FINITE DEFORMATION ANALYSIS
USING THE
FINITE ELEMENT METHOD

by
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ABSTRACT

An analysis of the finite deformation of an elastic body using the finite element method is investigated. The governing nonlinear equations of equilibrium are derived through the principle of virtual work using a Lagrangian description. A general incremental virtual work equation is obtained, and then linearized to permit the use of direct solution techniques. A residual loading term is defined which represents the nonsatisfaction of equilibrium of the solution obtained at the end of an increment using the linear incremental virtual work equation. The residual loading term is used to control the divergence of the linearized incremental solution from the exact equilibrium solution, through the self-correcting solution technique.

The finite element method is introduced in general for three dimensional analysis, and is then specialized for two dimensional, plane elasticity analysis. Two eight degree of freedom rectangular finite elements are developed using a bilinear assumed displacement field. The first element is numerically integrated using Gaussian quadrature, while the second employs a nonuniform integration scheme in order to improve this element's performance.

Four finite deformation problems are analysed using the procedure presented in this thesis, and the results are compared with available closed form solutions. The problems analysed are those of a uniformly loaded infinite plate strip having either simply supported longitudinal edges or fixed longitudinal edges, a cantilever beam under a uniformly distributed load, and lastly a cantilever beam with a parabolically distributed end load. Excellent agreement was obtained between the finite element analysis results and the closed form solutions.
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NOTATION

The specific use and meaning of all symbols used is given in the text of this thesis where they are first introduced.

The summation convention holds for subscripted variables with repeated lower case indices, it does not apply to repeated upper case indices or to superscripts. The range of summation is normally three, except where specifically indicated to be otherwise.
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1.1 Background

The analysis of finite deformation is becoming increasingly more important as structures are being designed for severe loading conditions, as new more flexible and ductile material and structural elements are employed, and as structures are being optimized for various considerations. The primary generator of interest in finite deformation thus far, has been the aerospace industry where a considerable amount of research has been done. The severe mechanical and thermal loads encountered, and the penalties to be paid for excess weight have led the aerospace industry to consider less rigid structures with the requirement of an accurate assessment of the ultimate load behaviour of the structure.

The ultimate load behaviour of a structure may be largely governed by finite deformation effects, which are nonlinear, and may be significantly different from that predicted by the usual small deformation analysis. If the ultimate load behaviour can be accurately obtained through a finite deformation analysis, then the structure may be more efficiently designed and its safety more reliably established.

There are no general methods to solve nonlinear boundary value problems, and only a few specialized and simple nonlinear problems can be solved by exact methods. The majority of practical problems are unapproachable by any of these methods. Thus, the nonlinear nature of the governing equations of equilibrium for finite deformation make it
necessary to resort to the various numerical approximation schemes available. The finite element method is one such technique which has been extensively used in linear analysis and is now being employed in nonlinear analysis [1, 2, 3]. Only the development of the high-speed, large capacity, digital computer has made this solution technique practical for the analysis of actual structures.

The nonlinearities in finite deformation can be conceived of as arising from two sources, geometric nonlinearity and material nonlinearity. Geometric nonlinearity is a result of large displacements that alter the distribution or magnitude of the loads on the structure, and the manner in which the structure responds to the loading. Material nonlinearities arise from nonlinear constitutive relationships or from nonconservative deformation such as elasto-plastic or viscoelastic deformation.

Geometric nonlinearities were first incorporated in structural analysis through the use of an initial stress matrix in an incremental approach. This accounted for the initial stresses at the beginning of each increment and was originally derived on the basis of physical intuition. The first use of the initial stress matrix in a linearized incremental analysis was reported by Turner, et al [4] for stringers and triangular membrane elements. Gallagher and Padlog [5], derived the geometric initial stress matrix from the expression of potential energy for beam columns. The development of the initial stress matrix was finally derived in a more consistent manner through the use of the Lagrangian or Green's strain tensor by Martin [6]. Subsequent analysis by Marcal [7] demonstrated the importance of additional terms, which are represented by an initial displacement matrix in an incremental solution.
In considering finite deformation it is important to use a consistent continuum mechanics formulation for the equations of equilibrium, and an effective and accurate solution technique. In the development of the equations of equilibrium two basic approaches exist, the Lagrangian and the Eulerian descriptions.

In the Lagrangian description the stresses, strains and displacements are referred to the initial state, and the nodal coordinates, when considering the finite element method, remain fixed throughout the analysis. The Lagrangian description is also known as the material description since any particle of the body or structure has the same coordinate throughout the deformation history. This is contrasted by the Eulerian description where stresses, strains and displacements are referred to the current state, and a convective coordinate system is used with the nodal coordinates being updated after each increment. The Eulerian or spatial description does not have a constant coordinate for any particle of the body or structure undergoing deformation. The Eulerian approach typified the earlier finite deformation analyses, while the more recent analyses have tended to use the Lagrangian description. The Lagrangian description has been used by Hibbit, Marcal and Rice [8], and Felippa and Sharifi [9]; whereas the Eulerian description has been used by Sharifi and Popov [10], and Yaghami and Popov [11]. Once the equations of equilibrium have been derived, then an effective and appropriate solution technique must be used, given specified loads and structural characteristics, to solve these equations in order to specify or predict the finite deformation behaviour of the structure.

Using the finite element method to approximate the structural response field, the solution of the equations of equilibrium is
accomplished by minimization techniques, static perturbation methods, incremental methods, iterative procedures, self-correcting formulas or predictor-corrector methods. An assessment and survey of these techniques as applied to the finite element method, in static nonlinear analysis, has been made by Tillerson, Stricklin and Haisler [12].

Most of the finite deformation analyses use an incremental approach where the nonlinear equations of equilibrium are linearized for a small increment of deformation. The nonlinear terms are assumed to be negligible for the small increment used, and hence can be neglected. The advantages of using linearized incremental equilibrium equations are, that a direct forward solution is available, the equations are readily programmed, and when considering material nonlinearity an incremental constitutive relationship is easily incorporated. Since the incremental methods seek to trace the deformation behaviour in small steps, this represents the most rational approach to path-dependent problems such as plasticity. A survey of the development of incremental methods in continuum mechanics is given by Yaghami [13].

At the present time there exists several general purpose nonlinear finite element programmes, that have been developed for the finite deformation problem. Among these are NONSAP from the University of California, Berkeley [14] and DYPLAS from the Franklin Institute Research Laboratories [15], both of which use an incremental procedure in a Lagrangian description.
1.2 Purpose and Scope

The purpose of this thesis is to present the development of the governing equilibrium equations for static, finite deformation in a Lagrangian description, and then to apply the finite element method to solve the incremental relationships obtained from these equations. The Lagrangian description will be contrasted with the Eulerian description to show the advantages and disadvantages of each approach.

The governing equilibrium equations in both the Eulerian and the Lagrangian descriptions will be developed through the principle of virtual work. They will be valid for arbitrary magnitudes of deformation and strain, and will not be restricted to conservative deformation.

The governing equations of equilibrium that are developed are nonlinear, therefore a linearized increment virtual work relationship will be derived between two arbitrary configurations or states. A residual loading expression will also be obtained which represents the error or nonsatisfaction of equilibrium at the end of each linear increment. This residual loading term will be used to evaluate and control the error involved in linearizing the incremental virtual work expression. The deformation of the structure is obtained by analyzing a succession of incremental steps until the desired magnitude of loading or displacement is achieved as a sum of all increments.

The finite element method will be used to solve the linearized incremental virtual work expression, and to evaluate the residual loading term. This, along with a self-correcting solution procedure to control the error arising from linearizing the incremental virtual work expression, will be used to analyze several finite deformation problems. The results of the application of this method of analysis will be compared to certain available closed-form solutions.
1.3 Limitations

Although the basic governing virtual work expressions are derived, and are valid for conservative and nonconservative deformation, the attention of this thesis will be confined to conservative deformation. Furthermore, it will be assumed that there exists a linear constitutive relationship between Kirchhoff stress and Lagrangian strain.

The finite element analysis will be performed for the two-dimensional cases of plane strain and plane stress, although the full three-dimensional procedure will be presented in general.
MATHEMATICAL PRELIMINARIES

2.1 General

The analysis of finite deformation requires a consistent mathematical approach in order to have a valid formulation of the problem. The Eulerian and Lagrangian descriptions are both valid for finite deformation, but in this thesis for reasons given in the following chapter, the Lagrangian description will be used.

In this chapter the kinematics of the Lagrangian description will be demonstrated and the definition of Green's strain tensor will be given. The definition of two stress tensors in the Lagrangian description and their relationship to the natural physical concept of stress will also be given.

2.2 Kinematics of the Lagrangian Description

In the Lagrangian description the motion of a three-dimensional body in its path of deformation is described in terms of a fixed rectangular coordinate system. Three configurations of a general body are shown in Fig. 1, the undeformed or reference configuration (\(a\)C), the current deformed configuration (\(1\)C), and a neighbouring deformed configuration (\(2\)C) with respect to the current deformed configuration (\(1\)C). In this thesis, left superscripts indicate the configuration of the body to which the quantities or expressions refer, while no left
THREE CONFIGURATIONS OF A GENERAL BODY

FIG. 1
superscript indicates incremental quantities between configurations $^1C$ and $^2C$. Since the configurations $^1C$ and $^2C$ are completely general, the increment represents the transition between any two configurations and is capable of assuming any magnitude of deformation.

A material point of the body, in terms of the fixed rectangular coordinate system, is described by:

1. the material coordinates $a_i (i = 1, 2, 3)$ in $^0C$.
2. the coordinates $\alpha x_i$ in $^aC$, given by

$$\alpha x_i = a_i + \alpha u_i \quad (i = 1, 2, 3)$$  \hspace{1cm} (2.1)

where $\alpha u_i (i = 1, 2, 3)$, are the vector components of the total displacement from the reference configuration $^0C$ to configuration $^aC$.

Deformation of the body is represented by strain, and in the Lagrangian description the strain tensor used is the Lagrangian, or Green's strain tensor, given for the material point $a_i$ in configuration $^aC$ by

$$\epsilon_{ij} = \frac{1}{2} \left[ \frac{\partial \alpha x_k}{\partial a_i} \frac{\partial \alpha x_k}{\partial a_j} - \delta_{ij} \right]$$  \hspace{1cm} (2.2)

where $\delta_{ij}$ is the Kronecker delta.

This strain tensor is developed by considering the transformation given by Eq. 2.1 which maps the position of a material point $^0P$ given by $a_i$ in configuration $^0C$, to a point $^aP$ in configuration $^aC$. 


\( \alpha \mathbf{C} \) given by \( \alpha x_i \). Both \( \alpha x_i \) and \( a_i \) are referred to the same set of rectangular Cartesian coordinates. Now consider a material point \( \alpha Q \) in configuration \( \alpha \mathbf{C} \) in the neighbourhood of \( \alpha P \), and whose coordinates are given by \( a_i + da_i \). Using the transformation of Eq. 2.1, then \( d^\alpha x_i \) is given by

\[
d^\alpha x_i = \frac{\partial x_i}{\partial a_j} da_j \quad (2.3)
\]

Then the coordinates of \( \alpha Q \) in configuration \( \alpha \mathbf{C} \) will be \( \alpha x_i + d^\alpha x_i \).

Taking the difference of the square of the length of line segments \( \alpha P \alpha Q \) and \( \alpha P \alpha Q \) then Green's strain tensor is defined such that

\[
d^\alpha x_i d^\alpha x_i - da_i da_i = 2 \epsilon_{ij} da_i da_j \quad (2.4)
\]

which gives Eq. 2.2. It should be noted that Green's strain tensor vanishes only if there is no change in the length of the line segment in the transformation between configurations \( \alpha \mathbf{C} \) and \( \alpha \mathbf{C} \). Thus the vanishing of Green's strain tensor indicates the absence of strain. It can also be seen that Green's strain tensor is symmetric from Eq. 2.2, that is,

\[
\epsilon_{ij} = \epsilon_{ji} \quad (2.5)
\]

Green's strain tensor given in Eq. 2.2 can also be written as

\[
\epsilon_{ij} = \frac{1}{2} \left[ \alpha u_{i,j} + \alpha u_{j,i} + \alpha u_{m,i} \alpha u_{m,j} \right] \quad (2.6)
\]
where a comma indicates differentiation with respect to the material coordinates in configuration $^1C$, that is
\[ \alpha u_{i,j} = \frac{\partial^\alpha u_i}{\partial a_j} \] (2.7)

The incremental form of Green's strain tensor between configurations $^1C$ and $^2C$ is defined as:
\[ \varepsilon_{ij} = \varepsilon_{ij}^{^2} - \varepsilon_{ij}^{^1} \] (2.8)

Substituting for $\alpha \varepsilon_{ij}$, using Eq. 2.6 and rearranging terms
\[ \varepsilon_{ij} = \frac{1}{2} \left[ u_{i,j} + u_{j,i} + 1 u_{m,i} u_{m,j} + u_{m,i} u_{m,j} \right] + \frac{1}{2} u_{m,i} u_{m,j} \] (2.9)

where
\[ u_i = u_i^{^2} - u_i^{^1} \] (2.10)

Knowing the configuration $^1C$, and hence displacements $u_i$ and the required material derivatives, the expression for the incremental Green's strain tensor can be decomposed into a linear component $\varepsilon_{ij}$, and a nonlinear component $\eta_{ij}$ for the increment of deformation. The two components are defined as
\[ 2\varepsilon_{ij} = \left[ u_{i,j} + u_{j,i} + 1 u_{m,i} u_{m,j} + u_{m,i} u_{m,j} \right] \] (2.11)
\[ 2\eta_{ij} = \left[ u_{m,i} u_{m,j} \right] \] (2.12)
Thus

\[ \varepsilon_{ij} = \varepsilon_{ij} + \tilde{\eta}_{ij} \] (2.13)

It should be noted that the incremental form of Green's strain tensor is still symmetric, and that the two components \( \varepsilon_{ij} \) and \( \tilde{\eta}_{ij} \) are also symmetric tensors.

The special case of small or infinitesimal strain is found from Eq. 2.6, by assuming that displacements and material derivatives are small and that the product \( u_{m,i} u_{m,j} \) can be neglected as a higher order term.

2.3 Stress in a Lagrangian Description

In the course of analyzing the deformation of a continuum it is necessary to relate stresses to strains. Since it is intended that the Green strain tensor be utilized, and it relates strains to the original undeformed or reference configuration, it would be convenient to describe the stress tensor also with respect to the undeformed configuration.

Two stress tensors that are defined with respect to the original undeformed configuration \( \mathcal{O} \), are the Lagrange and Kirchhoff stress tensors. They are defined by arbitrarily assigning a rule of correspondence between force vectors acting on the surface of an element of the continuum, in the deformed and undeformed configurations.

Following the development given by Fung [16], consider an element of a deformed solid which has a force vector of \( \overrightarrow{F^T} \) which acts on surface PQRS as shown in Fig. 2. A corresponding vector \( \overrightarrow{F^T_0} \) acts on the surface \( P_0Q_0R_0S_0 \) in the undeformed configuration. If stress vectors
FORCE VECTORS ACTING ON DEFORMED AND UNDEFORMED CONFIGURATIONS OF AN ELEMENT OF A SOLID BODY

FIG. 2
are then defined in each configuration as the limit of $\frac{\overrightarrow{dT}}{dS}$ and $\frac{\overrightarrow{dT_0}}{d^0 S}$ as $d^0 S$ and $dS$ go to zero, the areas of the surfaces PQRS and $P_0Q_0R_0S_0$ respectively, then stress tensors may be defined in each configuration.

Finally a rule of correspondence between $\overrightarrow{dT}$ and $\overrightarrow{dT_0}$ is necessary, and as stated before this is arbitrary but must be consistent. Therefore, two rules of correspondence are used, giving $\overrightarrow{dT}$ and $\overrightarrow{dT_0}$ in terms of their components, by

$$d^\alpha T_{0i}^{(L)} = d^\alpha T_{i}$$  \hspace{1cm} (2.14)

and

$$d^\alpha T_{0i}^{(K)} = \frac{\partial a_1}{\partial x_j} d^\alpha T_j$$  \hspace{1cm} (2.15)

The rule of correspondence given by Eq. 2.14 is the Lagrangian rule, and Eq. 2.15 gives the Kirchhoff rule. The Kirchhoff rule of correspondence uses the same transformation as that used for a line segment in going from the deformed to the undeformed configurations, as can be seen from

$$da_i = \frac{\partial a_1}{\partial x_j} d^\alpha x_j$$  \hspace{1cm} (2.16)

The correspondence of force vectors using the Lagrangian rule and the Kirchhoff rule between the deformed and undeformed configurations is shown in Fig. 3, for the two-dimensional case.

Three different stress tensors are now definable in terms of the force vector, the elemental surface area, and the unit outward normals associated with the deformed surface $^\alpha \psi$, and the undeformed surface, $^0 \psi$. 
Lagrange's and Kirchhoff's Rules of Correspondence for Force Vectors

Fig. 3
If the deformed configuration is considered first, the associated stress tensor is the Euler stress tensor, $\sigma_{ij}$, defined as

$$d^aT_i = \sigma_{\alpha ij} \alpha_j d^a{\sigma}$$  \hspace{1cm} (2.17)$$

Similarly if the Lagrange rule of correspondence is used, then a Lagrangian stress tensor $T_{ij}$, is defined by

$$d^aT_{ij}^{(L)} = \sigma_{\alpha ij} \alpha_j d^a{\sigma} = d^aT_i$$  \hspace{1cm} (2.18)$$

and if the Kirchhoff rule is used, then the Kirchhoff stress tensor $S_{ij}$, is defined by

$$d^aT_{ij}^{(K)} = \sigma_{\alpha ij} \alpha_j d^a{\sigma} = \frac{\partial a_i}{\partial x_k} d^aT_k$$  \hspace{1cm} (2.19)$$

The relationships between the three tensors are given by

$$\alpha_{T_{ij}} = \frac{\partial a_i}{\partial x_k} \alpha_{\sigma_{ij}}$$  \hspace{1cm} (2.20)$$

$$\alpha_{S_{ij}} = \frac{\partial a_i}{\partial x_k} \alpha_{S_{ij}}$$  \hspace{1cm} (2.21)$$

$$\alpha_{S_{ij}} = \frac{\partial a_i}{\partial x_k} \alpha_{T_{ij}}$$  \hspace{1cm} (2.22)$$

where $\alpha_{\rho}$ is the density in the deformed configuration and $\sigma_{\rho}$ is the density in the undeformed configuration.

It should be noted that the Eulerian stress tensor is symmetric, that is
by requirements for rotational equilibrium. Then it can be seen that the Lagrangian stress tensor is not in general symmetric by examining Eq. 2.20, while the Kirchhoff stress tensor is symmetric as a consequence of the symmetry of the Eulerian stress tensor and Eq. 2.21 relating the two tensors. The symmetry of the Kirchhoff stress tensor is represented by

\[ \alpha_{ij}^\sigma = \alpha_{ji}^\sigma \]  

(2.23)

The property of symmetry makes the Kirchhoff stress tensor convenient to use when the strain tensor is symmetric as in Green's strain tensor, especially with the use of a symmetric constitutive tensor.

Finally, the Eulerian stress tensor represents the actual physical concept of stress, therefore once the deformed configuration has been determined by a Lagrangian analysis then the Eulerian stresses should be calculated to give a physical representation of the state of stress. The Eulerian stresses may be obtained from the Kirchhoff stresses by the inverse of Eq. 2.21, given by

\[ \alpha_{ij}^\sigma = \frac{\partial \alpha_{x_1}}{\partial \alpha_{x_i}} \frac{\partial \alpha_{x_j}}{\partial \alpha_{m_k}} \alpha_{mj}^S \]  

(2.24)

(2.25)
3.1 General

In the analysis of finite deformation it is necessary to formulate the equilibrium equations in a consistent manner, based on either the Eulerian or the Lagrangian description.

The Eulerian description is attractive from the point of view that the associated stress tensor, the Euler stress tensor, is based on the deformed configuration and is a natural physical concept. However, it is usually desirable to define strain in terms of the original material points in the undeformed configuration, and thus it becomes difficult to relate stress to strain. The Eulerian description also presents a problem in an incremental variational analysis, in that the deformed volume and surface are not known at the beginning of the increment. Thus, the required surface and volume integrals may be only approximately evaluated.

In the Lagrangian description, the strain tensor is associated with the original material points but the associated stress tensors no longer represent the natural physical concept of stress. If a physical representation of the state of stress is required, then the Euler stress must be evaluated by means of a transformation; Eq. 2.25 gives the relationship between the Euler and Kirchhoff stress tensors. The Lagrangian description in an incremental variational analysis has the advantage that the volume and surface, over which the necessary integrals
are performed, are known a priori. Therefore, there is no approximation introduced here as in the Eulerian description. The Lagrangian description with its constant volume and surface characteristic, makes it possible that several of the integrals required for an incremental variational analysis need be evaluated only once as constants, or in terms of an increment parameter, thus saving computational expense.

Incremental variational equations of equilibrium based on the Lagrangian description and the principle of virtual displacements are developed in this chapter for the analysis of finite deformation. The expressions developed will be nonlinear for the increment, so in order to facilitate a direct solution the equations will be linearized. A residual loading term is developed that represents the nonsatisfaction of equilibrium at the end of each increment caused by neglecting the nonlinear incremental variational terms.

3.2 Principle of Virtual Work

The principle of virtual work represents a necessary condition for equilibrium of a body subjected to prescribed surface tractions, body forces and boundary conditions. There are two approaches to the principle of virtual work, through the use of virtual displacements or the use of virtual forces. The virtual displacement approach is used throughout this chapter.

Consider a body in static equilibrium when subjected to specified surface tractions, body forces and boundary conditions. The surface of the body is composed of two types of regions; the surface $S_o$ over which surface tractions are specified and the surface $S_u$ over which
displacements are specified. Assume a set of arbitrary displacements $\delta u_i$ that vanish on $S_u$, are compatible, and which do not change the magnitude or direction of the surface tractions or body forces. These arbitrary displacements are called kinematically admissible virtual displacements, and their magnitude is arbitrary. Then the work done by the surface tractions and body forces as they go through the virtual displacements is called the virtual work and is given in the Eulerian description by:

$$W^E = \int_S T_i \delta u_i \, dS + \int_V \rho F_i \delta u_i \, dV.$$  \hspace{1cm} (3.1)

where:

- $T_i$ are the components of the surface traction per unit area of the deformed configuration of the body.
- $F_i$ are the components of the body force per unit density.
- $\rho$ is the density in the deformed configuration
- $V$ is the deformed volume of the body
- $S$ is the deformed surface of the body.

Since the virtual displacements $\delta u_i$, to be kinematically admissible, vanish on $S_u$ the surface integral in Eq. 3.1 need only be evaluated on $S_o$.

Substituting for $T_i$ in Eq. 3.1,

$$T_i = \sigma_{ij} v_j.$$  \hspace{1cm} (3.2)
where \( \nu_j \) is the outward normal to the deformed surface and \( \sigma_{ij} \) is the Euler stress tensor, and considering only the surface integral first, transform by Gauss's theorem:

\[
\int_S T_i \delta u_i \, dS = \int_S \sigma_{ij} \nu_j \delta u_i \, dS
\]

\[
= \int_V (\sigma_{ij} \delta u_i)^{,j} \, dV \tag{3.3}
\]

Here the notation \( \frac{\partial}{\partial x_j} \) is used where \( x_j \) are the coordinates of the deformed body. (This will only be true for this section.) Therefore using the virtual work definition in Eq. 3.1, and Eq. 3.3:

\[
\int_S T_i \delta u_i \, dS + \int_V \rho F_i \delta u_i \, dV = \int_V (\sigma_{ij} \delta u_i)^{,j} \, dV + \int_V \rho F_i \delta u_i \, dV
\]

\[
= \int_V \sigma_{ij,j}^{,i} \delta u_i \, dV + \int_V \sigma_{ij} \delta u_i^{,j} \, dV + \int_V \rho F_i \delta u_i \, dV
\]

\[
= \left[ \sigma_{ij,j} + \rho F_i \right] \delta u_i \, dV + \int_V \sigma_{ij} \delta u_i^{,j} \, dV \tag{3.4}
\]

But the equation of equilibrium in the Eulerian description is

\[
\sigma_{ij,j} + \rho F_i = 0 \tag{3.5}
\]

Using the equation of equilibrium given by Eq. 3.5, Eq. 3.4 becomes:
This expression, Eq. 3.6, is the virtual displacement expression for equilibrium of a body under prescribed surface tractions and body forces in Eulerian description. If this equation is satisfied for all kinematically admissible virtual displacements then the body is in static equilibrium.

The Eulerian description is somewhat difficult to use in finite deformation, since it is usually desired to find the equilibrium configuration of a body under prescribed forces and therefore the deformed surface and volume will not be known a priori. Then the virtual displacement expression can be calculated only approximately by assuming a trial configuration in order to calculate the necessary surface and volume integrals. There arises therefore, the necessity of iterating to find the correct solution configuration to evaluate the virtual work expression properly.

If infinitesimal deformation is assumed then the difference between the undeformed and deformed configuration is sufficiently small that the integrals may be adequately evaluated using the initial configuration.

3.3 Virtual Work Using the Lagrangian Description

The virtual work expression in the Lagrangian description is developed in the same manner as in the Eulerian description. The virtual work done by the surface tractions and body forces going through an admissible set of arbitrary virtual displacements must be balanced by the internal virtual work.
In the Eulerian description the virtual work performed by the surface tractions and body forces, was defined as:

\[ a_V^{(E)} = \int_{a_S}^a \mathbf{T}_i \delta \mathbf{u}_i \, d^a S + \int_{a_V}^a \mathbf{f}_i \, \mathbf{\alpha} \mathbf{p} \, \delta \mathbf{u}_i \, d^a V \]  

(3.7)

where the left superscript indicates the appropriate configuration of the body.

To express the virtual work in the Lagrangian description, it is necessary to transform the integrals in Eq. 3.7, into integrals with respect to the undeformed configuration \( ^0C \).

By using Eq. 2.18 and 2.19 it can be seen that

\[ a_{T_i} \, d^a S = a_{T_{ji}} \, ^0v_j \, d^0 S \]  

(3.8)

and

\[ a_{T_i} \, d^a S = a_{S_{jk}} \, \frac{\partial a_{x_i}}{\partial a_k} \, ^0v_j \, d^0 S \]  

(3.9)

Therefore, the surface traction virtual work integral can be evaluated with respect to the undeformed configuration by virtue of the following equivalences:

\[ \int_{a_S}^a \mathbf{T}_i \delta \mathbf{u}_i \, d^a S = \int_{^0S}^a \mathbf{T}_{ji} \, ^0v_j \delta \mathbf{u}_i \, d^0 S \]  

(3.10)
or

\[ \int_{\alpha S} \alpha T \delta u_i \, d\alpha S = \int_{\alpha S} \alpha S \frac{\partial x_i}{\partial a_k} \delta v_j \, d\alpha S \quad (3.11) \]

Next considering the body force virtual work and requiring the conservation of mass and that the body force per unit mass remains constant, then

\[ \int_{\alpha V} \alpha F_i \alpha \delta u_i \, d\alpha V \quad (3.12) \]

In deriving the equivalence of these integrals in the two different configurations, the arbitrary nature of the virtual displacements has been used, therefore they do not need to be transformed in any manner.

Now requiring that the virtual work performed by a body be a constant for a specified set of surface tractions, body forces and boundary conditions regardless of the description used to evaluate it, that is

\[ W_v^{(L)} = W_v^{(E)} \quad (3.13) \]

then, the virtual work expression in the Lagrangian description is given by either of two equivalent forms:

\[ W_v^{(L)} = \int_{\alpha S} \alpha T_{ji} \delta u_i \, d\alpha S + \int_{\alpha V} \alpha F_i \delta u_i \, d\alpha V \quad (3.14) \]
and

\[ W_v^{(L)} = \int_{\partial V} \alpha_{ji} \frac{\partial x_j}{\partial a_i} \delta v_j \, d\sigma + \int_{\partial V} \alpha_{F_i} \delta u_i \, dV \quad (3.15) \]

Which of the two forms to be used to evaluate the virtual work of the surface tractions and body forces, depends on the nature of the surface tractions prescribed. Both forms are equivalent but one may be more convenient to use. For the derivation of the internal virtual work however, it is more convenient to use Eq. 3.15 since the Kirchhoff stress tensor \( S_{ij} \) is symmetric whereas the Lagrangian stress tensor \( T_{ij} \) is not in general.

First, applying Gauss's theorem to the surface integral in Eq. 3.15, and now using the notation \( j = \frac{\partial}{\partial a_j} \):

\[ \int_{\partial S} \alpha_{ji} \frac{\partial x_j}{\partial a_i} \delta v_j \, d\sigma = \int_{\partial V} \left[ \alpha_{ji} \frac{\partial x_j}{\partial a_i} \delta u_i \right] \, dV \]

\[ = \left[ \alpha_{ji} \frac{\partial x_j}{\partial a_i} \right] \delta u_i \, dV + \int_{\partial V} \alpha_{ji} \delta u_i \, dV \quad (3.16) \]

Therefore, substituting for the surface integral from Eq. 3.16, back into Eq. 3.15,

\[ W_v^{(L)} = \int_{\partial V} \left[ \left[ \alpha_{ji} \frac{\partial x_j}{\partial a_i} \right] \delta u_i \right] \, dV + \int_{\partial V} \alpha_{F_i} \delta u_i \, dV + \int_{\partial V} \alpha_{ji} \frac{\partial x_j}{\partial a_i} \delta u_i \, dV \quad (3.17) \]
But the equation of equilibrium in terms of the Kirchhoff stress tensor is given by Fung [17] as

\[
\begin{bmatrix}
\alpha_{S_{jk}} \\
\alpha_{X_{i'k}}
\end{bmatrix}
\begin{bmatrix}
\alpha_{\xi_{i'k}}
\end{bmatrix}
+ \alpha_{F_{i}} \delta_{\rho} = 0
\]  

(3.18)

therefore Eq. 3.17 becomes

\[
W_{V}^{(L)} = \int_{\partial V} \alpha_{S_{jk}} \alpha_{X_{i'k}} \delta_{\xi_{i'k}} \delta_{u_{i'j}} \, d\sigma
\]  

(3.19)

The virtual work expression in the Lagrangian reference frame can thus be expressed by either of the following two forms, found by combining Eq. 3.19 with Eq. 3.14 and 3.15 respectively:

\[
\int_{\partial S} \alpha_{T_{ji}} \delta_{u_{i}} \, d\sigma + \int_{\partial V} \alpha_{F_{i}} \delta_{\rho} \delta_{u_{i}} \, d\sigma = \int_{\partial V} \alpha_{S_{jk}} \alpha_{X_{i'k}} \delta_{\xi_{i'k}} \delta_{u_{i'j}} \, d\sigma
\]  

(3.20)

and

\[
\int_{\partial S} \alpha_{S_{jk}} \alpha_{X_{i'k}} \delta_{u_{i}} \, d\sigma + \int_{\partial V} \alpha_{F_{i}} \delta_{\rho} \delta_{u_{i}} \, d\sigma = \int_{\partial V} \alpha_{S_{jk}} \alpha_{X_{i'k}} \delta_{\xi_{i'k}} \delta_{u_{i'j}} \, d\sigma
\]  

(3.21)

Both of these virtual work expressions are exact and valid for arbitrarily large deformations and strains. Satisfaction of these expressions is equivalent to finding an equilibrium configuration of the body under the prescribed surface tractions and body forces. They are not restricted to any particular constitutive relationship and are valid for both conservative and nonconservative deformation.
3.4 Incremental Virtual Work Equation

Knowing the configuration of a body under any specified set of surface tractions it is desired to obtain an incremental virtual work equation that can be used to find a new neighbouring configuration of the body given incremental surface tractions and body forces. For convenience the incremental virtual work equation will be written for the deformation of a body between configuration \( \text{C} \) and \( \text{C}_2 \) (See Fig. 1), but is completely general since the two configurations used are arbitrary. The resulting expression can therefore be used to find the deformation between any given configuration and a new configuration resulting from incremental loads.

It is assumed that the configuration \( \text{C} \) is known and that the body undergoes a change in configuration, due to an incremental change in surface tractions and body forces, which results in the body being in configuration \( \text{C}_2 \). The incremental values of surface traction and body forces need not be monotonic with respect to past incremental values.

To derive the incremental equation, write the virtual work expressions of equilibrium in each of the two configurations \( \text{C}_1 \) and \( \text{C}_2 \), and then form the difference between the two. Thus for configuration \( \text{C}_2 \), using Eq. 3.20:

\[
\int_{\Omega}^{\text{C}_2} \mathbf{T}_{ji} \delta z_j \, d\mathbf{S} + \int_{\Omega}^{\text{C}_2} \mathbf{F}_i \delta u_i \, d\mathbf{V} = \int_{\Omega}^{\text{C}_2} \mathbf{S}_{jk} \delta u_{i',k} \, d\mathbf{V} \tag{3.22}
\]

and similarly for \( \text{C}_1 \)
\[
\left\{ T_{ji} \delta j_i d^\circ S \right\}_{\circ S} + \left\{ F_i \delta u_i d^\circ V \right\}_{\circ V} = \left\{ S_{jk} \delta u_{i',j} d^\circ V \right\}_{\circ V}
\]  

(3.23)

Now taking the difference of the two equations above

\[
\left\{ \left[ 2T_{ji} - 1T_{ji} \right] \delta j_i d^\circ S \right\}_{\circ S} + \left\{ \left[ 2F_i - 1F_i \right] \delta u_i d^\circ V \right\}_{\circ V}
\]

\[= \left\{ \left[ 2S_{jk} \delta x_{i',k} + S_{jk} \delta u_{i',j} \right] d^\circ V \right\}_{\circ V}
\]

(3.24)

This equation (Eq. 3.24), is made possible by the arbitrary nature of the admissible virtual displacements, and by the fact that the integrals are performed over the same undeformed reference configuration surface and volume. These together allow the corresponding integrals for each configuration to be combined. This would not be a valid procedure for the Eulerian description since the integrals are not evaluated for the same domain in the different configurations.

Now rearranging Eq. 3.24 using the definition of $\xi_{i}$ given in Eq. 2.1, and the convention that no left superscript indicates an incremental value between configurations $^1C$ and $^2C$:

\[
\left\{ T_{ji} \delta j_i d^\circ S \right\}_{\circ S} + \left\{ F_i \delta u_i d^\circ V \right\}_{\circ V}
\]

\[= \left\{ \left[ 2S_{jk} \delta x_{i',k} + S_{jk} \delta u_{i',j} \right] d^\circ V \right\}_{\circ V}
\]

\[= \left\{ S_{jk} \delta u_{i',j} d^\circ V \right\}_{\circ V} + \left\{ 2S_{jk} \delta u_{i',j} d^\circ V \right\}_{\circ V}
\]

(3.25)
Examine now, each of the two integrals on the right hand side of Eq. 3.25 in turn. First

\[
\int_{\mathcal{V}} S_{jk} \delta x_{i,k} \delta u_{i,j} \, d^3V = \int_{\mathcal{V}} S_{jk} (\delta_{ik} + l_{i,k}) \delta u_{i,j} \, d^3V \tag{3.26}
\]

Here \( \delta_{ik} \) is the Kroenecker delta which has the following properties:

\[
\begin{align*}
\delta_{ik} &= 1 \quad i = k \\
\delta_{ik} &= 0 \quad i \neq k 
\end{align*}
\tag{3.27}
\]

The Kroenecker delta should not be confused with the symbol preceding a virtual value.

Continuing:

\[
\int_{\mathcal{V}} S_{jk} \delta x_{i,k} \delta u_{i,j} \, d^3V = \frac{1}{2} \int_{\mathcal{V}} S_{jk} (\delta_{ik} + l_{i,k}) \delta u_{i,j} + S_{kj} (\delta_{ij} + l_{i,j}) \delta u_{i,k} \, d^3V \tag{3.28}
\]

by virtue of the interchangeability of dummy indices, that is, the equation below is an identity.

\[
S_{jk} (\delta_{ik} + l_{i,k}) \delta u_{i,j} = S_{kj} (\delta_{ij} + l_{i,j}) \delta u_{i,k} \tag{3.29}
\]
Now employing the symmetric property of the Kirchhoff stress tensor and the properties of the Kroenecker delta:

\[
\int_{V} S_{jk} \, \lambda_{i,k} \, \delta u_{i,j} \, d^3V = \int_{V} \frac{1}{2} \, S_{jk} (\delta u_{k,j} + u_{i,k} \delta u_{i,j} + \delta u_{j,k}) + u_{i,j} \delta u_{i,k} \, d^3V \quad (3.30)
\]

Taking the variation of Eq. 2.11, which represents the linear portion of the incremental Green's strain tensors:

\[
\delta e_{ij} = \frac{1}{2} \left[ \delta u_{i,j} + \delta u_{j,i} + u_{m,i} \delta u_{m,j} + \delta u_{m,i} u_{m,j} \right] \quad (3.31)
\]

Substituting from Eq. 3.31 into Eq. 3.30, gives finally

\[
\int_{V} S_{jk} \, \lambda_{i,k} \, \delta u_{i,j} \, d^3V = \int_{V} \delta e_{jk} \, d^3V \quad (3.32)
\]

The second integral on the right hand side of Eq. 3.25 can be modified, following the same arguments as used for the first integral considered, to give:

\[
\int_{V} 2 S_{jk} \, u_{i,k} \, \delta u_{i,j} \, d^3V = \int_{V} \frac{1}{2} \left[ 2 S_{jk} u_{i,k} \delta u_{i,j} + 2 S_{kj} u_{i,j} \delta u_{i,k} \right] d^3V \\
= \int_{V} 2 S_{jk} \left[ \frac{1}{2} (u_{i,k} \delta u_{i,j} + u_{i,j} \delta u_{i,k}) \right] d^3V \quad (3.33)
\]
Now taking the variation of the nonlinear component of the incremental Green's strain tensor given in Eq. 2.12

\[ \delta n_{ij} = \frac{1}{2} \left( u_{m',i} \delta u_{m',j} + \delta u_{m',i} u_{m',j} \right) \]  

(3.34)

Substituting this into Eq. 3.33 therefore,

\[ \int_{\delta V} 2S_{jk} u_{i',k} \delta u_{i',j} \, d\delta V = \int_{\delta V} 2S_{jk} \delta n_{jk} \, d\delta V \]  

(3.35)

Now the incremental virtual work equation, Eq. 3.25, can be written as:

\[ \int_{\delta S} T_{ji} \delta u_{j} \, d\delta S + \int_{\delta V} F_{i} \delta \rho \delta u_{i} \, d\delta V = \int_{\delta V} S_{jk} \delta e_{jk} \, d\delta V + \int_{\delta V} 2S_{jk} \delta n_{jk} \, d\delta V \]  

(3.36)

A similar expression can be obtained for using a surface traction specified such that the Kirchhoff stress tensor is more convenient to use, by following the same procedure as above beginning with Eq. 3.21. This equation will be given by:

\[ \int_{\delta S} (2S_{jk} x_{i',k} - 1S_{jk} x_{i',k}) \delta s_{j} \, d\delta S + \int_{\delta V} F_{i} \delta \rho \delta u_{i} \, d\delta V \]

\[ = \int_{\delta V} S_{jk} \delta e_{jk} \, d\delta V + \int_{\delta V} 2S_{jk} \delta n_{jk} \, d\delta V \]  

(3.37)
The surface traction integrals can be decomposed and rearranged to give

\[
\int_{S} \left( 2S_{jk}^{2} \delta_{i'k} - S_{jk}^{1} \delta_{i'k} \right) \nu_{j} \delta \mathbf{u}_{i} d\mathbf{S} = \int_{S} S_{jk}^{1} \delta_{i'k} \nu_{j} \delta \mathbf{u}_{i} d\mathbf{S} + \int_{S} 2S_{jk}^{2} \delta_{i'k} \nu_{j} \delta \mathbf{u}_{i} d\mathbf{S}
\]

(3.38)

It should be noted that the last integral on the right hand side of Eq. 3.38 is dependent on the deformation experienced by the surface of the body during the increment, and thus cannot be completely evaluated at the beginning of the increment.

The two incremental virtual work equations given above, Eq. 3.36 and 3.37, are exact and valid for any magnitude of deformation between configurations $^{1}C$ and $^{2}C$. If they are satisfied for arbitrary and admissible virtual displacements, they give the incremental deformation of the body for prescribed increments of surface tractions and body forces. Both equations are still nonlinear and hence they cannot be solved directly to yield the incremental deformation.

### 3.5 Linearized Incremental Virtual Work Equation and the Residual Loading Term

The incremental virtual work equation given by either Eq. 3.36 or Eq. 3.37 is nonlinear and as such cannot be solved directly. In this section these two equations will be separated into linear and nonlinear terms, and then linearized by discarding nonlinear terms. The linearized incremental virtual work equations derived are then not exact.
for the increment of deformation but it will be shown that the nonlinear terms discarded are of a higher order than the linear terms retained.

To examine the resulting degree of nonsatisfaction of equilibrium for the configuration obtained at the end of the increment, a residual loading term is derived. This residual loading term will then be used either to correct the solution, or used simply as an indication of the error resulting from the linearization.

Starting with Eq. 3.36, separate the linear and nonlinear integral terms, and place the nonlinear terms in brackets.

\[
\int_{S} T_{ji} \delta u_{i} dS + \int_{V} F_{i} \delta u_{i} dV = \int_{V} S_{jk} \delta e_{jk} dV + \int_{V} S_{jk} \delta \eta_{jk} dV + \int_{V} \left\{ S_{jk} \delta \eta_{jk} dV \right\}
\]  

(3.39)

In this case there is only one nonlinear integral, and its integrand can be seen to be the product of incremental Kirchhoff stresses and increments of the nonlinear portion of Green's strain tensor. Using Eq. 3.34, this is shown to be

\[
S_{jk} \delta \eta_{jk} = S_{jk} \left[ \frac{1}{2} \left( u_{m',j} \delta u_{m',k} + \delta u_{m',j} u_{m',k} \right) \right]
\]

(3.40)

Writing the integrands of the other two integrals on the right hand side of Eq. 3.39,
\[ l_{jk} \delta \eta_{jk} = l_{jk} \left[ \frac{1}{2} (u_{m,j} \delta u_{m,k} + \delta u_{m,j} u_{m,k}) \right] \] 
\( (3.41) \)

and using Eq. 3.31

\[ S_{jk} \delta e_{jk} = S_{jk} \left[ \frac{1}{2} (\delta u_{k,j} + u_{m,k} \delta u_{m,j} + \delta u_{j,k} + u_{m,j} \delta u_{m,k}) \right] \] 
\( (3.42) \)

Comparing the integrands it is evident that since the nonlinear term is a product of the incremental Kirchhoff stress and the nonlinear incremental portion of Green's strain tensor, it is in general less than the integrand represented in Eq. 3.41. Considering the integrand in Eq. 3.42, it is larger than the nonlinear integrand since it contains terms that are not products of two material derivatives, and also because it has derivatives of the total displacements included. Therefore the nonlinear integral may be neglected with respect to the other two linear integrals. It should be noted that the comparison of the integrands is made possible by the fact that the integrations are performed over the same domain \( \delta V \).

Neglecting the nonlinear integral, the linearized incremental virtual work equation is obtained as:

\[
\int_{\delta S} T_{ji} \delta u_i d^2S + \int_{\delta V} F_i \delta u_i d^\delta V = \int_{\delta V} S_{jk} \delta e_{jk} d^\delta V + \int_{\delta V} l_{jk} \delta \eta_{jk} d^\delta V
\]
\( (3.43) \)
Similarly, Eq. 3.37 can be separated into linear and nonlinear integral terms as,

\[
\left\{ S_{jk} \frac{\partial x_i}{\partial k} v_j u_i d^oS + \int_{\mathcal{V}} S_{jk} u_i \frac{\partial v_j}{\partial k} u_i d^oV + \int_{\mathcal{V}} F_i \frac{\partial u_i}{\partial k} d^oV \right\}
\]

\[
= \left\{ S_{jk} \delta e_{jk} d^o\mathcal{V} + \int_{\mathcal{V}} S_{jk} \delta v_{jk} d^o\mathcal{V} + \int_{\mathcal{V}} S_{jk} \delta \eta_{jk} d^o\mathcal{V} \right\} \tag{3.44}
\]

where the bracketed integral is the nonlinear term, and it is the same term as in the previous form, Eq. 3.39. By the same argument as advanced above, this nonlinear integral may be neglected in order to linearize the equation. Therefore:

\[
\left\{ S_{jk} \frac{\partial x_i}{\partial k} v_j u_i d^oS + \int_{\mathcal{V}} S_{jk} u_i \frac{\partial v_j}{\partial k} u_i d^oV + \int_{\mathcal{V}} F_i \frac{\partial u_i}{\partial k} d^oV \right\}
\]

\[
= \left\{ S_{jk} \delta e_{jk} d^o\mathcal{V} + \int_{\mathcal{V}} S_{jk} \delta v_{jk} d^o\mathcal{V} \right\} \tag{3.45}
\]

In the equation above, it should be noted that the second surface traction integral is linear even though it is dependent on the deformation. This is true because \(2S_{jk}\) will be prescribed on the surface at the beginning of the increment, however the integral value itself is not known at the beginning of the increment, and is dependent on the incremental deformation solution obtained.

By linearizing the incremental virtual work equations a direct solution may be obtained of the field or set of incremental displacements.
under prescribed incremental surface tractions and body forces. By examining the linearized equations it is evident that the configuration $^1C$ must be known since the solution is dependent on values of stresses and displacements in that configuration.

The set of incremental displacements and stresses derived through the use of a direct solution technique will, when added to the corresponding values of configuration $^1C$, give the approximation to configuration $^2C$. This approximate solution will tend to diverge from the exact solution depending on the nonlinearity of the true problem, and the size of increment in the surface tractions and body forces chosen.

If errors due to the direct solution technique are ignored for the present, the divergence in the solution will arise from two sources. The first source of divergence is caused by configuration $^1C$ being known only approximately, because it has been derived itself from a previous configuration by a linearized incremental equation. The second source of divergence occurs within the increment from $^1C$ to $^2C$, and is a result of discarding the nonlinear term in the incremental virtual work equation. As this term becomes significant with respect to the linear terms retained, the solution obtained will diverge from the true solution. In general the nonlinear term will become important for increments of large deformation and for highly nonlinear problems.

It is desirable that there be some indication of the accuracy of the approximate solution obtained at the end of an increment. The degree of nonsatisfaction of the exact virtual work equation by the approximate configuration obtained is chosen, and is called the residual loading term. The residual loading term for configuration $^aC$ is therefore
defined as

\[ \alpha_{R_c} = \int_{S} \alpha_{T_{ji}} \delta u_i \, d^S + \int_{V} \alpha_{F_i} \delta u_i \, d^V - \int_{V} \alpha_{S_{jk}} \delta u_i \, d^V \]

(3.46)

It can be seen from Eq. 3.20, that if upon substituting the configuration obtained at the end of an increment into Eq. 3.46, \( \alpha_{R_c} \) is zero, then the configuration represents an equilibrium configuration, and hence the exact solution. If \( \alpha_{R_c} \) is not equal to zero then equilibrium is not exactly satisfied.

Similarly, \( \alpha_{R_c} \) can be defined for the case where a Kirchhoff stress surface traction is used, by

\[ \alpha_{R_c} = \int_{S} \alpha_{S_{jk}} \alpha_{x_{i'k}} \delta u_i \, d^S + \int_{V} \alpha_{F_i} \delta u_i \, d^V - \int_{V} \alpha_{S_{jk}} \alpha_{x_{i'k}} \delta u_i \, d^V \]

(3.47)

This parameter \( \alpha_{R_c} \), the residual loading term, sometimes called the load correction term, represents in a sense the unbalanced loading of the body. By unbalanced loading, it is meant the virtual work done by the surface tractions and body forces on the body, that is not balanced by the virtual work performed by the stresses and strains within the body in the approximate configuration derived.

The residual loading for a single degree of freedom problem has a clear physical concept as shown in Fig. 4. The residual loading term here is the difference between the load applied and the actual load
ONE-DIMENSIONAL LOAD-DEFLECTION GRAPH SHOWING THE RESIDUAL LOADING PARAMETER

FIG. 4
required for the deformation calculated by the linear increment \( \Delta L \).

It therefore represents the amount of load that the structure has not deformed to accept. When the problem is multidimensional, the physical concept becomes obscure but the mathematical concept is still valid.

The residual loading term would be useful even if it was only an abstract indication of the amount of error involved in the approximate solution at the end of an increment. However, this term can be employed to control or reduce the error in finding equilibrium configurations. This will be examined in the following chapter.

### 3.6 Surface Trazions and Body Forces

The form of the linearized incremental virtual work equation that is chosen for any particular problem is primarily dependent upon the ease with which the surface tractions can be represented. It is possible through the use of the relationships between the three different types of stress tensor given in Eq. 2.20, 2.21 and 2.22 to use either of the two forms derived and given by Eq. 3.43 and 3.45. The transformations between the stress tensors are not easily made, and are dependent on the configuration of the surface at the end of the increment, therefore they cannot be specified at the beginning of the increment.

The alternative is to use surface tractions that behave during deformation in such a manner that they follow the Lagrange or Kirchhoff rule of correspondence as shown in Section 2.3.

A force vector that actually behaves according to the Lagrange rule of correspondence has the property that it has a parallel line of action with respect to its line of action on the surface \( ^o S \), and has the
same magnitude. It is therefore easy to represent any surface traction that keeps a parallel line of action and the same magnitude under deformation, by a Lagrangian stress tensor. This type of surface traction is the most commonly used, implicitly or explicitly, in available closed-form solutions for large deformations.

A force vector that changes according to the Kirchhoff rule of correspondence would be most easily represented by a Kirchhoff stress type of surface traction.

Finally a surface traction that is normal to the deformed surface and proportional to it is represented by an Eulerian stress tensor in the deformed configuration. To be used by either of the two linearized incremental virtual work equations this stress tensor must be transformed into either a Kirchhoff or Lagrange type of surface traction. This type of loading is the correct representation for a pressure loading of a surface.

The use of the body force term in the equations deserves some discussion. It would be possible by applying D'Alembert's principle to include inertia forces as part of the body forces, but this lies outside of the scope of this thesis. The body forces are usually neglected in the closed-form solutions available, and so will not be used in the numerical analyses performed for comparison purposes where this is true of the closed-form solution.

### 3.7 Summary

The virtual work expressions developed in this chapter use the Lagrangian description. The equations for any particular configuration
and the nonlinear incremental virtual work equation are exact, valid for any magnitude of deformation, and are independent of the particular constitutive relationships chosen. They are merely expressions of the equilibrium of a body subjected to surface tractions and body forces.

The nonlinear incremental virtual work equation was linearized by neglecting the nonlinear integral in the equation. This linearized incremental virtual work can be used to provide a direct solution for incremental deformation given the configuration at the beginning of the increment and the incremental values of the surface traction and body force. This direct solution thus gives an approximation to the exact equilibrium configuration. The approximate solution will tend to diverge from the exact solution depending on increment size and the degree of nonlinearity of the problem.

Using the approximate solution obtained for the configuration at the end of the increment, equilibrium can be checked by substituting into the exact virtual work expression for the configuration. The nonsatisfaction of the virtual work equation, which is equivalent to nonsatisfaction of equilibrium, is represented by the residual loading term.

The numerical techniques to solve the equations derived for finite deformation are given in the following chapters.
SOLUTION OF THE EQUILIBRIUM EQUATIONS

4.1 General

There are various solution strategies available to solve the equilibrium equations, derived in the previous chapter in the form of virtual work equations. These strategies will be differentiated by the manner in which divergence from the exact equilibrium configurations, caused by using a linear incremental virtual work equation, is controlled or reduced. By linearizing the incremental virtual work equation only a linear prediction or approximation of the true configuration at the end of an increment is obtained. As was shown in the previous chapter, the degree of nonsatisfaction of equilibrium is expressed by the residual loading term.

If no effort is made to control the divergence of the solution through the use of the residual loading term, then the solution strategy is known as an incremental method without equilibrium checks. Conversely, if the residual loading term is evaluated and used to modify the solution obtained in order to reduce or control the divergence from the true equilibrium configuration, then the solution strategy is known as an incremental method with equilibrium checks. These may be of an iterative nature with a tolerance level of divergence on some parameter of the system, or they may be what will be called self-correcting procedures.

No attempt is made in this chapter to consider the various direct methods that can be used to solve the governing nonlinear equations,
or the predictor-corrector strategies such as the Runge-Kutta and Euler procedures. This chapter will be restricted to the consideration of incremental methods with or without equilibrium checks.

In this chapter the recurrence relations for the various solution strategies will be written in matrix form, but will be illustrated graphically by reference to a single degree of freedom system. The stiffness matrix is used in a general manner to represent the relationship between an input and a response.

4.2 Incremental method without Equilibrium Checks

In this method, the surface tractions and body forces are applied in a sequence of increments that are assumed to be sufficiently small, such that the body may be assumed to respond linearly during each increment. The response characteristics of the body are determined by the configuration and material properties of the body at the beginning of each increment.

For each increment in the loads on the body, incremental displacements, stresses and strains are evaluated and used together with the configuration of the body at the beginning of the increment to define the configuration at the end of the load increment. New response characteristics of the body are then calculated based on the configuration just obtained, and the next increment of load is applied. This whole process is repeated until the sum of the loads applied in all the increments equals the desired total load for which the solution is required.

This technique has the disadvantage that there is no real estimate of its accuracy, since the linearized incremental equations do
not in general give a configuration at the end of the load increment that satisfies equilibrium. Since each increment is dependent on the accuracy of the initial configuration and represents only a linear approximation to the nonlinear response in the increment, the solution tends to "drift" or diverge from the exact solution. The divergence of the incremental method without equilibrium checks is shown in Fig. 5, for a simple single degree of freedom system. To ensure that a solution obtained in this manner is close to the exact solution, recourse must be made to solving the same problem repeatedly with successively smaller increments until two successive solutions converge within some tolerance. By reducing the increment size, and hence using more increments to follow a nonlinear response, it is expected that the cumulative error in the solution will decrease. In the limit as the increment size shrinks to zero and the number of increments approaches infinity, the solution should become exact. Since new response characteristics of the body must be calculated for the beginning of each increment, this procedure can become quite time consuming and costly for reasonably sized problems.

The advantages of this method are mostly to be found in the analysis of bodies having nonlinear material behaviour. This is true primarily of elasto-plastic materials, since plasticity laws are generally written in incremental form and thus are easily incorporated into an incremental method. Since plastic deformation is a path dependent phenomenon, it is an attractive feature of this method that by choosing increments of load sufficiently small, and given an adequate plasticity description, the deformation history of the body should be traceable. In this case, this solution method is superior to
DIVERGENCE OF THE INCREMENTAL METHOD WITHOUT EQUILIBRIUM CHECKS

FIG. 5
an iterative method which may either oscillate around the correct solution or cause a false deformation path to be obtained.

The two basic subdivisions of approach within the incremental methods without equilibrium checks are the tangent modulus method and the initial strain method. Another approach which is in essence the same as the initial strain method is the initial stress method. The tangent modulus and initial strain methods are differentiated by the manner in which the nonlinear aspects of the problem are dealt with.

For the tangent modulus method, considering a general nonlinear problem, the recurrence relations may be written for the $i^{th}$ increment as

$$\begin{bmatrix} K_L \end{bmatrix} + \begin{bmatrix} K_{NLM}(u_{i-1}) \end{bmatrix} + \begin{bmatrix} K_{NLG}(u_{i-1}) \end{bmatrix} \begin{bmatrix} \Delta u \end{bmatrix}_i = \begin{bmatrix} \Delta P \end{bmatrix}_i \tag{4.1}$$

with

$$\{u\}_i = \sum_{j=1}^{i} \{\Delta u\}_j \tag{4.2}$$

In Eq. 4.1, $[K_L]$ is the linear stiffness matrix, and $[K_{NLM}(u_{i-1})]$ and $[K_{NLG}(u_{i-1})]$ are the material and geometric nonlinearity stiffness matrices respectively. The term $u_{i-1}$ indicates that the two nonlinear matrices are functions of the configuration at the beginning of the increment. The vectors $\{\Delta u\}_i$ and $\{\Delta P\}_i$ are the incremental generalized displacements and forces respectively, and $\{u\}_i$ represents the total generalized displacements at the end of the $i^{th}$ increment.
In the tangent modulus method the evaluation of the combined
stiffness matrix in Eq. 4.1 for each increment and its subsequent
inversion, to solve for the incremental displacements, causes this
method to become very costly as the number of iterations required or the
size of the problem increases. It is to reduce the cost associated with
inverting a stiffness matrix for each increment that the initial strain
method was developed.

Considering again a general nonlinear problem, the recurrence
relations for the initial strain method are written as

\[
[K_L] \{\Delta u\}_i = \{\Delta P\}_i + \{Q_{NLG}\}_i - 1 + \{Q_{NLM}\}_i - 1
\]  

(4.3)

again using Eq. 4.2 to sum the incremental generalized displacements.
The vectors \{Q_{NLG}\}_i - 1 and \{Q_{NLM}\}_i - 1 are the pseudo-loads for geometric
and material nonlinearities respectively, and are based upon the
configuration at the beginning of the increment. They are derived by
considering their respective source of nonlinear deformation as an
initial strain for the next increment.

The advantage to be derived from the application of this method
is that only the matrix \[K_L\] need be inverted, and this is done only
once. The effort in evaluating the pseudo-load vectors of Eq. 4.3 is of
the same order as the effort required to evaluate the nonlinear stiffness
matrices in Eq. 4.1, so the main advantage is to be found in the
requirement to invert a stiffness matrix only once. The disadvantages of
this method as compared to the tangent modulus method, are that with a
reduction in load increment size it converges slower and also this method
may have numerical instabilities, especially when moderate geometric
nonlinearities are encountered [18].
The tangent modulus method appears to be more advantageous than the initial strain method as concerns convergence and numerical stability, but the cost is higher for a given number of increments used to apply the full load.

4.3 Incremental methods with Equilibrium Checks

4.3.1 General

The incremental methods without equilibrium checks do not in general give solutions that satisfy equilibrium. Cumulative errors may become significant as a result, and it then becomes necessary to solve the problem with smaller load steps, and hence more increments, until convergence of two successive solutions provides a degree of confidence in the results.

The incremental methods with equilibrium checks were devised in order that the divergence of the approximate solution could be evaluated and then reduced or controlled in some manner. At the same time the desirable aspects of an incremental formulation are maintained.

Once the divergence of the approximate solution from the exact solution is known in terms of the residual loading term, then the approximate solution is either modified by an iterative process, or else the residual loading term is used to modify the next incremental step. The latter approach is called a self-correcting method. Both approaches will be discussed in the following sections, and the appropriate recurrence relations given.
4.3.2 Iterative Methods

The iterative methods are used to modify the solution repeatedly within a particular increment, until some convergence criteria is satisfied. The ideal approach would be to iterate within the increment until the configuration acquired at the end of the increment exactly satisfies equilibrium. This would be observed by the vanishing of the residual loading term. Such an approach is rarely practical however, and thus the usual technique is to iterate until some convergence tolerance on the magnitude of the residual loading term is satisfied. This tolerance is chosen sufficiently small so that the approximate solution will be close enough to the exact solution for the purpose required. The convergence tolerance could be specified for any parameter of the configuration such as a stress or a displacement, but it is usual to use the residual loading term since an equilibrium configuration is sought.

Two different iterative methods that may be adopted are the Newton-Raphson method shown in Fig. 6, and the Modified Newton-Raphson method shown in Fig. 7. The difference between these two methods is in the manner in which the stiffness matrix is handled during the iterations to find the configuration at the end of an increment.

In the Newton-Raphson method, for each iteration the residual loading term is evaluated and the stiffness matrix updated. Then the residual loading term is applied to find an increase in the incremental deformations to bring the approximate configuration closer to the exact solution. This is repeated until the residual loading term is sufficiently small to satisfy the tolerance criteria. The procedure can be expressed for the first step in the iteration by
DEFLECTION

NEWTON-RAPHSON METHOD

FIG. 6
FIG. 7

MODIFIED NEWTON-RAPHSON METHOD

DEFLECTION
and for each iteration within the $i^{th}$ increment thereafter by

$$
\left[ \begin{array}{c}
K_L \\
K_{NL}\left( u_{i-1} + \Delta u_{i-1} \right)
\end{array} \right]
\left\{ \Delta u \right\}_i^j = \left\{ \Delta P \right\}_i^j \quad j = 1 \quad (4.4)
$$

where $j$ is the number of the iteration within the increment and

$$
\Delta u_i^j = \sum_{k=1}^{j} \Delta u_i^k
$$

(4.6)

$$
\left\{ R_c \right\}_i^j = \left\{ P \right\}_i^j - \left[ \begin{array}{c}
K_L \\
K_{NL}\left( u_{i-1} + \Delta u_{i-1} \right) + \Delta u_i^j
\end{array} \right] \left\{ u_{i-1} + \Delta u_i^j \right\}
$$

(4.7)

This procedure ceases when the residual loading term satisfies the tolerance required. Then the next increment of loading is applied. The matrix $[K_{NL}]$ includes both geometric and material nonlinearity.

The Modified Newton-Raphson method has the same format as above except that Eq. 4.5 is replaced by
As can be seen, this method does not use an updated stiffness matrix after each iteration, but rather uses the stiffness matrix derived at the beginning of the increment for all iterations. This saves calculation effort since the same stiffness matrix in Eq. 4.4 and 4.8 need only be inverted once for all iterations within the increment. In an extension to this method, the stiffness matrix for one increment is used for several successive increments until convergence deteriorates, then a new stiffness matrix is evaluated and inverted.

The Newton-Raphson method generally should converge in fewer iterations than the Modified Newton-Raphson method, but the latter may be more computationally efficient by not requiring the stiffness matrix to be evaluated and inverted after each iteration. Both methods would appear to give more informative results than an incremental method without equilibrium checks, simply because of the control on the degree of nonsatisfaction of equilibrium permitted. This is true at least for conservative deformation; in nonconservative deformation however, the iterative nature of the methods may adversely influence the solution by causing an oscillation of the iterations about the true solution.

4.3.3 Self-correcting Method

The self-correcting method is derived in the attempt to obtain the advantages of having an equilibrium check that is used to control the divergence of the solution, without the disadvantages of iterating. The particular procedure used in this thesis can be expressed by the
recurrence relation as

\[
\begin{bmatrix}
K_L + K_{NL}(u_{i-1})
\end{bmatrix}
\{\Delta u\}_i = \{\Delta P\}_i + \{R_{c_{i-1}}\}
\]  \hspace{1cm} (4.9)

where

\[
\{R_{c_{i-1}}\} = \{P\}_{i-1} - \begin{bmatrix}
K_L + K_{NL}(u_{i-1})
\end{bmatrix}
\{u\}_{i-1}
\]  \hspace{1cm} (4.10)

and where the total displacements are summed using Eq. 4.2 as before, and the total loads are summed similarly. The procedure is shown graphically in Fig. 8, for a single degree of freedom system.

The stiffness matrix in Eq. 4.9 is derived on the basis of the configuration at the beginning of the increment, the contribution \([K_{NL}(u_{i-1})]\) being a function of the deformation at the start of the increment. The distinguishing feature of this method is that the residual loading term from the previous increment, which is evaluated using Eq. 4.10, is added to the load specified for the present increment.

This procedure may be thought of as a one-step iteration of the Newton-Raphson method for the past increment, added into the present increment of deformation. Stricklin, et al [19] multiply the residual loading term in Eq. 4.9 by a scalar with a range of 1.0 to 1.2, in an attempt to improve the convergence of this method. The rationale behind this would appear to be that the trend in nonlinearity over the present increment of deformation will be similar in form to the past increment, so the factor applies an increased residual loading term to anticipate this. This technique of employing a scalar amplifier with the residual loading term will not be used or evaluated in this thesis.
FIG. 8

SELF-CORRECTING METHOD
4.4 Summary

Solution strategies for solving the linearized incremental virtual work equations have been presented. These are classified by whether or not there is a procedure to control or reduce the non-satisfaction of equilibrium caused by linearizing the virtual work equations.

Incremental methods with equilibrium checks include iterative and self-correcting procedures. The self-correcting procedure presents a combination of equilibrium control that is desirable without the cost and disadvantages of iterating. The self-correcting procedure will be used in this thesis in the numerical solutions presented in Chapter 7.
CONSTITUTIVE RELATIONSHIPS

5.1 General

In order to analyze the deformation of any body under prescribed surface tractions, body forces and boundary conditions using the incremental virtual work equations developed in Chapter 3, the Kirchhoff stress tensor within the body must be related to the Green's strain tensor. Without such a constitutive relationship, an analysis cannot proceed, since there would be then an infinite number of possible solutions for the deformation of the body.

The choice of a constitutive relationship should be such that the model, which is defined by the incremental virtual work equations, should represent accurately the characteristics of the real body. To this end, many constitutive relationships have been proposed. These include elasto-plastic relationships of various kinds, viscoelasticity, hyperelasticity, hypoelasticity, and elastic constitutive relationships.

In this thesis, it is intended that a linear elastic constitutive relationship between Kirchhoff stress and Green's strain tensors be adopted. Furthermore the relationship will be assumed to be that of Hooke's law for an isotropic material.

The development in the following section will be for a general elasticity case, including the linear elastic, isotropic relationship as a special case.
5.2 Elastic Constitutive Tensor

Assuming only elastic deformation for the present, the Kirchhoff stress tensor and Green's strain tensor may be related by

\[ aS_{ij} = aC_{ijkl} \varepsilon_{kl} \]  (5.1)

where \( aC_{ijkl} \) is the elastic constitutive tensor for configuration \( aC \). For the incremental Kirchhoff stress, \( S_{ij} \), between configurations \( 2C \) and \( 1C \), then

\[ S_{ij} = 2C_{ijkl} \varepsilon_{k\ell} - 1C_{ijkl} \varepsilon_{k\ell} \]

\[ = 2C_{ijkl} \varepsilon_{k\ell} + (2C_{ijkl} - 1C_{ijkl})\varepsilon_{k\ell} \]  (5.2)

The incremental constitutive relationship given in Eq. 5.2 is capable of representing nonlinear and nonisothermal constitutive relationships. The use of this relationship would however require iteration since the configuration \( 2C \) is not known a priori, and hence its elastic constitutive tensor \( 2C_{ijkl} \) is unknown. A reasonable simplification to overcome this problem is to assume that the change in the constitutive tensor for the increment is negligible, then

\[ S_{ij} = 1C_{ijkl} \varepsilon_{k\ell} \]  (5.3)

The residual loading term could then be employed to limit the error introduced by evaluating the stresses within the body at the end of the increment using Eq. 5.1.
A further simplification still needs to be made, since $e_{k\ell}$ is nonlinear for the increment of deformation. By introducing Eq. 5.3 into the linearized incremental virtual work equations they would again become nonlinear. Therefore only the linear portion of the incremental Green's strain tensor is used to evaluate the incremental Kirchhoff stress, thus modifying Eq. 5.3

$$S_{ij} = ^1C_{ij\ell\ell} e_{k\ell} + ^1C_{ijkl} \gamma_{k\ell}$$

and neglecting the nonlinear term as being of smaller order

$$S_{ij} = ^1C_{ij\ell\ell} e_{k\ell}$$

(5.5)

For the problems analyzed in this thesis, the constitutive relationship will be taken as being linearly elastic. Therefore the differentiation between constitutive tensors in different configurations disappears. The only approximation that remains is the necessary employment of only the linear portion of the incremental Green's strain tensor in Eq. 5.5. This is required in order to keep the incremental equations linear, so that a direct solution may be obtained. In the residual loading term however, this will be corrected for by using the full Green's strain tensor there.

The stress tensor $^1S_{jk}$ in the second integral on the right hand side of Eq. 3.43 will also be evaluated using the full Green's strain tensor and the constitutive relationship given in Eq. 5.1. This is possible because configuration $^1C$ is known at the beginning of the increment.
Now the choice of the elements of the constitutive tensor is still undetermined. It is intended that they be chosen so as to represent an isotropic material corresponding to Hooke's law. This can be expressed in indicial notation as

\[ C_{ijkl} = G \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + \delta_{ik} \delta_{kl} \frac{\nu E}{(1+\nu)(1-2\nu)} \]  

(5.6)

This gives a symmetric constitutive relationship, where \( G \) is the shear modulus, \( E \) is Young's modulus, and \( \nu \) is Poisson's ratio.
APPLICATION OF THE FINITE ELEMENT METHOD

6.1 General

The finite element method is introduced in this chapter and then applied to the incremental virtual work equation derived in Chapter 3, in order to provide a procedure for numerical analysis. The incremental virtual work equations will be recast in matrix form using the finite element method for the general three dimensional case, then specialized for two-dimensional analysis.

Constitutive relationships will be developed for plane strain and plane stress analysis, by assuming a linear elastic relationship between the Kirchhoff stress tensor and Green's strain tensor. The relationship used will be Hooke's law for an isotropic material.

Two eight degree of freedom rectangular finite elements will be derived using the assumed displacement approach. These elements will be formed using numerical integration, and only differ from each other in the integration scheme adopted. The first will be integrated using Gaussian quadrature and this element is then just the Melosh rectangle [20]. The second finite element will use a nonuniform integration scheme in an attempt to improve the accuracy obtained, for given computational effort, over that of the Melosh rectangle.

6.2 The Finite Element Method

The finite element method is a technique whereby a continuous solution of a problem over a specified domain may be represented by
piecewise continuous approximations. It is a purely topological approximation and is independent of the variational principles used, and the techniques used to minimize the error between the approximation and the actual continuous solution. The topological nature of the finite element method was presented by Oden [21], and the basic steps involved in the method were developed in the same paper.

The first step in any finite element analysis is to replace the specified domain of the continuous solution by another domain which can be divided into a finite number of subdomains called elements. These elements do not overlap. It is desirable to have this new domain be equivalent to the domain of the continuous solution, but this may not in all cases be possible. If this is the case then the difference between the two domains, called the error domain, should be kept as small as possible. This will be a problem encountered when trying to model complex domain boundaries with elements having a simpler geometric form. For example, in replacing the domain of the continuous solution where this domain has curved boundaries by a domain made up of finite elements having only straight boundaries, the new domain will only approximate that of the continuous solution. By using more elements of smaller dimension along the boundary, the error domain may be made to be as small as desired.

In the second step, the individual elements are assumed to be connected to adjacent elements only, and to these elements at only a finite number of discrete points called nodes. These nodes are generally located on the boundaries of the element since interior nodes may not be connected to any adjacent element without causing the elements to overlap in their domains. Interior nodes if they occur are usually statically condensed out of the element. Now the value of the solution and the derivatives, if applicable, at the nodes will be the
unknown parameters of the problem.

If the actual continuous solution on the specified domain is \( \omega \), then an approximate solution \( \omega^e \) is defined uniquely in each of the elements for any point \( p \) within the element by

\[
\omega^e(p) = \sum_{j=1}^{N} a_j^e \phi_j^e(p) \quad j = 1, \ldots, N \tag{6.1}
\]

where \( \phi_j^e(p) \) are known coordinate functions, or interpolation functions, defined in the domain of the element concerned only, \( a_j^e \) are the nodal values of the solution, and \( N \) is the number of nodal values, or degrees of freedom of the element. The coordinate functions are required to satisfy the condition that if the nodal coordinates of node \( n \) are given by \( x_n^1 \), then

\[
\phi_j^e(x_n^1) = \delta_j^n \tag{6.2}
\]

By satisfying this condition, the coordinate functions will be linearly independent throughout the element domain.

The third step is to combine all the elements to form the approximate domain by equating the nodal parameters of the corresponding nodes on the interelement boundaries of adjacent elements. The approximation \( \tilde{\omega} \) is now specified over the whole of the approximate domain. By combining the elements, there results a system of \( M \) linearly independent coordinate functions, and \( M \) unknown nodal parameters. The value \( M \) is numerically equal to the number of distinct nodes left in the problem after combining the elements, multiplied by the number of degrees of freedom per node, minus the constrained degrees of freedom representing
boundary conditions. Then the equations for determining the $M$ unknown nodal parameters may be obtained by using weighted residual methods such as the Galerkin method, least squares method, collection method, or in the case of a virtual work formulation by the $M$ independent equations that can be generated.

In using the finite element method, it is highly desirable that some form of convergence of the approximate solution to the real continuous solution be known to exist if certain readily identifiable criteria are satisfied. The two basic criteria upon which convergence proofs have been developed are the completeness of interpolation functions within the elements, and inter-element compatibility. The necessity and/or sufficiency of these two criteria to ensure convergence, has been a topic of interest and dispute since the first investigations were made into the convergence of the finite element method.

The criteria of completeness arises from the requirement that the highest derivative of the interpolation functions involved in the formulation of the energy or virtual work integrals must be continuous, and be able to take finite values within the element. This was enunciated by Oliveira [22, 23] in requiring displacement polynomial interpolation functions to be complete to order $p$, where $p$ is the order of the maximum derivative of displacements encountered in the energy integrals. Oliveira [23] in one of the first convergence proofs for the finite element method, concluded that only completeness of the interpolation functions was necessary to guarantee convergence in the limit as the element mesh is refined. The necessity of completeness for convergence gave a theoretical basis for an earlier intuitive requirement advanced by Melosh [24], that elements should be capable of exactly representing rigid body modes and constant strain states. The non-convergence of some
complete, but incompatible elements, indicates that more is required of an element to ensure convergence, than just completeness. Thus completeness is a necessary but not a sufficient condition for convergence of the finite element approximation to the exact continuous solution. Some use has been made of incomplete elements, and although they will not ultimately converge to the exact continuous solution, they may converge to a solution which is only slightly in error.[25].

The criteria of compatibility may be approached from different viewpoints, perhaps the most popular of which is through the use of the theorem of minimum potential energy when using displacement interpolation functions. Using this approach a compatible element is one which has a sufficient degree of inter-element continuity, such that the total potential energy of the system being analysed converges monotonically to a minimum as the subdivision of the domain or mesh is refined. The degree of compatibility sufficient to ensure this convergence, given completeness, has been obtained from convergence proofs. For compatibility, if a dependent variable enters the energy expressions with the highest derivative of order \( q \) \( (q > 0) \), then the \( q - 1 \) derivative of that variable must be continuous across inter-element boundaries [26].

For an element that satisfies both compatibility and completeness requirements as outlined here, monotonic convergence to the correct minimum potential energy is assured as the finite element mesh is refined. There is therefore a great advantage in using an element of this type, and an order of accuracy analysis is available to give an expected convergence rate, at least with respect to potential energy. McLay presents the basis of such an analysis beginning with Taylor’s theorem [27].
The use of complete but incompatible elements has been shown to have many advantages, since many of these elements appear to perform better than compatible elements. Until recently however, their convergence could not be expected, since no convergence proof existed. Now it would appear that if the element satisfies the patch test, then convergence can be expected although no order of accuracy analyses yet exist. Strang and Fix state that successful performance of an incompatible element in the patch test is both necessary and sufficient for convergence [28].

The application of the finite element method has therefore, the primary advantage of reducing the problem of obtaining a continuous solution having an infinite number of degrees of freedom, to that of an approximate solution expressed in terms of a finite number of degrees of freedom. This method permits the acquisition of solutions, although only approximate, to problems that are otherwise intractable.

6.3 Incremental Virtual Work Equations Incorporating the Finite Element Method

6.3.1 The Assumed Displacement Approach

The application of the finite element method to the incremental virtual work equations developed in Chapter 3, will be accomplished by adopting the assumed displacement approach. In this approach the continuum is first discretized into a number of finite elements. Within each element interpolation functions are prescribed which uniquely define the generalized displacements within the element in terms of the generalized nodal displacements. This is expressed by
\[ u_i(p) = N_{ij}(p) \, u_j^e \]  \hspace{1cm} (6.3)

where

- \( u_i(p) \) = generalized displacement at a point \( p \) in the domain of the finite element
- \( u_j^e \) = nodal generalized displacements of the finite element.
- \( N_{ij}(p) \) = interpolation or shape functions which are functions of the point \( p \), for the element.

Adopting matrix notation, this may be alternatively written as

\[ \{u\} = [N(p)] \, \{u\}^e \] \hspace{1cm} (6.4)

where

- \( \{u\} \) = vector of generalized displacements at point \( p \).
- \( [N(p)] \) = matrix of interpolation functions for point \( p \).
- \( \{u\}^e \) = vector of nodal generalized displacements for the element.

### 6.3.2 The Incremental Virtual Work Equations

The incremental virtual work equations developed in Chapter 3 and given by either Eq. 3.36 or 3.38, are exact expressions for the incremental deformation of a body. They are however nonlinear, and therefore the incremental expressions were linearized in order to be able to employ a direct solution method. The resulting linearized
equations are given by Eq. 3.43 and 3.45, and it is these equations that will be used along with the finite element procedure.

First, Eq. 3.43 will be rewritten in matrix form and then the finite element approximation will be introduced in general for the three-dimensional problem. Throughout this chapter, it will be assumed that the deformation is for the increment between two neighboring arbitrary configurations, from configuration \( C \) to configuration \( ^2C \). This is consistent with the increment of deformation chosen in Chapter 3.

Define the incremental Kirchhoff stress vector as

\[
\{S\}^T = \langle S_{11} \quad S_{12} \quad S_{13} \quad S_{21} \quad S_{22} \quad S_{23} \quad S_{31} \quad S_{32} \quad S_{33} \rangle
\]  

and the incremental linear component of Green's strain vector as

\[
\{e\}^T = \langle e_{11} \quad e_{12} \quad e_{13} \quad e_{21} \quad e_{22} \quad e_{23} \quad e_{31} \quad e_{32} \quad e_{33} \rangle
\]  

Similarly the incremental nonlinear component of Green's strain vector is defined as

\[
\{n\}^T = \langle n_{11} \quad n_{12} \quad n_{13} \quad n_{21} \quad n_{22} \quad n_{23} \quad n_{31} \quad n_{32} \quad n_{33} \rangle
\]  

Now the linear component of Green's strain vector for the increment, \( \{e\} \), can be separated into two vectors, one including the strictly linear terms \( \{e_L\} \), and the other representing the nonlinear effects of the initial displacements for the increment \( \{e_{NL}\} \). Both \( \{e_L\} \) and \( \{e_{NL}\} \) are linear for the increment, but the former is independent of previous deformation, while the latter depends on the deformed
configuration at the beginning of the increment. Thus

\[ \{e\} = \{e_L\} + \{e_{NL}\} \]  \hspace{1cm} (6.8)

where both \(\{e_L\}\) and \(\{e_{NL}\}\) are defined in an analogous manner to \(\{e\}\) in Eq. 6.6. The elements of the vectors \(\{e_L\}\) and \(\{e_{NL}\}\) are given by

\[ (e_{ij})_L = \frac{1}{2}(u_{i,j} + u_{j,i}) \]  \hspace{1cm} (6.9)

and

\[ (e_{ij})_{NL} = \frac{1}{2}(u_{k,i}u_{k,j} + u_{k,j}u_{k,i}) \]  \hspace{1cm} (6.10)

This can be seen to follow from the use of Eq. 2.11 and Eq. 6.8.

Now relating the strain vectors defined above, to the incremental three-dimensional displacement vector defined as

\[ \{u\}^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \]  \hspace{1cm} (6.11)

it is necessary to define two matrix operators \([B_L]\) and \([B_{NL}(1u)]\) by the relationships,

\[ \{e_L\} = [B_L]\{u\} \]  \hspace{1cm} (6.12)

\[ \{e_{NL}\} = [B_{NL}(1u)]\{u\} \]  \hspace{1cm} (6.13)

The notation \([B_{NL}(1u)]\) signifies that this operator matrix is a function of the initial displacements for the increment. Both these operators are given in full in Appendix A, and are derived from Eq. 6.9 and 6.12 for \([B_L]\) and Eq. 6.10 and 6.13 for \([B_{NL}(1u)]\). A relationship between these two operators is also examined, to reduce computational effort, in Appendix A.
Finally, by virtue of the decision to use a linear symmetric constitutive relationship as expressed in Eq. 5.5, the constitutive equation in matrix form is

$$\{S\} = [C] \{e\} \quad (6.14)$$

where \([C]\) is the constitutive matrix.

Considering each integral of Eq. 3.43 in turn, rewrite the equation in matrix form beginning with

$$\int_{\delta V} S_{jk} \delta e_{jk} \, d\delta V = \int_{\delta V} \{S\}^T \{\delta e\} \, d\delta V$$

$$= \int_{\delta V} \{e\}^T [C]^T \{\delta e\} \, d\delta V$$

$$= \int_{\delta V} \{\delta e\}^T [C] \{e\} \, d\delta V \quad (6.15)$$

Using Eq. 6.14 and 6.8, and the relationships given in Eq. 6.12 and 6.13 this can be written as

$$\int_{\delta V} S_{jk} \delta e_{jk} \, d\delta V = \int_{\delta V} \{\delta u\}^T [B_L]^T [C] [B_L] \{u\} \, d\delta V$$

$$+ \int_{\delta V} \{\delta u\}^T [B_{NL}(1u)]^T [C] [B_L] \{u\} \, d\delta V$$

$$+ \int_{\delta V} \{\delta u\}^T [B_L]^T [C] [B_{NL}(1u)] \{u\} \, d\delta V$$

$$+ \int_{\delta V} \{\delta u\}^T [B_{NL}(1u)]^T [C] [B_{NL}(1u)] \{u\} \, d\delta V \quad (6.16)$$
Considering for the moment only one finite element domain, then by using the expression in Eq. 6.4, and substituting this into Eq. 6.16,

\[ \int_{\Omega} \int_{\mathcal{J}_k} \delta e_{jk} \, d\mathcal{V} = \sum_{e} \{\delta u\}^e \begin{bmatrix} (\mathbf{N})^T \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \{u\}^e + \{\delta u\}^e \int_{\Omega} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \{u\}^e + \{\delta u\}^e \int_{\Omega} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (NL_{1u}) \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \{u\}^e + \{\delta u\}^e \int_{\Omega} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (NL_{1u}) \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \{u\}^e \]

(6.17)

where the right hand superscript \(e\) is used to indicate elemental nodal displacements as introduced in Eq. 6.4. The above equation can also be expressed in the form

\[ \int_{\Omega} \int_{\mathcal{J}_k} \delta e_{jk} \, d\mathcal{V} = \{\delta u\}^e \begin{bmatrix} (K_L) \end{bmatrix} \{u\}^e + \{\delta u\}^e \begin{bmatrix} (K_{NL1}) \end{bmatrix} \{u\}^e + \{\delta u\}^e \begin{bmatrix} (K_{NL2}) \end{bmatrix} \{u\}^e + \{\delta u\}^e \begin{bmatrix} (K_{NL3}) \end{bmatrix} \{u\}^e \]

(6.18)

where

\[ [K_L] = \int_{\Omega} \begin{bmatrix} (N)^T \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \]

(6.19)

\[ [K_{NL1}] = \int_{\Omega} \begin{bmatrix} (N)^T \end{bmatrix} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \]

(6.20)

\[ [K_{NL2}] = \int_{\Omega} \begin{bmatrix} (N)^T \end{bmatrix} \begin{bmatrix} (B_L)^T \end{bmatrix} \begin{bmatrix} (C) \end{bmatrix} \begin{bmatrix} (B_{NL})^T \end{bmatrix} \begin{bmatrix} (NL_{1u}) \end{bmatrix} \begin{bmatrix} (N) \end{bmatrix} \, d\mathcal{V} \]

(6.21)
The symbol $\approx$ is used here to indicate the approximation of the exact variational term, only in the sense that the finite element method introduces an approximation.

It should be noted that $[K_L]$ given by Eq. 6.19 is the usual small strain, small displacement stiffness matrix. Also it is worth noting that

$$[K_{NL1}] = [K_{NL2}]^T$$

(6.23)

since $[C]$ is symmetric. This means that only one of these two stiffness matrices need be evaluated, and then just transposed to give the other. Since $[K_L]$ and $[K_{NL3}]$ are both symmetric the addition of the four matrices will give a symmetric matrix.

Consider next the second integral on the right hand side of Eq. 3.43,

$$\int_{\Omega} \delta \eta_{jk} \ d\Omega = \int_{\Omega} \{1^S\}^T \{\delta \eta\} \ d\Omega$$

(6.24)

The vector of Kirchhoff stresses for configuration $1^C$, $\{1^S\}$, can be evaluated using the total Green's strain vector for the same configuration, $\{1^\varepsilon\}$. From Eq. 5.1, assuming the constitutive relationship to be constant throughout deformation, in matrix form

$$\{1^S\} = [C] \{1^\varepsilon\}$$

(6.25)
where

\[
\{\varepsilon^1\}^T = \begin{bmatrix} \varepsilon_{11}^1 & \varepsilon_{12}^1 & \varepsilon_{13}^1 \\ \varepsilon_{21}^1 & \varepsilon_{22}^1 & \varepsilon_{23}^1 \\ \varepsilon_{31}^1 & \varepsilon_{32}^1 & \varepsilon_{33}^1 \end{bmatrix}
\]  \hspace{1cm} (6.26)

and from Eq. 2.6

\[
\varepsilon_{ij}^1 = \frac{1}{2}(u_{i,j}^1 + u_{j,i}^1 + u_{k,i}^1 u_{k,j}^1)
\]  \hspace{1cm} (6.27)

Separating \(\{\varepsilon^1\}\), it can be seen that

\[
\{\varepsilon^1\} = \{\varepsilon^1_L\} + \{\eta^1\}
\]  \hspace{1cm} (6.28)

where \(\{\varepsilon^1_L\}\) and \(\{\eta^1\}\) are defined analogously to their incremental counterparts, only instead of incremental displacements \(u_i\), the total displacements for configuration \(1^C\), \(1u_i\), are used. The elements of \(\{\eta^1\}\) are given by

\[
\eta_{ij}^1 = \frac{1}{2}(u_{k,i}^1 u_{k,j}^1)
\]  \hspace{1cm} (6.29)

By comparing \(\{\eta^1\}\) to \(\varepsilon_{NL}\), it can be seen that

\[
\{\eta^1\} = \frac{1}{2} [B_{NL} (1u)] \{1u\}
\]  \hspace{1cm} (6.30)

and \(\{\varepsilon^1_L\}\) is given by

\[
\{\varepsilon^1_L\} = [B_L] \{1u\}
\]  \hspace{1cm} (6.31)
Therefore, Eq. (6.25) becomes

\[
\{1S\} = [C] [B_L] \{1u\} + \frac{1}{2}[C] [B_{NL}(1u)]'\{1u\} \tag{6.32}
\]

and applying the finite element method

\[
\{1S\} = [C] [B_L] [N] \{1u\}^e + \frac{1}{2}[C] [B_{NL}(1u)] [N] \{1u\}^e \tag{6.33}
\]

Thus knowing the nodal displacements for a configuration \(^1C\) of the finite element model, the full Green's strain and correct Kirchhoff stress vectors may be obtained.

Returning to Eq. 6.24, \(\{1S\}^T\) is now available through the use of Eq. 6.33 but the vector \(\{6\}n\) is not readily calculable. The product of the two vectors has to be recombined in a more readily useable form. To do this, define the vector of material derivatives of the incremental displacements as

\[
\{u_{i,j}\} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{6.34}
\]

and define an operator matrix \([L]\) such that

\[
\{u_{i,j}\} = [L] \{u\} \tag{6.35}
\]

The operator matrix \([L]\) is given fully in Appendix A. Now also define a new matrix composed of the elements of \(\{1S\}\) and called \([1ST]\), where this matrix is arranged such that
\[
\{1S\}^T \{\delta \eta\} = \{\delta u_{i,j}\}^T [1ST] \{u_{i,j}\}
\] (6.36)

The matrix \([1ST]\) is also given fully in Appendix A. Then inserting the result of Eq. 6.36 into Eq. 6.24, and also using Eq. 6.35

\[
\int_S \int \delta \eta_{jk} d^oV = \int_{\{\delta u\}^T [L]}^T [1ST] [L] \{u\} d^oV
\] (6.37)

Next considering just one finite element domain

\[
\int_S \int \delta \eta_{jk} d^oV = \int_{\{\delta u\}^e}^T [N]^T [L]^T [1ST] [L] [N] d^oV \{u\}^e
\]

\[
= \{\delta u\}^e [K_{IS}] \{u\}^e
\] (6.38)

where \([K_{IS}]\) is called the initial stress matrix, and is given by

\[
[K_{IS}] = \int_{\{\delta u\}^e}^T [N]^T [L]^T [1ST] [L] [N] d^oV
\] (6.39)

This matrix is called the initial stress matrix, since it represents the effect of the stresses present at the beginning of the increment on the incremental deformation. Note also that \([K_{IS}]\) is symmetric.

Now the right hand side of Eq. 3.43, using the finite element method may be given, for a single element, as

\[
\int_S \int e_{jk} d^oV + \int_S \int \delta \eta_{jk} d^oV \overset{\sim}{=} \{\delta u\}^e \left[ [K_L] + [K_{NL1}] + [K_{NL2}] + [K_{NL3}] + [K_{IS}] \right] \{u\}^e
\] (6.40)
Since $[K_L], [K_{NL3}]$ and $[K_{IS}]$ are all symmetric, and adding $[K_{NL1}]$ to its transpose $[K_{NL2}]$ gives a symmetric matrix, the stiffness matrix resultant is symmetric. This is expected, since the stiffness matrix for any linear increment should be symmetric by Betti's theorem.

Continuing with the loading integrals given on the left hand side of Eq. 3.43, by defining an incremental body force vector $\{F\}$ as

$$\{F\}^T = < F_1, F_2, F_3 >$$  \hspace{1cm} (6.41)

then

$$\int_\Omega \rho \delta u_i \delta F = \int_\Omega \rho \{F\}^T \{\delta u\} d\Omega$$  \hspace{1cm} (6.42)

similarly, considering the surface traction integral, define first an incremental surface traction vector $\{T^{(L)}\}$ where

$$\{T^{(L)}\}^T = < T_1^{(L)}, T_2^{(L)} , T_3^{(L)} >$$  \hspace{1cm} (6.43)

and the elements of this vector are given by

$$T_i^{(L)} = T_{ji} \delta v_j$$  \hspace{1cm} (6.44)

Then the surface traction integral becomes

$$\int_S T_{ji} \delta v_j \delta u_i dS = \int_S \{T^{(L)}\}^T \{\delta u\} dS$$  \hspace{1cm} (6.45)
Introducing the finite element method of approximation, then the left hand side of Eq. 3.43 is represented as

\[ \int_{S} T_{j}(\delta u_{j} \, dS) + \int_{V} F_{i} \, (\delta u_{i} \, dV) = \{\delta u\}^{T} \int_{S} [N]^{T} \{T(L)\} \, dS + \{\delta u\}^{T} \int_{V} [\rho]^{T} \{F\} \, dV \]  \hspace{1cm} (6.46)

by using Eq. 6.42 and 6.45, for a single element.

Finally the linearized incremental virtual work equation is written in matrix form, utilizing the finite element method, by combining the results of Eq. 6.40 and 6.46 to give, for a single element,

\[ \{\delta u\}^{T} \int_{S} [N]^{T} \{T(L)\} \, dS + \{\delta u\}^{T} \int_{V} [\rho]^{T} \{F\} \, dV \]

\[ = \{\delta u\}^{T} \left[ [K_{L}] + [K_{NL}] \right] \{u\}^{e} \]  \hspace{1cm} (6.47)

where

\[ [K_{NL}] = [K_{NL1}] + [K_{NL2}] + [K_{NL3}] + [K_{IS}] \]  \hspace{1cm} (6.48)

A similar approach could have been adopted starting with Eq. 3.45 which is essentially the same equation except for a surface traction defined in different terms. Defining a different incremental surface traction vector, than the one in Eq. 6.43, as

\[ \{T(K)\}^{T} = < T_{1}(K) \, T_{2}(K) \, T_{3}(K) > \]  \hspace{1cm} (6.49)
where the elements of the vector are given by

\[ T_i^{(K)} = S_{ji} \delta v_j \]  \hspace{1cm} (6.50)

and also defining

\[ [^1u_{i,j}] = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \]  \hspace{1cm} (6.51)

then Eq. 3.45 can be written for a single element as

\[
\{\delta u\}^T \int_{\Omega} [N]^T \{T(K)\} d\Omega \{\delta u\}^T + \{\delta u\}^T \int_{\Omega} [N]^T [^1u_{i,j}] \{T(K)\} d\Omega
\]

\[
+ \{\delta u\}^T \int_{\Omega} \{\kappa \} d\Omega
\]

\[ = \{\delta u\}^T \left[ [K_L] + [K_{NL}] \right] \{u\}^e \]  \hspace{1cm} (6.52)

The equations given above for the incremental deformation between configurations \( ^1C \) and \( ^2C \), employing the finite element method, are written for a single element. The equations representing the finite element model of the whole body are derived by combining the elemental equations. These equations are combined by equating the generalized displacements and summing the generalized forces at corresponding nodes. The resulting global system of equations is then solved for the incremental deformation of the body.
6.3.3 Residual Loading Term

The residual loading term was developed in Section 3.5 as a representation of the nonsatisfaction of equilibrium. In general the configuration of a body at the end of a load increment as found by using a linear incremental virtual work equation will not exactly satisfy equilibrium. The residual loading term when known can then be employed with any of the solution procedures shown in Chapter 4 that come under the heading of incremental methods with equilibrium checks. The residual loading term is given by either Eq. 3.46 or Eq. 3.47, the choice of which form to use being decided by the particular incremental virtual work equation employed.

If the problem is such that a Lagrangian surface traction vector is found to be easier to use, then the linear incremental virtual work equation will be that given in Eq. 3.43. The accompanying residual loading term is then given by Eq. 3.46. Rewriting this equation for the residual loading term in matrix form, for configuration \( l^C \), and introducing the finite element method for a single element,

\[
^{lR_c}e = \{\delta u\}^e^T \int_{S} [N]^T \{1^T(L)\} \, d^S \, + \{\delta u\}^e^T \int_{V} \rho [N]^T \{1^F\} \, d^V \\
- \{\delta u\}^e^T \int_{V} [N]^T [L]^T \{1^S\} \, d^V \\
- \{\delta u\}^e^T \int_{V} [N]^T [L]^T [^{1ST}] [L] [N] \, d^V \{1^u\}^e.
\]  

(6.53)

In this equation, \( \{1^S\} \) is evaluated through the use of Eq. 6.33. It should be noted that the last integral term on the right hand side of
Eq. 6.53 is the initial stress matrix $[K_{IS}]$ for the next increment of load in going from configuration $^1C$ to $^2C$. Thus the evaluation of the residual loading term contains one term that would have to be evaluated for the next load increment in any case.

Similarly, if the problem is such that the surface traction may be more easily expressed in terms of a Kirchhoff stress vector, then the incremental virtual work equation is given by Eq. 3.45. The corresponding residual loading term is then given by Eq. 3.47.

Proceeding as above

\[
^{1R}_{c} = \{\delta u\}^{T} \int_{S} [N]^{T} \{1T(K)^{T}\} dS + \{\delta u\}^{T} \int_{S} [N]^{T} \{1u_{i,j}\} \{1T(K)^{T}\} dS \\
+ \{\delta u\}^{T} \int_{\rho} [N]^{T} \{1F\} dV - \{\delta u\}^{T} \int_{\rho} [N]^{T} \{L\}^{T} \{1S\} dV \\
- \{\delta u\}^{T} \int_{V} [N]^{T} \{L\}^{T} \{1ST\} [L] [N] dV \{1u\}^{e} \quad (6.54)
\]

In Eq. 6.53 and 6.54, the values of the total stresses, displacements, surface tractions and body forces derived by the incremental analysis are used. However, the virtual displacements are completely arbitrary, and they need not be associated with the configuration for which the residual loading term is being evaluated. It is necessary though that the specified displacement boundary conditions are not changing as the body deforms for this to be true.
6.4 Two Dimensional Analysis

6.4.1 General

The analysis of the general deformation of a three dimensional body is a computationally expensive approach to any given problem. This approach may not be justifiably necessary for a rather large class of problems where, using certain simplifying assumptions, the original three dimensional problem may be reduced to a two dimensional one. The two basic assumptions that are made are those of a plane strain condition or of a plane stress condition.

For plane strain analysis it is assumed that the strain in the out-of-plane dimension is restrained, and that the deformation of the body is a function of planar coordinates only. On the other hand, for plane stress analysis it is the out-of-plane normal stress that is assumed to be zero, or at least negligible, and that only the planar shear and normal stresses specify the state of stress.

Both of these simplifying assumptions lead to the analysis of a significantly reduced problem from that of a general three dimensional problem. Care must be taken to choose the proper assumption corresponding to the problem being analysed. Actual physical problems will in general fall between the two extremes represented by these assumptions. In Chapter 7, the plane strain assumption will be used for a plate having an infinite length, and the plane stress assumption will be used for a cantilever beam.
These simplifying assumptions will be shown more clearly in the following sections, and the finite element incremental virtual work equations and residual loading term will be suitably modified for two dimensions.

6.4.2 Plane Strain

In a plane strain analysis the assumption made is that the out-of-plane strain is restrained or prevented and that the deformation of the body is a function of planar coordinates only. Therefore taking the out-of-plane dimension as being associated with the subscript 3, then

\[ \alpha u_{3,i} = \alpha u_{i,3} = 0 \] (6.55)

Referring to Eq. 2.6 which defines Green's strain tensor, this gives

\[ \alpha \varepsilon_{3,i} = \alpha \varepsilon_{i,3} = 0 \] (6.56)

and then using Eq. 2.8, which defines the incremental Green's strain,

\[ \varepsilon_{3,i} = \varepsilon_{i,3} = 0 \] (6.57)

As a consequence of the plane strain assumption given in Eq. 6.55, and the definition of an incremental quantity then

\[ u_{3,i} = u_{i,3} = 0 \] (6.58)
Now using Eq. 6.58 and the definitions of $e_{ij}$ and $\eta_{ij}$, given by Eq. 2.11 and Eq. 2.12 respectively, then

$$e_{13} = e_{31} = \eta_{13} = \eta_{31} = 0$$  \hspace{1cm} (6.59)

Since both $e_{ij}$ and $S_{ij}$ are symmetric tensors, it becomes economical to calculate only one of $e_{12}$ and $e_{21}$, and also only one of $S_{12}$ and $S_{21}$. Therefore, define the vector \{\overline{e}\}, where this vector in plain strain analysis contains all the non-zero $e_{ij}$ terms, as

$$\{\overline{e}\}^T = \langle e_{11} \ e_{22} \ 2e_{12} \rangle$$  \hspace{1cm} (6.60)

and then define the vector \{\overline{S}\} as

$$\{\overline{S}\}^T = \langle S_{11} \ S_{22} \ S_{12} \rangle$$  \hspace{1cm} (6.61)

The vector \{\overline{S}\} contains all the stress components that are multiplied by non-zero strains in the virtual work expressions. By defining \{\overline{S}\} and \{\overline{e}\} as above, the vector product is correct for the integral term in the incremental virtual work equation shown below

$$\int_V \{\overline{S}\}^T \{\delta e\} \ d\nu = \int_V \{\overline{S}\}^T \{\delta \overline{e}\} \ d\nu$$  \hspace{1cm} (6.62)

The scalar factor of two preceding the term $e_{12}$ in Eq. 6.60 is required in order to correctly account for the term $e_{21}$, which is not evaluated to save computational effort. This can be shown by considering all the non-zero terms in the vector products of the
integrands of Eq. 6.62 as follows,

\[
\{ S \}^T \{ \delta e \} = S_{11} \delta e_{11} + S_{12} \delta e_{12} + S_{21} \delta e_{21} + S_{22} \delta e_{22}
\]

\[
= S_{11} \delta e_{11} + S_{22} \delta e_{22} + S_{12} (2 \delta e_{12})
\]

\[
= \{ \delta \}^T \{ \delta e \}
\]

(6.63)

Only the symmetric properties of $S_{ij}$ and $\varepsilon_{ij}$ have been used to demonstrate this result.

Relating the incremental stress vector $\{ \delta \}^T$ to the incremental strain vector $\{ \varepsilon \}$ in a similar manner to Eq. 6.14,

\[
\{ \delta \} = [C] \{ \varepsilon \}
\]

(6.64)

where, as before, the incremental stresses are related to only the linear portion of the incremental Green's strain tensor. In the residual loading term however, the stresses will be evaluated using the complete Green's strain tensor.

Employing Hooke's law for an isotropic material, then the constitutive matrix $[C]$ for a plane stress analysis is given by the symmetric matrix,

\[
[C] = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 & \frac{\nu}{(1 - \nu)} & 0 \\
\frac{\nu}{(1 - \nu)} & 1 & 0 \\
0 & 0 & \frac{1 - 2\nu}{2(1 - \nu)}
\end{bmatrix}
\]

(6.65)
As before in the three dimensional analysis, the vector $\{\varepsilon\}$ may be decomposed into the sum of two vectors

$$\{\varepsilon\} = \{\varepsilon_L\} + \{\varepsilon_{NL}\} \tag{6.66}$$

where $\{\varepsilon_L\}$ is independent of previous deformation, and $\{\varepsilon_{NL}\}$ is dependent on the configuration of the body at the beginning of the particular increment being considered. In an analogous manner to Eq. 6.12 and Eq. 6.13, two matrix operators are defined such that

$$\{\varepsilon_L\} = [B_L]\{u\} \tag{6.67}$$

$$\{\varepsilon_{NL}\} = [B_{NL}(^1u)]\{u\} \tag{6.68}$$

where

$$\{u\}^T = <u_1 \ u_2> \tag{6.69}$$

The two matrix operators $[B_L]$ and $[B_{NL}(^1u)]$ are shown in full in Appendix B.

The vector of Kirchhoff stresses for configuration $^1C$, denoted by $\{^1\sigma\}$ where this vector is defined by

$$\{^1\sigma\}^T = <^1\sigma_{11} \ ^1\sigma_{22} \ ^1\sigma_{12}> \tag{6.70}$$
is evaluated using the full Green's strain vector for the same configuration, \( \{ \varepsilon' \} \), defined as

\[
\{ \varepsilon' \}^T = < \varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12} >
\]

They are related by the same constitutive matrix used in Eq. 6.64. The full Green's strain vector is evaluated by

\[
\{ \varepsilon' \} = [B_L] \{ \varepsilon' \} + \frac{1}{2} [B_{NL}(\varepsilon')] \{ \varepsilon' \} \]

and hence the Kirchhoff stress vector \( \{ \varepsilon' \} \) is given by

\[
\{ \varepsilon' \} = [C] [B_L]^T \{ \varepsilon' \} + \frac{1}{2} [C] [B_{NL}(\varepsilon')] \{ \varepsilon' \}
\]

where \( \{ \varepsilon' \} \) is defined in an analogous manner to \( \{ \varepsilon' \} \). The development of the above two equations, parallels the development in the three dimensional case, given in Eq. 6.28 through Eq. 6.32, of the corresponding three dimensional vectors.

Next, define in a similar manner to Eq. 6.34, the two dimensional vector of material derivatives of the incremental displacements as

\[
\{ u_{i,j} \}^T = < u_{i,1}, u_{1,2}, u_{2,1}, u_{2,2} >
\]

and then the matrix operator \( [L] \), defined similarly to \( [L] \) in Eq. 6.35, is given such that
\( \{\overline{u}_{i,j}\} = [\overline{L}] \{\overline{u}\} \) \hspace{1cm} (6.75)

Now the matrix \([\overline{L}ST]\) is assembled from the elements of \(\{L\}\) in order that the integrand product shown below, of the incremental virtual work equation, is correctly evaluated. That is

\[
\{L\}^T \{\delta \eta\} = \{\delta \overline{u}_{i,j}\}^T [\overline{L}ST] \{\overline{u}_{i,j}\}
\] \hspace{1cm} (6.76)

The matrix operator \([\overline{L}]\) and the matrix \([\overline{L}ST]\) defined above are shown fully in Appendix B.

Finally, a two dimensional body force vector is defined as

\[
\{a_F\}^T = \langle a_{F_1} a_{F_2} \rangle
\] \hspace{1cm} (6.77)

and the surface traction vector is given by either

\[
\{a_T(L)\}^T = \langle a_{T_1}(L) a_{T_2}(L) \rangle
\] \hspace{1cm} (6.78)

or

\[
\{a_T(K)\}^T = \langle a_{T_1}(K) a_{T_2}(K) \rangle
\] \hspace{1cm} (6.79)

depending on the particular problem being analyzed. The incremental forms of these vectors are similarly defined.

The incremental virtual work equation and the residual loading term for two dimensional plane strain analysis may now be written by substituting the two dimensional matrix operators, matrices, and vectors
defined in this section for their corresponding three dimensional counterparts in the equations given in sections 6.3.2 and 6.3.3. Thus the incremental virtual work equations for plane strain analysis are given, for a single element, by adapting Eq. 6.47 as

\[
\{\delta \bar{u}\}^T \int_{\Gamma_c} \left[ \overline{N} \right]^T \{\tau^{(L)}\} \ d^oS + \{\delta \bar{u}\}^T \int_{\Gamma_c} \left[ \overline{P} \right]^T \{\bar{F}\} \ d^oV \\
= \{\delta \bar{u}\}^T \left[ \left[ \overline{K}_L \right] + \left[ \overline{K}_{NL} \right] \right] \{ \bar{u} \}^e
\]

(6.80)

where \([\overline{K}_{NL}]\) and \([\overline{K}_L]\) are formulated identically to \([K_{NL}]\) and \([K_L]\), only with the corresponding two dimensional matrix operators, matrices, and vectors substituted for their three dimensional counterparts.

By first defining the matrix \([1\bar{u}_{i,j}]\) as

\[
[1\bar{u}_{i,j}] = \begin{bmatrix}
1u_{1,1} & 1u_{1,2} \\
1u_{2,1} & 1u_{2,2}
\end{bmatrix}
\]

(6.81)

then Eq. 6.52, can be similarly adapted to two dimensional plane strain as

\[
\{\delta \bar{u}\}^T \int_{\Gamma_c} \left[ \overline{N} \right]^T \{\tau^{(K)}\} \ d^oS + \{\delta \bar{u}\}^T \int_{\Gamma_c} \left[ \overline{1\bar{u}_{i,j}} \right]^T \{\tau^{(K)}\} \ d^oS \\
+ \{\delta \bar{u}\}^T \int_{\Gamma_c} \left[ \overline{P} \right]^T \{\bar{F}\} \ d^oV \\
= \{\delta \bar{u}\}^T \left[ \left[ \overline{K}_L \right] + \left[ \overline{K}_{NL} \right] \right] \{ \bar{u} \}^e
\]

(6.82)
The expression for the residual loading terms are similarly adapted from Eq. 6.53 and 6.54, and are not reproduced here.

6.4.3 Plane Stress

In plane stress analysis it is assumed that the out-of-plane normal stress is zero, or at least negligible, and that the state of stress is completely specified by the two planar normal stresses and the in-plane shear stress. The state of stress is assumed not to be a function of the out-of-plane dimension. Therefore taking the out-of-plane dimension as being associated with the subscript 3, then

\[ \alpha_{S_{13}} = \alpha_{S_{31}} = 0 \]  

(6.83)

and as a consequence

\[ S_{13} = S_{31} = 0 \]  

(6.84)

Now the vectors \( \{\bar{S}\} \) and \( \{\bar{S}^1\} \) as defined in the previous section contain all the non-zero incremental Kirchhoff stresses and total Kirchhoff stresses for configuration \( \text{I}^1\text{C} \), respectively. Although there are only three elements in these vectors, and four non-zero stresses, all the stresses are represented since \( S_{12} \) and \( S_{21} \) are equal by symmetry. It is therefore economical to retain only one of them and account for the other's influence by the inclusion of an appropriate scalar factor.

The vector \( \{\bar{e}\} \) as defined in the previous section contains all the strain components that are multiplied by non-zero stress components
in the formation of the virtual work expression. By retaining the factor of two preceding $e_{ij}$, the vector product of the integral term as shown below, is correctly evaluated.

$$
\int_{V} \{S\}^T \{\delta e\} \, dV = \int_{V} \{\delta \bar{S}\}^T \{\delta \bar{e}\} \, dV \tag{6.85}
$$

By using the same stress and strain vectors as for plane strain analysis, plane stress analysis is then identical with the exception that a different constitutive matrix is used.

Relating incremental Kirchhoff stresses to the incremental linear portion of Green's strain analogously to Eq. 6.64 in the plane strain analysis, then

$$
\{\bar{S}\} = [\bar{E}] \{\bar{e}\} \tag{6.86}
$$

where $[\bar{E}]$ is the plane stress constitutive matrix. Utilizing Hooke's law for an isotropic material, the constitutive matrix is given by

$$
[\bar{E}] = \frac{E}{(1 - \nu^2)} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{(1 - \nu)}{2}
\end{bmatrix} \tag{6.87}
$$

where $E$ is Young's modulus and $\nu$ is Poisson's ratio.

By using the same definition of stress and strain vectors as used in plane strain analysis, the plane stress analysis uses the same equations as derived in the previous section, with the exception that the
constitutive matrix $[E]$ is substituted for $[C]$. Thus a computer program can be used easily for both plane stress and plane strain analysis by simply inputting or choosing the appropriate constitutive matrix.

6.5 Eight Degree of Freedom Rectangular Finite Elements

6.5.1 The Assumed Displacement Field

The basic eight degree of freedom rectangular finite element, from which the two different numerically integrated elements will be derived, is shown in Fig. 9. The element is derived in terms of a local coordinate system $(\xi, \eta)$ with the origin located at the centroid of the element.

A bilinear displacement field is chosen where the generalized displacements $u_1$ and $u_2$ are given in terms of generalized displacement coefficients $a_i$, as shown below,

$$u_1 = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$u_2 = a_5 + a_6\xi + a_7\eta + a_8\xi\eta$$

(6.88)

or in matrix form

$$\{u\} = \begin{bmatrix} 1 & \xi & \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi & \eta \end{bmatrix} \{a_i\}$$

(6.89)

where
EIGHT DEGREE OF FREEDOM RECTANGULAR
FINITE ELEMENT

FIG. 9
In utilizing the finite element method however, it is necessary to have the generalized displacements \( \{u\} \) related to the nodal generalized displacements \( \{\bar{u}\}^e \), not the coefficients \( \{\alpha_i\} \) as in Eq. 6.89.

First define the vector of element nodal displacements as

\[
\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}
\]  (6.91)

where the right hand superscript is now used to indicate the node number. The elements of \( \{\bar{u}\}^e \) and their positive direction are as shown in Fig. 9. Now \( \{\bar{u}\}^e \) can be expressed in terms of \( \{\alpha_i\} \) by a transformation matrix \( [T] \) such that

\[
\{\bar{u}\}^e = [T] \{\alpha_i\}
\]  (6.92)

then obtaining the inverse of \([T]\),

\[
\{\alpha_i\} = [T^{-1}] \{\bar{u}\}^e
\]  (6.93)

Substituting for \( \{\alpha_i\} \) in Eq. 6.89, and carrying out the required matrix multiplication, the result is

\[
\{\bar{u}\} = [\bar{N}] \{\bar{u}\}^e
\]  (6.94)
and the interpolation functions, \( N_i \), are given by

\[
N_1 = \frac{(b - \xi) (c - \eta)}{4bc} \quad N_3 = \frac{(b + \xi) (c + \eta)}{4bc}
\]

\[
N_2 = \frac{(b + \xi) (c - \eta)}{4bc} \quad N_4 = \frac{(b - \xi) (c + \eta)}{4bc}
\]

This is the Melosh rectangle, and the interpolation functions given in Eq. 6.96 represent the application of Lagrange's interpolation formula for two dimensions [29].

The element as formulated satisfies both completeness and compatibility requirements for a two dimensional analysis of either plane strain or plane stress.

In evaluating the elemental stiffness matrices, load vectors, and residual loading vectors the virtual work integrands are functions of the local coordinates \( \xi \) and \( \eta \). To evaluate the integrals required for the finite element analysis, two numerical integration schemes are proposed and evaluated in the following sections. The first integration scheme uses Gaussian quadrature to evaluate the integrals. The second scheme proposed is a nonuniform or lower order integration, which is used in an attempt to improve the results obtained for a given number of elements.
6.5.2 Rectangular Finite Element Using Gaussian Quadrature

The integration of the expressions required in the virtual work equations, may be accomplished through the use of various numerical integration schemes. Of these, Gaussian quadrature requires the least number of integration points to construct and exactly integrate a polynomial of a given order. The evaluation of the integrand at each integration point, when using the finite element method, involves a considerable amount of complex calculations. Therefore, the fewer integration points which are required in using Gaussian quadrature, makes this method advantageous to use.

Gaussian quadrature is performed by evaluating the integrand at each integration point, multiplying this by a weighting value and then adding the result to the results from the other integration points. Considering first only one dimension, using \( n \) integration points a polynomial of degree \( 2n - 1 \) may be constructed and exactly integrated using Gaussian quadrature. The accuracy of any numerical technique is dependent on how well the substituted polynomial fits the real integrand. In this case if the real integrand is of equal or lower order than \( 2n - 1 \), then it will be exactly integrated, if not then the error is on the order of \( (\Delta)^{2n} \) where \( \Delta \) is the spacing of the integration points. [30, 31]

The integrals required for the virtual work equation are either surface or volume integrals. First considering the volume integrals, which include the linear and nonlinear stiffness matrices and the body force integral, they are of the form

\[
I = \int_{V} f(\xi, \eta) \, d^oV
\]  
(6.97)
Taking the thickness of each element as being constant, say \( t \), then

\[
I = t \int_{-c}^{c} \int_{-b}^{b} f(\xi, \eta) \, d\xi \, d\eta \quad (6.98)
\]

For the bilinear eight degree of freedom rectangle outlined in the previous section, the integrand will be composed of terms whose highest order is quadratic. Since one integration point will exactly integrate a linear integrand, and two integration points in each coordinate direction will exactly integrate a cubic polynomial integrand; two integration points are required in each direction. The integral in Eq. 6.98 is evaluated using Gaussian quadrature by

\[
I = (t \, b \, c) \sum_{i=1}^{2} \sum_{j=1}^{2} H_i \, H_j \, f(\xi_i, \eta_j) \quad (6.99)
\]

where the integration points \((\xi_i, \eta_j)\) and the weights \( H_i \) or \( H_j \), are given by

\[
\xi_i = (-1)^i \cdot (b) \left( \frac{1}{\sqrt{3}} \right) \\
\eta_j = (-1)^j \cdot (c) \left( \frac{1}{\sqrt{3}} \right) \\
H_1 = H_2 = 1.0
\]

The surface integrals are also numerically integrated using Gaussian quadrature. The integrals over the surface are of the form

\[
I = \int_{^d S} q(\xi, \eta) \, d^d S \quad (6.101)
\]
Again assuming constant element thickness $t$, and also that surface
tractions are applied only to the boundaries of the element then the
integral in Eq. 6.101 can be written for the bilinear rectangular finite
element as

$$ I = t \int_{-b}^{b} g(\xi, c) \, d\xi + t \int_{-c}^{c} g(b, \eta) \, d\eta + t \int_{-b}^{b} g(\xi, -c) \, d\xi $n
$$

(6.102)

If the integrand $g(\xi, \eta)$ is linear within a particular integral of Eq.
6.102 then only one Gauss integration point is required at mid-side; if
the integrand is either quadratic or cubic then two Gauss integration
points are required on the particular edge of the element. Considering
just one edge of the element to have a non-zero surface traction, say the
dge where $\eta = c$ and $-b \leq \xi \leq b$, then Eq. 6.102 reduces to

$$ I = t \int_{-b}^{b} g(\xi, c) \, d\xi $n
$$

(6.103)

If the integrand is linear then Gaussian quadrature gives

$$ I = t(2b) \, g(0, c) $n
$$

(6.104)

and if the integrand is quadratic or linear then

$$ I = t b \sum_{i=1}^{2} H_i \, g(\xi_i, c) $n
$$

(6.105)
where \( \xi_i \) and \( H_i \) are as given in Eq. 6.100.

When employing the Lagrangian surface traction integral as in Eq. 6.80, since the matrix of interpolation functions \([\tilde{N}]\) varies linearly along an edge, a one point Gaussian quadrature rule will be required if the surface traction \( \{\tilde{T}^\text{L}\} \) is constant along that edge, and two integration points will be required if \( \{\tilde{T}^\text{L}\} \) is linear or quadratic along that edge.

The same criteria apply to the first surface traction integral using the Kirchhoff stress tensor in Eq. 6.82, as for the Lagrangian integral described above. The second surface traction integral in this equation however, contains the product of \([\tilde{N}]\) which is linear on an edge, and the matrix \([^1\bar{u}_{i,j}]\) which is also linear along an edge. The integrand is obviously then quadratic when a constant surface traction along the edge is specified, and cubic when a linearly varying surface traction is used. Both of these cases require the use of a two point Gaussian quadrature rule. For a parabolically changing surface traction a three point Gaussian quadrature must be used.

Using two integration points in each coordinate direction in a Gaussian quadrature scheme, the linear stiffness matrix for an element was evaluated. This numerically obtained stiffness matrix was compared to the stiffness matrix for the same element obtained from a closed-form integration of the required integral. The two matrices were found to be identical to at least ten significant figures for all entries in the matrices. The value of ten significant figures was the arbitrary limit of agreement that was checked, and should not be construed as the actual limit of agreement of the numerically integrated scheme results to the closed form results.
The numerically integrated elemental stiffness matrix was checked in an eigenvalue routine where three zero eigenvalues were found, corresponding to three zero energy deformation modes, or rigid body modes as required. This element also successfully passed the patch test.

6.5.3 Rectangular Finite Element Using Nonuniform Numerical Integration

The motivation to develop an alternate integration technique to utilize in place of Gaussian quadrature, arises from the poor performance of the rectangular element in pure bending when the stiffness matrix is integrated using Gaussian quadrature. Cook [32], in discussing the poor performance of linear isoparametric elements, of which the rectangular bilinear element is a special case, shows that the problem arises from what he terms "parasitic" shear at the Gauss integration points.

Applying nodal displacements corresponding to pure bending, shear strains should be zero throughout the element. For the rectangular finite element of Section 6.5.1, shear strains will however, exist everywhere within the element except at the centroid of the element ($\xi=0$, $\eta=0$). This "parasitic" shear exists then at the Gauss integration points of any quadrature rule, other than a one-point rule, and makes the numerically integrated element too stiff in pure bending because the deformation pattern for bending requires the storage of shear strain energy as well as normal strain energy.

Therefore, to overcome this problem of "parasitic" shear at the Gauss integration points, a nonuniform integration scheme is proposed. In this nonuniform integration all terms used in the formation of the stiffness matrices that are associated with shear strain, are evaluated
at the centroid of the element. All other terms are evaluated at the usual Gauss integration points. Surface integrals are integrated as in the previous section.

The element linear stiffness matrix as formulated using this nonuniform integration scheme, was checked in an eigenvalue routine where three zero eigenvalues were found corresponding to the required rigid body modes. This element also successfully passed the patch test.

The rectangular finite element, when integrated using this nonuniform integration may be thought of as an element which is capable of only representing constant shear strain throughout the element domain. But in the limit, as the mesh is successively refined, the actual required shear strain distribution within an element domain will approach a constant value as is represented by this element using nonuniform integration.

6.5.4 Performance Comparison of the Two Rectangular Finite Elements

The two numerical integration schemes proposed in the preceeding sections give rectangular finite elements that have the required rigid body modes, and that also successfully pass the patch test. Convergence to the correct solution using either of these two finite elements is therefore assured, since both also satisfy the completeness requirements. In choosing between these two elements therefore, the criteria should be the degree of solution accuracy that is obtained for given computational effort. That is, the more desirable of the two finite elements would be the one that gives the closest approximation to the exact solution when each is used with the same degree of mesh refinement on a given problem.
The computational effort required in order to obtain the finite element solution to a given problem may be decomposed into two parts. The first part is the computational expense required to form the required elemental stiffness matrices. The finite element developed using the nonuniform integration scheme required slightly less computation than the Gaussian integrated element in this regard, since terms associated with shear strains are evaluated only once at the centroid of the element with nonuniform integration, and not at each of the four Gauss points separately. The saving in overall computational expense is minimal however, since the second part of the process of obtaining the finite element solution, that of assembling the global stiffness matrix and solving the global system of equations, represents by far the greatest portion of computation required. This second part of the computational expense is a function only of the mesh geometry and the total number of degrees of freedom, since both of the finite elements have the same elemental geometry and degrees of freedom. Thus the two elements may be assumed to require essentially the same computational effort for a specified finite element mesh in a given problem. Comparison of the performance of the two elements should therefore be on a basis of the accuracy of the solutions obtained with each element using the same mesh arrangement, when compared to the exact solution for the problem being considered.

The performance of the two finite elements was compared in the linear analysis of two different problems; a cantilever which has a parabolically varying end shear, and an infinite plate strip subjected to a uniform pressure. These two test examples were selected because they are similar to the problems which will be analysed for nonlinear
deformation. The finite element displaying the best performance in these tests should be utilized in the nonlinear analyses, since without at least a good representation of the linear response the nonlinear response will not be accurately determined using an incremental analysis.

The dimensions, loading, and material properties of the cantilever used to compare the performance of the two rectangular finite elements, are as shown in Fig. 10. This particular cantilever was chosen because solutions to this problem using other finite elements are available in the literature, in addition to a closed form solution from beam theory that gives an upper bound for the tip deflection of the cantilever. The cantilever shown in Fig. 10 was analysed using each of the two finite elements under consideration, in three different grids. These three finite element grids are shown in Fig. 11, along with the boundary conditions imposed. All nodes at the fixed end of the cantilever were fixed against either vertical or horizontal displacements, so as to correspond with the finite element solutions given in the literature for various elements. The exact plane stress elasticity solution does not exist for this case however, because of the restraint at the boundary in the vertical direction. The closed form solution to which the results are compared is derived from beam theory, and can be shown to provide an upper bound to the tip displacement of the cantilever with the boundary conditions as imposed on the finite element analysis.

The numerical results using the two rectangular elements in each of the three grids of Fig. 11, are tabulated in Table I. The comparison is based on the tip deflection, since in the nonlinear analysis tip deflection will be used to assess the accuracy of the method of analysis presented in this thesis. The results are also shown in Fig. 12,
\[ \tau = \frac{P}{8} \left( 1 - \frac{y^2}{36} \right) \]

\begin{align*}
E &= 30,000 \text{ ksi} \\
v &= 0.25 \\
t &= 1.0 \text{ in (t is the thickness of the beam)} \\
P &= 40 \text{ kips}
\end{align*}

CANTILEVER TEST

FIG. 10
CANTILEVER FINITE ELEMENT GRIDS

FIG. 11
<table>
<thead>
<tr>
<th>Element Integration</th>
<th>Grid</th>
<th>Degrees of Freedom (Net)</th>
<th>Tip Deflection $w$ (in.)</th>
<th>% Of Theory</th>
<th>% Error</th>
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<td>0.242424</td>
<td>68.1</td>
<td>31.9</td>
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<td>92.3</td>
<td>7.7</td>
</tr>
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<td>14.3</td>
</tr>
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</tr>
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<td>14.1</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
$w \text{ (Upper Bound)} = 0.35583$

Degrees of Freedom

CANTILEVER: FINITE ELEMENT COMPARISON

FIG. 12
where results for the same problem using the constant stress triangle and linear stress triangle are included for comparison purposes. The comparison is based on the total number of degrees of freedom used in obtaining the solution, and while this is valid for the two rectangular elements in comparing them to each other, it should be borne in mind when comparing the other two elements to them that other factors need to be considered to determine the amount of computation expended.

As can be seen from Table I and Fig. 12, the rectangular finite element using nonuniform integration exhibits significantly better performance than the Gaussian integrated element, especially when a coarse grid of elements was used. Both rectangular elements perform better than the constant stress triangle but not as well as the linear strain triangle, when compared on the basis of total number of degrees of freedom used in obtaining the solution. This comparison does not consider the computational expense of forming the stiffness matrices or the solution of the set of resulting equations to be different when using the different types and geometries of elements, which of course it is in general. The comparison of the two rectangular elements' performances on the basis of the number of degrees of freedom is valid for reasons outlined above.

The second problem analysed to compare the performance of the two rectangular elements was an infinite plate strip with simply supported longitudinal edges, subjected to a uniform pressure on the upper surface. The plate strip along with the dimensions, material properties and pressure used is shown in Fig. 13. The closed form linear solution to this problem is developed by Timoshenko and Woinowsky-Krieger [33].
L = 10.0 in.
t = 0.2 in.

E = $10^4$ ksi

ν = 0.25

q = 5.0 psi

$\frac{w_{\text{MAX}}}{t} = 0.457764$

SIMPLY SUPPORTED INFINITE PLATE STRIP
UNDER UNIFORM LOADING

FIG. 13
By arguments of symmetry, only half the width of the plate needs to be analysed. The two grids used for each of the two rectangular finite elements are shown in Fig. 14, with the boundary conditions imposed.

The numerical results using the two different rectangular finite elements are tabulated in Table II, and are also shown in Fig. 15. As can be seen, the nonuniformly integrated rectangular finite element again performs significantly better than the element that is integrated using the complete Gaussian quadrature.

On the basis of the results of these two test problems for comparing the performance of the two rectangular elements, the nonuniformly integrated finite element was chosen to be used in the nonlinear analyses of the following chapter.
FINITE ELEMENT GRIDS FOR THE INFINITE PLATE STRIP

FIG. 14

TABLE II

INFINITE PLATE STRIP FINITE ELEMENT RESULTS

<table>
<thead>
<tr>
<th>Element Integration</th>
<th>Grid</th>
<th>Degrees of Freedom (Net)</th>
<th>Deflection</th>
<th>% Of Theory</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>1</td>
<td>61</td>
<td>0.135474</td>
<td>29.6</td>
<td>70.4</td>
</tr>
<tr>
<td>Nonuniform</td>
<td>1</td>
<td>61</td>
<td>0.442357</td>
<td>96.6</td>
<td>3.4</td>
</tr>
<tr>
<td>Gaussian</td>
<td>2</td>
<td>121</td>
<td>0.283161</td>
<td>61.9</td>
<td>38.1</td>
</tr>
<tr>
<td>Nonuniform</td>
<td>2</td>
<td>121</td>
<td>0.443819</td>
<td>96.9</td>
<td>3.1</td>
</tr>
<tr>
<td>Theory</td>
<td></td>
<td></td>
<td>0.457764</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
INFINITE PLATE STRIP FINITE ELEMENT RESULTS

FIG. 15

Gaussian integrated rectangular element
Nonuniformly integrated rectangular element

Exact = 0.457764

Degrees of Freedom
APPLICATIONS TO NONLINEAR PROBLEMS

7.1 General

To evaluate the ability of the linearized virtual work equations, through the use of the finite element method and a self-correcting solution procedure, in accurately representing nonlinear deformation, several test problems are solved. These test problems were chosen because a corresponding analytical closed form solution is available for comparison.

The test problems used in this chapter generally fall into one of two categories. Firstly, there are those problems in which the nonlinear nature of the problem arises primarily from the build up of large strains, and hence correspondingly high stresses, without very large displacements of the body. The deformation behaviour of thin plates, where the maximum deflection is only on the order of the thickness of the plate is an example of this type. Secondly, the nonlinearity may arise from large displacements, but where the body still has only relatively small strains, as in the classic problem of the elastica.

For each test problem, the closed form solution available will be presented and discussed on the basis of its particular postulates and simplifications. Then the finite element solution will be obtained and compared to the corresponding closed form solution.

It is important to understand that in comparing the closed form and finite element solutions, the discrepancies between the two
solutions for a given problem may arise from several sources. Firstly, the closed form solution may have been obtained through the use of simplifications not required by the finite element analysis, the result being that two very similar but not identical problems are compared in their solutions. Secondly, the incremental Lagrangian analysis presented in this thesis uses a linear constitutive relationship between Kirchhoff stresses and Green's strains, whereas in a closed form solution the same constitutive law may be used to relate differently defined stresses and strains. Thirdly, each solution obtained at the end of an increment is only a linear approximation to the true nonlinear response during that increment. The factors affecting the accuracy of these linear incremental solutions were discussed in Chapter 4. Finally, the finite element procedure itself only provides an approximate solution of the Lagrangian virtual work expressions, the accuracy of which is limited by the capabilities of the element and the refinement of the mesh employed.

7.2 Elastic Infinite Plate Strip

7.2.1 General

The elastic finite deformation of an infinite plate strip is a classical problem which has a closed form solution given by Timoshenko and Woinowsky-Krieger [34]. This solution is derived for a uniformly loaded plate having an infinite length and a finite breadth. The solutions for both simply supported and fixed longitudinal edges have been obtained. The solution of the infinite plate strip is of practical value since Timoshenko shows that the solution for maximum deflections, stresses, and bending moments in a plate of finite length rapidly approach the values
given by the infinite plate strip solution when the length to breadth ratio is greater than about three [35].

The two plates to be analysed, one with simply supported longitudinal edges and the other with fixed longitudinal edges, are shown in Fig. 16, in their deformed and undeformed configurations. The length of the plate, out of the plane of the diagram, is taken as being infinite.

These two problems test the ability of the incremental Lagrangian virtual work equations, together with the finite element method and the self-correcting solution procedure, to accurately follow the nonlinear response of the plates, which is primarily due to the development of large membrane stresses. These large stresses develop even though the displacements of the plate remain relatively small with respect to the thickness of the plate. The simply supported plate will be analysed to a maximum deflection of just slightly less than the thickness of the plate, and the fixed edged plate to about half that value.

7.2.2 Infinite Plate Strip: Closed Form Solution

The closed form solution to this problem is obtained by considering an elemental strip of the plate. The governing differential equation for the plate is given by

\[ D \frac{d^2 w}{dx^2} = -M \]  

(7.1)

where \( D \) is the flexural rigidity of the plate and is given as

\[ D = \frac{E h^3}{12(1-\nu^2)} \]  

(7.2)
SIMPLY SUPPORTED INFINITE PLATE STRIP

FIG. 16(a)

FIXED EDGED INFINITE PLATE STRIP

FIG. 16(b)
In the two equations above, $M$ is the bending moment, $E$ is Young's modulus, $v$ is Poisson's ratio, and $w$ is the deflection of the plate normal to its original plane. Eq. 7.1 is an approximation of the Euler-Bernoulli law of bending, because the second derivative of the deflection is used to approximate the curvature of the plate. This assumption is valid where the slopes ($dw/dx$), of the plate remain small, which is true for this problem.

Considering first the simply supported plate, the bending moment at any cross section of the strip is given by

$$M = \frac{qLx}{2} - \frac{qx^2}{2} - Sw \quad (7.3)$$

where $S$ is the in-plane force preventing the edges of the plate from moving together, and $q$ is the intensity of the uniform pressure load as shown in Fig. 16(a). This relationship is derived by assuming that the uniform pressure loading does not change its line of action during deformation, and also that the magnitude of any loading on a segment is related to the original undeformed surface area. In reality a pressure loading would act on the deformed surface area and would be directed normally to the deflected surface at every point. The type of surface traction prescribed for the closed form solution is described correctly by the Lagrangian rule of correspondence, so a Lagrangian surface traction should be utilized in the finite element solution. Sharifi and Yates 36 analysed this problem, but specified a Kirchhoff surface traction vector.

Now, substituting Eq. 7.3 into Eq. 7.1, the governing differential equation becomes

$$\frac{d^2 w}{dx^2} + \frac{Sw}{D} = -\frac{qLx}{2D} + \frac{qx^2}{2D} \quad (7.4)$$
Adopting the notation

$$u^2 = \frac{S L^2}{4D} \quad (7.5)$$

then the general solution of Eq. 7.4 is given by

$$w = C_1 \sinh \left( \frac{2ux}{L} \right) + C_2 \cosh \left( \frac{2ux}{L} \right) + \frac{qL^2}{8u^2D} \left( Lx - x^2 - \frac{L^2}{2u^2} \right) \quad (7.6)$$

Applying the boundary conditions for the simply supported edges

$$w(0) = 0$$
$$w(L) = 0 \quad (7.7)$$

and simplifying, the solution for the deflection \( w \), is given as

$$w = \frac{qL^4}{16u^4D} \left[ \frac{\cosh \left( 1 - \frac{2x}{L} \right)}{\cosh u} - 1 \right] + \frac{qL^2x}{8u^2D} \left( L - x \right) \quad (7.8)$$

As can be seen, the deflection \( w \) is dependent on the parameter \( u \), which is a function of the in-plane force \( S \), as given in Eq. 7.5.

To determine the force \( S \), it is necessary to find the force required to prevent the ends of the plate from moving towards each other as the plate undergoes lateral deformation. To do this, the extension of the plate strip (\( \lambda \)), defined as the difference between the arc length of the middle surface of the plate and the chord length (\( L \)), is taken to be produced by the force \( S \).

The exact expression for determining \( \lambda \) is

$$\lambda = \int_{0}^{L} \sqrt{1 + \left( \frac{dw}{dx} \right)^2} \, dx - L \quad (7.9)$$
where the integral gives the arc length of the middle surface of the plate.
The integral in Eq. 7.9 is difficult to evaluate however, and so by assuming that

\[
\frac{dw}{dx} \ll 1
\]  

(7.10)

the expression in Eq. 7.9 for \( \lambda \) may be approximated to an acceptable degree of accuracy by

\[
\lambda = \frac{1}{2} \int_0^L \left( \frac{dw}{dx} \right)^2 dx
\]  

(7.11)

This assumption was also used in approximating the Euler-Bernoulli law of bending so is consistent with the accuracy assumed so far.

Now to find the force \( S \), assuming lateral strain in the infinite dimension of the plate to be restrained, the stress and strain may be related by

\[
\sigma_x = \frac{E}{(1 - \nu^2)} \varepsilon_x
\]  

(7.12)

and the approximations are made that

\[
S = \sigma_x h
\]  

\[
\varepsilon_x = \frac{\lambda}{L}
\]  

(7.13)

Using Eqs. 7.11, 7.12, and 7.13 the in-plane force \( S \) is finally given by

\[
S = \frac{E h}{2(1 - \nu^2) L} \int_0^L \left( \frac{dw}{dx} \right)^2 dx
\]  

(7.14)
Finally, substituting the expression for $w$ from Eq. 7.8 into Eq. 7.14, and substituting for $S$ in Eq. 7.14 using the relationship in Eq. 7.5, a transcendental equation of the parameter $u$ is obtained as

$$
\frac{135}{16u^9} \left[ \tanh u + \frac{u \tanh^2 u}{5} - u + \frac{2u^3}{15} \right] = \frac{E^2 h_8}{(1-\nu^2)^2 q L^8}
$$

(7.15)

For given material properties, uniform loading $q$, and $h/L$ ratio, the right hand side of Eq. 7.15 is defined. The parameter $u$ may then be obtained from this transcendental equation through the use of any of several iterative numerical methods. Once having evaluated the parameter $u$, then the deflection $w$ may be obtained for any cross section given by the coordinate $x$, by substituting for $u$ in Eq. 7.8.

By arguments of symmetry, it is obvious that the maximum deflection will occur at $x = L/2$, therefore substituting this into Eq. 7.8, the maximum deflection of the plate is given by

$$
W_{\text{MAX}} = \frac{5qL^4}{384D} \left[ \text{sech} u - 1 + \frac{u^2}{2} \right] \frac{\frac{5u^4}{24}}{\frac{\text{sech} u - 1 + \frac{u^2}{2}}{2}}
$$

(7.16)

For the fixed edge plate, the governing differential equation is still Eq. 7.1, but the now the bending moment at any cross section of the strip is given by

$$
M = \frac{qLx}{2} - \frac{qx^2}{2} - Sw + M_0
$$

(7.17)

where $M_0$ is the moment required at the supports to prevent rotation of
the plate at the supports. The elements of Eq. 7.17 are shown in Fig. 16(b). This equation has been derived using the same assumptions about the surface traction as in the simply supported case, therefore the same comments apply again in this case.

Following the same development as for the simply supported plate, and using the boundary conditions below along with the observation that the deflected surface is symmetrical with respect to the centre of the strip \((x = L/2)\),

\[
\begin{align*}
\frac{dw(0)}{dx} &= \frac{dw(L/2)}{dx} = 0 \\
w(0) &= 0
\end{align*}
\] (7.18)

then the following transcendental equation for the parameter \(u\) is found,

\[
\frac{135}{16u^9} \left[ \frac{2u^3}{15} - \frac{u^3}{5 \sinh^2 u} + 4u - \frac{3u^2}{5 \tanh u} \right] = \frac{E^2 h^8}{(1-v^2)^2 q^2 L^8} \] (7.19)

Having found the parameter \(u\) through the use of the transcendental equation, the maximum deflection of the fixed edge plate, at \(x = L/2\), is evaluated by

\[
W_{\text{MAX}} = \frac{q L^4}{384D} \left[ \frac{24}{u^3} \left( \frac{u + \text{csch} u}{2} - \coth u \right) \right] \] (7.20)

In summary, there are several observations to be made about the closed form solutions presented by Timoshenko. The first observation is that the surface tractions as employed in the differential equations should be described by a Lagrangian surface traction in the finite element analysis, so that the closed form problem and the finite element problem correspond in the loading applied. Secondly, the derivation of the closed
form solutions requires that the slopes of the plate \((dw/dx)\), remain small. This is not a serious restriction since large displacements would be required to give the slope magnitudes necessary to invalidate the solution. This would in turn require a material with an unusually large elastic deformation limit. Thirdly, care must be taken in choosing appropriate boundary conditions for the finite element analysis in order that they be consistent with the boundary conditions imposed in the closed form solutions. Therefore, in the case of the simply supported edge plate, in-plane and out-of-plane displacements of the middle surface at the supports must be prevented, but the edges must be free to rotate. For the plate with fixed edges, in addition to the constraints on the simply supported plate rotation of the plate at the supports must be prevented. It should be noted however, that in the closed form solutions to both problems, there is no special restraint at the supports to prevent any Poisson effects that might occur in the thickness dimension of the plates. Care should therefore be taken in not greater restraint in the finite element analysis. Fourthly, the constitutive relationship given in Eq. 7.12 is utilized with the stress and strain definitions given in Eq. 7.13. The effect of the approximations to the stress and strain is somewhat difficult to predict qualitatively. The stress \(\sigma_x\) as described is a Lagrangian type, but the strain definition is of a small strain type. Since the finite element analysis will relate Kirchhoff stresses and Lagrangian strains with essentially the same constitutive relationship, the closed form model and the finite element analysis model differ slightly. Finally, in the closed form solutions no account is made of the shear deformations of the plates under the uniform loading, whereas the finite element will automatically include shear deformation. Thus, if everything else was in exact correspondence, the deflection found by the finite element analysis
should be greater than that predicted by the closed form solution, by the amount of shear deformation of the plate.

The transcendental equations for both the simply supported edge plate, Eq. 7.15, and the fixed edge plate, Eq. 7.19, with the dimensions and material properties given below were solved.

\[
E = 10^7 \text{ psi (6.895 x 10}^{10} \text{ kPa )}
\]
\[
h = 0.2 \text{ in. (0.508 cm )}
\]
\[
L = 10.0 \text{ in. (25.4 cm )}
\]
\[
\nu = 0.25
\]

The numerical method employed to solve the transcendental equations was Newton's Method of Tangents. Having then obtained the parameter \( u \), for a range of load intensities, the maximum deflection of the plate at the centre of the span divided by the thickness of the plate \( (w_{\text{MAX}}/h) \), was evaluated. The numerical procedure required the parameter \( (w_{\text{MAX}}/h) \) to be obtained to six significant figure accuracy. The numerical results for both boundary conditions are given in Table III. The finite element analysis results will be compared against these closed form results.

7.2.3 Infinite Plate Strip: Finite Element Analysis

The finite element using the nonuniform integration scheme developed in the previous chapter, is utilized to obtain a numerical solution to the infinite plate strip problem. The plate is analysed by employing a grid of elements through the thickness of the plate. The elements are then used with a plain strain constitutive relationship to represent the condition of restrained strain in the infinite dimension. Only half of the plate span need be analysed by arguments of symmetry about the mid-span of the plate. Infinite plate strips
### TABLE III

**CLOSED FORM SOLUTION RESULTS FOR TWO INFINITE PLATE STRIPS**

<table>
<thead>
<tr>
<th>q (psi)</th>
<th>SIMPLY SUPPORTED EDGES</th>
<th>FIXED EDGES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w_{MAX} / h</td>
<td>w_{MAX} / h</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2 (13.79)</td>
<td>0.168562</td>
<td>0.036585</td>
</tr>
<tr>
<td>4 (27.58)</td>
<td>0.291145</td>
<td>0.072958</td>
</tr>
<tr>
<td>6 (41.37)</td>
<td>0.380915</td>
<td>0.108918</td>
</tr>
<tr>
<td>8 (55.16)</td>
<td>0.451510</td>
<td>0.144288</td>
</tr>
<tr>
<td>10 (68.95)</td>
<td>0.510024</td>
<td>0.178917</td>
</tr>
<tr>
<td>12 (82.74)</td>
<td>0.560287</td>
<td>0.212692</td>
</tr>
<tr>
<td>14 (96.53)</td>
<td>0.604556</td>
<td>0.245527</td>
</tr>
<tr>
<td>16 (110.32)</td>
<td>0.644265</td>
<td>0.277371</td>
</tr>
<tr>
<td>18 (124.11)</td>
<td>0.680381</td>
<td>0.308195</td>
</tr>
<tr>
<td>20 (137.90)</td>
<td>0.713587</td>
<td>0.337995</td>
</tr>
<tr>
<td>22 (151.68)</td>
<td>0.744382</td>
<td>0.366781</td>
</tr>
<tr>
<td>24 (165.47)</td>
<td>0.773146</td>
<td>0.394577</td>
</tr>
<tr>
<td>26 (179.26)</td>
<td>0.800170</td>
<td>0.421413</td>
</tr>
<tr>
<td>28 (193.05)</td>
<td>0.825687</td>
<td>0.447328</td>
</tr>
<tr>
<td>30 (206.84)</td>
<td>0.849884</td>
<td>0.472362</td>
</tr>
<tr>
<td>32 (220.63)</td>
<td>0.872914</td>
<td>0.496556</td>
</tr>
<tr>
<td>34 (234.42)</td>
<td>0.894903</td>
<td>0.519954</td>
</tr>
<tr>
<td>36 (248.21)</td>
<td>0.915958</td>
<td>0.542598</td>
</tr>
<tr>
<td>38 (262.00)</td>
<td>0.936170</td>
<td>0.564526</td>
</tr>
<tr>
<td>40 (275.79)</td>
<td>0.955615</td>
<td>0.585779</td>
</tr>
</tbody>
</table>
with both simply supported and with fixed supports are analysed, using the same dimensions and material properties as used in the previous section in obtaining the closed form solutions.

For the plate with simply supported edges, the finite element analysis was performed using the grid of elements shown in Fig. 17(a). The incremental virtual work equations and the self-correcting procedure formed the basis of the method of analysis. The problem was analysed twice, once using loading increments of 4 psi (27.58 kPa), up to a maximum loading of 40 psi (275.8 kPa), then a second analysis was made using a larger loading increment of 8 psi (55.16 kPa), up to the same maximum loading. The results obtained from these two analyses, for the deflection parameter \( w_{\text{MAX}}/h \), are given numerically in Table IV and shown graphically in Fig. 18, where they are compared with the closed form solution. It can be seen from the results obtained that the finite element analysis gives excellent results even for the larger loading increment analysis. In the 4 psi loading increment analysis it can be seen that the load correction applied to the solution has caused some oscillation about the closed form solution. The use of the smaller loading increment can also be seen to give a solution that more closely follows the closed form solution at every increment. It should be noted however, that the 8 psi increment analysis, except for the first load step which is really only a linear analysis, follows the nonlinear deformation path very well.

Next, the plate having fixed longitudinal edges was analysed using the same element grid as in the previous problem, but the boundary conditions at the support have been appropriately modified, as shown in Fig. 17(b). The numerical results obtained for \( w_{\text{MAX}}/h \) are given in
INFINITE PLATE STRIP WITH SIMPLY SUPPORTED EDGES
FINITE ELEMENT GRID

FIG. 17(a)

INFINITE PLATE STRIP WITH FIXED EDGES
FINITE ELEMENT GRID

FIG. 17(b)
TABLE IV

FINITE ELEMENT RESULTS FOR THE INFINITE PLATE STRIP
WITH SIMPLY SUPPORTED LONGITUDINAL EDGES

<table>
<thead>
<tr>
<th>q (psi)</th>
<th>( \frac{w_{\text{MAX}}}{h} ) \text{ Closed Form Solution}</th>
<th>Finite Element Analysis (Using Mid-depth Node)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>4 psi steps</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.291145</td>
<td>0.353943 (+21.6)</td>
</tr>
<tr>
<td>8</td>
<td>0.451510</td>
<td>0.422102 (-6.5)</td>
</tr>
<tr>
<td>12</td>
<td>0.560287</td>
<td>0.586580 (+4.7)</td>
</tr>
<tr>
<td>16</td>
<td>0.644265</td>
<td>0.644570 (+0.05)</td>
</tr>
<tr>
<td>20</td>
<td>0.713587</td>
<td>0.720350 (+0.95)</td>
</tr>
<tr>
<td>24</td>
<td>0.773146</td>
<td>0.777995 (+0.63)</td>
</tr>
<tr>
<td>28</td>
<td>0.825687</td>
<td>0.830960 (+0.64)</td>
</tr>
<tr>
<td>32</td>
<td>0.872914</td>
<td>0.878320 (+0.62)</td>
</tr>
<tr>
<td>36</td>
<td>0.915958</td>
<td>0.921635 (+0.62)</td>
</tr>
<tr>
<td>40</td>
<td>0.955615</td>
<td>0.961590 (+0.62)</td>
</tr>
</tbody>
</table>

Note: The numbers indicated in parentheses are the percentage absolute error of the finite element results when compared to the closed form solution.
FINITE ELEMENT RESULTS FOR THE INFINITE PLATE STRIP (SIMPLY SUPPORTED)

FIG. 18
Table V where they are compared to the closed form solution, and also shown graphically in Fig. 19 where the closed form solution is again shown. The fixed edge plate was analysed using the 4 psi and 8 psi increments of loading as was used for the simply supported plate. Again, excellent results were obtained using both increments of loading. That the 4 psi increment analysis is not significantly better than the 8 psi increment one, is primarily due to the nature of the nonlinear problem. This particular problem is not as highly nonlinear as is the simply supported case.

In both the simply supported and fixed boundary condition plates' analysis, it should be noted that the results were obtained using a reasonably coarse mesh of finite elements. The use of two elements through the thickness was required at least in the case of the simply supported plate to correctly model the boundary conditions. It is expected that the use of either more sophisticated elements with higher order shape functions, or the use of a greater number of the nonuniform rectangular elements developed in this thesis, would result in even more accurate solutions to the two problems posed.

7.3 The Elastica

7.3.1 General

The infinite plate proble of the previous section has a closed form solution that is obtained by solving the approximate differential equation given as Eq. 7.1. The approximation inherent in that equation is satisfactory when the slopes, and hence deflections, remain reasonably small. When the slopes and deflections become large however, then the exact differential equation of the deflection curve must be used. This
<table>
<thead>
<tr>
<th>q</th>
<th>( \frac{w_{\text{MAX}}}{h} ) Closed Form Solution</th>
<th>( \frac{w_{\text{MAX}}}{h} ) Finite Element Analysis (Using Mid-depth Node)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>4 psi steps</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8 psi steps</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.072958 (-3.2)</td>
<td>0.070619 (-3.2)</td>
</tr>
<tr>
<td>8</td>
<td>0.144288 (-3.3)</td>
<td>0.139505 (-3.3)</td>
</tr>
<tr>
<td>12</td>
<td>0.212692 (-2.9)</td>
<td>0.206531 (-2.9)</td>
</tr>
<tr>
<td>16</td>
<td>0.277371 (-2.7)</td>
<td>0.270013 (-2.7)</td>
</tr>
<tr>
<td>20</td>
<td>0.337995 (-2.5)</td>
<td>0.329659 (-2.5)</td>
</tr>
<tr>
<td>24</td>
<td>0.394577 (-2.3)</td>
<td>0.385444 (-2.3)</td>
</tr>
<tr>
<td>28</td>
<td>0.447328 (-2.2)</td>
<td>0.437562 (-2.2)</td>
</tr>
<tr>
<td>32</td>
<td>0.496556 (-2.1)</td>
<td>0.486300 (-2.1)</td>
</tr>
<tr>
<td>36</td>
<td>0.542598 (-2.0)</td>
<td>0.531970 (-2.0)</td>
</tr>
<tr>
<td>40</td>
<td>0.585779 (-1.9)</td>
<td>0.574880 (-1.9)</td>
</tr>
</tbody>
</table>

Note: The numbers indicated in parentheses are the percentage absolute error of the finite element results when compared to the closed form solution.
FINITE ELEMENT RESULTS FOR THE INFINITE PLATE STRIP (FIXED EDGES)

FIG. 19
is the Euler-Bernoulli law of bending expressed by

$$D \frac{d\theta}{ds} = -M$$

(7.21)

where $D$ is again the flexural rigidity, $M$ is the moment at the particular cross section, and $d\theta/ds$ represents the curvature of the deflection curve. The term $d\theta/ds$ is the rate of change of $\theta$, the angle of rotation of the deflection curve, with respect to $s$ which is a running coordinate along the deflection curve. Now, assuming that the constitutive relationship is that of a linear elastic material, the exact shape of the elastic curve satisfying Eq. 7.21 is called the elastica.

The mathematical solution of the elastica was first investigated by Bernoulli, Euler, Lagrange, and Plana. Many different types of beams and loading conditions have since been postulated and solved for, but only two will be used here to evaluate the ability of the incremental virtual work equations with the nonuniformly integrated finite element and the self-correcting solution procedure, in following large displacements. The two elastica solutions available that were chosen for comparison purposes are, a cantilever with a vertical tip load, and a cantilever with a uniformly distributed load. The method of obtaining the closed form solutions will be briefly discussed, then the finite element solutions will be obtained and compared to the closed form solutions.

7.3.2 Cantilever With A Vertical Tip Load: Closed Form Solution

Consider the cantilever AB shown in Fig. 20, having a vertical tip load $P$ which is assumed to produce large deflections and to deform the beam to a configuration AB'. The vertical and horizontal displace-
CANTILEVER WITH VERTICAL TIP LOAD UNDERGOING LARGE DISPLACEMENTS

FIG. 20
ments of the free end of the beam are denoted by $\delta_v$ and $\delta_h$ respectively, and the angle of rotation at the free end of the cantilever is denoted by $\theta_b$. Note that throughout the large deformations experienced, that the tip load $P$ does not change its line of action and its magnitude is independent of the deformation, and is thus properly described as a Lagrangian traction vector.

Assuming that the length of the deflection curve at $AB'$ is equal to the initial length $L$, that is no axial extension, then the expression for the bending moment can be derived and substituted into Eq. 7.21, the Euler-Bernoulli law of bending. The flexural rigidity in this case is given by

$$D = EI$$

(7.22)

where $I$ is the moment of inertia of the cross section. After applying boundary conditions and considerable manipulation of the governing equation, the solution is obtained in terms of elliptic integrals. The transcendental equation to be solved is

$$F(k) - F(k, \phi) = \sqrt{\frac{PL^2}{EI}}$$

(7.23)

where

$$k = \sqrt{\frac{1 + \sin \theta_b}{2}}$$

(7.24)

$$\phi = \arcsin \frac{1}{k^{1/2}}$$

and $F(k)$ is a complete elliptic integral of the first kind, while $F(k, \phi)$ is an elliptic integral of the first kind. They are defined by
\[ F(k) = \int_{0}^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \]  
\[ (7.25) \]

and

\[ F(k, \phi) = \int_{0}^{\phi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \]  
\[ (7.26) \]

Knowing the length of the cantilever \( L \), and flexural rigidity \( EI \), the transcendental equation given in Eq. 7.23 is solved by assuming a value of \( \theta_b \), and then finding the corresponding value of the load \( P \).

The equation for the vertical deflection of the end of the cantilever is given as

\[ \frac{\delta_v}{L} = 1 - \frac{4EI}{PL^2} \left[ E(k) - E(k, \phi) \right] \]  
\[ (7.27) \]

where \( E(k) \) is a complete elliptic integral of the second kind and \( E(k, \phi) \) is an elliptic integral of the second kind. Finally, the horizontal deflection of the cantilever is found from

\[ \frac{\delta_h}{L} = 1 - \frac{2EI \sin \theta_b}{PL^2} \]  
\[ (7.28) \]

Elliptic integrals of both kinds are well tabulated, and Rojahn[36] has calculated deflections and rotations of this cantilever in terms of the nondimensional parameter \( PL^2/EI \). The deflections and rotations are given also in their nondimensional form in Table VI, and these values will be used to compare the finite element results with.

7.3.3 Cantilever With A Vertical Tip Load: Finite Element Analysis

The finite element analysis was performed for a cantilever...
### Table VI

Angle of Rotation and Deflections of a Cantilever Beam with a Tip Load

<table>
<thead>
<tr>
<th>( \frac{PL^2}{EI} )</th>
<th>( \theta_b/\pi/2 )</th>
<th>( \delta_v/L )</th>
<th>( \delta_h/L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>0.079</td>
<td>0.083</td>
<td>0.004</td>
</tr>
<tr>
<td>0.50</td>
<td>0.156</td>
<td>0.162</td>
<td>0.016</td>
</tr>
<tr>
<td>0.75</td>
<td>0.228</td>
<td>0.235</td>
<td>0.034</td>
</tr>
<tr>
<td>1.0</td>
<td>0.294</td>
<td>0.302</td>
<td>0.056</td>
</tr>
<tr>
<td>2.0</td>
<td>0.498</td>
<td>0.494</td>
<td>0.160</td>
</tr>
<tr>
<td>3.0</td>
<td>0.628</td>
<td>0.603</td>
<td>0.255</td>
</tr>
<tr>
<td>4.0</td>
<td>0.714</td>
<td>0.670</td>
<td>0.329</td>
</tr>
<tr>
<td>5.0</td>
<td>0.774</td>
<td>0.714</td>
<td>0.388</td>
</tr>
<tr>
<td>6.0</td>
<td>0.817</td>
<td>0.744</td>
<td>0.434</td>
</tr>
<tr>
<td>7.0</td>
<td>0.874</td>
<td>0.785</td>
<td>0.472</td>
</tr>
<tr>
<td>8.0</td>
<td>0.894</td>
<td>0.799</td>
<td>0.531</td>
</tr>
<tr>
<td>10.0</td>
<td>0.911</td>
<td>0.811</td>
<td>0.555</td>
</tr>
<tr>
<td>( \infty )</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>
whose dimensions and material properties are given in Fig. 21(a). The finite element mesh used, and the boundary conditions imposed are shown in Fig. 21(b).

The problem was analysed using the incremental virtual work equations and the self-correcting solution procedure, along with the nonuniformly integrated rectangular finite element. Two solutions were obtained using two different loading increment magnitudes. Letting

\[ K = \frac{PL^2}{EI} \]  \hspace{1cm} (7.29)

Then, the two analyses were performed using increments of the load \( P \) corresponding to increments in the parameter \( K \) of 0.25 and 0.5.

The results of the two analyses using the different increments of \( K \), for the rotation and the vertical and horizontal displacements at the cantilever's free end are given in Tables VII, VIII, and IX, for a range of \( K \) values from 0.25 to 10.0. The horizontal and vertical deflections given are those of the mid-depth node at the free end of the cantilever. The rotation of the free end, \( \theta_b \), was evaluated by finding the angle that the line segment joining the two mid-depth nodes nearest the free end, forms with the horizontal. The finite element solution values for \( \theta_b \) are therefore an average of the rotation of the deflected middle surface over the segment from 9L/10 to L with the mesh used. The results for \( \theta_b, \delta_v, \) and \( \delta_h \) are also given graphically in Figs. 22, 23 and 24 respectively, along with the closed form solution.

It can be seen from the Tables and Figures given that the finite element solution results correspond fairly well with the closed form solution, especially when a smaller loading increment (0.25) is used. There is an oscillation of the finite element solution at low
E = 1.2 \times 10^4 \text{ psi}

v = 0.2

L = 10.0 \text{ in.}

h = 1.0 \text{ in.}

b = 1.0 \text{ in.}

CANTILEVER WITH A VERTICAL TIP LOAD

FIG. 21(a)

CANTILEVER WITH A VERTICAL TIP LOAD: FINITE ELEMENT GRID

FIG. 21(b)
## TABLE VII

FINITE ELEMENT RESULTS FOR CANTILEVER END ROTATION

<table>
<thead>
<tr>
<th>$K = \frac{pL^2}{EI}$</th>
<th>$\frac{\theta_b}{\pi/2}$ Closed Form Solution</th>
<th>$\frac{\theta_b}{\pi/2}$ Finite Element Results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$K$ Increment 0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>0.079</td>
<td>0.0783</td>
</tr>
<tr>
<td>0.50</td>
<td>0.156</td>
<td>0.1543</td>
</tr>
<tr>
<td>0.75</td>
<td>0.228</td>
<td>0.2867</td>
</tr>
<tr>
<td>1.0</td>
<td>0.294</td>
<td>0.1581</td>
</tr>
<tr>
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<td>0.2867</td>
</tr>
<tr>
<td>2.0</td>
<td>0.498</td>
<td>0.2981</td>
</tr>
<tr>
<td>2.5</td>
<td></td>
<td>0.4733</td>
</tr>
<tr>
<td>3.0</td>
<td>0.628</td>
<td>0.4781</td>
</tr>
<tr>
<td>3.5</td>
<td></td>
<td>0.5626</td>
</tr>
<tr>
<td>4.0</td>
<td>0.714</td>
<td>0.5842</td>
</tr>
<tr>
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<td>0.6839</td>
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<td>0.7548</td>
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<td>0.817</td>
<td>0.7636</td>
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</tr>
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<td>0.874</td>
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<td>0.894</td>
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<td>10.0</td>
<td>0.911</td>
<td>0.8857</td>
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</tbody>
</table>
FINITE ELEMENT RESULTS FOR CANTILEVER END ROTATION

FIG. 22

\[ \frac{\theta_b}{\pi/2} \]

Linear Solution

Nonlinear Closed Form Solution

\( \triangle \) K Increment = 0.5

\( \circ \) K Increment = 0.25
### TABLE VIII
FINITE ELEMENT RESULTS FOR CANTILEVER VERTICAL END DEFLECTIONS

<table>
<thead>
<tr>
<th>$K = \frac{PL^2}{EI}$</th>
<th>$\frac{\delta V}{L}$ Closed Form Solution</th>
<th>$\frac{\delta V}{L}$ Finite Element Results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K$ Increment 0.5</td>
<td>$K$ Increment 0.25</td>
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<tr>
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<td></td>
<td></td>
</tr>
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<td>0.0</td>
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<td>0</td>
</tr>
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<td>0.25</td>
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<td>0.5777</td>
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<td>0.670</td>
<td>0.6435</td>
</tr>
<tr>
<td>4.5</td>
<td>0.6695</td>
<td>0.6722</td>
</tr>
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<td>0.714</td>
<td>0.6955</td>
</tr>
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</tr>
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<td>0.744</td>
<td>0.7312</td>
</tr>
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<td>0.7463</td>
<td>0.7453</td>
</tr>
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<td>0.767</td>
<td>0.7577</td>
</tr>
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<td>0.7689</td>
<td>0.7687</td>
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<td>0.785</td>
<td>0.7784</td>
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<td>0.7951</td>
</tr>
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<td>0.8025</td>
<td>0.8058</td>
</tr>
<tr>
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<td>0.811</td>
<td>0.8090</td>
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</tbody>
</table>
FINITE ELEMENT RESULTS FOR CANTILEVER VERTICAL END DISPLACEMENTS

FIG. 23
### TABLE IX

**FINITE ELEMENT RESULTS FOR CANTILEVER HORIZONTAL END DEFLECTIONS**

<table>
<thead>
<tr>
<th>K = $\frac{PL^2}{EI}$</th>
<th>$\frac{\delta h}{L}$ Closed Form Solution</th>
<th>$\frac{\delta h}{L}$ Finite Element Results</th>
<th>K Increment 0.5</th>
<th>K Increment 0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0.0051</td>
<td>0</td>
</tr>
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<td>0.016</td>
<td>0.0166</td>
<td>0.0193</td>
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<tr>
<td>0.75</td>
<td>0.034</td>
<td>0.056</td>
<td>0.0298</td>
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</tr>
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<td>0.0298</td>
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<tr>
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</tr>
<tr>
<td>2.0</td>
<td>0.255</td>
<td>0.1365</td>
<td>0.1660</td>
<td>0.1660</td>
</tr>
<tr>
<td>2.5</td>
<td>0.329</td>
<td>0.1568</td>
<td>0.2116</td>
<td>0.2116</td>
</tr>
<tr>
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<td>0.388</td>
<td>0.2141</td>
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<td>0.2528</td>
</tr>
<tr>
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<td>0.3228</td>
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<td>0.3518</td>
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<tr>
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<td>0.504</td>
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<td>0.4007</td>
<td>0.4007</td>
</tr>
<tr>
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<td>0.531</td>
<td>0.4216</td>
<td>0.4217</td>
<td>0.4217</td>
</tr>
<tr>
<td>5.5</td>
<td>0.579</td>
<td>0.4360</td>
<td>0.4408</td>
<td>0.4408</td>
</tr>
<tr>
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<td>0.625</td>
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<td>0.4583</td>
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</tr>
<tr>
<td>6.5</td>
<td>0.679</td>
<td>0.4730</td>
<td>0.4744</td>
<td>0.4744</td>
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<td>0.704</td>
<td>0.4889</td>
<td>0.4893</td>
<td>0.4893</td>
</tr>
<tr>
<td>7.5</td>
<td>0.731</td>
<td>0.5027</td>
<td>0.5030</td>
<td>0.5030</td>
</tr>
<tr>
<td>8.0</td>
<td>0.755</td>
<td>0.5157</td>
<td>0.5158</td>
<td>0.5158</td>
</tr>
<tr>
<td>8.5</td>
<td>0.773</td>
<td>0.5277</td>
<td>0.5278</td>
<td>0.5278</td>
</tr>
<tr>
<td>9.0</td>
<td>0.804</td>
<td>0.5555</td>
<td>0.5555</td>
<td>0.5555</td>
</tr>
<tr>
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<td>0.831</td>
<td>0.5889</td>
<td>0.5889</td>
<td>0.5889</td>
</tr>
<tr>
<td>10.0</td>
<td>0.855</td>
<td>0.6215</td>
<td>0.6215</td>
<td>0.6215</td>
</tr>
</tbody>
</table>
Finite Element Results for Cantilever Horizontal End Displacements

FIG. 24
K values caused by the self-correcting solution procedure. The oscillation may be attributed to the effect of linearizing the virtual work equations, which caused part of the virtual work expressions to be neglected. The absence of the neglected portion appears to have caused the incremental stiffness matrices to be too stiff. It is only the action of the load correction term in the self-correcting solution procedure that prevents the solution obtained from diverging. The magnitude of this oscillation could be reduced either by using smaller increments of loading, as can be seen from the results shown here, or by iterating at each loading increment step to satisfy a criteria of convergence.

7.3.4 Cantilever With Uniform Loading: Closed Form Solution

The closed form solution for this problem shown in Fig. 25(a), was obtained by Holden[37]. The solution again starts from the Euler-Bernoulli law of bending, given in Eq. 7.21, from which, assuming a uniform loading intensity q, the following is obtained,

\[ \frac{d^2 \theta}{ds^2} = - \frac{q}{D} s \cos \theta \]  

The boundary conditions applied are

\[ \frac{d \theta}{ds} = 0 \quad \text{at} \quad s = 0 \]  
\[ \theta = 0 \quad \text{at} \quad s = L \]  

Holden then separates the second order equation, Eq. 7.30, into a system of two first order equations and integrates using numerical methods to get \( \theta \) as a function of \( s \) for prescribed \( q \). The deflections are obtained
CANTILEVER WITH UNIFORM LOADING

FIG. 25(a)

CANTILEVER WITH UNIFORM LOADING: FINITE ELEMENT GRID

FIG. 25(b)
by integrating along the length of the cantilever using the known values of \( \theta \), that were determined first. Holden used a fourth order Runge-Kutta procedure for the differential equations and a Simpson's rule for the integrations required. Holden assumes the cantilever to be inextensional.

Numerical results were not given by Holden in his paper, therefore his solution was obtained as accurately as possible by graphical means from a report by Bathe, et al \[38\], who also use this solution for comparison purposes.

7.3.5 Cantilever With Uniform Loading: Finite Element Analysis

The finite element solution to this problem was obtained using the grid of nonuniformly integrated rectangular finite elements as shown in Fig. 25(b). The tip deflection results were obtained using two different loading increment magnitudes, and the results are given in Table X. The results are given for a load parameter \( K' \), where \( K' \) is defined as

\[
K' = \frac{qL^3}{EI}
\]  

(7.32)

Two analyses were performed using increments of the uniform load \( q \) corresponding to increments of \( K' \) of 0.5 and 1.0. Both analyses included the range of \( K' \) from \( K' = 0 \) to \( K' = 10.0 \). The tip deflection values given are those of the mid-depth node. Another analysis was also performed, in which the load correction aspect of the other two analyses was not used. The results of this trial are also given in Table X. These three solutions are shown graphically in Fig. 26, where Holden's solution is given for comparison purposes.

As can be seen from Fig. 26, the incremental finite element
TABLE X

TIP DEFLECTIONS OF A UNIFORMLY LOADED CANTILEVER:
FINITE ELEMENT RESULTS

<table>
<thead>
<tr>
<th>K' Increment = 0.5</th>
<th>K' Increment = 1.0</th>
<th>K' Increment = 1.0 (No load correction)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.062479</td>
<td>0.124957</td>
</tr>
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TIP DEFORMATIONS OF A UNIFORMLY LOADED CANTILEVER:
FINITE ELEMENT RESULTS

FIG. 26
solutions are somewhat erratic in their incremental steps. The oscillations are due to the action of the load correction term in trying to prevent the solution from diverging from equilibrium states. The analysis of this problem using increments in $K'$ of 0.5, appears to be slightly less erratic than that using increments of 1.0, as should be expected. The rapid divergence of the finite element analysis using a solution procedure without equilibrium checks is clearly shown in Fig. 26, and thus shows the value of using the self-correcting solution procedure in the analysis of nonlinear problems.

The finite element analysis using increments of $K'$ of 0.5 shows slightly greater tip deflections than Holden's closed form solution predicts. In actual fact, an elasticity solution if it could be found, should predict greater tip deflections due to shear deformation and extension of the cantilever, which Holden's solution does not consider.

For this type of large deflection problem then, the incremental virtual work equations, the nonuniformly integrated rectangular finite element, and the self-correcting solution procedure combine to give an effective analysis technique.
CONCLUSIONS

The analysis of the finite deformation of linear elastic bodies has been approached in this thesis, through an incremental virtual work formulation using the Lagrangian description. To obtain numerical solutions to the incremental equations derived, the finite element method was utilized with a self-correcting solution technique. The particular finite element used was a nonuniformly numerically integrated, eight degree of freedom, bilinear rectangle.

The analytical procedure developed, comprised of the incremental virtual work equations, the finite element method and the self-correcting solution technique, has been used to obtain numerical solutions for four nonlinear finite deformation problems. The results of these analyses were compared against available closed form solutions. In all four problems, the analytical procedure developed gave solutions with excellent agreement to their respective closed form solutions.

The value of the self-correcting solution technique in preventing divergence of the incremental analysis from the correct nonlinear path was shown for the case of a uniformly loaded cantilever.

From the results obtained, and their excellent agreement to the closed form solutions, it is concluded that the analytical procedure as advanced in this thesis is an effective technique for obtaining the nonlinear finite deformation response of elastic bodies.

The computer program that was developed to implement this solution technique, will be made available to the Department of Civil Engineering, University of British Columbia.


17. IBID, pp. 440-441.


19. IBID.


26. IBID


34. IBID pp. 6-20

35. IBID pp. 118-131


APPENDIX A - THREE DIMENSIONAL ANALYSIS OPERATOR MATRICES

In section 6.3.2, use was made of three different operator matrices that because of their size were not shown there. These operator matrices will be shown in this appendix, and a relationship will be developed between them to reduce the computational effort required.

First, the operator matrix \([B_L] \) is required to operate on the vector \([u] \) to give \([e_L] \) as shown in Eq. 6.12, and the elements of \([e_L] \) are defined as in Eq. 6.9. Using these required relationships then,

\[
[B_L]^T = \frac{1}{2} \begin{bmatrix}
2 \frac{\partial}{\partial a_1} & \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_3} & 0 & 0 & \frac{\partial}{\partial a_3} & 0 & 0 \\
0 & \frac{\partial}{\partial a_1} & 0 & \frac{\partial}{\partial a_1} & 2 \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_3} & 0 & \frac{\partial}{\partial a_3} \\
0 & 0 & \frac{\partial}{\partial a_1} & 0 & 0 & \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_1} & 2 \frac{\partial}{\partial a_3}
\end{bmatrix} \tag{A.1}
\]

Secondly, the operator matrix \([B_{NL}(u)] \) is required to operate on the vector \([u] \) to give \([e_{NL}] \) as shown in Eq. 6.13, where the elements of \([e_{NL}] \) are defined by Eq. 6.10. Therefore, \([B_{NL}(u)] \) is given in matrix form, on the following page, as
\[ \begin{bmatrix}
2 \frac{\partial}{\partial a_1} 1u_1,1 \\
2 \frac{\partial}{\partial a_1} 1u_2,1 \\
2 \frac{\partial}{\partial a_1} 1u_3,1 \\
\end{bmatrix}
\]

\[ \begin{bmatrix}
(1u_{1,1}) \frac{\partial}{\partial a_2} + (1u_{1,2}) \frac{\partial}{\partial a_1} \\
(1u_{1,1}) \frac{\partial}{\partial a_3} + (1u_{1,3}) \frac{\partial}{\partial a_1} \\
(1u_{1,2}) \frac{\partial}{\partial a_1} + (1u_{1,1}) \frac{\partial}{\partial a_2} \\
(1u_{1,2}) \frac{\partial}{\partial a_3} + (1u_{1,3}) \frac{\partial}{\partial a_2} \\
(1u_{1,3}) \frac{\partial}{\partial a_1} + (1u_{1,1}) \frac{\partial}{\partial a_3} \\
(1u_{1,3}) \frac{\partial}{\partial a_2} + (1u_{1,2}) \frac{\partial}{\partial a_3} \\
2(1u_{1,3}) \frac{\partial}{\partial a_3} \\
2(1u_{2,3}) \frac{\partial}{\partial a_2} \\
2(1u_{3,2}) \frac{\partial}{\partial a_2} \\
\end{bmatrix}
\]

\[ [B_{NL}(1u)] = \frac{1}{2} \]

(A. 2)
Thirdly, the operator matrix \([L]\), as defined in Eq. 6.35 is given by

\[
[L]^T = \begin{bmatrix}
\frac{\partial}{\partial a_1} & \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial a_1} & \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial a_1} & \frac{\partial}{\partial a_2} & \frac{\partial}{\partial a_3}
\end{bmatrix}
\]  

(A.3)

Finally, the matrix of initial stresses as required by the relationship in Eq. 6.36 is given by

\[
[ST] = \begin{bmatrix}
^1S_{11} & ^1S_{12} & ^1S_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
^1S_{21} & ^1S_{22} & ^1S_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
^1S_{31} & ^1S_{32} & ^1S_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & ^1S_{11} & ^1S_{12} & ^1S_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & ^1S_{21} & ^1S_{22} & ^1S_{23} & 0 & 0 & 0 \\
0 & 0 & 0 & ^1S_{31} & ^1S_{32} & ^1S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ^1S_{11} & ^1S_{12} & ^1S_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & ^1S_{21} & ^1S_{22} & ^1S_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & ^1S_{31} & ^1S_{32} & ^1S_{33}
\end{bmatrix}
\]  

(A.4)

Now rather than calculate all the operator matrices separately, relationships will be established between them in order to avoid a duplication of effort. Beginning with the operator \([L]\), the operator \([B_L]\) may be obtained through the use of a transformation matrix \([A]\), that is
where the matrix \([A]\) is defined as

\[
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\]

Advantage may be gained by use of the fact that \([A]\) is symmetric about its major diagonal, in performing the required matrix multiplication.

Next, the operator \([B_{NL}^{(1)u}]\) may be readily obtained from the result of Eq. A.5, through the use of the matrix of initial derivatives, \([i_{1i,j}]\) defined in Eq. 6.51 for use in the Kirchhoff surface traction integral. The operator \([B_{NL}^{(1)u}]\) is obtained from the matrix product below,

\[
[B_{NL}^{(1)u}] [N] = [B_L] [N] [i_{1i,j}]^T
\]

Thus, beginning with the operator \([L]\), the other two required operators \([B_L]\) and \([B_{NL}^{(1)u}]\) are obtained in sequence using the transformation matrix \([A]\) and then the matrix of initial derivatives.
This approach should reduce the computational effort required in obtaining \( \mathbb{B}_L \) and \( \mathbb{B}_{NL}(^1u) \), by avoiding an unnecessary amount of duplication of calculations in forming these two operators.
In section 6.4.2, the operator matrices $[L]$, $[B]$, and $[B_{NL}(^1u)]$ as well as the matrix of initial stresses $[^1ST]$ for two dimensional analysis were defined. They were not shown fully in the main text due to their size but are given in full in this appendix. In addition relationships will be established between the three operator matrices in an analogous manner to those of Appendix A for three dimensional analysis.

The operator matrix $[B_L]$ is given by

$$
[B_L] = \begin{bmatrix}
\frac{\partial}{\partial a_1} & 0 \\
0 & \frac{\partial}{\partial a_2}
\end{bmatrix}
$$

(B. 1)

next $[B_{NL}(^1u)]$ is given by

$$
[B_{NL}(^1u)] = \begin{bmatrix}
(^1u_{1,1}) \frac{\partial}{\partial a_1} & (^1u_{2,1}) \frac{\partial}{\partial a_1} \\
(^1u_{1,2}) \frac{\partial}{\partial a_2} & ( ^1u_{2,2}) \frac{\partial}{\partial a_2} \\
(^1u_{1,1}) \frac{\partial}{\partial a_2} + (^1u_{1,2}) \frac{\partial}{\partial a_1} & ( ^1u_{2,1} \frac{\partial}{\partial a_2} + ( ^1u_{2,2}) \frac{\partial}{\partial a_1}
\end{bmatrix}
$$

(B. 2)
and finally \([L]\) is defined as

\[
[L] = \begin{bmatrix}
\frac{\partial}{\partial a_1} & 0 \\
\frac{\partial}{\partial a_2} & 0 \\
0 & \frac{\partial}{\partial a_1}
\end{bmatrix}
\]  

(B. 3)

Proceeding in an analogous manner to Appendix A, it may be simply shown that

\[
[B_L] [N] = [A] [L] [N]
\]  

(B. 4)

where

\[
[A] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]  

(B. 5)

Similarly

\[
[B_{NL}(1u)] [N] = [B_L] [N]
\]  

(B. 6)

The result of Eq. B. 6 may be alternately, and for computational purposes, more conveniently, obtained by
Finally the matrix of initial stresses is given by

\[
\begin{bmatrix}
^1\mathbf{u}_{1,1} & 0 & ^1\mathbf{u}_{2,1} & 0 \\
0 & ^1\mathbf{u}_{1,2} & 0 & ^1\mathbf{u}_{2,2} \\
^1\mathbf{u}_{1,2} & ^1\mathbf{u}_{1,1} & ^1\mathbf{u}_{2,2} & ^1\mathbf{u}_{2,1}
\end{bmatrix}
\]

\[(B. 7)\]

\[
\begin{bmatrix}
^1\mathbf{S}_{11} & ^1\mathbf{S}_{12} & 0 & 0 \\
^1\mathbf{S}_{12} & ^1\mathbf{S}_{22} & 0 & 0 \\
0 & 0 & ^1\mathbf{S}_{11} & ^1\mathbf{S}_{12} \\
0 & 0 & ^1\mathbf{S}_{12} & ^1\mathbf{S}_{22}
\end{bmatrix}
\]

\[(B. 8)\]