FINITE ELEMENT ANALYSIS OF PERIODIC VISCOUS FLOW IN A CONSTRICTED PIPE

by

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Abstract

The finite element method is extended to analyze axisymmetric periodic viscous flow in a constricted pipe. This method is used to solve the pulsatile fluid flow through an artery which may be stenosed due to impingement of extravascular masses or due to intravascular atherosclerotic plaques developed at the wall of the artery. The artery geometry is approximated as an axisymmetric channel. Blood is assumed to be a Newtonian fluid, i.e., it follows a linear relationship between the rate of shear strain and the shear stress.

The numerical model uses the finite element method based on velocity-pressure primitive variable representation of the Navier-Stokes equations. Eight-noded isoparametric elements with quadratic interpolation for velocities and bilinear for pressure are used. A truncated Fourier series is used to approximate the unsteadiness of flow and a modified method of averaging is used to obtain a periodic solution. Two non-dimensional parameters are used to characterize the flow: the frequency Reynolds number $R_\omega$, and the steady Reynolds number $R_e$. The pulsatile flow is modeled as a linear combination of steady, cosinusoidal and sinusoidal components. The magnitude of each component is determined by minimizing the error in approximation through Galerkin’s procedure. Results are obtained for various values of $R_\omega$ and $R_e$ for laminar flow. Results are also presented for limiting cases and, wherever possible, numerical results thus obtained are compared with analytical and experimental results published in the literature.
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Notation

Lower case Greek symbols

\( \alpha \)  
“periodicity factor”, real part of a complex number

\( \beta \)  
imaginary part of a complex number

\( \delta \)  
radius at the minimum section

\( \epsilon \)  
non-dimensional parameter for developing flow

\( \zeta \)  
vorticity of the flow

\( \eta \)  
non-dimensional radial coordinate

\( \theta \)  
angle of rotation

\( \kappa \)  
grid efficiency factor

\( \mu \)  
viscosity of the fluid

\( \nu \)  
kinematic viscosity

\( \xi \)  
non-dimensional axial coordinate

\( \rho \)  
density of the fluid

\( \sigma_\phi \)  
azimuthal stress in three dimensions

\( \sigma_r \)  
radial stress in three dimensions

\( \sigma_x \)  
avarial stress in two dimensions

\( \sigma_y \)  
radial stress in two dimensions

\( \sigma_z \)  
avarial stress in three dimensions

\( \tau_{\phi x} \)  
 shear stress on azimuthal face in radial direction

\( \tau_{r\phi} \)  
 shear stress on radial face in azimuthal direction

\( \tau_{rz} \)  
 shear stress on radial face in axial direction

\( \tau_{xy} \)  
 shear stress on axial face in radial direction

\( \phi \)  
azimuthal coordinate in three dimensions
\[ \psi \] stream function
\[ \omega \] frequency
\[ \omega^{-1} \] characteristic time

**Upper case Greek symbols**

\[ \Gamma \] fluid boundary
\[ \Gamma_s \] traction boundary
\[ \Gamma_u \] kinematic boundary
\[ \Gamma^e \] finite element boundary
\[ \Gamma^e_s \] finite element traction boundary
\[ \Gamma^e_u \] finite element kinematic boundary
\[ \Delta \] increment (prefix)
\[ \Omega \] fluid domain
\[ \Omega^e \] finite element domain

**Lower case Roman symbols**

\[ a \] pre-stenosis length, pre-orifice length
\[ a(y) \] steady part of pulsatile flow
\[ b \] post-stenosis length, post-orifice length
\[ b(y) \] magnitude of cosinusoidal part of pulsatile flow
\[ d \] nodal vector of unknowns
\[ f \] consistent load vector
\[ l \] length of stenosis, length of orifice
\[ l_E \] inlet length
\[ n_1 \] axial cosine of unit outward normal to the boundary
\[ n_2 \] radial cosine of unit outward normal to the boundary
\[ p \] pressure
\[ \Delta p_{add} \] additional pressure drop
\( r \) radial coordinate in three dimensions

\( r(x) \) instantaneous radius

\( t \) time

\( u \) axial component of velocity

\( u_0 \) characteristic velocity

\( v \) radial component of velocity

\( v_r \) radial component of velocity in three dimensions

\( v_\phi \) azimuthal component of velocity in three dimensions

\( v_z \) axial component of velocity in three dimensions

\( x \) axial coordinate in two dimensions

\( y \) radial coordinate in two dimensions

\( z \) axial coordinate in three dimensions

**Upper case Roman symbols**

\( A \) steady streaming solution

\( A_0(y) \) steady part of pulsatile approximation

\( B_0(y) \) magnitude of cosinusoidal part of pulsatile approximation

\( B(t) \) magnitude of cosinusoidal component

\( C_0(y) \) magnitude of sinusoidal part of pulsatile approximation

\( C(t) \) magnitude of sinusoidal component

\( C^0 \) continuity of zeroth order derivatives

\( C^1 \) continuity of first order derivatives

\( F_r \) body force in radial direction

\( F_\phi \) body force in azimuthal direction

\( F_z \) body force in axial direction

\( J \) Jacobian matrix

\( J_0 \) Bessel's function of the first kind of order zero
$K$ stiffness matrix

$M$ mass matrix

$M_j$ $j$-th bilinear isoparametric shape function

$N_i$ $i$-th quadratic isoparametric shape function

$Q(t)$ instantaneous rate of flow

$R$ radius of the pipe

$Re$ frequency Reynolds number

$R_0$ characteristic length

$R_e$ Reynolds number

$\mathcal{R}_0$ error in approximating Poisson's equation

$\mathcal{R}_1$ error in approximating $x$-momentum equation

$\mathcal{R}_2$ error in approximating $y$-momentum equation

$\mathcal{R}_3$ error in approximating continuity equation

$\mathcal{R}_\Gamma_1$ error in approximating $x$-component of traction boundary

$\mathcal{R}_\Gamma_2$ error in approximating $y$-component of traction boundary

$T$ tangent stiffness matrix

$U$ axial component of specified kinematic boundary

$U_0$ (maximum) entrance velocity

$V$ radial component of specified kinematic boundary

$X$ axial component of specified traction boundary

$Y$ radial component of specified traction boundary

$Y_0$ Bessel's function of the second kind of order zero

$Z_0$ development length

$\frac{D}{Dt}$ total material derivative $\equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$

differentiation

non-dimensional value

differentiation with respect to time
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To

My Parents

Indu and Madan Singh
1.1 General Remarks

Fluid mechanics poses one of the most difficult problems for applied mathematics: the solution of the Navier-Stokes equations and simplifications thereof for flow conditions dictated by practical applications of the laws of fluid mechanics and technological developments, rather than by academic questions. It was only after the advent of digital electronic computers, that the solutions for such problems could be developed with success. In the beginning of fluid mechanics research, limiting situations—offering permissible simplifications of the governing equations to solvable ones—were primarily the goal of research. Thus came the Prandtl’s lift line theory, the Prandtl-Meyer flow, or the Blasius series solution. Even though these problems were relatively quite simple, all of them already involved numerical work and a substantial amount of time in which those solutions were derived. Prandtl’s lifting line theory yielded an integro-differential equation for the circulation, the Prandtl-Meyer flow a transcendental equation in terms of Mach number to be solved for the Prandtl-Meyer angle and the Blasius series solution lead to a nonlinear two point boundary value problem. Then came the desk calculator which enabled the computation of more complex flows such as supersonic nozzle flow by the method of characteristics. The first electronic computers immediately enabled the solution of the initial value two point boundary value problem for two-dimensional boundary layers and of the potential equation for flows with rather arbitrary boundary shapes, a substantial step forward. Krause [11]
has compiled a short history of early works in this field.

With the advent of high speed digital computers, researchers started developing numerical techniques for the solving hitherto unsolvable differential equations. The basic idea behind most of these techniques is to try to 'simulate' the physical phenomena in some approximate way and to improve the approximation in successive attempts. Computational fluid dynamics has always been at the forefront of the development of these numerical techniques [12]. Several reasons have contributed to this leadership among many disciplines some of which are:

- the inherently nonlinear behavior of fluids (e.g., convection, turbulence) which must be accounted for,

- the mixed hyperbolic-elliptic character of the governing partial differential equations, and

- the need for engineers designing new airplanes to obtain much more accurate performance estimates and therefore more accurate results for the simulation than, say, their colleagues designing bridges.

Another reason that follows from above is the size of typical problems. A problem in structural mechanics is considered 'large' if it exceeds $5 \times 10^3$ nodes, whereas large in fluid problems means more than $5 \times 10^5$ grid points [12].

For about 25 years the computational fluid dynamics grew around Finite Difference Methods, as these were relatively simple to understand and code, easy to vectorize and the structured grids typically associated with them described appropriately the simple geometric complexity of the fields that were solved. However, as the computers became bigger and faster, attempts were made to simulate more and more complex flow domains and it soon became clear that structured grids were not flexible enough to describe these domains. It was at this point in time that unstructured grids and Finite Element Methods—a natural way of discretizing operators on
them—entered the scene. Since then the finite element methods have become a *de jure* standard techniques, although some researchers still prefer the finite difference methods.

Since the initial development efforts for the finite element methods for fluid mechanics, much attention has been devoted to the treatment of the primitive variable (velocities and pressure) formulations. The dominant approach used in the finite element methods is the mixed order interpolation scheme which attempts to eliminate the spurious pressure modes [31], i.e., the tendency to produce 'checkerboard' pressure distributions [24]. This scheme is analogous to the 'staggered grid' used in the finite difference methods. However, mixed order interpolation is not totally effective at eliminating the spurious pressure modes for simple elements, i.e., bilinear triangle or bilinear quadrilaterals where constant element pressures are used [24]. De Bremaecker [5] has suggested use of penalty functions to avoid this problem; Rice and Schnipke [24] have presented an equal order velocity pressure interpolation that does not show the spurious pressure modes at all; Zinser and Benim [37] have suggested a segregated velocity pressure formulation of Navier-Stokes equations which results in an approximate pressure equation and allows uncoupling of pressure and velocity solutions. Numerous other suggestions have been made. For the present investigation, we have used a mixed order interpolation with quadratic bilinear element where pressure is *not* constant over the element; along the element boundaries the variation is linear whereas in the domain the variation is linear plus one quadratic cross term. This eliminates the above mentioned ‘checkerboard’ pressure distributions.

### 1.2 Stenosis and Pulsatile Flow

The world "stenosis" is a generic medical term which means a narrowing of any body passage, tube, or orifice. Thus, an arterial stenosis refers to a narrowed or constricted segment of an artery and may be caused by impingement of extravascular
masses or due to intravascular atherosclerotic plaques which develop at the walls of
the artery. In the arterial systems of humans, it is quite common to find narrowings,
or stenoses, some of which are, at least approximately, axisymmetric or 'collar-like.'
Partial occlusion of blood vessels due to stenosis is one of the most frequently oc­
curring abnormalities in the cardiovascular system of humans; it is a well established
fact that about 75% of all deaths are caused due to circulatory diseases [2]. Of these,
atherosclerosis is the most frequent. Considerable studies related to the circulatory
flow in blood vessels were given major attention towards the beginning of the present
century. Quite a few analytical and experimental investigations related to blood flow
with different perspectives have already been carried out. Interest in studies of this
particular domain of biomechanics has increased with the discovery that many car­
diovascular diseases are associated with the flow conditions in the blood vessels; one
major type of flow disorder results from stenosis.

Diagnosis of arterial diseases is usually carried out by the method of X-ray angiog­
raphy which involve considerable risk of morbidity [3]. So any attempts to develop
non-invasive techniques for detecting arterial disease like stenosis are very worthwhile.
As a result, numerical modeling of arterial blood flow has long been of interest, but
development of realistic models has occurred only in recent times [22]. Causes of the
slow progress include nonlinearities in the model equations, unsteady flow in the vas­
cular system, and elastic properties of arteries. Most of the studies have been carried
out assuming that blood is a Newtonian fluid. However it has now been accepted that
blood behaves more like a non-Newtonian fluid under certain conditions, in particular
at low shear rate. Srivastava [28] has presented a study of the effects of stenosis on
the flow of blood when blood is represented by a couple stress fluid, a special type of
non-Newtonian fluid which takes particle size into account. This, however, is beyond
the scope of the current investigation.

Young [35] has presented a survey of some of the early works done in this field.
Over the years, Daly [4], Deshpande [6,7], Yoganathan et al. [34], Philpot et al. [21], Porenta, Young and Rogge [22], Forrester and Young [8,9] and others have shed considerable light on the complex problem of blood flow through an artery in the presence of stenoses. Daly studied the effects of pulsatile flow through stenosed canine femoral arteries for constrictions in the range 0-61% and presented a comparison of in-vivo and in-vitro results. Local flow reversal was observed in the wake of the stenosis during systole and during diastolic flow reversal. He also concluded that the rate of development of the local reverse flow is very sensitive to the height of the stenosis, a result supported by Forrester as well.

Deshpande obtained numerical solutions for steady flow through axisymmetric, contoured constrictions in a rigid tube utilizing full Navier-Stokes equations in cylindrical coordinates. Laminar boundary conditions, normally applicable only at $x = \pm \infty$, were applied at finite values of $x$ by using a transformation of the type $\eta = \tanh(\kappa x)$, where $\kappa$ is a constant chosen to group the grid points in an efficient manner. The stenosis geometry was modeled as a full cosine wave. Yoganathan et al. obtained steady flow velocity measurements in a pulmonary artery model with varying degrees of pulmonic stenosis. Philpot et al. used a 7-mW He-Ne laser as the light source for flow visualization in a Pulmonary artery model. They photographed the setup over the entire systolic period, thus observing the development and dissipation of the flow.

Porenta et al. developed a one dimensional analysis which included taper, branches and obstructions and used Kantovich-like numerical technique and yielded stable numerical solution after 3 cycles, independent of initial values. The arterial cross-section was of the form $a \cdot \exp(-\alpha x)$, where $\alpha$ is a small positive number. Low frequency behavior was approximated with seven cosine and seven sine components of the generalized Fourier series. Propagation of a pressure wave was evident from his results. Forrester modeled the stenosis as a full cosine wave and used the Karman-Pohlhausen
method as described by Schlichting [26] to model the flow using a five parameter fourth order polynomial and obtained an analytical solution. He has also reported some experimental results which supported his solution scheme. In particular, his experiments clearly reveal the development of separation downstream of the stenosis and the reattachment points as well as variation of incipient separation point and critical Reynolds number with height of stenosis.

1.3 Analysis in Time Domain

Pulsatile flow in the arteries is highly variable with time and to understand all the effects of stenosis, one has to solve the full Navier-Stokes equations for any time $t$. In general, this can be done by obtaining the finite element solution for the steady part and then carrying out simultaneous numerical integration, choosing a sufficiently small step so as not to allow divergence of the results. This, however, is extremely costly in terms of CPU time and storage since, at each step, we must solve for the effects of the quadratic, nonlinear, convection terms. McLachlan [14] and Nayfeh and Mook [17] have pointed out that the solution of the differential equations with such nonlinearities has a steady part as well as an oscillating part. The techniques available for determining the steady state behavior and the slowly varying periodic behavior of forced nonlinear systems can be divided into two groups: the first group includes the method of averaging and multiple scales. With this group one determines the equations describing phases and amplitudes which are to be solved simultaneously. The second group includes the Lindstédt-Poincaré technique and the method of harmonic balance. Details of these methods are presented in [14,17].

Pattani [19,20] has modified the method of averaging and shown that it can be used successfully for this kind of problem. Using the Floquet theory, he has also shown the solution to be stable. This method is used here.
1.4 Scope of Present Investigation

This thesis describes the development work on extending the finite element method to cover the effects of stenoses on pulsatile flow in arteries. The problem configuration of interest is that of an axisymmetric, or 'collar-like' artery. The pulsatile flow is approximated as a combination of steady, sinusoidal and cosinusoidal velocities.

In Chapter 2, the fundamentals of fluid flow in cylindrical coordinates are presented. The chapter begins by stating the governing differential equations for the fluid, derived in [26] based on the principle of conservation of mass and momentum, and choosing non-dimensional parameters to effectively characterize the flow through a pipe. Kinematic and traction boundary conditions are also presented and non-dimensionalized. In the last section, the Poisson's equation for stream function is derived.

Chapter 3 is about discretization of the various differential equations and the boundary conditions using a family of locally defined shape functions. Residuals are formed and Galerkin's procedure is used to minimize those residuals. The modified method of averaging is used to solve for the steady streaming part and slowly varying periodic parts. A Newton-Raphson iteration scheme is outlined for the solution of the resulting set of nonlinear, algebraic equations.

Numerical results are described in Chapter 4. Besides the pulsatile flow through stenoses, several other limiting problems are also solved to check the code. These results compare favorably with published analytical and experimental results in the literature. Streaklines and streamlines are plotted for different values of the non-dimensional parameters $Re$ and $Rw$. In the last section, conclusions and suggestions for further investigations are presented.
In this chapter, the governing equations for axisymmetric fluid flow are presented. The velocity-pressure primitive variable form of the Navier-Stokes equations for two-dimensional incompressible viscous flow is used. The equations of motion are non-dimensionalized to effectively characterize the flow situation. Finally, an approximate representation for pulsatile flow is presented. All the analysis is based on incompressibility assumptions which hold well for liquids.

2.1 Conservation Equations and Boundary Conditions

The basic equations governing viscous, incompressible fluid flow are the conservation equations: conservation of mass and conservation of momentum. These equations along with the specified boundary and initial conditions, in general, need to be solved over a domain $\Omega$ bounded by a contour $\Gamma$ which is composed of two distinct parts—kinematic boundary and traction boundary—denoted as $\Gamma_u$ and $\Gamma_s$, respectively, as shown in figure 2.1. If $r$, $\phi$ and $z$ denote the radial, azimuthal and axial coordinates, respectively, of a three-dimensional system of cylindrical coordinates and $v_r$, $v_\phi$, $v_z$ denote the velocity components in the respective directions, then the governing equations for incompressible fluid flow are given by Schlichting [26] as
Chapter 2: Derivation of Governing Equations

Figure 2.1: Fluid Domain

\[
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = \\
F_r - \frac{\partial p}{\partial r} + \mu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial v_r}{\partial \phi^2} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial^2 v_r}{\partial z^2} \right) 
\]

(2.1)

\[
\rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_r \frac{\partial v_\phi}{\partial z} \right) = \\
F_\phi - \frac{1}{r} \frac{\partial p}{\partial \phi} + \mu \left( \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi} + \frac{\partial^2 v_\phi}{\partial z^2} \right) 
\]

(2.2)

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z} \right) = \\
F_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{1}{r^2} \frac{\partial v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2} \right) 
\]

(2.3)

\[
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0. 
\]

(2.4)
Chapter 2: Derivation of Governing Equations

The stress components assume the form

\[
\begin{align*}
\sigma_r &= -p + 2\mu \frac{\partial v_r}{\partial r} \quad ; \quad \tau_{r\phi} = \mu \left[ r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \phi} \right] \\
\sigma_\phi &= -p + 2\mu \left( \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} \right) \quad ; \quad \tau_{\phi z} = \mu \left( \frac{\partial v_\phi}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \phi} \right) \\
\sigma_z &= -p + 2\mu \frac{\partial v_z}{\partial z} \quad ; \quad \tau_{rz} = \mu \left( \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right). 
\end{align*}
\] (2.5)

where,

- \( p \) is the fluid density,
- \( \nu = \frac{\mu}{\rho} \) is the kinematic viscosity,
- \( p \) is the fluid pressure,
- \( F_r, F_\phi \) and \( F_z \) are the body forces in the radial, azimuthal, and axial directions, respectively,
- \( \sigma_r, \sigma_\phi \) and \( \sigma_z \) are the stress in the radial, azimuthal and axial directions, respectively,
- and
- \( \tau_{r\phi}, \tau_{\phi z} \) and \( \tau_{rz} \) the corresponding shear stresses.

Equations for axisymmetric flow are obtained from equations (2.1-2.5) by letting

1. \( v_\phi \equiv 0 \),
2. \( \frac{\partial(\cdot)}{\partial \phi} \equiv 0 \), and
3. \( x, y \) refer to axial and radial directions, respectively.

Furthermore, for the present investigation the body forces are assumed to be zero.

This results in the following two momentum equations for the \( x \) and \( y \) directions, where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \) is the total material derivative.

\[
\begin{align*}
\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho} \frac{\partial u}{\partial y} \right) \quad (x, y) \in \Omega, \quad t > 0 \\
\frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{\rho} \frac{\partial v}{\partial x} - \frac{v}{y^2} \right) 
\end{align*}
\] (2.6)
Equation (2.4) reduces to

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\nu}{y} = 0 \quad (x, y) \in \Omega, \quad t > 0 \]  

(2.7)

which is the equation of continuity for axisymmetric flow.

Adding \( \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\nu}{y} \right) \) and \( \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\nu}{y} \right) \) to the right hand sides of the 1st and 2nd of equations (2.6) we obtain

\[ \begin{align*}
\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{1}{y} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \quad (x, y) \in \Omega, \quad t > 0. \\
\frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{2}{y} \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \right] 
\end{align*} \]  

(2.8)

Equations (2.8) are referred to as the Navier-Stokes equations and are preferred over equations (2.6) since the latter lead to symmetric matrices when integrated over a finite element. Of the total six stresses given by equations (2.5), only the following three are non-trivial. These are also referred to as constitutive relations.

\[ \begin{align*}
\sigma_x &= -p + 2\mu \frac{\partial u}{\partial x} \\
\sigma_y &= -p + 2\mu \frac{\partial v}{\partial y} \\
\tau_{xy} &= \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) 
\end{align*} \]  

(2.9)

Boundary conditions are given on the velocity components over \( \Gamma_u \), referred to as the kinematic boundary and on traction over \( \Gamma_s \), referred to as the mechanical boundary or traction boundary. These conditions are

\[ \begin{align*}
u = U, \quad v = V \quad (x, y) \in \Gamma_u, \quad t > 0 \\
\sigma_x n_1 + \tau_{xy} n_2 &= X \quad (x, y) \in \Gamma_s, \quad t > 0 \\
\tau_{xy} n_1 + \sigma_y n_2 &= Y
\end{align*} \]  

(2.10)
Substituting equations (2.9) into equations (2.10), we obtain the boundary conditions in terms of pressure and velocity components.

\[
\begin{align*}
    u &= U, \quad v = V \quad (x, y) \in \Gamma_u, \quad t > 0 \\
    &\left[ -p + 2\mu \frac{\partial u}{\partial x} \right] n_1 + \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] n_2 = X \\
    &\left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] n_1 + \left[ -p + 2\mu \frac{\partial v}{\partial y} \right] n_2 = Y
\end{align*}
\]

(2.11)

where,

- \( U, V \) are the specified velocities on \( \Gamma_u \),
- \( X, Y \) are the specified tractions on \( \Gamma_s \), and
- \( n_1, n_2 \) are the direction cosines of the unit outward normal to the boundary.

### 2.2 Non-dimensional Form of Governing Equations

The Navier-Stokes equations, the continuity equation and the boundary conditions can be non-dimensionalized by choosing suitable characteristic values for the coordinates, the velocities and the time-scale variables and defining non-dimensional parameters. Choosing the non-dimensional variables as

\[
\begin{align*}
    \tilde{x} &= \frac{x}{R_0} \quad \tilde{y} = \frac{y}{R_0} \quad \tilde{t} = \omega t \\
    \tilde{u} &= \frac{u}{u_0} \quad \tilde{v} = \frac{v}{u_0} \quad \tilde{p} = \frac{p}{\mu u_0 / R_0}
\end{align*}
\]

(2.12)

where, \( R_0, u_0 \) and \( \omega^{-1} \) are the characteristic values of the coordinates, the velocities and time-scale, respectively, and introducing equations (2.12) into equations (2.8) and (2.7), we obtain the non-dimensional form of the Navier-Stokes equations and the continuity equation for axisymmetric flow.
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\[ R_\omega \frac{\partial u}{\partial t} + R_e \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \left[ 2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{1}{y} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \]

\[ R_\omega \frac{\partial v}{\partial t} + R_e \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \left[ \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{2}{y} \left( \frac{\partial v}{\partial y} - \frac{v}{y} \right) \right] \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y} = 0 \]

(2.13)

All quantities are their respective non-dimensional values and the tilde has been omitted for convenience. The two non-dimensional parameters, \( R_\omega = \frac{\omega R_0^2}{\nu} \), the frequency Reynolds number and \( R_e = \frac{u_0 R_0}{\nu} \), the steady Reynolds number, completely characterize the flow. It should be noted that the pressure has been non-dimensionalized with respect to the characteristic shear stress rather than the dynamic pressure as it is anticipated that the flow domain would be a slow, shear dominated one.

Introducing equations (2.12) and the following additional non-dimensional variables

\[ \tilde{\sigma}_x = \frac{\sigma_x}{\mu u_0/R_0} \quad \tilde{\sigma}_y = \frac{\sigma_y}{\mu u_0/R_0} \quad \tilde{\tau}_{xy} = \frac{\tau_{xy}}{\mu u_0/R_0} \]

\[ \tilde{X} = \frac{X}{\mu u_0/R_0} \quad \tilde{Y} = \frac{Y}{\mu u_0/R_0} \quad \tilde{U} = \frac{U}{u_0} \quad \tilde{V} = \frac{V}{u_0} \]

into the constitutive equations (2.9) and boundary conditions (2.11), we obtain

\[ \sigma_x = -p + 2 \frac{\partial u}{\partial x} \]

\[ \sigma_y = -p + 2 \frac{\partial v}{\partial y} \]

\[ \tau_{xy} = \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]

(2.15)
and

\[
\begin{align*}
    u &= U, \quad v = V \quad (x, y) \in \Gamma_u, \quad t > 0 \\
    \left[-p + 2 \frac{\partial u}{\partial x}\right] n_1 + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] n_2 &= X \\
    \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] n_1 + \left[-p + 2 \frac{\partial v}{\partial y}\right] n_2 &= Y
\end{align*}
\]

(2.16)

where, again, all the quantities are their respective non-dimensional values and the tilde has been omitted for convenience.

### 2.3 Stream Function and Poisson’s Equation

Equations (2.13) must be solved for the boundary conditions (2.16) in order to find the velocities and the pressure at any point in the domain Ω. The first difficulty encountered by any solution method arises naturally from the nonlinear non-self-adjoint convective terms in the Navier-Stokes equations. A further significant difficulty arising in all numerical formulations of the governing equations in primitive variables \((u, v, p)\) representation is due to the interaction of the pressure and dilatation terms. This difficulty arises from the fact that the continuity equation merely represents a constraint on the divergence of the velocities rather than a full third equation coupling the pressure with the velocities. As a result, it is difficult to develop solution methods which are capable of determining the primitive variable \((u, v, p)\), simultaneously, with the same degree of accuracy—the pressure cannot be obtained equally accurately as the velocities—when a single restricted variational functional is constructed as the basis of formulation. Any effort to do so leads to spurious pressure modes, i.e., ‘checkerboard’ pressure distributions. Furthermore, an exact variational formulation is non-existent due to the presence of the non-self-adjoint, nonlinear, convective terms. Tuann and Olson [31,32] have shown that the only appearance of \(p\) in a restricted variational functional is coupled with the dilation term which is one order less than the velocities.
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One way to avoid this difficulty is to modify the formulation of primitive equations, so that either the system consists of the Navier-Stokes equations and an induced equation which involves pressure in a relatively dominant role, or else, the divergence constraint is satisfied explicitly and exactly, dropping the primitive variables approach. Alternatively, one fourth order equation can be obtained for the stream function. Relative merits of these formulations in avoiding the difficulties arising out of incomplete coupling between \(u, v\) and \(p\) are discussed in [31].

Another way to satisfy the continuity equation exactly is to define the velocities in terms of \(\psi\), the stream function, as

\[
\begin{align*}
    u &= \frac{1}{y} \frac{\partial \psi}{\partial y} \\
    v &= -\frac{1}{y} \frac{\partial \psi}{\partial x}.
\end{align*}
\]  

Hence,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y} = \frac{\partial}{\partial x} \left( \frac{1}{y} \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{1}{y} \frac{\partial \psi}{\partial x} \right) + \frac{1}{y} \left( -\frac{1}{y} \frac{\partial \psi}{\partial x} \right)
\]

\[
= \frac{1}{y} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{1}{y^2} \frac{\partial \psi}{\partial x} - \frac{1}{y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{1}{y^2} \frac{\partial \psi}{\partial x}
\]

\[
\equiv 0,
\]

i.e., the continuity equation is satisfied point-by-point. The vorticity \(\zeta\) is given as

\[
\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}.
\]  

Substituting equations (2.17) into (2.18), we obtain

\[
\frac{1}{y} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{y} \frac{\partial \psi}{\partial y} \right) = \zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}.
\]  

This is called Poisson's equation and is used to compute the stream function, once the primitive variables have been determined. (Right hand side is known.)
CHAPTER 3

Finite Element Formulation

In this chapter, the Navier-Stokes equations and the continuity equation (2.13) along with the boundary conditions (2.16) are discretized using the finite element method and a set of locally defined polynomial shape functions to represent the primitive variable $u$, $v$ and $p$. Galerkin's procedure is used to minimize the error in approximation by making the base vectors orthogonal to the residuals. Discretization of boundary conditions is also considered. A steady state periodic solution is obtained using the modified method of averaging and a Newton-Raphson iteration procedure is outlined for solving the resulting set of nonlinear algebraic equations. A truncated Fourier series is used to approximate the unsteady solution as a slowly varying periodic solution. The Galerkin's procedure is also used to find a representation of pulsatile flow. Brief details of computation of stream functions and plotting of streamlines and streaklines are also presented.

3.1 Discretized Form of Governing Equations

Olson and Tuann [31] have shown that the finite element interpolation for the pressure should be no higher than that for the dilation term $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ in order to avoid spurious singularities for the $(u, v, p)$ integrated formulation. Therefore, the finite element interpolation function for the pressure $p$ should be at least one degree less than that for the velocity components $(u, v)$. The highest order of derivative in the equations (2.13) is two for the velocity components $(u, v)$ and one for the pressure.
\( p \); as a result \( C^1 \) continuity is required of the interpolation functions for \((u,v)\). This difficulty can, however, be avoided by integrating the first two of equations (2.13) by parts to reduce the highest order of derivative to one for the velocity components \((u,v)\) and to zero for the pressure \(p\). Now, only \( C^0 \) continuity is required of the interpolation functions for equations (2.13). Furthermore, for the present investigation, isoparametric elements are used, i.e., the same shape functions are used for interpolating the variable \((u,v,p)\) and for transformation from \((\xi,\eta)\) natural coordinates to \((x,y)\) coordinates. Taking these conditions into consideration, curved isoparametric elements shown in figure (3.1), with quadratic interpolation for the velocities and bilinear interpolation for the pressure are used to carry out the finite element discretization of the governing equations.

The velocities \(u, v\) represented by

\[
\begin{align*}
    u &= N_i u_i(t) \\
    v &= N_i v_i(t)
\end{align*}
\]

where \(N_i\) are the quadratic isoparametric shape functions given as

\[
\begin{align*}
    N_1 &= -\frac{1}{4} (1 - \xi)(1 - \eta)(1 + \xi + \eta) \\
    N_2 &= -\frac{1}{4} (1 + \xi)(1 - \eta)(1 - \xi + \eta) \\
    N_3 &= -\frac{1}{4} (1 + \xi)(1 + \eta)(1 - \xi - \eta) \\
    N_4 &= -\frac{1}{4} (1 - \xi)(1 + \eta)(1 + \xi - \eta) \\
    N_5 &= \frac{1}{2} (1 - \xi^2)(1 - \eta) \\
    N_6 &= \frac{1}{2} (1 + \xi)(1 - \eta^2) \\
    N_7 &= \frac{1}{2} (1 - \xi^2)(1 + \eta) \\
    N_8 &= \frac{1}{2} (1 - \xi)(1 - \eta^2).
\end{align*}
\]
Figure 3.1: Isoparametric Element  (a) Element in $(\xi, \eta)$ space,  (b) Element in $(x, y)$ space.
Chapter 3: Finite Element Formulation

The pressure $p$ is represented by

$$p = M_j p_j(t) \quad j = 1, \ldots, 4$$

where $M_j$ are the bilinear isoparametric shape functions given as

$$
\begin{align*}
M_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
M_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
M_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
M_4 &= \frac{1}{4}(1 - \xi)(1 + \eta).
\end{align*}
$$

Substituting the interpolation functions (3.2) and (3.3) into the governing equations (2.13) and the boundary conditions (2.16), we obtain

$$
\begin{align*}
\mathcal{R}_1 &= R_w \frac{\partial N_j}{\partial t} u_j + R_e \left[ N_j \frac{\partial N_k}{\partial x} u_k + N_j \frac{\partial N_k}{\partial y} v_j \right] + \frac{\partial M_j}{\partial x} p_j \\
&\quad - \left[ 2 \frac{\partial^2 N_j}{\partial x^2} u_j + \frac{\partial^2 N_j}{\partial y^2} u_j + \frac{\partial^2 N_j}{\partial x \partial y} v_j + \frac{1}{y} \left( \frac{\partial N_j}{\partial y} u_j + \frac{\partial N_j}{\partial x} v_j \right) \right]
\end{align*}
$$

$$
\begin{align*}
\mathcal{R}_2 &= R_w \frac{\partial N_j}{\partial t} v_j + R_e \left[ N_j \frac{\partial N_k}{\partial x} u_k + N_j \frac{\partial N_k}{\partial y} v_j \right] + \frac{\partial M_j}{\partial y} p_j \\
&\quad - \left[ \frac{\partial^2 N_j}{\partial x^2} v_j + 2 \frac{\partial^2 N_j}{\partial y^2} v_j + \frac{\partial^2 N_j}{\partial x \partial y} u_j + \frac{2}{y} \left( \frac{\partial N_j}{\partial y} v_j - \frac{N_j}{y} v_j \right) \right]
\end{align*}
$$

$$
\mathcal{R}_3 = - \left( \frac{\partial N_j}{\partial x} u_j + \frac{\partial N_j}{\partial y} v_j + \frac{N_j}{y} v_j \right)
$$

$$
\begin{align*}
\mathcal{R}_{11} &= \left[ -M_j p_j + 2 \frac{\partial N_j}{\partial x} u_j \right] n_1 + \left[ \left( \frac{\partial N_j}{\partial y} u_j + \frac{\partial N_j}{\partial x} v_j \right) \right] n_2 - X
\end{align*}
$$

$$
\begin{align*}
\mathcal{R}_{12} &= \left[ \left( \frac{\partial N_j}{\partial y} u_j + \frac{\partial N_j}{\partial x} v_j \right) \right] n_1 + \left[ -M_j p_j + 2 \frac{\partial N_j}{\partial y} v_j \right] n_2 - Y
\end{align*}
$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are the errors in approximating the $x$-momentum equation, the $y$-momentum equation and the continuity equation, respectively; $\mathcal{R}_{11}$ and $\mathcal{R}_{12}$ are the errors in approximating the $x$-component and $y$-component of the traction boundary condition, respectively. These are called residuals and are identically zero.
only when the interpolation functions are capable of representing the exact solution. To minimize the errors, we make the residuals orthogonal to their base vectors—i.e., the shape functions \( N_i \) and \( M_i \)—over the finite element. This leads to

\[
\int \int_{\Omega} \mathcal{R}_1 N_i \, d\Omega + \int \int_{\Gamma} \mathcal{R}_1 N_i \, d\Gamma = 0
\]

\[
\int \int_{\Omega} \mathcal{R}_2 N_i \, d\Omega + \int \int_{\Gamma} \mathcal{R}_2 N_i \, d\Gamma = 0
\]

\[
\int \int_{\Omega} \mathcal{R}_3 M_i \, d\Omega = 0,
\]

where \( \Omega \) is the domain and \( \Gamma \) is the traction part of the boundary of the element under consideration. Substituting equations (3.4) into (3.5) and integrating the appropriate terms by parts, we obtain

\[
\begin{bmatrix}
m_{ij}^{uu} & 0 & 0 \\
0 & m_{ij}^{vv} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{u}_j \\
\dot{v}_j \\
\dot{p}_j
\end{bmatrix}
+
\begin{bmatrix}
 k_{ij}^{uu} & k_{ij}^{uu} & -p_{ij}^x \\
 k_{ij}^{uu} & k_{ij}^{uu} & -p_{ij}^y \\
-p_{ji}^x & -p_{ji}^y & 0
\end{bmatrix}
\begin{bmatrix}
u_j \\
v_j \\
p_j
\end{bmatrix}

+ R_s \left\{ \begin{array}{c}
\delta_{ijk} u_j u_k + \delta_{ijk} v_j u_k \\
\delta_{ijk} u_j v_k + \delta_{ijk} v_j v_k
\end{array} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathbf{f}
\]

(3.6)
where dot denotes differentiation with respect to time and the arrays are

\[
\begin{align*}
    m_{ij}^u &= \int_{\Omega_e} R_{\omega} N_i N_j \, d\Omega = m_{ij}^{uv} \\
    k_{ij}^{uv} &= \int_{\Omega_e} \left( 2 \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) \, d\Omega \\
    k_{ij}^{vv} &= \int_{\Omega_e} \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + 2 \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + \frac{2}{y^2} N_i N_j \right) \, d\Omega \\
    k_{ij}^{v} &= \int_{\Omega_e} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} \, d\Omega \\
    \delta_{ij}^x &= \int_{\Omega_e} N_i N_j \frac{\partial N_k}{\partial x} \, d\Omega \\
    \delta_{ij}^y &= \int_{\Omega_e} N_i N_j \frac{\partial N_k}{\partial y} \, d\Omega \\
    p_{li}^i &= \int_{\Omega_e} \frac{\partial N_i}{\partial x} M_j \, d\Omega \quad i = 1, \ldots, 8 \\
    p_{lj}^i &= \int_{\Omega_e} \left( \frac{\partial N_i}{\partial x} M_j + \frac{1}{y} N_i M_j \right) \, d\Omega \quad j = 1, \ldots, 4.
\end{align*}
\]

(3.7)

Vector \( \mathbf{f} \) on the right hand side of equation (3.6) is the consistent load vector and is given as

\[
\mathbf{f} = \left\{ \begin{array}{c} 
\int_{\Gamma_s} X N_i \, d\Gamma \\
\int_{\Gamma_s} Y N_i \, d\Gamma \\
0 
\end{array} \right\}
\]

### 3.2 Boundary Conditions

For all the nodes that are on the kinematic boundary \( \Gamma_u \), the nodal values are forced to the specified values of the corresponding variables. Thus, in the present investigation, the kinematic boundary conditions are satisfied exactly—node-by-node, that is. As is evident from equations (3.5), the traction boundary condition, in general, is satisfied only in the “mean”—\( \mathcal{R}_{\Gamma_1} \) and \( \mathcal{R}_{\Gamma_2} \) are not identically zero. The specified stresses \( \mathbf{X} \) and \( \mathbf{Y} \) are integrated over the traction boundary, \( \Gamma_s \), to give the consistent load vector \( \mathbf{f} \).
The computer program used for the present investigation is set up for homogeneous traction boundary conditions, \( \mathbf{X} = 0 \) and \( \mathbf{Y} = 0 \) only. Consequently, the consistent load vector \( \mathbf{f} = 0 \) and the traction boundary conditions are approximately satisfied. Equation (3.6) reduces to

\[
[M] \{ \dot{d} \} + [K] \{ d \} + R_e \begin{bmatrix}
\delta_{ijk} u_j u_k + \delta_{ijk} v_j v_k \\
0
\end{bmatrix} = 0
\]  

(3.8)

where,

\( M \) is the mass matrix given by

\[
M = \begin{bmatrix}
m_{ij}^{uu} & 0 & 0 \\
0 & m_{ij}^{uv} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\( K \) is the stiffness matrix given by

\[
K = \begin{bmatrix}
k_{ij}^{uu} & k_{ij}^{uv} & -p_{ij}^x \\
k_{ji}^{uv} & k_{ij}^{vv} & -p_{ij}^y \\
-p_{ji}^x & -p_{ji}^y & 0
\end{bmatrix}
\]

and

\[
\{ d \} = \begin{bmatrix}
i \dot{u}_j \\
i \dot{v}_j \\
i \ddot{p}_j
\end{bmatrix}
\]

is the nodal vector of unknowns.

### 3.3 Steady State Periodic Solution

Due to the presence of the quadratic nonlinear terms in the Navier-Stokes equations, the velocities as well as the pressure have a steady component and a time-dependent
component. One way to obtain the complete solution is to find a steady state finite element solution and then use simultaneous time-scale integration of the governing equations. This, however, is very costly and requires large storage and CPU time. Quite often the purpose is well served with an approximate slowly varying periodic solution. Nayfeh and Mook [17] have classified the available techniques for determining the steady state behavior of the forced oscillation of a nonlinear system into two groups: the first group includes the method of averaging and multiple scales and the second group includes such methods as Poincaré technique and the method of harmonic balance.

Pattani [19] has modified the method of averaging, as given by Nayfeh and Mook [17], to obtain the steady streaming part, as well as the slowly varying periodic part, of the solution from equations (3.8). The starting point in this method is to assume a form of the solution as

\[ d = A + B(t) \cos t + C(t) \sin t \]  \hspace{1cm} (3.9)

where \( B(t), C(t) \) are assumed to be slowly varying functions of non-dimensional time \( t \). The first term \( A \) represents the steady streaming part of the solution which naturally arises for systems of equations with quadratic nonlinearities as is encountered here. Alternatively, equation (3.9) can be interpreted as the first three terms of the Fourier expansion of \( d \). Differentiating equation (3.9), we obtain

\[ \dot{d} = -B(t) \sin t + C(t) \cos t + \dot{B}(t) \cos t + \dot{C}(t) \sin t \]  \hspace{1cm} (3.10)

To obtain an autonomous system of equations governing the amplitudes \( A, B(t) \) and \( C(t) \), equations (3.9) and (3.10) are first substituted into the equations (3.8). Then to obtain the three sets of equations for the three sets of unknowns \( A, B(t) \) and \( C(t) \), the resulting equations are

1. averaged over the period \( 2\pi \),
2. multiplied by \( \cos t \) and averaged over the period \( 2\pi \), and

3. multiplied by \( \sin t \) and averaged over the period \( 2\pi \).

By hypothesis, for small values of \( R_e \) and \( R_\omega \), \( B(t) \) and \( C(t) \) vary much more slowly with time than the functions \( \sin t \), \( \cos t \) and time \( t \) itself. Therefore, \( B(t) \), \( C(t) \), \( \dot{B}(t) \) and \( \dot{C}(t) \) are considered to be constant while performing the averaging. This process yields

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & M & 0 \\
0 & 0 & M
\end{bmatrix}
\begin{bmatrix}
\dot{A} \\
\dot{B} \\
\dot{C}
\end{bmatrix}
+
\begin{bmatrix}
K & 0 & 0 \\
0 & K & M \\
0 & -M & K
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta_{ijk}(A_i^u A_k^v + B_j^u B_k^v / 2 + C_j^u C_k^v / 2) + \delta_{ijk}(A_i^u A_k^v + B_j^u B_k^v / 2 + C_j^u C_k^v / 2) \\
\delta_{ijk}(A_i^u A_k^v + B_j^u B_k^v / 2 + C_j^u C_k^v / 2) + \delta_{ijk}(A_i^u A_k^v + B_j^u B_k^v / 2 + C_j^u C_k^v / 2) \\
0
\end{bmatrix}
\]

\[
\left\{\begin{array}{l}
\delta_{ijk}(A_i^u B_k^v + A_k^u B_i^v) + \delta_{ijk}(A_i^u B_k^v + A_k^u B_i^v) \\
\delta_{ijk}(A_i^u B_k^v + A_k^u B_i^v) + \delta_{ijk}(A_i^u B_k^v + A_k^u B_i^v) \\
0
\end{array}\right.
\]

\[
\begin{bmatrix}
\delta_{ijk}(A_i^u C_k^v + A_k^u C_i^v) + \delta_{ijk}(A_i^u C_k^v + A_k^u C_i^v) \\
\delta_{ijk}(A_i^u C_k^v + A_k^u C_i^v) + \delta_{ijk}(A_i^u C_k^v + A_k^u C_i^v) \\
0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[(3.11)\]
where
\[
A = \begin{bmatrix}
A^u_i \\
A^v_i \\
A^p_i
\end{bmatrix}, \\
B = \begin{bmatrix}
B^u_i \\
B^v_i \\
B^p_i
\end{bmatrix}, \\
C = \begin{bmatrix}
C^u_i \\
C^v_i \\
C^p_i
\end{bmatrix},
\]
are the average values over one period \((2\pi)\) of \(A(t), B(t)\) and \(C(t)\). The superscripts \(u, v\) and \(p\) indicate that the amplitudes are associated with the \(u, v\) and \(p\) degrees of freedom respectively.

The steady state solution corresponds to the singular points of the autonomous system of equations when \(\dot{B} = \dot{C} = 0\). This results in a set of nonlinear algebraic equations for \(A, B\) and \(C\) which are solved using the modified Newton-Raphson iteration procedure as outlined in Appendix A. The equations obtained for the increments to the solution vector are of the form

\[
[T]\{\Delta x\} = \{-f\}
\]

where,
- \([T]\) is the tangent stiffness matrix,
- \(\{\Delta x\}\) is the incremental solution vector and
- \(\{-f\}\) is the unbalanced load vector.

Details of equations (3.12) are given in Appendix B. The above matrices are calculated for each finite element and are assembled into the corresponding global matrices. This is carried out by programs AXILIN and AXINOLIN on the SUN 4/260 computer system and the resulting set of sparse, linear, algebraic equations are solved using the University of Waterloo solution package SPARSPAK.

### 3.4 Characterization of Arterial Fluid Flow

In the application of the principles and concepts of fluid mechanics to the stenosis problem, several general characteristics play a major role. These include the rheological properties of the fluid, the nature of flow (e.g., steady or unsteady, laminar...
or turbulent), the geometry of stenosis and the distribution of flow across the cross-section after it has sufficiently "stabilized" over a long development length. Blood consists of formed elements, red blood cells, white blood cells, platelets etc. suspended in plasma. Although plasma is a Newtonian fluid, Young [35] has reported that blood exhibits non-Newtonian behavior at low shear rates. However, at higher shear rates commonly found in larger arteries, blood is generally assumed to behave as a Newtonian fluid. The scope of the current investigation is to analyze the effects of stenosis on the fluid flow, without particular reference to the actual artery stenosis caused due to the impingement of extravascular masses, or due to the intravascular atherosclerotic plaques; therefore no effort is made to further justify the Newtonian fluid assumption. Rather, an attempt is made to represent the actual velocity distribution across the artery cross-section.

Arterial blood flow is highly pulsatile with the instantaneous flow rate varying over a wide range during a flow cycle. Young [35] has reported representative waveforms for the left circumflex coronary artery and the femoral artery which are reproduced here in figure (3.2). These are often approximated by the following equation.

\[
U_0(y, t) = a(y) + (b(y) \cos t) \cdot H(t)
\]

where,

\[
H(t) = \begin{cases} 
1 & 0 \leq t \leq \pi, \\
0 & \pi \leq t \leq \alpha \pi,
\end{cases}
\]

and \(H(t + \alpha) = H(t)\).

To get rid of the discontinuity arising from \(H(t)\) in figure (3.3), we approximate.
Figure 3.2: Typical arterial flow waveforms (Young, 1979)
$H(t)$

![Figure 3.3: Function $H(t)$](image)

$U_0(y, t)$ as

\[
U_0(y, t) \approx A_0(y) + B_0(y) \cos t + C_0(y) \sin t
\]

\[
= \begin{pmatrix}
1 & \cos t & \sin t
\end{pmatrix}
\begin{pmatrix}
A_0(y) \\
B_0(y) \\
C_0(y)
\end{pmatrix}
= \begin{pmatrix}
0 \\
\cos t \\
\sin t
\end{pmatrix}
\begin{pmatrix}
A_0(y) \\
B_0(y) \\
C_0(y)
\end{pmatrix}.
\]

$A_0(y), B_0(y)$ and $C_0(y)$ are determined by minimizing the error in approximation by Galerkin's procedure. The resulting residual

\[
\mathcal{R}_u = A_0(y) + B_0(y) \cos t + C_0(y) \sin t - [a(y) + (b(y) \cos t) \cdot H(t)]
\]

is made orthogonal to the base vectors $1, \cos t$ and $\sin t$. This gives

\[
\int_0^{\alpha \pi} \begin{pmatrix}
1 \\
\cos t \\
\sin t
\end{pmatrix}
\begin{pmatrix}
1 \\
\cos t \\
\sin t
\end{pmatrix}^T
\begin{pmatrix}
A_0 \\
B_0 \\
C_0
\end{pmatrix}
- \int_0^{\alpha \pi} \begin{pmatrix}
1 \\
\cos t \\
\sin t
\end{pmatrix}
\begin{pmatrix}
0 \\
\cos t \\
\sin t
\end{pmatrix}
(a + (b \cos t) \cdot H(t))
\]

\[
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]
or,
\[
\int_0^{\alpha \pi} \begin{bmatrix} 1 & \cos t & \sin t \\ \cos t & \cos^2 t & \sin t \cos t \\ \sin t & \sin t \cos t & \sin^2 t \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} dt = \begin{bmatrix} 0 \\ 1 \\ \cos t \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} \\
\begin{bmatrix} \sin t \\ \sin t \cos t \\ \sin^2 t \end{bmatrix} \begin{bmatrix} 1 \\ \cos t \\ \sin t \cos t \end{bmatrix}
\]

or,
\[
\begin{bmatrix} \alpha \pi & \sin \alpha \pi & -\cos \alpha \pi \\ \sin \alpha \pi & \alpha \pi & \sin^2 \alpha \pi \\ -\cos \alpha \pi & \frac{\sin^2 \alpha \pi}{2} & \alpha \pi \end{bmatrix}
\begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = a \begin{bmatrix} \alpha \pi \\ \sin \alpha \pi \\ -\cos \alpha \pi \end{bmatrix} + b \begin{bmatrix} 2 \\ \frac{\pi}{2} \end{bmatrix}
\]

Equation (3.13) can be solved for different values of \( \alpha \). In particular,

for \( \alpha = \frac{3\pi}{2} \),
\[
\begin{bmatrix} A_0(y) \\ B_0(y) \\ C_0(y) \end{bmatrix} = a(y) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b(y) \begin{bmatrix} 0.433897 \\ 0.0446932 \\ 0.675718 \end{bmatrix}
\]

for \( \alpha = 2\pi \),
\[
\begin{bmatrix} A_0(y) \\ B_0(y) \\ C_0(y) \end{bmatrix} = a(y) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b(y) \begin{bmatrix} \frac{1}{\pi} \\ 0 \\ \frac{1}{2} \end{bmatrix}
\]

and for \( \alpha = 3\pi \),
\[
\begin{bmatrix} A_0(y) \\ B_0(y) \\ C_0(y) \end{bmatrix} = a(y) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b(y) \begin{bmatrix} 0.18091221 \\ 0 \\ 0.29494257 \end{bmatrix}
\]

Both \( a(y) \) and \( b(y) \) represent the fully developed solutions for the steady state laminar flow and cosinusoidal laminar flow, respectively. Schlichting [26] has given
the exact solution for both these parameters. \( a(y) \) is a quadratic function of \( \frac{y}{R} \) and 
\( b(y) \) is a function of \( \frac{J_0(y\sqrt{-iR\omega})}{J_0(R\sqrt{-iR\omega})} \) where \( R \) is the radius and \( J_0 \) is Bessel's function of the first kind of order zero. For ease of programming, though, both \( a(y) \) and \( b(y) \) are assumed to vary parabolically across the cross-section. To compensate for this approximation, enough entrance length is provided to let the flow develop naturally.

### 3.5 Streamlines and Streaklines

A line in the fluid whose tangent is everywhere parallel to the fluid velocity \( \mathbf{u}(x, y, t) \) instantaneously, is a streamline. The family of streamlines at any time \( t \) are solutions of the equation

\[
\frac{1}{y} \frac{dx}{u(x,t)} = \frac{1}{y} \frac{dy}{v(x,t)}
\]

where \( u \) and \( v \) are the component of \( \mathbf{u}(x, y, t) \) velocity parallel to the axes \( x \) and \( y \), respectively. The path of a material element of fluid does not, in general, coincide with a streamline. The pathline and streamline are the same only when the fluid motion is steady, i.e., only the \( \mathbf{A} \) component is nonzero. A streakline is that line on which lie those fluid elements that, at some earlier instant, passed through a certain point of space. Thus, when a marking material is dropped in the flow, the visible path produced thus is a streakline.

The family of streamlines and streaklines are found by solving the Poisson's equation (2.19). After the governing equations for the primitive variables \((u, v, p)\) have been solved, the vorticity

\[
\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}
\]

can be computed at any point. In order to be consistent, the finite element interpolation functions chosen for the stream function \( \psi \) to discretize the Poisson's equation (2.19) are the same as those used for the velocities. This also simplifies the computation and setting up of the matrix equation. The stream function \( \psi \) is
Chapter 3: Finite Element Formulation

represented by

\[ \psi = N_i \psi_i \quad i = 1, 2, \ldots, 8 \quad (3.14) \]

where \( N_i \) are the shape functions given by equations (3.2) and \( \psi_i \) are the nodal values. Substituting equation (3.14) and

\[ \zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial N_j}{\partial y} u_j - \frac{\partial N_j}{\partial x} v_j \]

into equation (2.19), we obtain

\[ \mathcal{R}_\psi = \frac{1}{y} \left[ \frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_j}{\partial y^2} - \frac{1}{y} \frac{\partial N_j}{\partial y} \right] \psi_j - \left[ \frac{\partial N_j}{\partial y} u_j - \frac{\partial N_j}{\partial x} v_j \right], \]

where \( \mathcal{R}_\psi \) is the residual due to error in approximation. To minimize the error in approximation, we make \( \mathcal{R}_\psi \) orthogonal to the base vectors, i.e., the shape functions \( N_i \):

\[ \left\{ \int_{\Omega^e} \mathcal{R}_\psi N_i d\Omega = 0 \right\} \]

or,

\[ \left\{ \int_{\Omega^e} \int_{\Omega^e} \frac{1}{y} \left[ \frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_j}{\partial y^2} - \frac{1}{y} \frac{\partial N_j}{\partial y} \right] N_i \psi_j d\Omega - \int_{\Omega^e} \int_{\Omega^e} \left[ \frac{\partial N_j}{\partial y} u_j - \frac{\partial N_j}{\partial x} v_j \right] N_i d\Omega = 0 \right\}. \]

Integrating the appropriate terms by parts and simplifying, we obtain

\[ [Z_{ij}] \{ \psi_j \} = \{ P_i \} \quad (3.15) \]

where the arrays are

\[ Z_{ij} = \int_{\Omega^e} \left[ \frac{1}{y} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{1}{y} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] d\Omega \]

\[ P_i = -\int_{\Omega^e} \left[ \frac{\partial N_i}{\partial y} u_j - \frac{\partial N_i}{\partial x} v_j \right] N_i d\Omega + \int_{\Gamma^e y} \frac{1}{y} N_i g d\Gamma \]

\( \Omega^e \) is the domain of a single element, \( \Gamma^e \) is the traction part of the boundary of the element and

\[ g = \frac{\partial \psi}{\partial n} = \left[ \frac{\partial N_i}{\partial x} \psi_j \right] n_1 + \left[ \frac{\partial N_i}{\partial y} \psi_j \right] n_2 \]
is the tangential derivative of $\psi$ specified on the traction part of the boundary $\Gamma^t$. The above process is repeated for each element and the matrices are assembled into the corresponding global matrices. The resulting set of sparse, linear, algebraic equations are solved using the University of Waterloo solution package SPARSPAK. Substituting back the solution in equation (3.14), we obtain the finite element representation of the stream function $\psi$ in the fluid domain.
CHAPTER 4

Numerical Results

In this chapter numerical results are presented for different flow configurations. The results are obtained for various values of $R_e$ and $R_\omega$ in the viscous flow regime and, wherever possible, these results are compared with analytical, experimental or numerical results published in the literature. From the basic flow results, stream function values are computed and streamline and streakline plots are obtained. Streamlines are the stream function contours of the steady part of the velocity field and streaklines are the stream function contours of the total velocity field at one instant in time.

The steady state periodic solution of the problem corresponds to the singular points of the autonomous system of equations (3.11), that is, when $\dot{B} = \dot{C} = 0$. This results in a set of nonlinear algebraic equations in $A, B, C$. These equations are solved using the modified Newton-Raphson procedure as outlined in Appendix B.

All the computer programs are implemented on the Amdahl 5840 and on a departmental Sun 4/260 computer at the University of British Columbia. Double precision (Real*8) arithmetic is used throughout to reduce the effect of round-off errors. The resulting set of sparse, linear, algebraic equations are factorized and solved using the University of Waterloo solution package called SPARSPAK.

A 100 $\times$ 100 plot grid is used, in conjunction with the UBC surface visualization program *SURFACE, to obtain contour plots of streamlines and streaklines. Stream function values at the plot grid nodes are obtained by the procedure outlined in Appendix C.
4.1 Steady Flow Through a Pipe

4.1.1 Theoretical Results

The steady flow through a straight pipe of circular cross-section is the simplest case with rotational symmetry for which an exact solution can be obtained. Let the $x$-axis be selected along the axis of pipe, figure (4.1), and let $y$ denote the radial coordinate measured from the axis outwards. For fully developed flow the velocity component in the radial direction, denoted by $v$, is zero; the velocity component parallel to the axis, denoted by $u$, depends on $y$ alone and the pressure is constant at every cross-section. The governing equations (2.13) reduce to

\[
\frac{\partial^2 u}{\partial y^2} + \frac{1}{y} \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} = 0
\]

\[
-\frac{\partial p}{\partial y} = 0
\]

\[
\frac{\partial u}{\partial x} = 0.
\]

Figure 4.1: Parallel Flow with Parabolic velocity distribution

The second and third equations of (4.1) give $p = p(x)$ and $u = u(y)$. Rearranging the first equation, we obtain

\[
\frac{\partial^2 u}{\partial y^2} + \frac{1}{y} \frac{\partial u}{\partial y} = \frac{\partial p}{\partial x}.
\]
Chapter 4: Numerical Results

The left hand side of equation (4.2) is a function of \( y \) alone and the right hand side is a function of \( x \) alone; both these conditions can be true if and only if each side is equal to a constant. Hence,

\[
\frac{\partial^2 u}{\partial y^2} + \frac{1}{y} \frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} = \text{constant}
\]

or,

\[
\frac{1}{y} \frac{\partial}{\partial y} \left[ y \frac{\partial u}{\partial y} \right] = \frac{\partial p}{\partial y} = \text{constant}
\]

or,

\[
\frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} \frac{y^2}{2} + C_1
\]

or,

\[
u = \left( \frac{\partial p}{\partial x} \right) \frac{y^2}{4} + C_1 \ln y + C_2.
\]

For \( u \) to be bounded along the axis of the pipe, \( C_1 = 0 \) and for \( u = 0 \) at \( y = R \), \( C_2 = -\frac{R^2}{4} \). This gives

\[
u(y) = -\frac{1}{4} \left( \frac{\partial p}{\partial x} \right) \left[ R^2 - y^2 \right],
\]

where \( \left( \frac{\partial p}{\partial x} \right) \) is the pressure gradient which can be deduced from the given boundary conditions.

4.1.2 Finite Element Results

We note that the exact solution obtained above is quadratic in \( y \) for the velocity component \( u \) and linear in \( x \) for the pressure \( p \). This means that the quadratic isoparametric interpolation chosen for the current investigation should be capable of reproducing the results 'exactly'—up to the accuracy of the arithmetic used, that is. For this purpose, a single element, as shown in figure (4.2), is used to represent the fully developed flow. With node 1 as the origin, we choose the radius \( R = 1 \) and length \( l = 4 \). The following kinematic boundary conditions were applied:
• along \( x = 0 \): \( u(0,y) = u_0(y) = (1 - y^2) \) and \( v \equiv 0 \);

• along \( x = 4 \): \( v \equiv 0 \) and \( p \equiv 0 \);

• along \( y = 0 \): \( v \equiv 0 \) and \( u \equiv 0 \);

• along \( y = 1 \): \( u \equiv 0 \) and \( v \equiv 0 \).

Figure 4.2: One Element Representation of the Fully Developed Flow

The traction boundary condition along the centerline, \( y = 0 \), namely \( \frac{\partial u}{\partial y} = 0 \), was not enforced—it was left for the program to seek a satisfactory approximation to this condition. Also by letting \( u(0,y) = (1 - y^2) \), implicitly we let \( \frac{\partial p}{\partial x} = -4 \). This with the condition that \( p \equiv 0 \) at \( x = 4 \) gives \( p = (16 - 4x) \). The finite element results are given in table (4.1). They match the exact solution up to the precision of the Real*8 arithmetic used. No iterations were required to converge to the true solution.

4.2 Oscillating Flow Through a Pipe

4.2.1 Theoretical Results

The case of the flow of a fluid through a pipe under the influence of a periodic pressure difference affords another example where an exact solution can easily be computed. This type of flow occurs, e.g., under the influence of a reciprocating piston, and its
Table 4.1: Comparison of Finite Element Solution with Exact Solution

<table>
<thead>
<tr>
<th>Node</th>
<th>Coordinates</th>
<th>Variable</th>
<th>FE solution</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.0,0.0)</td>
<td>p</td>
<td>16.0000000001713</td>
<td>16.0</td>
</tr>
<tr>
<td>2</td>
<td>(4.0,0.0)</td>
<td>u</td>
<td>1.0000000000046</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>(0.0,1.0)</td>
<td>p</td>
<td>15.9999999999584</td>
<td>16.0</td>
</tr>
<tr>
<td>5</td>
<td>(2.0,0.0)</td>
<td>u</td>
<td>1.0000000000055</td>
<td>1.0</td>
</tr>
<tr>
<td>6</td>
<td>(4.0,0.5)</td>
<td>u</td>
<td>0.7500000000023</td>
<td>0.75</td>
</tr>
</tbody>
</table>

theory is presented in Schlichting [26]. It is assumed that the pipe is very long and circular in cross-section. Under the assumption that the pipe is very long, the flow is taken to be independent of x. This means that the axial velocity component \( u \) is also independent of x, i.e., \( u = u(y,t) \). This simplifies the continuity equation to

\[
\frac{\partial v}{\partial y} + \frac{v}{y} = 0
\]

or,

\[
\frac{1}{y} \frac{\partial}{\partial y} (yv) = 0
\]

or,

\[
yv \equiv \text{constant} = g(t)
\]

for some arbitrary function \( g(t) \). That is impossible unless \( v \equiv 0 \).

The \( y \)-momentum equation reduces to \( \frac{\partial p}{\partial y} = 0 \), i.e., \( p = p(x,t) \); the \( x \)-momentum equation reduces to

\[
R \omega \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{y} \frac{\partial u}{\partial y}
\]

which is 'exact' as it implies no additional simplifications. The boundary conditions are \( u(y,t) = 0 \) for \( y = R \) at the wall. Assume that the pressure gradient caused by the motion of the piston is harmonic and is given by

\[
-\frac{\partial p}{\partial x} = p_0 \cos t,
\]
where \( p_0 \) denotes a constant. For this problem it is convenient to use complex notation and to put

\[
-\frac{\partial p}{\partial x} = p_0 e^{it},
\]

attributing physical significance only to the real part (or, only to the imaginary part, if the pressure difference is sinusoidal). Note that the minus sign indicates that pressure drops in the positive \( x \) direction.

Assuming that the velocity function has the form \( u(y, t) = f(y)e^{it} \), we obtain,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= if(y)e^{it} \\
\frac{\partial u}{\partial y} &= f'(y)e^{it} \\
\frac{\partial^2 u}{\partial y^2} &= f''(y)e^{it}. \\
\end{align*}
\]

Substituting equations (4.4) into the equation (4.3), and canceling the common term \( e^{it} \) from both left hand side and right hand side, we obtain the following differential equation for the function \( f(y) \):

\[
f''(y) + \frac{1}{y}f'(y) + (-iR\omega)f(y) = -p_0.
\]

To transform the differential equation above, we use

\[
\eta = y\sqrt{(-iR\omega)}.
\]

\[
f'(y) = \frac{\partial f(\eta)}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \sqrt{(-iR\omega)} \frac{\partial f(\eta)}{\partial \eta},
\]

\[
f''(y) = (-iR\omega) \frac{\partial^2 f(\eta)}{\partial \eta^2}.
\]

This gives,

\[
\eta^2 \frac{\partial^2 f(\eta)}{\partial \eta^2} + \eta \frac{\partial f(\eta)}{\partial \eta} + (\eta^2 - 0^2) f(\eta) = \eta^2 \left(-i\frac{p_0}{R\omega}\right)
\]

which has the particular solution \( f_p(\eta) = \left(-i\frac{p_0}{R\omega}\right) \) and the homogeneous solution \( f_h(\eta) = AJ_0(\eta) + BY_0(\eta) \), where \( A \) and \( B \) are, in general, functions of time only, and
\( J_0 \) and \( Y_0 \) are the Bessel’s function of the first and second kind, respectively, of order zero. Adding the two solutions and switching back to the \( y \)-variable gives

\[
f(y) = -i \frac{p_0}{R_\omega} + AJ_0 \left( y\sqrt{-iR_\omega} \right) + BY_0 \left( y\sqrt{-iR_\omega} \right).
\]

For the solution to be bounded, \( B \equiv 0 \) since \( Y_0 \left( y\sqrt{-iR_\omega} \right) \) is unbounded at \( y = 0 \). \( A \) is determined using \( u = 0 \) at \( y = R \) at the wall. Hence,

\[
u(y, t) = \Re \left[ f(y) \cdot e^{it} \right] = \Re \left[ -i \frac{p_0}{R_\omega} e^{it} \left\{ 1 - \frac{J_0 \left( y\sqrt{-iR_\omega} \right)}{J_0 \left( R\sqrt{-iR_\omega} \right)} \right\} \right].
\]

The above can be further simplified by splitting the bracketed quantity into sine and cosine terms. Hence for cosinusoidal pressure difference \( -\frac{\partial p}{\partial x} = p_0 \cos t \),

\[
u(y, t) = \frac{p_0}{R_\omega} (\beta \cos t + \alpha \sin t)
\]

and for sinusoidal pressure difference \( -\frac{\partial p}{\partial x} = p_0 \sin t \),

\[
u(y, t) = \frac{p_0}{R_\omega} (-\alpha \cos t + \beta \sin t)
\]

where

\[
\alpha + i\beta = \left\{ 1 - \frac{J_0 \left( y\sqrt{-iR_\omega} \right)}{J_0 \left( R\sqrt{-iR_\omega} \right)} \right\}.
\]

A general discussion of the nature of the solution for arbitrary values of \( R_\omega \) is impossible owing to the presence of Bessel’s function with a complex argument, but for the two limiting cases of very low \( R_\omega \) and very large \( R_\omega \), the solution above can be simplified. For very small values of \( R_\omega \), the above can be simplified by retaining only the first two terms of the series expansion

\[
J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^m (m!)^2}.
\]

Similarly using the asymptotic expansion of the Bessel function

\[
J_0(z) \approx \sqrt{\frac{2}{\pi z}} e^{i z - 1/2}
\]
Figure 4.3: Flow in a pipe under oscillating pressure
we can obtain an expression for very large values of $R_\omega$.

In general, the velocity distribution across a cross-section of the pipe under the influence of cosinusoidal pressure difference is as shown in figure (4.3).

### 4.2.2 Finite Element Results

We choose a pipe of radius $R = 1$ and length $l = 4$ represented by a $5 \times 8$ grid as shown in figure (4.4). We compute the fully developed velocity distribution using the series solution presented in the last section and specify these values at the entrance $x = 0$. The program seeks its own approximation to the flow downstream and stabilizes before the exit $x = 4$. The velocity distributions thus obtained at the exit are compared with the exact solution.

\[
\begin{align*}
\bar{u} &= (\alpha \cos t + \beta \sin t) \\
v &= 0
\end{align*}
\]

\[p = 0\]

\[x = 0, \quad v = 0\]

\[y = 0, \quad u = 0\]

Figure 4.4: Finite element grid for oscillating flow through a pipe

The exact solution presented in the last section is independent of the Reynolds number $R_\omega$ and hence we expect the numerical solution to converge without any iterations; this indeed is the case. Furthermore, we choose $p_0 = 1.0$. Thus

\[
u(y, t) = \frac{1}{R_\omega} (\alpha \cos t + \beta \sin t)
\]

where $\alpha$ and $\beta$ are functions of $y$ and are computed using the series solution. The program is run for three different frequency Reynolds numbers $R_\omega = 1, 10$ and 20.
Chapter 4: Numerical Results

The results are compared in tables (4.2), (4.3) and (4.4).

Table 4.2: Comparison of exact and finite element solution for oscillating flow for $R_e = 1.0$

$R = 1.0$, $p_0 = 1.0$, $R_\omega = 1.0$

<table>
<thead>
<tr>
<th>$y$</th>
<th>Exact Solution</th>
<th>Finite Element Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>0.0</td>
<td>0.2419938822</td>
<td>0.0454856408</td>
</tr>
<tr>
<td>0.125</td>
<td>0.2382643889</td>
<td>0.0445439938</td>
</tr>
<tr>
<td>0.25</td>
<td>0.2270649264</td>
<td>0.0417627709</td>
</tr>
<tr>
<td>0.375</td>
<td>0.2083631157</td>
<td>0.0372732688</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1821068921</td>
<td>0.0312947819</td>
</tr>
<tr>
<td>0.625</td>
<td>0.1482273619</td>
<td>0.0241352748</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1066428141</td>
<td>0.0161922692</td>
</tr>
<tr>
<td>0.875</td>
<td>0.0572639012</td>
<td>0.0079538898</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 4.3: Comparison of exact and finite element solution for oscillating flow for $R_e = 10.0$

$R = 1.0$, $p_0 = 1.0$, $R_\omega = 10.0$

<table>
<thead>
<tr>
<th>$y$</th>
<th>Exact Solution</th>
<th>Finite Element Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0456409109</td>
<td>0.1109031291</td>
</tr>
<tr>
<td>0.125</td>
<td>0.0460493859</td>
<td>0.1091161974</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0470648468</td>
<td>0.1037100366</td>
</tr>
<tr>
<td>0.375</td>
<td>0.0480518359</td>
<td>0.0945759605</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0479364625</td>
<td>0.0816249082</td>
</tr>
<tr>
<td>0.625</td>
<td>0.0451974541</td>
<td>0.0649275362</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0378803750</td>
<td>0.0449123607</td>
</tr>
<tr>
<td>0.875</td>
<td>0.0236620859</td>
<td>0.0226194555</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Table 4.4: Comparison of exact and finite element solution for oscillating flow for $R_e = 20.0$

$$R = 1.0, p_0 = 1.0, R_\omega = 20.0$$

<table>
<thead>
<tr>
<th>$y$</th>
<th>Exact Solution</th>
<th>Finite Element Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0042213053</td>
<td>0.0041474265</td>
</tr>
<tr>
<td>0.125</td>
<td>0.0050090036</td>
<td>0.0049651334</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0072867953</td>
<td>0.0072308490</td>
</tr>
<tr>
<td>0.375</td>
<td>0.0107517756</td>
<td>0.0107261714</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0147491753</td>
<td>0.0147068992</td>
</tr>
<tr>
<td>0.625</td>
<td>0.0180745128</td>
<td>0.0180846562</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0187627664</td>
<td>0.0187522740</td>
</tr>
<tr>
<td>0.875</td>
<td>0.0139564340</td>
<td>0.0139112653</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

The solution becomes increasingly wavy in the $y$–direction as $R_\omega$ increases, and for this reason grid points lie closer to each other in this direction than along the pipe axis. The aspect ratio is 6.4 which is much less than 20 and hence no numerical difficulties arising out of the large aspect ratio were encountered. At low frequency Reynolds number, the cosinusoidal component is dominant—one full order of magnitude higher than the sinusoidal part. However, as shown in figure (4.5), with increasing frequency Reynolds number, the sinusoidal component increases whereas the cosinusoidal component decreases. At $R_\omega = 10$, both components are of the same order of magnitude, and at $R_\omega = 20$ the sinusoidal component becomes the dominant one.

From the velocity field results, stream function values at grid points were calculated. These values were then extrapolated to a 100 x 100 grid using the defined interpolation shape functions and streakline plots are shown in figures (4.6), (4.7),
Chapter 4: Numerical Results

Figure 4.5: Variation of cosinusoidal and sinusoidal components for unsteady parallel flow for $R_\omega = 1, 10$ and 20
(4.8) for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. Twenty contours spaced equally between the instantaneous minimum and the instantaneous maximum values of the stream function are plotted. Note that all the contours are almost completely horizontal indicating that the finite element representation of fully developed flow under cosinusoidal pressure is very good. The instantaneous rate of flow, denoted by $Q(t)$, is found by integrating the $u-$velocity across the cross-section of the pipe as shown below and is

$$Q(t) = \int_0^{R=1} u(y,t) \cdot (2\pi y) dy$$

$$= 2\pi \psi(1,t)$$

since $u(y,t) = \frac{1}{y} \frac{\partial \psi(y,t)}{\partial y}$.

Table 4.5: Instantaneous rate of flow through a pipe under cosinusoidal pressure difference

<table>
<thead>
<tr>
<th>Time</th>
<th>$R = 1.0$, $p_0 = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>Frequency</td>
</tr>
<tr>
<td></td>
<td>$R_\omega = 1.0$</td>
</tr>
<tr>
<td>0</td>
<td>0.38126</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>0.06357</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-0.38126</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>-0.06357</td>
</tr>
</tbody>
</table>

(Negative instantaneous rate of flow means that the flow is to the left)
Figure 4.6: Streaklines for $R_\omega = 1$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. 
Figure 4.7: Streaklines for $R_{\omega} = 10$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
$t = 0$

$\frac{\pi}{2}$

$\pi$

$\frac{3\pi}{2}$

Figure 4.8: Streaklines for $R_\omega = 20$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
4.3 Developing Flow in the Inlet Length of a Pipe

4.3.1 Theoretical Results

As a third example of axisymmetric flow, we consider the case of flow in the inlet length of a straight circular pipe. At a large distance from the inlet the velocity distribution across the cross-section becomes parabolic. We assume that the velocity in the inlet section is uniformly distributed over the circular cross-section of radius $R$ and that its magnitude is $U_0$. Owing to viscous friction, a boundary layer will be formed on the wall and the depth of the boundary layer will increase in the downstream direction. At the beginning, i.e., at small distances from the inlet section, the boundary layer will grow in the same way as it would along a flat plate at zero incidence [26]. The resulting velocity profile will consist of an axisymmetric boundary layer near the wall joined in the center by an area of constant velocity. Since the volume of the flow must be the same for every cross-section, the decrease in the rate of flow near the wall which is due to friction must be compensated by a corresponding increase near the axis. Thus the boundary layer is formed under the influence of an accelerated external flow, as distinct from the case of a flat plate. At larger distances from the inlet section the central region of constant velocity asymptotically shrinks to zero, and finally the velocity profile is transformed into the parabolic distribution of Poiseuille flow.

This process can be analyzed mathematically in one of two ways. First, the integration can be performed in the downstream direction so that the boundary layer growth is calculated for an accelerated external stream. Secondly, it is possible to analyze the progressive deviation of the profile from its asymptotic parabolic distribution, i.e., integration can proceed in the upstream direction. Having obtained both solutions, say in the form of series expansions, we can retain a sufficient number of terms in either of them, joining the two solutions at a section where both are still
applicable. In this way the solution for the whole inlet is obtained.

As shown in figure (4.9), this problem has a singularity at $x = 0, y = R$, namely

$$u(0^-, R) = U_0$$
$$u(0^+, R) = 0.$$

We start by satisfying the continuity equation:

$$\int_0^R (2\pi y) \cdot u(x, y) dy = U_0 \cdot (\pi R^2)$$

or,

$$\int_0^R u y dy = \frac{1}{2} U_0 R^2.$$

Next we introduce a non-dimensional parameter

$$\epsilon = \sqrt{\frac{1}{Re} \cdot \frac{x}{R}}$$

and expand $U(x) = u(x, 0)$ in a series form and balance the momentum equation. Further details are given in the book by Schlichting [26]. He has reported the following values for the inlet length $l_E$ and the additional pressure drop $\Delta p_{add}$. 
\[ l_E = 0.04(2R) \cdot R_e \]

\[ \Delta p_{add} = -0.3005R_e \]

Using equations (3.14) and (3.15), we can then compute stream function values.

At the entrance,

\[ \frac{1}{y} \frac{\partial \psi}{\partial x} = v \equiv 0 \]

and

\[ \frac{1}{y} \frac{\partial \psi}{\partial y} = u \equiv 1. \]

Hence,

\[ \psi = \frac{y^2}{2} + C. \]

Choosing \( \psi = 0 \) along the centerline of the pipe gives \( C = 0 \) and \( \psi_{\text{entrance}} = \frac{y^2}{2} \).

We also know that at the exit, the velocity is fully developed. Hence \( v = 0 \) and \( u = A(1 - y^2) \) where \( A \) is a constant. Again integrating the velocity fields and choosing \( \psi = 0 \) along the centerline, \( \psi_{\text{exit}} = \frac{A}{4} y^2(2 - y^2) \). Since the volume of flow must remain constant at any section, \( \psi_{\text{entrance}}(y = 1) = \psi_{\text{exit}}(y = 1) \). This gives \( A = 2 \), i.e., the centerline velocity is doubled.

### 4.3.2 Finite Element Results

As noted in subsection 4.3.1, this problem has a singularity at \( x = 0, y = R \). For this reason, the finite element grid used to solve this problem needs to be considerably finer near this point as compared to the grid downstream. A grid of \( 20 \times 10 \) elements shown in figure (4.10) is used with the length and width of successive elements chosen according to an arithmetic progression. Another patch of \( 4 \times 10 \) elements precedes the pipe entrance representing the undisturbed uniformly distributed flow of magnitude \( U_0 = 1.0 \). The finite element program is run for \( R_e = 0, 1, 10, 50, 100, 150, 200, 250, 300, 350 \) and 400 with a tolerance level of \( 1.0 \times 10^{-5} \). Results converge in 4
to 8 iterations with more iterations required as the Reynolds number is increased progressively. The distance along the centerline of the pipe between the entrance and a point $Z_0$ is called the development length where $Z_0$ is chosen such that

$$\left| \frac{u(\infty,0) - u(Z_0,0)}{u(\infty,0)} \right| \leq 0.001.$$

Twenty streamline contours equally spaced between the maximum and the minimum values of the stream function are plotted in figures (4.11), (4.12), (4.13) and (4.14). Again the $y-$dimension has been enlarged four times to clearly reveal the contours. Twenty streamline contours equally spaced between the maximum and the minimum values of the stream function are plotted in figures (4.11), (4.12), (4.13) and (4.14). Again the $y-$dimension has been enlarged four times to clearly reveal the contours. The relationship between $Z_0$ and $R_e$ is shown in figure (4.15). Change in $u$ velocity along the centerline as the flow develops is presented in figure (4.16) for different values of $R_e$.

### 4.4 Flow Through an Axisymmetric Orifice

A steady flow problem of interest to both engineers and mathematicians is that of a viscous, incompressible fluid through an orifice. Greenspan [10] used a finite difference method with a simple smoothing which yielded diagonally dominant systems of linear algebraic equations. Olson [18] used a pseudo-variational finite element theory with a high precision 18 degrees-of-freedom triangular element to get good results for Reynolds numbers up to $R_e = 20$. Figure (4.17) shows a step change in diameter of the pipe over the range $a \leq x \leq (a + 2l)$. For the purposes of comparison, we chose $R = 1.0$, $l = 0.5$, $\delta = 0.5$. $a = 5.25$ and $b = 15.55$ such that at the exit, the flow was fully developed.

The finite element grid used for a section bounded by the wall and the centerline is shown in figure (4.18). The rigid boundary conditions were imposed as follows:
Finite Element Grid for Developing Flow

\[ R = 1.0 \quad \text{length of pipe}=15.0 \]

Number of nodes = 789  
Number of elements = 280  
Number of equations for steady flow = 1665

\[ x-\text{axis scale:} \quad 1 \text{ cm} = 1.104 \text{ units} \]
\[ y-\text{axis scale:} \quad 1 \text{ cm} = 0.276 \text{ units} \]

Figure 4.10: Finite element grid for developing flow
Figure 4.11: Streamline contours for developing flow for $R_e = 0, 1$ and $10$
Figure 4.12: Streamline contours for developing flow for $Re = 50, 100$ and $150$.
Figure 4.13: Streamline contours for developing flow for $Re = 200, 250$ and $300$
Figure 4.14: Streamline contours for developing flow for $R_e = 350$ and 400
Figure 4.15: $\frac{Z_0}{R}$ versus $R_e$
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Figure 4.16: Velocity distribution along the centerline for different $R_e$
Figure 4.17: Flow through an axisymmetric orifice

- \( u = 4(1 - y^2) \), \( v \equiv 0 \) on the upstream edge (Poiseuille conditions),

- \( u \equiv 0 \), \( v \equiv 0 \) all along the wall,

- \( v \equiv 0 \) along the centerline, and

- \( v \equiv 0 \), \( p \equiv 0 \) on the downstream edge.

The natural boundary condition \( \frac{\partial u}{\partial y} \equiv 0 \) was left for the program to approximate along the centerline. Since at the entrance \( v \equiv 0 \),

\[
\psi(0,y) = \int_0^1 4(1 - y^2) \cdot y \; dy
\]

and thus \( \psi(0,1) = 1.0 \).

The program was run for \( R_e = 0 \) (linear), 1, 5, 10, 15 and 20 and the stream function solution was obtained from the primitive variable solution. Streamline contour plots based on a \( 100 \times 100 \) rectangular grid using the interpolation functions defined in Chapter 3, are shown in figures (4.19), (4.20) and (4.21). The first ten
Finite Element Grid for Orifice Flow

\[ a = 4.75 \quad b = 17.05 \quad l = 0.5 \]
\[ R = 1.0 \quad \delta = 0.5 \]

Number of nodes = 289
Number of elements = 80
Number of equations for steady flow = 479

- x-axis scale: 1 cm = 1.540 units
- y-axis scale: 1 cm = 0.385 units

Figure 4.18: Finite element grid for flow through an axisymmetric orifice
Figure 4.19: Streamline contours for orifice flow for $Re = 0$, 1 and 5
Figure 4.20: Streamline contours for orifice flow for $R_e = 10, 15$ and $20$
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Figure 4.21: Comparison of streamline contours for orifice flow for $R_e = 10$

(a) Greenspan [10], 1973

(b) Olson [18], 1975

(c) present study
contours have an increment of 0.1, the value of the 11th contour is $\psi = 1.001$ and the subsequent contours have an increment of 0.005. Again the $y$-dimension has been enlarged four times in the first two figures. In figure (4.21), the finite difference results of Greenspan [10], and the pseudo-variational finite element results of Olson [18] are shown along with the results of the present study. It is seen that the contours obtained using the 8-noded isoparametric elements compare well with those of Greenspan’s finite difference results and Olson’s high-precision finite element results using the same contour values thus indicating that the present axisymmetric program is accurately representing the phenomenon. The small bubble upstream of the step observed by Greenspan is missing here, probably because of grid coarseness.

4.5 Flow Through a Stenosed Pipe

In this section numerical results are presented for the parametric study performed for the case of flow through a stenosed pipe. A number of researchers have been working on this problem for the past few years and a brief summary of the past work is given in Chapter 1. The geometry of the stenosis, various geometrical parameters, the nature of the constriction and the choice of finite element element grids (coarse and fine) are presented in the first subsection. Results for the steady flow, and for periodic flow for $R_\omega = 5$, $R_\omega = 10$ and $R_\omega = 20$ are presented in the subsequent subsections. Shear stress along the wall of the pipe is determined by the method outlined in Appendix D.

4.5.1 Geometry of the Stenosis

Figure (4.22) represents a model of the stenosis. The various geometric parameters are as follows: $a$ is the length of the unconstricted region before the stenosis; $b$ is the length of the unconstricted region after the stenosis; $2l$ is the length of the stenosis;
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4.22: Geometry of Stenosis in an axisymmetric pipe

$r(x)$ is the effective radius of the pipe at any cross-section $x$; $R$ is the radius at any unconstricted cross-section; and $\delta$ is the maximum radial constriction so that the radius of the pipe at its minimum is $(R - \delta)$. Furthermore, the stenosis is symmetric about $x = (a + l)$. By suitably varying the $a$, $b$, $l$, $\delta$, and choosing appropriate representation of $r(x)$ we can represent almost any kind of stenosis. Chakravarty [2,3], Forrester [8,9], Deshpande [6] and many others have chosen to model the constriction with a half cosine wave, i.e.,

$$r(x) = \begin{cases} 
R - \frac{\delta}{2} \left[ 1 + \cos \left( \frac{\pi(x - (a + l))}{l} \right) \right] & a \leq x \leq (a + 2l); \\
R & \text{otherwise.}
\end{cases}$$

This model, however, has a drawback viz. the beginning and end of the stenosis are not smooth: that is, $\frac{\partial r}{\partial x}$ has a jump at $x = a$ and $x = (a + 2l)$. To get around this difficulty, some researchers use a full cosine wave to model the stenosis.

Another common technique is to assume that the variation of radius between $x = a$ and $x = (a + 2l)$ is circular, i.e., for some values for $\alpha$, $\beta$ and $\gamma$,

$$(x - \alpha)^2 + (y - \beta)^2 = \gamma^2.$$
To determine $\alpha$, $\beta$ and $\gamma$, we use $y = R$ at $x = a$ and $x = (a + 2l)$, and $y = (R - \delta)$ at $x = (a + l)$. This gives

$$r(x) = \begin{cases} \frac{R - \frac{(\delta^2 - l^2)}{2\delta}}{R} + \left\{ \frac{(\delta^2 + l^2)}{2\delta} \right\}^2 - \left[ x - (a + l) \right]^2 \frac{1}{2} & a \leq x \leq (a + 2l); \\ \text{otherwise}. \end{cases}$$

This model also suffers from the same drawback, although to a lesser extent. We have adopted this model for the present study.

The choice of $\delta$ is governed by the area reduction desired. Normally one restricts $\delta$ to a maximum of $0.5R$. We chose $\delta = 0.707R$; this results in a $50\%$ area reduction at the maximum constriction. Choosing the other parameters is not so easy; $b$, for example, should be long enough to allow the flow to develop again after separation. As a result, $b$ is highly dependent on the Reynolds number. Likewise, $a$ should be long enough to compensate for any separation occurring near the beginning of stenosis—especially at higher frequency Reynolds number, $R_w$. The half-length of the stenosis, $l$, affects the rate of change of diameter. Thus a very small $l$ means that the radius decreases, and subsequently increases, at a very fast rate and the resulting flow might be similar to the flow over an orifice.

The grids shown in figure (4.23) and figure (4.24) were used for this problem with the parameters as given in the table (4.6) below. The number of elements across the cross-section remains constant; between $a \leq x \leq (a + 2l)$, the width of each element is reduced by a factor of $\frac{r(x)}{R}$. The number of elements along the $y$—direction and in the pre-stenosis zone are chosen the same for both the grids. Over the length of the stenosis and in its wake, the grid points are closely spaced.

### 4.5.2 Steady Flow Through Stenosis

For the case of steady flow through a stenosed pipe, both grids were used. At the entrance section, $x = 0$, the flow is specified as fully developed, i.e., $u(0,y) = (1 -$.
Coarse Grid for Pulsatile Flow
(Grid 1)

\[ a = 5.0 \quad b = 5.0 \quad l = 3.0 \]
\[ R = 1.0 \quad \delta = 0.707 \]

Number of nodes = 381
Number of elements = 112
Number of equations for steady flow = 772
Number of equations for unsteady flow = 2316

\[ x\text{—axis scale: } 1 \text{ cm} = 1.052 \text{ units} \]
\[ y\text{—axis scale: } 1 \text{ cm} = 0.263 \text{ units} \]

Figure 4.23: Grid 1 used for analysis of stenotic flow
Chapter 4: Numerical Results

Fine Grid for Pulsatile Flow
(Grid 2)

\[ a = 5.0 \quad b = 12.0 \quad l = 3.0 \]
\[ R = 1.0 \quad \delta = 0.707 \]

Number of nodes = 719
Number of elements = 216
Number of equations for steady flow = 1487
\[ x-\text{axis scale:} \quad 1 \text{ cm} = 1.564 \text{ units} \]
\[ y-\text{axis scale:} \quad 1 \text{ cm} = 0.391 \text{ units} \]

Figure 4.24: Grid 2 used for analysis of stenotic flow
Table 4.6: Parameters for the finite element grids for stenotic flow

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Grid 1</th>
<th>Grid 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Designation</td>
<td>short/coarse</td>
<td>long/fine</td>
</tr>
<tr>
<td>No. of nodes</td>
<td>381</td>
<td>719</td>
</tr>
<tr>
<td>No. of elements in $x-$direction</td>
<td>14</td>
<td>27</td>
</tr>
<tr>
<td>No. of elements in $y-$direction</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Unconstricted radius $R$</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Pre-stenosis length $a$</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Post-stenosis length $b$</td>
<td>5.0</td>
<td>10.0</td>
</tr>
<tr>
<td>Half-stenosis length $l$</td>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Maximum constriction $\delta$</td>
<td>0.707</td>
<td>0.707</td>
</tr>
</tbody>
</table>

$y^2$) and $v \equiv 0$. Results converged for Reynolds numbers up to $R_e = 500$ for the coarse grid (grid 1). For Reynolds numbers higher than 500, grid 1 did not lead to converged solutions due to either, shorter post-stenosis length or due to inability of the finite elements to represent the rapid changes in primitive variables because of the coarseness of the grid; perhaps a combination of both the above. Grid 2, however, yielded converged solutions up to very high Reynolds number and in relatively less number of iterations. We present these results for Reynolds numbers up to $R_e = 2000$.

We note that the pre- and post-stenosis lengths for grid 1 are the same and hence the linear problem, i.e., the flow through the stenosis at zero Reynolds number, is symmetric about the line $x = (a+l)$. As such we expect the solution to be symmetric about this line; this indeed is the case. Furthermore, because of this symmetry, the $y-$component of fluid velocity along the grid line passing through $x = (a+l)$ should be zero. Since the area of cross-section is effectively halved at this point, $u$ must be doubled. This, too, is satisfied by the finite element results.

At higher Reynolds numbers, beginning at $R_e = 300$, a very characteristic region of separation is observed in the wake of the stenosis. From the streamline contours for the coarse grid, we can see the onset of the disturbance which subsequently leads
to the separation. Streamline contours for grid 2 very clearly reveal this phenomenon. At Reynolds numbers higher than $R_e = 2000$, the contours plots are a little zig-zagged which suggests that the results thus obtained are grid dependent. This numerical in-accuracy results from the fact that in the current formulation the continuity equation is not satisfied exactly, whereas the streamline formulation assumes it to be satisfied exactly. This can be avoided by analyzing the flow with a very fine grid.

Streamline contours for selected values of $R_e$ are given in figures (4.25-4.27) for the coarse grid and in figures (4.28-4.33) for the finer grid. A $100 \times 100$ rectangular grid in conjunction with the scheme outlined in Appendix C is used to plot the contours. Twenty streamline contours equally spaced between the minimum and the maximum values are plotted for grid 1. For the finer grid, contours 1-10 have an increment of 0.02, contours 11-14 have an increment of 0.01, contours 15-16 have an increment of 0.005 and subsequent contours have an increment of 0.0005. Maximum stream function values calculated at the rectangular grid points used for plotting for different values of $R_e$ are summarized in table (4.7).

Because of the no slip condition along the wall, the fluid experiences a high a shear stress there. The shear acting on the wall is equal and opposite of the shear acting on the fluid at $y = r(x)$. Computation of the shear stress from the primitive variables solution is similar to the computation of vorticity $\zeta$ and further details are given in Appendix D. At a fully developed, non-stenosed, cross-section the non-dimensional shear stress equals 2. Over the length of the stenosis, with the increase in velocity gradient, there is a corresponding increase in the shear stress. In the wake of the stenosis, as separation occurs, the velocity gradients, and consequently the shear stress changes sign. This is shown in figure (4.34) for the coarse grid and in figures (4.35) and (4.36) for the fine grid. Furthermore, up to $R_e = 300$, the results obtained using the coarse grid are very close to those obtained using the fine grid. This suggests that grid 1 is sufficiently accurate up to that Reynolds number.
Figure 4.25: Streamline contours for steady flow through stenosis for grid 1 for $R_e = 0$, 1 and 10
Figure 4.26: Streamline contours for steady flow through stenosis for grid 1 for $Re = 50, 100$ and $150$
Chapter 4: Numerical Results

Figure 4.27: Streamline contours for steady flow through stenosis for grid 1 for \( R_e = 200, 250 \) and 300
Figure 4.28: Streamline contours for steady flow through stenosis for grid 2 for $R_e = 0$, 10 and 50
Figure 4.29: Streamline contours for steady flow through stenosis for grid 2 for $Re = 100, 150$ and $200$
Figure 4.30: Streamline contours for steady flow through stenosis for grid 2 for $Re = 300, 400$ and $500$
Chapter 4: Numerical Results

RE = 600

Figure 4.31: Streamline contours for steady flow through stenosis for grid 2 for $Re = 600, 700$ and $800$
Figure 4.32: Streamline contours for steady flow through stenosis for grid 2 for $Re = 900$, 1000 and 1250.
Figure 4.33: Streamline contours for steady flow through stenosis for grid 2 for $R_e = 1500, 1750$ and $2000$
Figure 4.34: Shear stress along the wall of the pipe for grid 1 for $R_e = 0$ (linear), 50, 100, 200 and 300.
Figure 4.35: Shear stress along the wall of the pipe for grid 2 for $Re = 0$ (linear), 50, 100, 200 and 300
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Figure 4.36: Shear stress along the wall of the pipe for grid 2 for $Re = 400, 500, 1000, 1500$ and $2000$
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Table 4.7: Maximum stream function values for steady flow through the stenosis

<table>
<thead>
<tr>
<th>Reynolds Number $R_e$</th>
<th>Maximum Stream Function Value</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (linear)</td>
<td>0.2500</td>
<td>x 5.04</td>
</tr>
<tr>
<td>50</td>
<td>0.2500</td>
<td>0.99</td>
</tr>
<tr>
<td>100</td>
<td>0.2500</td>
<td>5.04</td>
</tr>
<tr>
<td>100</td>
<td>0.2500</td>
<td>0.99</td>
</tr>
<tr>
<td>200</td>
<td>0.2500</td>
<td>12.18</td>
</tr>
<tr>
<td>300</td>
<td>0.2503</td>
<td>0.93</td>
</tr>
<tr>
<td>400</td>
<td>0.2508</td>
<td>11.13</td>
</tr>
<tr>
<td>500</td>
<td>0.2513</td>
<td>0.89</td>
</tr>
<tr>
<td>600</td>
<td>0.2517</td>
<td>11.13</td>
</tr>
<tr>
<td>700</td>
<td>0.2522</td>
<td>0.88</td>
</tr>
<tr>
<td>800</td>
<td>0.2527</td>
<td>12.18</td>
</tr>
<tr>
<td>900</td>
<td>0.2531</td>
<td>0.88</td>
</tr>
<tr>
<td>1000</td>
<td>0.2535</td>
<td>12.18</td>
</tr>
<tr>
<td>1250</td>
<td>0.2543</td>
<td>0.87</td>
</tr>
<tr>
<td>1500</td>
<td>0.2549</td>
<td>12.18</td>
</tr>
<tr>
<td>1750</td>
<td>0.2554</td>
<td>0.86</td>
</tr>
<tr>
<td>2000</td>
<td>0.2559</td>
<td>12.39</td>
</tr>
</tbody>
</table>

4.5.3 Pulsatile Flow Through Stenosis

As pointed out in section 3.4, the pulsatile nature of arterial fluid flow is approximated by the first three components of the Fourier series. Thus for any non-zero frequency Reynolds number $R_\omega$, the problem size increases approximately nine-fold over that for the corresponding steady problem. For this reason, the finer grid (grid 2) could not be used for the results discussed in this sub-section; all the solutions were obtained using grid 1 with $a = 5.0$, $b = 5.0$, $l = 3.0$, $R = 1.0$ and $\delta = 0.707$. A parametric study was conducted for frequency Reynolds numbers $R_\omega = 5$, 10 and 20, and steady Reynolds number $R_e = 0$ (linear), 1, 5, 10, 20, 50, 100 and 150, thus resulting in a total of 24
cases. For Reynolds numbers higher than 150, grid 1 was not able to represent the flow satisfactorily and after a few iterations, the solution went divergent.

Stream function values are computed from the primitive variable solutions and a 100 × 100 rectangular grid in conjunction with the scheme outline in Appendix C is used to plot the contours. Twenty streakline contours equally spaced between the minimum and the maximum values are plotted at $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. Computation of the shear stress from the primitive variables solution is similar to the computation of vorticity $\zeta$; further details are given in Appendix D. Along the curved portion of the wall, axial stress $\sigma_x$, radial stress $\sigma_y$ and the shear stress $\tau_{xy}$ are computed. To find the shear stress acting on the wall, $\tau_{ew}$, Mohr’s circle construction is used.

For any fixed Reynolds number, when the steady and the periodic components of the pulsatile flow are additive, the “true” Reynolds number is higher and consequently the ‘deviation’ from the fully developed flow is more. Stream function contours shown in figures (4.37–4.60) clearly reveal the effects of the stenosis. The instantaneous shear values along the wall of the pipe are shown in figures (4.61–4.63) for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. At the beginning of the cycle, $t = 0$, the steady and the periodic components are additive and hence the shear stress and the stream function values along the wall are maximum at this moment. At quarter cycle, $t = \frac{\pi}{2}$, only the steady and the sinusoidal components are effective and hence the maximum shear stress and the maximum value of stream function along the wall drop; at half cycle, $t = \pi$, the cosinusoidal component is subtractive and hence the maximum shear stress and the maximum value of stream function along the wall is minimum at this instance. During the second half of the cycle, this process is reversed and the maximum value is again obtained at $t = 2\pi$. Due to the coarseness of the grid, however, the instantaneous shear stress values along the wall at various moments during a complete cycle showed a very small amount of oscillation near the minimum value. These oscillations, too, damp out at half cycle $t = \pi$, when the cosinusoidal component is subtractive.
Figure 4.37: Streaklines for pulsatile flow through stenosis for $R_\omega = 5$, $Re = 0$ (linear) for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$
Figure 4.38: Streaklines for pulsatile flow through stenosis for $R_\omega = 5$, $Re = 1$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$.
Figure 4.39: Streaklines for pulsatile flow through stenosis for \( R_w = 5, \, R_e = 5 \) for \( t = 0, \frac{\pi}{2}, \pi \) and \( \frac{3\pi}{2} \)
Figure 4.40: Streaklines for pulsatile flow through stenosis for $R_\omega = 5$, $Re = 10$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$
Figure 4.41: Streaklines for pulsatile flow through stenosis for $R_\omega = 5$, $Re = 20$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$.
Figure 4.42: Streaklines for pulsatile flow through stenosis for $R_w = 5$, $R_e = 50$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
Figure 4.43: Streaklines for pulsatile flow through stenosis for $R_w = 5$, $Re = 100$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$
Figure 4.44: Streaklines for pulsatile flow through stenosis for $R_w = 5$, $Re = 150$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
Figure 4.45: Streaklines for pulsatile flow through stenosis for \( R_\omega = 10, R_e = 0 \) (linear) for \( t = 0, \pi, \frac{3\pi}{2} \)
Figure 4.46: Streaklines for pulsatile flow through stenosis for $R_\omega = 10$, $Re = 1$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
Figure 4.47: Streaklines for pulsatile flow through stenosis for $R_e = 10$, $R_r = 5$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
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Figure 4.48: Streaklines for pulsatile flow through stenosis for $R_\omega = 10$, $R_e = 10$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$.
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Figure 4.49: Streaklines for pulsatile flow through stenosis for $R_\omega = 10$, $Re = 20$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$.
Figure 4.50: Streaklines for pulsatile flow through stenosis for \( R_\omega = 10, R_z = 50 \) for \( t = 0, \frac{\pi}{2}, \pi \) and \( \frac{3\pi}{2} \).
Figure 4.51: Streaklines for pulsatile flow through stenosis for $R_w = 10$, $Re = 100$ for $t = 0, \pi/2, \pi$ and $3\pi/2$
Figure 4.52: Streaklines for pulsatile flow through stenosis for $R_w = 10$, $Re = 150$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$.
Figure 4.53: Streaklines for pulsatile flow through stenosis for $R_o = 20$, $Re = 0$ (linear) for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
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Figure 4.54: Streaklines for pulsatile flow through stenosis for $R_\omega = 20, \ Re = 1$ for
$t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
Figure 4.55: Streaklines for pulsatile flow through stenosis for $R_w = 20$, $Re = 5$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$.
Figure 4.56: Streaklines for pulsatile flow through stenosis for $R_w = 20$, $Re = 10$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$.
Figure 4.57: Streaklines for pulsatile flow through stenosis for $R_\omega = 20$, $Re = 20$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$.
Figure 4.58: Streaklines for pulsatile flow through stenosis for $R_w = 20$, $R_e = 50$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$
Figure 4.59: Streaklines for pulsatile flow through stenosis for $R_\omega = 20$, $Re = 100$ for $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. 

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Figure 4.60: Streaklines for pulsatile flow through stenosis for $R_\omega = 20$, $Re = 150$ for $t = 0$, $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$.
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Figure 4.61: Instantaneous shear stress along the wall of the pipe for grid 1 for $R_\omega = 5$ and $Re = 0$ (linear), 10, 50, 100 and 150
Figure 4.62: Instantaneous shear stress along the wall of the pipe for grid 1 for $R_\omega = 10$ and $R_e = 0$ (linear), 10, 50, 100 and 150
Figure 4.63: Instantaneous shear stress along the wall of the pipe for grid 1 for $R_w = 20$ and $R_e = 0$ (linear), 10, 50, 100 and 150
CHAPTER 5

Conclusions

5.1 Concluding Remarks

The present method of representing the complete solution as a sum of steady, cosinusoidal and sinusoidal components seems to work very well. The modified method of averaging used is also observed to be quite accurate. In particular, the circular arc approximation of stenoses works well. The overall agreement between the present study and other published results in the literature is good. The modified Newton-Raphson procedure for solving the nonlinear algebraic equations is very successful and converges fast for the range of frequency Reynolds numbers and Reynolds numbers considered in this study.

The results obtained for the fully developed flow under the influence of a cosinusoidal pressure difference agree well with the numerical and graphical values reported by Schlichting. The streakline plots are almost completely horizontal indicating that even though the solution becomes increasingly wavy with increasing frequency Reynolds number, $R_e$, the finite element approximation is very good. For the developing flow, the program consistently exhibited good stability, fast convergence and good accuracy, even for higher Reynolds numbers. The effect of Reynolds number on the development length was in good agreement with the theoretical values reported in [26]. The additional pressure drop, $\Delta p_{add}$ obtained from the present study showed some discrepancies probably due to the inadequacies of the bilinear pressure shape functions and coarseness of the grid.
Chapter 5: Conclusions

The primitive variable solution as well as the stream function contour plots for axisymmetric flow through the orifice converged fast for Reynolds number up to $R_e = 30$ even though at the minimum section area of cross section of the pipe was reduced by 75% and the $u$-velocity was quadrupled. The stream function contours for $R_e = 10$, plotted to 1:1 scale, compares well with the pseudo-variational finite element results of Olson [18] and the finite difference results of Greenspan [10]. It may be noted that Greenspan's method converged in about 40 to 90 ‘outer’ iterations compared to 4 to 8 iterations required by the present method. Greenspan’s method also required a very fine finite difference mesh.

Steady results for flow through a stenosed pipe were obtained for two different grids and were used to check for the consistency of the results. All the major characteristics of the flow such as separation and re-attachment points, variation of shear stress along the wall, including the maximum and the minimum values, maximum stream function values, etc., show a monotonic improvement with the finer grid. The separation and the re-attachment points as obtained from the stream function results and from the shear stress results are in total agreement with each other.

For unsteady flow through a stenosed pipe only the coarse grid was used and satisfactory results were obtained for $R_e = 5, 10$ and 20, and for Reynolds numbers up to $R_e = 150$. For any fixed Reynolds number, when the steady and the periodic components of the pulsatile flow are additive, the “true” Reynolds number is higher and consequently the ‘deviation’ from the fully developed flow is more. Stream function contours clearly reveal the effects of the stenosis and instantaneous shear values along the wall of the pipe indicate the initiation of the separation phenomenon; for higher Reynolds numbers, the flow will separate and the region of back flow will have negative shear stress. Due to the coarseness of the grid, however, the instantaneous shear stress values along the wall at various moments during a complete cycle showed a very small amount of oscillation near the minimum value.
5.2 Suggestions for Further Developments

Some specific recommendations for further studies based on the work in this thesis are:

- perform the analysis with the transformations

\[ \eta = f(x, y) \]

...to map the curved boundary onto a straight line \( \xi = \xi_0 \), and

\[ \eta = \tanh(Kx) \]

...to map the infinite domain to the finite domain.

- perform the analysis for multiple stenoses in a series.

- conduct a grid refinement study to determine its effects on the numerical results.

- perform the analysis with high precision elements using the stream function and the vorticity as the primitive variables and compare and contrast it with the present study.
Bibliography


Bibliography


APPENDIX A

Newton-Raphson Procedure

The modified Newton-Raphson iteration procedure adopted in this study is outlined here. The scheme is outlined for a set of $n$ simultaneous nonlinear algebraic equations in $n$ unknowns $x_1, x_2, \ldots, x_n$. These equations can be written in the form

$$
\begin{align*}
    f_1(x_1, x_2, \ldots, x_n) &= 0 \\
    f_2(x_1, x_2, \ldots, x_n) &= 0 \\
    &\vdots \\
    f_n(x_1, x_2, \ldots, x_n) &= 0.
\end{align*}
$$

(A.1)

Expanding the $i$th equation around $x_{i0}, x_{20}, \ldots, x_{n0}$ using the Taylor series, we obtain

$$
\begin{align*}
    f_i(x_1, x_2, \ldots, x_n) &\approx f_i(x_{10}, x_{20}, \ldots, x_{n0}) + \left( \frac{\partial f_i}{\partial x_1} \right)_0 \Delta x_1 \\
    &\quad + \left( \frac{\partial f_i}{\partial x_2} \right)_0 \Delta x_2 + \cdots + \left( \frac{\partial f_i}{\partial x_n} \right)_0 \Delta x_n \approx 0
\end{align*}
$$

(A.2)

where the derivatives are evaluated at $(x_{10}, x_{20}, \ldots, x_{n0})$ and

$$
\begin{align*}
    \Delta x_1 &= x_1 - x_{10} \\
    \Delta x_2 &= x_2 - x_{20} \\
    &\vdots \\
    \Delta x_n &= x_n - x_{n0}.
\end{align*}
$$

(A.3)

Similarly, carrying out the expansion for all the equations in (A.1) and rearranging, we obtain
Equations (A.4) can be written in the form $[T]\{\Delta x\} = \{-f\}$ where the matrices denote the corresponding terms in equation (A.4). The algorithm for obtaining the solution vector $\{x\}$ is outlined below:

1. Guess the solution vector $\{x\}_0$.


3. Compute $\{-f\}$.

4. Obtain $\{\Delta x\}$ from $[T]\{\Delta x\} = \{-f\}$.

5. Obtain $\{x\}_1 = \{x\}_0 + \{\Delta x\}$.

6. If $\{x\}_1$ is close to $\{x\}_0$ within the specified tolerance go to 8, otherwise set $\{x\}_0 = \{x\}_1$.

7. If $[T]$ needs to be modified in this iteration go to 2, otherwise go to 3.

8. Set $\{x\} = \{x\}_1$ and stop.
APPENDIX B

Details of Tangent Stiffness Matrix and Nonlinear Load Vector

The left hand side of equation (3.12) can be re-written as

\[
[T] = \begin{bmatrix} K & 0 & 0 \\ 0 & K & M \\ 0 & -M & K \end{bmatrix} + \begin{bmatrix} A & B & C \\ D & A & 0 \\ E & 0 & A \end{bmatrix}
\]

where \(K\) and \(M\) are the stiffness and mass matrices. Sub-matrices \(A\) to \(E\), using index notation, are given as

\[
A = R_e \begin{bmatrix} \delta_{ijm} A^u_m + \delta_{ijm} A^u_m + \delta_{ijm} A^u_m & \delta_{ijm} A^u_m \\ \delta_{ijm} A^u_m & \delta_{ijm} A^u_m + \delta_{ijm} A^u_m + \delta_{ijm} A^u_m & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
B = \frac{R_e}{2} \begin{bmatrix} \delta_{ijm} B^u_m + \delta_{ijm} B^u_m + \delta_{ijm} B^u_m & \delta_{ijm} B^u_m \\ \delta_{ijm} B^u_m & \delta_{ijm} B^u_m + \delta_{ijm} B^u_m + \delta_{ijm} B^u_m & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
C = \frac{R_e}{2} \begin{bmatrix} \delta_{ijm} C^u_m + \delta_{ijm} C^u_m + \delta_{ijm} C^u_m & \delta_{ijm} C^u_m \\ \delta_{ijm} C^u_m & \delta_{ijm} C^u_m + \delta_{ijm} C^u_m + \delta_{ijm} C^u_m & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
Appendix B: Details of Tangent Stiffness Matrix and Nonlinear Load Vector

\[ D = R_e \begin{bmatrix} \delta_{ijm} B_m^u + \delta_{imj} B_m^u + \delta_{imj} B_m^u & \delta_{ijm} B_m^u & 0 \\ \delta_{ijm} B_m^u & \delta_{imj} B_m^u + \delta_{imj} B_m^u + \delta_{ijm} B_m^u & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ E = R_e \begin{bmatrix} \delta_{ijm} C_m^u + \delta_{imj} C_m^u + \delta_{imj} C_m^u & \delta_{ijm} C_m^u \\ \delta_{ijm} C_m^u & \delta_{imj} C_m^u + \delta_{imj} C_m^u + \delta_{ijm} C_m^u & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The incremental solution vector is \( \{ \Delta x \} = \begin{bmatrix} \Delta A \\ \Delta B \\ \Delta C \end{bmatrix} \) and the load vector is

\[ \{-f\} = \begin{bmatrix} K & 0 & 0 \\ 0 & K & M \\ 0 & -M & K \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \]

\[-R_e \begin{bmatrix} \delta_{ijk} (A_j^u A_k^u + B_j^y B_k^u/2 + C_j^v C_k^u/2) + \delta_{ijk} (A_j^u A_k^u + B_j^y B_k^u/2 + C_j^v C_k^u/2) \\ \delta_{ijk} (A_j^u A_k^u + B_j^y B_k^u/2 + C_j^v C_k^u/2) + \delta_{ijk} (A_j^u A_k^u + B_j^y B_k^u/2 + C_j^v C_k^u/2) \\ 0 \\ \delta_{ijk} (A_j^u B_k^u + A_k^u B_j^u) + \delta_{ijk} (A_j^u B_k^u + A_k^u B_j^u) \\ \delta_{ijk} (A_j^u B_k^u + A_k^u B_j^u) + \delta_{ijk} (A_j^u B_k^u + A_k^u B_j^u) \\ 0 \\ \delta_{ijk} (A_j^u C_k^u + A_k^u C_j^u) + \delta_{ijk} (A_j^u C_k^u + A_k^u C_j^u) \\ \delta_{ijk} (A_j^u C_k^u + A_k^u C_j^u) + \delta_{ijk} (A_j^u C_k^u + A_k^u C_j^u) \\ 0 \end{bmatrix} \]
APPENDIX C

Determination of the Stream Function Values

Consider an eight-noded isoparametric element as shown in the figure (3.1). The shape functions $N_i$ and $M_i$ for the element are presented in Chapter 3. The value of the stream function $\Psi$ and the coordinates $(x, y)$ at a point can be obtained in terms of the nodal values $\Psi_i$ and nodal coordinates $(x_i, y_i)$.

$$\Psi = N_i \Psi_i$$

$$x = N_i x_i \quad i = 1, 2, \ldots, 8$$

$$y = N_i y_i$$

The following procedure is adopted to obtain the value of $\Psi$ accurately at each node of the plotting grid which lies within the finite element.

1. Assume a quadratic interpolation for $\xi$ and $\eta$ of the form

$$\xi = A_1 + A_2 x + A_3 y + A_4 x^2 + A_5 xy + A_6 y^2 + A_7 x^2 y + A_8 y^2 x$$

$$\eta = B_1 + B_2 x + B_3 y + B_4 x^2 + B_5 xy + B_6 y^2 + B_7 x^2 y + B_8 y^2 x$$

(C.1)

The $(x, y)$ and $(\xi, \eta)$ values at each of the eight nodes on the element are known and hence eight equations each in $A_1, A_2, \ldots, A_8$ and $B_1, B_2, \ldots, B_8$ are obtained and solved for. Thus for any point $(x_o, y_o)$ in the finite element, using equation (C.1), corresponding $(\xi_o, \eta_o)$ coordinates can be calculated.

2. Determine the maximum and minimum value of $x$ and $y$ coordinates of the finite element. The approximate area occupied by this element is a rectangle of
length \( x_{\text{max}} - x_{\text{min}} \) along the \( x \)-axis and height \( y_{\text{max}} - y_{\text{min}} \) along \( y \)-axis; this approximation works reasonably well for an element which is not too skewed.

3. Determine \((\xi, \eta)\) value of each point of the plotting grid with coordinates \((x_p, y_p)\) within the finite element as outlined below:

(a) Take the initial guess value of \((\xi_p, \eta_p)\) to be that obtained from equation (C.1), i.e.,

\[
\begin{align*}
\xi_p &= A_1 + A_2 x_p + A_3 y_p + A_4 x_p^2 + A_5 x_p y_p + A_6 y_p^2 + A_7 x_p^2 y_p + A_8 y_p^2 x_p \\
\eta_p &= B_1 + B_2 x_p + B_3 y_p + B_4 x_p^2 + B_5 x_p y_p + B_6 y_p^2 + B_7 x_p^2 y_p + B_8 y_p^2 x_p
\end{align*}
\]

(b) Calculate the shape functions \(N_i\) for \(i = 1, 2, \ldots, 8\) for these values of \(\xi_p, \eta_p\) using the equations given in Chapter 3.

(c) Calculate the Jacobian \(J\).

\[
J = \begin{bmatrix}
\frac{\partial N_i}{\partial s} x_j & \frac{\partial N_i}{\partial t} x_j \\
\frac{\partial N_i}{\partial s} y_j & \frac{\partial N_i}{\partial t} y_j
\end{bmatrix}
\]

(d) Determine the \((x', y')\) coordinates of the point using the calculated value of shape functions.

\[
x' = N_j x_j \\
y' = N_j y_j
\]

\(j = 1, 2, \ldots, 8\)

(e) Determine the error in the \((x, y)\) coordinates.

\[
\Delta x = x_p - x'
\]

\[
\Delta y = y_p - y'
\]

(f) Determine the error in the \((\xi, \eta)\) coordinates.

\[
\begin{bmatrix}
\Delta \xi \\
\Delta \eta
\end{bmatrix} = [J]^{-1} \begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
\]
(g) The new value of \((\xi, \eta)\) is given as

\[
\begin{align*}
\xi' &= \xi_p + \Delta \xi \\
\eta' &= \eta_p + \Delta \eta.
\end{align*}
\]

With these new values of \((\xi, \eta)\), repeat steps (b) to (f) until desired convergence is achieved.

4. Once convergence has occurred in step 3, the stream function value can be obtained using \(\Psi = N_i \Psi_i\).
APPENDIX D

Determination of Stress Components

For axisymmetric flow, the three non-trivial components of the stress tensor in their non-dimensional form are given as

\[ \sigma_x = -p + 2 \frac{\partial u}{\partial x} \]
\[ \sigma_y = -p + 2 \frac{\partial v}{\partial y} \]
\[ \tau_{xy} = \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right). \]

Consider an eight-noded isoparametric element as shown in the figure (3.1). Using the shape functions \( N_i \) and \( M_i \) presented in Chapter 3, equations (D.1) are discretized as

\[ \sigma_x = \left[ -p_j M_j + 2 u_i \frac{\partial N_i}{\partial x} \right] \]
\[ \sigma_y = \left[ -p_j M_j + 2 v_i \frac{\partial N_i}{\partial y} \right] \quad i = 1, 2, \ldots, 8; \quad j = 1, 2, \ldots, 4. \]

\[ \tau_{xy} = \left[ v_i \frac{\partial N_i}{\partial x} + u_i \frac{\partial N_i}{\partial y} \right] \]

Stress components \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) at some point \( (x, y) \in \Omega \) can be obtained in terms of the nodal values of the primitive variables \((u, v, p)\). The following procedure is adopted to obtain the values of \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) accurately at any given point \((x_0, y_0)\).

1. Determine the element within which point \((x_0, y_0)\) lies.

\[ ^{1}\text{If the point lies on a boundary, stresses are obtained for each element that borders at the given point and a weighted average is obtained; in case of a stress discontinuity all the values are reported.} \]
2. Assume a quadratic interpolation for \( \xi \) and \( \eta \) of the form
\[
\begin{align*}
\xi &= A_1 + A_2 x + A_3 y + A_4 x^2 + A_5 xy + A_6 y^2 + A_7 x^2 y + A_8 y^2 x \\
\eta &= B_1 + B_2 x + B_3 y + B_4 x^2 + B_5 xy + B_6 y^2 + B_7 x^2 y + B_8 y^2 x
\end{align*}
\] (D.3)
The \((x, y)\) and \((\xi, \eta)\) values at each of the eight nodes on the element are known and hence eight equations each in \(A_1, A_2, \ldots, A_8\) and \(B_1, B_2, \ldots, B_8\) are obtained and solved for.

3. Determine \((\xi, \eta)\) value of each point of the plotting grid with coordinates \((x_o, y_o)\) within the finite element as outlined below:

(a) Take the initial guess value of \((\xi_o, \eta_o)\) to be that obtained from equation (D.3), i.e.,
\[
\begin{align*}
\xi_o &= A_1 + A_2 x_o + A_3 y_o + A_4 x_o^2 + A_5 x_o y_o + A_6 y_o^2 + A_7 x_o^2 y_o + A_8 y_o^2 x_o \\
\eta_o &= B_1 + B_2 x_o + B_3 y_o + B_4 x_o^2 + B_5 x_o y_o + B_6 y_o^2 + B_7 x_o^2 y_o + B_8 y_o^2 x_o
\end{align*}
\]
(b) Calculate the shape function derivatives \(\frac{\partial N_i}{\partial \xi}\) and \(\frac{\partial N_i}{\partial \eta}\) for \(i = 1, 2, \ldots, 8\) for these values of \(\xi_o, \eta_o\) using the equations given in Chapter 3.

(c) Calculate the Jacobian \(J\).
\[
J = \begin{bmatrix}
\frac{\partial N_i}{\partial x} x_i & \frac{\partial N_i}{\partial y} x_i \\
\frac{\partial N_i}{\partial y} y_i & \frac{\partial N_i}{\partial x} y_i
\end{bmatrix}
\]
i = 1, 2, \ldots, 8

(d) Determine the \((x', y')\) coordinates of the point using the calculated value of shape functions.
\[
\begin{align*}
x' &= N_i x_i \\
y' &= N_i y_i
\end{align*}
\]
i = 1, 2, \ldots, 8

(e) Determine the error in the \((x, y)\) coordinates.
\[
\begin{align*}
\Delta x &= x_o - x' \\
\Delta y &= y_o - y'
\end{align*}
\]
(f) Determine the error in the $(\xi, \eta)$ coordinates.

\[
\begin{pmatrix}
\Delta \xi \\
\Delta \eta
\end{pmatrix} = [J]^{-1} \begin{pmatrix}
\Delta x \\
\Delta y
\end{pmatrix}
\]

(g) The new value of $(\xi, \eta)$ is given as

\[
\xi' = \xi_0 + \Delta \xi \\
\eta' = \eta_0 + \Delta \eta.
\]

With these new values of $(\xi, \eta)$, repeat steps (b) to (f) until desired convergence is achieved.

4. Once convergence has occurred in step 3, compute $\frac{\partial N_i}{\partial x}$ and $\frac{\partial N_i}{\partial y}$ using

\[
\begin{pmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y}
\end{pmatrix} = [J]^{-1} \begin{pmatrix}
\frac{\partial \xi}{\partial x} \\
\frac{\partial \eta}{\partial x}
\end{pmatrix}.
\]

These derivatives can now be substituted into equations (D.2) to give stress results.

Figure D.1: Use of transformation $(x = x(x', y'); y = y(x', y'))$ to determine stresses along a curved wall.
To obtain stresses in any direction other than the global axes, we use the Mohr's circle construction. As shown in the figure (D.1), for equilibrium $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ acting on unit surface area must balance $\sigma_x'$ and $\tau_{x'y'}$ acting on $\sqrt{2}$ surface area. Thus

\[
\sigma_x' = \left( \frac{\sigma_x + \sigma_y}{2} \right) + \left( \frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
\sigma_y' = \left( \frac{\sigma_x + \sigma_y}{2} \right) - \left( \frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[
\tau_{x'y'} = -\left( \frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta.
\]