MATHEMATICAL UNDERSTANDING AND TONGAN BILINGUAL STUDENTS' LANGUAGE SWITCHING - *IS THERE A RELATIONSHIP?*

by

Sitaniselao Stan Manu

BSc. (Mathematics/Physics) 1993
The University of the South Pacific, Suva, Fiji.

MSc. (Mathematics) 1997
The University of Idaho, Moscow, ID, U.S.A.

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ABSTRACT

This study explores the relationship between Tongan bilingual students' language switching and their growth of mathematical understanding. The importance of this study lies not only in its ability to use the *Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding* as a theoretical tool for examining the relationship between language switching and growth of mathematical understanding, but also in its ability to demonstrate the theory's applicability and validity in a bilingual context.

Video case study was chosen as the most appropriate means of recording, collecting and examining the described relationship in a small-group setting. Two strands of data were collected between 2001 and 2002 from a selected number of bilingual students from five secondary schools in Tonga. Analysis of the students' language switching through the Constant Comparative Method resulted in the categorization of four main "forms" of language switching. These forms were identified, categorized, and developed from the data to provide a language for describing and accounting for the particular way Tongan students switch languages.

The evidence from the data clearly demonstrates how language switching both did and did not influence and was and was not influenced by the students' growth of understanding through the construction of mathematical meanings. At the same time, language switching was found to definitely enable the expression of growth of mathematical understanding. This study proposes that the effect of bilingual students' learning and development of understanding in mathematics is largely dependent on the kinds of mathematical images each bilingual student associates with his or her language. Therefore this study introduces the notion of "evocative" language switching, used for identifying, retrieving, and guiding one's existing understanding and ability to work with images. The evidence from this study is certainly applicable to other Tongan-type bilingual situations that involve individuals using words with no direct or precise translation between a dominant Western language and an indigenous language. Ultimately, the findings of this study challenge the assumption that Tongan-type bilingual students have enormous problems in the classroom. *Allowed the flexibility of language switching and thus access to appropriate terms and images in either language, they do not seem to be mathematically disadvantaged.*
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF MAPPINGS</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF GRAPHS</td>
<td>ix</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>x</td>
</tr>
<tr>
<td>LIST OF EXCERPTS</td>
<td>xi</td>
</tr>
<tr>
<td>LIST OF APPENDICES</td>
<td>xii</td>
</tr>
<tr>
<td>LIST OF ABBREVIATIONS</td>
<td>xiii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>xiv</td>
</tr>
</tbody>
</table>

**CHAPTER 1: PREVIEW and RESEARCH QUESTION**  
1.1 Introduction  
1.2 Formulating the Question  
1.3 Responding to the Question  

**CHAPTER 2: LITERATURE REVIEW**  
2.1 Introduction  
2.2 The Roots of the Study’s Theoretical Framework  
2.3 The Notion of “Understanding” in Mathematics Education  
2.4 The Interaction between Language, Mathematics, and Understanding  
2.5 Bilingualism in Mathematics Education  
2.6 The Notion of Language Switching  
2.7 Language Switching in Mathematics Education
CHAPTER 3: CONTEXT and THEORETICAL FRAMEWORK

3.1 Introduction

3.2 The Tongan Bilingual Context
   3.2.1 The First European Impact and its Effect on the Tongan Language
   3.2.2 Language in Tongan Education
   3.2.3 Mathematics Language in Tongan Mathematics Education

3.3 The Theoretical Framework
   3.3.1 Introduction
   3.3.2 The Development of the Pirie-Kieren Theory
   3.3.3 The Nature of the Pirie-Kieren Theory
      3.3.3.1 The Theory as “Levelled but Non-linear”
      3.3.3.2 The Theory as “Transcendentally Recursive”
      3.3.3.3 The Theory as a “Whole”
      3.3.3.4 The Theory as a “Dynamical Process”
   3.3.4 The Modes or Layers of Understanding
   3.3.5 The Two-Dimensional Model – A Mapping Technique
   3.3.6 The Special Features of the Pirie-Kieren Theory
      3.3.6.1 Folding Back
      3.3.6.2 Acting and Expressing Complementarity
      3.3.6.3 “Don’t-Need” Boundaries

3.4 The Appropriateness and Purpose of the Pirie-Kieren Theory

CHAPTER 4: METHODOLOGY, RESEARCH DESIGN, and METHOD

4.1 Introduction
4.2 Methodology: Video Case Study Research

4.2.1 Introduction

4.2.2 Case Study Research and the Notion of "Generalization"

4.3 Research Design

4.3.1 Introduction

4.3.2 Location

4.3.3 Participants

4.3.3.1 Selected Secondary Schools

4.3.3.2 Selected Forms (or Grades)

4.3.3.3 Selected Bilingual Students

4.3.4 Setting

4.3.4.1 Group Collaboration and Peer Discussion

4.3.5 Tasks

4.3.5.1 Topic of Investigation

4.3.5.2 Pictorial Sequence

4.3.5.3 The Set of Questions

4.3.5.4 The "Tongan" Task with Translated Set of Questions in Tongan

4.4 Method of Observation

4.4.1 Introduction

4.4.2 Video Analysis and the Notion of "Trustworthiness"

4.4.3 Advantages and Disadvantages of Video Study

4.5 Data Collection

4.5.1 Introduction

4.5.2 The Data Collection Technique(s)
6.2.1 Familiarization with the Data and the Role of Vprism 163
6.2.2 Timed-Activity Tracing and Flagging 164
6.3 Data Analysis: Categorization and Coding 165
6.4 Mapping using the Pirie-Kieren Model 168
6.5 The Data Analysis – Selected Groups and Individuals 169
6.5.1 Selai’s Growing Understanding of Patterns and Relations in Task 2 170
6.5.2 Malia’s Growing Understanding of Patterns and Relations in Task 3 189
6.5.3 Alaki and Malakai’s growing understanding of patterns and relations in Task 3 204
6.5.4 Christie, Ipeni, and Semi’s growing understanding of the topic in Task 3 217
6.5.5 Meki, Nanasi, and Rosina’s growing understanding of the topic in Task 3 225
6.5.6 Malia and Tupu’s growing understanding of the topic in the Tongan Task 229
6.5.7 Niko, Pola, and Seini’s growing understanding the topic in Task 3 232
6.5.8 Siona, Naati, and Samu’s growing understanding of the topic in Task 3 235
6.5.9 Malia and Tupu’s growing understanding of patterns and relations in Task 4 236
6.5.10 Hehea, Lani, and Leisi’s growing understanding of the topic in Task 1 239
6.6 Summary 243

CHAPTER 7: DISCUSSION and COMPARISONS 244
7.1 Introduction 244
7.2 Discussion: Relating Language Switching with Growth of Understanding 244
7.3 Responding to the Research Question 245
7.3.1 The Relationship between Language Switching and Image Making 246
7.3.2 The Relationship between Language Switching and the Move to Image Having 247
7.3.3 The Relationship between Language Switching and Image Having 249
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.3.4 The Relationship between Language Switching and Move to Property Noticing</td>
<td>151</td>
</tr>
<tr>
<td>7.3.5 The Relationship between Language Switching and Property Noticing</td>
<td>253</td>
</tr>
<tr>
<td>7.3.6 The Relationship between Language Switching and the Move to Formalising</td>
<td>255</td>
</tr>
<tr>
<td>7.3.7 The Relationship between Language Switching and Formalising</td>
<td>257</td>
</tr>
<tr>
<td>7.4 The Relationship between Language Switching and Folding Back</td>
<td>258</td>
</tr>
<tr>
<td>7.5 The Relationship between Language Switching and Acting and Expressing</td>
<td>263</td>
</tr>
<tr>
<td>7.6 Relating Growth of Understanding and Evocative Language Switching</td>
<td>266</td>
</tr>
<tr>
<td>7.7 The Tongan Task, Language Switching, and Growth of Understanding</td>
<td>272</td>
</tr>
<tr>
<td>CHAPTER 8: CONCLUSION AND IMPLICATIONS</td>
<td>273</td>
</tr>
<tr>
<td>8.1 Introduction</td>
<td>273</td>
</tr>
<tr>
<td>8.2 The Main Findings</td>
<td>273</td>
</tr>
<tr>
<td>8.3 Implications and Contributions of the Study</td>
<td>279</td>
</tr>
<tr>
<td>8.3.1 Contributions to the Pirie-Kieren Theory</td>
<td>279</td>
</tr>
<tr>
<td>8.3.2 Contributions to the Notion of &quot;Bilingualism&quot; and Bilingual Education</td>
<td>280</td>
</tr>
<tr>
<td>8.3.3 The Tongan Mathematical Language, Curriculum, and Bilingual Program</td>
<td>283</td>
</tr>
<tr>
<td>8.3.4 Implications for Teaching and Learning</td>
<td>285</td>
</tr>
<tr>
<td>8.4 Personal Reflections</td>
<td>288</td>
</tr>
<tr>
<td>8.5 Concluding Remarks</td>
<td>289</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>291</td>
</tr>
</tbody>
</table>
LIST OF MAPPINGS

**MAP 1:** Selai’s growing understanding of patterns and relations in Task 2  174

**MAP 2:** Malia’s growing understanding of patterns and relations in Task 3  196

**MAP 3:** Alaki’s and Malakai’s growing understanding of the topic in Task 3  209

**MAP 4:** Semi’s and Ipeni’s growing understanding of the topic in Task 3  223

LIST OF TABLES

<table>
<thead>
<tr>
<th>Table A</th>
<th>Tongan counting system for coconuts (niu)</th>
<th>56</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table B</td>
<td>Tongan discrete measurements for the length of “tapa” cloth</td>
<td>57</td>
</tr>
<tr>
<td>Table C</td>
<td>Number of selected participants from each of the participating schools</td>
<td>103</td>
</tr>
<tr>
<td>Table D</td>
<td>A typical transcribed video scene including “flags” of language switching</td>
<td>129</td>
</tr>
<tr>
<td>Table E</td>
<td>The categorization of word-mixing</td>
<td>134</td>
</tr>
<tr>
<td>Table F</td>
<td>Sample list of the kinds of language switching involved for one group</td>
<td>135</td>
</tr>
<tr>
<td>Table G</td>
<td>An initial list of general themes of language switching</td>
<td>138</td>
</tr>
<tr>
<td>Table H</td>
<td>Example of a timed-activity trace and “flagging” aspects of growth</td>
<td>164</td>
</tr>
</tbody>
</table>

LIST OF GRAPHS

**Graph A**  Percentage of language time used for Tongan and English (Grades 1-6)  61
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>Contrasting the SUP and CUP models of bilingualism (Baker, 2001)</td>
<td>48</td>
</tr>
<tr>
<td>Figure 2</td>
<td>The Pirie-Kieren model for the growth of mathematical understanding</td>
<td>70</td>
</tr>
<tr>
<td>Figure 3</td>
<td>Complementarities of Acting and Expressing in each of the layers</td>
<td>85</td>
</tr>
<tr>
<td>Figure 4</td>
<td>The pictorial sequences for all of the tasks using square blocks and cubes</td>
<td>110</td>
</tr>
<tr>
<td>Figure 5</td>
<td>The role of the image as a mediator in the translation between languages</td>
<td>139</td>
</tr>
<tr>
<td>Figure 6</td>
<td>A model for the bilingual individual’s shared underlying knowledge</td>
<td>145</td>
</tr>
<tr>
<td>Figure 7</td>
<td>The Tongan bilingual students’ types and forms of language switching</td>
<td>157</td>
</tr>
<tr>
<td>Figure 8</td>
<td>Relating bilingual students’ verbal expressions to their work with images</td>
<td>159</td>
</tr>
<tr>
<td>Figure 9</td>
<td>The continuing pictorial sequence in Task 2</td>
<td>170</td>
</tr>
<tr>
<td>Figure 10</td>
<td>Selai’s work indicates a common numerical difference of four blocks</td>
<td>172</td>
</tr>
<tr>
<td>Figure 11</td>
<td>Pictorial image of the pattern using added square blocks for each diagram</td>
<td>175</td>
</tr>
<tr>
<td>Figure 12</td>
<td>Pattern and relation between the numerical totals and diagram numbers</td>
<td>179</td>
</tr>
<tr>
<td>Figure 13</td>
<td>Continuing pictorial sequence of square blocks in Task 3</td>
<td>190</td>
</tr>
<tr>
<td>Figure 14</td>
<td>Tupu’s pictorial image – repeated addition of odd-number base layers</td>
<td>194</td>
</tr>
<tr>
<td>Figure 15</td>
<td>Malia’s pictorial image – addition of two extra square blocks at the base</td>
<td>194</td>
</tr>
<tr>
<td>Figure 16</td>
<td>Continuing pictorial sequence of square blocks in Task 3</td>
<td>205</td>
</tr>
<tr>
<td>Figure 17</td>
<td>Malakai’s pictorial image as a stack of vertical columns of square blocks</td>
<td>206</td>
</tr>
<tr>
<td>Figure 18</td>
<td>Alaki’s pictorial image of odd-numbered horizontal layers</td>
<td>207</td>
</tr>
<tr>
<td>Figure 19</td>
<td>Repeated stacking of odd-numbered base layers</td>
<td>208</td>
</tr>
<tr>
<td>Figure 20</td>
<td>Alaki’s pictorial images of the “extras” and relation to the “totals”</td>
<td>210</td>
</tr>
<tr>
<td>Figure 21</td>
<td>Ipeni recognizes the staircase pattern of each diagram as “sitepu” (steps)</td>
<td>219</td>
</tr>
<tr>
<td>Figure 22</td>
<td>Christie’s initial description of the 4th diagram as descending columns</td>
<td>220</td>
</tr>
<tr>
<td>Figure 23</td>
<td>Meki’s pictorial image as a stack of vertical columns of square blocks</td>
<td>226</td>
</tr>
<tr>
<td>Figure 24</td>
<td>Continuing pictorial sequence of square blocks in the Tongan Task</td>
<td>229</td>
</tr>
<tr>
<td>Figure 25</td>
<td>Image – addition of two extra square blocks at the ends of the base layers</td>
<td>233</td>
</tr>
<tr>
<td>Figure 26</td>
<td>The continuing pictorial sequence in Task 4</td>
<td>237</td>
</tr>
<tr>
<td>Figure 27</td>
<td>The continuing pictorial sequence in Task 1</td>
<td>239</td>
</tr>
<tr>
<td>Figure 28</td>
<td>Hehea’s structural rule for constructing each diagram</td>
<td>242</td>
</tr>
<tr>
<td>Figure 29</td>
<td>The schematic overview of the data analysis steps</td>
<td>245</td>
</tr>
</tbody>
</table>
## LIST OF EXCERPTS

<table>
<thead>
<tr>
<th>Excerpt</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Substitution with equivalent words</td>
<td>145</td>
</tr>
<tr>
<td>2</td>
<td>Borrowing of non-equivalent words</td>
<td>147</td>
</tr>
<tr>
<td>3</td>
<td>Substitution with Tonganised words</td>
<td>149</td>
</tr>
<tr>
<td>4</td>
<td>Translation – Repetition (direct) and Reformulation (indirect)</td>
<td>152</td>
</tr>
<tr>
<td>5</td>
<td>Shifting between two languages</td>
<td>154</td>
</tr>
<tr>
<td>6</td>
<td>Transcript of Selai’s image having in Task 2</td>
<td>171</td>
</tr>
<tr>
<td>7</td>
<td>Transcript of Selai’s reformulation in Task 2</td>
<td>176</td>
</tr>
<tr>
<td>8</td>
<td>Transcript of Selai’s property noticing in Task 2</td>
<td>180</td>
</tr>
<tr>
<td>9</td>
<td>Transcript of Selai’s generalizing pattern in Task 2</td>
<td>181</td>
</tr>
<tr>
<td>10</td>
<td>Transcript of Malia’s borrowing of “triangular” in Task 3</td>
<td>190</td>
</tr>
<tr>
<td>11</td>
<td>Transcript of Malia constructing image in Task 3</td>
<td>193</td>
</tr>
<tr>
<td>12</td>
<td>Transcript of Malia noticing “square numbers” in Task 3</td>
<td>199</td>
</tr>
<tr>
<td>13</td>
<td>Transcript of Malia generalizing a pattern in Task 3</td>
<td>201</td>
</tr>
<tr>
<td>14</td>
<td>Transcript of Malia calculating the 17th diagram in Task 3</td>
<td>203</td>
</tr>
<tr>
<td>15</td>
<td>Transcript of Alaki associating equivalent words in Task 3</td>
<td>210</td>
</tr>
<tr>
<td>16</td>
<td>Transcript of Malakai noticing property “square numbers” in Task 3</td>
<td>213</td>
</tr>
<tr>
<td>17</td>
<td>Transcript of Malakai generalizing “square numbers” in Task 3</td>
<td>214</td>
</tr>
<tr>
<td>18</td>
<td>Transcript of Malakai generalizing “add prime numbers” in Task 3</td>
<td>216</td>
</tr>
<tr>
<td>19</td>
<td>Transcript of Ipeni seeing pictorial image and Semi recognizing “square”</td>
<td>218</td>
</tr>
<tr>
<td>20</td>
<td>Transcript of Meki, Nanasi, and Rosina discussing meaning of prediction</td>
<td>227</td>
</tr>
<tr>
<td>21</td>
<td>Transcript of Malia and Tupu discussing “multiple” in Tongan Task</td>
<td>231</td>
</tr>
<tr>
<td>22</td>
<td>Transcript of Seini and Pola borrowing “prime” and “composite” numbers</td>
<td>233</td>
</tr>
<tr>
<td>23</td>
<td>Transcript of Naati and Siona arguing on translation in Task 3</td>
<td>235</td>
</tr>
<tr>
<td>24</td>
<td>Transcript of Malia and Tupu discussing “triangular” in Task 4</td>
<td>237</td>
</tr>
<tr>
<td>25</td>
<td>Transcript of Hehea and Lani discussing keywords in Task 1</td>
<td>240</td>
</tr>
</tbody>
</table>
LIST OF APPENDICES

Appendix 1: The Set of Questions (in English) for Task 1 to Task 4 308
Appendix 2: Tongan Translation of the Task Set of Questions 309
Appendix 3: Summary List of Notes 310
Appendix 4: Selai’s LHS Form 3 Group 1 Answer Sheet for Task 2 320
Appendix 5: Malia’s QSC Form 3 Group Answer Sheet for Task 3 321
Appendix 6: Alaki and Malakai’s TCA Form 3 Group Answer Sheet for Task 3 322
Appendix 7: Christie, Ipeni, and Semi’s LHS Form 3 Answer Sheet for Task 3 323
Appendix 8: Meki, Nanasi and Rosina’s AFC Form 2 Answer Sheet for Task 3 324
Appendix 9: Malia’s QSC Form 3 Group Answer Sheet for the Tongan Task 325
Appendix 10: Niko, Pola, and Seini’s AFC Form 2 Group 2 Answer Sheet for Task 3 326
Appendix 11: Siona, Naati, and Samu’s Form 2 Group Answer Sheet for Task 3 327
Appendix 12: Malia and Tupu’s Form 3 Group Answer Sheet for Task 4 328
Appendix 13: Hehea and Lani QSC Form 2 Group 2 in Task 1 329
Appendix 14: Map of the Tongan-Type Bilingual Societies in Oceania 330
(i) **Pirie-Kieren Layers of Understanding**

PK  Primitive Knowing
IM  Image Making
IH  Image Having
PN  Property Noticing
F   Formalising
O   Observing
S   Structuring
I   Inventising

(ii) **Acting and Expressing Complementarity**

ido  image doing
ire  image reviewing
ise  image seeing
isa  image saying
ppr  property predicting
pre  property recording
map  method applying
mju  method justifying
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The completion of this project comes with a great sense of relief after taking on what appeared, at times, to be an unattainable task. Yet, with all the challenges I faced, or whatever obstacles lay ahead, I managed to propel myself beyond my own perceived physical, mental, or emotional limits. I attribute this realisation of my inner potential to my strongest personal virtue: faith. Faith allowed me to continue on, despite life’s adversities, simply because I believed in myself, and more so, in God.

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memories. "Sepa" would have so easily filled my shoes, had he not been so unfortunate in his
life and in his ability to further his education. I wish that all of my grandparents were still around
to witness how far I have come in life, and to witness the fulfillment of a dream that was borne
out of their unconditional love, the sharing of their many gifts, and their passion for a better life.
1.1 Introduction

Since the 1970s, linguists have hotly debated the concept of "bilingualism", a concept characterized by an individual's use of two languages. Subsequent discussions have led toward the field of education, provoking ongoing dialogue among mathematics educators about the role and effect of bilingualism on students' mathematical performance and understanding. Such discussions have given rise to a variety of claims and "myths" about the nature of bilingualism and its effects – good or bad – on mathematics education.

In the past, educators have contended or simply assumed that bilingual students will find mathematics harder if the language of instruction is in their second language, and that such students are therefore naturally disadvantaged in mathematics in comparison to monolingual students (Gorgorió & Planas, 2001). Another naïve position, described by Clarkson and Dawe (1997), asserts that bilingual students' first language is irrelevant to their understanding of mathematics. As a result, educators have adhered to the belief that, in countries such as Tonga, students that are more competent in the dominant or second language are actually better educated and more intelligent than their indigenous peers (Fasi, 1999). This false belief has spurred a growing opposition among educators toward the use of indigenous languages in formal education, particularly at the secondary-school level; furthered by critics' claims that indigenous language learning is actually detrimental to a student's education (Fasi, 1999), particularly the learning of English (Cummins, 1981). Clarkson (1992) has raised questions about such longstanding educational misconceptions, by proposing further exploration of the potential
advantages of students' use of their native vernacular in academic situations, a move also supported by Setati (2004).

Research presented in this thesis challenges all of the preceding claims concerning the effect of bilingualism in mathematics education, in an effort to dispel existing cultural "myths" about the relationship between bilingualism and mathematics education in a Tongan-type bilingual context. The phrase, "Tongan-type bilingual context", refers to bilingual education systems similar to the one used in the South Pacific country of Tonga, and it reflects one of the many features of bilingual programs existing around the world. Indigenous groups such as the Tongans, which have emerged from a colonial past and are currently pursuing a cultural and linguistic renaissance, typically use a dominant second language, such as English, in their formal education. All of the island countries in the South Pacific (refer to map in Appendix 14) – currently associated with a dearth of published studies – offer clear examples of colonial languages, such as English (Tonga, Samoa, Fiji, Papua New Guinea, the Micronesian Islands, etc.), French (Tahiti, New Caledonia, and Wallis and Futuna), and Spanish (Easter Island [Rapanui] territories)1 being used primarily in secondary education. Such dominant languages are considered to have "superior" mathematics vocabulary than the Tongan-type bilingual students' first or indigenous language. Students who are taught mathematics through a dominant language like French or English, and who are learning mathematics within their own cultural environment, characterize the Tongan-type bilingual students. These students prefer to use their native language in mathematical activities, especially when they are engaged in discussions with their peers (Fasi, 1999). The inadequacy of indigenous languages in the language of mathematics and the students’ lack of proficiency in the language of instruction are two of the main reasons why these students, as well as teachers, switch languages during mathematical discourse

1 Lotherington (1997) offers a brief overview of the state of bilingual education in the South Pacific.
(Celedon, 1998). Thus, learning mathematics in and through a second-language context presents a double challenge for both teachers and students: the difficulty in learning the mathematics (and its vocabularies), and the prior need to understand the language of instruction (Adler, 1998). Ultimately, this dissertation attempts to question long-held misconceptions about bilingual learning contexts that have overlooked the fact that bilingual students' growth of mathematical understanding may be similar to monolingual students, and that bilingual students can voluntarily “swap” or switch languages in the process of talking about, or doing, mathematics.

In spite of these educational drawbacks and misconceptions about bilingual mathematics education, the National Council of Teachers of Mathematics (NCTM) has recognized that “students whose primary language is not the language of instruction have unique needs” (1989, p.142). Edwards (1999) noted that NCTM (1993) has further acknowledged and addressed the influences of language and culture of “minority” students by publishing a series of manuscripts to help “all readers develop a deeper understanding of, become more sensitive to, and stimulate a desire to learn more about Asian and Pacific Island students and their unique characteristics” (p. vi). The current study offers a new perspective about the unique characteristics of Tongan bilingual students.

Nevertheless, little attention has been focused on the ongoing problems faced by second-language learners in the field of mathematics education, demonstrated by the paucity of studies on this topic, particularly at the secondary level (Celedon, 1998). Yet the secondary-school level is a critical period for most Tongan-type bilingual students, especially for those with limited English proficiency, as they make the transition from instruction in one language at the primary-school level to another at the secondary-school level (Celedon, 1998; Fasi, 1999). It is at this transitional stage that research presented in this thesis investigates, within the Tongan-type
bilingual context, the nature of "growth of mathematical understanding", and how it relates to the notion of "language switching".

1.2 Formulating the Question

Language switching, or "code switching" as some linguists refer to it, is described by Baker (1993) as the way bilingual individuals alternate between two languages, whether in words, phrases or sentences, and is a term widely used worldwide (Clarkson & Dawe, 1997; Celedon, 1998; Qi, 1998; Fasi, 1999; Setati et al., 2002). Growth of mathematical understanding, characterized by Pirie and Kieren (1991a) as a "whole, dynamic, leveled but non-linear, transcendentally recursive process" (p. 1), is currently receiving much attention in the education community (Martin, Towers & Pirie, 2000; Powell, Francisco, Maher, 2001; Pirie, et al., 2001; Clark, 2001; Borgen & Manu, 2002). The prior mathematics and bilingual background of the author of this thesis allowed him to bring the two described phenomena together into a single thesis by asking: What is the relationship between Tongan bilingual students' language switching and their growth of mathematical understanding?

This study therefore appears to recall an earlier challenge by Cuevas (1984), who envisioned future research efforts being directed toward exploring the relationships among selected aspects of mathematical performance and understanding and various language skills, because up to the 1980s, the dearth of research on this kind of relationship "becomes almost a void when one restricts one's attention to students from a minority language group" (Cuevas, 1984, p. 140). The employment of language switching among Tongan-type bilingual students remains a challenging phenomenon for mathematics educators. What such switching entails, why it occurs, and how it
relates to aspects of the students’ growth of mathematical understanding, are all issues at the heart of the current study. The majority of the existing research in this area can be found in behavioral studies that assess and compare bilingual students’ levels of competence in both language and mathematics, without examining the qualitative nature of language switching and its effect on (growth of) mathematical understanding. Chapter 2 reviews the three main bodies of literature specifically relevant to the chosen research area: mathematical understanding, language switching, and the connection between the two phenomena.

Over the past 30 years or so, mathematics educators have been engaged in an ongoing discussion about what constitutes “mathematical understanding”. Some educators have been able to identify and label various kinds of understanding. Yet, these classifications are limited and insufficient to account for the author’s own mathematical understanding and recollections of valuable learning experiences in Tongan mathematics classrooms. As well, in accordance with Pirie’s (1988) analysis, such classifications fail to explain a learner’s observed levels of mathematical performance, or the complexity involved in individual mathematical understanding. Pirie’s perspective on this problem led her, along with Thomas Kieren, to develop the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding – a theoretical tool for observing and describing how mathematical understanding in a specific topic grows over time for a particular learner or group of learners. Chapter 3 looks at the details of this theoretical framework and how that choice, in turn, guided the methodology, and subsequently, the analytical process involved in the current study.

During the past 15 years, the emergence of the Pirie-Kieren theory has inspired a number of studies focusing on monolingual education, but no study has been done using the Pirie-Kieren framework in a bilingual situation. The research presented in this thesis redresses this omission.
A unique feature of this study therefore lies in the application of the Pirie-Kieren theory as a language for, and a way of observing and accounting for, the growth of mathematical understanding in a bilingual situation.

Three years ago, an ongoing interest in the Pirie-Kieren theory initially led the author to publish, along with a colleague, a case study that demonstrated incompatibilities between students’ written answers and observed evidence of their mathematical understanding (Borgen & Manu, 2002). But while that case study highlighted monolingual students’ evidence of mathematical (mis) understanding, it raised the possibility and challenge of finding an explanation for the relationship between (growth of) mathematical understanding and the students’ means of externalization – the language they used, coupled with the actions involved in the way they expressed their mathematical thoughts. Such curiosity guided the current research interest to pursue issues related to bilingual students’ language use and their mathematical understanding.

Because of the author’s own cultural, linguistic, and educational background, Tonga was considered the most appropriate and effective setting for this study. In order to understand the Tongan bilingual context, a brief historical account of the Tongan bilingual and mathematics education program is also offered in Chapter 3.

1.3 Responding to the Question

The complex and subtle connection between language switching and growth of mathematical understanding required an in-depth qualitative approach to research and data collection. In opting for a qualitative approach, “video case study” was chosen as the most appropriate means of
recording, collecting, and examining the described relationship in a small-group setting, a methodology chosen by previous researchers who have worked with the Pirie-Kieren theory (Towers, 1998; Martin, 1999; Thom, 2004). Chapter 4 focuses on the study’s methodology, research design, and the method of data collection used to address the research question.

For the current study, two strands of data were collected between 2001 and 2002 from five secondary schools in Tonga, where teachers and students use both Tongan and English in mathematics classrooms in various combinations. Although the use of both languages is common in the Tongan mathematics classroom, only one major research project has taken place in Tonga concerning the effects of bilingualism in mathematics education. Fasi’s (1999) comprehensive study on the relationship between language competence and mathematics achievement detected a significant correlation between the two variables, but failed to conclude whether one factor was a direct result of the other. The current study attempts to clarify this relationship through its response to the posed research question.

Subsequent chapters explore the results and findings of an analysis of the relationship between language switching and growth of mathematical understanding in detail. Chapter 5 summarizes a list of the identified categories (or “forms”) of language switching. Chapter 6 presents a detailed mapping and description of selected student groups and observed instances of their growth of mathematical understanding. Chapter 7 links the results and analyses of the previous two chapters in order to specifically address the research question. Finally, Chapter 8 summarizes the study’s findings based on the previous three chapters, responds to the research question, and considers the implications of the study for research, teaching, and learning.
2.1 Introduction

This chapter addresses the three main bodies of literature specifically relevant to the chosen research area: mathematical understanding, language switching, and the connection between these two. The link between mathematical understanding and language switching touches on the wider domains of bilingualism and mathematics education, as well as the role of language – in the sense of speech, or Halliday’s (1978) notion of “natural language” in its spoken form – in mathematical understanding.

The first main body of literature considers the notion of mathematical understanding, with a summary of the various philosophical beliefs, contemporary models and theories about, and of, mathematical understanding.

The second main body of literature looks at the connection between mathematics and the natural language, but examines that connection on a smaller scale. This review begins with a brief discussion of the role of language in mathematics education, and then it extends to the use of two languages in bilingual situations. The focus then shifts to examining the nature of mathematical understanding in bilingual situations and the existence of bilingualism within a mathematical context.

The third main body of literature explores the notion of language switching. This bilingual phenomenon has also emerged as a popular topic among mathematics educators, particular those
involved with bilingual situations. However, language switching rests in the domain of language (Halliday, 1978); it is regarded as a natural language process and not a mathematical switch of any kind, and therefore only plays a secondary role in this study.

2.2 The Roots of the Study’s Theoretical Framework

Previous studies by Towers (1998) and Martin (1999) have put forward similarly detailed discussions regarding the nature of growth of mathematical understanding, and their findings have been discussed in conference papers and various publications (Martin & Pirie, 1998; Martin, Towers & Pirie, 2000; Towers, 2001). But in order to understand the nature of growth of mathematical understanding, much of the attention in this section is directed toward the notion of “understanding”.

This study’s theoretical framework has its roots in constructivism, a theory of knowing and learning that has gained wide recognition for explaining the everyday reality of mathematics classrooms. The basic tenets of constructivist philosophy centre on a subjective construction of reality; constructivism therefore focuses on how ideas are created in the mind of the individual (Bauersfeld, 1988; Tall, 1991; Cobb, Yackel & Wood, 1992; Greeno, Collins & Resnick, 1996). Westrom (2001) describes constructivism plainly as follows:

It is based upon the observation and assumption that every learner must construct his or her own knowledge. A teacher cannot ‘put’ information into a learner’s brain; rather each learner makes his or her own mental constructions to build knowledge in his or her own mind. Of course these constructions are not the same for every student, nor do they always match what the teacher intended. The teacher is limited to placing information
and experiences in the student’s environment. The student then combines this information and experience with previously learned information and experience to create new knowledge (p. 1).

Constructivist philosophy lies contrary to the empiricist’s and behaviorist’s perspectives, whose ideas and beliefs are built exclusively on external observations of a stimulus and a response, so such a perspective “refuses to speculate about the internal workings of the mind” (Tall, 1991, p. 7). Yet, constructivists over the last two decades have gone on to suggest different perceptions of the individual’s “reality”. This expanded perception has led to the development of newer and more pragmatic ways of thinking, including “radical constructivism” and “social constructivism”, and later, the emergence of “enactivism”.

von Glasersfeld (1987) is a major proponent of the principles of radical constructivism, in which an individual can only know for certain that which he or she has created in his or her mind. This philosophy, which is a re-interpretation of Piaget’s notion of genetic epistemology (Martin, 1999), maintains that each individual’s experience can only be understood within a particular context; that is, it is an experience unique to that individual, and therefore inaccessible to others. von Glasersfeld (1995) outlines two basic principles of his “radical” model of constructivism: one, “knowledge is not passively received but built up by the cognizing subject”, and two, “the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality” (p. 18). The effect of this radical view involves reconstruction of basic concepts such as “knowledge”, “truth”, and “understanding” (von Glasersfeld, 1995).

For instance, radical constructivism does not claim “ontological truth” (von Glasersfeld, 1989) – a perspective that has come under criticism, not only in denying knowledge as a “true” representation of the individual’s world, but also in ignoring the social influences involved in the individual’s construction of his or her notion of the world (Ernest, 1991).
Social constructivists, on the other hand, embrace the role of social interaction in the learning process, and thus recognize the vital relationship between individual and collective knowledge (B. Davis, 1996); a factor social constructivists (e.g. Lerman, 1992; Ernest, 1991; 1994) claim is downplayed in the radical constructivist's view. Cobb (1995) asserts that social interaction constitutes a crucial source of opportunities to learn mathematics. Furthermore, Cobb (1995) argues that radical constructivism and social constructivism simply evolved to address different aspects of any learning situation.

The emergence of enactivism from constructivism offers an alternative, yet more holistic, way of explaining how individuals learn. Enactivism views understanding as an interactive process, as well as a continuously unfolding phenomenon, rather than a state to be achieved (B. Davis, 1996). Enactivist theory is based on the belief that, although understanding is still the individual's creation, the individual's mathematical understanding is linked with the environment in which that understanding was created (Varela, Thompson, & Rosch, 1991). Therefore, this understanding is manifested not by itself, or solely within the individual, but in relation to the mathematical space in which the individual is acting. Varela, Thompson, and Rosch (1991) first postulated the enactivist approach by drawing on modern and ancient philosophies, such as those found in biology, neuroscience, and Buddhism. These theorists went on to define and "situate cognition not as problem solving on the basis of representations, but as embodied action" (Towers, 1998, p. 11).

Enactivism challenges radical constructivism by contending that if one focuses entirely on the individual cognizing agent, then the "fluidity of the context", and the individual's role in the environment and larger community, are often unaccounted for (B. Davis, 1996, p. 8). This
perspective describes how “context is not merely a place which contains the student: the student literally is part of the context” (Davis, Sumatra & Kieren, 1996, p. 157). In addition, enactivism differs from social constructivism by focusing upon “who you are (a notion that subsumes the individual and the context), not just where you are (which considers the subject as separate from the environment)” (Towers, 1998, p. 12, original emphasis). This philosophical view recognizes how an individual’s knowledge depends on being in the world, inseparable from his or her body, language, and social environment (Martin, 1999). Hence, enactivism “allows a space for discussing understanding and cognition which recognizes the inter-dependence of all participants in an environment” (Towers, 1998, p. 12).

2.3 The Notion of “Understanding” in Mathematics Education

Since mathematics education emerged in the 1970s as a popular field for research, researchers have developed a growing interest in the notion of understanding in the context of mathematics. Skemp’s (1976) article on the existence of various types of mathematical understanding is widely credited with sparking subsequent discussions about this subject in the academic field. In 1976, Skemp considered “understanding” as one of two terms in mathematics (the other being “mathematics” itself) that often leads to serious confusion and misinterpretation among mathematics educators. Skemp explains the need to distinguish between two alternative meanings for the term “understanding” in mathematics, since it is the lack of such a distinction that lies at the root of the many misconceptions and difficulties students, teachers, and educators encounter in mathematics education. In explaining the notion of understanding, Skemp (1976) differentiates “relational” from “instrumental” understanding. Relational understanding describes an individual knowing both what to do and why, while instrumental understanding involves the
individual mentally applying specific mathematical rules without using his or her capacity to reason. Furthermore, Skemp's (1976) distinction between the two types of mathematical understanding leads to his detection (Skemp, 1978) of another subtle distinction between what he considers as two different subjects: "relational mathematics" and "instrumental mathematics". In Skemp's view, the word "mathematics" (as a subject) is often confused with the word "understanding" (as a way of thinking). To clarify the difference between these two terms, Skemp (1978) notes that the content of mathematics in relational and instrumental learning may be the same, but the knowledge involved in each process is so different, that they might as well be regarded as two different forms of mathematics. In addition, Skemp (1978) describes the nature of instrumental mathematics as a subject that is easily understood, and, because less knowledge is involved in the learning process, the paths to the "right" answer are relatively quick. Relational mathematics, however, is more adaptable to new tasks and can be easily remembered as parts of a whole connected body of knowledge that is effective as a goal in itself. In addition, relational mathematics suggests that the quality of any relational schema is "organic". In this instance, a schema refers to a conceptual structure an individual can use to execute an unlimited number of plans. The organic quality of such a structure acts as an agent for growth.

Towers, Martin, and Pirie (2000) note that a significant factor in Skemp's (1978) descriptions is the fact that the learner is seen as a cognizing individual, rather than a passive recipient of knowledge – a distinction welcomed by constructivists. The concept of "individuality" is demonstrated by Skemp's (1978) formulation of four main characteristics of relational learning, involving the gradual building up of a conceptual structure, as it can be distinguished from instrumental learning (p. 34):
1. The means toward understanding thereby become independent of particular ends.
2. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself.
3. The more complete one’s schema, the greater the feeling of confidence in one’s own ability to find new ways of “getting there” without outside help.
4. As the schemas enlarge (since a schema is never complete), the process often becomes self-continuing.

Using Skemp’s (1976) model, and combining it with Bruner’s (1960) distinction between analytical versus intuitive thinking, Byers and Herscovics (1977) proposed a tetrahedral model of understanding. Their model identified four complementary modes of understanding: instrumental, relational, intuitive, and formal. The “intuitive” mode involves the visual ability to solve a problem, while the “formal” mode represents the ability to connect mathematical notation and symbolism with relevant mathematical ideas. The nature of Byers and Herscovics’ (1977) tetrahedral model, and the interplay between its modes, reveals understanding as a complex, dynamic process, unlike Skemp’s multi-valued, static distinction (Martin, 1999).

Tall (1978) also emphasized the “dynamics” of mathematical understanding, critically responding to earlier models he claimed would create an unnecessary, and seemingly endless, list of categories. Tall suggested a “dynamic interpretation, which sees the various kinds of understanding as different facets of a single development” (1978, p. 2). From Tall’s perspective, mathematical understanding, therefore, arises from constantly changing mental patterns that characterize mathematical thinking. These changing mental patterns are called schemas, similar to Skemp’s (1978) earlier classification. While Tall (1978) insists on de-classifying unnecessary categories, Skemp (1979) extends upon his own existing model by adding a third mode of understanding, called “formal” (or logical) understanding, defined similarly to one of Byers and Herscovics’ (1977) tetrahedral categories. Skemp (1981) then goes on to identify within each
mode two complementary levels of thinking, which he calls "intuitive" and "reflective", and then later adds a fourth mode, that of "symbolic" understanding, to include understanding of the mathematical symbols and notations (Skemp, 1982).

Bergeron and Herscovics (1981) critique the early models of understanding as heavily oriented towards problem solving, and declare them inadequate in describing the knowledge involved in concept formation. Bergeron and Herscovics' (1981) initial model consists of four levels of understanding in the construction of mathematical concepts: intuitive understanding, initial conceptualization, abstraction, and formalization. In 1982, they go on to clarify each different level of understanding by characterizing initial conceptualization as "procedural understanding" and then re-defining this level of understanding to encompass both the acquisition of mathematical procedures as well as the ability to use these procedures appropriately. Further development of their initial model led Herscovics and Bergeron (1988) to eventually present a two-tiered, extended model of understanding, used to analyze the development of particular mathematical concepts. This extended model shows that "the understanding of a particular mathematical concept must rest on the understanding of the preliminary physical concepts" (Herscovics & Bergeron, 1988, p. 7).

Nickerson (1985) did not offer a model or theory of mathematical understanding, but instead discussed various aspects of understanding. Among his contentions, Nickerson noted that understanding is a context-dependent concept, in which an idea can be understood best from the context in which it is situated. But, he added, an incorrect answer is a reasonably good indication of a lack of understanding (Nickerson, 1985). Furthermore, understanding must also depend on the existing knowledge one has about the concepts involved, and requires not only having such knowledge, but also doing something useful with it. Understanding, for Nickerson (1987), can
also be seen as an active process, one that varies in its degree of complexity or completeness,² and which can be described or investigated in terms of the "breadth" and "depth" of a concept's connectedness (Nickerson, 1985). Such a proposal, in which "deep" understanding can be demonstrated in a variety of ways, is also the basis of the previously discussed Herscovics-Bergeron model.

Following Skemp’s initial definition (1976) of the dichotomy between instrumental and relational understanding, Nesher (1986) argued for the importance of the connection between learning algorithms (or algorithmic performance) and conceptual understanding, and the fact that the separation between the two is "impossible" at any stage of learning. Nesher (1986) concludes that there are different levels of understanding, and that there exists a relationship between understanding and knowing algorithmic procedures: one contributes to the other. Nesher’s (1986) notion of understanding is consistent with Nickerson’s (1985) proposal: that understanding is never going to be complete. Nesher also observed that the levels of understanding are not always ordered, and that no one can know precisely which stages lead to understanding.

By contrast, von Glasersfeld’s (1987) radical constructivist view considers conceptual structuring and re-organization as essential to the individual’s construction of knowledge. von Glasersfeld sees algorithmic operations, such as using multiplication tables successfully, as an inadequate demonstration of mathematical knowledge. He associates such types of knowledge with those learned through strict training – a process von Glasersfeld (1987) claims is mainly associated with the way animals learn. von Glasersfeld (1987) further argues that what

² Nickerson (1985) discusses the difficulty with the so-called 'Hegelian' theory that "in order to know all there is to know about anything, one must know all there is to know about everything, because everything is related to everything" (p. 231). This view, according to Nickerson, is not very helpful because it denies the reality of different degrees of complexity in various levels of understanding.
determines the value of the re-organized, conceptual structures is “their experiential adequacy, their goodness of fit with experience, their viability as a means for the solving of problems, among which is, of course, the never-ending problem of consistent organization that we call understanding” (p. 5, original emphasis). This view recalls Tall’s (1978) emphasis on the dynamic nature of understanding. von Glasersfeld (1987) further concludes that the “process of understanding…is analogous to the process of coming to know in the context of experience…it is a matter of building up, out of available elements, conceptual structures that fit into space as is left unencumbered by constraints” (p. 10).

Unlike von Glasersfeld’s (1987) radical view, the role of algorithms in mathematical thinking led Sfard (1991) to conclude that abstract concepts could be considered in two fundamentally different but complementary ways: “structurally” (as objects) and “operationally” (as processes). Sfard observes that the process of learning involves the interplay between these two incompatible approaches. She goes on to label three stages in concept development that she calls interiorization, condensation, and reification. “Interiorization” acquaints a learner with the processes that give rise to new concepts; “condensation” involves the ability of the learner in managing lengthy sequences of operations; and “reification” refers to a state of “ontological shift” in which the learner sees a familiar piece of mathematics in a totally new way (Sfard, 1991, p. 19).³

Sierpinska (1990), however, views understanding in an “active” sense, conceiving it as an act of grasping the meaning rather than a process or way of knowing. Moreover, Sierpinska sees understanding as an action that involves overcoming mathematical obstacles. Yet, Sierpinska

³ The next chapter describes how “reification” appears to resemble Pirie and Kieren’s notion of “don’t-need boundary” (refer to Section 3.3.6.3).
recognizes that the level of understanding changes with growth of knowledge as a process, rather than as an act. She goes on to distinguish two categorized levels of understanding, involving “intuition” and “logico-physical abstraction”, based on the previous work by Herscovics and Bergeron (1989). But Sierpinska’s (1990) work appears to overlook the question of how meanings are constructed. In her model, intuitive understanding is based essentially on visual perception, while logico-physical abstraction involves one’s awareness of one’s logico-physical invariants. Sierpinska (1990) then concludes that depth of understanding can be measured “by the number and quality of acts of understanding one has experienced, or by the number of epistemological obstacles one has overcome” (p. 35). Kieren, Pirie, and Reid (1994) view such a perspective of understanding as “a multileveled activity in which a person perceives and overcomes epistemological obstacles” in order to move on to a higher level (p. 49).

R. B. Davis (1984) initially built his own theory from the previous findings of Minsky and Papert (1972), suggesting that understanding occurs when a new idea is fitted into a larger framework of previously assembled ideas. This representational perspective, endorsed by mathematics educators such as Hiebert and Carpenter (1992), sees growth of understanding as a structured, connected network that grows bigger (in depth) and more organized (in complexity). R. B. Davis (1991) later surmised that a theory for the growth of mathematical understanding offered an exciting observation if one could speak about “the development of the students’ mathematical thinking” ⁴ (p. 232). From this observation, evaluation of one’s level of understanding moves from the view of taking “snapshots” of understanding at specific points in time, to mapping the growth of understanding of one’s mental representations over a period of time. To conduct such an evaluation, R. B. Davis (1991) prefers the direct observation of videotaped recordings in

⁴ R. B. Davis (1991) actually directed this comment toward the nature of the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding that is the theoretical framework for this study.
settings such as small group discussions. Furthermore, he contends that mathematical understanding, as Nickerson (1985) observed earlier, requires not only having connected knowledge, but also doing something useful with it (R. B. Davis, 1991). The notion of connectedness in one’s knowledge suggests that the more one knows about a particular concept, the more complex one’s knowledge is, and the better one understands the concept (Nickerson, 1985).

In the same manner, Hiebert and Carpenter (1992) later stated that the depth and strength of one’s understanding increases as the network of representations grows, and as the interconnections “become strengthened with reinforcing experiences and tighter network structuring” (p. 69). This representational view also sees growth (of understanding) in the manner in which internal and external representations (for a particular concept) constantly affect each other, and the network of representations becomes larger and better organized. Given that the connected network described in Hiebert and Carpenter’s (1992) model is continually undergoing re-organization and re-structuring, growth of understanding, then, can be considered dynamic in nature.

Duffin and Simpson (1995) initially defined understanding as the “awareness of internal mental structures” (p. 167). A “breakthrough” in their work, they claimed, came about through their interpretation of Sierpinska’s (1990) idea that an act of understanding took place through the use of network connections to solve problems. From there, Duffin and Simpson (1995) went on to name three components of understanding: building, having, and enacting. “Building” refers to the formation of the internal mental structures and the connections involved at a particular time during the learning process. Borrowing the idea from Nickerson (1985), Duffin and Simpson (1995) explained that a chain of connections from a single concept constitutes the learner’s
breadth and depth of understanding. "Having", however, is the structural state of the connections at any given time, while "enacting", means the use of the available connections at a particular moment to solve a problem or construct a response.\(^5\)

Throughout the discussion in mathematics education about the notion of understanding, the term "meaning" is used interchangeably in a mathematical context. B. Davis (1996) argues from an enactivist perspective that these two terms – understanding and meaning – can be separated by "rigid definition". B. Davis then describes how meaning is closely linked to "the objectifying realm of definition-seeking discussions", while understanding is closely aligned with "the interpretational realm of ever-evolving conversation (auditory sensory modalities)" (1996, p. 199). However, B. Davis (1996) argues that the distinction between understanding and meaning is less important than the common characteristic involved in each process: the "space of interpretation" (p. 199). For the most part, B. Davis accords a particular status to the word "meaning" within the field of mathematics education. He explains how the interpretations of "meaning" are as much denied in mathematics education as the "formulated" understanding is privileged, while the bodily and "unformulated" understandings are often neglected. In B. Davis' enactivist perspective, the bulk of meanings are "neither formulated nor strictly linguistic. They are, rather, lived through or enacted" (p. 205). As such, "meanings reside in the domain of language, and language, in turn, is generally cast as a mental (in contrast with a physical) capacity" (B. Davis, 1996, p. 205). However, the enactivist framework claims that understanding and meaning can exist not just in the words being spoken but also in the participants' joint action. Thus, according to B. Davis (1996), "the space of collective action is not merely a device in promoting individual sense-making; it is a location for (shared) meaning and understanding."

\(^5\) These three components of understanding appear to resemble three of Pirie and Kieren's modes of understanding: Image Making (building), Image Having (having) and Property Noticing (enacting). See description in Chapter 3 (Section 3.3.4). It is important to note, however, that Duffin and Simpson's concept of enacting is not, in any way, related to the philosophy of enactivism.
(p. 197), a fundamental notion in the enactivist philosophy. B. Davis’ (1996) description of understanding as a dynamic process may invoke a sense of social interaction, and is often considered to be “more immediate, more fluid, more negotiable” than meaning (p. 199). On the other hand, meaning is more authoritative than understanding, and has a sense of “intention, of directedness, of pointing, of referring to something in particular” (B. Davis, 1996, p. 198). B. Davis’ enactivist view concludes that, outside the teaching context, a mathematical idea is more concerned with meaning than with understanding on a broader, socially sanctioned level.

2.4 The Interaction between Language, Mathematics, and Understanding

Since the 1970s, extensive research and interest in mathematics education intensified, shifting the focus to the teaching and learning of mathematics. As a result, teachers and students have been in the spotlight in most discussions about mathematics education. As Austin and Howson (1979) observed, the purpose of these studies and discussions “centres upon attempts to understand how mathematics is created, taught, and learned most effectively” (p. 161). In addition, Austin and Howson identified from Streven’s (1974) UNESCO report the helpful distinction between three different levels of analysis: the language of the learner, the language of the teacher, and the language of mathematics. This distinction, according to Pirie and Schwarzenberger (1988), is essential to any study of the role of language in the development of mathematical concepts in a classroom setting. Other related factors have also been recognized for their major role in the learning and understanding of mathematics, including the impact of the classroom setting, available resources, the school environment, as well as the effect of the individual’s home environment. (See Ellerton and Clarkson (1996) for a comprehensive review of such interplay between language and mathematics learning in the mathematics classroom.)
Academic discussions about the language of mathematics or "mathematical language" often lead to confusion, because of the difficulty in distinguishing between mathematical understanding and language understanding. Mathematical language refers to the use of the natural or verbal language in mathematical discourse, rather than the mathematical contents involved. Mathematical language typically refers to the words and phrases as they are used specifically in mathematics, or as metaphors in a mathematical context. Central to the concept of mathematical language is the notion of a register, a term Halliday (1978) coined to describe "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (p. 195). While Halliday's notion of register appears to vary over time (Pimm, 1987), a mathematical register is described in this study as the collection of all the mathematical terms, including everyday words that often hold specific meanings within mathematical contexts; that is, the form of verbal language used in mathematics rather than the mathematical symbolism itself. For instance, terms in the mathematics register such as "square", "prime", and "odd" are often used in mathematical discourse, and the everyday words such as "increase", "line", and "period" often hold specific meanings within mathematical contexts.

Linguists and mathematics educators have initiated discussions about the relationship between students' natural language and their mathematical language whenever they are engaged in mathematical tasks. Language itself is the focus of diverse fields of study; some of which – for instance, semantics, semiotics, or functional analysis – can be related or applied to mathematics (Durkin, 1991). Durkin (1991), a mathematics educator whose research interest lies in language, asked why one needs to focus on, or at least refuse to ignore, language. After all, mathematicians are often known to work in abstract and highly symbolic representations, where precision and formalism are significant. Durkin (1991) then pointed out Pimm’s (1987) discussion on the role
of language in communicating mathematical ideas, because language reveals the significance of both people and communication in mathematics. In addition, Durkin (1991) said that to ignore the role of language:

certainly discounts many of the processes and problems of attaining any level of competence in mathematics: mathematics education begins and proceeds in language, it advances and stumbles because of language, and its outcomes are often assessed in language – the interweaving of mathematics and language is particularly intricate and intriguing (Durkin, 1991, p. 3).

The role of language in learning is vitally important, especially if the assumption holds true that students’ mathematical education takes place in language, and that it is therefore desirable to explore how this learning occurs and what problems and benefits it entails for education (Durkin, 1991). In this respect, viewing mathematics in terms of language use would shed some light on the issues of learning, teaching, and understanding mathematics (Pimm, 1987). Moreover, because mathematics education makes particular linguistic demands, such as explanation and justification, the act of learning mathematics could, in turn, make a more substantial contribution to learners’ general language development (Durkin, 1991). Language is critical to many of the processes of learning and instruction, such as cognitive and communicative functions (Hoyles, 1985, cited in Pirie & Schwarzenberger, 1988). Language enables one to articulate, discuss, construct, and re-present the ideas and problems within the field of mathematics.

In semantic theory, some theorists describe the principal function of language as “transmitting” meaning, whether mathematical or non-mathematical (Durkin, 1991). Such a function is disputed by the constructivistic ideology, in which the individual is seen as responsible to his or her own “subjective conceptualizations and re-presentations” (von Glasersfeld, 1995, p. 141). Others,
however, including notable linguistic theorists such as Halliday (1978), often refer to the role of language in “conveying” mathematical meaning – a way of reflecting the ambiguity, uniqueness, and complexity of any meaningful construction. Such language complexity involves words often used to convey not just one, but also a multitude, of mathematical meanings. Consequently, students with language difficulties sometimes fail to interpret the words and associated mathematical meanings their teachers intended them to learn. Furthermore, language is also said to “mediate” a mathematical experience between several individuals (Austin & Howson, 1979; Durkin, 1991), which appears to possess a social function (see also Gorgorió & Planas, 2001, Secada, 1992).

Social constructivists would argue that the construction of knowledge and understanding takes place through language whenever one takes into account the contribution of all participants in any social activity, the value of group discussion, and the social environment (Cobb, Yackel & Wood, 1992). This social constructivist view of language describes how language is used as a tool “to construct knowledge and regulate access to that knowledge” (Veel, 1999, p. 185). However, Brown (1999) in discussing the relationship between language and mathematics, views how language conditions all experiences of reality while, from other perspectives, language distorts the experience of reality.

For Durkin (1991), language brings together a great many dimensions of learning, including the syntactic, semantic, pragmatic, discourse, socio-linguistic factors, along with many other means of benefiting and enriching one’s view of mathematics education (Durkin, 1991). But language also brings its own rules and demands, which, according to Durkin (1991), may not always be in “perfect correspondence with the rules and demands of mathematics” (p. 14). In that sense, language can also present its semantic and lexical ambiguities and inconsistencies when it
involves mathematics, thereby misleading and confusing some students. The semantic hurdles develop because of shifts in the application of particular words, and because mathematical symbols and words can be considered *polysemous* (having multiple, systematically associated meanings). Similarly, potential lexical ambiguities include the notions of *homonymy* (words spelled and pronounced alike but different in meaning) and *homophony* (words pronounced alike but different in meaning or spelling), illustrating the potential for confusion and misunderstanding for language-deficient students trying to learn mathematics. All of these are consistent with Durkin’s (1991) description of varying levels of ambiguity in language use involving the “interplay of syntactic, semantic, inferential, temporal and contextual” clues (p. 9). Durkin’s conclusions are hardly surprising, because language is a natural human creation, and unlike mathematics, it is “not clear-cut or precise” and is “inherently messy” (Halliday, 1978, p. 203). However, it should also be clear, as Durkin (1991) pointed out, that language is seen as only one of a number of concerns in mathematics education.

Beyond the broad impact of language, others have focused narrowly upon the use of metaphors in mathematics education as the most potent and creative tool of any natural language (Pimm, 1987; Durkin, 1991; Pirie, 1998b; Lakoff & Núñez, 2000). Apart from highlighting differences in everyday word usage, metaphors are vital for the construction of meaning in mathematics (Pimm, 1987). In addition, much of everyday language is metaphorical in origin (Halliday, 1978), and so is the nature of the natural language in mathematical language. Halliday (1978) notes how mathematical expressions using the natural language “represent essentially concrete modes of meaning that take on a metaphorical guise when used to express abstract, formal relations” (p. 202). For Halliday, the difficulties of mathematical language relate to its extensive use of metaphorical and figurative concepts, such as the equivalence of the statement “four from six leaves two” with the numerical expression “6 - 4 = 2”. Nolder (1991), illustrates with an
example how metaphorical terms such as "staircases", "wings", and "triangles" are used to
describe cubes arranged symmetrically around a centre column of a tower pyramid; and she goes
on to observe that the use of some of these metaphorical terms in mathematics education can turn
out to be less helpful than others. Yet, in spite of the way such metaphorical expression pervades
many facets of mathematical teaching and learning, Nolder (1991) argues that the use of
metaphors usually goes unnoticed.

Although the relationship between language and mathematics has long been explored (Halliday,
1978; Pimm, 1987; Durkin & Shire, 1991), an apparent division between the two subjects
remains, along with a division between the way linguists and mathematics educators view each
other's roles. Bell and Woo (1998) claimed, for example, that a discussion of, or about, language
was absent from all 29 chapters of the 1992 American publication *Handbook of Research on
Mathematics Teaching and Learning*. Furthermore, throughout the book, "assertions of the
fundamental importance of language to mathematics teaching and learning tended to be sporadic
rather than thematic" (Bell & Woo, 1998, p. 51). Veels (1999) described the alleged division
more specifically when he wrote:

To the mathematician, the research on mathematical language in language education
journals frequently appears to be woefully inadequate in its understanding of
mathematical knowledge. To the linguist, the research in mathematics journals seems
horribly simplistic in the role it assigns to language in learning. The result of all this is
that language educators and mathematicians rarely talk to one another (p. 185).

Austin and Howson (1979), however, once suggested a logical way of bridging the described
division. In their extensive review of this topic, they proposed that mathematics educators should
pay attention to linguistics to better understand the language of mathematics, and to help make
the creation, teaching, and learning of mathematics effective (Austin & Howson, 1979). After all, mathematics education and language do have so much in common, and as a result, “co-operative research would pay dividends”, whereby each may have much to learn from, and contribute to, the other’s work (Austin & Howson, 1979, p. 162). In addition, to think about language and mathematics as separate entities seems inappropriate; rather, one should look at how “language is modified as a result of attempting to communicate mathematical ideas and perceptions which is of far greater import” (Pimm, 1987, p. 196).

In mathematics education, language comprehension within a mathematical context can be explored by asking the following question: “Is language understanding different from mathematical understanding?” The available literature suggests various answers to this question, while aiming to explain the relationship between language and mathematical understanding and the various theoretical and epistemological interpretations of language understanding. The answers provided by the current literature can be drawn from one of five areas: (i) the analysis of the various synonyms associated with the term “language understanding”; (ii) the understanding of the subjective meaning of words and symbols; (iii) the understanding of the objective meaning of words and symbols; (iv) the discussion about the depth of understanding within a particular language; and (v) the use of language in expression and external representation.

Language understanding in a mathematical context refers to how language is used, talked about, and analyzed within mathematics education. Various conceptual terms are used synonymously with language understanding: most notably are the notions of “meaning”, “sense-making”, “comprehension”, and “knowing”. These concepts are profoundly intertwined, and as a result, they float around interchangeably within the students’ (and teachers’) everyday and mathematical discourse. Students (and teachers alike) are often heard to say, “I understand the
"question", "I know what that word means", "I know (or understand) what you’re saying", "How do (or can) you comprehend the written instruction?", "What do you mean?", and so forth. It is possible to distinguish the subtle difference between these remarks by remaining aware of the fact that language understanding, as is the case with mathematical understanding, is composed of particular characteristics.

Similar to the process of mathematical understanding, understanding language – whether it is composed of a word, a phrase, or a sentence – is generally considered a context-dependent concept. This characteristic can be applied to the use of ordinary words, phrases, or algebraic symbols in mathematics. To understand a particular word being used in the natural language, it requires a context, which is often defined by the structure of the sentence or by the flow of the interaction. von Glasersfeld (1995) points out that almost every English word, including mathematical English words, contains more than one meaning while viewed in isolation. However, when such a word is spoken or written in a sentence, the context of communication usually eliminates all but one of the potential meanings. Consequently, the context or learning situation is central to determining the relationship between one’s mathematical understanding and the use of language in any mathematical discourse.

In the same sense as mathematical understanding, language understanding also varies in its degree of complexity or completeness. Nickerson’s (1985) concept of “connectedness” in mathematical understanding can be applied to language understanding: one has to have a depth of connectedness within a particular language; hence, the more one knows about a particular concept, the better one understands it. The depth or “growth” of language understanding reflects the complexity in one’s “language capacity” – the individual’s accumulated knowledge of the language. Nickerson (1985) associates an individual’s depth of understanding of a language with
how the individual uses a word, not only how he or she uses the word within a particular context, but also in his or her ability to use the word appropriately and meaningfully in a variety of ways.

The verbal connectedness outlined previously is closely related with Paivio’s (1971) notion of “verbal association”, which describes the depth of language understanding by the complexity of the interconnection within the individual’s language capacity. According to Paivio’s “Dual Coding” approach – which attempts to describe how language and images interconnect – language understanding appears to literally mean a measure of how many words come to mind whenever a particular word or image is activated. Paivio (1986) therefore describes verbal association in such a way that its internal structure varies and that the smaller units are “organized into larger units in sequential or successive fashion” (p. 58). This sequential organization gives rise to the depth of understanding and the spread of connection among related words, a process Paivio (1971) called “associative”.

Studies have shown that connected language understanding must be built on an individual’s prior or existing knowledge about the language, including the mathematical language (Baker, 1993; R. B. Davis, 1991). Hence, an individual can easily understand and make use of a mathematical concept in a particular language if the foundation of that language is sufficiently well developed (Baker, 1993), or if the process of learning language, including learning mathematics through language, is built upon a foundation of previously built-up understanding (R. B. Davis, 1991).

Given the relationship between a natural language and the mathematical language contained within it, the process of understanding the language is considered similar to understanding the language of mathematics. This process does not refer to understanding the mathematical concepts, ideas, or images that are attached to the mathematics terms. The concepts, ideas or
images are like mental representations: they are “not built out of words” (R. B. Davis, 1991, p. 227). But each term, whether mathematical or non-mathematical, has a particular meaning or purpose. Sierpinska (1990) defines one possible explanation for the way an individual demonstrates understanding of a term by directing mental activities toward some “object”; the object is then called the “meaning” of the word. In an individual’s mind, the objective meaning of words can transform into mental representations or images of the ideas denoted by words. The construction of mental representations, therefore, is one of the key features of the interpretation process. These mental representations are usually not about written words, but rather, about the ideas denoted by the words.

In reference to the use of words, Hiebert and Carpenter (1992) claim that in order for words to acquire specific meaning, an individual must connect his or her mental representations of the written words with his or her mental representations of concrete materials. This connection allows the individual to create subjective meaning for the words. From a constructivist’s point of view, the individual’s own construction in using a word points only to the representational meaning he or she associates with that word (von Glasersfeld, 1995). Therefore, according to von Glasersfeld (1995), “the meanings of whatever words one chooses are one’s own, and there is no way of [re-] presenting them” (p. 109). Moreover, for each individual, meaning and words are inexplicably intertwined, and in the mathematical context, an individual’s subjective interpretation of a word reflects in some way the objective meaning of the word as defined by its creator (Lakoff & Núñez, 2000). Any difference or mismatch between an individual’s construction of subjective and objective meanings can lead to misunderstanding. However, Pirie

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6 Post-defense note: a substantive issue, not discussed in detail in this thesis, deals with how some philosophers, following an interpretation of Wittgenstein, would argue that mathematics is the use of language, and developing mathematical understanding is the same as learning ways of talking (Barton, 2005, personal correspondence).
and Schwarzenberger (1988) warn that a mere mismatch of a particular language, symbol, or notation should not instantly be considered evidence of a lack of mathematical understanding.

The problem in language comprehension (whether it involves speaking, reading, or writing), or in difficulties understanding mathematical language, is often observed when students work with mathematical word problems. This kind of ambiguity is not new, especially with studies that cite difficulties when students work with simple relational English words such as “more” or “less” (Lean, Clements, & del Campo, 1990; MacGregor, 1993; Fasi, 1999). In fact, several studies have shown how students have difficulty in the transformation (or translation) of the relationships expressed in the natural language into corresponding mathematical relationships, and vice versa (Nickerson, 1985; Pimm, 1987; Fasi, 1999).

In algebra, the advantage of the conciseness and precision in mathematical representation is often overshadowed by the abstractness and ambiguities of the mathematical terms and notations. Nickerson (1985) demonstrates this situation by describing how students struggle to express verbally the mathematical meaning of the simple relationship denoted by the expression, “X = 2Y”. For example: “One X equals two Y’s”, “For every X there are two Y’s” or, “There are twice as many X’s as there are Y’s”. Moreover, Nickerson (1985) was also concerned with how certain algebraic tasks can be solved by purely syntactic methods that do not depend on comprehension of the meaning of the problem; thus alluding to the previously described separation between language and mathematics (Lakoff & Núñez, 2000).

Lakoff and Núñez (2000) distinguish between the mathematical concepts, the written mathematical symbols for these concepts, and the words used for these concepts. For Lakoff and Núñez, words are simply part of the natural language, but words are not part of mathematics.
Therefore, in their “embodied” view of mathematics, the mathematical terms such as “square” and “odd” are only meaningful by virtue of the mathematical concepts attached to them; that is, to understand a mathematical word is to associate it with a concept. Hence, in their analysis of the mathematical symbols, including words, Lakoff and Núñez (2000) conclude:

The meaning of mathematical symbols is not in the symbols alone and how they can be manipulated by rule. Nor is the meaning of symbols in the interpretation of the symbols in terms of set-theoretical models that are themselves uninterpreted. Ultimately, mathematical meaning is like everyday meaning. It is part of embodied cognition...From the perspective of embodied mathematics, ideas and understanding are what mathematics is centrally about (p. 49).

However, an individual’s understanding of a particular concept does not take place by reading a single word or text, but requires an active process of interpretation. A dialectic relationship between understanding and language exists through an explanation that is deeply reconciled to interpretation. In this regard, Sierpinska (1990) cites Ricoeur’s (1989) didactical presentation of such a dialectic relationship as “phases of a specific process”:

I propose to describe this dialectic as a passage, first, from understanding to explaining, and then from explaining to comprehending. At the beginning of this process understanding is a naïve grasping of the meaning of the text as a whole. By the second stage, as comprehending, it is an elaborate way of understanding based on explanatory procedures...In this way, explaining appears as a mediator between two phrases of understanding (cited in Sierpinska, 1990, p. 26).

Therefore, when an individual’s interpretation involves language, his or her understanding takes into consideration more than just the subjective meaning of the word or phrase, or even the subjective meaning of the sentence. Moreover, the depth of an individual’s interpretation often
includes more than the word(s) symbolizes and the concept(s) associated with it. Referring to words expressed through speech, von Glasersfeld (1995) quotes Vygotsky (1962), who stressed that “To understand another’s speech, it is not sufficient to understand his [or her] words – we must understand his thought. But even that is not enough – we must also know its motivation” (p. 151). Nevertheless, von Glasersfeld (1995) adds that the “result of an interpretation survives and is taken as the meaning, if it makes sense in the conceptual environment which the interpreter derives from the given words and the situational context in which they are now encountered” (von Glasersfeld, 1995, p. 142). Thus, another individual, based on the words being spoken and the context for communication, might represent meanings differently than the individual who is actually deciphering the words.

To broaden the discussion, Sierpinska (1990) gave an explicit explanation of the difference between the two statements relating sense and meaning: statement (a) refers to an objective description of meaning, while (b) reveals the subjective feeling or interpretation of the speaker. Sierpinska (1990) thus contends that the word “sense” has an objective meaning, demonstrated by the question, “In what sense are you using this word?” (p. 27) Since the question is answered with a sentence that contains the key word, mathematical meaning is therefore derived from context. The sentence itself, whose structure defines the mathematical function of the word, gives a mathematical sense to the word. Sierpinska adds that the sentence may exemplify, denote, state, or refer to something – its “reference” – that holds true (or false) in some mathematical reality. The meaning, then, of the mathematical word contains both the sense of the sentence and its reference.

Moreover, Sierpinska (1990) notes that Ricoeur’s (1989) “hermeneutic” philosophy – which is concerned with the interpretation of literary texts – defines the duality of sense and reference in
meaning. In other words, “sense” answers what the sentence says, and “reference” answers what the sentence is about. This “duality” can be observed, for instance, in the structure of the following statement: “Square the counting numbers to find the square numbers!” One method of writing this mathematical relationship in algebraic form is denoted by $a^2 = b$, where $a$ is an element of the counting numbers and $b$ a square number. The sense of the statement refers to the equality between the two sets of objects: square numbers and the square-product of counting numbers. The reference denotes that the statement holds true in the field of real numbers, but not in the field of complex numbers. According to Sierpinska, a combination of the notion of sense and reference will provide the objective meaning of the statement.

In addition, Sierpinska (1990), in her epistemological analysis of mathematical concepts, explains how language plays a role in mathematical meaning and understanding:

Suppose we start from the informal language of mathematics. Let us find words and expressions used in defining, describing, working with the concept we are analyzing. Let us find sentences which are the senses in which these words and expressions are used. Then let us seek the references of these sentences. And then see relations among all these senses and references. This analysis can lead us to a description of the meaning of the concept in question...Understanding the concept will then be conceived as the act of grasping this meaning (p. 27).

The social constructivist perspective goes further in relating language to learning. This perspective assumes learning is neither a static process, nor does it occur in isolation, but rather, learning is socially negotiated and expressed through language that focuses on explanation and clarification (Smith, 2000). It suggests that students in a social setting can interact through language during mathematical discourse. In addition, social constructivism views learning, and
hence, understanding mathematics through language, as a process of affecting one’s meaning. Many experienced language teachers have long sought to avoid passing on superficial verbal learning skills. It is all too easy to get a student to say something – or to write it – without their necessarily understanding it (R. B. Davis, 1991). Perhaps, for these reasons, some mathematics educators and academics go on to suggest that learning ought to be made less dependent on language; and that teachers of mathematics, in particular, should emphasize the importance of learning through concrete operations on objects (Halliday, 1978). However, Halliday (1978) warns that there is no point in trying to eliminate language from the learning process altogether, a problem seen in mathematics textbooks deliberately designed for teaching mathematics without words. Rather than engage in any such vain attempt, one “should seek equally positive ways of advancing those aspects of the learning process which are, essentially, linguistics” (Halliday, 1978, p. 203).

Verbalizing thought processes is a form of externalizing the internal dialogues or activities within the minds of individuals. It is then up to the individuals to construct their own conceptualizations and representations of whatever is being externalized. These external representations, through the use of natural language, may turn out to be the only way in which the individual’s thought processes can be revealed and his or her potential realized (Halliday, 1978). In fact, Brown (2001) argues that an individual’s framing of mathematical experience (including mathematical understanding) in words “should be seen as an integral part of the mathematics itself, inseparable from less visible cognitive activity” (p. 200).

The question of individual expressiveness is therefore a central issue in mathematics education, as demonstrated by Pimm’s (1987) question: “What is the connection between how things are described and how they are seen?” (p. 201). In fact, what an observer sees is a reflection of the
Indeed, a critical flaw in many structured mathematical learning situations lies in the gap between action and expression. On this point, Noss, Healy and Hoyles (1997) write:

Indeed, students who are able to apply a correct method to any number of specific cases often cannot articulate a general pattern or relationship in natural language, and expression in algebraic symbolism is still more problematic...the evidence suggests that algebraic formulation is often disconnected from the activity which precedes it (p. 203).

At the same time, the role of language as a cultural tool reflects how “thought and language enhance each other’s development” (Sierpinska, 1996, p. 42). Citing Piaget’s (1959) observation that “language is molded on the habits of thought”, Sierpinska (1996) argued that such a view expresses how language reflects thinking (p. 32). In addition, Vygotsky, according to Sierpinska (1996), was quite sensitive too, to the delicate relationship between language and thought, or as Sierpinska referred to it, the “subtle interplay between the natural language and spontaneous thought and scientific concepts” (1996, p. 41).

While this section offers a comprehensive look at many existing models and theories of, or about, the notion of “mathematical understanding”, Chapter 3 explains in detail an additional theory, known as the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding, and discusses why such theoretical framework is considered advantageous and more appropriate than others for this study. The next section of the current chapter, however, continues by looking at the notion of “bilingualism” in mathematics education.
2.5 Bilingualism in Mathematics Education

During the past three decades, a growing awareness of bilingualism has triggered considerable interest in the field of mathematics education, particularly with regard to the question of how mathematics can be learned and understood in two languages. In general, bilingual issues are gaining widespread attention in a variety of arenas, not just in education, but also in political, cultural, and social circles (Cummins, 1981), perhaps because virtually every country in the world represents bilingual populations. Baker (2001) noted that numerous researchers in the past have extensively researched and publicly discussed the complexities associated with bilingualism and bilingual education, and yet, these subjects continue to challenge, and occasionally confound, linguists and mathematics educators alike.

To add to the confusion, interested parties have yet to agree upon a precise definition of what it means to be “bilingual”, mainly because of its multi-dimensional nature. In fact, the difficulty in defining bilingualism has been the source of major differences, engaging researchers, educators, and even politicians, in public debates (Cummins, 1981; Baker, 2001). The only generally accepted principle regarding the term, is that any issue regarding bilingualism is characterized by the involvement of two languages coming in contact with each other through communication or interaction. Yet, the degree to which this “contact” is defined, or how these two languages interact, remains a central source of disagreement and discord.

In an effort to grapple with the broad nature of bilingualism, researchers have introduced varying typologies, largely to reflect an individual’s grasp of two languages, such as the distinction between “non-proficient/proficient”, “competent/incompetent”, “partial/dual”, “limited/full”, “semi-balanced/balanced”, “native/first/secondary”, “minority/dominant”, and so on (Lambert,
Over the years, mathematics researchers basing their studies on bilingual situations have tended to focus on diverse populations, or they have conducted their research in widely different contexts, making it difficult to generalize from one bilingual group to another (Barton, Fairhall, & Trinick, 1998). In general, the effect of bilingualism on mathematics has been widely discussed (Dawe, 1983; Cuevas, 1984; Leans, Clements, & del Campo, 1990; Clarkson, 1992; Clarkson & Galbraith, 1992; Clarkson & Thomas, 1993; Moschkovich, 1996, 1999; Clarkson & Dawe, 1997) and evaluated through testing and assessment tasks in mathematics (Cummins and Corson (1997) offers a broad review of existing bilingual programs in the world). The results of these assessment tests have always been presented as the bilingual students' “levels of performance” in either language or mathematics. A mathematical result of this nature is hardly a clear indication of a student’s mathematical understanding, since the student’s difficulty has typically been attributed to a lack of language understanding rather than a lack of mathematical understanding (Whang, 1996; Fasi, 1999). Furthermore, some researchers, such as Fasi (1999) and Cuevas (1984), have argued that the use of assessment in a language that the students do not understand is inappropriate, especially when assessment is applied to bilingual students in a minority language environment. As a result, students’ successful results on previous researchers’ assessments cannot be interpreted as definitive proof of their understanding of the mathematics, a fact that lays the groundwork for a study that can more clearly and conclusively demonstrate the nature of the bilingual students’ mathematical understanding.

In the past, most bilingual studies in mathematics education have been largely influenced by Cummins’ (1976, 1977, 1978, 1979, 1981) work, particularly in relation to his theoretical hypotheses on language acquisition and mathematical achievement. Previous studies used Cummins’ hypotheses to demonstrate that bilingual students’ competency in their first language
is significant for mastering both the second language, and the conceptual operations and
functions in mathematics (Cummins, 1979; Dawe, 1983; Brodie, 1989; Clarkson, 1992; Clarkson
& Dawe, 1997; Setati, 1998; Fasi, 1999). Various studies have also demonstrated the importance
of bilingual students' level of competence in their second language, which is usually the
language used for instruction (Cummins, 1979; Dawe, 1983; Cuevas, 1984; Leans, Clements, &
del Campo, 1990; Clarkson & Galbraith, 1992; Clarkson & Thomas, 1993; Fasi, 1999).

Other important research includes educational studies that report a sociological, linguistic, and
psychological interconnection between one's bilingual status and one's ability to achieve
mathematical understanding (Paivio & Desrochers, 1980). In any bilingual situation, each level
of analysis in the language of the learner, the language of the teacher, and the language of
mathematics (Austin & Howson, 1979) is concerned with at least two other layers – one for each
of the natural languages involved. Likewise, in a bilingual situation, the language of mathematics
can be examined by the way the mathematics is written, symbolized, expressed, taught, or
learned in two different sets of vernaculars.

As researchers to date have discovered, the complexities involved with bilingual learning
illustrate the need to investigate the role of language in mathematical understanding (Brown,
2001). In fact, researchers and mathematics educators interested in studying bilingual situations
have already been engaged in studying the relationship between natural language(s) and
students' mathematical understanding (e.g. Dale & Cuevas, 1987; Ellerton, 1989; Ellerton &
Clarkson, 1996); a sign of the attempt to bridge the division between linguists and mathematics
educators, a division Veel (1999) described earlier. Nevertheless, depending on the researcher or
educator's philosophical stance and his or her methodological approach, a range of perspectives
abound between the two disciplines (Austin & Howson, 1979; Durkin, 1991), perspectives that
will ultimately affect the research findings and implications. At the same time, evidence of the crucial link between language and mathematics suggests the potential for further research on this topic, particularly in the way such link relates to how bilingual students can best learn, understand, or be taught mathematics with or without the use of language(s).

Another meaningful challenge for mathematics educators and linguists relates to the development of new mathematical registers, tailored to be understood in a particular vernacular. Barton, Fairhall, and Trinick (1998), for instance, talk about how the development of a new Maori mathematical vocabulary during the past 20 years or so, a case that now demonstrates unforeseen semantic problems for new Maori speakers in mathematics education. In the Maori culture, and within other bilingual communities, both linguists and mathematics educators will inevitably face challenges in teaching, because “languages differ in their meanings, and in their structure and vocabulary, they may also differ in their paths towards mathematics, and in the ways in which mathematical concepts can most effectively be taught” (Halliday, 1978, p. 204). As a result, bilingual speakers, learners, and even teachers and educators, must pay attention to the unique characteristics of a particular language. Halliday’s (1978) semiotic approach raises this important point: namely, the need to ask which mathematical idea or concept is most easily conveyed when the medium of teaching or learning mathematics is in this or that particular language. The significance of such a perspective is reflected in Jones’ (1982) suggestion, that what one needs is knowledge of the aspects of that particular language that impinge directly on the learning or understanding of mathematics. Thus, both linguists and mathematics educators can learn and better understand each other if they jointly investigate the complexity of the process of learning and understanding mathematics through different languages (Austin & Howson, 1979).
The advantages of being bilingual have been confirmed in many studies conducted in various contexts and bilingual situations (Dawe, 1983; Clarkson, 1991, 1992; Secada, 1992; Stephens et al., 1993). At the same time, the findings of this group of researchers suggest that the language level of competence is vitally important to the bilingual students’ ability to construct images, and consequently, their ability to ascribe meanings. For second-language learners, “meaningful understanding of the language, which is a direct result of the level of competence and proficiency of the learner in that language, is as important in learning mathematics as the mathematics content itself” (Fasi, 1999, p. 60). The level of competence of the bilingual student in both languages has been the key to his or her advantage in learning mathematics. This concept is known as Cummins’ (1979) “Threshold Hypothesis”, which states that bilingual students must attain certain threshold levels of linguistic competence in both their languages in order to avoid cognitive disadvantages. In other words, the bilingual students’ level of competence in both languages mediates the effects of cognition on their mathematical thinking. Moreover, Cummins (1981) proposes a second hypothesis called the “Developmental Interdependence Hypothesis”, in which bilingual students’ level of competence in the second language is described as dependent on their competence in their first language.

The debate about the cognitive effect of bilingualism on bilingual students’ mathematical thinking has involved not only theoretical issues, but sociological and political issues as well. For example, social and political issues are involved in the French Immersion Program pioneered, praised, and supported in Canada (Cummins, 2000), while most states in the United States do not support its philosophy and benefits (Collins, 1997; Baker, 2001). Baker (2001) reported that bilingual programs in California were outlawed, and that public opinion remained against such programs, owing to the perception that bilingual programs failed to foster social issues such as “integration”.
In favour of bilingual education, the Maori education in New Zealand offers an example of another bilingual program that recognizes the benefits of bilingualism (Barton & Fairhall, 1995; Barton, Fairhall, & Trinick, 1995a, 1995b). A report by Ohia, Moloney, and Knight (1990) found "no evidence at all of Maori students being disadvantaged mathematically by being a bilingual unit" (cited in Fasi, 1999, p. 58), findings which, in fact, led to the development of the Maori's mathematical language. However, Barton, Fairhall, and Trinick (1998) presented a different scenario in which bilingual students' "thought patterns" were said to be dependent on language. In their view, if new Maori speakers did not develop both languages equally, they were in danger of having only one conceptual pathway because "true bilinguals have two conceptual pathways available" (Barton, Fairhall, & Trinick, 1998, p.7). Based on this proposition, Fasi (1999) concluded that if language changes, "distinctive thought patterns and conceptual pathways will also change" (p. 59). This line of thought resonated with the constructivists' view expressed by von Glasersfeld (1995), in which different languages determine different conceptualizations, in such a way that switching to another language may require "another way of seeing, feeling, and ultimately another way of conceptualizing experience" (p. 3). von Glasersfeld's view was reflected in Strevens' (1974) report, in which Strevens noted that there was a "major difference in mental preparation for mathematics learning" among different languages (cited in Fasi, 1999).

Other useful observations can be drawn from Adler's (1995, 1997, 1998, 1999) extensive research on multilingual issues in South Africa. Her studies were based on "sociocultural theory of mind", and some of her findings support Cummins' hypotheses that bilingualism does not impede mathematical learning. Cummins (1976, 1986) and Ben-Zeev (1977) had pioneered the idea that mathematics education benefited from bilingualism because it required educators (and learners) to construct more meaningful terms. The benefits of bilingualism were also based on
the underlying assumption that knowing two languages opened one to more experiences, and as a result, contributed to one’s mathematical knowledge. Fradd (1982) goes further to add that bilingualism has had positive effects on cognitive growth and divergent thinking; both of which are critical aspects in the individual’s mathematical experience. Others, such as Coffman and Cuevas (1979), Lean, Clements, and del Campo (1990), and Clarkson (1992) reported a higher performance for bilingual over monolingual students in (test) assessment tasks. Pimm (1987) and Cuevas (1984) concluded that an understanding of how the second language is learned might contribute to how one understands mathematics, if mathematics can indeed be considered a communicative language.

However, those who concern themselves (e.g. Andersson, 1977; Ransdell & Fischler, 1987) with the negative effects of bilingualism often direct their concerns toward the bilingual students’ deficiencies or incompetence with the language. They noted, in particular, that some of the linguistic disadvantages of bilingualism include the differences in lexicons and grammatical structures between two different languages (Austin & Howson, 1979). Skutnabb-Kangas (1986) discusses conflicting ideas about minority education. Secada’s (1992) extensive overview of bilingual studies in mathematics pointed to reports in which the low level of proficiency in English as a second language related negatively to achievement in mathematics. Such findings, of course, coincide with Cummins’ “Threshold Theory”, in which limited competence in both languages resulted in negative cognitive outcomes (Baker, 2001). Although there are still debates about what level of bilingualism results in positive or negative effects for students, Grosjean (1982) proposed that bilingualism may have no major effect at all – either positive or negative – on the cognitive and intellectual development of children, a proposition that will be addressed further in this study.
From the perspective of researchers chronicling the downside of bilingual education, Adler (1995, 1998) notes a "paradox" for multilingual mathematics teachers, based on the amount of time they spend in the mathematics classroom teaching language, rather than teaching mathematics content. Bilingual students (of the Tongan-type) face an equivalent challenge by having to learn a second language while learning mathematics at the same time. As a result, these bilingual students misunderstand the mathematical ideas, or over-emphasize vocabulary learning at the expense of mathematics learning (Adler, 1998). This problem is not surprising, because almost half of the difficulties in solving mathematics word problems are language-based (Whang, 1996). However, Adler (1998) claims that the "dilemma of mediation" or the pedagogical tension between teaching mathematics, and teaching the language in which mathematics is delivered, cannot be resolved.

As social constructivists claim, bilingual educators must also be cognizant of the cultural aspects of the languages involved (Smith, 2000). Crawford (1990), in a study of Australian aboriginal learners, acknowledged that many learning difficulties may be attributed not so much to a lack of language proficiency, but more to the cultural processes associated with the language. Brodie (1989) previously noted that the desirability of second-language learning in mathematics depended upon many factors, including: the emphasis on maintenance of first-language skills, the acknowledgement of students' natural modes of thought as determined by their first language and home culture, and the interaction of these factors with concept formation and mathematical symbolism. However, Adler (1995) pointed out that most studies are psycho-linguistic in nature, and that they do not examine the dynamics of bilingualism and mathematics in classroom settings. She further contends that the dynamics of teaching and learning mathematics in bilingual classrooms is about the "constant interplay in the cultural processes that constitute school mathematics learning between three analytically separable processes: proficiency in the
language of learning, access to the mathematics registers, and the social diversity and relations” of students (Adler, 1995, p. 265).

2.6 The Notion of Language Switching

The process of switching between two or more languages is commonly known in studies of linguistics as “code switching”. Ever since the concept of code switching appeared in the academic spotlight, researchers have focused on creating a number of related definitions and sub-categories, including the development of a special volume dedicated to the art of code switching (Eastman, 1992), along with other related literature and phenomena on the topic (Jacobson, 1998). As Halliday (1978) implied, the notion of code switching resides in the domain of language, not mathematics; a view shared by Lakoff and Núñez (2000). For that reason, it is important to consider first how code (or language) switching is interpreted in the field of linguistics, before examining how it is currently perceived, defined, and used in mathematics education.

While researchers have investigated code switching from a variety of perspectives, most notably for what Macswan (1999) identifies as its sociological and grammatical properties, its sociolinguistic dimension underscores the social nature of language use among code switchers. In the early 1970s, a few published linguistic studies on code switching already existed, but Myers-Scotton (1992) cites and credits the work of Blom and Grumperz (1972) for exposing and promoting the sociolinguistic dimensions of code switching. Blom and Grumperz’ (1972) study investigates code switching in the dialects of Norwegian in Hemnesberget, a Norwegian fishing village. Dialects, which are not considered types of code switching, are just “different ways of
saying the same thing and tend to differ in phonetics, phonology, lexicogrammar but not semantics” (Halliday, 1978, p. 35). Later, in 1982, Gumperz revised his model of code switching to cover its “situational”, “metaphorical”, and “conversational” aspects. Gumperz’s (1982) study sparked great interest among linguists, who then went on to focus on various important features of code switching, such as its functional nature.

The semantics issue associated with the term “code switching” poses a similar problem for linguists, because the term “understanding” defies easy definition for mathematics educators. Linguistic scholars are rarely in mutual agreement over which term to use at any given time. Halliday (1978), for example, refers to code switching as a particular form of shifting between languages that is actualized as a process within the individual. In this type of “code shift”, Halliday notes, a code switcher “moves from one code to another, and back, more or less rapidly, in the course of daily life, and often in the course of a single sentence” (1978, p. 65). According to Halliday’s definition (1978), Gumperz’ (1971) notion of a “code potential” refers to the individual’s “verbal repertoire”. Furthermore, Jacobson (1998) defines code switching as “a structured mechanism of selection of two or more languages in the construction of sentences” (p. 1). Myers-Scotton (1998), on the other hand, describes code switching slightly differently, as a product of how a bilingual individual produces monolingual utterances in either language, although the individual may speak fluently in only one language. In a more general way, Milroy and Muysken (1995) define code switching as “the alternative use by bilinguals of two or more languages in the same conversations” (p. 7).

It appears the key debate for many linguists revolves around the difficulty in identifying the role each language plays in the act of switching. Most linguists would agree (e.g. Poplack, 1980; Joshi, 1985; Myers-Scotton, 1993) that in any particular bilingual utterance, one language
dominates over the other. Myers-Scotton (1993) identifies a measure for this “dominance” to be the language with more morphemes – the smallest linguistic unit that has meaning or grammatical function. However, Sankoff and Poplack (1981) describe dominance in terms of which language can be attributed with the majority of phonological and morphological features of discourse. In either case, a language switching involves a “base language” – the dominant language – while the other will now be called the “embedded language” (also known as “donor language”). The embedded language refers to the language judged to be used in code switching, while the matrix language is the language code switching is judged to be coming from, and is recognized as a concept that needs meaningful clarification (Boechoten, 1998).

However, the derivations and various definitions of categories and sub-categories of code switching have provoked further confusion among educators. One of these sub-categories is the notion of “code-mixing”, which Kachru (1978) defines as “the relatively unconscious integration of elements from the donor language into the base language to form a single composite code” (cited in Gibbons, 1987). To add to the complexity of this topic, other researchers apply a variety of terminology in connection with the concept of code or language switching. Grosjean (1982, 1985), for instance, describes how a bilingual individual can have two language modes: “monolingual language mode”, when only one language is used while the other is “deactivated”, and “bilingual language mode”, where both languages are “activated”. Thus, for Grosjean, the word “mode” defines a type of mechanical, linguistic function when one or both languages are used. This traditional view recalls the nature of the “Balance Theory”, although Cummins (1979) calls it the “Separate Underlying Proficiency” (SUP) model (in contrast to his “Common Underlying Proficiency” model (CUP)). As Baker (2001) argues, the traditional balance theory is inappropriate because languages operate, as Cummins’ (1979) model suggests, from the same central operating system (see Figure 1).
Figure 1: Contrasting the SUP and CUP models of bilingualism (Baker, 2001)

For the purposes of this study, a crucial difference among linguists' definition of code switching comes from the use of the words “code” and “switching” themselves, because the term “code” varies so widely in its use by linguists and mathematics educators, and often suggests a technical connotation. In mathematics education, for instance, the term “code switching” has been used in different connotations. Zazkis' (2000) mathematical study on “code switching as a tool for learning mathematical language” distinguishes two different codes: everyday and formal mathematical language. Another contextual confusion is suggested by Setati’s (2004) description of a switch in a multilingual classroom comprising three different codes: languages, registers, and discourse. Thus, for methodological and theoretical reasons, mathematics educators such as Dawe (1983), Clarkson (1992), and Celedon (1998), prefer to use the phrase “language switching”, while others, such as Adler (1998), Setati (1998, 2002), and Moschkovich (1996) prefer to employ the linguistic terminology by using the phrase “code switching”. Dawe’s (1983) work with bilingual Punjabi, Mirpuri, Italian, and Jamaican students growing up in England provides evidence for the role of “language switching” in these students’ mathematical understanding. In his examination of these bilingual students’ mathematical, deductive reasoning abilities, Dawe (1983) noticed two processes of switching. He observed an “apparent switching
from one mode to the other during the reasoning process – often accompanied by a language switch as well” (p. 349). Dawe added that the complex nature of such a phenomenon makes it therefore very difficult to unravel or interpret.

Gibbons (1987) defines switching as “either situationally determined alternation between codes, or as a relatively more conscious exploitation of the alternative values represented by the two codes, usually for social or rhetorical effect” (p. 75). Aipolo and Holmes (1990), in their study of Tongans living in New Zealand, offer a definition of code switching as a general term to “cover the use of two codes in one situation without pre-judging whether such behavior is conscious or deliberate” (p. 519). As well, Grosjean (1985) describes a “language switch” as a shift from one base language to another. When the base language is established, he explains, “language-mixing” can occur, in which the bilingual individual brings in the other language (the donor). This process then divides into two types: “code-switching” and “borrowing”. “Code-switching” is characterized by a complete shift from one base language to another, while “borrowing” deals with adapting words or phrases into the base language. Grosjean (1985) also describes “code-changing” as a switch with at least a phrase or a sentence, while “code-mixing” – slightly different from his notion of “language-mixing” – involves inserting a single item (word) from one language into another. In addition, Jacobson (1998) brings similar or additional phrases to his list of switching terms, such as “code mixing”, “code alternation”, and “language mixing”. Some of these labels are broken down into sub-categories, such as “intersentential” and “intrasentential” code switching, which are used to distinguish structured from non-structured switching (Macswan, 1999).

The concept of “borrowing”, in particular, turns out to be a concern for some researchers who have argued that borrowing must be carefully distinguished from code switching (Pfaff, 1979;
Sankoff & Poplack, 1981), although others find it difficult to separate the two concepts (Hill & Hill, 1986). The degree to which bilingual speakers are aware of this process may differ with each borrowed item (Macswan, 1999). For example, Macswan (1999) explains how a monolingual English speaker might use the term “pork” without the slightest awareness that it was borrowed from the French (yet spelled “pore”) during the Norman Conquest. But the phrase “tour de force” may be used with full awareness that the expression is of French origin. This phenomenon also provokes a discussion about what it means to be “bilingual”, because, in some instances involving borrowing, for example, an English speaker using terms such as “pork” and “tour de force”, is not in fact knowledgeable about the French language or culture. To examine the key concept of borrowing further, the difference between the use of the terms “pork” and “tour de force” can be explained by the fact that borrowed words are considered either “marked” (as unconventional) or “unmarked” (as conventional) – a distinction made useful by Myers-Scotton (1992). In using both terms, borrowing is seen as filling “lexical gaps” in which Myers-Scotton (1992) describes as the conventional form of borrowing taking place without conscious awareness. This distinction is further discussed in the next chapter to explain how Tongans borrow English words through a process called “Tonganisation”, and how that act, in turn, integrates English words into the Tongan language.

Ultimately, in view of all the described definitions, defining code or language switching appears to be a matter of theoretical or personal preference, rather than a defining characteristic for the research itself. Nonetheless, researchers, such as Eastman, (1992) and Boztepe (2003), have been led to conclude that the efforts to distinguish these terms are “doomed”, based on the resulting confusion from all the defined categories and subcategories of code switching or language switching.
2.7 Language Switching in Mathematics Education

In mathematics education, the notion of language switching has become a major topic of investigation within any bilingual (or multilingual) setting. What language switching is, what it entails, what factors cause it, and how it affects or represents students' mathematical thinking, are among the many questions that provoke the interest of researchers and educators. In citing work by Bishop (1979) and Lean and Clements (1981) in developing countries such as Papua New Guinea, Dawe (1983) has recognized the importance of both linguistic and cultural factors in students' preferred modes of thought. Dawe expressed concern about the sociolinguistic view regarding "language distance" (the disparity and mismatch between two pairs of languages) as an important factor in any bilingual study. Berry (1985) picked up on the same concern, and he explained that it might be easier for bilingual students to function effectively in a second language semantically and culturally closer to their native language, than a language considered remote. One reason for this phenomenon may relate to Cummins' (1981) view that students' native language has a strong influence on their cognitive processes.

In addition, some studies have offered important information about the contextual and situational nature of language switching in mathematical discourse and cognitive activities. Moschkovich's (1996) study of Latino students in the United States and the nature of their mathematical and bilingual conversations recognizes how bilingual students switch languages between Spanish and English. Her study proposes a "situated model", in which language use and its relationship to mathematical learning depends on the learning situation. Although Moschkovich (1996) did not explicitly analyze language switching, she emphasizes the importance of the educational context in investigating the relationship between mathematical activity and the bilingual students' language choice.
Furthermore, Clarkson and Dawe's (1997) study of Vietnamese grade 4 students in Sydney and Melbourne, Australia, suggested the influence of the mathematical context and schooling environment upon these students' choice of language. These authors interviewed a sample of their participants to determine the central factors affecting their language switching. Clarkson and Dawe's study with these bilingual students mainly drew upon Cummins' theoretical framework. Among Clarkson and Dawe's findings was the view that the bilingual students' language choices were affected by competence in languages, semantic issues, and translation problems, affective responses, and memory factors. Clarkson and Dawe (1997) concluded that investigating language switching is very difficult, not because these two researchers lacked adequate methodological approaches, but because the nature of language switching is complex and "messy".

In other research, Qi (1998) investigated language switching for Chinese learners in Canada involving mathematical problem-solving tasks in a second language (L2), such as English. Noting the sociolinguistic view of code switching, Qi offered a psycholinguistic definition of language switching as "the act of switching from L2 to first language (L1) as the language of thinking in the cognitive process of a bilingual person in an L2 composing task" (p. 414). He used the word "composing" to refer to the thinking process involved in the task. Qi (1998) suggested that the demanded levels of knowledge might be the basic variable influencing language switching and the choice of language use in composing tasks.

Adler (1995, 1998) and Setati (1998, 2002, 2004) have identified various cues associated with the relationship between language switching and mathematical activities in multilingual settings. Setati (1998) also categorized types of switching in the classroom, although she defined each in
terms of the teachers’ purposeful use of language for “reformulation”, for “content of activity”, and for “translation”. Adler (1998) described such classroom-teacher situations in terms of a “dilemma of code-switching”: a dilemma of access to both mathematical and language meaning.

To focus on the Tongan setting used in this study, Fasi’s (1999) study with bilingual students in Tonga attempted to categorize language switching into four main groups. Each classification was defined based on the purpose of the switch. For example, “substitution” was used as a convenient way for Tongan students to benefit from the precision of the English words rather than using lengthy Tongan explanations. “Explanatory” involved phrases aimed at relating ideas, defining and clarifying tasks, and explaining mathematical work. “Reformulation” involved paraphrasing the given information, and “repetition” referred to direct translation between the two languages.

This study intends to expand upon and clarify further Fasi’s results on the types of language switching that take place in a learning environment, while at the same time, use the results of specific classroom tasks to answer the research question. Given the varying nature of research, both on bilingualism and the notion of language switching, it is important to clarify how each of these concepts are defined in the context of this study. In order to define “bilingualism” and “language switching”, it must first be clear that throughout this thesis, the word “language” refers to the natural or verbal language in the sense of speech (Halliday, 1978). Any other use will be specifically defined where needed, as was the case with the term “mathematical language”, defined in Section 2.4.

In this thesis, the term “bilingualism” takes on Cummins’ (1981) broad definition as “the production and/or comprehension of two languages by the same individual” (p. iii). While bilingual proficiency can be described at different levels, Cummins’ definition allows one to talk
about any individual’s use of two languages within the course of a verbal exchange without being concerned with how much or how little the individual comprehends in either language. Thus, in this thesis, a “bilingual” individual refers to any individual who is engaged in any production and/or comprehension of two languages. Such production can be expressed verbally through “language switching”, which refers to any alternation between two languages, ranging from mixing one or more words to changing in mid-sentence, or changing within larger blocks of words or clauses (Hoffman, 1999, cited in Baker, 1993). The decision to employ the term “language switching”, rather than “code switching”, aims to preserve the contextual and technical clarity of this study. A look at the bilingual context of this study and the study’s theoretical framework is offered in the next chapter.
CHAPTER 3: CONTEXT and THEORETICAL FRAMEWORK

3.1 Introduction

This chapter introduces the bilingual situation in Tonga and explores the role of language in the Tongan mathematics education program. The account of the Tongan context is based largely on a comprehensive study conducted in Tonga by Fasi (1999) on the effect of bilingualism on Tongan students' mathematical achievement. A brief historical account of the development and involvement of the two official languages in Tonga provided a basis for understanding the roles of these languages within the Tongan bilingual context and its mathematics education program.

3.2 The Tongan Bilingual Context

3.2.1 The First European Impact and its Effect on the Tongan Language

Tonga is one of the Polynesian islands in the South Pacific. Prior to European settlement, all Polynesian languages were considered only as oral languages; there were no written forms for any of these languages (Fasi, 1999). “Tongan”, the native and homogeneous language of the Tongans, is one of the ancient Polynesian languages belonging to the Austronesian family of languages known as “Oceanic”. While the first visits from the outside world were recorded by famous European navigators of the 17th and 18th centuries, it was Christian missionaries who first

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7 Fiefa’s (1981) article also provided a valuable source on “Education in Tonga”.
introduced Western education to Tonga, and later developed a written form of the Tongan language.

British missionaries were the first Christian missionaries to arrive in Tonga in 1814, followed by French Catholic priests about a decade later. Both of these missionary groups have had a profound effect on the development of the Tongan written language. Exposure to Christian religious reading – which enabled the native people to read the Bible in their own language – was the main purpose for teaching the Tongan written language prior to the introduction of Western education. The missionaries were less concerned with introducing an English mathematical language because the native people of Tonga had traditionally used their own mathematical system for counting, measurement, and categorization of objects, which they included in their everyday activities. **Table A** and **Table B**, below, illustrate examples of the counting and measurement systems the Tongans developed and still use today.
<table>
<thead>
<tr>
<th>Name of piece of “tapa”</th>
<th>Length in “langanga”</th>
<th>Approx. metric length (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fola’osi</td>
<td>4</td>
<td>02.4</td>
</tr>
<tr>
<td>Fatuua</td>
<td>8</td>
<td>04.8</td>
</tr>
<tr>
<td>Hongofulu’i ngatu (“tenth” ngatu)</td>
<td>10</td>
<td>06.0</td>
</tr>
<tr>
<td>Uofulu’i ngatu (“twentieth” ngatu)</td>
<td>20</td>
<td>12.0</td>
</tr>
<tr>
<td>Lautolu (“multiple of three”)</td>
<td>30</td>
<td>18.0</td>
</tr>
<tr>
<td>Laufa (“multiple of four”)</td>
<td>40</td>
<td>24.0</td>
</tr>
<tr>
<td>Launima (“multiple of five”)</td>
<td>50</td>
<td>30.0</td>
</tr>
<tr>
<td>Lautefuhi</td>
<td>100</td>
<td>60.0</td>
</tr>
</tbody>
</table>

Table B: Tongan discrete measurements for the length of “tapa” cloth

Under the influence of Christian missionaries and Catholic priests, and beginning with the introduction of the first Tongan orthography, the Tongan language continued to evolve along with the development of many subject areas, such as mathematics and science, and through growing interaction with the outside world. It was during this period that the missionaries realized that for some English words, no equivalent Tongan words existed. As a result, “borrowing” and “Tonganising” were the most common ways of incorporating English words into the Tongan language, and, in particular, to fill the lexical gaps in the Tongan language.

“Tonganisation” was and is still a common and convenient way of creating equivalent Tongan words from English. This process is also known as a form of “conventionalization”, in which English words are phonetically translated into Tongan and are generally used and accepted by the public in talking about everyday activities (Taumoefolau, 2004). As Fasi (1999) noted, some of these “imported” English words have become part of the everyday Tongan language, and no attempt has ever been made to examine more closely their necessity or the existence of

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8 Churchward (1953) discusses how special terms and special numerals in Tongan are used, with “more or less regularity” (because sometimes numerals are also used), when counting coconuts, yams, fish, and other objects. 9 Lotherington (1997) describes such forms of conventionalization as “pidginized English”, although this type of languaging is far more evident with Papua New Guineans than with Tongans.
corresponding Tongan words, if any had been or are still in existence. This practice of Tonganising foreign words is an unsatisfactory alternative to coining new words in Tongan, because no official rule or mechanism has ever been developed to regulate the importation or integration of foreign words into the Tongan language (Fasi, 1999). Consequently, Tonganising has created some social difficulties, particularly in the formal and informal uses of the Tongan language. For example, official government publications were often written in Tonganised words that were not connected to the common Tonganised language (e.g. *stamp*: official use, “sitamipa”, versus common use, “sitapa”); various local poets and musicians took pride in the “elegant” Tonganisation of English words in their compositions (e.g. *Britain*: common Tonganised word, “Pilitania”, versus the more elegant Tonganised word, “Polata’ane”); and various educational programs (government versus church schools, for instance) had their own way of Tonganising English words (e.g. *page*: government translation, “peesi”, versus Catholic translation, “pasina”). In mathematics education, there were also complications and confusion, and as Fasi (1999) noted, the teachers tended to believe that if the borrowed words were more English-sounding in the Tongan version, the students would find it easier to learn the equivalent English words. For instance, the Tonganising words, “sikuea” and “sifia” are said to evoke in many Tongan bilingual students the same images associated with their concepts of “square” and “sphere”, respectively (Fasi, 1999). (Further discussion in Chapter 5 will consider such Tonganised English mathematical terms as “borrowed” terms, if there were no equivalent words in Tongan.)

In an attempt to clarify the situation, bilingual Tongan-English dictionaries were developed in Tonga. Churchward (1953), under the commission of the local government, wrote the first official grammar guide and dictionary of the Tongan language, but published only the English-Tongan and Tongan-English format in translation form, rather than offering a dictionary with
definitions of Tongan words in Tongan. Decades later, as a result of educational system changes, as well as ongoing developments at the social, political, and economical level, the existing dictionary had to be expanded to account for the evolution of the Tongan language. A “functional” bilingual dictionary was therefore developed in 1977, based on the most commonly used Tongan words (Fasi, 1999). But, as Fasi (1999) noted, most Tongans acquired English only through the formal school system, and could not be considered functional in English until they were at the secondary level. Fasi then observed: “It is rather odd that in order to find the meaning of a word in Tongan, one has to learn first to read in a foreign language” (1999, p. 12).

Finally, beginning in 2004, a new Tongan Dictionary Project was commissioned for the development of the first monolingual Tongan-Tongan dictionary, to be completed by 2007, with a projected goal of collecting at least 20,000 Tongan words (Taumoefolau, 2004). This project has already faced many challenges, including having to avoid inventing “pure” Tongan words. Melenaite Taumoefolau, the head of the current project, announced that the project would include technical words (such as mathematics terms) “that have been conventionalized... and now have a public meaning” (Taumoefolau, 2004). Everyday Tonganised words such as “peteni” (pattern), “sitepu” (step), and “poloka” (block) came to be used in the everyday Tongan language as a result of this conventionalization process. So far, only about 6,000 words in total have been accumulated toward this project (Taumoefolau, 2004).

3.2.2 Language in Tongan Education

Today, the indigenous Tongan language remains the country’s national and homogeneous language. Tongan and English are the country’s two official languages. English, however, is the recommended (or expected) medium of instruction at the secondary and post-secondary school
level for all subjects except Tongan Studies, and Religious Studies in most church schools (which cover the majority of Tongan secondary schools).

Fasi (1999) noted that less than three decades ago, the Ministry of Education launched an integrated language program aimed at developing equal bilingual skills in Tongan and English for all Tongan students. The Ministry’s objective was to make all Tongan students “competent” in both languages, and to ensure that both languages were used effectively, in and out of school. This initiative reflected the Ministry’s dual effort – a dilemma similar to the Maori situation discussed in the previous chapter – to maintain the native language amid the onslaught of globalization, while improving English proficiency at the same time. From the establishment of the Education Act in 1947, the Tongan educational system, therefore, has evolved into the current language policy, summarized as follows:

i. Both Tongan and English are official languages, and used in public examinations.

ii. At the primary school level, both languages are used as the media of instruction with more Tongan at the lower levels (90 per cent at Class 1)\(^{11}\). The English component gradually increases to 50 per cent at Classes 4 to 6 (see graph in Graph A below).

iii. Both Tongan and English are required (or “compulsory”) subjects up to Form 5.

iv. The medium of instruction at the secondary level is English, except in Tongan and Religious Studies lessons.

\(^{10}\) In 1947, an Educational Act was established in which the primary schools would provide a general education in Tongan, the middle schools, a general elementary education using English as a medium of instruction, and the high schools, a general secondary education, mainly in English (Fasi, 1999).

\(^{11}\) Elementary school starts at Class 1 to 6 (the Canadian equivalents of grades 1 to 6), and then at the secondary school level, it starts at Form 1 to 6, (the Canadian equivalents of grades 7 to 12). The curriculum is not the same as in North America, but closely follows the New Zealand educational system, although the South Pacific Board of Education centred in Fiji now governs the curriculum.
Graph A: Percentage of language time used for Tongan and English from Grade 1 to Grade 6

In many secondary classrooms in Tonga, both Tongan and English are used in various switching combinations. It is widely recognized that the teacher and the students move back and forth freely between the two languages (Fasi, 1999). While language switching is a common arrangement, Fasi (1999) claims no research on language switching or bilingualism has taken place in Tonga prior to his study, nor has there been an investigation into the most effective classroom methods to be adopted in such a bilingual situation. Citing the Minister of Education’s Annual Report for 1995, Fasi (1999) further reports that the Ministry was confident the program would succeed by achieving competence in both languages by the end of Class 6 (grade 6), and that a smooth transition might also be achieved as the students moved through the middle or junior high school levels.

At the secondary level, the medium of instruction is English, and all mathematical texts are in English, particularly texts from other countries, such as New Zealand, that are frequently used at...
the senior levels. But the effects of the current education policy are most significant between Form 1 and Form 3 (grade 7 to grade 9), when the Tongan bilingual students begin an “intensive exposure” to their second language, English. This study focuses, in particular, on this junior high-school period of the Tongan students’ education, a time in which different levels of bilingualism are said to influence students’ cognitive growth and functioning (Cummins, 1976). Moreover, when the language of instruction changes from using primarily Tongan at the elementary school level, to using mostly English at the secondary school level, the students are said to encounter twice, or even three times, as many learning problems as those who grew up in an English-speaking environment (Fasi, 1999). The difficulties escalate when the learner’s first language is not the language of instruction, which is virtually always the case in Tonga.

3.2.3 Mathematics Language in Tongan Mathematics Education

The number system used in Tongan education today is very similar to the system used in Great Britain and other western countries, which evolved from traditional base-ten counting systems. In the late 1970s, Tongan educators developed and distributed a mathematics textbook using “invented” words for certain concepts and mathematical terms. As Fasi (1999) observed, “the work was so full of invented words that teachers and those who were interested in mathematics questioned whether they were teaching mathematics or a new language” (p. 23). For instance, Fasi explained how the word, “‘ulutefua” was created out of the native word stock and used for the corresponding mathematical word, “prime”. “‘Ulutefua” in Tongan means an only child, with the prefix “‘ulu” referring to a head, and “tefua” as the root Tongan word of the number “one”. Hence, “‘ulutefua” literally means “one head”. “Nomipa” was then used as a direct phonetic assimilation of the word, “number”. So “nomipa ‘ulutefua” was and is rarely used in the classroom to mean “a number that stands alone”, the translation used in Tongan for “prime
number”. In other words, the heavy English influence on the Tongan language has created double meanings for many words, creating potential confusion for students trying to understand the terminology – well before they can master the mathematical concepts themselves.

As a result of numerous linguistic misunderstandings, Fasi conceded that teachers and students were so confused, that mathematics was neither taught nor learned, and therefore, the mathematics-textbook project was soon abandoned. Consequently, Fasi argued that when learners enter school, the main problem they encounter is not the mathematical symbols and systems, but the language used in the classroom. The proponents of the currently commissioned dictionary project (mentioned earlier in Section 3.2.1) in Tonga recognized this problem by avoiding the creation of a “Tongan purist language dictionary”, as some people would have preferred (Taumoefolau, 2004).

As the Tongan language has evolved, other concessions have been made to incorporate the influence of English on the native vocabulary. For instance, the difficulties in finding a “real” Tongan mathematical vocabulary were in most cases resolved by “transliteration” (another term for conventionalization) of the English terms. The transliteration approach is still widely used today both in classroom teaching and in curriculum materials. Because English is the medium of instruction at the secondary school level, the mismatch between the language of mathematics and ordinary English plays a central role in the Tongan bilingual students’ (mis) understanding of written mathematics texts and other resources. Fasi (1999) discusses an example of this scenario using the Tongan word for circle, “fuapotopoto”, which in Tongan, encompasses all round shapes such as oval, ellipse, and even sphere. As a result of this confusion, transliterations were then used, and hence the notions of “circle” and “sphere” are commonly taught nowadays in the
mathematics classroom using the phonemic Tonganised words, “siakale” and “sifia”, respectively.

Another unique feature of the Tongan bilingual situation is the students’ everyday use of the mathematical terms in both Tongan and English. In this instance, the students’ prior learning and everyday experiences become sources of confusion when words are found to have different meanings in mathematical contexts. For instance, the Tonganised word “sikuea” (square) takes on various mathematical connotations – an area, a number, or a numerical property. However, the issue with finding proper Tongan words for many mathematical terms and concepts is complicated, not only by the lack of specific terms in the current Tongan language, but also, as Fasi (1999) argued, by the absence of many Westernized concepts in the life of the Tongan people. Concepts such as “probability”, “negative numbers”, and “absolute value”, for example, have no equivalent functions in the activities of the Tongan people. With regard to the concept of probability, for example, one teacher in this study recounted how difficult it is for him to teach and distinguish between the meanings of non-equivalent terms, such as “very likely”, “probable”, and “almost certain”, when these concepts have to be expressed and explained in Tongan.

In 1995, the Ministry of Education, in co-operation with the Australian government, launched a new unified mathematics program for Class 1 to 5 (grades 1 to 5) for phased implementation, beginning a year later, in 1996. This new program recognized the important role language played in the learning and teaching of mathematics. This step was significant because the emphasis on previous syllabuses was mainly on language and mathematics (Fasi, 1999). One of the aims of mathematics education, specified in this new syllabus, was that “students will develop appropriate vocabulary and language forms, first in Tongan, and later in English, for the
effective communication of mathematical ideas and experiences” (Tonga Ministry of Education, 1995a, p. 4). The early stages of secondary education, Form 1 to 3 (grades 7 to 9), are the most difficult for most students whose English proficiency is too underdeveloped for them to cope with the language demands of the classroom. To attend to this problem, the new program reviewed, revised, and recommended the following processes as essential for the teaching and learning of mathematics (Fasi 1999, p.28):

(i) Appropriate language should be developed and used by teachers and students during mathematical activities

(ii) Teachers must be familiar with accepted language patterns for different mathematical processes, and different age groups:
   a. In Classes 1 to 4 (grades 1 to 4), all teaching should be in Tongan.
   b. In Classes 5 to 6 (grades 5 to 6), teachers should teach in Tongan but should give the English words and pronunciation for important concepts, and students should write and say the words.
   c. In Forms 1 to 2 (grades 7 to 8), the teaching should be in English, but concepts and ideas should be explained in Tongan and in English.
   d. In Forms 3 to 5 (grades 9 to 11), all teaching should be in English.
   e. From Form 1 (grade 7) onward, an explanation in English that is not understood by students should be repeated in Tongan, and then repeated again in English.

(iii) Students should use both appropriate oral and written language to gain meaning from their mathematical learning experiences.

(iv) Teachers and students in all learning situations should use exemplary language modeling (in both English and Tongan).

The terms “appropriate language” and “accepted language patterns” indicate recognition of the existence and common use of language switching by teachers and students in the classrooms.
3.3 The Theoretical Framework

3.3.1 Introduction

In attempting to answer the research question first presented in Chapter 1, a suitable theoretical framework must be used to explain the nature of bilingual students' growth of mathematical understanding. While language switching may be identified through its explicit form of verbal expressions, the growth of mathematical understanding is not that explicit. The evidence of any growth of mathematical understanding is often observed as a mixture of both verbal and non-verbal actions. The Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding offers such a language as a way of examining understanding in action, of interpreting the observed cues, and of accounting for any evidence of growth of understanding.

The Pirie-Kieren theory was conceptualized and developed within a monolingual context, and has not been previously employed to investigate growth of mathematical understanding in the presence of two (or more) languages. This theory is nonetheless appropriate because it is a theory for the growth of mathematical understanding of a specific topic, by a specific person (that is, a group or individual), over time. A “specific person” implies the independent nature of the theory as a theoretical lens for observing growth of mathematical understanding, irrespective of what language(s) an individual or group of individuals possess. The Pirie-Kieren theory looks at how a learner gets to a particular mode of understanding through observations of the actions and language the learner uses, and how he or she comes to understand particular concepts in mathematics. A look at the theory’s history, philosophy, and practice may first be necessary to understand its roles in the interpretation process. Further discussion of the purpose and appropriateness of employing the Pirie-Kieren theory is offered in Section 3.4.
3.3.2 The Development of the Pirie-Kieren Theory

Since 1987, Susan Pirie and Tom Kieren have been exploring, and attempting to define, the complex notion of "understanding". Initially, Kieren (1988) worked on the area of elementary fractions, while Pirie (1987, 1988) was researching language in mathematics classrooms. Although Pirie and Kieren were working on different research areas in mathematics education, they were both interested in the notion of how one comes to understand anything at all (Pirie & Kieren, 1992a), rather than the behavioural ways of assessing what learners can and cannot do. Thus, instead of just looking at the notion of "understanding", Pirie and Kieren became more interested in investigating the "process of understanding". As a result, the Pirie-Kieren theory was developed, drawing on Pirie and Kieren’s interests and personal and academic experiences, to look at learners’ growth of mathematical understanding (Pirie, 2001, personal correspondence).

In her early work on “understanding”, Pirie (1988) found that a single category was inadequate to describe the complexity of a learner’s understanding. She observed mathematical understanding as a "whole dynamic process and not as a single or multi-valued acquisition" (Pirie & Kieren, 1994, p.165). In order to describe such understanding as a complex phenomenon, Pirie needed “an incisive way of viewing the whole process of gaining understanding” (Pirie & Kieren, 1989, p. 7). This led to the development of the Pirie-Kieren theory, initially based on the constructivist view of understanding as a continuing process of reflecting and reorganizing one’s conceptual structures (von Glasersfeld, 1987).
As the theory continued to evolve, and become increasingly complex in its language, Pirie and Kieren came to believe that although understanding was still the creation of the learner, other factors were viewed as inter-related within the space in which understanding was created (Varela, Thompson, & Rosch, 1991; Towers, Martin, & Pirie, 2000). The Pirie-Kieren theory therefore shares the enactivist view that observes learning and understanding as an interactive process (B. Davis, 1996; Kieren, Davis, Mason & Pirie, 1993). Citing B. Davis (1996), Martin (1999) describes this interactive process of understanding “not as a state to be achieved but as a dynamic and continuously unfolding phenomenon”, a view that recognizes the inter-dependence of all the participants in any particular environment (p. 35). Because learning was seen as a dynamic process, Martin (1999) said, “the creation of the diagrammatic model for the growth of mathematical understanding was therefore intended to provide a way of depicting this process” (p. 34). By adding this enactivist perspective to their initial constructivist approach, Pirie and Kieren, through an ongoing development of their theory, characterized growth of understanding as a “whole, dynamic, levelled but non-linear, transcendentally recursive process” (1991a, p. 1). The addition of the enactivist perspective does not change the nature of the Pirie-Kieren theory, but rather points out the evolving use of the language within the theory (Martin, 1999).

Since the Pirie-Kieren theory was first developed, various other researchers and former students have worked to expand and explore it further (Kieren, Davis, Mason & Pirie, 1993; Kieren, Pirie, & Reid, 1994; Martin & Pirie, 1998; Towers, 1998; Martin, 1999; Towers, Martin, & Pirie, 2000; Thom, 2004). The Pirie-Kieren theory has indeed expanded its application in both the research and in the mathematics education community. This study provides such an application and offers a new dimension to the power of the theory through using it to observe bilingual students’ growth of mathematical understanding. At the same time, some features of the Pirie-
Kieren theory remain open for further elaboration – a discussion reserved for the final chapter of this study.

3.3.3 The Nature of the Pirie-Kieren Theory

3.3.3.1 The Theory as “Levelled but Non-linear”

Drawing on Vitale’s (1988) work, Kieren and Pirie (1991a, 1991b) initially characterized mathematical understanding as “levelled” or “level-stepping”, because each level or “mode” of understanding was not exactly the same as the previous one. However, the word “level” carries with it the connotation of being hierarchical in structure. For Kieren and Pirie, understanding does not grow in a linear, hierarchical fashion or through attaining a particular level of understanding. They therefore do not believe that teaching can be implemented to achieve understanding at a certain level before the teacher moves on to the next task. This issue gives rise to Pirie and Kieren’s recent preference for using the more neutral and descriptive term, “layer” (Pirie & Kieren, 1994; Towers, Martin & Pirie, 2000; Pirie, 2001, personal correspondence), which not only refers to any modes of understanding, but also reflects the complexity of one’s paths of thinking within the Pirie-Kieren framework.

In the Pirie-Kieren theory, eight potential layers of understanding are possible, through either informal or formal actions, for a specific person and relating to a specific topic. Beginning with the innermost layer, these layers, described in detail in Section 3.3.4, are labeled: Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising (see diagrammatic model in Figure 2). These layers of “thinking sophistications” demonstrate the complexity of how learners come to understand mathematics,
rather than the sophistication or complexity of the mathematics involved. Each layer contains all previous layers (except for Primitive Knowing) and is embedded in all succeeding layers, thereby illustrating the fact that growth of understanding is "a non-linear, non-monotonic, process-in-action" (Kieren, Pirie, & Reid, 1994).

Figure 2: The Pirie-Kieren model for the growth of mathematical understanding

While the modes of understanding are described as layered, a path of the growth of understanding is observed to occur through a continuous back-and-forth movement through the modes of understanding, as illustrated later in the mapping process. In following such a path, learners remember and construct new understanding, based on their current and previous knowledge (Kieren & Pirie, 1991a; Pirie & Kieren, 1991b; Pirie, Martin, & Kieren, 1996).

Similarly, growth of understanding is observed to be "non-monotonic", because growth of

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12 Pirie and Kieren use this mapping technique to trace the pathway of growth of understanding (see the illustration in Figure 2). By tracing this pathway, the non-linear nature of growth of understanding is visible in two-dimensional form (see the detailed discussion in Section 3.3.5).
understanding does not progress outward, in a linear fashion, from one layer to another; understanding does not grow in a strictly hierarchical fashion.

3.3.3.2 The Theory as “Transcendentally Recursive”

Following the work of Maturana and Varela (1987), Kieren and Pirie (1991a, 1991b) see human knowledge of mathematics as “recursive” in nature since understanding at each layer is in some way defined in terms of itself (“self-similar”), yet each layer is not the same as the previous layer (“level-stepping”). Citing Maturana and Varela (1980), Kieren and Pirie (1991a) use the metaphor of “recursion” to mean that “the individual can distinguish between ideas seen as products of their own mental actions and those which appear to them to be driven by outside sources” (p. 81). Within this recursive structure, a learner is observed to always exhibit a different degree of understanding, in which each layer of understanding is contained within succeeding layers (Kieren & Pirie, 1991a). However, Kieren and Pirie (1991a) warned that mathematical understanding of a particular concept or topic cannot be fully attained at any particular layer; rather, it is an evolving process between layers of understanding. For instance, a learner who correctly applies his or her formal understanding of the quadratic formula does not necessary mean he or she fully understands the concept of quadratic functions or has attained a formal understanding of all quadratic properties.

In addition, Kieren and Pirie (1991a) take Margenau’s (1987) idea of growth of scientific constructs to describe new transcendent knowing, not as a simple extension of knowledge, but as learning compatible with the knowledge previously held. So the notion of “transcendentally recursive” refers to the structure of the theory repeating itself. A learner can keep going back, but it is never the same experience, because every time he or she re-visits a layer (either inner or
outer), the learner is re-visiting it with all the new understanding that he or she built up at the "return-to" layer during the process of growth of understanding. In other words, suppose a learner is working at an outer layer, "X". If the learner moves back to an inner layer, "Y", his or her existing understanding at that inner layer, Y, somehow becomes "thicker", and not at the same understanding as the original activity at that layer. Even if there was no prior activity at the inner layer, Y, or if the learner goes on to create new understanding at Y, his or her current understanding at Y is now shaped by his or her outer-layer knowing at X. Similarly, if the learner moves out again to the outer layer X, where he or she started, the learner's existing understanding at that outer layer is now shaped by his or her thick inner-layer knowing at Y. Therefore the learner's understanding is no longer the same as when he or she started, because the new inner-layer action simply alters the outer-layer knowing, and vice versa. The term, "thicker" means the learner has extended his or her understanding of the concept – possibly incorrectly through the introduction of misconceptions – and that the new "transcendent knowing frees one from the actions of the prior knowing" (Pirie & Kieren, 1989, p. 8).

3.3.3.3 The Theory as a "Whole"

The actions at each layer of the Pirie-Kieren theory do not constitute a complete understanding, but are simply named parts of a whole complex phenomenon. However, the recursive nature of the Pirie-Kieren theory is a way of describing a complex phenomenon, in which the "whole" at any time is structurally similar to, but not reducible to its previous states (Kieren & Pirie, 1991a). The Pirie-Kieren model therefore uses mapping techniques (to be discussed in Section 3.3.5) to illustrate not just how the path of one's growth of understanding unfolds, but also one's thinking sophistication or understanding as a whole complex phenomenon, and how each of the layers are dependent on the other.
The notion of recursion described earlier (in Section 3.3.3) is used as an attempt to understand a piece of mathematical knowledge or a problem-solving act as a “dynamic whole”, which means that “one can understand a piece of mathematics many ways at once” (Pirie & Kieren, 1992b, p. 244). To see the nature of the theory as a dynamic whole refers to both a condition and consequence of recursion called “level-connectedness”, which suggests that a learner can “drop” back to a previously developed layer of mathematical experience to re-construct the basis for a higher-layer experience. Because of this “flexibility”, which respects movement out to any of the other layers, the Pirie-Kieren theory demonstrates that it is possible, therefore, for a learner to understand a piece of mathematics in many ways at once (Pirie & Kieren, 1992b).

3.3.3.4 The Theory as a “Dynamical Process”

In their early writings, Pirie and Kieren (1989, 1991b) used the word “dynamic” for their theory in the sense of a dynamic within a group relationship. The word “dynamical” is now preferred, because it encompasses and reflects the interactive nature of learning and understanding, which is continuously affected by the environment (Pirie, 2001, personal correspondence). Thus, one cannot simply declare that the Pirie-Kieren view of growth of mathematical understanding is “not static”. The term, understanding, as it is portrayed in the Pirie-Kieren theory, refers to an ongoing process, and the process of re-organizing one’s knowledge structures within a specific environment characterizes the Pirie-Kieren theory as dynamical. In developing the theory, Pirie and Kieren (1994) characterized understanding to be “a dynamic process, not an acquisition of categories of knowing” (p. 187).
Likewise, in using the Pirie-Kieren diagrammatic model, the important aspect in any mapping is the \textit{dynamical} nature of what the path exemplifies in terms of one's growth of mathematical understanding. Although the Pirie-Kieren model contains embedded layers of understanding, Pirie and Kieren (1992b) claim that it is the dynamic within and between the layers that is critical to observe. "Acting and expressing complementarity" and "folding back" are two features that describe the back-and-forth movements within and between the layers of the Pirie-Kieren theory (see \textbf{Sections 3.3.6} for a discussion of these two features).

One feature of this \textit{dynamical} aspect can be seen through the interactions within the Pirie-Kieren modes of understanding, a system that can be seen as dynamic in the sense of what Capra (1996) called "ecological community". Capra (1996) describes the nature of an ecological community in the following way:

All members of an ecological community are inter-connected in a vast and intricate network of relationships, the web of life... The fact that the basic pattern of life is a network pattern means that the relationships among the members of an ecological community are non-linear, involving multiple feedback loops (p. 298).

This "ecological" view reflects the dynamic, back-and-forth flow that characterizes the uniqueness of the Pirie-Kieren theory. Words such as "connected", "disjoint", "(re-) organization", "structure", "(re) construct", "pattern", "root", "point", "inner", "outer", "layer", "path", "(re) tracing", all indicate a similarity of the Pirie-Kieren theory to that of a structural network. In addition, Capra's use of the words "non-linear" and "multiple feedback loops" resembles the recursive nature of the Pirie-Kieren theory; the term, "multiple feedback loops", of course, is also analogous to the Pirie-Kieren's notion of "folding back", although it may not reveal as much about the thickening effect involved in growth of understanding. The network
itself, in the sense of Capra's (1996) ecological idea of "network pattern", constantly undergoes re-structuring as the learner builds upon his or her understanding through a series of back-and-forth movements, thereby attaining thicker understanding. The Pirie-Kieren theory demonstrates the nature of understanding to be a constant, consistent re-organization of one's knowledge structures (Pirie & Kieren, 1994).

3.3.4 The Modes or Layers of Understanding

The Pirie-Kieren model is a diagrammatic representation consisting of a sequence of nested circles or layers, developed to represent the eight potential modes of understanding. Figure 2 (Section 3.3.3.1) shows a diagrammatic representation of the Pirie-Kieren Model. Although this two-dimensional representation of the model is not the theory, the diagram itself is a useful tool for mapping the path of one's growth of understanding. Further discussion of this mapping technique is contained in Section 3.3.5. A brief analysis of the layers within the model will be illustrated by using a hypothetical example of the way in which a student, say in grade 12, comes to understand the concept of linear functions. The eight potential modes of understanding of the Pirie-Kieren theory are offered, either as informal or formal actions, for a specific person, and relating to a specific topic.

Layer 1: Primitive Knowing

The process of a learner coming to understand a concept or idea starts at the Primitive Knowing layer. The word "primitive" does not imply prehistoric or low-level mathematics, but rather a sense of the starting point or "base knowledge" for the growth of understanding, "observed in a particular person on a particular topic at a particular time" (Kieren & Pirie, 1991b, p. 9). It is what a learner has in his or her mind when he or she enters a new mathematical situation, with
the exception of any existing knowledge, such as images about the topic to which he or she is about to be introduced. An observer (such as a teacher) can only assume what the learner can do initially, and in making such assumptions, the observer can then build on that supposition about the learner’s growth of initial understanding of that particular topic. Therefore, the observer cannot know in full what this primitive knowledge is, or is not, but his or her assumptions can largely be based on what the individual learner was exposed to in the previous classes or years of schooling.

For the concept of linear function, a grade 12 student may have certain “primitive” knowledge such as the language used to express this concept, and possibly mental images and existing knowledge of lines, points, graphs, or any concepts that were learned in previous lessons or at lower levels, say in grade 10 or 11. In this latter case, the student has some knowledge already in the other layers or modes of understanding; *Primitive Knowing* is everything a learner knows that is unrelated to the topic to be taught. Although this layer may seem uninteresting in relation to the other layers, Ausubel (1968) has noted how significant such primitive knowledge is (and any existing knowledge) in influencing how students learn and perform mathematical tasks. He said, “If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows” (p. vi).

Layer 2: Image Making

The labels for the next two layers are named using the word, “image”. A study of this nature focuses on identifying the mathematical images held, accessed, made, modified, and worked with by the students as they engage in the task. An “image” is defined as any physical or mental representation or idea that the students may have for the investigated topic. These images are often specific, mathematically limiting, and context dependent (Martin & LaCroix, 2004). Pirie
and Kieren (1994) suggest that the word "image" is less open to ambiguity and misunderstanding than, say, "idea". The participants' work with images (for the specified topic) in large part determines for them what is mathematically meaningful (or not meaningful) while they are working on the task.

The second layer is therefore called Image Making, and the learner acts on his or her Primitive Knowing to develop particular images for the given task. These images need not be visual or pictorial, and could be mental ideas about the specified topic. The learner has to engage in activities that will bring forth these particular images and reflect on those activities and on his or her previous abilities in order to use these images in new ways. Thus, this Image Making layer is dependent on the learner's primitive knowledge in order to bring about his or her growth of understanding. Using the example of the grade-12 topic of linear function, the learner may begin making an image by plotting linear graphs or relations. When the learner performs an action or activity, such as plotting a linear graph, he or she is working at the Image Making layer, building images of linear graphs based possibly on his or her Primitive Knowing.

Layer 3: Image Having
At the Image Having layer, a learner demonstrates the use of a mental construct about the topic and is able to use that particular image without doing the activity itself. Image Having implies that the learner is no longer tied to the particular action, but is now able to carry out the particular activity through internalization of the constructed images – a concept Martin (2001) called a "mental plan". As a result, the learner does not need to do whatever it was that he or she was doing before at the level of Image Making. Furthermore, the learner does not have to make every image possible on that particular topic in order to have a meaningful image. A statement such as, "It should look like a vertical line!" when asked about the shape of the line \( x = 2 \) demonstrates a
student constructing a mental picture of how the graph looks, without having to draw it. Such a statement reveals that the learner has now characterized, developed, and brought forth his or her sense of the meaning of the word “linear”. However, the learner’s meaning of the word “linear” may not be the correct one, or what is needed to accomplish the task at hand.

Layer 4: *Property Noticing*

The fourth layer is called *Property Noticing*, and it refers to a learner looking at his or her images and trying to identify context-specific, interconnected properties by manipulating and combining aspects of his or her constructed images. The word “interconnected” is used to represent the learner making distinctions, connections, and relationships concerning the particular example he or she is working on and is also able to articulate. These connections are based on knowledge from the previous three levels or knowledge that arose from manipulating his or her direct images (Pirie, 2001, personal correspondence). The learner consciously sees the relationships and makes the connections, answering questions such as, “How are the images connected?” For example, a student may compare the graphs of $y = 2x$ and $y = 6x$, and, after plotting the two lines, say, “Wow, $6x$ is a lot steeper going left to right!”, and then add, “I think $100x$ will be a lot steeper.” In this instance, the student speaks of specific lines (that is, $y = 6x$ and $y = 100x$) and a specific property (that is, “steeper going left to right!”). But then, someone else might say, “Well, but $y = -2x$ goes the other way around! What would happen, then, to its relationship with $y = -6x$?” Hence, based on such student responses, and by comparing and seeing features of each line, the observer is aware of a student working at the *Property Noticing* layer. Pirie has alleged that teachers quite often do not teach *Property Noticing*, but instead put more emphasis on constructing images without helping the students to connect the properties of their constructed images (2001, personal correspondence). As a result, students are not able to make connections
or to progress in their growth of understanding, not just within *Property Noticing*, but also with regard to that layer’s connections to other layers of understanding.

Layer 5: *Formalising*

*Formalising* refers to the next mode of understanding that involves a learner being able to articulate a general observation, which does not have to be stated algebraically. The learner abstracts from his or her noticed properties a method or generalization by stating the patterns or the rules, which are disconnected from specific examples, actions, or particular images. The learner, at the stage of *Formalising*, has therefore moved away from noticing specific properties, to generalizing and working with the concept as a formal object. It is at this layer that the learner may be capable of stating or recognizing formal mathematical definitions or algorithms, making the *Formalising* layer of understanding the focus of much teaching. Hence, the student can give some form of the general slope-intercept form \( y = ax + b \) of a linear function, a learner who knows that the magnitude or absolute value of “\( a \)” (the coefficient of “\( x \)” determines the steepness of the graph demonstrates a generalization of such a property for all linear functions of that algebraic form. Quite possibly, the learner may also *Formalise* that the constant value “\( b \)” determines the vertical translation of the linear graph, even without knowing whether \( b = -2 \) means moving the linear graph up or down.

Layer 6: *Observing*

In the case of linear functions, suppose a student has been taught the general algebraic forms: \( y = ax + b \), \( \frac{x}{a} + \frac{y}{b} = 1 \), or \( ax + by + c = 0 \). If the student is able to justify that these three algebraic equations are equivalent, and true for all linear functions, then his or her understanding of linear functions is beyond *Formalising*. Similar to *Property Noticing*, where a learner
constantly works with his or her images, *Observing* involves looking at all the generalized statements and making connections among them. Such ability demonstrates that the learner is ready to reflect upon, organize, and express such a formal activity as a *theorem*. It is a mode of understanding in which the learner is in a position to reference his or her own formal thinking, and he or she is able to see its consequences.

Layer 7: *Structuring*

An awareness of one’s assumptions, together with one’s observations, requires logical thinking in establishing relationships within one’s formal observations. A learner at this stage is said to be *Structuring*, a stage requiring formal mathematical proof. *Structuring* involves the learner being confident in justifying and verifying statements or proofs through logical or meta-mathematical arguments. Such a statement expresses mathematical structure, independent of physical or algorithmic actions. In the case of linear functions, when a student says that a line is only a special case of the polynomial functions with the highest degree of one, it is an indication of the logical argument present in *Structuring*.

Layer 8: *Inventising*

The outermost layer is called the *Inventising* layer. A learner working at this layer is capable of “breaking away” and developing a new creation, new concept, or even a new question that may also lead to the formulation of other new concepts. Within a given topic, the learner working at this layer is assumed to have a fully developed, *Structured* understanding that may therefore allow him or her to break away from *Structuring*, to *Inventising* new possibilities. Such a circumstance happens, therefore, when the learner knows all the rules and he or she wants to change something. So the learner would say, “What would happen if...?”
Using the example of linear functions, a grade 12 student who seeks new properties using complex numbers would be *Inventising* by pursuing possibilities other than the existing theories of, or about, linear functions. So, in a way, *Inventising* is not about a sophisticated piece of mathematics that is very hard to accomplish, but rather a mode of understanding a concept that may or may not make sense for the learner. Learners at any level, therefore, are capable of coming up with some amazing mathematical results or new discoveries if given the opportunity to *Inventise*, although such an outcome rarely occurs because it is not covered by the standard curriculum (Pirie, 2001, personal correspondence).

3.3.5 The Two-Dimensional Model – A Mapping Technique

Pirie and Kieren have also developed a technique they call “mapping” to allow an observer to trace the pathway of a learner’s growing understanding of a particular mathematical topic. Using the diagrammatic representation of their model, an observer can produce a “map” by drawing the learner’s growth of understanding, *as it is observed* (see Figure 2, Section 3.3.3.1). To produce such a map, the observer can plot points on the diagrammatic model and then, using connected or disconnected lines, and depending on how each of the successive events (represented by points) are related, show how the learner’s understanding of the specified topic grows. It is important to remember, however, that in using such a mapping process, the observer is limited to sometimes only being able to see with hindsight what goes on inside the learner’s thought processes.

In the Pirie-Kieren model, “connectedness” is associated with having a path between the layers or modes of understanding. Without such connected paths, learners who unsuccessfully build connections may develop what Pirie and Kieren (1994) call “disjointed” or “disconnected” knowing. While observing learners with no connections between their new understanding and
their existing knowledge, Pirie and Kieren (1994) hypothesize that these “students will be unable to successfully build further understanding based on this disjoint knowing until they have in fact constructed the connection for themselves” (p. 185, original emphasis). In this respect, the Pirie-Kieren theory, as illustrated pictorially by their model, is analogous to Capra’s (1996) ecological model of a connected network — a complex, dynamic whole, continually undergoing realignment and reconfiguration, as layers of thinking sophistication become thicker through ongoing recursion and back-and-forth movement.

3.3.6 The Special Features of the Pirie-Kieren Theory

As a theory for understanding, the uniqueness and strength of the Pirie-Kieren theory is characterized by its special features. These special features — folding back, acting and expressing complementarity, and “don’t-need” boundaries — are essential elements that add to the complexity of the theory. These features are examined in the context of this research study.

3.3.6.1 Folding Back

The metaphor of folding back is a very important notion in the Pirie-Kieren theory because it distinguishes this theory from any existing theory on mathematical understanding. One reason for this metaphor’s importance is that in order for understanding to grow and develop, folding back is “an intrinsic and necessary part of the process” (Towers, Martin, & Pirie, 2000, p. 226). Kieren and Pirie (1991b) define folding back in the following way: when a learner is faced with a challenge, one that is not immediately solvable, he or she is prompted to return to an inner mode of understanding in order to reconstruct, and to extend his or her currently inadequate inner-layer
The learner needs to *fold back* to an inner layer to extend his or her inadequate understanding in order to progress at an outer layer.

*Folding back* is not just going back; it carries with it the notion of superimposing one's current understanding on an earlier understanding, resulting in a thicker understanding at the "returned-to" layer. This returned-to layer is now informed and shaped by outer-layer awareness, and the inner-layer action is part of the recursive process necessary to build further upon the outer-layer knowing (Kieren & Pirie, 1991a, 199b). The returned-to layer is no longer identical to the original-layer understanding, because of the recursive nature and interdependence between each of the layers that enfold it. Martin (1999), identified four "forms" of *folding back*:

1. **Working at an Inner Layer:**
   This form of *folding back* involves a learner shifting to work in a less sophisticated, perhaps less formal, mathematical way and can involve the learner either extending his or her current understanding by changing his or her earlier constructs for the concept, or through the generation of new images. In either case, the less sophisticated, inner-layer understanding is thickened by the new constructed understanding and by the existing outer-layer knowing.

2. **Collecting at an Inner Layer:**
   *Folding back to collect* entails retrieving previous knowledge for a specific purpose and reviewing or "reading it anew" in light of the needs of current mathematical actions (Martin, 1999, p. 176). *Collecting* is not simply an act of recall – in which case, it would be called "remembering". *Collecting* has the thickening effect of folding back. *Folding back to collect* occurs when students "know that they know" what is needed, and yet their understanding is not sufficient for the automatic recall of usable knowledge.
iii. Moving Out of the Topic and Working There:
This form of folding back accounts for those cases involving a learner moving out of the current topic and working on a different mathematical area. This phenomenon takes place in instances involving the individual having an incomplete or inadequate Primitive Knowing from which to build or develop his or her understanding. Such "incompleteness" requires a stepping out of the current topic to permit the developing and thickening of concepts from a different mathematical area. The purpose of moving out of the topic and working there is to work on one's Primitive Knowing and then to be able to work with it at the outer or current-layer knowing of the topic being studied.

iv. Causing Discontinuity:
This form of folding back occurs when an external intervention (for example, a teacher or a peer) causes a learner to fold back to an inner layer of understanding and work there. But on the return to the inner layer, the learner essentially "begins again", making a new image unrelated to his or her existing understanding. Thus, the external intervention causes a break in the developing understanding of the learner. The key feature in this case concerns the learner's inability to perceive the relevance of the need to fold back to the problem he or she is working on.

3.3.6.2 Acting and Expressing Complementarity

Pirie and Kieren (1994) believe that each layer beyond Primitive Knowing is composed of a complementarity of acting and expressing, and that complementarity is necessary to complete a layer of understanding (see Figure 3). Furthermore, growth occurs through "first acting, then expressing, but more often through to-and-fro movement between these complementary aspects" (p. 5). Acting encompasses all previous understanding, either mental or physical activity, and expressing concerns making explicit and articulating to others or to oneself what was involved in that activity. Reflecting is a critical component of this complementarity. The process of reflecting incorporates looking at how previous understanding was generated.
The *acting/expressing* complementarity feature of the Pirie-Kieren theory is central to observing the students' verbal and nonverbal actions. Pirie and Kieren have yet to investigate this feature further (Pirie, personal correspondence). Currently, only the complementarity "pairs" in the first five layers have been identified and defined (see the definitions below). While this feature requires further investigation, this study attempts to investigate any observable relationship between acting and expressing through the act of language switching. (See Pirie & Kieren (1994) for more discussion and examples on these complementarity pairs.)

![Diagram showing complementarities of Acting and Expressing in each of the layers](Figure 3)

**Figure 3:** Complementarities of Acting and Expressing in each of the layers

The first pair of instances involving *acting* and *expressing* complementarity occurs at the *Image Making* layer and are called *image doing* and *image reviewing* respectively. In *image doing*, a learner is engaged in activities associated with constructing an image of the situation. In this situation, the learner sees his or her previous action(s) as completed, and rejects returning to it in
anything other than a rule-bound way. But in *image reviewing*, the learner reviews his or her previous work and attempts to make sense of the situation. Such a situation allows for a constructive alteration of previous behavior without yet seeing an image. The next pair of instances occurs at the *Image Having* layer and are called *image seeing* and *image saying* respectively. A learner able to demonstrate that he or she has now formed an image (though not necessarily a complete or correct one) is said to be *image seeing*. In this case, the learner has gone further in his or her understanding and, through reviewing, has constructed some image for the topic. He or she would be able to articulate that the new “point” does not fit with his or her image of the topic, but would not be able to explain why. In *image saying*, the learner is able to express aspects of the image. Such behaviour articulates on features of the image. At this level, the learner is in a position to talk about his or her actions and why “something” does not conform to his or her image. The third pair of instances occurs at the *Property Noticing* layer, and is called *property predicting* and *property recording* respectively. In *property predicting*, a learner is able to connect previous images and recognize a pattern. This process deals with distinguishing and connecting features of the learner’s constructed image(s), leading to a new kind of understanding. A learner who demonstrates property recording is able to express that he or she has an “organized” scheme(s) for the connection or pattern. Such behaviour articulates features of the observed pattern. The use of the word “recording” is not intended to imply that a written record be produced, but rather it must involve verbal or non-verbal explicit expression of some form. The last identified pair of instances occurs at the *Formalising* layer, and is called *method applying* and *method justifying* respectively. In *method applying*, a learner is able to apply his or her understanding of a particular concept in appropriate generalized circumstances, while demonstrating in *method justifying* that he or she is capable of justifying the generalization.
3.3.6.3 "Don’t-Need" Boundaries

The “don’t-need” boundaries are of importance to understanding the nature of the Pirie-Kieren theory. The need to articulate Formalised understanding explicitly reveals the important role of language in explaining the move from Property Noticing to Formalising. These boundaries are observable in the Pirie-Kieren model, and are illustrated by the bold rings in the two-dimensional representation (see Figure 2 and Figure 3). These bold rings are called “don’t-need” boundaries to convey the idea that beyond each of these boundaries, a learner is capable of operating in a particular mode of understanding without having to access previous modes, although these previous modes of understanding are embedded in the new level of understanding. With an image of a straight line, \( y = x \), a student does not need to draw \( y = 2x, y = 3x, \) etc., each time to check its linearity. Likewise, working with the Formalised expression \( y = ax + b \), the student does not need to check the formula with say, \( y = 3x + 4 \), to be sure of its linearity or accuracy.

As mentioned in the previous chapter (Section 2.3), the notion of “don’t-need” boundary resembles Sfard’s (1991) idea of “reification”, in which the learner suddenly sees the familiar use of mathematics in a new light. Thus, a learner who is working at each of these particular modes of understanding, namely Image Having, Formalising, and Structuring, is capable of working with concepts no longer tied to his or her previous modes of understanding (Pirie & Kieren, 1994).

The bold rings also illustrate the transcendent, recursive nature of the Pirie-Kieren theory; that “understanding is a transcendently growing whole” (Kieren & Pirie, 1991b, p. 21). The knowledge of a learner with connected understanding at a particular point in time is integrated with previous layers of knowing, and because of this interconnection, the learner’s previous
knowledge can be called into current-layer actions. In contrast to the bold-ring boundaries, a learner is constantly working between layers around a “do-need” boundary, one that is marked by a non-bold ring. For example, a learner who is observed to be working at the Observing layer is constantly working with his or her Formalised statements or understanding in the same manner as a learner who constantly accesses his or her images, while working at the Property Noticing layer.

3.4 The Appropriateness and Purpose of the Pirie-Kieren Theory

This chapter introduced the bilingual context of the study, and summarized the nature and features of the chosen theoretical framework. Ultimately, the importance of the chosen theoretical framework lies in its ability to explain and express in its own “language” the nature of a learner’s growth of understanding, irrespective of his or her cultural background. This usefulness of the Pirie-Kieren theory thus extends, in this study, to observing how bilingual students’ language switching interacts with their growth of mathematical understanding. The main question this study must therefore consider is the following: Why is the Pirie-Kieren theory appropriate in studying Tongan bilingual students’ growth of mathematical understanding?

To answer the preceding question: First, the Pirie-Kieren Theory is a theory developed for, and not of, mathematical understanding. A “theory of” is a theory that usually claims to explain everything. A “theory for” is an explanation, but it does not claim to be the only explanation. The Pirie-Kieren theory therefore offers a language for explaining, and a way of observing and accounting for the dynamical growth of mathematical understanding, which is appropriate to this study, taking place within a bilingual context. The theory has been given wide recognition as
exemplified by its influence in various curriculum reforms (Godino, 1996), and its application in the State of Wisconsin, where the Pirie-Kieren model is used by the mathematics teachers in a state-wide professional development program (Martin, 1999).

Second, the Pirie-Kieren theory is not just a theory for the growth of understanding in a general sense, but rather, a theory for the growth of understanding of a specific mathematical topic. This is an important distinction because one cannot just say, “Hey, the student understands!” The task of an observer is to focus on and specify exactly what it is a learner understands, and how. The Pirie-Kieren theory becomes a “lens”, and useful theoretical tool, for observing growth of mathematical understanding in any learning situation, with a chosen focus on the specified topic. It is therefore important to keep track of one’s growth of understanding in that specific topic; choosing another topic or focus would result in a different perspective or path of analysis.

Third, the Pirie-Kieren theory is a theory for the growth of understanding of a specific topic by a specific person. The word “person” is placed in inverted commas to indicate that the theory can be used to observe not just a single person’s growth of understanding, but also to analyze a certain group’s collective growth of understanding of any specific topic. The theory therefore includes the enactivist view of the “co-emergent of understandings”, to cite the role of others in the interactive process. It must be remembered the observer can only make inferences based on what is observed through the learner’s verbalization of his or her work, and any non-verbal externalizations that might be available, such as the learner’s worksheet, or body movements and gestures; the observer cannot make any inferences based on what is not observed, and therefore conclude what the learner does not know. More importantly, the dynamic of growth of understanding varies from one person to another for any specific topic at any given moment. Hence, different persons (or groups) will have different pathways of understanding.
Fourth, the Pirie-Kieren theory describes the growth of understanding of a specific topic, by a specific person, over time. Time, of course, does not necessarily refer to a long period of time, say, in years or months, but it can refer to growth of understanding over a period of hours, or even minutes. Most importantly, however, the Pirie-Kieren theory does not take a “snapshot” assessment or create a “still-photo” picture of someone’s growth of understanding.

Finally, Pirie and Kieren (1991a) recognize that language has an “orienting function” in the constructions and abstractions of mathematical knowledge, a view they cite as consistent with Maturana and Varela’s (1980) ideas. Pirie and Kieren do not believe that, during a course of interaction, language carries information between two individuals; rather, language “allows a child to orient herself to her own actions, particularly mental ones, thus facilitating abstraction and the recursive use of her own knowledge in the building of new patterns of knowledge” (Kieren & Pirie, 1991a, p. 81-82). This “facilitation” aids the personal internalizing of existing knowledge, and is part of the individual’s understanding (Kieren & Pirie, 1991a). Thus, language use can facilitate knowing and ready it for further, more elaborate knowledge building. The orienting function of language described here has a significant implication toward the current study because language can influence a learner’s path of growth of mathematical understanding in three different ways: (i) a provocative effect describes how the learner moves outward in his or her growth of understanding; (ii) an invocative effect describes how the learner folds back to work with inner-layer knowing; and (iii) a validating effect describes how the learner confirms his or her working within a particular layer. An external intervention can also influence a learner’s path of growth of mathematical understanding in one of the three described ways.
The Pirie-Kieren dynamical theory, therefore, can be used as a powerful theoretical tool in studying and understanding the notion of "mathematical understanding". However, it has never been a theory for mere assessment, because it does not look at understanding as only an outward progressing, nor as an acquisition of specific categories of knowing (Pirie & Kieren, 1994). It is not a theory for testing what a learner can or cannot do, because such a theory does not reveal how the learner got to his or her answer or a specific point in his or her work, or even how the learner got to that state of thinking. Rather, the Pirie-Kieren theory is a theory for looking at how the learner gets to a mode of understanding, and how he or she comes to understand the mathematics in question.
CHAPTER 4: METHODOLOGY, RESEARCH DESIGN, and METHOD

4.1 Introduction

This chapter focuses on the study's methodology, research design, and the method of data collection used to address the research question.

4.2 Methodology: Video Case Study Research

4.2.1 Introduction

In an effort to propose an appropriate methodology for the study, the research question was divided into two main phenomena: first, language switching, and second, growth of mathematical understanding. These two phenomena, in turn, steered the manner and direction of the investigation, and thus shaped the qualitative approach chosen for data collection and analysis. Unlike most linguistic studies, this study was not intended to evaluate bilingual students' language switching, nor was it an attempt to assess students' mathematical understanding through standard tests (or any such quantitative method). Instead, this study was intended to focus precisely on the relationship between language switching and growth of understanding for a particular mathematical topic. In order to investigate the relationship between these two phenomena, a qualitative approach had to be employed to investigate, in depth, the nature of a student's growth of mathematical understanding. Case-study research was deemed the most appropriate form of qualitative study, one that would involve a small number of
bilingual students, and examine how their language use (including language switching) related to their growth of mathematical understanding. This tactic, as Towers (1998) has demonstrated, provides a “deep and comprehensive description and interpretation” of the processes within a particular learning situation (p. 47). Because the Pirie-Kieren theory was chosen as the theoretical framework, a case-study approach seemed most appropriate for investigating such a complex process as the growth of mathematical understanding. Previous researchers have also dealt with this methodological choice while studying growth of mathematical understanding using the Pirie-Kieren model (Towers, 1998; Martin, 1999; Thom, 2004).

The methodology then selected for this study, termed “video case study”, refers to a case study based on video-recorded data. Because of the essential role video plays in this study, the label “video case study” was used to emphasize the primary role of video-recording technique. This emphasis has been documented elsewhere, in similar video case studies of this nature (Maher, Davis, & Alston, 1992; Towers, 1998; Martin, 1999; Pirie, 1996, 2001; Powell, Francisco, & Maher, 2001). Video case study is not a new paradigm in qualitative research, since it has been employed previously in various fields for various purposes (Wood, Cobb, & Yackel, 1991; Davis, Maher, & Martino, 1992; Erickson, 1992; Cobb & Whitenack, 1996; Goldman-Segall, 1998; Hall, 2000; Lesh & Lehrer, 2000). In addition, several case studies involving video recordings have been widely implemented and discussed in many different fields: in anthropology and ethnography (e.g. Clifford, 1986; Rollwagen, 1988; Atkinson & Hammersley, 1994; Goldman-Segall, 1998); in sociology and social sciences (e.g. Grimshaw, 1982; Chaplin, 1994); in health sciences (e.g. Gross, 1991; Bottorff, 1994); in the arts (e.g. Gernsey, 1997; Ohler, 2000); in humanities (e.g. Hall, 2000; Powell, Francisco, & Maher, 2001), and law and

13 But such a study should not be confused with a case study video, which is not a process, but a product material, such as an informative video production used for educational or professional workshops.
business (e.g. Westerfield, 1989; Hawkins, 2001), to name a few. As well, video case study methodology has been documented in various contexts, even within a single discipline, such as mathematics education. For instance, previous mathematics education studies have been based on groups' interactions (Davis & Maher, 1990; Pirie & Kieren, 1992b, 1994, 1999; Cobb & Whitenack, 1996), teacher interventions (Maher, Davis, & Alston, 1992; Towers, 1998; Towers, Martin, & Pirie, 2000), individuals' learning and understanding (Pirie & Kieren, 1992b; Martin, 1999; Hiebert & Carpenter, 1992). More specifically, video case study research has been used to observe students' growth of understanding (Maher, Davis, & Alston, 1991; Pirie & Kieren, 1992b, 1994), an area in which the research question for this thesis (on language switching and growth of mathematical understanding) is framed and situated.

Like all case studies, the advantage of using video case study involves what Geertz (1973) has called the quality of its “thick description” as opposed to the “thinner”, more scientific modes of description. Stake (2000) defines a case study to mean “both a process of inquiry about the case and the product of that inquiry” (p. 436). Similarly, a video case study is both the process of learning about the case – aspects of the research question – and the product of that learning through video recordings. While the case study gives a “thick description” of an event, video becomes the efficient vehicle for the collection, storage, and analysis of the evidence or data. What is more, video case study brings the case alive on recorded tape, enabling researchers to revisit and reflect on the recorded scene for clarification, an invaluable tool for anyone wishing to verify or extend their knowledge of existing research. Nonetheless, it is important to question the empirical nature of video case study, and whether or not it can be seen as a useful and appropriate tool for examining the relationship between bilingual students' language switching and their growth of mathematical understanding. To address this question, on one level, case study must be considered the most appropriate methodology, and on another level, the use of
video recording must also be deemed a trustworthy source of data collection. As the following arguments will demonstrate, the integration of a case-study methodology and data collection through videotape was the best possible choice for investigating the proposed research question.

4.2.2 Case Study Research and the Notion of “Generalization”

Case study has long been a standard and popular methodological research practice for studying the “particular”, a term often referring to a specific characteristic or feature, or the observation of an interesting phenomenon (Stake, 1994, 2000; Tellis, 1997). This notion is characteristic of any qualitative study, and is often mentioned in comparison with the quantitative notion of “generalization”, a term usually used to describe “something for all” (Yin, 1989). Case study research is also characterized, by its very nature, as an in-depth investigation of that particular phenomenon (Yin, 1989). For Stake (2000), the underlying epistemological question in any case study draws one’s attention to what can be observed and learned from a single case: a view that resembles Yin’s empirical approach, and one that is adopted in this study. Stake (2000) then describes case study research as merely “not a methodological choice but a choice of what is to be studied” (p. 435). Other researchers, notably Adelman, Jenkins, and Kemmis (1976) and Vaughan (1992), have focused on the particularity of case-study research, and its narrow applicability to the phenomenon being investigated. Yet, a dichotomy often arises whereby case study research emphasizes, on the one hand, its nature as a methodological technique, and on the other, its “particularity” with respect to the phenomenon being investigated. Born out of this debate is the emergence of video case study, in which its very nature and role in qualitative research combines both what is being investigated (the case) and how it is being investigated (the video). However, case study research may not only concern itself with a specific phenomenon.
The nature of each phenomenon can change when analyzed, observed, or conducted using different theoretical frameworks, different situations, or different methods of inquiry.

At the same time, educational researchers will continue to debate the nature and substance of a case study, or case study research (Tellis, 1997). Like others involved in image-based research, researchers conducting case studies are well aware of these studies' limited status – not because of the quality or the focus of researchers’ work, but because of the nature of their work (Prosser, 1998, p. 99). In the case of the current study, the limitations imposed by certain forms of qualitative study were unavoidable, so it was important to remain aware of such limitations while conducting the research, and to offer appropriate counteracting measures when necessary. One such limitation of the methodology, for instance, related to Towers’ (1998) observation, that the main objection to case study research is the statistically insignificant portion of a particular population being investigated. However, it was not the intention of this study to generalize. What is more, in defending such a position, other case study researchers claim they pride their findings upon knowing their cases extremely well, and as Towers (1998) further pointed out, in "recognizing the distinction between generalizing and over-generalizing" (p. 47, citing Wolcott, 1985). Such an emphasis allows for the selection of case study methodology with only a few participants because "one wishes to understand the particular in depth, not because one wants to know what is generally true of the many" (Merriam, 1988, p. 173). After all, committing oneself to generalization may draw the focus away from the features important for understanding the case (Stake, 2000). So case studies, including this one in particular, can be considered generalizable in so far as they are used in a general way to ask a question about a particular phenomenon, rather than in their claim to generalization of a particular concept as a conclusive

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14 Image-based research encompasses studies involving film, video, photography, cartoons, signs, symbols, and drawings (Prosser, 1998).
inquiry (Towers, 1998, citing Lincoln & Guba, 1985, p. 110). But the fact that such a study focuses on a statistically insignificant sample challenges all researchers to “openly acknowledge what they are attempting to do” (Towers, 1998, p. 47). By so doing, readers may then be less inclined to criticism and confusion, enabling the author to get his or her message across.

Yet, the concern toward attaining generalizability, or defining what is or is not generalizable, remains for some a weakness within qualitative research. The typical reaction to case study research is, “How can one generalize from a single case?” A notable counterpoint to this argument arises in the question of what exactly one generalizes from, a point made by Towers (1998) when she observed that “tensions are currently surfacing concerning what can and cannot be generated from” (p. 49). As well, Yin (1989) suggests a certain type of generalization can be made from case studies. He argues:

The short answer is that case studies, like experiments, are generalizable to theoretical propositions and not to populations or universes. In this sense, the case study, like the experiment, does not represent a “sample”, and the investigators goal is to expand and generalize theories (analytical generalization) and not to enumerate frequencies (statistical generalization). (Yin, 1989, p. 21)

In relation to the topic at hand, Stake (1994, 2000) also claims that while the search for particularity competes with the search for generalizability, the nature of some studies are unique and complex, which, statistically (and practically) speaking, cannot be compared generally but may lead into what one may call a generalized theory. Therefore, in an effort to seek such generalization in this study, Glaser and Strauss’ (1967) “constant comparative method” was employed, because it allowed for a general categorization of language switching to emerge from the analysis, while the analysis remained firmly grounded in the data. For the purposes of this
study, and to address longstanding concerns over what Yin (1989) called “sloppy”, “lack of rigor”, and researcher bias sometimes associated with case study research, certain features of the study were considered necessary. For instance, the case study approach took into account its role as an empirical inquiry, involving the use of multiple sources of evidence, and also the employment of the technique of “triangulation”. Such an approach recognized the complexity of investigating a contemporary phenomenon within a real-life context, and also improved the reliability of the study, in order to respond to traditional criticism about the trustworthiness of interpretations based solely on observational methods.

4.3 Research Design

4.3.1 Introduction

After deciding to use video case study with a relatively small number of students, it was important to choose the process of data collection. Various aspects of the research design were considered: location, setting, schedule, task, contact, participants, and so on. The prior consent of all participating parties – the government, the Ministry of Education, the schools, the parents, the teachers, and the students – had to be obtained before the study could take place. However, other aspects of the research process, such as drafted tasks and schedules, had to be continually revisited and revised or rescheduled, as the study progressed. Two separate studies were documented: one in 2001 and another in 2002, and these formed the two strands of the research.

Triangulation is a multi-method approach involving multiple investigators, multiple data sources, or multiple methods to confirm the emerging findings and to enhance the trustworthiness of the research (Merriam, 1988; Towers, 1998). Although this technique is sometimes associated with the notion of having “one truth to be told”, enactivism advocates such as B. Davis (1996) argued that the belief that reality is “out there” could be replaced “with an acknowledgement of the contingency of interpretation” (p. 103).
The first study was conducted in Tonga between September and October of 2001, to collect video recordings of secondary school students' work. The second Tongan study took place the following year, between September and October of 2002. Each study took five weeks to complete. Although the schedule was tight, it was easy to effectively organize video-recording schedules, given the small number of students used for the study group, and the strong support for the study from all participants, among them principals, teachers, and students in particular.

The decision to do a second study was guided mainly by what Glaser and Strauss (1967) would call the “emerging gaps” in data collection and data analysis used in the first study. Following the first study, and the researcher’s return to Canada, subsequent data analysis revealed the need for further investigation of the research question, thus necessitating a second trip to Tonga. The data from the second study helped justify the data already collected from the first study, and also clarified certain unanswered questions associated with the research. Hence, the second set of data brought more depth to the data analysis. In reviewing the original data, LHS Secondary School\textsuperscript{16}, for instance, was identified as a major additional source of data for the overall study because of the school’s unique characteristics. In the Tongan secondary school system, LHS Secondary School, arguably, had the most resources and was the best-equipped secondary school campus. In particular, LHS distinguished itself from other schools in Tonga through its promotion of the English language, based on the school’s strict, year-round enforcement of a “Speak English Only” policy. In addition, although LHS follows the Tonga secondary schools’ mathematics syllabus, it enjoys a broad exposure to Western culture, through its unique relationship with affiliated church schools in Hawaii; and the school possesses a large number of staff from overseas. In considering LHS’s exclusive approach, it was possible in this study to examine its mathematics students’ use of English and Tongan.

\textsuperscript{16} All school and participant names used in this study are pseudonyms.
4.3.2 Location

As previous chapters have made clear, this study focuses on the research question within the Tongan bilingual context. The researcher’s cultural upbringing and experience as a mathematics educator and former secondary school mathematics teacher in Tonga, together with a bilingual background in both English and his native Tongan language, made him an appropriate person to conduct the study. The required assistance and permission were easy to obtain, including the full support of both the Tongan government and the Ministry of Education. The Tongan bilingual context (described in Chapter 3) presented the opportunity to examine a unique population: an isolated ethnic group internally homogeneous in its culture, particularly with respect to its national language. Also, with the exception of a few foreign students, most Tongan secondary school students (about 12,000 in total each year) are only fluent in their native, and everyday, Tongan language. However, these students are expected to rely mainly on the use of their second language of English – the language of secondary-school instruction – inside the classroom during school hours. At the same time, according to Fasi (1999), these students prefer to use their native language for mathematical activities, especially when they are engaged in discussions with their peers.

4.3.3 Participants

The deliberate selection of the study’s participants (schools, grade-levels, and students) was based on what Glaser and Strauss (1967) called a matter of theoretical purpose and relevance. Determination of the “purpose” and “relevance” of each choice depended, in this study, on the nature of the targeted data and its application to the research question (Chapter 5 describes how
some of the collected data turned out to be irrelevant for the study). Although Glaser and Strauss used these two terms to define specific criteria for selecting “comparison groups”, their significance has a deeper implication for this study. In fact, Glaser and Strauss’ (1967) notion of the “constant comparative method” was employed throughout the data analysis, in the development of themes and emerging categories associated with the Tongan bilingual students’ acts of language switching. Further applications of Glaser and Strauss’ constant comparative method are discussed in the next chapter. The researcher’s role, therefore, was to choose particular schools and grade levels, and more importantly, to select bilingual students, based on their teachers’ recommendations, who would help “generate, to the fullest extent, as many properties of the categories as possible” (Glaser & Strauss, 1967, p. 49). Furthermore, it was important to determine the theoretical purpose of choosing the participating parties in order to maintain what Glaser and Strauss (1967) called the study’s “scope of population” and “conceptual level”. Theoretically, the study’s scope of population and the participants’ conceptual level are defined by the choice of bilingual context, schools, grade-levels, and individual students. Both conditions – scope of population and conceptual level – were determined by the purpose of the study. Glaser and Strauss (1967) emphasized in this situation that the “scope of a substantive theory can be carefully increased and controlled by such a conscious choice of groups” (p. 52).

4.3.3.1 Selected Secondary Schools

Considering that all the schools in Tonga followed the same mathematics syllabus set forth by the country’s Ministry of Education, it was appropriate to conduct the two studies within the main island of Tongatapu, where more than 70 percent of the country’s middle and secondary school students were located. There were two key reasons for selecting these participating
schools: accessibility and diversity. The first criteria, concerning access to the schools and its participants, played a major role in deciding whom to work with, as well as when, and where, to work. The five schools involved in the study were among the seven largest secondary schools in the country, all of which had Form 1 to Form 6 (grades 7 to 12) students. These participating schools represented a range of religious denominations, academic, and social influences, namely AFC (a Catholic school), LHS (a Latter Day Saints school), QSC (a Methodist school), TCA (a government school), and THS (a government school). Each of these schools supplied students for the study, but all of the schools conformed to the mathematics curriculum and syllabus. Since the selected secondary schools also represented the diverse social and demographic range of secondary schools in the country, the individual students presented some unique characteristics for the study. Administratively, the schools were run either by the government (TCA and THS) or by any of the established religious denominations (AFC, QSC, and LHS). Various gender-based secondary schools could also be found in Tonga, particularly in the main island, where AFC, LHS, and THS were secondary schools with mixed students (boys and girls), and TCA and QSC, both of which offered boarding schools, were an all-boys and all-girls school respectively. Also, within the main island of Tongatapu, some schools were located in or near the capital (AFC, QSC, and THS), while others were situated in rural areas (TCA and LHS).

4.3.3.2 Selected Forms (or Grades)

This study took note of the fact that the middle-school level is a critical period for most Tongan-type bilingual students, especially for those with limited English proficiency, as they make the transition from instruction in one language at the primary-school level to another at the secondary-school level (Celedon, 1998; Fasi, 1999). It was at this transitional stage that the nature of “growth of mathematical understanding”, and how it related to the notion of “language
switching" was investigated. The students were chosen from Form 1 to Form 3, because teachers and second-language learners in these age groups (11 to 14 years old) typically face serious communication problems in the classroom (Fasi, 1999). Judging from the Tongan Ministry of Education's guidelines outlined in Chapter 3, mathematics concepts are taught in Tongan up to Form 2 (grade 8) and then switched to English in Form 3 (grade 9). This middle-school shift in concept instruction is therefore a critical period for most of the Tongan bilingual students, particularly in mastering the mathematical language. As a result, some students struggle with the mathematical concepts when they switch languages (Fasi, 1999). Another problem is that while the language of instruction in the classroom is English, the majority of the Tongan secondary school students rarely use English outside their classrooms. In this study, the head of each school's mathematics department was responsible for choosing the mathematics teachers and classes involved in the research. **Table C** below shows the number and grade levels of participants selected from each of the schools:

**Strand One: FIRST STUDY (2001)**

<table>
<thead>
<tr>
<th>PARTICIPATING SCHOOLS</th>
<th>SELECTED FORMS</th>
<th>NUMBER OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>AFC</td>
<td>1, 2, 3</td>
<td>16</td>
</tr>
<tr>
<td>QSC</td>
<td>2, 3</td>
<td>6</td>
</tr>
<tr>
<td>TCA</td>
<td>2, 3</td>
<td>8</td>
</tr>
<tr>
<td>THS</td>
<td>2, 3</td>
<td>8</td>
</tr>
</tbody>
</table>

**Strand Two: SECOND STUDY (2002)**

<table>
<thead>
<tr>
<th>PARTICIPATING SCHOOLS</th>
<th>SELECTED FORMS</th>
<th>NUMBER OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>AFC</td>
<td>2, 3</td>
<td>12</td>
</tr>
<tr>
<td>LHS</td>
<td>2, 3</td>
<td>8</td>
</tr>
<tr>
<td>QSC</td>
<td>2, 3</td>
<td>9</td>
</tr>
</tbody>
</table>

**Table C:** Number of selected participants from each of the participating schools
4.3.3.3 Selected Bilingual Students

In each school, the mathematics teachers were responsible for identifying students in their classes to participate in the study. Participants in the study were chosen using a “purposeful sampling” method, basing the criteria for student selection on teachers’ identification of the following preferences:

(a) Currently in Forms 1 to 3 (11 to 14 years old);
(b) Considered native Tongans, fluent (since infancy) in their native Tongan language, and deemed to have started learning English in primary school;
(c) Keen to participate in group discussion;
(d) Capable of verbalizing their thought processes;
(e) Capable of switching languages (otherwise there was no point in selecting them);
(f) Confident about being video- and audio-taped;
(g) Most likely to be available when needed for any of the recorded sessions, especially outside school hours.

Based solely on the criteria cited above, students’ competency in either one of the languages or in mathematics was not a requirement for their selection.

4.3.4 Setting

The setting in both studies was designed not only to engage the bilingual students in peer discussion and group collaboration work, but also to create an environment in which they could comfortably switch languages as they wished. This group environment was deliberately created to focus on the Tongan bilingual students’ natural style of language switching. The word
“natural” in this case was associated with the desire to create a situation in which the Tongan bilingual students could switch languages voluntarily rather than in response to directions to do so. To adhere to the “natural” environment, a group setting was created with no external intervention or participation from either the students’ mathematics teacher or the researcher (see a further discussion on this topic in Section 4.5.3).

4.3.4.1 Group Collaboration and Peer Discussion

In order to record the students' work, a setting had to be created to allow observation of their verbal and non-verbal activities through the video camera lens. Group collaboration work provided an appealing way to observe students' interaction and their mathematical work in action (Pirie, 1991), and it emphasized the importance of meaningful discourse in the classroom (Fasi, 1999; NCTM, 1991). This idea is stated clearly in Standard 3 of the Professional Standards for Teaching School Mathematics (NCTM, 1991), which says:

> Whether working in small or large groups, students should speak to one another, aiming to convince or question their peers. Above all, the discourse should be focused on making sense of mathematical ideas, on using mathematical ideas sensibly in setting up and solving problems (p. 45).

Many educational researchers have focused on classroom talk and discussion (e.g. Pimm, 1987; 1991; Wood, 1990; Pirie, 1991, 1998a), and a great deal of study has been directed toward student-to-student talk and the process of group learning (e.g. Davis & Maher, 1990; Gooding & Stacey, 1993; Cobb & Whitenack, 1996). Although the terms “talk” and “discussion” do not necessarily refer to the same idea, it is essential that both activities are task-focused and that the style and level of their explicitness are taken into account (Pimm, 1987). According to Pimm,
“pupil talk” must be “focused, explicit, disembodied, and message-oriented” (1987, p. 42), while Pirie (1991) defines “peer discussion” as purposeful, mathematical talk in which there are genuine student contributions and interactions. Coincidentally, and conveniently for the purpose of this study, group work in a typical classroom setting is also strongly encouraged at the Tongan secondary-school level. Tonga’s Ministry of Education (1996) recognizes the benefits of placing students in small-group work to highlight the advantage of mathematics students sharing and working cooperatively, using language to refine and consolidate mathematical understanding, developing mathematical understanding through involvement; acquiring problem-solving strategies, and performing tasks more effectively.

In this study, students were assigned to work in groups of twos and threes, in order to maximize group discussion and individual participation; each group was encouraged to think out loud and to verbalize their thought processes while solving tasks. In addition, each group was asked to express any related work by writing or drawing on their worksheets. Thereafter, the non-participant role of both the researcher and the students’ teachers was observed in order to create a “free” discussion environment for the students.

4.3.5 Tasks

Considering the research question, and its chosen theoretical framework and methodology, the purpose of creating an appropriate task for the study was two-fold: to allow language switching among individual students by facilitating group discussion in both languages, and more importantly, to enable mathematical understanding of the specified topic to grow. It was therefore evident, based on the Pirie-Kieren theory as a theoretical framework, that a task or set of tasks required an investigative approach. Using this approach, the intention was not to present
the students with a task involving only one right answer, but to give them an open-ended problem in which they could explore various possibilities and argue for their own ideas and problem-solving methods. The set of questions cited below was used to verify the appropriateness of these tasks:

(i) Will the students’ mathematical thinking be externalized in some observable form?
(ii) Will a variety of strategies and approaches to the tasks likely emerge?
(iii) Will each task challenge the students’ growing understanding of the specified topic?
(iv) Will each task provide a context for seeing patterns and relationships?
(v) Will each student be able to make a connection with the previous mathematical knowledge and everyday experiences?
(vi) Will the students be likely to formulate conceptions and inquiries that set the stage for thinking about more complex ideas or sophisticated forms of thinking?
(vii) Will the tasks promote meaningful and rich mathematical discussion through debates, questions, conjectures, generalizations, and explorations?

4.3.5.1 Topic of Investigation

In Pirie and Kieren’s terms, the Property Noticing layer is a crucial mode of understanding mathematics that classroom teachers often neglect (Pirie, 2001, personal correspondence). One of the observations characterizing students working at the Property Noticing layer concerns their ability to distinguish relationships between images: students’ mental or physical representations in relation to their work on a particular topic. The topic, “patterns and relations”, was considered for the purposes of a study set in the Tongan educational context, because this topic encouraged possible work at the Property Noticing layer. In addition, Tonga’s Ministry of Education has clearly stated its assumption that Tongan bilingual students can learn mathematics by investigating “mathematical patterns, relationships, processes and problems” (1996, p. 2).
According to the ministry’s guidelines, “Students should be given opportunities to discover and create patterns, and to describe, record, and explain relationships contained in those patterns”. Thus, consistent with the Tongan mathematics syllabus for all grade levels, the topic, “patterns and relations” was chosen for investigating students’ growth of understanding in this study. What is more, patterns and relations are central features of any Tongan student’s everyday life; an attention to intricate patterns and designs being a uniquely rich element of the Tongan culture, as is displayed in various handicrafts, the most prominent of which are *tapa*-making and weaving. From the perspective of this study, the students’ prior knowledge of such shapes, patterns, and designs illustrates the primitive knowledge these students bring to the task, and provides a background for their growth of mathematical understanding. The employment of the Pirie-Kieren theory allows for observation and analysis of the students’ construction of mathematical images and a study of the way the students relate, create, and expand patterns and their existing knowledge.

4.3.5.2 Pictorial Sequence

The next important job was to construct a task that the students would be interested in exploring. After choosing patterns and relations as the topic for investigation, a simple pictorial sequence involving square blocks and cubes came to mind as a way of engaging students in their exploration. These kinds of pictorial representations have appeared previously in various studies (Pimm, 1987; Foreman, 1998; Thom, 1999). Each pictorial representation was set up to play a major role in the students’ image-construction processes. Although the tasks involved working with two-dimensional square blocks (first study) and solid cubes (second study), counting played a huge role in the students’ work, and the mathematics that came out of these tasks was already established as rich from a teaching perspective. In each task, students were shown a different set
of pictorial sequences of square blocks (see Figure 4), or what Foreman (1998) called “tile arrangements” and Pimm (1987) named “bricks”, to engage them initially in “visual thinking”. Maier (1985) described this visual thought process as:

thinking that draws upon the processes of perceiving, imaging, and portraying. Perceiving is becoming informed through the lenses, through sight, hearing, touch, taste, smell, and also through kinesthesia, the sensation of body movement and position. Imaging is experiencing a sense perception in our mind or body that, at the moment, is not physical reality. Portraying is depicting a perception by a sketch, diagram, model and some other representation (p. 3).

Foreman (1998) argues that the major premise of such visual thinking is that carefully designed tasks based on sensory experiences would:

enable students to develop meaningful mental and kinesthetic images of mathematical concepts and processes. These images help students understand, retain, and recall this information. Listening to others talk about their thought processes and mental images fascinates students and prompts new ideas. As students explore mathematics in this manner, ideas makes sense, math anxiety diminishes, and confidence grows (p. 142).

In addition, the tasks were intended to provide a mathematical-oriented environment, whereby the students were able to create, manipulate, test, and explore their ideas. Thus, through students’ actions, the gap between their physical world and their imaginary one would diminish, their experiences and the symbolic world of mathematics become connected, and hence, students’ mathematical understanding would be enhanced.
4.3.5.3 The Set of Questions

A set of questions was drafted in English for the two studies that would accompany the pictorial sequence. The role of this set of questions was mainly the following: (a) to guide and validate the work of the students with respect to the given pictorial sequence; (b) to allow analysis of the students' knowledge of the English language – the language used in their mathematics classes; (c) to engage them verbally in a group discussion. The word “prediction” was intentionally used to mean, “making an educated guess”. An educated guess in this task implies the need for an
individual to have spotted some pattern in the given pictorial sequence in order to offer a reason for the predicted answer; (d) to allow language switching; and (e) to occasion students’ growth of mathematical understanding in the specified topic. The English version of the set of questions which accompanied each pictorial sequence is shown below (refer also to Appendix 1):

For each sequence:
(1) Draw the 4th diagram in the sequence.
(2) How many extra squares (in addition to the 3rd diagram) did you draw?
(3) How many squares are there in total in each diagram?
(4) Predict first how many more square blocks will be needed for the 5th diagram.
   a. Draw it and check your prediction.
(5) Discuss the 6th diagram.
   a. Do you need to draw it in order to know how many square blocks will be needed?
(6) What can your group say about the 7th diagram?
   a. How many square blocks in total were needed?
(7) Can your group see a pattern in the number of squares you add each time?
(8) Can your group see a pattern in the total number of squares used in each diagram?
(9) Can your group make any prediction(s) for the 17th diagram?
(10) Can your group make any prediction(s) for the 60th diagram?

4.3.5.4 The “Tongan” Task with Translated Set of Questions in Tongan

The same set of questions was also translated into the students’ native Tongan language (see Appendix 2), together with a different, yet similar, pictorial sequence (see Figure 4). The purpose of this task was to see whether the Tongan students would still switch languages, and to observe any significant variation or similarity of such verbal action to their working with the English version of the questions. At the same time, the task attempted to look at the possible reason(s) for these bilingual students’ language switching. This objective stems from the claim
that Tongan students *think* in their native language while working with mathematics problems (Fasi, 1999), a claim that might have been significant in explaining why Tongan students would still switch languages.

**4.4 Method of Observation**

**4.4.1 Introduction**

The search for an appropriate method of data collection was driven by the following questions:

(i) How is it possible to capture, categorize, and analyze bilingual students’ acts of language switching?

(ii) How is it possible to engage and occasion bilingual students to grow in their mathematical understanding, and how can such processes be captured?\(^{17}\)

(iii) How is it possible to capture, define, and analyze the relationship between the investigated phenomena in (i) and (ii)?

From the outset, it was important to ask why video recording was appropriate in the context of this particular study and to probe the significance of using such a technology in a qualitative study of this nature. Researchers such as Cohen and Manion (1989) have noted that the heart of every case study lies not in the problem to be addressed or the researcher’s approach, but in the researcher’s method of observation. Because the method of research is so important, the employment of video as an observational tool in this study was central to the trustworthiness of the findings. This is because video, as Asch, Marshall, and Spier (1973) observed, could be to the researcher “what the telescope is to the astronomer or what the microscope is to the

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\(^{17}\) Towers (1998) defined the notion of “occasioning” as “a situation in which a growth of understanding is *allowed for*, not caused” (p. 229). This definition, however, refers to how a “teacher intervention” influences understanding.
biologist" (cited in Henley, 1998, p. 48). Observation may therefore be seen as the primary
function of video, yet video recordings are far more useful in many areas of scientific inquiries
than simply as observational tools. Unlike a telescope or a microscope, video records data
instantly, and new digital technologies these days are capable of transmitting data directly from
all kinds of technological tools.

4.4.2 Video Analysis and the Notion of “Trustworthiness”

Case studies, especially those involving video-recorded data, may not be replicable, a condition
often associated with any experimental study. Yet the most important factor in any qualitative
inquiry of this nature is a determination of the “trustworthiness” of the study – a term used, in
contrast to the quantitative notions of “reliability” and “validity”, to reflect whether or not the
study’s methodology, methods, data collection, analysis, or findings are demonstrably justified.
Clearly, the choice of methodology, and the reason for employing case study research in this
study, is important for any in-depth inquiry. Equally important is the method for collecting and
analyzing data, in order to establish trustworthiness and even to determine the “broad
suggests that in some case studies, certain circumstances might call for observation beyond
conventional strategies, such as the use of direct observation. Bottorff described such
circumstances, often characterized by situations such as the following:

when behaviors of interest are of very short duration, the distinctive character of
events change moment by moment, or more detailed and precise descriptions of
behaviors and/or processes than is possible with ordinary participant observation
are required (1994, p. 244).
Bottorff (1994) also argued that a participant observer could not simultaneously monitor all verbal and non-verbal behaviors, even failing to remember most of these instances when the “moment” had passed. It is for these kinds of reasons that the decision to employ video recording as a choice of observation strategy in this case-study research was based.

Since the introduction of film, along with the advancement of digital video and computer technology, the role of video in qualitative and, likewise, quantitative research, has become increasingly specialized and sophisticated. The evolution of the technology obviously shapes methodological strategies, effectively opening up new avenues for research. As a result, the integration of video technology within case study research can be considered a significant step toward attaining trustworthiness of a qualitative study. But, as Mignot (2000) asserts, the use of video case study, like many other methodologies, is not just a question of what resources are being used, but how and why they are being used for a particular case or purpose. Hence, Bottorff (1994) attributes successful video-recordings to careful planning and the expertise or competence of the researcher.

Ultimately, establishing the degree of trustworthiness is crucial to the success of all qualitative inquiries, particularly in how the notion of trustworthiness relates to the subjective interpretation of data. Adelman (1998) contends that arguments expressed against qualitative studies may be avoided by “honestly acknowledging our certainty and uncertainty in an inference, by being explicit about reasoning underlying our inference from low to a high level of generalization, and by studying the causal process” (p. 159). Towers (1998) goes further in this aspect to note Sanjek’s (1990) notion of “theoretical candour” which involves the explicit admission of the “problem addressed, the methods used and the interpretations made” as a way in which the trustworthiness of case study research may be strengthened (p. 51). In this study, the use of
video-recorded data, along with think-aloud, video-simulated-recall, and non-participant observer techniques (will be discussed in Sections 4.5.2 and 4.5.3), allowed the researcher to overcome the inherent weaknesses present in this and other forms of case-study research.

4.4.3 Advantages and Disadvantages of Video Study

Clearly, the primary function of video as a research tool is data collection and data analysis by scientists, researchers, and academics. Video has also been acknowledged in research as the primary data itself (Pirie, 1996b; Pirie, et al., 2001; Powell, Francisco, & Maher, 2001; Pirie, 2001, personal correspondence; Borgen and Manu, 2002). However, despite the prevalence of video data, little is known or written extensively about its nature in research (Hall, 2000). Nevertheless, the use of video data as a research tool does provide researchers with a number of distinct advantages over other methods of data collection.

The first advantage of video data over other data sources, such as a transcribed script, comes from video’s ability to capture moving pictures and sound, two valuable attributes in educational research (Bottorff, 1994; Towers, 1998), as opposed to the use of a still picture or photography. In any observational research, Eisner (1991) argues that sensitivity is required toward how something is said and done, in addition to what is said and done, in order to understand what is going on. The “what” may very well depend on the “how”, and video provides one with the opportunity to make such a critical distinction. For example, the benefit of using video data in educational research was demonstrated in a recent study conducted by Borgen and Manu (2002) analyzing the mathematical understanding demonstrated by a particular grade 12 monolingual mathematics student. On paper, the student had presented a picture-perfect solution to a calculus problem. Using a video recording of the student’s work, a detailed study of the process was
carried out to determine what this particular student understood in relation to her written work. In
the final analysis, an examination solely of the video transcript, or the simple evidence from the
student’s written work, would have lacked the details significant to determining the nature of the
student’s understanding (or lack thereof) of the problem and solution. The recorded video data
revealed clear evidence of the student’s lack of understanding about key aspects of the
mathematical problem.

The second advantage of video-recordings concerns what Grimshaw (1982) refers to as the
“density” of the video data. Even a two-minute videotape provides the viewer with a vast amount
of information that can be deemed much richer than other kinds of recordings, such as a
transcribed video or audio data. This characteristic of video data is very significant to this study,
because evidence of growth of understanding (such as an act of folding back) can occur in a
matter of seconds. However, a researcher must be aware that a video is there to capture and
follow, but not to dictate the observed event. In addition, even a video camera cannot record
everything there is to know, since any video-recorded material is a selective, rather than a
complete record of a particular event, and because all recording devices are subject to inherent
mechanical limitations (Bottorff, 1994). Thus, the researcher must make some effort to
compensate for such limitations through other supported means of recordings or strategies. For
example, researchers are encouraged to accompany video data with supplementary data such as
ethnographic field notes (journal entries), copies of the students’ written work, or records of
open-ended meetings with their subjects, all of which contribute rich data to the case.

The third advantage of video recordings is what Grimshaw (1982) refers to as the “permanence”
of its data. In this aspect, video preserves what Towers (1998) describes as the “natural
colouring” of an event, or what Prosser (1998) calls in film and photographic images “instant
appearances”. From this perspective, video data is an unalterable record of the actions in time and space, together with the recorded sounds, because the video-tape freezes, but does not alter the observed action, as it happens. Thus, video data allows a time delay for any observer for reviewing the stored recorded data in the analysis. At the same time, video data allows the researcher a degree of flexibility in the choice of what needs to be observed, recorded, analyzed, or reported. This advantage relates to what Prosser (1998) calls “reflexive data”, often used to describe the opportunity for post-event reflections. Thus, video data allows a researcher the luxury of re-visiting and re-viewing selected events as often as necessary in a variety of ways, and the capability of attending to the desired feature(s) of the tape at leisure (Bottorff, 1994; Pirie, 1996b; Martin, 1999). Consequently, video data delays decision-making (Towers, 1998), which Yin (1989) argues to be a key feature of any video case study research, since it permits the researcher to reflect more deeply before interpreting the data.

The time involved in reviewing video data may also present another of its major strengths. That is, any data presented on video, beyond mere verbal activity, may be considered relevant to the study (Pirie, 1996b). Pirie (1996b) reported in one of her studies that her classroom video recordings would seem to be the “least intrusive, yet most inclusive” way of studying a phenomenon, particularly when such data is coupled with a study of the students’ written work and any field notes of the classroom in general. If it is taken as a primary data source, video data contains the continuous sequence of action, both verbal and non-verbal, as it occurs in real time (Powell, Francisco, & Maher, 2001). Erickson and Wilson (1982) further extol such benefits by maintaining, “such records permit systematic analysis of verbal and non-verbal behaviours in the event recorded” (cited in Adelman, 1998, p. 154). In short, video data achieves contextual validity – also known as “contextual data” (Prosser, 1998) – based on its potential to capture continuity, movement, and contextual speech. In addition, Towers (1998) says that video data’s
subjectivity may offer its greatest advantage, through its ability to address concerns associated with "descriptive validity". She argues, "This category deals with what Maxwell calls the 'factual accuracy' of an account and, as such, concerns itself with errors of omission as well as commission" (Towers, 1998, p. 79). On this point, Pirie (1996b) noted that any data that is gathered or lost depends on the researcher, where he or she places the camera(s), or even the type of microphones he or she uses.

However, the use of videotaped recordings in case studies presents a few disadvantages, largely related to the expenditure of time and resources. For instance, the delayed approach to research by reviewing and revisiting taped events can be costly, tiresome, and time-consuming, and may even turn out to have no relevance to the study (Towers, 1998). In order to remain effective, the researcher needs to maintain his or her focus on the context of the study, keeping in mind its framework and overall purpose. As Bottorff (1994) warns, the "merits of filming with a fixed or movable camera are still under debate and may depend on the focus and purpose of the study, the type of setting, and the characteristics of the participants" (p. 249).

Such a focus, together with the video angle used, can freeze researchers to one particular setting of the observed event, and therefore the recorded video data is often considered to be selective, and lacking in contextual data. The contextual restrictions of video data are noted by Bottorff (1994), who assumed that video recordings are limited, but not "sensitive to the historical context of the observed behaviours" (p. 246). In short, researchers are apparently restricted in their view of the event to what is recorded through the video camera lens.

The effect of video on the subjects is another major concern. A relaxed environment is therefore important, given the generally accepted fact that the presence of the video camera in any video
case study is not invisible to those being videotaped, and also does affect the participants’
behaviour during the recorded session (Towers, 1998). This effect is known as “procedural
reactivity” (Prosser, 1998).

The subjective nature associated with video analysis presents another disadvantage in using
video data as a primary source of data. According to this argument, the dynamic nature of any
video image is widely open to subjective interpretation. An example of such potential
misinterpretation is noted by Goldman-Segall (1993), who asks: “is closing of the eyelid...a
twitch, a wink, or a conspiratorial communication?” (cited in Pirie, 1996b). In other words, the
dynamic nature of video calls into question the trustworthiness of both the data and the analysis.
Maxwell (1992) did raise a critical argument when he said that “validity” is not an inherent
property of a particular method, “but pertains to the data, accounts, or conclusions reached by
using that method in a particular context for a particular purpose” (cited in Towers, 1998). In this
regard, Goos and Galbraith (1996) quote Ginsburg, pointing out that, “in any form of research
the significance of the data must always be judged relative to the researcher’s explicit or implicit
theories and assumptions” (p. 236). Nonetheless, much of the criticism to date about qualitative
research has been directed towards the observer’s perception of what he or she sees on video,
regardless of the observer’s training in the interpretation process (Towers, 1998). Because the
interpretation of observed actions is a serious concern in video analysis, the researcher must be
sure to only make inferences based on participants’ verbal or non-verbal actions, without making
any assumptions about what the participants cannot do.
4.5 Data Collection

4.5.1 Introduction

During the first week of data collection with the targeted schools, the researcher organized the study’s daily and weekly recording schedules. Beginning in week two, the study focused on the chosen groups in one school, for one week, before the research moved on to another school for the following week. Most of the videotaping sessions took place outside normal classroom lessons, either during one of the school’s one-hour breaks, or for two hours after school, with the exception of the Saturday sessions at AFC and TCA. All necessary videotaping appointments were scheduled through prior arrangement with the teachers and the students, along with the prior approvals of school principals and each student’s guardian(s).

The video-recording sessions involved the use of a static digital video camera, set up about 10 yards away, and placed directly in front of the students, with a separate microphone wired from the camera to the students’ work-table, with the camera lens focused on the entire group. All of the students remained seated on one side of their work-table, facing the camera, without moving outside its frame. In addition, there was an interview prior to or after the recordings regarding the students’ language use in both Tongan and English in order to gain an understanding of their knowledge of both languages: which language they used at home or at school, which language they preferred for their mathematical work, which language they used with their peers or mathematics teacher, and which language they thought they preferred to use in thinking, etc.
4.5.2 The Data Collection Technique(s)

One way to gain trustworthiness in qualitative research is to conduct a collective case study or a joint study of a number of similar cases: in this instance, an examination of several individuals from five different secondary schools. Hence, as Towers (1998) suggested, a multiple case study offers an alternative approach to looking at a specific phenomenon. In addition, two techniques were employed to enhance the trustworthiness of the recorded data: (i) "think-aloud" and (ii) "video-stimulated recall".

In the first technique, video data is usually gathered in the recording phase by encouraging participants to verbalize their thought processes. This instant verbalization enabled the researcher to match what the students were saying, doing, and writing. In spite of what others might say against this process (for example, Marland, 1984, cited in Pirie, 1996b), this verbal revelation, coupled with the students' non-verbal activities, was the only form of external expression that enabled the observer to get as close as possible to the mind of the protagonists (Pirie & Kieren, 1994). Furthermore, Pirie (1996b) suggested that this think-out-loud process could be used in essentially two ways: (a) as a means of improving students' thinking abilities, and (b) as a means to study students' existing cognitive abilities, which is particularly effective in videotaping a single student. Pimm (1987) supported this claim, adding that thinking aloud helps "the pupils to clarify and organize the thoughts themselves" (p. 23). Moreover, thinking aloud can be enhanced through group-work settings involving the participants interacting with each other. The idea of situating students in collaborative group work allows them contentment and space, by working with their own peers.
With regard to the video-simulated-recall technique, the researcher conducted an interview while both he and the participant(s) watched the videotape, in order to clarify, if necessary, the participants’ initial thoughts and actions. This process had to be conducted immediately or shortly after the recorded session (Bloom, 1954; McConnell, 1985; both cited in Pirie, 1996b), lest participants find genuine recollection too difficult. Therefore, the video-stimulated sessions were never delayed longer than one day to ensure students clearly recalled what they had been thinking about at the time of the recording when they watched a replay of the recorded session. Thus, it was hoped that the participants were discouraged from explaining what they ought to have been thinking, which could have interfered with the interpretation process (Pirie, 1996b).

For the weekday groups, if a recording session was done during the break, a video-stimulated recall session then followed right after school in the late afternoon hours. This offered opportunities for the researcher to view the recorded tape between sessions, and to prepare questions needed to clarify aspects of the students’ work. If the recording was scheduled after school, a post-mortem on the recorded session had to be delayed until the next day. With the weekend groups, however, the video-stimulated recall sessions took place shortly after a lunch break during the Saturday sessions. Various other research protocols were in place to ensure the information was interpreted from all possible angles. During the video-stimulated sessions, a member of the group took charge of the video remote control and he or she (or any group member) could stop the video randomly to elaborate about a “special” moment that might offer some insight into their recorded mathematical activities. In these “post-mortem sessions”, the researcher (or the students’ teacher) also actively participated in the discussion and their questions either emerged during the video-stimulated session or the questions were prepared prior to the session. The researcher’s role was to query any activities related to students’
language use or mathematical understanding: particularly during incidents that showed evidence of language switching and/or growth of understanding of the specified topic.

4.5.3 The Ethical Issues and the Role of a Non-Participant Observer

In previous educational studies that focused on the process of the growth of understanding, either the researcher (e.g. Martin, 1999) or the teachers were actively involved (e.g. Towers, 1998) as "participant observers". Unlike these earlier studies, this study deliberately excluded outside interventions to allow students the freedom to discuss the tasks among themselves, in their own language, without being forced or coached to speak in a particular way, although the teachers were invited to participate, if they wished, in the video-stimulated recall sessions and to thus allay concerns over the trustworthiness of the study. In this case, the researcher’s decision to assume the role of “non-participant observer” did not preclude his impact on the integrity of the research. As Towers (1998) sensibly noted, “doing research involving human subjects is always a matter for ethical concern” (p. 61).

One key ethical aspect to consider was the avoidance of “harm” to participants, whether such harm took place bodily, mentally, or emotionally: a concern previously raised by Towers (1998). To avoid such “harm”, the interviews within video-stimulated sessions took place in an atmosphere of utmost respect, considered essential in a relatively isolated society that has yet to consider video technology a household item.

18 Atkinson and Hammersley (1994) refer this to a situation where “observation is carried out when the researcher (or teacher) is playing an established participant role in the scene studied” (p. 248).
4.5.4 The Supplementary Data

While the collected video data was treated as the primary source of evidence, supplementary data provided significant information for video data analysis. The supplementary data included the accompanying field notes and students’ work sheets. The field notes were recorded during the video-stimulated recall sessions, especially when the sessions were audio-recorded. The students’ work sheets were significant pieces of evidence in the video analysis, and because a different colored pen was used for each student, each student’s work was easily identified. After the task, a group meeting involving all participating students and their mathematics teachers was conducted and video-recorded to discuss issues related to the data, participants’ experiences, and the research question itself. In one case, some of the teachers from one school (AFC) from the Mathematics, Tongan, and English departments were invited to discuss with participating students their use of language switching in mathematical work. This discussion session proved to be very useful, offering insights into students’ learning experiences and teachers’ perceptions of the sources and reasons behind the students’ use of language switching in mathematical discourse. In addition, the Teachers’ and Pupils’ Guide Books in each of the grade levels were collected as references to previous work completed by each student and to gain a sense of the scope of the mathematics syllabus throughout the secondary school years.

4.5.5 The Collected Video Data

The bulk of the video-recorded data was based on students’ completion of the tasks. In the first study, a total of 24 hours of video-recorded tapes were collected from 16 different groups, together with an additional five videotaped hours and four audio-recorded hours of video-stimulated recall sessions. In the second study, a total of 26 hours of video-recorded tapes were
collected from 11 different groups, along with an additional five hours of video-recorded recall sessions. This video data was compressed, burned, and stored into mpeg-files. The analysis of this video data was then carried out using video-analytical software called Vprism. The following chapter discusses the role of Vprism in detail (see Section 5.2.2).

4.6 Summary

This chapter outlined the methodological approach, design, and method of data collection used to address the research question. These methodological issues were critically reviewed with respect to the actual data collection process, which took place in Tonga between 2001 and 2002. The next chapter begins with the first phase of the data analysis – categorization of the types and “forms” of language switching.
5.1 Introduction

The data analysis in this study was approached in order to identify and examine separately both aspects of language switching and growth of mathematical understanding, outlined in Chapters 5 and 6 respectively, with the results of two distinct analyses integrated into a holistic analysis in Chapter 7. Chapter 5 begins, therefore, with a look at the way various categories of language switching were identified, categorized, and developed from the data to provide a language for describing and accounting for the Tongan bilingual students' language switching. Various descriptive examples are drawn from the data to illustrate the Tongan bilingual students' distinct types and forms of language switching. This chapter concludes with a discussion on the data of a particular nature of language switching and how it relates to the bilingual students' mathematical activities, particularly in working with mathematical images.

5.2 Data Preparation for the Analysis

5.2.1 Familiarization with the Data

Before focusing on elements related to the research question – either language switching or growth of mathematical understanding – it was essential to develop an overall feeling and familiarity for the data itself, particularly the video-recorded footage. As a result, the researcher gave no attention at this stage to the question, the mathematical topic, or the theoretical
framework. Instead, the data was subjected to a “familiarization phase”, involving the viewing of each of the recorded digital videotapes at least twice,\(^\text{19}\) without pausing, to get a sense of each video-recorded session as a whole, and a sense of the context within which the students’ verbal and nonverbal actions took place. Following each video viewing, the follow-up data was also checked, which included a thorough review of the recorded video-stimulated recall sessions, field notes of those sessions, and the students’ worksheets.

5.2.2 The Role of Vprism in the Data Analysis

Because of the time-consuming nature of video-data viewing and analysis (Towers, 1998), a large amount of data was first stored and managed through computer video software called “Vprism”. Hence, all of the recorded digital tapes were first compressed into mpeg files. The use of the Vprism software allowed the researcher to time-code (using a built-in coding program), transcribe (using a separate, built-in transcribing program), and analyze all of the videotaped sessions within the same frame. This filing process allowed the researcher to link and code the video and its transcript for particular events or categories, thus synchronizing video playback with portions of the transcript by linking key portions of the video with the research text. As a result, the researcher used Vprism to build a history of each individual and group’s recorded verbal and nonverbal activities, in which all video clips were annotated with notes and transcripts, and later used to query video collections to find specific items of interest, examples, spoken utterances, notes, subjects, or events. Vprism therefore allowed the researcher to keep track of time, individual speakers, and their observed verbal and non-verbal actions all at once. The Vprism coding program was also used to allow various categories of language switching to

\(^{19}\) Each video-recorded tape involved at least two viewings prior to the actual data analysis. This process included: (i) watching each recorded tape at least once prior to each video-stimulated session; and (ii) viewing each tape without pausing during the compression of the recorded digital tapes to mpeg files.
be entered simultaneously without having to code the same video segments repeatedly. The capacity of Vprism as video computer software thus offered the researcher a flexible way of breaking down or “de-constructing” a particular video into smaller video segments and re-constructing those segments to provide a holistic analysis of the research subject in question. For each video-recorded tape, data preparation for analyzing the bilingual students’ language switching using Vprism involved two main processes: transcribing and flagging.

5.2.3 Transcribing Video Data

The need to thoroughly investigate students’ language switching called for a transcribed version of the recorded videotapes. Pirie (1991) described how the simplistic nature of a transcript data, “allows us to stop time, to wind back the clock and repeat transient utterances, and therefore enables us to form considerable hypotheses as to the thought processes of the participants at that moment” (p. 143). It was only through transcribing that an explicit and detailed analysis of students’ verbal expressions was brought to the fore, making the transcription a more effective working tool, rather than working directly with the video-recorded sound. Although the transcripts provided a more manageable way to analyze the students’ acts of language switching, a qualitative study of this nature recognized the importance of treating the original video data as the primary source (Pirie, 1996). Nonetheless, the transcribed video scripts provided an efficient and transparent way of identifying the bilingual students’ language switching, along with a necessary means of expressing the data for presentation purposes. The researcher’s fluency in Tongan and English allowed him to transcribe the majority of the recorded videos. The recorded video-stimulated recall sessions were used to help the researcher clarify what the students had done verbally and non-verbally. In addition, some of the students participating in the Tongan research would, on occasion, offer to watch themselves on video and offer their own
transcription of the events. The students’ assistance was helpful, and appreciated, but all of the transcripts still had to be double-checked by the researcher during subsequent video viewings.

**Table D**, below, features a two-minute transcribed video scene, as it appeared in Vprism for a particular group (LHS Form 3 Group 1) in Task 3. The Vprism scene (like **Table D**) was set up as follows: the “Unit” label showed the group and the tape used (if more than one tape was used for a particular group); the “Time Code” showed the exact time on the tape when a particular event occurred; the “Speaker” column was used to label each participating student using letter abbreviations; the “Transcript 1” column detailed each action (verbal and non-verbal); and “Transcript 2” was used as an alternate section, in which other features and analytical processes could be recorded, such as locating incidents of language switching.

**Unit: LHS Form 3 Task 4 (Tape 2)**

<table>
<thead>
<tr>
<th>Time Code</th>
<th>Speaker</th>
<th>Transcript 1</th>
<th>Transcript 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>03:16:30</td>
<td>S:</td>
<td>Can you see a pattern in the total number ---</td>
<td>Borrow the word “total” from question.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>--- of square blocks used in each diagram?</td>
<td></td>
</tr>
<tr>
<td>03:17:00</td>
<td>Oh, ko e ha e total?</td>
<td>(<strong>Pointing at the word “total” in the question.</strong>)</td>
<td></td>
</tr>
<tr>
<td>03:17:08</td>
<td>K:</td>
<td>Twenty-eight --- ua-valu.</td>
<td>Directly translate meaning of a number.</td>
</tr>
<tr>
<td>03:17:12</td>
<td>S:</td>
<td>Ko e ha c: taha, tolu, ono, hongofulu?</td>
<td>(<strong>Points at the written totals for Question 3.</strong>)</td>
</tr>
<tr>
<td>03:17:23</td>
<td>K:</td>
<td>Triangular numbers!</td>
<td>Borrowing a non-equivalent English term.</td>
</tr>
<tr>
<td>03:17:30</td>
<td>S:</td>
<td>Triangulars: tolu, ono, hongofulu, taha-nima. Pick up the word “triangulars” for a peer.</td>
<td></td>
</tr>
<tr>
<td>03:17:50</td>
<td>ALL:</td>
<td>“Yes, they are all triangular numbers.”</td>
<td></td>
</tr>
<tr>
<td>03:18:00</td>
<td>S:</td>
<td>Predict the total for the 17th diagram?</td>
<td>Reads question in English.</td>
</tr>
<tr>
<td>03:18:16</td>
<td>S:</td>
<td>Ko e me’a --- ko e ha koa?</td>
<td>Shifting from English to Tongan.</td>
</tr>
<tr>
<td>03:18:25</td>
<td>K:</td>
<td>Ko e guess!</td>
<td>Translation with equivalent word.</td>
</tr>
<tr>
<td>03:18:30</td>
<td>S:</td>
<td>Kesi --- fakafuofua?</td>
<td></td>
</tr>
</tbody>
</table>

**Table D**: A typical transcribed video scene including “flags” of language switching.
The analysis of language switching, at this stage, was based on the actual Tongan-English transcript, rather than the translated copy. The transcripts of the students’ Tongan utterances were later translated into English solely for presentation purposes. Once the observed verbal and nonverbal actions from the video were transcribed (using Transcript 1), each occurrence of language switching then needed to be identified. The language-switching pattern in the student’s language use was characterized by any alternation between Tongan and English, whether in words, phrases, or sentences. The video data collected from Tonga provided evidence comparable with Baker’s (1996) finding, in which the bilingual students’ verbal alternation ranged, “from one-word mixing, to switching in mid-sentence, to switching in larger square blocks” (p. 86). Each of the students’ verbal responses was considered an “utterance”, described as a word, or group of words that made up a unit of sense or meaning, usually separated from the next utterance by a pause (Sorsby & Martley, 1991).20 The preceding description offers each utterance as the verbal unit of analysis and the basis for categorizing the bilingual students’ language switching. In other words, a student who says, “Are those square numbers --- one, four, nine, sixteen?”21 indicates two utterances, separated by a pause in between. Thus, from a research perspective, isolating each incident of Tongan bilingual students’ language switching became relatively straightforward. The students’ verbal expressions were assumed to be instantaneous and explicit externalizations of their thought processes while working on tasks, starting from reading the task, executing it, and reporting on it, such that all thoughts were being used at the same time (Graham, 1997).

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20 There were instances involving single words or phrases as responses. Taken in context, and in relation to the preceding and following responses to the word, most of these instances were considered meaningful in and of themselves. This semantic consideration led to determining the types of language switching involved.
21 The underlined words in this example represent the Tongan utterances being translated into English, and the symbol “---” denotes a pause in the students’ verbal expression.
5.2.4 “Flagging” incidents of Language Switching

The identification of specific features and instances related to the students’ language switching involved a process called “flagging”. This process highlighted aspects of the bilingual students’ language switching that could quickly be attended to or identified. Using Vprism, the Transcript 2 section was used to enter the “flags” to provide an instant account of each incident of language switching. This flagging included incidents involving a student switching entirely from speech in one language to another. For example, student “S”, in Table D, above, showed such a shift from English [03:18:00] to Tongan [03:18:16], although there was no flagging when she used Tongan between the coded times [03:18:16] and [03:18:30]. However, flagging involved working with each piece of data in preparation for the next stage in the analysis: categorization.

5.3 Data Analysis: Categorization and Coding

Once the transcribed video data was prepared, the next stage in the analysis moved to the development and categorization of emerging “themes” of language switching. These themes constituted the needed language for talking about the particular way the participating Tongan students switched languages. Categorizing each incident of language switching was a demanding task, involving an ongoing development of general themes using the constant comparative approach with all available data. This approach, employed as a way of “grounding” the emerging and developing themes in the data, recognized the role of the researcher in purposefully constructing categories for the current study.
It is important to note here, that in determining how to organize the data, the concepts of “purpose” and “relevance” (discussed earlier in Section 4.3.3) offered the main reasons to exclude most of the THS data from the study. These excluded data contained nothing relevant in connection with the research question, and therefore the purpose of the study. The initial analysis after the first study showed that the collected data lacked evidence for language switching – a basic requirement in the investigation. Since THS is arguably the top academic secondary school in Tonga, its students can converse fluently (and perhaps think) in English, and thus do so for most of their group discussions. Thus these students could use English without any language switching, making related data irrelevant to the study. However, one of the students from THS did demonstrate language switching and this student’s activities are reported in this study.

The initial categorization process began by applying to the data an existing list of language-switching themes Fasi (1999) had developed in his study with Tongan bilingual students (previously discussed in Chapter 2, Section 2.7). Fasi’s data came from clinical interviews with students, and from observing their verbal responses from a psycholinguistic perspective. In this perspective, Fasi (1999) grouped instances of Tongan bilingual students’ language switching into four main themes – “substitution”, “explanatory”, “reformulation”, and “repetition” – each theme was defined based on the purpose of the switch. These general themes were soon found to be incomplete or inappropriate for the current study, acting instead to constrain the constant comparative process, rather than facilitate understanding (Towers, 1998). There were also difficulties in assigning certain incidents of language switching to one of Fasi’s categories. In addition, Fasi’s definitions of explanatory and substitution were found to be too vague, and too broad, for the purpose of the data analysis; thus allowing significant features of the words or phrases involved in switching to be easily overlooked. For instance, it became apparent that Fasi’s definition of “substitution” did not differentiate between what is distinguished in this
study to be three different types of words: equivalent, non-equivalent, and Tonganised words – a
distinction that became apparent in the course of the current study’s data analysis. Because Fasi’s
(1999) labels and definitions were deemed inapplicable to the current study, one of his labels
(“explanatory”) was rejected and others (“substitution”, “reformulation”, and “repetition”) were
redefined for the current study. It was then necessary to return to the data to construct new
themes, and to find commonalities among certain types of language switching exhibited by the
participating students in the current study.

5.3.1 Preliminary Categorization

At the beginning, it became clear that Tongan students frequently “mixed” English words in their
Tongan mathematical discourses. The analysis, therefore, shifted at this stage to look at the
nature of the specific words, phrases, or group of words used in any language-mixed situation.
While most of these situations involved mixing English and Tongan words, it was possible to
discern switches from Tongan to English, and from English to Tongan. In addition, it became
evident that students were using mathematical and non-mathematical terms. So an initial
categorization of the mixed words and phrases was divided into five classifications:

**Equivalent Words:** Words that were interchangeable or “translatable” between the two languages.

**Borrowed Words:** Words that were either extracted from the task (questions) or non-translatable.

**Concrete Words:** Words that referred to objects or events that were available to the senses.

**Functional Words:** Words that functioned as verbs, and were used to designate a particular process.

**Abstract Words:** Words that meant ideas or concepts that had no physical referents.

**Table E** shows an example of a list of words and phrases used in mixed sentences for one group
of Form 3 students (QSC Form 3) in working with Task 3:
### Table E: The categorization of word-mixing

While this list above shows an example of the type of words the participating students used when working with Task 3, the categories were found to be incomplete or inadequate in explaining some of the observed incidents of language switching. In particular, it turned out that these categories were not mutually exclusive, and hence needed to be redefined. For instance, the words “diagram” and “total” were initially defined as “borrowed words”, which included both non-equivalent words and words used in the questions. Most of these words were also equivalent words; some of which were also concrete words. In addition, the described categories above only
dealt with switching at the word-level, but did not account for switching at the sentence-level. It was therefore necessary to return again to the data to construct new themes, and to find commonalities among certain types of language switching exhibited by the students. This return involved reviewing all the incidents of language switching that were initially flagged in Vprism. As illustrated in Table D earlier, all of the flags in the Transcript 2 section set up the initial categorization process: what each particular switch entailed in terms of its content and language-switching pattern. Table F, below, shows a compiled list of the descriptive flags used in identifying the types of language switching involved for the same group, QSC Form 3, in Task 3:

**Table F:** Sample list of the kinds of language switching involved for one particular group (QSC Form 3)

<table>
<thead>
<tr>
<th>Type of Switching</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substituting an English word to emphasize a point.</td>
</tr>
<tr>
<td>Tonganising an English word in everyday discourse.</td>
</tr>
<tr>
<td>Relating mathematical work to a keyword in question.</td>
</tr>
<tr>
<td>Borrowing a word or phrase used by a peer.</td>
</tr>
<tr>
<td>Labeling in English the mathematical concept.</td>
</tr>
<tr>
<td>Integrating with a non-equivalent Tonganised word.</td>
</tr>
<tr>
<td>A peer recalling a concrete English word.</td>
</tr>
<tr>
<td>Recalling a translated-equivalent Tonganising word.</td>
</tr>
<tr>
<td>Indirectly translating a concrete Tonganising word.</td>
</tr>
<tr>
<td>Borrowing a non-translated equivalent English word.</td>
</tr>
<tr>
<td>Substituting with a Tonganising word.</td>
</tr>
<tr>
<td>Reformulating the Question in English.</td>
</tr>
<tr>
<td>Indirectly interpreting question to construct image.</td>
</tr>
<tr>
<td>Substituting with an everyday or common word.</td>
</tr>
<tr>
<td>Mixing an English word into a Tongan sentence.</td>
</tr>
<tr>
<td>Activating an image from a word or phrase.</td>
</tr>
<tr>
<td>Directly translating from English to Tongan.</td>
</tr>
<tr>
<td>Paraphrasing the English question in Tongan.</td>
</tr>
<tr>
<td>Defining and clarifying tasks using translation.</td>
</tr>
<tr>
<td>Changing language from English to Tongan.</td>
</tr>
<tr>
<td>Translating an answer from Tongan to English.</td>
</tr>
<tr>
<td>Determining the meaning of a keyword.</td>
</tr>
<tr>
<td>Translating a statement using Tonganised word.</td>
</tr>
<tr>
<td>Substituting with an equivalent English word.</td>
</tr>
<tr>
<td>A peer explaining constructed image in English.</td>
</tr>
<tr>
<td>Identifying a key word from the question.</td>
</tr>
<tr>
<td>Rephrasing the constructed answer in English.</td>
</tr>
<tr>
<td>Substituting with a key-word from the question.</td>
</tr>
<tr>
<td>Directly associating two English words.</td>
</tr>
<tr>
<td>Shifting in reading questions to work in Tongan.</td>
</tr>
<tr>
<td>Extracting a keyword from the question.</td>
</tr>
<tr>
<td>Associating a word or phrase with an image.</td>
</tr>
</tbody>
</table>

The type of compiled list shown in Table F, above, became the basis for using the constant comparative approach in generating the needed categories of language switching.
5.3.2 Coding the Themes using Vprism

During the ongoing process of language-switching data analysis, a clear structure began to emerge for labeling the nature of students' language-switching behaviours, including the reason for the switch, the sentence structure involved in the switch, and the content observed within each switch. Vprism was then used to code the data – a technical process of categorizing each act of language switching and any specific incident of interest for further analysis. In using the Vprism coding program, various event categories led to the generation of a list of “Event Types”, starting with the most frequently observed events of language switching. Each new category required the creation of a unique code, using up to five characters at a time. An example of the coding process used in this phase of the analysis is shown as follows:

- **“PTKQ”**: Peer Translates a Keyword from the Question.
- **“SRSE”**: Self-Replacing within a sentence a Substitution of an Equivalent word.
- **“SMLN”**: Self-Mixing within a sentence a Loaning of a Non-equivalent word.

By allowing for the systematic creation of a list of codes, the use of Vprism made the categorization process a lot easier and less time-consuming. Categorization was an ongoing and recurring process that involved re-visiting the existing themes with each new piece of data. The Vprism system enabled the researcher to quickly reference the old codes, whenever an event was categorized, or a new code added to the existing codes. Each previous category was revisited and modified when new evidence was observed, leading to the emergence of a new category or the grouping of the incident with a category on the existing list. This re-categorization process continued for each new incident of language switching until all pieces of data were exhausted,
and no new category emerged, a stage Glaser and Strauss (1967) call the “saturation” point.

Another advantage of working with Vprism was the case of “collapsing” of every identifiable act of language switching into a suitable set of general themes. In Table D, for example, the uses of the words “total” [03:17:00] and “predict” [03:18:09] were grouped under a general theme called “substitution” to indicate their use within each utterance. (The summary of the themes is discussed in Section 5.4.)

5.3.3 Clustering the Initial Themes

The categorization process described in the previous section eventually led to the emergence of a new list of types of language switching. Such a categorization process resulted in the creation of various general themes of language switching, observed occurring in both strands of data, and among all the participants, irrespective of their form-level or school of origin. Table G, below, shows a list of the generated themes at this stage of the analysis, identifying an initial generalization of the students’ language switching. The list offers a description of each general theme. These general themes were later refined (see next section), and further data analysis revealed a clearer picture of all the types of language switching that took place:

**Inducing** - mainly involving a switch at the word- or phrase-level when non-verbal actions or particular images directly activate specific labels or verbal representations in a particular language.

**Translation** - any language switching that deals with the accessing “routes” between two languages, resulting in either “direct” word-to-word encoding (if not involving images), or “indirect” encoding (if involving direct activation of images).
**Substitution** - a language switching that involves inserting an English word, for example, into a Tongan sentence or utterance. This type of language switching only occurs at the sentence level, as a word or phrase from one language is substituted into another language.

**Switching** - describes how bilingual individuals switch verbalization from one language to another in expressing their thought processes.

**Keywording** - describes the way students extract a particular word or phrase from an external source (e.g., from questions, or from peers’ language use).

**Table G:** An initial list of general themes of language switching for the Tongan bilingual students.

During the theme-labeling process, the use of the term “inducing” became problematic, because this term solely defines the relationship from nonverbal to verbal language, rather than also including a description of the reverse relationship. This limited definition prompted the introduction of another term, “activation”, considered to be a type of language switching, involving the interconnection between images and language. This new definition was later revised and incorporated into another type of language switching called “grouping”, which is discussed in Section 5.4.2. Another problematic term was “translation”, which was used in the literature in a wide variety of contexts. Two existing definitions of translation in bilingual mathematics education included: one, the direct and indirect relationships between two verbal languages (see Figure 5), and two, the correspondence between language and mathematics. The distinction between these two definitions rested with the “mediating” role of images (in the process of meaning construction), and whether or not images are considered in the translation process. It turned out that this basic distinction determined the significance of differentiating “translation” as a term for talking about language switching, and treating it as a process involving working with images.
Eventually, the initial list of general themes, described above, developed into the identified forms of language switching (see the next section). Although the term is not explicitly defined in Martin’s (1999) thesis, the word “form” is deliberately chosen to describe a pattern in the way the Tongan students switched languages, or as a term to indicate, within the Tongan context, a particular way the students switch in and out between two languages; thus, the preference of the word “form” over similar synonyms, such as “style”, “type”, or “kind”. Form is not just about a particular language-switching pattern, but it is also meant to reflect its purpose and effect on the students’ mathematical actions.

Nonetheless, the validity of these categorized themes was established largely through “triangulation”, whereby the developed themes were shared and discussed with other researchers, colleagues, and especially, committee members. In particular, the aforementioned people were given samples of the data and were asked to consider whether the themes were appropriate with respect to the transcribed excerpts. Initial findings from the study were also presented at two major conferences in North America22, and again, examples of transcribed excerpts were offered for the participants to differentiate the categorized themes for the types of

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22 See the references to a poster presentation at the PME-NA, Athens, GA (Manu 2002), and a short oral paper presented at the AERA meeting, San Diego, CA (Manu, 2004).
language switching involved. All of these activities contributed to the refinement of the defined themes and made it possible to decide, in the early stages, which themes should be deemed distinct categories and which could be included within other categories. This kind of “independent verification” by the aforementioned people offered validity to the language-switching categories included in the final list.

5.4 Definitions of General Themes

Three years of relentless engagement with the available data, trying to determine how the general themes of language switching were to be identified and categorized, prolonged the data analysis considerably. While using the constant comparative method, it became apparent that the Tongan bilingual students’ alternation patterns could occur at any point in the conversation (or speech) in two main structural ways:

STRUCTURE 1: A switch occurs within an utterance.

In this situation, a switch occurs within a single sentence, a verbal behaviour also described as “flip-flopping”, or “unstructured” (Macswan, 1999). It involves a single word or phrase from one language being inserted into an utterance in the other language.

STRUCTURE 2: A switch occurs between utterances.

In this situation, a switch occurs between sentences, a verbal behaviour also described as a “structured” alternation in utterances between Tongan and English. This situation occurred far less frequently and was very much dependent on the bilingual individual’s degree of proficiency in both languages.
Each structural type cited above consisted of a unique type of language-switching activity, which could be distinguished at either the word- or sentence-level. The nature of each switch depended on the purpose and need for its cause. The following statements [L1-L13] from the Tongan data illustrate STRUCTURE 1 – the insertions of English words and phrases, including Tonganised English words, within Tongan utterances:

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript 1 [Translated]</th>
<th>Transcript 2 [Transcribed]</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>“The composite numbers.”</td>
<td>“Ko e composite numbers.”</td>
</tr>
<tr>
<td>L2</td>
<td>“What is that pattern there?” [Tonganised word peteni.] “Ko e ha e peteni he?”</td>
<td>“A i ko e row.”</td>
</tr>
<tr>
<td>L3</td>
<td>“Just use row.”</td>
<td>“A i ko e row.”</td>
</tr>
<tr>
<td>L4</td>
<td>“So the extra square is how many? Two, eh?”</td>
<td>“A ia ko e fo‘i extra square ko e fiha? Ua, he?”</td>
</tr>
<tr>
<td>L5</td>
<td>“The pattern is prime numbers.”</td>
<td>“Ko e peteni ko e prime numbers.”</td>
</tr>
<tr>
<td>L6</td>
<td>“It’s one and three --- those two are prime.”</td>
<td>“Ko e taha mo e tolu -- ko kinau ia ko e prime.”</td>
</tr>
<tr>
<td>L7</td>
<td>“Just use block --- make another block.”</td>
<td>“A i ko e poloka --- make another block.”</td>
</tr>
<tr>
<td>L8</td>
<td>“The composite number --- what is it?”</td>
<td>“Ko e composite number --- ko e ha ia?”</td>
</tr>
<tr>
<td>L9</td>
<td>“Prime factor --- a number that has only two factors.”</td>
<td>“Prime factor, ko e mata‘ifika ‘oku ua pe hono fakitoa.”</td>
</tr>
<tr>
<td>L10</td>
<td>“Because so that we know --- the order of the pattern.”</td>
<td>“Because, ke tau ‘ilo ‘a e hokohoko ‘o e peteni.”</td>
</tr>
<tr>
<td>L11</td>
<td>“There is no such words as ‘intruction’!”</td>
<td>“Oku ‘ikai ke ‘i ai ha lea ko e ‘intruction’!”</td>
</tr>
<tr>
<td>L12</td>
<td>“What is ‘predict’?” Is it guess?”</td>
<td>“Ko e ha “predict”? Ko e guess?”</td>
</tr>
<tr>
<td>L13</td>
<td>“Oh, what is the total?”</td>
<td>“Oh, ko e ha e total?”</td>
</tr>
</tbody>
</table>

In contrast, the following statements [L14-L21] illustrate STRUCTURE 2 – the alternation between Tongan and English utterances:

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript 1 [Translated]</th>
<th>Transcript 2 [Transcribed]</th>
</tr>
</thead>
<tbody>
<tr>
<td>L14</td>
<td>“Triangular numbers --- three, six, ten, fifteen.”</td>
<td>“Triangular numbers --- tolu, ono, hongofulu, taha-nima.”</td>
</tr>
<tr>
<td>L15</td>
<td>“One by one, one; two by two, four; three by three, nine; four by four, sixteen --- square numbers.”</td>
<td>“Taha ‘a e taha, taha; ua ‘a e ua, fa; tolu ‘a e tolu hiva; fa ‘a e fa, taha-ono --- square numbers.”</td>
</tr>
<tr>
<td>L16</td>
<td>“Just use block --- make another block.”</td>
<td>“Ai ko e poloka --- make another block.”</td>
</tr>
<tr>
<td>L17</td>
<td>“Oh, I already knew it --- add the prime numbers.”</td>
<td>“Oh, ‘osi ‘ilo ia ‘e au -- add the prime numbers.”</td>
</tr>
<tr>
<td>L18</td>
<td>“Prime factor --- a number that has only two factors.”</td>
<td>“Prime factor, ko e mata‘ifika ‘oku ua pe hono fakitoa.”</td>
</tr>
</tbody>
</table>
To further understand the difference between the two described structural types of switching, a critical distinction — often at the centre of most debates about language-switching term definitions — must first be explained. Any act of language switching describes an interaction between two languages that come in contact within the course of a single conversation, whether at the word, or sentence, level. The Tongan data showed one language dominating the other within the course of a single conversation. This inter-language contact relationship is distinguished by one language being considered the primary or “base language”, and the other language being the secondary or “embedded language”. Although these two concepts continue to be debated, they are defined and used in this study specifically to identify language switching in particular situations. The terms “base language” and “embedded language” will be used throughout the thesis. Differentiating the base language from the embedded language led to labelling the two main structural types of language switching as “mixing” and “grouping”. These types and forms of language switching were then ready to be defined.

5.4.1 Types of Language Mixing

Language mixing is a type of language switching involving a language change within an utterance, which makes up a unit of sense or meaning. For example, the statement “Find the total number of square blocks needed — then draw the fifth diagram” illustrates a mixing in which the base language is Tongan (represented by the underlined translated English words), and the words “total number”, “draw”, and “fifth diagram” are embedded into a Tongan sentence. However, the
statement “Hanga pe ‘e koe ‘o square ‘a e fika ‘o e taiakalami ---” (“You just square the diagram number ---”) shows integration of the Tonganised word “taiakalami” (diagram) within a Tongan sentence, and insertion of the non-equivalent word “square”.

**Definition 1:** A “language mixing” refers to a language change at the word-level, whereby a word or a phrase from the embedded language is inserted or integrated into an utterance in the base language to make up a single unit of sense or meaning.

Since the structural change of this type of language switching occurred at the word-level, the analysis of this switching pattern (and how it related to the students’ mathematical activities) focused at the “micro” or word-level. Further analysis of the Tongan data revealed incidents involving a change between the role of the base language and the embedded language, particularly among those students proficient in both Tongan and English. Tolini and Mapa, for example, were Form 2 (grade 8) students who worked on Task 2. At one point during their group’s discussion of the task, Mapa used English as the base language while calculating the total for the 7th diagram to be “four sevens --- twenty-eight.” Then he turned to Tolini, and said, “Yeah --- you can tohi the pattern. Write the pattern.” In this instance, Mapa inserted the equivalent word, “tohi”, in place of the English word, “write”. Hence, Mapa’s example showed why the role of the base language could change from Tongan to English with Tongan bilingual students, as long as these students were capable of using English as a second language.

The insertion or integration of words and phrases in language mixing gives rise to two different types of language mixing, defined as forms of language switching: *borrowing* and *substitution*.

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23 The difference between an *act of insertion* and an *act of integration* involves the types of words used in *mixing*. If Tongan is the base language, insertion involves English and Tonganized words that are not part of the Tongan language, and integration involves Tonganized (English) words that have become part of the Tongan language. (See **Definition 4** to follow.)
**Definition 2:** *Substitution* is a type of language mixing in which an equivalent word (or phrase) from the embedded language is inserted or integrated into the base language to replace within an utterance the meaning associated with its equivalent word.

**Definition 3:** *Borrowing* denotes the process in which a non-equivalent word from the embedded language is loaned and inserted to act as a “stand-in” term within an utterance in the base language.

Equivalent words are directly interchangeable (or “translatable”) between the two languages, and they are assumed to refer to or embody the same meaning (idea, concept, or image) for a particular individual. *Substitution*, therefore, refers to those words that have equivalent words in the base language (although they may not necessarily evoke the same meaning between two different individuals), and hence do not really meet any lexical need in the base language (Myers-Scotton, 1993). The *substituted* words are viewed as being built-in and stored in the bilingual individual’s “underlying language capacity”, which refers to the individual’s available verbal repertoire. Therefore, these equivalent words are easily accessible whenever the individual chooses to use them in group discussion or mathematical discourse. During the study, bilingual students often used one language (English or Tongan) as the base language and then “threw-in” *substituted* words from the embedded language, without necessarily affecting their intended meaning. All of these equivalent words form part of the language foundation in which a bilingual individual verbalizes his or her cognitive processes, which is a component of the individual’s existing knowledge and *Primitive Knowing* that he or she brings to the task.

**Figure 6** illustrates how, in a mathematical context, the bilingual individual’s underlying mathematical knowledge – *Primitive Knowing* and existing knowledge of the topic – may be “shared” between his or her languages. This underlying mathematical knowledge connects the
verbal and non-verbal activities of the bilingual individual, and allows that individual to switch languages, while verbally expressing mathematical actions using equivalent and/or non-equal words.

![Diagram showing language capacity and shared underlying knowledge](image)

**Figure 6:** A model for the bilingual individual's language capacity and shared underlying knowledge

In Tongan Forms 1 to 3 (grades 7 to 9), *substituted* equivalent English words include mathematical-related words associated with the task, such as “equal”, “add”, “plus”, “number”, “question”, and “answer”, which the bilingual individual may have learned prior to the study or much earlier at the primary school level. The following example shows the use of *substitution* in language mixing.

**Excerpt 1:** Substitution with equivalent words

In Task 4, the QSC Form 2 group of Ana (A), Haiti (H), and Melisa (M) attempt to explain the pattern in the extra number of square blocks being added each time. All three students agree that their pictorial image for the pattern starts with two square blocks, and that the first diagram already existed, and therefore must not be counted. This problem still creates misunderstanding between the students, leading them to engage in an interesting discussion.

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24 This model resembles Cummins’ “Dual-Iceberg” model for the bilingual students’ “Common Underlying Proficiency” (Cummins & Swain, 1986).
<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Transcript 1 [Translated]</th>
<th>Transcript 2 [Transcribed]</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>H</td>
<td>Just begin from two.</td>
<td>Toki begin pe mei he two.</td>
</tr>
<tr>
<td>L2</td>
<td></td>
<td>Not beginning from that one there.</td>
<td>Oua 'e kamata ia mei he one ko 'ee.</td>
</tr>
<tr>
<td>L3</td>
<td></td>
<td>Look here: If it's plus one there [to the 1st];</td>
<td>Sio ki he: kapau 'e plus one he [to the 1st];</td>
</tr>
<tr>
<td>L4</td>
<td></td>
<td>--- it will get two [for the 2nd].</td>
<td>--- 'e ma'u ai e ua ia [for the 2nd].</td>
</tr>
<tr>
<td>L5</td>
<td></td>
<td>And plus two to the two, will get then a four.</td>
<td>Pea 'e plus two ki he ua 'o ma'u ai 'a e fa.</td>
</tr>
<tr>
<td>L6</td>
<td></td>
<td>Plus three to the four and get then a seven&quot;.</td>
<td>Plus three ki he fa 'o ma'u ai 'a e fitu ia.</td>
</tr>
<tr>
<td>L7</td>
<td>M</td>
<td>But what --- how did it get a one?&quot; [in 1st]</td>
<td>Ka ko e ha --- ne anga fefe 'ene ma'u e taha? [in 1st]</td>
</tr>
<tr>
<td>L8</td>
<td>H</td>
<td>It stands by itself.</td>
<td>'E tu'u pe ia ia!</td>
</tr>
<tr>
<td>L9</td>
<td>M</td>
<td>Will plus zero only!</td>
<td>'E plus noa pe ia!</td>
</tr>
<tr>
<td>L10</td>
<td></td>
<td>Will plus zero to get one, and then plus two.</td>
<td>E plus noa 'o ma'u ai e taha, pea plus two leva.</td>
</tr>
<tr>
<td>L11</td>
<td></td>
<td>Plus --- like this!</td>
<td>Plus --- pehe'i pe!</td>
</tr>
<tr>
<td>L13</td>
<td>A</td>
<td>No! Plus a zero will then get a one.</td>
<td>No! Plus e noa 'o ma'u ai e taha.</td>
</tr>
<tr>
<td>L14</td>
<td></td>
<td>Look here: one plus one</td>
<td>Sio ki he: taha plus taha ---</td>
</tr>
<tr>
<td>L15</td>
<td></td>
<td>--- zero plus zero, get then a one.</td>
<td>--- noa plus noa, ma'u ai 'a e taha.</td>
</tr>
<tr>
<td>L16</td>
<td>M</td>
<td>Zero plus one will then get a one.</td>
<td>Noa plus taha 'o ma'u ai 'a e taha.</td>
</tr>
<tr>
<td>L17</td>
<td>A</td>
<td>If it is ---</td>
<td>'O kapau ko e ---</td>
</tr>
<tr>
<td>L18</td>
<td>M</td>
<td>It's one plus two.</td>
<td>Ko e taha plus ua.</td>
</tr>
<tr>
<td>L19</td>
<td>A</td>
<td>--- zero times one will just equal zero.</td>
<td>--- noa times one 'e equal noa ai pe ia.</td>
</tr>
<tr>
<td>L20</td>
<td></td>
<td>Cause it's times; so that times is just multiply. He ko e times ia; 'a ia ko e times ia ko e multiply.</td>
<td></td>
</tr>
</tbody>
</table>

The results of their discussion lead to the substitution of English equivalent terms such as "begin" [L1], "one" [L2, L3], "two" [L5, L10], "three" [L6], "plus" [L3, L5, L6, L9-L18], "times" [L19, L20], "multiply" [L20], and "equal" [L19]. In particular, Melisa and Ana appear to pick up the term "plus" from Haiti, who was the first in the group [L5-L6] to use the word in this episode.

Unlike substitution, the kinds of words involved in borrowing are non-equivalent words because they are not directly translatable between the two languages. Like substituted words, the borrowed words, once conceptualized, are also built-in and stored in the bilingual individual’s
underlying language capacity. In Forms 1 to 3 (grades 7 to 9), for example, numerical words and concepts associated with the task, such as “prime” numbers, “triangular” numbers, and “composite” numbers, are borrowed from English and may have been learned previously in school. These non-equivalent words are part of the bilingual individual’s existing knowledge and Primitive Knowing prior to working on the task. The following example shows the use of borrowing in language mixing.

**Excerpt 2: Borrowing of non-equivalent words**

Another group of Form 2 (grade 8) AFC students, Seini (S), Niko (N), and Pola (P), illustrate, in Task 3, how the non-equivalent English words “composite” [L15, L16, L22] and “prime” [L19, L20] numbers are borrowed and used in their base language of Tongan. In this task, the group reads Question 8 and Seini immediately associates the question – pattern in the total [L5] – with the numerical totals her group had calculated for the first five diagrams. So she asks her peers for a label that describes the pattern in the set {1, 4, 9, 16, and 25} [L6]:

<table>
<thead>
<tr>
<th>Line Speaker</th>
<th>Transcript 1 [Translated]</th>
<th>Transcript 2 [Transcribed]</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1 S</td>
<td>(Reads) Can your group see a pattern in the total --- Can your group see a pattern in the total ---</td>
<td>--- number of square blocks in each diagram?</td>
</tr>
<tr>
<td>L2</td>
<td>Just hold on. Look here: it’s one, four then --- ‘Oleva hifo. Sio ki he: ko e taha, fa pea fiha?</td>
<td></td>
</tr>
<tr>
<td>L3 S</td>
<td>what? It’s one, four, nine, sixteen. What is it? Ko e taha, fa, hiva, taha-ono. Ko e ha ia?</td>
<td></td>
</tr>
<tr>
<td>L4</td>
<td>(Niko: Which one?) Pattern in total number. (Niko: ‘A e ha?) Pattern in total number.</td>
<td></td>
</tr>
<tr>
<td>L5 S</td>
<td>One, four, nine, sixteen, twenty-five --- Taha, fa, hiva, taha-ono, ua-nima ---</td>
<td></td>
</tr>
<tr>
<td>L6 N</td>
<td>Add that altogether?</td>
<td>Tanaki fakakatoa kinautolu?</td>
</tr>
<tr>
<td>L7 S</td>
<td>No, what is their pattern (“peteni”)?</td>
<td>‘Ikai, ko e ha ‘enau peteni?</td>
</tr>
<tr>
<td>L8 N</td>
<td>(Writes)“1, 4, 9, 16, 25”</td>
<td>(Writes) “1, 4, 9, 16, 25”</td>
</tr>
<tr>
<td>L9 S</td>
<td>What is that pattern (“peteni”) there?</td>
<td>Ko e ha e peteni he?</td>
</tr>
<tr>
<td>L10 N</td>
<td>(Niko: Which pattern (“peteni”)?)</td>
<td>(Niko: Peteni fe?)</td>
</tr>
<tr>
<td>L11 S</td>
<td>That one there: one, four, nine, sixteen ---</td>
<td>‘A e ko ee --- taha, fa, hiva, taha-ono ---</td>
</tr>
</tbody>
</table>
Seini’s question prompts Niko first to borrow the non-equivalent word “composite” [L15], and when Seini asks about the word [L16], Pola explains its meaning, using the borrowing of another non-equivalent word, “prime” [L19]. Seini and Pola later borrow the words “prime factors” [L19] and “composite” [L22], respectively.

The above example also shows the distinction between using Tonganised words through substitution and through borrowing. There are two Tonganised words used in the above episode: “peteni” (pattern) [L8, L10-L11] and “fakitoa” (factor) [L17, L21]. The word “peteni” or “pattern” has equivalent words in Tongan such as “sipinga” (design), “fakaholoholo” (sequence) and “fotunga” (appearance). Because the word “pattern” is present in the Tongan language, in fact, integrated into Tongan as the word “peteni” – that is, the word is used regularly in public – it can be considered a substituted Tonganised word. The word “fakitoa” (factor, in mathematical terms), however, has no equivalent word in Tongan. Hence, it is regarded as a borrowed word. While all Tonganised English words are characterized by their pronunciation in Tongan, the distinction between substitution and borrowing leads to the following definitions:

25 In a non-mathematical sense, the meaning of the everyday word “factor” can be expressed in Tongan, but in mathematics, “factor” has no equivalent word, and hence most students simply pronounce it in Tongan as “fakitoa".
**Definition 4:** A *substituted* Tonganised English word is either inserted or integrated within a Tongan utterance if the English word has an equivalent word in Tongan.

"Insertion" refers to a Tonganised equivalent word that has not been adopted into the Tongan language (e.g. tisikasi – discuss), while "integration" refers to a Tonganised equivalent word that has been adopted into the Tongan language (e.g. peteni – pattern).

**Definition 5:** A *borrowed* Tonganised English word is inserted within a Tongan utterance if the English word has no equivalent word in Tongan.

**Excerpt 3: Substitution with Tonganised words**

In this example, Alaki and his classmate, Malakai, explore the patterns and relations in the pictorial sequence given to them in Task 3. Up to this point, the two students have constructed various pictorial images for the topic by focusing on the “extras”, or the number of square blocks being added to the sequence each time. Alaki had just drawn the 4th diagram in the sequence, and this excerpt shows him explaining his work as he attempts to determine the number of extra square blocks they have added to the 3rd diagram to get the 4th diagram.

<table>
<thead>
<tr>
<th>Line</th>
<th>Transcript 1 [<em>Translated</em>]</th>
<th>Transcript 2 [<em>Transcribed</em>]</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>Those ‘blocks’ at the base there</td>
<td>'A e fo'i me'a 'i lalo ko ee (sketch base layer of 2nd diag.)</td>
</tr>
<tr>
<td>L2</td>
<td>--- which is the same thing (sketch base layer of 2nd)</td>
<td>--- 'a ia ko e fo'i me'a tatau pe (sketch base of 2nd)</td>
</tr>
<tr>
<td>L3</td>
<td>--- like that one (shifts and refers to 4th diag. drawn)</td>
<td>--- 'o hange ko ee (shifts and refers to the 4th diag. drawn)</td>
</tr>
<tr>
<td>L4</td>
<td>--- just do as row.</td>
<td>--- 'ai pe ko e row.</td>
</tr>
</tbody>
</table>

26 While there is no rule for controlling how people Tonganise foreign words, the commission of the new Tongan-Tongan dictionary (discussed in Chapter 2) will set out which foreign words, over time, have become part of the Tongan everyday language. Tonganised words in the examples given (and throughout this thesis) are those that have been pointed out to be, or not to be, on the approved list (refer to Taumoefolau, 2004).
In the process, Alaki uses the Tonganised words, “poloka” [L7, L9] and “sitepu” [L5, L6, L11, L13] to substitute for the English equivalent words, “block” and “step”, respectively. Incidents of Tonganising can also be identified when the students attach either a Tongan prefix or suffix to the English words to emphasize specific features or characteristics associated with the word. For example, words such as “discuss-’i” (discuss), “add-aki” (to add with), and “faka-square” (to make square) are commonly used by students, and at the same time, the use of these words reveals students’ internalization of these mathematical concepts in English. The addition, for example, of a suffix ending in “i” to an English verb generally expresses more definitely, or more emphatically, the idea of carrying the action through to completion (Churchward, 1953). Churchward (1953) describes how this suffix is by far the most common, and is even attached, in some instances, to words borrowed from English. The prefix “faka” is commonly used to indicate either “likeness” or “causation”, which in the example above, is attached to a borrowed word “square”. The suffix “’aki” is attached to an intransitive verb, which is more or less equivalent to an English preposition; in this case, the word, “with”. Churchward (1953) describes further the fundamental function of the Tongan suffixes, not because they make intransitive verbs transitive, but because they form new verbs with new meanings. The function of such
suffixes is significant in understanding how Tongan students Tonganise English words, and how such actions influence the students' mathematical meanings or expressions of those meanings.

5.4.2 Types of Language Grouping

Language grouping is a type of language switching involving language alternation between sets of utterances (or sentences), which make up a unit of sense or meaning. For example, the statement "Find the total number of square blocks needed --- then draw the fifth diagram" illustrates verbal grouping of a language switch between two clauses. This leads to the following definition:

**Definition 6:** "Language grouping" refers to a change in base language at the sentence-level, between groups of words or clauses that make up a unit of sense or meaning.

The analysis of this pattern focuses on bilingual students' language switching at the "macro" or sentence-level. The alternation between sets of utterances in language grouping gives rise to two different types of language mixing, defined also as forms of language switching: *translation* and *shifting*.

**Definition 7:** *Translation* refers to repeating current or existing information verbally from one language to another, whether through direct or indirect interconnections.

*Translation* therefore involves a change in base languages when re-expressing existing or current information. Given an instruction or problem in one language, *translation* functions in decoding information from one language to another. This decoding process involves direct or indirect
association between words as students attempt to make sense of the given or existing information. This sense-making ability depends on the bilingual student’s language capacity in each language. There are two types of translation: repetition and reformulation.

**Definition 8:** *Repetition* is a type of translation that refers to the direct one-to-one correspondence between two languages through their equivalent words.

Hence, *repetition*, also called “direct translation”, is represented verbally through word-to-word substitution at both the word- and sentence-levels.

**Definition 9:** *Reformulation* is a type of translation that refers to the paraphrasing of existing information from one language to another.

This paraphrasing process occurs at the sentence level, where a word (particularly a non-equivalent word), or a sentence, is restated indirectly; that is, *reformulation* is concerned with rephrasing existing information without direct word-to-word correspondence between the two languages. Like *direct translation*, the underlying factor in a bilingual individual’s ability to function effectively in two languages is the underlying knowledge he or she possesses in both languages.

**Excerpt 4:** Translation – Repetition and Reformulation

In the following example, Selai (S), a Form 3 (grade 9) student, works on Task 2 with Kepi (K), and attempts to translate, between Tongan and English, some of the words and phrases she uses while talking to herself and her group. Selai is asked to compare her drawing of the 5th diagram
with her initial prediction. She attempts to explain her image by talking about the common
difference between consecutive diagram totals, and the total number of square blocks for the first
five diagrams that she has already found.

Moreover, Selai is asked later on in the task if she needs to draw the 6th diagram. She replies,
"Yes", and then she proceeds to explain her answer, again showing evidence of her direct

translation of certain terms:

The above example illustrates various incidents of translation at both the word- and sentence-
levels. For example, Selai repeats the word "how" into Tongan ("angafefe", L2) and

reformulates her English statement [L7] into Tongan [L9], then back into English [L10]. In
addition, she shows repetition in expressing her thought [L13] and in directly translating her statement about the “order of the pattern” [L15-L16]. Not only does this case illustrate how Selai is faced with the task of expressing her constructed mathematical images through language, but how such expression leads her to switch languages. The ability to “shift” back and forth fluently between two languages reflects an exceptionally well-connected, shared, underlying knowledge within the bilingual individual. Such language ability often leads to the last form of language switching: shifting.

**Definition 10:** *Shifting* involves a change in the base language when expressing new information such as actions or understanding.

A verbal shift often moves a bilingual individual to operate cognitively with a new set of concepts and to function exclusively within a different set of semantic and grammatical rules associated with the new base or “target” language. The language limitations experienced by a bilingual individual using the target language can be evident in the way the individual operates within the language of mathematics, whether that language is in Tongan or English.

**Excerpt 5:** Shifting between two languages

Alaki (A) and Malakai are Form 3 (grade 9) students who have previously taken part in similar tasks. In this task (Task 3), they begin by attempting to answer the first question.

**Line Speaker Transcript 1 [Translated]**

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>A</td>
<td>Draw the fourth. Yes --- is that the first number?</td>
</tr>
<tr>
<td>L2</td>
<td></td>
<td><em>(Reads)</em> Draw the fourth diagram --- in the sequence.</td>
</tr>
</tbody>
</table>

**Transcript 2 [Transcribed]**

<table>
<thead>
<tr>
<th>Line</th>
<th>Speaker</th>
<th>Transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Draw the fourth [1]. Ko ia --- ko e fika 'uluaki ia? [2]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Draw the fourth diagram --- in the sequence. [3]</td>
</tr>
</tbody>
</table>
In this case (Task 3), Alaki quickly recalls the first question without directly reading it from the question sheet [L1]. He then reads it again [L2-L3], and immediately shifts to Tongan and explores the pattern in the pictorial sequence. Thus, Alaki appears to have translated the question indirectly through his mathematical actions. The observed language shifting is characterized by Alaki's work at a new level of understanding, illustrated by his construction of new images for the topic. Malakai, on the other hand, illustrates shifting from Tongan to English [L23], as he expresses his constructed numerical pattern, which he associates with the label "prime numbers", then shifts back to Tongan to explain the numerical set associated with his mathematical label.

Further analysis of this group's growth of mathematical understanding revealed why Malakai's use of the label prime numbers was incorrect, but he considered the term appropriate to him; hence his use of the term, prime numbers, did not interfere with his growth of mathematical understanding. The discussion of this example is detailed in the next chapter.

The nature of verbal shifting appears to resonate with von Glasersfeld's (1995) thinking. He contends that a shift in one's language or verbal mode of representation may ultimately alter that individual's way of seeing and feeling – and as a result – his or her conceptualization of the
problem, and hence, the solution. Thus, the critical aspect of language *shifting* lies in the assumption that different languages may be conceptualized differently and are often characterized by the addition of new information. This form of language switching also reflects the structural and semantic mismatch between two languages, and the cultural differences in personal experiences, which may explain why the study finds Tongan bilingual students *shift* mostly to the use of Tongan while working toward understanding at the informal modes of understanding.

*Figure 7* summarizes all the categorized types and forms of language switching for the Tongan students. These categories cannot claim to represent all instances of Tongan-type bilingual students' language switching. What can be stated is that the current study offers a collection of themes that describe the nature and variety of language switching seen specifically enacted by the participating Tongan students (see *Figure 7*).
**TYPES and FORMS of LANGUAGE SWITCHING**

- **Integration or insertion** of words or phrases (from an embedded language) into a base language to make up a single unit of sense or meaning

- **MIXING**
  - **SUBSTITUTION**
    - Insertion of an equivalent word within an utterance

  - **BORROWING**
    - Insertion of a non-equivalent word within an utterance

- **GROUPING**
  - **TRANSLATION**
    - Direct Translation
      - Repeating existing information between two languages

  - **SHIFTING**
    - Repeating existing information between two languages

- **Equivalent English/Tongan Words**
  - Directly interchangeable and "translatable"

- **Conventionalized Equivalent Words**
  - Integrated into the Tongan language

- **Non-Conventionalized Equivalent Words**
  - Inserted into the Tongan language

- **Non-conventionalized Tonganized**
  - A Tonganized word with no equivalent Tongan word

- **Non-Equivalent English Words**
  - Not directly translatable between the two languages

- **Repetition – Direct Translation**
  - Direct one-to-one correspondence between two languages

- **Reformulation – Indirect Translation**
  - Paraphrasing of existing information between languages

- **Follows rule of the "Target" Language**
5.5 Evocative Nature of Language Switching

So far, the types and forms of language switching have been described and identified by the language change in the words, phrases, or sentences used to verbalize cognitive processing. Acts of language switching alone are not considered significant to the growth of understanding of a particular mathematical topic, unless such acts are directly associated with the students' mathematical actions or work with images. The mathematical meaning or content of each act of language switching must be appropriate in order to be useful to the bilingual individual’s mathematical understanding. To understand how relationships between the two languages and the bilingual individual’s mathematical work may occur, a diagrammatical representation may be helpful.

Figure 8 illustrates the relationship between the Tongan bilingual students’ natural languages – English (E) and Tongan (T) – and their work with images (I) for the topic. Bilingual students’ verbal utterances were analyzed in relation to the specified topic of investigation. Other than non-verbal activities, such as drawing a particular image associated with the topic, the bilingual participants were observed to verbalize their work with images (I) for the topic using either the English language only (E-I), Tongan language only (T-I) or both languages in mixed utterances (E-I-T). According to this classification, when a bilingual student talked about other images that were not connected to anything about the specified topic (patterns and relations), he or she was observed to work within either area E, T, or E-T.
What was significant in this study, was whether verbal acts of language switching evoked aspects, features, or properties of specific images associated with the bilingual individual’s *Primitive Knowing*, existing knowledge, or current understanding of the topic. This “evocative” nature directed the bilingual individual to his or her previous, present, or future actions, and thus defined the nature of the observed mathematical action through language switching. The significant inter-connection between language switching and mathematical understanding was restricted to three main cases: when both English and Tongan were used concurrently while working with images (E-I-T), or when a structural change from using English only occurred while working with images (E-I), to using Tongan only when working with images (T-I), and vice-versa. In the first case, language mixing was often the norm in discussing aspects of the images in both languages, and in the second and third cases, a significant language switching occurred in language groupings, whether through shifting or translation.

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27 This representation was adapted from Paivio and Deschloser’s (1980) bilingual model, although their model treats images and the two languages as three distinct, separate components.
**Definition 11:** An *evocative* language switching activates mathematical actions or understandings that are different from the tasks a bilingual student was previously, or is currently, working on.

An *evocative language switching* existed in a language mixing only within the area labeled E-I-T, or by a language grouping (or alternation) between the areas E-I and T-I. This evocative nature of language switching showed how language switching functioned as a mechanism for identifying or calling forth specific aspects of the students' *Primitive Knowing*, existing knowledge or current understanding. These different kinds of "understandings" were often inaccessible through one language, and for some bilingual students, the only way to access understanding was by switching languages. Based on the interconnection symbolized by **Figure 8**, four kinds of "inter-connections" existed between the natural languages, Tongan or English, and mathematics. These inter-connections set the foundation for comprehending the relationship between language switching and the bilingual students' growth of mathematical understanding, a phenomenon discussed further in Chapter 7:

**Interconnection 1: Language-to-Language**

In this situation, the bilingual individual's emphasis is on the language involved (e.g. a particular word or phrase), and such an emphasis moves the bilingual individual to concentrate on the language rather than the mathematics. *Translation* is the norm in this situation. It involves restating (voluntarily or involuntarily when asked by someone) *existing* information from one language to another, whether non-mathematical or mathematical, with no immediate effect on the individual's mathematical activities for the specified topic.
Interconnection 2: Language-to-Mathematics

In this relationship, language comes first, before the mathematics, in activities such as reading the questions or interpreting what a peer is saying. But the difference between language and mathematics is brought together by their interconnection with the mathematical language being used. Borrowing keywords or phrases from the task, or keywords or phrases used by a peer, is usually observed in bilingual students’ language switching as they construct mathematical meaning. This “constructing” process is dependent on each bilingual individual’s language capacity, and particularly on his or her understanding of the associated language of mathematics.

Interconnection 3: Mathematics-to-Language

Unlike Interconnection 2, mathematics in this instance comes first, before language, in activities such as describing one’s existing or current mathematical understanding. But the differences between language and mathematics are also made clear through their interconnection in the mathematical language. The “verbalizing” process involved in this situation is dependent on each bilingual individual’s degree of understanding of the language of mathematics and how that language is used appropriately in verbally expressing mathematics through language switching.

Interconnection 4: Mathematics-to-Mathematics

Similarly with Interconnection 1, the distinction between mathematics and language is made apparent. The individual’s emphasis is on his or her work with mathematics. This emphasis leads to two different situations: One, mathematical understanding is observed to occur, but without language switching; and two, mathematical understanding is observed, but it does not bring about the acts of language switching. In this latter case, the acts of language switching are not necessary for the individual’s mathematical understanding of the topic. Both situations, however,
highlight the bilingual individual’s understanding and focus on the mathematics rather than the language.

5.6 Summary

Finally, following an analysis of the data from the perspective of language switching, the following categories were determined: Language mixing gives rise to two forms of language switching, substitution and borrowing. Language grouping involves two more forms of language switching, translation and shifting, in which translation occurs in two ways: repetition and reformulation. These categories of language switching were identified, categorized, and developed from the data to provide a language for describing and accounting for the Tongan students’ language switching. This “language” was used in discussing the relationship between the students’ acts of language switching and their growth of mathematical understanding, discussed in the next two chapters. Each of the forms of language switching was capable of activating mathematical actions or understandings different from the tasks the students were, or continued to work on for the study in question. The evocative nature of language switching defined each act of switching and its relationship with the growth of mathematical understanding.
CHAPTER 6: ANALYZING GROWTH OF MATHEMATICAL UNDERSTANDING

6.1 Introduction

This chapter outlines an analysis of the students' growth of mathematical understanding during their investigation of the topic, patterns and relations, as they worked on a particular set of tasks. The analysis was performed through the use of the Pirie-Kieren framework. The strength of this theoretical model became evident at this point in the study because, as Martin (1999) stated, “it is probably the most explicit and well-developed theory for describing the growth of mathematical understanding” (p. 267). This chapter, therefore, focuses on the concept of growth of mathematical understanding, as it takes place in this video case study of Tongan bilingual students, through the lens of the aforementioned theory.

6.2 Data Preparation for the Analysis

6.2.1 Familiarization with the Data and the Role of Vprism

Prior to analyzing the students’ growth of mathematical understanding, the researcher had developed a familiarity with the data as a whole through the analysis of students' language switching (see Chapter 5). What is more, the researcher had gained a sense of the mathematical context within which the students' verbal and nonverbal actions took place. For this second stage of analysis, Vprism was once again a valuable tool, used for time-coding, tracing, and analyzing all videotaped recordings. Yet while reviewing each video-recorded tape, initial data preparation
for analyzing the bilingual students' growth of mathematical understanding focused solely on
two main processes: *time-activity tracing* and *flagging*.

### 6.2.2 Timed-Activity Tracing and Flagging

The first step toward analyzing the students' growing understanding was to create what Pirie
(1996) called a "timed-activity trace". Unlike the language-switching analysis phase, involving
time-coding for each switch, a time-activity trace required the videotape be stopped at a regular
timed interval of three minutes (or so) for the researcher to make a brief, non-judgemental
written description of what was seen. This time-activity tracing process allowed the researcher to
easily pinpoint specific events of interest, and also provided what Pirie (1996) calls a "neutral
tool" and a "common reference" for others who might want to work with the same data in the
future. Using Vprism, a page of the timed-activity tracing is represented by the example shown
in **Table H**:

#### Table H: Example of a timed-activity trace and "flagging" of the critical moments.
Hence, timed-activity tracing involved outlining a brief description of the events observed on tape, so that the researcher might later browse through the written accounts and quickly pinpoint particular scenes of interest. For instance, in the above example, the location in the video unit “QSC Form 3 Task 1”, showing the students working on Questions 3 and 4, can easily be identified by the time codes [03:00:00] and [06:03:23] of Tape 1. The other key step associated with timed-activity tracing involved “flagging”, which was used in this stage of the analysis to locate and identify students’ critical moments in the observed mathematical activities. These “flagged moments” were noted and recorded in the written record as “flags” in the Transcript 2 column, so that each time a video was browsed, the researcher could easily identify specific aspects of the students’ mathematical activities. For instance, the flags in Table H show incidents at the time-coded interval beginning at [06:03:23], a moment when this particular group reflected on their previous constructions, discovered relationships between the diagram totals, and appeared to fold back to their primitive knowing. Thus, flagging involved working with each piece of data in preparation for the next stage of data analysis focusing on growth of mathematical understanding: categorization, a process marking the beginning of a thorough video analysis.

6.3 Data Analysis: Categorization and Coding

In this phase of the data analysis, categorization involved applying the layers and specific features of the Pirie-Kieren theory (explicitly defined in Chapter 3, Sections 3.3.4 and 3.3.6) to the data at hand. The intention of this application was not to expand on existing theory, but to use the Pirie-Kieren framework as an interpretative theoretical lens. Hence, categorization included,
for instance, identifying the current layer(s) of understanding within a particular interval, the images associated with the specified topic, and making note of any specific incidents of folding back or acting and expressing complementarity. In addition, Martin’s (1999) four main forms of folding back were also used as the language for describing and accounting for the observed invocative actions associated with the students’ growth of mathematical understanding. These forms of folding back were:

- Working at an inner layer with existing understanding.
- Collecting from an inner layer.
- Causing discontinuity.
- Moving out of the topic and working there.

Subjective interpretation of video data is contextual and situational. In this type of qualitative inquiry, Pirie (1996) warns, “one must be aware of the danger that only what is looked for may be seen, and what has been concluded, confirmed” (p. 4). Thus the researcher was highly conscious of the problem of subjective analysis, and thereby made a point of continually examining the data in light of the research question. The notion of trustworthiness in data interpretation, as discussed earlier in Section 4.4.2, was crucial in justifying and verifying evidence based on available data. Hence, a detailed analysis of each event involved multiple viewings and re-viewings of the videotapes and any of the supplementary data. The roles of the supplementary data, therefore, became significant when justifying and clarifying the students’ mathematical work.

Nonetheless, the video data remained the primary data source. The students’ observable, non-verbal actions were compared with the students’ worksheets and matched, where appropriate,
with their observed verbal actions. This verification process involved what Pirie (1996b) calls “informed others” – those knowledgeable about the theory – to provide a fresh analysis of the events in an attempt to make sense of what was seen on tape. Some of these informers included Dr. Susan Pirie and Dr. Lyndon Martin, who were available for ongoing consultation and data interpretation. In addition, fellow graduate students, who were familiar with the Pirie-Kieren theory or who had applied the theory to their own work, offered valuable insights during the process of data interpretation. The contributions of such “informed” researchers ultimately enhanced the trustworthiness of this study’s data.

Another crucial step in the process of data analysis involved data “coding”, an activity aided greatly by the use of Vprism. The codes chosen for analysis reflected each specific aspect or feature of the students’ growth of mathematical understanding. Such a system was an efficient means of generating and tracking categories of notable events using a list of “Event Types”. Some examples of the codes created, using acronyms that allowed for a maximum of five characters (when using Vprism), included the following

“FPNIM”: Folding Back from Property Noticing to Image Making.
“RIMCI”: Reviewing at Image Making layer the Construction of an Image.
“WPNCI”: Working at Property Noticing with Constructed Images.

The creation of a systematic list of codes allowed a new piece of data to be quickly categorized and assigned a new code, or a code that already existed on the list. The use of VPrism meant that old codes could be quickly referenced whenever an event was categorized or new codes were added. The code system led to the development of a uniform and consistent master set of codes. Once the data analysis – categorizing and coding – of each student’s growing understanding in
the specified topic was completed, the next important stage of mapping took place using the
Pirie-Kieren diagrammatic model.

6.4 Mapping using the Pirie-Kieren Model

*Mapping* is a technique used to visually trace the students' growth of mathematical
understanding as these instances are observed through the Pirie-Kieren framework. The
diagrammatic representation used in this study for the Pirie-Kieren model follows Towers'
(1998) approach in order to see clearly the pathways of one's growth of understanding within
and between the layers of understanding. Each numbered point on the mapping marks a
significant incident, such as change in growth of understanding of the specified topic, or an
aspect of the individual's mathematical understanding that is considered necessary to discuss in
light of the research question. However, these points are not intended to describe growth as
"snapshots" of the individual's mathematical understanding; rather, the points represent the
observed pathway of the targeted individuals or specific group's growth of understanding, and
the connection, or disconnection, between what the students do from one learning incidence to
another.

Within each analysis, the points are marked with encircled numbers on the Pirie-Kieren
mapping, in order to trace the pathway of the selected individual or group's growth of
understanding. In addition, notes are made of any significant interaction between growth of
mathematical understanding and language switching (for further discussion in Chapter 7). All of
the task work and original answer sheets are reproduced and presented in *Appendices 4 to 11.*
Each analysis hinges on the individual’s mathematical constructions and his or her development of mathematical images for the topic: patterns and relations. While an analysis of students’ language switching treated their utterances as the verbal units of analysis, images were considered the nonverbal units of analysis while observing students’ growth of mathematical understanding. Hence, images of patterns and relations were identified and grouped, based on the students’ work and their examination of the different features associated with the pictorial sequences provided for them. The students’ work led to their observing two inter-related aspects of the given sequences: extras and totals. The “extras” refer to the features of the given pictorial sequence that are associated with the differences in the number of square blocks between diagrams or between layers within a particular diagram, or are related to the additional square blocks or layers being added each time. The “totals” refer to the pattern in the diagrams as a whole, whether in terms of the number of square blocks involved in each diagram or in terms of its diagrammatic or spatial structure. This distinction, between the extras and the totals, turned out to be an important means of distinguishing, in each analysis, which of the Pirie-Kieren layers the students appeared to be working at during the task in question.

6.5 The Data Analysis – Selected Groups and Individuals

The work and detailed analysis of four selected groups offer a general picture of the participating bilingual students’ mathematical activities and their forms of language switching. In addition, the analysis and mapping of one example of a group working with the Tongan task is provided, along with five analyses in summarized forms from other pieces of data in the study that are intentionally chosen for further discussion in Chapter 7.
6.5.1 Selai’s Growing Understanding of Patterns and Relations in Task 2

This first analysis presents a detailed description of Selai’s growing understanding of patterns and relations over time, while working with Task 2 (see Figure 9; MAP 1 shows the path of Selai’s growth of mathematical understanding.) The comprehensive analysis that follows presents an intense look at the back-and-forth nature of growth of mathematical understanding, and how Selai’s growth of mathematical understanding precedes, and sometimes appears to bring about, her acts of language switching. In this example, Selai’s inadequate mathematical understanding prompts her to fold back with no relation to language switching. Selai was a Form 3 student who worked with two other students from her school, Kepi, and Alusa. Kepi plays a significant role in completing the task in question, while Alusa hardly participates, other than being asked occasionally to write out some of the group’s answers.

<table>
<thead>
<tr>
<th></th>
<th>1st diagram</th>
<th>2nd diagram</th>
<th>3rd diagram</th>
<th>4th diagram</th>
<th>5th diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Continuing pictorial sequence</strong></td>
<td>🟧</td>
<td>🟧 🟧</td>
<td>🟧 🟧 🟧</td>
<td>🟧 🟧 🟧 🟧</td>
<td>🟧 🟧 🟧 🟧 🟧</td>
</tr>
</tbody>
</table>

**Figure 9:** The continuing pictorial sequence in Task 2

Given the pictorial representations of each of the first three diagrams in Task 2 (see Figure 9), Selai immediately acts on her Primitive Knowing to construct images for the pattern without yet reading any of the questions. She points at each of the given diagrams and switches languages through substitution when she says, “First number --- so the diagram, there is one. Diagram two.
that's [pauses as if mentally counting] five --- number three?" Thus, through visual perception, Selai counts the number of square blocks in each of the first three diagrams in order to bring about her numerical image of the totals for the pattern. At this point, Selai engages in image doing [Point 1] (See MAP 1). She expresses her Image-Making activity through substitution of the words “diagram” and “one” [Note 1]. Kepi then prompts Selai to read Question 1 (Draw the 4th diagram in the sequence), and Selai follows that request by first reviewing [Point 2] her counting of the diagrams’ totals in Tongan, starting with the 1st diagram: “So there --- if that is one ---”.

Kepi (K) intervenes, but Selai (S) continues by pointing out the numerical totals of the first three diagrams as they continue discussing the task [Excerpt 6]29:

L1  S  If that’s one, five there, nine there --- so that’s it!”
L2  Add four every pattern. [Points to the diagrams in the sequence.]
L3  Meaning, every pattern must be increased only by four.
L4  If that’s it --- the fourth? It’s thirteen! [Points to empty 4th diagram.]
L5  K  [Kepi attempts to draw the 4th diagram but appears unsure as to how it looks.]
L6  S  So that pattern going up --- [Points to top vertical part of 3rd] three there --- that goes.
L7  The pattern going straight: one, two, three, four, five [points to vertical] Just read ---
L8  K  One, two, three, four, five, six, seven, eight --- uhehe!
L9  One, two, three, four, five, six, seven, eight, nine --- [Counts total drawn to nine.]
L10 S  Add then with a one from either side --- [Points out to Kepi one each on the sides.]
L11 K  All! [ Draws correctly the 4th diagram with 13 blocks.]
L12 S  Yes! [Means she agrees to add one on each end.]

28 In the translated versions, the underlined words represent the English translation of the students’ utterances in their native language, Tongan. Brief descriptions of the students’ nonverbal actions are italicized in brackets, while the symbol “——” denotes a pause between utterances.
29 Throughout this chapter, all of the excerpts from the video transcripts feature only the translated versions for presentational purposes.
Selai continues reviewing her construction of a numerical image for the pattern and relation [L1]. Her remark, "so that's it!" [L1], demonstrates how she has internalized an image for the pattern and relation, and she also sounds excited at the thought of seeing it. Selai’s subsequent statement [L2] is expressed verbally, through substitution, using the phrase “every pattern” [Note 2], for her image for the topic, based on the totals she has counted from the first three diagrams. These totals, “1, 5, and 9”, she writes underneath each corresponding diagram (see Figure 10). These totals do not represent an image (or different images) for the pattern. Selai’s numerical image for the pattern appears when she realizes almost immediately that there is a common difference of four between each successive total (see Figure 10). However, Selai has not explicitly shown that she has related this numerical difference [L2-L3] to the number of square blocks being added each time. Her understanding of the pattern is later evident when she and her peers are prompted to draw the 4th diagram.

![Diagram](image.png)

**Figure 10:** Selai’s work indicates a common numerical difference of four (4) between the totals

Nevertheless, Selai has moved out to *Image Having* [Point 3], showing she is capable of identifying the pattern through the internalization of her image. Selai continues immediately by using her image as a mental plan for constructing the total for the 4th diagram without actually having to build with square blocks. She says, “If that’s it --- the fourth? It’s thirteen” [L4], referring to her writing of “+ 4 = 13” alongside the total for the 3rd diagram (see Figure 12). What Selai sees is a relation to the particular task, based on the knowledge that arises directly from
manipulating the totals (image doing). It is important to note, however, that Selai’s use of the word “pattern” [L2] and “peteni” [L3, L6, L7] in this context means she is making a reference either to each diagram or to each total.

At this stage of the task work, Kepi attempts to draw the 4th diagram on the Answer Sheet. Selai has just figured out the numerical total for the 4th diagram, but both of them show no evidence of internalizing a clear pictorial image of the 4th diagram; for example, the two students do not have a sense of how many square blocks will appear on each side of the diagram. Both students appear, as a result, to fold back from Image Having to working at Image Making [Point 4], as they discuss how the 4th diagram is drawn. This folding back did not relate to any act of language switching [Note 3]. Counting from the central square block, Selai looks at the 3rd diagram and points out that there are three square blocks at the top half of the diagram’s vertical segment [L6], a segment she also counts in Tongan to have five square blocks in total [L7]. Kepi, however, first draws a duplicate of the 3rd diagram, and immediately Selai suggests adding an additional square block on either side of the vertical segment – one on top and another at the bottom to get the appropriate vertical component of the 4th diagram [L10]. Selai has therefore perceived her pictorial image for the pattern in such a way that the whole structure can be seen as a build-up of the individual components. Kepi then extends Selai’s idea to mean “All” [L11] by adding an additional square block to each side of the horizontal component. Selai agrees [L12] that in order to draw the subsequent diagrams, repeated additions of four square blocks are needed, using one square block per side for each four-sided structure (see the illustration in Figure 11). Both students have illustrated moving out again to working at the Image Having layer [Point 5], without any involvement of language switching [Note 4].
MAP 1: Selai's growing understanding of patterns and relations in Task 2

<table>
<thead>
<tr>
<th>PK</th>
<th>IM</th>
<th>IH</th>
<th>PN</th>
<th>F&lt;sup&gt;30&lt;/sup&gt;</th>
</tr>
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<tbody>
<tr>
<td>ido</td>
<td>ire</td>
<td>ise</td>
<td>isa</td>
<td>ppr</td>
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<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

Abbreviations used: Primitive Knowing (PK), Image Making (IM), Image Having (IH), Property Noticing (PN), Formalising (F), image doing (ido), image reviewing (ire), image seeing (ise), image saying (isa), property predicting (ppr), property recording (pre), method applying (map), and method justifying (mju).
Selai then goes on to answer Question 2 (How many additional square blocks added to the 3\textsuperscript{rd} diagram to get the 4\textsuperscript{th} diagram?), while continuing to reflect on her images at the Image Having layer [Point 6]. After pointing at the given pictorial sequence, Selai goes on to say, “That’s four. Four blocks --- Add a four --- Plus four! Add four to the nine to get that answer, which is thirteen”. Selai is observed here to reflect on her mental construct for the pattern and to continually translate her verbal description of the image while still working at Image Having [Note 5]. Selai continues using her numerical image after being asked in Question 3 to find the totals – 1, 5, 9, and 13 – for the first four diagrams.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{pictorial_image.png}
\caption{Pictorial image of the pattern using the added square blocks for each successive diagram}
\end{figure}

The group then reads Question 4, asking them to find the 5\textsuperscript{th} diagram, and Selai immediately replies, using her numerical image for the pattern, and pointing at the total they have calculated for the 4\textsuperscript{th} diagram: “If the fourth is thirteen --- add a four to thirteen to get then a seventeen!”

Then, throughout the discussion with her group, Selai uses only Tongan without any switching as she calculates the total for the 5\textsuperscript{th} diagram [Point 7]. Her remark, “Then add with a one there --- each pattern, on each side” also confirms her pictorial image for the pattern. Kepi then draws the 5\textsuperscript{th} diagram by first copying a duplicate of the 4\textsuperscript{th} diagram and then adding, as Selai had suggested: “a one to every pattern to get then a seventeen” (see Figure 11). Then Selai counts with Kepi the total of the 5\textsuperscript{th} diagram to be 17 and she responds, “Yes --- add four --- right!” Up
to this point, Selai continues to use her images at the Image Having layer and is able to express (mainly in Tongan) aspects of her images. Her substitution of the phrases “every pattern” and “each pattern” again refers to a specific diagram, and not a generalization of her understanding of the pattern [Note 6].

While still working on Question 4, the group, using their Answer Sheet, is now asked to comment on their drawing and what they have initially predicted for the total of the 5th diagram [Excerpt 7]:

L13 K, T [Read] How do you compare your prediction with your diagram? Explain any difference.
L14 S How --- how can I tell the difference? If there was a difference of that one.
L15 [Points to 5th diagram drawn.] The only difference, meaning that --- the difference.
L16 K Or is it adding four --- the difference only is a four!
L17 S The difference --- the difference is the answer --- especially for the difference.
L18 The difference is only the answer. [Continues writing answer to Question 4.]
L19 The difference only is the answer --- only the answer is change. [A: "-- only the answer."
L20 But the --- the answer --- only the answer is --- is change.
L21 But the increase in the pattern: that must stay the same. [Uses hand-motions to illustrate.]
L22 K Pattern is four or just adds four --- more than by four?
L23 S But the pattern should equal --- [Continues writing explanation to Question 4.]
L24 --- be the same like plus four [writes "+4"] to each pattern.

In this excerpt, Selai and Kepi deduce two different meanings of the question through the word “difference” [L13], using both English [L17-L18] and Tongan, with the equivalent word “faikehekehe” [L14, L16, L19]. Kepi views the difference between consecutive totals as the same [L16], while Selai sees the sequence of totals (or as she puts it, the “answer” or “pattern”) as different from each other [L19-L20]. Hence, the notion of “difference” takes on two different meanings [Note 7]. Selai, however, recognizes Kepi’s meaning as a common difference between the totals and she uses the Tongan word “lahiange” (increase), when she says: “But the increase
in the pattern; that must stay the same." [L21] Selai, in particular, demonstrates how her conscious translation (of her mathematical thinking) reveals her acts of Property Noticing [Point 8] [Note 8], as she continually connects between her image of the totals and her image of the extras, although she is still constrained by the use of specific numbers. This action appears to lead Selai toward the don't-need boundary between Property Noticing and Formalising. She then adds, "But the pattern should equal --- be the same like plus four to each pattern" [L21, L23-L24]. This general statement gives evidence to Selai's moving out to Formalisation and her use of reformulation in re-expressing her thoughts in English [Point 9] [Note 9]. Interestingly, her use of the word "pattern" [L23-L24] refers to two different meanings: the first, a reference to the general connection, and the second, a reference to each diagram. This difference, however, does not affect her current understanding [Note 10].

Selai goes on to answer Question 5, concerning why her group needs to draw the 6th diagram. She explains, "That we know the order --- meaning, order is what? The order --- the order of the pattern --- to know the order --- O-R-D-E-R --- the order of the pattern. It is true or not!" This repetition indicates Selai is saying that she can verify her group's answers (guesses or prior predictions) by using their drawings. Thus, in terms of Selai's growing understanding, her comments at this stage indicate that she has shifted her attention to her images associated with the totals, which she has not Formalised; thus prompting her to fold back to work at property recording [Point 10]. Her repetition, in this instance, did not invoke her folding back [Note 11].

Following Selai's remarks, the group discusses the 7th diagram in Question 6. Selai appears to continue working with her images for the pattern at the Property Noticing layer [Point 11]. She reflects on her previous construction, and borrows the word "formula": "So there is the fifth --- did we do the sixth? --- If that one is seventeen --- Just wait --- just do a formula". This prompts
Selai to join Kepi, who is currently using the trial-and-error approach to finding an arithmetic rule to generate the diagrams' totals. Selai has therefore momentarily disconnected herself from her current understanding at Property Noticing to fold back in order to work at the Image Making layer [Point 12]. Her borrowing of the word “formula” indicates a shift in her mathematical thinking to Image Making [Note 12]. She accesses her Primitive Knowing through the trial-and-error method to construct an arithmetic rule – a new image of the pattern and relation – for generating the diagram totals. So, in pointing at the 2nd diagram, Selai notes, “Two by five, ten. So then --- if the two by five is ten --- add with two --- it results in twelve if that’s it!”

In addition, both Selai and Kepi try a few combinations of multiplication and addition to specifically test the total for the 7th diagram, and none of their attempts prove successful. Kepi gives up, and moves back out to working again at the Property Noticing layer. He suggests to Selai, “You just add four”, referring to their previous outer layer knowing. Selai replies, “Yes!” and then she immediately attempts to generalize an algebraic relationship by shifting to English and saying, “So if that’s it --- do the formula --- x or is equal to ---”, an evidence of her moving out to work at the Property Noticing layer [Point 13]. Selai is provoked by her peer, Kepi, to move out to Property Noticing [Note 13]. However, Selai cannot continue with the task, and as a consequence, her group is not only unable to find an arithmetic rule, but also cannot make a generalization of their noticed property. Nonetheless, Kepi continues to work at the task by attempting to reflect on the group’s previous constructions: “What number is seventh?” Selai points at the empty space for the 5th diagram in the pictorial sequence and replies, “If seventeen is the fifth --- twenty-one is the sixth”. Kepi again adds to Selai’s statement, saying, “and the seventh is twenty-five”, to which Selai agrees. Both Selai and Kepi move on to discuss the relationship between the diagram numbers and the totals. Figure 12 illustrates how Selai and Kepi attempt to formulate an arithmetic rule by relating the diagram number to its respective
numerical total, in contrast to the general non-algebraic rule for constructing the subsequent diagrams by adding four each time.

Hence, Selai attempts to relate her images – diagram numbers and numerical totals – to find a general rule for calculating the totals. In this situation, Selai does not fold back to Image Making, but instead, she continues working at the Property Noticing layer by testing various combinations of the diagram numbers – an image associated with the counting numbers – by using multiplication by four, combined with subtraction by five, and then later with addition by one [Point 14]. Selai uses substitution, but it shows no effect on her learning process [Note 14]. Selai starts her work by picking a rule that worked previously for a particular diagram number, but then her approach does not work for the others. For example, Selai points at the 2nd diagram and says, “Yes, it’s true! Four by one is four, plus one.” While Selai continues her calculations, Kepi insists that, “You just make a formula to add four --- plus four!” So when Selai appears to give up on her trial-and-error approach, she then points at the 1st and 2nd diagrams again and says, “Four plus one, get then a five --- You just do it: four plus number --- Meaning, let’s see that: plus the four to the nine to get then a thirteen ---” (see Figure 12).
Then, Selai goes on to demonstrate property recording by adding, "Plus that --- there is no formula or to multiply something! Plus four to the --- to each --- the number --- and you get the answer." Selai concludes that the group should just use the step-by-step method of adding fours repeatedly [Point 15] She borrows the term "formula" and shifts to express in English aspects of her noticed property [Note 15]. Kepi then writes down their group discussion (and answer) to Question 6 as, “You have to + 4 to each of the number and you get the pattern”.

During her earlier manipulation of the images, however, Selai twice came close to achieving a general arithmetic rule. The first instance took place when she observed the 2nd diagram and said, “Four by one is four, plus one.” But then, she became doubtful about the differences between the other diagram totals, despite having found, through numerical manipulations, the total for the 2nd diagram. The second instance occurred when she considered the 5th diagram and explained, “Four by five it’s twenty --- add it ---”. She fell short of adding an extra one to get the 21 total square blocks for the 5th diagram. Selai appeared to lose track of her calculation, and was unable to connect her general method with her targeted arithmetic rule (see Figure 12). Later on, the group moved instead to Question 7 to decide if they had seen a pattern in the extra number of square blocks being added each time. The students exchanged ideas in the following manner [Excerpt 8]:

L25  S  Yes --- it’s because it increases. It means you don’t --- The answer is ---
L26  --- it means the answer won’t stay the same.
L27  [Uses hand to illustrate how the sequence totals increases.]
L28  The answer must continually increase --- the answer --- the result. [Points to sequence.]
L29  Or the answer that we get from --- from every different diagram.
L30  K  [Kepi adds explanation to answer in Answer Sheet.]
L31  S  Must be getting different answers. The reason, to explain the thing that we get for it ---
L32  --- different --- the number will be different. It means, only the answer will be different.
Selai’s explanation resembles the earlier thoughts she expressed while answering Question 4 [Excerpt 7]. Selai’s use of the word “lahiange” (increase) [L25, L28] in this context means a repeated addition of four; both as a constant numerical addition and also as a pictorial lengthening of the sides by four square blocks each time. Again, Selai’s use of the word “answer” refers to each of the diagram totals. This incident, however, demonstrates Selai’s continual reformulation of her understanding of pattern and relation in the extras, as she moves out to work at the Formalising layer [Point 16] [Note 16]. In addition, Selai offers the same explanation in Question 8 about observing, “the pattern in the total number of square blocks used in each diagram”. Selai begins by focusing on the word “total” [Excerpt 9]:

L33  S  It says whether we can see all the total ---
L34  --- total number of the pattern of the square then we use a diagram. Yes! We can do.
L35  K  We can add a four --- a four to that.
L36  S  “Explain your answer.” Add that --- means we add a four.
L37  [Again uses hand motions as ‘Alusa writes.]
L38  K  A four to that --- a four to the number.
L39  S  --- to each number in each pattern. Add a four to each pattern to get the answer.
L40  K  So it gets then a --- [Alusa writes answer to Question 8.]
L41  S  Every pattern --- each pattern --- If --- if we don’t add four ourselves then ---
L42  Won’t get the answer then! And the --- what’s the good word to use for one about ---
L43  --- increase? [Selai points to sequence and indicates increases between each diagrams.]
L44  K  Add four and then you get the next pattern --- next diagram!

The statements shown above once again demonstrate Selai’s continued Formalisation [Point 17] about the way each diagram must be constructed. This is not an algebraic generalization, yet Selai appears to be confident in stating her approach as a general method for constructing the pattern in repeated additions of four (square blocks). Her substitution of the phrases, “each pattern” [L39, L41] and “every pattern” [L41] refers to each diagram, and thus Selai concludes,
by *shifting* to Tongan, that without the repeated additions of four, the totals would not be accumulated correctly [L41-L42] [Note 17]. Following Selai’s efforts, the other member of the group, Alusa, writes down the answer they have discussed as, “Yes, plus 4 to every pattern to get the answers”, again illustrating their *Formalised* understanding of the pattern.

The group then moves to Question 9, asking them if they can predict the 17th diagram, and if they can, what they would declare as the rule or pattern for calculating the total for this diagram. Selai first suggests that there has to be a “formula” or a rule. Her inability to predict the total for the 17th diagram prompts Selai to fold back to *Property Noticing* and to work with her images [Point 18]. Selai’s invocative move was not tied to any language switching [Note 18]. She begins by reflecting on her previous constructions and tries to generalize an algebraic rule. Selai looks at the pictorial sequence and suggests, “if I just do it like this --- x plus four is equal” while she writes down “x + 4 =” where x refers to the diagrams’ totals. Next, Selai tests out a few numerical numbers using this algebraic relation to find a possible formula or connection between the diagram numbers and the totals. In the process, she tries a few multiplications of her own: “Just multiply something to that to get that answer”. Selai applies her idea using “times two”, but finds this potential rule insufficient for accumulating any of the numerical totals. Selai, therefore, has yet to *Formalise* an algebraic method for the relation, nor is she able to determine a correct arithmetic rule, while mainly verbalizing her thoughts in Tongan. She continues working at the *Property Noticing* layer [Point 19].

While Selai appears to be “stuck” in her current understanding, Kepi reminds her that “The pattern is just adding up --- go with the thing [*meaning the pattern*] --- Just add a four, get the answer”. As a result, Selai agrees again to their previous non-algebraic generalization, therefore moving out again to her existing *Formalised* understanding [Point 20], which she demonstrated
previously in answering Question 8. She says, “If that’s it, then it’s just $x$ plus four --- meaning, then the $x$; it represents a --- number from the thing --- pattern to each pattern! Just do $x$ plus four.” This generalization is represented by her use of the expression, “$x + 4$”, where $x$ again stands for the numerical totals of each diagram [Note 19]. Kepi concludes by adding, “Just do that. That’s just the rule.” Alusa then writes down the rule or pattern as, “Every pattern must add four to the total number.”

However, in applying their general rule, the students agree to start with the biggest total they had calculated – the 7th diagram with 25 total square blocks. As they continually add four to each total, Kepi suddenly suggests that by adding 17 twice, they might be able get the total for the 17th diagram. But then all the students agree that Kepi’s suggestion is wrong, because they predict the total should be bigger, or much more than 34 square blocks. In this situation, the group has folded back again to work at the Property Noticing layer [Point 21], with language switching not involved [Note 20]. While the students may have generalized a non-algebraic rule for the pattern, the rule deals with their images for the pattern in the extras, and not a general rule for the pattern in the totals, which they have not been able to find. As the students resume working on the task, Selai continues in Tongan with Kepi in using repeated additions of four as their way of finding the subsequent totals, up to the 10th diagram. She says: “Seventh is twenty-five --- [8th] is twenty-nine. Ninth is thirty-three.” But then Kepi claims the 7th diagram total to be 31 square blocks. Selai corrects him by counting with her fingers: “Thirty-three! Add four --- Twenty-nine [8th] --- thirty, thirty-one, thirty-two, thirty-three. [9th]” She then concludes that the 10th diagram has to be “Thirty-seven” square blocks. Selai holds back the group, saying, “Wait --- do that again!” and then she recalls her numerical image of the totals, again starting with 25 for the 7th, and

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[31] Variables such as $x$, $y$, and $z$, are not part of the Tongan alphabetical letters (which has only 16 of the 26 English letters), and Selai, in this case, borrows one ($x$) to represent her algebraic Formalisation.
continuing up to the 10\textsuperscript{th} diagram. In this instance, both students are reflecting on their previous constructions and using their organized scheme for the pattern to calculate the totals.

Selai and Kepi seem to be confident of their scheme for calculating the total for the 17\textsuperscript{th} diagram. But once Selai finds the 10\textsuperscript{th} diagram to have 33 square blocks in total, she suggests, “Add with twenty-five” (the total for the 7\textsuperscript{th} diagram). In this case, Selai notices yet another property of her numerical image of the totals that might serve as a shortcut for calculating the total for the 17\textsuperscript{th} diagram. She comes up with a total of 62, adding the totals of the 7\textsuperscript{th} and the 10\textsuperscript{th} diagrams as a short way of figuring out the 17\textsuperscript{th} diagram (see illustration below):

\begin{center}
\begin{tabular}{cccc}
\textbf{diagram number} & \textbf{7\textsuperscript{th} diagram} & \textbf{10\textsuperscript{th} diagram} & \textbf{17\textsuperscript{th} diagram} \\
numerical totals: & 25 & + & 37 \\
                                                                 &                  & = & 62 \\
\end{tabular}
\end{center}

In the situation described above, Selai demonstrates continued work at the Property Noticing layer [Point 22], without any connection to language switching [Note 21]. She extends her existing outer-layer knowing by changing her earlier constructs for the pattern and generating a new connection. Although Selai’s existing outer-layer knowing appears to be thickened by her inner-layer action at the Property Noticing layer, her short-cut method for calculating the total is incorrect. Nonetheless, Selai’s new understanding prompts Kepi to notice further properties using his existing knowledge. Kepi insists the total cannot be 62, because it “should not be an even number”. Kepi’s statement and substitution of the phrase “even number” indicates Property-Noticing activity in his understanding of the nature of diagram totals: that the pattern will result in odd-numbered and not even-numbered totals [Note 22]. Because of Kepi’s observation, the students start again with the total of the 10\textsuperscript{th} diagram, and by repeatedly adding
fours each time, they manage to come up with the correct total for the 17th diagram – 65 square blocks.

In the next incident, having calculated the total for the 17th diagram to be 65, Kepi then recalls Selai’s earlier total of 62. He suggests, “just add a three there” to make up the total of 65. Selai replies, “Let’s do that [repeats addition] --- I think our work is wrong”, to which Kepi agrees. The two students’ verbal exchanges appear to momentarily prompt Selai to recall previous constructions, as she says, “Just wait: if the third --- Wait: if the seventh diagram is twenty-five; sixth, twenty-nine ---”. Thus it is clear that the two students continue to work with their images at Property Noticing, by using the trial-and-error approach to finding an arithmetic general rule [Point 23]. Both students manipulate their images by relating each diagram number (in the sequence) and its corresponding total through various combinations of multiplications and additions. At first, Selai appears to think she has now gotten the formula. Selai writes the expression “2(2 + x)” and resumes by saying, “Wait: two times two add with x. So then the x; it’s representing the diagram”. Selai’s discussion with Kepi moves back and forth between her images without her being able to make a connection between the images. The two students’ attempt to solve the problem again proves unsuccessful as both Kepi and Selai agree that the formula “2(2 + x)” cannot produce the correct total for the 17th diagram by replacing x with 17. Their unsuccessful attempt is once again due to their lack of mathematical abstraction, and not because of language switching [Note 23].

As a result of their difficulties, Kepi suggests that they ought to continue using the “old pattern: keep adding the four”. Selai agrees, “The rule only is every pattern must add four --- but the reason, it will take long if I sit and add it.” However, Selai still suspects that an arithmetic rule exists: “But there exist a method but it’s...difficult”, to which Kepi responds that a repeated
addition of fours is “only long but easily understood.” As a result, the group adopts the total of “65” for the 17th diagram and Selai offers their explanation using the general method of “every pattern must add four to the total number…[it] means, that we get a definite answer in the diagram.” This implies that Selai has moved out again to Formalising her pattern in the extras through shifting [Point 24] [Note 24]. Kepi agrees with Selai that they would not have gotten the correct answer had they made the additions themselves: “If we didn’t add ourselves --- to add four, the pattern won’t be correct. Need that we add four always --- always add four so that we get the pattern.” Again, Kepi, like Selai, chooses to use this repeated-addition method, without any multiplication, as the group’s generalized rule for calculating the totals. This shows, therefore, that both Selai and Kepi have been unable to make a link between their repeated-addition method and their attempt to generalize through multiplication.

As they move on to Question 10, the group discusses the 60th diagram. After reading the question, Selai again challenges the group regarding the usefulness of a general arithmetic formula (different from the repeated additions of four) for bigger diagrams: “There is a rule, but -- wait, let’s think about it to get that rule --- and then we use this new rule for the seventeenth.” This prompts both students to fold back again to Property Noticing [Point 25], where they continue manipulating their images using various combinations of multiplications and additions to find a connection between each diagram ordinal number and its total. Language switching plays no role as the students use only Tongan [Note 25]. Kepi uses his numerical images for the pattern and suddenly realizes that he has finally generated some of the totals in the diagrams: “Four by one is four, add with a one is five. Four by two is eight, add with a one is nine. Four by three, twelve --- add with a one is thirteen. So it’s --- so it’s four by four, sixteen --- add with a one is seventeen! Four by five is twenty, add with a one, twenty-one.” Kepi has now associated repeated addition with multiplication by expressing observed connections in Tongan [Note 26].
Selai notices that Kepi's new arithmetic rule appears to be correct and she therefore joins him in using it to check some of the totals. She then adds, "Eh! Wait --- and if that's it." Selai then proceeds to apply Kepi's new rule to the 17th diagram – multiply the diagram number by four and add one – by first attempting to multiply 17 by four. Thus, this incident gives rise to Selai's continued work at Property Noticing [Point 26], rather than Formalising, as she expresses the specific organized scheme Kepi has just noticed.

It is interesting to note that while applying the rule to the 17th diagram, Selai folds back temporarily to access her Primitive Knowing [Point 27], in collecting her base knowledge of multiplication by four. This incident shows how folding back is accompanied by the students' use of their first language in working with their informal activities [Note 27]. Selai says, "Four times seventeen --- get then --- four by seven, twenty-four?" Then she gets stuck and decides to re-collect her knowledge of multiplication by four, starting with "four by four, sixteen; four by five, twenty; four by six, twenty-four --- four by seven, twenty-eight." Selai appears unable to access this knowledge immediately, and hence she needs to fold back and collect her base knowledge of multiplication and work with it in order to come up with the correct product. Selai has therefore moved out of the topic to do a different mathematical task of multiplication by four. However, the product of "17 x 4" was mistakenly multiplied for a product of 58 instead of 68. At the same time, Selai moves back out again to property recording [Point 28], involving the addition of an extra unit, using Kepi's new rule, in order to get a total of 59. Kepi asks, "Fifty-nine is correct then?" Then Selai replies, "Just wait: there is an answer as fifty-nine --- in what we did --- and may be our counting was wrong." Because of her expressed uncertainty, Selai prompts the group to reflect on their numerical images of the pattern by saying, "Just wait: just try that --- the seventh diagram. Forty-nine --- fifty-three --- fifty-three add with four". Kepi replies that the sum only adds up to 57, and that he is still certain that his "previous try is
correct”. He therefore checks the totals again using his new arithmetic rule, which still appear to be correct, because his arithmetic rule appears to generate the totals.

At this point in the group’s activities, Selai has not demonstrated moving out to generalization; instead she continues working at Property Noticing as she notices [Point 29] that her earlier result, which she wrote as “(4 x 17) + 1”, is wrong. This observation is not tied to language switching [Note 28]. Yet, Selai realizes that to find the appropriate total number of square blocks for the 17th diagram, they ought to have multiplied four by 16 instead of 17, and then followed by adding the extra one. She reasons, “If it is our multiplication of the seventeen [17th diagram], there, it will get then the answer of that --- of the eighteenth pattern [18th diagram].” This reasoning indicates how Selai gets so close to crossing over to Formalising, and yet, she is observed to be working at the Property Noticing layer because of her reliance on specific numbers for justification of the pattern. Kepi agrees with Selai’s deductions, and so Selai, in the process of multiplying 16 by four, folds back again temporarily (as she had done before) to her Primitive Knowing [Point 30] without language switching [Note 29]. She says: “Multiply the sixteen: sixteen times four --- four by six, twenty-four, is it? Four by four is sixteen; four by five, twenty; four by six, twenty-four.” She then goes back out again to Property Noticing [Point 31] to work on the product of 16 times four until she finally gets the right total for the 17th diagram: “Sixty-four! Add with one --- That’s it! Get the answer --- sixty-five! Get the answer! That’s it!”

Finally, Selai then proceeds to use the group’s arithmetic rule to find the correct total for the 60th diagram. Her follow-up explanation of the rule, “It’s four multiply to the next diagram, plus one ---” suggests that Selai has made the leap from relying on a specific diagram number at Property Noticing, to applying the rule to any diagram at Formalising [Point 32]. In this situation, Selai’s expression of the general rule in English, particular her use of the phrase “next diagram”, is not
correct, but based on her application of the rule to the 17th and 60th diagrams, Selai may have meant to say the “preceding diagram” [Note 30]. So when Kepi asks her about the 60th diagram, Selai appears confident with their generalized rule and quickly applies it to the 60th diagram, “Times four the fifty-nine.” In a similar way, Selai folds back again to access her primitive knowledge about multiplication by four [Point 33], an act that is unrelated to her language use [Note 31]. Thus, in calculating the product of four times nine, Selai starts with the results she obtained earlier: “seven fours, twenty-eight; eight fours, thirty-two; nine fours, thirty-six.” By knowing the product of nine and four, Selai collects that piece of knowledge and uses it to extend her current outer layer knowing [Formalising, Point 34] to find the correct total for the 60th diagram as “two-thirty-seven”. Selai writes “(4 \times 59 + 1) = 237”, then explains the generalized rule as: “Multiply --- the additional there of the pattern --- which is four --- to the diagram behind it to get then the answer before”. Alusa then writes out, in Tongan, the group’s explanation for the rule, saying, “Means to multiply the increase (4) to the diagram behind it [preceding it] to get the next diagram”. Significantly, since the students cannot find the proper English word for “preceding”, they retreat to using Tongan in explaining their rule (see the remark at Note 30). Selai later adds to Alusa’s written explanation, the phrase, “and plus one” to complete their general rule (see Appendix 4) [Point 35]. Feeling confident they have answered the questions to their own satisfaction, the group then moves on to do the next task.

6.5.2 Malia’s Growing Understanding of Patterns and Relations in Task 3

The second piece of analysis focuses on Malia’s growing understanding of patterns and relations over time, while working on Task 3 (see Figure 13; MAP 2 shows her mapping). Malia’s mapping in this example illustrates how her language switching enables her to express her growth of mathematical understanding. Malia was another of the Form 3 students. The analysis
includes an acknowledgement and a discussion of the contributions made by Malia’s peers: Tupu and Kelela. These two students took part in collaborative work, particularly Tupu, who continuously shared her ideas with Malia as they built on each other’s understanding of the specified topic. Kelela’s participation in the task work was minimal.

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**Figure 13:** Continuing pictorial sequence of square blocks in Task 3

As soon as the group is given the Task Sheet, the students Kelela (K), Malia (M), and Tupu (T), immediately read Question 1 – Draw the 4th diagram in the sequence [Excerpt 10]:

L1 K Triangular! [Looks at the given pictorial sequence.]
L2 T One, two, three, four [Points & counts totals in 2nd diagram.]
L3 M One, two, three, four, five, six, seven, eight, nine. [Counts up from base of 3rd diagram.]
L4 M It is not triangular --- [As she looks at T].
L5 K Triangular!
L6 M Look at this: one, two, three, four --- four plus five. [Counts 2nd diagram & points to its base.]
L7 K Four plus five [look at M] --- Four plus five. [Uses pen to sketch “4+5” on top of the table.]
L8 T Five? [Look at the base of the 3rd diagram.]
L9 M, T One, two, three, four, five. [Both T and M count the bottom layer of the 3rd diagram.]
L10 T Oh, yes!
L11 K That’s it!
L12 M Wait, wait, wait! Just add-up the odd numbers. That’s it --- it’s not triangular numbers.

A follow-up interview with the group reveals that the given pictorial sequence immediately evokes, in Kelela, a geometrical pattern of triangular shapes the moment he utters aloud the word
“triangular” [L1]. While Tupu is counting (in Tongan) the number of square blocks in the 2nd and 3rd diagrams [L2-L3], Malia immediately responds to Kelela’s statement, saying, “It’s not triangular” [L4]. Borrowing Kelela’s term, Malia apparently misunderstands her, thinking Kelela is referring to a numerical pattern in the sequence: a set of triangular numbers [Note 33]. Despite their seeming disagreement, both students appear to be involved in image seeing [Point 1], although they have yet to articulate why their respective images fit, or do not fit, with the other’s.

Malia’s thinking aloud does not appear to include her working at Image Making, but that does not mean she does not engage in such an activity. Malia might have been Image Making silently, or simply following Tupu’s counting [L2-L3]. Alternatively, she may already have had some existing images, other than the numerical triangular numbers, for the pattern, which may be why she seems opposed [L4] to Kelela’s suggestions [L1, L5]. By calculating the total number of square blocks in the 2nd diagram to be four, Malia adds “four plus five” [L6], a reference that relates the 2nd to the 3rd diagram through the extra base layer. At the same time, Malia’s statement appears to be reflecting on her own construction [Point 2], a process Pirie and Kieren describe as a component of acting (which, at this layer, is image seeing), since it incorporates the process of looking at how an individual’s previous understanding was generated. When Tupu checks [L8] what Malia means, Malia points at the base layer of the 3rd diagram and counts the total to be five square blocks [L9].

Malia continues her observation as if she has a numerical image of the extras for the pattern: “Just add-up the odd numbers. That’s it --- it’s not triangular numbers” [L12]. This statement apparently reveals five interrelated learning insights about Malia’s current understanding: (i) it shows Malia has constructed a numerical image of the extras (set of odd numbers); (ii) it reveals
how Malia uses that image as a basis for constructing the numerical sequence (add up the odd numbers); (iii) it shows that Malia switches through substitution to using the phrase “odd numbers” to identify the numerical set involved; (iv) it reveals Malia comparing her image (odd numbers) with an existing image (triangular numbers), evoked by Kelela’s comment; and (v) it shows Malia switching again through borrowing with the non-equivalent term, “triangular numbers” [Note 34].

It is apparent, however, all three students must have perceived a picture of the given diagrammatic sequence, but have yet to observe how each of these diagrams is constructed pictorially, or how each layer is structurally aligned or different from the other. Having observed these distinctions would certainly present the students with various images of patterns and relations to talk about regarding the given pictorial sequence, and thus put them in a position to move out further to Property Noticing. However, only Malia appears, at this stage, to have developed a particular numerical image for the pattern along the layers, although she has yet to explicitly express her understanding of it. Thus, at this stage, Malia’s images can be deemed partially complete for two reasons: (i) Malia has yet to construct the 4th diagram to indicate if she has indeed associated the set of odd numbers with the pattern along the base layers or within the horizontal layering of each diagram; and (ii) Malia later argues in the next extract against Tupu’s pictorial image for the pattern that represents the odd numbers in the same way she expected them to be represented. Nevertheless, Malia’s statement [L12] shows she has moved beyond image seeing to image saying [Point 3], which involves her articulating features of her numerical image (add up the odd numbers), and is therefore able to make the claim that Kelela’s image of “triangular” does not conform with her image of the term or the pattern. In the next extract, Malia provides more evidence of her view of her pictorial images. Her discussion with the group
shows her continued work at the *Image Having* layer, as they re-read the first question [*Excerpt 11*]:

L13  K   Right. Do that first! *[Pointing to the first question.]*
L14  K, T  Read the question first: “Draw the fourth diagram in the sequence.” *[Points to first question.]*
L15  T   So do that *[points to whole 3rd diagram]* and then add to it one, two, three --- *[Points to base.]*
L16  --- four, five, six, seven. *[Counts and “sketches” the number of 7 blocks for the bottom layer.]*
L17  M   No. will plus one, plus two. *[Points to additional blocks on each side of the 3rd diagram base.]*
L18  T   Will plus seven; will plus seven. Look here.
L19  M   Yes, yes, yes! Look at this: five, [base layer of 3rd] six, seven to that [add blocks to each side].
L20  K   One, three, five, seven *[then repeat]* --- seven --- one, three, five, seven.
L21  T   *[Draws the 4th diagram on Question Sheet while K & M talk about how many blocks are drawn.]*
L22  M   One, two, three *[Points to middle layer of 3rd diagram with three blocks as T draws 4th diagram.]*
L23  K   Shade --- No; don’t shade it yourself! *[As T draws second layer of 4th diagram with 3 blocks.]*
L24  M   Four --- five there [3rd layer] --- you “over” by one. *[Points to add one on each side of 3rd layer.]*
L25  K, T  One, two, three, four, five --- *[Both count the number of blocks as T draws 3rd layer of 4th diag.]*
L26  K   --- six, seven! *[Getting ahead of T’s drawing.]*
L27  K   Over, over here --- just over. *[Reference K made to drawing the “extra” block on the side.]*
L28  K, T  One, two, three, four, five, six --- seven. *[As T finishes drawing the base layer of 4th diagram]*
L29  M   *[Reads] “How many extra square blocks that is, in addition to the 3rd diagram, did you draw?”*
L30  K   Seven!
L31  K, T  [Counts] One, two, three, four, five, six, seven --- seven! *[As T points & counts the base of 4th.]*
L32  M   *[Writes down answer as “7 blocks”.]*

As the above excerpt illustrates, the group goes back to discussing the first two questions about the 4th diagram. This episode provides further evidence of Malia developing her own pictorial images of the extras for the pattern through interaction with her peers. Since the students have not constructed the 4th diagram, Tupu begins by *folding back to Image Making* in Tongan [L15-L16]. Malia reacts to Tupu’s pictorial construction as evidence of her *image reviewing*, by expressing her pictorial construction through *substitution* of the word “plus” [L17] [Point 4] [Note 35]. Immediately, Tupu realizes that the next base or bottom layer (for the 4th diagram) would have to contain seven square blocks [L18]. Tupu’s pictorial image for the pattern [L15-
L16] resembles the numerical image of the extras Malia had identified earlier along the base layers [L12]: a sequence of odd numbers. Malia’s response [L19] indicates seeing Tupu’s pictorial image for the pattern as a progressive sequence of base layers being added underneath each subsequent diagram (see Figure 14 below).

Yet, Malia has not explicitly related the two described images as equivalent (that would be Property Noticing) and although she does appear to agree with Tupu’s pictorial image of the extras for the pattern [L19], her pictorial image of the extras considers the pattern visually, in terms of additional square blocks being added on either side of the base layer [L17, L19, L22, L24] (see Figure 15 below). During the students’ discussion, and Malia’s movement out to the Image Having layer [Point 5], Malia uses the Tonganised word “"ova” (over) [L24] through substitution to describe both the activity involved, and the distinctiveness of her pictorial image for the pattern [Note 36].

It is therefore interesting to note the way in which Malia suddenly transforms the added base layers as a sequence of odd numbers [L12] to that of two additional square blocks being added
repeatedly on the sides [L17, L19, L22, L24], a shift from a numerical to a pictorial image. Malia’s *folding back* to join Tupu’s *Image-Making* activity indicates the disconnection between her numerical image and her pictorial image. Arguably, the students have seen the same pictorial image of the totals – for instance, the spatial shape of each diagram – but each student produced different pictorial images of the extras. For Malia, the distinction from her peers, it seems, is not about what is involved at the base, which she agrees with Tupu is seven square blocks, but how the base is constructed: by adding one square block each time at each end of the previous base layer. Nonetheless, throughout this episode, Malia is observed to move back out again to *Image Having*.

As they work through the task at hand, the group then moves on to answering Question 3: Can you find the totals in the first four diagrams? Malia begins working on the question by writing the subheadings of the totals for the first four diagrams. Then, Tupu suddenly points to the last two questions (about the 17th and the 60th diagrams in Questions 9 and 10) and indicates these questions’ similarity to the current question (about the 4th diagram). She says, “It’s like: seventeen diagram *only* and sixty, eh? All they do is just change the picture.” [L36-L37]. Malia replies: “So it was *that* first [referring to the first task], second *that* [referring to the second task], and third sequence *there*”; pointing this time to the pictorial sequence given in Task 3 (see Figure 13). Tupu seems to understand Malia’s idea, and she says, “Oh, that’s what they do!”

While these exchanges show how Malia and Tupu relate between the set of questions and between the different tasks, they do not connect to the students’ specific images for the pattern as the two students continue working at the *Image Having* layer [Point 6]. Malia uses *substitution* (“first”, “second”, “third”) and *borrowing* (“sequence”) in expressing her thoughts [Note 37].
In the next stage of group discussions, Malia continues with Question 3 to discuss (in Tongan) and write the numerical totals for the first four diagrams. Then, moving on to Question 4, the group is asked to predict how many square blocks (in total) will be needed for the 5th diagram.

32 In Point 15, it was Tupu who folded back, not Malia, and it was not clear as to whether Malia made use of that collected piece of knowledge to verify her calculation.
Tupu immediately replies that an extension of her image along the base layers would include nine square blocks for the 5th diagram. While Kelela supports Tupu, Malia realizes that the question is being asked about the totals, and not about the extras. Malia points at the 5th diagram and says, “In total --- all that”. Quickly, Tupu picks up on Malia’s mental construct of the pattern, but she adds incorrectly, “Nine plus seventeen”, using the connection between the extras and the totals. Malia corrects her, saying it should be, “Nine plus sixteen”, and therefore the two students agree that the 5th diagram’s total has to be “Twenty-five”. The mathematical activities involved in the preceding situation indicate a continuation of the students’ working at the Image Having layer [Point 7], in spite of Tupu’s arithmetic difficulties in calculating the total. Malia extends her image along the base layers – add up the odd numbers – to get the total for the next diagram, and it allows both her and Tupu to mentally construct the same numerical total for the 5th diagram. In addition, Malia writes down their answer for the question as 25 square blocks, and then she correctly draws her pictorial image for the 5th diagram. Thus, Malia continues to work at the Image Having layer, enabling her to carry out activities through internalization of her images without having to do the initial actions that bring forth these images. Moreover, language switching throughout this episode does not play a significant role in the students’ mathematical activity [Note 38].

The students are now in the process of answering Question 5 about the 6th diagram. While they recognize the similarity of the related set of questions to the previous tasks they have done, the students also realize how the sequence (and therefore the pattern) is different in each task; a similar discussion to the one raised earlier by Tupu (see the incident at Note 37). However, Malia asks about the additional layer being added to the 4th diagram. Tupu replies, using her pictorial image of the extras as seven, and then adds on to that image, “then plus nine there ---” (for the 5th diagram), as a progression of odd numbers along the base layers. Thus, Malia simply
extends her image along the base layers to find the number of square blocks needed for the 5th diagram. She finishes drawing the 5th diagram and moves on to tackle the 6th diagram. After reading Question 5, Malia points to the pictorial sequence and says, “So the sixth diagram --- that will be eleven!” In Pirie-Kieren’s terms, Malia has internalized a mental plan for constructing the continuing sequence for the pattern and continues to extend that mental construct along the base layers, both numerically and pictorially (that is, as a base layer of 11 square blocks). Tupu follows Malia by adding, “So you just plus eleven to the twenty-five, you get --- thirty-six”, again demonstrating her mental construct for calculating the numerical totals of the sequence. This episode again shows Malia extending her numerical image of the extras for the pattern at the Image Having layer [Point 8].

In her continued work with Question 5, Malia reflects upon her numerical image of the pattern along the base layers. With her head resting on the table, Malia looks at their worksheet, then suddenly notices a specific property associated with the numerical image. She says, “So it’s adding the odd numbers. Yes, add the odd numbers” by substituting the phrase “odd numbers” [Note 39]. It is at this moment that Malia consciously sees a new property associated with her image of the extras for the pattern. As a result, she verifies her noticed property by checking the numerical pattern in the extras (base layers). Hence, Malia has moved out to work at the Property Noticing layer [Point 9], where she looks at her images and constructs a context-specific property. She goes on to explain this property when she grabs the worksheet and says, “Five to that [points to 3rd diagram]; seven to that [4th diagram]; nine to the fifth diagram”. Tupu picks up on that image and extends it further to find 11 square blocks for the 6th diagram’s base layer.
Then immediately after reading Question 6, Malia says, “Seventh diagram --- so that is thirteen”. Malia continues to simply “add the odd numbers” successively, based on the pattern she noticed earlier. Together with Tupu, Malia mentally constructs the total for the 7th diagram at “forty-nine”. Again, Malia appears to continue working at the Property Noticing layer [Point 10], using their image associated with the extras. Tupu agrees and uses that image too, to do an arithmetic calculation (in vertical form) of the total number of square blocks needed for the 7th diagram. While thinking aloud, Tupu writes down 25, and then adds 11 vertically to the sequence. Tupu then adds 13 to the previous sum (36) and gets the final total of forty-nine (49) square blocks for the 7th diagram. Malia then agrees that 49 square blocks are needed for the 7th diagram.

The next excerpt [Excerpt 12] illustrates evidence of another new piece of understanding as Malia continues working at the Property Noticing layer. While Tupu writes down their answer to Question 6, Malia checks with her to find out which diagram number corresponds to the recently calculated total.

L33 M  So what number is that? [Referring to the total of the 7th diagram.]
L34 T  Seventh diagram.
L35 M  Are those square numbers? [Points to pictorial sequence.]
L36 K  What?
L37 T  No!
L38 M  Yes!
L39 T  Seven times seven, forty-nine? [Points to Question 6 with the 7th diagram.]
L40 M  Three by three is nine; four by four is sixteen; five by five is twenty-five.
L41 M  (Smiles as she points out to T each of the 3rd and 4th diagrams and its numbers in sequence.)
L42 T  Those are square numbers.
L43 K  Yes, it’s one, two, three, four. (Counts 2nd diagram then repeat on table.) One, two, three, four.
L44 T  Okay, your turn (and shift worksheet to M).
L45 M  Do like this: seven square. [Writes “(7^2)” besides T’s written answer “= 49 blocks.”]
When Tupu replies that she is working on the 7th diagram [L34], Malia points to the sequence and asks, “Are those square numbers?” [L35] Malia therefore notices another pattern in her numerical image of the totals, which she expresses through borrowing [Note 40]; hence she is demonstrating new evidence of property predicting [Point 11]. In this case, Malia has yet to Formalise this new property because she only uses specific numbers [L39] to illustrate this property, rather than a generalization. Kelela and Tupu initially appear to be unsure [L36-L37] about what Malia has just asked [L35]. But with a smile on her face, Malia responds to Tupu’s remark [L39] by working with the images that bring about her new discovery at the Property Noticing layer. She expresses her new finding by using her hand to point to each of the totals and the corresponding diagram numbers in the sequence, saying, “Three by three is nine; four by four is sixteen; five by five is twenty-five —” [L40-L41] [Note 41].

To reinforce her noticed property, Malia recalls, from her primitive knowing, the notational form of $x^2$ for square numbers. This notational form, which is part of Malia’s primitive knowing, is immediately accessible to her in symbolically expressing her noticed property. Malia then writes down her answer of “$7^2$” beside Tupu’s written answer of “49 blocks” for Question 6 [L45]. This mathematical form has brought about a different approach to the problem – a symbolic representation of her noticed property – allowing Malia to represent the relationship between the diagram numbers and their corresponding totals [Note 42]. Malia’s act of recall, however, is quite different from collecting, which involves retrieving previous knowledge for a specific purpose and re-viewing or reading that information anew to apply it toward current mathematical actions, and has the “thickening effect” of folding back (Martin, 1999). In this instance, the diagram’s ordinal numbers suddenly come into the picture as soon as the students realize the property of the square numbers within their numerical totals. All of the preceding student
activities and statements indicate that Malia, as well as Tupu [L42], continually work on aspects of their noticed property [property recording, Point 12].

After the group reads Question 7 (Can you see a pattern in the number of extra square blocks you add each time?), the students exchange ideas in the following manner [Excerpt 13]:

L46  ALL  Number Seven. [Reads Question 7 as K grabs worksheet to write the next answer.]
L47  K    Square numbers. [Grabs worksheet to write next answer.]
L48  M    Yes!
L49  K    Yes! [Writes down “Yes” as part of answer to Question 7.]
L50  T    What’s the pattern?
L51  M    It’s extra ---
L52  K    Square number!
L53  M    No, it’s extra --- extra blocks.
L54  T    Oh, it’s the next odd number. It’s odd number.
L55  M    Extra blocks use odd numbers. [While K writes it down]
L56  K    Which is?
L57  T    It means, like: one plus three to get the second diagram. [Points to 1st and 2nd diagrams.]
L58  M, T  Four plus five to get the --- (Points at the 3rd diagram.)
L59  K    --- second diagram!
L60  T    And so the next --- yes! [as she looks over to K’s written answer.]
L61  M    Number eight. [T reads Question 8.]
L62  M    [Immediately answers] Yes! [K writes down “Yes” as part of answer to Question 8.]
L63  K, T  Yes!
L64  M    Square numbers!
L65  T    Square numbers --- you just square the --- [As K writes “square numbers” only.]
L66  M    --- square the number of the diagram.
L67  T    You square the number of the diagram. [Takes the Answer Sheet, writes answer down.]
L68  K, T  You square the number of the diagram to get the sequence. [Reads aloud as she writes.]

Kelela appears to think the pattern in the number of extra square blocks being added is associated with “square numbers” [L47, L52], although her immediate answer suggests she might not have
paid close attention to the word, "extra". As a result, Kelela immediately writes down, "Yes", as part of the group’s answer to Question 7 [L49]. In this instance, Kelela might have been thinking Malia was agreeing with her when she heard Malia saying, "Yes". [L48] Malia’s immediate response [L48], however, is directed toward the question being read, rather than agreeing with Kelela’s instant reply [L47, L52]. The students’ mutual misunderstanding is evident later during Malia’s reply to Kelela, after Tupu asks, “What’s the pattern?” [L50], leading Malia to correct Kelela by explaining that the pattern refers to the extra number of square blocks [L53] [Note 43].

In terms of her assessment of the extras or additionals, Malia replies, “Extra blocks use odd numbers” [L55]. Malia is observed to come close to crossing over the don’t-need boundary to Formalising, but her explanation of the pattern to Tupu [L57-L58] still illustrates dependence on specific numbers [Point 13]. Then, in answering Question 8, Malia is observed to generalize an algorithmic formula or rule for finding the total number of square blocks in each diagram [Point 14] when she again responds: “Square numbers --- square the number of the diagram” [L64, L66] [Note 44]. Malia is therefore capable of stating and recognizing a formal algorithmic method, not only for the number of extra square blocks being added, but also in finding the totals for each diagram. Such a Formalisation is disconnected from any specific example, action, or image.

Tupu, on the other hand, expresses her generalization of the pattern as well [L54, L65, L67-L68], an indication that she has also moved out to work at the Formalising layer. The analysis of these two students’ actions later becomes clear when they attempt to predict the totals for higher diagrams in the sequence, such as the 17th and 60th diagrams (see Excerpt 14 below). Malia is no longer dependent on the specific numbers, but she instead steps out to Formalise her understanding of the pattern. Interestingly, the students Formalise their understanding of the pattern by using the diagrams’ ordinal numbers in the sequence as an image, and then relate it to
the diagrams' totals. Previously, however, no relationship was made between the diagrams' numbers and any of the pictorial images constructed by the students. With the exception of their numerical image of the totals, the students did not mention the numerical image associated with the extra number of square blocks (odd numbers) in relation to their generalized rule for the totals. The last extract [Excerpt 14], indicated below, shows how the students apply their formal understanding of the pattern while calculating the diagram totals.

L69  K So that to get the difference?
L70  M Sequence not difference!
L71  K Because she said “difference”!
L72  T I said “sequence”!
L73  K You said “difference’!
L74  M Nine! [Reads Question 9.] Yes!
L75  T Yes! You multiply that. Seventeen times seventeen.
L76  M [Takes sheet to write answer] Yes!
L77  K Seventeen times seventeen?
L78  T Seven by seven --- something, one-five-nine --- something --- one-fifty-nine!
L79  M Seventeen squared? Which is?
L80  T Seventeen by seventeen.
L81  M [Reads aloud in Tongan as she calculates seventeen squared in vertical form.]
L82  T Explain your answer. [After calculating the total for the 17th diagram as “289”.]
L83  M [Writes down “squared 17 by it self”.]
L84  Number ten. (Reads Question 10.)
L85  Yes! Three by six --- twelve; one-two-zero-zero! [As she initially writes “1200 blocks”]
L86  K One-two-zero-zero.
L87  T Explain your answer. You square the number.
L88  K You square it!
L89  T Six by six --- thirty-six!
L90  No, no, no --- you’re not doing it right! Thirty-six!
L91  T Remember in counting the seconds?
L92  K It’s sixty minutes.
L93  M Six by six --- thirty-six-zero-zero. [Corrected her answer to “3600 blocks”.
L94  [Writes answer: “Yes, 3600 blocks. Multiply 60 by itself, you get 3600 blocks.”]
As the above example demonstrates, the beginning of this student discussion shows a move beyond the mathematical topic to clarify an issue associated with language, rather than mathematics; the unrelated discussion does not interfere with the students’ mathematical understanding [L69-L73] [Note 45]. Following this discussion, the group moves on to find the numerical totals for the 17th and the 60th diagrams. Using the rule they have generalized in the previous episode, Malia and Tupu realize they only need to square each diagram’s number in order to find the total number of square blocks required [L74-L76, L79-L81, L83, L87]. These two students then correctly calculate the totals for the 17th and 60th diagrams [L82, L93-L94]. As a result, Malia and Tupu are seen to continue working at the Formalising layer. Furthermore, it is worth mentioning that Tupu appears to fold back to Primitive Knowing by re-collecting a piece of her primitive knowledge of the relationship in the units of time [L91] as a way of checking the students’ prediction of the total for the 60th diagram; that is, the product of squaring 60 is equivalent to the number of seconds in a minute [L92]. In this case, Tupu collects an existing piece of understanding [Point 15] and works with it at the Formalising layer [Point 16] to verify Malia’s calculation of the total for the product of 60 times 60 [Note 46].

6.5.3 Alaki and Malakai’s growing understanding of patterns and relations in Task 3

The analysis in this section provides a detailed account of Alaki and Malakai’s growing understanding of patterns and relations while working on Task 3 (see Figure 16). MAP 3 shows their mapping. This example shows two significant pieces of evidence: one, that Malakai’s mathematical understanding continues to grow despite his use of wrong mathematical labels to express his mathematical image; and two, that Alaki’s “disconnected understanding” (refer to
discussion in Section 3.3.5) is unrelated to any act of language switching. The students’
discussions throughout their work show a preference to work mainly in their native language,
Tongan. In an interview, and according to their teacher, neither student is considered a fluent
English speaker, although they are considered capable of understanding the nature and
requirements of the task.

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**Figure 16:** Continuing pictorial sequence of square blocks in Task 3

Alaki and Malakai are given the first three diagrams of a continuing sequence in diagrammatic
form (see Figure 16). Alaki quickly reads the first question aloud—draw the fourth diagram in
the sequence—then immediately shifts languages to work in Tongan as he points out the total of
the first two diagrams [Note 47]: “So it’s one there [1st diagram] --- four there [2nd diagram] ---
Hold on --- one there; one, two, three, four --- add three to that [1st diagram]?” Alaki’s actions
indicate evidence of image doing [Point 1], as he first engages directly, through counting, in an
activity associated with constructing an image of what he sees, and then he goes on to review
[Point 2] his work in an attempt to make sense of it. This reviewing involves counting, with
Alaki assigning a one-to-one correspondence between the numerals (counting numbers) and the
designated objects (square blocks). This process of cardinalizing the number of square blocks in
each diagram allows Alaki to deduce a difference of three square blocks between the first and
second diagram.
In terms of Malakai’s work on the task, he responds to Alaki in Tongan and recommends drawing the 4th diagram right away, as if he has figured out visually a mental construct of the pattern: “Do it quickly --- you just draw it [4th diagram] --- you don’t have to waste time.” Then Malakai makes a quick sketch of the 4th diagram and explains, in Tongan, his pictorial image of the pattern as a set of ascending and descending vertical columns of square blocks. Starting with four square blocks in the middle, Malakai explains as he draws his pictorial image of the pattern, “Look here: just do it like this: four, three, two, one [middle column to left – see left arrow] --- three, two, one [right columns – see right arrow] --- finish!” Figure 17 represents the kind of vertical layering Malakai describes as he draws the 4th diagram.

Figure 17: Malakai’s pictorial image as a stack of vertical columns of square blocks

In response to Malakai’s image seeing [Point 3], Alaki explains, in Tongan, his different image for the pattern using the base layers. He says, “Five there [base of 3rd diagram] --- so it’s five there and seven at the bottom there [base of 4th diagram]. Right? Seven there and then five there, and three there. Right? Yes!” Figure 18 below shows Alaki’s constructed representation, an image Malakai agrees with for the first four diagrams. At this point, both students have now seen and formed a pictorial image associated with the extras for the pattern and are both able to articulate its features, and are thus in a position to talk about why a different image may not conform to the image they have. These deductions are evidence of the students moving out in
their growth of mathematical understanding to work at the Image Having layer [Point 4], while working only in Tongan [Note 48].

\[
\begin{array}{cccccc}
1^{st} \text{ diagram} & 2^{nd} \text{ diagram} & 3^{rd} \text{ diagram} & 4^{th} \text{ diagram} & 5^{th} \text{ diagram} \\
\begin{array}{c}
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**Figure 18:** Alaki’s pictorial image of odd-numbered horizontal layers

Next, speaking only in Tongan, Alaki expands his image of the pattern to find a total of seven square blocks for the base layer of the 4\textsuperscript{th} diagram (see Figure 18). He then uses his pyramidal image to draw the 4\textsuperscript{th} diagram by stacking the horizontal layers together, arranged appropriately from bottom to top (see Figure 19). Later on, Alaki’s mental construct becomes obvious in his construction of the other diagrams. Although no language switching is involved throughout this incident, Alaki uses Tongan words such as “lalo” (base or bottom) and “hoko” (next) to describe how his stack of horizontal layers is added up to build the 4\textsuperscript{th} diagram. However, in the previously described incident, there is insufficient evidence to suggest both students have explored the similarities between their images. If they had done so, they would have been said to move out and work at the Property Noticing layer. In particular, Alaki, continued to work mainly with his images at the image saying layer [Point 5], without language switching [Note 49].

The students move on to the task of finding how many extra square blocks are added to the 3\textsuperscript{rd} diagram in order to construct the 4\textsuperscript{th} diagram. After reading Question 2, Alaki extracts the key phrase, “extra square” by saying “Extra square --- so it’s seven there [base of 4\textsuperscript{th} diagram]; and then nine the base there [5\textsuperscript{th} diagram] --- and then eleven the base there [6\textsuperscript{th} diagram]. Alaki
reflects and articulates on his pictorial image for the pattern along the base layers from the 4th to the 6th diagram, as evidence of his continued working at the *Image Having* layer [Point 6]. Not only does he extend his image for the pattern along the base layers, but Alaki expresses how to use that image in constructing each of the first four diagrams: "Look here: add three to that [to get 2nd diagram]; and add five to that [3rd diagram]; and add seven to that [4th diagram]" (see Figure 19 below). In this incident, Alaki modifies his image for the pattern in a number of ways. He extends his pictorial (and associated numerical) image along the base layers from the 4th diagram to the 6th diagram, and also enlarges his stack of horizontal overlapping layers from the 4th to the 5th diagram (see Figure 19). Throughout this construction, Alaki continues to articulate (image saying) his mathematical constructs in Tongan [Point 7].

![Figure 19: Repeated stacking of odd-numbered base layers](image)

During his work at the *Image Having* layer, Alaki consciously sees the relationship between the diagram layers, and concludes that the pattern in his pictorial images – the common difference – along the base layers and between the horizontal layers is "Add two extra squares" (see Figure 20). Alaki has moved out to *Property Noticing* [Point 8] by constructing a context-specific property, based on his knowledge in manipulating and combining aspects of his pictorial images.
MAP 3: Malakai’s and Alaki’s growing understanding of patterns and relations in Task 3

Pathway of Malakai’s growing understanding of patterns and relations traces through:

Points 3, 4, 10, 11, 14, 15, 16, 17, 18, 19.

Pathway of Alaki’s growing understanding of patterns and relations traces through:

Points 1, 2, 4, 5, 6, 7, 8, 9, 12, 13, 17, 18, 19.

--- (dotted line) represents “disconnected” understanding
In this instance, Alaki associates the key phrase, “extra square”, derived from the question, with the common difference (of two square blocks) between any two consecutive layers. This extracted phrase appears, therefore, to dictate the way Alaki approaches his images, and at the same time directs him toward outer-layer thinking sophistication [Note 50].

While Malakai silently figures out the task (evident because of his attentiveness to Alaki’s work and the response Malakai reveals later on), Alaki continues articulating aspects of his images. Alaki describes the pattern in the base layers, saying “Extra it’s three, five, seven --- another row, eh? Right?” But Malakai responds, “Extra square is five”, an indication of the kind of image he has, which he later identifies in relation to the pattern along the base layers. However, Alaki continues describing his images in Tongan, and then he sketches the base layer of the 3rd diagram and asks Malakai, “What is it called that one at the bottom there?” [Note 51]. Malakai (M) responds, “Base!” Still, Alaki (A) is not satisfied with Malakai’s response because he continues to use equivalent English words, such as “row” and “step” as metaphors for the extra square blocks being added on both ends of the base layers [Excerpt 15]:

```
L1  A  Those things at the base there --- [Sketch base layer of the 2nd diagram.]
L2  --- which is the same thing ---
L3  --- like that one. [Shifts and refers to the 4th diagram drawn.]
L4  --- just do as row.
L5  To make another step --- it’s a step there --- [Points to the 2nd diagram.]
```
Alaki continues to work at the *Property Noticing* layer [Point 9], but still struggles to find the appropriate English label for his noticed property, although he seems comfortable expressing it in Tongan [L1-L3, L12-L18]. Alaki continually describes features of his images using, in this case, the words “row” [L4] and “step” [L5] as metaphors for the extra square blocks being added on both ends of the base layers. He associates the Tonganised equivalent words “sitepu” (step) and “poloka” (block) with the phrase “poloka fakamuimuitaha” (last block) [L12] and “sitepu faka’osi” (last step) [L14, L16] [Note 52]. However, Malakai’s statement, “Hold on while I think ---” [L9], shows stepping back and looking for any connection within or among his image(s). This stepping-back process allows Malakai to notice a numerical pattern along the base layers when he declares, “I already know it --- add the prime numbers”. This evidence of moving out to *Property Noticing* [Point 10] is accompanied by a shift in language to using the non-equivalent English word, “prime”. However, Malakai’s mathematical meaning for this label becomes apparent – he sees a set of odd numbers – in relation to Alaki’s image of the extras (see Figure
18). Malakai demonstrates how his more formal understanding of concepts at the *Property Noticing* layer informs and assists with the less formal, specific actions at the *Image Having* layer, even though he is using a wrong English word [Note 53]. As a result, Malakai is able to extend his current understanding by manipulating his earlier constructs of the topic and generating a new numerical relationship [Point 11].

By contrast, Alaki does not explicitly acknowledge Malakai’s finding, as he continues to express his understanding of the pattern and relation in English by answering Question 2. He says: “Add two to the last step --- *it’s one step* --- the last step of the ordinary. Then total them and get --- then plus them --- then add and get the total --- total block”. Malakai is not involved verbally, as he appears to continue “thinking” while Alaki moves to answering the next question: Can you find the total number of square blocks in the first four diagrams in the sequence? In the process, Malakai *shifts* languages to using Tongan. Alaki then goes on to discuss the total of the 5th diagram with Malakai, and together the two students come up with a total of “ua-nima --- twenty five square blocks”. Alaki demonstrates relating and combining his images of the pattern, using the sequence of odd-number base layers to find each total, as further evidence of his continued work at the *Property Noticing* layer [Point 12]. It is important to note here that Alaki probably does not see the numerical pattern in the base layers as “odd numbered”; otherwise, he would likely have reacted to Malakai’s image of “prime numbers”. Certainly, Alaki does not see the connection between his pictorial image of the extras and Malakai’s numerical image. Hence, the two students’ pathways of understanding patterns and relations in the given pictorial sequence are quite different in a variety of ways.

In Alaki’s next attempt to draw the 5th diagram, he finds himself unable to sketch an image for the pattern. This prompts him to *fold back* from *Property Noticing*, where he was currently
In this instance, Malakai finds a relationship between the diagrams' ordinal numbers and their corresponding totals [L22-L26], again illustrating continued working with his numerical images at the Property Noticing layer [Point 14]. He identifies this noticed property of the numerical
totals as “square numbers” [Note 55], then quickly applies it to the 17\textsuperscript{th} diagram [L31], and later verifies the total for the 6\textsuperscript{th} diagram [L34] as evidence of a constructed, organized scheme for the pattern [\textit{property recording}, Point 15]. But while Malakai is referring to the total of the 6\textsuperscript{th} diagram for Question 5 [L34], his statement confuses Alaki [L35], partly because he thinks Malakai means to apply the total for the 5\textsuperscript{th} diagram [Note 56]. The two students then continue using Malakai’s newly found arithmetic rule to find out the total for the 7\textsuperscript{th} diagram, with no evidence that Alaki has yet understood why the rule applies. At this point, they are asked to explain their answer. Malakai simply says, “No need to draw it!” [Note 57] Such a declaration implies, perhaps, Malakai’s readiness to move outward in his growth of understanding to \textit{Formalising}, a progression supported by the students’ answer to Question 8. Malakai claims further to have seen the pattern in the totals, and he shows evidence of a cognitive shift in his mathematical thinking.

However, the difficulty Alaki faced earlier in expressing his pictorial image of the extras appears to cause him to persuade Malakai to defer answering Question 7 about the pattern in the extra number of square blocks to a later time. Instead, the students move on to Questions 9 and 10 – predicting the totals for the 17\textsuperscript{th} and 60\textsuperscript{th} diagrams [Excerpt 17]:

\begin{verbatim}
L40  M  Nine --- [Reads] Can you make any predictions for the 17\textsuperscript{th} diagram?
L41  A, M  Square the seventeen.
L42  M  What number? Seventeen by seventeen.
L43  A  [After doing the arithmetic multiplication in tabular form.]
L44  M  Seven by seven, forty-nine:
L45  A  One by seven --- eleven [add seven and carried four];
L46  --- one, seven, zero.
L47  M  Nine, eighty-two [add down columns from right to left];
L48  --- two eighty-nine.
L49  M  Do that --- seventeen times seventeen --- just the same.
\end{verbatim}
L50 M Square! [Writes as “289. Explanation is, $17^2 = 289$.”]
L51 A Square --- square the seventeenth diagram.
L52 A What number is that?
L53 M Ten --- “Can you group predict the 60th diagram?”
L54 A Sixty by sixty.
L55 A Sixty times sixty --- six by zero is zero --- zero ---
L56 --- six by zero is zero --- six by six is thirty-six ---
L57 --- three hundreds --- three-sixty --- three thousands.
L58 A Three thousands, six hundreds; three thousands, six hundreds
L59 [Finishes calculation in vertical form then writes “square the diagram number by itself.”]
L60 Finish? [Malakai finishes answer as “3600, $60^2$ or square 60”.

In this extract, the students shift back and forth, between reading the question in English, to calculating the totals in Tongan: the language they feel most comfortable in expressing themselves [Note 58]. In particular, the Tonganised word, “sikuea” (square) is used by both students in expressing the rule, “Square seventeen” [L41], but is later borrowed as “square” [L50-L51]. Alaki and Malakai are observed to work at the Property Noticing layer [Point 16], through their application of the arithmetic rule for calculating the totals for the 17th and 60th diagrams. In this case, their collaborative work is illustrated by the intersection of their pathways of understanding about the topic in their group mapping [see MAP 3, Point 16]. Alaki’s understanding of the pattern comes from using Malakai’s understanding of the totals (square numbers), although Alaki has yet to show evidence of seeing any connection between their images associated with the extras. This evidence is represented in MAP 3 (using dotted lines between marked points “13” and “17”) to show how Alaki’s path of growth of understanding is disconnected. While working at the Formalising layer, Alaki explains and translates their answer as “square the diagram number by itself”, a generalized statement of their understanding at the Formalising layer [Point 17] [Note 59]. However, the two students have not generalized the pattern in the extras. They now turn their attention to look at their images associated with the
In this episode, both students rely on their previous constructions to describe the pattern in the extras; hence, they fold back to working at the Property Noticing layer [Point 18]. Alaki, after reading the question [L61], shifts languages to discuss the pattern in the extras in Tongan [L62-L65]. Alaki’s language shifting is not related to his folding back [Note 60]. He uses the Tonganised word, “sitepu” [L63-L64] metaphorically to describe the common difference of two extra square blocks. He follows this explanation through with a translation of his current understanding into English [L66]. However, Malakai reflects on his own prior constructions using the borrowed language he established earlier (Note 61): “You just add the prime number to the last row” [L67]. As he writes down an explanation of their answer, Malakai also draws a
tabular form of the added prime numbers, showing how they are added for the first four diagrams. This tabular illustration shows how Malakai *borrows* the mathematical label “prime numbers” to represent “the last row or the base”. Alaki strongly disagrees with Malakai’s answer [L68], then they both express their respective images, and Malakai uses his image to answer Question 7 – the pattern in the extras: “Add two to the last row or the base is a prime number.”

The two students then move back further to discussing Question 2: How many extra square blocks were added, in addition to the 3rd diagram, in order to draw the 4th diagram? The students show continued work at the Property Noticing layer [Point 19] as they reflect on their previous constructions. Their answer to Question 2 shows a similar response to Question 7, when they are asked to describe the pattern in the extra number of square blocks. While Alaki attempts to explain their answer in English (“Plus two --- add two to the last step and total them”), Malakai says, “Don’t know what the English translation of the words we’ve used!” [Note 62], and the two students wrap up their work for the task.

6.5.4 Christie, Ipeni, and Semi’s growing understanding of patterns and relations in Task 3

This analysis looks at growth of understanding of patterns and relations for the Form 3 students Ipeni (I) and Semi (S), working together with another peer, Christie (C), on Task 3 (see Figure 21 below). MAP 4 shows their mapping. This mapping not only illustrates how Semi is able to express his growth of mathematical understanding through language switching, but it also shows how each student builds on the other’s mathematical understanding through a shared understanding of the mathematical language. When the group receives the Task Sheet, they immediately respond to the pictorial sequence [Excerpt 19]:

Hey, those are steps! [Points to the given diagrams.]

No --- one, two, three, four, five --- [Counts number of blocks.]

--- Nine there [Uses pen to point to the 3rd diagram] --- it's square! [Pause]

[Uses pen to point to each diagram] One by one is one; two by two is four; three by three, nine.

[Semi points to 4th diagram] --- four by four, sixteen; five by five, twenty-five.

Look at this [points pen to each of the first three diagrams]: one, four, nine --- it's square.

[Reads] Draw the fourth --- do [draw diagram] Christie!

Fourth? Sixteen? [Voice-tone reflects need for confirmation & uncertainty in what to do.]

That's four there --- [Sketches middle column to indicate vertical arrangement of blocks.]

--- and how many there? [Sketches line to indicate next column.]

--- that four there [indicate middle column again] ---

--- and add with a one --- [That is, put another one on top for the next diagram.]

--- and all the way down to bottom. [Points to another one on top, all the way down to the side.]

One --- [Appears to point to the far side, moving his pen in as he slowly sketches.]

Fourth?

--- and three --- [pause].

Four up, and three, and two, and one down? [Again with a tone for approval.]

Just four and like this: that's four there; then three; then two.

[Sketches 4th diagram, focusing on the central column, and going down the side.]

Ipeni’s spontaneous reaction to the given pictorial sequence reveals one of the existing images he associates with the pattern. He looks over to the given pictorial sequence and grabs the Task Sheet out of Semi’s hand, stating, “Hey, those are steps!” [L1]. His use of the Tonganised word, “sitepu” (step), is a metaphorical reference to the structural pattern of the diagram, shaped by the corner blocks around the edges (see Figure 21). Ipeni then pauses and says, “No” [L2], before starting to count the total number of square blocks [L2], indicating his cognitive shift from perceiving an instant pictorial image of the pattern (image seeing) [Point 1], to folding back and working through counting in constructing an image (image doing) [Point 2] for the pattern. This invocative action is not related to any act of language switching [Note 63].
Figure 21: Ipeni immediately recognizes the staircase pattern of each diagram as "sitepu" (steps)

At this point, Semi watches attentively, while holding a pen, which he places on the Task Sheet, sitting in front of Ipeni. The movement of Semi's pen makes it appear as though he is counting the number of square blocks in the 3rd diagram. When he appears to be done counting the 3rd diagram, he quickly adds, "Nine there --- it's square!" [L3] Semi's declaration shows that he has constructed a numerical image associated with the totals, which he recognizes and labels as part of the "square" numbers (image seeing) [Point 3] [Note 64]. Semi simply lifts up his pen and moves from one diagram to another as he says, One by one is one [1st diagram]; two by two is four [2nd diagram]; three by three, nine [3rd diagram] --- [L4]. There is no double-movement in his pen to indicate that he is relating the diagram number and the total for each diagram; otherwise he is said to be moving out to Property Noticing. However, Christie picks up on this relationship and recites the sequence further for the 4th and 5th diagrams by adding, "Four by four, sixteen; five by five, twenty-five" [L5]. Both students express the property of the square numbers in Tongan [Point 4] [Note 65]. Semi then goes back, this time at a slower pace, by reflecting on his earlier counting by pointing out to Ipeni the total number of square blocks in the first three diagrams: "One, four, nine --- it's square." [L6] Semi sees a numerical image in the totals (1, 4, 9), associates it with the label "square" numbers, then expresses a property of the numbers, but not the image, which would have identified him as moving out to work at Property Noticing. The conclusions that can be drawn from this episode, and throughout the evidence shown on videotape, is that none of the students appeared to be thinking of the diagram numbers and their connection to the totals.
In the next sequence of events, Semi appears to read the first question, then turns to Christie and asks her to draw the diagram associated with the question [L7]. Christie’s reply [L8] suggests she is not so sure how the diagram is going to be drawn. Christie recalls the total for the 4th diagram [L8], but she needs affirmation from her peers about the appropriateness of her total. Semi turns to Christie and sketches his picture of the 4th diagram [L9-L14]. He sketches a vertical line indicating his focus on the middle column, rather than the sequence of numbers. He says, “That’s four there ---” [L9] then goes on to add, “--- and how many there?” [L10]. Semi’s sketching appears to mean that the next column to the side of the middle column has three square blocks. He goes back again to point out the middle column with four square blocks [L11]. He says, “And add with a one --- and all the way down to bottom” [L12-L13], as though Semi is actually suggesting a pattern involving decreasing increments of one square block. Throughout this sequence of events, Semi has the pictorial image of the next pattern as follows: put one on the middle column and one on each of the other columns, going all the way down the sides to the bottom (see Figure 22).

![Figure 22: Christie’s initial description of the 4th diagram as descending columns of square blocks](image)

Hence, Semi demonstrates continued working at the *Image Having* layer (*image seeing*), and he is able not only to extend his image for the pattern but also able to articulate his image (*image saying*) [Point 5]. Semi does not see his pictorial image as Alaki did (Section 6.5.3) along the base layers. In this example, Semi sees how to build the next column, but does not relate it to anything other than the previous column. While Christie waits for him, Semi continues, using his
pen on the table, as he sketches (and seemingly counts at the same time) the number of square blocks in each column, starting, it looks, from his far-right-side column, going in toward the middle column. At the same time, he slowly says out loud, with pauses in between, “One ---” [L13], and Christie asks “Fourth?”, and Semi continues, “and three ---” [L14]. Semi’s slow approach to reading the totals [L14, L16] does not seem to mean he is unable to see an image, but he is probably trying to compare his numerical image (square numbers) with the drawing. But while Semi pauses as he checks his work, Christie appears to see the pictorial image for drawing the 4th diagram [L17], and that is when Ipeni intervenes [L18] by showing Semi and Christie a drawing of the 4th diagram [Point 6], similar to the diagram Semi may have envisioned [Note 66].

For the next part of the task, Semi goes on to read Question 2, and Ipeni answers that drawing the 4th diagram requires seven extra square blocks in addition to the 3rd diagram. Ipeni recalls the question – how many extra square blocks in addition to the 3rd diagram did you draw (for the 4th diagram)? – and emphasizes the keyword “extra”, explaining the term in Tongan as, “how many blocks you’ll be drawing for the 3rd diagram”. Ipeni goes on to answer Question 3 by listing the numerical totals for the first four diagrams “One there, four there, nine, sixteen --- write: first diagram, one; second diagram, four; third diagram, nine; fourth diagram, sixteen”, as he shifts languages. Evidence from this incident points to Ipeni’s continued working at the Image Having layer with his pictorial image [Point 7].

But when Semi reads Question 4 about predicting the total for the 5th diagram, he quickly disconnects himself from working with pictorial images to working with numerical images by applying his earlier numerical image of the totals (square numbers). Semi immediately responds, “twenty-five”, and when Christie asks if his response refers to the total number of square blocks,
Semi raises his voice and says, “twenty-five”, as if to affirm his prediction. Then, when the students read Question 5 (the prediction for the 6th diagram), Ipeni immediately answers “thirty-six”. Christie asks through repetition, “Where did that thirty-six come from? Where did you get that thirty-six?” [Note 67]. Ipeni answers, “Square!” In this situation, Ipeni’s immediate answer suggests that he is associating each square number with the diagram number, rather than a recitation of the sequence of square numbers. In this situation, both students move out to Property Noticing, and in the case of Ipeni, he borrows the word “square” to express his rule [Point 8] [Note 68]. When the students are asked if they need to draw the 6th diagram, in order to determine how many square blocks are needed, they answer, “No”, and Ipeni explains, “because...we already know it.” The students move on to make a similar prediction about the total for the 7th diagram, when they are prompted to do so in Question 6. Immediately after reading the question, Semi says, “forty-nine ---”, an indication of the students’ ability to free themselves from having to draw each diagram to abstracting a numerical total, showing a possible readiness to generalize their mathematical property (or different properties).

The students’ “readiness” to move out to Formalising later becomes evident when Semi goes on to read Question 7 about the pattern in the number of extra square blocks. Both Christie and Ipeni misunderstand the question, assuming it is asking them to generalize the pattern in the numerical totals. Christie quickly explains the generalized formula using the rule, “Yes, we square the diagram. Just yes as the short answer. Square the diagram”, while Ipeni answers with, “Square the number” (of the diagram?). Hence, these two students’ responses indicate a shift in their mode of understanding toward Formalised ideas [Point 9] [Note 69]. However, Semi reminds them of the question: “Look there: do number eight later but that one: can your --- extra --- is extra square --- . Semi’s reminder, and the fact that Christie and Ipeni have not explored explicitly the pattern in the extras, moves their mathematical thinking to reflecting on their image.
of the additional squares; thus Christie and Ipeni fold back to working with their images at the Property Noticing layer [Point 10]. Again, language switching is not involved in this incident [Note 70].

**MAP 4: Semi’s and Ipeni’s growing understanding of patterns and relations in Task 3**

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<td>ppr</td>
</tr>
<tr>
<td>1</td>
<td>Ipeni sees pictorial image as “sitepu” (step). [Note 63]</td>
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<tr>
<td>2</td>
<td>Ipeni folds back to construct pictorial image.</td>
<td></td>
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<tr>
<td>3</td>
<td>Semi sees numerical image in the totals as “square numbers”. [Note 64]</td>
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<tr>
<td>4</td>
<td>Students reflect on previous construction.</td>
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<td>5</td>
<td>Semi continues working at Image Having.</td>
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<tr>
<td>6</td>
<td>Ipeni intervenes with description of pictorial image. [Note 66]</td>
<td></td>
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<tr>
<td>7</td>
<td>Ipeni continues working at Image Having.</td>
<td></td>
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<tr>
<td>8</td>
<td>Ipeni and Semi borrow “square” and applies to prediction.</td>
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<tr>
<td>9</td>
<td>Ipeni generalises pattern in extras. [Note 69]</td>
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<tr>
<td>10</td>
<td>Ipeni and Christie fold back to work on totals.</td>
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<tr>
<td>11</td>
<td>Ipeni continues working at property noticing. [Note 71]</td>
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<tr>
<td>12</td>
<td>13 Semi applies rule to 17th diagram. [Note 72]</td>
<td></td>
<td></td>
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<tr>
<td>13</td>
<td>“Just square the diagram number”.</td>
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</tbody>
</table>

Pathway of Ipeni’s growing understanding of patterns and relations traces through:  
*Points 1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14.*

Pathway of Semi’s growing understanding of patterns and relations traces through:  
*Points 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14.*
In response to Semi’s challenge (that is, Question 7), Ipeni immediately explains the pattern in the extras: “That question there is a yes --- look there: every diagram, you add to it an odd number, but it’s going up. Look here: three, add to that [2nd diagram] a five --- add to that [3rd diagram] a seven --- add to that [4th diagram] a nine”. Ipeni’s response appears to be a formal mathematical statement. His statement, “every diagram, you add to it an odd number” seems to be a general statement about the numerical pattern in the extras.

But Ipeni’s reliance on his previous constructions indicates he is not totally disconnected from specific actions and images, and therefore he cannot be characterized as crossing the “don’t-need” boundary to work at the stage of Formalising. Hence, Ipeni is observed to continue working at the Property Noticing layer, where he demonstrates his ability to express a pattern in his numerical images [property recording, see Point 11]. Interestingly, Ipeni explicitly discussed the pattern in the extras just once before, during his explanation of the difference between the 3rd and 4th diagram as seven square blocks. However, the absence of Ipeni’s expression of the numerical pattern in the extras as a sequence of “odd numbers” suggests such a sequence may have occurred to him before; his actions show he must have internalized the numerical pattern in the extras with regard to the added base layers. So when Christie asks about an explanation for the pattern in the extras, Ipeni responds, “As the diagram goes up, odd number goes too ---” as evidence of his ability to articulate (through substitution) features of his noticed property [property recording, Point 12] [Note 71]. Christie then writes down the group’s answer accordingly, by expressing the group’s organized scheme for the pattern.

As the task continues, Semi then reads Question 9 to predict the total for the 17th diagram, and Ipeni quickly responds, “Square the seventeen --- seventeen times seventeen.” Christie reads the same question and writes the product as “17² = 17 x 17 = 289”, then explains “Just square the
number of diagram.” With this general statement, the students are reiterating their Formalised ideas [Point 13]. Moreover, Semi turns to answer Question 10 by predicting the total for the 60th diagram, and he immediately predicts the total to be, “Sixty square --- three thousands, six-hundreds. Sixty square --- three-six-zero-zero” [Note 72]. Christie writes the same explanation of their Formalised rule [Point 14] – “Just square the diagram number”.

6.5.5 Meki, Nanasi, and Rosina’s growing understanding of patterns and relations in Task 3

This section represents the first of a set of brief analysis reports without the complete mappings. The following analysis focuses on the Form 2 group of Meki, Nanasi, and Rosina. This example demonstrates how a lack of language understanding and a lack of mathematical understanding constrain the students’ growth of mathematical understanding, because of one student’s (Nanasi’s) interpretation of the term “prediction”. Once the group is given the Task Sheet (refer to Figures 16 and 21 for Task 3), Rosina immediately starts counting each of the given diagrams in Tongan. While they initially work at Image Making, all of the students shift from reading the question in English to working in Tongan. As they continue with Question 2, the students momentarily shift languages to using only English [Note 73]. When they re-read Question 2, both Rosina and Nanasi immediately answer, “We plus seven!” and “Seven extra square!”, respectively. But when asked in Question 3 to find the total number of square blocks in each of the first four diagrams, Nanasi slightly misunderstands the request and counts instead (in English) the total number of square blocks altogether, for all four diagrams, to get 30 [Note 74]. Despite Nanasi’s language barrier, she and her group move forward to work at the Image Having layer. As they continue with the task, the students then quickly move to Question 4 to predict how many square blocks will be needed for the 5th diagram. Rosina reformulates the question, saying, “So our predict or what is how many square blocks ---”. Meki asks, “Is diagram fifth
what we need to predict?” Nanasi answers, “Guess! Mine’s twenty-one. How many is yours?”, to which Rosina replies, “Ua-fitu --- twenty-seven!” Meki appears to disagree with the way the other two students answer the question, but retreats to Nanasi’s demand with her own random guess of 20 square blocks [Note 75]. Rosina then reads the second part of the question, and rephrases it: “To draw --- it says we draw and check --- who will draw?” She then directs Nanasi to draw the 5th diagram but Nanasi appears uncertain about how the diagram should look, thus prompting her to fold back, without the involvement of language switching, from Image Having to Image Making [Note 76]. However, Meki intervenes and explains in Tongan the structure of the diagram (image saying), saying, “Look here: Draw this way: a five going straight; next is a four --- a four, a three; a two; a one ---” (see Figure 23).

![Figure 23: Meki’s pictorial image as a stack of vertical columns of square blocks](image_url)

Nanasi then draws the 5th diagram to indicate 25 square blocks. Immediately, Nanasi turns to Rosina and congratulates her for making the closest guess, which in this case, is the nearest total for the 5th diagram, “You are the one [with the right guess] --- twenty-five!” [Note 77]. The group then read Question 5, asking them to discuss the 6th diagram. Rosina counts the base layer of the 5th diagram (in English) to find nine square blocks, then she adds, “So plus eleven” (for the 6th diagram). This evidence points to Rosina’s ability to extend aspects of her numerical image for the pattern (along the base layers), and demonstrates continued work at the Image Having layer. The group goes on to answer Question 6 – discussing the 7th diagram. In this instance, Meki first offers a strategy of dividing up all the questions into three parts so that each of them
can solve the problems and write the answers separately in order to speed the process. Nanasi, however, reminds Meki, "But the prediction is a must for all of us to participate." Meki agrees: "Yes, except the prediction only ---" [Note78].

Later, when Nanasi continues with Question 8 by discussing any pattern in the total number of square blocks used in each diagram. Meki, however, is still confused by their previous pictorial construction for the 7th diagram. So Nanasi reflects on their prior constructions and explains to Meki (in Tongan), using substitution for such phrases as "fifth diagram", "sixth diagram", "seventh diagram", and "peteni" (pattern). Then, after re-reading Question 8, Nanasi shifts languages to explain the structure of the 7th diagram in Tongan [Note 79].

Nanasi then goes back to re-read Question 8, and asks Rosina to explain what the question means. Rosina explains, "It means whether we see the pattern --- the totals there ---.", pointing at the pictorial sequence. Nanasi asks, "So we add them all together?", and Rosina answers, "No! It asks what is the pattern?" Then Nanasi responds, "It's plus the odd numbers." Meki then re-affirms their answer to Question 7 and writes, "We add the odd numbers." The students appear, therefore, to continue to work at the Property Noticing layer [Note 80].

Meki (M), Nanasi (N), and Rosina (R) then move to Question 9 to predict the total for the 17th diagram, which leads to the following exchanges [Excerpt 20]:

L1   N  My prediction is forty-four [changes mind] --- forty-five.
L2   R  Mine is fifty-one. [Meki appears to mentally calculate prediction.]
L3   N  Meki, guess before you work it out. Don't do it like that.
L4   R  First do your prediction --- before you do it like that.
L5   N  Do your guess --- you, you always begin by doing it [calculation] ---
--- and your prediction is wrong. Do your prediction so that we add that.

[pause] --- thirty-nine!

What is our explain?

Explain our answer.

All of --- they are all odd numbers.

Explain your answer. What does explain mean?

After all, they tell us there to make a guess ---

--- and then they asks again for explanation of our answer.

[referring to the Task questions] is wrong ---

because I believe his copy is wrong ---

--- maybe there is another question there to draw again and check.

As they have with earlier diagrams, Nanasi and Rosina choose to randomly pick a total for the 17th diagram. Nanasi answers, “My prediction is forty-four --- [changes mind] forty-five” [L1]. Rosina replies “Mine is fifty-one” [L2]. But Meki appears to mentally calculate her prediction. Nanasi takes note of Meki’s mental calculation and urges Meki to “guess before you work it out!” [L3-L4]. Meki then offers her guess as, “thirty-nine” [L7]. Nanasi and Rosina attempt to explain their guesses [L8-L9], to which Nanasi suggests, “they are all odd numbers” [L10]. However, Meki expresses confusion about whether the question was meant to ask about random guessing. Meki maintains that it would not make much sense if the question asks to guess a random number and then goes on to explain the guessing [L11-L13]. In other words, prediction, for Meki, involves what she was trying to do earlier – make a prior mental calculation – a different act than the kind of random guessing Nanasi was suggesting. However, Nanasi still insists, “We just guess!” [L14] and continues, “We just say that ‘We guess it!’ because I believe his copy [the researcher’s question] is wrong; maybe there is another question there to draw again and check” [L16-L18]. Nanasi’s comment suggests she expects the questions to be similar to Question 4, in which the students were asked to predict first, draw, and then check their
answers. Nevertheless, Meki stops probing the meaning of the question, and all three students randomly guess the total for the 17th diagram to be 51, 45, and 39, and then write explanations for their prediction with the statement, “We guess it!” They go on to repeat their approach in answering Question 10 by guessing 171, 221, and 201, for the total of the 60th diagram, without actually calculating the total. Throughout this incident, the students, apart from their generalization of the pattern in their images of the extras, continue to work at the Property Noticing layer. Their translation of the meaning of prediction appears to restrict their mathematical thinking to random guessing, and hence, does not allow them to make the generalization of the pattern in their images of the totals [Note 81]. A further analysis of this situation is discussed in the next chapter.

6.5.6 Malia and Tupu’s growing understanding of patterns and relations in the Tongan Task

In this analysis, Malia’s growth of understanding of patterns and relations is mapped, based on her group work in the Tongan Task33 (see Figure 24).

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<th>3rd diagram</th>
<th>4th diagram</th>
<th>5th diagram</th>
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<tbody>
<tr>
<td>Continuing pictorial sequence</td>
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**Figure 24**: Continuing pictorial sequence of square blocks in the Tongan Task

It is important to note that this task was given after the students had completed the English tasks. During their discussion of the task and through language switching, the students used words that

---

33 All of the text and questions in this task were posed in Tongan (see Appendix 2 for a copy of the task).
were relevant and were probably, as the evidence shows, influenced by the text of the English tasks. In this case, the group spends about two minutes on the task, discussing in Tongan how to draw the 4th diagram. Then Malia suddenly shifts to say in English, “Add three --- add three. One plus three equals four. Four plus three equals seven. Seven plus three equals ten.” She looks over to Tupu and says in Tongan “Look?”, pointing at the pictorial sequence, and Tupu nods her head in approval. Malia has therefore moved over from acting, in seeing the common difference between consecutive diagrams to image saying in expressing aspects of her image [Note 82].

Tupu reads Question 2 – “Ko e ha e fo’i poloka tapafa-tatau makehe ‘e fiha (tanaki atu ki he fakatata fika 3) Na’a ke taa?” – asking about the extra (“makehe”) square blocks added to the 3rd diagram to find the 4th diagram. Malia immediately associates the question with a word she recalls from the previous (written English) tasks [Note 83]. She asks, “So that is the extra, right?” Tupu adds, “So the extra is three.” Malia agrees it is “three blocks”, a confirmation of her image (image seeing) and her ability to express an aspect of it (image saying). The two students’ substitution of the word “extra” serves a particular purpose as a keyword that the students access in reference to their construction of an image.

The students continue for the next four minutes to discuss the task in Tongan as they extend their image for the pattern to find the totals for the 5th, 6th, and 7th diagrams. This extension involves adding three each time to the previous diagram to get the next diagram: their mental plan for calculating the totals. In discussing the 4th to the 7th diagrams, the group is observed to move freely between image seeing to image saying without language switching [Note 84]. Then, when the students move on to answering Question 7 – the pattern in the extra number of square blocks being added each time – Malia begins by reflecting on some of the totals, and when Tupu, asks, “What’s the pattern?”, Malia verifies with Tupu the meaning of the question by asking, “So that is the extra, right?” Malia’s substitution of the keyword “extra” identifies specific aspect of their
images – the extras – that they need to focus upon to determine the pattern [Note 85]. Thus, both Malia and Tupu agree that the extra number of square blocks is three. Kelela writes their explanation in Tongan as the group’s answer to Question 7. The students continue reflecting on their previous constructions at the Image Having layer.

The group then reads Question 8 and discusses the pattern in the total number of square blocks used in each diagram. Previously, the students had constructed an image for the pattern using the number of additional square blocks (extras). In this case, Tupu reflects on the total, beginning with four square blocks for the 2nd diagram, and leading up to 16 for the 6th diagram. The following exchanges then occur between Malia (M) and Tupu (T) [Excerpt 21]:

L1  T  Hold on. How is it? That's four there [2nd] and seven there [3rd] ---
L2  --- and ten there [4th], and thirteen there [5th], and sixteen there [6th].
L3  So they are odd numbers.
L4  Is there a multiple? There is no multiple involved, right?"
L5  M  There is no multiple of anything.
L6  T  Multiple of two?
L7  M  No, it is even --- even, odd, even, odd, even --- it's alternating.
L8  T  It's even numbers.

In this case, both students are recalling their Primitive Knowing associated with various numerical properties, such as “multiple”, “even”, and “odd”, in order to come up with an explanation for the pattern in the totals. Malia continues explaining the pattern to Tupu as an alternation between even and odd numbers, “until it finishes”, while Tupu writes down his answer in Tongan [Note 86].
In working on Question 9 – the prediction for the 17th diagram – Malia and Tupu proceed to work at the Property Noticing layer. This time, the two students attempt to manipulate their images – a set of counting numbers and an ordered set of totals – to come up with a rule for calculating the total for the 17th diagram. In the process, both students exchange ideas in Tongan until Malia suddenly substitutes the word “plus” in order to explain her reasoning. Malia uses a set of numbers to construct a rule for generating the diagrams’ totals and she explains the relationship: “So it’s like this: two by two --- so there is no addition; three times two is six, plus one; four times two, plus two; five times two plus three.” Throughout their calculations of the totals for the 17th and 60th diagrams, Malia and Tupu do not state any general statement about their rule or express any Formalised understanding as to how to calculate the totals. Because of the absence of a generalized rule, and the evidence of these students’ continued reliance on using specific numbers, it suggests the students have been continually working at the Property Noticing layer [Note 87].

6.5.7 Niko, Pola, and Seini’s growing understanding of patterns and relations in Task 3

This brief analysis report focuses on a group of Form 2 students, Seini, Niko, and Pola, and their work in Task 3. In the first 12 minutes of this group’s work, the students discuss how to construct the 4th, 5th, 6th, and 7th diagrams. During this period, the students count the total of each diagram and construct numerical and pictorial images of patterns and relations. These images are mainly associated with finding a way to add appropriate base layers of square blocks to build the sequence from one diagram to another. In order to complete this task, the students find that the common difference between successive base layers is two (square blocks) – the image they use in constructing the diagrams (see Figure 25).
The students are observed to progress slowly from working at *Image Making* to working at the *Image Having* layer. While most of the group’s discussions are in Tongan, the group occasionally switches languages but their actions are observed be unrelated to their move between and within their work at the *Image Making* and *Image Having* layers [Note 88]. Moreover, the group figures out that the numerical set associated with the number of square blocks in the bases of the first seven diagrams – that is, 1, 3, 5, 7, 9, 11, and 13 – corresponds to the set of odd numbers. In determining the answer for the pattern in the extras (Question 7), Seini explains, “pattern --- what is the addition? Addition is odd number.” Seini’s substitution of the words “addition” and “odd number” indicates her ability to recognize specific features of her numerical image as she moves out to working at the *Property Noticing* layer [Note 89]. The group – Niko (N), Pola (P), and Seini (S) – then moves to answer Question 8, to discuss the pattern in the totals, and Seini begins by reading the question aloud [Excerpt 22]:

```
L1  S  [Reads] Can your group see a pattern in the total number of square blocks in each diagram?
L2  S  Just hold one. Look here: it’s one, four then what? It’s one, four, nine, sixteen. What is it?
L3  N  (Niko: Which one?) Pattern in total number.
L4  S  One, four, nine, sixteen, twenty-five.
L5  N  Add that all together?
L6  S  No, what is their pattern (“peteni”)?
L7  N  [Writes] “1, 4, 9, 16, 25”.
L8  S  What is that pattern (“peteni”) there?
L9  N  Which pattern (“peteni”)?
L10 S  That one there: one, four, nine, sixteen.
```
In this instance, Seini, after recalling the totals for the first four diagrams [L2], asks for a mathematical label to describe the pattern in the numerical totals [L3, L4, L6, L8]. Niko is the first to suggest the pattern is an alternation between “fika tauhoa” (even numbers) and “ta’etauhoa” (odd). His use of the Tongan words is interesting because most of the students in the study mainly used the English words [Note 90]. Nevertheless, Niko appears to recall the term, “composite number” (L13), although he later admits he initially forgot the meaning of the term. But when Seini asks about the term (L14), it triggers Pola to access his Primitive Knowing in defining a contrasting borrowed term, “prime”, using the Tonganised borrowed word, “fakitoa” (factor) [L15]. Pola’s example immediately prompts Seini to access her Primitive Knowing in defining the term, “prime number” [L16-L17] [Note 91]. Pola then uses Seini’s primitive knowledge to finally express examples of composite numbers [L18]. As a result, Pola proceeds to write their answer for the pattern in the totals as “1, 4, 6, 19, 25. Composite numbers.” [Note 92] The group moves on to find the totals for the 17th and 60th diagrams by repeatedly adding appropriate sets of consecutive odd numbers as their rule. Their reliance on each specific diagram number in order to determine the set of odd numbers involved suggests that the students have not progressed over the don’t-need boundary to Formalising.
The next brief analysis report involves a group of three Form 2 students, Samu, Naati, and Siona, while they work on Task 3. This example illustrates students moving out of the topic and discussing issues associated with language that are unrelated to their current mathematical understanding. These students start by counting each of the given diagrams in Tongan. Samu leads the group, and appears to do most of the work. He constructs a numerical image along the base layers in the order 1, 3, 5, and 7, for the first four diagrams, determined by the common difference of two square blocks. Naati and Siona appear to follow along with Samu’s work, and they move on to draw, using Samu’s pictorial image of the extras, the 4th and 5th diagrams correctly. At this time, the students are observed to be working at Image Having. They construct a numerical and pictorial image for the pattern along the base layers (extras), and as a result, they are able to draw the 4th and 5th diagrams correctly with 16 and 25 total square blocks, respectively. Then, the group is asked, in the second part of Question 4, how they compare their prior calculation of the total for the 5th diagram with their drawing. Samu recites his explanation as he writes (in English): “Plus odd number according to the introduction that gives on the paper.” According to the students’ follow-up interview, Samu’s use of the phrase “introduction that gives on the paper” refers to the first three diagrams in the given pictorial sequence. It turns out that Samu was unsure about the appropriate use of the word “introduction”, and so he asked his peers about its meaning. Samu’s (Sa) query led to a “heated” discussion between Naati (N) and Siona (Si) [Excerpt 23]:

L1  Sa  What does introduction mean?
L2  Si  Instruction --- introduction is instruction.
L3  Sa  The introduction?
The analysis of the preceding example indicates these students switched languages (and engaged in group discussions) to move out of the topic and to discuss an issue in language, unrelated to their mathematical work [Note 93]. The group then proceeded with their current work at the Property Noticing layer.

6.5.9 Malia and Tupu's growing understanding of patterns and relations in Task 4

In this section, the same Form 3 group of Malia, Tupu, and Kelela are analyzed for their growing understanding of patterns and relations in Task 4 (see Figure 26). This is another example of
how students’ language switching evokes inappropriate images, thus leading the students on a
different path of understanding.

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<th>4th diagram</th>
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Figure 26: The continuing pictorial sequence in Task 4

With this task, the group is observed to progress smoothly in their understanding of the topic,
beginning with their initial work at the Image Making layer. Language switching is frequently
involved during the students’ work. As the students build the 4th to the 7th diagrams, they identify
various pictorial and numerical images for the topic – evidence of their working at the Image
Having layer. In particular, the students see the numerical image for the pattern in the extras as a
set of “counting numbers” (1, 2, 3, 4, ...), which they easily arrange pictorially in descending
order from left to right [Note 94]. The students use this numerical image to answer Question 7 by
describing the pattern in the extras as, “You add the counting number to the number of the
diagram to get the sequence” [Note 95]. Then, after determining the totals for the first four
diagrams, the students recognize the pattern in the totals as “triangular numbers” (1, 3, 6, 10,
15...): their answer to Question 8 [Note 96]. At this point, the students have moved out in their
understanding of patterns and relations to working at the Property Noticing layer. The group –
Kelela (K), Malia (M) and Tupu (T) – then proceeds to answer Question 9, asking if they can
make any predictions for the 17th diagram [Excerpt 24]:

L1 T Oh, you just predict --- guess!
L2 M Prediction! Plus how many?
You just times three because it's triangular number.

That's why we guess it, so it's not.

Seventeen time [with Tupu] three equals?

Three times three equal nine.

No! Three times seventeen. [Writes calculations for 17 x 3.]

Three by seven, twenty-one --- three by one is three.

Three times seven equal twenty-one.

Hold one, carry two; three by one [appears to mentally calculate product] ---

--- five --- fifty-one.

Explain your answer. Because --- it's triangular number. What we do is --- [pause].

Three is a --- [pause].

Three is a tri --- [pause].

Triangular number?

That three, it means the three-sidedness of the triangle. [Draws a triangle on the table.]

Yes, because three is how many sides a triangular have.

Three is the total sides of a triangle. [Kelela writes explanation for their answer.]

Tupu immediately associates “predict” with “guess” [L1], a comparison she makes for the first time, unlike her assumptions during previous work on the other tasks. Yet, in this situation, Tupu appears to mean “prediction” in the sense of “approximation” rather than random guessing [L4] [Note 97]. Malia, however, appears to think about how many counting numbers they would have to add up to get the total for the 17th diagram [L2], a method the group had used earlier in calculating the 4th to the 7th diagrams. However, Tupu’s use of the word, “triangular” leads her to formulate a rule, a way of “guessing” an approximation for the total [Note 98]. Tupu immediately applies this rule to calculating the total for the 17th diagram [L7-L11]. In explaining their answer, Malia replies, “That three it means the three-sidedness of the triangle” [L16]. Tupu supports this explanation by adding, “Because three is how many sides a triangular have” [L17].

Because the recording of this particular incident ends at that point, insufficient evidence is available to claim that this group did indeed Formalise their new understanding. They used a
specific diagram for their newfound rule, without expressing that rule explicitly as a general rule for all diagrams. Hence, in this incident, the group is observed to continue working at the Property Noticing layer, very close to moving across the don’t-need boundary toward Formalising.

6.5.10 Hehea, Lani, and Leisi’s growing understanding of patterns and relations in Task 1

In this final, brief report, the Form 2 group of Hehea, Lani, and Leisi are working on Task 1 (see Figure 27). This example shows two students disagreeing about the meaning of the question for Task 1, and the confusion that arises from their differing interpretation of the question, originating from these two students emphasizing separate sections of the question and its wording.

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<th>3rd diagram</th>
<th>4th diagram</th>
<th>5th diagram</th>
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<tr>
<td><strong>Continuing pictorial sequence</strong></td>
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**Figure 27:** The continuing pictorial sequence in Task 1

This group begins the task by discussing Question 1, asking them to draw the 4th diagram in the sequence. Hehea states that she does not know the meaning of the word “sequence”, but she takes it, as Lani guesses, to mean the given set of pictorial diagrams in Task 1 [Note 99]. The group is observed to work initially at the Image Making layer in constructing the 4th diagram for the given pictorial sequence. Based on the first three diagrams, Lani draws the correct 4th diagram with seven square blocks in total. In this extract, the group – Hehea (He), Lani (La), and
Leisi (Le) – then moves on to discuss Question 2 regarding the additional number of square blocks they have drawn in addition to the 3rd diagram [Excerpt 25]:

L1  La  So it’s: How many square blocks did you draw? How many --- seven!
L2  He  Look at this: How many extra --- extra --- may be the meaning of extra is that new “thing” that you added again there.
L3  He  Look here: that is in addition to the third diagram did you draw?
L4  La  I think may be the last three, will be one, two, three; sixth diagram --- isn’t it?
L5  Le  That’s five there!
L6  La  Look here: one, three, five, seven, nine --- eleven --- it’s two!
L7  He  That’s it, we meant it’s plus two.
    Meaning, look here: we see that one --- which is adding one.
L8  La  So it means they already draw those three, and then we’re supposed to ---
    draw the next three --- so that is eleven.
L9  He  How many extra --- that’s the thing that --- we have not found --- extra square blocks,
    that is in addition to the third diagram.
L10 Le  The extra?
L11 La  Did you draw --- so it means: it’s up to us --- to draw the three diagram there.
L12 He  That’s not the important thing. The important thing is the extra --- extra square blocks.
L13 La  Extra --- right. We need to explain extra.
L14 He  It refers to the number there that is --- added to the first diagram to get the fourth ---
    --- or does it means the --- or how many?
L15 He  The extra maybe meant the additionsals.
L16 La  Hold on! Leave it there so that we move on.
L17 He  How many square blocks are there in total --- it means what you have added on.

When she is prompted to do so in Question 2, Lani answers correctly at first, saying there are “seven” additional square blocks for the 3rd diagram [L1]. At this moment, Hehea is observed to focus on the word “extra” in the question, and her emphasis on the word eventually shapes her mathematical thinking. Not only does she extract the word “extra” from the question and use it in her Tongan discourse, she uses it as the basis for her correct interpretation of the question [L2-L3]. Hehea points out and reads the first part of the question, focusing on the word “extra”, and
saying that the term means to her that the new blocks are being added to the sequence. But when Hehea pauses, then again continues reading the second half of the question [L3], her interpretation conflicts with Lani’s translation of the question’s emphasis: to Lani, the question is asking them to draw the next three diagrams in the pictorial sequence [L4]. While appearing unaware of Leisi’s substituted statement [L5], Lani reflects on the numerical totals for the diagrams, and she is able to see the difference of two square blocks between consecutive diagrams [L6]. This last piece of evidence shows that Lani is constructing an image for herself about the pattern in the extras.

In the next part of the analysis, Hehea appears to agree with Lani by substituting “plus two” in her explanation for the answer [L7]. Still, Lani insists that the question means they are to draw the next three diagrams in the sequence [L8]. Lani then predicts the total number of square blocks for the 6th diagram to be 11 [L8]. However, Hehea insists that they have not found the “extra” square blocks, or determined how many square blocks must be added “in addition to the third diagram” [L9]. Leisi steps in and asks Helilala what she means by “extra” [L10]. Lani’s response [L11] indicates her interpretation and continuing focus on drawing the next three diagrams in the sequence. It is at this point that Hehea explains that the extracted word “drawing” is “not the important thing ---[but] (pointing at the word “extra”) that is the important thing, the extra – extra square blocks” [L12]. Lani agrees that they ought to explain what the word “extra” actually means in the context of the question, to which Hehea offers her explanation [L14-L15]. Lani appears to “give up” momentarily and suggests moving on instead to the next question, to which Hehea agrees, and she goes on to read the next question [L17]. The students’ confusion arises from their interpreting the question differently by focusing on different parts of the question’s wording [Note 100].
As the group moves on to find the totals for the 5th, 6th, and 7th diagrams, Hehea is able to come up with a plan for calculating the totals by using their image for the extras. Hehea is able to make the connection between the diagrams’ totals and the structural pattern of the sequence. Hence, the students have moved out to work at the *Property Noticing* layer. In this situation, Hehea finds that in order to calculate the total of each diagram, all the group needs to do is to add the diagram number (representing the longest side of the diagram) with one less than that number (representing the shortest side of the diagram) [*Note 101*]. *Figure 28*, below, illustrates the structural pattern and numerical rule Hehea uses in calculating the total for the 5th, 6th, and 7th diagrams. The students explain their answers for the 6th diagram by saying, “We need to know, to see the way it stands, and to know how many blocks will be needed to add to the next diagram.” Thus, their explanation reveals the structural dependency of their pictorial image. At the same time, their final explanation of the total for the 7th diagram is: “We have to add two square blocks for each diagram.”

![Figure 28: Hehea’s structural rule for constructing each diagram](image)

The group then moves to answering Questions 7 and 8 about the patterns in the extras and the totals. During their discussion, they decide to say that as the sequence increases, the total number of square blocks increases. They are observed to continue working at the *Property Noticing* layer. Quickly, the students move on to Question 9, asking them to predict the total for the 17th diagram. Hehea immediately suggests, “It must be seventeen the bottom and sixteen the top ---”, an explanation that reflects their structural image for the pattern. Immediately, Lani suggests a
very different rule that she has deduced, based on their previous work. She says, "**Just use seventeen plus two --- nine-teen.**" Hehea checks with her by adding, "So your thing is plus two for each diagram?" Lani replies, "yes, you just write, it's seventeen plus two." Lani goes on to explain her rationale by pointing to the first two diagrams, and saying, "The first to third [diagram], that difference is two ---" This suggests that Lani uses their image of the extras, which the group then refers to by way of explaining the total for the 7th diagram. This new rule leads the students to calculate totals for the 17th (19 square blocks) and 60th (62 square blocks) diagrams. In these calculations, the group are observed to be dependent on specific numbers, and therefore show no evidence of Formalising their understanding, and they are thus unaware of their incorrect totals [Note 102]. Interestingly, the students did not apply this rule in order to check their previous answers for the first seven diagrams: answers the group obtained correctly.

6.6 Summary

All of the results from this chapter's analysis of each selected individual and group's growth of mathematical understanding of topic, patterns and relations, will be used, together with the analysis of their language switching (in Chapter 5), to respond in the next chapter to the research question posed in Chapter 1. Each of the marked "Notes" throughout all of the analyses of this chapter represents a particular incident where the students' language switching and their growth of mathematical understanding were observed either to interact or not interact. These marked incidents will be the evidence for discussing the relationship between the two said phenomena.
CHAPTER 7: DISCUSSION and COMPARISONS

7.1 Introduction

This chapter concludes the analysis at the heart of this thesis by responding to the research question posed in Chapter 1. Using the data collected from Tonga, this study focused on the following question: What is the relationship between bilingual students' language switching and their growth of mathematical understanding? This question asked not just when and if bilingual students switch languages, but whether such verbal activities have any connection to their growth of mathematical understanding. This chapter considers how the question might be answered by describing clearly and accurately the nature of the Tongan bilingual students' language switching and the growth of mathematical understanding, and then discusses how those verbal actions might affect or contribute to students' mathematical learning.

7.2 Discussion: Relating Language Switching with Growth of Mathematical Understanding

After identifying key features of the data in Chapters 5 and 6, the last phase of analysis involved observing connections between the bilingual students' language switching and their growth of mathematical understanding. This process featured, in Pirie-Kieren's terms, a form of property-noticing: examining the two sets of analyses, and then trying to identify interconnected properties between language switching and growth of mathematical understanding. In order to understand how the researcher arrived at his conclusions for this study, it is important to examine – and lend some degree of validity to – the initial phase of data analysis. During the introductory
phases of data interrogation, critical events in both language switching and growth of understanding were identified, categorized, and analyzed. The question of validity was discussed in Chapter 4 (Section 4.4.3). The schematic outline (Figure 29), below, summarizes the analytical steps followed during the study in question.

![Diagram](image)

**Figure 29:** The schematic overview of the data analysis steps

### 7.3 Responding to the Research Question

To talk about the relationship between language switching and growth of mathematical understanding using the language and principles behind the Pirie-Kieren theory, three critical moments had to be considered, as outlined in the mappings in Chapter 6: One, instances involving the students working and extending their current understanding within a particular layer of understanding; two, instances of the students extending their current understanding toward outer layer knowing; and three, instances of the students folding back to inner layer actions. By focusing on a change in the students’ understanding, and considering the events
leading up to or following such an observed change, it was possible to develop a clear sense of how language switching related to growth of mathematical understanding.

7.3.1 The Relationship between Language Switching and Image Making

This section focuses primarily on the students' language switching within the Image Making layer. This study found that all forms of switching occurred at this layer. Some students immediately engaged at Image Making activities in their first language as soon as they were given the task (see Notes 1, 32, 88). Other students shifted to using Tongan as soon as they read the first question (see Notes 47, 63, 73). What was clear from the data is that the Tongan students preferred to use their first language in any informal activities such as Image Making, but would eventually switch to English if they lacked the words to describe aspects of their mathematical activities. In addition to the students' tendency to shift, the students engaged in substitution and borrowing. Because their work at the Image Making level involved accessing Primitive Knowing, the students' language capacity in both languages was part of their Primitive Knowing prior to working on the task in question. Evidence from the data showed that this language capacity allowed the students to substitute between equivalent words (see Notes 1, 35, 87) or borrow non-equivalent words (see Hehea's use of the word "sequence" in Note 98). Borrowing, in particular, is an act that characterizes the Tongan-type bilingual students, with their use of "imported" words in order to adequately express their thoughts.

The students' preference for their first language for group discussions is also demonstrated by their constant translation – repetition or reformulation – of the questions from English to Tongan. Because most of the data analysis presented in this study was associated with the students' work in Task 3, little explicit translation was evident, because the students had already
figured out what they should tackle first, based on the previous tasks. Hehea and Lani’s example (refer to Section 6.5.10), however, illustrates a group who worked in Task 1 and immediately tried to find the meaning of the first question (see Note 99).

7.3.2 The Relationship between Language Switching and the Move to Image Having

This section solely considers the outward movement from *Image Making* to *Image Having*. The don’t-need boundary between *Image Making* and *Image Having* conveys the idea that once a learner internalizes a mental construct, or an image for the topic, he or she is capable of using that image – physical, visual, or mental – without having to do the activity that brought it forth (Pirie & Kieren, 1991). Within the boundary between *Image Making* and *Image Having*, the data showed evidence of students moving toward the use of a mental construct about the topic, without any language switching involved in their activities (see Notes 4, 48, 63, 87). Language switching was involved, however, in certain instances concerning the expression of a new mental construct. However, in these situations (see Notes 2, 36, 82), language switching was not considered a factor in their growth of mathematical understanding. For example, Selai’s remark “So that’s it! Add four every pattern ---” (see Note 2, Section 6.5.1) demonstrated a verbal expression of her new understanding. Selai’s substitution of the phrase, “every pattern” was a reference to each diagram following her observation of the image. Her statement, “So that’s it!” sounded like an “Aha!” experience, which signaled the moment in which Selai saw her image for the pattern, after reviewing her prior *Image Making* activities. Such “Aha” moments often mark students’ new understanding – new insights – as demonstrated by Selai’s subsequent verbalization of her understanding through the substitution of the phrase, “every pattern”. Malia Barnes (2000) calls an “Aha” experience a “magical moment.” He recalls how Polya (1981) speaks of “a sudden clarification” and Davis and Hersch (1980) dub such an experience “the flash of insight.” See also the article by Liljedahl (2002).
showed a similar reaction while working with the Tongan Task (see Note 82, Section 6.5.6). She instantly expressed her image seeing by shifting to using English and saying, “Add three --- add three ---”, for the common difference between each of the consecutive diagrams. In none of these instances did the language switching provoke growth of mathematical understanding.

In contrast, two occasions (see Notes 34, 64) offered evidence of language switching enabling the expression of growth of mathematical understanding. Each instance – Malia’s substitution of the term “odd numbers” (see Note 34) and Semi’s borrowing of the term “square” (see Note 64) – allowed the students to explore new understandings associated with their images. Semi appeared to mentally count the totals for the first three diagrams, then said out loud, “Nine there [total of the 3rd diagram] --- it’s square!” (See Section 6.5.4). The numerical totals, 1, 4, 9, instantly evoked the mathematical label “square” within Semi’s existing knowledge about patterns and relations. Thus, for Semi, language switching followed a moment of new understanding in the form of a new image.

The students also showed instances that involved image seeing bringing about language switching (see Notes 32, 51, 72, 63, 83). In such cases, evocative language switching instantaneously allowed a particular individual to recall from memory a particular image associated with a given task. Thus, the students were observed to “jump” into working at the Image Having layer whenever aspects of their existing knowledge were activated through language switching. Through this transition, the students were observed to have experienced sudden awareness about their new understanding. For instance, Kelela’s immediate reaction to the triangular shape of the given pictorial sequence in Task 3 (see Note 32) showed such a “flash” of new understanding – a moment of realization that something was different or new. Ipeni’s remark about the same set of diagrams as “sitepu” (steps), illustrated the application of
everyday experiences toward grasping a mathematical image (see Note 63). However, in both of these examples, the spatial images Kelela and Ipeni immediately associated with their existing knowledge did not turn out be viable images for their exploration of the patterns and relations in the task. They did not use the spatial images, and in Ipeni’s case, the decision led him to fold back to Image Making (see Note 63).

However, the students’ actual growth of mathematical understanding in this situation was not deemed to be dependent on their switching. In other words, the students’ growth of understanding of the topic would still have happened had they chosen not to switch languages, or if they had used a wrong mathematical label, the latter exemplified by Malakai (see Note 53). Instead, the use of the English mathematical terms in the Tongan-type bilingual context gave the students the precise and abstract mathematical meanings they needed in order to work on the tasks at hand. Such a reliance on English can be seen with the use of the term, “square numbers” (with its Tonganised form, “sikueba”). In addition, English is often used with the term, “odd numbers”, despite the fact that the latter has an equivalent word in Tongan, “fika ta’etauhoa” (refer to Niko’s use of the term in Note 90), a term rarely heard at the Tongan senior secondary school level. Perhaps the rare use of such terms can be directly attributable to the current policy governing the transition from the teaching of mathematics in Tongan at the primary school level to teaching mathematics primarily in English at the secondary level.

7.3.3 The Relationship between Language Switching and Image Having

In connection to the relationship between language switching and Image Having, students’ work at the Image Having layer demonstrates the use of a mental construct about the topic and the ability to use that particular image without having to do the activity itself. The data showed
instances where no language switching was involved while the students worked at the *Image Having* layer (see Notes 48, 49, 66, 73, 77). In such cases, the students were observed to move between *image seeing* and *image saying* without switching languages. However, language switching was observed also to occur during the process of verbalizing aspects of the students’ images (see Notes 4, 5, 6, 36, 37, 38, 51). In addition, some students were able to articulate the features of the images they were working on and explain why “something” did not conform to their images. This articulation and the need for verbal expression involved the continual association of equivalent words through *substitution* (at the word-level) and *translation* (at the sentence-level) as the students verbalized their mathematical constructions. Again, the students’ understanding of both languages seemed to allow them to switch languages based on individual preference or classroom requirements.

One episode, for instance, involving Selai working on Task 3, demonstrated her continual *translation* of her thoughts while she expressed her images (see Note 4). She stated, “That’s four --- four blocks --- add a four --- plus four.” In this case, Selai was reflecting on her previous construction – her numerical image associated with the common difference of four between diagrams. Selai continued to describe her image using the phrases, “every pattern” and “each pattern”, with the word “pattern” referring to each diagram (see Note 6). Similarly, Malia *substituted* the words, “plus” and “over”, to characterize the uniqueness of her pictorial image of the extras (see Note 36). Malakai, on the other hand, was observed to work at the *Image Having* layer when his peer, Alaki, prompted him about using an English label for his pictorial image (see Note 51). Malakai, in return, replied with the word, “base”, in reference to what he thought Alaki was pointing to – the bottom layer of the diagram.
Another significant piece of evidence from the Tongan data was the statement made by Malia, who used language switching to talk about her numerical image for the pattern. Malia’s statement, “Just add-up the odd numbers --- That’s it, it’s not triangular numbers!” (see Note 34), again demonstrated the evocative nature of language switching and its relation to growth of mathematical understanding. In this incident, Malia (see MAP 2 in Section 6.5.2) demonstrated working at image saying, as she expressed why Kelela’s image did not conform to her image. While Malia’s reaction turned out to be a misunderstanding about Kelela’s statement (an evidence discussed further in Section 7.6), Malia nevertheless, expressed aspects of her image through language switching. Her statement suggested a new kind of understanding, in which she appeared to validate her understanding through the substitution of the phrase, “odd numbers” and the borrowing of the phrase, “triangular numbers”. In both switches, Malia relied on using English mathematical terms to clearly distinguish her own mathematical construction in relation to Kelela’s image.

### 7.3.4 The Relationship between Language Switching and the Move to Property Noticing

The understanding of a learner at a particular point in time is integrated with previous layers of knowing, and because of this connectedness, the learner’s previous knowledge can be called into current-knowing actions. Hence, Pirie and Kieren (1991) described a learner in this situation as constantly working between layers around a do-need boundary. In this study, the data showed evidence of the role of language switching in the movement between such layers. In particular, the data demonstrated that the move from Image Having to Property Noticing could occur without the need, or use, of language switching (see Notes 13, 67, 87, 90). However, the students were involved in several instances that showed language switching interacting with the growth of
mathematical understanding across this do-need boundary. The following examples are used to examine the relationship between language switching and the move toward Property Noticing.

The discussion in Excerpt 7 (see Section 6.5.1) showed Selai’s continual and seemingly conscious translation of her thoughts leading her to Property Noticing, as she repeatedly expressed her perceived connections between her image of the totals and the image of the extras (see Notes 5, 8). The students also demonstrated instances of substitution, which they used to express their mathematical thinking. Malia, for example, recalled, through substitution, her numerical image of “odd numbers”, an act that signaled a new understanding of a property in her numerical image of the extras along the base layers (see Note 39). Substitution was also involved whenever the students detected specific keywords in the questions, leading them to notice specific properties in their images. Alaki, for example, extracted the key phrase, “extra square”, from Question 2, and through substitution, he expressed specific features of his images (see Note 50). This example shows how substitution of the keywords associated with the question directed and dictated the students’ mathematical construction, and eventually influenced the pathway of their growth of mathematical understanding.

The data also showed instances involving either shifting or borrowing, in which the students expressed recognition of new properties through their careful observation and work with images. Malakai, for example, expressed his awareness of a new property associated with a numerical image through shifting (see Note 53). More significantly, Malakai’s comment, “I already know it!” signaled a new understanding, which he expressed by shifting to say, “add the prime numbers”. Malakai’s shifting to English, however, might largely be attributed to his lexical need to borrow the term, “prime numbers”. But, as this example showed, despite Malakai’s incorrect use of the term, the term enabled him to express his numerical image, which turned out to be
associated with odd numbers. His growth of mathematical understanding was not affected. In a different, yet related act of borrowing at *Property Noticing*, a discussion between Malia and Tupu (see Excerpt 21, Section 6.5.6), involved their examination of their constructed set of numerical values for the totals. Tupu looked over to say, “So, they are odd numbers. Is there a multiple?” (See Note 86). Her peer, Malia, responded with a “No”, and then offered a description of the pattern as an alternation between odd and even numbers. In this regard, *borrowing* the word “multiple” appeared to lead to a new understanding for Malia. Tupu’s question prompted her to recall existing knowledge or to fold back to collect a piece of *Primitive Knowing*. In spite of the subtle differences between each of the learning scenarios described above, the evidence from the data underlined the fact that some acts of language switching enabled growth of mathematical understanding, and, conversely, growth of mathematical understanding was observed to bring about language switching.

7.3.5 The Relationship between Language Switching and Property Noticing

Observation of the students’ work at the *Property Noticing* layer was associated with their use of images, as they manipulated and compared these images, and explored the inter-connections and properties of the images. Again, the data showed instances where no language switching was involved while the students worked at the *Property Noticing* layer (see Notes 13, 20, 23, 25-28, 41, 100). In each of these instances involving *Property Noticing*, the students were observed to work and move between property predicting and property recording, without language switching. However, language switching did sometimes occur while the students were in the process of verbalizing their mathematical thinking. Such instances involved *translation* (see Notes 78, 81), *borrowing* (see Notes 40, 55, 61, 68, 96), and *substitution* (see Notes 21, 22, 52, 79, 80, 85, 87, 94, 95). All of these substitution examples involved the use of the terms, “even”
and "odd" numbers, and in which case, it was observed that growth of mathematical understanding preceded language switching. At this layer, the students demonstrated shifting in attempting to express mathematical thinking in English (see Notes 13, 24) or in Tongan (see Note 41). Selai's attempt (see Note 13) to formulate an algebraic expression for her observed relationship showed her need to use English in expressing her mathematical abstraction.

Furthermore, several other examples from the data indicated instances of students' growth of mathematical understanding at Property Noticing, influenced by their language switching. For instance, Nanasi's translation of "prediction" to mean random guessing (see Notes 78, 81) appeared to "block" the students from progressing toward their formal understanding of the topic in question, a point discussed further in Section 7.6, a section that addresses the evocative nature of language switching. Yet, Malia's use of the term, "square numbers" (see Note 40), showed her arrival at another new understanding as she noticed a specific property associated with the diagram totals. She was observed to have made the move toward Property Noticing, based on her work with images associated with the extras, when she suddenly turned to her peer, Tupu, and asked about the totals, "Are those square numbers?" (See Note 44). Her new understanding about the pattern in the totals indicated further property predicting; thus, extending her knowledge of the patterns and relations. In this case, Malia's growth of mathematical understanding was provoked further and allowed her to later Formalise a general rule for calculating the diagram totals. Similarly, Malakai (see Note 55) and Ipeni (see Note 68) continued to build their understanding based on their noticing the relationship between the diagram numbers and the totals (square numbers).
7.3.6 The Relationship between Language Switching and the Move to Formalising

The move from Property Noticing to Formalising involves a learner being able to abstract a general method or express a “for all” statement about the pattern in his or her noticed properties. The learner must explicitly state a general idea about his or her noticed properties, with a clear indication of the use of the rule in a general sense “without recourse to any specific image or idea” (Pirie, personal correspondence). Only after such a general rule was identified, would an observer be in a position to claim that the learner has moved out in his or her understanding to Formalising.

The data also contained instances showing no language switching involved in connection with the students’ move toward Formalising (see Notes 23, 101). In these instances, the students worked in their first language without any switching to English, and they were able to make the leap in their understanding of the topic from Property Noticing to Formalising. Hence, growth of mathematical understanding can occur without language switching. However, in a few instances, students did show evidence of moving toward Formalising, as expressed through language switching (see Notes 8, 43, 44, 57, 70, 71, 61, 98). All forms of language switching were observed to be involved in the way the students expressed their Formalised ideas. Thus, the degree of language switching involved ranged from no observed relationship between the acts of language switching and the students’ mathematical understanding, to instances showing language switching apparently enabling growth of mathematical understanding.

By contrast, instances of translation and shifting were most apparent in the data, partly because the students were required to write their answers and explanations for the task in English, even though they spent a great deal of time working in Tongan. For example, during the episode (see
Excerpt 7) in which Selai consciously translated her mathematical thinking (see Notes 8, 9), while continually connecting the relationship between her images, Selai’s actions demonstrated her Property-Noticing activity leading her to Formalising when she added, “But the pattern should equal --- be the same like plus four to each pattern [diagram]” (see Note 9). Selai’s general statement, along with her reformulation in order to reiterate her thoughts in English, indicated her move out to Formalisation. The act of translation also played a significant role in Tupu and Malia’s interpretation of the word “triangular” – associated with their noticed property of the pattern in the totals as a set of triangular numbers – to mean a general rule involving the multiplication of each diagram number by three (see Note 98). Section 7.6 outlines in detail the way language switching, as it does in the preceding example, evoked different, and sometimes inappropriate, images or properties for the students. In this situation, language switching unfavourably affected the students’ growth of mathematical understanding.

As well, shifting played a key role in the relationship between language switching and moving out to Formalising. For instance, Meki’s generalized statement, “We add the odd numbers” (see Note 80) illustrated this form of shifting, along with Malakai’s answer, “No need to draw it” (see Note 57), an indication of his readiness to move out to Formalisation after he had discovered the rule for calculating the totals. The students showed evidence of substitution and borrowing when they expressed moving out to Formalisation. Whether they used substitution or borrowing, the students tended to use English words that best suited the mathematics they were working with for the task in question. Ipeni’s description of the rule for the totals, “every diagram, you add to it an odd number” at first appeared to be a general statement using substitution (see Note 71), but his reliance on a specific diagram indicated he was just very close to crossing the don’t-need boundary. Selai, by contrast, demonstrated borrowing through algebraic generalization of her group’s pattern in the extras (see Note 19). Thus, these examples showed that all forms of
language switching are involved when students verbally express and actively demonstrate moving out to *Formalising*. At the same time, these examples showed that not all instances of language switching enable students to move toward *Formalising*.

7.3.7 The Relationship between Language Switching and Formalising

In keeping with the previous discussion, the data shows evidence of the students using all forms of language switching while they were observed to be working at the *Formalising* layer. While they were working with a generalized rule, the students chose to do their calculations in Tongan. Malakai and Alaki always *shifted* to using Tongan after reading the questions in English (see Note 58); and so did Malia and Tupu (see Note 46). But because of the language used for the questions, and the need to explain their answers in English, the students were forced to *shift* or *translate* explanations of their mathematical work or their answers into English. Such actions were evident in the group work of Malia and Tupu (see Note 44), Alaki and Malakai (see Note 59), and Semi, Christie, and Ipeni (see Note 72). Selai, however, was able to explain the generalization of her group’s non-algebraic rule in Tongan, without having to switch languages (see Notes 9, 16). But in a different situation, she was engaged in algebraic manipulation (see Note 19) as she relied on her *borrowing* of the variable “x”. Selai’s example is an exception. However, it is important to note that the act of *shifting* can offer a misleading view of what an individual is actually able to understand in mathematics. For example, the exchange between Kepi and Selai toward the end of their work in Task 2 (see Note 30) showed Selai’s use of the phrase “next diagram” actually meant “preceding diagram”. After her group calculated the total for the 60th diagram, Selai attempted to explain the answer by adding, “*We just say we times four to the --- to the next --- by next it means --- preceding it ---*”, in which she used the appropriate *positional* Tongan word “kimu’a” (before). Kepi, however, used the contrasting Tongan word,
“kimui”(after), to express the same idea, as he explained in an earlier example involving calculating the total for the 60th diagram. Kepi said, “Multiply four to --- just multiply four to the number, after [kimui] --- that is, if given to find the sixtieth diagram --- just multiply four to the number after sixty --- [which] is fifty-nine.” Thus, Kepi’s subsequent example shows the way he viewed the rule similarly to Selai, despite the different words (both in English and in Tongan) each student chose to express their Formalised rule. But, as previously mentioned, the English-only task requirement forced the students to initially express their Formalisation in their second language.

7.4 The Relationship between Language Switching and Folding Back

In order to accurately explore the precise relationship between folding back and language switching examined in this study, Martin’s categories of folding back were used to identify invocative movements within the growth of mathematical understanding. Martin’s (1999) thesis establishes the notion of folding back as a sound and useful phenomenon by elaborating on the folding-back feature of the Pirie-Kieren theory. The current Tongan data showed 19 observed instances of folding back. Of these 20 instances, 12 were coded as working at the inner layer with existing understanding, five were coded as collecting, three were coded as moving out of the topic and working there, and none were observed as causing discontinuity.

Out of the observed 12 incidents of working at an inner layer, language switching was observed to take place on four occasions (see Notes 11, 12, 18, 35). During one such incident (see Note 35), language switching involved a form of substitution, as Malia folded back from image seeing to image reviewing, while substituting the word “plus” in her expression. Malia’s substitution did
not occur out of necessity, or because language switching depended on it. Rather, Malia used substitution to access both languages – which were part of her Primitive Knowing – in an act of language mixing to express her inner-layer activity. In this situation, folding back was not influenced by her substitution. On two other occasions, Selai and her peer, Kepi, were observed to be working at the Property Noticing layer and the Formalising layer when they decided to fold back to work with their earlier constructs. In both instances, Selai’s use of the word “formula” signaled an intentional shift in her mode of understanding, leading her to fold back from working at Property Noticing to Image Making (see Notes 11-12), and from Formalising to working at the Property Noticing layer (see Note 18). Both invocative moves showed Selai’s intention to find a rule for calculating the totals. Yet Selai’s folding back was not dependent on her borrowing the word “formula”. This non-equivalent word was borrowed mainly for expressing Selai’s thoughts.

During the next stage of the task, the idea of working with the established “formula” provoked Selai to disconnect herself from her Image Making activity, and move back out to Property Noticing layer to work with the existing algebraic expression, “x + 4”, associated with her image of the extras (see Note 13). Selai went on to challenge her group about the usefulness of a general arithmetic formula (different from using the repeated additions of four) while she was working in Question 10 to find the total for the 60th diagram. She said to her group: “There is a rule, but --- wait, let’s think about it to get that rule --- and then we use this new rule for the seventeenth.” To explain her conclusions, Selai used the related Tongan word “lao” (rule) (see Note 25) in a similar way to her earlier use of the word “formula” – an indication of her intention and purpose. Hence, through these examples, the study found that working at an inner layer with existing understanding was not influenced by language switching, and that either working at an
inner layer with existing understanding or language switching could have occurred without the other.

Further data analysis revealed that language switching was only involved with two coded incidents of collecting. In the first instance (see Note 46), Tupu (in Section 6.5.2) folded back and collected a useful piece of Primitive Knowing involving units of time. Her suggestion, "remember in counting seconds?" prompted Kelela to add, "It's sixty minutes!" It is not clear if Tupu's invocative act affected Malia, who was, at the time, calculating the product of 60 times 60 as the total for the 60th diagram. What is clear, however, is that the shared understanding between Kelela and Tupu was appropriate to verify the needed product, had Malia been unable to come up with the correct total. It is apparent that the two students were aware of what to collect, and yet the knowledge they needed was not immediately accessible for them to remember the product of 60 times 60. Furthermore, Tupu and Kelela's shifting had no effect on their act of collecting, except as a way of representing their thoughts through languages. In other words, the students' decision to shift – or not – would have made no difference to what either of them thought about the task at any given moment.

Other evidence from the data showed how language switching – in this case, borrowing – led to folding back. In a second incident (see Note 91), Seini’s question led to Pola’s collecting a piece of Primitive Knowing that eventually played a significant part in helping Seini progress in her understanding. Seini had only been looking for a mathematical label for the numerical set {1, 4, 9, 16, 25}. But Pola’s act of folding back – collecting a piece of his Primitive Knowing about “prime numbers” – led Seini to a new understanding involving composite numbers, as numbers with more than two factors. As a result of their borrowing and interaction, both Seini and Pola’s understanding of patterns and relations were able to grow. Pola’s invocative action turned out to
be effective, as both Pola and Seini were able to express further properties of their numerical image (associated with the totals). In this example, while the *borrowing* of the terms “fakitoa” (factor), “prime”, and “composite” was based on the inadequacy of the current Tongan language, the mathematics that followed were dependent on the images and properties the students associated with the described terms. Further discussion of this phenomenon is offered in Section 7.6, regarding the evocative nature of language switching and the role of *translation*. What can be seen in this example, however, is that *borrowing* enables folding back and consequently the two students’ growth of mathematical understanding was dependent on the images that were evoked.

It is not surprising, however, that the notion of *causing discontinuity* was absent from the work of the participants. This absence does not preclude such acts of folding back from existing within the bilingual environment, but the context of this study meant that the students were not inclined to discuss a different mathematical topic or perform work unrelated to their existing tasks. While an analysis of such acts of folding back is beyond the scope of this thesis, the notion of *causing discontinuity* is hypothesized to be unrelated to language switching. The only finding in this study is the fact that language switching occurred without *causing discontinuity*. Consequently, none of the acts of language switching invoked the students’ current understanding toward working on a new activity or problem disconnected from their existing understanding. There were cases where the students switched languages while debating meanings of words and other language issues unrelated to their current mathematical activities, and the students were occasionally observed to move out of the topic in both their actions and discussions. However, most of these instances involved the students trying to better understand the language, rather than the mathematics. In one case, Malia and her group had just *Formalised* a rule for calculating the totals (see Note 44), and Kelela mistakenly overheard Malia saying “difference”, not
“sequence”, leading the group to argue over the two words before returning to read the next question (see Note 45). In this example, however, the students’ digression from the topic had no effect on their current mathematical understanding. Similarly, Alaki (see Note 52) struggled at length to find an appropriate English word to describe his image. His efforts included prompting his peer, Malakai, for a label (see Note 51). Furthermore, Naati and Siona’s extensive discussion about the words “introduction”, “instruction”, and “intruction” (see Excerpt 23, Section 6.5.8) demonstrated the students moving out of the topic and working there (see Note 93). Again, the students’ lack of understanding of the language, rather than the mathematics, was the main reason for switching languages and engaging in their group discussion. Consequently, such acts of folding back can be deemed ineffective since they did not affect the students’ current understanding of the task and the mathematics involved.

The mapping of Alaki and Malakai (see MAP 3, Section 6.5.3) shows how the pathways of Alaki’s growth of mathematical understanding were observed to be disconnected between his formal understanding of the topic and his image associated with the extras. In this example, Alaki followed his peer’s work and was unable to make the connection between his image and that of his peer’s. In this situation, Alaki’s discontinuous understanding was not influenced by language switching, nor did it bring about acts of language switching.

Shifting to express formal (or informal) mathematical understanding in English was usually associated with the demand of the task (for second-language learners), and the inadequacy of the individual’s native language to express mathematical concepts and ideas. Such characteristics were evident in the work of Malia (see Note 44), and Semi and Christie (see Note 72), who continually translated and shifted between English and Tongan.
In short, the data showed that language switching can occur without folding back, and that folding back can occur without language switching. In addition, all of the previously described examples demonstrated that not all incidents of folding back bring about language switching, and that not all incidents of language switching enable folding back. It is crucial, therefore, for educators to distinguish from among the various sources of misunderstanding, often associated with the following: (i) inadequate understanding of the mathematics; (ii) inadequate understanding of the mathematical language; and (iii) inadequate understanding of the language in general.

7.5 The Relationship between Language Switching and Acting and Expressing

One dimension of the Pirie-Kieren theory, known as acting and expressing complementarity, describes the nature of growth of understanding within each layer. As Pirie and Kieren (1994) explain, expressing does not mean, in its literal sense, verbal externalization. In addition, the theory also refers to a phenomenon called reflecting, which describes a component of acting, whereby a learner looks at how his or her previous understanding was generated. This study identifies these acting/expressing categories of understanding in the Tongan data. All of the Pirie-Kieren mappings in Sections 6.5.1, 6.5.2, 6.5.3, and 6.5.4 illustrate movements between acting and expressing. However, it was unclear from the outset of this study whether language switching would differentiate the two kinds of understanding, or whether language switching would be evident in one kind of understanding and not the other. In this study, no clear evidence arose of any relationship between students' language switching and the back-and-forth movement between their acting and expressing activities.
To demonstrate this principle in the data, for example, Malakai’s mapping (see Section 6.5.3, MAP 3), showed him working at the Property Noticing layer between the marked points “10” (located as acting) and “16” (located as expressing). Malakai switched languages through shifting (Point 10, see Note 57), borrowing (see Note 55), and substitution (Point 16): all taking place within Malakai’s work at the Property Noticing layer. Between the two marked incidents, Malakai, who urged his peer, Alaki, to continue working on the task while he “thought” about the pattern, came up with two distinct properties. The first noticed property was the pattern in the extras, which Malakai expressed as, “Add the prime numbers” (Point 10). Further analysis indicated Malakai was in fact working with odd numbers. Nevertheless, Malakai went on to notice another property: this time, the relationship between the diagram ordinal numbers and the totals, which Malakai declared to be “square numbers” (Point 16).

The example of another student presents an alternative way of examining the relationship between language switching and acting and expressing complementarity. For instance, Selai’s mapping (MAP 1, see Section 6.5.1) shows movements between acting and expressing within Property Noticing in her understanding of the topic, during her group discussion of Question 4 to Question 6 (see the marked points 8 to 15). Selai engaged in translation at the Property Noticing layer. She continually translated (repetition) her thoughts between English and Tongan while property predicting (Point 8, see Note 8), and then later went on to translate while she was observed to be property recording (Point 10, see Note 11). Selai’s mapping also showed evidence of reformulation when she moved out to Formalise the pattern in the extras (Point 9, see Note 9). But when Selai resumed property predicting, she borrowed and substituted English words (Points 11, 12, see Note 12, see also Note 15). Then, when her group moved to answer Question 9, asking them to predict the total for the 17th diagram, Selai was observed to work mainly at the property predicting level (refer to MAP 1, and marked points 18 to 25). Within
this period of engaging in *property predicting*, Selai verbalized her thoughts through *shifting* (Point 18-20, see Note 18), *borrowing* (Point 18, see Note 18), *substitution* (Point 23, see Note 23), and sometimes, without any language switching at all (Points 22, 25, see Notes 21, 25).

Selai’s actions show that, even when a student engages in either *acting* or *expressing*, language switching can take on any form, as the individual chooses to verbalize his or her thoughts and current understanding. This finding about the multifaceted nature of language switching is consistent with previously discussed results, as well as the phenomenon of folding back. Thus, a Tongan-type bilingual student who prefers to use his native language would use his or her second language (say, English), for two main reasons: accessibility of the individual’s underlying language capacity, as expressed primarily through *substitution* and *shifting*, and inadequacy of the current language, as expressed through *borrowing* and *translation*. In either case, the individual used both languages in expressing and provoking aspects of his or her growth of mathematical understanding.

Recapping the above discussion, certain points have been picked from two of the mappings to illustrate two general findings: (1) Malakai’s example is used to show that students used all forms of language switching while moving back and forth between *acting* and *expressing* within a particular layer (*Property Noticing*); and (2) Selai’s mapping shows how students used various forms of language switching within *property predicting* (*acting*), and even one more form of language switching (*translation*) between *acting* and *expressing*. This does not explain all possibilities but it gives an idea of the lack of influence between language switching and growth of mathematical understanding within the complementarities.
7.6 Relating Growth of Mathematical Understanding and Evocative Language Switching

An evocative language switching activates mathematical actions or understanding distinct from the tasks a student was or is currently working on. This activation, in a mathematical context, refers to the interconnection between language and mathematical images. There are two situations in which language switching directly relates to mathematical images: (i) a word, phrase, or sentence can be assigned to a specific image; or (ii) an image can be attached to a specific word, phrase, or sentence.

The first case is illustrated by the association of mathematical images to words, such as examples involving Nanasi’s interpretation of the word “prediction” (see Note 81), Tupu’s interpretation of “triangular” (see Notes 96-98), and Hehea and Lani’s discussion of the meaning of Question 3 (see Note 100). In the second case, the association of words to mathematical images was evident in various examples, such as Malakai’s use of “prime numbers” and “square numbers” for his numerical images of the extras and totals (see Notes 55, 61), and Semi and Ipeni’s use of “square” (see Notes 64, 68). These situations can be referred to as the verbal expression and construction of mathematical meaning from and through language.

To clarify the notion of verbal expression: an individual would use his or her underlying language capacity to convey his or her thought processes through language. So when the students do not have the necessary language to express their mathematical understanding, they either borrow or shift to express their understanding in their second language, even if words are used incorrectly. It is important to note that when students struggle to express their understanding, it does not mean their mathematical understanding is affected. For example, Selai struggled to express her Formalisation in English (see Note 31). Selai knew the rule, evident in her
calculation for the diagram totals, but she could not find the right English words to express it. She used the phrase “next diagram” incorrectly, meaning to say “preceding diagram”. Selai’s misunderstanding was obvious when she shifted to explain the rule in Tongan, using the appropriate Tongan word “kimu’a”. However, her peer, Kepi, came up with the opposite word, “kimui” although he, too, knew and used the rule correctly. Thus, in either case, these students’ inadequate grasp of the language did not prevent them from understanding the mathematics.

Another difficulty often faced by bilingual students involves incorrect mathematical labeling for images. In one instance, Malakai incorrectly borrowed the phrase “prime numbers”, but his understanding of the pattern suggested he did not recognize his mistake (see Note 53). Thereby, Malakai assumed his mathematical label was the appropriate one. Pola’s incorrect use of the label “composite numbers” again did not inhibit his growth of mathematical understanding (see Note 92). Thus, the students were still able to understand the problem, but their incorrect mathematical labels could create misunderstandings for others: among them, teachers in particular. These students’ use of incorrect labels, despite their ability to solve mathematical problems, reflects the kinds of challenges typically faced by Tongan-type bilingual students.

To examine further the concept of construction: mathematical images are attached to specific words, phrases, or sentences. Consequently, bilingual students’ mathematical work, and by extension their growth of mathematical understanding, is shaped and directed by the construction of mathematical meaning from language. In this instance, the effect of an individual’s language switching on his or her growth of mathematical understanding depends on what image (or aspects of the image) is being attached to or evoked by the switch. Students construct meaning from language: whether the meaning comes from an instruction, a question, a problem, or a task, such a discovery will ultimately shape the way the students conduct their mathematical activities.
In these mathematical activities, constructing mathematical meaning is associated with the role of translation, whether in reformulating the given information from one language to another, or in direct translation (repetition) between two languages. When students work with written questions, translation can direct them to particular images. In this situation, growth of mathematical understanding will depend on how students translate the given information.

In this study, some of the students translated Question 3, asking them how many square blocks each diagram contained in total, to mean adding together all of the totals of the diagrams. For instance, in doing Task 3, Nanasi added all of the totals of the first five diagrams as 25 square blocks (see Note 74). A similar situation occurred during the discussion between Lani and Hehea (see Note 100): each student came up with a different translation of the question, but eventually saw the same sequence in two different ways. When Leisi asked what the word “extra” meant, Lani’s response indicated her interpretation and continuing focus on drawing the next three diagrams in the sequence. Then, Hehea explained that the extracted word, “drawing” was “not the important thing ---[but] (pointing at the word “extra”) that is the important thing, the extra – extra square blocks”. Lani agreed that they ought to explain what the word “extra” actually meant in the context of the question, to which Hehea offered her explanation. Lani appeared to “give up” momentarily and instead suggested moving on to the next question, to which Hehea agreed. Hehea then went on to read the next question. In the scenario described above, the students’ growth of understanding discontinued momentarily in relation to this particular image, until it was re-visited when the students returned to the task. This example illustrates the importance of both students’ agreement about how to construct each diagram, and their differences did not derail their growing understanding of the topic, except with regard to their interpretation of the exact meaning of the question. The example of Hehea and Lani clearly distinguishes between language understanding and mathematical understanding: while the
students may focus on different aspects of the question, their divergent approaches do not prevent them from understanding the mathematics.

Similarly, the use of specific keywords can have a profound effect on the students' translation of the question. In his task-work, Alaki (see Note 50) showed how his Image Making activity revolved around the keywords “extra square”, which allowed him to eventually move out in his understanding to Property Noticing. In this situation, Alaki’s substitution enabled his growth of understanding. Another example featured Meki, Nanasi, and Rosina (see Note 78), who adopted the same method they had used in previous English tasks involving the simple guessing of random numbers for the totals. Nanasi’s translation of the term, “prediction”, as the random guessing of numbers, demonstrated a combination of language and mathematical-meaning construction, and thus affected how these students approached their mathematical work in the task. Nanasi’s misinterpretation inhibited her group’s growth of understanding because they did not go on to Formalise their understanding of the pattern, even though they had already constructed relevant images for the pattern (see Notes 75, 78, 81). In other words, Nanasi’s translation held back her group from pursuing further properties of their images; thus preventing the group from Formalising. As a result, it can be said that the group’s growth of understanding depended upon Nanasi’s interpretation (mathematical meaning) and translation (language meaning) of the word, prediction. In this case, Nanasi’s translation led her to interpret her mathematical meaning for “prediction”, which then affected her understanding of the mathematics involved. Consequently, Nanasi’s understanding of the term, prediction, influenced not just her, but her group’s growth of mathematical understanding. Nanasi’s example showed how inappropriate translation could block growth of mathematical understanding. However, a group of monolingual English-speaking students described in some of Pirie’s data (personal correspondence) showed a similar lack of understanding toward the word, “predict”, which led to
similar problems involving random guessing. It might be inferred, therefore, that the concept of prediction is not well understood by students of this age, rather than the word itself. Such a lack of life experience might well be attributed to Tupu (see Note 97), who associated the word “predict” with “guess”, but used the rule, “multiply the diagram by three” as an approximation, based on her interpretation of the concept of “triangular numbers”.

In another instance, borrowing a non-equivalent word can lead to a misunderstanding when two students have two or more different meanings associated with the word. For instance, Malia and Kelela developed a misunderstanding over the meaning of the word “triangular” (see Notes 32-34). In the Tongan language, triangular had an equivalent word in Tongan when it referred to a two-dimensional shape (“tapa-tolu”, meaning three-sided), yet no equivalent word existed in Tongan for the numerical set {1, 3, 6, 10, 15, ...}. Thus, it can be said that Kelela was substituting in her language switching, while Malia was borrowing. The distinction between substituting and borrowing is significant, because a single English word, such as triangular, takes on not just two different meanings in Tongan, but also uses two different types of language mixing. Translating a word such as “triangular” into Tongan would differentiate the two meanings, but may also lead to misinterpretation. The same group demonstrated such an error in Task 4 with their use of the word “triangular”, by using its substituted meaning to generalize an arithmetical rule for the totals, rather than its borrowed meaning (see Notes 96-98). In this case, Tupu appeared to assume that triangular referred to multiplying each diagram number by three, in terms of the numerical set. This latter incident shows that the students did Formalise their understanding of the pattern, although they had mathematical misunderstandings owing to language difficulties. It was important, therefore, to take a closer look at the nature and content of the switch and how appropriate it was to the individual’s current understanding.
An observation also worth noting from these exchanges concerned the differences in the types of language switching used in each case. “Odd” and “even” numbers are simply labels – mathematical ones in this case – that students recall and use through substitution to describe the totals. But the word “multiple” is quite different: it is a borrowed word, since it has no direct equivalent in Tongan. “Multiple” is also a label, which in this case does not refer to an object, but a process. As a result, the use of the word, “multiple,” makes it appear the students were folding back to collect, had one of them demonstrated working with that knowledge. But neither student showed he or she was working with that knowledge. And therefore the students’ responses reflected a mere tapping into their Primitive Knowing without thickening their present understanding. For instance, when Tupu asked if the totals were a “multiple of two”, Malia replied, “No!”, and she explained the image as an alternating pattern of “odd” and “even” numbers, in spite of the fact that some of the totals (such as 4, 10, 16) were multiples of two. Following the discussion between Tupu and Malia, however, a new understanding of the pattern emerged for these two students, although their deduction was not a correct one, nor was it complete. This form of understanding emerged not just from the knowledge that arose from directly working with their images, but as a by-product of their interaction: a phenomenon similar to that recognized and described by Davis (1995; 1996). In this case, Malia and Tupu’s current mathematical activity evoked aspects of their Primitive Knowing and existing knowledge through language switching (see Note 86). They used the existing knowledge associated with the task and accessed their Primitive Knowing through language switching in their attempt to describe the pattern in the totals. Thus, in this case, language switching, along with group discussion, became a useful tool for accessing Primitive Knowing and furthering mathematical understanding.
7.7 The Tongan Task, Language Switching, and Growth of Understanding

The importance of the Tongan task was to see whether the Tongan bilingual students would still be switching to using English, and if they did, what kind of switching would be involved and what its effect would be on the growth of understanding. As the example of Malia and Tupu demonstrated (Section 6.5.6), some students still switched to using English, partly because the substituted or borrowed English words had become part of their underlying language capacity, and were therefore easily accessible to them. This evidence invalidates the notion that Tongan students use only Tongan when a mathematical task is presented to them in Tongan.

The final chapter attempts to paint a coherent picture of the relationship between language switching and the growth of students' mathematical understanding and examines this picture against some of the existing bilingual research.
CHAPTER 8: CONCLUSION AND IMPLICATIONS

8.1 Introduction

To conclude the work contained in this thesis, this chapter attempts to paint a coherent picture of the relationship between language switching and the growth of mathematical understanding, and responds to issues and myths often associated with bilingualism in mathematics education. In addition, this chapter describes the researcher’s personal experiences, and considers the implications of the study for future research, and for teaching and learning in bilingual situations.

8.2 The Main Findings

The last three chapters responded to the research question by describing the nature of the Tongan bilingual students’ language switching and of their growth of mathematical understanding, and then discussed how those verbal actions affected or contributed to the students’ mathematical learning. The categorized forms of language switching provided a useful way of describing the pattern in the Tongan students’ language use. The students involved in this study were observed to switch languages frequently in each of the four categorized forms, both in working at a particular layer of understanding and in crossing a particular boundary between layers of understanding. In addition, the study showed that language switching is a significant aspect of the students’ mathematical activities, and in their understanding of the specified topic. The evidence from the data clearly demonstrated how language switching influenced growth of understanding through the construction of mathematical meanings. At the same time, language switching was found to enable the expression of growth of mathematical understanding. The study therefore reveals various relationships between language switching and growth of
mathematical understanding, ranging from no interaction to a significant interconnection between the two phenomena. A look at the main findings reveals the following conclusions:

At Image Making, students preferred to work in their first language. The students shifted to using Tongan, they translated questions into Tongan, they substituted-in English words, and they borrowed English words if they lacked the words to describe aspects of their mathematical activities. Because of their preference to work in their own vernacular, the students had to translate the questions from English to Tongan. This translation shaped how they approached the tasks, and in spite of how incorrect such a translation turned out to be for some of the students, it did not necessarily imply a lack of mathematical understanding on their part.

Nevertheless, the interconnection between what the students did mathematically and their language switching rested in their use of both languages in constructing mathematical meaning from language and in expressing their work and mathematical thoughts through language. The fact that the students constantly accessed their Primitive Knowing while working at the Image Making layer allowed them to choose between using only one or both languages in verbalizing their thoughts.

Students who demonstrated the act of moving to Image Having and working there, also showed connections between language switching and growth of mathematical understanding in the specified topic. Crossing the don’t-need boundary between Image Making and Image Having is indeed a significant leap in the students’ understanding as they relied on viable images without having to go back to Image Making. The study showed that not all incidents of language switching enabled movement from Image Making to having viable images at Image Having. Moreover, growth of understanding as observed in this movement could occur without language switching, and conversely, language switching, in all the categorized forms, can also occur
without such movement. Having said that, the data showed evidence where growth of mathematical understanding brought about language switching. In this situation, growth of mathematical understanding preceded language switching, which, for some students, enabled them to express their current understanding without the limitation of using just one language.

In moving out further to work at the *Property Noticing* layer, the role of language switching was observed to be connected with evoking various aspects of the students' images. The interaction of language switching and growth of mathematical understanding was observed across the do-need boundary between *Image Having* and *Property Noticing*. In this movement, language switching not only enabled expression of *Property-Noticing* activities but also enabled growth of mathematical understanding when working at this layer. Various examples in the data showed the students' working with images as they used terms associated with numerical concepts; these terms enabled the students to express their growth of mathematical understanding. Furthermore, language switching was observed to influence growth of mathematical understanding. For instance, Alaki's use of the keyword “extra square” allowed him to examine specific features of his images. Selai's example showed how her conscious *translation* led her eventually to *Formalise* her understanding associated with her images of the extras. By contrast, other incidents showed how students' language switching through *translation* and interpretation of specific keywords constrained their growth of understanding. The example of Nanasi's interpretation of “prediction” showed her combining a lack of language understanding and a lack of mathematical understanding in order to determine what prediction meant *mathematically*.

Language switching also interacted with growth of mathematical understanding during instances of the students moving out and working at the *Formalising* layer. Yet, the data did not show the dominance of one form of language switching over another when students worked at the
Formalising layer. What the study did show was that language switching allowed the expression of Formalised understanding. In addition, the need to express explanations in English prompted the students to translate and shift to expressing their generalizations in English. In some instances, the switch from Tongan to English did not match the students’ actual understanding of the task. The example of Selai and Kepi’s attempt to explain in English their general pattern for constructing the diagram totals illustrates this disparity, which on the surface, appears to mean the opposite of what they intended to say. Such difficulties with language can pose a problem, particularly at the Formalising layer, where an explicit form of verbalization is needed to show clear indication of the students’ Formalised understanding. The students in this study showed that in a Tongan-type bilingual situation, when formal mathematical understanding had to be expressed in English, borrowing and shifting was sometimes needed to express ideas that were verbally inexpressible in the students’ first language. It is important to note, however, that a lack of language understanding did not necessarily mean a lack of mathematical understanding.

The evocative nature of language switching was particularly important when students were observed to work in a particular layer, and when they moved across layers, because new actions and understanding are activated through language switching that are distinct from the tasks the students were previously, or were currently, working on. In working at a particular layer, language switching calls up specific images. As the data showed, most images were appropriate, while some turned out to be inappropriate. When the images were appropriate, it allowed the students to extend their understanding at a particular layer, or they provoked students in their understanding, while the inappropriate images resulted in the students either folding back or in their inability to progress forward. Folding back, as illustrated in this study, can be beneficial, and sometimes necessary, in order for students to progress in their understanding.
The data showed evidence of how each of the forms of language switching allowed the students the expression of their growth of mathematical understanding. At the same time, the interaction between an individual and his or her environment (such as the questions involved in the task and discussions with peers) played a significant role in the individual’s growth of understanding. However, the individual’s growth of understanding was observed to be dependent on, but not determined by, the actions of his or her peers or the demands of the questions. The data presented examples where these two external factors contributed in both provoking and invoking the individual’s growth of understanding. Such influences were seen in how some students folded back to work at an inner layer with outer-layer knowing (for example, the effect of Pola’s act of folding back on Seini’s growth of understanding), and in other cases, where the students moved outward from their current-layer understanding (for example, when Selai Formalised her understanding, after Kepi figured out the arithmetical rule for calculating the totals). A similar example also featured one student (Alaki), who was observed to discontinue working with his current images (in the extras) at Property Noticing and moving out instead to work with a pattern in the totals at the Formalising layer as a result of his peer’s (Malakai) generalization. In this latter example, however, Alaki was ultimately unsuccessful in making the connection between his images: those associated with repeated addition and those associated with multiplication.

The data clearly showed no relationship between language switching and folding back and working at an inner layer with existing understanding. This finding applies to moving from working with formal (or outer-layer) understanding to informal (inner-layer) understanding, in the Pirie-Kieren sense. However, the provocative move from informal to formal mathematical understanding is sometimes dependent on the evocative nature of language switching, as previously discussed. This evocative characteristic explains why the other three forms of folding back – collecting, causing discontinuity, and moving out of the topic and working there – can be
related to language switching. The *collecting* example of Pola and Seini illustrated how language switching, through *borrowing*, enabled both the invocative action and the expression of that action. The evidence associated with *moving out of the topic and working there* was largely associated with the students' lack of language understanding, but not with their understanding of the mathematics. Because of the separation between language understanding and mathematical understanding, the students involved in this study were observed to engage in *ineffective folding back* whenever they moved out to discuss language issues. The fact that language switching was found to be capable of evoking new understanding raises a challenge pertaining to how language switching is utilized or related to these specific forms of *folding back*. After all, since language switching can evoke new images, it is possible that language switching can also call up new concepts unrelated to the topic a learner is working with at the time. It is therefore very likely that language switching would allow the learner to *move out of the topic* and work in the realm of the new concept, or that switching would *cause discontinuity* as the student explores the new concept, unrelated to his or her existing understanding. Thus, despite the absence from the data of instances of *causing discontinuity*, all the forms of *folding back* would warrant further research in this context of language switching.

Lastly, Chapter 3 discussed how *acting and expressing* complementarity has yet to be developed further within the monolingual context. However, this feature has been used to describe and account for the students' complex mathematical activities. Towers' (1998) study on teacher interventions showed how the *acting and expressing* complementarities feature could be realized in a monolingual learning situation. While extensive research on this feature within the bilingual context is beyond the scope of this thesis, *acting and expressing* complementarities are nevertheless an interesting aspect of the students' growth of mathematical understanding, through their verbal and non-verbal actions. This study, however, found no specific connection
with language switching and *acting and expressing* complementarities, based on any of the mappings in the data. The results of such observations, in fact, showed no clear evidence of whether language switching depended on either *acting or expressing*, or whether it was related to the move between the two complementarities. At the same time, all of the mappings exhibited aspects of both *acting and expressing*, without a specific connection to a particular form of language switching. However, further research in this area is warranted, in order to examine how language switching is related to the *expressing* component, in the sense that a bilingual individual makes explicit in representing to others and to him or herself what kinds of understanding are involved in a particular activity.

### 8.3 Implications and Contributions of the Study

#### 8.3.1 Contributions to the Pirie-Kieren Theory

This research has taken a detailed look at the Tongan bilingual students’ growing understanding of patterns and relations, and examined its relationship with the students’ acts of language switching using the Pirie-Kieren approach as a theoretical lens. In his study of monolingual students, Martin (1999) asserted that “explaining and accounting for mathematical understanding is a complex and challenging problem” (p. 261). While Martin’s assertion appears to be magnified in any bilingual situation, the employment of the Pirie-Kieren theory was ideal for allowing this study to respond to the research question at hand, particularly because of the theory’s alternative view toward observing, describing, and accounting for any evidence of (growth of) mathematical understanding in bilingual students. This alternative view disregarded the necessity called upon by educators and researchers such as Clarkson (1992), Clarkson and Galbraith (1992), and Clarkson and Dawe (1997) to categorize each bilingual student differently
according to his or her proficiency in both languages. The current study did not concern itself with language proficiency. Through the Pirie-Kieren theory, this study looks at not just when, and if, the students switched languages, but whether their switching influenced their growth of mathematical understanding. Hence, the importance of this study lies not only in its ability to use the Pirie-Kieren theory as a theoretical tool for examining the relationship between language switching and growth of mathematical understanding, but also in its ability to demonstrate the theory's applicability and validity in a bilingual context. This study, however, suggests a need for further research on the questions of both folding back and acting and expressing complementarity and how these activities relate to language switching or how they are experienced within a bilingual context.

8.3.2 Contributions to the Notion of “Bilingualism” and Bilingual Education

This study is clearly applicable to other Tongan-type bilingual situations that involve individuals using words with no direct or precise translation between a dominant Western language and an indigenous language. However, the extent of this study’s application to other bilingual situations involving pairs of advanced languages such as English and French (as in Canada), or Spanish and English (as in the United States) remains limited, because the historical growth of these languages into modern-day Western languages took place at the same time as images were developed in association with these languages, including the development of mathematical images. English words, for example, were invented to associate with Western everyday life, or to carry images from other root languages such as Arabic, Latin, or Greek. Although mismatches still exist between various developed languages, the historically associated images remain within their respective cultures. By contrast, typically underdeveloped countries with indigenous languages, such as Tongan, have suddenly encountered a different language for ideas that were
previously undeveloped within their respective cultures. In Tonga, mathematical concepts such as "absolute numbers", "negative numbers", and "triangular numbers", for example, have yet to be associated with people's everyday activities (Fasi, 1999). As is common with countries encountering importation of other dominant, foreign languages, Tongan people tend to Tonganise, or borrow, English words. This study, nevertheless, is limited in its scope to focusing on bilingual populations undergoing a transition between their traditional use of indigenous languages and the imposition of Western language-learning in the classroom (such as English) – a setting that tends to lead to the phenomenon known as language switching, and one which allows for the study of such switching and its relationship to the growth of mathematical understanding.

At the same time, it is important to remember that this study presents aspects of the Tongan cultural and social environment that offer unique mathematical experiences for their students. These experiences, coupled with the degree of "richness" of the Tongan language, determine how each individual student copes with language as a tool of thought, and as a means of communication. In Tonga, the relationship between language and images is very much intertwined with the environmental upbringing of the students, which has evolved over time. For example, Chapter 3 presented culturally unique ways in which Tongans count objects of various types, and these activities remain a part of their everyday lives. Barton, Fairhall, and Trinick (1998) talk about similar mathematical experiences in connection with the Maoris' unique way of life. Interestingly, in Tonga, such ethno-mathematical knowledge is included in the syllabus for teaching Tongan Study, but has yet to be included in the curriculum for Mathematics.

While English and French are among the academic languages in which mathematics can be discussed at high degrees of abstraction, a significant issue in mathematics education involves
whether mathematics should be taught in indigenous languages (Barton, Fairhall, & Trinick, 1998). This issue has been a concern in bilingual mathematics education systems of the Tongan-type, and as Barton, Fairhall, and Trinick (1998) observe, “both language and mathematics are caught between opposing forces: the risk of becoming fossilized on the one hand and the danger of being corrupted on the other” (p. 3). These bilingual educators believe it is possible to balance the need to preserve native languages and mathematical concepts, while exposing local cultures to “advanced” Westernized education systems, but such a balance would involve changes in mathematics as a result of the language change. This educational quandary has become the central focus in the development of the Maori mathematics vocabularies over the last 25 years, involving a great deal of involvement through the research of Barton, Fairhall, and Trinick. In the case of the Maori people, their language has developed together with its own mathematical vocabulary during the last two decades by deliberately creating Maori words that would evoke the same images as the English words or images associated with their own cultural experiences (Barton, Fairhall, Trinick, 1995; 1998). The dilemma faced by researchers involves the development of a sufficient mathematics register to preserve the existing and ongoing evolution of mathematical knowledge. As Barton, Fairhall, and Trinick (1998) pointed out, the key is to find a balance between “fossilization” and “corruption” factors common to any small-group indigenous language such as Tongan or Maori. The development of a sufficient mathematics register is supported by the fact that any natural language is a living entity (Halliday, 1978). While this plan seems promising, the danger of such a development, as Barton, Fairhall and Trinick (1998) noted, relates largely to moving a language, such as Tongan or any other indigenous language, toward English (or any other dominant language) modes and conventions. Tongan students’ favoured use of Tonganising English words into the Tongan language, in the absence of a standard rule for assimilating such a practice, poses serious concerns that demand close attention by all language (and mathematics) educators in Tonga.
In this study, acts of borrowing and shifting to a second language do not necessarily mean that the individual’s thought patterns are affected. It means only that bilingual individuals, under the limitations of their base language, lack the availability and quick, convenient access to language needed to express their thoughts (or images), and that this problem is less of a consideration, say, for native English speakers. More importantly, a Tongan-type bilingual individual’s limited language, and lack of understanding of the second language, does not necessarily mean he or she lacks mathematical understanding. This study proposes that the effect of bilingual students’ learning and development of understanding in mathematics is largely dependent on the kinds of mathematical images each bilingual student associates with his or her language, and it demonstrates the significant role images play in the bilingual students’ thought processes, and how these images can be implemented in classroom teaching.

8.3.3 The Tongan Mathematical Language, Curriculum, and Bilingual Program

The Maori example presents an evolving language scenario that is not far from the Tongan vision that would see the education system developing its own mathematical vocabulary. After all, the culture and language of the Tongans and Maoris evolved from the same Polynesian roots. However, each culture does differ greatly from the other in the current language- and social-environment associated with the students’ everyday experiences, especially because the Maori situation reveals a minority group existing within a larger Western environment, and thus not geographically independent. The current issue in Tonga’s mathematics education system primarily begs the question of how teachers can best teach mathematics in order for students to learn and understand its concepts effectively, and to determine the language that is best suited to such a task (Barton, Fairhall, & Trinick, 1998). This question relates directly to this study, not
because its findings concern primarily the preservation of an indigenous language *per se*, but because determining the best approach to Tongan mathematics education concerns the use of language, whether in English or Tongan, within a mathematical context. The study shows there is a “check-and-balance” between the role(s) the English and Tongan languages play in the students’ growth of mathematical understanding. Thus, it presents a challenge for teachers in mathematics classrooms: both in terms of the language they choose in presenting the mathematics material, and in the way they *listen* to students, who may or may not switch languages in order to enhance their growth of mathematical understanding.

Ultimately, the findings from this study present implications for application toward the curriculum and bilingual education program in Tonga. Mathematics education problems within the Tongan education system obviously show the need to revisit the development of the vocabulary used for its mathematics curriculum. The example of the Maori language renaissance is worth pondering, given how closely each culture mirrors the other in its history and language development. The Tongan government’s commissioning last year of the first Tongan dictionary is a much welcomed initiative toward maintaining and advancing the native language in order to keep up with the pace of technological change. The teaching of mathematics in a Tongan-type bilingual classroom ought to be re-examined as well, as the study shows that growth of mathematical understanding is context-based and that it can take place successfully without language switching. At the same time, the study shows how language switching can be a valuable part of the students’ mathematical activities, particularly in areas where certain images are only be accessible or expressed verbally through a particular language. These findings suggest the need to find a more effective educational approach toward using two languages in the Tongan classroom environment, without allowing educators to be tainted by the belief that
students understand mathematics better when they limit their language use to a particular language.

8.3.4 Implications for Teaching and Learning

The fact that the growth of mathematical understanding takes place in the absence of language switching, and the fact that language switching occurs without growth of mathematical understanding, also suggests many implications for educators and educational resources. Thus, in teaching mathematics, teachers' language and textbook content must reflect the separate, yet interconnected influences of mathematical understanding and language understanding. In the learning of mathematics, teachers should emphasize the role of images in the construction of understanding for a particular topic. According to the research uncovered by this thesis, the relationship between language switching and growth of mathematical understanding will determine the appropriate language to use with bilingual students for particular mathematical activities. Teachers who know best which specific mathematical areas (concepts, ideas, or images) are accessible or expressed verbally through a particular language will be a step closer to choosing the best language of instruction. Thus, once the myths have been dispelled about the disadvantages of teaching students in their native language, teachers will find themselves far better prepared in classrooms when they discover the advantage of teaching in two languages. At the same time, teachers must learn to appreciate how and when it is appropriate to use either the students' first or second language, if they are to offer their bilingual students the greatest opportunities for classroom learning. Although many foreign mathematics teachers in Tonga may not be required to actually teach in two languages, for those whose goal it is to teach mathematics, it is important that such non-native teachers understand the significance of
focusing on the mathematical ideas, concepts, and images, while at the same time introducing the
students to the mathematical conventions and terminologies of the target language.

Pirie’s (1996a) study, appropriately titled, “Is Anybody Listening?” talks about the essential need
for educators to listen to what students actually say, how they say it, and when they say it. For
instance, Malia’s misunderstanding of her peer, Kelela’s, use of the word “triangular” illustrates
how the listener needs to clearly discern what the students mean, and not necessarily what they
say. After all, a grasp of mathematics rests in the ideas and not in the words (Lakoff & Nunez,
2000). But it is this kind of literal misunderstanding that teachers often assume when they “hear”
wrong mathematical labels and consequently “believe” they have encountered a lack of
mathematical understanding. Thus, teachers ought to pay attention to distinguishing wrong
mathematical labels from a lack of mathematical understanding, particularly in bilingual
situations where an act of language switching can be easily overheard, identified, and likely evoke
instantaneous images different from one “hearer” to another. The use of video technology
in this study provided numerous re-viewing opportunities toward the effort to be a better
“listener”. In fact, improved listening skills enabled the researcher of this study to distinguish
between instances of the students using the wrong mathematical labels and students simply
exhibiting the wrong (or a lack of) mathematical understanding.

Another serious issue for teachers to ponder in bilingual situations involves students’ common
misconceptions about particular mathematical concepts. The evidence from the data, particularly
the students’ interpretation of the notion of “prediction”, demonstrated how their growth of
mathematical understanding may be similar to monolingual students, because monolingual
English-speaking students seem to commonly misinterpret this term as well (studies by Pirie
(1996) and Thom (1999)). The Tongan data provided another interesting distinction that is often
neglected in teaching the concept of multiplication: The term “multiply” in Tongan is translated as “liunga”. But in the process of doing multiplication, Tongan students used the prepositions ‘e and ‘a e interchangeably in the same way the word, “times”, is used in English. Thus, saying “five times two” in Tongan can either mean “nima ‘e ua” or “nima ‘a e ua”. When the students involved in this study were asked if they were able to identify the subtle distinction between their uses of the two prepositions, they all said, “No”. However, in English, the Tongan phrase “nima ‘e ua” means “two fives”, while the phrase “nima ‘a e ua” means “five by two”. The first phrase refers to an image of multiplication as a repeated addition, while the second phrase refers to an image of multiplication as an area. With the latter, Churchward (1953) noted that the preposition “‘a e” does not have an equivalent word in English, except on occasion, “by”, as is shown in the example given above. The potential problems and misunderstandings associated with such subtle differences means that teachers and students need to carefully watch the way they use language to express their mathematical images, and be aware of how images can be perceived differently through language.

Images, therefore, are fundamental to understanding the nature of one’s growth of mathematical understanding of a specified topic: what Pimm (1987) called the “stuff of mathematics”. Without an image for a particular topic, whether the image is a physical or mental one, a student will struggle to make sense of the mathematical ideas. To be useful, such images must be appropriate and be constructed purposefully, depending on the individual’s current understanding (Martin & LaCroix, 2004). If, in a bilingual situation, mathematics is the goal of teaching, then mathematical ideas, concepts, and images, have to remain the teacher’s focus in order to properly teach this subject. This proposal echoes a similar concern voiced by Moschkovich (1999) in her study of Hispanic students in the United States.
8.4 Personal Reflections

This study has provided numerous opportunities to reflect upon personal teaching and learning experiences, and more importantly, to consider mathematical understanding as "always under construction" (Kieren, Pirie, & Reid, 1994, p. 49), and to explore how such understanding relates to the learning and teaching of mathematics in a bilingual context. Although very little was known about the subject matter at the study's outset, prior educational and bilingual experiences in learning, teaching, and everyday activities offered a useful guide for this research experience. In particular, the employment of the Pirie-Kieren theory, along with the ongoing supervision of Dr. Susan Pirie, and the close support and guidance of other faculty members and graduate students interested in exploring the notion of mathematical understanding, has provided a valuable learning experience and paved the way for tremendous personal growth.

Prior to this study, the researcher's informal and formal education sparked a fascination with the power of mathematics, but left behind a sense of ignorance about the significant role language played in the process of coming to know and understand mathematics. Throughout his elementary and secondary education, the researcher relied upon a strong knowledge in his native language as a basis for understanding the English language, an experience apparently consistent with Cummins' (1979) findings. English was never a strong subject, offering perhaps an extra incentive to excel in mathematics - a "playground" suited with symbols, notations, and terms more accessible to a non-proficient speaker of English. This study, however, provided a chance to glimpse the role a weakness in language can play in mathematical learning, an experience so close to home for this researcher.
8.5 Concluding Remarks

This study identifies and introduces the notion of evocative language switching: how particular acts of language switching are related to the bilingual individual's mathematical understanding, particularly in working with mathematical images. Furthermore, evocative language switching, when it is coupled with images, is said to influence or be influenced by the bilingual individual's mathematical actions, because aspects and features of images can be manifested through language. Just as Cummins (1981) and Dawe (1983) discovered, language switching as an explicit act provides a precise means or a "conceptual peg" for identifying, retrieving, and guiding one's existing understanding and ability to work with images. At the same time, language switching can be a resourceful tool for conveying mathematical meanings among individuals: a tool that bilingual teachers are advised to consider using in appropriate contexts. The challenge Pirie (1996a) put forward based on her study of monolingual students speaks volumes about the important role of language switching in bilingual classrooms. She suggests that since "mathematics is about precision of thought", which is "best expressed through precision of language", one needs to be "drawn to the need to attend to the subtle power of the individual words that students use." Pirie's study showed that such insights could be gained "by consciously and specifically paying close attention to the oral communication of the student, in other words, by actively listening, not simply hearing what they say" (Pirie, 1996a, p. 115).

Through the detailed mappings of the Tongan bilingual students' growth of mathematical understanding, this study disputes two main beliefs: one, that bilingual students find mathematics harder than monolingual students and, two; that Tongan-type bilingual students are naturally disadvantaged in comparison to monolingual students in their learning and understanding of mathematics. These beliefs support the notion that monolingual students are
better off than bilingual students when mathematics is taught or learned in only one language. At first glance, it seems likely that if an individual does not understand the language, then he or she will find it hard to understand the mathematics. However, these beliefs disregard one critical, common factor: the role of the language used to evoke relevant images. The disparity in mathematical richness between the two languages, influencing the choice of images associated with particular words or phrases, is a crucial factor to consider in examining how Tongan-type bilingual students might react to a particular language. As a result, all studies in bilingual situations are contextual and situational, and as this study shows, language switching too, varies among bilingual students in how it is used, and how it affects growth of mathematical understanding.

Finally, the difference between inadequate mathematical understanding, as opposed to inadequate language understanding, presents a key distinction that often creates confusion among educators. While the Tongan-type bilingual students may be disadvantaged in understanding English compared to English speakers, that does not necessarily imply a subsequent disadvantage in mathematical understanding. In this study, the ability to distinguish mathematical understanding from language understanding offers a significant contribution to understanding the teaching and learning of mathematics in a bilingual situation; it allows educators to see what appropriate action(s) should be taken, and what appropriate form of language switching can overcome such language or mathematical obstacles. Ultimately, the findings of this study challenge the assumption that bilingual students have enormous problems in the classroom. In fact, if Tongan-type bilingual students are allowed the flexibility of language switching, and thus access to appropriate terms and images in either language, it is clear that these students can be declared in no way mathematically disadvantaged from their monolingual counterparts.


For each sequence:

(1) Draw the 4th diagram in the sequence.

(2) How many extra squares (in addition to the 3rd diagram) did you draw?

(3) How many squares are there in total in each diagram?

(4) Predict first how many more square blocks will be needed for the 5th diagram.
   a. Draw it and check your prediction.

(5) Discuss the 6th diagram.
   a. Do you need to draw it in order to know how many square blocks will be needed?

(6) What can your group say about the 7th diagram?
   a. How many square blocks in total were needed?

(7) Can your group see a pattern in the number of squares you add each time?

(8) Can your group see a pattern in the total number of squares used in each diagram?

(9) Can your group make any prediction(s) for the 17th diagram?

(10) Can your group make any prediction(s) for the 60th diagram?
Appendix 2: Tongan Translation of the Task Set of Questions

LANGUAGE-SWITCHING AND BILINGUAL STUDENT’S GROWTH OF MATHEMATICAL UNDERSTANDING

The Mathematics Problem-Solving Task 1

Purpose:
I am interested in finding out the effect of language-switching in bilingual students’ growth of mathematical understanding. The data collected will form part of my (Mr. Manu’s) Ph.D. thesis.

<table>
<thead>
<tr>
<th>Sipinga</th>
<th>Fakaholoholo</th>
<th>Fika 1</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>fakatātā 1</td>
<td>fakatātā 2 (x^2)</td>
<td>fakatātā 3 (x^{2+1})</td>
<td>fakatātā 4 (x^{2+4})</td>
<td>fakatātā 5 (x^{2+5})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>149 2</td>
<td>4 + 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ngaahi fakahinokino pea mo e ngaahi fehu'i:
'Oku 'oatu 'a e fakatata fika 1, 2, pea mo e 3, 'i he sipinga fakaholoholo 'oku hā atu. Fekau'aki pea mo e fo'i sipinga fakaholoholo ko 'eni, tali ange 'a e ngaahi fehu'i ko 'eni, pea tohi kotoa 'a ho'omou tali 'i he la'i PEPA-TALI 'oku 'oatu.

(1) Tā 'a e fakatātā fika 4 'o e sipinga fakaholoholo 'oku hā atu.'

(2) Ko e fo'i poloka tapa芳-tatau makehe 'e fiha (tānaki atu ki he fakatātā fika 3) na'a ke taa?

(3) Ko e fo'i poloka tapa芳-tatau 'e fiha fakatata 'i he ngaahi fo'i fakatataad takitaha?

(4) Tomu'a fakakaukau'i angē pe ko e fo'i poloka tapa芳-tatau 'e fiha (fakakatā) 'e fiema'u ki he fakatātā fika 5?
  - Tā pea mou vakai'i angē 'a ho'omou tomu'a fakakaukai?

(5) Fevahevahe'aki ange fekau'aki pea mo e fakatātā fika 6.
  - 'Oku fiema'u nai ke mou tā 'a e fakatātā ke lava ai ke mou 'ilo'i 'a e lahi 'o e fo'i poloka tapa芳-tatau makehe 'e fiema'u?

(6) Ko e hā e lau 'a ho'omou kupu'ki he fakatātā fika 7? Ko e fo'i poloka tapa芳-tatau 'e fiha nai 'oku fiema'u?

(7) 'E malava 'e ho'omou kupu' o tala ha pēteni (sipinga) 'i he lahi 'o e fo'i poloka tapa芳-tatau makehe 'oku toutou tānaki atu 'i he taimi takitaha? Fakamatala'i 'a ho'omou tali.

(8) 'E malava 'e ho'omou kupu' o tala ha pēteni (sipinga) 'i he lahi fakakātoa 'o e fo'i poloka tapa芳-tatau na'e ngāue'aki ki he fo'i fakatātā takitaha? Fakamatala'i 'a ho'omou tali.

(9) 'E malava 'e ho'omou kupu' o tomu'a fakakaukau'i 'a e fo'i fakatātā fika 17?
  Fakamatala'i 'a ho'omou tali.

(10) 'E malava 'e ho'omou kupu' o tomu'a fakakaukau'i 'a e fo'i fakatātā fika 60?
  Fakamatala'i 'a ho'omou tali.
Appendix 3: SUMMARY LIST OF NOTES

Note 1: Selai immediately acts on her *Primitive Knowing* to construct images for the pattern without previously reading any of the questions. Selai engages in *image doing*. She expresses her *Image-Making* activity through *substitution* of the words “diagram” and “one”, with no effect on her growth of mathematical understanding.

Note 2: Selai sees the image in the extras when she says “*So that’s it! Add four every pattern.*” Her remark: “*so that’s it!*” demonstrates how she has internalized an image for the pattern and relation, and she also sounds excited at the thought of *seeing* that image. Her *substitution* follows her growth as she expresses her image for the pattern. She continues working in Tongan while applying her image to calculate the total for the 4th diagram.

Note 3: Selai and Kepi appear to lack a pictorial image for the pattern in order for them to draw the 4th diagram. Both students appear, as a result, to *fold back* from *Image Having* to working at *Image Making*, as they discuss how the 4th diagram is drawn. This *folding back* activity did not relate to any act of language switching.

Note 4: After *folding back*, Selai and Kepi construct a pictorial image for the pattern in terms of the extras. They work in Tongan and hence show no tendency to use language switching. As a consequence, both students move out again, without any involvement of language switching, to working at the *Image Having* layer.

Note 5: Selai is observed here to *reflect* on her mental construct for the pattern and to continually *translate* (through repetition) her verbal description of her numerical image associated with the extras while still working at *Image Having*.

Note 6: Selai continues to use her images at the *Image Having* layer and is able to express (mainly in Tongan) aspects of her images. Her *substitution* of the phrases, “*every pattern*”, and, “*each pattern*”, again refers to a specific diagram, and not a generalization of her understanding of the pattern.

Note 7: Kepi and Selai see two different images in answering Question 4. In this case, both students demonstrate contrasting associations between their images, and the word, “*difference*”. Their separate interpretations do not imply lack of mathematical understanding; just that they have viewed the question differently, based on the original request they received for the task: “*Explain any difference.*”

Note 8: Selai consciously *translates* her mathematical thinking as she continually connects between her images – evidence of her moving out to *property predicting*.

Note 9: Selai’s *translation* in this case appears to enable her growth of mathematical understanding. Her mathematical understanding leads her toward the *don’t-need boundary*, as she continues to “juggle” *translation* back and forth between English and Tongan. However, most of Selai’s verbalizations are in Tongan, until she *shifts* to *Formalise* her understanding in English.
Note 10: Selai adds, “But the pattern should equal --- be the same like plus four to each pattern”. This general statement gives evidence of Selai’s moving out to Formalisation and her use of reformulation in re-expressing her thoughts in English. Interestingly, her use of the word “pattern” refers to two different meanings: the first, a reference to the general connection, and the second, a reference to each diagram. This difference does not affect her understanding.

Note 11: Selai shows repetition in folding back to Property Noticing work with her images for the totals. Her comments indicate that she has shifted her attention to her images associated with the totals, which she has not Formalised; thus prompting her to fold back to work at property recording. Her repetition, in this instance, does not invoke her folding back.

Note 12: Selai uses mainly Tongan to express herself. She is prompted to join Kepi, who is currently using the trial-and-error approach to finding an arithmetic rule to generate the diagrams’ totals. Selai momentarily disconnects herself from her current understanding at Property Noticing to fold back in order to work at the Image Making layer. Her borrowing of the word “formula” indicates, but does not cause, a shift in her mathematical thinking to Image Making.

Note 13: Selai is provoked by her peer, Kepi, to move back out to Property Noticing and she continues working with her existing understanding at the outer layer. She uses borrowing of the word “formula” and shifting to express algebraic expressions in English.

Note 14: Selai continues working, mainly in Tongan, at the Property Noticing layer, by testing various combinations of the diagram numbers – an image associated with the counting numbers – by using multiplication by four, combined with subtraction by five, and then later with addition by one. Selai uses substitution, but it shows no effect on her learning process.

Note 15: Selai concludes that the group should just use the step-by-step method of adding fours repeatedly. She borrows the term “formula” and shifts to express aspects of her noticed property in English.

Note 16: Selai continually reformulates her understanding of the pattern and relation in the extras, as she moves out to work at the Formalising layer.

Note 17: Selai appears to be confident in stating her approach as a general method for constructing the pattern in repeated additions of four (square blocks). Her substitution of the phrases, “each pattern”, and, “every pattern”, refers to each diagram, which allows her to express her Formalisation. Selai concludes, by shifting to Tongan, that without the repeated additions of four, the totals would not be accumulated correctly.

Note 18: Selai’s inability to predict the total for the 17th diagram prompts her to fold back to Property Noticing and to work with her images. Selai’s invocative move is not tied to any language switching.

Note 19: While disconnecting herself from Property Noticing, Selai moves back out to generalization with the extras. She borrows the variable “x” to express her generalization algebraically. Her generalization is represented by her use of the expression, “x + 4”, where x again stands for the numerical totals of each diagram.
Note 20: As the group continually adds four to each total, Kepi suddenly suggests adding 17 twice to get the total for the 17th diagram. But then all the students agree that Kepi's suggestion is wrong, because they predict that the total should be bigger, or much more than 34 square blocks. In this situation, the group has folded back again to work at the Property Noticing layer, with language switching involved.

Note 21: Selai demonstrates continued work at the Property Noticing layer, without any connection to language switching. She extends her existing outer-layer knowing by changing her earlier constructs for the pattern and generating a new connection.

Note 22: Kepi's statement and substitution of the phrase “even number” indicates Property-Noticing activity in his understanding of the nature of diagram totals: that the pattern will result in odd-numbered, and not even-numbered, totals.

Note 23: Selai’s discussion with Kepi moves back and forth between her images without her being able to make a connection between the images. The two students cannot formulate a rule for calculating the totals. Their unsuccessful attempt can be attributed to their lack of mathematical abstraction, and not associated with language switching.

Note 24: The group adopts the total of 65 for the 17th diagram and Selai offers their explanation using the general method of “every pattern must add four to the total number...[it] means, that we get a definite answer in the diagram.” Selai has moved out again to Formalising her pattern in the extras through shifting.

Note 25: Both students are prompted to fold back again to Property Noticing, as they continue manipulating their images using various combinations of multiplications and additions to find a connection between each diagram ordinal number and its total. Language switching plays no role as the students use only Tongan.

Note 26: Kepi associates repeated addition with multiplication by expressing in Tongan the observed connections between the diagram numbers and the totals.

Note 27: Selai folds back temporarily to access her Primitive Knowing, in collecting her base knowledge of multiplication by four. This incident shows how folding back is accompanied by the students' use of their first language in working with their informal activities.

Note 28: Selai has not demonstrated moving out to generalization; instead she continues working at Property Noticing as she notices that her earlier result, which she wrote as “(4 x 17) + 1”, is wrong. This observation is not tied to language switching.

Note 29: Selai, in the process of multiplying 16 by four, folds back again temporarily (as she had done before) to her Primitive Knowing without language switching. Language switching is not involved in this case.

Note 30: Selai and Kepi have a problem trying to find the right English word for “kimu’a” (preceding) to describe their generalized rule. Their use of the incorrect phrase “next diagram” does not mean they have misunderstood the mathematics, because they correctly apply the rule to finding the totals for the 17th and 60th diagrams.

Note 31: Selai folds back again to access her primitive knowledge about multiplication by four, an act that is unrelated to her language use.
Note 32: Tupu works (*Image Making*) in Tongan while her peers are observed to be *image seeing*. A follow-up interview with the group reveals that the given pictorial sequence immediately evokes, in Kelela, a geometrical pattern of triangular shapes the moment he utters aloud the word “triangular”.

Note 33: *Borrowing* Kelela’s term, Malia apparently misunderstands him, thinking Kelela is referring to a numerical image in the sequence: a set of triangular numbers.

Note 34: Malia continues her observation as if she has a numerical image of the extras for the pattern: “*Just add-up the odd numbers. That’s it --- it’s not triangular numbers.*” Malia’s statements show that she switches through *substitution* to using the phrase “odd numbers” to identify the numerical set involved; (iv) it reveals Malia comparing her image (odd numbers) with an existing image (triangular numbers), evoked by Kelela’s comment.

Note 35: Malia shows further evidence of developing her own pictorial images of the extras for the pattern, through interaction with her peers. Since the students have not constructed the 4th diagram, Tupu begins by *folding back to* *Image Making* in Tongan. Malia reacts to Tupu’s pictorial construction as evidence of her *image reviewing*, by expressing her pictorial construction through *substitution* of the word “plus”.

Note 36: During the students’ discussion, and Malia’s movement out to the *Image Having* layer, Malia uses the Tonganised word “*ova*” (over) through *substitution* to describe both the activity involved, and the distinctiveness of her pictorial image for the pattern.

Note 37: Tupu suddenly points at the last two questions (about the 17th and the 60th diagrams in Questions 9 and 10) and indicates their similarity to the current question (about the 4th diagram). While these exchanges show how Malia and Tupu relate the set of questions and the different tasks, they do not connect to the students’ specific images for the pattern, since the two students continue working at the *Image Having* layer. Malia uses *substitution* (“first”, “second”, “third”) and *borrowing* (“sequence”) in expressing her thoughts.

Note 38: Malia continues to work at the *Image Having* layer, enabling her to carry out activities through internalization of her images without having to do the initial actions that bring forth these images. Moreover, language switching throughout this episode does not play a significant role in the students’ mathematical activities.

Note 39: Malia says, “*So it’s adding the odd numbers. Yes, add the odd numbers*” by *substituting* the phrase “odd numbers”. It is at this moment that Malia consciously sees a new property associated with her image of the extras for the pattern.

Note 40: When Tupu replies that she is working on the 7th diagram, Malia points to the sequence and asks, “*Are those square numbers?*” Malia therefore notices another pattern in her numerical image of the totals, which she expresses through *borrowing*; hence she is demonstrating new evidence of *property predicting*.

Note 41: Malia *shifts* to Tongan to express her new finding by using her hand to point to each of the totals and the corresponding diagram numbers in the sequence, saying, “*Three by three is nine; four by four is sixteen; five by five is twenty-five ---*”. 
Note 42: Malia’s act of recall (“seven square”) is quite different from collecting, which involves retrieving previous knowledge for a specific purpose. The mathematical form of “$7^2$” has brought about a different approach to the problem – a symbolic representation of Malia’s noticed property – allowing her to represent the relationship between the diagram numbers and their corresponding totals.

Note 43: Malia notices mistakes with her peer’s (Tupu’s) answer to the question using the keyword “extra”. In terms of her assessment of the extras or additionals, Malia replies, “Extra blocks use odd numbers”. Malia is observed to come close to crossing over the don’t-need boundary to Formalising, but her explanation of the pattern to Tupu still illustrates dependence on specific numbers.

Note 44: Malia shifts to generalized statement in English, provoked by Question 8. She is observed to generalize an algorithmic formula or rule for finding the total number of square blocks in each diagram when she again responds: “Square numbers --- square the number of the diagram”. Such a Formalisation is disconnected from any specific example, action, or image.

Note 45: The beginning of the group discussion shows a move beyond the mathematical topic to clarify an issue associated with language, rather than mathematics; the unrelated discussion does not interfere with the students’ mathematical understanding. The group moves out of the topic to talk about language while comparing the terms, “sequence” and “difference”.

Note 46: Tupu appears to fold back to Primitive Knowing by re-collecting a piece of her primitive knowledge of the relationship in the units of time as a way of checking the students’ prediction of the total for the 60th diagram; that is, the product of squaring 60 is equivalent to the number of seconds in a minute.

Note 47: Alaki quickly reads the first question aloud – draw the fourth diagram in the sequence – then immediately shifts languages to work in Tongan as he points out the total of the first two diagrams.

Note 48: At this point, both students have now seen and formed a pictorial image associated with the extras for the pattern and are both able to articulate its features, and are thus in a position to talk about why a different image may not conform to the image they have. These deductions are evidence of the students moving out in their growth of mathematical understanding to work at the Image Having layer, while working only in Tongan.

Note 49: Alaki expresses aspects of his pictorial image in Tongan and continues to work mainly with his images at the image saying layer, without language switching.

Note 50: Alaki associates the key phrase, “extra square”, from the question, with the common difference (of two square blocks) between any two consecutive layers. This extracted phrase appears, therefore, to dictate the way Alaki approaches his images, and at the same time directs him toward outer-layer understanding.

Note 51: Alaki continues describing his images in Tongan, and then he sketches the base layer of the 3rd diagram and asks Malakai, “What is it called that one at the bottom there?”. Malakai (M) responds, “Base!” Still, Alaki (A) is not satisfied with Malakai’s response, because he continues to use equivalent English words, such as “row” and “step” as metaphors for the extra square blocks being added on both ends of the base layers.
Note 52: Alaki associates the Tonganised equivalent words “sitepu” (step) and “poloka” (block) with the phrase “poloka fakamuimuitaha” (last block) and “sitepu faka’osi” (last step).

Note 53: Malakai steps back and looks for any connection within or among his image(s), which allows him to notice a numerical pattern along the base layers when he declares, “I already know it --- add the prime numbers”. This evidence of moving out to Property Noticing is accompanied by a shift in language to using the non-equivalent English word, “prime”. It turns out to be a wrong mathematical label.

Note 54: Alaki is faced with a challenge that is not immediately solvable, prompting him to fold back to an inner mode of understanding in order to re-construct and extend his currently inadequate mathematical understanding. Language switching is not involved in this incident, as Alaki speaks only in Tongan.

Note 55: Malakai finds a relationship between the diagrams’ ordinal numbers and their corresponding totals, again illustrating continued working with his numerical images at the Property Noticing layer. He identifies this noticed property of the numerical totals as “square numbers”, and then he quickly applies this rule to the 17th diagram.

Note 56: Malakai quickly applies his newfound rule to the 17th diagram, and later verifies the total for the 6th diagram. But while Malakai is referring to the total of the 6th diagram for Question 5, his statement confuses Alaki, partly because he thinks Malakai means to apply the total for the 5th diagram.

Note 57: The two students continue using Malakai’s newly found arithmetic rule to find out the total for the 7th diagram, with no evidence that Alaki has yet understood why the rule applies. At this point, they are asked to explain their answer. Malakai simply says, “No need to draw it!” Such a declaration implies, perhaps, Malakai’s readiness to move outward in his growth of understanding to Formalising.

Note 58: The students shift back and forth, between reading the question in English, to calculating the totals in Tongan. In particular, the Tonganised word, “sikuea” (square) is used by both students in expressing the rule, “Square seventeen”, but is later borrowed as “square”. Alaki shows “disconnected understanding” because he does not quite understand Malakai’s image, and that he has not connected between his image of the extras and the totals (see Note 59).

Note 59: This evidence is represented in MAP 3 (using dotted lines between the marked points “13” and “17”) to show how Alaki’s path of growth of understanding is disconnected. While working at the Formalising layer, Alaki explains and translates the group’s answer as “square the diagram number by itself”, a generalized statement of their understanding at the Formalising layer.

Note 60: Both students rely on their previous constructions to describe the pattern in the extras; hence, they fold back to working at the Property Noticing layer. Alaki, after reading the question, shifts languages to discuss the pattern in the extras in Tongan. Alaki’s language shifting is not related to his folding back.

Note 61: Malakai reflects on his own prior constructions and expresses the pattern in the numerical image using the borrowed language he established earlier: “You just add the prime number to the last row.”
Note 62: While Alaki attempts to explain their answer to Question 2 in English ("Plus two -- add two to the last step and total them"), Malakai says, "Don’t know what the English translation of the words we've used!" Malakai expresses, in Tongan, a difficulty with the translation.

Note 63: Ipeni’s spontaneous reaction to the given pictorial sequence reveals the existing image he associates with the pattern, "Hey, those are steps!" Ipeni then pauses and starts counting the total number of square blocks, indicating his cognitive shift from image seeing to folding back and work in constructing an image (image doing) for the pattern. This invocative action is not related to language switching.

Note 64: When he appears to be done counting the numbers in the 3rd diagram, he quickly adds, "Nine there --- it’s square!" Semi’s declaration shows that he has constructed a numerical image associated with the totals, which he recognizes and labels as part of the “square” numbers (image seeing).

Note 65: Semi expresses property of the set of numbers (totals) in Tongan by reciting the sequence of square numbers. Christie joins him to express the property of the square numbers in Tongan. Semi then goes back, this time at a slower pace, by reflecting on his earlier counting by pointing out to Ipeni the total number of square blocks in the first three diagrams: "One, four, nine --- it’s square."

Note 66: All three students work at Image Having in Tongan as they attempt to describe the pictorial image. Their actions show no involvement of language switching.

Note 67: Semi use of the image to construct the 5th diagram suggests his immediate association with the diagram number. Christie shows repetition in confirming her peer’s work.

Note 68: Ipeni’s immediate answer appears to be his association of each square number (total) with the diagram number, and no longer a recitation of the sequence of square numbers. In this situation, both students move out to Property Noticing, and in Ipeni’s case, he borrows the word “square” to express his rule.

Note 69: Ipeni and Christie mistakenly formalise a rule for the totals, but not the extras. Christie quickly explains the generalized formula using the rule, “Yes, we square the diagram. Just yes as the short answer. Square the diagram”, while Ipeni answers with, “Square the number” (of the diagram). The two students’ responses indicate a shift in their mode of understanding toward Formalised ideas.

Note 70: Christie and Ipeni have not explicitly explored the pattern in the extras. Hence, prompted by Semi, Christie, and Ipeni fold back to working with their images at the Property Noticing layer. Language switching is not involved in this incident.

Note 71: When Christie asks about an explanation for the pattern in the extras, Ipeni responds, “As the diagram goes up, odd number goes too ---” as evidence of his ability to articulate (through substitution) features of his noticed property.

Note 72: Semi turns to answer Question 10 by predicting the total for the 60th diagram, and he immediately predicts the total to be, “Sixty square --- three thousands, six-hundreds. Sixty square --- three-six-zero-zero”. Christie reiterates and writes the same explanation of their Formalised rule -- “Just square the diagram number”.

Note 73: While they initially work at Image Making, all of the students shift from reading the question in English to working in Tongan. As they continue with Question 2, the students momentarily shift languages to using only English.

Note 74: When asked to find the total number of square blocks in each of the first four diagrams, Nanasi misunderstands the request and counts instead (in English) the total number of square blocks altogether, for all four diagrams. Despite Nanasi's language barrier, she moves forward to work at the Image Having layer.

Note 75: Meki joins Nanasi in guessing random totals, when they are asked to predict the totals. Meki appears to disagree with the way the other two students answer the question, but retreats to Nanasi's demand, with her own random guess of 20 square blocks.

Note 76: Rosina directs Nanasi to draw the 5th diagram, but Nanasi appears uncertain about how the diagram should look, thus prompting her to fold back, without the involvement of language switching, from Image Having to Image Making.

Note 77: The group continues making guesses for the 5th diagram. Nanasi turns to Rosina and congratulates her for making the closest guess, which in this case, is the nearest total for the 5th diagram: "You are the one [with the right guess] --- twenty-five!"

Note 78: The group continues translating their predictions to mean random guessing. Meki first offers a strategy of dividing up all the questions into three parts, so that each of them can solve the problems and write the answers separately in order to speed the process. Nanasi, however, reminds Meki: "But the prediction is a must for all of us to participate." Meki agrees: "Yes, except the prediction only."

Note 79: Nanasi reflects on their prior constructions and explains to Meki (in Tongan), using substitution for such phrases as "fifth diagram", "sixth diagram", "seventh diagram", and "peteni" (pattern). Then, after re-reading Question 8, Nanasi shifts languages to explain the structure of the 7th diagram in Tongan.

Note 80: Nanasi responds, "It's plus the odd numbers." Meki then reaffirms their answer to Question 7 and writes, "We add the odd numbers." The students appear, therefore, to continue to work at the Property Noticing layer.

Note 81: Apart from their generalization of the pattern in their images of the extras, the students continue to work at the Property Noticing layer. Their translation of the meaning of "prediction" appears to constrain their mathematical thinking to random guessing, and hence, does not allow them to make the generalization of the pattern in their images of the totals.

Note 82: Malia explains her image in English, "Add three --- add three ---" then shifts to expressing her image in Tongan. She looks over to Tupu and says in Tongan "Look?", pointing at the pictorial sequence, and Tupu nods her head in approval. Malia has therefore moved over from acting, in seeing the common difference between consecutive diagrams, to image saying in expressing aspects of her image.
Note 83: Malia immediately associates the question with a word she recalls from the previous (written English) tasks. She asks, “So that is the extra, right?” Tupu adds, “So the extra is three.” Malia’s answer confirms her image (image seeing) and her ability to express an aspect of it (image saying). The two students’ substitution of the word “extra” serves a particular purpose as a keyword that the students access in reference to their construction of an image.

Note 84: In discussing the set of diagrams, ranging from the 4th to the 7th, the group is observed to move freely between image seeing to image saying, without language switching.

Note 85: Malia verifies the meaning of the question with Tupu by asking, “So that is the extra, right?” Malia’s substitution of the keyword “extra” identifies specific aspects of their images that the two students need to focus upon to determine the pattern.

Note 86: In this case, both students are recalling their Primitive Knowing, associated with various numerical properties, such as “multiple”, “even”, and “odd”, in order to come up with an explanation for the pattern in the totals. Malia continues explaining the pattern to Tupu as an alternation between even and odd numbers, “until it finishes”, while Tupu writes down his answer in Tongan.

Note 87: Malia and Tupou continue working, using substitution of the term, “plus”. No effect of language switching is observed to take place during the group’s working within and between layers (Malia). Throughout their calculations of the totals for the 17th and 60th diagrams, Malia and Tupu do not state any general statement about their rule or express any Formalised understanding as to how to calculate the totals.

Note 88: While most of the group’s discussions are in Tongan, the group occasionally switches languages. But their actions are observed to be unrelated to their movement between and within their work at the Image Making and Image Having layers.

Note 89: Seini’s substitution of the word, “addition”, and the phrase, “odd number”, indicates her ability to recognize specific features of her numerical image as she moves out to working at the Property Noticing layer.

Note 90: Niko is the first to suggest the pattern is an alternation between “fika tauhoa” (even numbers) and “ta’etauhoa” (odd). His use of the Tongan words is interesting, because most of the students in the study mainly used the English words.

Note 91: When Seini asks about the term “composite numbers”, it triggers Pola to access his Primitive Knowing in defining a contrasting borrowed term “prime”, using the Tonganised borrowed word, “fakitoa” (factor). Pola’s example immediately prompts Seini to access her Primitive Knowing in defining “prime number”.

Note 92: Pola then uses Seini’s primitive knowledge to finally express examples of composite numbers [L18]. As a result, Pola proceeds to write their answer for the pattern in the totals as “1, 4, 6, 19, 25. Composite numbers.” His association of the term “composite numbers” with the set {1, 4, 6, 9, 25, …} is incorrect.
Note 93: Natini and Siona move out of the topic to discuss meanings of “introduction” and “instruction”. The analysis of this example indicates that these students switched languages (and engaged in group discussions) to move out of the topic and to discuss an issue in language unrelated to their mathematical work.

Note 94: As the students build the 4th to the 7th diagrams, they identify various pictorial and numerical images for the topic: evidence of their working at the Image Having layer. In particular, the students see the numerical image for the pattern in the extras as a set of “counting numbers” (1, 2, 3, 4, ...), which they easily arrange pictorially in descending order from left to right.

Note 95: The group describes the pattern in image for the extras. The students use this numerical image to answer Question 7, by describing the pattern in the extras as, “You add the counting number to the number of the diagram to get the sequence”.

Note 96: After determining the totals for the first four diagrams, the students recognize the pattern in the totals as “triangular numbers” (1, 3, 6, 10, 15...): their answer to Question 8.

Note 97: Tupu immediately associates “predict” with “guess”, a comparison she makes for the first time, unlike her assumptions during her previous work on the other tasks. Yet, in this situation, Tupu appears to mean “prediction” in the sense of “approximation”, rather than random guessing.

Note 98: Tupu suggests formalizing a rule using “triangular” to mean multiplying the diagram number by three. Tupu’s use of the word, “triangular” leads her to formulate a rule, a way of “guessing” an approximation for the total. Tupu immediately applies this rule to calculating the total for the 17th diagram.

Note 99: The group begins the task by discussing Question 1 – draw the 4th diagram. Hehea states that she does not know the meaning of the word “sequence”, but she takes it, as Lani guesses, to mean the given set of pictorial diagrams in Task 1.

Note 100: Hehea and Lani disagree, and their confusion arises from their differing interpretations of the question, by focusing on separate parts of the question’s wording. Hehea insists on emphasizing how many square blocks must be added “in addition to the third diagram” while Lani’s response indicates her interpretation and focus on drawing the next three diagrams in the sequence.

Note 101: Hehea finds that in order to calculate the total of each diagram, they need to add the diagram number (representing the longest side of the diagram) with one less than that number (representing the shortest side of the diagram).

Note 102: Latu suggests a rule by adding two to the diagram number to find the total for each diagram. Neither student is aware that the rule is incorrect, and do not relate the rule to their previous totals. Language switching has no relationship to their move to Formalising.
Question (1): Draw 4th diagram

Question (2): Extra square blocks = 4
How did you figure that out? Take 4 less 1. 7 less 4 is 1.

Question (3): Square blocks in total in each diagram
1st diagram 2nd diagram 3rd diagram 4th diagram
1 5 9 13

Question (4): Square blocks needed for the 5th diagram = 17
Draw 5th diagram

How do you compare your prediction with your drawing?
Explain especially if there is any difference: The difference is only in the answer is change but the pattern should be the same. Take 4 to each pattern.

Question (5): Do you need to draw the 6th diagram? Yes
Because to know the order of the pattern it is true or not.

Question (6): Square blocks needed for the 6th diagram = 25
Group discussion: You have to F 4 to each
if the number and you get you next answer.

Question (7): Pattern in the extra number of square blocks you add each time – WRITE PATTERN with EXPLANATION: Yes, Plus 4 to every pattern to get the answers.

Question (8): Pattern in the total number of square blocks used in each diagram – WRITE PATTERN with EXPLANATION: Yes, Plus 4 to the total number of square blocks used in each diagram.

Question (9): Prediction(s) for the 17th diagram = 163
Rule or pattern: Every pattern must add four to the total.
Explanation: To get the real answer of the pattern at each diagram, plus 4 to the answer of the previous diagram + 1.

Question (10): Prediction(s) for the 60th diagram = 237
Rule or pattern: (4 × 59 + 1) = 237
Explanation: Using the formula 4 × 59 + 1, to make a 60th diagram here and plus one.

Explanation for patterns or rules in (7) and (8):
4 × 4 + 1 = 17 8 = (4 × 7) + 1 = 28

Using your pattern (or rule), what can your group say about other diagrams?
Suggest a number put it in, but if the diagram is correct and we should find the number before the diagram plus one (4 × 4 + 1) = 17.

Prediction(s) for the 100th diagram = (99 × 4 + 1) = 396 + 1 = 397
Reason: We used a quick way but it's very easy and we do it in hurry way.
LANGUAGE-SWITCHING AND BILINGUAL STUDENT’S GROWTH OF MATHEMATICAL UNDERSTANDING

The Mathematics Problem-Solving Task 3

Purpose:
I am interested in finding out the effect of language-switching in bilingual students’ growth of mathematical understanding. The data collected will form part of my (Mr. Manu’s) Ph.D. thesis.

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<th>2nd diagram</th>
<th>3rd diagram</th>
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Number and write ALL your answers to the questions and SHOW ALL your work on this sheet.

Name: ____________________ School: ____________________ Form: F3

ANSWER & WORK SHEET: Problem-Solving Task 3

2) 7 blocks.

3) 1st diagram: 1
   2nd 3rd 4th
   = 1 = 4 = 9 = 16.

4) 85

5) No.

6) \( \frac{25}{11} + \frac{36}{13} = 49 \) blocks \( \cdot (7^2) \)

7) Yes, extra blocks use odd numbers.

8) Yes, square numbers, you square the number of the diagram to get the sequence.

9) Yes, \( 17^2 = 289 \) squared 11 by itself.

10) Yes, 3600 blocks.

To multiply 60 by itself, you get 3600 blocks.
Appendix 6: Alaki and Malakai's TCA Form 3 Group Answer Sheet for Task 3

Instructions and Questions:

You have been given the 1st, 2nd, and 3rd diagram in a continuing sequence. For this sequence, answer the following questions and write all your answers in the ANSWER SHEET provided:

(1) Draw the 4th diagram in the sequence.

(2) How many extra square blocks (that is, in addition to the 3rd diagram) did you draw?

(3) How many square blocks are there in total in each diagram?

Number and write ALL your answers to the questions and SHOW ALL your work on this sheet.

Name: ___ School: TCA Form: 3

ANSWER & WORK SHEET: Problem-Solving Task 3

1. Diagrams:
   - 1st diagram
   - 2nd diagram
   - 3rd diagram
   - 4th diagram
   - 5th diagram

2. Number of square blocks in each diagram:
   - 1st diagram: 3
   - 2nd diagram: 4
   - 3rd diagram: 9
   - 4th diagram: 16
   - 5th diagram: 25

3. 6th diagram:
   - 6th diagram: 36 square block

4. 7th diagram:
   - 7th diagram: 49 square block

5. Yes, in each diagram starting from the first diagram the last row or the base is a prime number (2)

6. Yes, the pattern is the square number eg (1, 4, 9, 16, 25, ...)

7. 479, Explanation is, 17² = 289

8. 3600, 60² or square 60
Appendix 7: Christie, Ipeni & Semi's LHS Form 3 Answer Sheet for Task 3

ANSWER SHEET: Problem-Solving Task 3

Instructions: Number clearly and write ALL your answers to the questions in the spaces provided and SHOW ALL your group work on this sheet (front and back). You may use the blank papers and cubes provided for your personal work.

1. 7 extra square blocks.
2. 1st diagram = 1  2nd diagram = 4  3rd diagram = 9  4th diagram = 16.
3. 36
   No, because it's already shown on number 4th.
4. 49 square box.
5. Yes, every diagram we add the odd number, as the diagram goes up the more the blocks are come.
6. Yes, we only square the diagram given.
7. \(17^2 = 17 \times 17 = 289\)
   Just square the number of diagram.
8. \(60^2 = 3600\)
   Just square the diagram number.
9. Yes \(= 100^2 = 10,000\)
   Square the number that given by diagram.
The Mathematics Problem-Solving Task 3

Purpose:
I am interested in finding out the effect of language-switching in bilingual students' growth of mathematical understanding. The data collected will form part of my (Mr. Mann's) Ph.D. thesis.

Instructions and Questions:
You have been given the 1st, 2nd, and 3rd diagram in a continuing sequence. For this sequence, answer the following questions and write all your answers in the ANSWER SHEET provided:

1. Draw the 4th diagram in the sequence.
2. How many extra square blocks (that is, in addition to the 3rd diagram) did you draw?
3. How many square blocks are there in total in each diagram?
4. Predict first how many square blocks (in total) will be needed for the 5th diagram.
   - Draw it and check your prediction.
5. Discuss the 6th diagram.
   - Do you need to draw it in order to know how many extra square blocks will be needed?
6. What can your group say about the 7th diagram? How many square blocks are needed?
7. Can your group see a pattern in the number of extra square blocks you add each time?
   Explain your answer.
8. Can your group see a pattern in the total number of square blocks used in each diagram?
   Explain your answer.
9. Can your group make any predictions for the 17th diagram? Explain your answer.
10. Can your group make any predictions for the 60th diagram? Explain your answer.
**The Mathematics Problem-Solving Task 1**

**Purpose:**
I am interested in finding out the effect of *language-switching* in bilingual students’ growth of mathematical understanding. The data collected will form part of my (Mr. Manu’s) Ph.D. thesis.

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<td>7 + 3 = 10</td>
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</tbody>
</table>

Fakafika pe tohi KOTOA 'a ho'omou tali ki he ngaahi fehu'i pea mo FAKAHAA'I KOTOA 'a ho'omou ngaue 'i he la'i pepa pe ko 'eni:

**Hingoa:** ........................................... **'Apiako:** .......................... **Fooum:** F3

**LA'I PEPA-TALI PE'A MO E NGAUE:** Palopalema Fika 1

1. **8)** 3 piloka.

2. fakatātā 1 = 1
   - 2 = 4
   - 3 = 7
   - 4 = 10

3. 5 + 9
   \[ \frac{5 + 9}{16} = \text{Iakai} \]

4. 6. 19

5. 7. 10, tana'ui e foa'ipuka e 3 ki he fo'i toliki 'e

6. 8. **3**

7. **10**. Oka matala tawhoa pe'a takeatawhoa o pehepehe a1 pe ki hehe 'osi,'

8. **10**. Liunga 82 i 17 e 2 o ranaki hia i e 5 i mau ai e 84 + 8 = 149

9. **10**. 60 liunga 82 e 3 o mau ai e 120 tana teki a ni no 58 i mau ai e 178.
Write ALL your answers to the questions in the spaces provided and SHOW ALL your group work, (front and back). You may use the blank papers provided for your personal work.

Question (6): Square blocks needed for the 7th diagram = 25

Group discussion: Keep using odd numbers.

Extra square blocks = 7

How did you figure that out? Adding one to the side plus with the two on the side.

Square blocks in total in each diagram
1st diagram 2nd diagram 3rd diagram 4th diagram
1 4 9 16

Square blocks needed for the 5th diagram = 25

Draw 6th diagram

How do you compare your prediction with your drawing?
Explanation: Some as the top explanation.

Do you need to draw the 6th diagram? Yes
WHY? So that we can count it.

Prediction(s) for the 100th diagram =
REASON: We keep using odd numbers then we pile or add it together to get the answer.

Rule or pattern: odd numbers

Rule or pattern: same as the top explanation

Using your pattern (or rule), what can your group say about other diagrams?
It will be faster than drawing the whole blocks.

Rule or pattern: Write pattern

Pattern in the total number of square blocks used in each diagram - WRITE PATTERN
with EXPLANATION: 1, 4, 9, 16, 25, Composite number

Pattern in the extra number of square blocks you add each time - WRITE PATTERN
with EXPLANATION: 1, 3, 5, 7, 9, add numbers

Pattern in the extra number of square blocks you add each time - WRITE PATTERN
with EXPLANATION: 1, 3, 5, 7, 9, add numbers

Question (7): Pattern in the extra number of square blocks you add each time - WRITE PATTERN
with EXPLANATION: 1, 3, 5, 7, 9, add numbers

Question (8): Pattern in the extra number of square blocks you add each time - WRITE PATTERN
with EXPLANATION: 1, 3, 5, 7, 9, add numbers

Question (9): Prediction(s) for the 17th diagram =

Rule or pattern: Write the odd No.

Explanation: To find what's the next block or finding the next blocks.

Question (10): Prediction(s) for the 60th diagram =

Rule or pattern: Write the odd No.

Explanation: Same as the top explanation.

Question (11): Explanation for patterns or rules in (7) and (8): We use the pattern to find the next blocks instead of drawing the picture.
ANSWER SHEET: Problem-Solving Task

**Answer Sheet: Problem-Solving Task**

**Instructions:** Write ALL your answers to the questions in the spaces provided and SHOW ALL your group work on this sheet (front and back). You may use the blank paper provided for your personal work.

**Question (1):** Draw the 6th diagram

![Diagram 6]

**Question (2):** Extra square blocks = 4

How did you figure that out? Add them by 4

**Question (3):** Square blocks in total in each diagram

<table>
<thead>
<tr>
<th>1st diagram</th>
<th>2nd diagram</th>
<th>3rd diagram</th>
<th>4th diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
</tr>
</tbody>
</table>

**Question (4):** Square blocks needed for the 5th diagram = 4

Draw the 5th diagram

![Diagram 5]

**Question (5):** Do you need to draw the 6th diagram? Yes.

Why? Because it is the pattern followed.

**Question (6):** Square blocks needed for the 7th diagram = 49

Group discussion: The odd number starts from 1.

**Question (7):** Pattern in the extra number of square blocks you add each time = WRITE PATTERN with EXPLANATION:

| 1 | 3 | 5 | 7 |

**Question (8):** Pattern in the total number of square blocks used in each diagram = WRITE PATTERN with EXPLANATION:

**Question (9):** Prediction(s) for the 17th diagram = 289

Rule or pattern: 17x17

Explanation: Use the number that gives the result in the example: if it is 10, we add 10.

**Question (10):** Prediction(s) for the 60th diagram = 3600

Rule or pattern: 60x60

Explanation: Use 60 to multiply with 60.

**Question (11):** Explanation for patterns or rules in (7) and (8):

Using your pattern (or rule), what can your group say about other diagrams?

Prediction(s) for the 100th diagram = 10000

**Reason:** Because 100 x 100 is equal to 10000
Appendix 12: Malia and Tupu's Form 3 Group Answer Sheet for Task 4

LANGUAGE-SWITCHING AND BILINGUAL STUDENT'S GROWTH OF MATHEMATICAL UNDERSTANDING

**Instruction:**

Number and write ALL your answers to the questions **and** SHOW ALL your work on this sheet.

**Name:** ___________________________ **School:** Qsc **Form:** F:

**ANSWER & WORK SHEET: Problem-Solving Task 4**

1. 4 blocks.

2. 1st diagram 1st and 4th
   
   \[ 1 + 3 + 6 = 10 \]

3. 15 blocks.

4. No. plus 6 to the 5th diagram.

5. 28 blocks.

6. Yes, you add the counting number to the number of the diagram to get the sequence.

7. Yes, they are all triangular numbers.

8. 17 times 3 equals 51.
   
   **Because 3 is the total sides of a triangle.**
Question (1): Draw the 4th diagram

Question (2): Extra square blocks = 2.

How did you figure that out? Because we have to add 2 square blocks for each diagram.

Question (3): Square blocks in total in each diagram

<table>
<thead>
<tr>
<th>Diagram</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

Question (4): Square blocks needed for the 5th diagram = 9 square blocks.

Draw the 5th diagram

How do you compare your prediction with your drawing? Explain especially if there is any difference: No difference because our prediction and drawing are same.

Question (5): Do you need to draw the 6th diagram? Yes.

Why? Because we need to know and to see the way it stands and to know how many blocks are needed to add to the next diagram.

Question (6): Square blocks needed for the 7th diagram = 13 square blocks.

Group discussion: We have to add 2 square blocks for each diagram.

Question (7): Pattern in the extra number of square blocks you add each time - WRITE PATTERN with explanation: Yes, because we see that when the number of diagram is bigger the number of square blocks will be bigger.

Question (8): Pattern in the total number of square blocks used in each diagram - WRITE PATTERN with explanation: Yes, because we see that every diagram in the front is bigger than the one in the back.

Question (9): Prediction(s) for the 19th diagram = 17.

Rule or pattern: $17 + 2 = 19$.

Explanation: Because when we see every diagram, we add 2 square blocks, the one in the front has 2 blocks bigger than the one in the back.

Question (10): Prediction(s) for the 60th diagram = 69.

Rule or pattern: $60 + 2 = 62$.

Explanation: Because when we see every diagram, we add 2 square blocks, the one in the front has 2 blocks bigger than the one in the back.

Question (11): Explanation for patterns or rules in (7) and (8): Yes, we've seen that if you work out the other questions or you will get the same answer.

Using your pattern or rule, what can your group say about other diagrams? They will have the same pattern by adding two square blocks to another.

Prediction(s) for the 100th diagram = 102.

Reason: Because you will always add 2 square blocks for each diagram.
Appendix 14: Map of the Tongan-Type Bilingual Societies in Oceania (Fasi, 1999)