THE EFFECT OF PROBLEM CONTEXT UPON THE PROBLEM SOLVING PROCESSES USED BY FIELD DEPENDENT AND INDEPENDENT STUDENTS: A CLINICAL STUDY

by

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Abstract

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It was the purpose of this study to analyze the processes students used in solving mathematical word problems and to determine the effect of problem context on these processes. A concomitant purpose was to determine whether students who differ in their degree of field independence, differ in the processes they use in solving mathematical problems.

Forty subjects of both sexes, who were completing a grade eleven academic mathematics program were randomly selected from 14 Algebra II classes. The subjects were of average ability for students on this program (IQ range 115-125). The subjects were tested individually, using Witkin's Embedded Figures Test. They were matched on this variable and randomly assigned to one of two groups. One group was given problems using a real world setting, while the other group was given the same problems using a mathematical setting. The subjects were individually interviewed and asked to think aloud as they solved five mathematical word problems.

To analyze the problem solving procedures the subjects' tape recorded protocols of the interviews were coded by means of a system based on a model of mathematical problem solving by MacPherson. The coding system had two parts: a coding matrix used to sequentially code problem solving behavior, and a summary sheet for compiling information obtained from the coding matrix as well as other data related to the subjects' problem solving behavior. The coding system had intercoder reliability of .80 and intracoder reliability of .86.

Problem context proved to be unrelated to the heuristics used. Both the total number of heuristics used and the number of different heuristics used were not influenced by problem setting. Subjects working problems in the math world setting had a slightly more difficult time understanding the problems, but performed as well as the other group.

Within the IQ range, 115 to 125, field independence had a marked effect on the use of heuristics and on the number of correct solutions obtained. The field independent subjects used a greater variety of heuristics ($r = .33$) in attacking and solving problems. They were more willing to change their mode of attack ($r = .27$) and they obtained a greater number of correct solutions ($r = .30$) than their field dependent counterparts.

Both total number of heuristics as well as number of different heuristics, accounted for a significant ($P < .01$) amount of the variance in number of correct solutions. In particular, heuristics accounted for an additional 21% of variance not accounted for by core procedures (algorithms, diagramming, equations, and guessing). The heuristics used by the subjects in this study added to their ability to solve problems beyond their mathematical core knowledge.

The number of times a subject attempted to solve a problem was found to be unrelated to obtaining a correct solution, while changing one's mode of attack in solving a problem was significantly ($P < .01$) related to obtaining a correct solution.

When the subjects were grouped by problem context, both groups exhibited the same general pattern of problem solving behavior. The real world students expressed concern for solutions obtained using heuristics while those students in the math world setting expressed none. However, expression of concern for solution was unrelated to the correctness of the solution.

When grouped by field independence a difference was observed in the overall pattern of sequential moves in the problem solving process. These differences among the groups were not tested for statistical significance. The field independent student moved more freely across all procedures coded. He was more concerned with his work, and continually checked both the procedures he was using as well as his solution. The field independent student was more willing to check his work, usually by retracing his steps, whereas the field dependent student usually reread the problem.

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Chapter I

THE PROBLEM

The International Commission of Mathematics, which met in Moscow in the summer of 1966, chose three topics of particular importance for discussion. These topics were: the university programs in mathematics for physicists, the use of the axiomatic method in the teaching of mathematics in the secondary school, and the role of problems in developing students' mathematical activity. The report to the commission by the Conference Board of Mathematical Sciences (Begle, 1966) indicated that the creation of problem sets should be a body of problem material which will provide the environment for imaginative and creative thinking. The Cambridge Conference on School Mathematics (Goals for School Mathematics, 1963) has noted that "the composition of problem sequence is one of the largest and one of the most urgent tasks in curricular development [p. 28]."

Even with the growing emphasis on solving more complex and challenging problems, little has been done in developing methods that the teacher may use to improve the problem solving skills of his students. In fact Dessart and Frandsen (1972) have suggested that the "major question of how problems are solved. . .[is one]. . .of the topics for which a more intensive search for fundamental principles might begin in hopes of building foundations to support valid research [p. 1191]." In view of the limited knowledge of how students solve problems, precursors to any large scale study in the teaching of problem solving should be clinical studies of individual subjects (Kilpatrick, 1969,p. 179).

Purpose of the Study

The question of how mathematical problem solving may be improved has attracted considerable attention among mathematics educators. In most problem solving tasks, there is no simple relationship between the solution obtained by an individual and the process used to achieve it. Measures such as the correctness of a solution or the time needed to achieve a correct solution do not indicate the particular procedures the individual used in reaching a solution. In order to study the problem solving process techniques must be used which allow the subject to generate an observable sequence of behavior (Kilpatrick, 1967, p. 4).

It is a purpose of this study to analyze an individual's behavior during the solution of a mathematical problem and to determine the effect of varying the problem context on this behavior. The experimenter will also try to determine if students differing in their degree of field independence, differ in their behavior while solving mathematical problems. Field independence is a construct which provides information about individual differences. Spitler (1970, pp. 53-57) contends that field independent students are more able to deal with complex geometric problems and with problems which deal with patterned stimuli than field dependent students. This would seem to imply that the field-independent-dependent construct is an important variable in mathematical problem solving.

Analyzing the Problem Solving Process

A search of the literature reveals that many techniques have been used to study the problem solving process. One of the principal objectives which must be sought by the investigator is that of obtaining and maintaining observable behavior. It is not enough to try to detect which problem solving methods are used by studying the results of the students' work on their final papers. The investigator must try to elicit an observable response from the student even though this type of behavior may be difficult for the student working a novel problem.

Psychologists have used numerous devices and techniques in attempts to encourage their subjects to externalize their thought processes. Comprehensive reviews can be found in Lucas (1972) and Kilpatrick (1967). In view of the general purpose of this study a method well suited for externalizing the problem solving procedure was suggested by Kilpatrick (1967), who observed:

> There is one method of getting a subject to produce sequentially-linked, observable behavior that requires neither skill in self-observation nor the manipulation of mechanical devices: have the subject think aloud as he works [p. 6].

Thinking aloud requires only that the subject give an account of his mental activity as best he can. However, this method may have certain limitations. Thoughts may come and go too quickly to be verbalized. Subjects may also tend to remain silent during moments of deepest thought. More serious, however, is the possibility that a subject may solve a problem in a different manner when asked to verbalize his thoughts than when he is left alone to solve it silently (Kilpatrick, 1967, p. 7).

Two research studies have examined the effect of requiring subjects to think aloud when solving problems. Flaherty (1973) found no significant difference with respect to problem solving test scores between subjects who were required to think aloud and those subjects who were not required to verbalize overtly. However, there was a significant difference between the two groups in the area of computational errors. Flaherty indicated that:

> Perhaps the tendency of overt verbalization subjects to make more computational errors than the non-verbalization subjects may be attributed to their being somewhat distracted by the requirement to think aloud [p. 1767]

Roth (1966) found there was no significant difference in the number of correct solutions or the time required to find a solution between subjects who were required to think aloud while solving reasoning problems and subjects not required to think aloud.

Information-processors have made frequent use of the thinking aloud procedure. Paige and Simon (1966) used the protocols of subjects asked to think aloud to identify some of the important processes that a person must use in order to do algebra word problems successfully.

The method of thinking aloud is both productive and easy to use, only requiring the subject to work on the solution to a mathematical problem and to tell about it as he goes along. If the method is used intelligently and conscientiously, keeping in mind its limitations, it can provide information about the detailed process of thought. The problem is not so much to collect the data as it is to know what to do with them (Miller, Galanter, and Pribram, 1960, p. 193).

The Need for a Model of Problem Solving

It is apparent that a framework for classifying and analyzing the data is necessary. In fact, before undertaking a study in problem solving, the investigator must decide which procedures to examine and which to ignore. This choice of factors will not only determine the usefulness of the study, but also its limitations.

In order to minimize the chances of overlooking or not being able to account for a procedure used by a subject, the investigator must be aware of the model he is using. Begle and Wilson (1970) described a model as "providing an organizational framework; it represents a categorization system with some stated rules and relationships for using the system [p. 372]." The functions of a model should be to make clear to others what one has in mind and so aid communication, to distinguish between definition and empirical propositions, and to provide a means of data organization and interpretation (Kaplan, 1964, ch. 7).

With more emphasis being placed on the role of problems in both the curriculum and in research, it is necessary to know more about the problem solving process than the usual three or four "steps" suggested by models in this area. The following statement made by Johnson (1944) which summarized the knowledge of problem solving in the forties is still relevent today:

> Problem solving begins with the initial orientation and ends with the closing judgment, but between these bounds almost anything can happen, in any sequence [p. 203].

The writings of George Polya (1957, 1962, 1965) have added a great deal to the knowledge of problem solving in mathematics. Polya describes mathematical problem solving in terms of "methods and rules of discovery and invention" which he calls heuristics (1957, p. 112). However, it is still "between these bounds" where a more carefully constructed model is needed. The procedures given in the model must be broad enough to account for all the observed processes and yet not so broad that no real distinction can be made. The model used in this study appears to be such a model. The model is primarily the work of Eric MacPherson, presently Dean of the School of Education, University of Manitoba.

<u>MacPherson's Model for Mathematical Problem Solving</u>

For the purpose of discussion from a problem solving point of view it is convenient to view the discipline of mathematics as having three facets: Application, Core and Discovery (MacPherson, 1970, 1973).

APPLICATION	CORE	DISCOVERY
	facts	
lore	categories	heuristics
	algorithms	
	notation	

FIGURE 1

THREE FACETS OF THE DISCIPLINE OF MATHEMATICS

Core. The core is that part of the discipline which is not under active question at the moment. It consists of categories, notation, facts and algorithms. The categories are the conventionally agreed upon groupings of facts and algorithms. In mathematics such groupings are broadly termed algebra, geometry, analysis, topology, and more particularly, number systems, integers and the like. Associated with these categories is a set of facts which are propositions about the data not under active question; and algorithms are the procedures for answering questions, which, when applied correctly, guarantee a solution in a finite number of steps. Associated with each of these is usually found well-defined notation (MacPherson, 1973).

Lore. Lore is the set of acts of faith by which mathematical systems are tied to the real world. The efficacy of lore is not determined by the logical consistency of any algorithms used, but by the justification of the acts of faith (MacPherson, 1970, 1973).

Discovery. A discipline invariably has a set of procedures for adding to its core. Such procedures are termed heuristics (MacPherson, 1970, 1973). That is: a heuristic is a general, non-core strategy which is used for the purpose of discovering some order of mathematical generalization in a novel situation.

Operationally, heuristics differ from algorithms in that heuristics are characterized by an uncertainty as to the choice of the initial data and the efficacy of the procedure, whereas algorithms have a guarantee of producing a solution when applied correctly.

It seems useful to consider four hierarchical categories of heuristics: Low, Cases, Middle, and General. Lower heuristics lead more directly to the use of algorithms and in general do not create new problems, whereas the higher heuristics usually create new problems and then lead to use of the lower heuristics or core procedures (MacPherson, 1973). The heuristics from MacPherson's model are listed in Figure 2 and defined as follows:

Smoothing[1] - The heuristic of smoothing is used when the problem is altered in order to obtain some isomorphism between the new problem and a mathematical system. For example, in the problem:

> What is the longest piece of metal rod which can be placed in a box of dimension 3 inches by 4 inches by 12 inches?

One might "idealize" or smooth the box to a rectangular parallelepiped and the metal rod to a line segment.

Another example, in solving the problem:

> The point A is 50 units from a straight line CD, and B is a point 80 units from CD. Find the point X on CD so that the distance from A to X to B is as small as possible.

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[1]The definitions for the twelve heuristics in the model were determined in consultation with MacPherson.

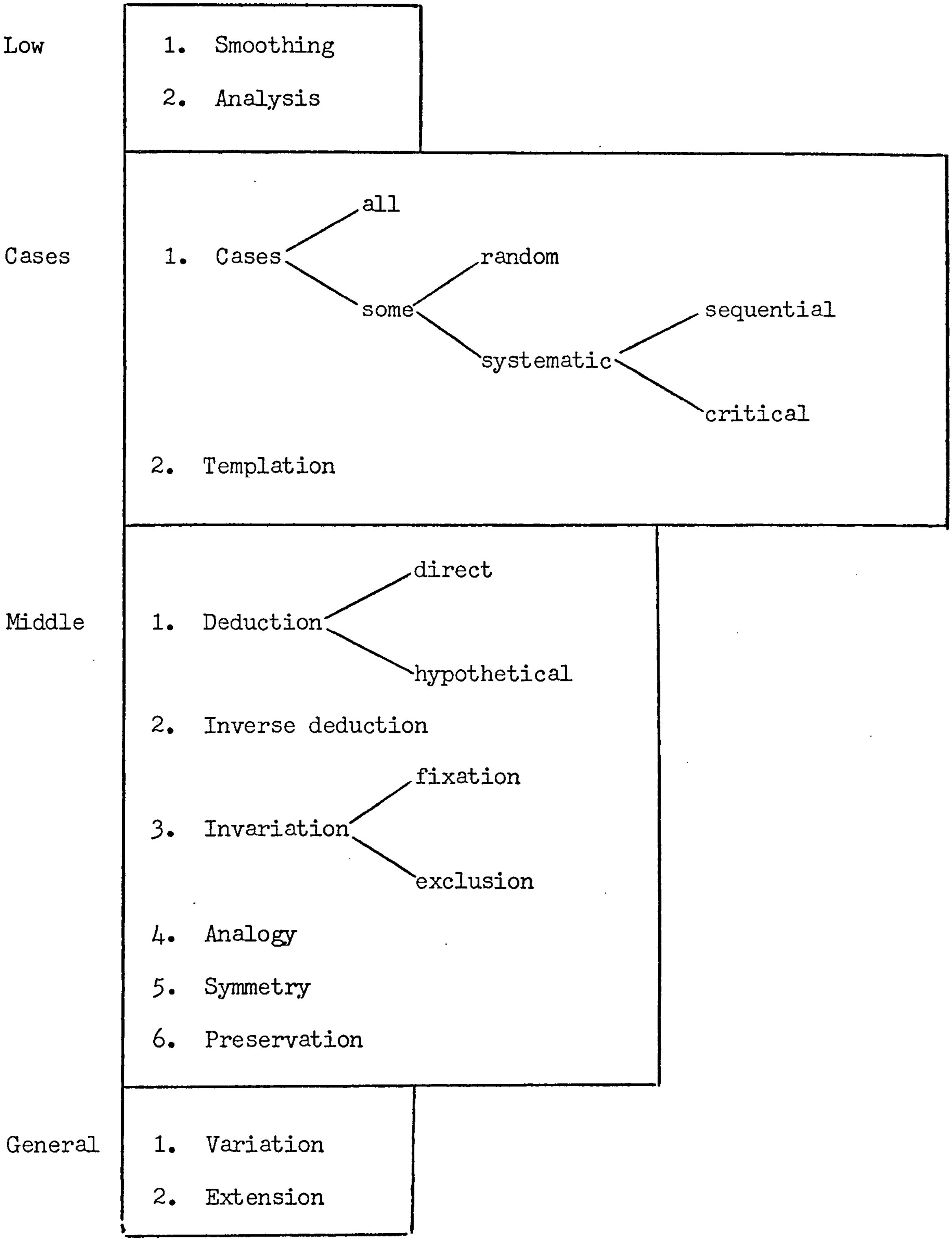
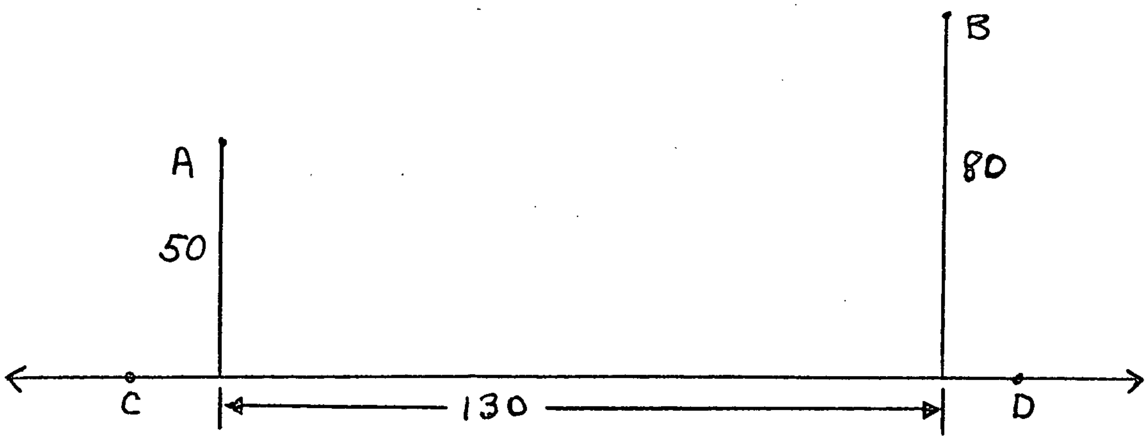


FIGURE 2

THE HEURISTICS FROM MACPHERSON'S MODEL



One could divide each of the measurements by 10, creating a similar figure with dimensions of 5, 13, and 8. This may simplify the arithmetic involved in obtaining a solution. Once a solution for the simplified problem is found, the solution to the original problem can be found by multiplying the simpler solution by ten.

Analysis - The heuristic of analysis is used when the problem is separated or broken up into subproblems. The solutions obtained from the subproblems are then used in solving the original problem. One solution for the problem:

> What is the maximum perimeter you can obtain by arranging one hundred one inch squares by following the rule that each time a new square is added at least one of its sides must be placed against one side of a previously arranged square.

is to arrange the squares in a straight row. When determining the perimeter, one might consider the end squares separate from the others, thus creating two subproblems. One finding the perimeter for the two end squares involving three sides and a subproblem for finding the perimeter of the remaining squares involving two sides.

Cases - A case is renaming the variables in a problem as constants. Using the heuristics of cases is to consider two or more cases in one of the following manners:

(a) To consider all possible cases, usually if there is only a small number.

(b) To consider cases at random.

(c) To consider cases in some systematic fashion, such as sequentially or to determine the critical cases.

and then to recognize a pattern; that is, to recognize the characteristics shared by the data collected from the cases considered and the properties and procedures from core. Consider this example of the use of systematic cases:

Find the sum of the first one hundred odd natural numbers.

The following cases (involving the number of natural numbers in the sum) could be considered in sequence:

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

There is a relationship between the number of terms in the sum and the square root of the term on the right of the equation. Hence, the solution to the problem is 100 squared.

One can also use sequential cases when all of the possible solutions are recognized. For example:

A lady gave the postage stamp clerk a one dollar bill and said, "Give me some two-cent stamps, ten times as many one-cent stamps, and the balance in fives." How can the clerk fulfill this puzzling request?

In answering this question, one could write the equation $12X + 5Y = 100$, for which there are eight possible values for X. All cases could be considered, or systematic cases for $X = 1, 2, 3, \ldots, 8$ could be used until a solution is obtained.

Sequential cases can be used in finding a^0 where $a \neq 0$ if one is familiar with the use of positive exponents. That is:

Find the value of a^0 where $a \neq 0$.

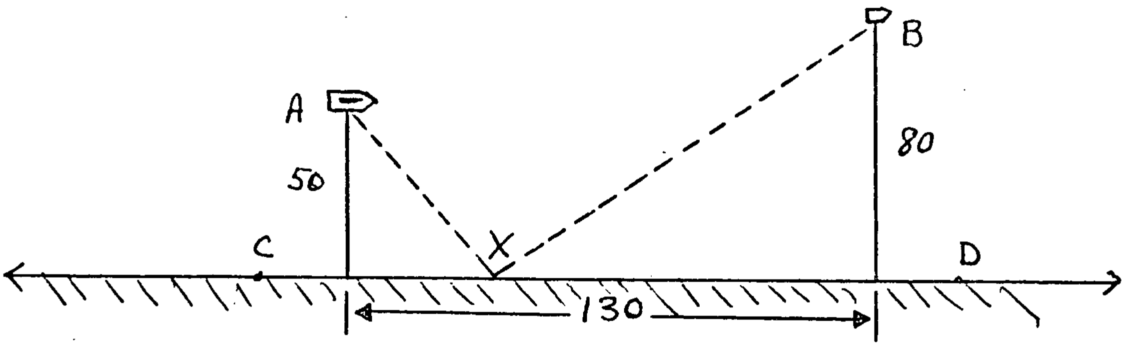
The following cases could be considered in sequence using a base of 2:

$$2^4 = 16$$
$$2^3 = 8$$
$$2^2 = 4$$
$$2^1 = 2$$

As each exponent decreases by 1, the value on the right of the equation is one-half the preceding value. If the pattern is to hold, then $2^0 = 1$. Other bases can be considered, either at random or systematically obtaining a similar result. Hence $a^0 = 1$ for $a \neq 0$.

<u>Templation</u> - The heuristic of templation is recalling a category of content which is related to the problem being solved. This includes the recall of such things as algorithms, problem types, procedures, theorems, and properties related to the problem or problem area. The purpose of templating is to recall related core material which can be used in solving the problem. For example:

A yacht is moored at A, 50 metres away from a straight sea wall, CD. The captain of the yacht wishes to row to the sea wall to collect a passenger and then to a speed-boat moored at B, 80 metres from the wall. Where should the passenger meet the captain to make the route as short as possible?

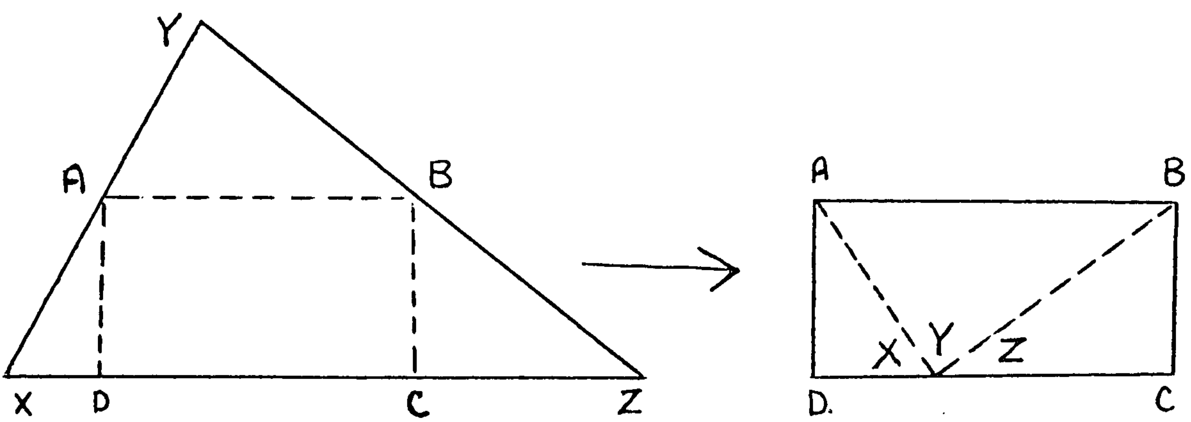


One might pick a point X between C and D, then looking at the two triangles, consider the theorems related to right triangles to see if any information about the position of the point X can be obtained.

Deduction - The heuristic of deduction can be sub-classified into two heuristics, direct and hypothetical. To use direct deduction is to ask what consequences can be implied from a given premise or set of premises. Hypothetical deduction is asking what consequences can be implied from a premise or set of premises assumed by the problem solver. In either instance, an attempt must be made to answer the questions posed.

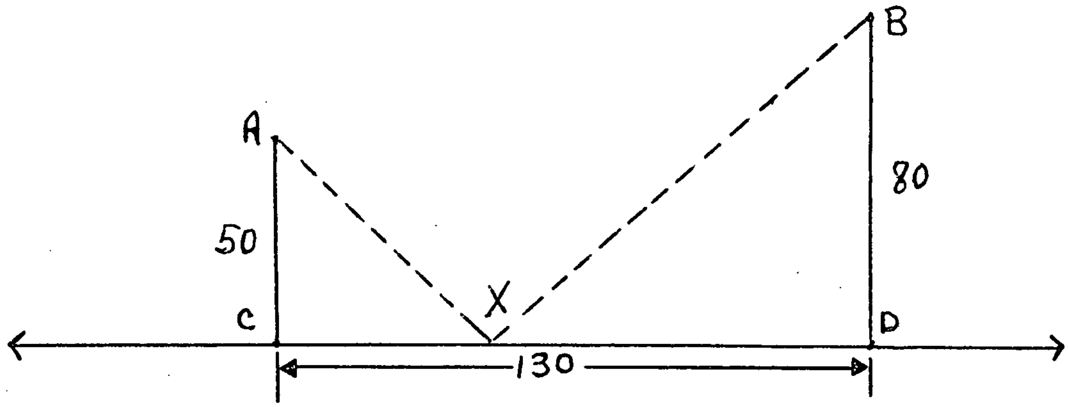
For an example of direct deduction, assume the following data, or set of premises are given:

> Given triangle XYZ with A the midpoint of XY and B the midpoint of YZ, let C and D be points on XZ so that quadrilateral ABCD can be formed by folding along AD so that X is a point on CD, fold on BC so that Z is a point on CD, and fold on AB.



What consequences can be implied by the set of premises? ABCD is a rectangle. The sum of angles X, Y, and Z is 180 degrees. The area of quadrilateral ABCD is equal to one-half the area of triangle XYZ. These are examples of some of the implications which could be made.

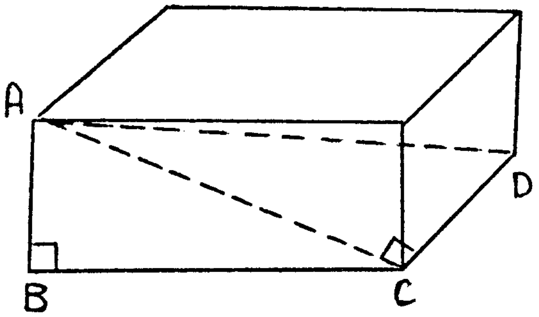
Hypothetical deduction may be used in solving the previous yacht problem. If X is the point on the sea wall where the captain picks up the passenger, then two right triangles are formed. A solution to the problem can be obtained by minimizing the sum of the lengths of their hypotenuse. If one assumes the hypotenuse of a right triangle is twice



the longest side (which it is not), what can be concluded in this problem? If CX is less than 50m, then the length of hypotenuse AX is 100m and XB is 2(130 - CX)m. The sum of AX and BX is increased as the point X is moved closer to C. If XD is less than 80m, the length of XB will be 160m and AX will have length 2(130 - XD)m. Hence the sum of AX and XB will increase as the point X is moved towards the point D. So one could conclude the minimum of the sum, AX + XB will occur when X is 50m from C.

<u>Inverse Deduction</u> - The heuristic of inverse deduction is used when one assumes (or one is given) a conclusion, and asks what premises or antecedents imply it. Inverse deduction means 'working the problem backwards'. For example:

What is the longest piece of metal rod which can be placed in a box of dimension 3 inches by 4 inches by 12 inches?



The longest piece of metal rod which will fit into the box has the length AD. Now AD is the hypotenuse of right triangle CDA, and if the length of DC and AC were known, then the Pythagorean Theorem could be applied to find AD. CD is known to be 4 inches long, so the length of AC must be found. But AC is the hypotenuse of right triangle BCA and the lengths of AB and BC are known to be 3 and 12 inches respectively. So by applying the Pythagorean Theorem to triangle BCA, AC can be found.

Invariation - The heuristic of invariation can be subclassified into two heuristics, fixation and exclusion. In using invariation either a variable is renamed as a constant (fixation), or a variable is excluded in the problem (exclusion). Then an attempt is made to solve the new problem and its solution studied in order to gain some insight into the given problem.

The heuristic of invariation - fixation can be used in solving the following problem:

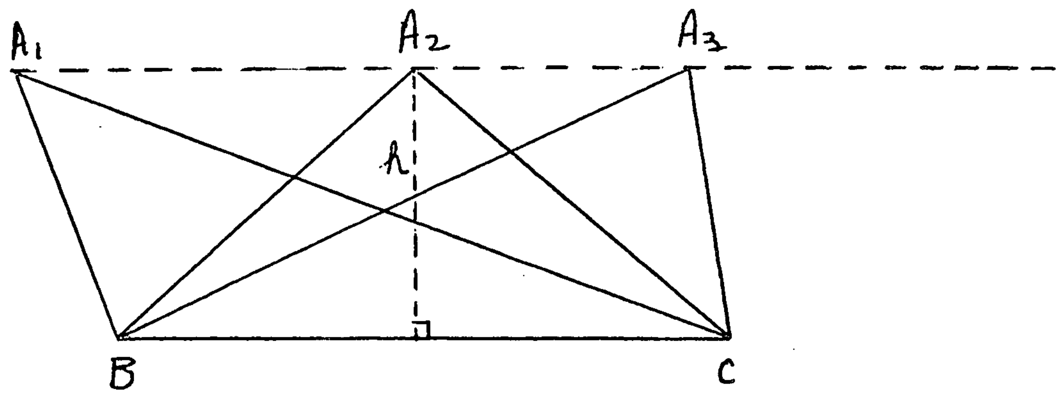
Derive a formula to find the roots of the general cubic $aX^3 + bX^2 + cX + d$, where a, b, c, and d are real numbers.

For this problem, one or more of the constants is fixed at zero and the roots of the new function are found. Then the solution of this new problem is studied in hopes of gaining some insight into solving the original problem.

The heuristic of invariation-exclusion can be used in solving the following geometry problem:

> Construct a triangle ABC given the length of the side BC, the measure of angle A, and the length of the altitude h from A.

In this problem, one might exclude the measure of angle A and formulate the new problem of constructing a triangle ABC given BC and the altitude h from A. Again the new problem may be solved by some method and its solution studied to gain some insight into the original problem.



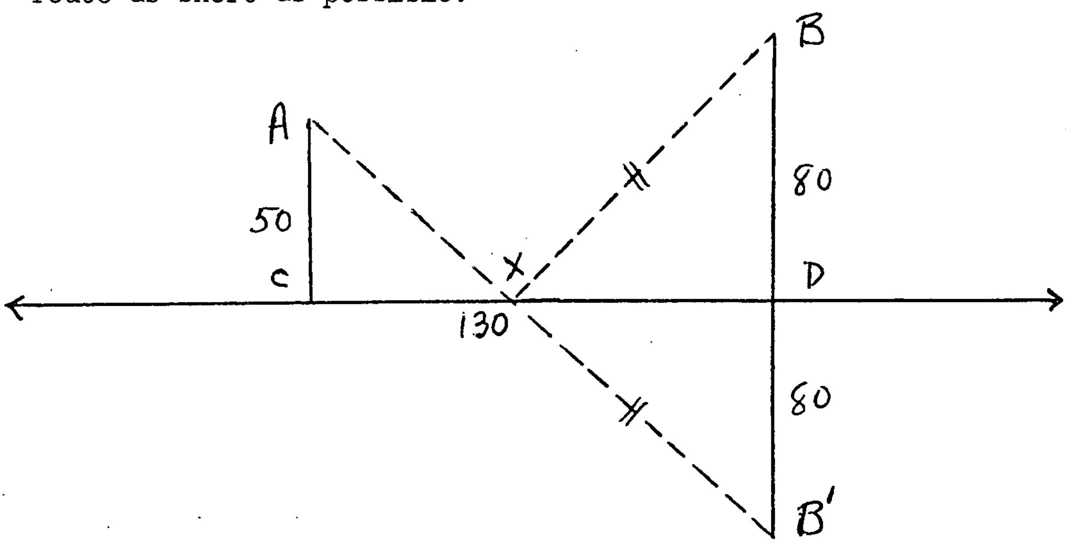
Analogy - If there is a "quasi" isomorphism between the problem and a mathematical system previously studied then the heuristic of analogy can be used. The heuristic of analogy involves asking questions and considering properties based on the recognition of the isomorphism. Analogy can be used to solve the following problem by recalling an analogous situation from two-dimensional geometry for which the solution is known:

> What is the greatest distance between two points of a rectangular solid of dimensions 3 by 4 by 12?

If one considers the similarity between this problem and that of finding the length of a diagonal of a rectangle of dimension a by b, then, since the length of the diagonal is $\sqrt{a^2 + b^2}$, the length of the diagonal of this rectangular solid could be $\sqrt[3]{3^3 + 4^3 + 12^3}$, $\sqrt{3^2 + 4^2 + 12^2}$, or $\sqrt{3^3 + 4^3 + 12^3}$.

Symmetry - To use the heuristic of symmetry is to make use of the inherent or constructed symmetry in a problem. For example:

> A yacht is moored at A, 50 meters away from a straight sea wall, CD. The captain of the yacht wishes to row to the sea wall to collect a passenger and then row to a speed-boat moored at B, 80 meters from the wall. Where should the passenger meet the captain to make the route as short as possible?



One might take advantage of the fact that once the captain has moved from point A to the sea wall CD at X, the distance from X to B is the same as the distance from X to a point B' (where B' is the reflection of B through the line CD.) Since the shortest distance between two points is a straight path, the point X is easily found.

In the following example, the problem is stated symmetrically in terms of the sets A and B:

> Let A and B be sets such that for any set, C, $A \cap C = B \cap C$ and $A \cup C = B \cup C$. Prove $A = B$.

The role of A and B can be interchanged without changing the problem.

So if one shows A is a subset of B, then by the symmetry, B will be a subset of A. Since two sets are equal if and only if they are subsets of each other, A = B.

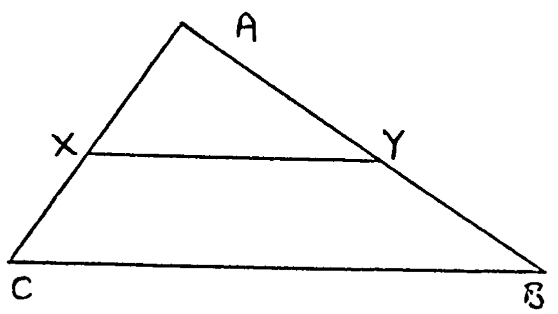
<u>Preservation</u> - The heuristic of preservation is used when a conscious effort is made to preserve the properties of a mathematical system when its domain is extended (isomorphically) to form a new system. For example in the extension of the real numbers to the complex, one wishes to preserve the field properties of the real number system. The heuristic of preservation is used if the commutative, associative and distributive properties are taken into account when defining addition and multiplication for the complex numbers so as to preserve the field properties.

The heuristic of preservation can be used in solving the following problem:

Find a definition of a^0 where $a \neq 0$.

One of the properties of exponents is $b^X/b^Y = b^{X-Y}$, for $X \neq Y$. If a conscious effort is made to preserve this property when defining a^0, then the heuristic of preservation has been used.

<u>Variation</u> - In using the heuristic of variation the constraints on one of the variables of the problem are relaxed in order to study the effect on the other variables. As an example from geometry, let X and Y be the midpoints of AC and AB respectively in triangle ABC. Then



one can conclude that XY is parallel to BC and the length of XY is $\frac{1}{2}$ the length of BC. The heuristic of variation is used if one relaxes the condition that X and Y are midpoints and studies the effect on the relationships between XY and BC.

Extension - The heuristic of extension is posing a problem that is an extension or a generalization of the given problem. Given the problem, "Find a natural number which is both a square and a cube.", one uses the heuristic of extension when posing the question, 'can all natural numbers which are both a square and a cube be characterized?'.

For the purposes of this study, the sequence of events used while solving a mathematical problem is characterized by Figure 3. A brief description of the model follows:

Willingness: The subject's willingness to accept the problem and once he has started to solve it, his willingness to continue until he has a solution.

Sieve: The mathematical content and processes (core) which are familiar to the problem solver.

Upon presentation of the problem, the problem is either accepted or rejected (willingness box). If the problem is accepted, then the individual immediately tries to recall whatever content he knows that is related to the problem (sieve) in hopes of recognizing a pattern which will enable him to obtain a solution. If the exit is from the sieve to pattern, one of three things can happen depending upon the confidence (probability) the individual has in the pattern he has obtained: 1) Keeping the pattern in mind, he could return to the sieve and try to obtain a better fit of the pattern to the core and problem. 2) If the individual has little confidence in the pattern he could return to the willingness box and decide to either continue working on the problem or to quit without a solution. 3) He could exit to the use of core algorithms. Two confidence levels or probabilities (indicated by dotted line) are assigned to the sequence of procedures used for every problem, one concerning the fit of the pattern and the other concerning the accuracy of the algorithm. These

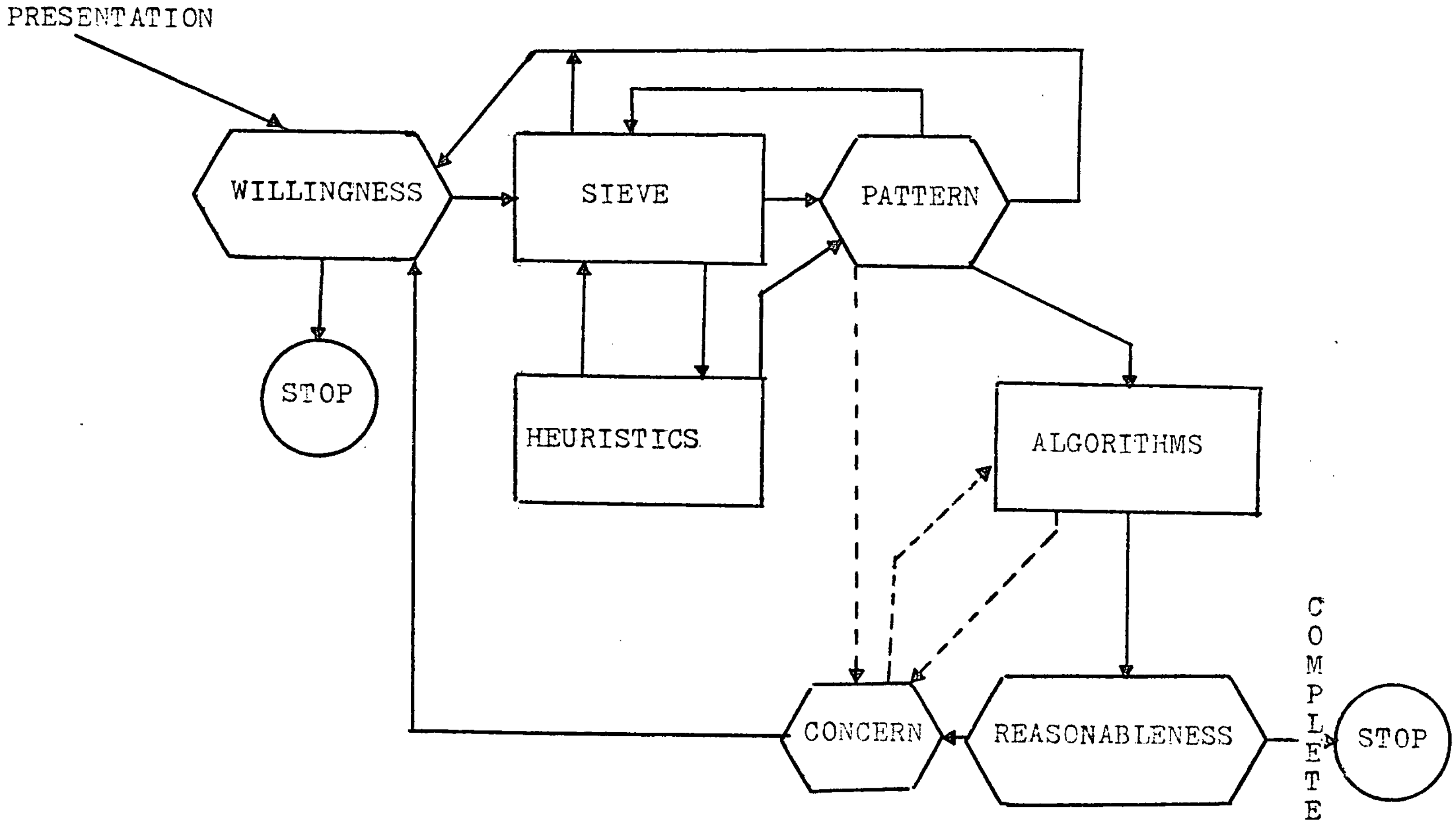


FIGURE 3

A MODEL OF THE EVENTS INVOLVED IN
MATHEMATICAL PROBLEM SOLVING

probabilities determine the possible routes from the concern box. Similarly with the application of an algorithm, a reasonableness is attached to the solution. If there is a high probability attached to the pattern and yet, there is some concern for the use of an algorithm, the problem solver may return directly to the algorithm box. If however, there is some concern for a particular pattern then he may return to the sieve (via the willingness box) and try to obtain a better fit. If the outcome of the application of the algorithm is reasonable, then the individual will exit with a solution. However, if no exit is made from the sieve to pattern, then he must either return to the willingness box or to the use of heuristics. The lower heuristics such as smoothing and analysis appear to be closely related to the sieve so there may be a considerable movement between the sieve and heuristics. The use of the lower heuristics may also lead more directly to the sieve and pattern whereas the use of higher heuristics usually creates new problems and then leads to the use of the lower heuristics.

Definition of Terms

Certain terms occur throughout the study and are defined here.

The term degree of field independence refers to the relative position of the student's score in the distribution of the experimental sample scores on the Embedded Figures Test (Witkin, 1969). See Chapter II for further discussion of the construct of field independence. Conventionally a student is field independent if his score is above the sample median on the Embedded Figures Test (EFT) and field dependent if his score is below the median. To test one of the hypotheses in this study, a subject is field independent if his score on the EFT is in the top third of the scores for the sample and field dependent if his score is in the bottom one-third. The term problem refers to " a situation in which one must give a response (that is, when he seeks satisfaction) and has no habitual response which will give this satisfaction [Cronbach, 1948, p. 32]." This definition implies that what is a problem for one student may not be a problem for another. It is not the posing of a question that makes it a problem, but the willingness of the individual to accept it as something he must try to solve. Furthermore, a question such as:

> What is the greatest distance between two points in a rectangular solid of dimension 3 inches by 4 inches by 12 inches?

is not a problem to someone who is familiar with three-dimensional geometry and recalls a formula for finding the length of the diagonal of a rectangular parallelepiped. However, it may be a problem for the student who has studied only plane geometry.

The term problem context refers to the setting in which the problem is stated, mathematical vs. real or physical world. The vocabulary used in the statement of the problem should be familiar to a student who is taking a grade eleven mathematics. The terms "mathematical" and "real" refer to the situation and the vocabulary used in the problem. A problem stated in each context is given below:

> Mathematical World or Math World: What is the greatest distance between two points in a rectangular solid of dimension 3 units by 4 units by 12 units?
>
> Real World: What is the longest piece of metal rod which can be placed in a box of dimension 3 inches by 4 inches by 12 inches?

The term <u>algorithm</u> refers to a systematic procedure which, if carried out correctly, guarantees a solution in a finite number of steps. Algorithms vary in complexity from very simple such as the addition of two digit whole numbers without regrouping, to complex and involved procedures, like using the elimination method to find the solution of a system of linear equations. An example of an algorithm from the grade eleven curriculum is the procedure used to find the roots of the quadratic $4x^2 - 16x - 9$ by completing the square.

The term <u>heuristic</u> refers to the twelve procedures defined in the model. A heuristic differs from an algorithm in that heuristics have a low probability of guaranteeing any success in solving the problem. For example, the renaming of a variable as a constant in order to study its effects on the other variables in the problem does not guarantee that one will find a solution, but it may give some insight into the problem.

The Problem

In this study, the writer attempted to determine, by use of a clinical procedure, whether there were any significant differences in the problem solving behavior among subjects working problems in either a mathematical or real world setting. An attempt was also made to investigate the relationship between these problem solving behaviors, in terms of heuristic and nonheuristic procedures, and how these behaviors relate to field independence.

The present study has as its central theme, the use of heuristics in mathematical problem solving. Little is known about how students attack and solve mathematical problems and what part, if any, heuristics play in this procedure. In order to gain some insight into this area, a study which is exploratory in nature was designed. The general aims of the study were:

1. To develop a workable system, based on the model of MacPherson, for coding the audio tape recorded protocols of subjects asked to think aloud as they solve mathematical world problems.
2. To investigate the relationship of the problem solving behaviors derived from the coding system, to one another.
3. To investigate the effect of problem context and field independence on problem solving behaviors.

In light of these aims, the following questions were asked:

1. What procedures from core and what heuristics are used by students in attacking and solving mathematical word problems?
2. Does the context of the problem influence heuristic usage?
3. What effect does field-dependence-independence have on the problem solving process?
4. Do selected groups of individuals exhibit patterns of heuristic usage when solving mathematical word problems?

General Hypotheses

Some of the hypotheses that were tested in order to help the investigator gain some insight to the answers for the above questions are as follows:

H1: Problem context will not affect the total number of heuristics used by a student.

H2: Problem context will not affect the number of different heuristics used by a student.

H3: Problem context will not affect the number of correct solutions obtained by a student.

H4: Field independence will not affect the number of times heuristics are used by a student.

H5: Field independence will not affect the number of different heuristics used by a student.

H6: Field independence will not affect the number of correct solutions obtained by a student.

H7: The number of times heuristics are used by a student is unrelated to the number of correct solutions he obtains.

H8: The number of different heuristics used by a student is unrelated to the number of correct solutions he obtains.

H9: The sequence of heuristics used by the subjects will not be affected by problem context.

H10: The sequence of heuristics used by the subjects will not be affected by field independence.

Significance of the Study

Kilpatrick (1969) stated in a recent review of mathematical problem solving,

> ...the researcher...who chooses to investigate problem solving in mathematics is probably best advised to undertake clinical studies of individual subjects... because our ignorance in this area demands clinical studies as precursors to larger efforts [p. 179].

The significance of this study is apparent when one considers that so few attempts in research have been made to analyze individual behavior during the problem solving process. This behavior will be analyzed in terms of a problem solving model developed by MacPherson. Until more is known about the way these heuristics are employed little can be gained by way of insight into an adequate teaching methodology to enhance problem solving skills.

This study is also significant in that it makes use of complex problems usually not found in today's textbooks. With a greater emphasis being placed on challenging problems in the curriculum, there is a need for research which makes use of such problems. In Kilpatrick's words (1967):

> The mathematics educator, in particular, sometimes complains that the kind of complex, challenging problems that are the most difficult to learn how to solve rarely appear in the research literature. He questions how much one can extrapolate from findings based on lever pulling and card sorting to the processes that underlie the search for an elegant geometric proof or the production of an equation describing a physical situation. The analogies may be direct or they may be exceedingly subtle and complicated. We have no way of knowing because complex problems have so seldom been used in research [p. 1].

Finally, the most important contribution of this study may be that of adding to the current selection, as small as it may be, of theoretical models of mathematics problem solving. This model can be used as a framework for extending research in mathematical problem solving and perhaps as a mode for teaching mathematics.

Assumptions and Limitations

Five basic assumptions were made in beginning this study. It was assumed that the heuristics from MacPherson's model could be observed and used to classify problem solving procedures. It was assumed the problems used in this study would elicit the use of heuristics. Also, it was assumed that the students would be willing to try to solve the five problems assigned to them, and that they would verbalize their thoughts. It was also assumed that students in grade eleven were mature enough, both emotionally and mathematically, to participate in this clinical study.

During the pilot study these assumptions were examined and found to be tenable. The students in the pilot study were willing to work on all assigned problems and to verbalize their thoughts. In modifying the coding system it was found that the heuristics could be observed and used to classify problem solving procedures.

The limitations of this study are attributable to:

(a) The problems used. The problems were not selected from a specific category, such as algebra or geometry, but were selected from those used in the pilot study using the following four criteria:

1. They were problems not usually found in the school curriculum
2. They could be solved by about half the students
3. They could be solved in a number of different ways
4. They elicited the use of heuristics

A set of different problems could have produced different data.

(b) The sample of subjects used. The subjects were selected from the middle IQ range of academic grade eleven mathematics students. The data are based on a relatively homogeneous group of students. It is quite possible that, if the study had included a greater variety of individuals, a greater variety of heuristics might have been observed.

(c) The method of data collection. Having the subjects think aloud as they solve mathematical problems may cause them to commit errors they normally would not; in fact they may solve the problem in a different manner when asked to verbalize than when left alone to solve it.

Chapter II

REVIEW OF RELATED LITERATURE

This chapter will examine some of the literature relevant to the study. Rather than attempt a comprehensive survey of recent problem solving research (see Kleinmuntz, 1966; Kilpatrick, 1969; Jerman, 1971), the discussion will be confined to two relevant topics. The first concerns the problem solving process. A series of models of the problem solving process will be discussed with the related research. The second topic deals with the consistent or inconsistent problem solving behaviors exhibited by subjects. Recent research in which the field independence construct is used as an aptitude variable in a problem solving setting will be cited.

Models of Problem Solving

Over the years, scholars have sought to shed light upon the problem solving process by specifying the sequence of behaviors through which one might proceed in solving a problem (Davis, 1973, p. 15). One example is given by Johnson (1944) in which he identifies three processes or groups of processes which, he says regularly occur during problem solving: (1) orientation to the problem, (2) producing relevant material, and (3) judging.

Dewey (1933) gives a slightly different analysis. As he defines reflective thinking, there is little difference between it and problem solving. Hence, his analysis of reflective thinking can be taken as an analysis of the process of problem solving (Henderson and Pingry, 1953).

Dewey (1933, pp. 107-116) outlines five phases of reflective thinking:

1. "A felt difficulty", a question for which the answer must be sought
2. Location and definition of the difficulty
3. "The identification of various hypotheses...to initiate and guide observation and other operations in collection of factual material [p. 107]."
4. Elaboration of each hypothesis by reasoning and the testing of the hypothesis
5. Acting on the basis of the particular hypothesis selected in step four to see whether the results theoretically indicated actually occur.

Dewey notes these five phases or stages of thought do not follow one another in a set order. Each step in thinking does something to the formation of a suggestion to change it into an idea or hypothesis. Each step also does something to promote the location and definition of the problem. Each improvement in the idea leads to new observations that yield new facts to help judge the relevance of facts already at hand. The elaboration of an hypothesis does not wait until the problem has been defined, but it may come at any intermediate time. As well, any evaluation need not be the final step in the process. It may be introductory to new observations or suggestions, according to what happens as a consequence of it (Dewey, 1933, p. 115).

Other examples of the suggested "steps" in problem solving follow:

Burt (1928)	Gray (1935)
1. Occurrence of a perplexity	1. Sensitivity to the problem
2. Clarification of the perplexity	2. Knowledge of the problem conditions, recognition of significant information

3. Appearance of suggested solutions

4. Deducing implications of suggested solutions

5. Verifying action or observation.

Humphrey (1948)

Directed thinking involves:

1. A problem situation

2. Motivating factors

3. Trial and error

4. Use of association and images

5. A flash of insight (the place of 3, 4, and 5 varies with the problem)

6. Some applications in action.

Bloom and Brader (1950)

"problem solving characteristics" are:

1. Understanding of the nature of the problem

2. Understanding of the ideas contained in the problem

3. General approach to the solution of the problem

4. Attitude towards the solution of the problem.

3. Suggested solution or hypothesis

4. Subjective evaluation, does the proposed solution work?

5. Objective test

6. Conclusion or generalization.

Burock (1950)

1. Clear formulation of the problem

2. Preliminary survey of all concepts of the material

3. Analysis into major variables

4. Locating a crucial aspect of the problem

5. Application of past experience

6. Varied trials

7. Control

8. Elimination of sources of error

9. Visualization.

Kingsley and Garry (1957)

1. A difficulty is felt

2. The problem is clarified and defined

3. A search for clues is made

4. Various suggestions appear and are tried out

5. A suggested solution is accepted

6. The solution is tested.

Both Brownell (1942, p. 432) and Kilpatrick (1967, p. 20) caution against the tendency to misuse conceptual frameworks such as those above by assuming that in reality, problem solving occurs in well defined sequential stages. In the 1930's students were taught to use the technique of formal analysis to solve arithmetic problems. Students were encouraged to ask themselves a sequence of questions, such as "(1) What is the given? (2) What is to be found? (3) What is to be done? (4) What is a close estimate of the answer? [Burch, 1953, p. 47]." These questions were to be asked before any computational work was done. Brownell (1942) indicates that formal analysis, "represents a logical pattern of thinking which may or may not characterize expert thinking on the part of adults, but certainly has not yet been shown to characterize good thinking on the part of children [p. 432]."

Formal analysis has been shown to be ineffective and troublesome for the student. To study the effect of formal analysis, Burch (1953), conducted a study using 305 elementary school children who had been trained through the use of formal analysis to solve arithmetic problems. He found that students generally attained higher scores on the test which did not require formal analysis than on one which did. In an attempt to determine whether students used formal analysis even if not required to do so, Burch selected 51 students from the sample and asked them to think aloud as they solved a series of arithmetic problems. Out of approximately five hundred problems attempted during the interviews, formal steps were used in only two instances. Furthermore, when students were questioned following the interview, each of them indicated that he never used formal analysis except when required to do so. Many reported that when they tried to use formal analysis, they became confused. Burch noted that "one explanation of the inadequacy of formal analysis may be that it fragments the problem into isolated parts [p. 47]."

One of the best known models of problem solving was given by Wallas (1926). Wallas' stages of problem solving are: (1) Preparation, clarifying and defining the problem, (2) incubation, unconscious mental activity, (3) inspiration, solution appears suddenly, and (4) verification, checking the solution. The stages of incubation and inspiration, by definition, are unobservable mental processes. However, both Polya (1945, 1962, 1965) and Hadamard (1945) readily acknowledge the unconscious, preconscious or sometimes "fringe-conscious" activity leading to the solution of a difficult problem. Green (1966) observed that the Wallas model may characterize the research efforts of scientists working on difficult problems, but it fails to describe the high school student trying to solve an algebra problem.

Wickelgren's (1974) model of mathematical problem solving has been influenced by current work in artificial intelligence and computer simulation of human problem solving done by Newell and Simon. The principal aim of the Wickelgren model is to present the elementary principles necessary to solve mathematical problems of either the "to find" or the "to prove" character, but not problems of defining "mathematically interesting" axiom systems (Wickelgren, 1974, p. 2).

The procedures in Wickelgren's model include inference, trial and error, state evaluation, subgoal, contradiction, working backwards, and recall of related problems. These procedures are to be used by the problem solver only if he can't solve the problem by the direct use of algorithms. A description of Wickelgren's model follows:

> Inference: To draw inferences from explicitly and implicitly presented information that satisfy one or both of the following two criteria: (a) the inferences have frequently been made in the past from the same type of information; (b) the inferences are concerned with properties (variables, terms, expressions, and so on) that appear in the goal, the given,

or inferences from the goal and the givens [p. 23].

Drawing inferences is the first problem solving procedure employed in attempting to solve a problem. The goal or the givens are essentially expanded by bringing to bear all of the knowledge the problem solver has concerning the problem.

Random Trial and Error: to apply the allowable operations to the given in a random fashion [p. 46].

Systematic Trial and Error: to produce a mutually exclusive and exhaustive listing of all sequences of actions up to some maximum length [p. 47].

Classificatory Trial and Error: to organize sequences of actions into classes that are equivalent with respect to the solution of the problem [p. 47].

In this case if one sequence of actions within a class will solve (not solve) the problem, then all the other sequences of actions within the same class will probably also solve (not solve) the problem.

State Evaluation and Hill Climbing: This method has two parts: (a) defining an 'evaluation function' over all states (the set of all the expressions that exist in the world of the problem at a given time) including the goal state and (b) choosing actions at any given state to achieve a next state with an evaluation closer to that of the goal. Picking an action on the basis of such a local evaluation of its consequences is known as 'hill climbing' [p. 67].

An example of state evaluation and hill climbing is given in the following example:

You have a pile of 24 coins. Twenty-three of these coins have the same weight, and one is heavier. Your task is to determine which coin is heavier and to do so in the minimum number of weighings. You are given a beam balance (scale), which will compare the weight of any two sets of coins out of the total set of 24 coins.

A suitable evaluation function for solving this problem would be the number of coins whose classification as heavy or light is known. At the beginning of the problem, the value of the function is zero, since none of the 24 coins is known to be either heavy or light. In the goal state, the heavy-light classification of all 24 coins is known, so the value of the function is 24.

Thus, a hill climbing approach would choose an action at each node that maximized the number of coins whose heavy-light classification is known [p. 71].

Subgoal: to analyze a problem into subproblems or to break it into parts [p. 91].

The purpose of this procedure is to replace a single difficult problem with two or more simpler problems.

Contradiction: Proving the goal could not possibly be obtained from the givens [p. 109].

The method of contradiction can be applied in the following four ways:

Indirect Proof: to assume the contrary is true and show that the contrary statement in combination with the givens, results in a contradiction [p.111].

Multiple Choice - Small Search Space: in problems involving a small set of alternative goals, to systematically apply the method of contradiction to every alternative goal [p. 115].

Classificatory Contradiction - Large Search Space: to devise an effective search procedure that contradicts large classes of alternative goals simultaneously [p. 126].

Classificatory Contradiction - Infinite Search Space: to devise an effective search procedure that contradicts infinitely large classes on the basis of some common property [p. 133].

Working Backwards: to guess a preceding statement or statements that, taken together, would imply the goal statement [p. 138].

Wickelgren describes four fundamental types of relationships between problems that can be used by the problem solver.

Equivalent Problems: to recognize that problems differ only with respect to the names attached to different elements, but whose relations and operations are identical [p. 156].

Similar Problems: to recognize that two problems share common elements, and then to recall the methods used to solve the similar problem [p. 153].

Simpler Problems: to pose and solve or recall a problem which is simpler or a special case of the more complex problem [p. 157].

> <u>More Complex Problems</u>: Posing a problem that is more complex than the given problem and in which the given problem is embedded. Then solve the more complex problem [p. 166].

In MacPherson's (1973) terms, Wickelgren's model consists of procedures from both core and heuristics. His definitions and examples of random trial and error, subgoals, working backwards, and complex problems correspond to the heuristics of random cases, analysis, inverse deduction, and extension, respectively. Rather than procedures for solving problems, state evaluation and hill climbing, systematic and classificatory trial and error, and the methods of contradiction appear to be overall methods or plans of attacking particular types of problems.

This model may have some implications for teaching problem solving but as a framework for classifying data it is quite restricted. Some of the procedures, such as subgoals and working backwards, are useful in solving many kinds of problems and should be included in a classificatory scheme. However, many of Wickelgren's procedures appear to be too specific to be used to characterize problem solving in any general sense. A framework for classifying data should be <u>flexible</u> enough to accommodate broad strategies of problem solving as well as narrower tactics used to realize these strategies (Kilpatrick, 1967).

Schwieger (1974) describes a model of mathematical problem solving based on the identification and description of eight basic components generated from the literature. These basic components are:

> <u>Classify</u>: to recognize pertinent characteristics and attributes of mathematical problems or expressions and to specify the class or classes to which they belong [p. 38].

> <u>Deduce</u>: to relate a set of statements so that acceptance of the statements and their interrelationships dictates a particular conclusion [p. 41].

> <u>Estimate</u>: to use available mathematical information to make a judgement of measurement or of a result of calculation [p. 44].

Generate Pattern: to put known or available mathematical data into a systematic arrangement [p. 47].

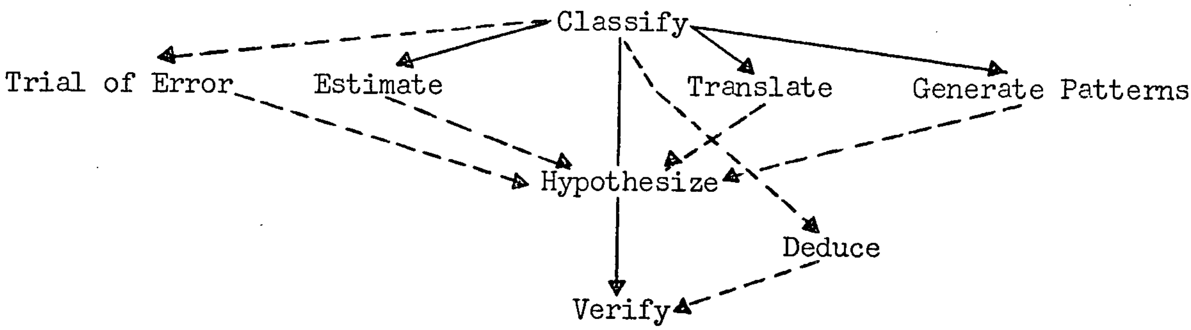
Hypothesize: to recognize or to generate conditional relationships between mathematical statements [p. 50].

Translate: to substitute for one mathematical form, an equivalent representation [p. 53].

Trial and Error: to apply knowledge to a mathematical problem in an unorganized manner [p. 56].

Verify: to apply data to a hypothesis in testing its validity [p. 59].

Schwieger also identified the following hierarchy among the basic components (p. 80). A solid arrow indicates that the ability at the tail of the arrow is a prerequisite to the ability at the head of the arrow. The dashed arrows indicate some of the more common task-specific



prerequisites, (i.e. "ability to hypothesize may depend on prior generation of patterns."). A problem solver may return to a component higher on the diagram at any time.

Schwieger (pp. 36-39) claims "These eight components are considered to be basic to any thinking in mathematical problem solving, and any mathematical problem solving can be explained in terms of them.", and that, in fact, these basic components are independent of each other. However, there is very little evidence to support this claim of independence. Schwieger reported the analysis of only two problems coded using this model.

If the model is to be used to analyze problem solving, some of the components appear to be not defined clearly enough for reliable coding. This is especially true for the components of deduce, hypothesize, and estimate.

The current interest in the use of "heuristics" in mathematical problem solving is due principally to Polya (1957, 1962, 1965). Polya's model includes a variety of procedures, both general and specific, for solving mathematical problems in a number of content areas. A portion of Polya's model is presented in terms of a list of questions one asks himself as he tries to solve a problem. He postulates that these correspond to mental action. Polya's list includes different forms of questioning geared to defining and approaching difficult and unfamiliar mathematical tasks. Polya's (1957) mathematical checklist includes:

Understanding the Problem

What is the unknown? What are the data? What is the condition? Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
Draw a figure. Introduce suitable notation. Separate the various parts of the condition. Can you write them down?

Devising a Plan

Have you seen it before? Or have you seen the same problem in a slightly different form? Do you know a related problem? Do you know a theorem that could be useful?
Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.
Here is a problem related to yours and solved before.
Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
Could you restate the problem? Could you restate it still differently?
Go back to definitions.
If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem?
Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you

derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other? Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Carrying Out the Plan

Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?

Looking Back

Can you check the result? Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result, or the method, for some other problem [p. xvi, xvii].

The model consists of four phases, but it is not to be implied that in reality these phases always occur in sequence, or that a given problem solver will exhibit behavior characterized by every phase during the solution of a problem.

This model is expanded, (Polya, 1957) in a "Short Dictionary of Heuristics" and in two of Polya's other books (Polya, 1962, 1965). The model includes procedures from drawing figures to generalization. Most of these are either expansions of the questions from the list or are explained in terms of combinations of questions from the list. These procedures include generalization, variation of the problem, and working backwards.

Difficulties in attempting to use Polya's checklist to analyze the protocols of fifty-six eighth grade students of above average ability were indicated by Kilpatrick (1967) who stated:

Attempts to apply the checklist to several protocols from the pilot study demonstrated clearly that whatever merits Polya's list has for teaching problem solving, it is of limited usefulness, as it stands, for characterizing the behavior of these subjects. Many of the categories were unoccupied: subjects seemingly did not exhibit behavior even remotely resembling actions suggested by the heuristic questions. For example, no subject

> asked themselves aloud whether they were using all of the essential notions of the problem. Furthermore, the categories were not defined clearly enough for reliable coding [p. 44].

Kilpatrick (1967) used a modified checklist and "Process-sequence" system based on Polya's model to analyze the protocols of the subjects in his study. Non-sequential behavior, such as "draw figure", "recalls same or related problem", "uses successive approximation", etc., was checked on the checklist the first time it was observed, and repeated occurrences were not recorded. The sequential behavior was translated into coding symbols and recorded in their order of occurrence. These procedures were:

R Reading and trying to understand the problem

D Deduction from condition

E Setting up equation

T Trial and error

C Checking solution

The three symbols D, E, and T were followed by a number from 1 to 5 that represented the outcome of the process. For example, D1 stands for an incomplete deduction. A full explanation of this coding system can be found in Kilpatrick's (1967, Appendix F) dissertation.

When Kilpatrick applied this coding system to the students' tape recorded protocols, only a few of the procedures from Polya's model were observed. For instance, few students varied the conditions of the problem or attempted to derive a solution by another method. Most of the subjects drew figures while solving the problems, but the frequency with which figures were drawn was unrelated to the other procedures or to success in solving the problems. Both trial and error ($P < .05$) and successive approximation ($P < .01$) were correlated with successful problem solving. Checking, as identified by the coding system, was also related ($P < .05$) to the number of correct solutions.

Kilpatrick also compared the results of the coding system for each subject with his performance on a battery of tests from the National Longitudinal Study of Mathematical Abilities (NLSMA) files. He found that subjects who attempted to set up and use equations were significantly superior to those students who did not use equations on measures of mathematics achievement, general reasoning, word fluency, quantitative ability, and reflective-impulsive style. Of those subjects who did not use equations, those that used trial and error were higher than those students who did not use trial and error in achievement and quantitative ability. The subjects who used the least trial and error and did not use equations had the most trouble with word problems, spent the least amount of time on them, and got the fewest number of correct solutions.

Using a modified version of Kilpatrick's coding system based on Polya's model, Webb (1975) analyzed the problem solving ability of forty second year high school algebra students. The students were interviewed individually and asked to think aloud as they solved eight mathematical problems. The protocols were recorded and coded at a later date. Each student was given a total score from zero to five for each of the problems. The score was based on the sum of subscores for approach, plan, and result. Using regression analysis, Webb found that the mathematical achievement component accounted for 50% of the variance in the total score and the heuristic strategy components accounted for an additional 13% of the variance.

Kantowski (1975) analyzed the problem solving ability of eight above average ability ninth grade algebra students as they learned to solve problems in geometry. Her study was comprised of four phases: a pretest, a readiness instruction phase in the use of selected heuristics of Polya's, an instruction in geometry phase using teaching strategies based on the same heuristics, and a posttest. During each of the phases, the subjects were asked to think aloud as they solved problems and their protocols were recorded and then analyzed using a modified version of Kilpatrick's coding scheme. A score was assigned to each problem based on the procedures used by the subject as well as the solution.

Kantowski's study was a clinical exploratory study to determine the procedures used by students as they learned to solve problems in geometry. The objective of the study was to seek regularities that would generate hypotheses for further experimental study and not to state conclusions. Kantowski found that the increased use of heuristics and the development of problem solving ability were positively correlated. The use of "looking back" strategies did not increase as problem solving ability developed nor did it appear to be related to success in problem solving. Kantowski noted the level of rigor required in the use of these strategies may be beyond students who are just beginning to study in a content area.

Polya's model has been modified and used in research to analyze problem solving. MacPherson's model of heuristics may be considered as a refinement of Polya's model. Each heuristic can be accounted for by a combination of procedures from Polya's work. For example, the heuristic of invariation-fixation can be described in terms of Polya's list. The heuristic of fixation is the renaming of a variable as a constant and then attempting to solve the new problem and study it's solution in order to gain some insight into the given problem. In terms of Polya's model, the following questions would have to be asked: If you cannot solve the proposed problem try to solve some related problem first. Could you change the unknown or the data (rename a variable as a constant)? Carry out your plan (on the new problem) of the solution. Can you use the result, or the method, for some other problem (in this case the original one)? One of the major advantages of using MacPherson's model to analyze problem solving may be the relatively small number of heuristics, (twelve), as compared to Polya's checklist of thirty-six questions.

Models of problem solving vary in complexity from the 3 or 4 steps of "formal analysis" to those of Wickelgren, Schwieger, and Polya who attempt detailed descriptions of procedures used in mathematical problem solving. Of these models, Polya's has been used in recent research to analyze problem solving. The model was modified by Kilpatrick, with other modifications by Webb and Kantowski, and used at different grade levels.

Wickelgren's model seems inappropriate to use for analyzing problem solving. His model includes some procedures similar to Polya's but many of them are overall plans for attacking problems rather than problem solving procedures. Many of the procedures in Schwieger's model appear to overlap and are not clearly defined enough to use in analyzing problem solving.

This review of the literature has raised several important questions in terms of the usability of models of mathematical problem solving for analyzing problem solving. These questions are:

1. Are the procedures from the model used by the subjects?
2. Do the procedures from the model describe the problem solving process, i.e., are the procedures in the model broad enough to account for all of the processes used by the subject and yet not so broad that no real distinction can be made?
3. Are the procedures from the model defined clearly enough to be coded reliably?

In terms of the usability of MacPherson's model to analyze mathematical problem solving, an attempt was made to answer each of these questions.

Field Independence

In the last decade, psychologists have attempted to revive and extend the study of dominant patterns or modes of cognitive behavior. One speaks of these dominant patterns or modes as cognitive style (Spitler, 1970, p. 1) and classifies together individuals who typically use the same style. The study of cognitive styles, which began in observations of how individuals perceive and categorize information, has gradually broadened to include the operation of these styles in intellectual tasks such as problem-solving (Kilpatrick, 1967, pp. 20-21).

Modern experimental work on cognitive style began with Witkin and his colleagues (Witkin, Dyk, Faterson, Goodenough, and Karp, 1962), who found that people differ in the way they orient themselves in space. When a subject is seated in a tilted chair and the room is tilted independently of the chair, the conflicting visual cues and bodily sensations often make it difficult for him to bring either his body or the room into a vertical position. Subjects seem to make consistent errors on this task and their score is correlated with their performance on the Embedded Figures Test (EFT), in which they are asked to find a particular simple figure within a larger complex design (Witkin, Ch. 4).

A subject whose score is above the median for a given sample is said to be field independent and those whose score is below the median are said to be field dependent. Field independence-dependence is an index of perceptual components. Field independence represents the ability to overcome an embedding context and perceive an item as distinct from its background.

Research indicates (Witkin, 1962, Walsh, 1974) that the construct of field independence is not significantly correlated with IQ level. Witkin found significant correlation between the construct of field independence and a group of subtests of WISC (Wechsler Intelligence Scale for Children); Block Design, Object Assembly and Picture Completion; however, there was non-significant correlation between the construct and verbal comprehension and arithmetic subtest scores of WISC. Hence "...intelligence test scores cannot be interpreted to mean field-independent children are of generally superior intelligence [Witkin, 1962, p. 70]."

Research has shown that field independence is related to mathematical achievement. In a study of 100 grade nine boys, Rosenfeld (1958) examined the relationship of mathematical ability as measured on scores of the Progressive Achievement Test and performance on the EFT. Significant Correlations of -.56, -.32, -.64, ($P < .05$), were found relating field independent scores on the EFT (used time as subject's score: field independent below the median, field dependent above the median) and the total mathematics score, mathematical reasoning score, and the mathematical fundamentals score, respectively. The poor mathematics students were more field dependent than the good mathematics students.

Few studies have investigated the relationship between field independence and problem solving in mathematics. However, the field independent construct has been used in studying problem solving in other areas. Saarni (1972) investigated differences in problem solving as a function of cognitive style. She proposed that Piaget's development of logical thinking would provide an overall framework for understanding problem solving performance and that Witkin's field independence would provide information about individual differences in problem solving behavior within each Piagetian developmental level. Sixty-four students, eight male and eight female, per grade were randomly selected from grades six through nine. Two Productive Thinking problems, "The Missing Jewel" and "The Old Black House", constituted the problem solving tasks. The subjects' performance on each problem was scored in four categories: (a) number of relevant clues cited (puzzling facts in second problem), (b) number of correct analytic choices made, (c) number of plausible ideas generated for solution, and (d) score of 1 to 5 for speed and adequacy of attainment of the correct solution. The nesting of field independence within Piagetian levels did not yield any significant ($P < .05$) differences in problem solving performance. Saarni concluded that:

> The construct field independence appears to have doubtful implications for complex problem solving performance. The analysis indicate that field independence within each Piagetian level does not affect complex, multi-step problem solving performance as manifested in the Productive Thinking problems. This does not invalidate the role field independence might have in determining performance on problems which are more perceptually bound and/or relatively non-verbal [p.22].

Farr (1968) conducted a study to determine whether field independence and problem difficulty were related to problem solving performance on organized and disorganized reorganization-type problems. Two hundred ninety eight students between the ages of 18 and 24, enrolled in undergraduate education courses were used in the study. The students were asked to solve anagram problems representing verbal, and match stick problems representing non-verbal, reorganization-type problems. Each type of problem was presented in organized and disorganized forms, with easy and difficult problems in each form. She found that when mathematics aptitude was held constant, field independent students generally received significantly higher scores than field dependent students on non-verbal but not on verbal problems, regardless of problem organization or difficulty.

However, both Walsh (1974) and Cooperman (1974) found field independent students performed significantly better than field dependent students in solving anagram problems of moderate difficulty. The students in Walsh's study were 12 and 13 years old while those in Cooperman's study were 10 years old.

The results of research involving field independence and problem solving in a non-mathematical setting are mixed. Both Saarni and Farr found non-significant differences between field independence and verbal problem solving. However, studies by Walsh and Cooperman found field independent students performed significantly better than field dependent subjects on verbal problems. The results of Farr's study also indicate that with mathematical ability held constant, field independence is significantly related to problem solving in a non-verbal setting.

Two research studies have examined the relationship of field independence and a subject's ability to change his mode of attack in solving non-mathematical problems.

The Einstellung problems of Luchins (1942) have been used to study the effects of the "set" of a subject upon his problem solving behavior. A set is defined as "the tendency of an individual to perseverate in a given mode of attack [Buetzkow, 1951, p. 219]." The problems require an individual to obtain a given quantity of liquid by using various combinations of three jars, A, B, and C, with given volumes. The initial five problems which all have the same solution, A + B - 2C, establish a "set". A measure of the strength of the "set" is contained in the succeeding two problems which can be solved by a simple direct procedure, A - C, or by the complex manner used to solve the first five problems. Finally, the terminal problem which can only be solved by the simpler method, is used as a measure of ability to overcome the "set". Guetzkow (1951) carried out a study using Luchins' Einstellung problems. He divided his subjects into two groups according to their set-breaking ability. The set breakers were those who adopted the set by using the (A + B - 2C) method of solution, rather than the (A - C) method on the critical problems, but who were able to break the set on the terminal problem. The non-set breakers were those subjects who were unable to break the set on the terminal problem. He found the set-breakers did significantly better than the non-set-breakers on Thurstone's Gottschaldt Figures Test.[2] Guetzkow also found significant relationship between Thurstone's Gottschaldt Figures Test and the time required to solve the terminal problem.

Goodman (1960), conducted a similar study using college students. These students received the EFT, Thurstone's Gottschaldt Figures Test, and the Einstellung test. No significant difference in performance of either of the perceptual tests were found between students who solved the critical Einstellung problems by the short method and those who solved them by the long method. However, significant correlations were found between both the EFT and Thurstone's Gottschaldt Figures Test and the time required to solve the terminal Einstellung problem.

These results, confirming and extending the findings of Guetzkow, indicate that set-breaking ability in the Einstellung situation is related to field independence.

A multivariate design was used by Dodson (1972) to test the relevance

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[2]Witkin's Embedded Figures Test was adapted from Thurstone's Gottschaldt Figures Test. Color was added to Gottschaldt's black and white outline complex figures to make them more difficult. Studies carried on in the 1950's have all shown significant correlations between these two tests (See Witkin 1962, p. 40).

of 77 concomitant variables for the ability to solve insightful mathematics problems and to determine which of these variables discriminate best among ability groups. Insightful problems are problems which cannot be solved by simple recall from memory or standard computational algorithms, nor does the solution depend on a special trick (Dodson, 1973, p. 3).

A random sample of 1123 grade eleven students was selected from those students participating in NLSMA who were currently enrolled in mathematics. All of the data for each subject were obtained from NLSMA. The students were placed in one of six ability groups depending upon their score on a test of insightful mathematics problem solving.

The 77 descriptor variables were classified into five major categories: (1) mathematics aptitude and achievement variables, (2) psychological variables (e. g. attitudes, anxiety, and cognitive factors), (3) teacher background variables, (4) school and community variables and (5) mathematics curriculum variables.

Dodson found the mathematical achievement variables to be the strongest discriminators among ability groups and that the cognitive variables were second strongest. Of the cognitive variables, Dodson found the best insightful mathematics problem solvers tended to have the highest scores on the reasoning tests - verbal and logical reasoning as well as the numerical reasoning test.

Field independence was found to discriminate as well ($P < .001$) among the ability groups as did the poorest reasoning variables and was included in the composite list of the strongest characteristics of a successful insightful mathematics problem solver (Dodson, 1973, p. 122).

Analyzing the problem solving procedures of 40 second year high school algebra students, Webb (1974) supported the finding of Dodson that mathematical achievement is the strongest component in accounting for problem solving ability. Using regression analysis, Webb found that mathematical achievement accounts for 50% of the variance in the total scores from his problem solving inventory. With mathematical achievement, verbal reasoning and negative anxiety entered in the regression equation first, field independence did not add significantly to the amount of variance on the total score already accounted for.

Research findings involving the field independent construct and problem solving in non-mathematical settings are mixed, but in general favor the field independent subject. Field independent students are more able to overcome a problem solving set than their field dependent counterparts. This would seem to be an important characteristic of a good problem solver in mathematics, that is, when solving a problem if a mode of attack is not leading to a solution, rather than continue to use this mode, change the procedures being used and try a different method of attacking the problem.

Field independence has been shown to relate significantly to mathematical reasoning and insightful problem solving. The characteristics associated with a field independent student are: (1) ability to resist distraction, (2) ability to identify the critical elements of a problem, (3) ability to separate parts from the whole and recombine them to form a new whole, (4) ability to remain independent of irrelevant elements, and (5) ability to overcome a problem set (Witkin, 1962; Romberg and Wilson, 1969).

In conclusion, the literature from two major areas has influenced the design of the present study. The literature on models of problem solving has raised three important questions in terms of the usability of models of mathematical problem solving for analyzing problem solving behavior.

1. Are the procedures from the model used by subjects in solving mathematical problems?
2. Can the procedures from the model be coded reliably?
3. Do the procedures from the model describe the problem solving process?

In terms of the usability of MacPherson's model for analyzing problem solving an attempt was made to answer each of these questions.

The characteristics associated with a field independent student indicate that these students are better problem solvers than the field dependent students. In terms of the outcome of this study, the field independent student is expected to use a greater variety of heuristics from MacPherson's model, to be more willing to change procedures if he is not being successful in solving a problem, and to obtain more correct solutions than the field dependent student.

Chapter III

PROCEDURES

Two major considerations in analyzing problem solving behavior are the choice of subjects and the choice of problems. If the subjects are too widely diverse in their background and abilities, there is little hope of observing any common patterns of thinking. The choice of the problem material used must be made with the subjects in mind. If the problems are too hard there will be little problem solving behavior observed. On the other hand, if the problems are too easy, they may not elicit any problem solving behavior at all (Kilpatrick, 1967, p. 32).

Subjects

The subjects selected for this study were all completing the eleventh grade academic mathematics program. The sample consisted of boys and girls, of average ability for those students enrolled in this program, who were participating in a one semester algebra course. A popular modern algebra and trigonometry text was used for this course. At the time of the problem solving interviews the students were studying the chapter related to the quadratic function. One of the aims of the study was to assess the heuristics used by students in solving word problems. Since a large portion of the content of second year algebra deals with word problems, students who have completed the course should have a sufficient background for dealing with word problems.

An average IQ range was selected in order to control the IQ variable and hopefully still obtain a fairly wide range of EFT scores. A pilot study had shown that students in this ability range were not unduly threatened by the task of solving difficult mathematical problems in an interview situation.

To determine those students of average ability in the academic mathematical program, a sample of 150 students was randomly drawn from the Algebra II classes of the schools involved in the study. The mean IQ score on the California Test of Mental Maturity for this sample was 119.8 with a standard deviation of 10.4. A student was defined to be of average ability if his IQ was within one-half standard deviation of the mean.

A total of 40 students, in the IQ range 115 to 125, was randomly selected from 14 Algebra II classes in three senior secondary schools (grades 11 and 12) in the greater Vancouver area. Three students were selected from each of twelve classes and two students from each of the other two classes. The classes were taught by ten different teachers, no teacher having more than two classes.

The mean IQ for the students in the study was 119.5 with a standard deviation of 10.2. Ages of the students at the time of the interview ranged from 16 years, 4 months, to 17 years, 4 months, with a median age of 16 years, 11 months. The sample contained 22 boys and 18 girls.

Pilot Study

The pilot study was conducted from December, 1973 to March, 1974, in a senior high school in the same general area as the schools used in the main study. The initial phase of the pilot study involved the use of written problem sets to aid in the selection of the problems to be used in the study. This phase was followed by a series of pilot interviews. The major aims of the pilot study were:

1. Determine if eleventh grade students of average ability could respond well to the interview procedure
2. Select the problems to be used in the study
3. Modify Kilpatrick's coding system or develop a new coding system using MacPherson's model for problem solving
4. Develop an interview format
5. Practice the interviewing and coding techniques

Twelve mathematical problems selected from the areas of number theory, geometry, and algebra were used in the initial testing of the pilot study. The problems were stated in both the real world and math world setting. These problems, selected on the basis of their potential to elicit the use of certain heuristics, were those that the experimenter felt could be solved by approximately half of the students participating in the study. The problems were then randomly divided into sets of three in the same setting and administered as written examinations through December, 1973, and January, 1974, to 90 students who would complete the Algebra II course in February. Throughout the period, 11 new problems were added to the list and many others were deleted because they proved to be too difficult. The students were asked to work on a problem as long as they wished and not to erase anything. Each student was asked to record the amount of time he spent on each problem. Appendix A contains the problems used in the pilot study.

Fifteen problems were selected from this list as possible candidates for the main study. Over the next three months, twelve students were interviewed using from 3 to 8 of the 15 problems. Each interview was designed to generate as much observable problem solving behavior as possible. A room was provided at the high school for the interview sessions. Subjects were told that the session would be tape recorded and they should think aloud while working on the problems. The students were also instructed not to erase anything and if a diagram was to be modified, to draw a new one. All instructions, except the problem statements, were communicated verbally by the interviewer. The problems were typewritten and presented to the students one at a time.

It was determined that grade eleven students of average ability are not unduly threatened by the task of solving difficult mathematical problems in the interview situation. Also, they are able to verbalize their thoughts, especially if they have some success and understand what the interviewer expects. The interview format was revised to take this into account.

The need for several practice problems to acclimate the subjects to solving problems aloud became evident in the pilot study when the students involved became accustomed to thinking aloud only after attempting one or two problems.

The list of problems was reduced to 5 and two practice problems were added (see Appendix B). This allowed the subject sufficient time to work on the problems and still complete the interview in one session. Most students were willing to work for up to $2\frac{1}{2}$ hours if given a short rest period between problems.

Four of the problems were selected from the pilot study because they did elicit the use of different heuristics and the fifth problem was added in an effort to obtain some use of the heuristic of symmetry (problem #1, Appendix B). However, all five problems could be solved in a number of different ways.

A final four students were interviewed using the 7 problems, two students doing problems in each of the settings, as a check on the problems and the interview method. The protocols were also analyzed using the modified coding system.

The Coding System

One of the main reasons for spacing the pilot interview sessions over the three month period was to allow ample time to evaluate and modify the coding system. The initial coding forms were quite similar to Kilpatrick's (1967, pp. 50-53). Both the checklist and the process sequence had been modified to take into account the procedures from MacPherson's model.

After each interview period during the pilot study, the investigator used both the tape recorded protocols as well as all written work to help analyze the subjects' problem solving behavior. The problem solving behaviors were analyzed in terms of process-sequence coding symbols and other events were recorded by checking off appropriate categories on the checklist.

During the attempts to apply the coding system to protocols from the pilot study, difficulties and shortcomings became apparent. Many of the categories on the checklist were not used. In some instances, the process sequence became very long and complex with some procedures nested in others. More importantly, it became evident that the position of the procedure in the sequence of the problem solving process was more important than its frequency of occurance.

The final coding system, used in the analysis of this study, combined both the check list and coding sequence into a complete sequential coding system. The use of a matrix provided the researcher with a convenient device for recording the procedures obtained from the tape recorded protocols. The coding matrix, (see Figure 4), accounts for different

Reading problem																											
Request defn. of terms																											
Recall same problem																											
Recall related problem																											
Recall problem type																											
Recall related fact																											
Draw diagram																											
Modify diagram																											
Identify variable																											
Setting up equations																											
Algorithms-algebraic																											
Algorithms-arithmetic																											
Guessing																											
Smoothing																											
Analysis																											
Templation																											
Cases-all																											
Cases-random																											
Cases-systematic																											
Cases-critical																											
Cases-sequential																											
Deduction																											
Inverse deduction																											
Invariation																											
Analogy																											
Symmetry																											
Obtain solution																											
Checking part																											
Checking solution																											
by subst. in equation																											
by retracing steps																											
by reasnble/realistic																											
uncodable																											
Exp. concern-method																											
Exp. concren-algorithm																											
Exp. concern-equation																											
Exp. concern-solution																											
Work stopped-soln.																											
Work stopped-no soln.																											

FIGURE 4

THE CODING FORM
(Reduced by 33%)

procedures used simultaneously; procedures which are nested in others, such as the use of different algorithms and diagrams, which might take place while a subject is using the heuristic of random cases; and the sequential order of all procedures. At a glance one can tell the methods used by a subject, the order in which he used them, and how successful he was.

To take into account the sequence of events used in solving a problem, see Figure 3, p. 19, the procedures were grouped in the coding matrix as follows:

Preparation

Reading problem
Request definition of terms

Sieve

Recall same problem
Recall related problem
Recall problem type
Recall related fact
Draw diagram
Modify diagram
Identify variable
Setting up equations
Algorithms-algebraic
Algorithms-arithmetic
Guessing

Heuristics

Smoothing
Analysis
Templation
Cases-all
Cases-random
Cases-systematic
Cases-critical
Cases-sequential
Deduction
Inverse deduction
Invariation
Analogy
Symmetry

Solution

Obtain solution

Reasonableness

Checking part
Checking solution
by substituting in equations
by retracing steps
by reasonable/realistic
uncodable

Concern

Express concern about method
Express concern about algorithm
Express concern about equation
Express concern about solution

Work Stopped

Work stopped-solution
Work stopped-no solution

As the coder identifies these processes from a subject's protocol, he enters a check ($\surd$) in the appropriate row and first empty column. If the process is from the heuristic category or the sieve (except recall of same problem related problem, problem type) he enters either a 1, 2, or 3, depending upon the outcome:

1 Incomplete

2 Incorrect

3 Correct

For example, a 2 in the algorithm-algebraic row indicates an error in the use of the algorithm, whereas, a 2 for one of the heuristics, analysis, for example, indicates the outcome of the problem after the use of analysis is incorrect, not that the subject committed an error in the heuristic itself.

If two or more procedures occur simultaneously then a check or number is entered in the same column for each one. If a procedure is carried on longer than a column, it is enclosed in a box with the outcome entered in the last column enclosed. For example, suppose a subject is using random cases and considers three cases along with using algorithm, checking part of his work and recalling related fact. In this instance the "block" for random cases would cover ten columns. See the example in Figure 5.

Most of the time the subject, in checking his work or solution, did so by substituting in an equation, retracing steps or checking to see if his result was reasonable, or realistic. If a subject was checking his solution, a check would go in the appropriate column in the checking solution row and a check or numbers in the row which indicated the procedure he used.

According to MacPherson's model, a probability (amount of concern) is attached to the outcome from the pattern box, application of an algorithm, and the solution obtained by the problem solver. To discern some feeling for these probabilities the concern expressed by a subject was coded in one of the four categories.

As an illustration of the coding system, two sample protocols follow, with the coded forms given in Figures 5 and 6.

Problem #3. A rectangular lawn is to be formed so that one side of a barn serves as one side of the rectangle. You have 300 feet of wire to enclose the other three sides. What are the dimensions of the rectangle if the area is to be a maximum size? Why does your solution give the maximum area?

Protocol	Column #
(Student #36, problem #3) A rectangular lawn is to be formed so that one side of a barn serves as one side of the rectangle. You have 300 feet of wire to enclose the other three sides. What are the dimensions of the rectangle if the area is to be of maximum size? Why does your solution give the maximum area? I don't know. O.K., I've got a rectangular lawn...is to be formed so that one side of the barn serves as one side of the lawn. O.K.,	1

let's draw a rectangle...O.K. barn is here. Covers 2
that whole side. You have 300 feet of wire to enclose
the other three sides. O.K. Let's see - a rectangular
lawn is to be formed so that...I don't know 3
how big this rectangle has to be, but I have to get
the most size out of it. That's 300 feet. O.K. -
Let's see, no equation for this one. I'll scrap the
equation. O.K., Now, if it's going to need more 4
area for my barn, I think I'd use somewhat of a
square, or as close to a square as possible. O.K., 5
How big is my barn? Let's imagine a 50 foot length 6
across there, 50 down here - right, O.K., 50 and 50
from there is 250 so 125 feet long. Now am I using 7
as much area as I could? Now, let's see, area is a
125 times 50. O.K., so 0 is 5 is 5 is 2, 2 over 8
there is 12, is 1 over the 6, 6250 square feet. O.K., 9
if I had a 100 there, so 100, 100, 100, O.K., so 10
100 times 100 is 0, 0, 0 is 10,000 square feet. How am 11
I going to get a rectangular lawn? How rectangular
is rectangular? Now, that's the most I can have for
a square. Now, 0,0,1 that's right. That would be 12
the maximum for a square. I want a rectangle. O.K.,
I'll give my barn 90 feet, O,K., 90 and 90 is 180, 13
O.K., that's 120, that's 60 times 60, 60 times 90 14
is 0, nope, wrong. A...I don't know how I'm going 15
to do this... 16

Problem #4: What is the longest piece of metal rod which can be placed in a box of dimensions 3 inches by 4 inches by 12 inches?

Protocol Column #

(Student #24, problem #4) What is the longest piece of
metal rod which can be placed in a box of dimensions 3 1
inches by 4 inches by 12 inches? It's not hard to see
the longest piece of metal rod will be from opposite
corners, from the bottom opposite corner of one to the
top opposite corner of the other. Now the question is 2
how long that is? I know its longer than 12. Let's see
if I can work out some kind of triangle here. The two 3
ends of the boxes like the ends of the box are 3 by 4,
now that would mean that, A, I'm trying to form 2 tri-
angles here to find out, so I can find out, adjacent, or
complementary, or corresponding parts so I can get that
long rod as a corresponding part in this box. Now, if I 4
had a 3 by 4 inch box, I'm assuming this box is rectan-
gular in shape. That, I'm just putting in the angles 5
here. If there is right angles in the box, side-angle 6
side, I forgot how to do that, oh yeah, I want to find 7
out in that box, I want to find out right on the two
ends, a line going right through the middle to the oppo-
site sides so that one side would connect with the line
I have put as the imaginary rod and therefore form a
triangle, well, I said the altitude of the box is 4
inches, and the width is 3, therefore, I formed a

triangle with two legs of 3 inches and 4 inches and a
hypotenuse that I do not know, and since it is a right
angle triangle, since I'm assuming its a rectangle, a, 8
4 squared is equal to the hypotenuse squared, I'll say
H squared, but should be C squared that the old equation.
So that's 9 plus 16 is equal to 25 and that is 9
5 squared. Therefore, 5 is equal to the length of that
line, 5 inches. Now, I've found out that one side of 10
my imaginary triangle is 5 inches. Now I know the other
side is 12 inches because of the thing. Again, I am
assuming I have a right angle triangle, because of the
angles of the box. Therefore, all I would do is 5 11
squared plus 12 squared is equal to C squared. 5 12
squared is 25 and 12 squared is 144 is equal to 169.
Now, I know 169 is 13 squared. Therefore, 13 inches 13
must be the length of that rod. Let me think. 13
inches I'm saying. That means I've formed a triangle
from one part of that box. I'll say its 13 inches. It
must be because, like I tried to form a whole triangle
and trying to find the sides of the triangle so that
I could assume what the last side is by mathematical
equalities...So 25, 5 squared, that one side plus 12
squared which is 144 is equal to 169. And 169 is
13 squared. Therefore the answer is 13, 13 inches. 14, 15

A coding form summary sheet was also used to summarize the major parts from the coding form and to count errors and procedures used. The number of cycles and changes was also noted here. A cycle occurs each time the subject attempts to solve the problem. These are determined from the subject's tape recorded protocol for the problem either by the subject himself, indicating that he is going to start the problem over or from the sequence of procedures used. Usually the subject will read the problem each time he cycles. If a subject alters his method of attack, either by changing the core procedures or heuristics he is using or by changing his overall plan, then a change occurs. A full explanation of the coding system is given in Appendix C.

The Interview Procedure

The Embedded Figures Test(EFT), was administered during the last half of April, 1974, and the problem solving interviews were held

Reading problem	√		√																								
Request defn. of terms																											
Recall same problem																											
Recall related problem																											
Recall problem type																											
Recall related fact					2			3																			
Draw diagram		√																									
Modify diagram																											
Identify variable																											
Setting up equations																											
Algorithms-algebraic																											
Algorithms-arithmetic							3		3		3			3	1												
Guessing																											
Smoothing		3																									
Analysis																											
Templation				3																							
Cases-all																											
Cases-random						√				√			√			1											
Cases-systematic																											
Cases-critical																											
Cases-sequential																											
Deduction																											
Inverse deduction																											
Invariation																											
Analogy																											
Symmetry																											
Obtain solution																											
Checking part												√															
Checking solution																											
by subst. in equation																											
by retracing steps												√															
by reasnble/realistic																											
uncodable																											
Exp. concern-method																											
Exp. concren-algorithm																											
Exp. concern-equation																											
Exp. concern-solution																											
Work stopped-soln.																											
Work stopped-no soln.																	√										

FIGURE 5

EXAMPLE OF CODING SYSTEM
STUDENT # 36, PROBLEM # 3
(Reduced by 33%)

Reading problem	✓																										
Request defn. of terms																											
Recall same problem																											
Recall related problem																											
Recall problem type																											
Recall related fact		3																									
Draw diagram					✓						✓																
Modify diagram						✓																					
Identify variable																											
Setting up equations									3			3															
Algorithms-algebraic										3			3														
Algorithms-arithmetic																											
Guessing																											
Smoothing					3						3																
Analysis																											
Templation			3				2	3																			
Cases-all																											
Cases-random																											
Cases-systematic																											
Cases-critical																											
Cases-sequential																											
Deduction																											
Inverse deduction				3																							
Invariation																											
Analogy																											
Symmetry																											
Obtain solution													✓														
Checking part																											
Checking solution														✓													
by subst. in equation																											
by retracing steps														✓													
by reasnble/realistic																											
uncodable																											
Exp. concern-method																											
Exp. concren-algorithm																											
Exp. concern-equation																											
Exp. concern-solution																											
Work stopped-soln.															✓												
Work stopped-no soln.																											

FIGURE 6

EXAMPLE OF CODING SYSTEM
STUDENT # 24, PROBLEM # 4
(Reduced by 33%)

during May and June, 1974, in the senior secondary schools the subjects attended. Each school made a room available for both testing periods. All EFT's and interviews were conducted by the experimenter.

The subjects were told that at a later date they would spend about $2\frac{1}{2}$ hours solving mathematical problems and thinking aloud while they worked on them. It was emphasized that the purpose was to learn more about how grade eleven students solved problems and was not to make an individual diagnosis or evaluation. It was also emphasized to each individual that the outcome of the interview had nothing to do with his grade in Algebra II nor would the information be made available to his mathematics teacher. The subjects were told that the interviews would be audio tape recorded to assist the interviewer in determining what procedures were used.

The EFT was administered individually to the subjects in a 50 minute period. The test material consists of a set of 12 cards with complex figures and a set of 8 cards with simple figures. The task is to find a given simple figure which is embedded in a complex pattern. The subject is asked to describe the complex figure in any way he wishes. Then the simple figure is shown to the subject and he is asked to find it in the complex design. When he indicates he has found it, the interviewer stops timing and the subject outlines the figure with a stylus. If he is correct, the time is noted and the next problem is presented. If he is incorrect the subject is told he is wrong and may continue to look for the sample figure. A maximum of 180 seconds is allowed for each of the 12 problems (See Witkin, Oltman, Raskin, and Karp, 1971, pp. 16, 17).

After all 40 subjects had taken the EFT, the scores were rank ordered and paired. Each subject from a pair was randomly assigned to one of two groups: those working problems in the real world setting and those working problems in the math world setting.

The problem sets along with the instructions that were given to the students can be found in Appendix B. The problems were ordered so that each of them appeared in positions 1 to 5 exactly 4 times and each problem was preceded by the same problem 4 times and followed the same problem 4 times. This gave a total of twenty different orderings of the five problems. The orderings were randomly assigned to the twenty pairs of subjects. Each student was given the same two warm up problems (except for setting) in the same order.

The problem solving interviews were held from May 6, to June 7, 1974. The subjects were dismissed from classes for either the morning or afternoon. All subjects from the same mathematics class were interviewed in consecutive sessions in order to minimize discussion between the subjects. No interviews were held on Friday afternoons.

The interviews were tape recorded with the microphone in full view of the subject. The subject was asked to do his thinking aloud, to say everything that came to mind. He was also instructed to verbalize all writing and diagrams. Whenever the subject fell silent for ten seconds or more, the interviewer would ask, "What are you thinking now?" or "Can you tell me what you are thinking about?" or some similar probe.

The subjects were given the sheet of directions and asked to read them. The importance of thinking aloud was stressed. The subject was told that he was free to work the problems any way he saw fit. It was indicated that the only kind of question the interviewer would answer would be to define any terms used in the statement of the problem that were unfamiliar to the subject. It was also stated that any discussion of the problem with the interviewer, including the correctness of any solutions obtained by the subjects, would take place after the interview was over. Once a subject started work on a problem he was asked to continue to work on the problem and not to return to it at a later time.

After the instructions were carefully presented to the subject, he was given the first practice problem to solve. If the subject had trouble expressing himself or fell silent too often during the time he was working on this problem, the interviewer would remind the subject of the purpose of the interview and give some instructions and examples. For example, the interviewer might work part of the problem thinking aloud, or if the subject had been writing equations, the interviewer would go back over the work and verbalize what the subject might have been thinking. By the end of the second problem, all the subjects were talking most of the time and indicating what they were doing.

As soon as the subject had finished practicing on the two sample problems and had discussed any remaining questions on format, he was given the first of the five problems. Each problem was typewritten at the top of the page with space underneath for the subject's work. Extra paper was also available.

While the subject was engaged in working the given problem, the interviewer observed the student and if the subject fell silent, he was encouraged to vocalize as much of his thinking as possible. When each problem was completed, the subject had the option of either stopping temporarily to rest or proceding to the next problem. Most subjects preferred to work continuously on the problem set.

Once the interview session was completed and the subject had left, the interviewer played the tape recording back, matching the protocol with the written work. This was done so the two could be combined when the coding took place.

The Coding Procedure

As a final check on the coding system and for practice coding, the writer coded 40 problems chosen from the main study. These included all five problems taken from both the real world and math world settings. At the time it was felt that the coding system was indeed accounting for the procedures used by the subjects in the various problems.

The five problems were randomly ordered as follows: 3, 1, 4, 5, 2. Then the protocols of all 40 subjects for problem number 3 were randomly assigned. These protocols were then coded by the writer using the final coding form. The procedure was repeated for the other 4 problems, giving a total of 200 protocols. Each problem was coded on a separate coding form.

The coding was done using the tape recorded protocols and the written work of each subject.

To assess the reliability of the coding, both intercoder reliability and intracoder reliability of the coding were used. A second coder had been trained during the pilot study. She and the writer spent nine hours working together coding sixteen problems. The second coder then coded another 14 problems alone. Like the writer, she was very familiar with the heuristics used in the study and had had several years experience teaching secondary mathematics.

For each of the 5 problems, 4 protocols were selected at random. These were then coded by both the writer and the second coder. The results of these codings were compared to the writer's original codings of the same protocols. Four types of coding errors were identified. These are illustrated in Figure 7 and described here:

1) A Blank occurs when a procedure is coded by one coder and not the other. The assumption is that this is the only error. That is, in the example coder A missed the procedure of identifying a variable which was coded by coder B.

2) Calling a procedure by a different name occurs when a procedure is coded differently by the two coders.

3) Coding procedures in different order occurs when the order of two or more successive procedures is reversed by one of the coders. It is assumed that the only error made is the reversal of the procedures. In the example, recall of related fact and templation were reversed.

4) The last error, cases, occurs when one coder indicates the use of the heuristics of cases, stops, and then codes it starting again and the other coder codes it as a continuous process.

A summary of the procedures coded differently, under each of the categories, by the two coders, is found in Table 1 and the code-recode by the writer is found in Table 2.

As a measure of reliability, the percentage of items coded identically was calculated using a formula derived from McGrew (1971, p. 24). The formula used in this study is a more conservative measure than that of McGrew because it takes into account the number of items coded identically by both coders only once.

Let the number of items coded by A which agree with those coded by B equal a, and the remainder equal a'.

Then a + a' = total coded by A.

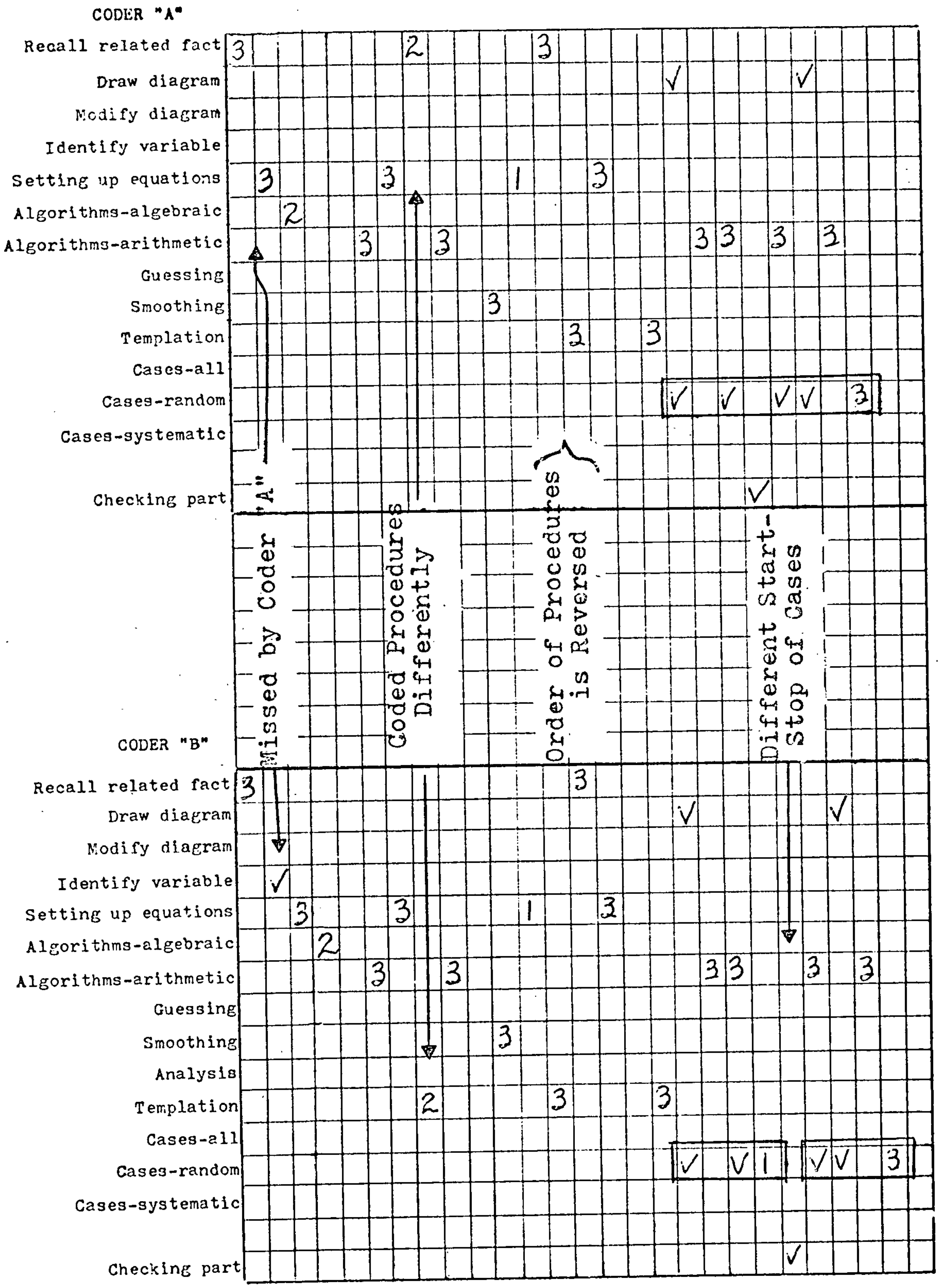


FIGURE 7

EXAMPLES OF CODER ERROR
(Reduced by 33%)

TABLE 1

TYPES AND NUMBERS OF DISAGREEMENTS
BETWEEN TWO DIFFERENT CODERS

FIRST CODER	SECOND CODER
BLANKS	
Reading Problem -2	Reading Problem -3
Modify Diagram -3	Recall Related Fact -2
Identify Variable -2	Modify Diagram -4
Setting up Equations -4	Identify Variable -2
Guessing -3	Setting up Equations -1
Templation -2	Guessing -4
Analysis -1	Smoothing -2
Concern -2	Templation -1
	Checking Part -2
	Concern -3
CODING PROCESS DIFFERENTLY	
Templation	Recall Related Fact -2
Setting up Equations	Algebraic Algorithms -1
Diagram	Modify Diagram -1
Checking Part	Concern -2
Checking Part	Cases(All) -1
Templation	Cases(Random) -1
Arithmetic Algorithms	Algebraic Algorithms -2
ORDER REVERSED	
Templation	Recall Related Fact -2
Modify Diagram - Smoothing	Cases(Start) -1
CASES	
Continuous	Start-Stop -3

TABLE 2

TYPES AND NUMBERS OF DISAGREEMENTS
BETWEEN CODE AND RECODE

FIRST CODE	RECODE
BLANKS	
Reading Problem -3	Reading Problem -1
Modify Diagram -2	Recall Problem Type -1
Identify Variable -1	Recall Related Fact -4
Setting up Equations -2	Identify Variable -2
Arithmetic Algorithms -1	Setting up Equations -1
Guessing -1	Arithmetic Algorithms -3
Checking Part -2	Guessing -2
Concern -1	Templation -1
	Checking Part -2
	Concern -3
CODING PROCESS DIFFERENTLY	
Templation	Recall Related Fact -2
Obtain Solution	Checking Part -1
Checking Part	Arithmetic Algorithms -1
Recall Related Fact	Templation -1
Checking Part	Concern -1
ORDER REVERSED	
Smoothing	Diagram -1
Recall Related Fact	Templation -1
CASES	
Continuous	Start-Stop -1

Similarly, $a + b'$ = total coded by B.

Then $a + a' + b'$ = total number of different items coded by both A and B, call this sum T.

Then $a/T = y$, the percentage of items coded identically by both coders is the measure of reliability.

For intercoder reliability:

$a = 294, \quad a' = 40, \quad b' = 35, \quad T = 369$

The percent of items coded identically by two different coders is 80.

For intracoder reliability:

$a = 307, \quad a' = 27, \quad b' = 21, \quad T = 355$

The percent of items coded identically on the code/recode by the same coder is 86.

These reliabilities were felt to be sufficiently high. There are several important factors which may have contributed to the reliabilities being high. Both coders were very familiar with the second year algebra course, both having several years teaching experience at this level. Both coders were very familiar with the coding system and with the heuristics. In addition a great deal of time was spent on the coding itself.

MacPherson's model appears to be a reliable model to use for analyzing mathematical problem solving. It can be used effectively by a person who: (1) knows and understands the heuristics well, (2) knows the core and is familiar with the background of the subjects, and (3) is familiar with this coding system.

Chapter IV

ANALYSIS AND RESULTS

In this study two phases were involved in the analysis of the data. The first phase consisted of the analysis of the hypotheses stated in Chapter I. The hypotheses are restated below. Hypotheses 1-8 were tested statistically. A regression analysis procedure was designed to test hypotheses 1-6. Hypotheses 7 and 8 were tested using Pearson product-moment correlation coefficients. Hypotheses 9 and 10 were not tested statistically. A Flanders (See Amidon and Hough, 1967) type interaction matrix (a process matrix) was used to gain some insight into the sequence of procedures used by the students in this study. The second phase involved past hoc analysis of the data obtained from the coding system. Problem solving behaviors were examined in terms of their intercorrelations as well as with the use of regression analysis.

Research Hypotheses

H1: Problem context will not contribute to the number of times heuristics are used by a student.

H2: Problem context will not contribute to the number of different heuristics used by a student.

H3: Problem context will not contribute to the number of correct solutions obtained by a student.

H4: Field independence will not contribute to the number of times heuristics are used by a student.

H5: Field independence will not contribute to the number of different heuristics used by a student.

H6: Field independence will not contribute to the number of correct solutions obtained by a student.

H7: There is no correlation between the number of times heuristics are used by a student and the number of correct solutions obtained.

H8: There is no correlation between the number of different heuristics used by a student and the number of correct solutions he obtained.

H9: Problem context will not observably affect the sequence of procedures used by a subject.

H10: Field independence will not observably affect the sequence of procedures used by a subject.

Method of Analysis

The regression analysis approach employed for this study has been described by many writers (e.g. Kerlinger and Pedhazur, 1973). Walberg (1971) describes three advantages that regression analysis has over the conventional analysis of variance: (1) the use of continuous variables, (2) less data processing time, and (3) direct comprehensive estimates of the magnitude and significance of the independent variables effects on the dependent variable. The first and third of these are especially important to this study. With the use of continuous variables and a small sample size (N=40), precision in grouping scores into two or three levels could be lost and the sample size is too small to maintain cell sizes if more levels were added.

In an exploratory study such as this one, the third advantage appears to have direct application, that is, the use of the multiple regression coefficient: when squared it reveals directly how much variance in the dependent variable is associated with or accounted for by the independent variables.

A separate regression equation was defined for the analysis of each of the hypotheses 1 to 6. The regression analyses were performed using The Triangular Regression Package (TRIP) (Bjerring and Seagraves, 1974) available at the Computing Center of the University of British Columbia.

Although in some cases the relationship between the variables was logarithmic in nature, the range of the independent variables was so small that very little statistical difference (F- probabilities agreed in first two decimal places) was noted between a logarithmic and linear model. Hence a linear relationship was assumed between the variables.

Determining the appropriate significance level for meaningful interpretation must be considered carefully. This is an exploratory study using MacPherson's model to determine if any difference exists between groups of individuals in methods of attacking and solving mathematical problems. As an exploratory study, if differences do exist, a significance level which is not too stringent should be chosen otherwise important implications may be lost. A significance level of .10 has been chosen to test the hypotheses of this study. The probability of making a Type 1 error by rejecting the null hypothesis will be given for each statistical test.

Besides the hypothesized variables, two variables of interest to the writer were obtained from the protocols of the subjects. These are the number of times a subject attempted to begin a problem again (cycles) and the number of times he changed his method of attack. It appeared from the protocols that some subjects would attempt a problem a number of times and continue to do the same thing each time while others tried a variety of methods. Many of these changes involved the use of heuristics while others were more directly related to core.

In an attempt to gain some insight into the sequence of procedures used by the subject, a process matrix was developed from the coding systems. This process matrix contains six major categories for analyzing the procedures. They are: (1) Core, (2) Heuristics, (3) Solution,

(4) Checking, (5) Concern, and (6) Work stopped (See Figure 8).

The data are taken from the coding sheets and entered in the process matrix (Figures 8 and 9) in the appropriate row and column. For example: if a student reads the problem and then draws a diagram, one is added to the entry in the first row, fourth column. After drawing the diagram, if the student uses templation, one is added to the entry in the fourth row, thirteenth column. This system does not account for the procedures which are nested in others, such as the use of algorithms or diagrams the subject may be using while he is using random or systematic cases. Also, when a subject is checking either part of his work or his solution, the method he is using is not noted in the matrix.

Different parts of the matrix indicate different kinds of problem solving procedures. These areas are identified in Figure 8 and described as follows:

AREA A. This area indicates the problem solving procedures related to the core. All procedures are from core to core.

AREA B. The cells in this area include all of the core procedures which are followed by the use of a heuristic.

AREA C. This group of cells includes all of the core followed by the use of a checking procedure.

AREA D. This area indicates the concern which is shown after the use of a core procedure.

AREA E. This area consists of the heuristics which are followed by the use of a core procedure.

AREA F. This area indicates the use of a heuristic followed by a heuristic.

AREA G. These cells include the checking procedures which follow the use of a heuristic.

AREA H. This group of cells indicates the concern shown after the use of a heuristic.

AREA I. This area represents the core procedures used after a solution was obtained.

AREA J. This shows all of the heuristics used after a solution was

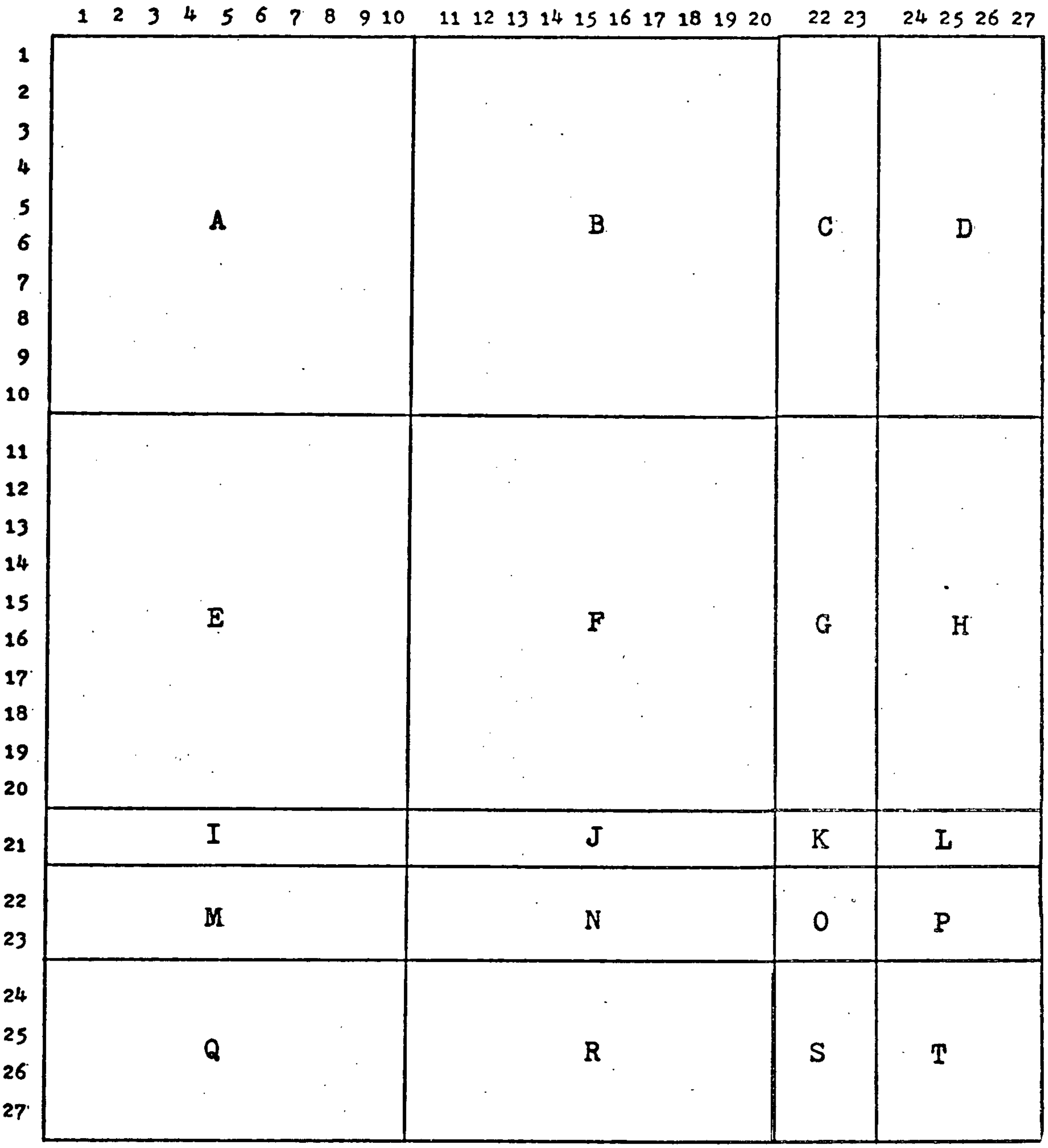


FIGURE 8

THE PROCESS MATRIX WITH
IDENIFIED AREAS

FIGURE 9

PROCESS MATRIX CATEGORIES

CATEGORIES	
Core	1. Reading the problem 2. Recall problem (includes: Recall same problem Recall related problem Recall problem type) 3. Recall related fact 4. Draws diagram 5. Modify diagram 6. Identify variable 7. Setting up equations 8. Algorithms-algebraic 9. Algorithms-arithmetic 10. Guessing
Heuristics	11. Smoothing 12. Analysis 13. Templation 14. All cases 15. Random cases 16. Systematic cases (includes: Systematic cases Sequential cases) 17. Critical cases 18. Deduction 19. Inverse deduction 20. Variation
Solution	21. Obtain solution
Checking	22. Checking part 23. Checking solution
Concern	24. Express concern about method 25. Express concern about algorithm 26. Express concern about equation 27. Express concern about solution
Work Stopped	28. Work stopped-solution 29. Work stopped-no solution

obtained.

AREA K. The cells in this area indicate the obtaining of a solution followed by a checking procedure.

AREA L. This area indicates the concern shown following a solution.

AREA M. These cells represent the use of a checking procedure followed by the use of core.

AREA N. This indicates the use of heuristics following a checking procedure.

AREA O. The cells in this area represent the use of two consecutive checking procedures.

AREA P. This area indicates the concern shown following a checking procedure.

AREA Q. These cells show the use of a core procedure following an expression of concern.

AREA R. The cells in this area show the heuristics used following some type of concern.

AREA S. These cells indicate concern followed by a checking procedure.

AREA T. These cells include the consecutive expression of concern.

The matrix indicates the amount and to some extent a pattern of the procedures used to solve mathematical problems according to the categories in the process matrix. The matrix also indicates the number of times that subjects change core and heuristics and the amount of procedures used within each category.

Students working problems in the mathematical world setting were coded 1 and those working problems in the real world setting were coded 2. "Total Heuristics" is the total number of all heuristics coded for all five problems. "Different Heuristics" is the number of different heuristics coded for all five problems and the correct solution is the total number of correct solutions for the five problems. The means and standard deviations of the hypothesized variables are given in Table 3. The reader should be aware of the distribution of some variables in interpreting the results from the regression analysis. Appendix D contains the histograms for several selected variables. Extreme scores may have influenced the

TABLE 3

MEANS AND STANDARD DEVIATIONS FOR
MEASURES OF HYPOTHESIZED VARIABLES (N = 40)

Variable	Mean	Standard Deviation
Field Independence	143.53	20.03
Total Heuristics	8.60	7.90
Different Heuristics	3.00	1.70
Cycles	10.37	4.02
Changes	1.85	2.30
Correct Solution	1.15	.89

statistical results, but because of the small sample size (N = 40) these scores were not deleted from the statistical analysis. Table 6 shows the intercorrelations between these variables.

It was noted in Chapter III that one of the reasons for choosing these particular problems was that they could be solved by about half of the subjects. The mean for the correct number of solutions is 1.15 which implies the problems may have been more difficult than the pilot study indicated. Only problems 2 and 5 were answered correctly by about half of the subjects.

The discussion of the results of the analysis is divided into three areas. The influence of problem context is examined first along with the related hypotheses. The second area deals with field independence and its related hypotheses. In the third area, the results of behaviors derived from the coding system are discussed. The section deals with the influence of both core and heuristic procedures on problem solving.

<u>Results of Analysis - Problem Context</u>

Problem context appears to have had little effect on the problem solving procedures used by the subjects in this study. The results of the regression analysis performed with problem context as the independent variable are given in Table 4. The results of hypotheses 1, 2, 3, and 9 are given as follows:

Hypothesis 1: No significant difference (P = .5195) was found in the total number of different heuristics used by students solving problems in a real world setting and students solving problems in a mathematical setting.

Hypothesis 2: No significant difference (P = .1556) was found in the number of correct solutions obtained by students solving problems in a real world setting and students solving problems in a math world setting. Although there was no significant difference for the hypothesis tested, the .1556 probability level may be sufficient to indicate a trend. The mean for correct solutions for the real world subjects is 1.35 compared to .95 for the math world subjects.

The results of post hoc analyses using linear regression indicated that problem context did not affect the number of cycles or changes made by a student. No significant difference (P = .6232) was found between the number of times a student is willing to attack a problem (cycles) and the problem context. Nor was there a significant difference (P = .4225) between the number of times a student changes his method of attacking a problem and the problem context.

The process matrices for hypothesis 9 are given in Figures 10 and 11 with the percent of procedures used in each category given in Figures 12 and 13. Hypothesis 9 was not tested for statistical significance. In evaluating hypothesis 9, the following major areas from the process matrices were considered:

1. Moves from core to core or to heuristics (Areas A and B)
2. Moves from heuristics to core or to heuristics (Areas E and F)

TABLE 4

RESULTS OF [1]REGRESSION ANALYSIS WITH PROBLEM CONTEXT AS INDEPENDENT VARIABLE (N=40)

DEPENDENT VARIABLE	SOURCE OF VARIATION	F - VALUE TO ENTER/REMOVE	[2]F - PROB.	R WITH DEPENDENT VARIABLE	[3]RSQ
TOTAL HEURISTICS	PROBLEM CONTEXT	.4478	.5145	.1077	.0116
DIFFERENT HEURISTICS		.9762	.3312	.1581	.0250
CYCLES		.2533	.6232	.0802	.0064
CHANGES		.6722	.4225	.1319	.0174
CORRECT SOLUTION		2.061	.1556	.2267	.0514

[1]Each of These is a Separate Simple Regression.

[2]Probability of Making a Type 1 Error by Rejecting Null Hypothesis, that is, Claiming Statistical Significance.

[3]The Proportion of Variance in the Dependent Variable Accounted for by Problem Context.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	27		18	71	12	27	87	7	20	26			12		9	2					7	14	8	2		1	7	6	7
2																													
3	8		3	9	2	5	4	4	10	5			3		4	1	1				2	2	2						
4	39		13	19	7	3	4		11	8	7	1	9		5	3			1		2	4	4				1	5	1
5	4		1	4	3	2			4	6	2		5								4	1	1	1				1	
6	10		4		1		26	4					4													1			1
7	44		5	2		3	48	61	3	7			8		6					1		6		2		4	1		2
8	19				1	3	12	62	5	5			1		1	2		1			8	24		3		3			3
9	10		8	8	2		2	3	43	7	2		6		3			2			31	4	2		4		3	2	
10	9			4	4		3	2	21	2					1						32						3	1	
11	2			3	2	1			2				2																1
12	1												1																
13	13		3	8	2	6	6	1	4	8		1	4		5	2		1				2	1	3				1	5
14																													
15	3		1	1		1	1	1					1			1					22	2	3	1			2	3	
16	1			1					1				1								6	1			1	1	1		
17																1													
18								1	1				2									1							
19				1																									
20													1																
21	21		3	5	2				5		1		4		4							2	27	1			18	22	
22	20		3	5			8	3	7	2			6								3	3		1		2	3	5	2
23	14		2	2	1				3				1		4	1		1			2		4				8	17	
24	3						1	2	1	1			4									1			1				1
25	1																				1	3		1					
26	7		1					2		1					1							1							
27	9			4			1		1	4	1		1			1						2	8			1	2	13	1

FIGURE 10

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES FOR 100 MATH WORLD PROBLEMS (N=20)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	21	3	12	64	6	31	31	2	15	35	9		7		15	2					2	6	15	2			4	10	3
2					1	2		1	1																				
3	2			2	1	6	9		3	2			8		2						1	1			1				
4	24	1	9	17	4	4	1		13	15	10	1	5		3							3	2	1			1	1	1
5	2		2	2	1		1		6	5		1	2		1						1	1	3						
6	8		1	1		1	41	1			1				2	1						1							
7	19		3			5	36	56	2	5	1		2		2	1						17		1		7			
8	8	1	1				8	64	6	5			1		4						16	14		5	1	1	1		1
9	12		2	4	2		1		67	6	3		6		7	1		1			42	11	2	4	3		5	1	1
10	6			3	2	1		3	24		1	3	1								35	9		1			3		
11	5		2	3	4				3	3	1		9		4							1							
12	1						1		7																				
13	10		1	5	1	1	3		11	4	3		1		4	1		1	1				1	2				2	2
14																													
15	8		1	2		2	6		2	3	2	2	1		2	4	2	1			18		3	1	1			2	5
16	1		1	1					2		1		1			2					5	1		1					
17															1														
18	1				1				1																				
19											1																		
20																													
21	19		2	2	2				2				2		4	1						3	49	2			12	22	
22	18			3		4	14	5	9	3	1		2		5	3					2	1	1	1	1	1	1	4	
23	15		1	4	1	1	3		3	4		1	3		8							3	4	1		1	9	22	2
24	4				1		1	2	2	1	1	1			1							4					1	3	
25				1				1	2													2						1	
26	2						3	3							1							1							
27	9			2	1				1	1			3		2								5					11	2

FIGURE 11

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES FOR 100 REAL WORLD PROBLEMS (N=20)

	1 2 3 4 5 6 7 8 9 10	11 12 13 14 15 16 17 18 19 20	22 23	24 25 26 27
1 2 3 4 5 6 7 8 9 10	62	7	5	2
11 12 13 14 15 16 17 18 19 20	4	2	1	1
21	2	1	2	1
22 23	5	1		1
24 25 26 27	3	1	1	

FIGURE 12

PERCENT OF PROCEDURES USED IN EACH CATEGORY - MATH WORLD (N=20)

	1 2 3 4 5 6 7 8 9 10	11 12 13 14 15 16 17 18 19 20	22 23	24 25 26 27
1 2 3 4 5 6 7 8 9 10	54	7	3	3
11 12 13 14 15 16 17 18 19 20	7	3	1	
21	2	1	4	1
22 23	6	2	1	1
24 25 26 27	3	1	1	

FIGURE 13

PERCENT OF PROCEDURES USED IN EACH CATEGORY - REAL WORLD (N=20)

3. Moves concerned with checking (Areas C, G, M, and N)
4. Moves concerned with expression of concern (Areas D, H, Q, and R).

In comparing these areas, for the two groups (math world and real world), the researcher considered the percentage of moves in each area for each group as well as comparing the number and types of moves for each procedure. That is, for a given procedure, is there a procedure (or procedures) which students have a tendency to use next?

Hypothesis 9: The pattern of procedures used by subjects in the two groups, real world and math world, was <u>not</u> observably different.

Although the subjects working problems in the math world setting used a higher percent of core related procedures as compared to the subjects in the real world setting (See Area A and percentages of core related areas in Figures 12 and 13), these differences are not great. Two exceptions involve the number of times subjects move from reading (#1) to setting up equations (#7) and from setting up equations to reading the problem. In both of these instances the math world subjects made more moves than the real world subjects, 44 to 19 and 87 to 37 respectively. These differences come mainly from problem number 2, where the number of moves from equations to reading and reading to equations for the math world and real world is 31 to 13 and 66 to 23 respectively.

With over half of the moves in Area A, the divisions in the process matrix may not have been fine enough to detect any differences between the two groups. Also a few students may have contributed the majority of moves in some areas. Rather than counting the total number of moves for each subject, a second process matrix was obtained where a move for each subject was only counted once for each problem (see Figures 14 and 15). The entries in these matrices indicate the number of problems in which a particular move was made. Five areas from the core were identified:

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	16	11	55	34	32	5	6	13	2	5	14	2	4
RECALL	2		3	14	2		6	2		1		1	
DIAGRAM	24	8	18	6	20	8	6	4		4	5	1	1
EQUATION	21	4	1	51	4	2	2	4	2	15		6	
GUESS	6		5	1		1	1			9		1	3
SMOOTHING	4	2	6		3	1	9	4		1			
TEMPLATION	10	1	6	4	4	2	1	3	1		1	2	
RANDOM CASES	7	1	2	8	3	2	1	2	4		2	2	
SYSTEMATIC CASES	1	1	1			1	1		1	1		1	
OBTAIN SOLUTION	19	2	4				2	4	1	3	38	2	11
CHECKING PART	13		3	13	3	1	2	5	3	1	1	3	1
CHECKING SOLUTION	12	1	5	3	4		3	6		1	4	2	6
CONCERN PROCESS	6		2	3	1	1		2		7			1
CONCERN SOLUTION	9		3		1		3	2			5		

FIGURE 14

PROCESS MATRIX:FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES, COUNTED ONCE PER PROBLEM, FOR 100 MATH WORLD PROBLEMS (N=20)

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	21	16	65	54	25		11	8	2	13	8	3	6
RECALL	8	3	11	8	5		2	4	1	2	2		
DIAGRAM	35	14	22	9	12	7	12	5	3	4	5	1	1
EQUATION	33	9	3	47	7		11	6		5		7	1
GUESS	9		7	3	2			1					3
SMOOTHING	2		4	1			2						
TEMPLATION	12	1	7	10	8		4	5	2	2	1	1	
RANDOM CASES	3	1	1	2			1		1	2	3	1	2
SYSTEMATIC CASES	1		1				1			1		2	1
OBTAIN SOLUTION	19	3	6			1	4	4		2	25	1	14
CHECKING PART	14	3	4	6	2		6			3		3	3
CHECKING SOLUTION	12	2	3				1	4	1		4		8
CONCERN PROCESS	11	1		1	2		3	1		5		2	
CONCERN SOLUTION	9		4	1	4	1	1		1	2	6	1	2

FIGURE 15

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES, COUNTED ONCE PER PROBLEM, FOR 100 REAL WORLD PROBLEMS (N=20) reading, recall related fact, working with a diagram (includes diagram and modify diagram), writing equations (includes identify variable and setting up equations), and guessing. Four of the heuristics: smoothing, templation, random cases, and systematic cases) were used across the five problems and are identified in the matrix. The other areas included were obtaining a solution, checking part, checking solution, concern for process (includes concern for method, algorithm, and equation), and concern for solution. The percentage of moves was obtained for each of these procedures across each row (see Figures 16 and 17). For example in Figure 16, in 28% of the problems, working with a diagram followed reading the problem.

The subjects in the real world setting used the heuristic of smoothing across almost all of the procedures in the process matrix while the subjects in the math world setting used it mainly in connection with diagramming. The math world subjects made a greater percentage of moves to reading from almost every process. Kilpatrick (1967, p. 64) indicated the number of times a subject reads the problem is a measure of his difficulty in understanding the problem. It appears as though subjects in both problem settings had difficulty in understanding the problems, with those in the math world setting being the hardest to understand. Also, the real world subjects seemed to be more confident in the solutions they obtained using the heuristics, obtaining 21 solutions using random and systematic cases compared to 25 for the math world subjects. The math world subjects expressed concern for solutions obtained using both of these procedures while the real world subjects expressed none.

Both groups of subjects moved to and from the use of templation and random cases, using all the procedures in the process matrix. Both of these groups stopped work with a solution using approximately the same procedures. In general, the overall pattern of moves (except for smoothing) for both groups of subjects is the same.

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	9	7	28	23	11		5	3	1	1	3	4	3
RECALL	17	7	24	17	11		4	9	2	4	4		
DIAGRAM	27	11	17	7	9	5	9	4	2	3	4	1	1
EQUATION	26	7	2	38	6		9	5		4		6	1
GUESS	36		28	12	8			4					12
SMOOTHING	22		44	11			22						
TEMPLATION	23	2	13	19	15		8	9	4	4	2	2	
RANDOM CASES	18	6	6	12			6		6	12	18	6	12
SYSTEMATIC CASES	14		14				14			14		29	14
OBTAIN SOLUTION	24	4	8				5	5		3	32	1	18
CHECKING PART	32	7	9	14	5		14			7		7	7
CHECKING SOLUTION	35	6	9				3	12	3		12		24
CONCERN PROCESS	42	4		4	8		12	4		19		8	
CONCERN SOLUTION	28		12	3	12	3	3		3	6	19	3	6

FIGURE 16

ROW PERCENTAGE OF MOVES:PROCESS MATRIX FOR MATH WORLD PROBLEMS (N=20)

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	8	6	28	17	16	5	3	7	1	3	7	1	2
RECALL	6		9	44	6		19	6		3		3	
DIAGRAM	23	8	17	6	19	8	6	4		4	5	1	1
EQUATION	19	4	5	49	4	2	2	4	2	14		5	
GUESS	22		19	4		4	4			33		4	11
SMOOTHING	13	7	20		10	3	30	13		3			
TEMPLATION	29	3	17	11	11	6	3	9	3		3	6	
RANDOM CASES	21	3	6	24	9	6	3	6	11		6	6	
SYSTEMATIC CASES	12	12	12			12	12		12	12		12	
OBTAIN SOLUTION	22	2	5				2	5	1	3	44	2	13
CHECKING PART	25		6	25	6	2	4	9	6	2	2	6	2
CHECKING SOLUTION	26	2	11	6	9		6	13		2	9	4	13
CONCERN PROCESS	26		9	13	4	4		9		30			6
CONCERN SOLUTION	39		13		4		13	9			22		

FIGURE 17

ROW PERCENTAGE OF MOVES:PROCESS MATRIX FOR REAL WORLD PROBLEMS (N=20)

<u>Results of Analysis</u> - <u>Field Independence</u>

With IQ restricted to one standard deviation, 115 to 125, the field independent-dependent variable had a significant effect on many of the variables considered in this study. The results of the regression analysis performed with field independence as the independent variable are given in Table 5. The results of hypotheses 4, 5, 6, and 10 are given below:

Hypothesis 4: <u>A significant difference</u> (P=.0110) was found in the total number of heuristics used by field independent students over field dependent students. By itself the field independence variable accounts for 16% of the variance of the total number of heuristics.

Hypothesis 5: <u>A significant difference</u> (P=.0385) was found in the number of different heuristics used by field independent students over field dependent students. By itself the field independence variable accounts for 11% of the variance of the number of different heuristics used.

Hypothesis 6: <u>A significant difference</u> (P=.0575) was found in the number of correct solutions obtained by field independent students over field dependent students. By itself the field independent variable accounted for 9% of the variance of the number of correct solutions.

Post hoc analyses revealed that there was no significant difference (P=.3263) between the number of times a subject is willing to attack a problem (cycles) and field independence. However, field independence did have a significant effect (P=.0892) on the number of changes he makes in attacking a problem. Field independence accounts for 7% of the variance in the number of changes.

TABLE 5

RESULTS OF [1]REGRESSION ANALYSIS WITH FIELD INDEPENDENCE AS INDEPENDENT VARIABLE (N=40)

DEPENDENT VARIABLE	SOURCE OF VARIATION	F - VALUE TO ENTER/REMOVE	[2]F - PROB.	R WITH DEPENDENT VARIABLE	[3]RSQ
TOTAL HEURISTICS	FIELD INDEPENDENCE	7.084	.0110	.3964	.1571
DIFFERENT HEURISTICS		4.495	.0385	.3252	.1058
CYCLES		.9957	.3263	.1597	.0255
CHANGES		2.970	.0892	.2693	.0725
CORRECT SOLUTION		3.745	.0575	.2995	.0879

[1]Each of These is a Separate Simple Regression.

[2]Probability of Making a Type 1 Error by Rejecting Null Hypothesis, that is, Claiming Statistical Signifance.

[3]The Proportion of Variance in the Dependent Variable Accounted for by Field Independence.

In testing hypothesis 10, the subjects were divided into thirds according to their score on the EFT. The top third (14 subjects) formed the field independent group and the bottom third (14 subjects) formed the field dependent group. Those subjects in the middle range of scores were excluded from this analysis. This was done in order to obtain a greater difference in the degree of field independence between subjects classified as field independent and those classified as field dependent. Hypothesis 10 was <u>not</u> tested for statistical significance. If field independence does effect the pattern of procedures used, the difference may be easier to observe using these two groups of subjects rather than the entire sample.

To test hypothesis 10 a pair of process matrices were formed using the results from the coding forms for the top 14 field independent subjects for one matrix (Figure 18) and the bottom 14 field dependent subjects for the other matrix (Figure 19). The percentage of moves in each area is given in Figures 20 and 21. In evaluating hypothesis 10, the following major areas from the process matrix were considered:

1. Moves from core to core or to heuristics (Areas A and B)
2. Moves from heuristics to core or to heuristics (Areas E and F)
3. Moves concerned with checking (Areas C, G, M and N)
4. Moves concerned with expression of concern (Areas D, H, Q and R).

In comparing these areas for the two groups (field independent and field dependent) the researcher considered the percentage of moves in each area for each group as well as comparing the number and types of moves for each procedure. That is, for a given procedure is there a procedure (or procedures) which students have a tendency to use next?

Hypothesis 10: The pattern of procedures used by the top 14 field independent subjects is <u>observably different</u> than that of the bottom

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	9	3	11	42	5	16	41	3	9	16	4		8		10	1					2	9	4				5	4	
2					1	2																							
3	3		1	5	1	5	6	2	6	2			8		2	1					2	2	1	1					
4	17		12	10	4	2	2		13	7	8		5		4	3						5	4				1	2	
5	1		3	2	1	1	1		5	2			3								3		1	1					
6	1		3	1		1	27	3					2			1						1				1			
7	25		3			2	30	39	2	6			4		3	1				1		7		1		7			1
8	10					2	7	33	1	3					3	1		1			8	12		3		2	1		3
9			6	7	1		1	1	65	8	3		7		5			3			29	7	1	4	6		3	2	
10	5			3	1	1	1	2	18	1	1										24	4		1			3	1	
11	2			2	5	1			5	1	1		4		4							1							
12									3																				
13	5		4	3	2	2	1	1	10	7	3		3		6	1		2	1			1	1	4				2	4
14																													
15	6		1	2		2	3		2	2	2	1	1		2	1		1			17		1	2	1		1	3	2
16	1			1					2				1								8	2		1	1	1	1		
17																													
18	1				1			1	2				2									1							
19											1																		
20													1																
21	13		1	7		3	7	2	5	3			3		4	3					4	3		1	1		3	3	1
22	13		1	7		3	7	2	6	3			3		4	3					4	3		1	1		3	3	1
23	5		1	4	2	1	1		1	3		1	2		6	1		1			1		4	1		1	4	14	
24	4						1	1	2	1	1	1	3		1							4			1			3	
25				1					2												1	5		1				1	
26	4						3	2		1					2							1							
27	7			4					1	3	1		4			1							8			1		4	

FIGURE 18

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES FOR FIELD INDEPENDENT STUDENTS (N=14)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	25		9	45	4	23	52	3	15	32	1		6		11	1					2	6	10	2		1	6	4	8
2									1																				
3	4		1	1	1	4	3	1	4	4			1		3						1								
4	25	1	6	4	1	3	3		2	5	1	1	2		1								2	1				3	
5	2			1					3	4	1											1	1					1	
6	11		1				22	1					1		1														1
7	24		3	1		3	28	50	2	4	1		2		1							7		1		2	1		
8	13		1			1	7	57	6	6			1		1						11	16		2		2			1
9	12		2	1			1	2	22	3	1		4		3	1					22	5	1		1		3		1
10	6				3			3	16	1		2	1		1						29	2					1		
11	2			2						1			2																
12	1								2				1																
13	9			3	1	3	2		2	2		1			1	2						1						1	1
14																													
15	3					1	3	1					1			1	1				12	2	2					1	1
16			1						1		1		1			2					1								
17															1														
18																													
19																													
20																													
21	16		1	2	3				2				2		2							2	23				10	16	
22	17		1				6	2	4	1	1		3									1	1	1				5	1
23	16		1	1			2		2	1			1		3						1		2				4	10	1
24	2							2	1	1																			1
25	1																												
26	1		1					3																					
27	6				1		1															1	3				2	11	2

FIGURE 19

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES FOR FIELD DEPENDENT STUDENTS (N=14)

	1 2 3 4 5 6 7 8 9 10	11 12 13 14 15 16 17 18 19 20	22 23	24 25 26 27
1 2 3 4 5 6 7 8 9 10	49	8	5	3
11 12 13 14 15 16 17 18 19 20	9	3	1	1
21	2	1	3	1
22 23	5	2	1	1
24 25 26 27	3	1	2	

FIGURE 20

PERCENT OF PROCEDURES USED IN EACH CATEGORY - FIELD INDEPENDENCE (N=14)

	1 2 3 4 5 6 7 8 9 10	11 12 13 14 15 16 17 18 19 20	22 23	24 25 26 27
1 2 3 4 5 6 7 8 9 10	64	5	5	2
11 12 13 14 15 16 17 18 19 20	4	2	1	
21	3		3	1
22 23	6	1		1
24 25 26 27	2			

FIGURE 21

PERCENT OF PROCEDURES USED IN EACH CATEGORY - FIELD DEPENDENCE (N=14)

14 field dependent subjects. The field dependent subjects appear to be more ·e oriented with a higher percent of procedures in Area A. The field independent subjects are more consistently concerned with their work (see Areas D, H, Q, R, S, T). In the use of heuristics, the field independent subjects make greater use of systematic cases (#16) and give what little evidence there is of the higher heuristics. Both groups appear to use both templation (#13) and random cases (#15) across all procedures.

Again, a second process matrix was obtained for each group where the entries in the matrix indicate the number of problems in which a particular move is made. (See Figures 22 and 23). The row percentages for these matrices are given in Figures 24 and 25.

The field dependent subjects appear to have had great difficulty in understanding the problems with 31 to 57 percent of their moves to reading from a majority of the procedures. The majority of their moves are to diagrams, equations or templation. Half of their responses to concern is to reread the problem hoping to gain some insight.

The field independent subject moves more freely to a greater variety of procedures and to the four heuristics. He is more concerned with his work, continually checking both his work and solution. When concerned about the process he uses or his solution, he is more willing to check his procedures or solutions.

In general, the field independent subject moves between all the procedures in the matrix while the field dependent subject spends a great deal of time in the core areas.

<u>Results of Analysis - Core and Heuristic Procedures</u>

The correlations for hypotheses 7 and 8 are given in Table 6. An F-Test was used to determine if these correlations were significantly different from 0.

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	6	9	36	28	14	3	7	9	1	8	4		4
RECALL	3	1	6	9	2		6	2	1	2	1	1	
DIAGRAM	13	12	14	6	9	7	8	4	3	4	5	1	1
EQUATION	13	6	1	39	5		6	3	2	6		7	
GUESS	5		3	2	1	1				4		1	3
SMOOTHING			6	1	1	1	4	4		1			
TEMPLATION	5	2	4	3	7	2	3	5	1	1	1	1	
RANDOM CASES	5	1	2	5	2	2	1	2	4		1	3	1
SYSTEMATIC CASES	1		1				1			2		3	1
OBTAIN SOLUTION	11	2	5			1	2	4	1	3	23	3	9
CHECKING PART	9	1	6	8	3		3	4	3	3		2	3
CHECKING SOLUTION	5	1	6	2	3		2	5	1		4	2	4
CONCERN PROCESS	8		1	3	2	1	2	2		10		2	
CONCERN SOLUTION	7		4		3	1	4		1		7	1	

FIGURE 22

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES, COUNTED ONCE PER PROBLEM, FOR FIELD INDEPENDENT STUDENTS (N=14)

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	20	9	40	34	30	2	5	9	1	5	9	3	6
RECALL	4	1	2	7	4		1	3					
DIAGRAM	24	6	6	6	8	3	2	1		1	3	1	
EQUATION	26	4	1	32	4	1	3	2		7		3	1
GUESS	6		3		1		1	1		2			1
SMOOTHING	3		2		1		2						
TEMPLATION	9		4	3	2			1	2	1		1	
RANDOM CASES	3			4			1		1	2	2		
SYSTEMATIC CASES		1				1	1		1				
OBTAIN SOLUTION	16	1	4				2	2		2	19		8
CHECKING PART	11	1		5	1	1	3			1	1	1	
CHECKING SOLUTION	11	1	1	1	1		1	3			2		3
CONCERN PROCESS	4	1			1			1					
CONCERN SOLUTION	6		1	1						1	2		2

FIGURE 23

PROCESS MATRIX: FREQUENCY DISTRIBUTION OF CHANGE OF PROBLEM SOLVING PROCEDURES, COUNTED ONCE PER PROBLEM, FOR FIELD DEPENDENT STUDENTS (N=14)

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	5	7	28	22	11	2	5	7	1	6	3		3
RECALL	9	3	18	26	6		15	6	3	6	3	3	
DIAGRAM	14	13	16	7	10	8	9	4	3	4	6	1	1
EQUATION	15	7	1	44	6		7	3	2	7		8	
GUESS	25		15	10	5	5				20		5	15
SMOOTHING			33	6	6	6	22	22		6			
TEMPLATION	14	6	11	9	20	6	9	14	3	3	3	3	
RANDOM CASES	17	3	7	17	7	7	3	7	14		3	10	3
SYSTEMATIC CASES	11		11				11			22		33	11
OBTAIN SOLUTION	17	3	8			2	3	6	2	5	36	5	14
CHECKING PART	20	2	13	17	7		7	9	7	7		4	7
CHECKING SOLUTION	14	3	17	6	9		6	14	3		11	6	11
CONCERN PROCESS	26		3	10	6	3	6	6		32		6	
CONCERN SOLUTION	25		14		11	4	14		4		25	4	

FIGURE 24

ROW PERCENTAGE OF MOVES:PROCESS MATRIX FOR FIELD INDEPENDENT STUDENTS (N=14)

	READING	RECALL	DIAGRAM	EQUATION	GUESS	SMOOTHING	TEMPLATION	RANDOM CASES	SYSTEMATIC CASES	CHECKING PART	CHECKING SOLUTION	CONCERN PROCESS	CONCERN SOLUTION
READING	12	5	23	20	17	1	3	5	1	3	5	2	3
RECALL	18	5	9	32	18		5	14					
DIAGRAM	39	10	10	10	13	5	3	2		2	5	2	
EQUATION	31	5	1	38	5	1	4	2		8		4	1
GUESS	40		20		7		7	7		13			7
SMOOTHING	38		25		13		25						
TEMPLATION	39		17	13	9			4	9	4		4	
RANDOM CASES	23			31			8		8	15	15		
SYSTEMATIC CASES		25				25	25		25				
OBTAIN SOLUTION	30	2	7				4	4		4	35		15
CHECKING PART	44	4		20	4	4	12			4	4	4	
CHECKING SOLUTION	46	4	4	4	4		4	13			8		
CONCERN PROCESS	57	14			14		14						
CONCERN SOLUTION	46		8	8						8	15		15

FIGURE 25

ROW PERCENTAGE OF MOVES:PROCESS MATRIX FOR
FIELD DEPENDENT STUDENTS (N=14)

TABLE 6

INTERCORRELATIONS AMOUNG MEASURES OF
HYPOTHESIZED VARIABLES (N=40)

	2	3	4	5	6	7	8
1. PROBLEM CONTEXT[a]	.01	.11	.16	-.08	.13	.23	-.02
2. FIELD INDEPENDENCE		.40*	.33*	.16	.27	.30	.23
3. TOTAL HEURISTICS			.84**	.48**	.80**	.49**	.08
4. DIFFERENT HEURISTICS				.43**	.75**	.47**	.19
5. CYCLES					.52**	.14	.12
6. CHANGES						.51**	.20
7. CORRECT SOLUTION							.18
8. OFFERED SOLUTION							

Note- Entries are Product Moment Correlations. For a Dichotomous Variable this is Equivalent to the Point-Biserial Correlation Coefficent.

[a]Dichotomous Variable. $^{*}R_{crit}=.3121$ @ $P=.05$ $^{**}R_{crit}=.4026$ @ $P=.01$

Hypothesis 7: A <u>significant</u> <u>positive</u> correlation (P=.0013) was found between the total number of heuristics used by a student and the number of correct solutions he obtained.

Hypothesis 8: A <u>significant</u> <u>positive</u> correlation (P=.0022) was found between the number of different heuristics used by a student and the number of correct solutions he obtained.

Post hoc analysis was performed, using linear regression, to determine the amount of variance in the number of correct solutions which could be accounted for by the heuristics, cycles and changes. These results are given in Table 7. The total number of heuristics accounts for 24% of the variance in correct solutions while the number of different heuristics used accounted for 22%. The number of cycles (the number of times a subject attempts to solve a problem) accounts for less than 2% of the variance, while the change of procedures accounts for 26% of the variance of correct solutions.

Table 8 lists the means and standard deviations and Table 9 lists the intercorrelations between pairs of variables derived from the coding system for the total sample of forty subjects. It should be noted that a correlation coefficient, which is statistically significant, provides additional insight into the data, but does not allow causal inference to be made.

Most of the significant correlations have obvious interpretations. The large number of significant correlations with cycles and changes reflect the fact that subjects who are attempting to begin problems again or change their procedures are performing more processes. As would be expected, the core areas are significantly correlated. Those procedures which are algebraic such as, identifying variables, setting up equations

TABLE 7

RESULTS OF [1]REGRESSION ANALYSIS WITH CORRECT SOLUTION AS DEPENDENT VARIABLE (N=40)

DEPENDENT VARIABLE	SOURCE OF VARIATION	F - VALUE TO ENTER/REMOVE	[2]F - PROB.	R WITH DEPENDENT VARIABLE	[3]RSQ
CORRECT SOLUTION	TOTAL HEURISTICS	11.84	.0015	.4873	.2375
	DIFFERENT HEURISTICS	10.53	.0025	.4658	.2170
	CHANGES	13.33	.0009	.5096	.2597
	CYCLES	.7424	.3986	.1386	.0192

[1]Each of These is a Separate Simple Regression.

[2]Probability of Making a Type 1 Error by Rejecting Null Hypothesis, that is Claiming Statistical Significance.

[3]The Proportion of Variance in the Dependent Variable Accounted for by Independent Variable.

TABLE 8

MEANS AND STANDARD DEVIATIONS FOR PROCEDURES DERIVED FROM CODING SYSTEM (N=40)

Variable	Mean	Standard Deviation
Reading	17.85	6.93
Recall Fact	3.25	2.68
Draw Diagram	8.12	6.80
Modify Diagram	2.32	2.29
Identify Variable	2.80	2.33
Set up Equations	9.08	6.85
Algebraic Algorithms	8.28	10.53
Arithmetic Algorithms	31.10	35.68
Guessing	4.90	2.63
Smoothing	1.08	1.99
Templation	3.75	3.54
Random Cases	2.35	2.51
Systematic Cases	.95	1.95
Obtain Solution	6.15	2.69
Checking Part	5.98	4.44
Checking Solution	4.42	3.42
Concern Process	2.32	3.12
Concern Solution	2.52	3.13
Cycles	10.38	4.12
Changes	1.85	2.30
Stops without Solution	1.12	1.22
Correct Solution	1.15	.89

Note- Appendix D Contains Histograms for Some Selected Variables.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	
1.READING PROBLEM	.20	.26	.25	.34*	.28	-.01	-.09	-.19	-.19	.02	.07	-.08	.14	.30	.17	.05	.26	.26	-.08	.25	-.13	1
2.RECALL RELATED FACT		.51**	.48**	.50**	.39*	.18	.23	.18	.13	.49**	.20	.22	.26	.60**	-.02	.45**	.20	.36*	.38*	-.03	.06	2
3.DRAW DIAGRAM			.54**	.44**	.13	-.02	.46**	.18	.50**	.38*	.39*	.43**	.37*	.50**	.13	.29	.32*	.51**	.62**	-.08	.41**	3
4.MODIFY DIAGRAM				.24	.09	.04	.05	.44**	.18	.34*	.14	.21	.18	.15	.22	.12	.45**	.18	.21	.02	.24	4
5.IDENTIFY VARIABLE					.56**	.59**	.30	-.02	.30	.47**	.25	.32*	.28	.62**	.05	.38*	.10	.60**	.45**	.20	.14	5
6.SETTING UP EQUATIONS						.80**	.01	.10	-.06	.17	-.04	.11	.21	.39*	-.06	.22	.20	.60**	.15	.09	-.13	6
7.ALGORITHMS-ALGEBRAIC							-.11	-.08	-.13	.07	-.12	.00	.15	.31*	-.02	.14	.12	.43**	-.08	.13	-.16	7
8.ALGORITHMS-ARITHMETIC								.14	.62**	.19	.79**	.89**	.59**	.67**	.22	.54**	.23	.70**	.84**	-.27	.47**	8
9.GUESSING									.19	.17	.03	.21	.41**	.20	.40*	.27	.63**	.32*	.18	-.18	-.03	9
10.SMOOTHING										.36*	.48**	.56**	.36*	.42**	.21	.32*	.16	.49**	.66**	-.18	.44**	10
11.TEMPLATION											.14	.08	-.05	.42**	-.08	.48**	.04	.32*	.36*	.26	.16	11
12.CASES-RANDOM												.62**	.57**	.56**	.29	.52**	.18	.58**	.62**	-.21	.58**	12
13.CASES-SYSTEMATIC													.58**	.52**	.21	.40*	.34	.68**	.81**	-.30	.30	13
14.OBTAIN SOLUTION														.60**	.61**	.30	.56**	.62**	.45**	-.53**	.22	14
15.CHECKING PART															.18	.75**	.33*	.74**	.57**	-.02	.22	15
16.CHECKING SOLUTION																-.02	.59**	.25	.01	-.28	.21	16
17.CONCERN-PROCESS																	.21	.60**	.49**	.16	.15	17
18.CONCERN-SOLUTION																		.49**	.14	-.12	-.07	18
19.CYCLES																			.52**	-.12	.14	19
20.CHANGES																				-.20	.51**	20
21.WORK STOP-NO SOLN.																					-.18	21
22.CORRECT SOLUTION																						22

$^{*}R_{crit}$=.3121 @ P=.05 $^{**}R_{crit}$=.4026 @ P=.01

TABLE 9

INTERCORRELATION BETWEEN PAIRS OF VARIABLES FROM CODING SYSTEM (N=40) and algebraic algorithms are all correlated. As well, those from the geometric area such as drawing diagrams, modifying diagrams and arithmetic algorithms (mainly from the use of formulas) are correlated. Checking part of work is positively correlated with most of the process variables, whereas, checking solutions is significantly correlated with those procedures pertaining to solutions or which were used to obtain most of the solutions. However, expressing concern for solution is unrelated to obtaining a correct solution.

Four procedures from the coding form accounted for 86% of the solutions obtained: algebraic algorithm (10%), arithmetic algorithm (31%), guessing (28%), and random cases (17%), (with systematic cases, 22%). Of these, guessing and algebraic algorithms are unrelated to correct solution, whereas both arithmetic algorithms and random cases are significantly correlated with correct solution. A detailed analysis of the coding forms shows that more than half of the correct solutions (26) were obtained by random cases followed by arithmetic algorithms (14) with guessing accounting for only 6 correct solutions and algebraic algorithms for 1. Both arithmetic algorithms and diagrams were used considerably with both random and systematic cases as well as with other core procedures.

To gain some further insight into the relationship between these variables, a regression analysis was performed using correct solution as the dependent variable. The variables from the core area were entered into the regression equation first, followed by the heuristic strategies (variables within core and heuristics were allowed to enter the regression equation free). The core procedures account for 40% of the variance in correct solution, in addition heuristics accounted for another 21% with systematic cases accounting for 13% of this (see Table 10). When the

TABLE 10

REGRESSION ANALYSIS ON CORRECT SOLUTION WITH CORE ENTERED BEFORE HEURISTICS (N=40)

SOURCE OF VARIANCE	MULTIPLE R	RSQ[2]	ΔRSQ	F - VALUE TO ENTER/REMOVE	F-PROB[1]
CORE					
Arith. Algorithms	.4742	.2249	.2249	11.0246	.0021
Draw Diagram	.5242	.2748	.0499	2.5477	.1151
Reading	.5597	.3133	.0384	2.0153	.1608
Recall Fact	.5829	.3398	.0266	1.4085	.2418
Guessing	.5981	.3577	.0178	.9442	.3402
Modify Diagram	.6322	.3997	.0420	2.3098	.1343
Alg. Algorithms	.6340	.4019	.0022	.1194	.7292
Set up Equations	.6343	.4024	.0005	.0248	.8496
Iden. Variables	.6348	.4026	.0002	.0088	.8877
HEURISTICS					
Systematic Cases	.7552	.5307	.1281	11.3247	.0023
Random Cases	.7712	.5948	.0641	1.6888	.2018
Templation	.7778	.6058	.0110	.6986	.4154
Smoothing	.7800	.6084	.0026	.2226	.6453

[1]Probability of Making a Type 1 Error by Rejecting Null Hypothesis, i.e. Claming Statistical Significance.

[2]The Amount of Variance in Correct Solution Accounted for by the Independent Variable.

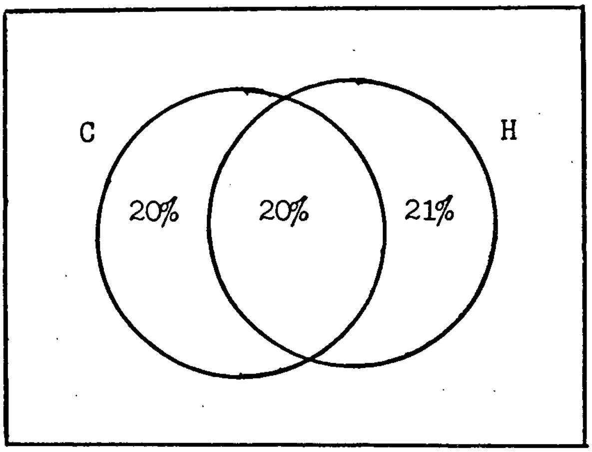
TABLE 11

REGRESSION ANALYSIS ON CORRECT SOLUTION WITH HEURISTICS ENTERED BEFORE CORE (N=40)

SOURCE OF VARIANCE	MULTIPLE R	RSQ^2	ΔRSQ	F - VALUE TO ENTER/REMOVE	F-PROB[1]
HEURISTICS					
Random Cases	.5830	.3399	.3399	19.5656	.0001
Smoothing	.6133	.3760	.0361	2.1427	.1484
Systematic Cases	.6365	.4051	.0291	1.7580	.1903
Templation	.6366	.4052	.0001	.0096	.8852
CORE					
Draw Diagram	.6596	.4351	.0298	1.7946	.1862
Reading	.6952	.4833	.0482	3.0808	.0849
Arith. Algorithms	.7093	.5031	.0198	1.2731	.2670
Modify Diagram	.7305	.5336	.0306	2.0315	.1606
Guessing	.7481	.5596	.0260	1.7700	.1905
Recall Fact	.7658	.5865	.0269	1.8846	.1772
Set up Equations	.7783	.6058	.0193	1.3706	.2504
Alg. Algorithms	.7793	.6074	.0016	.1089	.7398
Iden. Variable	.7800	.6084	.0010	.0666	.7867

[1]Probability of Making a Type 1 Error by Rejecting Null Hypothesis, i.e. Claming Statistical Significance.

[2]The Amount of Variance in Correct Solution Accounted for by the Independent Variable.



The area within the rectangle represents the total variance of correct solution. Circles C and H represent the percentages of variance of correct solution which are contributed by core and heuristics, respectively.

FIGURE 26

RELATIVE CONTRIBUTIONS OF CORE AND HEURISTICS TO CORRECT SOLUTION

order was reversed in the regression equation (see Table 11), the heuristics accounted for 41% of the variance. The core procedures added an additional 20%. It appears as though for these subjects, the influence of core and heuristics procedures is equivalent (see Figure 26).

CHAPTER V

CONCLUSIONS

This study was undertaken to gain some insight into mathematical problem solving. Confining attention to mathematical word problems, an attempt was made to understand something of the solution techniques used by a group of senior high school students. Specifically, what core procedures and heuristics do they use? What effect does problem context have on these procedures? Does field independence influence heuristic usage? What patterns of heuristic usage do specific groups of individuals exhibit? To help answer these questions, a coding system was developed in a form which allowed comparison across individuals not only of the procedures they used, but also of the sequence in which they were applied. The coding system appears to be both useful and usable. It's application to the data in this study has yielded many interesting results and implications for further research.

Summary of the Experimental Study

Forty students who were just completing the grade eleven academic program in mathematics were randomly selected from fourteen Algebra II classes in three senior secondary schools. The subjects were of average mental ability (IQ range: 115 to 125) for this population. The subjects were given the Embedded Figures Test individually and the scores rank-ordered and paired. The subjects from each pair were randomly assigned to one of two groups, one group working problems in a mathematical setting, the other in a real world setting. They were then interviewed individually and asked to think aloud as they solved a selection of mathematical word problems.

A coding system was developed based on the model of heuristics of MacPherson and was used to analyze the tape recorded protocols of the subjects. The coding system not only indicated the procedures used by the subjects, but the sequence in which they were applied.

When the coding system was applied to the subjects' tape recorded protocols, only a few of the heuristics described by MacPherson were observed sufficiently often to be used in the sequential analysis. Most of the subjects were very core oriented and made limited use of the heuristics.

The data collected from the coding forms along with the use of regression analysis were used in answering the first three questions posed in Chapter I. The information gained from the process matrices was used in answering the last question. Consider the four questions in order:

1. <u>What procedures from core and what heuristics are used by students in attacking and solving mathematical word problems</u>?

All of the core procedures identified in the coding system were used by the forty subjects. As would be expected, most subjects attempted to use algebraic operations to solve the problems by identifying variables and attempting to set up some equations.

The heuristics of templation and random cases were used by three-fourths of the subjects on at least one problem. Over half of the students used each of these heuristics more than once. Smoothing, analysis, and sequential cases were used by one-fourth of the students, with analysis used only on problem number 5. Some evidence was obtained of the use of critical cases, all cases, hypothetical deduction, inverse deduction (used by four students on problem number 4), and variation. The other heuristics were not observed.

Of the forty-seven correct solutions obtained, 26 were through the use of random or systematic cases, arithmetic algorithms mainly accounted for 14, guessing for 6, and algebraic algorithms for 1. The use of diagrams was related to the number of correct solutions as well as many of the heuristics. Four of the problems were geometric in nature, and most of the smoothing and templation was related to geometric concepts. Many of the students used diagrams with both random and systematic cases and drew a separate diagram for each case.

Both the total number of heuristics used by a subject as well as the number of different heuristics used, account for a significant amount of the variance in the number of correct solutions. In particular, heuristics accounted for an additional 21% of the variance not accounted for by core procedures with systematic cases accounting for a statistically significant amount of this (13%). Hence, the heuristics used by the subjects in this study added to their ability to solve problems beyond their mathematical core knowledge.

Students who used a wide range of heuristics were on the average, better problem solvers. The subjects who used a wider range of heuristics used them more effectively. These students were more willing to change their mode of attack or the procedures they were using if they were not having success in obtaining a solution. Changing one's mode of attack in solving a problem was significantly related to obtaining a correct solution.

2. <u>Does the context of the problem influence heuristic usage?</u>

Problem context, as identified in this study, proved to be unrelated to the heuristics used. Both the total number of heuristics used and the number of different heuristics used were not influenced by the problem setting. Although in each case, the means of the variables favor the real world setting, the differences are not great. Subjects working problems in the mathematical world setting had a slightly harder time understanding the problems, but performed as well as the other group.

3. What effect does field-dependence-independence have on the problem solving process?

The subjects used in this study were chosen from the middle IQ range of students taking Algebra II. Within this IQ range, field independence had a marked effect on the use of heuristics and on the number of correct solutions obtained.

Field dependent students appear to be very oriented to the use of core procedures. They exhibit some use of the lower heuristics, but in general they attacked most problems from an algebraic point of view. Most of them drew a diagram for a problem, but this was used in order to help set up equations. Once they had attempted to solve a problem without success, most continued in the same vein using exactly the same procedures as before. Evidence of this is exemplified in the solving of the stamp problem (problem #2) in which a single equation with two unknowns can be written, and hence, there appears to be a need for a second equation. The field dependent students continued to do so until they obtained one which was incorrect or dependent on the first equation. The field independent students soon realized a second equation could not be obtained and changed their mode of attack usually to cases.

What little evidence there was of the use of the higher heuristics came from the field independent students. These students used a greater variety of both core and heuristics in attacking and solving problems. They were much more willing to change their mode of attack. They obtained a greater number of correct solutions. From the view point of procedures used, as well as correct solutions, the field independent student is a better problem solver.

4. Do selected groups of individuals exhibit patterns of heuristic usage when solving mathematical word problems?

Two different criteria were used to group subjects in an attempt to answer this question, by problem context and by degree of field independence. Students grouped by problem context exhibited the same general patterns of problem solving procedures, but when grouped by field independence showed an observable difference. These differences were not tested for statistical significance.

When grouped by problem context, both groups exhibited the same general pattern of problem solving moves involving core procedures and heuristics except for smoothing. Subjects in the real world setting moved to smoothing from a variety of procedures whereas those in the math world setting used smoothing mainly through diagramming. Both groups of students moved to and from the use of templation and random cases to all of the core procedures.

Both the real world and math world groups obtained solutions using the same procedures and stopped work on the problems following the same moves. The students who encountered real world problems expressed concern for solutions obtained using heuristics while those students in the math world setting expressed none. However, expression of concern for solution was found to be unrelated to correct solution.

When grouped by field independence a difference was found in the overall pattern of sequential moves in the problem solving process. Field dependent students had greater difficulty in understanding the problems. In general they were very core oriented, making the majority of their moves from core procedures to core procedures. The core procedures used most often were writing an equation and some use of diagrams. Field dependent students exhibited some use of random cases, but mainly used the heuristic of templation. In general, they were poorer at templating than the field independent student.

The field independent student moved more freely between the core and heuristic procedures. He used four of the heuristics: smoothing, templation, random and systematic cases, across all the procedures in the process matrix. He was more concerned with his work because he was continually checking both the procedures he was using as well as his solution. When concerned with his solution the field independent student was more willing to check it, usually by retracing his steps, whereas the field dependent student reread the problem. Perhaps the field dependent student was not sure of what the problem was asking.

In general, the field independent student was a better problem solver. He used a greater variety of heuristics and moved more freely across all the core and heuristic procedures. The field dependent student was very core oriented in the procedures he used.

Summary and Conclusions for The Model and Coding System

The heuristics defined in this model account for the non-core procedures used by the forty subjects in the study. They are broad enough to account for all of the observed processes and yet, not so broad that no real distinction can be made. The heuristics appear to be the natural kinds of procedures that would be used in attacking and solving problems.

The use of cases proved to be an important procedure in obtaining solutions. One of the strong advantages of this model over those discussed in Chapter II, is the distinction between the different kinds of "trial and error" or cases. Over three-fourths of the students involved in this study used cases of some kind in an attempt to obtain a solution to a problem. It is important to distinguish the type of sequence they are using. Is the student just trying some numbers in a completely random fashion in hopes of hitting on the solution or is he attempting to find a solution in some kind of systematic fashion? Is he perhaps using the information obtained from one case to "improve" the next or applying the data from the problem in a sequential fashion? The student might determine that there can be only a small finite number of possible solutions (as in problem # 2) and hence, consider all possible cases. All of these types of cases have their place in problem solving. They are different procedures and should be considered as such.

As would be expected, the subjects in this study were very oriented to using core procedures. Although they did exhibit use of the lower heuristics, they spent the majority of their time and effort in attempting to solve the problem using only core material. Even though fewer correct solutions were obtained by these students than those in the pilot studies, the students were still motivated and appeared to do their best in working all the problems.

The ineffective use of the heuristic of templation, which was the most used heuristic, revealed that students do not know what core material they can bring to bear on a particular problem. They can recall general categories of information; sometimes correctly, sometimes not; but cannot recall specific details. In particular, only four students could recall the Pythagorean Theorem. Many of the other subjects knew there was some relationship among the sides of a right triangle, while others did not even comment on the relationship. The way templation is defined, it gives no evidence of whether the subject is looking at the given information in the problem and then trying to roam over some content area to help bridge the gap to the conclusion or working the other way, looking at the conclusion of the problem and trying to determine which content will help him achieve it. This distinction may reveal some important characteristic of the problem solver. Does he attack the problem in a forward manner only, or from the conclusion, working backwards (very little evidence was obtained of this) or is it a combination of both?

As core oriented as students are, templation appears to be an important heuristic and should be redefined as follows:

Direct Templation - The heuristic of direct templation is used when a premise or set of premises is given or assumed. The use of the heuristic involves recalling a category of content which is related to the premises given. This includes the recall of such things as algorithms, problem types, procedures, theorems, and properties related to the premises. The purpose of direct templation is to recall core material which is related to the premises in order to help bridge the gap to the conclusion by expanding knowledge about the premises.

Inverse Templation - The heuristic of inverse templation is used when a conclusion is given or assumed. The use of the heuristic involves recalling a category of content which is related to the conclusion of the problem. This includes the recall of such things as algorithms, problem types, procedures, theorems, and properties related to the conclusion. The purpose of inverse templation is to recall core material which is related to the conclusion in order to expand knowledge about the conclusion and determine content that will help achieve it.

Very little evidence was obtained to support any ordering of the heuristics (see Figure 2, p. 8). Few of the heuristics in the middle and general categories were observed. There appeared to be considerable movement between the sieve (core) and the lower heuristics. From those higher heuristics which were observed, students using them did move to a lower heuristic or back to core. Subjects did move from both the sieve and heuristics to obtaining a solution (Figure 3, p. 19).

Some evidence was obtained which related to expression of concern, but this expression was unrelated to obtaining a solution. Concern for process was related to both core and heuristic procedures.

The coding system proved to be reliable and capable of identifying individual differences. The system not only identifies the procedures used, but also indicates the sequence in which they are applied. In order to be able to identify problem solving styles, the sequence in which procedures are used will certainly be a principal source of information.

Implications for Education

The findings of this study indicate that students who are completing an Algebra II course have some facility with the heuristics defined in MacPherson's model. In particular, some evidence was obtained to suggest that students who use the heuristics of random and systematic cases, when faced with a difficult mathematical problem, tend to be successful in solving the problem. Many of the students who failed to use cases may have known how to use this technique, but avoided it because they thought it was inappropriate. In fact, most of the students who used cases did so as a last resort with some implying that, although this technique was not acceptable to use as a procedure in the classroom, they would at least "try some numbers" to see what would happen. Many mathematics teachers who are eager to have their students learn the "right" way of solving problems, penalize the students who resort to the use of cases.

Polya (1962, p. 26) suggests that the teacher should not discourage his students from using trial and error (cases) - but to the contrary, he should encourage the intelligent use of this fundamental procedure. Many of the good problem solvers in this study did use both random and systematic cases effectively. Some of the students did not understand how to use the method. When trying to use cases, these students gained very little information about the problem. It would be advantageous to both mathematics teachers and their students if more consideration were given to the use of cases as an acceptable problem solving procedure.

The heuristic of templation was used by three-fourths of the students and yet had no effect on the solutions obtained. A reason for this may be that students don't have to template for the problems they are asked to do in the classroom. In most textbooks the material required for a problem is given in the two or three pages immediately preceeding the exercise. In general, students don't have to recall material from the previous year or even the previous chapter. Most exercises involve the direct application of the contents of a few pages from the textbook. The students in this study had very poor recall of the core material they had studied and very little idea of what core material they could use in a problem. This is not surprising since most students are told what to do by both the text and the teacher. If the creation of new problem sets is as important as suggested in the opening remarks of this paper, then not only are curriculum developers going to have to devote more time to the creation of imaginative and creative problems, but also to the placement of these problems in the curriculum. Problems should not always be placed in the exercises immediately following the presentation of the material needed to solve them, but they should be included in exercises at a later date.

Another of the findings of this study is that those students who change their mode of attack are more successful in obtaining a solution. Most teachers do tell their students to try another method if they are having difficulty in solving a problem, but if students are going to be able to change their mode of attack, they have to be able to recall the core material they have studied (template) or be able to attack the problem by using various heuristics.

Henderson and Pingry (1953, p. 238) state that practice in solving problems and a conscious awareness of the problem solving process will improve problem solving performance. Polya proposes a program for teaching problem solving which consists of two aspects: abundant experience in solving problems and serious study of the solution process. He (Polya, 1962, p. v) expresses the need for the first of these as follows: "Solving problems is a practical art, like swimming or skiing or playing the piano; you can learn it only by imitation and practice."

Polya also warns that in problem solving, imitation and practice are not sufficient. Not only must problems be solved, but the learners' attention must be directed to the methods used. These must be general enough so that they become available for use in solving similar problems in the future.

The heuristics in MacPherson's model are of a general nature. They appear to be the natural kinds of procedures that would be used in attacking and solving mathematical problems. If the programs of Henderson, Pingry and Polya do indeed have merit for teaching problem solving, then perhaps the heuristics from this model should be included.

Problem solving, as considered in this study with students of average ability, is related to field independence. Not only does the field dependent student obtain fewer correct solutions, he is also less willing to change his method of attack once he starts to solve a problem. Spitler (1970) discussed five procedures which may be used by the teacher to help students learn to solve problems using different modes of attack.

1. Near the end of each set of problems which are designed to provide practice in a given procedure, include several problems which require other procedures.
2. Problems which can be or must be done using different procedures should be interspersed throughout a set of problems.
3. The teacher should regularly discuss the influences of patterned responses with his students.
4. Superfluous information should be included in problems throughout the exercises.
5. When practicing complex problem solving, problems which the student recognizes that he could solve by a less complex procedure, should be included.

Spitler additionally recommends that field dependent students be given problem sets which require many different methods of solution.

The recommendations in this section require that the mathematics teacher continually assign problems which require different procedures to obtain a solution and that these procedures be discussed with the students. Also the field independent-dependent construct may be useful in grouping students for teaching mathematical problem solving because the problem solving behavior of students in these two groups indicate they may have a need for different teaching strategies.

Limitations of the Study

There are several limitations of the present study. The first three limitations were noted in Chapter I as incoming limitations of the study. These limitations are restated here along with other limitations which were determined as the study was carried out.

All of the subjects were of average ability for subjects on an academic mathematics program, therefore, regularities observed give no evidence that above average or weak students would exhibit the same problem solving behavior. The subjects were randomly selected from the academic grade eleven program (Algebra II) and so any generalizations made based upon these results are limited to this or a similar population of students.

An obvious limitation is the selection of problems used in the study. The data for each student were collected from one session in which the student was asked to solve five problems. An individual may have exhibited different problem solving characteristics if asked to attempt a different set of problems. The problems used in this study were not selected from any particular core area nor were they of any particular type, although four of them involve a maximum or minimum solution.

There were also limitations resulting from the use of the "think aloud" method in determining the procedures used by the students in solving problems. The method itself may be unreliable since an individual might remain silent during moments of deepest thought. Moreover, a verbalized solution could be essentially different from one affected silently. The presence of an observer might inhibit a problem solver in such a way that he might not attempt solutions which he felt would be considered foolish by someone else. These limitations were noted by both Kilpatrick (1967) and Kantowski (1975).

There were also limitations in the coding system used. The subjects in the study were very oriented to using core procedures and spent a great deal of time attempting to understand the problems. Perhaps if the core procedures used in the coding system had included a finer division in the reading and preparation categories, some important problem solving characteristics may have been identified.

Finally, the analysis of the sequential data was of limited use. Sequences were only analyzed in pairs of consecutive moves and this analysis only involved a non-statistical comparison of percentages. If problem solving styles are to be identified, the sequence of procedures will most certainly offer many important clues.

There is also a limitation in terms of the interpretation of the results from the regression analysis. Extreme scores on some of the variables may have influenced the statistical results. However, because of the small sample size (N = 40), these scores were not deleted from the statistical analysis. Appendix D contains the histograms for several of these variables.

Implications for Research

One practical difficulty in undertaking studies in which the problem solving process is investigated is that of determining the procedures used. The use of the "think aloud" method as a data gathering technique is gaining in popularity. However, with the exception of the studies by Roth (1966) and Flaherty (1973), little research has been done to determine the reliability of this method. Do students attack problems differently when asked to verbalize their thoughts than when left alone to solve the problem? This and other questions related to the "think aloud" method need further investigation.

Krutetskii (1969) believes that the problem solving processes of students with above average or low mathematical aptitude differ from those students with average mathematical aptitudes. Therefore, this study should be replicated using students of higher and lower ability. It is possible that the problems selected for the lower ability students would have to be different. Since the procedures used in solving problems in different content areas may vary, this study should be repeated using content areas such as algebra, geometry, number theory, and calculus to determine if regularities exist in heuristic usage across areas of content as well as to suggest hypotheses specific to a content area.

Since the procedures used in problem solving may differ as students gain more experience in problem solving, exploratory studies such as this one should be done using students at various stages of mathematical sophistication. In particular, this study gives very little information on the use of the higher heuristics, therefore, it should be replicated using students who are very mathematically mature. Also the study may be replicated using open ended problems to determine if they will elicit the use of higher heuristics. One interesting aspect of MacPherson's model is the ordering of the heuristics (see Figure 2, p. 8) into four categories.

In order to determine if the ordering of the heuristics in MacPherson's model does indeed hold, a study will have to be carried out using subjects who might demonstrate the use of these heuristics. Also, a modification of the model, addition or deletion of some heuristics, could be carried out if the data obtained from such a study so indicate.

The coding system proved to be very reliable when used by coders who were both very familiar with the heuristics and the coding system. The system was designed primarily with the problems used in this study in mind and hence needs much refinement, especially in the core area. The coding system should be applied to new students and new sets of problems to see if the differences observed in the present study are maintained. If the system is used with students who are more mathematically sophisticated than the ones in this study, some of the categories might show greater frequency of usage.

The coding system allows for all of the procedures used to be coded in sequence. Some attempt was made to analyze the sequential data, but the information obtained was limited. If any problem solving styles are going to be identified, the sequence in which procedures are used will be a major source of information. More research is needed in developing and refining methods of analyzing sequential information.

This study did not include an instructional phase because it was desired to observe what heuristics were being used by students without any special instruction. Research is needed which includes instruction on the use of heuristics to determine if such instruction will increase the effectiveness of the heuristics beyond that accounted for by the core procedures. Research is also needed to determine the effect of instruction on the problem solving processes used by field independent and field dependent students. The students' knowledge of both core and heuristics, no matter how limited, should be taken into account in designing experimental studies which include an instructional phase. Exploratory studies such as Lucas (1972) and Kantowski (1975) where analysis of the problem solving process was undertaken during the study indicate that instruction in heuristics not only affects problem solving performance, but also has a positive effect on the procedures used by the students. A planned program of process research, where the role of heuristics in problem solving is systematically investigated is needed.

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Appendix A

PROBLEMS USED IN PILOT STUDY

A bookshelf whose length (in whole inches) lies between 30 and 50 inches holds exactly 5 poetry books each <u>a</u> inches thick, 3 history books each <u>b</u> inches thick and 5 dictionaries each <u>c</u> inches thick; (<u>a</u>, <u>b</u>, <u>c</u>, are unequal integers). It could instead hold exactly 4 poetry books, 5 history books, and 4 dictionaries. If instead, 7 poetry books and 4 history books are placed on it, how many dictionaries must then be added to fill the shelf exactly? Find also the thickness of each book and the length of the shelf.

A lady gave the postage stamp clerk a one dollar bill and said, "Give me some two-cent stamps, ten times as many one-cent stamps, and the balance in fives." How can the clerk fulfil this puzzling request?

A rectangular lawn is to be formed so that one side of a barn serves as one side of the rectangle. You have 300 feet of wire to enclose the other three sides. What are the dimensions of the rectangle if the area is to be of maximum size?

A group of students was surveyed to ascertain their preferences for sport watching on television. The three sports selected for the survey were football, golf, and hockey. The data were collected and organized as follows:
7 watched none of the sports
10 watched golf
8 watched hockey
6 watched football
1 watched football only
4 watched football and golf
5 watched golf and hockey
2 watched all three sports.

Questions:

a) How many students answered the survey?
b) How many students watched hockey only?
c) How many students watched hockey and football?

Find the sum of all the odd integers (positive) less than 100.

A circular table rests in a corner, touching both walls of a room. A point on the edge of the table is 8 inches from one wall and 9 inches from the other wall. Find the diameter of the table.

Find the solution set of $X + Y = XY$ where X and Y are real numbers.

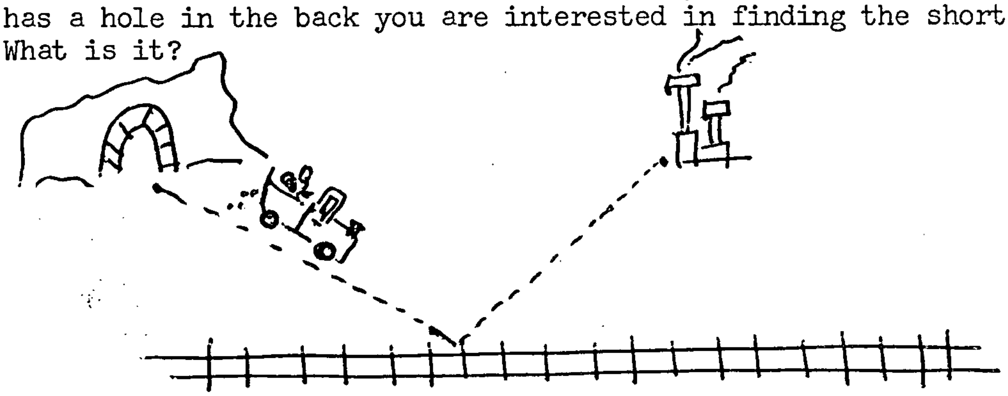
Find all X such that $|X - 1|^2 - |X - 1| = 12$.

What is the longest piece of metal rod which can be placed in a box of dimension 3 inches by 4 inches by 12 inches?

Imagine a row of 1000 lockers, all closed, and a line of 1000 men. Suppose the first man goes along and opens every locker. Suppose the second man goes along and shuts every other locker. Suppose the third man goes along and changes the state of every third locker (if it's open, he shuts it, and vice versa). Suppose the fourth man goes along and changes the state of every fourth locker, and so on, until all the men have passed by all the lockers. Which lockers are open in the end?

What is the maximum number of people you can seat at one time using 100 card tables provided you can only seat one person on a side of a table and the tables are arranged so that each time a new table is added, at least one of its sides must be placed against one side of a previously arranged table?

Suppose you have a gold mine and are hauling gold to the smelter in your truck which has a hole in the back. In order to get to the smelter you must drive from your mine to a railroad track and then to the smelter which is on the same side of the track as your mine. Since your truck has a hole in the back you are interested in finding the shortest route. What is it?



A real estate agency offers you a choice of two triangular pieces of land. One piece has dimensions 30, 25, and 40 feet; the other has dimensions 75, 90, 120 feet. The price of the larger piece is 4 times the price of the smaller piece. Which is the better buy?

A barrel of honey weighs 50 pounds. The same barrel with kerosene in it weighs 35 pounds. If honey is twice as heavy as kerosene, how much does the empty barrel weigh?

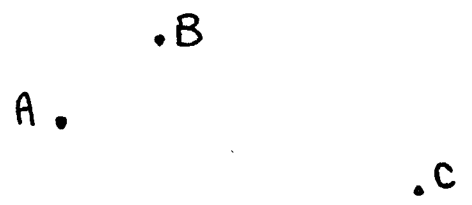
A farmer has hens and rabbits. These animals have 50 heads and 140 feet. How many hens and how many rabbits has the farmer?

In the following addition problem each letter stands for a different digit. If a letter appears more than once, it represents the same digit. What are the numbers if the letter "o" stands for 6?

$$\begin{array}{r} \text{MOM} \\ +\ \text{POP} \\ \hline \text{LOVE} \end{array}$$

Given a circle and a point in its plane. Outline the major steps you would use to construct a tangent to the circle through the point.

Three boys lit a firework and immediately ran off at the same speed in different directions. The figure below shows their position when the firework exploded. Where did the firework explode?

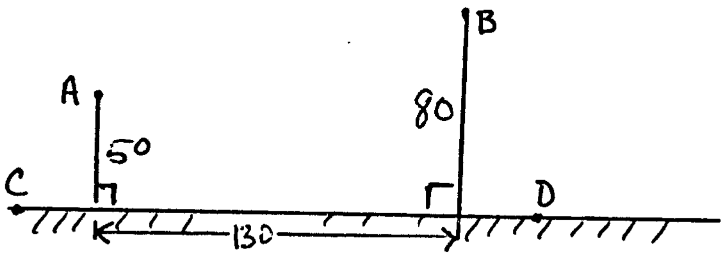


Prove that no term in the sequence 11, 111, 1111, 11111, ... is the square of an integer.

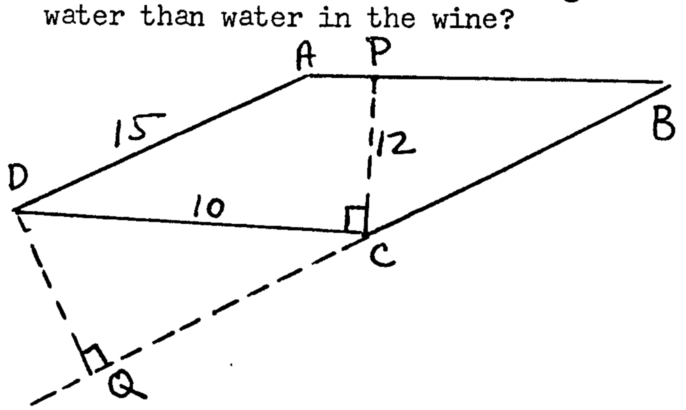
Given a regular hexagon and a point in its plane. Draw a straight line through the given point that divides the hexagon into two parts of equal area. Can you do the problem for any regular polygon? Other closed geometric figures?

Find the set of positive integers which have less than 5 divisors.

A yacht is moored at A, 50 meters away from a straight sea wall CD. The captain of the yacht wishes to row to the sea wall to collect a passenger and then to a speed-boat moored at B, 80 meters from the wall. Where should the passenger meet the captain to make this route as short as possible?



There are 2 glasses: One containing 10 spoonsful of wine and the other 10 of water. A spoonful of wine is taken from the first glass, put in the second glass, and mixed round. Then a spoonful of the mixture is taken and put back in the first glass. Is there now more wine in the water than water in the wine?



Given that ABCD is a parallelogram,
AB perpendicular to PC,
DQ perpendicular to BC,
Find the length of DQ.

Appendix B

PROBLEMS USED IN THE STUDY

Instructions

The purpose of this interview is to obtain some information on the ways in which students who are completing grade eleven Algebra solve mathematical problems. This is for my own information and will have absolutely nothing to do with your grade in Algebra.

What you are asked to do is to work a small set of problems, and to <u>think aloud</u> as you are working on each problem. This means that in addition to writing your steps on paper, you are being asked to <u>say everything that you are thinking about</u> while you work each problem. This includes all the little things that pop into your mind --whether you use them or not. You will be tape-recorded while you do this. The reason for recording your work is so that I can get a clearer picture of <u>what</u> thought processes you use and <u>when</u> you use them while solving a problem.

There are several other instructions I hope you will follow while working the problems:

1) Read each problem out loud before beginning to work it.

2) Write down everything you usually write while solving a problem. This includes scratch work, diagrams, equations, calculations, etc.; use as much paper as you need; talk as you write.

3) Do not erase anything; if you decide not to use something you've already written down, draw a line through it and then write down the correction.

4) I am <u>more</u> interested in how you go about solving a problem than in your solution. So please <u>do not ask</u> if any of your work is correct until you have completed the interview.

Since other students will be participating in these interviews, please do not discuss the problems or the interview with anyone. This would invalidate the results for the whole school. If someone has questions, tell him to see me. Your cooperation will be greatly appreciated.

Mathematical World Problems

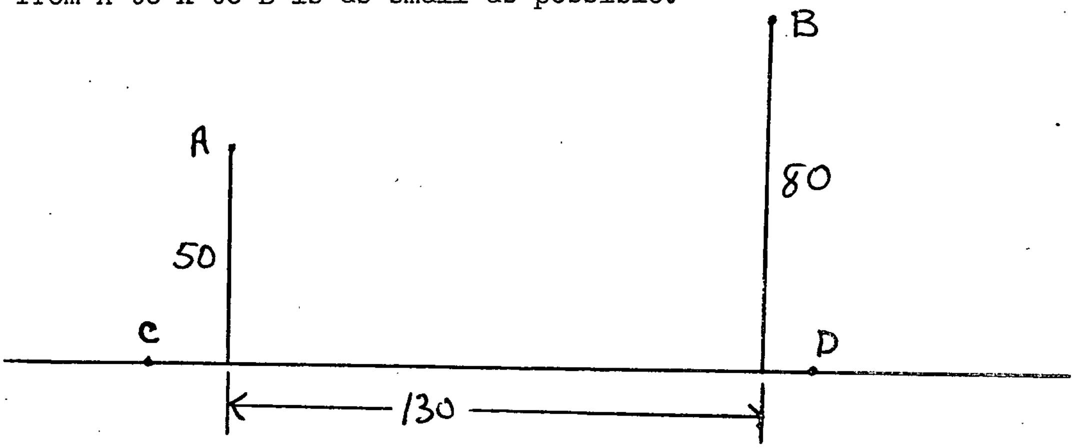
Warm-up Problems

A. The sum of two positive integers is 23. The sum of the first integer and twice the second is 34. Find the numbers.

B. T, D, H, J, and B are real numbers with the following relationships:
T is 20 less than D,
D is 50 greater than H,
H is 20 less than B,
J is 30 greater than T.
Which number is the largest? Which is second, third, fourth, and fifth largest?

Main Problems

1. The point A is 50 units from a straight line CD, and B is a point 80 units from CD. Find the point X on CD so that the distance from A to X to B is as small as possible.



2. The sum of a positive integer and twice another is subtracted from 100 and the difference is divisible by 5. The first integer is 10 times the second. What are the numbers?

3. A rectangle ABCD is formed so that one of its sides CD is on a line L. The sum of the lengths of the other sides, AB, AD, and BC, is 300 units. What are the dimensions of the rectangle if the area is to be of maximum size? Why does your solution give the maximum area?

4. What is the greatest distance between two points in a rectangular solid of dimension 3 units by 4 units by 12 units?

5. What is the maximum perimeter you can obtain by arranging one hundred one-inch squares? Each time a new square is added at least one of its sides must be placed against one side of a previously arranged square. Why does your solution give the maximum perimeter?

Real World Problems

Warm-up Problems

A. Jabbar scored 23 times in a basketball game. He scored 34 points, two for each field goal and one for each free throw. How many field goals did he make? How many free throws?

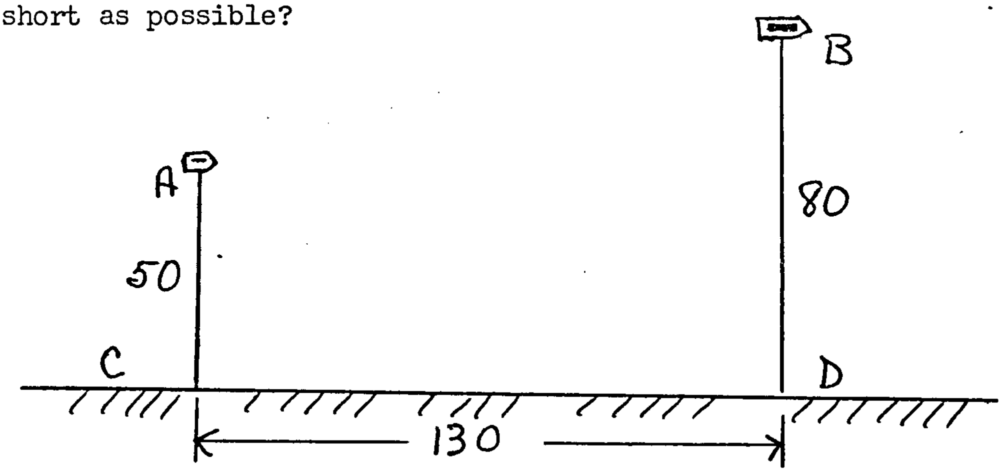
B. Tom, Dick, Harry, Joe, and Bill are lined up in these positions midway through a track meet:

Tom is 20 yards behind Dick,
Dick is 50 yards ahead of Harry,
Harry is 20 yards behind Bill,
Joe is 30 yards ahead of Tom,

At this point in the race, who is winning? who is second, third, fourth and fifth?

Main Problems

1. A yacht is moored at A, 50 meters away from a straight sea wall CD. The captain of the yacht wishes to row to the sea wall to collect a passenger and then to a speedboat moored at B, 80 meters from the wall. Where should the passenger meet the captain to make the route as short as possible?



2. A lady gave the postage stamp clerk a one dollar bill and said, "Give me some two-cent stamps, ten times as many one-cent stamps, and the balance in fives." How can the clerk fulfil this puzzling request?

3. A rectangular lawn is to be formed so that one side of a barn serves as one side of the rectangle. You have 300 feet of wire to enclose the other three sides. What are the dimensions of the rectangle if the area is to be of maximum size? Why does your solution give the maximum area?

4. What is the longest piece of metal rod which can be placed in a box of dimension 3 inches by 4 inches by 12 inches?

5. What is the maximum number of people you can seat at one time using

100 square card table? The tables are arranged so that each time a new table is added, at least one of its sides must be placed against one side of a previously arranged table. Why does your solution give the maximum seating plan?

Appendix C

THE CODING SYSTEM

This coding system is based on MacPherson's model of mathematical problem solving. It is designed for coding problem solving behavior of subjects who are asked to think aloud as they solve mathematical word problems.

The problem solving interview is audio tape recorded and the subject is asked to do all of his thinking aloud. He is instructed to verbalize all his writing and diagrams. Whenever the subject falls silent, he is reminded to do his thinking aloud.

The problems are presented to the subject one at a time. Each problem is typewritten at the top of the page with space underneath for the subject to work. Once the interview is complete, the subject's protocol is matched with his written work by marking the appropriate footage from the tape on his written work. The coding is done using both the subject's written work and the tape recorded protocol.

A subject's protocol for each problem is coded on the coding form (see next two pages). A matrix is used to sequentially code problem solving behavior. A summary form is used to compile information obtained from the coding matrix as well as recording other data related to the subject's problem solving behavior.

The procedures from the coding form are divided into five categories: core, heuristics, solution, reasonableness and concern. As the coder identifies the procedure from the subject's protocol, he enters a check (✓) in the appropriate row and first empty column. If the procedure is

Reading problem																											
Request defn. of terms																											
Recall same problem																											
Recall related problem																											
Recall problem type																											
Recall related fact																											
Draw diagram																											
Modify diagram																											
Identify variable																											
Setting up equations																											
Algorithms-algebraic																											
Algorithms-arithmetic																											
Guessing																											
Smoothing																											
Analysis																											
Templation																											
Cases-all																											
Cases-random																											
Cases-systematic																											
Cases-critical																											
Cases-sequential																											
Deduction																											
Inverse deduction																											
Invariation																											
Analogy																											
Symmetry																											
Obtain solution																											
Checking part																											
Checking solution																											
by subst. in equation																											
by retracing steps																											
by reasnble/realistic																											
uncodable																											
Exp. concern-method																											
Exp. concren-algorithm																											
Exp. concern-equation																											
Exp. concern-solution																											
Work stopped-soln.																											
Work stopped-no soln.																											

THE CODING FORM

CODING FORM SUMMARY

Number______ Problem#______ Tape#______ Tape Reading______

Solution: Correct____ Time spent on problem_________

Incorrect____ Errors: Arithmetic_________

None____ Algebraic_________

Other_________

# of Cycles_____ # of Changes_____

Heuristics used	# of times	Core procedures used	# of times

Remarks:

from the heuristic or core category (except recall of same problem, related problem, or problem type; draw diagram; modify diagram), he enters a 1, 2, or 3, depending upon the outcome:

1	Incomplete
2	Incorrect
3	Correct

The outcome from a core procedure is coded in terms of the correctness of the application of the procedure. For example, a 2 in the algorithm-arithmetic row indicates an error in the use of the algorithm, whereas, a 1 in the setting up equation row indicates the subject has written an incomplete equation. For the heuristic category, the outcome is coded in terms of the outcome of the problem or sub-problem after the use of the heuristic, not that the heuristic is incorrect. For example, a 2 in the analysis row indicates the outcome of the problem after the use of analysis is incorrect, not that the subject committed an error in the heuristic itself.

If two or more procedures occur simultaneously then a check or number is entered in the same column for each procedure. If a procedure is carried on longer than a column, it is enclosed in a box with the outcome entered in the last column enclosed. For example, a subject is using random cases and considers three cases along with using algorithms and drawing diagrams. In this instance the "block" for random cases would cover more than one column.

Core The core category includes behavior related to the subject's use of core procedures. All subjects start with reading problem and a check is entered in the first column. Reading the problem is also checked each time the subject re-reads the problem or part of the problem. If he continues to re-read the problem, with no other activity interspersed, only one check is entered in reading problem. Request definition of terms is checked if a subject asks the interviewer to define a term used in the statement of the problem.

Recall same problem or related problem is checked if the subject mentions that he has seen the problem or a similar problem before. If the subject classifies the problem by type, such as a work problem or a mixture problem, recall problem type is checked. If the subject merely mentions a related mathematical topic such as "this is a geometric problem" or "I'll have to write an equation for this", the recall categories are not checked. If a subject mentions a specific fact which is related to the problem, recall related fact is coded with a 1, 2, or 3, depending on the correctness of the fact. For example a 2 is entered if a subject mentions the Pythagorean Theorem as "the side opposite the right angle is twice as long as the longest of the other two sides", and recalls no other properties of right triangles.

Each time a subject draws a new figure, draw diagram is checked. If the subject only draws a few lines and immediately erases them, the category is not checked. Once a figure is drawn, if the subject alters it in any way by deleting or adding a line (or lines), modify diagram is checked.

If a subject identifies a variable, either verbally or written, identify variable is coded. The subject need not use the variable in an equation for this category to be checked. If a subject writes an equation and does not identify what the variables represent, identify variable is not coded. Setting up equations is coded if the subject writes any part of an equation. A subject need not identify the variables in the equation, however, if he does, the outcome assigned to setting up equations will depend on both the equations written and the variables identified by the subject. For example, in the stamp problem (problem #2, Appendix B), if X is the number of 2 cent stamps, 10X the number of 1 cent stamps and 5X the number of 5 cent stamps (identify variable would be coded 2) the equation $10X + 2X + 5(5X) = 100$ would be coded 3, the equation $10X + 2X + 5X = 100$ would be coded 2. If an algorithm is applied to an equation (the equation could be coded 1, 2, or 3), algorithms-algebraic is coded. If a subject copies an equation over on his paper with no change intended, no code is entered.

Algorithms-arithmetic is coded each time the subject uses or attempts to use an arithmetic algorithm. If the subject uses two or more algorithms consecutively, such as the subtraction of two pairs of numbers, followed by the multiplication of their differences, arithmetic-algorithms is only coded once. If the algorithm is performed mentally by the subject with no difficulty, such as one half of 500 is 250, it is not coded. Guessing is coded each time the subject selects a solution to the problem or a subproblem (such as an algorithm or equation) by guessing, provided the subject does not indicate some intention of further trials.

Heuristics This category includes the heuristic procedures used by the subjects. Smoothing is coded each time the subject disregards irrelevant information in the problem. Smoothing is coded the first time a subject draws a diagram but not for similar figures drawn after. For example, in the rod in the box, problem (#4, Appendix B) the subject can smooth the box to a rectangular parallelepiped, however, if he draws a rectangle, smoothing is coded as 2 and a check is entered in draws diagram. If a subject considers a special case such as using 50 feet of fencing in a problem rather than 300, smoothing is coded.

Analysis is coded whenever the subject breaks the problem into subproblems and attempts to solve the subproblems. If the subject merely mentions that the problem can be solved in terms of two or more subproblems, no code is made.

Templation is coded each time the subject mentions or recalls a category of content, i.e. recalls part of the core, which is related to the problem being solved. If the subject mentions only one specific fact, such as recalling the Pythagorean Theorem, recall related fact is coded. In a problem which involves a right triangle, if the subject mentions some of the theorems, definitions, or properties of a right triangle, then templation is coded. If the subject recalls any properties of the right triangle and uses any of them in solving the problem, a 3 is coded. If the information used is incorrect, a 2 is coded and if the subject does not use the information obtained, a 1 is coded.

The heuristics of cases are coded each time the subject either expresses an intention to use or uses two or more cases in one of the following manners: Cases-all is coded if the subject considers all possible cases in a problem where only a finite number of cases exist or the subject states there are only a finite number. Cases-random is coded if the subject uses cases in a random fashion with no apparant use made of the information obtained from previous trials or cases. If the subject's remarks indicate that he is using information from previous cases then cases-systematic or cases-sequential is coded. If the data are applied in a sequential manner, such as in solving the equation $12X + 5Y = 100$ by letting $X = 1, 2, 3, \ldots$, or finding the maximum of $X(300 - 2X)$ by letting $X = 50, 60, 70, \ldots$; cases-sequential is coded. If the subject is using cases in a non-sequential increasing or decreasing manner or using information from the previous cases such as finding the square root of 160 by refining each case to obtain a value closer to the square root, Cases-systematic is coded. Cases-critical is coded if the subject states a bound or bounds for the variables when using cases. For example, in solving the equation $12X + 5Y = 100$ where X and Y are positive integers, if the subject is using random or systematic cases and states "the largest value X can be is 8" then a 3 is coded for cases-critical.

Deduction includes both hypothetical and direct deduction. Deduction is coded whenever a subject's remarks indicate he has assumed a premise or is using data obtained from the problem as a premise and is attempting to determine as many implications as possible. Deduction is not coded if the subject states a single logical implication. Inverse deduction is coded if the subject's comments indicate he is intentionally attempting to work the problem backwards. For example, if a subject indicates he can find a formula for the general cubic equation $aX^3 + bX^2 + cX + d$, provided he can solve $aX^3 + cX + d = 0$, the inverse deduction would be coded depending on the outcome.

Invariation is coded whenever a subject's comments and actions indicate that he is intentionally excluding or fixing (renaming a constant) a variable, then attempting to solve the new problem and use its solution to help him solve the original problem. If a subject fails to solve the new problem or use its solution after indicating an intent to do so, a 1 is coded for invariation.

Analogy is coded if the subject recalls an analogous mathematical situation and intentionally makes use of its properties in solving the problem. If a subject recalls a non-mathematical situation such as "this is similar to a boat moving up and down on a river", recall problem type is coded. The heuristic of symmetry is coded if the actions and comments of the subject indicate he is intentionally making use of the inherent or constructed symmetry in the problem.

Solution The category of obtain solution is checked each time the subject obtains a solution to the problem.

Reasonableness The reasonableness category pertains to checking part of the subject's work and checking his solution. If a subject's actions and comments indicate he is checking part of his work, checking part is coded and the procedure or procedures he used are coded in the same column. If he indicates he is checking his solution, checking solution is coded and the procedure or procedures he used are coded in the same column. These procedures may include any procedures from the coding system. Four other categories are included in the coding system for checking. Checking by substituting in equation is checked if the subject replaces a variable in an equation with a value he believes to be a solution to the equation. Checking by retracing steps is checked if the subject explicitly repeats an operation or series of operations after he obtains a solution or indicates he is checking over his work. Checking by reasonable/realistic is coded if the subject indicates that he has tested whether his solution is reasonable either in terms of the problems or in terms of the "real world". If the subject checks his solution by reading the problem, checking by reasonable/realistic is coded. Uncodable is checked if the subject's comments indicate he has checked his work but does not indicate how e.g. "Oh! That can't be right" or "Let's see – that's not right".

Concern The concern category includes several kinds of comments that the subject may make about the procedures he is using or about his solution. Expresses concern about method is checked if the subject explicitly expresses concern about the procedures he is using. For example, the subject might say, "I'm not sure this problem can be solved by using two equations." or "This may not work, but I'll try some numbers anyway." Statements indicating that he does not understand the problem or that he does not know how to solve it are not coded. Expresses concern about algorithm is checked whenever the subject indicates a concern about the algorithm he has chosen. This includes concern about the process involved in carrying out the algorithm. However, an expression of concern such as "I'm not sure this problem can be solved with an algorithm" is coded concern about method. Expresses concern about equation is coded if the subject expresses concern about the equation or equations he has written or is attempting to write. An expression of concern such as "I'm not sure this problem can be solved using equations" is coded concern about method. Expresses concern about solution is coded whenever the subject's remark indicates that he doubts the correctness of his solution. Remarks that there is no solution are not coded.

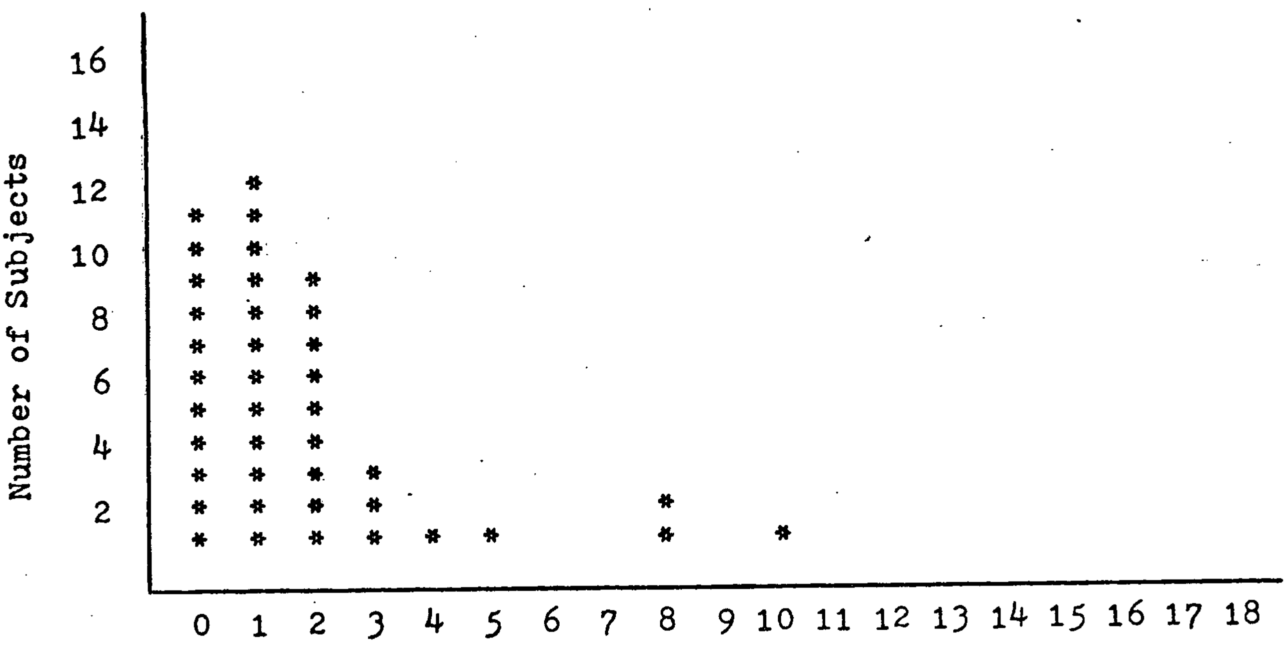
Work stopped-solution is coded if the subject stops work on the problem with a solution. If the subject quits working on the problem without a solution then work stopped-no solution is coded.

The summary form summarizes data from the coding form as well as recording other data related to the subject's problem solving behavior. Most entries on the summary form are easily identified. Errors, both arithmetic and algebraic, are counted from the coding form. An entry coded either 1 or 2 is counted as an error. A cycle is counted each time the subject's remarks indicate he is attempting to re-attack the problem. A cycle is also counted for the first attack on the problem. A change is counted each time the subject changes his mode of attacking the problem after a cycle. A subject changes his mode of attack if he uses a procedure from either core or heuristics that he did not use on his previous attempt at solving the problem.

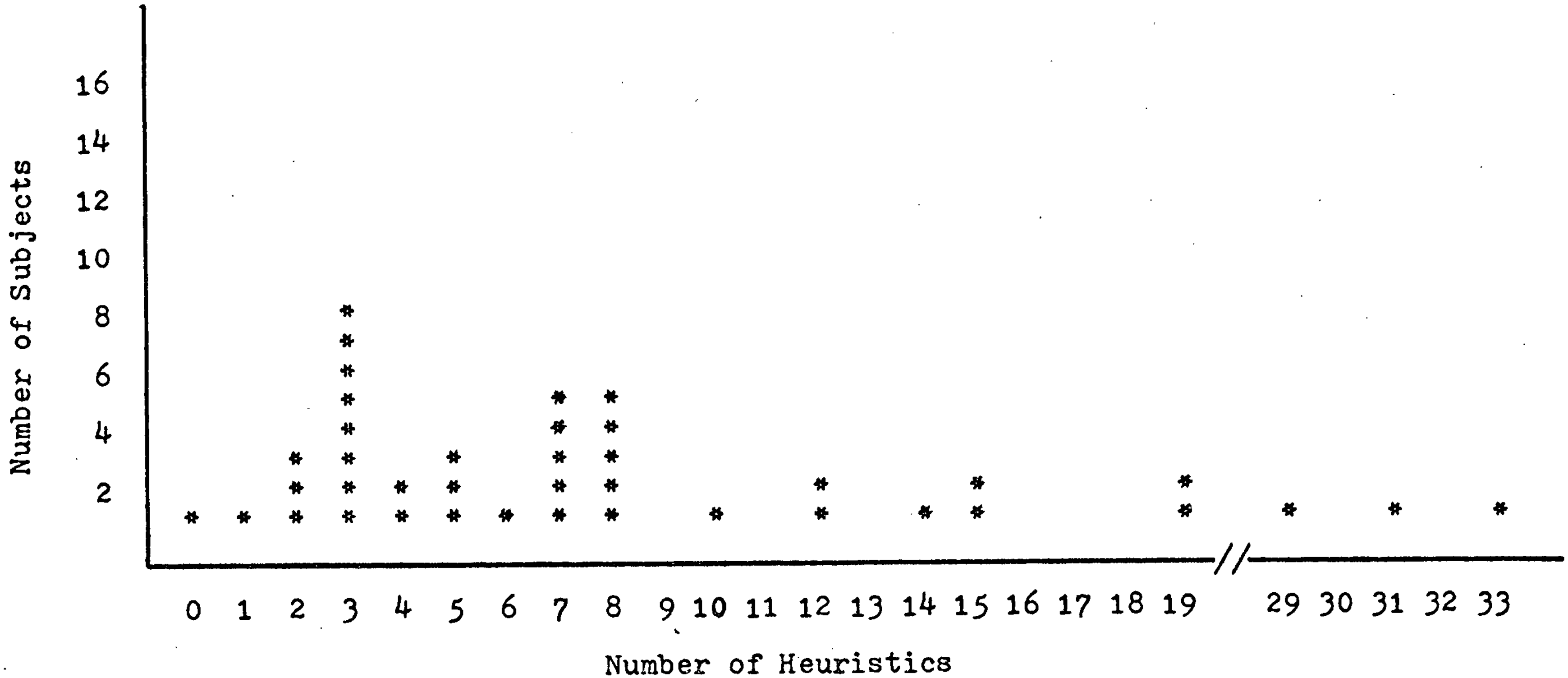
As illustrations of how the coding system operates, the written protocols for two problems are given in Chapter III together with the completed coding forms.

Appendix D

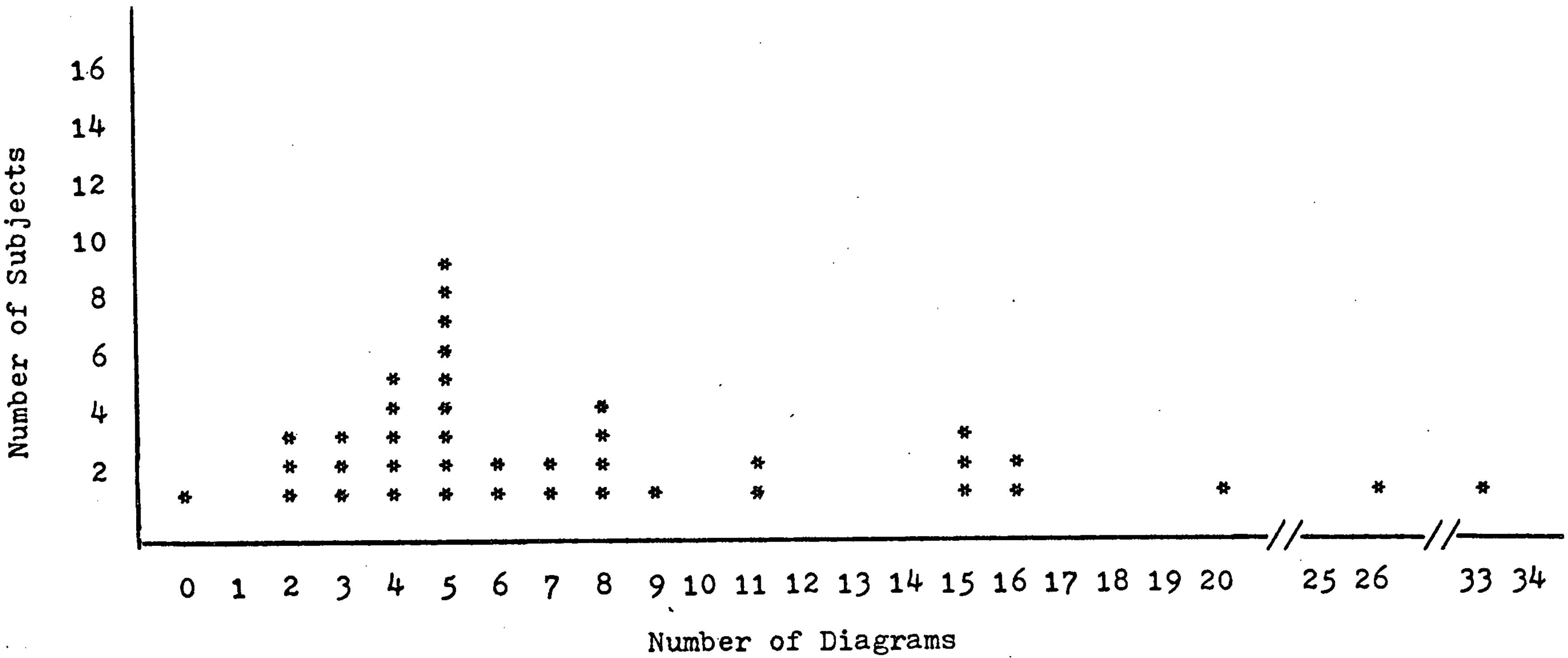
HISTOGRAMS OF SELECTED VARIABLES



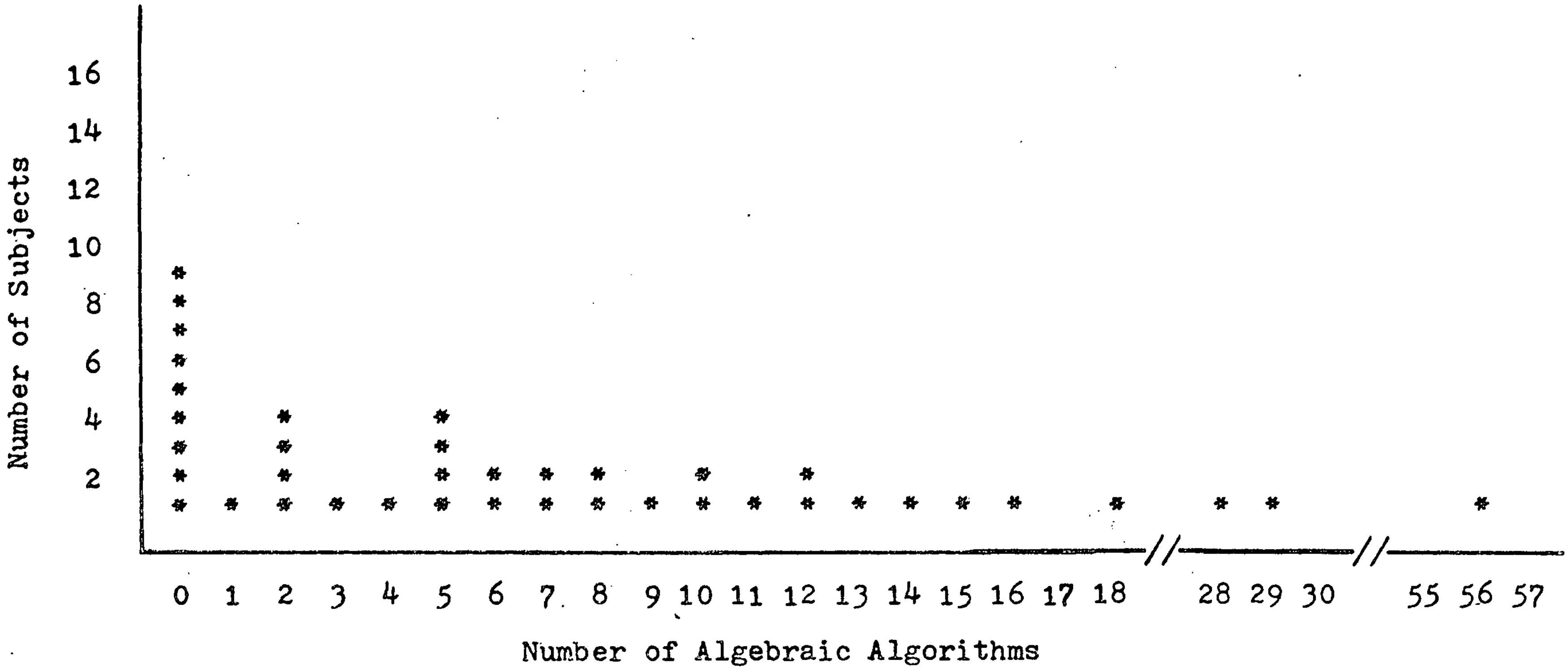
A Histogram Representing the Distribution of the Number of Changes Made by the Subjects in the Sample (N=40)



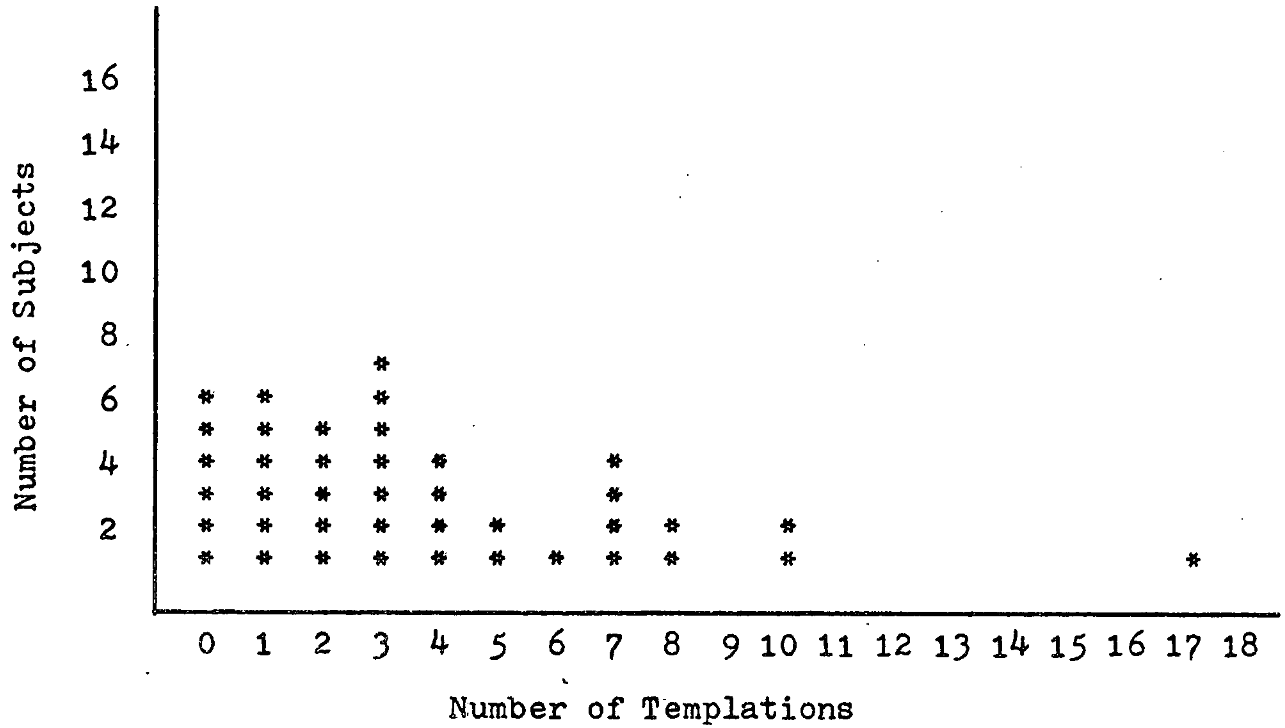
A Histogram Representing the Distribution of the
Number of Times Heuristics were used by the Subjects
(N=40)



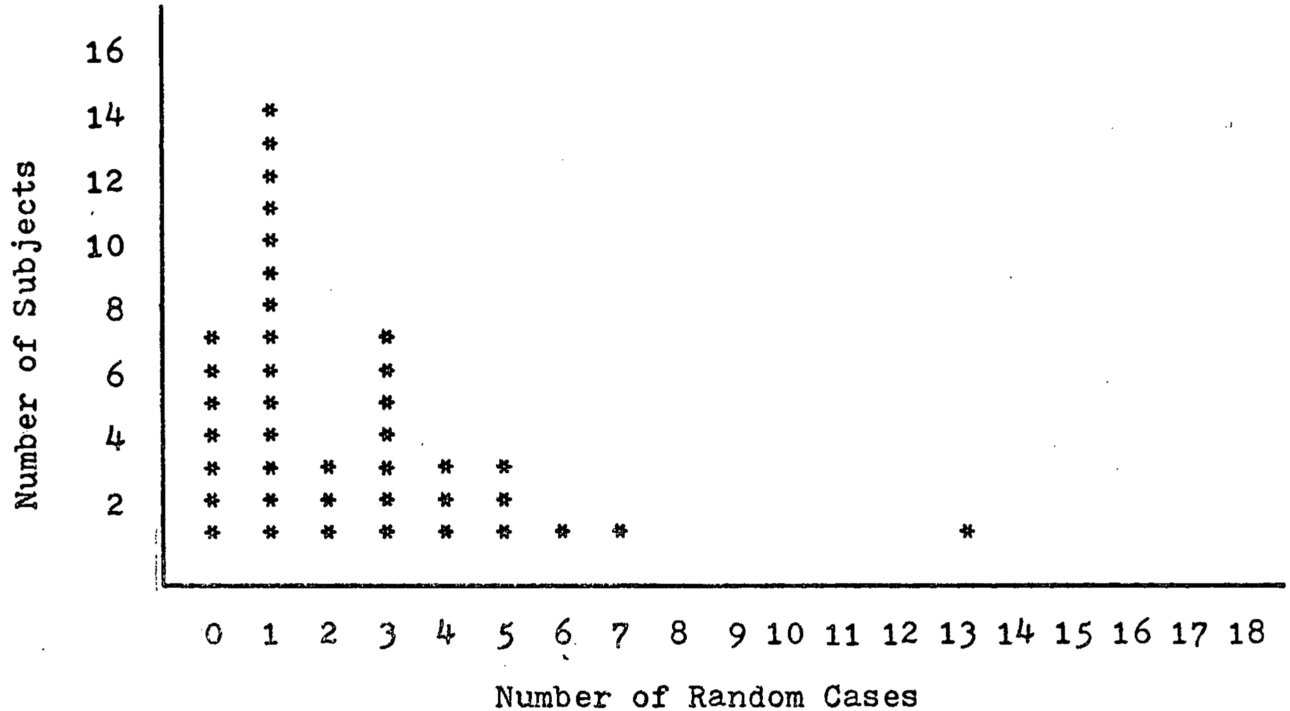
A Histogram Representing the Distribution of the Number of Times Diagrams were Drawn by the Subjects (N=40)



A Histogram Representing the Distribution of the
Number of Times Algebraic Algorithms were used by the Subjects
(N=40)



A Histogram Representing the Distribution of the Number of Times Templation was used by the Subjects (N=40)



A Histogram Representing the Distribution of the Number of Times Random Cases were used by the Subjects (N=40)