:

a) Select appropriate cross section of earth structure to be used in analysis and model with a finite element grid.

b) Determine appropriate static and dynamic properties of the soil to used in static and dynamic finite element analysis.

c) Determine the static stresses which existed before the earthquake. Insitu stresses are evaluated using a convention static finite element analysis.

d) Select a design earthquake time history.

e) Compute the dynamic shear stress time histories for selected elements within the cross section using a two-dimensional computer response analysis.

f) Subject representative samples of the soil structure material to the combined effects of insitu static stresses and the superimposed dynamic shear stresses and determine their effects in terms of development of potential strains.

g) From the knowledge of the soil deformation and
shown in Figure 2-4.

These curves are used in an iterative procedure known as the equivalent linear method. In this method, successive linear problems are solved until the modulus and damping corresponds to the average dynamic shear strain (usually taken as 65 percent of the max shear strain). The final linear solution with strain-compatible soil properties is taken as an approximation to a true non-linear response. The equivalent linear solution corresponds to the assumption of an elliptic hysteresis loop for cyclic loading. Currently, the computer programs QUAD4, LUSH and FLUSH based on the equivalent linear method, are used in two dimensional dynamic response analysis. QUAD4 and LUSH are the most frequently used programs for soil structures such as slopes and earth dams. FLUSH is commonly used for the solution of dynamic soil structures interaction problems such as the response of embedded nuclear reactor structures to earthquake loading.

Since the final iteration with strain compatible soil properties is purely elastic, the permanent deformations caused by earthquake shaking cannot be computed directly from this type of analysis. As in all elastic analyses the final deformations return to zero. To circumvent this shortcoming, a laboratory procedure has been developed (steps f to h) to predict strains from computed dynamic shear stress histories. While the computed elastic strains bear no relation to strains in the field they are used for deriving the strain compatible soil
FIG. 2-4 TYPICAL SHEAR MODULI AND DAMPING RATIOS FOR SANDS
properties. Stresses obtained from the analysis using these strain compatible properties are assumed to be representative of stresses in the ground. As indicated in step f, the computed stresses are used to estimate permanent deformations. Herein lies a serious inconsistency in the Seed procedure; stresses computed are considered accurate while the strains are not, obviously there is a one to one correspondence.

There are two other possible shortcomings to this method. First, equivalent linear methods may overestimate the response of earth structure due to a pseudo resonance effect. This can occur when the predominant period of the earthquake motion coincides with the natural period of the earth structure. In pure linear elastic systems, resonance is a real and possible event. It cannot occur in materials, where the stiffness properties are highly non-linear strain dependant. Second, it is not uncommon that computed stresses exceed the dynamic strength resistance of the soil. This cannot actually happen as at most the dynamic stress can equal the dynamic resistance.

The ultimate aim of a dynamic response analysis is to be able to predict permanent deformation. This can only be done directly by a non-linear analysis in the time domain. A non-linear finite element analysis is presented in the following two chapters.
CHAPTER 3
THE FINITE ELEMENT METHOD

3.1 Introduction

Several mathematical techniques have been developed to evaluate the seismic performance of earth embankments, and structures founded on soil deposits. The finite element method has been commonly used in geotechnical engineering problems due to the ease and accuracy with which geometry and varying soil properties can be modelled. More recently, the method has been used to evaluate the effects of nonlinearity in stress-strain soil behavior, initial static stresses, and boundary conditions. As the finite element method is the only available technique which will allow a rigorous assessment of the non-linearity of soil, this method will adopted in this thesis.

The finite element method has been developed as a consequence of the advent of high-speed digital computers and has been extremely successful in solving many static and dynamic problems in continuum mechanics. Its application to static analyses of elastic continua has been described by Wilson and Clough (1962) and its extension to dynamic analyses by Clough and Penzien (1975). This method will be briefly outlined here.

The finite element method may be described as a numerical discretization procedure whereby continuum is idealized as an assemblage of discrete elements. The numerical technique allows an infinite degree of freedom system to be transformed to a
finite degree of freedom system. Proper modeling of the system in terms in element size and selection will permit accurate predictions of displacements, stresses, and strains of the actual continuum.

The displacement formulation of the finite element method is employed herein. This assumes an internal displacement distribution within an element in terms of the nodal displacements such that certain required conditions on compatibility and completeness are satisfied. Once the displacement field has been assumed for any element of prescribed geometry and once the constitutive properties of the elements are determined, it is possible with the aid of the virtual work theorem to derive the stiffness matrix of the element. The stiffness matrix represents the stiffness properties associated with the displacements at the nodes of the element. The stiffness matrix of the entire continuum is obtained by the proper addition of individual element stiffness matrices by the direct stiffness method. Solution of the stiffness equations satisfies equilibrium (in a global sense.)

The advantage of this discrete mathematical formulation is that for the dynamic problem the equilibrium of the system may be expressed by a set of ordinary differential equations rather than a set of partial differential equations, while in a static problem the partial differential equations are reduced to a set of algebraic equations. If the displacements of all nodal points in the complete assemblage are designated by the vector
\{r\}, the corresponding nodal forces by the vector \{R\}, and the stiffness matrix of the entire system by the matrix \([K]\); the static equilibrium equation may be expressed in the form

\[ [K] \{r\} = \{R\} \] (3-1)

The finite element method has many advantages over other numerical techniques. Different material properties can be prescribed from one element to the next and/or can have varying properties within the element themselves. Any displacement, stress, or coupled boundary condition can be handled, and the boundaries can be very irregular in shape. It can be shown that for elastic systems by using elements with properly selected displacement functions, the finite element method converges to the exact solution as the number of elements used to model a system increases. This indicates that any desired degree of accuracy can be obtained. The governing constraint here being the computational costs associated with the increased numerical operations performed on larger stiffness matrices and vectors.

Two dimensional systems may be represented by elements of various shapes. The simplest, the three node constant strain triangle (CST), has been used extensively by earlier investigators. Displacements within this element are assumed to vary linearly through the element. This linear displacement function results in constant strain and stress within the element, hence its name CST. Higher order displacement
displacement functions for triangles, and quadrilateral elements have been introduced into soil dynamic analyses by Finn and Miller (1971), and Seed (1969) respectively.

The plane quadrilateral isoparametric element is used in the present analysis. The plane quadrilateral element possesses eight degree of freedom, namely two translational degrees of freedom at each of the four corner nodes. The term 'isoparametric' is derived from the use of the same interpolation functions to map the quadrilateral element shape as are used to define displacements within the element. The quadrilateral element need not be rectangular but can be of any arbitrary shape. This feature allows irregular shaped structures to be modelled without difficulty. As well, the four node element may be specified such that the location of any two adjacent nodes is the same, making the element triangular. The stiffness matrix of a triangular element, obtained treating it as a four node element and following the procedure for isoparametric elements, reduces to that of a CST. Or in other words, by specifying the node number of the third node to be the same of the node number of the fourth node, the element is actually a CST. From this result, quadrilaterals and/or CST's can be used in the analysis. The formulation of the stiffness matrix for plane quadrilateral isoparametric element is presented in Appendix I.

The dynamic finite element analysis presented in this thesis consists of three major parts
a. Modelling the constitutive behavior of soil materials.

b. Formulation of stiffness, mass and damping matrices

c. Solution of the equations of motion

Parts a and b will be discussed in the remaining sections of this chapter. Part c will be discussed in chapter 4.
3.2 Constitutive relationships

A fundamental assumption made in the analysis is that soil behaves isotropically. This allows two-dimensional stress-strain behavior to be described by two elastic parameters during each increment. In the present analysis, the shear modulus $G$ and the bulk modulus $B$ were selected as the elastic parameters for the following reasons.

Dynamic loading is essentially due to vertically propagating shear waves. The shear waves induce dynamic shear stresses and shear deformations on the deposit. Therefore, shear deformations make up a great part of the overall displacements. Bearing in mind that the 'failure condition' should be based on the magnitude of residual displacements that can be tolerated. A proper stress-strain law should control shear deformation in order that a prediction on failure can be made. Based on this argument, it was decided that shear modulus be taken as one of the elastic parameters, which can be reduced if near failure conditions occurs using a simple hyperbolic model.

During dynamic loading a soil element often experiences near failure condition, i.e., reaches near maximum shear strength. During this condition, higher shear deformation can occur without an accompanying increase in volumetric strain as the bulk modulus remains constant. This behavior is observed in laboratory tests on clay and sand. In a stress-strain model, $E$ and $\nu$ can be used as the variables. And from elasticity, we
know,

\[
B = \frac{E}{3(1-2\nu)} \quad (3-2)
\]

Near the failure condition, where \(E\) is reducing, if \(\nu\) is kept constant, from equation (3-2) \(B\) will reduce, resulting in an increase in volumetric strain. This can be avoided by varying \(\nu\) to keep \(B\) constant (Byrne et al 1982). It is simpler to use the bulk modulus directly as one the elastic parameters. As well, relationships for determining bulk modulus values have been well developed from extensive investigations by Duncan et. al. (1980).

In order to arrive at expressions describing the stress-strain behavior of an idealized soil element under general loading it is assumed that hyperbolic shear stress-shear strain relationship similar to that used by Martin, Finn and Seed (1975), and that volume change behavior as proposed by Duncan et. al. (1980) is followed. Tangent values of shear modulus is varied with the level of both the shear strain and the mean normal stress. Values of bulk modulus is varied with the level of mean normal stress only. These relationships are discussed in the following sections.
3.2.1 Bulk Modulus stress dependency

Studies by Duncan et. al., (1980) have shown that the volume change behavior of most soils can be modelled reasonably accurately by assuming that the bulk modulus of the soil varies with confining pressure, and is independent of the percentage of strength mobilized. Values of B have been found to increase with increasing confining pressure and in terms of the mean normal pressure can be approximated by the equation,

\[ B = K_b \cdot P_o \left( \frac{\sigma_m}{P_o} \right)^m \] (3-3)

where,

- \( B \) = tangent bulk modulus
- \( K_b \) = bulk modulus parameter
- \( m \) = bulk modulus exponent
- \( \sigma_m \) = mean normal effective stress
- \( P_o \) = atmospheric pressure, included to have a non-dimensional equation, expressed in units consistent with \( \sigma_m \) and \( B \).

The tangent bulk modulus is defined by

\[ B = \frac{\Delta \sigma_1 + \Delta \sigma_2 + \Delta \sigma_3}{3 \Delta \epsilon_v} \] (3-4)

where \( \Delta \sigma_1 \), \( \Delta \sigma_2 \), and \( \Delta \sigma_3 \) are the changes in the values of the principal stresses, and \( \Delta \epsilon_v \) is the corresponding change in volumetric strain, which for the plane strain condition is,

\[ \Delta \epsilon_v = \Delta \epsilon_x + \Delta \epsilon_y \] (3-5)

alternatively equation (3-4) can be written,
\[ B = \frac{\Delta \sigma_m}{\Delta \varepsilon_N} \]  

(3-6)

where, \( \Delta \sigma_m \) is the change in mean normal stress.

As will be discussed in greater detail in chapter 4, the non-linear response of earth structure to seismic motions is solved using the Newmark step-by-step integration procedure. This procedure computes displacements and strains at the end of consecutive time intervals \( \Delta t \) assuming the material properties have been held constant during the increment. At the end of a time interval the values of bulk moduli must be updated for the effects of stress change. The bulk modulus for an soil element at any time can be determined as follows:

1. At time \( T \), the bulk modulus \( B \) from equation 3-3 as \( \varepsilon_N \) is known.

2. During time step \( \Delta t \) using bulk modulus values calculate \( \varepsilon_N \) at time \( T + \Delta t \).

3. At time \( T + \Delta t \) calculate the new bulk modulus \( B \).
3.2.2 **Hysteretic hyperbolic Shear stress-strain Relationship**

A comprehensive survey of the factors affecting the shear moduli of soils and expressions for determining this property have been presented by Hardin and Drnevich (1970). In their study, an empirical equation was presented to determine the values of maximum shear modulus $G_{\text{max}}$. Their equation is as follows:

$$G_{\text{max}} = 320.8 \cdot P_0 \left( \frac{2.973-e}{e} \right)^2 \cdot (\text{OCR})^a \left( \frac{a_m}{P_0} \right)^{1/2}$$  \hspace{1cm} (3-7)

where,

- $G_{\text{max}}$ = Maximum shear modulus
- $e$ = void ratio
- OCR = overconsolidation ratio
- $a$ = parameter that depends on the plasticity index of the soil
- $a_m$ = mean normal effective stress.

For clays, Seed and Idriss used an equation of the form,

$$G_{\text{max}} = (S_u) \cdot \text{(constant)}$$  \hspace{1cm} (3-8)

where $S_u$ is the undrained shearing strength of the clay

Laboratory and in-situ test data performed by several investigators have found the constant value to vary from 1000 to 3000.

For the initial loading phase the hyperbolic shear stress-strain relationship formulated by Kondner and Zelasko (1963) to
model the response of granular soil in simple shear is used. The same hyperbolic relationship is used for clays, as cyclic triaxial tests performed by Idriss et. al. (1978) on clays has shown this behavior. The effect of the static shear stress is included. For the assumed isotropic material, in general two dimensional loading, the initial response up to the first reversal to given by the equation, and is shown in Figure 3-1. The starting point on the stress-strain curve is \((0, \tau_{st})\).

\[
\tau = \frac{G_{max} \gamma_{max}}{1 + \frac{G_{max} \gamma_{max}}{\tau_{ult} + \frac{\tau_{st}}{R_f}}} + \tau_{st}
\]  

(3-9)

where,

\(\tau\) = shear stress

\(\gamma_{max}\) = max shear strain

\(\tau_{ult}\) = the shear strength of the soil

\(\tau_{st}\) = the static shear stress of the soil

\(R_f\) = the failure ratio

The - sign and + sign are applied to loading in the positive and the negative direction respectively.

\(\gamma_{max}\) is the second invariant of strain, as determined by,

\[
\gamma_{max} = \pm \sqrt{\left(\gamma_{xy}\right)^2 + \left(\varepsilon_x - \varepsilon_y\right)^2}
\]

(3-10)

where,

\(\varepsilon_x\) = the normal strain in the x-direction

\(\varepsilon_y\) = the normal strain in the y-direction

\(\gamma_y\) = the shear strain in the x-y plane.

\(\gamma_{max}\) is given the same sign as \(\gamma_{xy}\).
FIG. 3-1 HYPERBOLIC SHEAR STRESS-SHEAR STRAIN RELATIONSHIP
For unloading and reloading, the Masing (1926) type of hysteretic behavior has been used by Lee (1977), in the development of a one-dimensional dynamic effective stress analysis for saturated sand deposits. Briefly described here, the Masing criterion assumes that if the initial loading curve, or skeleton curve, which is described by equation (3-9) can be represented by,

\[ \tau = f(\gamma_{\text{max}}, \tau_{\text{st}}) \]  

where \( \tau_{\text{st}} \) is set to zero  \hspace{1cm} (3-11)

Then the unloading or reloading curve can be obtained from,

\[ \frac{\tau - \tau_r}{2} = f(\gamma_{\text{max}} - \gamma_r) \quad (3-12) \]

where \((\gamma_r, \tau_r)\) is the last reversal point in the stress-strain plot. Geometrically equation (3-12) means that the unloading and reloading branches of a hysteretic loop are the same skeleton curve with both the stress and strain scales increased by a factor of two and the origin translated to the reversal point. The tangent modulus after stress reversal is equal to \( G_{\text{max}} \). Further to this Lee assumes that, if the stress-strain curve described by equation (3-12) intersects an extension of the skeleton curve, the stress-strain path follows the skeleton curve until there is a reversal of loading again. If the stress strain curve intersects the curve of the previous load cycle, the stress-strain path then follows the latter stress-strain curve.

The stress-strain model requires that previous stress reversal points for each soil layer or element be recorded, in
order to determine whether intersection of the skeleton or previous unloading or reloading stress strain curves have occurred. Because very little experimental work has been performed on sands or clays to verify the Lee's assumptions, it may be more reasonable to simplify the reloading and unloading behavior.

In the present analysis, if the initial loading as described by equation (3-9) is represented by,

\[ \tau = f(\gamma_{\text{max}}, \pm \tau_{\text{st}}) \]  

Then it is assumed that the unloading and reloading curve can be obtained from,

\[ \tau = f(\gamma_{\text{max}} - \gamma_r, \pm \tau_r) \]  

Consider a soil element being strained from its undeformed state to a shear strain level \( \gamma_A \), with corresponding shear stress \( \tau_A \). The element is now unloaded from that point. The stress-strain diagram is shown in Figure 3-2. Curve OA is given by (3-9) while curve AB is given by equation (3-14), where the reversal point \((\gamma_r, \tau_r)\) is \((\gamma_A, \tau_A)\). Equation (3-14) has the effect of resetting the origin of the stress-strain curve at the reversal point \((\gamma_A, \tau_A)\), and the strength of the soil equal to the value \(\tau_{\text{ult}} + \tau_A\). Upon another reversal at point B the new strength asymptote is equal to \(\tau_{\text{ult}} + \tau_B\). This simplified hyperbolic unloading and reloading is essentially hysteretic in nature.
FIG. 3-2 HYPERBOLIC RELATIONSHIP UNDER GENERAL LOADING
The shear modulus at any time, is the value of the tangent of the stress-strain curve at the stress-strain point corresponding to that time. The tangent shear modulus is calculated by evaluating the differential of the equation describing the initial or after reversal hyperbolic curve, whichever is appropriate at that instant in time.

3.3 Formulation of structure stiffness, mass and damping matrices

3.3.1 Partial Stiffness Matrices

An element stiffness matrix is, for a given geometry, a linear function of the terms in the stress-strain matrix \([D]\), \((\{\delta \sigma\} = [D]\{\delta \varepsilon\})\). In general the \(D\) matrix is a full 6 by 6 matrix, with 36 independent terms. For stability, the element stiffness matrix must be positive definite, this requires the \([D]\) matrix to be as well. As mentioned earlier, the present analysis considers isotropic soil behavior under the restricted but practical case of plane strain. Thus only 6 of the 21 independent terms (which may be expressed in terms of two material parameters) are relevant in the present study. The isotropic plane strain linear elastic relation between stress and strain can be written as:

\[
\{\sigma\} = [D]\{\varepsilon\} \quad (3-15)
\]

For an increment change this can be written as:

\[
\{\Delta \sigma\} = [D]\{\Delta \varepsilon\} \quad (3-16)
\]

where
\{\Delta \varepsilon\} the incremental strain vector \((\Delta \varepsilon_x, \Delta \varepsilon_y, \Delta \gamma_{xy})\)

\{\Delta \sigma\} is the incremental stress vector \((\Delta \sigma_x, \Delta \sigma_y, \Delta \tau_{xy})\)

For the plane strain condition the \([d]\) matrix is commonly written as,

\[
[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\]

(3-17)

where \(D = \) Young's modulus and \(\nu = \) Poisson's ratio.

For the simple isotropic case where there is only one independent modulus \(E\), the Young's modulus with a constant Poisson's ratio (Duncan et. al. 1980), non-linear analysis programs did not do any more than merely change the \(D\) matrix by multiplying by a constant. The stiffness matrix of an element is commonly written as:

\[
[k] = \int_{\text{Area}} [B]^T [D] [B] d\text{Area}
\]

(3-18)

\([B]\) is the displacement matrix for the isoparametric element, Appendix I.

Likewise, it can be seen that the new stiffness matrix at each load step is obtained merely by multiplying by a constant. A slightly more complex case is isotropic conditions with varying Poisson's ratio and Young's modulus. The new \([D]\) cannot be obtained directly by a multiplication of the old \([D]\). From
equation (3-18), each element stiffness matrix must be regenerated at each load step. For the type of analysis undertaken, this would be a quite expensive process. E and \( \nu \) are not independent terms of the \([D]\) matrix: we cannot write,

\[
[D] = E \begin{bmatrix} \frac{dD}{dE} \\ \frac{dD}{d\nu} \end{bmatrix} + \nu \begin{bmatrix} \frac{dD}{dE} \\ \frac{dD}{d\nu} \end{bmatrix}
\]

We can, however, write,

\[
[D] = B \begin{bmatrix} \frac{dD}{dB} \\ \frac{dD}{dG} \end{bmatrix} + G \begin{bmatrix} \frac{dD}{dB} \\ \frac{dD}{dG} \end{bmatrix}
\] (3-19)

where \( G \) and \( B \) are the shear and bulk modulus, and,

\[
\begin{bmatrix} \frac{dD}{dB} \\ \frac{dD}{dG} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \frac{dD}{dB} \\ \frac{dD}{dG} \end{bmatrix} = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(3-20),(3-21)

The above matrices are commonly referred to as the constitutive patterns for the G, B model.

Since the element stiffness matrix is a linear function of the terms in \( D \), and since the terms in \( D \) are a linear function of the independent moduli \( G \) & \( B \), it follows that the element stiffness matrix (for a given geometry) is a linear function of the independent moduli \( G \) & \( B \). Thus we can write, for the G-B model,

\[
\]

(3-22)

Where \( G \) and \( B \) are tangent values at the beginning of an increment and evaluated according to constitutive relationships developed in chap 3.2.
$S_B$ is the element 'partial stiffness matrix' for $B=1$ and $G=0$, and $S_G$ is element partial stiffness matrix for $B=0$ and $G=1$. $S_B$ and $S_G$ are obtained from:

\[
[S_B] = \int_{\text{Area}} [B] \left[ \frac{dD}{dB} \right] [B] \text{dArea} \quad (3-23)
\]

\[
[S_G] = \int_{\text{Area}} [B] \left[ \frac{dD}{dG} \right] [B] \text{dArea} \quad (3-24)
\]

For isoparametric quadrilaterals, the integrals (3-23) and (3-24) have to be evaluated numerically. The procedure is outlined in Appendix I.

The advantage of the partial stiffness approach is that the shear and bulk stiffness matrices for each element need be evaluated just once. At each load step, an element stiffness matrix is obtained by first multiplying their partial stiffness matrices by their respective modulus values and adding the two matrices, as indicated by equation (3-22).

It should be noted that for isoparametric elements the strains and therefore the stresses vary over the element and consequently the moduli are not constant throughout the element. If this was taken into consideration the partial stiffness approach would not be valid and the re-generation of the element stiffness matrix at each step would be required. Instead the stresses and strains at the element centroid were taken as being representative of the values in the entire element and the
moduli appropriate only to the centroid was used to describe the stress-strain behavior of the entire element. This assumption allows partial element stiffness matrices to be represented in their present form, equations (3-23) and (3-24).

3.3.2 Mass Matrix

It is possible to develop a mass matrix for each finite element which is consistent with the adopted displacement interpolation function and mass distribution within the element (Cooke 1975). The resulting mass matrix is like its element stiffness matrix in that it is banded and possesses coupling properties. A lumped mass approximation, where the mass of the elements is assumed to be concentrated at the nodal points, leads to a diagonal mass matrix and no coupling. When the diagonal mass matrix is used, Lsymer (1979) has observed that the rotational inertia of the individual elements is overestimated, which results in an underestimation of the highest natural frequencies of the system. On other hand, the consistent mass matrix leads to an overestimation of the same order of magnitude of the highest natural frequencies. For site response problems where the response is governed greatly by the lower natural frequencies either approach for deriving a mass matrix may be suitable. Good results have been shown by Penzien (1969) when using the lumped mass approximation. It has the advantages of savings in computer storage and computation time required for matrix calculations.
For the present analysis, one-third of the mass of each triangular element and one-fourth of the mass of each quadrilateral element are lumped at their respective nodes. The mass of any one node is the sum of the contributions of its surrounding elements to that particular node. The matrix $[M]$ is as follows:

$$[M] = \begin{bmatrix}
  m_1 & 0 & 0 & 0 & 0 & 0 & -0 \\
  0 & m_2 & 0 & 0 & 0 & 0 & -0 \\
  0 & 0 & m_3 & 0 & 0 & 0 & -0 \\
  0 & 0 & 0 & m_4 & 0 & 0 & -0 \\
  0 & 0 & 0 & 0 & m_5 & 0 & -0 \\
  0 & 0 & 0 & 0 & 0 & m_6 & -0 \\
  -0 & -0 & -0 & -0 & -0 & -0 & -0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -m
\end{bmatrix}$$

where $m$ is the mass of the node associated with the $i$th degree of freedom, and $n$ is the total number of degrees of freedom.

3.3.3 Damping Matrix

When vibrational energy is being transmitted through a material medium, a portion of its energy is dissipated internally due to a number of mechanisms. One of these is a viscous type of damping. This dissipation of energy causes a decrease in the amplitude of vibration and can be broadly termed 'material damping'. In the equivalent linear analyses by Seed
(1969) and Lsymer (1969) where non-linear hysteretic material is represented by a linear visco-elastic model, material damping has to be introduced artificially through a frequency dependent viscous type damping. In truly non-linear analysis, as is in the present analysis, where a hysteretic stress-strain relationship is used, artificial viscous damping is no longer needed to model the different types of material damping. Damping is introduced into the present analysis to describe the actual viscous effects due to the presence of water in the soil grains. Studies by Finn et. al., (1979) have shown that some degree of viscous damping is required to stabilize systems at reversal points, where there are abrupt changes in modulus values.

Damping expressions introduced by Rayleigh to produce orthogonality in modal superposition solutions to dynamic structure response, have been used in equivalent linear analyses by Seed (1960). A Rayleigh type damping expression (although the orthogonality feature is not important in the present analysis) is used to model viscous effects in the present analysis. The following relationship is used for each element;

\[
[c]_e = a [m]_e + b [k]_e
\]  

(3-26)

in which \([c]_e\), \([m]_e\) and \([k]_e\) are the damping, mass and stiffness matrices respectively for element \(e\). The element stiffness matrix corresponds to its value at time= 0. The parameters \(a\) and \(b\) are given by:
\[ a = \lambda_e \omega_1 \]  
\[ b = \lambda_e \omega_1 \] (3-27) (3-28)

The value of \( \lambda_e \) expressed in percentage of critical damping represents the damping ratio for element \( e \). The parameter \( \omega_1 \) is equal to the fundamental frequency of the system and is evaluated by the analysis at time \( t=0 \) (or based on \( G_{\text{max}} \)). The complete damping matrix \([C]\) of the entire structure is obtained from the individual element damping matrices by the direct stiffness method. Given the form of equation (3-26), element damping matrices and therefore the structure damping matrix is assumed to remain constant during dynamic loading.
CHAPTER 4
NUMERICAL ANALYSIS OF NON-LINEAR DYNAMIC RESPONSE

4.1 General

In this chapter the numerical technique used to solve the finite element modelling of the non-linear dynamic response of an earth structure to earthquake motions is presented. In many practical cases a state of plane strain can be assumed so that for analysis purposes three dimensional structures can idealized by a finite element system representing the cross section of the structure. Modelling of the cross section by a finite element system of triangles and quadrilaterals requires, the formulation of constitutive stress strain behavior of the soil material, and the derivation of the stiffness and damping properties. This has been outlined in chapter 3.

In earthquake response analyses of linear structures, many early investigators have used the mode-superposition method, (Wilson (1962), Clough and Penzien (1975)). This method involves the solution of the characteristic value problem represented by the free vibration response of the system, followed by the transformation of the displacements to the mode shapes of the system. This procedure uncouples the response of the system, so that the response of each mode may be evaluated independently of the others. The second method of dynamic analysis is called the step-by-step method, and involves the direct numerical integration of the equilibrium equations in their original form.
The main advantage of the mode superposition method is that the response of a system may be obtained with good accuracy by considering only a few of the lower normal modes, while in the step-by-step method all generalized coordinates must be retained, Penzien (1969). On the other hand, the evaluation of the characteristic value problem and transformation to the mode shapes are major computational problems not required in the step-by-step method. Recent investigators (Seed and Idriss 1969) have used an equivalent linear viscoelastic iterative method to approximate the non-linear solution. The dynamic response solved in the frequency domain is based on the assumption of linear structural behavior, cannot be used for a non-linear system.

The step-by-step integration method, on the other hand can be applied to non-linear systems. In this approach, the response is calculated for a short time increment $\Delta t$, starting with known conditions, to evaluate the conditions at a later time. The incremental linear nature of the system is considered by assuming linear behavior throughout each successive time step, and by making proper modifications to the linear properties prior to each step. In the present study, where the stiffness properties of the structure behave non-linearly, being strain and stress dependent, the step-by-step integration method is used.

The complete response is obtained by using the known displacement, velocity and acceleration at the end of one time
interval as the initial conditions for the next interval. The process is continued step-by-step from initial static conditions to the completion of seismic motions. The dynamic equilibrium condition is satisfied at the beginning and end of each time interval.

The equations of motions for non-linear structures, together with their solution by a step-by-step integration procedure will be presented in the following sections.

4.2 Equation of motion

As mentioned previously it is assumed that the earthquake ground motion is identical at all points along the base of the structure. Spatial variations in the ground motion are not considered in the present analysis.

The dynamic equation of motion of nodal points above the rigid base for the finite element system when subjected to earthquake ground motion can be expressed in the matrix form:

\[ [M] \{\ddot{x}\} + [C] \{\dot{x}\} + [K] \{x\} = \{R\} \]  \hspace{1cm} (4-1)

Details of the derivation is given in Zienkiewicz (1971) and Newmark and Rosenblueth (1971). The matrices are defined as follows:

- \([M]\) = the mass matrix (section 3.3.2)
- \([C]\) = the damping matrix (section 3.3.2)
- \([K]\) = the stiffness matrix (section 3.3.2)
\{x\} = \text{vector of nodal displacements relative to base}
\{R\} = \text{inertia force vector}

For the lumped mass system the inertia force vector \{R\} is

\[ \{R\} = -\{M\}_x \ddot{x}_b - \{M\}_y \ddot{y}_b \quad (4-2) \]

where,

\[ \{M\}_x = (m, 0, m, 0, ... , 0, 0) \quad (4-3) \]

and

\[ \{M\}_y = (0, m, 0, m, ... , 0, 0) \quad (4-4) \]

\{M\}_x \text{ and } \{M\}_y \text{ are the vectors of the mass of the nodes associated with the x and y degrees of freedom respectively (m is the mass of the node associated with the ith degree of freedom which is in the x direction). Therefore, } \{M\}_x + \{M\}_y \text{ is diagonal of the mass matrix } [M]. \ddot{x}_b \text{ and } \ddot{y}_b \text{ are respectively the horizontal and vertical components of the ground acceleration.}

The stiffness matrix for each finite element during any time interval of the step by step method is obtained by the procedure described in chapter 3.3 and in appendix I. The complete stiffness matrix [K] of the entire structure is obtained from the individual element stiffness matrices by the direct stiffness method. In a similar manner the structure damping matrix is obtained. These matrices are of order N x N,
where $N$ is the number of degrees of freedom. Since the matrices are banded and symmetric, only matrices of size $N \times M$ need be considered, where $M$ is the half bandwidth of the structure. This greatly reduces the computer storage requirements.

### 4.3 Incremental equation of motion

Equation (4-1) must be satisfied at every instant in time.

Let $T = t + \Delta t$, where $\Delta t$ is a small time interval then,

$$\begin{align*}
[M]_t \{\ddot{x}\}_t + [C]_t \{\dot{x}\}_t + [K]_t \{x\}_t &= \{R\}_t \\
[M]_T \{\ddot{x}\}_T + [C]_T \{\dot{x}\}_T + [K]_T \{x\}_T &= \{R\}_T
\end{align*}$$

(4-5)

(4-6)

The subscript refers to the instant of time at which the particular quantity takes on its value. Since the mass matrix is constant matrix,

$$[M]_T = [M]_t = [M]$$

so that,

$$[M]_T \{\ddot{x}\}_T - [M]_t \{\ddot{x}\}_t = [M]_T (\{\ddot{x}\}_T - \{\ddot{x}\}_t) = [M] \{\Delta \ddot{x}\}$$

(4-7)

where,

$$\{\ddot{x}\}_T - \{\ddot{x}\}_t = \{\Delta \ddot{x}\}_T$$

(4-8)

and the damping matrix is assumed to remain constant throughout the analysis,

$$[C]_T \{\ddot{x}\}_T - [C]_t \{\ddot{x}\}_t = [C]_T (\{\ddot{x}\}_T - \{\ddot{x}\}_t) = [C] \{\Delta \ddot{x}\}_T$$

(4-9)

where,
\[
([\dot{x}]_T - \{\dot{x}\}_t) = \{\Delta \dot{x}\}_T
\]  

(4-10)

However, the \([K]\) matrix depends on the level of shear strain in the soil, as a result of the non-linear stress-strain relation, and thus equations similar to (4-7) and (4-9) can not be obtained for this matrix in a straightforward manner. There are two ways to arrive at expressions similar to (4-9) using approximation methods. A crude method is to replace \([K]_T\) with \([K]\), so that,

\[
[K]_T\{x\}_T - [K]_t\{x\}_t = [K]_t([x]_T - [x]_t) = [K]_t\{\Delta x\}
\]  

(4-11)

where,

\[
([x]_T - [x]_t) = \{\Delta x\}_T
\]

The tangent stiffness properties defined at the beginning of the time interval are used. During any time interval the \([k]\) matrix is changing as the stiffness properties change with strain. The \([K]\) matrix should represent some average stiffness properties during the time interval. This can be only done by iteration because the displacement at the end of the time increment depend on these properties. As will be explained in chapter 4.3, iteration is not always desired. For this reason the approximate method is used in the present analysis. Subtracting equation (4-5) from equation (4-6) and taking note of equations (4-7), (4-9) and (4-11), gives the incremental form of the equation of motion for the time interval starting at time \(t\), which is as follows:
Matrix equation (4-12) which is a set of second order differential equations, may be reduced to a recurrence equation if an assumption is made regarding the variation of the acceleration of each node within the time interval ∆t. To this end, numerical procedures developed by Newmark (1959) can be applied to the equation so that unknown displacements \{\ddot{x}\}_t, \{\dot{x}\}_t, \{x\}_t at time T, can be expressed in terms of known displacements \{\ddot{x}\}, \{\dot{x}\}, \{x\}_t at time t.

4.4 Step by Step Integration

In Newmark's β method of step by step integration two parameters, α and β are used so that the velocity and displacement at time T can be expressed in terms of the acceleration, velocity and displacement at time t, and of unknown acceleration at time T. The equations for velocity and displacement at time T are as follows:

\[
\{\dot{x}\}_T = \{\dot{x}\}_t + (1-\alpha)\Delta t\{\ddot{x}\}_t + \alpha\Delta t\{\ddot{x}\}_T
\]  
\[\text{(4-13)}\]

\[
\{x\}_T = \{x\}_t + \Delta t\{\dot{x}\}_t + (.5-\beta)(\Delta t)^2\{\ddot{x}\}_t + \beta(\Delta t)^2\{\ddot{x}\}_T
\]
\[\text{(4-14)}\]

\(\Delta t\) is refered to as the 'time step' of in the integration.

Newmark (1959) proposed for a unconditionally stable integration procedure that \(\alpha = 1/2\) and \(\beta = 1/4\). This
corresponds to a constant average acceleration method of integration. When values of \( a = 1/2 \) and \( \beta = 1/6 \) are used, a linear variation of acceleration is assumed over the time increment. The linear variation of acceleration method was proposed by Wilson and Clough (1962). Both approaches have been incorporated into the computer program and are options available to the user.

The linear variation of acceleration method may lead to an instability of the solution. This instability is usually dependent on the size of the time step \( \Delta t \) used in the integration, material properties, and size of the finite element grid. The Wilson \( \theta \) method which provides stability in the solution (Wilson et al, 1973) has been written into the computer program.

If the following simplifying expressions are used,

\[
{a}_i = \Delta t \{\ddot{x}\}_i, \quad (4-15)
\]

\[
{b}_i = \Delta t \{\ddot{x}\}_i + .5(\Delta t)^2 \{\ddot{x}\}_i, \quad (4-16)
\]

then equation (4-13) and equation (4-14) can be written as:

\[
\{\Delta \ddot{x}\}_T = \{a\}_i + a\Delta t \{\Delta \ddot{x}\}_T. \quad (4-17)
\]

\[
\{\Delta x\}_T = \{b\}_i + \beta(\Delta t)^2 \{\Delta \ddot{x}\}_T \quad (4-18)
\]

Equations (4-17) and (4-18) are substituted into equation (4-12). Transferring all forms associated with known values to the right hand side leads to the following equation,

\[
[D]_T\{\Delta \ddot{x}\}_T = [P]_T \quad (4-19)
\]
where,

\[ [D]_T = [M] + a\Delta t[C] + \beta(\Delta t)^2[K], \]

and

\[ [P]_T = \{R\}_T - \{R\}, - [C]\{a\}, - [K]\{b\}, \]

Equation (4-19) is equivalent to a static increment equilibrium relationship and may be solved by matrix inversion and multiplication. Consequently,

\[ \{\Delta\dot{x}\}_T = [D]_T^{-1}[P]_T \]

with incremental displacements \(\{\Delta x\}_T\) and velocities \(\{\Delta\dot{x}\}_T\) obtained from equations (4-17) and (4-18).

This numerical analysis procedure includes two significant assumptions: (1) the acceleration is assumed to vary in some described manner, (as determined by the values of the \(a\) and \(\beta\)), during the time step and (2) the stiffness properties of the structure remain constant during any time step and are equal to its values at the beginning of the time step. Neither of the two assumptions are strictly correct but can be viewed as good approximations when the time step \(\Delta t\) is chosen to be small. Because the structure damping matrix is assumed to remain constant for the duration of the analysis, inherent errors arising from assumption (2) are not present in the modeling of damping properties.
As discussed in Chapter 3.3, the stiffness property of the soil is determined by the tangent shear modulus which is varied with the level of shear strain and mean normal stress, and by the tangent bulk modulus which is varied with the mean normal stress only. Element shear strain and mean normal stress values calculated at the end of the time step are used to compute moduli according to their constitutive relations. The \([K]\) matrix for the next time step is based on these newly calculated moduli. In this way the inelastic behavior of soil is modelled by an incremental linear approach.

In non-linear problems true convergence seldom occurs, in that the incremental forces applied are not equilibrated by the incremental stresses. In order to resolve this inconsistency an error correction is incorporated into the analysis. This is discussed in the next section.

4.5 Error Correction

A tangent stiffness method has been used to formulate the incremental finite element matrix equation (4-12). With this method there are basically three approaches that can be taken.

1. Accept the stresses:
   Accept \( \{\delta \sigma\} = [D_L]\{\delta \epsilon\} \), where \([D_L]\) is the tangent stress-strain matrix used for the last iteration, see equation (3-20). Calculate the new \([D_L]\) based on the stress level attained. In this approach equilibrium is maintained
throughout the analysis, however the true stress-strain curve is not followed. The resulting inconsistency between the computed strain and exact strains is accepted.

2. Accept the strains:

Accept strains computed from nodal displacements obtained by the step by step integration procedure and calculate the new tangent modulus based on the strain level. Equilibrium is violated, but the true stress-strain curve is followed.

3. Accept the strains and apply correction forces:

The true curve is followed, and any error in equilibrium at the end of the time step is calculated and applied at the beginning of the next time step. In this way any errors in equilibrium do not accumulate.

All the above methods can benefit from multiple iterations. For example, from the initial tangent moduli used for the first integration of the time step, the tangent moduli at the end of the time step can be obtained. Iteration is then performed on the time step using the average of the two tangent moduli. This approach may be described as a step-secant approach. While iterating is desirable in terms of improving the accuracy of the incremental approach, the number of additional iterations at each time step would proportionally increase the cost of the analysis. This may prove to be expensive when a large number of elements is involved. (Although there is a trade off here in that one can generally use larger time increments if iterations are used.) For this reason, iteration is not considered in the
present analysis. Approach (3) has the advantage of following the actual stress-strain curve while at the same time satisfying equilibrium. In this sense, and when multiple iterations are not performed, approach (3) is the most accurate method. This approach has been built into the analysis and is discussed here in greater detail.

The strains determined in each integration are accepted as the true strains and the stress-strain relations are used to determine the 'true restoring' force \([K]_T \{x\}_T\). However these restoring forces do not necessarily satisfy the equilibrium equation (4-3). An set of artificial 'external' forces, \(\{P\}\), defined by the following can be applied to the system so that equilibrium is restored,

\[
\{P_{err}\} = \{R\}_T - [M]\{\ddot{x}\}_T - [C] \{\dot{x}\}_T - [K]_T \{x\}_T
\] (4-23)

The \(\{P_{err}\}\) is added to \(\{P\}_T\) for the next time increment of the analysis. All terms in the above equation are computed in a straightforward manner. The manner in which the term \([K]_T \{x\}_T\) is evaluated deserves comment. All other terms in the above equation are computed in a straightforward manner.

A basic assumption of the finite element method is that all forces are transmitted at the nodal points. The total force at any one node is the sum of the contributions of its surrounding elements to that particular node. The restoring forces \([K]_T \{x\}_T\) then, are nodal forces representing the sum of the stresses of the elements. The state of stress within an element can be
represented by a statically equivalent system of nodal point forces. From the stresses \( \sigma \), nodal forces are obtained by,

\[
\{ P \} = \iiint [B] d\xi d\eta \cdot \{ \sigma \}
\]  

(4-24)

As was the decision earlier in evaluating the stiffness matrix for an element, Chapter 3.3., the stresses and strains at the centroid of the element are taken to be representative of the average stresses and strains in the element. In this case, the stress vector can be removed from within the integral sign, as shown above.

The total force at any one node is the sum of the contributions of its surrounding elements to that particular node. The restoring forces \([K]_T \{ x \}_T\) then, are nodal forces representing the sum of the stresses of the elements. The manner in which element stresses are calculated is described below.

From the accepted strains the corresponding stresses can be determined as follows:

1. Calculate incremental strains:

\[
\{ \Delta \varepsilon \}_i = [B_i] \{ \Delta u \}_i
\]

(4-25)

where,

\( \{ \Delta u \}_i \) is the vector of incremental nodal displacements associated with element \( i \).

\([B_i]\) is the value of the strain displacement matrix element \( i \), at its centroid, see Appendix I.
\{\Delta \varepsilon\}_i \text{, the incremental strain vector (} \Delta \xi, \Delta \xi_y, \Delta \gamma_y \text{) for element } i.

2. Calculate increment stresses:
   \[
   \{\Delta \sigma\}_i = [D_L]\{\Delta \varepsilon\}_i
   \]  \hspace{1cm} (4-26)
   where,
   
   \[D_L\] is the stress-strain matrix, based on tangent shear and bulk moduli, of element \(i\), during the interaction time step, see equation (3-20).
   \{\Delta \sigma\}_i \text{ is the incremental stress vector (} \Delta \sigma_x, \Delta \sigma_y, \Delta \tau_{xy} \text{) for element } i.

3. Calculate total element stresses:
   For any element, stresses at time \(T\) are equal to the summation of the incremental stresses up to time \(T\).
   They are obtained as follows:
   \[
   \{\sigma\}_T = \{\sigma\}_t + \{\Delta \sigma\}_T
   \]  \hspace{1cm} (4-27)
   where,
   \(T = t + \Delta t\)
   \{\sigma\}_t \text{ is the total stress } (\sigma_x, \sigma_y, \tau_{xy}), \text{ at time } t
   \{\Delta \sigma\}_T \text{ is the incremental stress for the last time step

4.6 Procedure for strain reversal occurrence

A complete description of the stress-strain relationship under general loading and unloading has been given in Chapter 3.2.2. On unloading or reloading of a soil element, the shear stress-shear strain behavior of the material is governed by a
newly defined curve, equation (3-14), where its origin is located at the shear stress shear strain reversal point, \((\gamma, \tau_r)\), see Figure 3-2. For two reasons it is important that the analysis can determine when strain reversal occurs: (1) upon reversal a new stress-strain branch is followed, and (2) there is a discontinuity of shear modulus at any strain reversal location.

Shear strain, as defined in Chap 3.2.2 is re-written here for convenience,

\[
\gamma_{\text{max}} = \pm \sqrt{(\gamma_{xy})^2 + (\varepsilon_x - \varepsilon_y)^2} \quad (4-28)
\]

The rate of shear strain, differentiating equation (4-29) with respect to time, is defined by,

\[
\dot{\gamma}_{\text{max}} = \frac{(\gamma_{xy})(\gamma_{xy}) + (\varepsilon_x - \varepsilon_y)(\dot{\varepsilon_x} - \dot{\varepsilon_y})}{\gamma_{\text{max}}} \quad (4-30)
\]

Based on displacement and velocity values \(\{x\}\) and \(\{\dot{x}\}\), element strains and shear strain rates are calculated by,

\[
\{\varepsilon\}_i = [B_i]\{u\}_i \quad (4-31)
\]

where,

\(\{u\}_i\) is the vector of nodal displacements associated with element \(i\).

\(\{\varepsilon\}_i\) is the strain vector \((\varepsilon_x, \varepsilon_y, \gamma_{xy})\) for element \(i\).

and,

\[
\{\dot{\varepsilon}\}_i = [B_i]\{\dot{u}\}_i \quad (4-32)
\]

where,
\{\dot{u}\}_i is the vector of nodal velocities associated with element \(i\).

\{\dot{\varepsilon}\}_i = \text{ the rate of strain vector } (\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\gamma}_{xy}) \text{ for element } i.

Strain reversal can be defined as a decrease from the most recent \(|\gamma_{\text{max}}|\) value, see Figure 3-2. A local maximum or minimum in the value of shear strain occurs when \(\gamma_{\text{max}}\) is equal to zero. Therefore, strain reversal occurs in an element when there is a change in sign in the rate of shear strain \(\dot{\gamma}_{\text{max}}\) from one time step to the next time step.

Let \(T = t + \Delta t\)

Strain reversal occurs when,

\[\dot{\gamma}_{\text{max}T} \cdot \dot{\gamma}_{\text{maxT}} < 0 \quad (4-33)\]

where, the subscript refers to the instant time at which the particular quantity takes on its value.

Now that a criterion for strain reversal has been established, one further point should be considered. Since a reversal point is characterized by a discontinuity in modulus, it may be desirable to reduce the time interval \(\Delta t\) of the analysis with subsequent iteration when they occur, in order to obtain a better accuracy when turning these 'points'. Ideally the time step of the analysis should be reduced sufficiently in order to land right on them. With earlier investigations by Salgado (1980) on a one degree of freedom, elastic-plastic system, this was easily achieved. With multi-degree of freedom
systems, where more than one strain reversal (in different elements or layers) may occur within a time interval and not necessarily at precisely the same time, and where strain reversal in one part of the system and may later in the same time interval cause strain reversal in another part of the system, cost considerations make this approach impractical.

A reasonable approach which does require landing on reversal points would be to make a prediction regarding the segments of the time interval for which strain reversal has and has not occurred. Within the time interval of strain reversal, this can be approximated by assuming a linear variation in acceleration. The time of reversal can be expressed as,

$$\Delta t_1 = \frac{\gamma_{\text{max}T}}{\gamma_{\text{max}T} - \gamma_{\text{max}1}} \cdot \Delta t \quad (4-34)$$

and, \( \Delta t_2 = \Delta t - \Delta t_1 \quad (4-35) \)

where,

- \( \Delta t_1 \) is the time segment 'before' strain reversal
- \( \Delta t_2 \) is the time segment 'after' strain reversal

This approximation can be used on strain reversal elements, with iteration performed for the time interval segment corresponding to the strain reversed element with the shortest \( \Delta t_1 \) time interval. After the integration for this time segment has been completed, strain reversed modulus is assigned to the appropriate element, and the integration is performed on the remaining portion of the time interval. The strain reversal check procedure and subsequent segmenting of \( \Delta t_2 \) further into
two smaller segments if reversal occurs, is performed on $\Delta t_2$. This type of approach has been cost efficient when one dimensional soil layer systems (max degree of freedom= 20, halfbandwidth= 2) have been analysed, with the limiting of the size of $\Delta t_1$ and $\Delta t_2$ to $\Delta t/10$, (Lee, 1977). This approach can be exorbitantly expensive for typically large degree of freedom systems that result from finite element modelling.

A crude approximation to this approach can be made by estimating, on strain reversal(s), whether reversal occurred closer to the beginning or the end of the time interval. As determined from equation (4-34),

if, $\Delta t_1 < .5\cdot\Delta t$, assume strain reversal at $t$
if, $\Delta t_1 \geq .5\cdot\Delta t$, assume strain reversal at $T$

If $\Delta t_1 < .5\cdot\Delta t$ for any strain reversed elements, strain reversed moduli are assigned to these elements and the complete time step $\Delta t$ is repeated. Any additional strain reversed elements are assumed to have reversed at the end of the time step and are assigned the appropriate moduli. When strain reversal occurs closer to the end of the time interval, the strain reversed moduli are not assigned to these elements until the next time step. Iteration in this case is not necessary. With this crude approach multiple subiterations of any time step are avoided, and at most, a time step is repeated once. This procedure to handle the occurrences of strain reversal has been adopted in the present analysis.
4.7 Summary of procedure

A brief outline of solution scheme for any given time increment is given as follows:

1. Based on the current values of $\gamma$, $\tau$, at time $t$, the tangent shear modulus $G$ and tangent bulk modulus $B$ are calculated as described in chapter 3.2.1 and 3.2.2 respectively.

2. The matrix $[K]$ in equation (4-20) and (4-21) is then updated, see chapter 3.3.

3. Calculate $\{a\}_t$ and $\{b\}_t$, equations (4-15) and (4-16), substitute into equation (4-21) to set up $[P]$ and set up $[D]$.

4. Solve for $\{\Delta \ddot{x}\}_T$ using equation (4-22), and subsequently $\{\Delta \dot{x}\}_T$ and $\{\Delta x\}_T$ from equations (4-17) and (4-18).

5. Acceleration, velocity and displacements of the nodes at the end of the increment are obtained from:

   \[
   \{\ddot{x}\}_T = \{\ddot{x}\}_t + \{\Delta \ddot{x}\}_T \\
   \{\dot{x}\}_T = \{\dot{x}\}_t + \{\Delta \dot{x}\}_T \\
   \{x\}_T = \{x\}_t + \{\Delta x\}_T
   \]

6. Based on displacement and velocity values at the end of increment $\{x\}_T$ and $\{\dot{x}\}_T$, element strains and shear strain rates are calculated by,

   \[
   \{\varepsilon\} = [B_i]\{u\} \tag{4-31}
   \]

   and,

   \[
   \{\dot{\varepsilon}\} = [B_i]\{\dot{u}\} \tag{4-32}
   \]

7. Check to see if there are any strain reversals. Strain reversal is determined by,
a change in sign of:

\[ \gamma_{\text{max}}, \gamma_{\text{max}}^T \]

If there are strain reversals and they occur at the beginning of the time step, the integration is repeated for the time step, by repeating steps 1 to 8.

8. Calculate stresses according to the appropriate stress-strain law.

9. In order to bring the stress-strain point closer to the actual stress-strain curve, an artificial 'external' force, \( \{P_{\text{err}}\} \), defined by equation (4-23),

\[ \{P_{\text{err}}\} = \{R\}_T - [M]\{\dot{x}\}_T - [C]\{\dot{x}\}_T - [K]\{x\}_T \]

is calculated and added to \( \{P\} \) in step 3 for the next increment.

When step 9 has been completed, the analysis for one time increment is finished. The entire process may be repeated for the next time interval. The process can be carried out consecutively for any desired number of time increments; thus the complete response history can be evaluated for the non-linear system.

A computer program has been written based on this solution scheme and the overall flow chart is shown in figure 4-1.
Read in Data

Initialize Values

Start of do loop

Assign BULK and SHEAR Modulus for each Element at each time step

Build Master STIFFNESS and DAMPING Matrices \([M] & [C]\) at each time step

Set up Dynamic eqns \([D] & \{R\}\)

Solve for DISP., VEL., ACC.

Calculate STRAINS at the centroid of each Element

Are there any strain reversals

Yes
FIG 4-1 PROGRAM FLOWCHART
CHAPTER 5
APPLICATIONS OF THE NON-LINEAR FINITE ELEMENT METHOD

5.1 General

The non-linear dynamic response analysis developed in earlier chapters was applied to a number of soil structure systems. The same structures were also analysed using Newmark type methods to estimate movements so that a comparison could be made on the prediction of permanent displacements.

If comparison of results are to be meaningful, the analyses should be based on similar stiffness and damping characteristics. In the Newmark analysis, (Chapter 2.2) the soil is assumed to behave in a rigid plastic manner. To closely match the rigid plastic behavior, the non-linear finite element method employs a linear elastic-plastic shear stress - shear strain law. This is done by setting the $R_f$ value in the hyperbolic relationship equal to 0.01 or less. Material damping in the form of plastic deformation is inherent in both approaches. In the finite element analysis viscous damping must be included for numerical reasons. A nominal but small value of 2 percent of critical damping is used to stabilize the solution at turning points of stress-strain curves. The Makdisi-Seed procedure for predicting displacements is also performed. The matching of stiffness and damping characteristics cannot be strictly done in this case, as the dynamic response of earth structures is evaluated using an equivalent linear method. While comparison may not be as valid, Makdisi and Seed have
shown good agreement with the rigid block analogy. In view of this, their approach is included in the comparative study.

A hypothetical clay slope structure and clay dam were used in the study. Each has a height of 150 ft. and slopes of 2 to 1. The yield acceleration value of the two different structures, obtained from pseudo-static analyses (see Chapter 5.3.1) are the same. The Newmark analysis implies that given the same value of yield acceleration, the predicted displacements of the two structures will be the same. This cannot be strictly correct as there is no consideration taken for geometrical differences. The influence of the remaining soil mass of the structure on the slide mass is not inherently considered in the Newmark analysis. The displacement fields may be significantly different. In order to investigate the possible differences, the non-linear finite element method is applied to the two different soil structures.

In all the methods of analysis, the San Fernando earthquake February 9, 1971 N21E component recorded at Lake Hughes Station 12 in California was used as the digitized rigid base motion. The acceleration time history and acceleration response spectra are shown on Figure 5-1. The earthquake had a maximum acceleration of .35g and a predominant period of .18 sec. The Lake Hughes record was obtained on rock and is an appropriate free field motion to use when the base of an earth structure rest on rock. The most significant motions occur in the first 15 seconds of the record. As a large percentage of the
FIG. 5-1 ACCELERATION RESPONSE SPECTRA 
AND TIME HISTORY FOR 
SAN FERNANDO EARTHQUAKE
permanent deformation should occur within this time, (for structures subjected to this motion) a duration of 15 seconds is used in the analyses. The time interval of the digitized earthquake record is used as the time step ($\Delta t = 0.02$) of the dynamic finite element analyses.

As a preliminary verification of the non-linear method, a single quadrilateral finite element comparison with Newmark's procedure is presented in the next section.

5.2 Single element comparison with Newmark analysis

A single quadrilateral finite element resting on a rigid base with assigned static and dynamic soil properties are shown in Figure 5-2. Nodes 3 and 4 are allowed to translate in the horizontal direction only. The element behaves in a linear elastic-plastic manner to shear as noted in the earlier section of this chapter. The shear stress - shear strain characteristics of the element are shown in Figure 5-3. The static shear stress $Q_s$ can be considered a 'static bias', whereby on shaking, the accumulation of displacement will tend to be in the direction of the static force. The formulation of the mass matrix assumes the mass of the element is lumped at the nodes, see Chapter 3.3.2. The finite element under these conditions is essentially a lumped mass, spring dashpot mechanical model, where the spring and dashpot replace the shear stiffness and viscous damping of the element respectively. The movement of the lumped masses at the top nodes of the statically
FIG. 5-2 SINGLE FINITE ELEMENT MODEL

FIG. 5-3 SHEAR DEFORMATION RELATIONSHIP
biased finite element should be very similar to the sliding in the Newmark analogy of a rigid block on a inclined plane. Therefore a valid comparison can be made.

The single finite element connected to a rigid base is subjected to the San Fernando earthquake motion scaled to a maximum acceleration of .50g. The maximum displacement of the top nodes for different static shear values in the finite element method are compared with displacements obtained by Newmark (1965) in Figure 5-4. The Newmark prediction of displacements were computed for a range of resistance coefficient "N" values using the scaled San Fernando earthquake motion. The procedure is described in Chapter 2.2. The resistance coefficient corresponds to the yield acceleration of the finite element. The yield acceleration is related to its static shear stress and was computed as follows:

N was defined by Newmark as being the coefficient of acceleration acting in the proper direction to cause sliding (or in this case yielding of the finite element). hence:

\[(Q_y - Q_{st})L = NW\]  \(\text{(5-1)}\)

where \(Q_y\) = yield strength (50psf)
\(L\) = Length of element (10ft)
\(N\) = Newmark's coefficient
\(W\) = weight of the top half of the element
\((2.5 \times 10 \times 100 \text{ lb/ft}^3)\)
\(Q_{st}\) = static shear stress
FIG. 5-4 SINGLE FINITE ELEMENT
NEWMARK DISPLACEMENT COMPARISON

LEGEND

- SFE
- Newmark

\[ N = \frac{\text{Max Resistance Coeff}}{\text{Max Earthquake Acc}} \]
Typical time histories of the earthquake induced displacements computed using the two procedures are shown in Figure 5-5 for comparison.

A static bias of 43.75 lbs is used in the finite element method, corresponding to the Newmark N value of .025. As expected the displacement accumulates with time and the maximum values occur at the end of the shaking period. It may be seen that the analysis presented herein predicts displacements that are in
good agreement with the simpler Newmark rigid plastic model.

5.3 **Dynamic Response of Clay Structures**

The finite element representations of the cross sections of the clay slope structure and clay dam are shown on Figure 5-6 and Figure 5-7 respectively. The structures are assumed to rest on a rigid rock base. The dynamic response procedure requires that the pre-earthquake stresses be determined beforehand. The static analysis is performed using the finite element program SOILSTRESS developed by Byrne (1981). For each structure, the same finite element grid was used for both the static and dynamic analysis.

The non-linear nature of static shear stress-strain behavior is modelled assuming the Duncan and Chang (1970) hyperbolic relationship. A secant modulus as determined by the hyperbolic relationship is used in a one step linear elastic iterative approach. According to this method, the linear problem is solved until agreement is obtained between the strains used to compute secant moduli in each soil element and the strains developed using these assumed secant moduli.

Static and dynamic soil parameters used for the analysis of the clay structures in later sections of this chapter are shown in Table I. For saturated undrained normally consolidated clays, Duncan 1980 suggested $E_i$ (initial Young's modulus) is roughly equal to $600S_u$. Assuming a Poisson ratio .50, the initial static shear modulus $G_i$ is equal to $200S_u$. Laboratory
FIG. 5-6 FINITE ELEMENT GRID OF CLAY SLOPE STRUCTURE
Fig. 5-7 Finite Element Grid of Clay Dam
and field studies have shown that for the same soil under the
same conditions, maximum dynamic shear modulus $G_{\text{max}}$ is greater
than the static equivalent $G_i$ by a factor of 4 to 5. Accordingly, the dynamic maximum shear modulus is assigned the
value $1000S_u$. This is in agreement with the Seed et. al.,
(1969) findings, where laboratory results have shown $G_{\text{max}}$ ranges
1000 to $3000S_u$. The shear strength $S_u$ is assumed constant
(therefore making $G_{\text{max}}$ constant) throughout the clay slope and
dam structures. In the dynamic analysis, the initial value of
bulk modulus is calculated from the $G_{\text{max}}$ value and an assumed
Poisson ratio equal to .45.

### TABLE I

<table>
<thead>
<tr>
<th>Analysis</th>
<th>Shear Strength (lb/ft$^2$)</th>
<th>Unit Wt. (lb/ft$^3$)</th>
<th>Poisson Ratio</th>
<th>Failure Ratio</th>
<th>Shear modulus parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>STATIC</td>
<td>2940.</td>
<td>125.</td>
<td>.49</td>
<td>0.9</td>
<td>$n= 0$ $Kg= 278.$ $G_i/S_u= 200.$</td>
</tr>
<tr>
<td>DYNAMIC</td>
<td>2940.</td>
<td>125.</td>
<td>.45</td>
<td>.01</td>
<td>hyperbolic $G_{\text{max}}/S_u= 1000.$</td>
</tr>
</tbody>
</table>

In the non-linear method, the static stresses obtained from
the SOILSTRESS program are assumed to be the initial stress
state for the dynamic analysis. Finite elements are assigned a
static bias on their shear stress - shear strain curve equal to
their static shear stress. The initial shear modulus is the
It has been observed in the field that seismic loading of earth structures cause longitudinal vertical tensile cracks. The presence of a tensile crack allows free movement in a direction away and perpendicular to the crack plane. This free horizontal movement corresponds to free volume change. The approach in which tensile cracks are modelled is limited by the isotropic assumptions of the present analysis. Tensile crack behavior may only be properly modelled in an anisotropic manner, where Young's modulus is reduced in the direction of free movement and is kept at the original value in the direction of the crack plane. For an isotropic material, this effect can be achieved by softening of the stiffness moduli $G$ and $B$. While a vertical tensile crack may allow horizontal movement at the crack location within an element, shear stress can be carried throughout the intact remainder of the element. Therefore it would not be strictly correct to reduce $G$ throughout the entire element. As well, reducing $G$ to significantly low values would actually model a liquified condition, which is not the intent. An attempt to incorporate tensile crack behavior into the analysis was carried out by reducing only the bulk modulus values, in elements where tensile stress occur. Starting from the initial static stress state, if during shaking the minor principal stress $\sigma_3$ becomes tensile, the bulk modulus is assigned a value of 1 percent of its initial value. The bulk modulus of the failed element is left at the reduced value for the remainder of the earthquake loading.
5.3.1 **Comparison with the Newmark and the Makdisi-Seed Analysis**

The non-linear dynamic response of a clay slope structure and dam subjected to the San Fernando earthquake motion is compared against results obtained from procedures developed by both Newmark, and Makdisi-Seed.

Newmark's resistance coefficient "N" is the value of the horizontal seismic coefficient determined from a pseudo-static conventional slope stability analysis which will give a factor of safety equal to unity. Failure surface, and soil properties used in the slope stability analyses of the clay structures are shown in Figure 5-8 and Figure 5-9. The resistance coefficient (or yield acceleration) values are coincidently equal to $0.10g$ for the two clay structures. Computations of displacements using Newmark's procedure are performed using the San Fernando earthquake scaled to maximum acceleration values ranging from 0.4 to 1.0g., for the determined yield acceleration value of $0.10g$.

Refinements to allow for variation in acceleration throughout the slope and slide mass as prescribed by the Makdisi-Seed analysis are made as follows. The response of the structures when subjected to the San Fernando earthquake is evaluated using the dynamic finite element program QUAD4. The analysis employs the use of shear modulus reduction and damping ratios curves which are built into the computer program. No attempt is made to modify these curves. The soil elements are
FIG. 5-8 SLOPE STABILITY ANALYSIS ON CLAY SLOPE STRUCTURE

125pcf $\phi = 0.0$

c = 2940 psf

Slide mass

N = 0.10

NW

F.S. = 1.0

Center of Failure Circle
FIG. 5-9 SLOPE STABILITY ANALYSIS ON CLAY DAM

125pcf $\phi = 0.0$
c = 2940 psf

Slide mass
$N = 0.10$

NW
F.S. = 1.0

Center of Failure Circle
are assigned the same $G_{\text{max}}$ value of $1000S_u$. The time history of the average acceleration of the slide mass obtained from the dynamic response analysis is used to estimate permanent displacement of the slide mass. The average acceleration is the weighted average of the acceleration values of the nodes within the slide mass as given by,

$$a(t) = \frac{\sum m_i \cdot x_i(t)}{\sum m_i}$$

where $m_i$ and $x_i$ are the mass and acceleration at node $i$. The summation is taken over all the nodes within the slide mass.

The displacement for both the Newmark and Makdisi-Seed approaches are determined by integrating the effective acceleration in excess of the yield acceleration value ($0.10g$) for the duration of the earthquake motion.

In the finite element method each mode within the domain displaces a different amount. In order to compare with the Newmark and Makdisi-Seed methods an average of the final displacements of the nodes within the slide mass is used.

5.3.1.1 Clay Slope Comparison

The results from the Newmark type methods and the non-linear finite element analysis of the clay slope are in good agreement as shown in Figure 5-10. Typical displacement time histories computed using the Newmark analysis and non-linear finite element method under the same earthquake conditions are shown in Figure 5-11. The non-linear finite element
FIG 5-10 CLAY SLOPE
NON-LINEAR FINITE ELEMENT-
MAKDISI SEED-NEWMARK
DISPLACEMENT COMPARISON

\[ N = \frac{\text{Max Resistance Coeff}}{\text{Max Earthquake Acc}} \]
FIG 5-11  CLAY SLOPE  
NON-LINEAR FINITE ELEMENT  
NEWMARK ANALYSIS  
DISPLACEMENT HISTORY COMPARISON
displacement history shows an some uphill motion which the Newmark analysis ignores, and my be one reason why the Newmark displacements are larger. For the clay slope structure, the typical displacement pattern as shown by the displaced grid in Figure 5-12, is of translation or sliding of the entire embankment along a horizontal slip plane. The circular arc failure does not seem apparent. The yield acceleration obtained from a pseudo static analysis is for the failure surface with the minimum factor of safety. Therefore the yield acceleration for any other failure surface (in this case of a sliding block) would be somewhat higher value. A higher yield acceleration value would result in lower Newmark predicted displacements.

From the formulation of the dynamic finite element analysis, the forces on a structure caused by the rigid base acceleration is represented by inertia loads at the free nodes, (Chapter 3). Inertia forces for any given row of elements should be greatest for the row of elements just above the rigid base. With this consideration, the predicted behavior of sliding along the bottom row of elements (Figure 5-12) is reasonable. As well, horizontal sliding of embankment slopes have been observed by Seed (1979).

The results of the non-linear finite element analysis indicate that in general, tensile stresses first occur in elements located at the top of the embankment, and progress vertically downwards from there. This is in agreement with field observations, where vertical tensile cracks are known to
FIG. 5-12 DISPLACED GRID OF CLAY SLOPE STRUCTURE

Displacements
Magnified 30 times

Elastic-Plastic
$T = 15 \text{ secs} \quad \alpha_{\text{max}} = 0.75g$
occur in seismically loaded slopes and embankments, (Seed et al., 1973). With these tensile cracks, there is potential for additional deformations. As observed in Figure 5-12, the tensile failed elements in the upper section of the slope structure do not result in additional relative displacement of these elements. Displacement, as noted previously, is mainly attributed to the shear movement in the bottom row of elements. The isotropic assumptions in the treatment of tensile cracks in the present analysis may account for the lack of agreement between observed and predicted behavior. Though, it should be noted that with the development of tensile cracks, any static bias present may be reduced. Therefore in an anisotropic analysis, additional displacements where tensile cracks develop may not be as significant, as there is an accompanying reduction in static bias. Overall, deformations may not be substantially different than that predicted in the present analysis. There is another important consideration. Static gravity loads of the structure during the earthquake loadings is carried by dynamically softened soil material. Under these conditions, the earthquake loading will cause additional static displacements. While this may explain the observed field displacements, the static effect has not been formulated into the analysis. For this reason, agreement may not be possible.

5.3.1.2 Clay Dam Comparison

It is observed in Figure 5-13 that the non-linear method predicts markedly lower displacements for the clay dam than does
FIG 5-13  CLAY DAM
NON-LINEAR FINITE ELEMENT-
MAKDISI SEED - NEWMARK
DISPLACEMENT COMPARISON

\[ \frac{N}{A} = \frac{\text{Max Resistance Coeff}}{\text{Max Earthquake Acc}} \]
the Newmark or Makdisi-Seed method. As indicated by the displacement history in Figure 5-14, the non-linear finite element method predicts that the dam sliding mass undergoes significant negative displacements. The negative displacements can be explained in this way. It is reasonable to assume that during shaking, outward movement of a slide mass on the other embankment slope should occur. The outward movement of a slide mass on one slope will allow movement of the other slide mass in its inward direction, as resistance to movement in that direction is greatly reduced due to temporary failure of the outwarding slide mass. If this is the case, there is a tendency for overall outward displacement of the slide mass to be reduced. A typical displacement pattern of the clay dam after earthquake loading Figure 5-15, shows net outward movement of the each dam slope.

In addition, lower displacement estimates may be expected as the overall static bias on the clay dam is zero. The total of the shear stresses on the dam slopes are essentially equal and opposite. Permanent displacements should be lower than in the clay slope structure, where there is a net static bias. The Newmark type estimates of permanent displacement are overly conservative under these conditions. Displacements for symmetrical resistance (Newmark, 1965) are an order of magnitude lower than for unsymmetrical resistance. Comparison with symmetrical displacements may be more applicable in the clay dam where net static bias is zero.
FIG. 5-14 CLAY DAM DISPLACEMENT TIME HISTORY
FIG. 5-15 DISPLACED GRID OF CLAY DAM
5.3.2 Comparison of Hyperbolic and Elastic-Plastic Shear stress-strain Laws

As outlined in Chapter 3, in shear, soil behaves in a hyperbolic and hysteretic manner. The elastic-plastic shear law, while used for comparative purposes earlier in this chapter, is a simple approximation to the hyperbolic relationship. The elastic-plastic curve approximation to the hyperbolic curve is shown in Figure 5-16. Using the hyperbolic relationship, \( R_f = 1.0 \) the dynamic analysis will closer model and predict the displacements. For this reason, the clay slope structure is re-analysed, and comparison of the simpler and actual relationship is made.

The hyperbolic and elastic-plastic predictions of displacement are shown on Figure 5-17. As might be expected the two stress strain law results agree quite closely for the large disturbances where their stiffness are similar, while for small disturbances the displacements predicted by the hyperbolic shear law are greater which reflects this laws lesser stiffness at small shear strains. Typical displacement time histories are shown on Figure 5-18.
FIG. 5-16 ELASTIC PLASTIC APPROXIMATION OF HYPERBOLIC CURVE
FIG. 5-17  HYPERBOLIC ELASTIC PLASTIC DISPLACEMENT COMPARISON
CLAY SLOPE
FIG. 5-18 HYPERBOLIC ELASTIC PLASTIC
DISPLACEMENT HISTORY COMPARISON
(Clay Slope)
CHAPTER 6
SUMMARY AND CONCLUSIONS

6.1 Summary

A two-dimensional finite element method of analysis for predicting the stress and permanent displacements of earth structures to seismic loading is presented. The inelastic behavior of the soil is modelled by an incremental linear approach in which the tangent shear modulus is varied with the level of the shear strain. The shear stress strain relation was modelled by hyperbolic loading and unloading curves leading to a Masing type of energy dissipation, and by the more classical elastic-plastic constitutive law.

The equations of motion of the structure are solved using the Newmark step-by-step time integration procedure. At each time step the tangent shear and bulk modulus are evaluated. Hysteretic damping as a result of the stress-strain loading and unloading curves is inherent in the model. Viscous damping may also be included.

The non-linear dynamic response analysis was applied to a number of soil structures. The same structures were also analysed using Newmark type approaches so that a comparison could be made on the prediction of permanent displacements. In order that comparison be meaningful, the non-linear analysis is based on an elastic-plastic shear law. In order to closer model and predict displacements a clay slope was re-analysed using the
While the Newmark methods gives a single value estimate of permanent displacements, the move rigorous multi-degree of freedom analysis is desirable as the distribution of displacements within the structures can be obtained.

6.2 Conclusions

The conclusions reached from this research are as follows:

1) The Newmark methods and the non-linear finite element method for predicting single value estimates of permanent displacements of the clay slope structure are in good agreement. However, the non-linear method predicts movement along a horizontal failure surface which is different than the Newmark assumed circular arc failure and more in keeping with observed movements.

2) For prediction of displacement of dam slopes, the Newmark methods are overly conservative.

3) The simpler elastic-plastic shear law and the actual hyperbolic relationship gives displacement estimates that are of the same order.

6.3 Suggestions for Further Research

Incorporating a pore pressure model into the analysis will allow the evaluation of the very important problem of
liquefaction potential. The effective stress analysis can be used to predict displacements fields of saturated cohesionless soil structures.

Tensile crack behavior may only be properly modelled in an anisotropic manner, where Young's modulus is reduced in the direction of free movement (away and perpendicular to the crack plane) and is kept at the original value in the direction of the crack plane.

Static gravity loads carried by dynamically softened soil during earthquake loading may cause additional displacements. The effect should be modelled and included in the analysis.

If possible, correlation study with observed field data should be carried out.
BIBLIOGRAPHY


APPENDIX I

Formulation of the Element Stiffness Matrix for Plane Isoparametric Quadrilateral Elements

Consider a quadrilateral having eight degrees of freedom namely u and v at each of the four corner nodes i. An element that has straight sides but is otherwise of arbitrary shape Figure I-1a may be considered as a distortion of a 'parent' rectangular element, Figure I-1b.

A mapping function is expressed as,

\[
\begin{bmatrix}
  x \\ y
\end{bmatrix} = 
\begin{bmatrix}
  N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
  0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix} 
\begin{bmatrix}
  x_1 \\ y_1 \\ x_2 \\ \vdots \\ \vdots \\ y_4
\end{bmatrix}
\]

where,

\[
N_1 = \frac{(1 - \xi)(1 - \eta)}{4}, \quad N_2 = \frac{(1 + \xi)(1 - \eta)}{4}
\]

\[
N_3 = \frac{(1 + \xi)(1 + \eta)}{4}, \quad N_4 = \frac{(1 - \xi)(1 + \eta)}{4}
\]

This mapping relates a unit square in the \(\xi\eta\) coordinates to points in the quadrilateral in xy coordinates whose size and shape are determined by the eight nodal coordinates \(x_1, y_1, x_2, y_2, \ldots, y_4\).
FIG 1-1a QUADRILATERAL ELEMENT

FIG 1-1b UNIT SQUARE
The axes \( \xi \eta \) are in general not orthogonal. They are orthogonal for a rectangular element, in which case they are merely dimensionless forms of rectangular centroidal coordinates. The 2 by 2 rectangular element, for which we may write \( \xi = x \) and \( \eta = y \), is a convenient special case to refer when studying the following development.

Displacements within the element are defined by the same interpolation functions as used to define the element shape and hence the name isoparametric. Thus,

\[
\{f\} = \{u \ v\} = [N]\{u_1 \ v_1 \ u_2 ... v_4\} = [N]\{d\} \quad (I-3)
\]

where \([N]\) is the rectangular matrix of the \( N \) equations as indicated in the equation \((I-1)\), and \([d]\) is the vector of nodal displacements. The displacements \( u \) and \( v \) are directed parallel to \( x \) and \( y \), and not parallel to the \( \xi \) and \( \eta \) axes.

Note that for any point in the unit in \( \xi \eta \) coordinates, equation \( I-1 \) gives a point \((x,y)\) in the real coordinates and equation \( I-3 \) will give the displacements \( u \) and \( v \) in terms of the nodal displacements. Thus the displacements of every point \((x,y)\) in the real coordinates is defined in terms of the nodal displacements.

Steps taken to formulate the element stiffness matrix according to the approach outlined by Cooke (1975) are detailed in the remainder of this appendix. Because it is impossibly tedious to write the shape functions in terms of \( x \) and \( y \), the formulation will be carried out in terms of the isoparametric
coordinates $\xi\eta$. Equations (I-1), (I-3) are rewritten for convenience in the form,

\[
x = \sum N_i x_i \quad y = \sum N_i y_i \\
u = \sum N_i u_i \quad v = \sum N_i v_i
\]

(I-4)

A relationship between derivatives in the two coordinates, from the chain rule of differentiation, is as follows:

\[
\begin{align*}
\begin{bmatrix}
( ,)_{\xi} \\
( ,)_{\eta}
\end{bmatrix} &=
\begin{bmatrix}
x, & y, \\
x, & y
\end{bmatrix}
\begin{bmatrix}
( ,)_{x} \\
( ,)_{y}
\end{bmatrix} \\
&= [J]
\begin{bmatrix}
( ,)_{x} \\
( ,)_{y}
\end{bmatrix}
\end{align*}
\]

(I-5)

where commas denote partial differentiation. From equation (I-4) $[J]$ is the Jacobian matrix,

\[
[J] = \begin{bmatrix}
N_{1,\xi} & N_{2,\xi} & N_{3,\xi} & N_{4,\xi} \\
N_{1,\eta} & N_{2,\eta} & N_{3,\eta} & N_{4,\eta}
\end{bmatrix}
\begin{bmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \\
x_4 & y_4
\end{bmatrix}
\]

(I-6)

where, for example,

\[
N_{1,\xi} = -(1 - \eta) \frac{4}{4}
\]

(I-7)

Using the inverse relation of equation (I-4) and letting $[J^*] = [J]^{-1}$, differentials in the $xy$ system may be written as differentials in the $\xi\eta$ system as,

\[
\begin{bmatrix}
u, \\
v
\end{bmatrix} = \begin{bmatrix}
J_{1,1} & J_{1,2} & 0 & 0 \\
J_{2,1} & J_{2,2} & 0 & 0 \\
0 & 0 & J_{1,1} & J_{1,2} \\
0 & 0 & J_{2,1} & J_{2,2}
\end{bmatrix}
\begin{bmatrix}
u, \\
v
\end{bmatrix}
\]

(I-8)
The strain-displacement relation may be written as,

\[
\{\varepsilon\} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ v_x \end{bmatrix}
\] (I-9)

and, from equation (I-4),

\[
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v \end{bmatrix} = \begin{bmatrix} N_{i,1} & 0 \\ N_{i,2} & 0 \\ 0 & N_{i,3} \\ 0 & N_{i,4} \end{bmatrix} \begin{bmatrix} N_{1,1} \\ N_{1,2} \\ N_{1,3} \\ N_{1,4} \end{bmatrix}
\] (I-10)

Substitution of equation (I-8) into equation (I-9), and thereafter substitution of equation (I-10) into the result, yields the relation \(\{\varepsilon\} = [B]\{d\}\).

The derivation of the element stiffness matrix using the principle of stationary potential energy is well described by Cooke (1975), and Zienkiewicz (1979) and is obtained by the integration of the expression \(\int \int [B]^T[D][B]dx\,dy\). Where \([D]\) is the elasticity matrix, and \([B]\) is as established above, expressed in \(\xi\eta\) coordinates. The elasticity matrix \([D]\) is determined from equation (3-20), and the constitutive patterns for the \(G, B\) model, equations (3-21), (3-22). For the present analysis the stiffness matrix is expressed in terms of a shear and bulk, partial stiffness matrices, see Chap 3.3.1 for further details. The element 'partial' stiffness matrices are obtained by integration in the \(\xi\eta\) coordinate system using the following
transformation relationship.

\[
[k] = \iint [B]^T[D][B]dxdy = \iint [B]^T[D][B]\det[J]d\xi d\eta \quad (I-11)
\]

The determinant of \([J]\) is a magnification factor that yields area \(dxdy\) from area \(d\xi d\eta\), and which is a function of position with the element.

Having expressed all matrices in terms of \(\xi\) and \(\eta\), the integration can be carried in \(\xi\eta\) coordinates. However, in general the integration cannot be performed exactly due to the complexity of the polynomials in \(\xi\) and \(\eta\) that appear in the denominator of \([J^*]\). Hence, numerical integration is commonly used.

The Gauss method of numerical evaluation of the definite integral is used in the present analysis. In two dimensions the quadrature formula is obtained by integrating first with respect to one coordinate and then with respect to the other,

\[
[k] = \int_0^1 \int_{\xi=0}^1 f(\xi, \eta)\,d\xi d\eta = \int_0^1 \sum_{i=1}^n W_i f(\xi, \eta)\,d\eta
= \int_{\eta=0}^1 \sum_{i=1}^n W_i f(\xi, \eta)\,d\eta = \sum_{i=1}^n W_i f(\xi, \eta)
\]

(I-12)

where \(f(\xi, \eta)\) represents the expression \([B]^T[D][B]\det[J]\).

The integral is evaluated as the summation of values of the function at selected sampling points, each multiplied by an appropriate 'weight' \(W\). The Gauss method locates the sampling points so that for a given number of them, greatest accuracy is obtained. Sampling points are located symmetrically with respect to the center of the element. In general, the Gauss
quadrature using 'n' points is exact if the integral is a polynomial of degree 2n-1 or less. In using n points, the given function \( f(\xi, \eta) \) is effectively replaced by a polynomial of degree 2n-1.

A sequence of solutions to a problem may be generated by using successively finer meshes of elements. The sequence may be expected to converge to the correct result if the assumed element displacement fields satisfy the criteria of; invariance, continuity of displacements within elements, rigid body modes, constant strain behavior, and interelement compatibility. The validity of the isoparametric element is examined with regard to convergence by Cooke, with the element displacement fields shown to satisfy the above criteria. But as well, with numerically integrated elements, there is one addition consideration. That is, convergence to the correct solution is possible only if the numerical integration of the element stiffness integral is exact. This condition is relaxed and can be re-stated; numerically integrated elements yield convergence toward correct results as the mesh is refined if numerical integration is adequate to evaluate the element area exactly. This statement can be explained by noting that, from the stationary potential energy derivation, formation of a stiffness matrix is essentially the same as integration of a strain energy expression. As a mesh is refined and a constant strain condition comes to prevail in each element, the strain energy expression for an element assumes the form,

\[
\int ([\epsilon][D][\epsilon]) d\xi d\eta = \int (\text{constants}) \det[J] d\xi d\eta \quad (I-13)
\]
Hence, in the limit the strain energy of the structure is correctly assumed if the volume of each element is correctly assessed.

Examination of the Jacobian determinant yields the number of Gauss points needed to obtain the area of the particular elements. For a plane quadratic element \( \det[J] \) contain terms \( \xi^3 \) and \( \eta^3 \), hence a 2 by 2 Gauss rule (four points) is the minimum that can be accepted. It can be shown (Cooke 1975) that \( W \) for each Gauss point is 1.0 and the Gauss point values are \( \pm \frac{1}{3} \approx \pm 0.57735 \ldots \) Thus equation (I-11) can be written,

\[
[k] = (1.)(1.)f(\xi_1, \eta_1) + (1.)(1.)f(\xi_2, \eta_1) + (1.)(1.)f(\xi_1, \eta_2) + (1.)(1.)f(\xi_2, \eta_2)
\]

(I-14)

where \( \xi_1, \eta_1 = 0.57735 \ldots, \) and \( \xi_2, \eta_2 = -0.57735 \ldots \)