PERFORMANCE ASSESSMENT OF FEEDBACK CONTROLLER OUTPUT AND PROCESS OUTPUT

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Abstract

Performance assessment tools are very useful in providing information about the control loop performance. To maintain high process performance, effective methodologies to measure the actual performance of the control loops are urgently needed. Since the industry normally has a fixed set-point for each loop, most of the control action is to reject disturbance, i.e. regulatory control.

The most popular performance index for regulatory control nowadays is the Harris index due to its simplicity of usage. The Harris index requires only routine operating data and prior knowledge of the process delay. It compares the variance of the process output with the benchmark of minimum variance. An application of the Harris index to PID controller tuning is discussed in this thesis.

A new input/output index (I/O) is developed in this thesis. It requires more knowledge about the process model than the Harris index, but provides more information about the performance of a closed-loop system. The key feature of this I/O index is that the user can specify a benchmark which considers the balance between process output variance and the controller output variance. However, in this case, the process model needs to be identified so that the upper bound of the controller output variance and the lower bound of the process output variance can be obtained to set the benchmark for thorough performance assessment of the control system.

In industry, during closed-loop control, both controller output and process output variances are desired to be as little as possible. Though Minimum Variance Control (MVC) may minimize the process output variance, it is not desirable in many cases due to excessive control action and sensitivity to changes in process dynamics. Therefore,
balancing the process output and process input variances becomes very important. This new performance index uses the upper bound of the controller output and lower bound of the process output from MVC. In fact, it can indicate graphically the optimal operating point for each loop. Hence, this $I/O$ index is more practical and reliable.

An extensive simulation study was performed to examine the operation, usefulness and limitations of the performance indices. These simulations confirm the benefit of using the new $I/O$ index for assessing regulatory control.
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Chapter 1

Introduction

1.1 Background

In a typical process industry, there are hundreds to thousands of control loops. Every controller is expensive to implement, requiring hardware, software, and engineering services. It is not uncommon for controllers to perform poorly in process control loops. Some controllers cannot reach their design potential after implementation, some lose their effectiveness over time from a lack of maintenance. Other controllers, due to valve wear, fouling of the equipment, variation of the process dynamics and nature of the disturbance change, are no longer suitable for the new conditions. These factors lead to poor performance, resulting in an increase of energy consumption, waste raw materials and non-uniform end products.

Poor performance is basically a maintenance problem. With limited manpower, control audits are rarely performed routinely. Poor control performance is often reported subjectively. It is not usually detected until severe oscillations or process upsets occur. Hence, it is almost impossible to ensure that all loops are operating satisfactorily. It would be time consuming to investigate each loop individually without an easy and efficient assessment tool. Given the limited technical resources available to support control systems, the performance analysis/diagnosis methodology must be reliable, computationally simple, and readily interpretable. Recently, many performance assessment tools have been developed to detect poor performance, diagnose the underlying cause of this
abnormal behavior and take remedial actions. In particular, performance indices are practical and meaningful as performance analysis tools.

A review of existing methods for assessing the performance is presented here. The most basic form for performance assessment includes observation of process operation, recording the control action and process output, and calculation of the performance index. Using performance analysis tools to improve process performance is very attractive and efficient, since it is easy to implement and only requires the addition of software code programming to process the plant measurements.

1.2 Summary of Existing Performance Analysis Methods

To be effective, a plant-wide control monitoring and performance assessment package should have the following properties (Harris et. al., 1997) [1]: i) automated background operation, ii) scheduled remote collection of control loop data, iii) theoretically sound, efficient, and automated computational procedures, iv) the ability to identify and report problems by exception with preliminary diagnosis, v) acceptable false alarm and detection rates, and vi) an intuitive user interface. Together, these properties form the basis of a powerful process control performance monitoring system.

The topic discussed here focuses on property iii) above, i.e. the background theory and associated computational procedures required to obtain meaningful performance information from large amounts of measured data.

A common way to quantify controller performance is by computing the variance in terms of deviation from the target. Minimum variance of the process output can be estimated from routine operating data when the time delay is known (Harris, 1989) [2]. In reality, it is rarely desirable to implement a minimum variance controller which is nevertheless an ideal benchmark as it gives the best control. To use the resources more
efficiently, frequency domain and settling time performance characteristics, which might be considered at the design stage, are discussed in Astrom (1991) [3].

Many authors have introduced a number of performance indices to provide an indication of the departure of current performance from minimum variance control. As indicated in Desborough and Harris (1992) [4], it is more useful to replace the process output variance by the mean square error of $y_t$ which accounts for offset. The spectral interpretation for this normalized index has also been discussed in this paper. Harris et al. (1996b) [5] and Kozub (1996) [6] also utilize an extended horizon performance index which gives a proportion of the variance arising from non-zero impulse coefficients. The advantage of this extended horizon performance index is that it does not require a precise estimate of the process delay.

Desborough and Harris (1993) [7] have developed a method that analyzes the variance procedure for feedforward/feedback control schemes. Using this approach, it is possible to assess the effectiveness of the feedforward and feedback elements of the control schemes. This can yield considerable insight into factors affecting the dynamic behavior of the process.

Huang, Shah and Kwok (1997) [8, 9] have developed a filtering and correlation based method - FCOR, to estimate the minimum variance and normalized Harris index for MIMO systems. A pre-whitening filter is used to obtain disturbance series. A cross-correlation function is then calculated, leading directly to the calculation of the performance index.

1.3 Motivation

Among all previous research work which, most of the methods only analyze the process output performance, which minimize the process output variance without considering
Chapter 1. Introduction

the controller output performance. Though Linear Quadratic Gaussian (LQG) control algorithm does take into account the controller output variance, it requires more prior knowledge of the process model.

The Harris index is popularly used as the performance index for stochastic control nowadays. By investigating performances both for process input and output through simulation, it was found that only using the Harris index is not sufficient to describe the control loop performance. Essentially, the Harris index cannot indicate whether the control system is running optimally or not. Achieving the same Harris index may result in different control laws, i.e. the controller output may have different variances for the same process output variance. The larger value of the controller output variance is not at the optimal running point for process control. This is because the Harris index didn’t take into account the controller output. Another drawback of the Harris index is that the relationship between this index and the controller parameters is not straightforward.

A new performance index - I/O index - is proposed here which provides more information about the performance of a feedback control system. It combines the performance measurements of the controller output and process output. By using a modified benchmark from MVC (minimum variance control), it can be ensured that the control system is running optimally, which means consuming less controller energy, while obtaining desired process output variance. It also allows a user to specify a benchmark that considers the balance between the controller output variance and the process output variance for a specific control loop. The goal of using this performance index is to provide thorough information about a process during closed-loop operation. This information can be used to confirm if the process is operating as expected and also to highlight any changes in the process.

In this index, more emphasis is put on balancing process input and process output variances. This is because the original Harris index only compares the actual output
variance with the theoretical minimum variance. The reason that MVC cannot be widely used is that even though the output can reach minimum variance, the control action will probably be too aggressive to be implemented. In other words, the controller output magnitude will be too large when trying to eliminate the disturbance. Therefore, the process input will oscillate significantly to bring the output back to the setpoint. This type of controller behavior should be avoided. A trade-off between process input and output variance should be considered in performance assessment.

The prior information required for this performance analysis method is the knowledge of the process model. Though this requirement is more than what the Harris index requires, it provides more information. Therefore, performance assessment through this new $I/O$ index can avoid the energy wastage and be more practical.

The class of processes considered in this project is restricted to SISO, discrete-time, linear and time invariant systems.

1.4 Objectives of the Thesis

The objective of this thesis is to provide process control engineers with a useful performance analysis tool. To be more specific, it is to define a new performance measure which takes into account both controller and process output. It also attempts to design a new performance measure which allows users to specify an achievable performance rather than ideal minimum variance by investigating the minimum variance that PID controllers can achieve.

1.5 Organization of the Thesis

The outline of the thesis is as follows: In chapter 2, literature in the field of performance assessment is reviewed. It covers performance analysis tools for both servo control and
regulatory control purposes. In chapter 3, a closed-loop performance assessment using the process output variance and time delay is discussed. Further research about the Harris index on its limitations and application to PID controller tuning is discussed, followed by a few examples. In chapter 4, a new I/O index diagram is designed. It includes the estimation of the upper bound of the controller output variance and the definition of the I/O index. Chapter 5 is about the demonstration of the I/O index, using computer simulations as well as industrial data. Chapter 6 concludes and indicates the contributions of this thesis and outlines the future research directions.

1.6 Contributions

The major contributions of this research are as follows:

1. Developed a tuning method for regulatory control which relates PID controller parameters to the achievable minimum variance for the process output.

2. Developed a new input/output (I/O) index that considers both the process output performance and the corresponding controller output performance of a closed-loop system. This index has the advantage of using the achievable minimum variance as a benchmark instead of the ideal one. It also shows the performance measurements graphically. The prior knowledge needs operating data from the control loop and model identification by switching between two different controllers in closed loop.

3. Demonstrated the new I/O index by performing computer simulations and an experimental study of a level control system.
2.1 Introduction

Controllers are implemented with many design objectives, including setpoint tracking, disturbance rejection, constraint handling, and surge attenuation. Assessing performance with contradictory and mismatched criteria degrades rather than improves performance.

The performance of a control loop can be assessed as servo performance or regulatory performance, in terms of different control purposes. Servo performance indicates how well the controller is doing to keep the process output tracking the setpoint. On the other hand, regulatory performance indicates how well the controller can eliminate the disturbance. The discussion on performance assessment will be divided into the following two sections: servo performance and regulatory performance. Besides, performance monitoring systems as complete analyzing, detecting and alarming tools will be reviewed in this chapter later.

2.2 Servo Performance

Regarding servo performance, there are several well-developed criteria to measure it, such as overshoot, rising time, settling time, etc (Kuo, 1982) [10]. Recently, Astrom (1990) [11] introduced some other criteria to measure the servo performance, such as bandwidth, peak error, maximum time and integral gain from the response to a unit-step process input. The paper considered problems where the process is described by
linearized models with actuators which saturate. Although the technique for the performance assessment is straightforward, it requires more information on process dynamics, disturbances and regulator complexity. Also in that paper it is limited to PID control algorithm. Zhuang and Atherton (1991) [12] introduced an integral performance criteria for optimal PID controller settings. The integral squared error criterion (ISE) has a disadvantage that its minimization often results in a relatively oscillatory step response because large errors occurring within a small time contribute significantly to the performance index. Therefore, the authors proposed the time moment weighted ISE criterion. The results clearly show that the proposed time moment weighted ISE provides a better tuning.

Swanda and Seborg (1997) [13] developed a new performance index, the normalized settling time to characterize the performance of PID-type feedback control loops. The index is determined by normalizing the settling time of a setpoint by the apparent time delay of the process. The advantage of this index is that it is insensitive to model order and model type for a wide range of transfer function models. It can be used on-line to detect poorly performing control loops along with setpoint overshoot.

2.3 Regulatory Performance

In reality, most of the control loops in industry are running at steady state, as such most of the performance assessment is considered only for regulatory control. The following performance assessment is restricted to regulatory performance.

Control loop performance assessment here is a measure of performance of the control loop relative to some predefined benchmark standards. One of the first performance assessment application was reported by Astrom in 1967 [14]. One-step ahead prediction variance was used as a benchmark standard. This application though simple, had the
disadvantage of impracticality on processes with a time delay.

2.3.1 Performance Indices Based on Minimum Variance

Astrom (1967, 1970) [14, 15], Astrom and Wittenmark (1973) [16], Box and Jenkins (1976) [17], Harris (1989) [2] and Stanfelj et al. (1993) [18] have suggested the use of an autocorrelation function to calculate the loss function of the achievable optimal minimum variance, which can be the benchmark standard to measure the loop performance. The technique is quite elegant, that is because the autocorrelation function up to lag \( d - 1 \) (where \( d \) is the process time delay) can be easily estimated. From the autocorrelation function figure, minimum variance or the achievable minimum variance can be obtained.

Harris (1989) [2] formalized a performance index. Instead of using an autocorrelation function, he introduced a time series analysis to calculate the achievable minimum variance. The feedback-invariant property of the minimum variance term can be used as the benchmark for assessing loop performance.

After that, Desborough and Harris (1992) [4] modified the original performance index into the normalized performance index. This normalized performance index is a scalar between zero and one, with zero indicating that the process is operating under minimum variance control, i.e. at its optimal performance bound. Desborough and Harris (1993) [7] extended the application of the index to feedforward/feedback control loops. In their work, a scheme was developed for analyzing linear dynamic MISO systems where there is no cross-correlation among the disturbances (i.e. process input disturbance and process output disturbance). The overall performance bound they developed, can be decomposed into the best possible bounds for each of the controllers viz. feedforward and feedback controllers. This accounts for the performance limitations imposed by the disturbances of the process.

Kozub and Garcia (1993) [19] have reported a similar measure of the performance
which they defined as Closed Loop Potential (CLP). Lynch and Dumont (1993) [20] also applied a similar idea to control loop performance monitoring.

These research results were limited to SISO, linear and time invariant systems. For more sophisticated control systems, for example, non-linear, MIMO, and non-minimum phase, the original Harris index cannot assess the performance well.

2.3.2 Application of the Harris Index

Recently, several researchers have extended Harris's performance assessment concept to make it more applicable. Tyler and Morari (1995) [21] have applied the idea to SISO unstable systems and systems with general non-invertible dynamics when the locations of the poles and zeros outside the unit circle are known. The extension of this idea has also been studied by Huang et al. (1995) [22]. It is a new approach on loop performance analysis of MIMO processes, based on Filtering and CORrelation (FCOR) analysis of the process output and filtered data. This algorithm is simple and efficient. However, it requires some prior knowledge or estimation of the time-delay or interactor matrix of the MIMO process.

Isaksson (1996) [23] introduced a set of alternative indices which take into account the controller structure limitations (such as PI, PID, Dahlin etc.) and intended control task (such as stochastic control, servo control, regulatory control). A drawback with this set of indices is that, as compared to the original Harris index which uses minimum variance control, they require an identified process model and knowledge of the current controller setting for calculating the index.

2.3.3 Concluding Remarks

Essentially, all these developments of the performance index are based on the same theory, the index was basically defined to be the ratio:
Variance of the error between the controlled variable and its target
Minimum variance achievable

This method does not require an external signal to perturb the closed-loop system, the process delay should be known and a time series model has to be fitted to closed-loop process data.

Ideally, minimum variance control is the best possible control since no other controller is able to provide a lower variance than that. Although this performance index provides very valuable information on the lower bound of performance, minimum variance control is not desirable in practical situations. The reason is that a minimum variance controller demands excessive control action and the closed-loop has poor robustness.

2.4 Other Approaches for Performance Assessment

In 1995, Tyler and Morari [24] presented an approach which expressed good performance as constraints on the impulse response coefficients. A generalized likelihood ratio test with suitable threshold was used to determine if the constraints were met. However, the drawback of using this approach requires the knowledge of the impulse response of the closed-loop transfer function from disturbance to the output. Furthermore, the impulse response coefficients depend not only on the plant and the controller, but also on the disturbance generator.

2.4.1 LQ Performance Index

In general, strict quality specification resulting in smaller variation in the process output will require more control effort. This minimum variance benchmark doesn’t explicitly take into account the control effort. Therefore, performance evaluation with control
action constraints needs to be taken into account. Hence, LQ control becomes more and more popular in the research community.

Kammer et al. (1996) [25] searched for similar performance measurement that could be extended to LQ control. In that paper, a test of optimality for linear quadratic control is presented. The advantage is that it does not require parametric model fitting for the process being controlled. However, it has to use external excitation. From this test, it is possible to determine the closed-loop pole positions which would have been obtained by using the LQ optimal controller. The key point is to illustrate the feasibility of the method.

Astrom and Wittenmark (1990) [26] explained how to find the weighting matrices in the LQ performance index. One way to decide the weights is to choose the diagonal elements as the inverse value of the square of the allowed deviations. Another way is to consider only penalties on the state variables and the constraints on the control deviations.

Huang (1997) [8] in his thesis proposed the solution (achievable performance) for LQG benchmark performance assessment. A tradeoff curve was shown. By varying \( \lambda \), various optimal solutions of \( E[y^2] \) and \( E[u^2] \) can be calculated. Thus a curve with the optimal \( E[u^2] \) as the abscissa and \( E[y^2] \) as the ordinate is generated. This curve therefore represents the bound of the performance and can be used for performance assessment purpose.

Stuckman and Stuckman (1993) [27] presented a method of determining the weighting matrices for the best optimal LQ control. The advantage of this technique is that it uses a design which is simultaneously optimal in the quadratic sense as well as in terms of a separate, more meaningful performance criterion developed by the designer.
2.4.2 Applications of the LQ Performance Index

Hagiwara et al. (1996) [28] proposed a new method to design a two-degree-of-freedom robust servo system for step references and step disturbances. In this method, the tracking characteristics for step references are determined with respect to a quadratic-integral performance index, while the feedback characteristics for step disturbances are determined with respect to another quadratic performance index where the frequency-dependent weighting matrices are introduced. This method can reduce the sensitivity in the low frequency range without deteriorating the characteristics in the high frequency range.

Lee and Wu (1993) [29] studied a time-weighted performance index for optimal discrete time linear time-invariant systems. The performance index consists of two parts. One is used to penalize the sustained error while the other one is used to improve the robustness of the closed-loop system. This method also ensures all closed-loop poles lie inside the region.

Veillitte (1995) [30] introduced a procedure for the design of reliable LQ state-feedback control which guaranteed both stability and known quadratic performance bound, despite any outages within a selected subset of actuators.

2.4.3 Concluding Remarks

LQ Performance Index gives a better assessment for closed-loop systems since it considers both the output and the input variations. However, the difficulties to compute LQ performance index are as follows:

1. It needs prior knowledge of the process model, while the Harris index only needs the disturbance dynamics and process delay.

2. In LQ performance index, the weight of the control input is not easy to choose. The initial control weighting is usually "guessed" and then the controller is implemented
If the control action is too high, the control weighting needs to be increased and in turn the controller has to be redesigned. This procedure is repeated until proper control is achieved.

2.5 Performance Monitoring System

For oscillation detection in the control loop, Hagglund (1994) [31] presented a procedure for automatic monitoring of control loop performance. The Control Loop Performance Monitor (CLPM) detects oscillations in the control loop. This CLPM is quite robust because the theory is based on monitoring the integrated absolute value of the control error (IAE) between successive zero crossings. The assumption for this method allows to detect any shape of oscillations - only the measured signal needs to deviate significantly from the set point for sufficient times during certain supervision period.

In 1996, Owen et al. [32] developed a prototype on-line system for automatic detection and location of malfunctioning control loops. They extended the single variable analysis for loop performance to a multi-variable analysis which took into account the propagation of disturbances from the malfunctioning loop to other normal functioning loops. The authors claimed that the package required very little prior information and was able to detect the presence of upset conditions. Excess variation caused by non-linearities due to sensor or valve failures would also be taken into account in the calculation of the performance index. However, no detail was written in the literature because the package was considered proprietary.

Harris et al. (1996) [33] reviewed SISO control performance assessment and detailed the development of an expert system that quantifies control loop performance on an ongoing basis. It uses a normalize performance index to quantify the loop's performance. This expert system is an interface for users to acquire, examine and store process data.
2.6 Conclusion

The existing methods to assess/measure the control loop performance have been briefly reviewed in this chapter. Some extensions of the performance analysis indices and performance monitoring systems have been reported. They allow better detection and identification of poor performance, as well as diagnosis of malfunctioning in industrial control systems. Most of the performance measuring indices can be sorted into two categories: one for servo performance assessment and the other for regulatory performance assessment according to different control purposes. Care must be taken to select appropriate criteria for assessing individual controllers. This will guarantee that the benefits upon which the controller was initially justified will continue to be realized.
Chapter 3

Minimum Variance Performance Index

3.1 Introduction

Modern manufacturing facilities have many automatic control loops. It is nearly impossible to monitor the performance of more than a few of the most critical control loops without some formalized assessment tools. In order to identify the poorly performing control loop and be able to improve performance, it is necessary to correctly diagnose the underlying problem. The emerging area of performance assessment provides a means of diagnosing control loop performance using time series and digital signal processing techniques.

As mentioned earlier in chapter 2, in terms of performance assessment, many methods have been investigated since the 1960's. Most of those methods require some priori knowledge about the process model, noise model and/or time delay of the system.

In 1992, Desborough and Harris [4] developed a normalized index for analyzing the performance of regulatory control in a closed-loop system. This index (known as the Harris index) uses minimum variance control as a benchmark. It is a scalar between 0 and 1, as 0 indicates the output variance reaching ideal minimum variance and 1 indicates an unstable/unacceptable response of the process output which has a huge variance. The major advantage of using the Harris index is that instead of knowing much information about the process model and noise properties, it only requires routine operating data and little prior knowledge of the time delay in plant processes. This
method brings operation personnel great convenience to estimate the ideal minimum variance of the process output. Therefore, the Harris index gives the user a certain number which measures how far the current process variance is to ideal minimum variance by simply collecting the closed-loop data from process outputs. In this chapter, the derivation of the Harris index will be reviewed followed by a discussion on the limitations and application of the Harris index. A novel method to calculate PID tuning parameters for near minimum variance control will be described later.

3.2 Definition of the Harris Index

As an emerging technique in performance assessment of control systems, the Harris index is derived in this section for future discussions to follow. The idea of using minimum variance as a performance measurement was first proposed in 1989 [2]. The normalized Harris index was then derived in 1992 [4] by Desborough and Harris.

Several assumptions were introduced in this performance assessment technique:

- If a controller is a minimum variance controller, the true autocorrelation functions of process output $y$ are zero for lags $\geq d$, where $d$ is the number of whole periods of delay. This is because in reality, models always have some mismatches with the real processes which may result in the autocorrelation functions not being zero for lags $\geq d$. The Harris index is extremely sensitive to poor estimates of delay $d$.

- The process output or its transformation can be adequately described by a linear transfer function with additive disturbance. The Harris index is not applicable to non-linear systems.

The following model used for describing a process is assumed to be ARIMA:

$$y(t) - \mu = \frac{B(z^{-1})}{A(z^{-1})} z^{-d} u(t) + \frac{C(z^{-1})}{D'(z^{-1})} \nabla w(t)$$  \hspace{1cm} (3.1)
Chapter 3. *Minimum Variance Performance Index*

where

\[ y(t) \] - controlled variable/process output.
\[ \mu \] - mean of \( y(t) \).
\[ z^{-1} \] - backward shift operator which is defined such that \( z^{-1}y(t) = y(t - 1) \).
\[ B(z^{-1}), A(z^{-1}) \] - polynomials of order \((p,q)\) respectively in backward shift operator \( z^{-1} \), \( A(z^{-1}) \) is monic.
\[ d \] - the number of whole periods of delay including a zero-order hold.
\[ u(t) \] - manipulated variable-controller output.
\[ C(z^{-1}), D'(z^{-1}) \] - stable monic polynomials.
\[ \nabla = 1 - z^{-1} \]
\[ \{w(t)\} \] - a sequence of independently and identically distributed random variables.

The controller is represented by \( G_c(z^{-1}) \) which has the following relation with the process input and output.

\[
u(t) = G_c(z^{-1})(y_{sp} - y(t)) \quad (3.2)
\]

After substituting equation (3.2) into equation (3.1), a closed-loop model can be written in the following format:

\[
y(t) - \mu = \frac{B(z^{-1})}{A(z^{-1})}z^{-d}G_c(y_{sp} - y(t)) + \frac{C(z^{-1})}{D(z^{-1})}w(t) \quad (3.3)
\]

where \( D'(z^{-1})\nabla \) is equal to \( D(z^{-1}) \). Ideally, the mean of the output \( \mu \) equals setpoint \( y_{sp} \).
For regulatory control, only the deviation of a variable is concerned. Therefore, the process output deviation can be defined as

\[ \tilde{y}(t) = y(t) - y_{sp} \]  

(3.4)

Combining equation (3.3) and equation (3.4) gives

\[ A(z^{-1})D(z^{-1})\tilde{y}(t) = -B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})\tilde{y}(t) + A(z^{-1})C(z^{-1})w(t) \]  

(3.5)

re-arranging the terms alike yields

\[ [A(z^{-1})D(z^{-1}) + B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})]\tilde{y}(t) = A(z^{-1})C(z^{-1})w(t) \]  

(3.6)

Then,

\[ \tilde{y}(t) = \frac{A(z^{-1})C(z^{-1})}{A(z^{-1})D(z^{-1}) + B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})}w(t) \]

\[ = \frac{C(z^{-1})}{D(z^{-1}) + \frac{B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})}{A(z^{-1})}}w(t) \]

\[ = \psi(z^{-1})w(t) \]  

(3.7)

where

\[ \psi(z^{-1}) = \frac{A(z^{-1})C(z^{-1})}{A(z^{-1})D(z^{-1}) + B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})} \]

(3.8)

\( \psi(z^{-1}) \) is a monic polynomial since both \( A(z^{-1})C(z^{-1}) \) and \( A(z^{-1})D(z^{-1}) \) are monic. It can be broken into two parts such that

\[ \tilde{y}(t) = F(z^{-1})w(t) + \psi_1(z^{-1})w(t - d) \]

\[ = e(t) + \hat{y}(t) \]  

(3.9)

where

\[ F(z^{-1}) = 1 + f_1z^{-1} + f_2z^{-2} + \cdots + f_{d-1}z^{-d+1} \]  

(3.10)
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and the degree of the monic polynomial \( F(z^{-1}) \) is \( d-1 \). On the right hand side of equation \((3.9)\), \( e(t) \) can be interpreted as the \( d \)-step ahead prediction error from the future noise which cannot be eliminated by the controller. \( \hat{y}(t) \) is called the \( d \)-step predictor of \( y(t) \). The coefficients of \( F(z^{-1}) \) can be obtained by solving the Diophantine identity/long division shown as follows:

\[
\frac{C(z^{-1})}{D(z^{-1}) + B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1})z^{-d}}{D(z^{-1}) + B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})} \tag{3.11}
\]

Obviously, \( \psi_1(z^{-1}) \) in equation \((3.9)\) is equal to

\[
\frac{G(z^{-1})}{D(z^{-1}) + B(z^{-1})D(z^{-1})z^{-d}G_c(z^{-1})}.
\]

So far, the variance of \( \hat{y}(t) \) can be given by

\[
Var(\hat{y}(t)) = Var(e(t)) + Var(\hat{y}(t)) + 2Cov\{e(t), \hat{y}(t)\} \tag{3.12}
\]

Since \( w(t) \) is a sequence of independent random variables, \( Cov\{e(t), \hat{y}(t)\} \), which is equivalent to \( Cov\{F(z^{-1})w(t), \psi_1(z^{-1})w(t-d)\} \), vanishes.

As long as the model equation \((3.1)\) is known, a controller can be designed to minimize the variance of \( \hat{y}(t) \). For minimum variance (MV) control, the controller forces the output to the setpoint in \( d \) steps, which can be explained as follows.

\[
\hat{y}(t) = \frac{B(z^{-1})}{A(z^{-1})}u(t-d) + \frac{C(z^{-1})}{D(z^{-1})}w(t) = \frac{B(z^{-1})}{A(z^{-1})}u(t-d) + F(z^{-1})w(t) + \frac{G'(z^{-1})}{D(z^{-1})}w(t-d) \tag{3.13}
\]

Here the Diophantine equation was again used for \( \frac{C(z^{-1})}{D(z^{-1})} \),

\[
\frac{C(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G'(z^{-1})}{D(z^{-1})} \tag{3.14}
\]

Notice the polynomial \( F(z^{-1}) \) in equation \((3.14)\) (open-loop model) is the same as in equation 3.9 (closed-loop model). This is because the future noise term won't be affected/eliminated by the controller form, i.e. \( F(z^{-1}) \) is the same in open-loop model and closed-loop model.
Therefore, the controller will force the predictor to go to zero since the setpoint is considered to be zero, i.e.

\[
\hat{y}(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t - d) + \frac{G'(z^{-1})}{D(z^{-1})} w(t - d) = 0
\]

(3.15)

As a result, the variance of \( \hat{y}(t) \) achieves the minimum variance \( \delta_{MV}^2 \).

\[
Var(\hat{y}(t)) = Var(e(t)) = \delta_{MV}^2
\]

(3.16)

If the controller is not a MV controller, the variance of the predictor \( \delta_y^2 \) is non-zero. The process output variance will always be greater than the minimum variance.

\[
Var(\hat{y}(t)) = \delta_y^2 = Var(e(t)) + Var(\hat{y}(t)) = \delta_{MV}^2 + \delta_y^2
\]

(3.17)

### 3.3 Estimation of Ideal Minimum Variance

The Harris index uses minimum variance as the benchmark, hence, how to calculate the minimum variance is a point of discussion. Theoretically, the minimum variance \( \delta_{MV}^2 \) can be calculated from term \( e(t) \) which demands to know the coefficients in equation (3.10) and the variance of the white noise \( w(t) \). In this section, by knowing only the time delay of the process and the closed-loop operating data collected from the output, the minimum variance will be estimated through the least squares method.

Recall the closed-loop model equation (3.7) and equation (3.9):

\[
\hat{y}(t) = \psi(z^{-1}) w(t)
\]

\[
= F(z^{-1}) w(t) + \psi_1(z^{-1}) w(t - d)
\]

(3.18)
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Backward shifting equation (3.18) by step \( d \), one obtains

\[
\hat{y}(t - d) = \psi(z^{-1})w(t - d)
\]

\[
w(t - d) = \frac{\hat{y}(t - d)}{\psi(z^{-1})}
\]

Substituting equation (3.19) into equation (3.9) gives

\[
\hat{y}(t) = F(z^{-1})w(t) + \frac{\psi_1(z^{-1})}{\psi(z^{-1})}\hat{y}(t - d)
\]  \hspace{1cm} (3.20)

Since all system dynamics are assumed to be stable in closed loop, \( \frac{\psi_1(z^{-1})}{\psi(z^{-1})} \) must form a convergent infinite series addition such that,

\[
\frac{\psi_1(z^{-1})}{\psi(z^{-1})} = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots
\]

\[
= \sum_{k=0}^{\infty} \alpha_k z^{-k}
\]

(3.21)

Notice that

\[
\sum_{k=0}^{\infty} \alpha_k < \infty
\]

To simplify the calculation, in practice, the infinite series \( \alpha_k \) is truncated after \( m \) terms. Substituting equation (3.21) into equation (3.20) yields.

\[
\hat{y}(t) = F(z^{-1})w(t) + \sum_{k=1}^{\infty} \alpha_k \hat{y}(t - d - k + 1)
\]

\[
\approx F(z^{-1})w(t) + \sum_{k=1}^{m} \alpha_k \hat{y}(t - d - k + 1)
\]  \hspace{1cm} (3.22)

A lagged regression of closed-loop data \( y_i \) can be fitted only in matrix-vector notation.

\[
Y = X\bar{\alpha} + R
\]
where,

\[
Y = \begin{bmatrix}
\tilde{y}(n) \\
\tilde{y}(n - 1) \\
\vdots \\
\tilde{y}(d + m)
\end{bmatrix}
\]

\[
X = \begin{bmatrix}
\tilde{y}(n - d) & \tilde{y}(n - d - 1) & \cdots & \tilde{y}(n - d - m + 1) \\
\tilde{y}(n - d - 1) & \tilde{y}(n - d - 2) & \cdots & \tilde{y}(n - d - m) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{y}(m) & \tilde{y}(m - 1) & \cdots & \tilde{y}(1)
\end{bmatrix}
\]

\[
\alpha = \begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
r_n \\
r_{n-1} \\
\vdots \\
r_{d+m}
\end{bmatrix}
\]

\(R\) is the error vector. Using the least squares estimate algorithm, vector \(\alpha\) can be solved by

\[
\alpha = [X^TX]^{-1}X^TY
\]

The minimum variance can be estimated from the error vector \(E\) which contains all the unrejectable future noise variance.

\[
\delta^2_{MV} = \frac{R^TR}{n - d - 2m + 1}
\]
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\[ (Y - X_{\alpha})^T (Y - X_{\alpha}) \]

In equation (3.23), the denominator at the right hand side is from the degree of freedom according to the least squares algorithm. The reason for subtracting 2m is that one m is from m data points lost, the other m is due to m number of elements in matrix \(\alpha\).

In order to measure the performance of the process output by the Harris index, the real process output variance needs to be measured to compare with the benchmark - minimum variance. To do so, in practice, one would simply calculate the mean square error \(\text{mse}(y)\) which is defined as follows.

\[
\text{mse}(y) = E[y - y_{sp}]^2
\]

\[
= E[y - E(y) + E(y) - y_{sp}]^2
\]

(3.24)

The expectation of process output \(E(y)\) can be considered as the average of the output \(\bar{y}\). To calculate mean square error of data \(\{y_i\}\), the following step is taken:

\[
\text{mse}(\bar{y}) = \frac{\sum_{i=d+m}^{n} (\bar{y}(i))^2}{n - d - m + 1}
\]

\[
= \frac{\sum_{i=d+m}^{n} (y(i) - \bar{y} + \bar{y} - y_{sp})^2}{n - d - m + 1}
\]

\[
= \frac{\sum_{i=d+m}^{n} (y(i) - \bar{y})^2}{n - d - m + 1} + (\bar{y} - y_{sp})^2
\]

(3.25)

Note that all data used to calculate \(\text{mse}(\bar{y})\) are mean-centered. To choose the proper number \(m\) in equation (3.25), normally one can start from 5 and up until the normalized performance index estimate shows no appreciable change.

Therefore, the normalized Harris index is defined as follows:

\[
\eta(d) = 1 - \frac{\hat{\sigma}^2_{MV}}{\text{mse}(\bar{y})}
\]

(3.26)
Note that all the data used to calculate $\delta^2_{MV}$ is mean-centered.

In summary, the actual procedure to estimate $\delta^2_{MV}$ is as follows:

1. Collect data $y(i)$ and $y_{sp}(i)$ that does not include scheduled setpoint changes.
2. Obtain $\tilde{y}(i)$ by subtracting $y(i) - y_{sp}(i)$.
3. Calculate $mse(\tilde{y}(i))$.
4. Mean-center data $\tilde{y}(i)$.
5. Calculate $\delta^2_{MV}$.

3.4 Application to Tuning PID Controller Parameters

The development and interpretation of the normalized performance index - Harris index - has been briefly reviewed. It is computationally simple, yet not applicable to PID controller tuning. As PID controllers still have the dominant role in the industry, this section will discuss how to relate the controller tuning to the Harris index.

3.4.1 The Discrete Control system

With the advent of the digital computer, more and more important control loops are under digital feedback. It provides an opportunity to a process engineer to carry more sophisticated control designs. Since the Harris index was derived on a linear, single input single output discrete control model, the following section will also start with a discrete control system.
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**Control system model**

The block diagram of a feedback control system is shown as follows.

![Block Diagram of a Feedback Control Loop](image)

Figure 1.1. Block Diagram of a Feedback Control Loop.

The process model is assumed to be a Box and Jenkins model.

\[
y(t + d) - \mu = \frac{B(z^{-1})}{A(z^{-1})} u(t) + \frac{C(z^{-1})}{D'(z^{-1})} w(t + d) \quad (3.27)
\]

where

- \( y(t) \) - controlled variable/process output.
- \( \mu \) - mean of \( y(t) \).
- \( z^{-1} \) - backward shift operator which is defined such that \( z^{-1} y(t) = y(t - 1) \).
- \( B(z^{-1}), A(z^{-1}) \) - polynomials of order \( (p,q) \) respectively in backward shift operator \( z^{-1} \); \( A(z^{-1}) \) is monic.
- \( d \) - the number of whole periods of delay including a zero-order hold.
- \( u(t) \) - manipulated variable/controller output.
- \( C(z^{-1}), D'(z^{-1}) \) - stable monic polynomials.
\[ \nabla = 1 - z^{-1} \]
\[ \{w(t)\} \quad \text{a sequence of independently} \]
\[ \text{and identically distributed random variables.} \]

\[ D'(z^{-1}) \nabla \] is equal to \( D(z^{-1}) \). Ideally, the mean of the output \( \mu \) equals setpoint \( y_{sp} \). \( \bar{y}(t) \) can be defined as \( y(t) - y_{sp} \).

**Closed-loop Model**

As discussed earlier, in order to design the minimum variance controller which forces the output to the setpoint in \( d \) steps, the disturbance term has to be separated into two parts: one is the future noise (cannot be observed or eliminated by the controller), the other is the predictable part due to past disturbances (can be eliminated by the controller). To do so, the Diophantine equation is used for the disturbance model.

\[ \frac{C(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1})}{D(z^{-1})} z^{-d} \]  \hspace{1cm} (3.28)

Substituting \( \bar{y}(t) \) into the open-loop model 3.27 gives

\[ \ddot{y}(t + d) = \frac{B(z^{-1})}{A(z^{-1})} u(t) + \frac{C(z^{-1})}{D(z^{-1})} w(t + d) \]  \hspace{1cm} (3.29)

By combining equation (3.28) and equation (3.29), one obtains

\[ \ddot{y}(t + d) = \frac{B(z^{-1})}{A(z^{-1})} u(t) + F(z^{-1})w(t + d) + \frac{G(z^{-1})}{D(z^{-1})} w(t) \]  \hspace{1cm} (3.30)

By backward shifting equation (3.30) by \( d \) steps and dividing both sides by \( \frac{D(z^{-1})}{C(z^{-1})} \),

\[ w(t) = [\ddot{y}(t) - \frac{B(z^{-1})}{A(z^{-1})} u(t - d)] \frac{D(z^{-1})}{C(z^{-1})} \]  \hspace{1cm} (3.31)
Substituting equation (3.31) into equation (3.30),

\[
\hat{y}(t + d) = \frac{B(z^{-1})}{A(z^{-1})} u(t) + \frac{G(z^{-1})}{D(z^{-1})} [\hat{y}(t) - \frac{B(z^{-1})}{A(z^{-1})} u(t - d)] \frac{D(z^{-1})}{C(z^{-1})} + F(z^{-1})w(t + d)
\]

(3.32)

By combining the terms alike, one obtains

\[
\hat{y}(t + d) = \frac{B(z^{-1})}{A(z^{-1})} [1 - \frac{G(z^{-1})}{C(z^{-1})} z^{-d}] u(t) + \frac{G(z^{-1})}{C(z^{-1})} \hat{y}(t) + F(z^{-1})w(t + d)
\]

(3.33)

Since the last term on the right hand side of equation (3.33) is the future noise term (which is the source of the minimum variance), the controller does not have any impact on it. As for the first two terms, one can combine them together as follows:

\[
\hat{y}(t + d) = \frac{B(z^{-1})F(z^{-1})D(z^{-1})}{C(z^{-1})A(z^{-1})} u(t) + \frac{G(z^{-1})}{C(z^{-1})} \hat{y}(t) + F(z^{-1})w(t + d)
\]

\[= \frac{B(z^{-1})F(z^{-1})D(z^{-1})u(t) + A(z^{-1})G(z^{-1})\hat{y}(t)}{A(z^{-1})C(z^{-1})} + F(z^{-1})w(t + d)\]

(3.34)

To simplify the discussion in the sequel, it is better to redefine the following polynomials:

\[
a(z^{-1}) = B(z^{-1})F(z^{-1})D(z^{-1})
\]

(3.35)

\[
b(z^{-1}) = A(z^{-1})G(z^{-1})
\]

(3.36)

\[
c(z^{-1}) = A(z^{-1})C(z^{-1})
\]

(3.37)

\[
e(t) = F(z^{-1})w(t)
\]

(3.38)

Consider a general linear feedback controller,

\[
u(t) = \frac{-R_1(z^{-1})}{R_2(z^{-1})} \hat{y}(t)
\]

(3.39)

Three important equations can be obtained from the original equation (3.34):
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1. In the first equation, \( \hat{y}(t) \) is represented by two variables: \( u(t-d) \), \( \hat{y}(t-d) \), plus the future noise term \( e(t) \). The predictor is \( \hat{y}_1(t) \).

\[
\hat{y}(t) = \frac{B(z^{-1})F(z^{-1})D(z^{-1})u(t-d) + A(z^{-1})G(z^{-1})\hat{y}(t-d) + F(z^{-1})w(t)}{A(z^{-1})C(z^{-1})} \\
= \frac{a(z^{-1})u(t-d) + b(z^{-1})\hat{y}(t-d)}{c(z^{-1})} + e(t) \\
= \hat{y}_1(t) + e(t) \tag{3.40}
\]

2. In the second equation, substituting the controller from equation (3.39) into equation (3.34), \( \hat{y}(t) \) can only be represented by variable, \( u(t-d) \), plus the future noise term \( e(t) \). The predictor is \( \hat{y}_2(t) \).

\[
\hat{y}(t) = \frac{B(z^{-1})F(z^{-1})D(z^{-1})R_1(z^{-1}) - A(z^{-1})G(z^{-1})R_2(z^{-1})}{A(z^{-1})C(z^{-1})R_1(z^{-1})}u(t-d) + F(z^{-1})w(t) \\
= \frac{a(z^{-1})R_1(z^{-1}) - b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_1(z^{-1})}u(t-d) + e(t) \\
= \hat{y}_2(t) + e(t) \tag{3.41}
\]

3. In the third equation, \( \hat{y}(t) \) is only represented by variable, \( \hat{y}(t-d) \), plus the future noise term \( e(t) \). The predictor is \( \hat{y}_3(t) \).

\[
\hat{y}(t) = \frac{-B(z^{-1})F(z^{-1})D(z^{-1})R_1(z^{-1}) + A(z^{-1})G(z^{-1})R_2(z^{-1})}{A(z^{-1})C(z^{-1})R_2(z^{-1})}\hat{y}(t-d) + F(z^{-1})w(t) \\
= \frac{-a(z^{-1})R_1(z^{-1}) + b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_2(z^{-1})}\hat{y}(t-d) + e(t) \\
= \hat{y}_3(t) + e(t) \tag{3.42}
\]

These three equations are important because they present the closed-loop model in different predictor forms of the process output, but they are the same in the sense they repeat the same closed loop. In the first predictor \( \hat{y}_1 \), the numerator is the equation of the minimum variance controller. This provides a short cut to the design of a minimum
variance controller. Both the second predictor $\hat{y}_2$ and the third predictor $\hat{y}_3$ use the controller in their prediction equations. This fact helps to analyze non-identifiability of closed-loop data. With closed-loop data, the predictor form of the Box-Jenkins model can be identified. Therefore, by using the predictor form for the closed-loop model, it is easier and more practical to analyze the performance of the loop. Several closed-loop identification methods will be introduced in the next section.

### 3.4.2 Closed-Loop Identification

The Box-Jenkins model is a well-known and parsimonious model for a discrete control system. But due to the fact that it is a sum of two rational functions, the model is difficult to identify. There have been algorithms suggested to identify this model.

The question of closed-loop data began with a paper by Akaike (1967). In this paper, by using cross-spectral method, Akaike showed that if closed-loop data are used, one might not get the transfer function of the process dynamics but the inverse transfer function of the controller.

In 1975, Ljung et al. (1974) [35] studied identifiability of closed-loop data and suggested that by shifting controller parameters in different control laws, identifiability can be achieved as in open loop. The number of control laws or controllers must be greater than the ratio of number of input to number of output variables. This is for multivariable systems. For single input single output systems, one just needs to shift between two control laws. The conclusion involving the prediction error method was drawn as “Direct identification with a prediction error method can be used exactly as in the open-loop case; the fact that the system operates in closed-loop causes no extra difficulty” [35].

In 1981, Gevers, and Anderson [36] presented a new approach to the problem. They claimed that under feedback control, the white noise drives both the input and output variable time series, therefore these variables can be put in a vector form and treated as a
vector time series. This vector form gives a joint input-output model which contains both
the controller and the transfer function of the process dynamics. This joint input-output
model can be identified by a factorization of the joint spectral density matrix. However,
with no dither signal this method becomes an indirect method.

In 1988, Zervos et al. [37] introduced a new method called Laguerre expansion to
identify the plant and obtain optimal PID tuning constants. The plant in their study
is a linear, continuous-time plant. It is assumed initially to be under stable, closed-loop
control. The closed-loop plant is modeled by a Laguerre series expansion, rather than by a
fixed-structure transfer function, because of the requirements of robustness with minimal
prior information. It is seen that the estimates of the Laguerre coefficients are unbiased
even for a truncated series. Also a fairly large number of terms can be used without
undue computational burden. The step response of a closed-loop system is identified
by Laguerre expansion in that paper. It offers certain advantage over ARMA models,
namely lack of bias in estimates, structural flexibility and the ability to precompute the
regressor.

Overall, closed-loop identification methods have been studied extensively by many
researchers. It has become more comprehensive and important. However, closed-loop
identification is not the major scope of this thesis, but its results are being used in the
study of performance tuning.

In practice, the standard function \( b_j \) in the system identification toolbox of the MAT-
LAB software package can be used to identify a Box-Jenkins model. This software func-
tion uses the method of prediction error which interprets the white noise driving the
disturbance model as the prediction error. This white noise has minimum variance prop-
erty and so identification is done via minimization of the sum of squares of its values.
The theory is from Ljung et al. (1974) [35]. Basic concepts are shown as follows:
A linear, discrete time, stochastic system, $s$, is considered which has the general form

$$y(t) = G_s(z^{-1})u(t) + H_s(z^{-1})e(t)$$  \hspace{1cm} (3.43)

where the output, $y(t)$, is a vector of dimension $n_y$, the input, $u(t)$, has dimension $n_u$ and $e(t)$ is a sequence of independent, random vectors with zero-mean value. It has the same dimension as $y(t)$. The initial values for matrix $G_s$ and $H_s$ are 0 and $I$ respectively.

The controller is defined as

$$u(t) = F(z^{-1})y(t)$$ \hspace{1cm} (3.44)

The model functions $G_s(z^{-1})$ and $H_s(z^{-1})$ are parametrized in a suitable manner by a parameter vector $\theta$. Therefore, the model corresponding to a certain value of $\theta$ is denoted by $m(\theta)$ and is given by

$$y(t) = G_{m(\theta)}(z^{-1})u(t) + H_{m(\theta)}(z^{-1})e(t)$$ \hspace{1cm} (3.45)

The identification problem is to determine the parameter $\theta$ so that $m(\theta)$ suitably describes the system $s$ given by equation (3.43).

A common way of parametrizing $G$ and $H$ is to consider vector difference equation models.

$$A_{m(\theta)}(z^{-1})y(t) = B_{m(\theta)}(z^{-1})u(t) + C_{m(\theta)}(z^{-1})e(t)$$ \hspace{1cm} (3.46)

where $A_{m(\theta)}$, $B_{m(\theta)}$ and $C_{m(\theta)}$ are matrix polynomials.

The identification method is denoted by $j$. The concept of identifiability is introduced as follows:

$$D_T(s, m) = \{ \theta | G_{m(\theta)} = G_s; H_{m(\theta)} = H_s \}$$ \hspace{1cm} (3.47)

as estimate time $N$ goes to infinity.
Definitions

1. The system $s$ is said to be system identifiable (SI) if $\theta$ can be found to meet $D_T(s,m)$ as estimate time $N \to \infty$.

2. The system $s$ is said to be strongly system identifiable (SSI) if it is SI for all $m$ such that $D_T(s,m)$ is non-empty.

3. The system $s$ is said to be parameter identifiable (PI) if it is SI and $D_T(s,m)$ consists of only one element.

As long as the system meets the condition for SSI, the fact that the system may operate in closed loop does not add difficulties to the identification of the models.

Identification in closed-loop using predictor error method

One possibility to achieve SSI in close-loop operation is to add extra perturbations to the input or to add noise in the regulator. However, in some cases this may not be permitted since it may excessively increase the variance of the input. It is often possible to find several linear regulators that are acceptable for production quality. According to the theory, SSI can be obtained by switching between several regulators. The required number of regulators depends only on the number of inputs and outputs of the system. A direct identification with a prediction error method in closed loop can be used exactly as in the open-loop case. The cost function (prediction error)

$$Q_N(s,m(\theta)) = \frac{1}{N} \sum_{i=1}^{N} [y(t) - \hat{y}(t|t-1;m(\theta))]^T [y(t) - \hat{y}(t|t-1;m(\theta))] \tag{3.48}$$

is minimized. The minimizing element $\theta$ is estimated. Here the predictor can be derived from equation (3.45) as follows:

$$\hat{y}(t|t-1;m(\theta)) = (I - H_{m(\theta)}^{-1}) y(t) + H_{m(\theta)}^{-1} G_{m(\theta)} u(t) \tag{3.49}$$
By substituting the controller form (equation 3.44) into equation (3.49), the following predictor form can be obtained,

\[
\dot{y}(t|t-1;m(\theta)) = (I - H_{m(\theta)}^{-1} + H_{m(\theta)}^{-1} G_{m(\theta)} F)y(t) = Ly(t)
\]  

(3.50)

Suppose the regulator switches between the \( r \) different control laws and each regulator is used \( r_i \) part of the total time, then SSI can be achieved only if the following equation can be satisfied:

\[
D_T(s,m) = \{\theta|L^{(i)} = \hat{L}^{(i)}, i = 1, 2, 3, ..., r\}
\]  

(3.51)

From equation (3.50), one can see that equation (3.51) is essentially the same as the following equation:

\[
[H^{-1} - H^{-1}; H^{-1} G - \hat{H}^{-1} \hat{G}] R_r = [0, ..., 0]
\]  

(3.52)

where

\[
R_r = \begin{bmatrix}
I & I & \ldots & I \\
F^{(1)} & F^{(2)} & \ldots & F^{(1)}
\end{bmatrix}
\]  

(3.53)

The dimensions of the matrices in equation (3.52) are \( n_y(n_y + n_u) \), \( (n_y + n_u)(n_y r) \) and \( n_y(n_y r) \), respectively. If

\[
\text{rank } R_r = n_y + n_u
\]  

(3.54)

then equation (3.52) implies that

\[
\hat{H}^{-1} - H^{-1} = 0
\]  

(3.55)

\[
H^{-1} G - \hat{H}^{-1} \hat{G} = 0
\]  

(3.56)
Thus equation (3.54) is a sufficient condition for SSI. A necessary condition for equation (3.54) to hold is

\[ rn_y \geq n_y + n_u; \text{ therefore } r \geq 1 + \frac{n_u}{n_y} \quad (3.57) \]

The conclusion can be drawn as follows: when the regulator switches at least \( r(r \geq 1 + \frac{n_u}{n_y}) \) linear feedback laws, it is possible to guarantee strong system identifiability (SSI). Then the input-output data collected during closed-loop control can be used to identify the models in a direct manner.

### 3.4.3 Estimation of Controller Parameters under MVC

In the previous closed-loop model (3.40), (3.41) and (3.42), the \( e(t) \) term is actually the future noise which generates the minimum variance of the control loop. Polynomials \( a(z^{-1}), b(z^{-1}) \) and \( c(z^{-1}) \) can be identified from the closed-loop data using the prediction error method. In this case, the controller form is a time series model. Variance of the output \( \hat{y}(t) \) can be represented by either

\[
\text{var}(\hat{y}(t)) = \text{var}\left( \frac{a(z^{-1})R_1(z^{-1}) - b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_1(z^{-1})} u(t - d) \right) + \text{var}(e_t) \quad (3.58)
\]

or

\[
\text{var}(\hat{y}(t)) = \text{var}\left( \frac{-a(z^{-1})R_1(z^{-1}) + b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_2(z^{-1})} \hat{y}(t - d) \right) + \text{var}(e_t) \quad (3.59)
\]

From the latter equation, the ratio between the minimum variance and the variance of the process output \( \frac{\text{var}(e(t))}{\text{var}(\hat{y}(t))} \) can be seen to be related to the quantity of the time series model

\[
\frac{\text{var}(e(t))}{\text{var}(\hat{y}(t))} = 1 - \text{var}\left( \frac{-a(z^{-1})R_1(z^{-1}) + b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_2(z^{-1})} \hat{y}(t - d) \right)
\]
Recall the normalized Harris Index equation (3.26),

\[ \eta(d) = 1 - \frac{\delta_{MV}^2}{mse(y)} = 1 - \frac{var(e(t))}{var(\hat{y}(t))} \]

By comparing the Harris index definition with \( \frac{var(e(t))}{var(\hat{y}(t))} \), two observations can be made:

1. Since \( a(z^{-1}) \), \( b(z^{-1}) \), and \( c(z^{-1}) \) are all model parameters, for a linear time-invariant process, \( \eta(d) \) only depends on controller parameters. Therefore,

\[ A = \frac{-a(z^{-1})R_1(z^{-1}) + b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_2(z^{-1})} \]

\( A \) is called the amplification factor. This is because it amplifies the output variance higher than the minimum variance except when the controller is in the minimum variance form. When the controller \( \frac{R_1(z^{-1})}{R_2(z^{-1})} \) is equal to \( \frac{b(z^{-1})}{a(z^{-1})} \), the amplification factor is exactly zero and \( var(y(t)) = var(e(t)) = MV \). Also, other amplification factors result in higher output variance than MV by either detuning or overtuning the controller.

2. Though Harris (1992) already introduced the method of minimum variance estimation from the closed-loop output data (delay has to be known as a prerequisite), the process input or controller output variance under minimum variance control cannot be calculated by Harris’ approach. Nevertheless, the input variance under MV control should be referred to as the upper bound for control action. The reason behind this is that if the control action variance exceeds this bound, it will not improve the output performance but waste more energy and result in higher output variance which is not desirable.

From equation (3.41), if the controller is minimum variance controller, the input
variance will be:

\[ \text{var}(u(t)) = \text{var} \left( \frac{b(z^{-1})}{a(z^{-1})} \hat{y}(t) \right) \] (3.60)

By knowing the time series model \( \frac{b(z^{-1})}{a(z^{-1})} \) and minimum variance of the output, the upper bound of the control action can be computed. This upper bound can then be used as another performance measure for the overall controller performance. Also, by knowing the "best" PID tuning for a certain process, one can assess the performance of a PID controller against an achievable target rather than MV. This "best" PID is discussed in the next section.

### 3.4.4 MV (minimum variance) PID Controller

By using the least squares method to estimate the minimum variance of the process output, the Harris index can be calculated for any closed-loop process. However, it doesn't indicate how to tune the controller to approach the minimum variance for the process output. Since PID controllers are still widely used in the industry, in this section, PID controller tuning will be discussed specifically. By using the proper identification method mentioned in section 3.4.2, one is able to obtain a PID controller which achieves the possible minimum variance for the process output, i.e., the possible minimum Harris index in PID controller form.

Recall equation (3.59),

\[ \text{var}(\hat{y}(t)) = \text{var} \left( \frac{-a(z^{-1})R_1(z^{-1}) + b(z^{-1})R_2(z^{-1})}{c(z^{-1})R_2(z^{-1})} \hat{y}(t - d) \right) + \text{var}(e_t) \] (3.61)

A standard PID controller in discrete form can be defined as

\[
G_c = \frac{R_1(z^{-1})}{R_2(z^{-1})} = \frac{r_0 + r_1z^{-1} + r_2z^{-2}}{1 - z^{-1}}
\] (3.62)
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Here \( R_2(z^{-1}) = 1 - z^{-1} \). Therefore

\[
\nabla u(t) = -(r_0 + r_1 z^{-1} + r_2 z^{-2}) \hat{y}(t)
\]

(3.63)

Substituting equation (3.63) into equation (3.61) gives

\[
\text{var}(\hat{y}(t)) = \text{var} \left( \frac{-a_1(z^{-1}) R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(t - d) \right) + \text{var}(e_t)
\]

(3.64)

where \( a_1(z^{-1}) = \frac{a(z^{-1})}{V} \), and \( R_2(z^{-1}) = \nabla \).

As mentioned earlier in section 3.2, \( \text{var}(e_t) \) is actually the process minimum variance which can be estimated by knowing the process output operating data and time delay. Therefore, variance of \( \hat{y}(t) \) in equation (3.64) can be estimated by

\[
\text{var}(\hat{y}(t)) \approx \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1}) R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(i) \right]^2 
\]

(3.65)

Equation (3.64) becomes

\[
\text{var}(\hat{y}(t)) \approx \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1}) R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(i) \right]^2 + MV
\]

(3.66)

For minimum variance control, \( \text{var}(\hat{y}(t+d)) = MV \), implying \( \text{var}(\hat{y}(t)) = 0 \). To obtain the minimum achievable variance, the PID controller should minimize the following cost function

\[
\text{var}(\hat{y}(t)) \approx \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1}) R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(i) \right]^2
\]

(3.67)

Then the controller parameters \( r_0, r_1 \) and \( r_2 \) can be obtained by differentiating \( \text{var}(\hat{y}(t)) \) with respect to \( r_0, r_1 \) and \( r_2 \).

\[
\frac{\partial \text{var}(\hat{y}(t))}{\partial r_0} = 2 \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1}) R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(i) \right] \frac{-a_1(z^{-1})}{c(z^{-1})} \hat{y}(i)
\]

(3.68)

To find the solution for the controller parameters, it would be simpler to consider a linear form of the derivation by eliminating \( c(z^{-1}) \) in the denominator. Since polynomial
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\( c(z^{-1}) \) is monic and contains no zero outside the unit circle, \( \frac{1}{c(z^{-1})} \) can be approximated by a sum of finite coefficients in the backward shift operator.

\[
\frac{1}{c(z^{-1})} = d(z^{-1}) = 1 + d_1(z^{-1}) + d_2(z^{-2}) + d_3(z^{-3}) + \cdots \quad (3.69)
\]

After \( m \) terms, the coefficients are close to zero which can be truncated:

\[
\frac{1}{c(z^{-1})} \approx d(z^{-1}) \approx 1 + d_1(z^{-1}) + d_2(z^{-2}) + d_3(z^{-3}) + \ldots + d_m(z^{-m}) \quad (3.70)
\]

Equation (3.66) becomes

\[
\frac{\partial \text{var}(\hat{y}(t))}{\partial r_0} = 2 \frac{1}{N} \sum_{i=1}^{N} \left[ -a_1(z^{-1})d(z^{-1})R_1(z^{-1}) + b(z^{-1})d(z^{-1}) \right] \hat{y}(i) \left[ -a_1(z^{-1})d(z^{-1}) \right] \hat{\gamma}(i) \quad (3.71)
\]

To simplify the calculation, let's define the following polynomials

\[
a^*(z^{-1}) = a_1(z^{-1})d(z^{-1}) \quad (3.72)
\]
\[
b^*(z^{-1}) = b(z^{-1})d(z^{-1}) \quad (3.73)
\]

\[
\frac{\partial \text{var}(\hat{y}(t))}{\partial r_0} = 2 \frac{1}{N} \sum_{i=1}^{N} \left[ (-a^*(z^{-1}))(r_0\hat{y}(i) + r_1\hat{y}(i - 1)) + r_2\hat{\gamma}(i - 2) + b^*(z^{-1})\hat{\gamma}(i) \right] \left[ -a^*(z^{-1})\hat{\gamma}(i) \right] \quad (3.74)
\]

To obtain the parameter \( r_0 \), setting the above equation (3.74) to zero and simplifying it gives

\[
\sum_{i=1}^{N} (a^*(z^{-1}))^2 [r_0\hat{y}(i)\hat{\gamma}(i) + r_1\hat{y}(i - 1)\hat{\gamma}(i) + r_2\hat{\gamma}(i - 2)\hat{\gamma}(i)] = \sum_{i=1}^{N} [b^*(z^{-1})\hat{\gamma}(i)a^*(z^{-1})\hat{\gamma}(i)] \quad (3.75)
\]
Then
\[
\frac{\partial \text{var}(\hat{y}(t))}{\partial r_1} = 2 \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1})R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(i) \right] \left[ \frac{-a_1(z^{-1})}{c(z^{-1})} \hat{y}(i - 1) \right]
\]
(3.76)

Same simplification procedure for \( \frac{\partial \text{var}(\hat{y}(t))}{\partial r_1} \).

\[
\sum_{i=1}^{N} (a^*(z^{-1}))^2 [r_0\hat{y}(i)\hat{y}(i - 1) + r_1\hat{y}(i - 1)\hat{y}(i - 1) + r_2\hat{y}(i - 2)\hat{y}(i - 1)]
\]
\[= \sum_{i=1}^{N} [b^*(z^{-1})\hat{y}(i)a^*(z^{-1})\hat{y}(i - 1)]
\]
(3.77)

Again
\[
\frac{\partial \text{var}(\hat{y}(t))}{\partial r_2} = 2 \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1})R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \hat{y}(i) \right] \left[ \frac{-a_1(z^{-1})}{c(z^{-1})} \hat{y}(i - 2) \right]
\]
(3.78)

\[
\sum_{i=1}^{N} (a^*(z^{-1}))^2 [r_0\hat{y}(i)\hat{y}(i - 2) + r_1\hat{y}(i - 1)\hat{y}(i - 2) + r_2\hat{y}(i - 2)\hat{y}(i - 2)]
\]
\[= \sum_{i=1}^{N} [b^*(z^{-1})\hat{y}(i)a^*(z^{-1})\hat{y}(i - 2)]
\]
(3.79)

Combining equations (3.75), (3.77) and (3.79) gives the following matrix-vector equation,

\[PR_1 = Q\]

where
\[
P = \begin{bmatrix}
\sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i)\hat{y}(i) & \sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i)\hat{y}(i - 1) & \sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i)\hat{y}(i - 2) \\
\sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i)\hat{y}(i - 1) & \sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i - 1)\hat{y}(i - 1) & \sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i - 1)\hat{y}(i - 2) \\
\sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i)\hat{y}(i - 2) & \sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i - 1)\hat{y}(i - 2) & \sum_{i=1}^{N} (a^*(z^{-1}))^2 \hat{y}(i - 2)\hat{y}(i - 2)
\end{bmatrix}
\]
(3.80)
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\[ R_1 = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix} \] (3.81)

\[ Q = \begin{bmatrix} \sum_{i=1}^{N} a^*(z^{-1})\bar{y}(i)b^*(z^{-1})\bar{y}(i) \\ \sum_{i=1}^{N} a^*(z^{-1})\bar{y}(i-1)b^*(z^{-1})\bar{y}(i) \\ \sum_{i=1}^{N} a^*(z^{-1})\bar{y}(i-1)b^*(z^{-1})\bar{y}(i) \end{bmatrix} \] (3.82)

Since matrix \( P \) is symmetric, \( R_1 = P^{-1}Q \).

Matrix \( R \) gives the achievable minimum variance controller in PID form, which means that for the limited PID controller structure, this set of parameters can achieve the possible minimum variance of the process output.

3.4.5 Examples

In this section, a few examples will be used to demonstrate the theory that by knowing the closed-loop operating data and identifying the process model, an optimal MV PID controller can be obtained, giving the achievable minimum variance of the process output when the controller is in PID form. By using equation (3.82), a MATLAB program included in the appendix A was written to calculate the optimal MV PID controller parameters in this section.

Example 1.

In the first example, the following process model was considered:

\[ y(t) = \frac{0.25}{1 - 0.9z^{-1} + 0.2^{-2}}u(t - 1) + \frac{1}{1 - z^{-1}}w(t) \] (3.83)
where \( w(t) \) is a white noise with variance of 0.01. The process is a second-order model without any time delay. The noise model is a pure integration which generates a Gaussian type noise.

According to equation (3.27)

\[
B(z^{-1}) = 0.25
\]

(3.84)

\[
A(z^{-1}) = 1 - 0.9z^{-1} + 0.2^{-2}
\]

(3.85)

\[
C(z^{-1}) = 1
\]

(3.86)

\[
D(z^{-1}) = 1 - z^{-1}
\]

(3.87)

Given the following Diophantine equation,

\[
\frac{C(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1})z^{-1}}{D(z^{-1})}
\]

\[
= \frac{1}{1-z^{-1}} = 1 + \frac{z^{-1}}{1-z^{-1}}
\]

(3.88)

Therefore,

\[
a(z^{-1}) = B(z^{-1})F(z^{-1})D(z^{-1}) = 0.25(1 - z^{-1})
\]

(3.89)

\[
a_1(z^{-1}) = \frac{a(z^{-1})}{\sqrt{w}} = 0.25
\]

(3.90)

\[
b(z^{-1}) = A(z^{-1})G(z^{-1}) = 1 - 0.9z^{-1} + 0.2^{-2}
\]

(3.91)

\[
c(z^{-1}) = A(z^{-1})C(z^{-1}) = 1 - 0.9z^{-1} + 0.2^{-2}
\]

(3.92)

\[
d(z^{-1}) = F(z^{-1}) = 1
\]

(3.93)
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The following non-minimum variance PID controller was selected for generating closed-loop data:

\[ G_c = \frac{3.5 - 3.9z^{-1} + 0.7z^{-2}}{1 - z^{-1}} \]

This is equivalent to

\[ K_c = 2.5; \tau_i = 0.83; \tau_d = 0.028 \]

when sampling is 0.1 seconds.

Figure 3.1 shows the closed-loop dynamics of this simulation.

By using the algorithm developed in section 3.4.4, the MV PID controller parameters: \( r_0, r_1 \) and \( r_1 \) can be obtained. Vector \( R_1 \) was calculated as follows:

\[ R_1 = \begin{bmatrix} 4.0 \\ -3.6 \\ 0.8 \end{bmatrix} \quad (3.94) \]

Implementing this MV PID controller

\[ G_c = \frac{R_1(z^{-1})}{1 - z^{-1}} = \frac{4.0 - 3.6z^{-1} + 0.8z^{-2}}{1 - z^{-1}} \]

into the closed-loop system, the simulation results in figure 3.2 show that this controller does achieve the minimum variance of the process output which is 0.01.

Theoretically, minimum variance of the process output \( \sigma_{y_{\text{mv}}}^2 \) can be obtained by the following equation

\[ \sigma_{y_{\text{mv}}}^2 = (1 + f_1^2 + f_2^2 + f_3^2 + ... f_{d-1}^2) \sigma_{w_i}^2 \quad (3.95) \]

Coefficients \( f_i \) are from the polynomial \( F(z^{-1}) \) in equation (3.9):

\[ \hat{y}(t) = F(z^{-1})w(t) + \psi_1(z^{-1})w(t - d) \]

\( \sigma_{w_i} \) is the variance of the white noise. In this case, since \( F = 1 \), the theoretical MV can be calculated to be 0.01 which is exactly the same as that obtained from the simulation.
Figure 3.1: Closed-loop dynamics under a non-minimum variance PID control
Figure 3.2: Closed-loop dynamics under MV PID control
Example 2.

The second example is slightly different from the first one, being that the order of polynomial \( B(z^{-1}) \) is increased to 2. In this case, the minimum variance controller structure will not match the standard PID form. To achieve an optimal PID minimum variance controller, the following steps are taken.

The process model is as follows,

\[
y(t) = \frac{0.25 - 0.07z^{-1}}{1 - 0.9z^{-1} + 0.2z^{-2}}u(t - 1) + \frac{1}{1 - z^{-1}}w(t)
\]

where \( w(t) \) is a white noise with variance of 0.01.

According to equation (3.27)

\[
B(z^{-1}) = 0.25 - 0.07z^{-1} \tag{3.97}
\]

\[
A(z^{-1}) = 1 - 0.9z^{-1} + 0.2^{-2} \tag{3.98}
\]

\[
C(z^{-1}) = 1 \tag{3.99}
\]

\[
D(z^{-1}) = 1 - z^{-1} \tag{3.100}
\]

Given the following Diophantine equation,

\[
\frac{C(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1})}{D(z^{-1})} = \frac{1}{1 - z^{-1}} = 1 + \frac{z^{-1}}{1 - z^{-1}} \tag{3.101}
\]

Therefore,

\[
a(z^{-1}) = B(z^{-1})F(z^{-1})D(z^{-1}) = (0.25 - 0.07z^{-1})(1 + z^{-1}) \tag{3.102}
\]
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\[ a_1(z^{-1}) = \frac{a(z^{-1})}{\sqrt{}} = 0.25 - 0.07z^{-1} \]  \hspace{1cm} (3.103)

\[ b(z^{-1}) = A(z^{-1})G(z^{-1}) = 1 - 0.9z^{-1} + 2^{-2} \]  \hspace{1cm} (3.104)

\[ c(z^{-1}) = A(z^{-1})C(z^{-1}) = 1 - 0.9z^{-1} + 2^{-2} \]  \hspace{1cm} (3.105)

\[ d(z^{-1}) = F(z^{-1}) = 1 \]  \hspace{1cm} (3.106)

The following non-minimum variance PID controller was selected for generating closed-loop data:

\[ G_c = \frac{3.5 - 3.9z^{-1} + 0.7z^{-2}}{1 - z^{-1}} \]

This is equivalent to

\[ K_c = 2.5; \tau_i = 0.83; \tau_d = 0.028 \]

when sampling is 0.1 seconds. The process output variance is measured to be 0.0126.

By using the algorithm developed in section 3.4.4, the MV PID controller parameters: \( r_0, r_1 \) and \( r_1 \) can be obtained. Vector \( R_1 \) was calculated as follows:

\[ R_1 = \begin{bmatrix} 3.88 \\ -4.2 \\ 1.33 \end{bmatrix} \]  \hspace{1cm} (3.107)

Implement this MV PID controller

\[ G_c = \frac{R_1(z^{-1})}{1 - z^{-1}} = \frac{3.88 - 4.2z^{-1} + 1.33^{-2}}{1 - z^{-1}} \]

into the closed-loop system. The simulation results in figure 3.4 shows that this controller does achieve the minimum variance of process output which is 0.0103 which is much less than the one in figure 3.3.
Figure 3.3: Closed-loop dynamics under a non-minimum variance PID control
Figure 3.4: Closed-loop dynamics under MV PID control
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Theoretically, minimum variance of the process output $\sigma_{y_{min}}^2$ can be obtained by

$$
\sigma_{y_{min}}^2 = (1 + f_1^2 + f_2^2 + f_3^2 + \cdots + f_{d-1}^2)\sigma_{w_i}^2
$$

(3.108)

Coefficients $f_i$ are from the polynomial $F(z^{-1})$ in equation (3.9). $\sigma_{w_i}^2$ is the variance of the white noise. In this case, since $F = 1$, the theoretical MV can be calculated to be 0.01 which is exactly the same as that obtained from the simulation.

3.5 Limitations of the Harris Index

There are many performance measures that might be used to detect changes in the performance characteristics of a control system. In selecting a performance measure, it is important to select a measure appropriate for the control task. It is also important to consider not just the information returned by the performance measure, but also its computational complexity and requirement for some prior process knowledge.

When using the Harris index to assess control loop performance, there are few limitations discussed as follows:

1. The Harris index is quite sensitive to the time delay of the process model. Either over-estimating or under-estimating the time delay will influence the estimated ideal minimum variance significantly.

2. By knowing time delay and operating data, one can calculate the ideal minimum variance. However, the estimated ideal minimum variance may not be achievable physically. It is probably due to the limitation of the controller structure, process properties or the controller output could be unacceptably aggressive etc. In that case, from a practical point of view, the Harris index may not be as meaningful to operating personnel as one would like.
3. Although the Harris index provides a measure of the difference between the operating process output variance and the ideal one, it does not indicate how much the controller should be tuned according to the obtained value. In other words, the relationship between the Harris index and controller tuning is not straightforward.

4. Ettaleb et al. (1997) [34] have also discussed a problem about the Harris index. Because this technique requires little knowledge of the process, only two extreme situations are well defined. The first situation is when the output variance is close to minimum variance which corresponds to good control loop performance. The second situation is when the output variance is very far from the minimum variance which corresponds to very poor control loop performance. However, the output variance is, in most cases, between these two extreme situations. Then it is necessary to judge whether a control loop performance is acceptable or not. The limit between a satisfactory and unsatisfactory performance remains unknown.

5. Even though the Harris index measures the process output variance to assess the performance for regulatory control, it does not take into account the controller output. Studies done for this project have shown that different control laws may result in the same Harris index. In other word, the same Harris index can have different solutions for the controller output. The following section will focus on this statement, using a mathematical equation and simulation results to demonstrate it.

3.5.1 Non-Unique Controller Output for the Harris Index

The major disadvantage of the Harris index discussed here is the lack of controller output assessment. The following section will show that for the same Harris index, the solution for the controller output performance is not unique. In other words, a small Harris
index measured from the process output which means good process output performance may not guarantee the optimal performance for controller output as well. There will be multiple PID controller tunings for the same performance. As the objective of control is to reject the stochastic disturbance using as little energy as possible (here the energy is measured by the variance of the controller output), there will be a relatively optimal controller output performance (smaller variance for the controller output) for the same process output variance.

In this section, for a certain amount of process output variance, solutions for the control law will be solved in a mathematical equation.

Recall the variance of the process output equation (3.65)

\[ \text{var}(\tilde{y}(t)) \approx \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1})R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \tilde{y}(i) \right]^2 \]

As mentioned above, for the same \( \text{var}(\tilde{y}(t)) \), solution for the controller parameters represented in vector \( R_1(z^{-1}) \) is not unique. To prove this, the following equation can be extracted by taking a constant \( c \) equal to \( \text{var}(\tilde{y}(t)) * N \), such that

\[ c = \sum_{i=1}^{N} \left[ \frac{-a_1(z^{-1})R_1(z^{-1}) + b(z^{-1})}{c(z^{-1})} \tilde{y}(i) \right]^2 \]

Following the same simplification procedure for \( \frac{1}{c(z^{-1})} \approx d(z^{-1}) \), one obtains

\[ c = \sum_{i=1}^{N} \left[ (-a_1(z^{-1})d(z^{-1})R_1(z^{-1)) + b(z^{-1})d(z^{-1}))\tilde{y}(i) \right]^2 \]

\[ = \sum_{i=1}^{N} \left[ (-a_1^*(z^{-1}))r_0\tilde{y}(i) + r_1\tilde{y}(i + 1) + r_2\tilde{y}(i + 2)) + b^*(z^{-1})\tilde{y}(i) \right]^2 \]

\[ = \left[ -r_0 \sum_{i=1}^{N} a_1^* z^{-i} \tilde{y}(t - i) - r_1 \sum_{i=1}^{N} a_1^* z^{-i-1} \tilde{y}(t - i) \right. \]

\[ - \left. r_2 \sum_{i=1}^{N} a_1^* z^{-i-2} \tilde{y}(t - i) + \sum_{i=1}^{N} b_1^* z^{-i} \tilde{y}(t - i) \right]^2 \]

Here \( N \) is the total number of collected data, \( y(t) \) is the current process output. \( a_1^* \) and \( b_1^* \) are the coefficients of the polynomials \( a^*(z^{-1}) \) and \( b^*(z^{-1}) \) respectively.
This is a bilinear equation for each variable $r_0$, $r_1$, and $r_2$. Therefore, the solution for set of $r_0$, $r_1$, and $r_2$ is not unique. This proves that for the same variance of process output, the controller form can be different, hence the performance of the controller output will be different as well. The Harris index cannot give any information about which controller form is better.

### 3.5.2 Simulation Results

Besides the mathematical proof, a few examples are shown in this section to illustrate limitation of the Harris index.

**Example 1. Mesh Plot**

The process model is described as follows:

\[
y(t) = \frac{0.168z^{-1}u(t)}{1 - 0.908z^{-1}} + \frac{1}{(1 - z^{-1})(1 - 0.30z^{-1} - 0.17z^{-2})}w(t)
\]  
(3.113)

with the estimated variance of the white noise sequence $w(t)$ being 2.37 and the setpoint being zero.

A discrete PID controller in $k_c$ (proportional), $k_i$ (integral), and $k_d$ (derivative) form can be represented as follows (assuming $T_s = 1$)

\[
\frac{R_1(z^{-1})}{R_2(z^{-1})} = \frac{r_0 + r_1z^{-1} + r_2z^{-2}}{1 - z^{-1}}
\]

with

\[
r_0 = (k_c + k_i + k_d); r_1 = -(k_c + 2kd); r_2 = kd
\]  
(3.114)

Since the setpoint is zero, $u(t)$ can be substituted by

\[
u(t) = \frac{R_1(z^{-1})}{R_2(z^{-1})}y(t)
\]
Then, the closed-loop model becomes

\[ y(t) = \frac{1 - 0.908z^{-1}}{1 - m_1 z^{-1} - m_2 z^{-2} - m_3 z^{-3} - m_4 z^{-4} - m_5 z^{-5}} w(t) \]  

(3.115)

where

\[ m_1 = 2.208 - 0.168 \times r_0 \]
\[ m_2 = -1.310 + 0.05 \times r_0 - 0.168 \times r_1 \]
\[ m_3 = -0.052 + 0.028 \times r_0 + 0.05 \times r_1 - 0.168 \times r_2 \]
\[ m_4 = 0.154 + 0.028 \times r_1 + 0.05 \times r_2 \]
\[ m_5 = 0.028 \times r_2 \]

To simplify the calculation, let’s assume \( kd \) is equal to zero (PI controller). By varying \( ki \) from 1 to 4, while \( kc \) varies in the range 4 to 9, one can compute the variance of the process output in closed-loop. Using the program \( r\_cont.m \) and \( armavar.m \) in appendix A, a three-dimensional plot, as shown in figure 3.5, for \( \text{var}(y) \) versus \( kc \) and \( ki \) is obtained.

In this figure, for a constant variance of the process output \( \text{var}(y) \) (excluding minimum variance), there are different sets of solutions for \( kc \) and \( ki \). It should be noted that if \( kd \) is not zero, more combinations of the tuning parameters can achieve the same variance. By projecting this mesh plot to a two-dimensional plan, one can see the contour plot, as in figure 3.6, which shows that the same variance of process output can be obtained by different \( kc \) and \( ki \) along the elliptical contour.
Figure 3.5: Three-dimensional process output variance versus controller tuning parameters

Figure 3.6: Variance contour for different tuning values
Example 2.

The closed-loop simulation model is described in figure 3.7. The process is a first-order model with a time delay of 2 including a zero-order hold. The noise model is an ARMA model with a white noise variance of 0.01. The controller is a non-minimum variance PID.

When the setpoint is set to zero, the PID controller is working as a regulator. By changing the proportional gain from 0.1 to 3, while the integral and derivative gains are fixed at 0.4 and 0.1 respectively, the controller output variance is increased. The question here is “With more energy being consumed by the controller, will it improve the process output performance?” In the closed-loop simulation, the process output and controller output data have been collected to calculate the variances which are plotted in figure 3.8.

Figure 3.8 shows that as the controller output variance increases, the process output
Figure 3.8: Process output variance versus controller output variance in closed loop variance drops until the achievable minimum variance is reached before increasing again. This implies that a higher energy consumption by the controller will not improve the process output and will eventually deteriorate it. This region should definitely be avoided in close-loop control. Therefore, if the Harris index only measures the process output, it doesn’t indicate whether the controller is working efficiently or not.

3.6 Conclusions

In this chapter, the Harris index which is still being commonly used in the industry to analyze the process output performance has been reviewed. Though it is computationally simple to estimate the minimum variance and calculate the performance index, its application for controller assessment is limited. Therefore, a novel PID tuning strategy to achieve near minimum variance control has been proposed and demonstrated. Furthermore, the major drawback for the Harris index is that it doesn’t take into account
the controller output performance which is equally important in assessing control performance especially for PID controllers. The solution for this problem will be addressed in the next chapter.
Chapter 4

Performance Assessment Diagram

4.1 Introduction

Though the Harris index is an easy method to measure the performance of the closed-loop output, the major limitation for this index is that it doesn't take into account the controller output. This directly causes adverse performance in closed-loop control. As explained in section 3.5.1, different control laws may result in the same variance of the process output. This fact indicates that for different variances of the controller output, the resulting Harris index may be the same number to reflect the performance except when the loop is under minimum variance control (i.e., the controller output variance, $\sigma^2_{u,mv}$, is unique when the system is under MV control). It raises the question of which control law costs less energy for the same result in the process output variance when the system is not under MV control. High energy (variance) of the controller output not only affects the wear and tear of the instruments involved, but also causes potential instability within the system or strong interactions with other loops. Also, the calculation of the variances of the process input and output variables of a stochastic system under feedback control is important, because these values are used to evaluate the optimal controller prior to its implementation. A new input/output (I/O) index will be introduced in this chapter to address the shortcomings of the Harris index. During closed-loop control, by switching between two different controllers, one is able to identify the process model using the prediction error method. With this prior knowledge, the benchmark for controller
and process output can be estimated to define the new I/O index. This I/O index is able to indicate whether the closed loop is running at the optimal point or not. Also, for a specific control loop, the user can define certain achievable targets for the process and controller output variances. The I/O index can then be used to measure how good the current system is running according to the target.

In this chapter, the new index (I/O index) is developed to address the limitation of the Harris index on the controller output. The first section will present the method to estimate the upper bound for the controller output. The second section will introduce the definition of the new I/O index. The best presentation of the new index is by means of a performance diagram. Some examples will be shown to illustrate the efficiency of the I/O index.

4.2 Controller Output Upper Bound

Section 3.5.1 has shown that achieving the minimum variance of the process output requires a certain amount of energy from the controller. A greater amount of energy will not reduce the process output variance. Therefore, when the process output reaches MV, the controller output variance can be considered as the upper bound for the controller output. The control target should include limiting the controller output variance to the amount needed for MV control.

4.2.1 Estimation of Controller Output Variance Under MV Control

When the control loop is under MV control, from the first predictor form in equation (3.40),

\[
\tilde{y}(t) = \frac{B(z^{-1})F(z^{-1})D(z^{-1})u(t - d) + A(z^{-1})G(z^{-1})\tilde{y}(t - d)}{A(z^{-1})C(z^{-1})} + F(z^{-1})w(t)
\]
one can conclude that to achieve minimum variance control, the control law should force the predictor \( \hat{y}_1(t) \) to go to zero, so that the output \( \hat{y}(t) \) only contains the future noise term, \( e(t) \).

\[
\hat{y}_1(t) = \frac{a(z^{-1})u(t-d) + b(z^{-1})\hat{y}(t-d)}{c(z^{-1})} = 0
\]

To do so, the requirement for the controller output \( u(t) \) will obviously be set to equal \( -\frac{b(z^{-1})\hat{y}(t)}{a(z^{-1})} \). Therefore, the variance of the controller output under MV control \( \sigma_{u,mv}^2 \) can be calculated as

\[
\sigma_{u,mv}^2 = Var \left( \frac{b(z^{-1})\hat{y}(t)}{a(z^{-1})} \right) \quad (4.116)
\]

Since \( \hat{y}(t) \) is equal to \( e(t) \), \( \hat{y}(t) \) can be substituted by \( F(z^{-1})w(t) \) (in equation (3.40)) as well.

\[
\sigma_{u,mv}^2 = Var \left( \frac{b(z^{-1})\hat{y}(t)}{a(z^{-1})} F(z^{-1})w(t) \right) \quad (4.117)
\]

In equation (4.117), \( w(t) \) is a sequence of independently and identically distributed random variables. All the covariances between the variables in this sequence are equal to zero. Therefore the variance of the controller output in equation (4.117) can be calculated the same way as the minimum variance of the process output \( \sigma_{y,mv} \) in equation (3.108):

\[
\sigma_{y,mv} = (1 + f_1^2 + f_2^2 + f_3^2 + ... f_{d-1}^2)\sigma_{w_k}
\]

The only difference is that in equation (3.108), coefficients \( f_i \) are from the polynomial \( F(z^{-1}) \), but to calculate \( \sigma_{u,mv}^2 \), the coefficients are from polynomial \( \frac{b(z^{-1})}{a(z^{-1})} F(z^{-1}) \).
Since polynomial $a(z^{-1})$ is stable, $\frac{b(z^{-1})}{a(z^{-1})}F(z^{-1})$ must form a convergent infinite series such that,

$$\frac{b(z^{-1})}{a(z^{-1})}F(z^{-1}) = \theta_0 + \theta_1 z^{-1} + \theta_2 z^{-2} + \cdots$$

$$= \sum_{k=0}^{\infty} \theta_k z^{-k} \quad (4.118)$$

Notice that

$$\sum_{k=0}^{\infty} \theta_k < \infty$$

To simplify the calculation, in practice, the infinite series $\theta_k$ is truncated after $n$ terms. Therefore, $\sigma_{u,mv}^2$ can be represented as follows:

$$\sigma_{u,mv}^2 = (1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \cdots \theta_n^2) \sigma_{w_t}$$  \hspace{1cm} (4.119)$$

Since polynomials $a(z^{-1})$ and $b(z^{-1})$ are already known, to obtain the series $\theta_k$, $F(z^{-1})$ has to be calculated so that $(1 + \theta_1^2 + \theta_2^2 + \theta_3^2 + \cdots \theta_n^2)$ can be computed. Besides, the variance of white noise $\sigma_{w_t}$ has to be estimated to obtain the final $\sigma_{u,mv}^2$ value. Estimation of polynomial $F(z^{-1})$ and variance of the white noise $\sigma_{w_t}$ will be discussed in the following subsection.

### 4.2.2 Estimation of $F(z^{-1})$ and the Variance of White Noise $\sigma_{w_t}$

To calculate the variance of the controller output under MV control, it is necessary to estimated the polynomial $F(z^{-1})$ and the variance of the white noise coming into the control system.

When the system is under MV control, the following relationship exists,

$$e(t) = F(z^{-1})w(t)$$
Now define the autocovariances for time series $e(t)$,

\[
\gamma_0 = \text{Cov}(e(t), e(t)) = \text{Cov}(F(z^{-1})w(t), F(z^{-1})w(t)) \]

\[
= \begin{bmatrix} 1 & f_1 & f_2 & \cdots & f_{d-1} \end{bmatrix} \begin{bmatrix} \sigma_{w_t} \end{bmatrix}
\]

\[
= (1 + f_1^2 + f_2^2 + \cdots + f_{d-1}^2)\sigma_{w_t}
\]

(4.120)

\[
\gamma_1 = \text{Cov}(e(t), e(t-1)) = \text{Cov}(F(z^{-1})w(t), F(z^{-1})w(t-1))
\]

\[
= \begin{bmatrix} f_1 & f_2 & \cdots & f_{d-1} \end{bmatrix} \begin{bmatrix} \sigma_{w_{t-1}} \end{bmatrix}
\]

\[
= (f_1 + f_2f_1 + \cdots + f_{d-1}f_{d-2})\sigma_{w_t}
\]

(4.121)

\[
\gamma_2 = \text{Cov}(e(t), e(t-2)) = \text{Cov}(F(z^{-1})w(t), F(z^{-1})w(t-2))
\]

\[
= \begin{bmatrix} f_2 & f_3 & \cdots & f_{d-1} \end{bmatrix} \begin{bmatrix} \sigma_{w_{t-2}} \end{bmatrix}
\]

\[
= (f_2 + f_3f_1 + \cdots + f_{d-1}f_{d-3})\sigma_{w_t}
\]

(4.122)

and continue to

\[
\gamma_{d-2} = \text{Cov}(e(t), e(t-d+2)) = \text{Cov}(F(z^{-1})w(t), F(z^{-1})w(t-d+2))
\]
Chapter 4. Performance Assessment Diagram

\[
\begin{bmatrix}
    f_{d-2} & f_{d-1}
\end{bmatrix}
\begin{bmatrix}
    1 \\
    f_1
\end{bmatrix}
\sigma_{w_{t-d+2}}
\]

\[= (f_{d-2} + f_{d-1} f_1) \sigma_{w_t} \quad (4.123)\]

\[
\gamma_{d-1} = \text{Cov}(e(t), e(t - d + 1)) = \text{Cov}(F(z^{-1})w(t), F(z^{-1})w(t - d + 1))
\]

\[= \begin{bmatrix} f_{d-1} \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \sigma_{w_{t-d+1}}
\]

\[= f_{d-1} \sigma_{w_t} \quad (4.124)\]

After \(d - 1\) steps, the autocovariance of time series \(e(t)\) becomes zero. \(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{d-2}\) and \(\gamma_{d-1}\) can be obtained directly from the time series \(e(t)\). From section 3.3, using the least squares algorithm, the minimum variance of the process output \(\delta_{MV}^2\) in equation (3.23) and the error vector \(R\), which represents the future noise term \(e(t)\), can be estimated from the closed-loop operating data.

\[
R = \begin{bmatrix}
    r_n \\
    r_{n-1} \\
    \vdots \\
    r_{d+m}
\end{bmatrix} = Y - X\alpha
\]

Therefore, for the total of \(d\) number of equations listed above (from 4.120 to 4.124), an equivalent number of unknown variables \(f_1, f_2, \ldots, f_{d-1}\) and \(\sigma_{w_t}\) can be computed. All the variables needed for estimating the upper bound controller variance are now available. As a result, the upper bound of the controller output variance can be estimated by closed-loop operating data.
4.2.3 Example

The process block diagram is shown in figure 4.9. The process is a first-order model with a time delay of 2. The noise model is an ARMA model. The controller is a non-minimum variance PID controller. From the process model

\[
\dot{y}(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t - d) + \frac{C(z^{-1})}{D(z^{-1})} w(t)
\]

\[
= \frac{0.2}{1 - 0.8z^{-1}} u(t - 3) + \frac{1 - 0.2z^{-1}}{1 - 0.8z^{-1}} w(t)
\]

The MV controller can be obtained as follows,

\[
\frac{C(z^{-1})}{D(z^{-1})} = \frac{C(z^{-1})}{A(z^{-1})} \\
= \frac{1 - 0.2z^{-1}}{1 - 0.8z^{-1}} \\
= F(z^{-1}) + \frac{G(z^{-1})}{A(z^{-1})} z^{-d} \\
= 1 + 0.6z^{-1} + 0.48z^{-2} + \frac{0.384}{1 - 0.8z^{-1}}z^{-3}
\]
The MV controller form will be

\[ G_c = \frac{G(z^{-1})}{B(z^{-1})F(z^{-1})} \]

\[ = \frac{0.384}{0.2(1 + 0.6z^{-1} + 0.48z^{-2})} \]

\[ = \frac{1.92}{1 + 0.6z^{-1} + 0.48z^{-2}} \]  (4.127)

When implementing the MV controller in equation (4.127) into the process in figure 4.9, the process output reaches minimum variance. Figure 4.10 shows the closed-loop dynamics of the process under MV control. The controller output, however, is aggressive and spiky due to the random walk disturbance. Since the control action is weakly stationary, the incremental control action \( \Delta u \) will be considered instead of the position control action. The variance of \( \Delta u \) under MV control is calculated to be 0.037 here.

In order to demonstrate the estimation of the controller upper bound output variance using closed-loop operating data, the same process under PID control with \( K_c = 0.5 \), \( K_i = 0.3 \) and \( K_d = 0.075 \) was simulated. The closed-loop responses are shown in figure 4.11.

By collecting the data from process output shown in figure 4.11 and using the algorithm described in section 4.2.2, the covariance array is calculated:

\[
\begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2
\end{bmatrix} = 
\begin{bmatrix}
0.0161 \\
0.1189 \\
0.0053
\end{bmatrix}
\]

The coefficients for \( F(z^{-1}) \) are estimated to be

\[ f_1 = 0.5862; f_2 = 0.5343 \]

Three MATLAB functions, \texttt{var umv.m}, \texttt{lse.f.m} and \texttt{ma.id.m}, used for the calculation of the covariance array are included in the appendix A. The controller output upper bound
Figure 4.10: Closed-loop dynamics under MV control
Figure 4.11: Closed-loop dynamics under a non-minimum variance PID control
variance is estimated to be 0.0365. It can be concluded that this estimated number is quite close to the actual one, 0.037. The control action variance (considered to be $\Delta u$ here) under the PID control is 0.0325 which is obviously lower than the upper bound. Thus, this example has illustrated that the upper bound for the controller output variance can be successfully estimated without implementing MV control.

### 4.3 Definition of the $I/O$ Index

The $I/O$ index is an index which measures both the process output and controller output. The objective of the $I/O$ index is to measure whether the closed-loop is running optimally or not. This is to compensate the drawback of the Harris index which does not reflect whether the minimum possible controller energy is used to obtain the same process output variance. Therefore, this $I/O$ index measures the closed-loop performance according to both the lower bound of process output variance and the upper bound of controller output variance.

By collecting the closed-loop operating data from the controller output and process output, variances for these two sets of data can be calculated easily as $\sigma_u$ and $\sigma_y$ respectively.

The $I/O$ index is defined as follows,

$$I = 1 - \frac{\sigma_u^2}{\sigma_{u,mv}^2} \quad (4.128)$$

$$O = 1 - \frac{\sigma_{y,mv}^2}{\sigma_y^2} \quad (4.129)$$

where $\sigma_u^2$ is the upper bound of controller output variance estimated with the assumption that the system was under MV control and $\sigma_{y,mv}^2$ is the theoretical process output minimum variance. It contains two parts: $I$ index represents the controller output which
measures how much energy the controller consumes (or how much variance the controller generates) by comparing it with the upper bound of the controller output variance. If \( I \) is less than zero, it indicates that the controller is unnecessarily wasting energy or generating higher variance.

\( O \) index represents the process output performance which measures how far the process output variance is from the minimum variance by comparing it with the lower bound of the process output variance. If \( O \) is equal to zero, it indicates that the process output reaches the theoretical minimum variance. \( O \) close to 1 indicates very poor performance.

Figure 4.12 shows the \( I/O \) index in the form of a performance diagram.

![Figure 4.12: I/O index diagram](image)

In this diagram, the \( I/O \) index is only located within the dashed frame. \( I \) will always be less than 1 and \( O \) will vary between 0 and 1 according to the definition. An index that falls within the shaded region indicates that the current closed-loop control system is running in an undesirable condition. It is consuming unnecessarily higher energy
Chapter 4. Performance Assessment Diagram

compared to the one under minimum variance control. \( I \) being greater or equal to zero (the right-hand side region) is desirable for closed-loop performance. Theoretically, the optimal point is at the origin \((0,0)\) which represents minimum variance control for both controller output and process output. However, in reality the minimum variance control is normally unachievable or undesirable due to the limited structure for the controller or the too aggressive control action. Therefore, a user can specify a certain performance target which can be represented by the line parallel to the \( O \) axis. The exact location of this practical optimal operating point depends on the individual process. For example, by using the optimal PID controller developed in chapter 3, one is able to calculate this optimal point for the PID control system. When there is no control for the system which is equivalent to open-loop condition, the \( I/O \) index moves along the line \( I = 1 \) depending on the open-loop process output variance.

If the controller output has a limitation for a certain variation range, instead of using \( \sigma_{umv}^2 \), one can use a user-specified \( \sigma_{su}^2 \) as a benchmark for the \( I \) index. However, normally process output variance is desired to be as small as possible during regulatory control. Therefore, \( \sigma_{ymv}^2 \) can still be used as the benchmark for the \( O \) index.

No matter which benchmark for controller output is used (\( \sigma_{umv}^2 \) or \( \sigma_{su}^2 \)), the conclusion is the same. Obviously, if \( I \) and \( O \) are quite large, an increase in the controller output variance \( \sigma_u^2 \) will be needed to drive the \( I/O \) index close to the practical optimal operating point. The operating personnel is then recommended to retune the controller by increasing the controller gain to reach smaller \( I \) and \( O \) but not to the shaded region. This is an advantage to have the input variance constrained.

Another thing to be noted is when the disturbance is non-stationary, the corresponding PID controller will also have non-stationary movements. In that case, the controller output cannot be assessed by calculating the variance directly. Rather the incremental controller outputs have to be obtained and used as a criteria instead of the controller
output position.

4.4 Conclusions

A new performance I/O index in the form of a performance diagram is proposed in this chapter. To calculate this index, the upper bound of the controller output under minimum variance control has to be defined first. The I/O index diagram is defined as a two-dimensional map, where the I axis represents the performance measure of the controller output, and the O axis represents the performance measurements of the process output. The theoretical optimal operating point is located at the origin (0, 0). The practical operating point can be defined by the user or calculated through the optimal PID controller developed in Chapter 3 if the controller structure is limited to a PID type.
5.1 Introduction

In this chapter, two control systems are used to demonstrate the optimal PID controller and the new performance input/output (I/O) index. The first section includes an experiment of a level control system. The control loop is to keep the liquid level in a tank at a desired value even though process upsets occur. The experiment is designed to illustrate the change of the I/O index as a function of the operating parameters. It illustrates that with the optimal PID controller tuning, both the process output and controller output achieve the desired variance. Therefore, the I/O index does indicate the improvement of controller performance. The second section uses a set of real data from a coat weight control system to demonstrate the performance of the overall control system. Furthermore, by using the same data, the process and noise models were identified. The optimal PID controller was derived and the simulation results show that the performance measured by the I/O index has been improved tremendously.

5.2 Level Control System

The apparatus set up diagram for this level control system is shown in figure 5.13. Water is pumped from a reservoir through a control valve and rotameter(s) into the acrylic plastic tank. By suitably setting the shut-off valves, water can pass through the tank. Discharge from the tank is by gravity through flow orifices. The water subsequently
returns to the reservoir.

![Diagram of Liquid Level Automatic Control System](image)

Figure 5.13: Liquid level automatical control system

The control system block diagram is shown in figure 5.14. A transfer function which relates the liquid height in the tank to the flowrates in and out was derived theoretically and experimentally. The details are given in the next sections.

### 5.2.1 Theoretical Model

A mass balance upon the tank, followed by the introduction of deviation variables and Laplace transformation of the equation yields a first order transfer function describing the dynamic behaviour. (Refer to figure 5.15 for notation).

\[
H(t) = h(t) - h_{ss}(t) \tag{5.130}
\]

\[
Q_{in}(t) = q_{ss}(t) - q_{inss}(t) \tag{5.131}
\]
Figure 5.14: Apparatus set up diagram for level control system

\[ H(t) = h(t) - h_{\text{set}}(t) \]
\[ Q(t) = q_{\text{in}}(t) - q_{\text{inst}}(t) \]

Figure 5.15: Notation for level control system
The volumetric balance is:

\[ q_{\text{in}} - q_{\text{orifice}} = A \frac{dh}{dt} \]  

(5.132)

where \( q_{\text{orifice}} \) is equal to \( \frac{h}{R_{\text{orifice}}} \).

By introducing deviation variables (\( \text{ss} \) indicates steady state), the following equation can be obtained:

\[ (q_{\text{in}} - q_{\text{in,ss}}) \cdot \frac{1}{R_{\text{orifice}}} (h - h_{\text{ss}}) = A \frac{d(h - h_{\text{ss}})}{dt} \]  

(5.133)

Using Laplace transform and rearranging the equation, one obtains

\[ \frac{H(s)}{Q_{\text{in}}(s)} = \frac{R}{\tau s + 1} \]  

(5.134)

where \( \tau \) is equal to \( AR \).

In deriving this equation it is assumed that the outlet flow from the tank varies linearly with the height of fluid in the tank. This assumption can be justified by the fact that \( R_{\text{orifice}} \) is estimated from experimental data. Control valve gain \( k_v \) is calculated as follows from figure 5.16

\[ k_v = \frac{\Delta \text{flowrate}(L/s)}{\Delta \text{controller output}} = -0.33L/min\% \]  

(5.135)

Sensor gain \( k_m \) is calculated from figure 5.17.

\[ k_m = \frac{\Delta \text{controller input}}{\Delta \text{height(cm)}} = 1.18%/cm \]  

(5.136)

The orifice resistance is calculated as \( 27.41 \text{cm.min}/L \) from figure 5.18.

The cross-sectional area of the tank is equal to \( 366cm^2 \). Therefore, the process time constant \( \tau \) is calculated as

\[
\tau = \text{orifice resistance} \times \text{cross section area of the tank} \\
= 27.41 \text{cm.min}/L \times 366cm^2 \times \frac{1L}{1000cm^3} \\
= 10.03min
\]  

(5.137)
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Figure 5.16: Calibration curve of control valve gain, $K_v$
Figure 5.17: Calibration curve of sensor/transmitter gain, $K_m$
Figure 5.18: Calibration curve of orifice resistance, $R$
Since the control valve gain can be considered as part of the process, the process model can be described as follows

\[ G_p(s) = k_v \frac{R}{\tau s + 1} = -0.33 \cdot \frac{27.41}{10.03s + 1} \approx \frac{-9.05}{10s + 1} \]  \hspace{1cm} (5.138)

Where the unit of gain is cm*min/L and the unit of time constant is min. If the sampling time is chosen to be 15 seconds, the discrete time model will be

\[ G_p(z^{-1}) = \frac{-0.2263z^{-1}}{1 - 0.975z^{-1}} \]  \hspace{1cm} (5.139)

### 5.2.2 Experimental Model

In reality, there is always some disturbance in the real process. Therefore, the model was estimated from the operating data collected during an open-loop test. Figure 5.19 shows the open-loop process input and output dynamics.

Using the prediction error method and using a sampling time of 15 seconds, one is able to identify the process and noise models in discrete time as follows.

\[ B(z^{-1}) = -0.2 \]  \hspace{1cm} (5.140)

\[ A(z^{-1}) = 1 - 0.979z^{-1} \]  \hspace{1cm} (5.141)

\[ C(z^{-1}) = 1 + 0.08z^{-1} \]  \hspace{1cm} (5.142)

\[ D(z^{-1}) = 1 - z^{-1} \]  \hspace{1cm} (5.143)

Since the system has a very small time delay, the discrete time delay was selected to be 1.
Figure 5.19: Liquid level open loop dynamics
Therefore, the model is found to be as follows,

\[ y(t) = \frac{-0.2}{1 - 0.979z^{-1}}u(t - 1) + \frac{1 + 0.082z^{-1}}{1 - z^{-1}}e(t) \]  
(5.144)

where \( e(t) \) is the white noise with a variance of 0.0952. By using the Diophantine equation, the disturbance model can be separated into two parts.

\[
\frac{G(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1})z^{-1}}{D(z^{-1})} \\
= 1 + \frac{1.082}{1 - z^{-1}}
\]  
(5.145)

Therefore, according to equation (3.35), (3.36) and (3.37),

\[ a(z^{-1}) = B(z^{-1})F(z^{-1})D(z^{-1}) = -0.2(1 - z^{-1}) \]  
(5.146)

\[ a_1(z^{-1}) = B(z^{-1})F(z^{-1}) = -0.2 \]  
(5.147)

\[ b(z^{-1}) = A(z^{-1})G(z^{-1}) = (1 - 0.979z^{-1})1.082 \]  
\[ = 1.082 - 1.06z^{-1} \]  
(5.148)

\[ c(z^{-1}) = A(z^{-1})C(z^{-1}) = (1 - 0.979z^{-1})(1 + 0.082z^{-1}) \]  
\[ = 1 - 0.897z^{-1} - 0.08z^{-2} \]  
(5.149)

The optimal PID that generates near minimum variance performance is estimated as follows using the MATLAB program pidxon.m in appendix A:

\[ G_c(z^{-1}) = \frac{-4.87 + 5.4343z^{-1} - 0.6962z^{-2}}{1 - z^{-1}} \]  
(5.150)

Converting this optimal PID controller into continuous time, one obtains the following PID structure:

\[ G_c(s) = -4.04(1 + \frac{1}{7.475s} + 0.043s) \]  
(5.151)
the integral time of 7.47 minutes is very reasonable compared to the process time constant of 10 minutes. The derivative time could have been neglected because the system delay is quite insignificant.

When implementing the optimal PID controller into the system, the process output and controller output are shown on figure 5.20.

By measuring the PID controller output and process output, the process output reaches a variance of 0.0992. The variance of the incremental controller output is 4.92. By comparing the operating data under the optimal PID control with the theoretical minimum variance process output which is 0.0952 and the variance of the incremental controller output of 5.03, one is able to calculate the I/O index for this PID control performance. \( I \) is equal to 0.022 and \( O \) is equal to 0.04. Theoretically, for any first-order process with no delay, the optimal PID will be exactly the same as the minimum variance controller. However, due to model identification errors, one would not obtain the perfect model. Therefore, the designed optimal PID will not perform like a minimum variance controller. From the I/O index calculated from this PID control loop, which is located at \((0.022, 0.04)\), one can see that it is very close to the minimum variance control point of \((0, 0)\). It shows that the performance of the control system is quite good.

When the system is in open loop, the process output variance is equal to 4.03 and the process input (controller output) variance is zero. Therefore, the I/O index during open loop is at \((1, 0.976)\).

Using a non-minimum variance PI controller

\[
G_c = -6(1 + \frac{1}{10s})
\]  
(5.152)

one can see the closed-loop dynamics in figure 5.21. The process output variance is equal to 0.0985. The incremental controller output variance increased significantly to 3.974 which exceeds the one required for MV control 5.03. Therefore, the I/O index is located
Figure 5.20: Level control system closed-loop dynamics under optimal PID control
Figure 5.21: Level control system closed-loop dynamics under a non-minimum variance PI control
on the left-hand side of the performance diagram plot at \((-0.629, 0.034)\). This indicates that the controller is performing poorly and therefore wasting energy.

Again, using another non-minimum variance PI controller

\[ G_c = -3(1 + \frac{1}{10s}) \]  

one can see the closed-loop dynamics in figure 5.22. The process output variance is equal to 0.1206. The incremental controller output variance is 8.194. Therefore, the I/O index is located at \((0.772, 0.21)\) in the performance diagram plot in figure 5.23. This indicates the poor performance for both the process output and controller output which can be improved by tuning the PID controller properly.

Closed-loop Identification

If using a non-minimum PI controller,

\[ G_c = 4\left(1 + \frac{1}{10s}\right) \]  

by changing the setpoint from 0 to 1, to 0.2 and then to 0.8, a set of closed-loop data shown in figure 5.24 can be collected to identify the process and noise models.

The model is found to be as follows while the sampling time is 15 seconds,

\[ y(t) = \frac{-0.197}{1 - 0.975^{-1}} u(t - 1) + \frac{1 + 0.06 z^{-1}}{1 - z^{-1}} e(t) \]  

where \(e(t)\) is the white noise with a variance of 0.1. By using the Diophantine equation, the disturbance model can be separated into two parts.

\[
\frac{C(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1}) z^{-1}}{D(z^{-1})} \\
= 1 + \frac{1.06}{1 - z^{-1}}
\]  

(5.156)
Figure 5.22: Level control system closed-loop dynamics under a non-minimum variance PI control
Therefore, according to equation (3.35), (3.36) and (3.37),

\[ a(z^{-1}) = B(z^{-1})F(z^{-1})D(z^{-1}) = -0.197(1 - z^{-1}) \]  \hspace{1cm} (5.157)

\[ a_1(z^{-1}) = B(z^{-1})F(z^{-1}) = -0.197 \]  \hspace{1cm} (5.158)

\[ b(z^{-1}) = A(z^{-1})G(z^{-1}) = (1 - 0.975z^{-1})1.06 \]
\[ = 1.06 - 1.0335z^{-1} \]  \hspace{1cm} (5.159)

\[ c(z^{-1}) = A(z^{-1})C(z^{-1}) = (1 - 0.975z^{-1})(1 + 0.06z^{-1}) \]
\[ = 1 - 0.915z^{-1} - 0.059z^{-2} \]  \hspace{1cm} (5.160)

The optimal PID that generates near minimum variance performance is estimated as follows using the MATLAB program pid_con.m in appendix A:

\[ G_c(z^{-1}) = \frac{-5.38 + 5.24z^{-1} - 0.009z^{-2}}{1 - z^{-1}} \]  \hspace{1cm} (5.161)
Figure 5.24: Level control system closed-loop dynamics under a non-minimum variance PI control
Converting this optimal PID controller into continuous time, one obtains the following PID structure:

\[ G_c(s) = -5.24(1 + \frac{1}{9.36s} + 0.0004s) \]  

(5.162)

This PID is still quite reasonable though it is obtained from the closed-loop identification. Compared to the one obtained from open-loop bump tests, equation (5.151), the integral time is even closer to the theoretical time constant of the process 10 min. It also shows that the derivative is not necessary for this level control system since \( \tau_d \) is very small.

### 5.3 Demonstration Using Industrial Data

The industrial data used here to demonstrate the optimal PID controller and I/O index is from a coat weight control loop provided by a consulting company. The process output - coat weight- is manipulated through a blade load. The disturbance is assumed to be integrated white noise. The control parameters are self-adjusting and depend on the operating conditions. The system was most of the time under closed-loop control. Data was collected every 44 seconds.

Figure 5.25 shows the data collected from the coat weight and blade load respectively during closed-loop control. Using the prediction error method and using a sampling time of 15 seconds, one is able to identify the process and disturbance models in discrete time as follows.

\[ B(z^{-1}) = -8.822 \]  

(5.163)

\[ A(z^{-1}) = 1 - 0.223z^{-1} - 0.255z^{-2} \]  

(5.164)

\[ C(z^{-1}) = 1 - 0.25z^{-1} \]  

(5.165)
Figure 5.25: Closed-loop operating data from coat weight control system
Chapter 5. Demonstration of the I/O Index

\[ D(z^{-1}) = 1 - z^{-1} \quad \text{(5.166)} \]

The process time delay is found to be 1.

Therefore, the model is described as

\[ y(t) = \frac{-8.822}{1 - 0.223z^{-1} - 0.255z^{-2}} u(t - 1) + \frac{1 - 0.25z^{-1}}{1 - z^{-1}} e(t) \quad \text{(5.167)} \]

where \( e(t) \) is the white noise with a variance of 0.0082.

By using the Diophantine equation, the disturbance model can be separated into two parts:

\[ \frac{C(z^{-1})}{D(z^{-1})} = F(z^{-1}) + \frac{G(z^{-1})z^{-1}}{D(z^{-1})} \]

\[ = 1 + \frac{0.75z^{-1}}{1 - z^{-1}} \quad \text{(5.168)} \]

Therefore, according to equation (3.35), (3.36) and (3.37)

\[ a(z^{-1}) = B(z^{-1})F(z^{-1})D(z^{-1}) = -8.822(1 - z^{-1}) \quad \text{(5.169)} \]

\[ a_1(z^{-1}) = B(z^{-1})F(z^{-1}) = -8.822 \quad \text{(5.170)} \]

\[ b(z^{-1}) = A(z^{-1})G(z^{-1}) = (1 - 0.223z^{-1} - 0.255z^{-2})0.75 \]

\[ = 0.75 - 0.1673z^{-1} + 0.1913z^{-2} \quad \text{(5.171)} \]

\[ c(z^{-1}) = A(z^{-1})C(z^{-1}) = (1 - 0.223z^{-1} - 0.255z^{-2})(1 - 0.25z^{-1}) \]

\[ = 1 - 0.473z^{-1} + 0.281z^{-2} - 0.0563z^{-3} \quad \text{(5.172)} \]

The optimal PID is found to be

\[ G_c = \frac{-0.085 + 0.019z^{-1} - 0.0217z^{-2}}{1 - z^{-1}} \quad \text{(5.173)} \]
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The continuous form is

\[ G_c = -0.02(1 + \frac{1}{0.346s} + 0.814s) \]  \hspace{1cm} (5.174)

with the time unit of minute.

Since implementation of the theoretical PID control is impossible for this coat weight process, closed-loop simulation is used to demonstrate the results. The Block diagram of this control loop simulation is shown in figure 5.26.

\[ \text{Figure 5.26: Block diagram of the coat weight closed-loop simulation under optimal PID control} \]

The process output and controller output, when implementing the optimal PID controller into the system, are shown in figure 5.27.

By using the PID controller, the process output reaches a minimum variance of 0.0082. The variance of the incremental controller output is \(7.16 \times 10^{-4}\). Comparing them with the original operating data from the process output which has a variance of 0.0171 and the incremental controller output variance of \(5.95 \times 10^{-5}\), one is able to calculate the I/O
Figure 5.27: Coat weight closed-loop dynamics under optimal PID control
index for different control performances. The $I/O$ index performance diagram for this coat weight control loop is shown in figure 5.28.

Without any control, the process output variance is equal to the noise variance 0.197 which gives the $O$ index equal to 0.9584. The controller output variance is 0 which has $I$ index of 1. The $I/O$ index for the original operating data is calculated as $(0.917, 0.4795)$. Under the optimal PID control, the closed-loop achieves the best performance which gives the $I/O$ index at $(0, 0)$. By tuning the normal PID to optimal PID, one can easily see that the $I/O$ index moves towards $(0, 0)$ in the diagram. This indicates the improvement of the control performance.

5.4 Conclusions

In this chapter, the experiment of a level control system and industrial data from a coat weight control system were used to illustrate the optimal PID controller and the $I/O$ performance index. By collecting the data from the controller output and process output of
the operating systems, the process and disturbance models were identified as Box-Jenkins model. Then, by using the strategy developed in chapter 3, the optimal PID which gives the best achievable performance was estimated. Both level control experiment and coat weight simulation verified the performance of the estimated optimal PID controller. The new performance $I/O$ index developed in chapter 4 was also calculated for these two systems and presented in the form of an $I/O$ index diagram.
Chapter 6

Conclusions and Recommendations

In this thesis, some basic performance analysis tools were reviewed. A few questions regarding the regulatory control performance analysis were discussed. A convenient and clear performance index is of great importance for modern industrial process control systems, since thousands of control loops need to be monitored and analyzed on time. The conventional performance analysis tools are not sufficient to reflect the performance clearly and practically. The current commonly used performance index is the Harris index which has certain limitations. These limitations include lack of controller output performance and controller re-tuning. A new algorithm to derive an optimal PID controller for regulatory control purpose was developed. The important issues about the performance index has been taken into account. Therefore, a new I/O index was proposed in this thesis which results in a more practical and complete assessment of control loop performance analysis.

The current work can be summarized as follows:

1. For regulatory control, an optimal PID controller is derived in the sense of obtaining the achievable minimum variance for the process output. Since the most popular used controller is limited to the PID type, the best feasible minimum variance instead of the theoretical minimum variance as a benchmark for performance assessment is more practical and useful. Although the process model needs to be identified, the algorithm is comparatively simple and it is based on minimizing the variance of the process output by taking differential with respect to the controller
parameters. By collecting the closed-loop operating data, the process model including the time delay and noise properties can be estimated. By using the Diophantine equation, the prediction form of the control loop can be obtained directly. Therefore, the process output related to the noise properties can be extracted, leading to the necessary optimal PID controller. Furthermore, the achievable minimum variance of the process output can be obtained when the controller structure is limited to a PID form. This benchmark makes the performance assessment to be more practical and straightforward for control personnel. The other advantage of deriving an optimal PID controller is that it provides the control personnel certain direction to tune the control system to achieve better performance.

2. As a performance index used for regulatory control, the Harris index has the limitation of not taking into account the controller output. From the study of this thesis, it clearly shows that with the same amount of the variance for the process output (the same Harris index the same performance), the controller output variance can be very different. Sometimes, this causes unnecessary energy wastage for the controller or unnecessarily large controller output variance. Therefore, an upper bound of the controller output is very useful in guaranteeing that the process operates efficiently. It is calculated by estimating the controller output variance under minimum variance control where maximum controller energy is required. Combining this factor with the process output, the concept of the new I/O index has two parts. One is the $I$ index which measures the process input (controller output) performance. A negative value of $I$ indicates that the system is running in a very undesirable region where it is using more energy than required. The other is the $O$ index which compares the process output variance with the benchmark of minimum variance. A two-dimensional $I/O$ index diagram was constructed. This
graphical plot shows how the overall control loop performance looks.

The performance assessment for regulatory control has been well developed and understood for single-input single-output, linear Box-Jenkins process model. Some future research suggestions are:

1. Extend the work to multi-input multi-output systems. Although the author sees no major obstacles in doing so for most of the topics covered in this thesis, some efforts are needed to identify the proper process model when there are some feedforward control or coupled loops etc.

2. Extend the work for integrating and non-linear systems. For an integrating system, since the integrator $\frac{1}{s}$ will magnify the noise, it becomes much more difficult to identify the proper dynamics of the process for regulatory control. Also for a non-linear system, it is not sufficient to describe the system by a rational function of polynomials. Nonlinear control systems have not been well described even mathematically.

3. For certain types of process models, the optimal PID derived using the algorithm in chapter 3 may not be physically implementable. Therefore, certain constraints may be needed to ensure that the integral and derivative time constants are positive. However, the concept to derive a controller by minimizing the variance of the process output will be the same.
Appendix A

Computer Programs

In this section, programs written to verify the theory presented in the thesis will be included.

function [F,vw]=lse_f(y,b,n,m);

% Routine to calculate the variance of white noise and the monic moving average polynomial.

% Calling sequence:

% function [F,vw]=lse_f(y,b,n,m);

% Input arguments:
% y : the closed-loop data.
% b : time delay of the process including discretization delay.
% n : number of the samples.
% m : number of coefficients of alpha we want to choose to estimate alpha.

% Output arguments:
% F : y(t) = F*w(t) + alpha* y(t-b); where F is moving average model of future noise. w(t) is white noise. Define ee=F*w.
% vw : variance of the white noise.
Appendix A. Computer Programs

\% Least squares estimation for a time series model of output \( y(t) \).
\% \( y(t) = \text{Sum } (k \text{ from } 1 \text{ to } m) \alpha(k) \times y(t-b-k+1) + \epsilon \).

\text{yy}(1,1) = \text{y}(n,1);
for \( l=1:n-b-m \)
\text{yy}(l+1,1) = \text{y}(n-l,1);
end

\( g = 0; \)
for \( r=1:n-m-b+1 \)
\( x=n-b-g; \)
for \( c=1:m \)
\text{X}(r,c) = \text{y}(x,1);
\text{x} = \text{x} - 1;
end
\text{g} = \text{g} + 1;
end
\text{M} = \text{X}' \times \text{X};
\text{N} = \text{inv}(\text{M});
\alpha = \text{N} \times \text{X}' \times \text{yy};
\text{mv} = (\text{yy} - \text{X} \times \alpha)' \times (\text{yy} - \text{X} \times \alpha) / (n-b-2*m+1)
\% Calculate F and the variance of white noise \( w(t) \).
\text{ee} = (\text{yy} - \text{X} \times \alpha);

\% calculate autocovariances of \text{ee}.
for i=0:b-1
    cm=cov(ee(i+1:n-m-2,1),ee(1:n-m-2-i,1)) % cm is the covariance matrix.
    gamma(i+1,1)=cm(2,1);
end

gamma
pause
[F,var_w]=ma_id(gamma);
vw=var_w;

function [F,var_w]=ma_id(gamma);

% Identification algorithm for a Moving Average time series given the autocovariances.
% Calling sequence:
% [F,var_w]=ma_id(gamma);
% Input arguments:
% gamma : The autocovariances starting from lag 0 to d=b-1 where d is the pure process delay.
% Output arguments:
% F1 : The moving average parameter vector.
% var_w : The variance of the white noise.
\[
q = \text{length}(\gamma) - 1;
\]

\[
b = \text{zeros}(q, q);
\]

\[
F = \text{zeros}(q, 1);
\]

\[
nF = \text{zeros}(q, 1);
\]

\[
gm = \frac{\gamma(2:q+1, 1)}{\gamma(1, 1)};
\]

\[
\text{err} = 1;
\]

\[
\text{iter} = 1;
\]

\[
\text{while } (\text{iter} < 100 \& \& \text{err} > 1e-11)
\]

\[
\text{for } i = 1:q-1
\]

\[
\text{for } j = 1:q-1
\]

\[
m = i + j;
\]

\[
\text{if } (m \leq q)
\]

\[
b(i, j) = F(m, 1);
\]

\[
\text{end};
\]

\[
\text{end};
\]

\[
\text{end};
\]

\[
nF = b*F - (1+F'*F)*gm;
\]

\[
\text{iter} = \text{iter} + 1;
\]

\[
\text{err} = (nF-F)'*(nF-F)/q;
\]

\[
F = nF;
\]

\[
\text{end};
\]

\[
F1 = [1; -F];
\]

\%
Notice the original F doesn't have the first coefficient 1, also it is negative
l=length(F1);
if l==1
    var_w=gamma(1,1);
else
    var_w=gamma(1,1)/(1+F'*F);
end

function [G1]=pid_con(a,b,c,y,n)

% % % Routine to calculate the matrix E which is used in E*G1=F to obtain the optimal
% % PID controller parameters G1.
%
% % % Input variables :
% % a, b, c : polynomials identified from the open loop or closed-loop data.
% Note here 'a' doesn't contain integral which assumes that D has
% an integral. Parameters put as column.
% y : open loop or close loop output data.
% n : the number of samples
% n1 : The order or degree of the quotient polynomial from 1/c.
% Output variables:
% G1 : The optimal PID controller parameters.
%
% PID controller :
\%
G1/G2 = \[g_0 + g_1 z^{-1} + g_2 z^{-2}\]/[1 - z^{-1}] .
\%
Therefore G1 = \[g_0 \ g_1 \ g_2\] .
\%
\%
Equation \(E*G1 = F\).
\%
\%
\%
\%
\%
E = \sum_i(1^n)\{a*\ y_i*y_i\} \sum_i(1^n)\{a*\ y_i-1*y_i\} \sum_i(1^n)\{a*\ y_i-2*y_i\}
\%
\%
\%
\%
\%
F = [\sum_i(1^n)\{b*\ y_i*y_i\} \sum_i(1^n)\{b*\ y_i-1*y_i\} \sum_i(1^n)\{a*\ y_i-2*y_i\}]
\%
\%
Here \(1/c = d, \ a = a*d, \ b = b*d\) from equation (3).
\%
\%
Since \(E\) is full rank, \(G1 = E^{(-1)}*F\).
\%
\[d\] = polydiv (c,[1],n);
[a1] = polymul(a,d);
[b1] = polymul(b,d);
e11=0;e12=0;e13=0;e22=0;e23=0;e33=0;f1=0;f2=0;f3=0;
for t=n:2*n
    sum1=0;sum2=0;sum3=0;sum4=0;
    for i=0:n;
if(i<t)
    sum1=sum1+a1(i+1,1)*y(t-i,1);
    sum4=sum4+b1(i+1,1)*y(t-i,1);
end
end
for j=0:n;
  if(j<(t-1))
    sum2=sum2+a1(j+1,1)*y(t-j-1,1);
  end
end
for j=0:n;
  if(j<(t-2))
    sum3=sum3+a1(j+1,1)*y(t-j-2,1);
  end
end

e11=e11+sum1*sum1;
e12=e12+sum1*sum2;
e13=e13+sum1*sum3;
e22=e22+sum2*sum2;
e23=e23+sum2*sum3;
e33=e33+sum3*sum3;
f1=f1+sum1*sum4;
f2=f2+sum2*sum4;
\[ f_3 = f_3 + \sum_3 \times \sum_4; \]

end

\[ E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix}; \]

% Calculate matrix F.
\[ F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}; \]

% Calculate the optimal PID controller parameter.
\[ G_1 = \text{inv}(E) \times F; \]

function \[ \text{[quot,rem]} = \text{polydiv}(\text{den}, \text{num}, n) \]

% % Routine to calculate the coefficients of % the quotient and remainder polynomials % of 2 polynomials.
% % Calling sequence:
% % % [quot,rem]=polydiv(den,num,n); % % % Input variables:
% num : The numerator monic polynomial.
% den : The denominator monic polynomial. Note: if the
Appendix A. Computer Programs

% polynomial is not monic, factor the first coefficient out.
% n : The order or degree of the
quotient polynomial.
%
% Output variables:
% quot : The quotient polynomial.
% rem : The remainder polynomial.
%
%%%%%%%%

if den(1,1)==1 & num(1,1)==1
    p=length(den)-1;
    q=length(num)-1;
    xquot=zeros(n,1);
    if (q>1)
        xnum=-num(2:q+1,1);
    else
        q=0;
        xnum(1,1)=0;
    end;
    if (p<0)
        error('The denominator is a scalar');
    end;
    xden=-den(2:p+1,1);
    xquot(1,1)=xnum(1,1)-xden(1,1);
    for j=2:1:n,
xquot(j,1)=0.0;
for i=1:j-1,
    if (i<=p)
        xquot(j,1)=xquot(j,1)+xden(i,1)*xquot(j-i,1);
    end;
end;
if (j<=q)
    xquot(j,1)=xquot(j,1)+xnum(j,1);
end;
if (j<=p)
    xquot(j,1)=xquot(j,1)-xden(j,1);
end;
end;
quot=[1 -xquot']';
[rem1]=polymul(den,quot);
m=max(length(rem1),length(num));
rem2=zeros(m,1);
rem2(1:q+1,1)=num;
rem0=rem2-rem1;
rem=rem0(n+2:m,1);
else
    r=num(1,1)/den(1,1);
    den=den/den(1,1);num=num/num(1,1);
    p=length(den)-1;
    q=length(num)-1;
xquot=zeros(n,1);
if (q>=1)
    xnum=-num(2:q+1,1);
else
    q=0;
    xnum(1,1)=0;
end;
if (p<=0)
    error('The denominator is a scalar');
end;
xden=-den(2:p+1,1);
xquot(1,1)=xnum(1,1)-xden(1,1);
for j=2:1:n,
    xquot(j,1)=0.0;
    for i=1:j-1,
        if (i<=p)
            xquot(j,1)=xquot(j,1)+xden(i,1)*xquot(j-i,1);
        end;
    end;
    if (j<=q)
        xquot(j,1)=xquot(j,1)+xnum(j,1);
    end;
    if (j<=p)
        xquot(j,1)=xquot(j,1)-xden(j,1);  
    end;
end;
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```
quot=[1 -xquot'];
[rem1]=polymul(den,quot);
m=max(length(rem1),length(num));
rem2=zeros(m,1);
rem2(1:q+1,1)=num;
rem0=rem2-rem1;
rem=rem0(n+2:m,1);

quot=quot*r;
rem=rem*r;
end

function [c]=polymul(a,b);

% Function to multiply two polynomials
% Calling sequence:
% [c]=polymul(a,b);
% Input argument:
% a : Column of input polynomial coefficients.
% b : Column of input polynomial coefficients.
% Output argument:
```
c : Column of resultant polynomial coefficients.  \( c = a*b \).
end;

% Routine to calculate the contour for
% the same value of the process output variance.
% Inputs arguments:
% PID controller form as follows.
% \[ G_c = \frac{(c_1+2c_2+c_3B^2)}{(1-B)}; \]
% \[ c_1 = (kc+ki+kd); c_2 = -(kc+2kd); c_3 = kd; \]
%
% Process model
% \[ et = \frac{(1-0.908B)\cdot at}{(1-m_1B-m_2B^2-m_3B^3-m_4B^4-m_5B^5)} \]
% Here
% \( B \) is backward shift operator.
% \( et \) is the error at the output,
% \( at \) is the white noise sequence with variance of 2.37
% \[ m_1 = 2.208-0.168c_1; m_2 = -1.310+0.5c_1-0.168c_2; \]
% \[ m_3 = -0.052+0.028c_1+0.05c_2-0.168c_3; \]
% \[ m_4 = 0.154+0.028c_2+0.05c_3; m_5 = 0.028c_3; \]
%
% clg;
clear;
kd=0;
i=0;
for ki=1:0.1:4;
i=i+1;
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vki(i,1)=ki;
j=0;
for kc=4:0.1:9;
    j=j+1;
    vkc(j,1)=kc;
    c1 = (kc+ki+kd);
    c2 = -(kc+2*kd);
    c3 = kd;
    m1 = 2.208-0.168*c1;
    m2 = -1.310+0.05*c1-0.168*c2;
    m3 = -0.052+0.028*c2+0.05*c2-0.168*c3;
    m4 = 0.154+0.028*c2+0.05*c3;
    m5 = 0.028*c3;
    theta=[1 -0.908];
    phi=[1 -m1 -m2 -m3 -m4 -m5];
    vy(j,i)=armavar(theta,phi,2.37);
end
end
mesh(vki,vkc,vy);
contour(vki,vkc,vy);
xlabel('ki');
ylabel('kc');
ylabel('var(y)');
title('kd = 0');
end
function [vu]=var_umv(a,b,y,d,n,n1);

% Routine to estimate the variance of the controller output when the process is in MV(minimum variance) control.
% This variance is the practical meaningful maximum variance for control output.
% Input variables:
% a, b: polynomials identified from the open loop or closed-loop data.
% if using PID controller, because the controller output is non-stationary, we calculate variance of u(t)-u(t-1).
% y: the output data.
% d+1: time delay of discrete process model.
% n: number of samples.
% n1: the order or the degree of the quotient polynomial. n1 is also the number of coefficients in f estimated in moving average model of white noise n1 should be large enough to reduce the inaccuracy.

% Output variables:
% vu: for PID controller, it is the practical maximum variance for \( \Delta u \), otherwise it is the maximum controller output variance.

[f,vw]=lse_f(y,d,n,n1);
% f is the moving average model, vw is the variance of white noise.
[b1]=polymul(b,f);
sum=0;
if length(a) ==1
    b1=b1/a;
    l=length(b1);
    for i=1:l
        sum=sum+b1(i,1)^2;
    end
else
    [b2]=polydiv(a,b1,n1);
    m=length(b2);
    for i=1:m
        sum=sum+b2(i,1)^2;
    end
end
vu=sum*vw;
Bibliography


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