A CONTRIBUTION TO THE COMPUTER AIDED DESIGN OF OPTIMIZED STRUCTURES FOR THE STEEL INDUSTRY

BY

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Abstract

A practical method of incorporating realistic flexible connections including the effect of connection sizes and shear deflection in plane frame analysis is presented. The general algorithm can be easily implemented in a standard plane frame analysis program and once implemented it can be an ideal tool for production work in the steel industry. In this approach connection stiffness is programmed directly into the analysis by utilizing the connection moment-rotation equations developed by Frye and Morris but it may also be entered separately as data. Nonlinear connection analysis is carried out by the procedure outlined by Frye and Morris. Practical application of this method of analysis is demonstrated by modifying a standard plane frame analysis program to include the effect of flexible connections. The validity of the modified program, CPlane, was verified against the findings of Moncarz and Gerstle.

Using CPlane, a simple plane frame structure was analyzed under various lateral load intensities for different connection assumptions. It was found that the inclusion of connection behavior significantly altered the internal force distribution and design of the structure.

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1 Introduction

In the conventional analysis of framed steel structures, member end-connections are idealized as finite points and are assumed to behave as perfectly hinged or perfectly rigid (Figure 1). In general these idealizations are contrary to actual connection configuration and connection behaviour but are adopted because of the simplicity in analysis and design. Typically connections are about five percent the length of a member and they all possess certain amount of flexibility in rotation (Figure 2). Although connections may appear to be small in size and contribute little to the overall weight of a structure, their behaviour can significantly alter the internal force distribution and hence affecting the overall design of a structure. The common assumption of perfectly rigid connections neglecting connection flexibility and connection sizes may lead to underestimation of the sway of bare frames and overestimation of the forces at the connections resulting in overly heavy columns and connections. Further, connections have a relatively high labour content and they can represent a substantial portion of the overall cost of a structure.

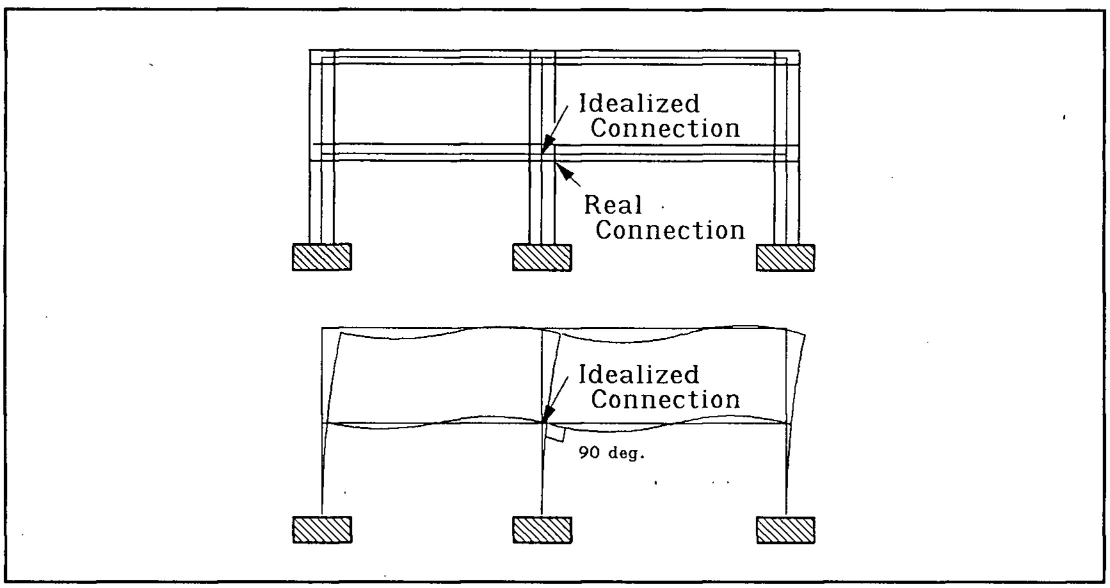

Figure 1 An Idealized Structure

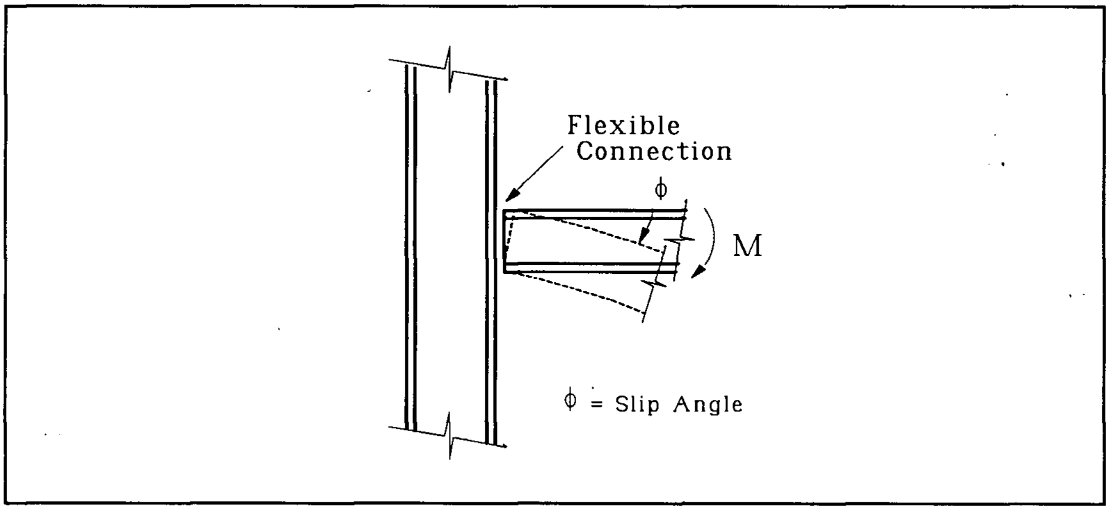

Figure 2 A Real Connection

The topic of connection behaviour has been researched as early as 1917 [1]. More recent research include studies performed by Douty in 1964 [2], Popov and Pinkney in 1969 [3], Frye and Morris in 1975 [4], and Stelmack, Marley and Gerstle in 1986 [5]. Test result [5] indicates that connection response is nonlinear in nature, but can be approximated as linearly elastic within the working range of the frames. Many analytical methods of modeling connections have been proposed over the years. One commonly accepted method is the use of a member with rigid ends (Figure 3). This method correctly accounts for connection sizes, but it neglects the rotational flexibility of connections. Gere and Weaver [6], Morforton and Wu [7], and Livesley [8] have each presented methods of modeling linear elastic connections. However their methods all neglect the effect of connection sizes and shear deflection. More elaborate methods of modeling nonlinear connection response have been presented by Romstad and Subramania [9], and Moncarz and Gerstle [10]. Unfortunately their methods also neglect the effect of connection sizes and shear deflection. In addition practical implementation of their methods involve extensive and expensive programming work. A practical method of incorporating connection behaviour in plane frame analysis for office use is much in need.

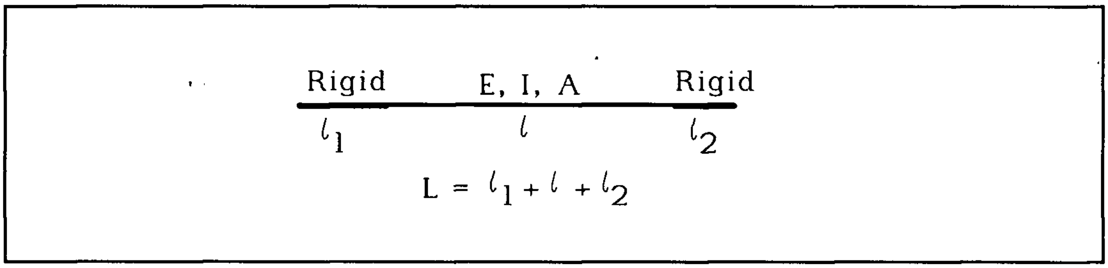

Figure 3 A Member with Rigid Ends

The object of this work is twofold. The first is to present a practical method of modifying an existing linear elastic plane frame analysis program to perform refined flexible frame analysis for office use. The second objective is to demonstrate the usefulness of such a method of analysis by applying it to a typical unbraced steel frame and comparing its response with the response from commonly accepted methods of analysis. It is hoped that this will serve to encourage the use of this type of analysis in practice in the future.

2 Refined Member-Connection Model

Conventional structural analysis assumes that connections have negligible dimensions and behave as perfectly rigid or perfectly hinged. But real connections have finite dimensions and possess some degree of flexibility whether they are rigid or hinged, and their behaviour is rather complex. To be perfectly precise one should really distinguish between joint flexibility and connection flexibility. Joint flexibility refers to the ability of the joint to deform in shear or in bending (Figure 4) while connection flexibility refers to the slip in rotation between a joint and the member end (Figure 5).

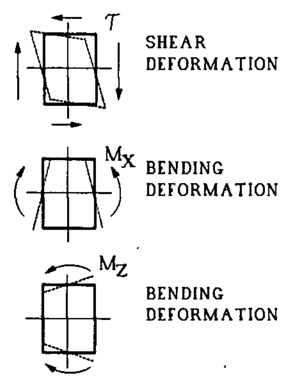

Figure 4 Joint Flexibility

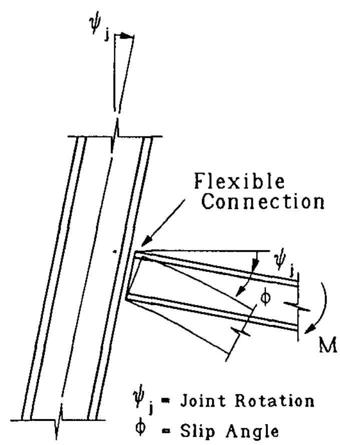

Figure 5 Connection Flexibility

A properly detailed joint should not undergo significant deformation in shear or in bending; in practice joints which are weak in shear are strengthened with the addition stiffeners (Figure 6). Therefore the effect of joint flexibility is usually small and it will not be considered herein.

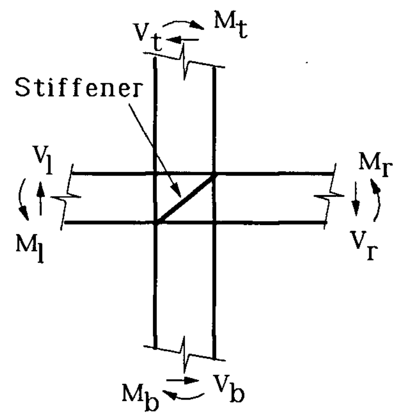

Figure 6 Joint Stiffener

However, connections may possess significant flexibility in rotation [1,2,3,4,5] and its effect should be considered. The degree of rotational flexibility exhibited by a connection depends very much on the type of connection in question; a bolted or lightly welded connection is likely to be more flexible than a fully welded connection [4]. It is desirable to have a computer model capable of accounting for the effect of connection sizes as well as connection flexibility.

Many investigators [1,2,3,4,5] have studied the behaviour of flexible connections. The most comprehensive formulation of flexible connection behaviour is perhaps the one presented by Frye and Morris [4]. They tested a wide variety of connections under different monotonic loading conditions. Their result indicates that the response of flexible connection is nonlinear in nature. However, Moncarz and Gerstle [10] observed that flexible connections may be satisfactorily modeled by assuming linear elastic connection response. They compared analytically nonlinear connection response with linear connection response (Figure 7) and their conclusion :

> *"The assumption of linear response of flexible connections seems reasonable and appears to give a good prediction of the bare frame response".*[1]

They further remarked that the sequence of load application only played a minor influence in the sways of the multistory frames which they investigated.

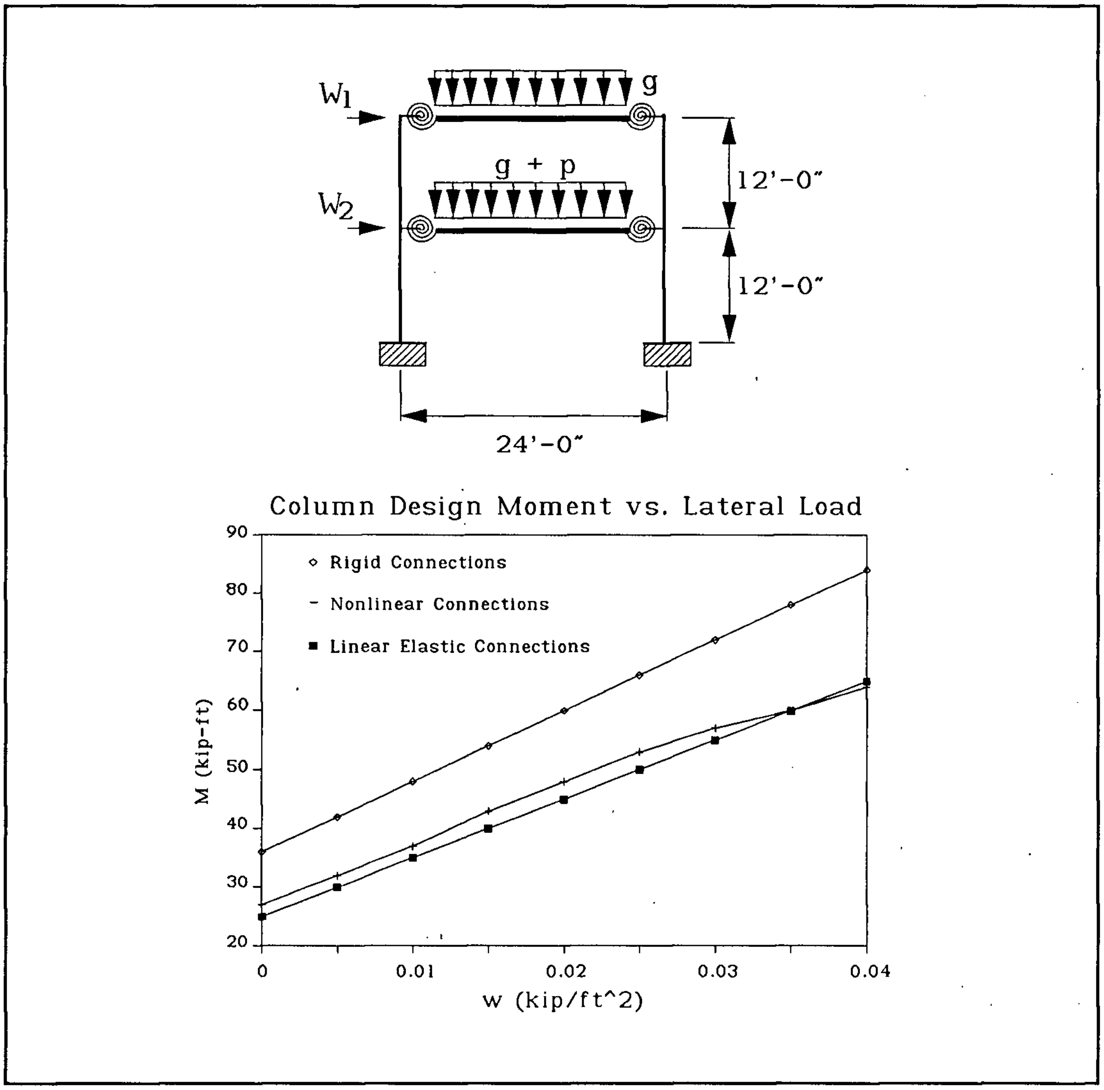

Figure 7 Nonlinear versus Linear Connection Response

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1 Moncarz, P.D. and Gerstle K.H., [Ref 10] p.1440.

The findings of Moncarz and Gerstle [10] are later confirmed by the experimental result of Stelmack, Marley and Gerstle [5]. Stelmack, Marley and Gerstle performed a total of 10 tests of 2 frame configurations (Figure 8) and here are some of their observations :

> "1. *The connection response remained essentially linear elastic within the working range of the frames. Accordingly linear elastic frame analysis is adequate for predicting frame response to service loads,*
>
> 2. *No evidence of incremental deflections or other instabilities was observed under a significant number of cycles at high loads".*[2]

In light of the analytical and experimental evidence presented, it appears reasonable to assume linear elastic connection response for linear elastic structural analysis.

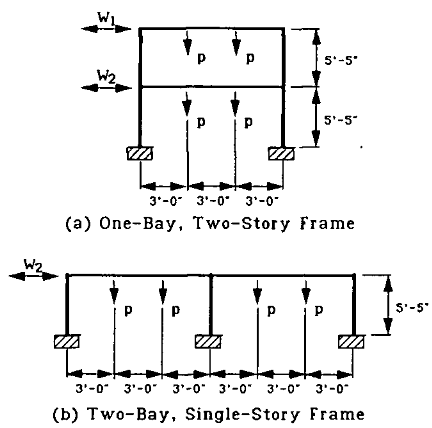

Figure 8 Test Frames of Moncarz, Marley and Gerstle

Several different approaches may be adopted to model linear elastic connections. The simplest and most obvious one is to treat each connection as four elements joined together rigidly at one point like a cross (Figure 9). Connection behaviour may be modeled by specifying appropriate value of length, EI_x[3] and EA[4] for each element. This method can be applied to any standard

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2 Stelmack, T.W., Marley, M.J., and Gerstle, K.H., [Ref.5] p.1586.

3 EI_x represents the bending stiffness of the connection.

4 EA represents the axial stiffness of the connection.

structural analysis program. Unfortunately it also introduces 12 extra degrees of freedom and 4 extra elements at each connection. This greatly reduces the size of structure which can be analyzed on a computer and renders it impractical.

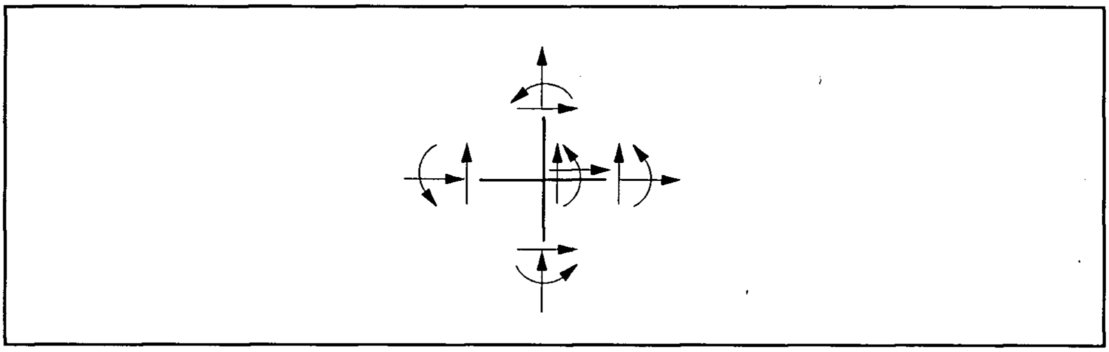


Figure 9 A Four-Element Connection Model

An alternative approach is to introduce a special connection element [11] at each connection (Figure 10). Connection behaviour may be modeled by assigning appropriate value of length, height and EI_x[5] to each special element. This method introduces only 1 extra degree of freedom and 1 extra element at each connection, but it is still inconvenient for practical use because one has to specify explicitly how the member elements are connected with the connection elements. If one wishes to alter the structural configuration the connection sequence would have to be specified again.

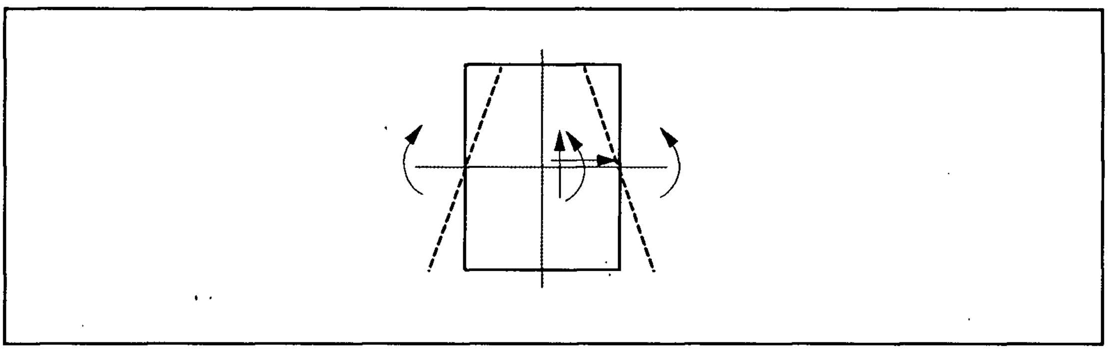


Figure 10 A Special Connection Element Model

5 EI_x represents the bending stiffness of the connection.

A better approach is to incorporate the connection elements directly into the member element (Figure 11). This can be achieved by combining two connection elements together with a member element and then removing the six interior degrees of freedom by means of static condensation. This method is fine except for the fact that its implementation involves substantial modification to a standard structural analysis program which can be both expensive and time consuming.

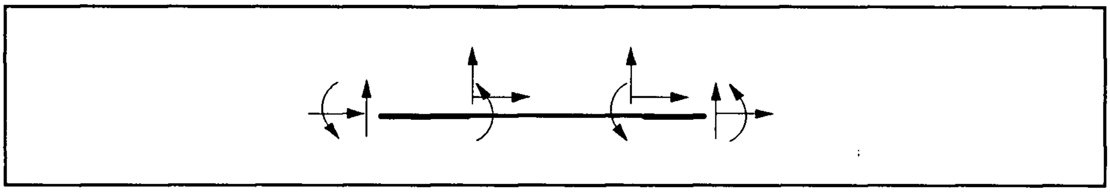


Figure 11 A Member-Connection Model by Static Condensation

It seems best to take a direct approach in modeling connection response. Figure 12 depicts the proposed refined member-connection model; it consists of a member element and two rigid end pieces connected together by two rotational springs. The rigid end pieces model the effect of connection sizes and the two rotational springs model the slip between the connections and the member ends (Figure 6). This approach is deemed direct because each element of the model represents a specific aspect of connection behaviour. There are several advantages to this model. First of all it does not introduce any additional degrees of freedom to the system. Secondly, connection elements are directly incorporated into the member elements and the tedious task of joining member elements to connection elements is avoided. Lastly this model can be easily implemented in a standard plane frame analysis program as will be shown in later sections.

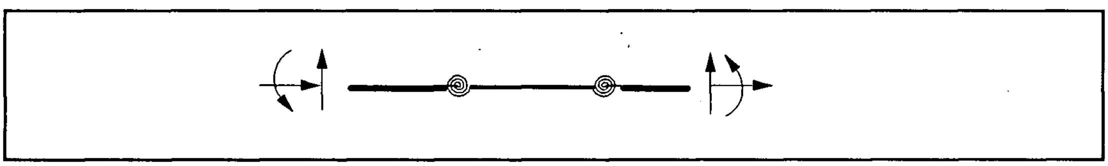


Figure 12 The Refined Member-Connection Model

3 General Procedure for Assembling Refined Member Stiffness Matrix

The first step is to derive a local member stiffness matrix, $\{k\}$ which includes the effect of connection flexibility and shear deflection. The effect of connection sizes will be added on later. One should note that the axial and bending component of a member stiffness matrix are uncoupled and they could be separated into a local axial stiffness matrix $\{k_a\}$ and a local bending stiffness matrix $\{k_b\}$.

$$\{k\} = \{k_a\} + \{k_b\} \qquad [3\text{-}1]$$

Connection flexibility and shear deflection only affect the local bending stiffness matrix $\{k_b\}$, thus only $\{k_b\}$ needs to be derived. The adopted sign convention is illustrated in Figure 13; The d's and f's denote the local degrees of freedom and member end-forces associated with the member.

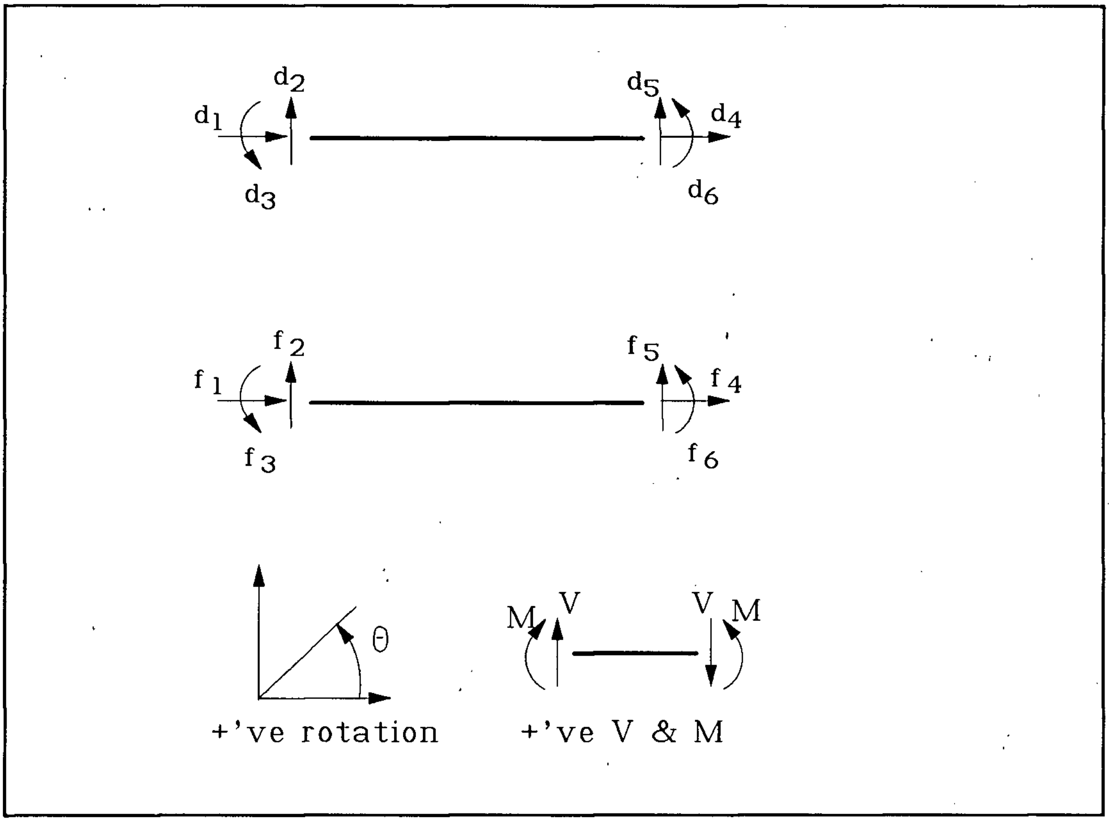


Figure 13 The Adopted Sign Convention

As explained earlier in Section 2 connection flexibility may be modeled by means of two rotational springs, one at each end of the member. Connection behaviour is governed by the rotational stiffness of the springs, K_1 and K_2 such that:

$$\phi_1 = -\frac{f_3}{K_1} \qquad [3\text{-}2a]$$

$$\phi_2 = -\frac{f_6}{K_2} \qquad [3\text{-}2b]$$

where

ϕ_1, ϕ_2 = member-end rotations,

f_3, f_6 = member-end forces,

K_1, K_2 = spring stiffness constants.

If the connections are perfectly rigid (Figure 14a), the slope of member cross section at either end of the member ψ_{m_1}, ψ_{m_2} must equal to the corresponding joint rotation, ψ_{j_1}, ψ_{j_2}.

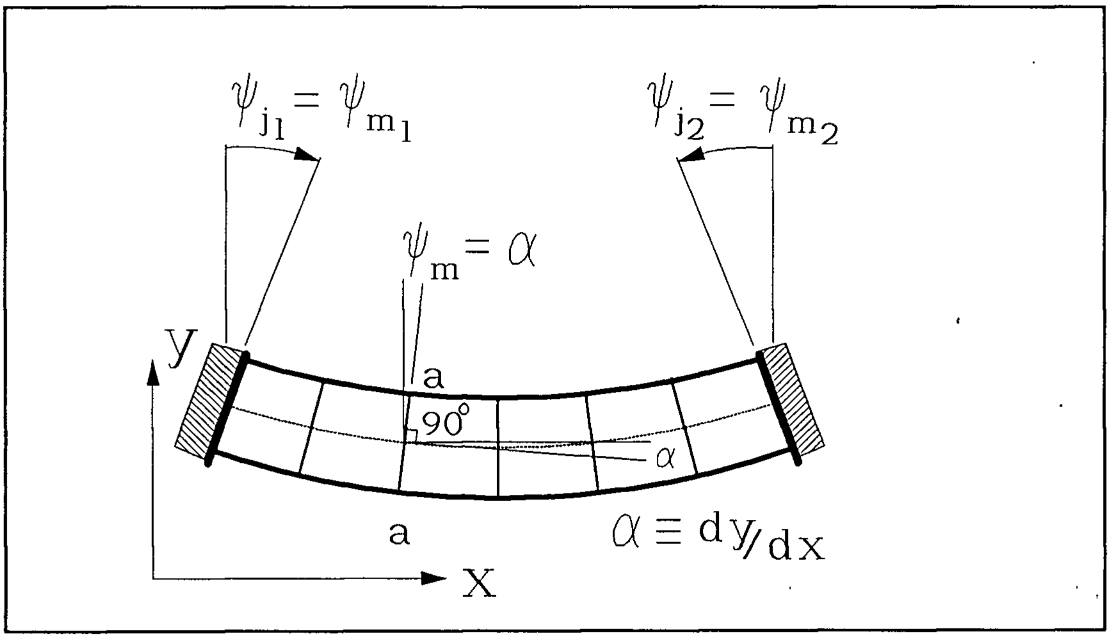


Figure 14a The Relationship between Joint and Member End Rotations for Rigid Connections

If the connections are flexible (Figure 14b) and the slip in rotation between connections and member ends are denoted as ϕ_1, ϕ_2, the relationship between ψ_j's and ψ_m's now become:

$$\psi_{m_1} = \psi_{j_1} + \phi_1 \qquad [3\text{-}3a]$$

$$\psi_{m_2} = \psi_{j_2} + \phi_2 \qquad [3\text{-}3b]$$

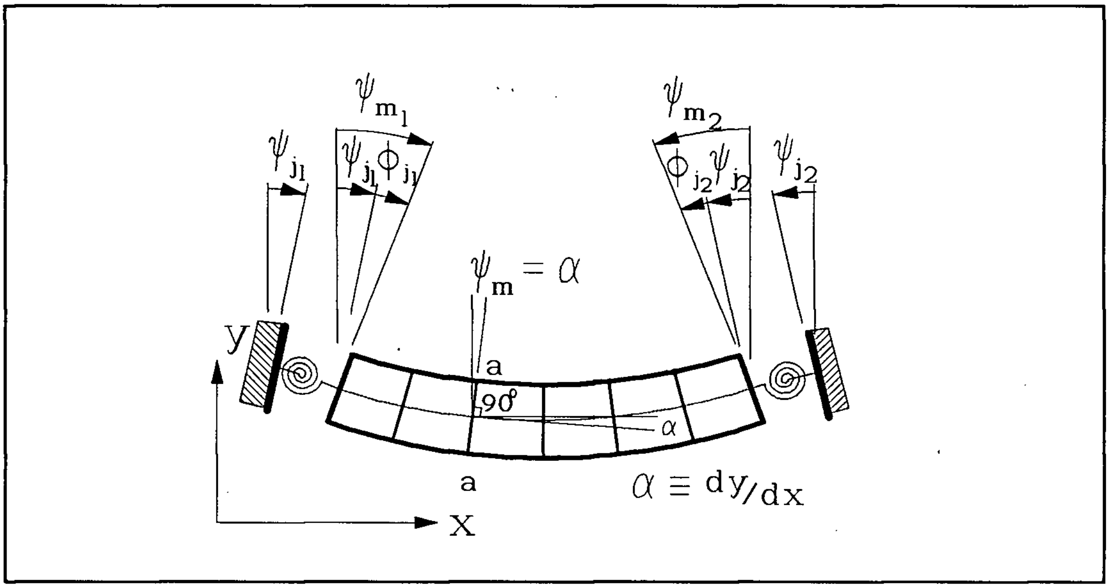


Figure 14b The Relationship between Joint and Member End Rotations for Flexible Connections

The method of modeling shear deflection will now be explained. Consider a beam (Figure 15a) with a series of lines a-a painted on its surface at right angle to the neutral axis. These pained lines represent the cross sections along the beam. The same beam is shown in its deformed position after a uniformly distributed load is applied to it (Figure 15b). Under the engineering beam theory the lines a-a must remain at right angle to the neutral axis such that their rotation denoted as α is dy/dx. In other words, the slope of member cross sections with respect to the vertical axis, ψ_m must equal to the slope of the neutral axis of the member, α.

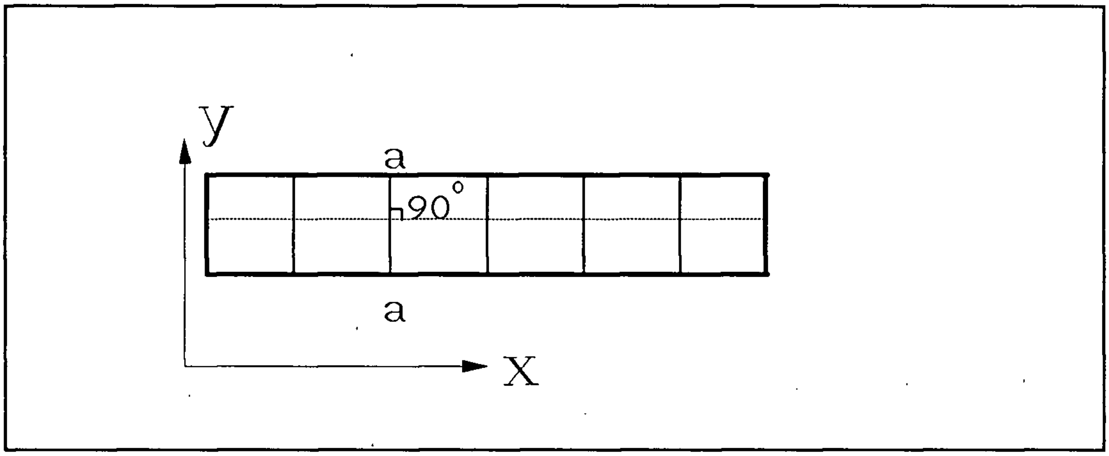


Figure 15a A Simply Supported Beam

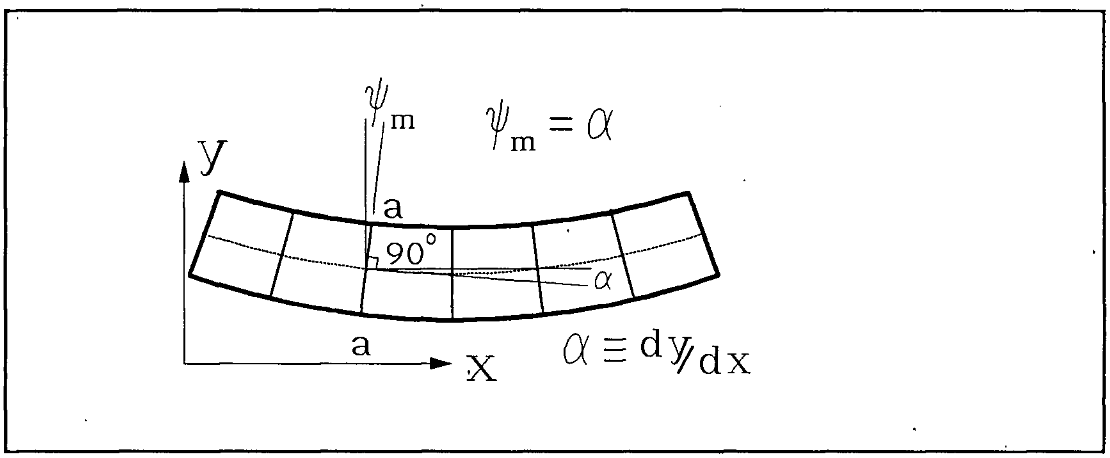


Figure 15b A Simply Supported Beam under a Uniformly Distributed Load

However shear stresses in the beam will cause the lines to rotate to the dotted positions a'-a' (Figure 16) and the cross sections will no longer be at right angle to the neutral axis. The rotation of the cross sections from a-a to a'-a' is defined as the shear strain, γ.

$$\gamma = -\frac{V}{(A_v \cdot G)} \qquad [3\text{-}4]$$

where

V = shear force at member cross section,

A_v = shear area,

G = shear modulus.

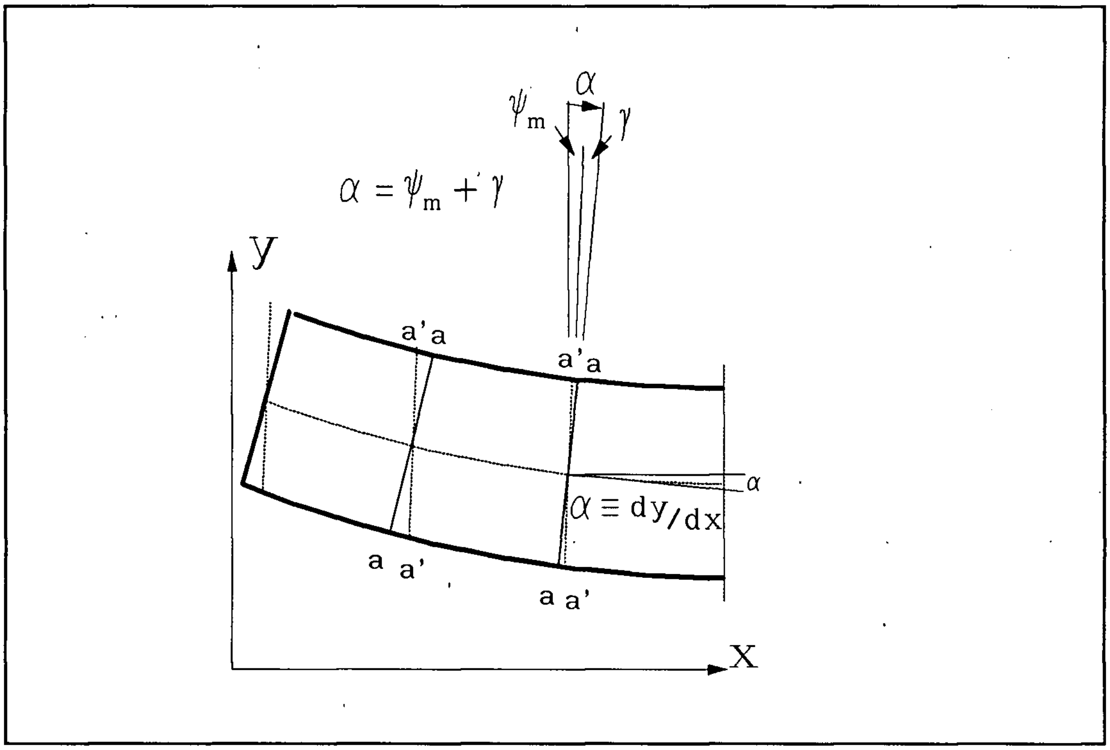


Figure 16 The Effect of Shear Strain on Beam Deflection

This would mean that:

$$\psi_m = \alpha - \gamma \qquad [3\text{-}5]$$

where

ψ_m	=	slope of member cross section,
α	=	rotation of neutral axis, dy/dx,
γ	=	shear strain.

Combining the result of [3-3] and [3-5]:

$$\alpha_1 = \psi_{j_1} + \phi_1 + \gamma_1 \qquad [3\text{-}6a]$$

$$\alpha_2 = \psi_{j_2} + \phi_2 + \gamma_2 \qquad [3\text{-}6b]$$

The combined effect of connection flexibility and shear deflection will be included in the derivation if the the boundary conditions as prescribed by [3-6a,b] are enforced at both ends of the member.

Having derived $\{k_b\}$, the next step is to introduce the rigid end pieces to model connection sizes. This is done by means of transformation. Figure 17a depicts a typical refined member in its deformed position. The relationship between the displacements of the flexibly connected member denoted as d's and that of the refined member denoted as D's is summarized in the same figure.

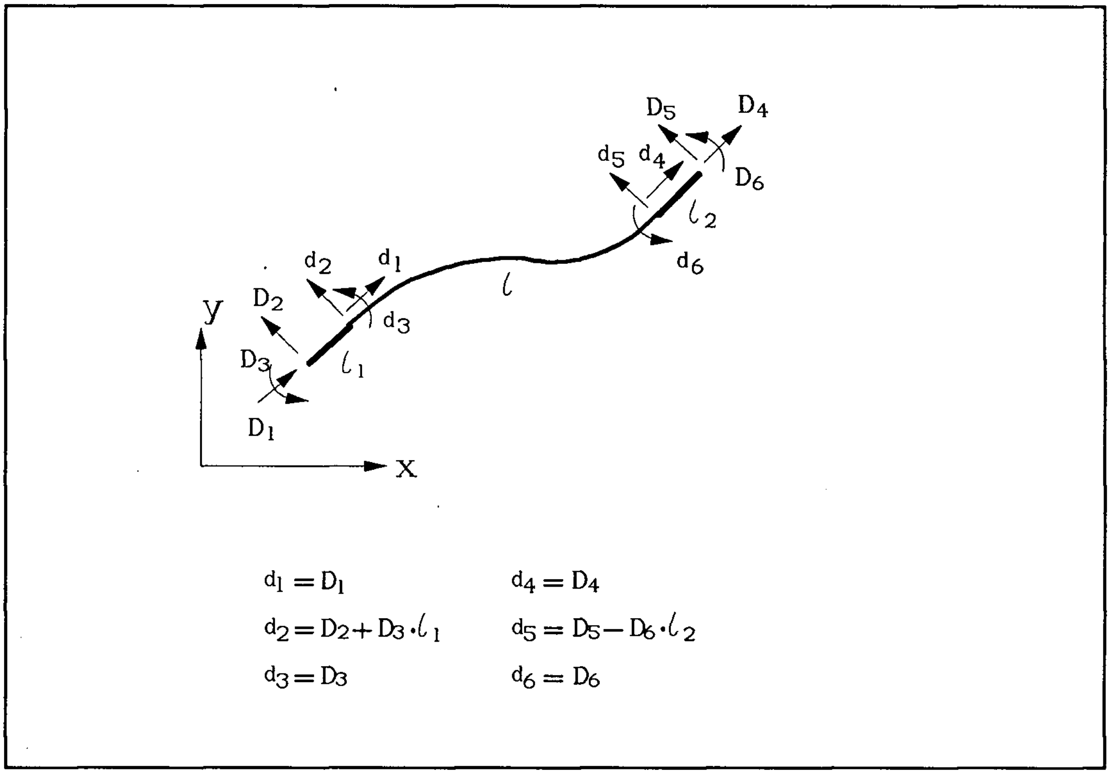


Figure 17a A Typical Refined Member in Its Deformed Position

Arranging these relationships in matrix notation:

$$\{d\} = \{T_c\} \cdot \{D\} \tag{3-7}$$

where

$\{d\}$ = displacement vector of flexibly connected member

$\{T_c\}$ = displacement transformation matrix

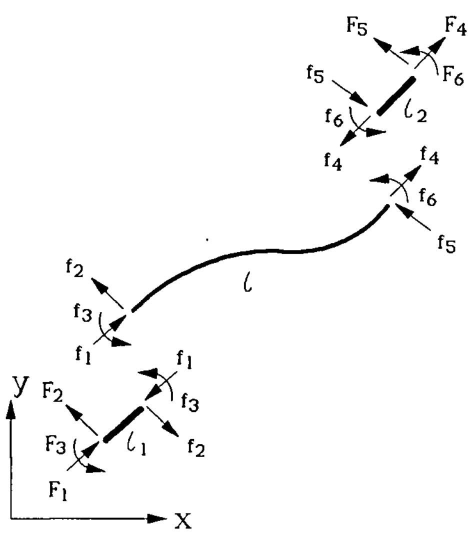
$\{D\}$ = displacement vector of refined flexibly connected member

The matrices of [3-7] are summarized in Figure 17b.

$$\{D\} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{pmatrix} \quad \{T_c\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & l_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -l_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \{d\} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix}$$

Figure 17b Coordinate Transformation

Similarly, Figure 18a depicts the relationships between the forces of the flexibly connected member denoted as f's and that of the refined member denoted as F's.



$F_1 = f_1$ $\quad F_4 = f_4$

$F_2 = f_2$ $\quad F_5 = f_5$

$F_3 = f_3 + f_2 \cdot l_1$ $\quad F_6 = f_6 - f_5 \cdot l_2$

Figure 18a Transfer of Forces

Arranging these relationships in matrix notation:

$$\{F\} = \{T_f\} \cdot \{f\} \qquad [3\text{-}8]$$

where

$\{f\}$ = force vector of flexibly connected member

$\{T_f\}$ = force transformation matrix

$\{F\}$ = force vector of refined flexibly connected member

The matrices of [3-8] are summarized in Figure 18b.

$$\{F\} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{pmatrix} \quad \{T_f\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & l_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -l_2 & 1 \end{pmatrix} \quad \{f\} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}$$

Figure 18b Transfer of Forces

Noting that:

$$\{T_c\} = \{T_f\}^T$$

one could define:

$$\{T\} = \{T_c\} = \{T_f\}^T \qquad [3\text{-}9]$$

From theory of elasticity:

$$\{f\} = \{k\} \cdot \{d\} \qquad [3\text{-}10]$$

Substituting [3-6] and [3-8] into [3-9]:

$$\{f\} = \{k\} \cdot \{T\} \cdot \{D\} \qquad [3\text{-}11a]$$

Pre-multiplying both sides of [3-11a] by $\{T\}^T$ and substituting [3-8]:

$$\{F\} = \{T\}^T \cdot \{k\} \cdot \{T\} \cdot \{D\} \qquad [3\text{-}11b]$$

Define:

$$\{\bar{K}\} = \{T\}^T \cdot \{k\} \cdot \{T\} \qquad [3\text{-}12]$$

where

$\{\bar{K}\}$ = the refined stiffness matrix in local coordinates.

Finally, the refined stiffness matrix in global coordinates may be obtained by applying the standard transformation procedure:

$$\{K\} = \{T_r\}^T \cdot \{\bar{K}\} \cdot \{T_r\}$$

or $$\{T_r\}^T \cdot \{T\}^T \cdot \{k\} \cdot \{T\} \cdot \{T_r\} \qquad [3\text{-}13]$$

where

$$\{T_r\} = \begin{pmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \{T\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & l_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -l_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note :

l_1, l_2 = length of connection (rigid end pieces),

l = length of member,

L = total span = $l_1 + l + l_2$,

Δx = change in x form joint 1 to joint 2,

Δy = change in y from joint 1 to joint 2,

c = $\cos(\Delta x/L)$,

s = $\sin(\Delta y/L)$.

3.1 Assembling the Refined Fix-Fix Member Stiffness Matrix

3.1.1 The Local Bending Stiffness Matrix of Flexibly Connected Fix-Fix Members

The derivation of $\{k_{b11}\}$ as outlined in Section 3 may be carried out by any classical methods. The method of conjugate beam is used here out of personal preference. The adopted sign convention is illustrated in Figure 19. The relationship between the boundary conditions of a real beam and the corresponding support conditions of the conjugate beam is summarized in Figure 20.

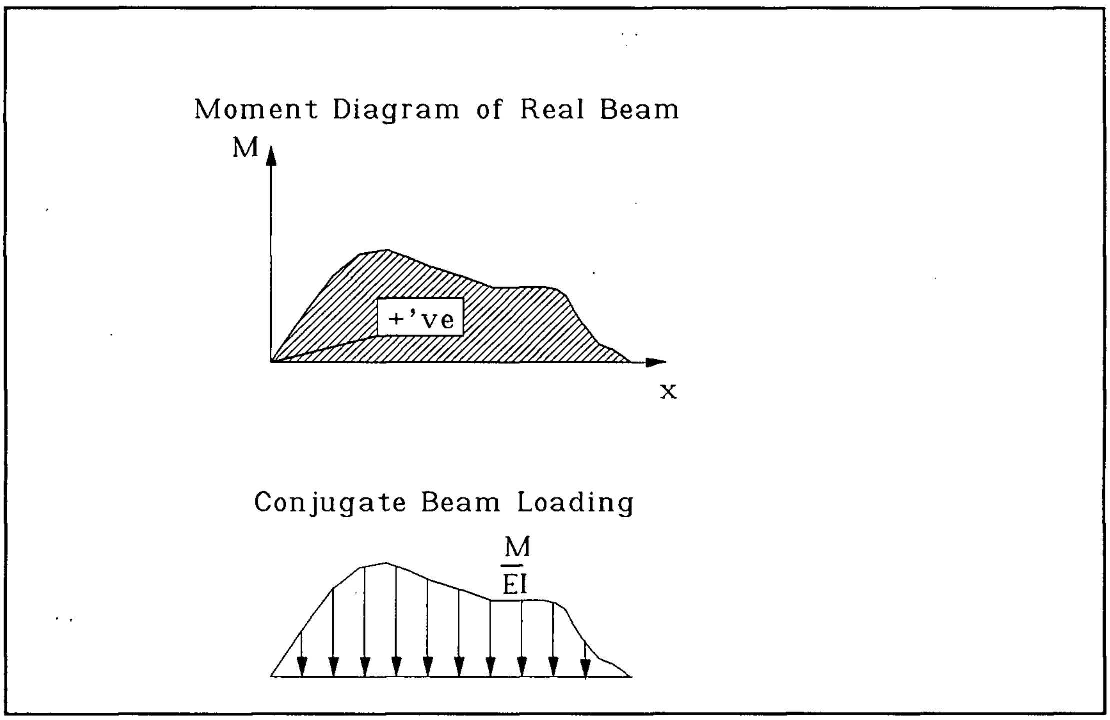


Figure 19 The Adopted Sign Convention of Conjugate Beam

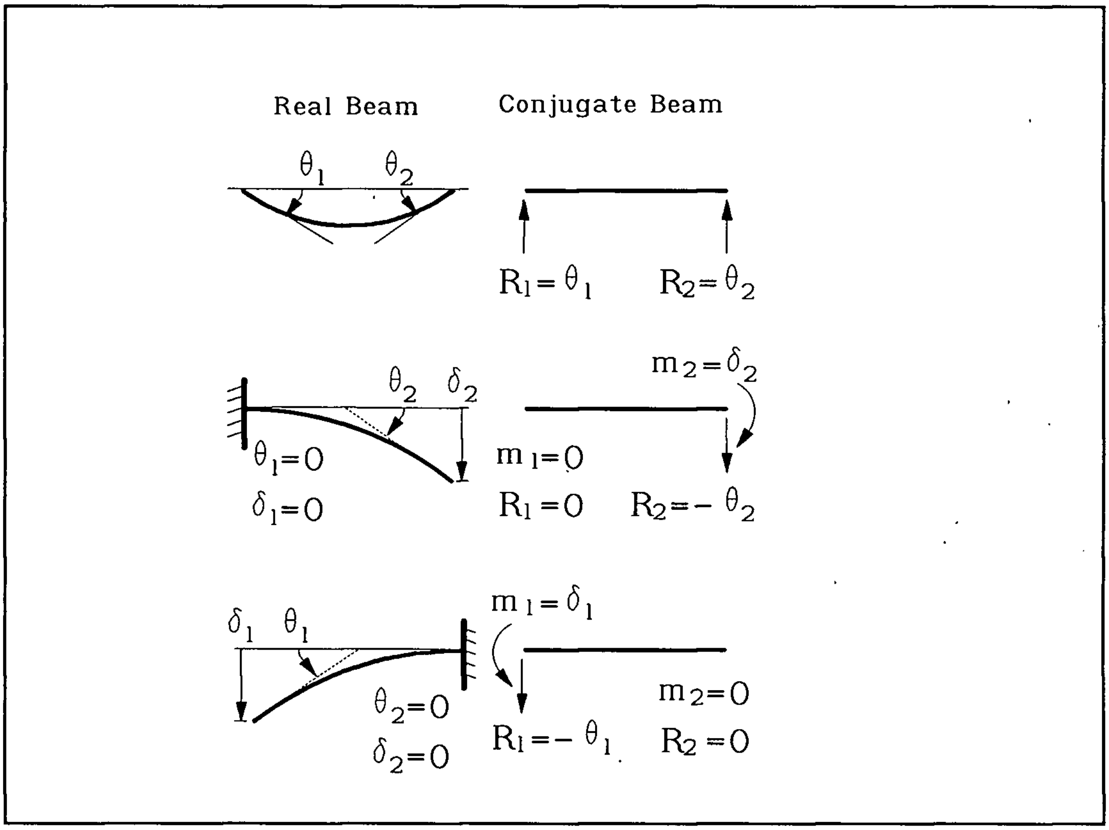


Figure 20 Boundary Condition of Real Beam versus Support Condition of Conjugate Beam

The first and fourth column of $\{k_{b11}\}$ are zero vectors because axial displacements do not affect $\{k_{b11}\}$.

The second column of $\{k_{b11}\}$ is obtained by setting d_2 equal to 1 (Figure 21). This is a 2° indeterminate system. If the two fixed-end moments, f_3 and f_6 are chosen to be the redundant, the conjugate beam will be loaded as shown in Figure 22.

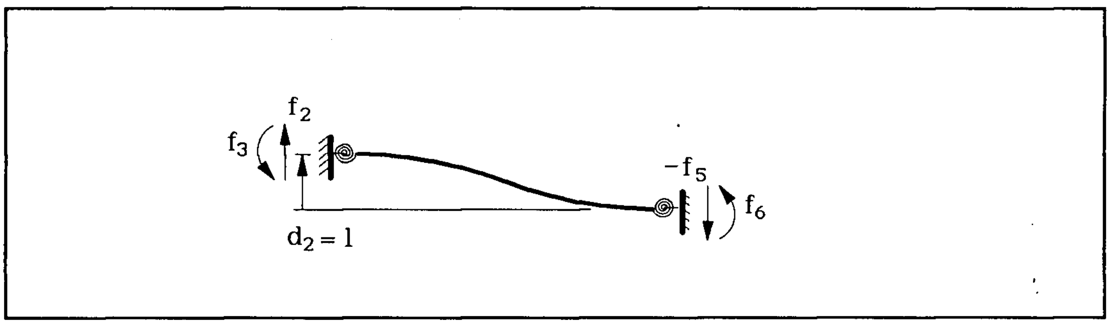


Figure 21 Deriving the Second Column of $\{k_{b11}\}$

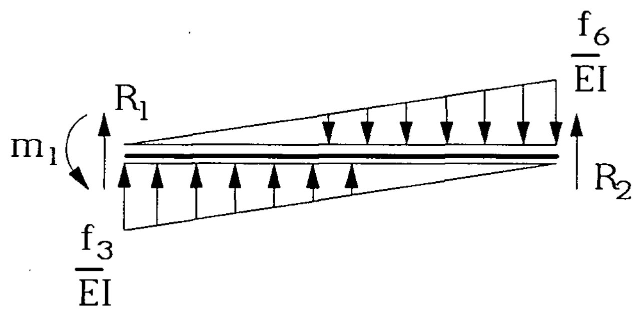


Figure 22 Conjugate Beam Load : Fix-Fix Member, d_2=1

Because the shears at the ends of the conjugate beam are equal to the net angle changes at the ends of the actual beam, one could write from [3-6]:

$$R_1 = \alpha_1 = \psi_{j1} + \gamma_1 + \phi_1 \quad [3\text{-}14a]$$

$$R_2 = \alpha_2 = \psi_{j2} + \gamma_2 + \phi_2 \quad [3\text{-}14b]$$

Also, the moment m_1 of the conjugate beam equals to the deflection d_2 of the real beam. Therefore,

$$m_1 = 1 \quad [3\text{-}14c]$$

Substituting [3-2] and [3-4] into [3-14],

$$R_1 = -\left(\psi_{j1} - \frac{V}{A_v \cdot G} - \frac{f_3}{K_1} \right) \quad [3\text{-}15a]$$

Note : The -'ve sign is due to the conjugate beam's definition of counter clockwise rotation as positive rotation (See Figure 13).

$$R_2 = \left(\psi_{j2} - \frac{V}{A_v \cdot G} - \frac{f_6}{K_2} \right) \quad [3\text{-}15b]$$

But,

$$d_3 = \psi_{j1} = 0,$$
$$d_6 = \psi_{j2} = 0$$

Therefore [3-15] becomes,

$$R_1 = \frac{V}{A_v \cdot G} + \frac{f_3}{K_1} \qquad [3\text{-}16a]$$

$$R_2 = -\frac{V}{A_v \cdot G} - \frac{f_6}{K_2} \qquad [3\text{-}16b]$$

From equilibrium of conjugate beam:

$$R_1 = -\frac{f_3 \cdot l}{3 \cdot EI} + \frac{f_6 \cdot l}{6 \cdot EI} + \frac{m_1}{l} \qquad [3\text{-}17a]$$

$$R_2 = \frac{f_6 \cdot l}{3 \cdot EI} - \frac{f_3 \cdot l}{6 \cdot EI} - \frac{m_1}{l} \qquad [3\text{-}17b]$$

where

$$m_1 = 1 \qquad \text{from } [3\text{-}14c]$$

Combining [3-16a] and [3-17a] and collecting terms,

$$f_3 \cdot \left(\frac{1}{K_1} + \frac{l}{3 \cdot EI} \right) = \frac{f_6 \cdot l}{6 \cdot EI} - \frac{V}{A_v \cdot G} + \frac{1}{l}$$

Multiplying both sides by:

$$\frac{3 \cdot EI}{l}$$

and defining dimensionless constants:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2} \qquad [3\text{-}18]$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1} \qquad [3\text{-}19]$$

Resulting in:

$$\frac{f_3}{\nu_1} = \frac{f_6}{2} - \frac{V \cdot g \cdot l}{4} + \frac{3 \cdot EI}{l^2} \qquad [3\text{-}20]$$

Similarly, combining [3-16b] and [3-17b] and collecting terms,

$$-f_6 \cdot \left(\frac{1}{K_2} + \frac{l}{3 \cdot EI} \right) = -\frac{f_3 \cdot l}{6 \cdot EI} + \frac{V}{A_v \cdot G} - \frac{1}{l}$$

Multiplying both sides by:

$$\frac{3 \cdot EI}{l}$$

and defining dimensionless constants:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l} \right)^{-1} \qquad [3\text{-}21]$$

Resulting in:

$$\frac{f_6}{\nu_2} = \frac{f_3}{2} - \frac{V \cdot g \cdot l}{4} + \frac{3 \cdot EI}{l^2} \qquad [3\text{-}22]$$

From equilibrium of real beam:

$$V = \frac{f_3 + f_6}{l} \qquad [3\text{-}23]$$

Substituting [3-23] into [3-20],[3-22] and collecting terms,

$$f_3 = \left(\frac{4 \cdot \nu_1}{4 + \nu_1 \cdot g} \right) \cdot \left(\frac{3 \cdot EI}{l^2} + \frac{f_6}{4} \cdot (2 - g) \right) \qquad [3\text{-}24]$$

$$f_6 = \left(\frac{4 \cdot \nu_2}{4 + \nu_2 \cdot g} \right) \cdot \left(\frac{3 \cdot EI}{l^2} + \frac{f_3}{4} \cdot (2 - g) \right) \qquad [3\text{-}25]$$

Substituting [3-25] into [3-24],

$$f_3 = \left(\frac{4 \cdot \nu_1}{4 + \nu_1 \cdot g} \right) \cdot (\frac{3 \cdot EI}{l^2} +$$

$$\left[\left(\frac{4 \cdot \nu_2}{4 + \nu_2 \cdot g} \right) \cdot \left(\frac{3 \cdot EI}{l^2} + \frac{f_3}{4} \cdot (2 - g) \right) \right] \cdot \ \frac{(2 - g)}{4} \ )$$

Collecting terms,

$$f_3 \cdot \left(\frac{(4+\nu_1 \cdot g) \cdot (4+\nu_2 \cdot g) - \nu_1 \cdot \nu_2 \cdot (2-g)^2}{4 \cdot \nu_1 \cdot (4+\nu_2 \cdot g)} \right) =$$

$$\frac{3 \cdot EI}{l^2} \cdot \left(\frac{4 \cdot (4+\nu_2) + 8 \cdot \nu_2 - 4 \cdot \nu_2 \cdot g}{4 \cdot (4+\nu_2 \cdot g)} \right)$$

or,

$$f_3 \cdot \left(1 + g \cdot \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \right) = \frac{6 \cdot EI}{l^2} \cdot \left(\frac{\nu_1 \cdot (2+\nu_2)}{4 - \nu_1 \cdot \nu_2} \right)$$

Defining dimensionless constants:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad [3\text{-}26]$$

$$C_2 = \nu_1 \cdot \frac{2+\nu_2}{4 - \nu_1 \cdot \nu_2} \qquad [3\text{-}27]$$

Therefore,

$$f_3 = \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot \left(\frac{1}{1 + g \cdot C_1} \right) \qquad [3\text{-}28]$$

Similarly, substituting [3-24] into [3-25] and collecting terms:

$$f_6 = \left(\frac{4 \cdot \nu_1}{4 + \nu_1 \cdot g} \right) \cdot \Bigg(\frac{3 \cdot EI}{l^2} +$$

$$\left[\left(\frac{4 \cdot \nu_2}{4 + \nu_2 \cdot g} \right) \cdot \left(\frac{3 \cdot EI}{l^2} + \frac{f_3}{4} \cdot (2-g) \right) \right] \cdot \frac{(2-g)}{4} \Bigg)$$

or,

$$f_6 \cdot \left(1 + g \cdot \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \right) = \frac{6 \cdot EI}{l^2} \cdot \left(\nu_2 \cdot \frac{(2+\nu_1)}{4 - \nu_1 \cdot \nu_2} \right)$$

Defining dimensionless constants:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2} \qquad [3\text{-}29]$$

Therefore,

$$f_6 = \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot \left(\frac{1}{1 + g \cdot C_1} \right) \qquad [3\text{-}30]$$

Substituting [3-28], [3-30] into [3-23],

$$V = \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot \left(\frac{1}{1 + g \cdot C_1} \right) \qquad [3\text{-}31]$$

Introducing another dimensionless constant to [3-28], [3-30] and [3-31],

$$S_1 = \frac{1}{1 + g \cdot C_1} \qquad [3\text{-}32]$$

f_3, f_6 and V now become:

$$f_3 = \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \qquad [3\text{-}33]$$

$$f_6 = \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \qquad [3\text{-}34]$$

$$V = \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 \qquad [3\text{-}35]$$

In summary the second column of $\{k_{b11}\}$ is:

$$\left\{k_{b11}\right\}_{i2} = \begin{pmatrix} 0 \\ \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 \\ \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ 0 \\ \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 \\ \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \end{pmatrix}$$

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad \nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2} \qquad \nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2} \qquad g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$S_1 = \frac{1}{1 + g \cdot C_1}$$

The fifth column of $\{k_{b11}\}$ is obtained by setting d_5 equal to 1 (Figure 23), and it can be derived in a similar fashion as before.

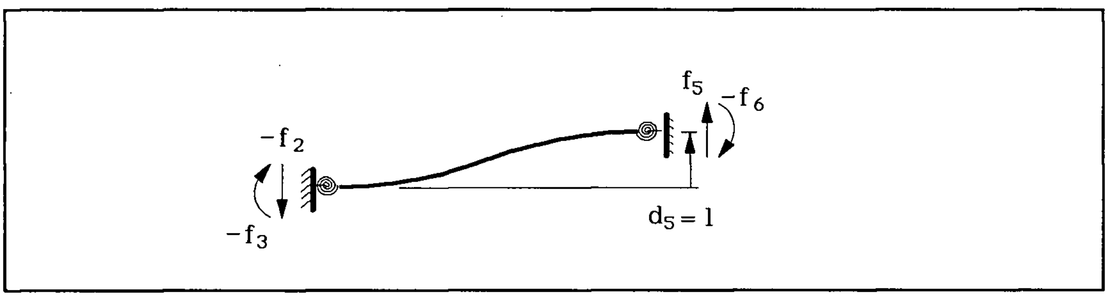


Figure 23 Deriving the Fifth Column of $\{k_{b11}\}$

But noting that Figure 23 is a mirror reflection of Figure 21, the column vector can be written directly as:

$$\{k_{b11}\}_{i5} = \begin{pmatrix} 0 \\ -\dfrac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 \\ -\dfrac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ 0 \\ \dfrac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 \\ -\dfrac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \end{pmatrix}$$

The third column of $\{k_{b11}\}$ is obtained by setting d_3 equal to 1 (Figure 24). Once again this is a 2° indeterminate system. If the two fixed-end moments, f_3 and f_6 are chosen to be the redundant, the conjugate beam will be loaded as shown in Figure 25.

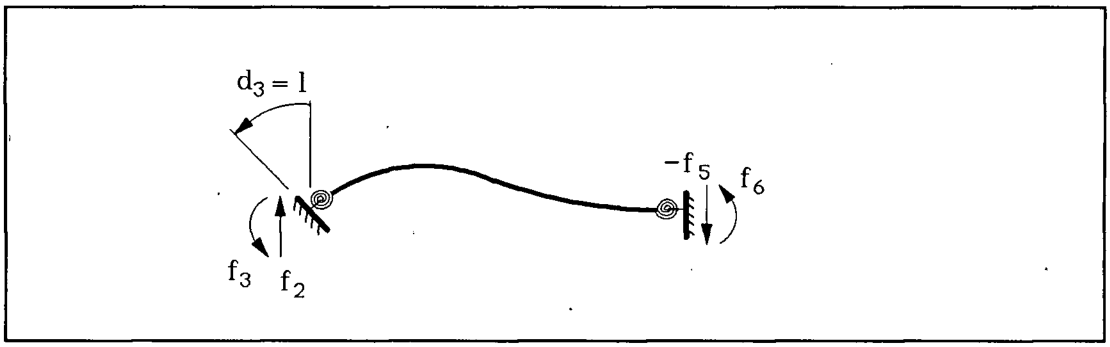


Figure 24 Deriving the Third Column of $\{k_{b11}\}$

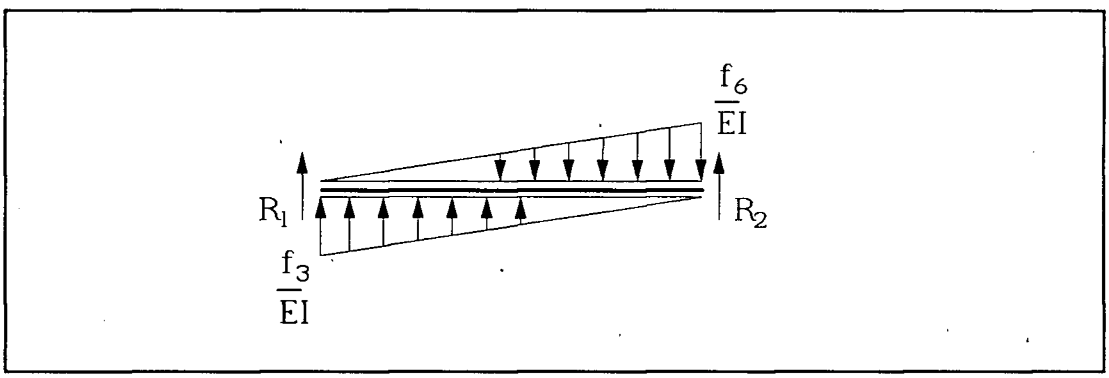


Figure 25 Conjugate Beam Load : Fix-Fix member, d_3=1

Because the shears at the ends of the conjugate beam are equal to the net angle changes at the ends of the actual beam, one could write:

$$R_1 = \alpha_1 = \psi_{j1} + \gamma_1 + \phi_1 \qquad [3\text{-}36a]$$

$$R_2 = \alpha_2 = \psi_{j2} + \gamma_2 + \phi_2 \qquad [3\text{-}36b]$$

Substituting [3-2] and [3-4] into [3-36],

$$R_1 = -\left(\psi_{j1} - \frac{V}{A_v \cdot G} - \frac{f_3}{K_1} \right) \qquad [3\text{-}37a]$$

Note : The -'ve sign is due to the conjugate beam's definition of counter clockwise rotation as positive rotation (See Figure 13).

$$R_2 = \left(\psi_{j2} - \frac{V}{A_v \cdot G} - \frac{f_6}{K_2} \right) \qquad [3\text{-}37b]$$

But,

$$d_3 = \psi_{j1} = 1,$$

$$d_6 = \psi_{j2} = 0 \cdot$$

Therefore [3-37] becomes,

$$R_1 = -1 + \frac{V}{A_v \cdot G} + \frac{f_3}{K_1} \qquad [3\text{-}38a]$$

$$R_2 = -\frac{V}{A_v \cdot G} - \frac{f_6}{K_2} \qquad [3\text{-}38b]$$

From equilibrium of conjugate beam:

$$R_1 = -\frac{f_3 \cdot l}{3 \cdot EI} + \frac{f_6 \cdot l}{6 \cdot EI} \qquad [3\text{-}39a]$$

$$R_2 = \frac{f_6 \cdot l}{3 \cdot EI} - \frac{f_3 \cdot l}{6 \cdot EI} \qquad [3\text{-}39b]$$

Combining [2-38a] and [2-39a] and collecting terms,

$$f_3 \cdot \left(\frac{1}{K_1} + \frac{l}{3 \cdot EI} \right) = \frac{f_6 \cdot l}{6 \cdot EI} - \frac{V}{A_v \cdot G} + 1$$

Multiplying both sides by:

$$\frac{3 \cdot EI}{l}$$

and defining dimensionless constants as in [3-18], [3-19]:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1}$$

Resulting in:

$$\frac{f_3}{\nu_1} = \frac{f_6}{2} - \frac{V \cdot g \cdot l}{4} + \frac{3 \cdot EI}{l} \qquad [3\text{-}40]$$

Similarly, combining [3-38b] and [3-39b] and collecting terms,

$$-f_6 \cdot \left(\frac{1}{K_2} + \frac{l}{3 \cdot EI} \right) = -\frac{f_3 \cdot L}{6 \cdot EI} + \frac{V}{A_v \cdot G}$$

Multiplying both sides by:

$$\frac{3 \cdot EI}{l}$$

and defining dimensionless constants as in [3-18], [3-21]:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l} \right)^{-1}$$

Resulting in:

$$\frac{f_6}{\nu_2} = \frac{f_3}{2} - \frac{V \cdot g \cdot l}{4} \qquad [3\text{-}41]$$

From equilibrium of real beam:

$$V = \frac{f_3 + f_6}{l} \qquad [3\text{-}42]$$

Substituting [2-42] into [2-40],[2-41] and collecting terms,

$$f_3 = \left(\frac{4 \cdot \nu_1}{4 + \nu_1 \cdot g}\right) \cdot \left(\frac{3 \cdot EI}{l} + \frac{F_6}{4} \cdot (2 - g)\right) \qquad [3\text{-}43]$$

$$f_6 = -\frac{\nu_2 \cdot f_3}{4 + \nu_2 \cdot g)} \cdot (2 - g) \qquad [3\text{-}44]$$

Substituting [3-44] into [3-43],

$$f_3 = \left(\frac{4 \cdot \nu_1}{4 + \nu_1 \cdot g}\right) \cdot \left(\frac{3 \cdot EI}{l} + \frac{\nu_2 \cdot f_3}{4 + \nu_2 \cdot g} \cdot (2 - g) \cdot \frac{(2 - g)}{4}\right)$$

Collecting terms,

$$f_3 \cdot \left(\frac{(4 + \nu_1 \cdot g) \cdot (4 + \nu_2 \cdot g) - \nu_1 \cdot \nu_2 \cdot (2 - g)^2}{4 \cdot \nu_1 \cdot (4 + \nu_2 \cdot g)}\right) = \frac{3 \cdot EI}{l}$$

or,

$$f_3 \cdot \left(1 + g \cdot \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}\right) = \frac{4 \cdot EI}{l} \cdot \left(\frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2}\right) \cdot (1 + \nu_2 \cdot g/4)$$

Defining dimensionless constants:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad [3\text{-}45]$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2} \qquad [3\text{-}46]$$

Therefore,

$$f_3 = \frac{4 \cdot EI}{l} \cdot C_3 \cdot \left(\frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1}\right) \qquad [3\text{-}47]$$

Similarly, substituting [2-43] into [2-44] and collecting terms:

$$f_6 \cdot \left(1 + g \cdot \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \right) = \frac{3 \cdot EI}{l} \cdot \frac{\nu_1 \cdot \nu_2 \cdot 2 \cdot (1 - g/2)}{4 - \nu_1 \cdot \nu_2}$$

Defining dimensionless constants:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad [3\text{-}48]$$

Therefore,

$$f_6 = \frac{2 \cdot EI}{l} \cdot C_5 \cdot \left(\frac{1 - g/2}{1 + g \cdot C_1} \right) \qquad [3\text{-}49]$$

Substituting [2-28], [2-30] into [2-23],

$$V = \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot \left(\frac{1}{1 + g \cdot C_1} \right) \qquad [3\text{-}50]$$

Introducing dimensionless constants to [3-47], [3-49] and [3-50],

$$S_2 = \frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1} \qquad [3\text{-}51]$$

$$S_3 = \frac{1 - g/2}{1 + g \cdot C_1} \qquad [3\text{-}52]$$

f_3, f_6 and V now become:

$$f_3 = \frac{4 \cdot EI}{l} \cdot C_3 \cdot S_2 \qquad [3\text{-}53]$$

$$f_6 = \frac{2 \cdot EI}{l} \cdot C_5 \cdot S_3 \qquad [3\text{-}54]$$

$$V = \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \qquad [3\text{-}55]$$

In summary the third column of $\{k_{b11}\}$ is:

$$\{k_{b11}\}_{i3} = \begin{pmatrix} 0 \\ \dfrac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ \dfrac{4 \cdot EI}{l} \cdot C_3 \cdot S_2 \\ 0 \\ -\dfrac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ \dfrac{2 \cdot EI}{l} \cdot C_5 \cdot S_3 \end{pmatrix}$$

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$S_1 = \frac{1}{1 + g \cdot C_1}$$

$$S_2 = \frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1}$$

$$S_3 = \frac{1 - g/2}{1 + g \cdot C_1}$$

$$S_4 = \frac{1 + \nu_1 \cdot g/4}{1 + g \cdot C_1}$$

The sixth column of $\{k_{b11}\}$ is obtained by setting d_6 equal to 1 , and it can be derived in a similar fashion as before. The result is shown below:

$$\{k_{b11}\}_{i6} = \begin{pmatrix} 0 \\ \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \\ \frac{2 \cdot EI}{l} \cdot C_5 \cdot S_3 \\ 0 \\ -\frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \\ \frac{4 \cdot EI}{l} \cdot C_6 \cdot S_4 \end{pmatrix}$$

Where

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$S_4 = \frac{1 + \nu_1 \cdot g/4}{1 + g \cdot C_1}$$

In summary:

$$\{k_{b11}\}_{ii} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 & \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ 0 & \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 & \frac{4 \cdot EI}{l} \cdot C_3 \cdot S_2 \end{pmatrix}$$

$$\{k_{b11}\}_{ij} == \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 & \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ 0 & -\frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 & \frac{2 \cdot EI}{l} \cdot C_5 \cdot S_3 \end{pmatrix}$$

$$\{k_{b11}\}_{ij} = \{k_{b11}\}_{ji}^T$$

$$\left\{k_{b11}\right\}_{jj}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{12\cdot EI}{l^3}\cdot C_1\cdot S_1 & -\dfrac{6\cdot EI}{l^2}\cdot C_4\cdot S_1 \\ 0 & -\dfrac{6\cdot EI}{l^2}\cdot C_4\cdot S_1 & \dfrac{4\cdot EI}{l}\cdot C_6\cdot S_4 \end{pmatrix}$$

where

$$C_1=\frac{\nu_1+\nu_2+\nu_1\cdot\nu_2}{4-\nu_1\cdot\nu_2}$$

$$C_2=\frac{\nu_1\cdot(2+\nu_2)}{4-\nu_1\cdot\nu_2}$$

$$C_3=\frac{3\cdot\nu_1}{4-\nu_1\cdot\nu_2}$$

$$C_4=\frac{\nu_2\cdot(2+\nu_1)}{4-\nu_1\cdot\nu_2}$$

$$C_5=\frac{3\cdot\nu_1\cdot\nu_2}{4-\nu_1\cdot\nu_2}$$

$$C_6=\frac{3\cdot\nu_2}{4-\nu_1\cdot\nu_2}$$

$$\nu_1=\left(1+\frac{3\cdot EI}{K_1\cdot l}\right)^{-1}$$

$$\nu_2=\left(1+\frac{3\cdot EI}{K_2\cdot l}\right)^{-1}$$

$$g=\frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_1=\frac{1}{1+g\cdot C_1}$$

$$S_2=\frac{1+\nu_2\cdot g/4}{1+g\cdot C_1}$$

$$S_3=\frac{1-g/2}{1+g\cdot C_1}$$

$$S_4=\frac{1+\nu_1\cdot g/4}{1+g\cdot C_1}$$

3.1.2 Introducing the Effect of Connection Sizes

The effect of connection sizes is introduced by means of transformation as outlined in [3-12]:

$$\{\overline{K}_{11}\} = \{T\}^T \cdot \{k_{11}\} \cdot \{T\}$$

Where

$\{k_{11}\}$	=	the local member stiffness matrix including the effect of flexible connections and shear deflections,
$\{T\}$	=	the transformation matrix defined by [3-9],
$\{\overline{K}_{11}\}$	=	the refined stiffness matrix in local coordinates.

The local member stiffness matrix $\{k_{11}\}$ comprises of $\{k_a\}$ and $\{k_{b11}\}$. $\{k_a\}$ is the standard pin-pin member matrix and $\{k_{b11}\}$ has just been presented in the previous section. The full $\{k_{11}\}$ matrix is shown in Figure 26.

$$\{k_{11}\}_{ii} = \begin{pmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 & \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ 0 & \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 & \frac{4 \cdot EI}{l} \cdot C_3 \cdot S_2 \end{pmatrix}$$

$$\{k_{11}\}_{ij} = \begin{pmatrix} -\frac{AE}{l} & 0 & 0 \\ 0 & -\frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 & \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot S_1 \\ 0 & -\frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 & \frac{2 \cdot EI}{l} \cdot C_5 \cdot S_3 \end{pmatrix}$$

$$\{k'_{11}\}_{ij} = \{k_{11}\}^T_{ji}$$

$$\{k_{11}\}_{jj} = \begin{pmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 & -\frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \\ 0 & -\frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 & \frac{4 \cdot EI}{l} \cdot C_6 \cdot S_4 \end{pmatrix}$$

Figure 26 The $\{k_{11}\}$ matrix

The local refined member stiffness matrix $\{\overline{K}_{11}\}$ is illustrated in Figure 27.

$$\{\overline{K}_{11}\}_{ii} = \begin{bmatrix} \dfrac{AE}{l} & 0 & 0 \\ 0 & \dfrac{12\cdot EI}{l^3}\cdot C_1\cdot S_1 & \begin{array}{l} \dfrac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1 \\ +\dfrac{6\cdot EI}{l^2}\cdot C_2\cdot S_1 \end{array} \\ 0 & \begin{array}{l} \dfrac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1 \\ +\dfrac{6\cdot EI}{l^2}\cdot C_2\cdot S_1 \end{array} & \begin{array}{l} \dfrac{4\cdot EI}{l}\cdot C_3\cdot S_2 \\ +\dfrac{12\cdot EI\cdot l_1}{l^2}\cdot C_2\cdot S_1 \\ +\dfrac{12\cdot EI}{l^3}\cdot \; l_1\ ^2 \cdot C_1\cdot S_1 \end{array} \end{bmatrix}$$

$$\{\overline{K}_{11}\}_{ij} = \begin{bmatrix} -\dfrac{AE}{l} & 0 & 0 \\ 0 & -\dfrac{12\cdot EI}{l^3}\cdot C_1\cdot S_1 & \begin{array}{l} \dfrac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1 \\ +\dfrac{6\cdot EI}{l^2}\cdot C_4\cdot S_1 \end{array} \\ 0 & \begin{array}{l} -\dfrac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1 \\ -\dfrac{6\cdot EI}{l^2}\cdot C_2\cdot S_1 \end{array} & \begin{array}{l} \dfrac{2\cdot EI}{l}\cdot C_5\cdot S_3 \\ +\dfrac{6\cdot EI\cdot l_1}{l^2}\cdot C_4\cdot S_1 \\ +\dfrac{6\cdot EI\cdot l_2}{l^2}\cdot C_2\cdot S_1 \\ +\dfrac{12\cdot EI\cdot l_1\cdot l_2}{l^3}\cdot C_1\cdot S_1 \end{array} \end{bmatrix}$$

$$\{\overline{K}_{11}\}_{ij} = \{\overline{K}_{11}\}_{ji}^{T}$$

Figure 27 The Local Refined Member Stiffness Matrix of Fix-Fix Members

$$
\{K_{11}\}_{jj} = \begin{bmatrix}
\frac{AE}{l} & 0 & 0 \\
0 & \frac{12 \cdot EI}{l^3} \cdot C_1 \cdot S_1 & -\frac{12 \cdot EI \cdot l_2}{l^3} \cdot C_1 \cdot S_1 - \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 \\
0 & -\frac{12 \cdot EI \cdot l_2}{l^3} \cdot C_1 \cdot S_1 - \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot S_1 & \frac{4 \cdot EI}{l} \cdot C_6 \cdot S_4 + \frac{12 \cdot EI \cdot l_2}{l^2} \cdot C_4 \cdot S_1 + \frac{12 \cdot EI}{l^3} \cdot l_2^{\;2} \cdot C_1 \cdot S_1
\end{bmatrix}
$$

Figure 27(cont'd) The Local Refined Member Stiffness Matrix of Fix-Fix Members

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$S_1 = \frac{1}{1 + g \cdot C_1}$$

$$S_2 = \frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1}$$

$$S_3 = \frac{1 - g/2}{1 + g \cdot C_1}$$

$$S_4 = \frac{1 + \nu_1 \cdot g/4}{1 + g \cdot C_1}$$

l_1, l_2 = length of connection

l = length of member

L = total span = $l_1 + l + l_2$

3.1.3 Transforming to Glocal Coordinates

Finally, the global refined member stiffness matrix $\{K_{11}\}$ is obtained by means of rotational transformation as outlined by [3-13]:

$$\{K_{11}\} = \{T_r\}^T \cdot \{\overline{K}_{11}\} \cdot \{T_r\}$$

$\{K_{11}\}$ is illustrated in Figure 28.

$$\{K_{11}\}_{ii} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c\cdot s & -\frac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1\cdot s - \frac{6\cdot EI}{l^2}\cdot C_2\cdot S_1\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c^2 & \frac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1\cdot c + \frac{6\cdot EI}{l^2}\cdot C_2\cdot S_1\cdot c \\ -\frac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1\cdot s - \frac{6\cdot EI}{l^2}\cdot C_2\cdot S_1\cdot s & \frac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1\cdot c + \frac{6\cdot EI}{l^2}\cdot C_2\cdot S_1\cdot c & \frac{4\cdot EI}{l}\cdot C_3\cdot S_2 + \frac{12\cdot EI\cdot l_1}{l^2}\cdot C_2\cdot S_1 + \frac{12\cdot EI}{l^3}\cdot l_1^{\,2}\cdot C_1\cdot S_1 \end{bmatrix}$$

Figure 28 The $\{K_{11}\}$ matrix

$$
\{K_{11}\}_{ij} = \begin{bmatrix}
-\frac{AE}{l}\cdot c^2 - \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot s^2 & -\frac{AE}{l}\cdot c\cdot s + \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c\cdot s & -\frac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1\cdot s - \frac{6\cdot EI}{l^2}\cdot C_4\cdot S_1\cdot s \\
-\frac{AE}{l}\cdot c\cdot s + \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c\cdot s & -\frac{AE}{l}\cdot s^2 - \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c^2 & \frac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1\cdot c + \frac{6\cdot EI}{l^2}\cdot C_4\cdot S_1\cdot c \\
\frac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1\cdot s + \frac{6\cdot EI}{l^2}\cdot C_2\cdot S_1\cdot s & -\frac{12\cdot EI\cdot l_1}{l^3}\cdot C_1\cdot S_1\cdot c - \frac{6\cdot EI}{l^2}\cdot C_2\cdot S_1\cdot c & \frac{2\cdot EI}{l}\cdot C_5\cdot S_3 + \frac{6\cdot EI\cdot l_1}{l^2}\cdot C_4\cdot S_1 + \frac{6\cdot EI\cdot l_2}{l^2}\cdot C_2\cdot S_1 + \frac{12\cdot EI\cdot l_1\cdot l_2}{l^3}\cdot C_1\cdot S_1
\end{bmatrix}
$$

$$\{K_{11}\}_{ij} = \{K_{11}\}_{ji}^T$$

Figure 28(cont'd) The $\{K_{11}\}$ matrix

$$
\{K_{11}\}_{jj} = \begin{bmatrix}
\frac{AE}{l}\cdot c^2 + \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c\cdot s & \frac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1\cdot s + \frac{6\cdot EI}{l^2}\cdot C_4\cdot S_1\cdot s \\
\frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{12\cdot EI}{l^3}\cdot C_1\cdot S_1\cdot c^2 & -\frac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1\cdot c - \frac{6\cdot EI}{l^2}\cdot C_4\cdot S_1\cdot c \\
\frac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1\cdot s + \frac{6\cdot EI}{l^2}\cdot C_4\cdot S_1\cdot s & -\frac{12\cdot EI\cdot l_2}{l^3}\cdot C_1\cdot S_1\cdot c - \frac{6\cdot EI}{l^2}\cdot C_4\cdot S_1\cdot c & \frac{4\cdot EI}{l}\cdot C_6\cdot S_4 + \frac{12\cdot EI\cdot l_2}{l^2}\cdot C_4\cdot S_1 + \frac{12\cdot EI}{l^3}\cdot l_2^{\,2}\cdot C_1\cdot S_1
\end{bmatrix}
$$

Figure 28(cont'd) The $\{K_{11}\}$ matrix

where

$$C_1 = \frac{\nu_1+\nu_2+\nu_1\cdot\nu_2}{4-\nu_1\cdot\nu_2}$$

$$C_2 = \frac{\nu_1\cdot(2+\nu_2)}{4-\nu_1\cdot\nu_2}$$

$$C_3 = \frac{3\cdot\nu_1}{4-\nu_1\cdot\nu_2}$$

$$C_4 = \frac{\nu_2\cdot(2+\nu_1)}{4-\nu_1\cdot\nu_2}$$

$$C_5 = \frac{3\cdot\nu_1\cdot\nu_2}{4-\nu_1\cdot\nu_2}$$

$$C_6 = \frac{3\cdot\nu_2}{4-\nu_1\cdot\nu_2}$$

$$\nu_1 = \left(1+\frac{3\cdot EI}{K_1\cdot l}\right)^{-1}$$

$$\nu_2 = \left(1+\frac{3\cdot EI}{K_2\cdot l}\right)^{-1}$$

$$g = \frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_1 = \frac{1}{1+g\cdot C_1}$$

$$S_2 = \frac{1+\nu_2\cdot g/4}{1+g\cdot C_1}$$

$$S_3 = \frac{1-g/2}{1+g\cdot C_1}$$

$$S_4 = \frac{1+\nu_1\cdot g/4}{1+g\cdot C_1}$$

Note :

l_1, l_2 = length of connection (rigid end pieces),

l = length of member,

L = total span = $l_1 + l + l_2$,

Δx = change in x form joint 1 to joint 2,

Δy = change in y from joint 1 to joint 2,

c = $\cos(\Delta x/L)$,

s = $\sin(\Delta y/L)$.

3.1.4 Verifying the Refined Fix-Fix Member Stiffness Matrix

3.1.4.1 Morforton and Wu's Derivation

Morforton and Wu [7] also used the conjugate beam method to derive the local stiffness matrix for flexibly connected members but neglecting the effect of connection sizes and shear deflection (Figure 29). The stiffness matrix $\{K_{11}\}$ presented in the previous section should reduce to that of Morforton and Wu if the effect of connection sizes and shear deflection are eliminated. This is done by setting:

$$l_1, l_2 = 0,$$
$$g = 0.$$

and considering local coordinates:

$$c = 1,$$
$$s = 0.$$

The dimensionless constants now become:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad \nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2} \qquad \nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2} \qquad S_1 = \frac{1}{1 + g \cdot C_1} = 1$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2} \qquad S_2 = \frac{1 + \nu_2 \cdot (g/4)}{1 + g \cdot C_1} = 1$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad S_3 = \frac{1 - g/2}{1 + g \cdot C_1} = 1$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad S_4 = \frac{1 + \nu_1 \cdot (g/4)}{1 + g \cdot C_1} = 1$$

One sees that the stiffness matrix presented in the previous section reduces to that of Morforton and Wu.

$$\{k_{11}\}_{ii} = \begin{pmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{12 \cdot EI}{l^3} \cdot C_1 & \frac{6 \cdot EI}{l^2} \cdot C_2 \\ 0 & \frac{6 \cdot EI}{l^2} \cdot C_2 & \frac{4 \cdot EI}{l} \cdot C_3 \end{pmatrix}$$

$$\{k_{11}\}_{ij} = \begin{pmatrix} -\frac{AE}{l} & 0 & 0 \\ 0 & -\frac{12 \cdot EI}{l^3} \cdot C_1 & \frac{6 \cdot EI}{l^2} \cdot C_2 \\ 0 & -\frac{6 \cdot EI}{l^2} \cdot C_4 & \frac{2 \cdot EI}{l} \cdot C_5 \end{pmatrix}$$

$$\{k_{11}\}_{ij} = \{k_{11}\}_{ji}^{T}$$

$$\{k_{11}\}_{jj} = \begin{pmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{12 \cdot EI}{l^3} \cdot C_1 & -\frac{6 \cdot EI}{l^2} \cdot C_4 \\ 0 & -\frac{6 \cdot EI}{l^2} \cdot C_4 & \frac{4 \cdot EI}{l} \cdot C_6 \end{pmatrix}$$

Figure 29 Morforton and Wu's $\{k_{11}\}$ Matrix

Where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} \qquad \nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2} \qquad \nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

3.1.4.2 The Stiffness Matrix of Members with Rigid Ends

Another useful verification of $\{K_{11}\}$ is to compare it with the stiffness matrix of a member with rigid ends (Figure 30). This can be done simply by setting:

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1} = 1$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1} = 1$$

The dimensionless constants now become:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} = 1 \qquad g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_1 = \frac{1}{1 + g}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_2 = \frac{1 + g/4}{1 + g}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_3 = \frac{1 - g/2}{1 + g}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_4 = \frac{1 + g/4}{1 + g}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} = 1$$

One sees that $\{K_{11}\}$ correctly reduces to the stiffness matrix of members with rigid ends.

$$\{K_{11}\}_{ii} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & -\frac{12\cdot EI\cdot l_1}{l^3}\cdot S_1\cdot s - \frac{6\cdot EI}{l^2}\cdot S_1\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c^2 & \frac{12\cdot EI\cdot l_1}{l^3}\cdot S_1\cdot c + \frac{6\cdot EI}{l^2}\cdot S_1\cdot c \\ -\frac{12\cdot EI\cdot l_1}{l^3}\cdot S_1\cdot s - \frac{6\cdot EI}{l^2}\cdot S_1\cdot s & \frac{12\cdot EI\cdot l_1}{l^3}\cdot S_1\cdot c + \frac{6\cdot EI}{l^2}\cdot C_2\cdot c & \frac{4\cdot EI}{l}\cdot S_2 + \frac{12\cdot EI\cdot l_1}{l^2}\cdot S_1 + \frac{12\cdot EI}{l^3}\cdot l_1{}^2\cdot S_1 \end{bmatrix}$$

$$\{K_{11}\}_{ij} = \begin{bmatrix} -\frac{AE}{l}\cdot c^2 - \frac{12\cdot EI}{l^3}\cdot S_1\cdot s^2 & -\frac{AE}{l}\cdot c\cdot s + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & -\frac{12\cdot EI\cdot l_2}{l^3}\cdot S_1\cdot s - \frac{6\cdot EI}{l^2}\cdot S_1\cdot s \\ -\frac{AE}{l}\cdot c\cdot s + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & -\frac{AE}{l}\cdot s^2 - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c^2 & \frac{12\cdot EI\cdot l_2}{l^3}\cdot S_1\cdot c + \frac{6\cdot EI}{l^2}\cdot S_1\cdot c \\ \frac{12\cdot EI\cdot l_1}{l^3}\cdot S_1\cdot s + \frac{6\cdot EI}{l^2}\cdot S_1\cdot s & -\frac{12\cdot EI\cdot l_1}{l^3}\cdot S_1\cdot c - \frac{6\cdot EI}{l^2}\cdot S_1\cdot c & \frac{2\cdot EI}{l}\cdot S_3 + \frac{6\cdot EI\cdot l_1}{l^2}\cdot S_1 + \frac{6\cdot EI\cdot l_2}{l^2}\cdot S_1 + \frac{12\cdot EI\cdot l_1\,l_2}{l^3}\cdot S_1 \end{bmatrix}$$

Figure 30 The $\{K_{11}\}$ Matrix of Members with Rigid Ends

$$\{K_{11}\}_{ij} = \{K_{11}\}_{ji}^T$$

$$\{K_{11}\}_{jj} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & \frac{12\cdot EI\cdot l_2}{l^3}\cdot S_1\cdot s + \frac{6\cdot EI}{l^2}\cdot S_1\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c^2 & -\frac{12\cdot EI\cdot l_2}{l^3}\cdot S_1\cdot c - \frac{6\cdot EI}{l^2}\cdot S_1\cdot c \\ \frac{12\cdot EI\cdot l_2}{l^3}\cdot S_1\cdot s + \frac{6\cdot EI}{l^2}\cdot S_1\cdot s & -\frac{12\cdot EI\cdot l_2}{l^3}\cdot S_1\cdot c - \frac{6\cdot EI}{l^2}\cdot S_1\cdot c & \frac{4\cdot EI}{l}\cdot S_4 + \frac{12\cdot EI\cdot l_2}{l^2}\cdot S_1 + \frac{12\cdot EI}{l^3}\cdot l_2^{\,2}\cdot S_1 \end{bmatrix}$$

Figure 30(cont'd) The (K_{11}) Matrix of Members with Rigid Ends

where

$$g = \frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_1 = \frac{1}{1+g}$$

$$S_2 = \frac{1+g/4}{1+g}$$

$$S_3 = \frac{1-g/2}{1+g}$$

$$S_4 = \frac{1+g/4}{1+g}$$

3.1.4.3 The Conventional Fix-Fix Member Stiffness Matrix

One final check is to compare the derived $\{K_{11}\}$ with the conventional $\{K_{11}\}$ (Figure 31). This can be done simply by setting:

$$l_1, l_2 = 0,$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1} = 1$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1} = 1$$

The dimensionless constants now become:

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} = 1 \qquad g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_1 = \frac{1}{1 + g}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_2 = \frac{1 + g/4}{1 + g}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_3 = \frac{1 - g/2}{1 + g}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} = 1 \qquad S_4 = \frac{1 + g/4}{1 + g}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2} = 1$$

One sees that $\{K_{11}\}$ correctly reduces to the conventional member stiffness matrix.

$$\{K_{11}\}_{ii} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & -\frac{6\cdot EI}{l^2}\cdot S_1\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c^2 & \frac{6\cdot EI}{l^2}\cdot S_1\cdot c \\ -\frac{6\cdot EI}{l^2}\cdot S_1\cdot s & \frac{6\cdot EI}{l^2}\cdot S_1\cdot c & \frac{4\cdot EI}{l}\cdot S_2 \end{bmatrix}$$

$$\{K_{11}\}_{ij} = \begin{bmatrix} -\frac{AE}{l}\cdot c^2 - \frac{12\cdot EI}{l^3}\cdot S_1\cdot s^2 & -\frac{AE}{l}\cdot c\cdot s + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & -\frac{6\cdot EI}{l^2}\cdot S_1\cdot s \\ -\frac{AE}{l}\cdot c\cdot s + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & -\frac{AE}{l}\cdot s^2 - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c^2 & \frac{6\cdot EI}{l^2}\cdot S_1\cdot c \\ \frac{6\cdot EI}{l^2}\cdot S_1\cdot s & -\frac{6\cdot EI}{l^2}\cdot S_1\cdot c & \frac{2\cdot EI}{l}\cdot S_3 \end{bmatrix}$$

$$\{K_{11}\}_{ij} = \{K_{11}\}_{ji}^{T}$$

Figure 31 The Conventional $\{K_{11}\}$ Matrix

$$\{K_{11}\}_{jj} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & \frac{6\cdot EI}{l^2}\cdot S_1\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{12\cdot EI}{l^3}\cdot S_1\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{12\cdot EI}{l^3}\cdot S_1\cdot c^2 & -\frac{6\cdot EI}{l^2}\cdot S_1\cdot c \\ \frac{6\cdot EI}{l^2}\cdot S_1\cdot s & -\frac{6\cdot EI}{l^2}\cdot S_1\cdot c & \frac{4\cdot EI}{l}\cdot S_4 \end{bmatrix}$$

Figure 31(cont'd) The Conventional $\{K_{11}\}$ Matrix

where

$$g = \frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_1 = \frac{1}{1+g}$$

$$S_2 = \frac{1+g/4}{1+g}$$

$$S_3 = \frac{1-g/2}{1+g}$$

$$S_4 = \frac{1+g/4}{1+g}$$

3.2 Assembling the Refined Fix-Pin Member Stiffness Matrix

3.2.1 The Local Bending Stiffness Matrix of Flexibly Connected Fix-Pin Members

The derivation of $\{k_{b10}\}$ is carried out in the same manner as in the previous section. The first and fourth column of $\{k_{b10}\}$ are zero vectors because axial displacements do not affect $\{k_{b10}\}$.

The second column of $\{k_{b10}\}$ is obtained by setting d_2 equal to 1 (Figure 32a). This is a 1° indeterminate system. If the fixed-end moment, f_3 is chosen to be the redundant, the conjugate beam will be loaded as shown in Figure 32b.

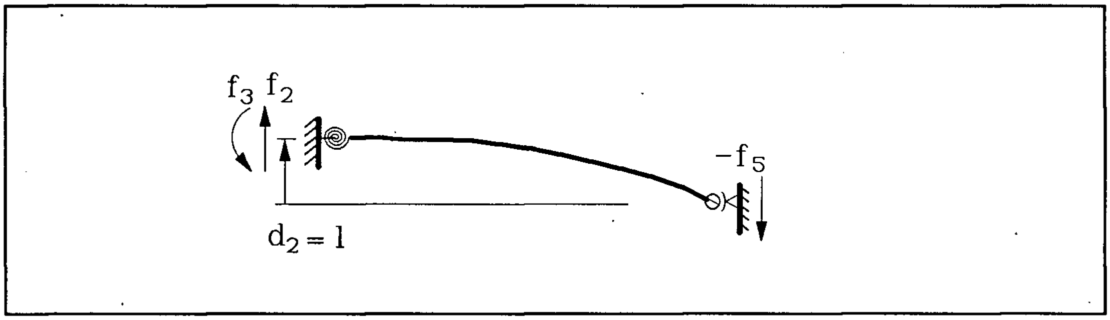


Figure 32a Deriving the Second Column of $\{k_{b10}\}$

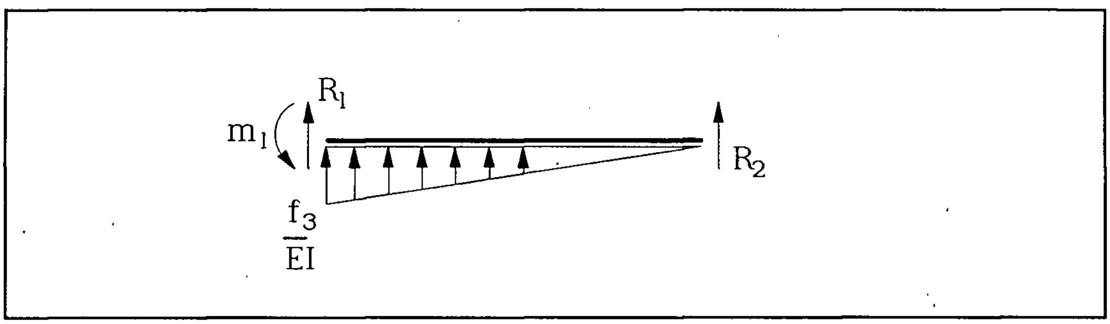


Figure 32b Conjugate Beam Load : Fix-Pin Member, d_2=1

Because the shears at the ends of the conjugate beam are equal to the net angle changes at the ends of the actual beam, one could write from [3-6]:

$$R_1 = \alpha_1 = \psi_{j1} + \gamma_1 + \phi_1 \qquad [3\text{-}56]$$

Also, the moment m_1 of the conjugate beam equals to the deflection d_2 of the real beam. Therefore,

$$m_1 = 1 \qquad [3\text{-}57]$$

Substituting [3-2] and [3-4] into [3-56],

$$R_1 = -\left(\psi_{j1} - \frac{V}{A_v \cdot G} - \frac{f_3}{K_1} \right) \qquad [3\text{-}58]$$

Note : The -'ve sign is due to the conjugate beam's definition of counter clockwise rotation as positive rotation (See Figure 13).

But,

$$d_3 = \psi_{j1} = 0,$$

Therefore [3-58] becomes,

$$R_1 = \frac{V}{A_v \cdot G} + \frac{f_3}{K_1} \qquad [3\text{-}59]$$

From equilibrium of conjugate beam:

$$R_1 = -\frac{f_3 \cdot l}{3 \cdot EI} + \frac{m_1}{l} \qquad [3\text{-}60]$$

where

$$m_1 = 1 \qquad \text{from } [3\text{-}57]$$

Combining [3-59] and [3-60] and collecting terms,

$$f_3 \cdot \left(\frac{1}{K_1} + \frac{l}{3 \cdot EI} \right) = -\frac{V}{A_v \cdot G} + \frac{1}{l}$$

Multiplying both sides by:

$$\frac{3 \cdot EI}{l}$$

and defining dimensionless constants:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2} \qquad [3\text{-}61]$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1} \qquad [3\text{-}62]$$

Resulting in:

$$\frac{f_3}{\nu_1} = -\frac{V \cdot g \cdot l}{4} + \frac{3 \cdot EI}{l^2} \qquad [3\text{-}63]$$

From equilibrium of real beam:

$$V = \frac{f_3}{l} \qquad [3\text{-}64]$$

Substituting [3-64] into [3-63] and collecting terms

$$f_3 = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right) \qquad [3\text{-}65]$$

Substituting [3-65] into [3-64],

$$V = \frac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right) \qquad [3\text{-}66]$$

Introducing a dimensionless constant to [3-65] and [3-66],

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4} \qquad [3\text{-}67]$$

f_3, and V now become:

$$f_3 = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 \qquad [3\text{-}68]$$

$$V = \frac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 \qquad [3\text{-}69]$$

In summary the second column of $\{k_{b10}\}$ is:

$$\{k_{b10}\}_{i2} = \begin{pmatrix} 0 \\ \frac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 \\ \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 \\ 0 \\ -\frac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 \\ 0 \end{pmatrix}$$

where

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

The fifth column of $\{k_{b10}\}$ is obtained by setting d_5 equal to 1 (Figure 33), and it can be derived in a similar fashion as before.

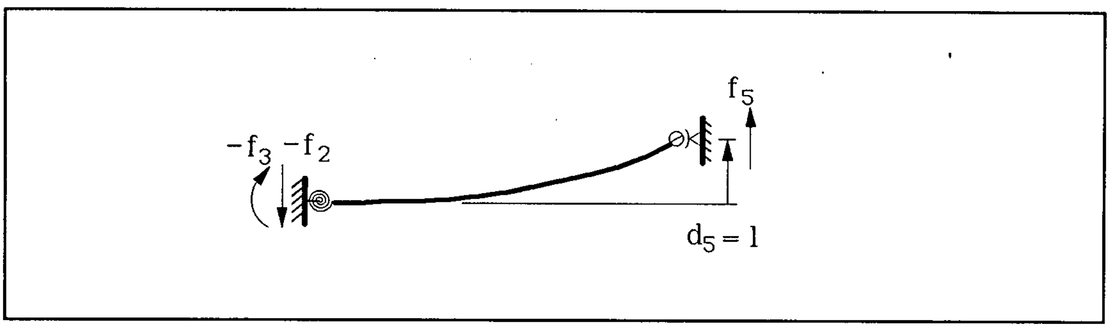


Figure 33 Deriving the Fifth Column of $\{k_{b10}\}$

But noting that Figure 33 is a mirror reflection of Figure 32, the column vector can be written directly as:

$$\{k_{b10}\}_{i5} = \begin{pmatrix} 0 \\ -\frac{3 \cdot EI}{l^3} \cdot v_1 \cdot S_5 \\ -\frac{3 \cdot EI}{l^2} \cdot v_1 \cdot S_5 \\ 0 \\ \frac{3 \cdot EI}{l^3} \cdot v_1 \cdot S_5 \\ 0 \end{pmatrix}$$

The third column of $\{k_{b10}\}$ is obtained by setting d_3 equal to 1 (Figure 34a). Once again this is a 1° indeterminate system. If the fixed-end moment, f_6 is chosen to be the redundant, the conjugate beam will be loaded as shown in Figure 34b.

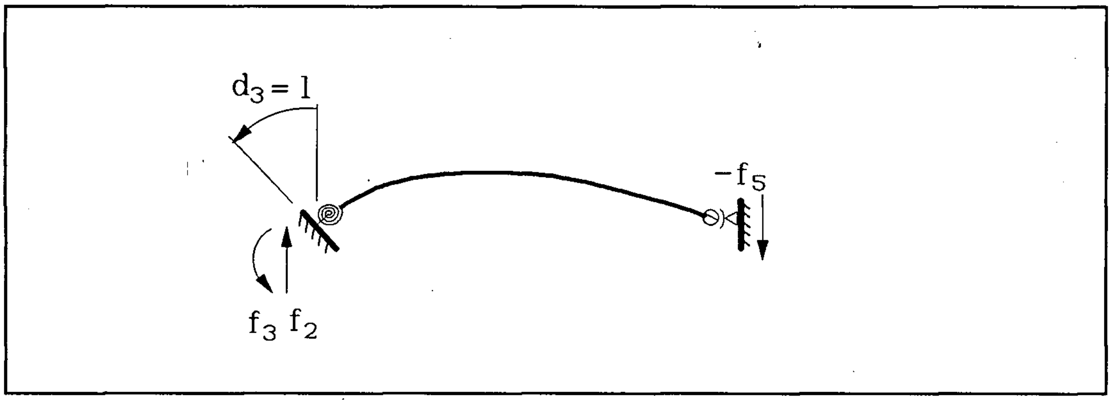


Figure 34a Deriving the Third Column of $\{k_{b10}\}$

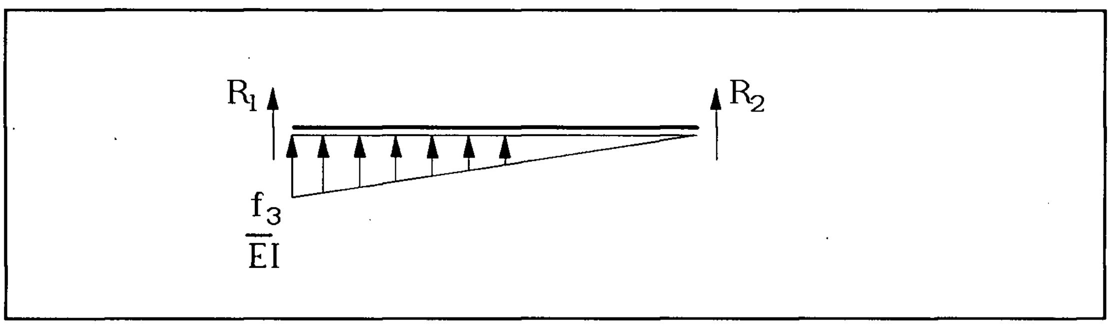


Figure 34b Conjugate Beam Load : Fix-Pin Member, d_3=1

Because the shears at the ends of the conjugate beam are equal to the net angle changes at the ends of the actual beam, one could write:

$$R_1 = \alpha_1 = \psi_{j1} + \gamma_1 + \phi_1 \qquad [3\text{-}70]$$

Substituting [3-2] and [3-4] into [3-70],

$$R_1 = -\left(\psi_{j1} - \frac{V}{A_v \cdot G} - \frac{f_6}{K_1} \right) \qquad [3\text{-}71]$$

Note : The -'ve sign is due to the conjugate beam's definition of counter clockwise rotation as positive rotation (See Figure 13).

But,

$$d_3 = \psi_{j1} = 1\,,$$

Therefore [2-71] becomes,

$$R_1 = -1 + \frac{V}{A_v \cdot G} + \frac{f_3}{K_1} \qquad [3\text{-}72]$$

From equilibrium of conjugate beam:

$$R_1 = -\frac{f_3 \cdot l}{3 \cdot EI} \qquad [3\text{-}73]$$

Combining [3-72] and [3-73] and collecting terms,

$$f_3 \cdot \left(\frac{1}{K_1} + \frac{l}{3 \cdot EI} \right) = -\frac{V}{A_v \cdot G} + 1$$

Multiplying both sides by:

$$\frac{3 \cdot EI}{l}$$

and defining dimensionless constants as in [3-61], [3-62]:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

Resulting in:

$$\frac{f_3}{\nu_1} = -\frac{V \cdot g \cdot l}{4} + \frac{3 \cdot EI}{l} \qquad [3\text{-}74]$$

From equilibrium of real beam:

$$V = \frac{f_3}{l} \qquad [3\text{-}75]$$

Substituting [3-75] into [3-74] and collecting terms,

$$f_3 = \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4}\right) \qquad [3\text{-}76]$$

Substituting [3-74] into [3-75],

$$V = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4}\right) \qquad [3\text{-}77]$$

Introducing a dimensionless constant as in [3-65] and [3-66],

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4}$$

f_3, and V now become:

$$f_3 = \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot S_5 \qquad [3\text{-}78]$$

$$V = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 \qquad [3\text{-}79]$$

In summary the third column of $\{k_{b10}\}$ is:

$$\{k_{b10}\}_{i3} = \begin{pmatrix} 0 \\ \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 \\ \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot S_5 \\ 0 \\ -\frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 \\ 0 \end{pmatrix}$$

where

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

The sixth column of $\{k_{b10}\}$ is a zero vector because of the pin connection.

In summary:

$$\{k_{b10}\}_{ii} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 & \dfrac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 \\ 0 & \dfrac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 & \dfrac{3 \cdot EI}{l} \cdot \nu_1 \cdot S_5 \end{pmatrix}$$

$$\{k_{b10}\}_{ij} == \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 & 0 \\ 0 & -\dfrac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot S_5 & 0 \end{pmatrix}$$

$$\{k_{b10}\}_{ij} = \{k_{b11}\}_{ji}^{T}$$

$$\{k_{b10}\}_{jj} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\dfrac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4}$$

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

3.2.2 Introducing the Effect of Connection Sizes

The effect of connection sizes is introduced by means of transformation as outlined in [3-12]:

$$\{\overline{K}_{10}\} = \{T\}^T \cdot \{k_{10}\} \cdot \{T\}$$

Where

$\{k_{10}\}$ = the local member stiffness matrix including the effect of flexible connections and shear deflections,

$\{T\}$ = the transformation matrix defined by [3-9],

$\{\overline{K}_{10}\}$ = the refined stiffness matrix in local coordinates.

The local member stiffness matrix $\{k_{b10}\}$ comprises of $\{k_a\}$ and $\{k_{b10}\}$. $\{k_a\}$ is the standard pin-pin member matrix and $\{k_{b10}\}$ has just been presented in the previous section. The full $\{k_{10}\}$ matrix is shown in Figure 35.

$$\{k_{10}\}_{ii} = \begin{pmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{3\cdot EI}{l^3}\cdot v_1 \cdot S_5 & \frac{3\cdot EI}{l^2}\cdot v_1 \cdot S_5 \\ 0 & \frac{3\cdot EI}{l^2}\cdot v_1 \cdot S_5 & \frac{3\cdot EI}{l}\cdot v_1 \cdot S_5 \end{pmatrix}$$

$$\{k_{11}\}_{ij} = \begin{pmatrix} -\frac{AE}{l} & 0 & 0 \\ 0 & -\frac{3\cdot EI}{l^3}\cdot v_1 \cdot S_5 & 0 \\ 0 & -\frac{3\cdot EI}{l^2}\cdot v_1 \cdot S_5 & 0 \end{pmatrix}$$

$$\{k_{11}\}_{ij} = \{k_{11}\}_{ji}^T$$

$$\{k_{11}\}_{jj} = \begin{pmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & -\frac{3\cdot EI}{l^3}\cdot v_1 \cdot S_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 35 The $\{k_{10}\}$ matrix

The local refined member stiffness matrix $\{\overline{K}_{10}\}$ is illustrated in Figure 36.

$$\{\overline{K}_{10}\}_{ii} = \begin{bmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{3\cdot EI}{l^3}\cdot v_1\cdot S_5 & \frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5 + \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5 \\ 0 & \frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5 + \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5 & \frac{3\cdot EI}{l}\cdot v_1\cdot S_5 + \frac{6\cdot EI\cdot l_1}{l^2}\cdot v_1\cdot S_5 + \frac{3\cdot EI}{l^3}\cdot l_1^{\,2}\cdot v_1\cdot S_5 \end{bmatrix}$$

$$\{\overline{K}_{10}\}_{ij} = \begin{bmatrix} -\frac{AE}{l} & 0 & 0 \\ 0 & -\frac{3\cdot EI}{l^3}\cdot v_1\cdot S_5 & \frac{3\cdot EI\cdot l_2}{l^3}\cdot v_1\cdot S_5 \\ 0 & -\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5 - \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5 & \frac{3\cdot EI\cdot l_2}{l^2}\cdot v_1\cdot S_5 + \frac{3\cdot EI\cdot l_1\cdot l_2}{l^3}\cdot v_1\cdot S_5 \end{bmatrix}$$

$$\{\overline{K}_{10}\}_{ij} = \{\overline{K}_{10}\}_{ji}^{T}$$

Figure 36 The Local Refined Member Stiffness Matrix of Fix-Pin Members

$$\{K_{10}\}_{jj} = \begin{bmatrix} \frac{AE}{l} & 0 & 0 \\ 0 & \frac{3 \cdot EI}{l^3} \cdot \nu_1 \cdot S_5 & -\frac{3 \cdot EI \cdot l_2}{l^3} \cdot \nu_1 \cdot S_1 \\ 0 & -\frac{3 \cdot EI \cdot l_2}{l^3} \cdot \nu_1 \cdot S_5 & \frac{3 \cdot EI}{l^3} \cdot l_2^{\ 2} \cdot \nu_1 \cdot S_5 \end{bmatrix}$$

Figure 36(cont'd) The Local Refined Member Stiffness Matrix of Fix-Pin Members

where

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4}$$

l_1, l_2 = length of connection

l = length of member

L = total span = $l_1 + l + l_2$

3.2.3 Transforming to Global Coordinates

Finally, the refined member stiffness matrix $\{K_{10}\}$ in global coordinates is obtained by means of rotational transformation as outlined by [3-13]:

$$\{K_{10}\} = \{T_r\}^T \cdot \{\bar{K}_{10}\} \cdot \{T_r\}$$

$\{K_{10}\}$ is illustrated in Figure 37.

$$\{K_{10}\}_{ii} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{3\cdot EI}{l^3}\cdot v_1\cdot S_5\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot v_1\cdot S_5\cdot c\cdot s & -\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5\cdot s - \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot v_1\cdot S_5\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{3\cdot EI}{l^3}\cdot v_1\cdot S_5\cdot c^2 & \frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5\cdot c + \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5\cdot c \\ -\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5\cdot s - \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5\cdot s & \frac{3\cdot EI\cdot l_1}{l^3}\cdot v_1\cdot S_5\cdot c + \frac{3\cdot EI}{l^2}\cdot v_1\cdot S_5\cdot c & \frac{3\cdot EI}{l}\cdot v_1\cdot S_5 + \frac{6\cdot EI\cdot l_1}{l^2}\cdot v_1\cdot S_5 + \frac{3\cdot EI}{l^3}\cdot l_1^{\,2}\cdot v_1\cdot S_5 \end{bmatrix}$$

Figure 37 The $\{K_{10}\}$ Matrix

$$\{K_{10}\}_{ij} = \begin{bmatrix} -\frac{AE}{l}\cdot c^2 - \frac{3\cdot EI}{l^3}\cdot \nu_1 \cdot S_5 \cdot s^2 & -\frac{AE}{l}\cdot c \cdot s + \frac{3\cdot EI}{l^3}\cdot \nu_1 \cdot S_5 \cdot c \cdot s & -\frac{3\cdot EI\cdot l_2}{l^3}\cdot \nu_1 \cdot S_5 \cdot s \\ -\frac{AE}{l}\cdot c \cdot s + \frac{3\cdot EI}{l^3}\cdot \nu_1 \cdot S_5 \cdot c \cdot s & -\frac{AE}{l}\cdot s^2 - \frac{3\cdot EI}{l^3}\cdot \nu_1 \cdot S_5 \cdot c^2 & \frac{3\cdot EI\cdot l_2}{l^3}\cdot \nu_1 \cdot S_5 \cdot c \\ \frac{3\cdot EI\cdot l_1}{l^3}\cdot \nu_1 \cdot S_5 \cdot s + \frac{3\cdot EI}{l^2}\cdot \nu_1 \cdot S_5 \cdot s & -\frac{3\cdot EI\cdot l_1}{l^3}\cdot \nu_1 \cdot S_5 \cdot c - \frac{3\cdot EI}{l^2}\cdot \nu_1 \cdot S_5 \cdot c & \frac{3\cdot EI\cdot l_2}{l^2}\cdot \nu_1 \cdot S_5 + \frac{3\cdot EI\cdot l_1\ l_2}{l^3}\cdot \nu_1 \cdot S_5 \end{bmatrix}$$

$$\{K_{10}\}_{ij} = \{K_{10}\}_{ji}^T$$

Figure 37(cont'd) The $\{K_{10}\}$ Matrix

$$
\{K_{10}\}_{jj} = \begin{bmatrix}
\frac{AE}{l}\cdot c^2 + \frac{3\cdot EI}{l^3}\cdot\nu_1\cdot S_5\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot\nu_1\cdot S_5\cdot c\cdot s & \frac{3\cdot EI\cdot l_2}{l^3}\cdot\nu_1\cdot S_5\cdot s \\
\frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot\nu_1\cdot S_5\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{3\cdot EI}{l^3}\cdot\nu_1\cdot S_5\cdot c^2 & -\frac{3\cdot EI\cdot l_2}{l^3}\cdot\nu_1\cdot S_5\cdot c \\
\frac{3\cdot EI\cdot l_2}{l^3}\cdot\nu_1\cdot S_5\cdot s & -\frac{3\cdot EI\cdot l_2}{l^3}\cdot\nu_1\cdot S_5\cdot c & \frac{3\cdot EI}{l^3}\cdot l_2{}^2\cdot\nu_1\cdot S_5
\end{bmatrix}
$$

Figure 37(cont'd) The $\{K_{10}\}$ Matrix

where

$$\nu_1 = \left(1 + \frac{3\cdot EI}{K_1\cdot l}\right)^{-1}$$

$$g = \frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_5 = \frac{1}{1+\nu_1\cdot g/4}$$

Note :

l_1, l_2 = length of connection (rigid end pieces),

l = length of member,

L = total span = $l_1 + l + l_2$,

Δx = change in x form joint 1 to joint 2,

Δy = change in y from joint 1 to joint 2,

c = $\cos(\Delta x/L)$,

s = $\sin(\Delta y/L)$.

3.2.4 Verifying the Fix-Pin Member Stiffness Matrix

As a check of the above derivation it is useful to compare the derived $\{K_{10}\}$ with the conventional $\{K_{10}\}$ (Figure 38). This can be done simply by setting:

$$l_1, l_2 = 0,$$

$$\nu_1 = \left(1 + 3 \cdot \frac{EI}{K_1} \cdot l\right)^{-1} = 1.$$

The dimensionless constants now become:

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

$$S_5 = \frac{1}{1 + \nu_1 \cdot g/4}$$

One sees that $\{K_{10}\}$ correctly reduces to the conventional fix-pin member stiffness matrix.

$$\{K_{10}\}_{ii} = \begin{bmatrix} \frac{AE}{l} \cdot c^2 + \frac{3 \cdot EI}{l^3} \cdot S_5 \cdot s^2 & \frac{AE}{l} \cdot c \cdot s - \frac{3 \cdot EI}{l^3} \cdot S_5 \cdot c \cdot s & -\frac{3 \cdot EI}{l^2} \cdot S_5 \cdot s \\ \frac{AE}{l} \cdot c \cdot s - \frac{3 \cdot EI}{l^3} \cdot S_5 \cdot c \cdot s & \frac{AE}{l} \cdot s^2 + \frac{3 \cdot EI}{l^3} \cdot S_5 \cdot c^2 & \frac{3 \cdot EI}{l^2} \cdot S_5 \cdot c \\ -\frac{3 \cdot EI}{l^2} \cdot S_5 \cdot s & \frac{3 \cdot EI}{l^2} \cdot S_5 \cdot c & \frac{3 \cdot EI}{l} \cdot S_5 \end{bmatrix}$$

Figure 38 The Conventional $\{K_{10}\}$ Matrix

$$\{K_{10}\}_{ij} = \begin{bmatrix} -\frac{AE}{l}\cdot c^2 - \frac{3\cdot EI}{l^3}\cdot S_5\cdot s^2 & -\frac{AE}{l}\cdot c\cdot s + \frac{3\cdot EI}{l^3}\cdot S_5\cdot c\cdot s & 0 \\ -\frac{AE}{l}\cdot c\cdot s + \frac{3\cdot EI}{l^3}\cdot S_5\cdot c\cdot s & -\frac{AE}{l}\cdot s^2 - \frac{3\cdot EI}{l^3}\cdot S_5\cdot c^2 & 0 \\ \frac{3\cdot EI}{l^2}\cdot S_5\cdot s & -\frac{3\cdot EI}{l^2}\cdot S_5\cdot c & 0 \end{bmatrix}$$

$$\{K_{10}\}_{ij} = \{K_{10}\}_{ji}^{T}$$

$$\{K_{10}\}_{jj} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{3\cdot EI}{l^3}\cdot S_5\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot S_5\cdot c\cdot s & 0 \\ \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot S_5\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{3\cdot EI}{l^3}\cdot S_5\cdot c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Figure 38(cont'd) The Conventional $\{K_{10}\}$ Matrix

where

$$g = \frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_5 = \frac{1}{1 + g/4}$$

Note :

l_1, l_2	=	length of connection (rigid end pieces),
l	=	length of member,
L	=	total span = $l_1 + l + l_2$,
Δx	=	change in x form joint 1 to joint 2,
Δy	=	change in y from joint 1 to joint 2,
c	=	$\cos(\Delta x/L)$,
s	=	$\sin(\Delta y/L)$.

3.3 Assembling the Refined Pin-Fix Member Stiffness Matrix

The $\{K_{01}\}$ matrix is derived in the exact same manner as presented in the previous section therefore only the result will be quoted below (Figure 39):

$$
\{K_{01}\}_{ii} = \begin{bmatrix}
\begin{array}{l} \frac{AE}{l}\cdot c^2 \\ +\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot s^2 \end{array} &
\begin{array}{l} \frac{AE}{l}\cdot c\cdot s \\ -\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c\cdot s \end{array} &
-\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_2\cdot S_6\cdot s \\[2ex]
\begin{array}{l} \frac{AE}{l}\cdot c\cdot s \\ -\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c\cdot s \end{array} &
\begin{array}{l} \frac{AE}{l}\cdot s^2 \\ +\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c^2 \end{array} &
\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_2\cdot S_6\cdot c \\[2ex]
-\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_2\cdot S_6\cdot s &
\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_2\cdot S_6\cdot c &
\frac{3\cdot EI}{l^3}\cdot l_1{}^2\cdot v_2\cdot S_6
\end{bmatrix}
$$

$$
\{K_{01}\}_{ij} = \begin{bmatrix}
\begin{array}{l} -\frac{AE}{l}\cdot c^2 \\ -\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot s^2 \end{array} &
\begin{array}{l} -\frac{AE}{l}\cdot c\cdot s \\ +\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c\cdot s \end{array} &
\begin{array}{l} -\frac{3\cdot EI}{l^2}\cdot v_2\cdot S_6\cdot s \\ -\frac{3\cdot EI\cdot l_2}{l^3}\cdot v_2\cdot S_6\cdot s \end{array} \\[2ex]
\begin{array}{l} -\frac{AE}{l}\cdot c\cdot s \\ +\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c\cdot s \end{array} &
\begin{array}{l} -\frac{AE}{l}\cdot s^2 \\ -\frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c^2 \end{array} &
\begin{array}{l} \frac{3\cdot EI}{l^2}\cdot v_2\cdot S_6\cdot c \\ +\frac{3\cdot EI\cdot l_2}{l^3}\cdot v_2\cdot S_6\cdot c \end{array} \\[2ex]
\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_2\cdot S_6\cdot s &
-\frac{3\cdot EI\cdot l_1}{l^3}\cdot v_2\cdot S_6\cdot c &
\begin{array}{l} \frac{3\cdot EI\cdot l_1}{l^2}\cdot v_2\cdot S_6 \\ +\frac{3\cdot EI\cdot l_1\ l_2}{l^3}\cdot v_2\cdot S_6 \end{array}
\end{bmatrix}
$$

Figure 39 The $\{K_{01}\}$ Matrix

$$\{K_{01}\}_{ij} = \{K_{01}\}_{ji}^{T}$$

$$\{K_{01}\}_{jj} = \begin{bmatrix} \frac{AE}{l}\cdot c^2 + \frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot s^2 & \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c\cdot s & \frac{3\cdot EI}{l^2}\cdot v_2\cdot S_6\cdot s + \frac{3\cdot EI\cdot l_2}{l^3}\cdot v_2\cdot S_6\cdot s \\ \frac{AE}{l}\cdot c\cdot s - \frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c\cdot s & \frac{AE}{l}\cdot s^2 + \frac{3\cdot EI}{l^3}\cdot v_2\cdot S_6\cdot c^2 & -\frac{3\cdot EI}{l^2}\cdot v_2\cdot S_6\cdot c - \frac{3\cdot EI\cdot l_2}{l^3}\cdot v_2\cdot S_6\cdot c \\ \frac{3\cdot EI}{l^2}\cdot v_2\cdot S_6\cdot s + \frac{3\cdot EI\cdot l_2}{l^3}\cdot v_2\cdot S_6\cdot s & -\frac{3\cdot EI}{l^2}\cdot v_2\cdot S_6\cdot c - \frac{3\cdot EI\cdot l_2}{l^3}\cdot v_2\cdot S_6\cdot c & \frac{3\cdot EI}{l}\cdot v_2\cdot S_6 + \frac{6\cdot EI\cdot l_2}{l^2}\cdot v_2\cdot S_6\cdot s + \frac{3\cdot EI}{l^3}\cdot l_2^{\,2}\cdot v_2\cdot S_6 \end{bmatrix}$$

Figure 39(cont'd) The $\{K_{01}\}$ Matrix

where

$$v_2 = \left(1 + \frac{3\cdot EI}{K_2\cdot l}\right)^{-1}$$

$$g = \frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

$$S_6 = \frac{1}{1 + v_2\cdot g/4}$$

Note :

l_1, l_2 = length of connection (rigid end pieces),

l	=	length of member,
L	=	total span = $l_1 + l + l_2$,
Δx	=	change in x form joint 1 to joint 2,
Δy	=	change in y from joint 1 to joint 2,
c	=	$\cos(\Delta x/L)$,
s	=	$\sin(\Delta y/L)$.

3.4 Assembling the Refined Pin-Pin Member Stiffness Matrix

Since connection sizes and connection flexibility have no effect on the axial forces of a member, $\{K_{00}\}$ matrix is the same as the conventional pin-pin member stiffness matrix (Figure 40).

$$\{K_{00}\} = \frac{AE}{l} \cdot \begin{pmatrix} c\cdot c & c\cdot s & 0 & -c\cdot c & -c\cdot s & 0 \\ c\cdot s & s\cdot s & 0 & -c\cdot s & -s\cdot s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c\cdot c & -c\cdot s & 0 & c\cdot c & c\cdot s & 0 \\ -c\cdot s & -s\cdot s & 0 & c\cdot s & s\cdot s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 40 The $\{K_{00}\}$ Matrix

Note :

l = length of member,

Δx = change in x form joint 1 to joint 2,

Δy = change in y from joint 1 to joint 2,

c = $\cos(\Delta x/L)$,

s = $\sin(\Delta y/L)$.

4 Modified Fixed End Forces

Connection sizes and connection flexibility also affect the fixed end forces. The method of modifying fixed end forces for various loading conditions are presented in the following sections.

4.1 Uniformly Distrinuted Load on Flexibly Connected Fix-Fix Members

The first step is to derive the fixed end forces for a flexibly connected fix-fix member under uniformly distributed load. The effect of connection sizes can be introduced by means of transfer of forces at a later stage.

The fixed end forces of a flexibly connected fix-fix member is obtained by combining the following three load cases (Figure 41a,b,c).

$$\text{Case a} \; + \; \beta_b \cdot \text{Case b} \; + \; \beta_c \cdot \text{Case c}$$

Case a

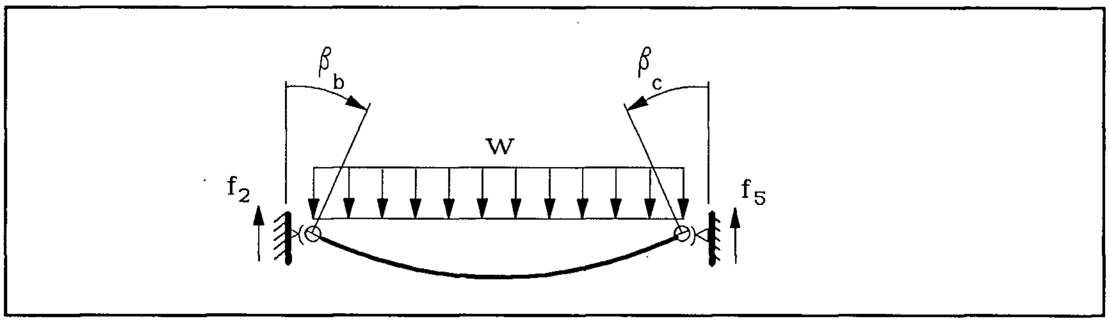


Figure 41a A Simply Supported Beam under a Uniformly Distributed Load

$$f_2 = f_5 = \frac{w \cdot l}{2}$$

$$\beta_b = \beta_c = \frac{w}{24 \cdot EI} \cdot l^3$$

Case b

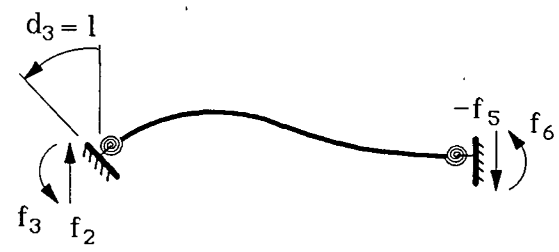


Figure 41b A Fix-Fix Beam with d_3=1

From the third column of $\{k_{b11}\}$: d_3=1, see [3-53],[3-54], and [3-55]

$$f_2 = \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_3 = \frac{4 \cdot EI}{l} \cdot C_3 \cdot \left(\frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1} \right)$$

$$f_5 = -\frac{6 \cdot EI}{l^2} \cdot C_2 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_6 = \frac{2 \cdot EI}{l} \cdot C_5 \cdot \left(\frac{1 - g/2}{1 + g \cdot C_1} \right)$$

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

for $d_3=\beta_b$

$$f_2=\beta_b\cdot\frac{6\cdot EI}{l^2}\cdot C_2\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_3=\beta_b\cdot\frac{4\cdot EI}{l}\cdot C_3\cdot\left(\frac{1+\nu_2\cdot g/4}{1+g\cdot C_1}\right)$$

$$f_5=-\beta_b\cdot\frac{6\cdot EI}{l^2}\cdot C_2\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_6=\beta_b\cdot\frac{2\cdot EI}{l}\cdot C_5\cdot\left(\frac{1-g/2}{1+g\cdot C_1}\right)$$

Case c

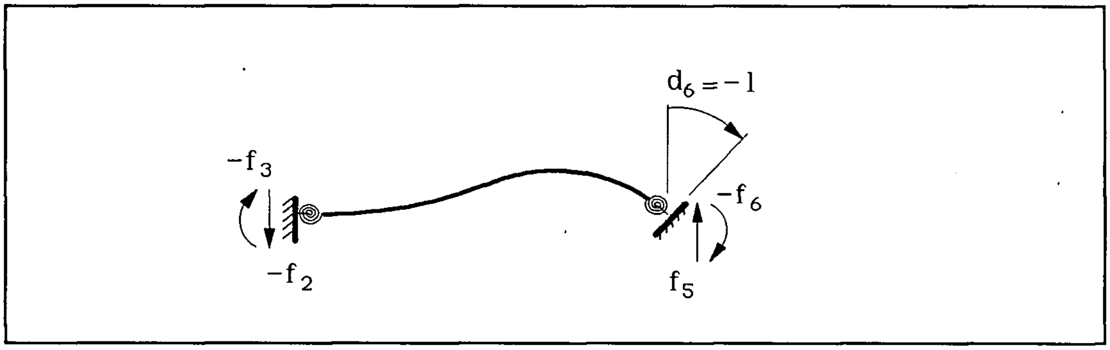


Figure 41c A Fix-Fix Beam with d_6=-1

From the sixth column of $\{k_{b11}\}$: $d_6=1$

$$f_2=\frac{6\cdot EI}{l^2}\cdot C_4\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_3=\frac{2\cdot EI}{l}\cdot C_5\cdot\left(\frac{1+\nu_2\cdot g/4}{1+g\cdot C_1}\right)$$

$$f_5=-\frac{6\cdot EI}{l^2}\cdot C_4\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_6=\frac{4\cdot EI}{l}\cdot C_6\cdot\left(\frac{1-g/2}{1+g\cdot C_1}\right)$$

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

for $d_6 = -\beta_c$

$$f_2 = -\beta_c \cdot \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_3 = -\beta_c \cdot \frac{2 \cdot EI}{l} \cdot C_5 \cdot \left(\frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1} \right)$$

$$f_5 = \beta_c \cdot \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_6 = -\beta_c \cdot \frac{4 \cdot EI}{l} \cdot C_6 \cdot \left(\frac{1 - g/2}{1 + g \cdot C_1} \right)$$

Therefore,

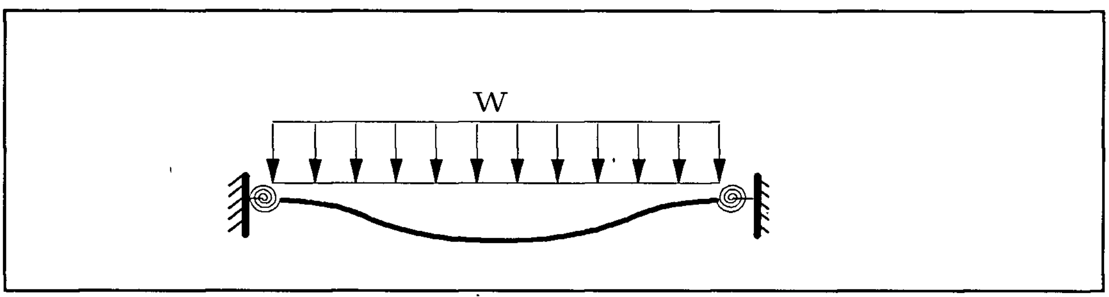


Figure 41d A Fix-Fix Beam under a Uniformly Distributed Load

$$f_2 = \frac{w \cdot l}{2} \cdot \left(1 + \left(\frac{\nu_1 - \nu_2}{4 - \nu_1 \cdot \nu_2} \cdot \frac{1}{1 + g \cdot C_1}\right)\right)$$

$$f_3 = \frac{w \cdot l^2}{12} \cdot \frac{3 \cdot \nu_1 \cdot (2 - \nu_2 \cdot (1 - g))}{(4 - \nu_1 \cdot \nu_2) \cdot (1 + g \cdot C_1)}$$

$$f_5 = \frac{w \cdot l}{2} \cdot \left(1 + \left(\frac{\nu_2 - \nu_1}{4 - \nu_1 \cdot \nu_2} \cdot \frac{1}{1 + g \cdot C_1}\right)\right)$$

$$f_6 = \frac{w \cdot l^2}{12} \cdot \frac{3 \cdot \nu_2 \cdot (2 - \nu_1 \cdot (1 - g))}{(4 - \nu_1 \cdot \nu_2) \cdot (1 + g \cdot C_1)}$$

The effect of connection sizes is introduced by means of transfer of forces as outlined in [3-8] (Figure 18b).

In summary,

$$F_2 = \frac{w \cdot l}{2} \cdot \left(1 + \left(\frac{\nu_1 - \nu_2}{4 - \nu_1 \cdot \nu_2} \cdot \frac{1}{1 + g \cdot C_1}\right)\right)$$

$$F_3 = \frac{w \cdot l^2}{12} \cdot \frac{3 \cdot \nu_1 \cdot (2 - \nu_2 \cdot (1 - g))}{(4 - \nu_1 \cdot \nu_2) \cdot (1 + g \cdot C_1)} + F_2 \cdot l_1$$

$$F_5 = \frac{w \cdot l}{2} \cdot \left(1 + \left(\frac{\nu_2 - \nu_1}{4 - \nu_1 \cdot \nu_2} \cdot \frac{1}{1 + g \cdot C_1}\right)\right)$$

$$F_6 = \frac{w \cdot l^2}{12} \cdot \frac{3 \cdot \nu_2 \cdot (2 - \nu_1 \cdot (1 - g))}{(4 - \nu_1 \cdot \nu_2) \cdot (1 + g \cdot C_1)} - F_5 \cdot l_2$$

Where

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l}\right)^{-1}$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l}\right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

l_1, l_2 = length of connection

l = length of member

L = total span = $l_1 + l + l_2$

4.2 Uniformly Distributed Load on Flexibly Connected Fin-Pin Members

The fixed end forces for a fix-pin member under uniformly distributed load is derived by the method of superposition as presented in the previous section (Figure 42).

$$\text{Case a} \;+\; \beta_b \cdot \text{Case b}$$

Case a

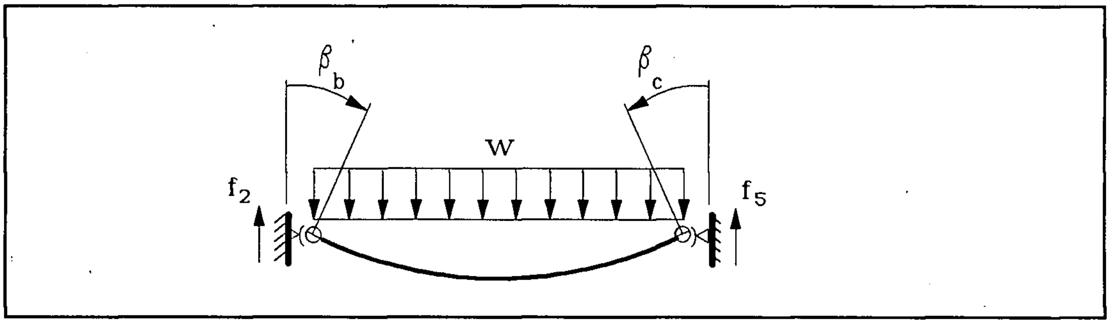


Figure 42a A Simply Supported Beam under a Uniformly Distributed Load

$$f_2 = f_5 = \frac{w \cdot l}{2}$$

$$\beta_b = \beta_c = \frac{w}{24 \cdot EI} \cdot l^3$$

Case b

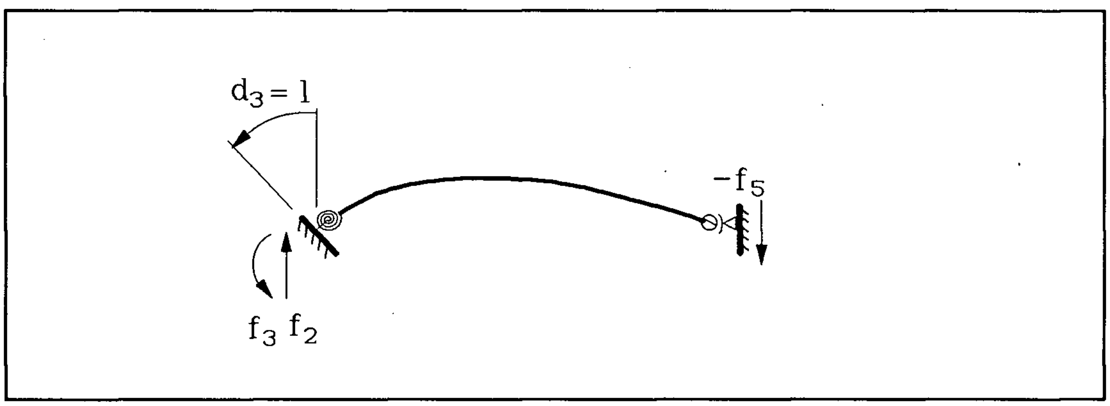


Figure 42b A Fix-Pin Beam with d_3=1

From the third column of (k_{b10}): $d_3=1$, see [3-78] and [3-79]

$$f_2 = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_3 = \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_5 = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

For $d_3=\beta_b$

$$f_2 = \beta_b \cdot \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_3 = \beta_b \cdot \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_5 = \beta_b \cdot \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

Therefore,

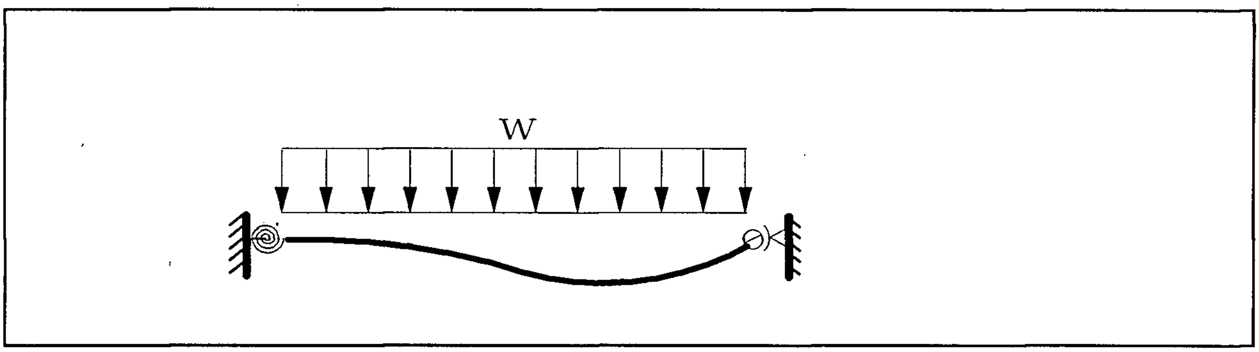


Figure 42c A Fix-Pin Beam under a Uniformly Distributed Load

$$f_2 = \frac{w \cdot l}{8} \cdot \left(\frac{4 + \nu_1 \cdot (1 + g)}{1 + \nu_1 \cdot g/4} \right)$$

$$f_3 = \frac{w \cdot l}{8} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_5 = \frac{w \cdot l}{8} \cdot \left(\frac{4 - \nu_1 \cdot (1 - g)}{1 + \nu_1 \cdot g/4} \right)$$

The effect of connection sizes is introduced by means of transfer of forces as outlined in [3-8] (Figure 18b).

In summary,

$$F_2 = \frac{w \cdot l}{8} \cdot \left(\frac{4 + \nu_1 \cdot (1 + g)}{1 + \nu_1 \cdot g/4} \right)$$

$$F_3 = \frac{w \cdot l}{8} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right) + F_2 \cdot l_1$$

$$F_5 = \frac{w \cdot l}{8} \cdot \left(\frac{4 - \nu_1 \cdot (1 - g)}{1 + \nu_1 \cdot g/4} \right)$$

$$F_6 = -F_5 \cdot l_2$$

Where

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

l_1, l_2 = length of connection (rigid end pieces)

l = length of member

L = total span = $l_1 + l + l_2$

4.3 Uniformly Distributed Load on Flexibly Connected Pin-Fix Members

The fixed end forces for a pin-fix member under uniformly distributed load are analogous to that of a fix-pin member (Figure 43). Therefore only the result will be quoted here.

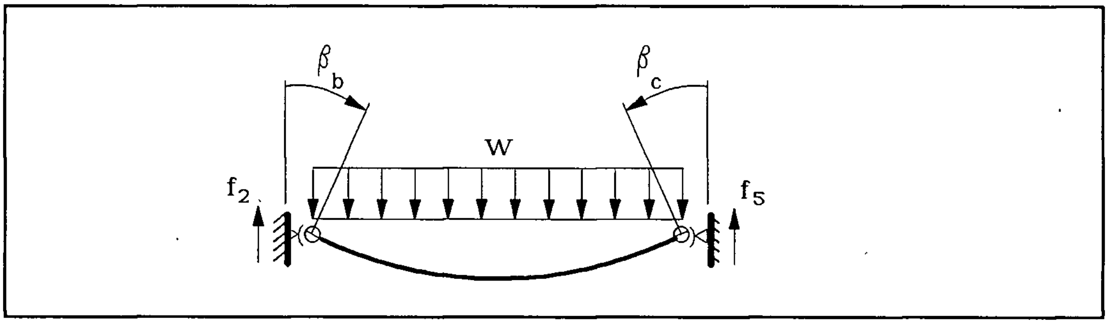


Figure 43a A Simply Supported Beam under a Uniformly Distributed Load

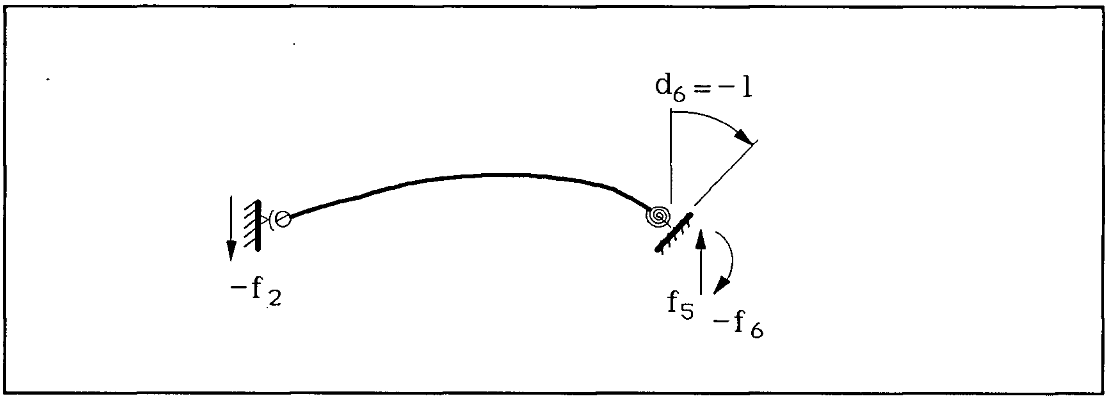


Figure 43b A Pin-Fix Beam with d_6=-1

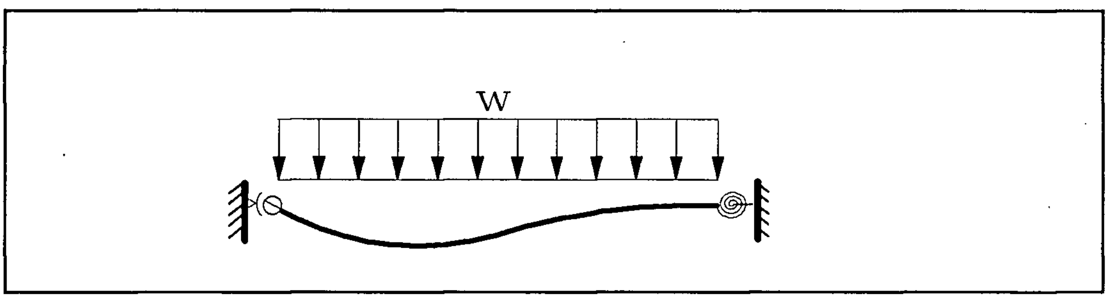


Figure 43c A Pin-Fix Beam under a Uniformly Distributed Load

$$F_2 = \frac{w \cdot l}{8} \cdot \left(\frac{4 - \nu_2 \cdot (1 + g)}{1 + \nu_2 \cdot g/4} \right)$$

$$F_3 = F_2 \cdot l_1$$

$$F_5 = \frac{w \cdot l}{8} \cdot \left(\frac{4 + \nu_2 \cdot (1 - g)}{1 + \nu_2 \cdot g/4} \right)$$

$$F_6 = \frac{w \cdot l}{8} \cdot \nu_2 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right) - F_5 \cdot l_2$$

Where

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

l_1, l_2 = length of connection (rigid end pieces)

l = length of member

L = total span = $l_1 + l + l_2$

4.4 Point Load on Flexibly Connected Fix-Fix Members

Once again the fixed end forces for a flexibly connected fix-fix member under a point load is derived first. The effect of connection sizes will be introduced by means of transfer of forces at a later stage.

The fixed end forces of a flexibly connected fix-fix member is obtained by combining the following three load cases (Figure 44).

$$\text{Case a} + \beta_b \cdot \text{Case b} + \beta_c \cdot \text{Case c}$$

Case a

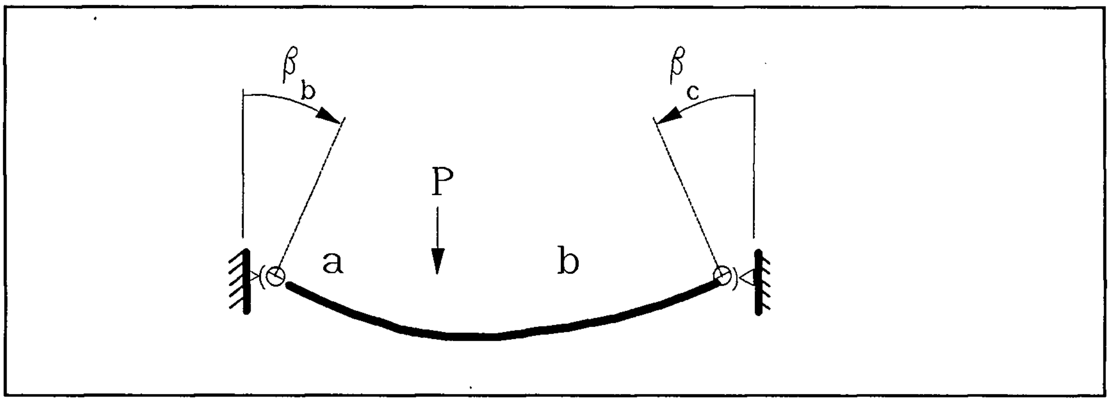


Figure 44a A Simply Supported Beam under a Point Load

$$f_2 = \frac{P \cdot b}{l}$$

$$f_4 = \frac{P \cdot a}{l}$$

$$\beta_b = \frac{P \cdot a \cdot b}{6 \cdot EI \cdot l} \cdot (b + l)$$

$$\beta_c = \frac{P \cdot a \cdot b}{6 \cdot EI \cdot l} \cdot (a + l)$$

Case b

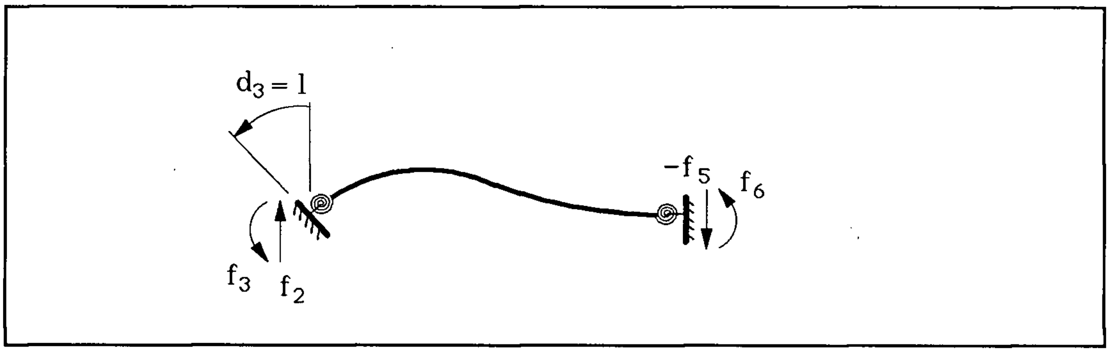


Figure 44b A Fix-Fix Beam with d_3=1

From the third column of $\{k_{b11}\}$: d_3=1, see [3-53],[3-54], and [3-55]

$$f_2 = \frac{6 \cdot EI}{l^2} \cdot C_2 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_3 = \frac{4 \cdot EI}{l} \cdot C_3 \cdot \left(\frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1} \right)$$

$$f_5 = -\frac{6 \cdot EI}{l^2} \cdot C_2 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_6 = \frac{2 \cdot EI}{l} \cdot C_5 \cdot \left(\frac{1 - g/2}{1 + g \cdot C_1} \right)$$

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_2 = \frac{\nu_1 \cdot (2 + \nu_2)}{4 - \nu_1 \cdot \nu_2}$$

$$C_3 = \frac{3 \cdot \nu_1}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

For $d_3=\beta_b$

$$f_2=\beta_b\cdot\frac{6\cdot EI}{l^2}\cdot C_2\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_3=\beta_b\cdot\frac{4\cdot EI}{l}\cdot C_3\cdot\left(\frac{1+\nu_2\cdot g/4}{1+g\cdot C_1}\right)$$

$$f_5=-\beta_b\cdot\frac{6\cdot EI}{l^2}\cdot C_2\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_6=\beta_b\cdot\frac{2\cdot EI}{l}\cdot C_5\cdot\left(\frac{1-g/2}{1+g\cdot C_1}\right)$$

Case c

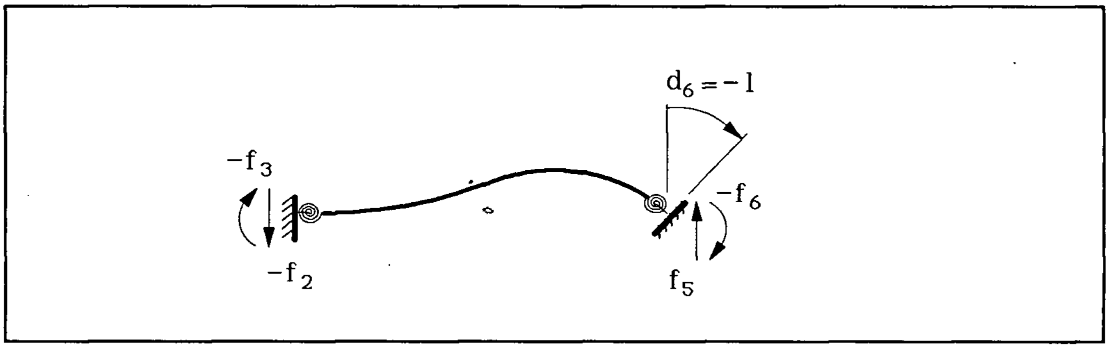


Figure 44c A Fix-Fix Beam with d_6=-1

From the sixth column of {kb11}: d_6=1

$$f_2=\frac{6\cdot EI}{l^2}\cdot C_4\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_3=\frac{2\cdot EI}{l}\cdot C_5\cdot\left(\frac{1+\nu_2\cdot g/4}{1+g\cdot C_1}\right)$$

$$f_5=-\frac{6\cdot EI}{l^2}\cdot C_4\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_6=\frac{4\cdot EI}{l}\cdot C_6\cdot\left(\frac{1-g/2}{1+g\cdot C_1}\right)$$

where

$$C_1 = \frac{\nu_1 + \nu_2 + \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_4 = \frac{\nu_2 \cdot (2 + \nu_1)}{4 - \nu_1 \cdot \nu_2}$$

$$C_5 = \frac{3 \cdot \nu_1 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

$$C_6 = \frac{3 \cdot \nu_2}{4 - \nu_1 \cdot \nu_2}$$

For $d_6 = -\beta_c$

$$f_2 = -\beta_c \cdot \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_3 = -\beta_c \cdot \frac{2 \cdot EI}{l} \cdot C_5 \cdot \left(\frac{1 + \nu_2 \cdot g/4}{1 + g \cdot C_1} \right)$$

$$f_5 = \beta_c \cdot \frac{6 \cdot EI}{l^2} \cdot C_4 \cdot \left(\frac{1}{1 + g \cdot C_1} \right)$$

$$f_6 = -\beta_c \cdot \frac{4 \cdot EI}{l} \cdot C_6 \cdot \left(\frac{1 - g/2}{1 + g \cdot C_1} \right)$$

Therefore,

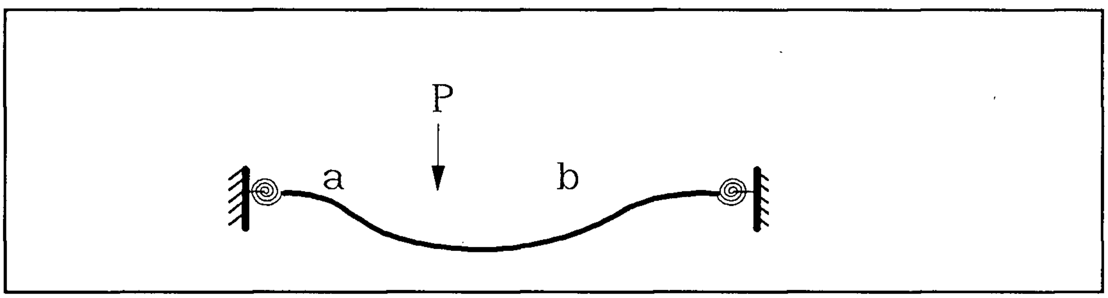


Figure 44d A Fix-Fix Beam under a Point Load

$$f_2=\frac{P\cdot b}{l}\cdot\left(1+\frac{a}{l^2}\cdot\frac{4\cdot l(\nu_1-\nu_2)-\nu_1\cdot a\cdot(2+\nu_2)+\nu_2\cdot b\cdot(2+\nu_1)}{(4-\nu_1\cdot\nu_2)\cdot(1+g\cdot C_1)}\right)$$

$$f_3=\frac{P\cdot a\cdot b^2}{l^2}\cdot\left(\frac{\nu_1\cdot l\cdot(4+\nu_2\cdot(1.5\cdot g-1))-\nu_1\cdot a\cdot(2+\nu_2)}{b\cdot(4-\nu_1\cdot\nu_2)}\right)\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

$$f_5=\frac{P\cdot a}{l}\cdot\left(1+\frac{b}{l^2}\cdot\frac{4\cdot l(\nu_2-\nu_1)+\nu_1\cdot a\cdot(2+\nu_2)-\nu_2\cdot b\cdot(2+\nu_1)}{(4-\nu_1\cdot\nu_2)\cdot(1+g\cdot C_1)}\right)$$

$$f_6=\frac{P\cdot a^2\cdot b}{l^2}\cdot\left(\frac{\nu_2\cdot l\cdot(4+\nu_1\cdot(1.5\cdot g-1))-\nu_2\cdot b\cdot(2+\nu_1)}{a\cdot(4-\nu_1\cdot\nu_2)}\right)\cdot\left(\frac{1}{1+g\cdot C_1}\right)$$

The effect of connection sizes is introduced by means of transfer of forces as outlined in [3-8] (Figure 18b).

In summary,

$$F_2=\frac{P\cdot b}{l}\cdot\left(1+\frac{a}{l^2}\cdot\frac{4\cdot l(\nu_1-\nu_2)-\nu_1\cdot a\cdot(2+\nu_2)+\nu_2\cdot b\cdot(2+\nu_1)}{(4-\nu_1\cdot\nu_2)\cdot(1+g\cdot C_1)}\right)$$

$$F_3=\frac{P\cdot a\cdot b^2}{l^2}\cdot\left(\frac{\nu_1\cdot l\cdot(4+\nu_2\cdot(1.5\cdot g-1))-\nu_1\cdot a\cdot(2+\nu_2)}{b\cdot(4-\nu_1\cdot\nu_2)}\right)\cdot\left(\frac{1}{1+g\cdot C_1}\right)+F_2\cdot l_1$$

$$F_5=\frac{P\cdot a}{l}\cdot\left(1+\frac{b}{l^2}\cdot\frac{4\cdot l(\nu_2-\nu_1)+\nu_1\cdot a\cdot(2+\nu_2)-\nu_2\cdot b\cdot(2+\nu_1)}{(4-\nu_1\cdot\nu_2)\cdot(1+g\cdot C_1)}\right)$$

$$F_6=\frac{P\cdot a^2\cdot b}{l^2}\cdot\left(\frac{\nu_2\cdot l\cdot(4+\nu_1\cdot(1.5\cdot g-1))-\nu_2\cdot b\cdot(2+\nu_1)}{a\cdot(4-\nu_1\cdot\nu_2)}\right)\cdot\left(\frac{1}{1+g\cdot C_1}\right)-F_5\cdot l_2$$

Where

$$\nu_1=\left(1+\frac{3\cdot EI}{K_1\cdot l}\right)^{-1}$$

$$\nu_2=\left(1+\frac{3\cdot EI}{K_2\cdot l}\right)^{-1}$$

$$g=\frac{12\cdot EI}{A_v\cdot G\cdot l^2}$$

l_1, l_2 = length of connection

l = length of member

L = total span = $l_1 + l + l_2$

4.5 Point Load on Flexibly Connected Fix-Pin Members

The fixed end forces for a fix-pin member under a point load is derived by the method of superposition as presented in the previous section (Figure 45).

Case a + $\beta_b \cdot$ Case b

Case a

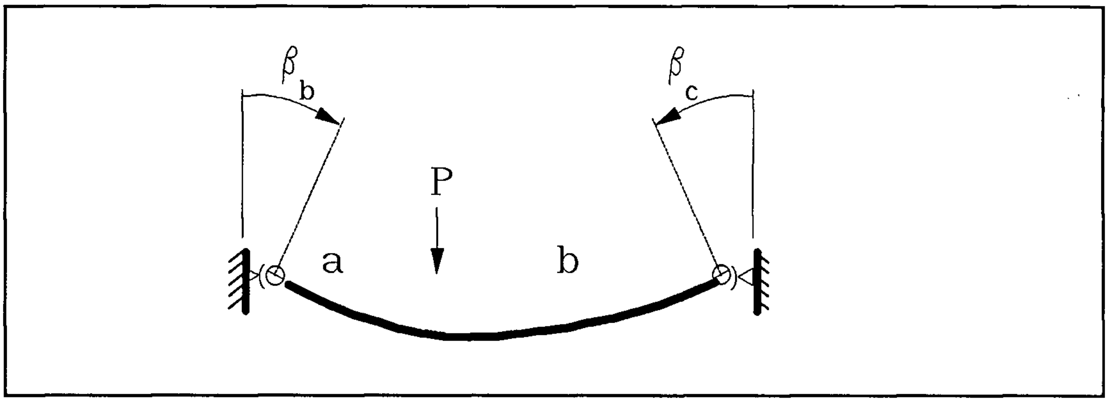


Figure 45a A Simply Supported Beam under a Point Load

For simply supported beams under point load

$$f_2 = \frac{P \cdot b}{l}$$

$$f_4 = \frac{P \cdot a}{l}$$

$$\beta_b = \frac{P \cdot a \cdot b}{6 \cdot EI \cdot l} \cdot (b + l)$$

$$\beta_c = \frac{P \cdot a \cdot b}{6 \cdot EI \cdot l} \cdot (a + l)$$

Case b

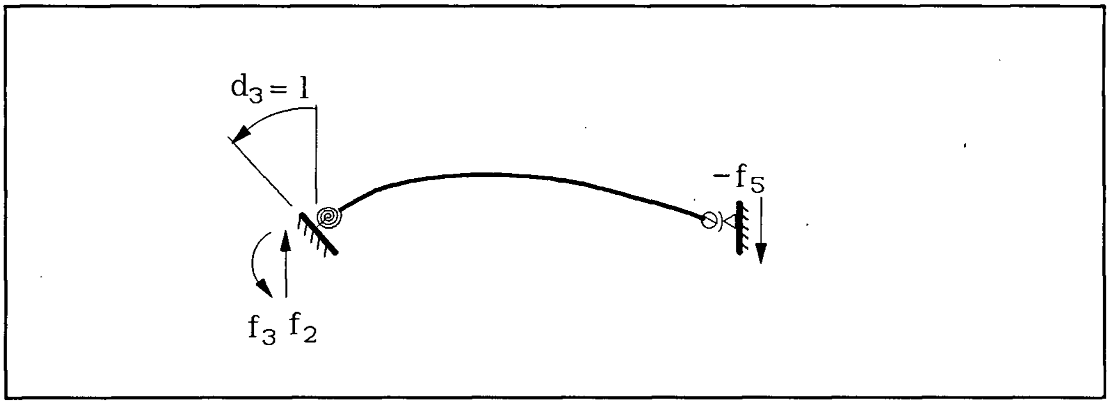


Figure 45b A Fix-Pin Beam with d_3=1

From the third column of $\{k_{b10}\}$: d_3=1, see [3-78] and [3-79]

$$f_2 = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_3 = \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_5 = \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

For $d_3 = \beta_b$

$$f_2 = \beta_b \cdot \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_3 = \beta_b \cdot \frac{3 \cdot EI}{l} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_5 = \beta_b \cdot \frac{3 \cdot EI}{l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

Therefore,

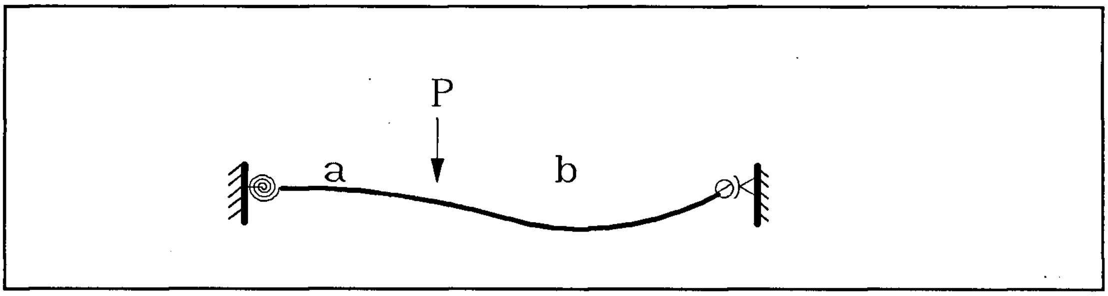


Figure 45c A Fix-Pin Beam under a Point Load

$$f_2 = \frac{P \cdot b}{l} + \frac{P \cdot a \cdot b \cdot (a + 2 \cdot b)}{2 \cdot l^3} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_3 = \frac{P \cdot a \cdot b \cdot (a + 2 \cdot b)}{2 \cdot l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$f_5 = \frac{P \cdot a}{l} - \frac{P \cdot a \cdot b \cdot (a + 2 \cdot b)}{2 \cdot l^3} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

The effect of connection sizes is introduced by means of transfer of forces as outlined in [3-8] (Figure 18b).

In summary,

$$F_2 = \frac{P \cdot b}{l} + \frac{P \cdot a \cdot b \cdot (a + 2 \cdot b)}{2 \cdot l^3} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$F_3 = \frac{P \cdot a \cdot b \cdot (a + 2 \cdot b)}{2 \cdot l^2} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right) + F_2 \cdot l_1$$

$$F_5 = \frac{P \cdot a}{l} - \frac{P \cdot a \cdot b \cdot (a + 2 \cdot b)}{2 \cdot l^3} \cdot \nu_1 \cdot \left(\frac{1}{1 + \nu_1 \cdot g/4} \right)$$

$$F_6 = -F_5 \cdot l_2$$

Where

$$\nu_1 = \left(1 + \frac{3 \cdot EI}{K_1 \cdot l} \right)^{-1}$$

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

l_1, l_2 = length of connection

l = length of member

L = total span = $l_1 + l + l_2$

4.6 Point Load on Flexibly Connected Pin-Fix Members

The fixed end forces for a pin-fix member under a point load are analogous to that of a fix-pin member (Figure 46). Therefore only the result will be quoted here.

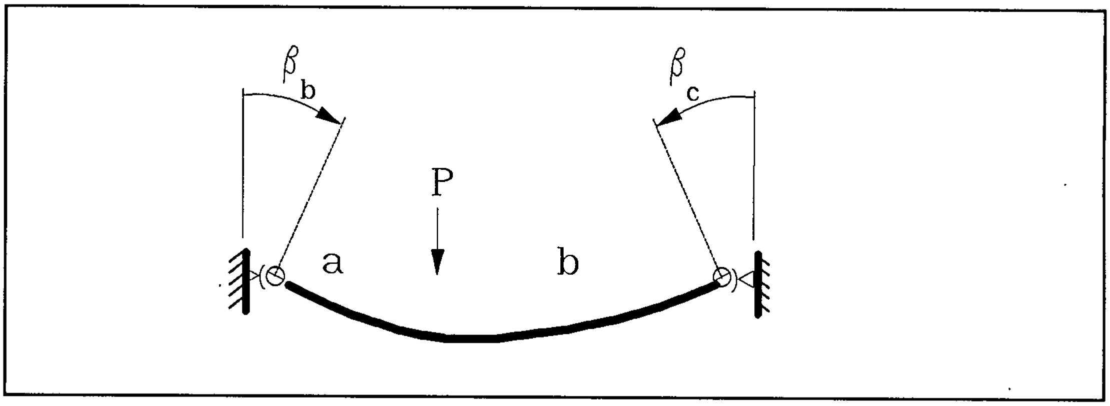


Figure 46a A Simply Supported Beam under a Point Load

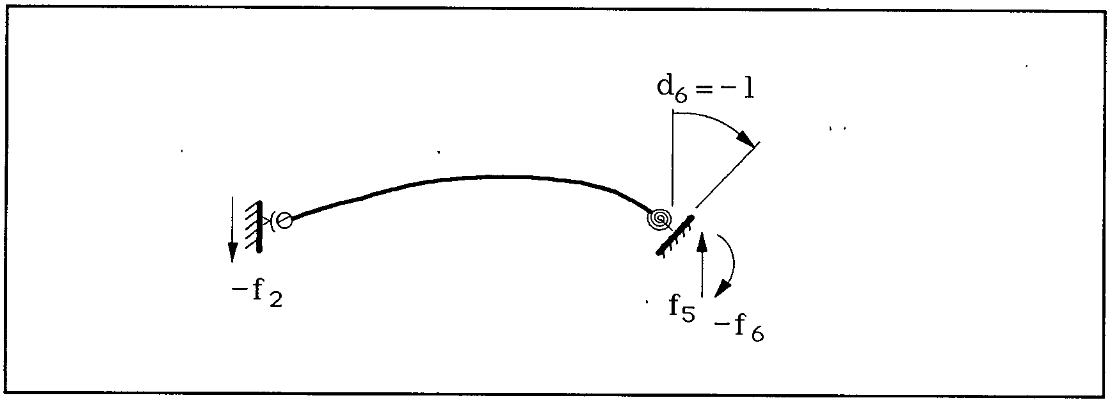


Figure 46b A Pin-Fix Beam with d_6=-1

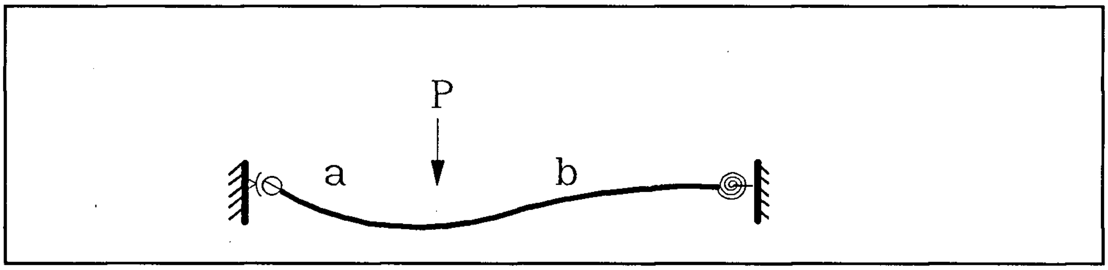


Figure 46c A Pin-Fix Beam under a Point Load

$$F_2 = \frac{P \cdot b}{l} - \frac{P \cdot a \cdot b \cdot (b + 2 \cdot a)}{2 \cdot l^3} \cdot \nu_2 \cdot \left(\frac{1}{1 + \nu_2 \cdot g/4} \right)$$

$$F_3 = F_2 \cdot l_1$$

$$F_5 = \frac{P \cdot a}{l} + \frac{P \cdot a \cdot b \cdot (b + 2 \cdot a)}{2 \cdot l^3} \cdot \nu_2 \cdot \left(\frac{1}{1 + \nu_2 \cdot g/4} \right)$$

$$F_6 = \frac{P \cdot a \cdot b \ \cdot (b + 2 \cdot a)}{2 \cdot l^2} \cdot \nu_2 \cdot \left(\frac{1}{1 + \nu_2 \cdot g/4} \right) - F_5 \cdot l_2$$

Where

$$\nu_2 = \left(1 + \frac{3 \cdot EI}{K_2 \cdot l} \right)^{-1}$$

$$g = \frac{12 \cdot EI}{A_v \cdot G \cdot l^2}$$

l_1, l_2 = length of connection (rigid end pieces)

l = length of member

L = total span = $l_1 + l + l_2$

5 Member Forces Calculation

With the addition of connections at either end of a member, nodal displacements now become the end displacements of the connections and not the member itself. Therefore member forces must be calculated differently. By adopting the sign convention illustrated in Figure 47 and recalling the member forces relationships established in Section 3 (See Figure 18a), one can relate forces at connection ends and member ends as follows:

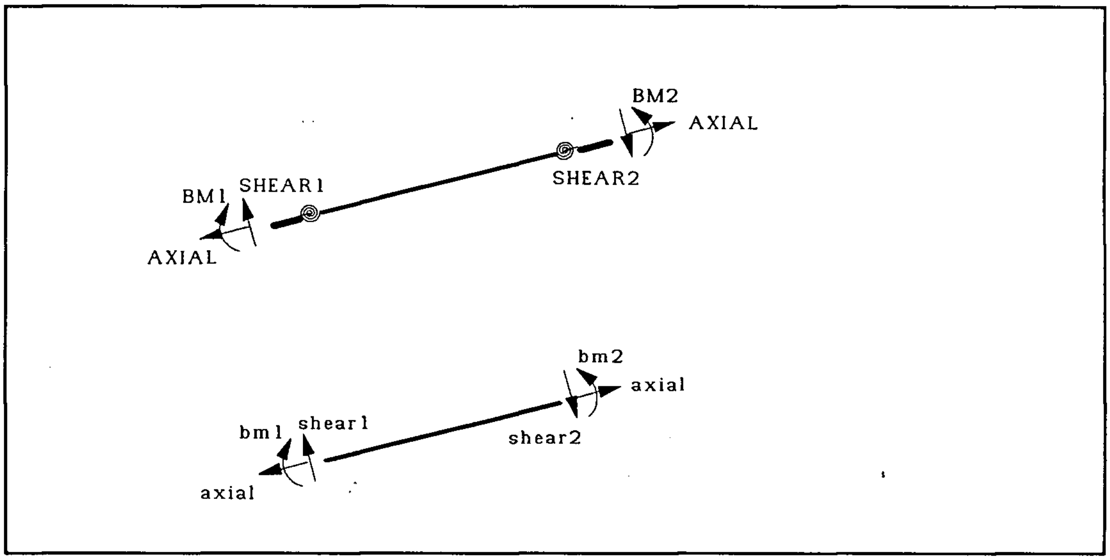


Figure 47 Member Forces Sign Convention

$$F_1 = -AXIAL \qquad f_1 = -axial$$
$$F_2 = SH1 \qquad f_2 = shear1$$
$$F_3 = -BM1 \qquad f_3 = -bm1$$
$$F_4 = AXIAL \qquad f_4 = axial$$
$$F_5 = -SH2 \qquad f_5 = -shear2$$
$$F_6 = BM2 \qquad f_6 = bm2$$

Recalling the transfer of forces relationships of [3-8] and reorganizing:

$$axial = AXIAL \qquad [5\text{-}1]$$

$$shear1 = SH1 \qquad [5\text{-}2]$$

$$shear2 = SH2 \qquad [5\text{-}3]$$

$$bm1 = BM1 + shear1 \cdot l_1 \qquad [5\text{-}4]$$

$$bm2 = BM2 - shear2 \cdot l_2 \qquad [5\text{-}5]$$

Noting that connections are perfectly rigid, the axial force can be written directly the same as before:

$$axial = \frac{AE}{l} \cdot \left(\left(d_4 - d_1\right) \cdot c + \left(d_5 - d_2\right) \cdot s\right)$$

where

l_1, l_2 = length of connection (rigid end pieces),

l = length of member,

L = total span = $l_1 + l + l_2$,

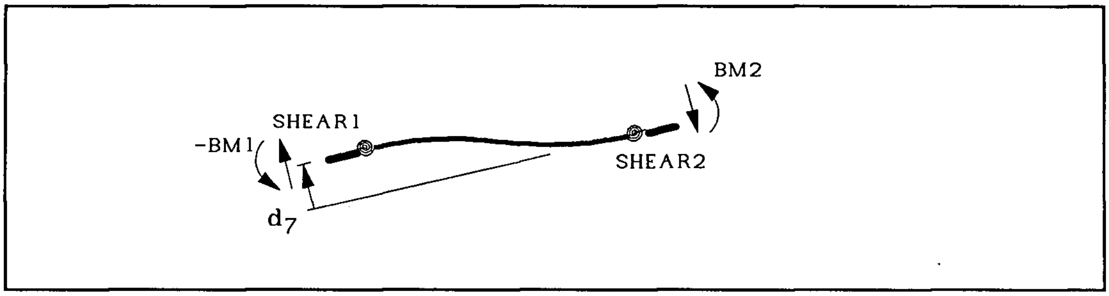
Δx = change in x form joint 1 to joint 2,

Δy = change in y from joint 1 to joint 2,

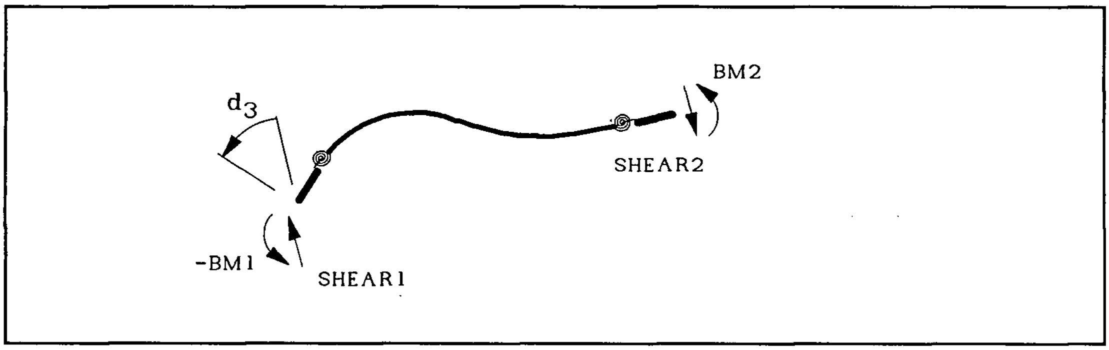
c = $\cos(\Delta x / L)$,

s = $\sin(\Delta y / L)$.

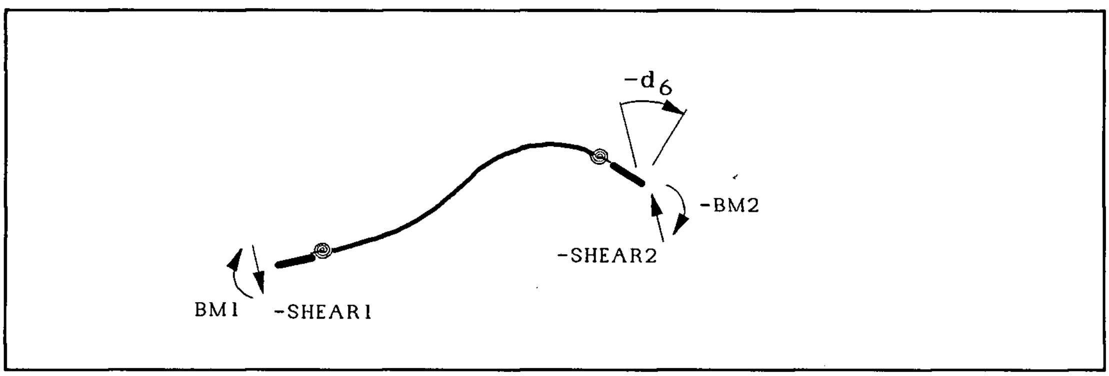
Shears and moments at connection ends are obtained by reorganizing the six deflections into the three cases shown below:



case 1



case 2

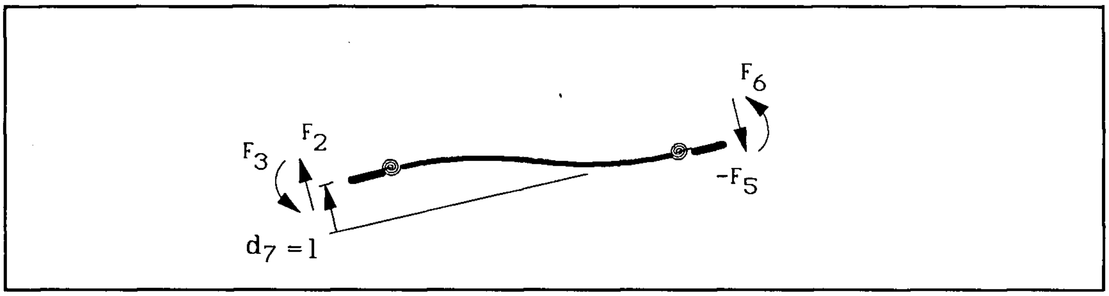


case 3

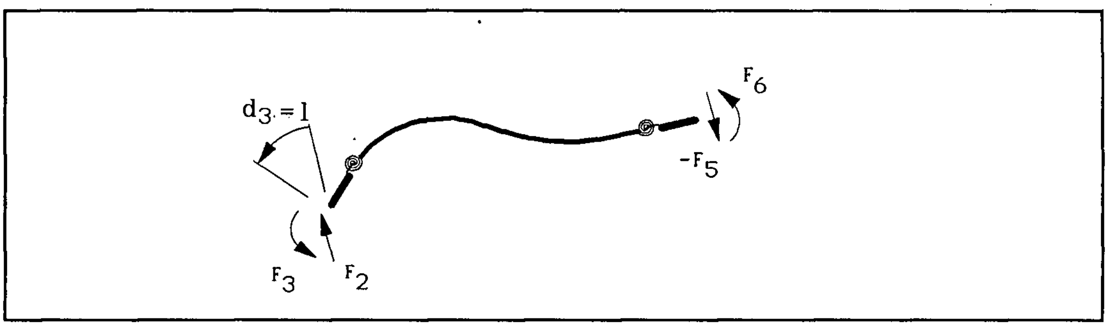
Figure 48 Calculating Shears and Moments

The above three cases have already been solved for unit deflections in Section 3. Therefore the shears and moments at connection ends can be obtained simply by means of superposition:

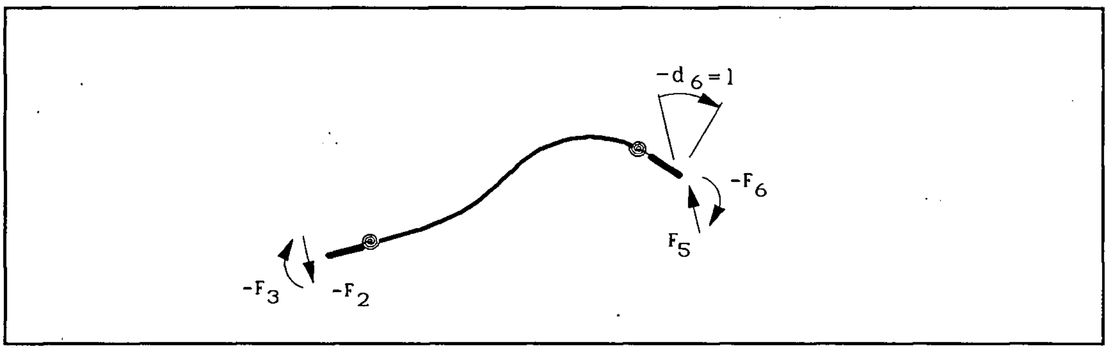
$$d_7 \cdot case\ a + d_3 \cdot case\ b + d_6 \cdot case\ c$$



case a



case b



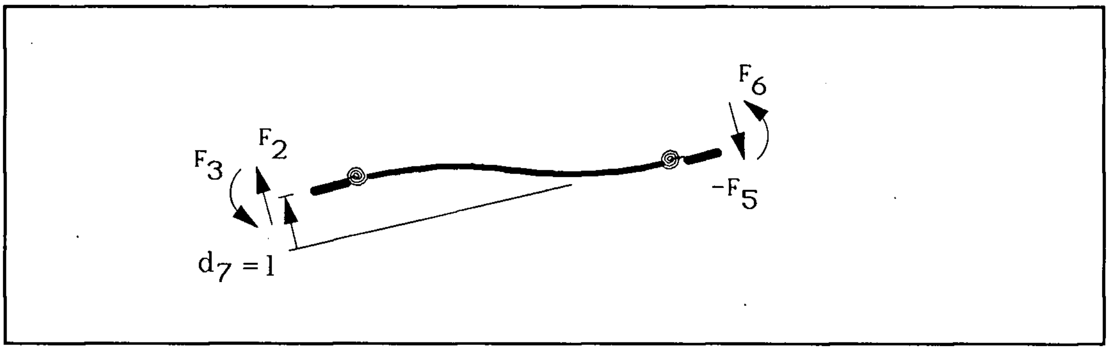
case c

Figure 49 Calculating Shears and Moments at Connection Ends by Superposition

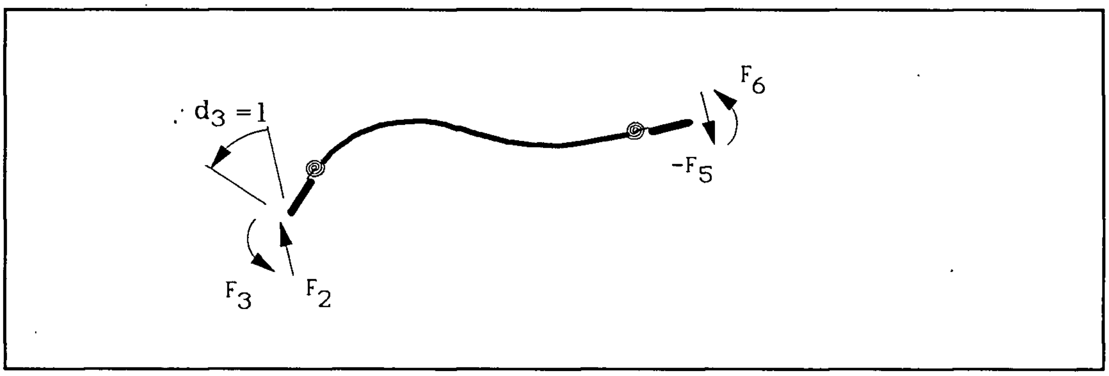
Once the forces at connection ends are calculated, the member forces can be obtained by [5-1] to [5-5].

5.1 Calculating Shears and Moments of Fix-Fix Members

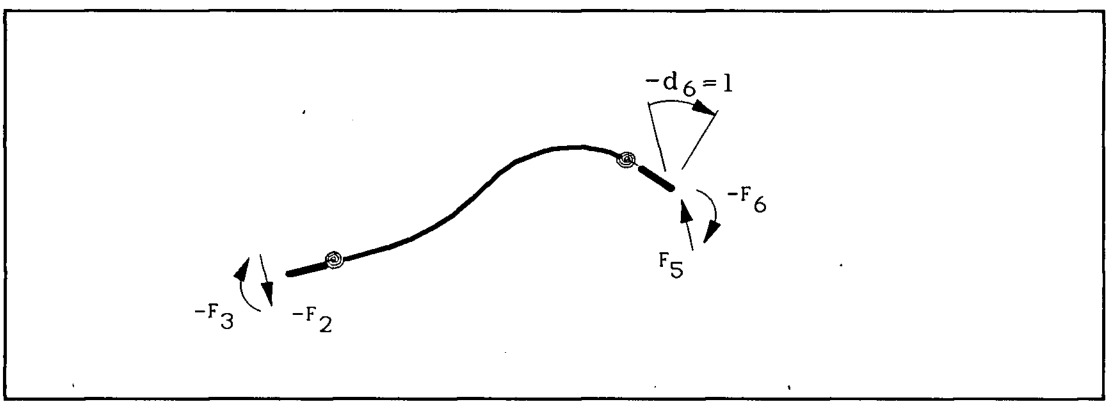
The three cases used for calculating shears and moments of fix-fix members have already been solved in Section 3.1 and are summarized below in Figure 50.



case a



case b



case c

Figure 50 Shears and Moments of Fix-Fix Members by Superposition

By superposition:

$$SH1 = \left(\frac{12EI}{l^3} \cdot C_1 \cdot S_1 \right) \cdot d_7$$
$$+ \left(\frac{6EI}{l^2} \cdot C_2 \cdot S_1 + \frac{12EI}{l^3} \cdot l_1 \cdot C_1 \cdot S_1 \right) \cdot d_3$$
$$+ \left(\frac{6EI}{l^2} \cdot C_4 \cdot S_1 + \frac{12EI}{l^3} \cdot l_2 \cdot C_1 \cdot S_1 \right) \cdot d_6$$

[5-6]

and,

$$BM1 = -\left(\frac{6EI}{l^2} \cdot C_2 \cdot S_1 + \frac{12EI}{l^3} \cdot l_1 \cdot C_1 \cdot S_1 \right) \cdot d_7$$
$$-\left(\frac{4EI}{l} \cdot C_3 \cdot S_2 + \frac{12EI}{l^2} \cdot l_1 \cdot C_2 \cdot S_1 + \frac{12EI}{l^3} \cdot l_1{}^2 \cdot C_1 \cdot S_1 \right) \cdot d_3$$
$$-(\frac{2EI}{l} \cdot C_5 \cdot S_3 + \frac{6EI}{l^2} \cdot l_1 \cdot C_4 \cdot S_1 + \frac{6EI}{l^2} \cdot l_2 \cdot C_2 \cdot S_1$$
$$+ \frac{12EI}{l^2} \cdot l_1 \cdot l_2 \cdot C_1 \cdot S_1) \cdot d_6$$

rearranging and noting [5-6]

$$BM1 = -\left(\frac{6EI}{l^2} \cdot C_2 \cdot S_1 \right) \cdot d_7$$
$$-\left(\frac{4EI}{l} \cdot C_3 \cdot S_2 + \frac{6EI}{l^2} \cdot l_1 \cdot C_2 \cdot S_1 \right) \cdot d_3$$
$$-\left(\frac{2EI}{l} \cdot C_5 \cdot S_3 + \frac{6EI}{l^2} \cdot l_2 \cdot C_2 \cdot S_1 \right) \cdot d_6 - SH1 \cdot l_1$$

[5-7]

Member force shear1 is calculated from [5-2]:

$$shear1 = \left(\frac{12EI}{l^3}\cdot C_1\cdot S_1\right)\cdot d_7 + \left(\frac{6EI}{l^2}\cdot C_2\cdot S_1 + \frac{12EI}{l^3}\cdot l_1\cdot C_1\cdot S_1\right)\cdot d_3 + \left(\frac{6EI}{l^2}\cdot C_4\cdot S_1 + \frac{12EI}{l^3}\cdot l_2\cdot C_1\cdot S_1\right)\cdot d_6 \qquad [5\text{-}8]$$

From equilibrium,

$$shear2 = shear1 \qquad [5\text{-}9]$$

From [5-4] and noting that shear1=SH1, bm1 reduces to:

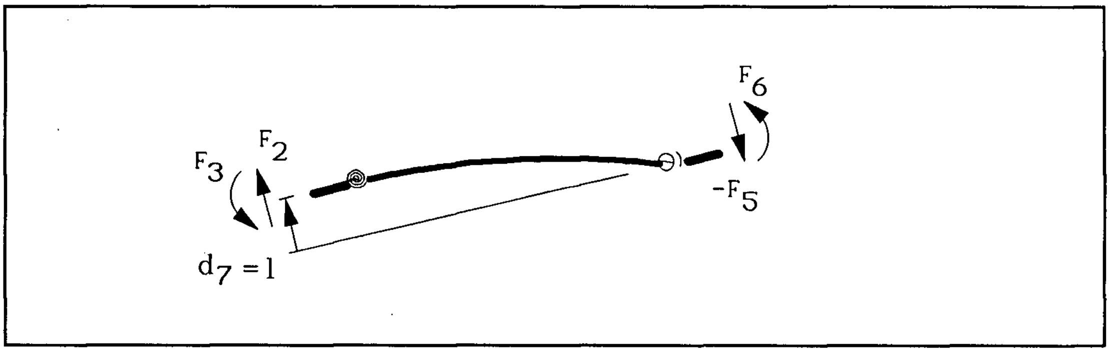
$$bm1 = -\left(\frac{6EI}{l^2}\cdot C_2\cdot S_1\right)\cdot d_7 - \left(\frac{4EI}{l}\cdot C_3\cdot S_2 + \frac{6EI}{l^2}\cdot l_1\cdot C_2\cdot S_1\right)\cdot d_3 - \left(\frac{2EI}{l}\cdot C_5\cdot S_3 + \frac{6EI}{l^2}\cdot l_2\cdot C_2\cdot S_1\right)\cdot d_6 \qquad [5\text{-}10]$$

From equilibrium,

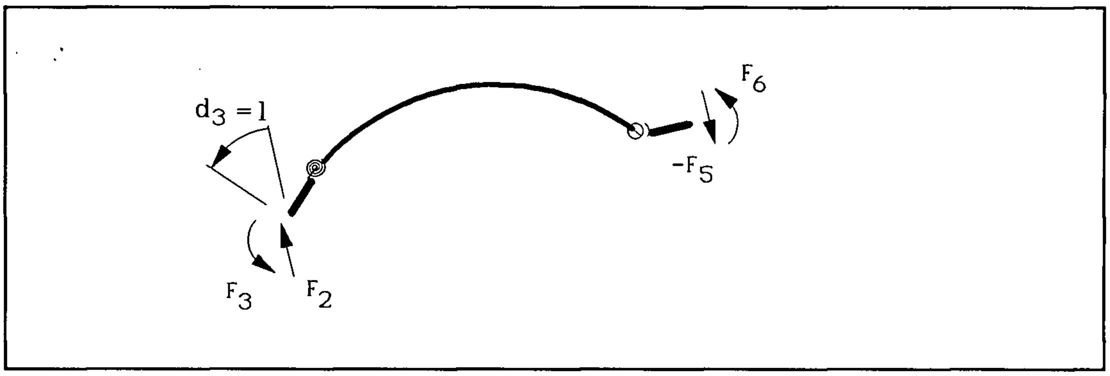
$$bm2 = bm1 + shear1\cdot l_1 \qquad [5\text{-}11]$$

5.2 Calculating Shears and Moments OF Fix-Pin Members

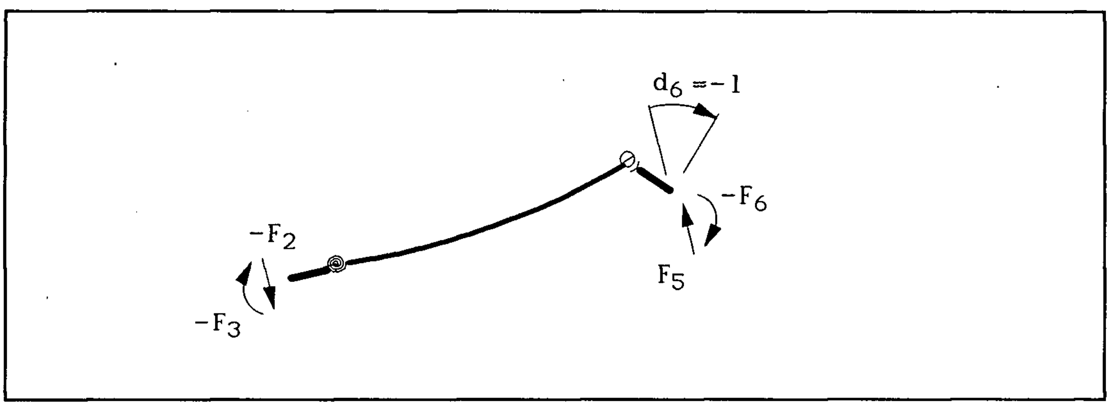
The three cases used for calculating shears and moments of fix-pin members have already been solved in Section 3.2 and are summarized below in Figure 51.



case a



case b



case c

Figure 51 Shears and Moments of Fix-Pin Members by Superposition

By superposition:

$$SH1 = \left(3\frac{EI}{l^3}\cdot v_1\cdot S_5\right)\cdot d_7 + \left(\frac{3EI}{l^2}\cdot v_1\cdot S_5 + \frac{3EI}{l^3}\cdot l_1\cdot v_1\cdot S_5\right)\cdot d_3 + \left(\frac{3EI}{l^3}\cdot l_2\cdot v_1\cdot S_5\right)\cdot d_6$$

[5-12]

and,

$$BM1 = -\left(\frac{3EI}{l^2}\cdot v_1\cdot S_5 + \frac{3EI}{l^3}\cdot l_1\cdot v_1\cdot S_5\right)\cdot d_7 - \left(\frac{3EI}{l}\cdot v_1\cdot S_5 + \frac{6EI}{l^2}\cdot l_1\cdot v_1\cdot S_5 + \frac{3EI}{l^3}\cdot l_1^{\,2}\cdot v_1\cdot S_5\right)\cdot d_3 - \left(\frac{3EI}{l^2}\cdot l_2\cdot v_1\cdot S_5 + \frac{3EI}{l^3}\cdot l_1\cdot l_2\cdot v_1\cdot S_5\right)\cdot d_6$$

rearranging and noting [5-6]

$$BM1 = -\left(\frac{3EI}{l^2}\cdot v_1\cdot S_5\right)\cdot d_7 - \left(\frac{3EI}{l}\cdot v_1\cdot S_5 + \frac{3EI}{l^2}\cdot l_1\cdot v_1\cdot S_5\right)\cdot d_3 - \left(\frac{3EI}{l^2}\cdot l_2\cdot v_1\cdot S_5\right)\cdot d_6 - SH1\cdot l_1$$

[5-13]

Member force shear1 is calculated from [5-2]:

$$shear\,1 = \left(\frac{3EI}{l^3} \cdot \nu_1 \cdot S_5 \right) \cdot d_7$$
$$+ \left(\frac{3EI}{l^2} \cdot \nu_1 \cdot S_5 + \frac{3EI}{l^3} \cdot l_1 \cdot \nu_1 \cdot S_5 \right) \cdot d_3$$
$$+ \left(\frac{3EI}{l^3} \cdot l_2 \cdot \nu_1 \cdot S_5 \right) \cdot d_6$$

[5-14]

From equilibrium,

$$shear\,2 = shear\,1 \qquad [5\text{-}15]$$

From [5-4] and noting that shear1=SH1, bm1 reduces to:

$$bm\,1 = -\left(\frac{3EI}{l^2} \cdot \nu_1 \cdot S_5 \right) \cdot d_7$$
$$-\left(\frac{3EI}{l} \cdot \nu_1 \cdot S_5 + \frac{3EI}{l^2} \cdot l_1 \cdot \nu_1 \cdot S_5 \right) \cdot d_3$$
$$-\left(\frac{3EI}{l^2} \cdot l_2 \cdot \nu_1 \cdot S_5 \right) \cdot d_6$$

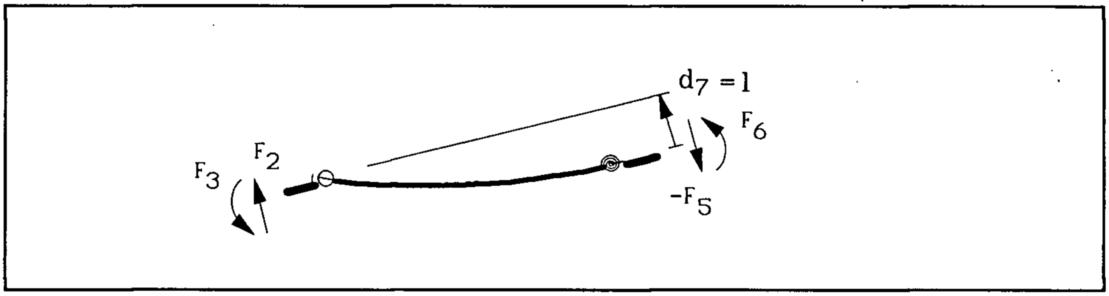
[5-16]

By definition,

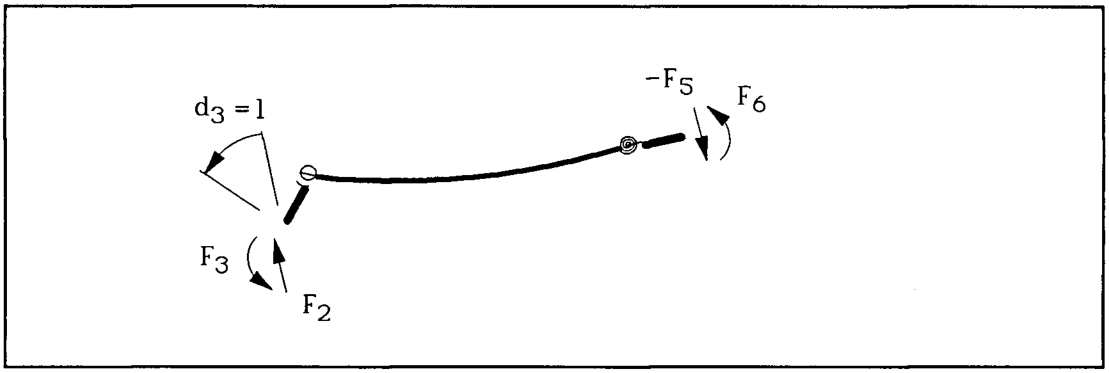
$$bm\,2 = 0 \qquad [5\text{-}17]$$

5.3 Calculating Shears and Moments OF Pin-Fix Members

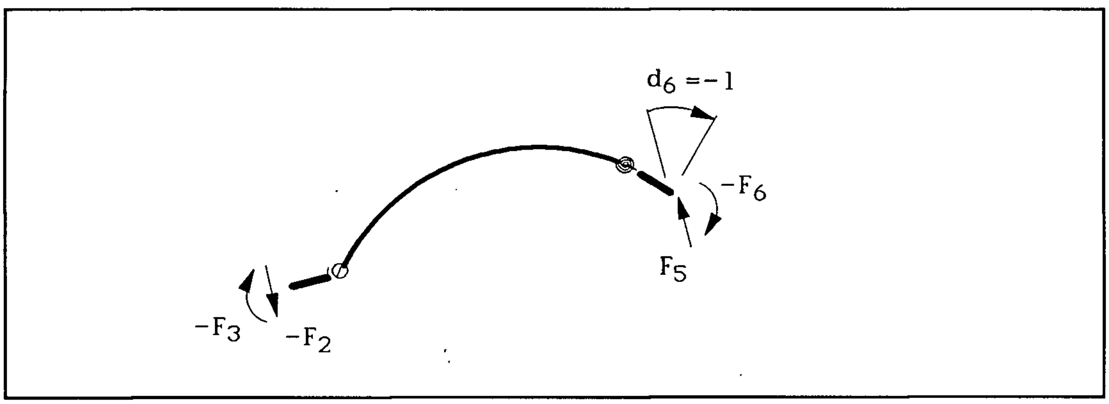
The three cases used for calculating shears and moments of pin-fix members have already been solved in Section 3.3 and are summarized below in Figure 52.



case a



case b



case c

Figure 52 Shears and Moments of Pin-Fix Members by Superposition

By superposition:

$$SH1=\left(3\frac{EI}{l^3}\cdot v_2\cdot S_6\right)\cdot d_7$$
$$+\left(\frac{3EI}{l^3}\cdot l_1\cdot v_2\cdot S_6\right)\cdot d_3$$
$$+\left(\frac{3EI}{l^2}\cdot v_2\cdot S_6+\frac{3EI}{l^3}\cdot l_2\cdot v_2\cdot S_6\right)\cdot d_6$$

[5-18]

and,

$$BM2=\left(\frac{3EI}{l^2}\cdot v_2\cdot S_6+\frac{3EI}{l^3}\cdot l_2\cdot v_2\cdot S_6\right)\cdot d_7$$
$$+\left(\frac{3EI}{l^2}\cdot l_1\cdot v_2\cdot S_6+\frac{3EI}{l^3}\cdot l_1\cdot l_2\cdot v_2\cdot S_6\right)\cdot d_3$$
$$+\left(\frac{3EI}{l}\cdot v_2\cdot S_6+\frac{6EI}{l^2}\cdot l_2\cdot v_2\cdot S_6+\frac{3EI}{l^3}\cdot l_2{}^2\cdot v_2\cdot S_6\right)\cdot d_6$$

rearranging and noting [5-6]

$$BM2=\left(\frac{3EI}{l^2}\cdot v_2\cdot S_6\right)\cdot d_7$$
$$+\left(\frac{3EI}{l^2}\cdot l_1\cdot v_2\cdot S_6\right)\cdot d_3$$
$$+\left(\frac{3EI}{l}\cdot v_2\cdot S_6+\frac{3EI}{l^2}\cdot l_2\cdot v_2\cdot S_6\right)\cdot d_6+SH2\cdot l_2$$

[5-19]

Member force shear1 is calculated from [5-2]:

$$shear1 = \left(\frac{3EI}{l^3} \cdot v_2 \cdot S_6 \right) \cdot d_7 + \left(\frac{3EI}{l^3} \cdot l_1 \cdot v_2 \cdot S_6 \right) \cdot d_3 + \left(\frac{3EI}{l^2} \cdot v_2 \cdot S_6 + \frac{3EI}{l^3} \cdot l_2 \cdot v_2 \cdot S_6 \right) \cdot d_6 \quad [5\text{-}20]$$

From equilibrium,

$$shear2 = shear1 \quad [5\text{-}21]$$

From [5-5] and noting that shear2=SH2, bm2 reduces to:

$$bm2 = \left(\frac{3EI}{l^2} \cdot v_2 \cdot S_6 \right) \cdot d_7 + \left(\frac{3EI}{l^2} \cdot l_1 \cdot v_2 \cdot S_6 \right) \cdot d_3 + \left(\frac{3EI}{l} \cdot v_2 \cdot S_6 + \frac{3EI}{l^2} \cdot l_2 \cdot v_2 \cdot S_6 \right) \cdot d_6 \quad [5\text{-}22]$$

By definition,

$$bm1 = 0 \quad [5\text{-}23]$$

6 Connection Stiffness

While the derivation of the stiffness matrix, fixed end forces and member forces is complete, there still remains the question of what stiffness values to use for the various types of connections in analysis. In 1969 Somner[16] devised a standardized procedure for expressing the moment-rotation characteristics for all connections of a given type in a non-dimensional form. A few years later, Frye and Morris[4] utilized the same procedure to develop a set of dimensionless equations which express the moment-rotation relationships of the seven most commonly used connections (Figure 53, Table 1). The general moment-rotation relationship is of the form:

$$\phi = c_1 \cdot (kM) + c_2 \cdot (kM)^3 + c_3 \cdot (kM)^5$$

and,

$$k = p_2^a \cdot p_3^b \cdot p_4^c \cdot p_5^d$$

where

c_1, c_2, c_3, p_1, p_2, p_3 and p_4 are constants which depend on the connection type.

These moment-rotation relationships are very compact and well suited for programming. It will be shown in the next section how these relationships can be incorporated into a plane frame analysis program.

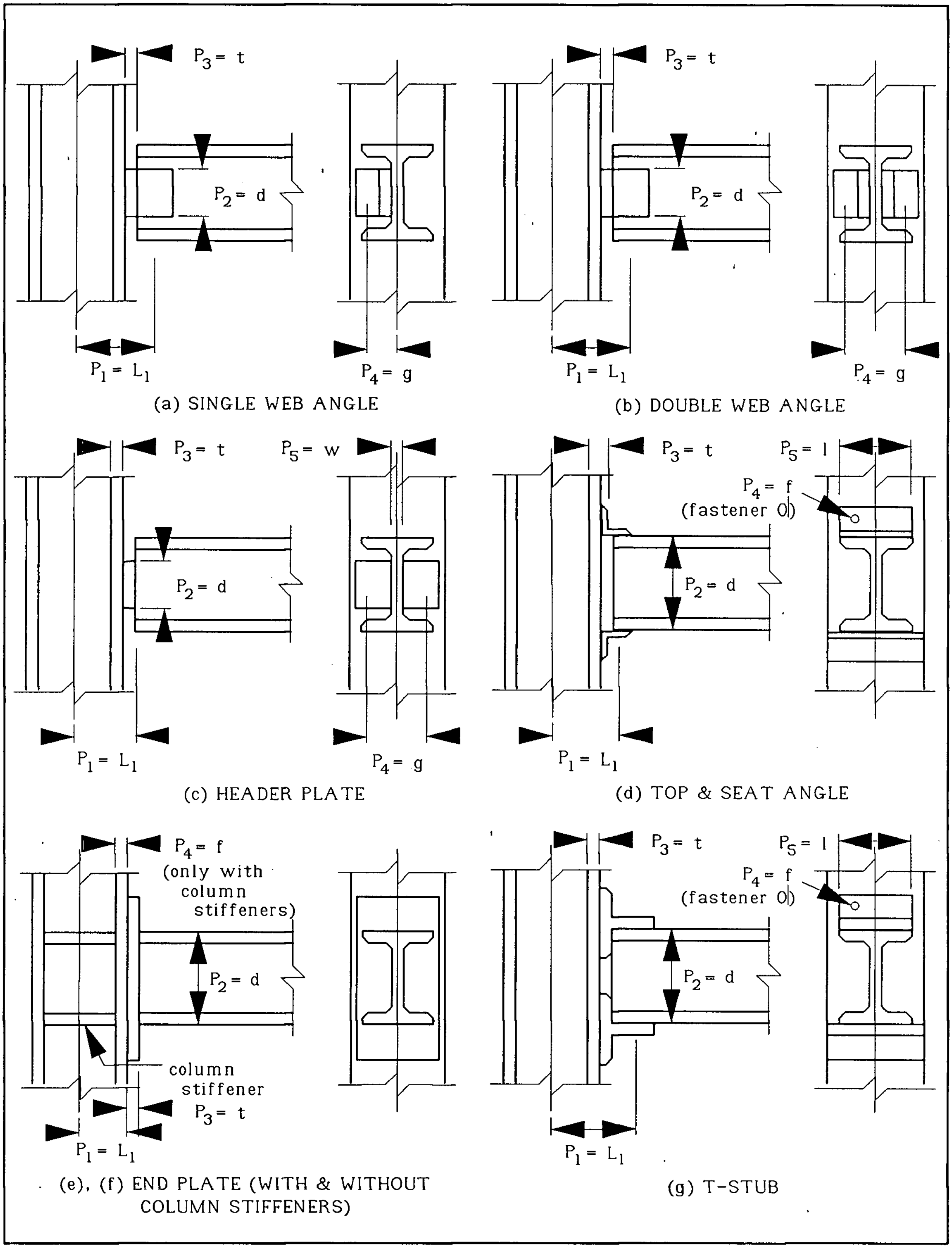


Figure 53 Common Types of Connections and Their Standardization Parameters

Table 1 Standardized Connection Moment-Rotation Functions

Connection Type	Standardized Function, ϕ			Standardization Constant, k			
	C_1	C_2	C_3	a	b	c	d
single web angle	4.28E-03	1.45E-09	1.51E-16	-2.40	-1.81	+0.15	0.00
double web angle	3.66E-04	1.15E-06	4.57E-08	-2.40	-1.81	+0.15	0.00
header plate	5.10E-05	6.20E-10	2.40E-13	-2.30	-1.60	+1.60	+0.50
top & seat angle	8.46E-04	1.01E-04	1.24E-08	-1.50	-0.50	-1.10	-0.70
end plate (without column stiffener)	1.83E-03	-1.04E-04	6.38E-06	-2.40	-0.40	+1.10	0.00
end plate (with column stiffener)	1.79E-03	1.76E-04	2.04E-04	-2.40	-0.60	0.00	0.00
t-stub	2.10E-04	6.20E-06	7.60E-09	-1.50	-0.50	-1.10	-0.70

7 Programming Details

This section discusses in details some practical aspects of implementing the algorithm described above in an existing plane frame analysis program. The general strategy is laid out in Figure 54. For linear connection behaviour only four of the six sections of the program requires modifications. For nonlinear connection behaviour, additional modification has to be made to the output section.

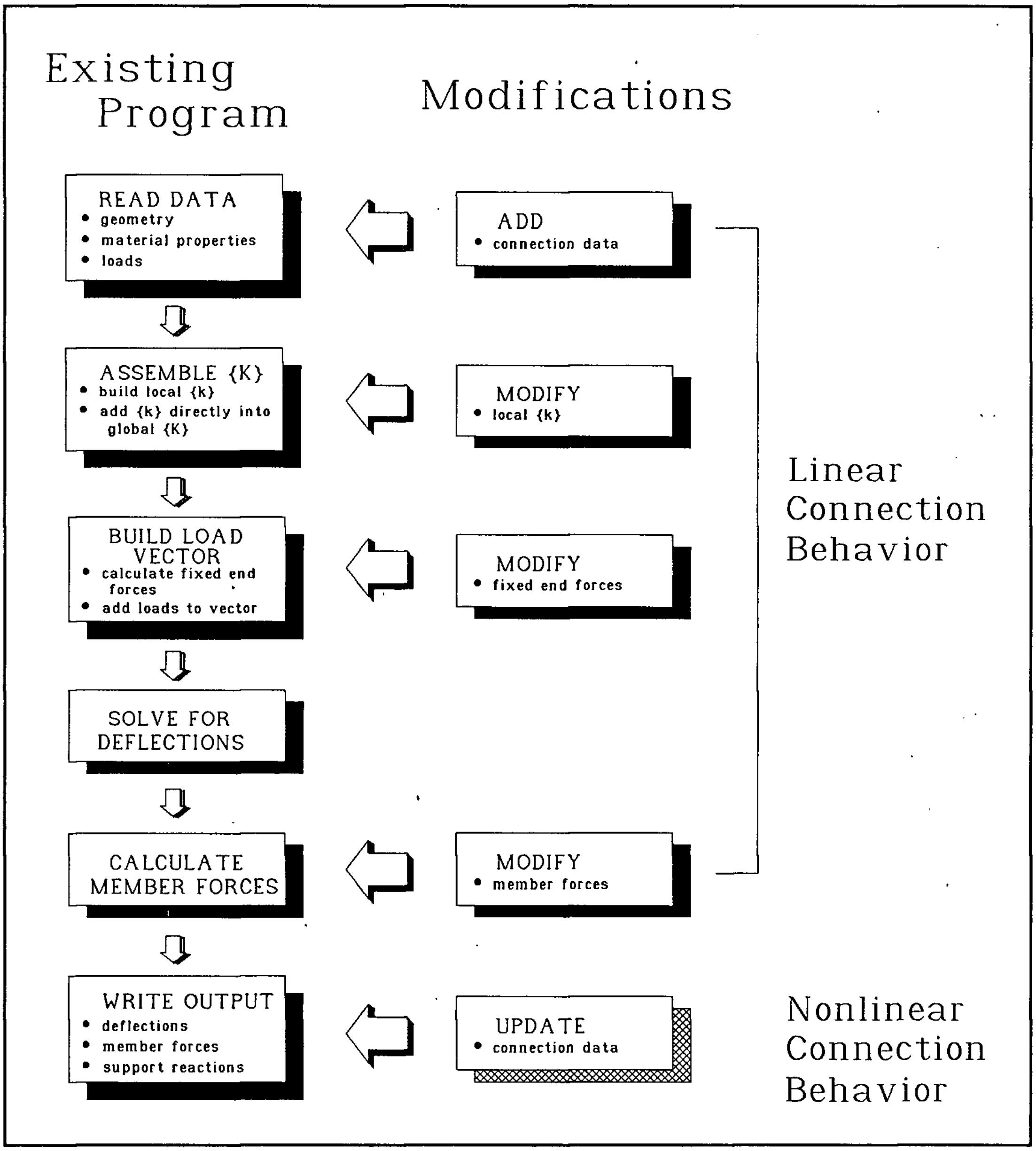


Figure 54 Incorporating Flexible Connections in Plane Frame Analysis

7.1 Modifying Input Format

The input section of the program needs be modified to accommodate the additional connection data. Each connection is identified by the joint and member which it is associated with and the connection type. A maximum of six parameters should be allotted for describing a particular connection (Figure 55) with parameter Mi representing the moment at the particular connection, parameter p1 describing the length of the connection and parameter p2, p3, p4, and p5 containing detail information about the connection (Figure 53).

Connection Data								
n_num	m_num	typ	Mi	p1	p2	p3	p4	p5

Figure 55 Input Format for Connection Data

7.2 Modifying Stiffness Matrix

The global stiffness matrix is assembled directly from the local stiffness matrices of individual members. However, before a local stiffness matrix can be formulated, connection stiffness must first be obtained. This can be accomplished simply by adding to the program a subroutine which reads in connection properties directly as data or calculates connection stiffness according to the connection parameters entered. A sample subroutine which incorporates the equations as outlined by Frye and Morris [4] for calculating connection stiffness is listed in Appendix A.

Table 2 Common Connection Types

Type	Description	k value
1	perfectly hinged	0
2	single web angle	calculated
3	double web angle	calculated
4	header plate	calculated
5	top & seat angle	calculated
6	end plate (without column stiffeners)	calculated
7	end plate (with column stiffeners)	calculated
8	t-stub	calculated
9	other	user specified
10	perfectly rigid	-1E+038

Type 9 connection is to provide user with the option of using his own stiffness values and also to accommodate connections which are not covered in the list. The special value of -1E+38 is adopted to represent the stiffness value of a rigid connection.

The local stiffness matrix can be modified as discussed in Section 3. Sample listing of the modified local stiffness assembly routine can be found in Appendix B. The only point to note is that when defining the dimensionless constants v_1 and v_2, one should be careful to check for the value of k. If k equals -1E+38 for a rigid connection, the program should automatically assign the value of 1 to the corresponding v_1 and v_2.

7.3 Modifying Fixed End Forces

The fixed end forces can be modified as discussed in Section 4. Sample listing of the modified fixed end forces can be found in Appendix C.

7.4 Calculating Member Forces

The member forces can be modified as discussed in Section 5. Sample listing of the modified fixed end forces can be found in Appendix D.

7.5 Modeling Nonlinear Connection Response

The procedure outlined here is the same one presented by Frye and Morris [4] and it is based on the premise that correct structural response and member forces can be obtained from a single linear analysis provided the correct connection stiffness are used. The procedure is therefore a simple iterative process which guesses at the connection stiffness characteristics in a structure at every cycle. When the connection stiffness characteristics converge to sufficient accuracy then these values are used to perform a linear analysis to calculate the correct deflections and member forces for a structure with nonlinear connections.

Figure 56 depicts the typical moment-rotation relationship of a nonlinear connection:

$$\phi = f(M)$$

where

f(M) is a nonlinear function of the moment acting on the connection.

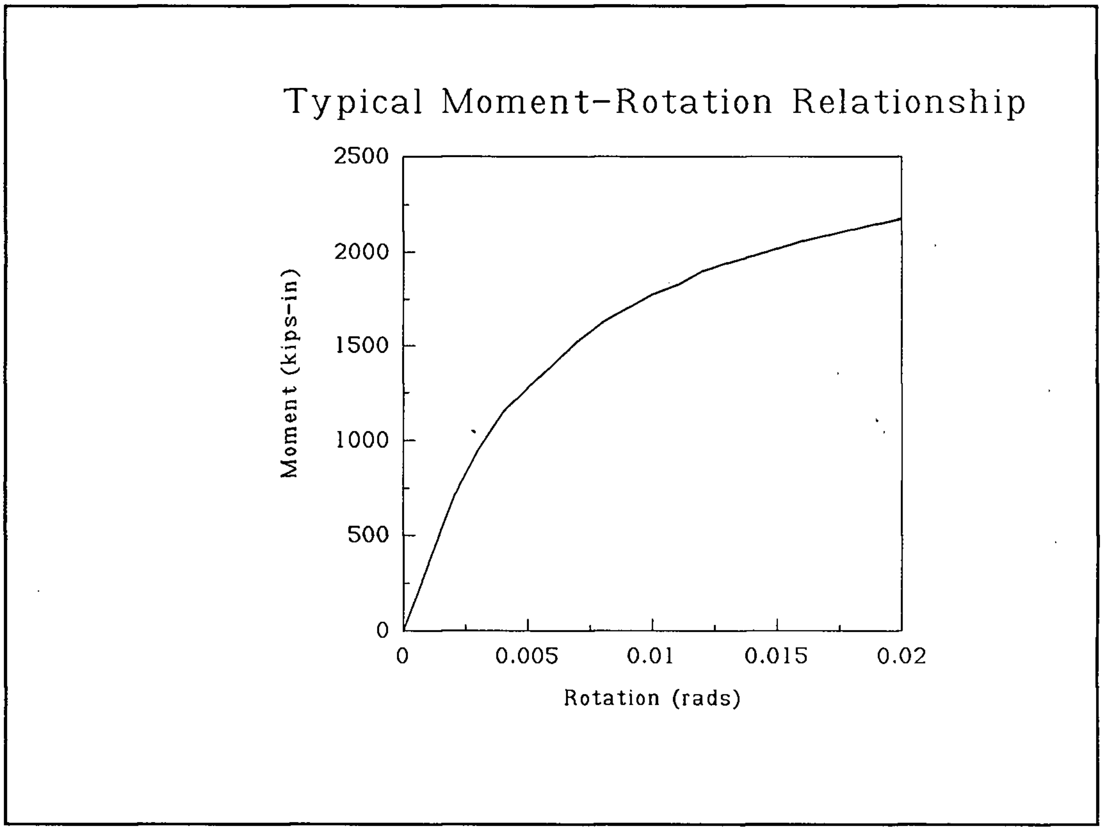


Figure 56 Typical Moment-Rotation Relationship of a Nonlinear Connection

One begins the iterative procedure by assuming M=0, and perform a linear analysis on the structure. This is equivalent to setting the stiffness of the connection equal to the initial tangent of the moment-rotation relationship.

$$k_o = \frac{1}{f'(M=0)}$$

From the analysis one gets a new value of M:

$$M = M_1$$

With this one can calculate:

$$\phi_1 = f(M = M_1),$$
$$k_1 = \frac{M_1}{\phi_1}$$

If k_o and k_1 are sufficiently close then the iterative procedure can be terminated. Otherwise one can repeat the analysis using k_1 to obtain a new value of k_2 and then compare the value of k_1 to k_2. This process is repeated until the stiffness value, k, at each connection stabilizes at a certain value with sufficient accuracy. Figure 57 depicts the convergence of k for a typical nonlinear connection.

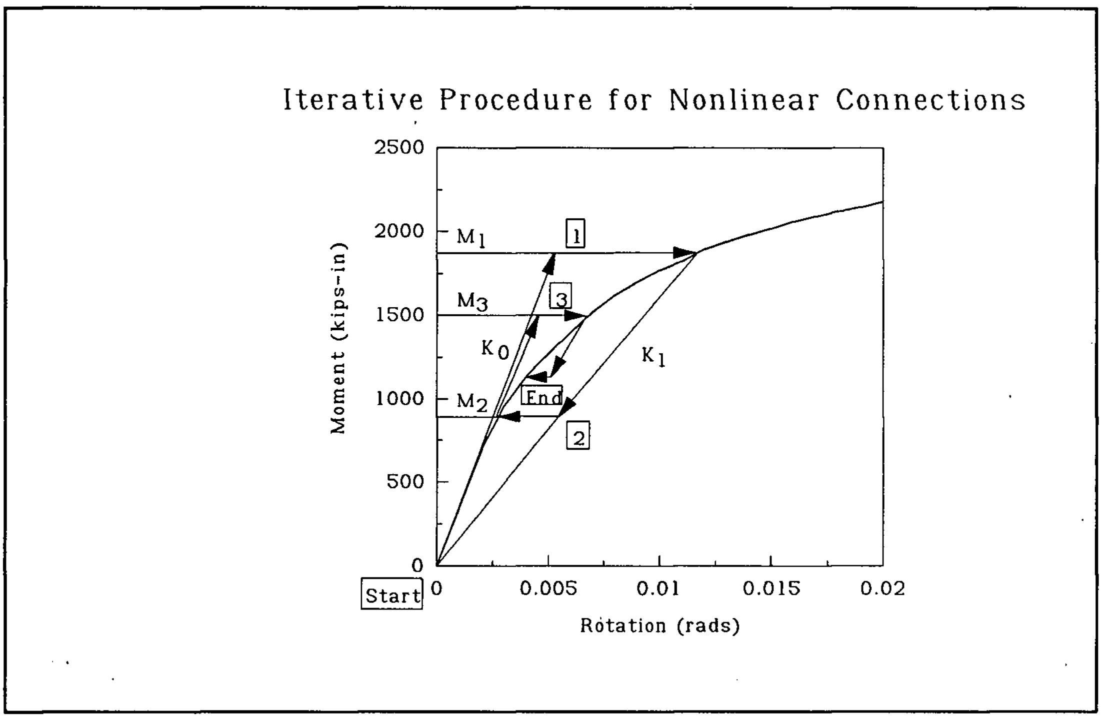


Figure 57 The Convergence of k for Nonlinear Connections

The algorithm for the iterative procedure described above is extremely easy to implement. One needs only to update the connection stiffness of the structure at the end of each cycle of analysis and to add a statement for comparing the values of k.

8 Analysis of a Simple Plane Frame Structure with Flexible Connections

Using the method outlined in previous sections, a standard plane frame analysis program was modified to include the effect of flexible connections. As means of verifying the validity of the resulting program, CPlane, the simple plane frame structure (Figure 58a) previously investigated by Moncarz and Gerstle [10] was analyzed for comparison purposes. The experimentally determined moment-rotation curves for the upper and lower connections are shown in Figure 58b along with the approximations used by Moncarz and Gerstle and the ones used by CPlane. The service load conditions are as follow:

(1) Dead load, g = 1.86 kips/ft or 0.155 kips/in,

(2) Live load, l = 1.20 kips/ft or 0.100 kips/in,

(3) Lateral load intensity, w = 0.00, 0.01, 0.02, 0.03, 0.04 kips/sq ft

Resulting in: W_1 = 0.00, 2.88, 5.76, 8.46, 11.52 kips/sq ft

W_2 = 0.00, 1.44, 2.88, 4.32, 5.74 kips/sq ft

Details of the results are discussed in Section 8.1.

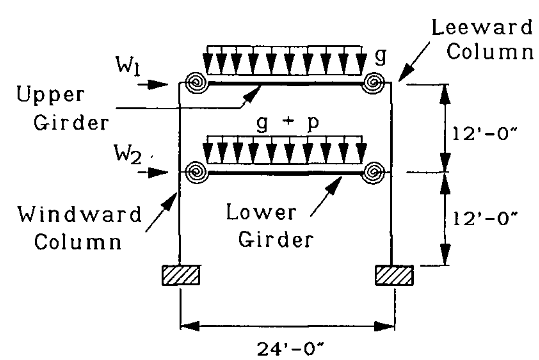


Figure 58a A Simple Plane Frame

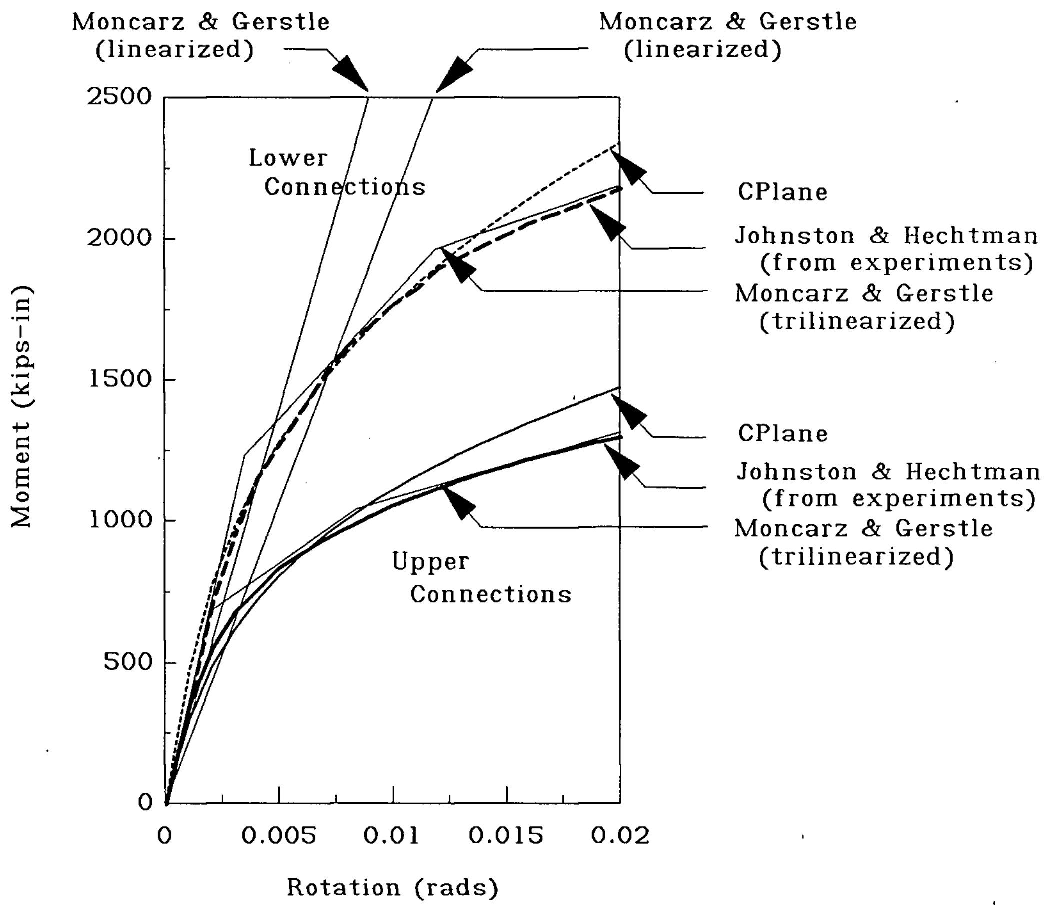
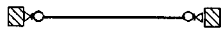
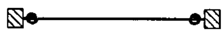
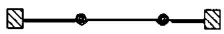
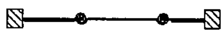
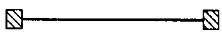
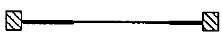


Figure 58b Connection Properties

In order to investigate the effect of flexible connections on internal force distribution of a structure, the same frame as illustrated in Figure 58a was analyzed using various member-connection models with different assumptions on connection behavior. Details of the study are discussed in Sections 8.2 to 8.5. The list of connection models and assumptions used in the study is summarized in Table 3.

Table 3 List of Girder-Column Connection Design Assumptions

CONNECTION TYPE	MODEL	CONNECTION LENGTHS	CONNECTION BEHAVIOR
Perfectly Pinned (PIN)		N/A	Pin
Linear Flexible Connection (LF)		N/A	Linear Elastic
Nonlinear Flexible Connection (NLF)		N/A	Nonlinear
Linear Refined Connection (LR)		ℓ_1, ℓ_2	Linear Elastic
Nonlinear Refined Connection (NLR)		ℓ_1, ℓ_2	Nonlinear
Perfectly Rigid (RIGID)		N/A	Rigid
Rigid Ends (RGEND)		ℓ_1, ℓ_2	Rigid

8.1 Verfication of CPlane

Moncarz and Gerstle [10] investigated the effect of flexible connections on structural response under the assumption of linear and nonlinear connection behavior. However, their formulation neglects the effect of connection sizes and treats connections as point connections. For verification purposes the same set of connection assumptions was used by CPlane and the member forces from the two studies are shown in Figure 59a and Figure 59b.

The column design moments from the two studies are almost identical. It is of interest to note that the assumption of linear connection behavior produced nearly the same result as nonlinear connection behavior.

The critical girder design moments from Moncarz and Gerstle and CPlane follow the same pattern but the values from CPlane are consistently higher by about 5% (Figure 59b). What appears surprising is that the response for rigid connections from the two studies also differ by about the same amount. Surely if identical parameters were used, the rigid connection response from both studies should be the same. Therefore by making the rigid connection response the benchmark for comparison, the results from Moncarz and Gerstle were normalized accordingly and Figure 60 shows the plot of normalized critical girder moments against lateral load intensity. It is clear from the plot that the results from the two studies are practically identical after normalization. The apparent discrepancy probably stems from the fact that the particular girder under investigation is not available from any Canadian steel mills and the section properties selected are slightly different from the ones used by Moncarz and Gerstle. As an independent check, the same structure was analyzed under the assumption of rigid connections by another structural analysis program, LPS, from the Institute für Baustatik der Universität Stuttgart, West Germany. The results from LPS for rigid connections are in exact agreement with that of CPlane (Figure 59b).

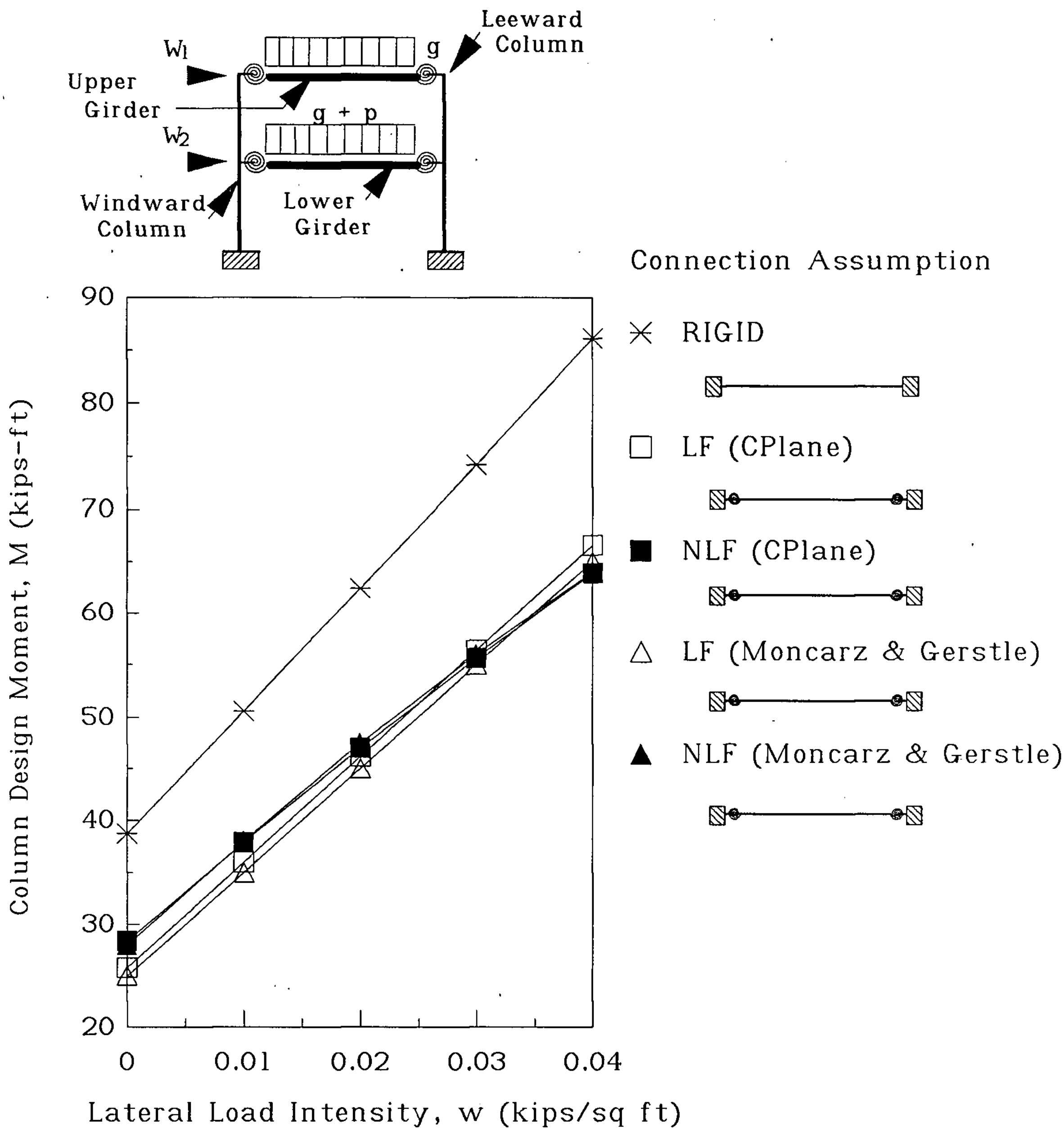


Figure 59a Leeward Column Top Moment versus Lateral Load Intensity

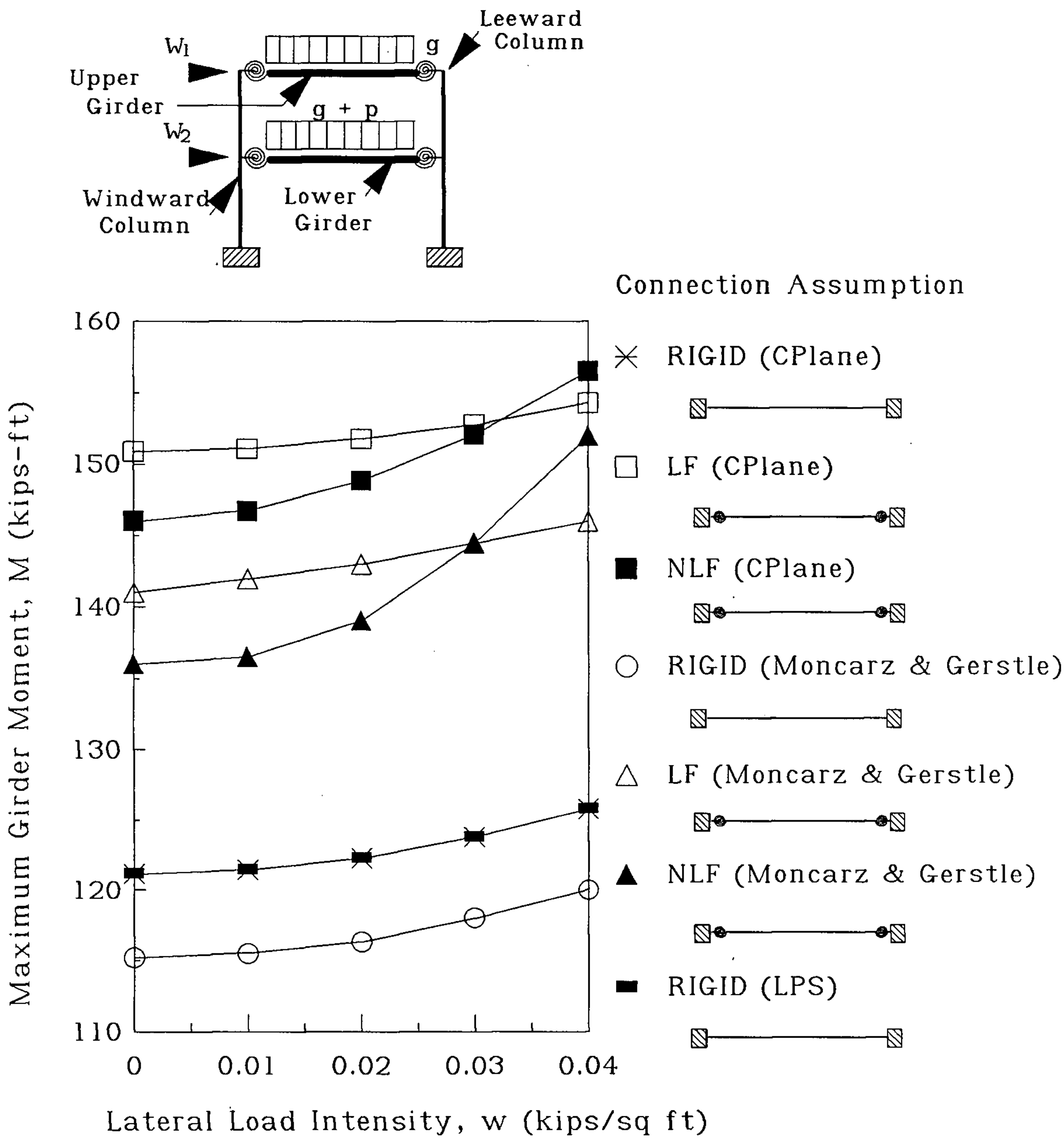


Figure 59b Critical Moment of Lower Girder versus Lateral Load Intensity

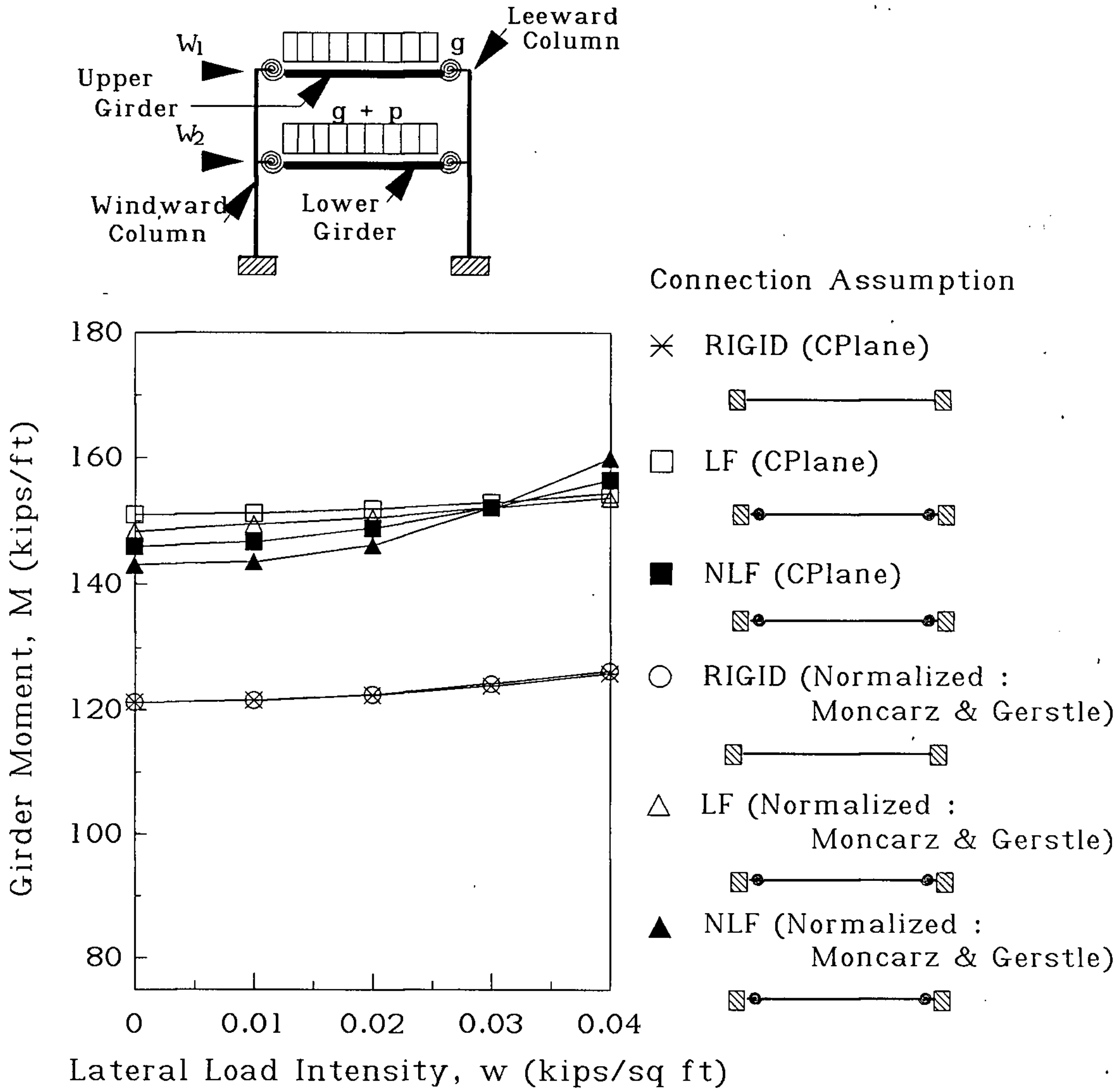


Figure 60 Normalized Moment of Lower Girder versus Lateral Load Intensity

From the close agreement between the normalized moments of Moncarz and Gerstle and that of CPlane and the excellent correlation between LPS and CPlane, it appears reasonable to conclude that the method of incorporating connections in plane frame analysis outlined in this paper is good and gives result comparable to that of Moncarz and Gerstle.

8.2 Girder and Connection Moments

The structure as illustrated in Figure 58a was analyzed again using the various connection models and assumptions described in Table 3 to study the effect of flexible connections on maximum critical girder moments and girder end moments. Figure 61 and Figure 62 plot the critical girder moments and girder end moments of the lower girder against lateral load intensity for the various connection assumptions.

As expected, the critical girder moment varies nonlinearly with lateral load intensity for all connection assumptions with the exception of pin connections. The apparent nonlinearity of the critical girder moment is caused by the variable location of the critical section for differenet lateral load intensity. From Figure 61 it is evident that the common assumption of pin and rigid connections leads to substantial overestimation and underestimation of critical girder moments respectively. The assumption of flexible connections neglecting the effect of connection sizes leads to underestimation of girder stiffness resulting in somewhat lower critical girder moments. Since critical girder moments govern the design of girders, one expects the assumption of pin and rigid connections will lead to conservative and unconservative design of girders respectively and the assumption of flexible connections neglecting connection sizes will result in slightly unconservative design. By making the response from nonlinear refined connections the basis for comparison, Figure 63 shows the difference in girder design moments for the different connection assumptions with lateral load intensity, w, equals 0.02 kips/sq ft.

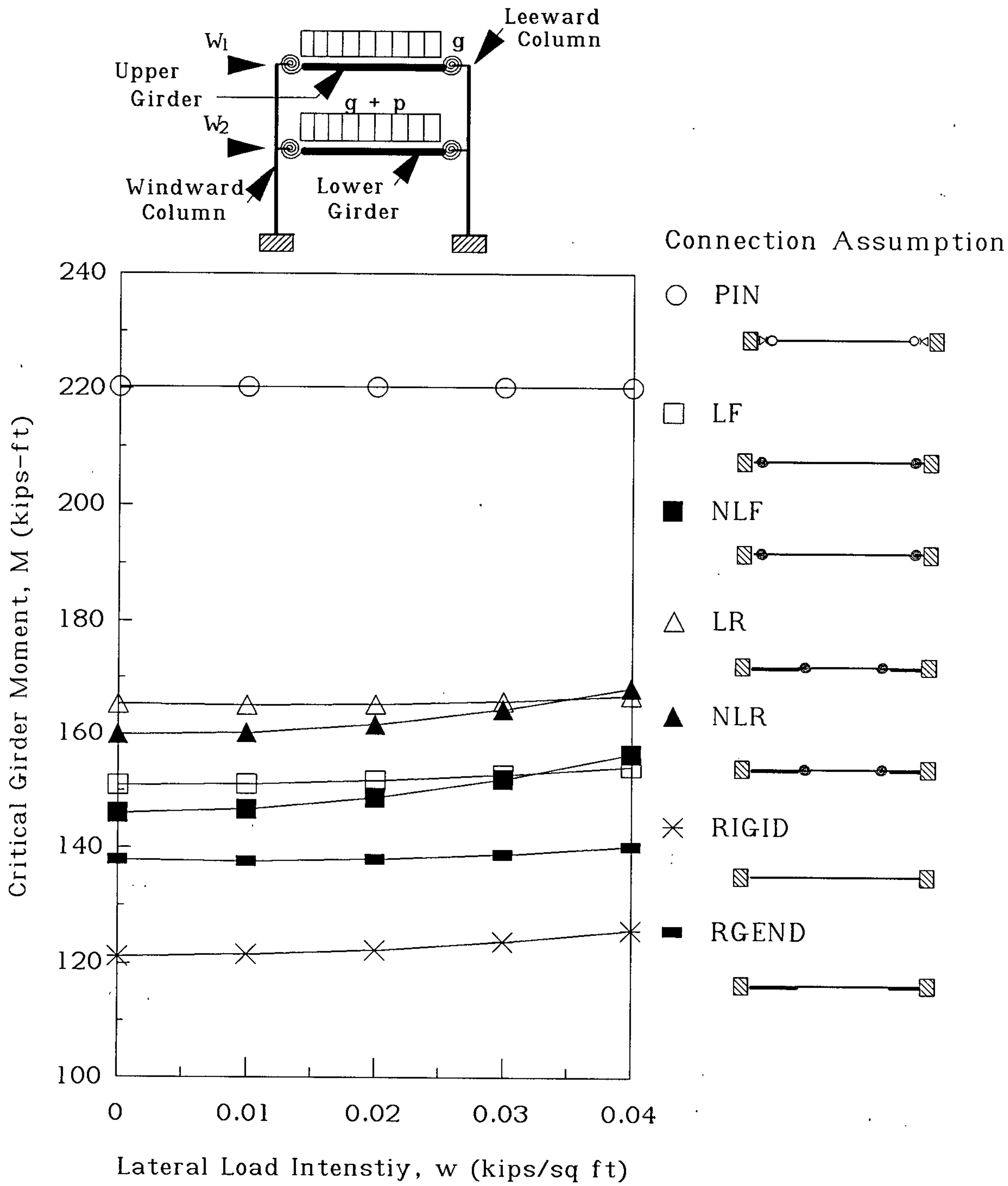


Figure 61 Critical Moment of Lower Girder versus Lateral Load Intensity for Various Connection Assumptions

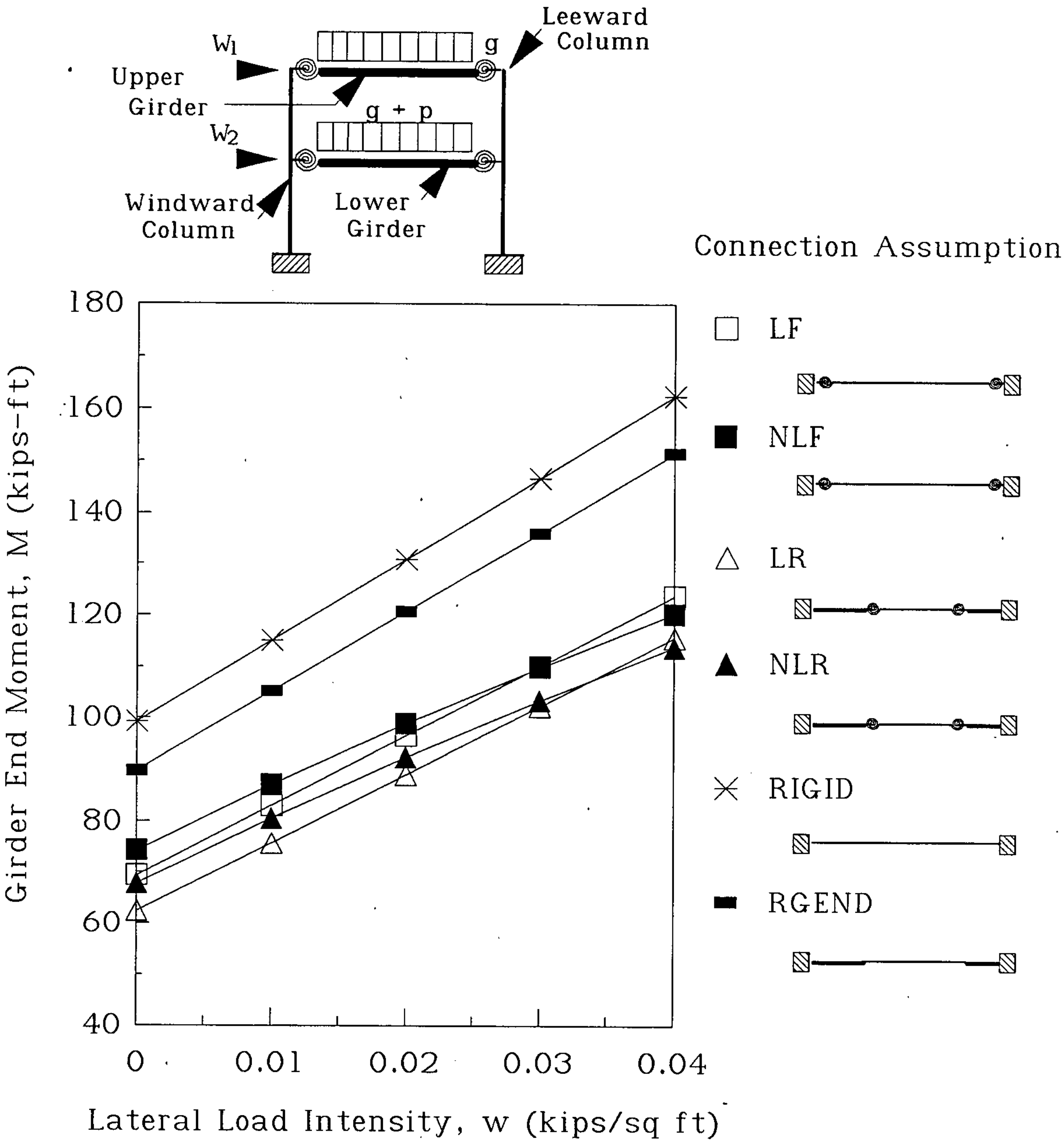


Figure 62 Girder End Moment of Lower Girder versus Lateral Load Intensity for Various Connection Assumptions

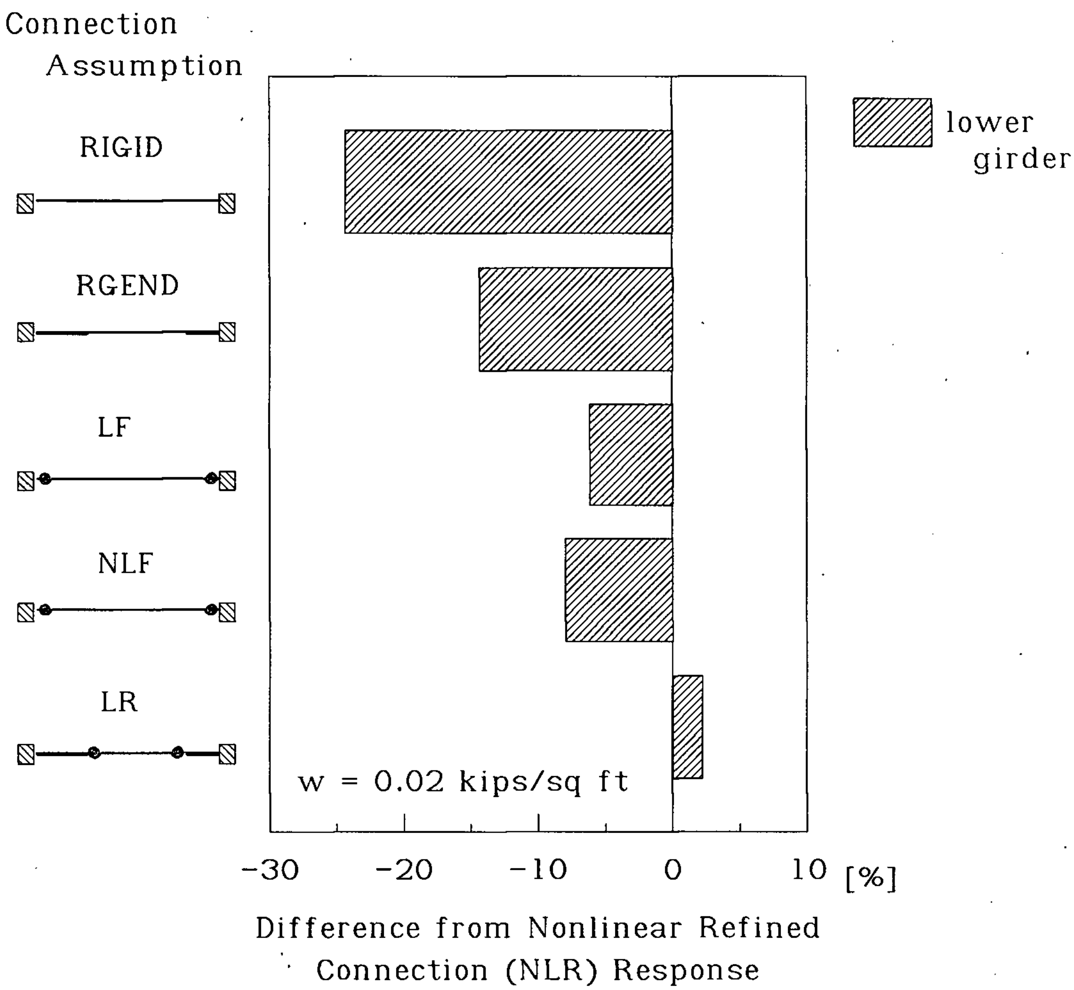


Figure 63 Difference in Lower Girder Design Moments for Various Connection Assumptions, w=0.02

From Figure 62 it is clear that the assumption of rigid connections leads to substantial overestimation of girder end moments while the assumption of flexible connections neglecting connection sizes leads to slight overestimation of girder end moments. Since girder end moments are used in the design of connections, one expects the assumption of rigid connections will lead to conservative design while the assumption of flexible connections neglecting connection sizes will lead to slightly conservative design. Once again by making the response from nonlinear refined connections the basis for comparison, Figure 64 shows the difference in connectionn design moments for the different connection assumptions with lateral load intensity, w, equals 0.02 kips/sq ft.

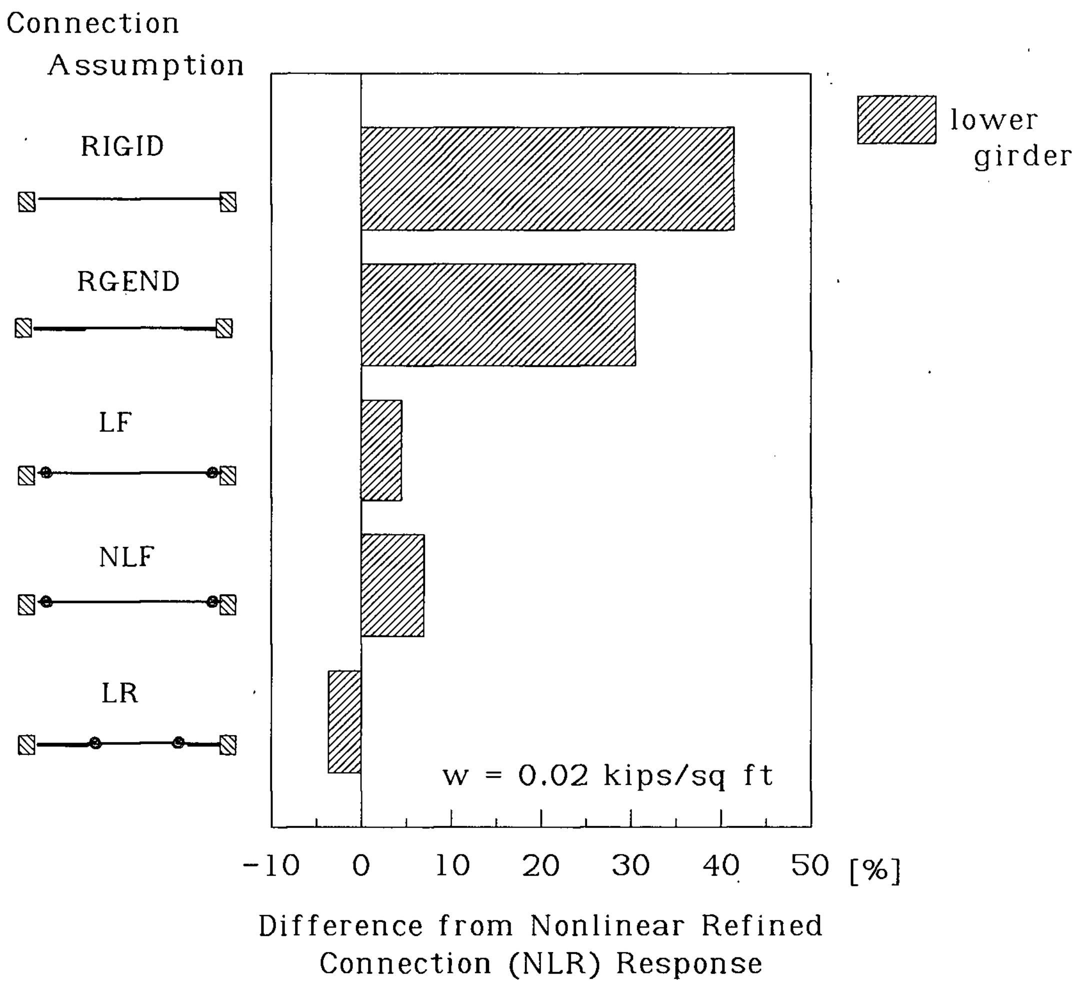


Figure 64 Difference in Lower Connection Design Moments for Various Connection Assumptions, w=0.02

8.3 Column Moments

Figure 65 plots the moments of the leeward lower column top against lateral load intensity. The results show that the assumption of rigid connections leads to substantial overestimation of column moments. However, in practice this is not as significant as it appears because columns are usually designed according to the magnitude of axial forces and the magnitude of column end moments are not quite as important. Nevertheless it is interesting to observe how the different connection assumptions affect these moments.

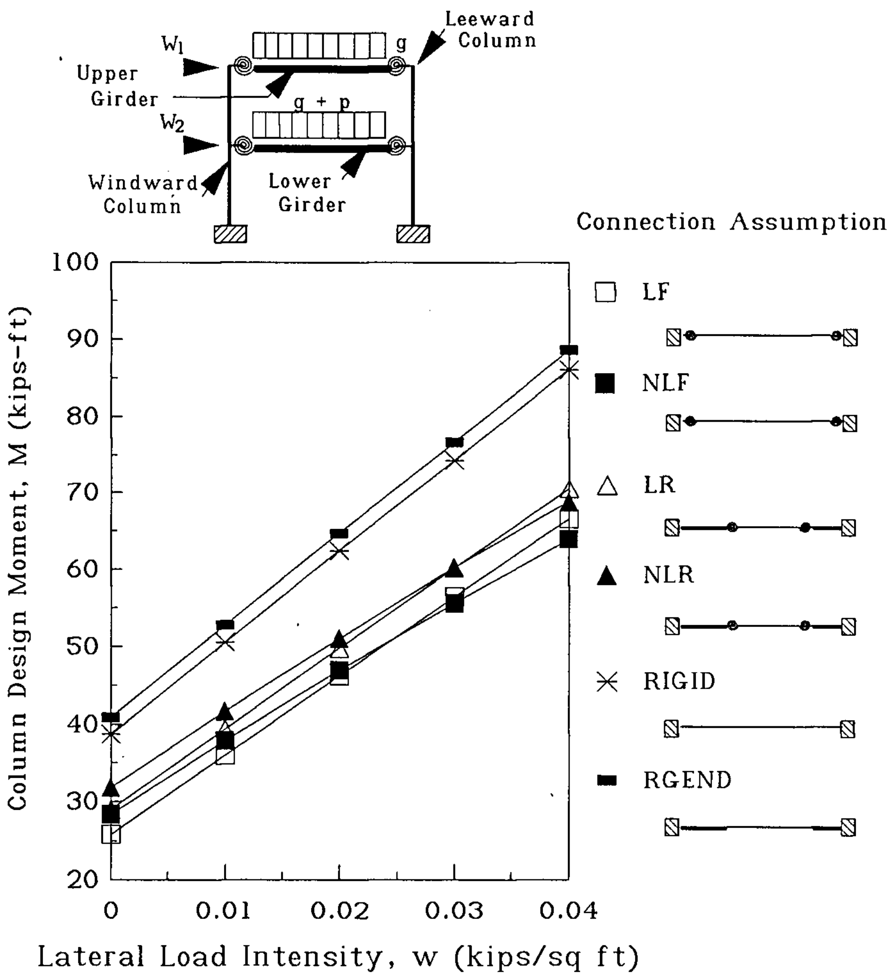


Figure 65 Leeward Lower Column Top Moment versus Lateral Load Intensity for Various Connection Assumptions

8.4 Maximum Top Story Sway

The maximum top story sway varies linearly with lateral load intensity as illustrated in Figure 66. As expected, the assumption of rigid connections results in stiffer structures and leads to underestimation of bare frame sways while the assumption of flexible connections neglecting connection sizes results in more flexible structures and leads to overestimation of bare frame sways.

By making the nonlinear refined connections the basis for comparison, Figure 67 shows the difference in maximum top story sway for the different connection assumptions with the lateral load intensity, w, equals 0.04 kips/sq ft.

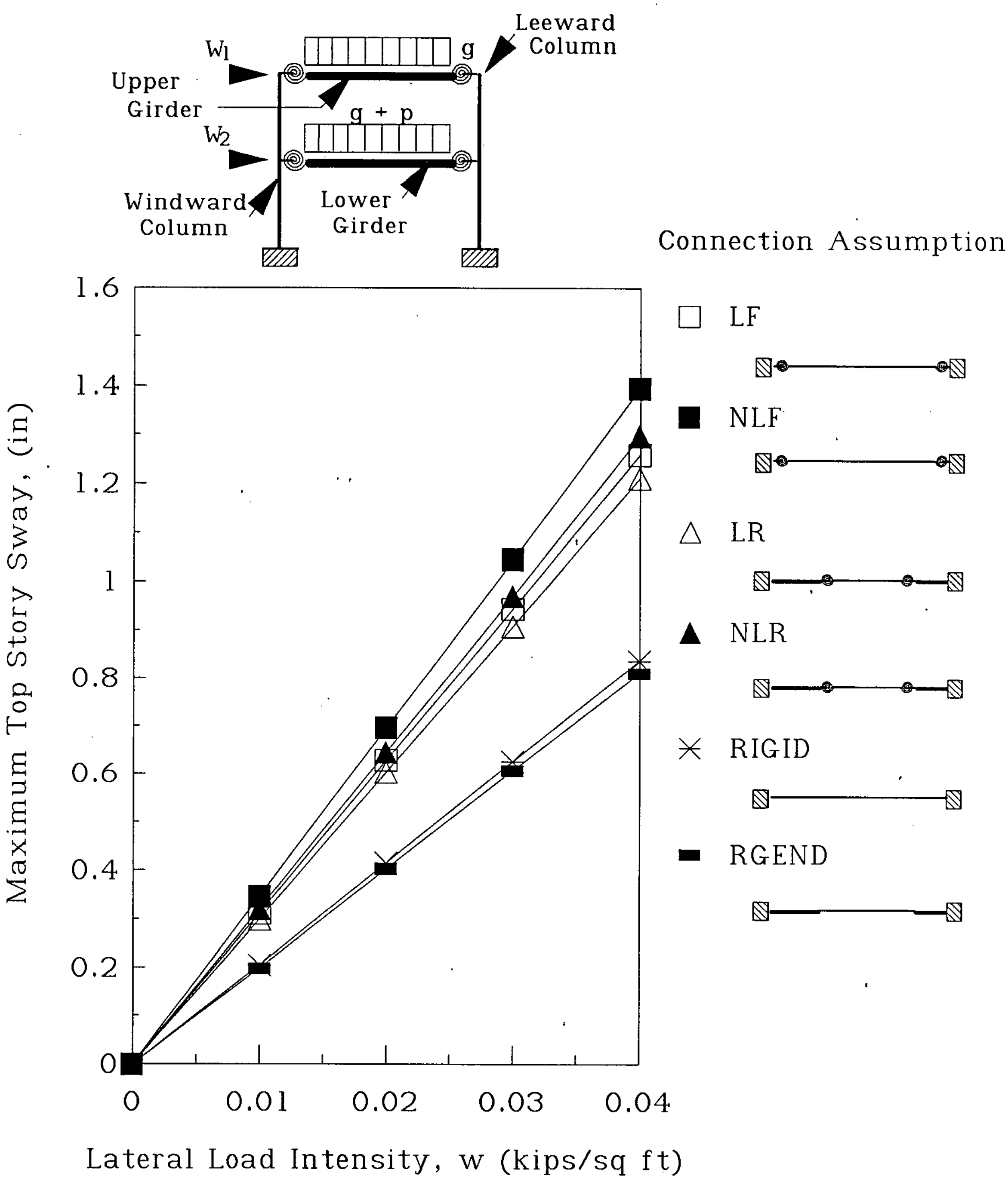


Figure 66 Maximum Top Story Sway versus Lateral Load Intensity for Various Connection Assumptions

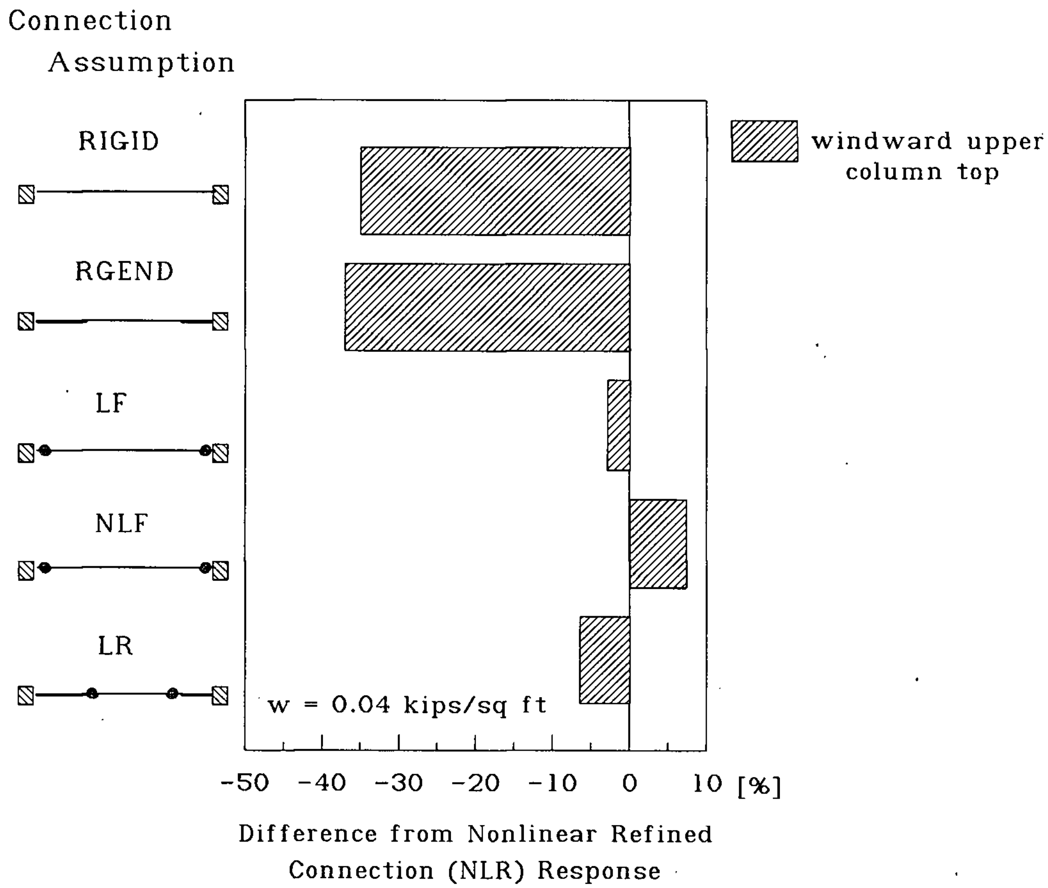


Figure 67 Difference in Maximum Top Story Sways for Various Connection Assumptions, w=0.04

8.5 Linear versus Nonlinear Connection Behavior

From Figure 63, Figure 64, Figure 65 and Figure 67, it appears that the assumption of linear connection response yields satisfactory result provided the connection moment-rotation relationships are correctly linearized. However linearization of these moment-rotation relatioships is a subjective process and good results depend on judgement and experience. Poor linearization of these relationships could only lead to incorrect structural response and therefore caution should be exercised.

9 Conclusion

A practical method of incorporating realistic flexible connections in plane frame analysis has been presented. This method requires modification to the input format, local stiffness matrix, fixed end forces and member forces of a standard plane frame analysis program. Connection stiffness is programmed directly into the analysis by utilizing the connection moment-rotation equations developed by Frye and Morris [4] but may also be entered as data separately. The algorithm presented is very general and it can be used to model linear as well as nonlinear connection response.

A standard plane frame analysis program was modified accordingly and the resulting program, CPlane, was used to analyze the simple plane frame structure previously investigated by Moncarz and Gerstle [10]. The results from CPlane were found to be comparable to that of Moncarz and Gerstle.

The same structure was analyzed again using different connection models and different assumptions on connection behavior for various lateral load intensities. It was found that the inclusion of flexible connections in analysis significantly alter the internal force distribution of a structure. By making the response from nonlinear refined connections the basis for comparison, here are the findings:

Girder Design

1. Pin connections lead to overdesign of girders.
2. Flexible connections neglecting connection sizes lead to slightly unconservative design.
3. Rigid or Rigid-end connections lead to unconservative design of girders.

Connection Design

4. Linear flexible connections neglecting connection sizes lead to slightly conservative design.

5. Rigid or Rigid-end connections lead to conservative design of connections.

Column Design

6. While the column design moments vary substantially for different connection assumptions, this is not significant because in design it is usually the magnitude of axial forces and not the magnitude of column end moments which governs.

Maximum Top Story Sway

7. Flexible connections neglecting connection sizes lead to overestimation of building sway.

8. Rigid or Rigid-end connections lead to underestimation of building sway.

Linear versus Nonlinear Connection Behavior

9. Proper linearization of connection moment-rotation relationships yields satisfactory structural response. However the linearization process is subjective and the quality of the result depends strongly on judgement and experience.

Acknowledgment

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Appendices

```
#include <math.h>
double ConnectKi(typ,mi,p2,p3,p4,p5)
int typ;
float mi,p2,p3,p4,p5;
{
   /*  Local Variables  */
   float c4,c5,c6,y1,y2,y3,y4;
   double k,phi,kmi,ki;
   k=ki=kmi=0.0;
   mi=fabs(mi);
   switch(typ)
   {
     case 1:    /* perfectly hinged connection */
            ki=0.0;
            break;
     case 2:    /* single web connection */
            y1=-2.4;
            y2=-1.81;
            y3=0.15;
            k=pow(p2,y1)*pow(p3,y2)*pow(p4,y3);
            c4=4.28e-3;
            c5=1.45e-9;
            c6=1.51e-16;
            kmi=k*mi;
            break;
     case 3:    /* double web connection */
            y1=-2.4;
            y2=-1.81;
            y3=0.15;
            k=pow(p2,y1)*pow(p3,y2)*pow(p4,y3);
            c4=3.66e-4;
            c5=1.15e-6;
            c6=4.57e-8;
            kmi=k*mi;
            break;
     case 4:    /* header plate connection */
            y1=-2.3;
            y2=-1.60;
            y3=1.6;
            y4=0.5;
            k=pow(p2,y1)*pow(p3,y2)*pow(p4,y3)*pow(p5,y4);
            c4=5.10e-5;
            c5=6.20e-10;
            c6=2.40e-13;
            kmi=k*mi;
            break;
     case 5:    /* top & seat angle connection */
            y1=-1.5;
            y2=-0.5;
            y3=-1.1;
            y4=-0.7;
            k=pow(p2,y1)*pow(p3,y2)*pow(p4,y3)*pow(p5,y4);
            c4=8.46e-4;
            c5=1.01e-4;
            c6=1.24e-8;
            kmi=k*mi;
            break;
     case 6:    /* end plate connection (without column stiffeners) */
            y1=-2.4;
            y2=-0.4;
```

```
          y3=1.1;
          k=pow(p2,y1)*pow(p3,y2)*pow(p4,y3);
          c4=1.83e-3;
          c5=-1.04e-4;
          c6=6.38e-6;
          kmi=k*mi;
          break;
      case 7:    /* end plate connection (with column stiffeners) */
          y1=-2.4;
          y2=-0.6;
          k=pow(p2,y1)*pow(p3,y2);
          c4=1.79e-3;
          c5=1.76e-4;
          c6=2.04e-4;
          kmi=k*mi;
          break;
      case 8:    /* t-stub connection */
          y1=-1.5;
          y2=-0.5;
          y3=-1.1;
          y4=-0.7;
          k=pow(p2,y1)*pow(p3,y2)*pow(p4,y3)*pow(p5,y4);
          c4=2.1e-4;
          c5=6.2e-6;
          c6=7.60e-9;
          kmi=k*mi;
          break;
      case 9:    /* other */
          ki=p2;
          break;
      case 10:   /* perfectly rigid */
          ki=-1.00e+38;
          break;
    }
    if (typ>1 && typ <9)
    {
      kmi=k*mi;
      phi=c4*kmi+c5*pow(kmi,3.)+c6*pow(kmi,5.);
      if (phi<=1.e-30)
        ki=1./(k*c4);
      else
        ki=mi/phi;
    }
    return(ki);
}
```

```
#include <stdio.h>
#include <math.h>
#include "Globals.h"
#include "inout.h"
extern float E,G;
extern struct BEAMSTIFFNESS theStorage;
struct BEAMSTIFFNESS *BeamStiffnessFor(TPBeam)
struct BEAM *TPBeam;
{
    /* Local Variables */
    struct BEAMSTIFFNESS *Local = &theStorage;
    int DOF[6],k12;
    float l1,l2;
    double K[22];
    double c,cc,s,ss,cs,xm,ym,l,dm,xm2,ym2,dm2,g,nu1,nu2;
    double C1,C2,C3,C4,C5,C6,S1,S2,S3,S4,S5,S6,f1,f2,f3,f4,k1,k2;
    int i;
    c=cc=1.;
    s=ss=cs=0.;
    DOF[0] = TPBeam->node1->DOF[0];
    DOF[1] = TPBeam->node1->DOF[1];
    DOF[2] = TPBeam->node1->DOF[2];
    DOF[3] = TPBeam->node2->DOF[0];
    DOF[4] = TPBeam->node2->DOF[1];
    DOF[5] = TPBeam->node2->DOF[2];
    k12 = TPBeam->k12;
    xm = (TPBeam->node2->X) - (TPBeam->node1->X);
    ym = (TPBeam->node2->Y) - (TPBeam->node1->Y);
    xm2 = xm*xm;
    ym2 = ym*ym;
    dm2 = xm2+ym2;
    dm  = sqrt(dm2);
    c=xm/dm;
    cc=c*c;
    s=ym/dm;
    ss=s*s;
           /*  Connection Sizes    */
    if (TPBeam->connect1==NULL)
      l1=0.;
    else
      l1=(TPBeam->connect1->p1);
    if (TPBeam->connect2==NULL)
      l2=0.;
    else
      l2=(TPBeam->connect2->p1);
    l=dm-l1-l2;
           /*  Flexible Connection */
    k1=k2=-1.00e+38;
    if (TPBeam->connect1==NULL)
      nu1=1;
    else
    {
      k1=(TPBeam->connect1->Ki);
      if (k1>0. && k1<=1.e-30)
        nu1=0.;
      else
        if (k1==-1.00e+38)
          nu1=1.;
        else
          nu1=1./(1.+((3*E*(TPBeam->inertia))/(k1*l)));
```

```
  }
if (TPBeam->connect2==NULL)
  nu2=1.;
else
{
  k2=(TPBeam->connect2->Ki);
  if (k2>0. && k2<=1.e-30)
    nu2=0.;
  else
    if (k2==-1.00e+38)
      nu2=1.;
    else
      nu2=1./(1.+((3*E*(TPBeam->inertia))/(k2*l)));
  }
        /*  Shear Deflection  */
g = 0.0;
if ((TPBeam->ashear) != 0.0 && G != 0.0)
  g=12.*E*(TPBeam->inertia)/((TPBeam->ashear)*G*l*l);

/*
Storage Allocation for K
1   1  2  3  4  5  6
2      7  8  9 10 11
3        12 13 14 15
4           16 17 18
5              19 20
6                 21
    1  2  3  4  5  6

Member Degrees of freedom  DOF
              1         4
              |         |
      0 ---> %=========% ---> 3
             '->2       '->5

(+)ve: right, up, ccw
*/

     /***  Defining constants for pin_pin member  ***/
fl=(TPBeam->area)*E/l;
  /* fill in pin_pin section of member stiffness matrix */
K[1]  =  fl*cc;
K[2]  =  fl*cs;
K[3]  =  0.0;
K[4]  = -fl*cc;
K[5]  = -fl*cs;
K[6]  =  0.0;
K[7]  =  fl*ss;
K[8]  =  0.0;
K[9]  = -fl*cs;
K[10] = -fl*ss;
K[11] =  0.0;
K[12] =  0.0;
K[13] =  0.0;
K[14] =  0.0;
K[15] =  0.0;
K[16] =  fl*cc;
K[17] =  fl*cs;
K[18] =  0.0;
K[19] =  fl*ss;
K[20] =  0.0;
K[21] =  0.0;
```

```
if (k12 != pin_pin)
{
      /*** Defining constants for fix_fix member   ***/
  if (k12 == fix_fix)
  {
    C1 = (nu1+nu2+nu1*nu2)/(4-nu1*nu2);
    C2 =  nu1*(2+nu2)/(4-nu1*nu2);
    C3 =  3*nu1/(4-nu1*nu2);
    C4 =  nu2*(2+nu1)/(4-nu1*nu2);
    C5 =  3*nu1*nu2/(4-nu1*nu2);
    C6 =  3*nu2/(4-nu1*nu2);
    S1 =  1/(1+g*C1);
    S2 = (1+g*nu2/4)/(1+g*C1);
    S3 = (1-g/2)/(1+g*C1);
    S4 = (1+g*nu1/4)/(1+g*C1);
    f1 = 12.*E*(TPBeam->inertia)/(l*l*l)*C1*S1;
    f2 = 12.*E*(TPBeam->inertia)/(l*l)*C2*S1;
    f3 = 12.*E*(TPBeam->inertia)/(l*l)*C4*S1;
    f4 = 4.*E*(TPBeam->inertia)/l;
  }
  else
  {
   if (k12 == fix_pin)
   {
     S5 = 1/(1+g*nu1/4);
     f1 = 3.*E*(TPBeam->inertia)/(l*l*l)*nu1*S5;
     f2 = 6*E*(TPBeam->inertia)/(l*l)*nu1*S5;
     f3 = 0;
     f4 = 3*E*(TPBeam->inertia)/l;
    }
   else
     if (k12 == pin_fix)
     {
       S6 = 1/(1+g*nu2/4);
       f1 = 3.*E*(TPBeam->inertia)/(l*l*l)*nu2*S6;
       f2 = 0;
       f3 = 6*E*(TPBeam->inertia)/(l*l)*nu2*S6;
       f4 = 3*E*(TPBeam->inertia)/l;
      }
   }
/* fill in terms which are common to pin_fix, fix_pin and fix_fix members
*/
    K[1]  += f1*ss;
    K[2]  -= f1*cs;
    K[3]  -= f1*l1*s;
    K[4]  -= f1*ss;
    K[5]  += f1*cs;
    K[6]  -= f1*l2*s;
    K[7]  += f1*cc;
    K[8]  += f1*l1*c;
    K[9]  += f1*cs;
    K[10] -= f1*cc;
    K[11] += f1*l2*c;
    K[12] += f1*l1*l1;
    K[13] += f1*l1*s;
    K[14] -= f1*l1*c;
    K[15] += f1*l1*l2;
    K[16] += f1*ss;
    K[17] -= f1*cs;
    K[18] += f1*l2*s;
    K[19] += f1*cc;
    K[20] -= f1*l2*c;
    K[21] += f1*l2*l2;
       /* fill in remaining fix_fix terms */
```

```
 if (k12 == fix_fix)
 {
   K[3]  -= 0.5*f2*s;
   K[6]  -= 0.5*f3*s;
   K[8]  += 0.5*f2*c;
   K[11] += 0.5*f3*c;
   K[12] += f4*C3*S2 + f2*l1;
   K[13] += 0.5*f2*s;
   K[14] -= 0.5*f2*c;
   K[15] += 0.5*f4*C5*S3 + 0.5*f3*l1 + 0.5*f2*l2;
   K[18] += 0.5*f3*s;
   K[20] -= 0.5*f3*c;
   K[21] += f4*C6*S4 + f3*l2;
  }
    /* fill in remaining fix_pin terms */
    else
      if (k12 == fix_pin)
      {
        K[3]  -= 0.5*f2*s;
        K[8]  += 0.5*f2*c;
        K[12] += f4*nu1*S5;
        K[13] += 0.5*f2*s;
        K[14] -= 0.5*f2*c;
        K[15] += 0.5*f2*l2;
       }
    /* fill in remaining pin_fix terms */
      if (k12 == pin_fix)
      {
        K[6]  -= 0.5*f3*s;
        K[11] += 0.5*f3*c;
        K[15] += 0.5*f3*l1;
        K[18] += 0.5*f3*s;
        K[20] -= 0.5*f3*c;
        K[21] += f4*nu2*S6 + f3*l2;
       }
 }
 for (i=1; i<=21; ++i)
 {
   Local->K[i] = K[i];
  }
 for (i=0; i<=5; ++i) Local->DOF[i] = DOF[i];
 return(Local);
}
```

```
void PerpBeamLoad (shear1, shear2, bm1, bm2, dm, dm2, TPBLoad, TPBeam)
float *shear1;
float *shear2;
float *bm1;
float *bm2;
float dm;
double dm2;
struct BLOAD *TPBLoad;
struct BEAM *TPBeam;
{
    float w12, w18, b, l1, l2, l, k1, k2;
    float w, P, a;
    double nu1, nu2, C0, C1, g, S1, A1, A2, A3, A4, Rx1, Rx2, Ry1, Ry2;
      /*  Connection Sizes  */
    l=sqrt(dm2);
    if (TPBeam->connect1==NULL)
      l1=0.;
    else
      l1=(TPBeam->connect1->pl);
    if (TPBeam->connect2==NULL)
      l2=0.;
    else
      l2=(TPBeam->connect2->pl);
    l=dm-l1-l2;
      /*  Flexible Connection */
    if (TPBeam->connect1==NULL)
      nu1=1.;
    else
    {
      k1=(TPBeam->connect1->Ki);
      if (k1==0.0)
        nu1=0.;
      else
        if (k1==-1.00e+38)
          nu1=1.;
        else
          nu1=1/(1+((3*E*(TPBeam->inertia))/(k1*l)));
    }
    if (TPBeam->connect2==NULL)
      nu2=1.;
    else
    {
      k2=(TPBeam->connect2->Ki);
      if (k2==0.0)
        nu2=0.;
      else
        if (k2==-1.00e+38)
          nu2=1.;
        else
          nu2=1/(1+((3.*E*(TPBeam->inertia))/(k2*l)));
    }
      /*  Shear Deflection  */
    g=0.0;
    if ((TPBeam->ashear) != 0.0 && G != 0.0)
      g=12.*E*(TPBeam->inertia)/((TPBeam->ashear)*G*l*l);
```

```
    w=(TPBLoad->w)*dm/l;
    P=TPBLoad->P;
    a=TPBLoad->a-l1;
    b = l-a;
    switch (TPBeam->k12)
    {
    case fix_fix:
         C0  = 1/(4-nu1*nu2);
         C1  = (nu1+nu2+nu1*nu2)/(4-nu1*nu2);
         S1  = 1/(1+g*C1);
         if (P != 0.0)
         {
           /* calculate the Beam Forces for final output */
           A1  =
1+a/(l*l)*(4*l*(nu1-nu2)-nu1*a*(2+nu2)+nu2*b*(2+nu1))*C0*S1;
           A2  =
1+b/(l*l)*(4*l*(nu2-nu1)+nu1*a*(2+nu2)-nu2*b*(2+nu1))*C0*S1;
           A3  = (nu1*l*(4+nu2*(1.5*g-1))-nu1*a*(2+nu2))/b*C0*S1;
           A4  = (nu2*l*(4+nu1*(1.5*g-1))-nu2*b*(2+nu1))/a*C0*S1;
           Ry1 = P*b/l*A1;
           Ry2 = P*a/l*A2;
           *shear1 += Ry1;
           *shear2 -= Ry2;
           *bm1    -= P*a*b*b/(l*l)*A3+Ry1*l1;
           *bm2    -= P*a*a*b/(l*l)*A4+Ry2*l2;
         }
         if (w != 0.0)
         {
           /* calculate the Beam Forces for final output */
           wl2 = w*l/2.0;
           C1  = (nu1+nu2+nu1*nu2)/(4-nu1*nu2);
           A1  = 1+(nu1-nu2)/(4-nu1*nu2)/(1+g*C1);
           A2  = 1+(nu2-nu1)/(4-nu1*nu2)/(1+g*C1);
           A3  = 3*nu1*(2-nu2*(1-g))*C0*S1;
           A4  = 3*nu2*(2-nu1*(1-g))*C0*S1;
           Ry1 = wl2*A1;
           Ry2 = wl2*A2;
           *shear1 += Ry1;
           *shear2 -= Ry2;
           *bm1    -= wl2*l/6.0*A3+Ry1*l1;
           *bm2    -= wl2*l/6.0*A4+Ry2*l2;
         }
         break;

   case pin_fix:
         if (P != 0.0)
         {
           /* calculate the Beam Forces for final output */
           A1  = nu2/(1+g*nu2/4);
           A2  = a*b*(b+2.0*a)/(2.0*l*l*l);
           Ry1 = P*a/l + P*A2*A1;
           Ry2 = P*b/l - P*A2*A1;
           *shear1 += Ry1;
           *shear2 -= Ry2;
           *bm1    -= Ry1*l1;
           *bm2    -= P*A2*l+Ry2*l2;
         }
         if (w != 0.0)
         {
           /* calculate the Beam Forces for final output */
           wl8 = w*l/8.0;
           A1  = (4-nu2*(1-g))/(1+g*nu2/4);
           A2  = (4+nu2*(1+g))/(1+g*nu2/4);
           A3  = nu2/(1+g*nu2/4);
```

```
        Ryl = wl8*Al;
        Ry2 = wl8*A2;
        *shearl += Ryl;
        *shear2 -= Ry2;
        *bml    -= Ryl*ll;
        *bm2    -= wl8*l*A3+Ry2*l2;
      }
      break;

  case fix_pin:
      if (P != 0.0)
      {
         /* calculate the Beam Forces for final output */
        Al  = nul/(1+g*nul/4);
        A2  = a*b*(a+2.0*b)/(2.0*l*l*l);
        Ryl = P*b/l + P*A2*Al;
        Ry2 = P*a/l - P*A2*Al;
        *shearl += Ryl;
        *shear2 -= Ry2;
        *bml    -= P*A2*l+Ryl*ll;
        *bm2    -= Ry2*l2;
      }
      if (w != 0.0)
      {
         /* calculate the Beam Forces for final output */
        wl8 = w*l/8.0;
        Al  = (4+nul*(l+g))/(1+g*nul/4);
        A2  = (4-nul*(l-g))/(1+g*nul/4);
        A3  = nul/(1+g*nul/4);
        Ryl = wl8*Al;
        Ry2 = wl8*A2;
        *shearl += Ryl;
        *shear2 -= Ry2;
        *bml    -= wl8*l*A3+Ryl*ll;
        *bm2    -= Ry2*l2;
      }
      break;
   case pin_pin:
      if (P != 0.0)
      {
         /* calculate the Beam Forces for final output */
         *shearl = P*b/l;
         *shear2 =-P*a/l;
      }
      if (w != 0.0)
      {
         /* calculate the Beam Forces for final output */
         wl2 = w*l/2.0;
         *shearl += wl2;
         *shear2 -= wl2;
      }
      break;
   }
}
```

```
void XYdirBeamLoad (px1, px2, pm1, pm2, xym,  dm2, TPBLoad, TPBeam)
float *px1;
float *px2;
float *pm1;
float *pm2;
double xym, dm2;
struct BLOAD *TPBLoad;
struct BEAM  *TPBeam;
{
    float wl2, wxy2, wl8, wxy8, b, l1, l2, l, k1, k2;
    float w, P, a;
    double dm, nu1, nu2, C0, C1, g, S1, A1, A2, A3, A4, Rxy1, Rxy2;
    dm=sqrt(dm2);
       /*  Connection Sizes  */
    if (TPBeam->connect1==NULL)
      l1=0.;
    else
      l1=(TPBeam->connect1->p1)*xym/dm;
    if (TPBeam->connect2==NULL)
      l2=0.;
    else
      l2=(TPBeam->connect2->p1)*xym/dm;
    l=dm-l1-l2;
       /*  Flexible Connection */
    if (TPBeam->connect1==NULL)
      nu1=1;
    else
    {
      k1=(TPBeam->connect1->Ki);
      if (k1==0.0)
        nu1=0.;
      else
        if (k1==-1.00e+38)
          nu1=1.;
        else
          nu1=1/(1+((3*E*(TPBeam->inertia))/(k1*l)));
     }
    if (TPBeam->connect2==NULL)
      nu2=1.;
    else
    {
      k2=(TPBeam->connect2->Ki);
      if (k2==0.0)
        nu2=0.;
      else
        if (k2==-1.00e+38)
          nu2=1.;
        else
          nu2=1/(1+((3*E*(TPBeam->inertia))/(k2*l)));
     }
       /*  Shear Deflection  */
    g=0.0;
    if ((TPBeam->ashear) != 0.0 && G != 0.0)
      g=12.*E*(TPBeam->inertia)/((TPBeam->ashear)*G*l*l);

    w=(TPBLoad->w)*dm/l;
    P=TPBLoad->P;
    a=TPBLoad->a-l1;
    b = l-a;

    switch (TPBeam->k12)
    {
```

```
case fix_fix:
   C0  = 1/(4-nu1*nu2);
   C1  = (nu1+nu2+nu1*nu2)/(4-nu1*nu2);
   S1  = 1/(1+g*C1);
   if (P != 0.0)
   {
     /* calculate the structure applied loads */
     if (l>0.0)
     {
       A1   = 1+a/(l*l)*(4*l*(nu1-nu2)
              -nu1*a*(2+nu2)+nu2*b*(2+nu1))*C0*S1;
       A2   = 1+b/(l*l)*(4*l*(nu2-nu1)
              +nu1*a*(2+nu2)-nu2*b*(2+nu1))*C0*S1;
       A3   = (nu1*l*(4+nu2*(1.5*g-1))-nu1*a*(2+nu2))/b*C0*S1;
       A4   = (nu2*l*(4+nu1*(1.5*g-1))-nu2*b*(2+nu1))/a*C0*S1;
       *px1 += P*b/l*A1;
       *px2 += P*a/l*A2;
       Rxy1  = *px1;
       Rxy2  = *px2;
       *pm1 -= sign(xym)*P*a*b*b/(l*l)*A3+Rxy1*l1;
       *pm2 += sign(xym)*P*a*a*b/(l*l)*A4+Rxy2*l2;
     }
     else
     {
       *px1 += P*b/l;
       *px2 += P - (*px1);
     }
   }
   if (w!=0.0)
   {
     if (l>0.0)
     {
       /* calculate the structure applied loads */
       wl2  = w*l/2.0;
       wxy2 = w*l/2.0*sign(xym);
       C1   = (nu1+nu2+nu1*nu2)/(4-nu1*nu2);
       A1   = 1+(nu1-nu2)/(4-nu1*nu2)/(1+g*C1);
       A2   = 1+(nu2-nu1)/(4-nu1*nu2)/(1+g*C1);
       A3   = 3*nu1*(2-nu2*(1-g))*C0*S1;
       A4   = 3*nu2*(2-nu1*(1-g))*C0*S1;
       *px1 += wl2*A1;
       *px2 += wl2*A2;
       Rxy1  = *px1;
       Rxy2  = *px2;
       *pm1 -= wxy2*l/6.0*A3+Rxy1*l1;
       *pm2 += wxy2*l/6.0*A4+Rxy2*l2;
     }
     else
     {
       *px1 += w*l/2.;
       *px2 += *px1;
     }
   }
   break;
case pin_fix:
   if (P != 0.0)
   {
     if (l>0.0)
     {
       /* calculate the structure applied loads */
       A1   = nu2/(1+g*nu2/4);
       A2   = a*b*(b+2.0*a)/(2.0*l*l*l);
       Rxy1 = P*a/l + P*A2*A1;
       Rxy2 = P*b/l - P*A2*A1;
```

```
          *px1 += Rxy1;
          *px2 += Rxy2;
          *pm1 -= Rxy1*l1;
          *pm2 += sign(xym)*P*A2*l+Rxy2*l2;
        }
        else
        {
          *px1 = P*b/l;
          *px2 = P - (*px1);
        }
      }
      if (w != 0.0)
      {
        if (l>0.0)
        {
          /* calculate the structure applied loads */
          wl8 = w*l/8.0;
          wxy8 = w*l/8.0*sign(xym);
          A1  = (4-nu2*(1-g))/(1+g*nu2/4);
          A2  = (4+nu2*(1+g))/(1+g*nu2/4);
          A3  = nu2/(1+g*nu2/4);
          Rxy1 = wl8*A1;
          Rxy2 = wl8*A2;
          *px1 += Rxy1;
          *px2 += Rxy2;
          *pm1 -= Rxy1*l1;
          *pm2 += wxy8*l*A3+Rxy2*l2;
         }
        else
        {
          *px1 += w*l/2.;
          *px2 += *px1;
        }
      }
      break;
   case fix_pin:
      if (P != 0.0)
      {
        if (l>0.0)
        {
           /* calculate the structure applied loads */
          A1  = nu1/(1+g*nu1/4);
          A2  = a*b*(a+2.0*b)/(2.0*l*l*l);
          Rxy1 = P*b/l + P*A2*A1;
          Rxy2 = P*a/l - P*A2*A1;
          *px1 += Rxy1;
          *px2 += Rxy2;
          *pm1 -= sign(xym)*P*A2*l+Rxy1*l1;
          *pm2 += Rxy2*l2;
        }
        else
        {
           *px1 = P*b/l;
           *px2 = P - (*px1);
        }
      }
      if (w != 0.0)
      {
        if (l>0.0)
        {
          /* calculate the structure applied loads */
          wl8  = w*l/8.0;
          wxy8 = w*sign(xym)*l/8.0;
          A1  = (4+nu1*(1+g))/(1+g*nu1/4);
```

```
        A2   = (4-nul*(1-g))/(1+g*nul/4);
        A3   = nul/(1+g*nul/4);
        Rxy1 = w18*A1;
        Rxy2 = w18*A2;
        *px1 += Rxy1;
        *px2 += Rxy2;
        *pm1 -= wxy8*1*A3+Rxy1*11;
        *pm2 += Rxy2*12;
      }
      else
      {
        *px1 += w*1/2.;
        *px2 += *px1;
      }
    }
    break;
  case pin_pin:
    if (P != 0.0)
    {
      if (l>0.0)
      {
         /* calculate the structure applied loads */
         *px1 = P*b/l;
         *px2 = P*a/l;
      }
      else
      {
         *px1 = P*b/l;
         *px2 = P - (*px1);
      }
    }
    if (w != 0.0)
    {
      if (l>0.0)
      {
        /* calculate the structure applied loads */
        wl2    = w*l/2.0;
        *px1 += wl2;
        *px2 += wl2;
      }
      else
      {
        *px1 += w*l/2.;
        *px2 += *px1;
      }
    }
    break;
  }
}
```

```
void Write(NodeVec, MaxNodeNum, BeamVec, MaxBeamNum, MaxSpringNum)
struct NODE *NodeVec[];
struct BEAM *BeamVec[];
int MaxNodeNum, MaxBeamNum;
{
     /*  Local Variables  */
    int  DOF1,DOF2,DOF3,bof,boflen,pcode,eof,eoflen,col,row,i,j;
    int  cnheader,k12,fmt,colwl,colwlen,width,Typ;
    double d1,d2,d3,d4,d5,d6,d7,c,s,C1,C2,C3,C4,C5,S1,S2,S3,S5,S6;
    float k1,k2,df1,df2,df3,axial1,axial2,shear1,shear2,bm1,bm2;
    float area,ashear,inertia,axial,shear,bm,kc,p2,p3,p4,p5;
    double nu1,nu2,xm,ym,dm,xm2,ym2,dm2,l,l1,l2,g,ks1,ks2,ks3;
    struct BEAM *TPBeam;
    struct CONNECT *connectn;
    struct forces *reactns, *end_rctns;

     /*  Functions used   */
    void flt_rec(int, int, float, int, FILE *);
    void header(int, int, char *, char *, char *, char *, char *, char *,
char *, FILE *);
    void int_rec(int, int, int, FILE *);
    void lbl_rec(int, int, char *, char, FILE *);
    void wrtint(int, FILE *);
    double ConnectKi(int, float, float, float, float, float);
     /*  Defining constants  */
    row=col=0;                     /*  define column A & row 1 */
    bof=0;                         /*  opcode for BOF marker */
    boflen=2;                      /*  length of BOF marker  */
    pcode=1030;                    /*  product code          */
    colwl=8;                       /*  opcode for setting column width */
    colwlen=3;                     /*  length of column width record */
    eof=1;                         /*  opcode for EOF marker */
    eoflen=0;                      /*  length of EOF marder  */
     /*
          ### Write output ###
     */
    wrtint(bof,fptarget);
    wrtint(boflen,fptarget);
    wrtint(pcode,fptarget);
    for (col=0; col<=6; col++)
      {
       wrtint(colwl,fptarget);
       wrtint(colwlen,fptarget);
       wrtint(col,fptarget);
       if (col<=0)
         width=6;
       else
         {
          if (col==3 || col==6)
            width=13;
          else
            width=10;
          }
       fputc(width,fptarget);
       }
    col=0;
```

```
 /*   ###  Write TITLE   ###    */
lbl_rec(col,row,ver,left,fptarget);
row++;
lbl_rec(col,row,title,left,fptarget);
row+=2;
reactns = beg_rctns;    /* initialize linked list of support reactions
*/
 /*   ###  Write NODE displacements  ###  */
lbl_rec(col,row,nodedisp,left,fptarget);
row++;
header(2,row,"node","x_disp","y_disp","rotation","blank",
                                   "blank","blank",fptarget);
row++;
i=1;
while (i<=MaxNodeNum)
  {
    if (NodeVec[i] != NULL)
      {
       int_rec(0,row,NodeVec[i]->num,fptarget);
       fmt=0x92;
       d1=NodeVec[i]->d[0];
       d2=NodeVec[i]->d[1];
       d3=NodeVec[i]->d[2];
       df1=(float)d1;
       df2=(float)d2;
       df3=(float)d3;
       flt_rec(1,row,df1,fmt,fptarget);
       flt_rec(2,row,df2,fmt,fptarget);
       flt_rec(3,row,df3,fmt,fptarget);
       row++;

       /*  identify support nodes  */
       DOF1=NodeVec[i]->DOF[0];
       DOF2=NodeVec[i]->DOF[1];
       DOF3=NodeVec[i]->DOF[2];
       if (DOF1 ==0 || DOF2 == 0 || DOF3 == 0)
         {
          reactns->num=NodeVec[i]->num;
          reactns++;
          }
      }
    i++;
  }
end_rctns=reactns;
reactns=beg_rctns;
row++;
 /*   ###  Write MEMBER forces  ###  */
lbl_rec(col,row,beamforce,left,fptarget);
row++;
header(2,row,"member","axial1","shear1","bm1","axial2",
                            "shear2","bm2",fptarget);
row++;
i=1;
while (i<=MaxBeamNum)
  {
    if (BeamVec[i] != NULL)
    {
      TPBeam = BeamVec[i];
      d1 = BeamVec[i]->node1->d[0];
      d2 = BeamVec[i]->node1->d[1];
      d3 = BeamVec[i]->node1->d[2];
      d4 = BeamVec[i]->node2->d[0];
      d5 = BeamVec[i]->node2->d[1];
      d6 = BeamVec[i]->node2->d[2];
```

```
k12= BeamVec[i]->k12;
area     = BeamVec[i]->area;
ashear   = BeamVec[i]->ashear;
inertia = BeamVec[i]->inertia;
xm = (BeamVec[i]->node2->X) - (BeamVec[i]->node1->X);
ym = (BeamVec[i]->node2->Y) - (BeamVec[i]->node1->Y);
xm2 = xm*xm;
ym2 = ym*ym;
dm2 = xm2+ym2;
dm  = sqrt(dm2);
c   = xm/dm;
s   = ym/dm;
/*  Connection Sizes  */
if (TPBeam->connect1==NULL)
  l1=0.;
else
  l1=(TPBeam->connect1->p1);
if (TPBeam->connect2==NULL)
  l2=0.;
else
  l2=(TPBeam->connect2->p1);
l = dm-l1-l2;
/*  Flexible Connection */
k1=k2=-1.00e+38;
if (TPBeam->connect1==NULL)
  nu1=1;
else
{
  k1=(TPBeam->connect1->Ki);
  if (k1>0. && k1<=1.e-30)
    nu1=0.;
  else
    if (k1==-1.00e+38)
      nu1=1.;
    else
      nu1=1./(1.+((3*E*(TPBeam->inertia))/(k1*l)));
}
if (TPBeam->connect2==NULL)
  nu2=1.;
else
{
  k2=(TPBeam->connect2->Ki);
  if (k2>0. && k2<=1.e-30)
    nu2=0.;
  else
    if (k2==-1.00e+38)
      nu2=1.;
    else
      nu2=1./(1.+((3*E*(TPBeam->inertia))/(k2*l)));
 }
 axial1=axial2 = ((d4-d1)*c+(d5-d2)*s)*area*E/l;
 if (k12 == pin_pin)
    shear1 = shear2 = bm1 = bm2 = 0.0;
 else
 {
   d7 = (d2-d5)*c+(d4-d1)*s;
   /*  Shear Deflection */
   if (G == 0.0 || ashear == 0.0)
     g=0.0;
   else
     g=12.*E*inertia/(ashear*G*l*l);
```

```
            if (k12 == fix_fix)
            {
              C1 = (nu1+nu2+nu1*nu2)/(4-nu1*nu2);
              C2 =  nu1*(2+nu2)/(4-nu1*nu2);
              C3 =  3*nu1/(4-nu1*nu2);
              C4 =  nu2*(2+nu1)/(4-nu1*nu2);
              C5 =  3*nu1*nu2/(4-nu1*nu2);
              S1 =  1/(1+g*C1);
              S2 = (1+g*nu2/4)/(1+g*C1);
              S3 = (1-g/2)/(1+g*C1);
              shear1 =((2.0*C1/l)*d7+(C2+2*l1*C1/l)*d3
                       +(C4+2*l2*C1/l)*d6)*6*E*inertia*S1/(l*l);
              shear2 = shear1;
              bm1   = -((6.0*C2*S1/l)*d7+(4.0*C3*S2+6.0*l1*C2*S1/l)*d3
                       +(2.0*C5*S3+6.0*l2*C2*S1/l)*d6)*E*inertia/l;
              bm2   = bm1+shear1*l;
             }
             if (k12 == pin_fix)
             {
               S6 = 1/(1+g*nu2/4);
               shear1 = (d7/l+l1/l*d3+(1+l2/l)*d6)*3*E*iner-
tia*nu2*S6/(l*l);
               bm1    = 0.0;
               shear2 = shear1;
               bm2    = (d7/l+l1/l*d3+(1+l2/l)*d6)*3*E*inertia*nu2*S6/l;
              }
             if (k12 == fix_pin)
             {
               S5 = 1/(1+g*nu1/4);
               shear1 = (d7/l+(1+l1/l)*d3+l2/l*d6)*3*E*iner-
tia*nu1*S5/(l*l);
               bm1    = -(d7/l+(1+l1/l)*d3+l2/l*d6)*3*E*inertia*nu1*S5/l;
               shear2 = shear1;
               bm2    = 0.0;
              }
           }
          /* if there were loads on the beam the end forces will be
          atached to the beam in the BFORCE structure. Add these to
          structural forces to get resultant */
          if (BeamVec[i]->bforce!=NULL)
          {
            axial1 +=BeamVec[i]->bforce->axial1;
            axial2 +=BeamVec[i]->bforce->axial2;
            bm1    +=(BeamVec[i]->bforce->bm1)
                     +((BeamVec[i]->bforce->shear1)*l1);
            bm2    +=(BeamVec[i]->bforce->bm2)
                     -((BeamVec[i]->bforce->shear2)*l2);
            shear1 +=BeamVec[i]->bforce->shear1;
            shear2 +=BeamVec[i]->bforce->shear2;
          }
          /* write out resultant beam forces */
          int_rec(0,row,BeamVec[i]->num,fptarget);
          fmt=0x81;
          flt_rec(1,row,axial1,fmt,fptarget);
          flt_rec(2,row,shear1,fmt,fptarget);
          flt_rec(3,row,bm1,fmt,fptarget);
          flt_rec(4,row,axial2,fmt,fptarget);
          flt_rec(5,row,shear2,fmt,fptarget);
          flt_rec(6,row,bm2,fmt,fptarget);
          row++;
```

```
        /*  Assign appropriate forces to Connections  */
        if (TPBeam->connect1 != NULL)
        {
          connectn = TPBeam->connect1;
          connectn->Vc = shear1;
          connectn->Mc = bm1;
          Typ = connectn->Typ;
          p2  = connectn->p2;
          p3  = connectn->p3;
          p4  = connectn->p4;
          p5  = connectn->p5;
          kc = (float) ConnectKi(Typ,bm1,p2,p3,p4,p5);
          connectn->Kc = kc;
         }
        if (TPBeam->connect2 != NULL)
        {
          connectn = TPBeam->connect2;
          connectn->Vc = shear2;
          connectn->Mc = bm2;
          Typ = connectn->Typ;
          p2  = connectn->p2;
          p3  = connectn->p3;
          p4  = connectn->p4;
          p5  = connectn->p5;
          kc = (float) ConnectKi(Typ,bm2,p2,p3,p4,p5);
          connectn->Kc = kc;
         }
            /*  sum all beam forces at node supports  */
        reactns=beg_rctns;
        while (reactns != end_rctns)
          {
           if (BeamVec[i]->node1->num==reactns->num)
             {
              axial=axial1*(-1.);
              shear=shear1;
              reactns->M+=bm1*(-1.)+shear*l1;
              reactns->Rx+=axial*c-shear*s;
              reactns->Ry+=axial*s+shear*c;
              }
           if (BeamVec[i]->node2->num==reactns->num)
             {
              axial=axial2;
              shear=shear2*(-1.);
              reactns->M+=bm2-shear*l2;
              reactns->Rx+=axial*c-shear*s;
              reactns->Ry+=axial*s+shear*c;
              }
           reactns++;
          }
      }
    i++;
  }
/*   ###  Write CONNECTION forces   ###   */
 cnheader=0;
 i=1;
 while (i<=MaxNodeNum)
 {
   if (NodeVec[i] != NULL)
   {
     j=1;
     while (j <= MaxBeamNum)
     {
       if (BeamVec[j] != NULL)
       {
```

```
        TPBeam = BeamVec[j];
        if (TPBeam->connect1->Jn==NodeVec[i]->num ||
            TPBeam->connect2->Jn==NodeVec[i]->num)
        {
          if (cnheader==0)
          {
            row++;
            lbl_rec(col,row,connforce,left,fptarget);
            row++;
            header(2,row,"n_num","m_num","Vc","bmc","Kc",
                                   "bmi","Ki",fptarget);
            row++;
            cnheader=1;
          }
          if (TPBeam->connect1->Jn==NodeVec[i]->num)
            connectn=TPBeam->connect1;
          else
            connectn=TPBeam->connect2;
          int_rec(0,row,connectn->Jn,fptarget);
          int_rec(1,row,connectn->Mn,fptarget);
          fmt=0x81;
          flt_rec(2,row,connectn->Vc,fmt,fptarget);
          flt_rec(3,row,connectn->Mc,fmt,fptarget);
          fmt=0x92;
          if ((connectn->Kc) < 0.)
            lbl_rec(4,row,"rigid",centre,fptarget);
          else
            flt_rec(4,row,connectn->Kc,fmt,fptarget);
          fmt=0x81;
          flt_rec(5,row,connectn->Mi,fmt,fptarget);
          fmt=0x92;
          if ((connectn->Ki) < 0.)
            lbl_rec(6,row,"rigid",centre,fptarget);
          else
            flt_rec(6,row,connectn->Ki,fmt,fptarget);
          row++;
         }
        }
        j++;
      }
   }
   i++;
}
row++;
 /*  ###  Write Support Reactions  ###  */
lbl_rec(col,row,reactions,left,fptarget);
row++;
header(2,row,"node","Rx","Ry","M","blank","blank","blank",fptarget);
row++;
reactns=beg_rctns;
while (reactns != end_rctns)
  {
   int_rec(0,row,reactns->num,fptarget);
   fmt=0x81;
   flt_rec(1,row,reactns->Rx,fmt,fptarget);
   flt_rec(2,row,reactns->Ry,fmt,fptarget);
   flt_rec(3,row,reactns->M,fmt,fptarget);
   row++;
   reactns++;
   }
row++;
```

```
  /*  ###  Write Spring Forces  ###  */
  if (MaxSpringNum>0)
    {
     lbl_rec(col,row,springforce,left,fptarget);
     row++;
header(2,row,"node","Rsx","Rsy","Ms","blank","blank","blank",fptarget);
     row++;
     i=1;
     while (i<=MaxSpringNum)
       {
         if (NodeVec[i] != NULL)
           {
            if (NodeVec[i]->spring!=NULL)
              {
               ks1=NodeVec[i]->spring->Ks[0];
               ks2=NodeVec[i]->spring->Ks[1];
               ks3=NodeVec[i]->spring->Ks[2];
               axial1=(float)(ks1*NodeVec[i]->d[0]*(-1.0L));
               axial2=(float)(ks2*NodeVec[i]->d[1]*(-1.0L));
               bm1=(float)(ks3*NodeVec[i]->d[2]*(-1.0L));
               int_rec(0,row,NodeVec[i]->num,fptarget);
               fmt=0x81;
               flt_rec(1,row,axial1,fmt,fptarget);
               flt_rec(2,row,axial2,fmt,fptarget);
               flt_rec(3,row,bm1,fmt,fptarget);
               row++;
               }
           }
         i++;
       }
     }
   /*  ###  EOF marker  ###  */
  wrtint(eof,fptarget);
  wrtint(eoflen,fptarget);
  fclose(fptarget);
}
```

CPlane

Highlight:

--- a **structural analysis program as an add-on for spreadsheets**

--- includes the effect of **flexible connections**
linear and **nonlinear**

--- simple geometry generation and post processing
with spreadsheets

--- ease of use

--- written in C

compatible with spreadsheets:

LOTUS 1-2-3 (TM), SYMPHONY (TM), QUATTRO (TM).

Introduction

The elastic structural analysis program "CPlane" was developed with the intention of devising a practical method of incorporating behavior of flexible connections in analysis of steel structures. The program is designed for use with personal computers equiped with a math coproccessor and requires DOS or OS/2 operating systems. Any personal computer capable of running LOTUS 1-2-3 (TM) or SYMPHONY (TM) should have no problem in running "CPlane".

"CPlane" works independently from Lotus 1-2-3, Release 2.0 and later. However, it requires input (source) data and produces output (target) data in LOTUS worksheet file format (files with extension ".WK1"). "CPlane" operates with all IBM PC's or compatibles and no specific requirements are necessary. The problem solution module is centered around a unique stiffness matrix program which is written in the C language. It interfaces with the spreadsheet program directly without going through cumbersome data file conversions. The spreadsheet allows for greatest simplicity and flexibility in creating input data and evaluating numerical results. A set of pre- and post-processor templates is provided to guide the user in the use of "CPlane". User can easily implement this analysis module in his custom designed spreadsheet templates as will be shown in an example later.

"CPlane" can solve static linear elastic structural systems with a multitude of load types. The user should be familiar with basic engineering and computer terminology to use this program most effectively. We suggest that new users read parts of this manual to become familiar with "CPlane"'s capabilities. In most cases, a thumbing through the manual is suffice.

"CPlane" accepts any consistent set of units. Geometry plots can be viewed on the screen when the preset templates are used for creating the program input. Hardcopy plots can be made using the spreadsheet routines (PRINTGRAPH, etc.). Once the structure's geometry is described, it is advisable to perform a geometry check in order to help to locate any errors that were made during the input process. Loads and springs can be specified in various ways and they can be automatically generated, which is very helpful for sloping distributed loads or elastic foundations.

The brief manual consists mainly of a basic introduction to the program. The data input is unparalleled in terms of flexibility. This manual assumes that you are a current 1-2-3 user, familiar with its basic functions and operations. It also assumes that you are familiar with your computer, its keyboard and operating system.

Hard Disk Installation of CPLANE ACCESS SYSTEM

1. Place floppy disk "CPLANE" into drive A
   and start the installation routine:

   A:\>**installh**[enter]

2. Change the LOTUS 1-2-3 default directory to "C:\PLANE" and
   update the directory.
   (command sequence in LOTUS: "/wgddC:\PLANE[Enter]uq")

3. Include locations of the LOTUS system and "CPLANE"
   in the path of your AUTOEXEC.BAT file. If you don't
   have an AUTOEXEC.BAT, rename the one supplied:

   C:\>**ren planeaut.bat autoexec.bat**[enter]

4. Reboot your system [Crtl]-[Alt]-[Del].

Start the "**CPLANE ACCESS SYSTEM**" from the root directory C:

C:\>**caccess**[enter]

From now on a menu system will guide you.

Floppy Disk Installation of CPLANE ACCESS SYSTEM

1. Create your own "work disk". Place an empty floppy disk, which has been previously formatted, into drive B and the program disk "CPLANE" in drive A. Start the installation routine:

   A:\>**installf**[enter]

2. Replace the "CPLANE" disk by your Lotus program disk in drive A, start 1-2-3, change the default directory to "B:\", and update the directory.
   (command sequence in LOTUS: "/wgddB:\[Enter]uq")

3. Quit LOTUS.

Start the "**CPLANE ACCESS SYSTEM**" from B:

   B:\>**caccess**[enter]

From now on a menu system will guide you.

Using "CPLANE"

There are two ways to conduct a structural analysis with "CPLANE" and LOTUS 1-2-3 (TM):

A. **CPLANE ACCESS SYSTEM**: (installation required)

Menu driven pre- and postprocessor templates have been prepared and can be used as is or easily customized.

(hard disk)

```
C:\>caccess[enter]
```

or (floppy disk)

```
B:\>caccess[enter]
```

From then on a menu system will guide you through all modules including data generation, storing, plotting, viewing, printing, analyzing, reducing, etc...

The more advanced user might want to develop his own pre- or postprocessor or prefers to work without any menu system. Then the following procedure is proposed:

B. **GENERIC PROCEDURE**: (no installation required, just copy CPLANE.EXE to your work disk or directory)

1. Load LOTUS 1-2-3 and create your input values for the structure and loads, safe the worksheet under a descriptive name (i.e. FRAME1.WK1). Quit LOTUS.
2. Start "CPLANE" by

   CPLANE[enter]

   and respond to "CPLANE"'s prompts (it will ask you for the name of the source data file and target data file (including file extensions))

or

   CPLANE frame1.wk1 result1.wk1[enter]

   The latter procedure enables you either to create a batch file containing several input and output files or to use "CPLANE" as the analysis engine for your custom pre- or postprocessors.

3. Load LOTUS 1-2-3 and retrieve the target file(s). Now you can use the output data to plot, to print etc.

Input File Structure for "Cplane":

```
CPlane
title

nodes
n_num     ix   iy   im    xcoor     ycoor     _____     ngen

members
m_num     fm   j1   j2    Area      Ashea     I         jstep

material
_____     ___  ___  ___   E         G         wt        ____

springs
s_num     ___  ___  ___   Sx        Sy        Sm        sgen

nodal_loads
l_num     ___  ___  ___   Px        Py        Pm        lgen

member_loads
i_num     dir  ___  ___   w         P         a         igen

connection data
n_num  m_num typ  Mi      p1        p2        p3        p4       p5

echo

iterate

endata
```

List of Symbols:

CPlane
title (one line of text)

nodes
n_num, ix, iy, im, xcoor, ycoor, ngen (one row for each node)

```
n_num  : nodal number
ix     : freedom in x-direction
iy     : freedom in y-direction
im     : freedom to rotate
         if = 0: fixed
         if = 1: free
         if = J: linked to same degree of freedom as node J
xcoor  : nodal coordinate x
ycoor  : nodal coordinate y
ngen   : generation of n_num in ngen-steps,
         same degrees of freedom,
         linear interpolation of geometry,
         generation ends at node from next row
```

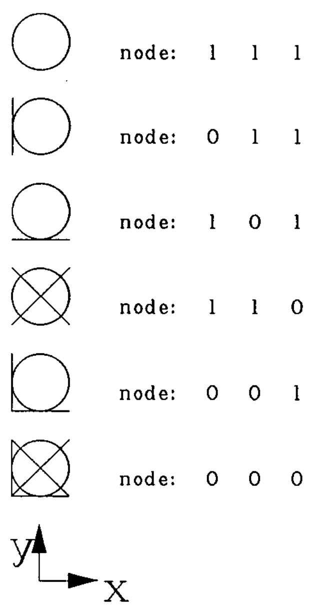

Fig. 1: Nodes and Coding of Fixities

members
m_num,fm,j1,j2,Area,Ashea,I,jstep (one row for each member)

```
m_num  : member number
fm     : end condition of member
         if =  p: member is pinned - pinned
         if = fp: member is fixed - pinned
         if = pf: member is pinned - fixed
         if =  f: member is fixed - fixed
j1     : lesser nodal number
j2     : greater nodal number
Area   : total cross sectional area
Ashea  : cross sectional area to carry shear,
         if this cell is empty or equal to zero
         analysis is done without consideration of
         shear deformation (simplified stiffness analysis)
I      : moment of inertia
jstep  : generation of j1 in jsteps,
         keeps difference between j1 and j2 constant,
         linear interpolation of Area, Ashea and I,
         ends at member form next row
```

Local +-direction is lower nodal to higher nodal number. +-member force is tension, +-member bending exerts tension in upper fibres.

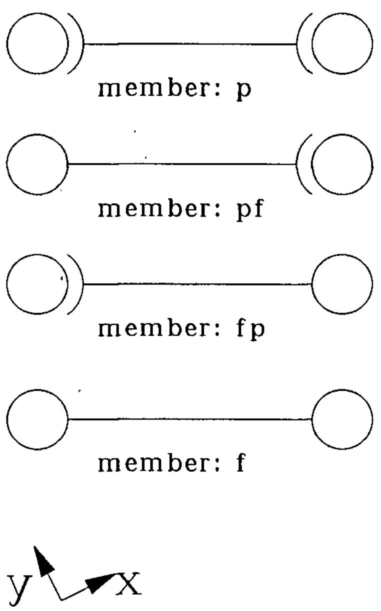

Fig. 2: Members and Coding of End Fixities

material
E,G,wt

E	: modulus of elasticity
G	: shear modulus, if this cell is empty or equal to zero, shear deformation is not considered
wt	: weight of material (note: weight is in the current version of "plane2.0" not considered)

springs
s_num, Sx, Sy, Sm, sgen (one row for each spring)

s_num	: node that spring is attached to
Sx	: spring stiffness in x-direction, positive if +x-displacement causes tensile (+) reaction in spring
Sy	: spring stiffness in y-direction, positive if +y-displacement causes tensile (+) reaction in spring
Sm	: rotational stiffness, positive if counter-clockwise rotation causes counter-clockwise moment in spring
sgen	: generation of n_num in sgen-steps, linear interpolation of stiffness

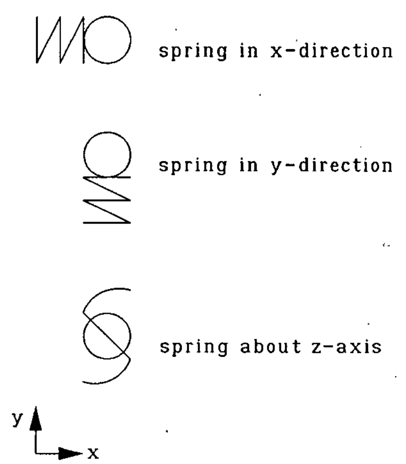

Fig. 3: Elementary Springs,
Combination with each other and
with partially fixed nodes possible

```
nodal_loads
l_num, Px, Py, Pm, lgen (one row for each loaded node)

l_num   : number of loaded node
Px      : load in x-direction
Py      : load in y-direction
Pm      : moment about node
lgen    : generation of n_num in lgen-steps, linear interpol. of loads
```

```
member_loads
i_num, dir, w, P, a, igen (one row for each loaded member)

i_num   : member number the load is applied to
dir     : direction the load is applied to
          if =  x: in x-direction (positive)
          if =  y: in y-direction (positive)
          if =  p: perpendicular to member (positive when
                clockwise about lower node)
w       : uniformly distributed load
          if direc = p: load = w
          if direc = x: load = vertical projection of w
          if direc = y: load = horizontal projection of w
P       : concentrated load
          if direc = p: load perpendicular
          if direc = x: load in x-direction
          if direc = y: load in y-direction
a       : distance from lower node of member to location
          of conc. load
          if direc = p: a = dist. along member from lower node
                            to concentrated load
          if direc = x: a = dist. along y-axis from lower node
                            to concentrated load
          if direc = y: a = dist. along x-axis from lower node
                            to concentrated load
igen    : generation of i_num in igen-steps,
          linear interpolation of loads and load locations,
          the parameter direc of this line is used for
          all generated nodes;
          w, P, and a are incremented between the loads
          given in this line and the next line
```

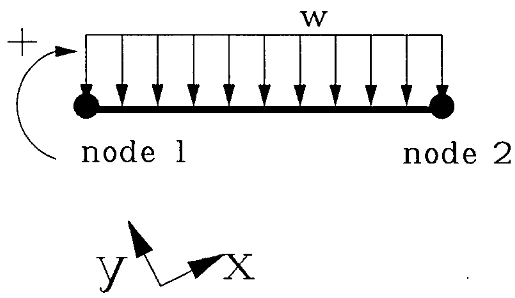

Fig. 4: Uniformly Distributed Load Perpendicular to Member, direc=p

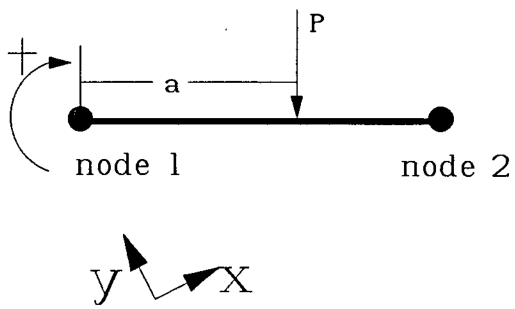

Fig. 5: Concentrated Load Perpendicular to Member, direc=p

```
connections
n_num, m_num, typ, Mi, p1, p2, p3, p4, p5

n_num   : joint at which connection is located
m_num   : member which the connection is connected to
typ     : connection type ( 1. perfectly hinged,
                            2. single web angle,
                            3. double web angle,
                            4. header plate,
                            5. top & seat angle,
                            6. end plate (without column stiffeners),
                            7. end plate (with column stiffeners),
                            8. t-stub,
                            9. other, (note: connection stiffness is
entered
                                       as parameter p2)
                           10. perfectly rigid.)

Mi      : connection moment (optional)
          default uses initial tangent stiffness value
p1-p5   : connection parameters.  Please refer to Figure 6 for details.
```

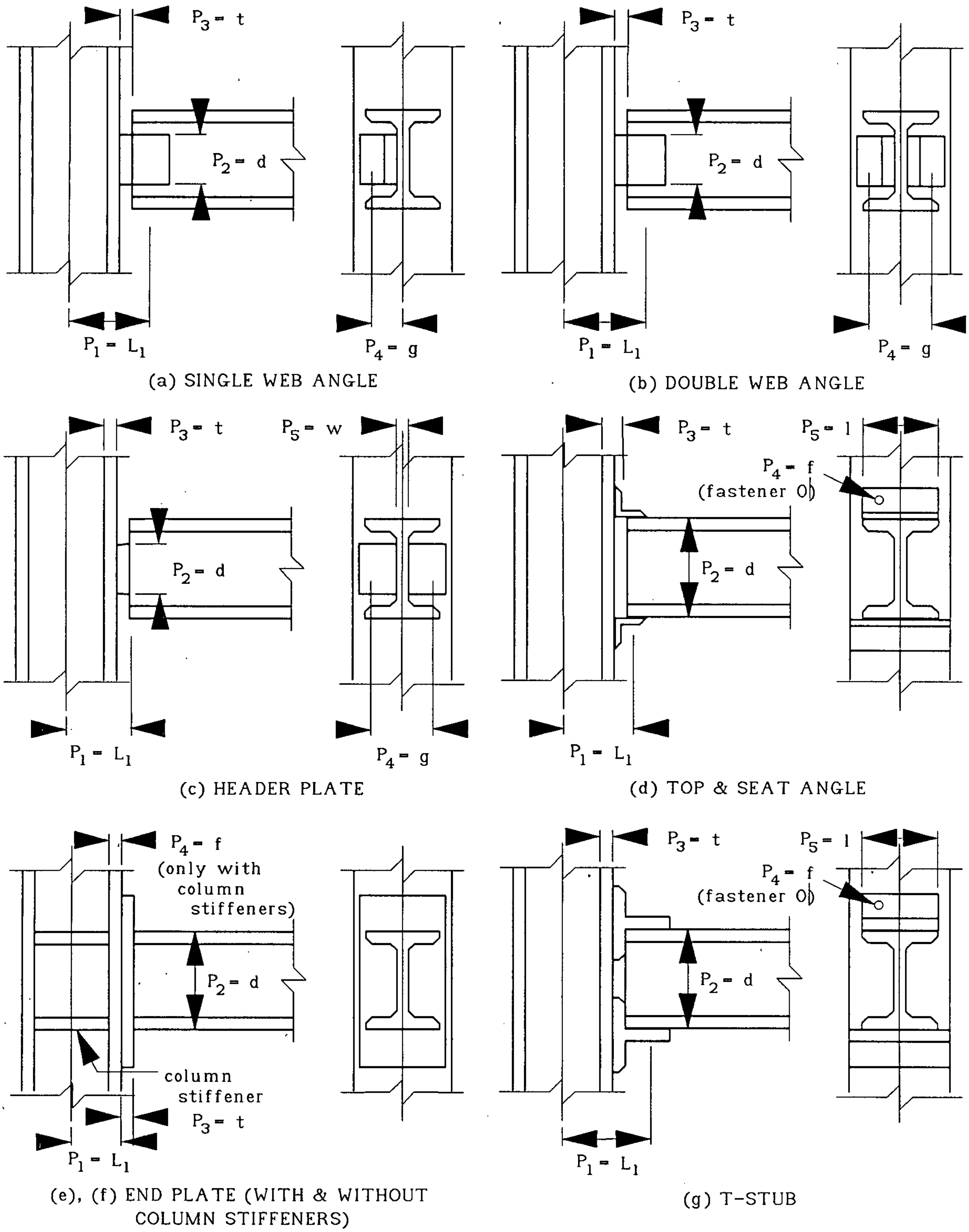

Fig. 6: Connection Types and Their Standardization Parameters

echo
indicator for creation of echofile if this keyword exists in the source file then ECHO.WK1 file will be created

iterate
indicator for nonlinear connections if this keyword exists program automatically creates and updates the intermediate data file ITERATE.WK1 for iteration purposes.

endata
indicator for end of data if this keyword exists in the source file then spreadsheet may contain other comments, macros or data (for i.e. plotting)

Example

Source File:

```
CPlane
nlflex2: steel frame investigated by Moncarz & Gerstle, w=0.02 kips/sq
ft.

nodes
   1    0    0    0        0        0
   2    1    1    1        0      144
   3    1    1    1        0      288
   4    1    1    1      288      288
   5    1    1    1      288      144
   6    0    0    0      288        0

members
   1    f    1    2     9.71     0.00   170.00
   2    f    5    6     9.71     0.00   170.00
   3    f    2    5    13.00     0.00   843.00
   4    f    2    3     9.71     0.00   170.00
   5    f    4    5     9.71     0.00   170.00
   6    f    3    4     9.12     0.00   375.00

material
                       30000

member loads
   3    y             -0.255
   6    y             -0.155

nodal loads
   2                    5.76
   3                    2.88

connections
   2    3    5    0    4.865    20.66    1.222    1.125    6.50
   3    6    5    0    4.865    15.88    1.065    1.250    5.52
   4    6    5    0    4.865    15.88    1.065    1.250    5.52
   5    3    5    0    4.865    20.66    1.222    1.125    6.50

iterate

echo
```