THE EFFECT OF TEACHING HEURISTICS ON THE ABILITY OF GRADE
TEN STUDENTS TO SOLVE NOVEL MATHEMATICAL PROBLEMS

by

Lionel Pereira-Mendoza

B. Sc., Southampton University, 1966

M. Sc., Southampton University, 1968

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Department of MATHEMATICS EDUCATION

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date July 10/1975

Abstract

Research Supervisor: Dr. Gail J. Spitler

The study investigated whether it was possible to teach students to apply at least one of the heuristics of examination of cases or analogy to novel mathematical problems, and whether learning to apply heuristics could be better accomplished by having students focus solely on the heuristics, rather than also involve them with the simultaneous learning of mathematical content. Two factors were considered in the design of the study. The first factor considered employing treatments designed to meet different objectives: one instructional treatment was designed to teach heuristics only (H); another combined the instruction in heuristics with the teaching of content (HC); and a third treatment utilized content only (C), without reference to heuristics. The latter treatment was used as an experimental control. The second factor concerned employing distinct instructional vehicles (materials) for the treatments, namely algebraic (A), geometric (G) and mathematically neutral (N) materials.

A sample of grade ten boys was selected for the study. Nine self-instructional booklets were prepared, corresponding to each of the treatment x vehicle combinations. The subjects were randomly assigned to one of the nine groups. At the end of the ten day instructional period (one class period per day) two transfer tests, one algebraic and the other geometric, were administered to all subjects.

Data from 189 subjects (21 per group) were analysed by ANOVA. On the algebraic test the H treatment was significantly better than both the HC and C treatments, and there was no significant difference between the HC and C treatments. The conclusion was that only the H treatment can be claimed to be effective in teaching students to apply at least one heuristic to a novel algebraic problem. On the geometric test both the H and HC treatments were significantly better than the C treatment, and there was no significant difference between the H and HC treatments. The conclusion was that both the H and HC treatments were effective in teaching students to apply at least one heuristic to a novel geometric problem. An analysis of the pattern of heuristic application on the two tests revealed that while both heuristics were employed on the algebraic test there was little evidence of analogy being used on the geometric test. The investigator suggested that the effectiveness of the HC treatment on the geometric test may have been due to the fact that the HC treatment was more effective in teaching examination of cases than teaching analogy, and the geometric test proved more amenable to the application of examination of cases.

There were no significant differences involving the vehicles. Thus it seemed reasonable to conclude that the instructional vehicle employed was not an important factor in whether students learned to apply heuristics.

Finally, an exploratory non-parametric (χ^2) analysis for the H treatment indicated that subjects who were in the top third on either mathematical or reading achievement (or both) scored substantially higher than the bottom third.

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CHAPTER I

The Problem

Background

In a world where there is rapid growth in technology and knowledge, society is faced with the challenge of educating individuals who will be able to function in a continually changing environment. Educators are faced with the problem of identifying and teaching the competencies needed to deal with the changing world. Many educators have discussed the challenge of providing an education that will meet the needs of future generations, and one concept that was common to many of their writings is that of 'strategies of inquiry'. For example, when Downey (1965, p. 22) discussed the implications of the knowledge explosion on a secondary school education he stated:

For it is assumed that if students acquire a mastery and comprehension of the broad structure in a subject area as well as facility with the strategy of inquiry appropriate to the particular subject discipline, they will be in a good position to continue learning as future experiences demand.

Bernard (1972, p. 284) advocated that since the school cannot anticipate the problems students will encounter in the future, approaches to, rather than answers to problems should be of serious concern to teachers. Heathers (1965, p. 2) suggested that "problem-solving thinking or inquiry is generally considered to be the core of the educational process, and the chief mark of the educated person".

The general concern among educators with the role of

strategies of inquiry in the educational process is reflected in the mathematics education literature. For example, Rosenbloom (1966, p. 130-131) indicated the significance of strategies of inquiry in mathematics (heuristics) when he wrote:

One of the major social functions of the mathematics curriculum is to teach one of the principal methods of acquiring new knowledge. To find out whether we are achieving this objective, we must see whether the student can solve problems for which he has received no specific instruction.

One of the leading advocates of the teaching of heuristics in mathematics education is George Polya. Polya (1957, 1962, 1965) has written many articles and texts concerned with the teaching of heuristics, and in these he presented specific questions that can be employed by students for acquiring new knowledge. A few of these questions include:

What is the unknown? What are the data? What are the conditions?...

Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem?...

(Polya, 1957, pp. xvi - xvii)

A succinct summary of many educators' views on strategies of inquiry (heuristics) is embodied in the following statement by Richards (1968, p. 46).

In the most rapidly changing society in history, teachers can no longer foresee what a student will need to know. All the teacher knows is that education will be a lifelong activity, so the most important things he can teach are methods of acquiring new knowledge.

[italics not in original]

Purpose of the Study

The purpose of this study is to explore one component of problem solving, namely whether it is possible to teach students to apply mathematical heuristics to novel mathematical problems. Two factors are considered in the design of the study. The first factor considers the effect of utilizing treatments designed to meet different instructional objectives: one treatment designed to teach heuristics only and another that combined instruction in heuristics with the teaching of content, on students' ability to apply a heuristic. The second factor considers the effect of employing distinct instructional settings, algebraic, geometric and mathematically neutral on students' ability to apply a heuristic.

An explicit statement of the research hypotheses is included at the conclusion of this chapter, after a discussion of the model and terminology employed in the study.

The Role of Frameworks in Research

The initial task in the development of a research study in a particular area is the identification of significant questions related to that area. Either a theory or a model can provide an investigator with a valuable aid in the search for and classification of these questions. Snow (1973, p. 77) suggested that theories and models are

extremely valuable to those pursuing research on teaching. Travers (1969, p. 18) noted that the extensive use of models in scientific work "derives largely from their value in suggesting experiments and studies and their long history as a means of achieving important knowledge". As well as acting as a catalyst in the identification of significant research questions, models can also serve other useful purposes. Nuthall and Snook (1973) suggested that models provide a framework for the interpretation and organization of data. Kaplan (1964) indicated that models can be useful in communication as well as in the organization of data.

Many mathematics educators have alluded to the need for a framework in research. For example, Romberg and Vere DeVault (1967, p. 95) noted that the first step towards identifying needed research in the area of mathematics education is to organize a model of the various curriculum components. Becker (1970, p. 21) made a plea for more theory-related research. Pingry (1967, p. 44) stated the need for a framework when he wrote:

The proponents of research directed towards model building and theory were of the opinion that real progress in research in mathematics education would not be made until such a time that research could be related to a theory of mathematics education, even if the theory were a very crude one.

The next section contains the model within which this study is embedded.

A Model of Mathematics

MacPherson (1970, 1971a) characterized a discipline as having three components. Application, Core and Discovery. Since this study is concerned with mathematical problem solving the discipline of mathematics was chosen as an exemplar of the model.

Figure 1

A Model of Mathematics

MacPherson defined these components as follows.

Application

Application incorporates the set of acts of faith (lore) by which systems developed within the discipline are tied to the physical world. Since there is no logical connection between systems in the discipline of mathematics and the physical world, whenever one attempts to solve

physical world problems by mathematical means lore is called into play. For example, consider the following problem:

John has three marbles. Paul gives John four marbles. How many marbles does John now have?

The translation of this problem into the corresponding mathematical equation, $3 + 4 = ?$, requires an act of faith, namely that $3 + 4 = ?$ is a valid mathematical model for the physical world problem.

Core

The core consists of three components: facts, algorithms and categories.

Facts are propositions about data which are not under active consideration as to validity. Examples of these are:

$2 + 3 = 5$; $2 \times 6 = 12$; $5 < 10$; the Fundamental Theorem of Arithmetic; the axioms of Euclidean Geometry.

Algorithms are procedures for the answering of questions which, if applied correctly, will guarantee a solution in a finite number of steps. Some typical algorithms are:

The decomposition algorithm for subtraction;
the Euclidean method for finding the greatest common divisor of two numbers; the Gaussian elimination method for the solution of a system of equations.

Categories are mathematical frameworks for organizing information. For example:

The natural numbers; Non-Euclidean geometries; algebra; geometry; topology.

Discovery

Discovery incorporates the set of techniques, heuristics, by which mathematical discoveries are made concerning physical world or mathematical problems. Any material so discovered may become part of the core of the discipline.

The term heuristic has been utilized by many mathematics educators. Kilpatrick (1967, p. 19) defined a heuristic as "any device, technique, rule of thumb, etc. that improves problem-solving performance." Wilson (1967, p. 3) referred to a heuristic as "a decision mechanism, a way of behaving, which usually leads to desired outcomes, but with no guarantee of success." Polya (1957, p. 112) stated that the "aim of heuristic is to study the methods of discovery and invention." Polya (1957, p. 113) continued by defining the term heuristic reasoning:

Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem.

One strand that links all these definitions is that a heuristic is a flexible procedure that does not guarantee a solution. The investigator feels that the following definition, which is employed in this study, is in accord with those stated previously:

A Mathematical Heuristic is a procedure which is used for the purpose of discovering mathematical relationships in a novel problem, and whose application does not guarantee success.

The novel problem in which a heuristic is employed may be totally mathematical in nature, or be associated with a physical world situation.

Although the term heuristic is in general use by educators the literature contains few sources of mathematical heuristics. Polya (1957), Smith and Henderson (1959), MacPherson (1971b) and Rousseau (1971) all have suggestions concerning specific techniques that can be employed in solving novel problems, and the heuristics used in this experiment, namely analogy and examination (enumeration) of cases are common to the sources.

The term analogy is in common use in the English language. Webster's dictionary (1967, p. 32) defined analogy as "resemblance in some particulars between things otherwise unlike." Polya (1954, p. 32) defined analogy as "a sort of similarity," while Dinsmore (1971, p. 17) defined it "as the perception of a similarity between the problem to be solved and one that has already been solved." Basically all these definitions describe analogy in terms of a perceived similarity between things. In this study analogy is defined as follows:

Analogy is the heuristic by which a perceived similarity between familiar systems or experiences is utilized in order to raise questions concerning a novel mathematical problem.

The following examples of mathematical problems where analogy can be of value are included to clarify the definition.

(a) The student has discussed modulo 5 arithmetic in some detail and is then presented with modulo 6. An analogy between the new situation and modulo 5 would enable the student to raise questions concerning the structure of the new system. For example, the analogy would raise questions concerning the commutativity of addition, the existence of an additive identity, as well as many other field properties. Thus, the analogy with modulo 5 provides the student with a possible basis from which to analyse the new problem.

(b) In discussing the heuristic of analogy Polya (1957, p. 37) noted that a rectangular parallelogram is analogous to the three dimensional figure, the rectangular parallelepiped, in that the relationship between the sides of the parallelogram are like those between the faces of the parallelepiped. Thus, when faced with the three dimensional figure for the first time employing an analogy with the parallelogram would raise questions such as whether opposite faces are congruent, or opposite solid angles are congruent.

Examining cases is often referred to in the mathematics education literature. Smith and Henderson (1959) discussed the value of enumeration of cases in formulating conjectures. Leask (1968, p 8) defined simple enumeration of cases as consisting "of the presentation of many instances of the generalization to be discovered." Leask noted that the students can form hypotheses based on the cases and then test them to determine their validity. For this study examination of cases is defined as:

Examination of Cases is the heuristic by which particular cases are examined in order to make discoveries concerning a novel mathematical problem.

The cases may be examined in a systematic or random manner. Furthermore, this examination of cases may be employed to formulate a conjecture, or to test the validity of a conjecture developed from the use of examination of cases or another heuristic, such as analogy. When validating a conjecture by examination of cases one will either find cases that support the conjecture, or produce a counterexample necessitating its rejection. The following examples are of mathematical problems where examination of cases can be of value.

(a) A high school student has been presented with the problem of finding a general formula for the following series.

$$1 + 3 + 5 + 7 + 9 + \dots + (2n-1)$$

The student can apply the heuristic of examination of cases by substituting different values for n .

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

$$1 + 3 + 5 + 7 + 9 = 25$$

This systematic examination may suggest the following pattern, namely that the sum of n terms is n^2 . The application of the heuristic only enables the student to make a reasonable guess that the sum is n^2 , and does not constitute a mathematical proof of the result.

(b) An elementary student is exploring the sum of the interior angles of a triangle. By experimenting with different triangles, selected either systematically or randomly, he is able to conclude that the sum of the interior angles of a triangle is 180° . Again this examination of cases does not provide a proof, in a mathematical sense, but only reasonable evidence of the result.

A general discussion of the significance of these heuristics in mathematical problem solving is included in the review of the literature. (Chapter II)

The nature of heuristics is that they are applied to novel mathematical problems. Problem, like analogy, is in common use in the English language. Some research studies defined problem implicitly in terms of the particular

problems investigated in the study, while others explicitly stated what was meant by the word problem. Duncker (1945, p. 1) suggested that "a problem arises when a living creature has a goal but does not know how this goal is to be reached." Maier's (1970) conception of a problem involved the existence of an obstacle, or obstacles between the problem solver and the goal. Polya (1962, p.117) did not define problem, per se, but suggested what it means to have a problem:

Thus to have a problem means: to search consciously for some action appropriate to attain a clearly conceived, but not immediately attainable, aim.

[italics in original]

In each of these definitions there is either explicitly stated, or implied, one of the essential components of any definition of a problem, namely the existence of a barrier that prevents the problem solver from reaching the desired goal. The existence of such a barrier means that a problem must be more than just the routine application of knowledge, skills or algorithms. For example, Davis (1973, p. 21) noted that simple questions such as 'Who invented the Ford?' could not be considered problems for the average adult. Hints on Problem Solving (National Council of Teachers of Mathematics, 1969, p. 1) referred to the non-routine nature of a mathematical problem.

In mathematics the usually accepted concept of a problem differentiates between situations such as this assignment [refers to the assigning of exercises from the textbook - the investigator] and those that require certain behaviors beyond the routine application of an established procedure. A true problem in mathematics can be thought of as a situation that is novel for the individual called upon to solve it.

The sample definitions of a problem cited previously, together with the terminology of the model led to the following definition of problem.

A problem is a mathematical task for which the learner has no available facts, algorithms or categories that can be applied directly to the task in order to reach the desired goal.

This definition of problem is not meant as all embracing, but rather delineates the interpretation of the word problem that formed the basis for developing the tests for this study.

Definition of Terms

The discussion of the model includes many definitions that are used throughout this study and, for convenience they are restated in this section.

A mathematical heuristic is a procedure which is used for the purpose of discovering mathematical relationships in a novel problem, and whose application does not guarantee success.

Analogy is the heuristic by which a perceived similarity between familiar systems or experiences is utilized in order to raise questions concerning a novel mathematical problem.

Examination of cases is the heuristic by which particular cases are examined in order to make discoveries concerning a novel mathematical problem.

A problem is a mathematical task for which the learner has no available facts, algorithms or categories that can be applied directly to the task in order to reach the desired goal.

One implication of these definitions is that any mathematical task that could be called a problem would be considered as novel for the problem solver, and require the application of a heuristic, or heuristics in attempting a solution. For such problems, problem solving is defined as follows:

Problem solving is the search for and application of heuristics, facts, algorithms and categories that aid the learner in reaching the desired goal.

The model, together with the terms defined in this section provided the investigator with the framework within which to develop and state the hypotheses.

Three questions arising from the model

The structure of the model suggests certain questions concerning the teaching of heuristics. The three questions which directly related to the formation of the hypotheses are discussed in this section.

(1) Is it possible to teach students to apply heuristics to novel mathematical problems?

(2) Would it be more efficacious to teach heuristics only or combine instruction in heuristics with the teaching of content?

(3) Would the ability to learn how to apply heuristics depend on the materials selected for their introduction?

The model specifies the existence of a set of heuristics. This study selected the heuristics of analogy and examination of cases and investigated whether it is possible to teach students to apply at least one of these heuristics to novel mathematical problems. The first two questions led to the selection of two treatments, heuristics only and heuristics plus content. The heuristics only treatment was designed to teach only the heuristics, while the heuristics plus content treatment was designed to combine instruction in heuristics with the teaching of content. A third treatment utilized content only, with reference to heuristics, and was used as an experimental control.

The core contains many categories such as algebra and

geometry. Question 3 concerns whether one category is more effective for introducing heuristics than another. Since the secondary school mathematics curriculum is primarily based on the categories of algebra and geometry these two were utilized as instructional vehicles for the heuristics. Furthermore, a student is also exposed to mathematics through a variety of experiences outside a mathematics course, such as science courses, newspapers and commercial transactions. Thus a third vehicle, referred to as mathematically neutral was employed. This resulted in a total of nine groups, corresponding to the cells in figure 2.

Figure 2
Experimental Groups

Vehicles

		Neutral	Algebraic	Geometric
T r e a t m e n t s	Heuristics	H - N	H - A	H - G
	Heuristics plus content	HC - N	HC - A	HC - G
	Content only	C - N	C - A	C - G

For the purpose of later discussion the nine groups have been identified by a two part code. The H, HC and C denote the three treatments: heuristics only (H); heuristics plus content (HC) and content only (C). The N, A and G denote the three vehicles: mathematically neutral (N), algebraic (A) and geometric (G). Further details on these groups can be found in chapter III - Design and Procedure.

The H, HC and C are used separately to denote the three treatments, irrespective of the vehicle. For example, the group receiving the H treatment consists of all subjects in H - N, H - A and H - G.

Research Hypotheses

In each of hypotheses 1 through 5, the dependent variable is a measure of ability to apply at least one of the heuristics (analogy or examination of cases) to a novel mathematics problem:

Hypothesis 1. A group taught heuristics only (H) will score higher than a group taught content only (C).

Hypothesis 2. A group taught heuristics plus content (HC) will score higher than a group taught content only (C).

Hypothesis 3. A group taught heuristics only (H) will score higher than a group taught heuristics plus content (HC).

Hypothesis 4. The mean score of a heuristics only group (H - N, H - A, H - G) as compared to the mean score of the corresponding content only group (C - N, C - A, C - G) will be independent of the vehicle.

Hypothesis 5. The mean score of a heuristics plus content group (HC - N, HC - A, HC - G) as compared to the mean score of the corresponding content only group (C - N, C - A, C - G) will be independent of the vehicles.

CHAPTER II

Related Literature

Introduction

This review of the literature is organized in the following manner: (1) literature on the problem solving process; (2) historical and education views of analogy and examination of cases; and (3) related research.

The Problem Solving Process

Although educators have discussed the role of heuristics in problem solving there have been few research studies that have investigated general heuristics. Wittrock (1964, p. 61) had suggested that researchers should direct their attention to teaching children how to make discoveries. He wrote:

Instead of asking if learning by discovery is or is not as effective as some other way to learn, one ought now to start to ask by what sequence of directed or undirected experiences can we teach children to learn to discover.

Shulman (1968, p. 89) observed that there was a lack of research on general heuristics when he wrote:

Notably absent are studies which deal with the question of whether general techniques, strategies, and heuristics of discovery can be learned - by discovery or any other manner - which will transfer across grossly different kinds of tasks.

One approach taken by educators and psychologists has been to investigate the nature of the problem solving

process. Their efforts have resulted in analyses of the problem solving process. For example, Bloom and Broder (1950, p. 25) suggested that the following four components are involved in the problem solving process.

1. Understanding of the nature of the problem
2. Understanding of the ideas contained in the problem
3. General approach to the solution of problems
4. Attitude towards the solution of problems.

Garry and Kingsley (1970, p. 464) suggested that there are three steps in problem solving: the search phase, the functional solution phase, and verification of the final solution. Gagné (1966, p. 138) included the stages of defining the problem by distinguishing essential features, searching for and formulating hypotheses, and verifying the solution in his analysis of the problem solving process. While these analyses give an overall framework for examining the nature of the problem solving process, they do not include specific techniques that can be applied by the learner to make discoveries about novel problems. For the mathematics educator the works of Polya (1954a, 1954b, 1957, 1962, 1965) provide a major source of specific questions that can be applied to novel mathematical problems. In How To Solve It Polya (1957, pp. xvi - xvii) listed four stages in the problem solving process.

UNDERSTANDING THE PROBLEM

What is the unknown? What are the data? ...

DEVISING A PLAN

Have you seen it before? Or have you seen the same problem in a slightly different form? ...

Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve part of the problem? ...

CARRYING OUT THE PLAN

Carry out your plan of the solution, ...

LOOKING BACK

Can you check the results? ...

The major headings are general, but the various suggestions under these headings contain a series of specific questions that a problem solver can use when faced with a novel problem. The significant contribution of Polya to the study of heuristics in mathematics is evidenced by the fact that much of the research concerned with mathematical heuristics is based upon Polya's work. For example, Ashton (1962) employed the questions in Polya's heuristic process as the basis of her heuristic treatment in the study of problem solving by ninth grade algebra students. A major component of Kilpatrick's (1967) study concerned the development of a system for identifying characteristics of the problem solving process. Polya's heuristic questions and suggestions formed the initial basis for developing the Kilpatrick categorization system.

Smith and Henderson (1959) have also developed categories of problem solving thought processes. They listed five types of evidence for probable inference that are useful in developing and testing generalizations. These are the methods of simple enumeration, analogy, extending a pattern of thought, hunches and their relation to proof, and testing the hunches (Smith and Henderson, 1959, p. 121). Two additional sources were available to the investigator, namely unpublished articles by MacPherson (1971b) and Rousseau (1971). In selecting the heuristics of analogy and examination of cases the investigator chose two heuristics that were common to all four sources (Polya, Smith and Henderson, MacPherson, Rousseau).

Analogy and Examination of Cases

Analogy

The historical significance of analogy as a heuristic cannot be overestimated. The classic reference to the value of analogy can be attributed to Henri Poincaré. In developing the theory of Fuchsian Functions Poincaré was reported to have said (Hadamard, 1949, p. 13):

I want to represent these functions by the quotient of two series; this idea was perfectly conscious and deliberate; the analogy with elliptic functions guided me.

Polya (1954a, pp. 18,19) noted that Euler employed an analogy to conjecture a sum for the infinite series $\sum 1/n^2$.

The value of analogy is not limited to the discipline of mathematics. For example, Maxwell (Niven, 1890, pp. 156 - 7) discussed the importance of physical analogies in the mathematical sciences, where one is involved in analogies between physical laws and the laws of number. Brumbaugh (1968, p. 6) indicated how Plato used analogy in his arguments. He wrote:

If we can find numbers falling in the same classes as the entities we want to study, the relations of these entities can be illustrated and investigated by observing the correlated relations of the analogous classes of numbers.

Deltheil (1971, p. 37), after mentioning the works of Kepler and Poincaré concluded:

The establishment of such comparisons is the very essence of the genius for discovery and the goal of a mode of intuition of the first importance, analogical intuition.

The key historical significance of analogy as a heuristic has been summarized in the following brief but concise quotation of Polya (1954a, p. 17), who said of analogy that it "seems to have a share of all discoveries, but in some it has the lion's share".

Many educators have discussed the role of analogy in teaching. In illustrating the value of analogy as a problem solving technique Smith and Henderson (1959, p. 124) wrote:

Analogies between the two subjects [plane and solid geometry - the investigator] facilitate learning, provide a hierarchy of relationships that are more easily remembered, and suggest many conjectures that may later be proved or disproved.

Kline (1964, p. 457) suggested that once the ideas of Non-Euclidean Geometries have been grasped there is value in "seeking the analogues of theorems that held in Euclidean Geometry". Moser (1968, pp. 374 - 6) pointed to the value of analogy in developing relationships between the graphs of various conics and appropriate absolute value equations. For example, he considered the relationship between the graphs of $y = x^2$ and $y = |x|$; and $(x/2)^2 + (y/2)^2 = 1$ and $|x/2| + |y/2| = 1$. Finally, Balk (1971, p. 135) stated that:

It is well known that heuristic methods, and above all analogy and induction, play an important role in any specialized field.

Balk (1971, pp. 135 - 6) continued by considering examples of where analogy could be of value in the formulation of conjectures. For example, he suggested how the propositions that if the sum of the digits of a number is divisible by 3 (or 9) then the number is divisible by 3 (or 9), could lead to the analogous proposition that if the sum of the digits is divisible by 27 the number is divisible by 27. The fact that this particular conjecture is invalid does not minimize the value of analogy as a mathematical heuristic.

These examples provide an historical perspective of analogy as well as an indication of the educational significance of the heuristic. It seems reasonable to conclude from this literature that analogy is a heuristic worth teaching.

Examination of Cases

Direct historical evidence of mathematicians applying examination of cases is practically non-existent. The only direct reference is to Nichomachus (Heath, 1931, p. 68) considering the series of odd numbers:

1, 3, 5, 7, 9, 11, 13, ...

Nichomachus wrote that 1 was a cube; the sum of the next two, $3 + 5$ was a cube; the sum of the next three was a cube; and so on; and he noted that such information could be utilized to deduce the formula for the sum of the first n cubes.

Indirect evidence of the utilization of examination of cases is found in Gauss's diary. There is a note to the effect that every positive integer is the sum of three triangular numbers (Bell, 1937, p. 228).

The lack of direct historical evidence of mathematicians applying the heuristic of examination of cases could well be attributed to the nature of published mathematics. Mathematical publications consist of the final products, usually in the form of proofs of a mathematical endeavor. Since examination of cases would be most appropriate in the exploratory phase of discovery, reference to the heuristic would not appear in the final literature. In particular, the cases may have been the first steps leading to a proof by mathematical induction. Hence, only

the final form of proof would be published and not the mathematician's investigation of different cases. Hardy's comment (1940, p. 114) on proof by complete enumeration of cases may reflect a prevailing attitude to references to the utilization of the heuristic:

... but it is just the 'proof' by enumeration of cases' (and cases which do not, at bottom, differ at all profoundly) which a real mathematician tends to despise.

Although there is little historical evidence of the use of examination of cases, educators consider the heuristic to be of significant value. Smith and Henderson (1959) included enumeration of cases in their five types of evidence for probable inference that should be encouraged in the secondary school. Kinsella (1970, p. 244) suggested that "the discovery of relations usually involves the strategy of studying special cases in an inductive manner until a pattern is perceived". Szabo (1967, p. 840) gave the following impression concerning the heuristic:

We also feel that students can learn how [italics in original] to search for patterns and will be able to use this knowledge in solving new problems.

Brown and Walter (1972, pp. 42 - 44) demonstrated how specific and general cases can be employed to formulate conjectures and questions. They asked students who were familiar with the following formula for generating Pythagorean Triples

$$a = m^2 - n^2 \quad ; \quad b = 2mn \quad ; \quad c = m^2 + n^2 \quad .$$

to use the formula to pose questions and formulate conjectures concerning Pythagorean triples. The questions raised include:

How are a , b and c affected by the evenness and oddness of m and n ? What happens to a , b and c if m and n are consecutive?...

The examination of specific triples produced conjectures, which were often stated in the form of questions such as:

Is c always odd?; If a is odd, then $c = b + 1$;
For a fixed, are b and c unique?...

Scott (1971, pp. 38 - 45) showed how the examination of the number of cells formed by the diagonals of different polygons could lead to an investigation of different patterns. For example, what is the pattern connecting the number of sides of a polygon and the number of cells formed by the diagonals.

Figure 3

Number of Cells

4 cells

11 cells

It seems reasonable to conclude that educators consider both analogy and examination of cases to be valuable techniques for problem solving. The historical evidence related to analogy suggests that this heuristic plays a significant role in mathematical discovery, while educators have long promoted the teaching of both analogy and systematic examination of cases.

Related Research

At the present time there is little research on teaching students to apply mathematical heuristics, per se. However, teaching students to transfer learning to new situations has been investigated by both educators and psychologists. The following review of the research is divided into two sections. The first section reviews studies that have compared discovery, as a teaching technique, with other forms of instruction. The second section considers studies where the major objective was to develop students' ability to apply general problem solving procedures (including heuristics) to new mathematical situations.

Discovery Versus Other Instructional Techniques

The studies reported here all include an investigation of the relative effectiveness of the instructional techniques in teaching students to transfer learning to new

situations. This review is limited to studies involving mathematical content.

Kersh (1962) used a programmed booklet to teach 90 high school students two rules for addition of series.

Kersh (1962, p. 66) described the rules as follows:

1. Odd Numbers rule. The sum of any series of consecutive odd numbers beginning with 1 is equal to the square of the number of figures in the series. (For example, 1, 3, 5, 7, is such a series; there are four numbers, so 4×4 is 16, the sum.)
2. Constant Differences rule. The sum of any series of numbers in which the difference between the numbers is constant is equal to one-half the product of the number of figures and the sum of the first and last numbers. (For example, 2, 3, 4, 5, is such a series; 2 and 5 is 7; there are four figures, so 4×7 is 28; half 28 is 14 which is the sum.)

The students were told the rules and given practice in their application. On completing the initial booklet the students were randomly assigned to one of three treatment groups, namely directed learning (DL), guided discovery (GD) and rote learning (RL). The DL group received an additional programmed booklet in which the rules were explained. The GD group was taught by a questioning technique in which the students were guided towards an explanation of the rules. The RL group was given no additional instruction and was incorporated in the experiment as a control group. After the instructional period each group of 30 was divided into three subgroups (10 per group). Different subgroups from each treatment were tested after 3 days, 2 weeks and 6 weeks.

The testing arrangement is summarized in figure 4.

Figure 4

Testing Arrangement for Treatment Groups

	<u>Group</u>			<u>Tested after</u>
	<u>DL</u>	<u>GD</u>	<u>RL</u>	
Sub- groups	10	10	10	3 days
	10	10	10	2 weeks
	10	10	10	6 weeks

The criterion test consisted of two word problems that could be solved by applying the rules in the instructional units. The number of subjects who used the appropriate rule in the solution was the index of transfer. With this index Kersh found significant differences between treatments and between test periods on one of the two test problems. He stated that perhaps the most striking result, and certainly unanticipated, was that the RL group "was found to be consistently superior in every respect to the other treatment groups" (Kersh, 1962, p. 68). In terms of the index of transfer the RL group performed best, the DL group worst and the GD group in between. This study is important for two reasons. First, since the addition rules in the experiment were new to all students the results indicate that all three treatments registered some degree of transfer (ability to apply the rules to a new problem). Second,

the results indicate that it is not necessary to teach by discovery to obtain transfer.

Ray (1961) compared discovery teaching with direct instruction on criteria of initial learning, retention and transfer to new but related problems. He used two experimental groups, direct instruction (DI) and directed discovery (DD), while a third group, which received no instruction was used as a control (C). Ray also divided his sample into three ability groups based on high, medium and low intelligence. The sample of 135 grade 9 boys were randomly assigned to one of the six experimental treatment x intelligence groups (18 per group), or to a control x intelligence group (9 per group). The instructional period consisted of 47 minutes of instruction on the physical structure of a ~~micrometer~~ and how to use it. The students were taught in groups of nine, with three students being selected from each of the ability levels. Students were tested one week after the instructional period was completed and further retention and transfer tests were administered six weeks later. The transfer test involved items "that attempted to measure application of knowledge gained to new but related situations" (Ray, 1961, p. 275). He found a significant difference between treatments on both the initial (one week) and delayed (six week) transfer tests, with the results statistically favouring the DD treatment. The

results also showed a significant difference on the intelligence factor with the results favouring the high ability students. Ray's conclusions are different from those in the Kersh (1962) experiment, in that Ray's results indicate that the discovery group performed better than the programmed group (direct instruction). However, like Kersh, the data shows that both experimental groups evidenced some degree of transfer.

Gagné and Brown (1961) employed three instructional techniques for teaching students how to sum number series. Three programmed units were developed, consisting of an introductory programme which was identical for all three units, followed by materials designed to correspond to three treatments, discovery (D), guided discovery (GD) and rule and example (R&E). Thirty-three grade 9 and 10 boys were randomly assigned to the three treatments (11 per group). The transfer test comprised four problems on summation of series, with each problem involving a series that was new to the students. The test was administered individually to each student. Using three measures of effectiveness of the programmes, namely time required to solve the problems, number of hints required and a weighted combination of time and hints, Gagné and Brown concluded that the GD treatment was the most effective in teaching students to transfer, the R&E treatment least effective with the D treatment in between.

Eldredge (1965a, 1965b) reported studies that were based on Gagné and Brown's experiment. In the first study Eldredge replicated the Gagné and Brown experiment, employing two of the three treatments, namely GD and R&E. The replication differed from the original experiment in that Eldredge used a mixed sample, two classes each with 13 boys and 13 girls, and different hints on the transfer test, since the hints used in the original experiment were not available. One class received the GD treatment and the other class the R&E treatment. Eldredge found only partial support for the results of Gagné and Brown. For example, the analysis of the four transfer problems showed a significant difference favouring the GD treatment on one problem, another problem favoured the GD treatment and the other two problems favoured the R&E treatment, although none of the last three differences were statistically significant. The differences between the Gagné and Brown and Eldredge results could be due to many factors, such as different samples and different hints.

In a second study Eldredge employed a modified version of the Gagné and Brown materials that had been developed by Della-Piana and Edlredge (1965a, 1965b). The modified materials consisted of two programmes, a guided discovery programme (GD) and a rule and example programme (R&E). The main differences between these two programmes were in the sequencing of the frames, and the conditions

under which the programmes prompt the students for the rule for summing the series. A total of 97 grade 9 students (43 boys, 54 girls) were used in the experiment. As in the original experiments (Gagné and Brown (1961), Eldredge (1965a)) the transfer task consisted of four unfamiliar problems on the summation of series. The instructional period consisted of three sessions, immediately followed by a transfer test, and then the transfer test was readministered four weeks later. Using intelligence and time taken to complete the programme as covariates, the results indicated that the GD group performed significantly better than the R&E group. A two-way analysis of variance using treatments as one factor and ability (high, low) as the other indicated that the high ability group performed significantly better than the low ability group, and again a significant difference between treatments was found. Identical conclusions followed from the data analysis of the delayed (four week) transfer test. However, if initial performance was used as a covariate in the analysis the differences disappeared, lending credence to the hypothesis that the differences on the delayed transfer test were just a reflection on the initial differences produced by the programmes.

In the studies reviewed to this point the transfer tests consisted of items that required the application of knowledge acquired in the instructional period to problems

that were new to the student but directly related to the instructional materials. Two overall conclusions can be drawn from these studies that relate to the present study. First, the results of the studies reviewed show that it is possible to teach students to apply what has been learned in a mathematical context to new, but directly related mathematical problems. Second, some of the studies (Gagné and Brown (1961), Eldredge (1965a,1965b)) indicate that using self-instructional materials can produce transfer. The present study can be considered an extension of these experiments in that it requires students to apply what has been learned, the heuristics, to transfer tasks where the mathematical content is unrelated to that used in the instructional materials. Furthermore, the present study employs self-instructional booklets and the evidence indicates that this mode of instruction can lead to successful transfer.

The following studies are of particular significance since one (Worthen (1965)) included an investigation of transfer to unrelated problems, and in the other Kersh (1964) searched for evidence of heuristics in the students solutions to the problems.

Worthen (1965) conducted an experiment involving two experimental treatments, discovery (D) and expository (E). The instructional content consisted of a development of

integers, addition and multiplication of integers, the distributive principle and exponential notation with integral powers. The sample consisted of 432 students, 14 grade 6 classes and 2 grade 5 classes in 8 schools, with an additional 106 students (3 grade 6 classes) in a ninth school used for control purposes. Eight teachers were involved in the study, each teaching two classes, one by treatment D and the other by treatment E. At the end of the six week instructional period a variety of criterion tests were administered, including attitude tests, content tests and transfer tests. The three tests germane to the present study are the concept transfer test (cTT), number sequences and series test (nsTT) and a test involving the associative principle (apTT). On the cTT data Worthen employed a two-way analysis of covariance with main factors teachers and treatments, and covariates intelligence, arithmetic problem solving, arithmetic computation and the score on the achievement test on the instructional materials. Worthen concluded that the D group performed better than (p < .08) the E group. He also found significant teacher effects and interaction effects (p < .001). Using the same covariates but employing a one-way ANACOVA with the D, E and C groups he found that the D group performed significantly better than the E group (p < .05), the D group performed significantly better than the C group (p < .01), and there was no significant difference between the C and E groups.

The cTT included new problems that pertained to the instructional materials and as such the test items were extensions of the materials. Worthen's results are consistent with the previous studies in that they show evidence of transfer to new but directly related problems. The nsTT and apTT's consisted of problems on content that was unrelated to the instructional materials. The nsTT consisted of questions on number sequences and series in which the students had to find the pattern and rule. The two-way analysis again showed significant teacher and interaction effects. This analysis also indicated that the D group performed significantly better than the E group ($p < .05$). It appears that Worthen used the words 'transfer of heuristics' to refer to the transfer of ability to make discoveries in new problem situations, in this case to search for and find a pattern. In the apTT the problems involved the associative principle, a principle not taught during the instructional period. The students were faced with problems in which discovering this principle would provide a shortcut to the solution. For example, in finding the sum of $975 + (25 + 983)$ the associative principle would be of great value. Using a grading scheme in which students were given one point for a correct answer and an additional four points if their work showed evidence of the principle, the D group performed significantly better than the E group ($p < .025$). Worthen's results

indicate that the discovery group learned how to search for a pattern or rule significantly better than the expository group. Thus, this experiment provides evidence that an instructional method, learning by discovery, resulted in transfer of heuristics to new problems that were unrelated to the instructional materials.

Kersh (1964) undertook a study to compare highly directed (programmed) learning with non-directed (discovery) learning. Three instructional units were designed with the primary objective of teaching the distributive principle to fifth graders. In the free discovery unit (D) each student was encouraged to discover the principle independently and then show the experimenter that he had learned the principle. The experimenter worked with the students individually, not allowing verbalization of the principle until all the students demonstrated by example that they had learned the distributive principle. The programmed discovery unit (PD) gave guidance designed to aid students to search for the principle. Verbalized approval by the experimenter was used to reinforce the students. A final programme, referred to as programmed guidance (PG) was developed to teach the same principle, but without employing the discovery method. The PG treatment did not encourage the searching behaviours fostered in the D and PD programmes. All students received instruction until they met a predetermined criterion,

namely that they were initially required to correctly answer five of six distributive principle questions. If they failed they were retaught, and the final result was that four students were dropped from the study for failure to meet the criterion. The posttests included a test of new learning, which was administered within 24 hours of a student completing the instructional materials. This test was administered individually to each student, and the students were required to "think out loud", or indicate their thinking by scratch work. Kersh identified problem solving behaviours such as 'search for patterns', 'checking for exceptions to patterns', and 'checking to see if statements were true or false'. In terms of new learning he found significant differences between the treatments on 'search for patterns'. The D group had the greatest number of students who evidenced this behaviour (27 of 30), followed by the PG group (21 of 30) and the least number in the PD group (17 of 30). No significant differences were found on the other criteria of new learning. For example, on 'checking for exceptions to patterns' 11 students in the D group used the technique, 13 in the PG group and 12 in the PD group. Kersh's study shows that the D treatment was more effective than the others in teaching students to apply the problem solving heuristic of searching for patterns.

To summarize, two major conclusions can be drawn from the studies reported in this section that have relevance for

the present study. The studies indicate that (1) it is possible to teach students to transfer knowledge to new but directly related problems, and (2) this transfer can be facilitated by self-instructional materials. Furthermore, Worthen's study (1965) indicates that students can be taught to approach problems that are not directly related to the instructional materials, a result that is particularly significant in light of the fact that the transfer tests to be employed in this study will be totally unrelated to the instructional materials. Kersh's study also produced evidence of the transfer of the heuristic of searching for a pattern, which is a component of examination of cases.

Teaching Heuristic Procedures

Schaaf (1954) designed a course which had as its major objective the development of students' ability to generalize. A group of ninth grade students at a university school acted as the experimental group. Data from 127 ninth grade students (in six classes in the Public School System) were employed for control purposes. The experimental period lasted one year. The content studied by the experimental group included numbers and operations; graphs and formulas; equations and problem solving; proportion and indirect measurement; and statistics. The instructional procedure for the experimental course was designed to encourage students to extend previous mathematical

learning, and to discover new principles from past and present experiences. Throughout the course, emphasis was placed on student discovery, with the teacher acting as a guide in helping the students to discover generalizations. The methods of generalization that were taught included simple enumeration, analogy, extension of form (extrapolation and interpolation), deduction, variation and inverse deduction. A pretest of the ability to generalize indicated that the experimental group was significantly better at generalizing than the control group. At the end of the instructional year the generalization test was readministered. Additional information on the experimental group was available from sources such as student notebooks, Schaaf's own notes on the classroom meetings and observer reports (an observer visited the classroom periodically throughout the experimental period). Schaaf's results indicate that on some measures of the ability to generalize the experimental group made significantly greater gains than the control group. For example, the experimental group's gain in generalizing by extrapolation and interpolation in situations where such generalization was justified was significantly greater than the control group's gain. On other measures he found significant gains by both groups, but no significant difference between the gain scores. He suggested that the lack of significance between the gain scores may have been due to the experimental group having

an initially higher score, giving them a more restricted range for improvement, or that the test was not sensitive enough in measuring the gain. Schaaf's results indicate that exposure to heuristics such as analogy, enumeration and inverse deduction significantly increased students' ability to generalize to new situations. Furthermore, on certain measures this gain was significantly greater than that of a control group.

The following four studies (Ashton (1962), Libeskind (1971), Lucas (1972), Goldberg (1974)) all based their instructional procedures on Polya's heuristic process. They employed his questions and questioning technique in the instructional materials (see Polya, How to Solve It, 1957).

Ashton (1962) selected ten grade nine algebra classes, in five different schools, for her study. Each school contained two classes, both being instructed by the same teacher. In each pair of classes one was an experimental (heuristic) group and the other a control (textbook) group. The instructional period lasted ten weeks, during which time whenever the heuristic group was faced with a problem they were to ask themselves, or were asked questions from Polya's work, such as "What is the unknown? What are the data? What are the conditions?". An examination of Ashton's procedure indicates that her instructional sessions involved

more than just Polya's questions. For example, students in the heuristic group were allowed to write their own questions to clarify concepts. Furthermore, before anyone in a heuristic class was permitted to answer a question at least two of the students had to restate the question, so that by this "translation" the students familiarized themselves with algebra as a language. During the experimental period the control classes continued with their normal classroom programme. At the end of the ten week instructional period a problem solving test consisting of word problems was administered. Ashton found significant results favouring the heuristic group. Thus, her study provides evidence that the teaching of heuristic procedures based on the work of Polya results in significantly better problem solving performance.

Libeskind (1971) employed Polya's heuristic approach as the basis for developing a high school unit on number theory. In order to keep the prerequisites required by the students to a minimum the development and proof of theorems was restricted to the domain of whole numbers. The unit was taught by Libeskind to ten students, three of whom had completed grade 9, another three had completed grade 10, and the last four had completed grade 11. The instructional period consisted of 25 sessions each of 50 minutes, with each session being immediately followed by a 30 minute

study period. As part of his evaluation of the unit he investigated the question of whether students could apply some of the proof techniques developed in the unit to new problems. He concluded that the students were able to transfer their knowledge to new problems. This conclusion was not based on any statistical testing, but he used as the justification the fact that the students were able to solve test problems given throughout the experimental period. For example, Libeskind noted that the students were able to prove that if $2b \mid a$ and $5 \mid b$ then $10 \mid a$ ($2b \mid a$ and $5 \mid b$ then $10 \mid a$). This was a new problem in that the students had not seen it before, although the instructional unit had included proofs of general results such as if a divides b and b divides c then a divides c ($a \mid b$ and $b \mid c$ then $a \mid c$). As further evidence of transfer he cited the fact that the students were able to solve a new problem involving a proof that there exists a prime greater than 10^6 . The unit had included a proof of the general result that there exists no greatest prime. Libeskind's examples of transfer involved the application of proof techniques to new problems that were closely related to the instructional materials. Thus his results indicate that training in heuristics will produce transfer to related problems. Clearly this conclusion is limited by the small sample.

Goldberg's study (1974) also used materials on number theory. Two sets of self-instructional booklets were

written. Both sets consisted of seven booklets on number theory, one set with emphasis on heuristics and the other set without instruction in heuristics. Nine classes containing 238 undergraduates met for 12 sessions (75 minutes per session) with two sessions per week over a six week instructional period. The students worked on the booklets for seven of these sessions. The nine classes were randomly assigned to one of three treatments. In the reinforced heuristics treatment (RH) the classes used the heuristics booklets. Furthermore, when homework was discussed, or new concepts considered in the sessions when booklets were not being employed, the instructor utilized the heuristics as often as possible. In the non-reinforced heuristics treatment (H) the heuristics booklets were used but the heuristics not reinforced in class. In the non-heuristic treatment (C) the students were given the non-heuristic booklets and the instructor did not employ the heuristics in class. The tests that were administered at the end of the experimental period included a test of understanding of number theory concepts (concepts I) and a test to measure the ability to construct proofs (proofs I). Five weeks later delayed posttests of concepts (concepts II) and proofs (proofs II) were given. Part of these delayed tests were parallel to the original tests, and part dealt with material covered in the intervening period. There was no significant difference between the RH and H groups

although the trend indicated that the RH group performed better than the H group. Based on Stanford Achievement Test data on mathematical achievement the sample was divided into thirds (high, medium, low) and the scores on the posttests were divided in a similar manner (high, medium, low). A Chi-square analysis indicated significant differences for high ability students on the concepts tests. These results favoured students in the RH treatment. The results also favoured this group on constructing proofs, although the results were not significant. Furthermore, high ability students who received the C treatment performed significantly better on understanding concepts and tended to construct proofs better than their counterparts who received the H treatment. A coding system similar to Kilpatrick's (1967) checklist was devised to identify heuristic usage and was applied to the proofs written by the students in the top third on the proofs posttest. The results give an indication that the students who received the RH treatment used heuristics more often than the students who were in the control group. The paper gives no additional information on the use of heuristics. Overall, Goldberg's results seem to indicate that high ability students in the RH treatment benefited most from the instruction.

The studies by Ashton (1962), Libeskind (1971) and Goldberg (1974) all based their instructional materials on Polya's heuristic process. These studies indicate that

instruction in heuristics tends to improve problem solving performance when measured by tests of the ability to apply what had been learned (content) to new but related problems. The next study differs from these three in that Lucas (1972, 1974) specifically investigated whether students could apply heuristics to new problems.

Lucas (1972, 1974) investigated the teaching of heuristics to university students. He taught calculus to two classes, one class (H) received the experimental programme consisting of an instructional approach that exposed students to, and emphasized heuristics, while the other class (C) received instruction in the same content but with no exposure to the heuristics. The sample selected for study consisted of 30 students, 17 in the H group and 13 in the C group. The instructional period lasted eight weeks. At the end of the instructional period a problem solving test consisting of word problems was administered individually to each student. The students 'thought out loud' while solving the problems, and Lucas made notes as well as recording the sessions. It should be noted that these problems could be solved by calculus techniques, although they were amenable to solution by other methods, such as graphing appropriate equations. In analysing the data Lucas was looking for evidence of a variety of heuristic strategies in the students' solutions. These strategies included using mnemonic notation, using diagrams, checking to see if

the results were reasonable, and using analogy. These heuristics were derived from Polya's work. He found significant differences favouring the heuristics treatment on variables such as mnemonic notation, planning the problem and use of the results of related problems. Certain variables are of particular significance for the present dissertation and are discussed in detail. Uses method of a related problem referred to the strategy by which a related problem could be of value in "... providing a method which either may be applied directly to the given problem or may supply a cue for devising an analogous plan of attack". (Lucas, 1972, p.137) If a student specified a result of another problem then used the result in solving the new problem he received credit for evidence of the heuristic, uses result of related problem. Reasoning by analogy involved the problem solver searching for similarities between the given problem and a related situation. He found significant differences between the H and C groups on both uses methods of related problems and uses results of related problems, both differences favouring the H group. Of the 30 subjects tested there were only four who exhibited evidence of reasoning by analogy and all four subjects were in the H group. This study is important for two reasons. First, Lucas looked for evidence of particular heuristics in students' solutions. Second, the conclusion involving the use of related problems in solving new problems suggests

the possibility of teaching the heuristic of analogy as defined in this study (see page 14). Lucas noted limitations such as small sample size and the fact that students were not randomly assigned to treatments or any attempt to generalize his results. However, his findings do lend support to the hypothesis that heuristics can be taught.

These last four studies (Ashton (1962), Libeskind (1971), Goldberg (1974) and Lucas (1972, 1974)) indicate that the teaching of heuristics tends to improve problem solving performance. Furthermore, Lucas's results, while limited, indicate the possibility of students transferring heuristics, not just core material to new problems.

The remaining studies reviewed here have all employed a variety of strategies to promote mathematical transfer.

Price (1965) conducted a study of the effect of discovery on achievement and critical thinking of tenth grade students. He utilized three groups: the experimental group (E) containing 22 students; the experimental-transfer group (T) containing 23 students; and a control group (C) of 18 students. The instructional period lasted 15 weeks, during which time the experimental group received mathematics instruction on topics that included set theory and properties of number systems. The programme for the E group consisted of question and answer sessions between the

teacher and the students (a discovery oriented approach). The T group, in addition to this teaching were given a single study guide on inference before the lessons (discussions) on extension of pattern. This study material was of an historical nature, and was followed by specific questions that the student was to answer. Price (1965, p.72) said of this period of instruction:

During the classroom discussion which took place regarding inference there were many interesting observations made by the students who had never before really stopped to think about this kind of material.

They discovered that from insufficient data incorrect or many different inferences may be made.

During the experiment the control group received their normal classroom instruction. Based on the results of the California Test of Mathematical Achievement Price concluded that there were no statistically significant differences between the experimental groups and control group on achievement. However, both experimental groups did significantly better than the control on a Survey of Algebra Aptitude, but only the T group gained significantly on the Watson-Critical Thinking Appraisal Test.

Post (1967) presented a "Structure of the Problem-Solving Process" experimental unit to 74 grade seven mathematics students. Corresponding to these 74 students were an equal number in a control group, who had been matched on problem solving ability. He also divided the sections into high intelligence (I.Q. above 110) and low

intelligence (below 110). The outline of his problem solving process consisted of the following 4 categories:

(a) Recognition of the existence of a problem; (b) Analysis phase; (c) Prediction phase; and (d) Verification phase.

The instructional material was presented to the experimental group in three lessons on three successive days. A follow-up period of six weeks was employed, during which time the teacher emphasized this structure where appropriate. The vehicle employed for the introduction of the problem solving process was not purely mathematical. For example, the following problem is a non-mathematical example used in the unit.

A boy comes home from school and wants to go to a dance in the evening. He has certain chores to do which will prevent him from going to the dance. How to solve the boy's dilemma is discussed in terms of Post's problem solving structure. On a mathematical problem solving test Post found no significant difference between the experimental and control groups gain scores.

This investigator suggests the following possible reasons for the lack of significance. First, the short instructional period, only three days, gave little time for the students to fully assimilate the structure of the problem solving process. Second, the instructional unit was only partially mathematical in nature, thus requiring

the students to learn both the problem solving process in a non-mathematical situation and then apply it to a mathematical problem solving test.

Wills (1967) explicitly considered the transfer of problem solving ability gained through learning by discovery. His sample consisted of 561 high school students in 24 classes. Eight classes were randomly assigned to two experimental groups, Covert group (CG) and Overt group (OG), and one control group (C). Over a period of two weeks the experimental groups received workbooks as well as instruction from the teachers. Wills (1967, p. 155) described the role of the teacher in the two experimental groups as follows:

Covert Group

Whenever a criterion problem arises in the Covert group, the teacher follows it with simpler problems, makes a table, and continues asking questions until the students' answers indicate that they have acquired a generalization. The teacher does not discuss why the easier problems are presented. No effort is made to discuss the general method of solving such problems... In the Covert group the teacher leads the student through several solutions as does the instructional unit...

Overt Group

Whenever a criterion problem arises, the teacher tells the students that an effective way to solve a problem is to solve several simpler problems of the same type and try to find a pattern which can be applied to the more difficult problem.

All three groups were given the same pretest and posttest, and the function of the control group seems to

have been to control for growth over the instructional period. On the basis of the analysis, which used the pretest scores as a covariate, Wills concluded that the experimental groups performed significantly better than the control, while there was no significant difference between the two experimental groups. The mean performance of the two experimental groups indicated a better performance from the overt group. The problem solving test was aimed towards problems that involved searching for a pattern. Some of the questions involved asking the students to find patterns in problem situations that were unlike those in the instructional materials. For example, one problem involved patterns associated with Pascal's triangle. Other problems involved patterns in situations that can be considered extensions of the instructional materials. Wills's results indicate that the discovery approach resulted in better problem solving performance on problems that involved trying to find patterns.

Leask (1968) investigated the teaching of simple enumeration as a strategy for discovery. The sample consisted of six grade twelve classes with a total of 158 students. The students had been randomly assigned to classes at the beginning of the year, and three classes were assigned to each treatment, expository (E) and simple enumeration (S). The latter was described as follows:

This consists of the presentation of many instances of the generalization to be discovered. The students form hypotheses based on the examples and test these to determine which is correct. One counterexample is sufficient to warrant rejection of the hypothesis.

(Leask, 1968, p. 8)

In both treatments the experimental period consisted of seven one hour sessions, with the materials consisting of the development of arithmetic and geometric progressions. Both achievement and transfer tests were administered to the students. Leask (1968, p. 34) described the transfer test as follows:

It was a twenty-item test based on mathematical materials somewhat related to the sequences but requiring a transfer and extension of the knowledge acquired. The items were comprised of examples from which students were required to generalize.

Her results indicate that the S group performed significantly better than the C group on the mathematical transfer test, while there was no significant difference between the groups in mathematical achievement. An examination of the role of intelligence in the conclusions indicates that the improvement in transfer was most significant for the medium intelligence range.

German (1973) conducted a study of mathematical transfer in the elementary school. He used two experimental programmes: the Productive Thinking Programme of Crutchfield and Covington, and a Modified Wanted-Given Programme developed by German. The German programme used

mathematical examples to develop problem solving skills. The sample selected for the study was eight classes of fifth grade students (261 students), with three classes being assigned to each of the experimental programmes and the other two classes received their normal classroom instruction and acted as a control group. The experimental materials were presented in self-instructional booklets, with the total instructional period being 16 days. The tests included a word problem test administered immediately after the instructional period and a follow-up test given 7 weeks later. Two NLSMA tests, Working with Numbers and Five Dots were also administered. Jerman found no significant differences between the treatments. However, he noted that the word problems had proved more difficult than anticipated. It appears that this difficulty was in terms of the computation involved. He proceeded to analyse the word problem data according to whether subjects used a correct procedure to obtain the solution. That is, whether the method would have resulted in a correct solution if no computational errors were made. With this new criterion Jerman found significant differences favouring the experimental treatments on both the immediate word problem test and the retention test given seven weeks later. On the first test there was a significant sex-treatment interaction favouring boys, a result not repeated

on the retention test. The fact that this result was not repeated may have been due to the improved performance of the students on the delayed test, which was a parallel version of the original word problem test.

Summary of the Related Literature

The literature is reflected in three facets of the design of the study. First, the literature indicates that educators consider analogy and examination of cases to be important mathematical heuristics (for example, see Smith and Henderson (1959), Polya (1954a), MacPherson (1971b)). Therefore, it was reasonable to select these heuristics to teach to students.

Second, it is noted in chapter III (p. 61) that booklets were to be employed in order to randomize the sample and control the instructional sequence. Studies (for example, see Gagné and Brown (1961), Eldredge (1965a) and Jerman (1973)) provide evidence that utilizing self-instructional materials as the instructional medium is a feasible technique for promoting transfer.

Finally, there were a few studies specifically concerned with the teaching of heuristics. Studies (for example, see Libeskind (1971) and Goldberg (1974)) indicate that teaching involving heuristic procedures can result in transfer of knowledge to problems that are related to the

instructional materials. Other studies (for example, see Schaaf (1954), Wills (1967), Leask (1968) and Worthen (1965)) provide evidence that students can apply general problem solving techniques to new problems. The present study can be considered both an extension and refinement of current research. It is a refinement in that the instructional materials were restricted to the heuristics of analogy and examination of cases, and an extension in that it investigated the transfer of heuristics (as opposed to content) to novel mathematical problems.

CHAPTER III

Design and ProcedureDesign

This study was designed to investigate the teaching of heuristics. Two factors were considered: the first factor concerned the employment of treatments with different objectives; the second factor with employing distinct instructional vehicles. In order to investigate the hypotheses related to these factors a total of nine groups were devised, as described in chapter I.

Figure 2

Experimental Groups

Vehicles

		Neutral	Algebraic	Geometric
T r e a t m e n t s	Heuristics only	H - N	H - A	H - G
	Heuristics plus content	HC - N	HC - A	HC - G
	Content only	C - N	C - A	C - G

For the purpose of later discussion the nine groups have been identified by a two part code. The H, HC and C denote the three treatments: heuristics only (H), heuristics plus content (HC) and content only (C). The N, A and G denote the three vehicles: mathematically neutral (N), algebraic (A) and geometric (G). The H, HC and C are used separately to denote the three treatments, irrespective of the vehicle. For example, the group receiving the H treatment consists of all subjects in H - N, H - A and H - G.

Employing three treatments and three vehicles resulted in nine experimental groups, one corresponding to each of the cells in figure 2. The following is an explanation of the nature of the instructional materials for these groups.

Instructional Materials

Neutral Vehicle (H - N, HC - N, C - N). The materials for these groups developed mathematical topics as they applied to the physical world. The word neutral was employed to indicate that these materials, although mathematical in nature, were not associated with any core category of the discipline of mathematics, such as algebra or geometry.

Algebraic Vehicles (H - A, HC - A, C - A). Both the topics and treatment of topics in this vehicle can be classified as algebraic. The selection of, and approach to

topics emphasized the use of symbols and operations on symbols.

Geometric Vehicle (H - G, HC - G, C - G). Both the topics and treatment of the topics in this vehicle can be classified as geometric. The emphasis in this vehicle was on the use of geometric figures, diagrammatic representations and geometric operations such as flipping and rotating.

Heuristics only (H - N, H - A, H - G). The materials for these groups were designed so that the heuristics were explicitly stated and emphasized, and the only objective for the student was learning to apply the heuristics.

Heuristics plus content (HC - N, HC - A, HC - G). The materials for these groups were designed so that the heuristics were explicitly stated, but emphasis was placed on both the heuristics and content.

Content only (C - N, C - A, C - G). The materials for these groups were designed so that the only objective for the student was the learning of the content. The content only groups were included in the design as control groups.

A total of nine distinct instructional units were required. For example, the materials for the group that received heuristics plus content instruction in an algebraic setting, HC - A, consisted of an algebraic unit with emphasis on both heuristics and other mathematical content. The materials for the C - G group consisted of a geometric unit designed to teach content only.

Booklets

Booklets were used so that subjects could be randomly assigned to one of the nine experimental groups as well as to control the instructional sequence. All nine booklets were designed so that each could be completed in ten days (one class period per day). Each of the booklets dealt with three topics. Three days were allotted to each topic and one additional day for review. The vehicles comprised the following topics:

Figure 5

The Booklets

Vehicles

T o p i c s		Neutral	Algebraic	Geometric
	Topic 1	Seating Plans	Permutations	Arrangements
	Topic 2	Tiling	Mappings	Flips & Turns
	Topic 3	Electrical Networks	Binary Networks	Networks

The overall organization of the booklets was as follows:

Topic 1:

Day 1 - Material presented and a quiz included at the end of the period. The quiz to be marked and returned the next day, together with the solutions.

Day 2 - Followed the identical pattern to day 1.

Day 3 - Material presented but no quiz.

Topic 2:

Days 4 - 6 followed the same pattern as days 1 - 3.

Topic 3:

Days 7 - 9 followed the same pattern as days 1 - 3.

Day 10 was allowed for review. No new material was included.

In addition to employing the same organizations, either identical or similar exercises, wording and examples were used in all the booklets. This meant that the nine instructional units were equivalent, varying only as the different objectives and the natures of a neutral, algebraic and geometric vehicle required. A detailed discussion of the development of the materials can be found later in this chapter, under the heading, 'Development of Materials'. Sample pages from the booklets are included in appendices A - I.

Instruments

Two tests were developed, one algebraic and the other geometric. The instruments both followed identical format, consisting of a novel mathematical problem followed by an open ended question.

Figure 6

Structure of the Tests

The test items employed operations invented specifically for the study. Details of the development of the instruments can be found in this chapter (see pages 82-3).

Grading Scheme

The grading scheme for the tests consisted of assigning a subject's response to one of six categories.

They are:

- 0 - No evidence of the use of either examination of cases or analogy.
- 1 - The subject has used examination of cases only. He has not attempted to look for a pattern.
- 2 - The subject has both examined cases and looked for a pattern.
- 3 - The subject has used analogy only. There is no evidence that he has tried to test his guess.

- 4 - The subject has used analogy and examination of cases, but as distinct techniques. There is no evidence that the subject has combined the two heuristics.
- 5 - The subject has used both heuristics. Furthermore, he has combined the two in solving the problem.

Procedure

The procedure is reported in two parts. First a brief summary of the pilot studies, and second the main study.

Pilot Studies

The development of the materials dictated the need for pilot studies to be undertaken. The following is a brief statement of the purposes of the pilots, and a summary of the nature of the modifications resulting from the pilots.

The objectives of the pilot studies were to determine: (1) the readability of the materials; (2) the difficulty of the concepts discussed in the materials; (3) if the length of an individual's day's material was reasonable; and (4) the difficulty of the exercises and examples.

The results of the pilot studies, together with suggestions from the investigator's committee, necessitated certain minor modifications of the materials. For example, these modifications included removing exercises because a particular day's work required too much time; inserting a

A presentation employing booklets had the effect of controlling teacher variability (in presenting the instructional units) which no amount of pre-experimental training could have totally eliminated.

It should be noted that randomly assigning treatments to classrooms would have resulted in the treatments being nested within classrooms, and consequently it would have been impossible to differentiate between treatment and classroom effects.

Instructions to Subjects. The day prior to subjects' starting their respective treatments a note was read explaining that the subjects were about to take part in an experiment (see appendix J). The major points in the instructions were:

(a) The subjects were informed that they were involved in an experiment and that booklets would be handed out.

(b) The subjects were informed that the booklets would be collected at the end of each period, and there would be no further work on the booklets until the next day. They were also asked not to discuss the content of the booklets with anyone. By collecting the materials at the end of each experimental session and assigning no work on the booklets outside class, the investigator hoped to minimize any discussion among subjects.

Instructional Period. On the first day of the instructional period each subject was presented with a folder. The folder contained the pages to be covered in the first period.' At the end of the period the booklets were collected, the quiz marked, and the next day's pages added. This procedure was followed throughout the instructional period, with the obvious restriction on those days on which there was no quiz. About halfway through each day's instructional period the teacher advised the subjects that they had twenty minutes remaining. On days on which there was a quiz, about ten minutes from the end of the period the teacher suggested that subjects who had not started the quiz should do so. If a student completed a particular day's material early he was not given any new material from the booklet.

No mark or letter grade was assigned to a subject's quiz since it was felt that this form of feedback might effect a subject's performance, confounding any attempt to interpret the results. The classes met at various times throughout the day enabling the investigator to visit all classes while the experiment was in progress. The teachers did not answer subjects' queries, other than directional type questions, such as whether one could write on the back of a page. For more involved questions concerning the experimental units the teacher suggested that a subject continue until the investigator arrived. Thus it was possible to control the teacher input into the experimental

process. At the conclusion of the ten instructional days two tests were administered to the subjects.

Administration of the Tests. On the day immediately following the last instructional period the algebraic test was administered to all subjects, and the geometric test administered the next day. On each of the test days, the experimental periods were concentrated in the afternoon. Since the school allowed three minutes for subjects to change classrooms between periods this arrangement reduced the possibility of discussions between subjects who had completed the test and those who would write it later in the day.

The experimental procedure is summarized in figure 8.

Figure 8

Grading. Each subject's response was assigned to one of six categories (see pages 63-4). A team of judges, consisting of the investigator and two other mathematics education professors, scored the tests. Each judge was presented with a copy of the grading scheme (see appendix M), together with sample responses obtained from subjects not involved in this study. A series of meetings to discuss the grading scheme was held prior to the scoring of the experimental data. Then, each judge graded all the experimental tests individually and recorded his results on class lists. No judge was aware of the other judges' categorizations of the tests. No comments or marks of any form were written on the tests. The three lists were then collated and the tests assigned to one of two categories; those on which all three judges agreed, and those about which there was disagreement. Meetings were then held to try to obtain consensus on those tests about which there had been disagreement. Throughout the discussions the investigator introduced tests on which all three judges had agreed in order to verify that the discussions were not causing alterations in the individuals' perceptions of the categories. The percentages of agreement can be found in table I.

Subjects

The subjects consisted of 294 grade ten boys from an all male high school in Eastern Canada. The choice of an

all male sample, with its obvious limitations on the generalizability of the results requires some explanation. Since the sample was to be divided into nine groups, allowing for experimental mortality (Campbell and Stanley, 1963, p. 5) the investigator wished to obtain as large a sample as possible. The boys' school was one of the few that contained a large number (in excess of 300) grade ten students.

Development of Materials

Development of the Booklets

The topics used in the three vehicles were listed in figure 5 which, for convenience, is repeated below:

Figure 5
The Booklets
Vehicles

T O P I C S		Neutral	Algebraic	Geometric
	Topic 1	Seating Plans	Permutations	Arrangements
	Topic 2	Tiling	Mappings	Flips & Turns
	Topic 3	Electrical Networks	Binary Networks	Networks

Since all the booklets were in excess of 100 pages, the complete materials have not been included. However, sample pages from the booklets can be found in appendices A - I.

The booklets were designed to be structurally and mathematically equivalent. This equivalence was of two types. First, equivalence between vehicles was obtained by selecting topics such as seating plans (mathematically neutral), permutations (algebraic) and arrangements (geometric) that had identical mathematical structures. Furthermore, where possible examples and exercises were translated into the terminology appropriate to a particular vehicle. Second, equivalence between treatments was obtained by, wherever possible, employing identical exercises and examples in different treatments. All booklets contained approximately the same number of exercises and examples.

Equivalence between vehicles

This equivalence is best described by considering individual topics.

Figure 9

Seating Plans considers different methods of seating people in a row, and round a circular table (See appendices A, B and C). A typical question involves seating three

people in a row. The three people are Mr. Blade,
Mr. Johnson and Mr. Hamilton.

Three chairs

The question posed to the students is:

How many different ways can you seat the three men?

Permutations considers different ways of permuting letters and numbers, and then proceeds to the concept of circular permutations, which is an adaption of the concept of cyclic permutations (See appendices D, E and F). The question corresponding to that given in the seating plans booklet is:

Consider the set consisting of b, j and h. Using all three letters, how many different possible permutations can you find?

Arrangements considers different methods of arranging geometric figures in a line, and then proceeds to consider different circular patterns (See appendices G, H and I). The question corresponding to those stated previously is:

Consider the set consisting of a triangle, square and circle. Using all three figures how many different possible arrangements can you find?

Seating people in a row and arranging geometric figures in a line are equivalent operations to permuting

letters. Similarly, seating people round a circular table and arranging geometric figures in a circular pattern are equivalent to organizing circular permutations. Thus, the first three topics are neutral, algebraic and geometric models of the same mathematical structure. Corresponding exercises and examples were reworded in the terminology appropriate to a given vehicle. The questions indicate the nature of this rewording. The translation of an exercise or example from vehicle to vehicle means that each topic contains approximately the same number of exercises and examples. The only differences between vehicles are those inherent in the nature of an algebraic, geometric and neutral approach to the same mathematical structure. Figure 10 summarizes the translations.

Figure 10

Equivalence - Topic 1

Neutral (Seating Plans)	Algebraic (Permutations)	Geometric (Arrangements)
<p>Sample question</p> <p>Translation</p> <p>Mr.Blade,Mr.Johnson and Mr.Hamilton are to be seated in a row.</p> <p> </p> <p>How many different ways can you seat the three men?</p>	<p>Sample question</p> <p>Translation</p> <p>Consider the set consisting of b, j and h.</p> <p>Using all three letters how many different possible permutations can you find?</p>	<p>Sample question</p> <p>Translation</p> <p>Consider the set consisting of a triangle, square and circle.</p> <p> </p> <p>Using all three figures how many different possible arrangements can you find?</p>

Figure 11

Topic 2

Tiling considers fitting tiles of various shapes and colours into appropriately shaped spaces. Different positions for the tiles, as well as movements and combinations of movements required to transpose a tile from one position to another are surveyed (See appendices A, B and C). A typical situation involves fitting an equilateral triangular tile into a congruent space and then rotating from one position to another. The following are examples of two positions and a rotation.

Figure 12

Mappings considers sets of three and four letters, and operations and combining operations that interchange letters. (See appendices D, E and F). The situation corresponding to that given in the tiling section is:

Figure 13

M_1 is the algebraic equivalent of rotation 1. Thus, after applying rotation 1 (or M_1) to position 1, B occupies the position previously held by A, C occupies the position previously held by B, and A occupies the position previously held by C. In both cases the new arrangement is labelled position 2.

Flips and Turns considers flips and turns (rotations) of different polygons and how these movements can be combined to produce new positions. The geometric figures correspond to those in the tiling section (See appendices G, H and I). The situation that corresponds to that presented in the tiling section involves the rotation of an equilateral triangle, rather than an equilateral triangular tile.

To summarize, topic 2 consists of three models (neutral, algebraic and geometric) of the same mathematical structure, and corresponding examples are exercises were translated into the terminology appropriate to a particular vehicle. Figure 14 contains the different models.

Figure 14

Equivalence - Topic 2

Figure 15

Topic 3

The titles themselves reflect the equivalent nature of the topics. Each considers combining switches to form networks, and then develops the concept of equivalent networks. In the sections on electrical networks the switches are electrical and the development is in terms of whether current flows or does not flow through the network. (See appendices A, B and C). Binary switches involves the idea of switches that have two possible values, 0 or 1, and the development centers on whether a network has the value I or II (See appendices D, E and F). Finally, geometric networks interpret a network as a combination of paths that are either connected or disconnected, and the development deals with the conditions under which a network is, or is not complete (See appendices G, H and I). As before, appropriate exercises and examples were translated into the terminology appropriate to a particular vehicle. Figure 16 summarizes topic 3.

Figure 16

Equivalence - Topic 3

Neutral (Electrical Networks)	Algebraic (Binary Networks)	Geometric (Networks)
<p>Sample network</p> <p>A and B are switches</p> <p>A open B closed</p> <p>Current does not flow</p>	<p>Sample network</p> <p>A and B are switches</p> <p>A has value 0 B has value 1</p> <p>Network has value II</p>	<p>Sample network</p> <p>A and B are switches</p> <p>A disconnected B connected</p> <p>Network is not complete</p>

The result of designing the materials to obtain equivalence between vehicles is that booklets for different vehicles contain corresponding topics that are models of the same mathematical structure; where possible corresponding booklets utilize equivalent exercises and examples; and the booklets contain approximately the same number of exercises. The differences between vehicles are those that result from the nature of a mathematically neutral, algebraic and geometric model and approach to the same mathematical structure.

Equivalence between Treatments

The three treatments are heuristics only, heuristics plus content and content only. Where appropriate the identical exercise or example, or translation of the exercise or example are utilized in all three treatments. For example, the reader should examine the development of the seating plans sections (See appendices A, B and C). In the heuristics only and heuristics plus content booklets the solutions to exercises often contain an explanation of how examination of cases or analogy was employed in finding the solution. In the content only booklets the identical exercise does not include any mention of the heuristics in the solution. The differentiation between the heuristics only and heuristics plus content booklets occurs in the different emphases in the materials. For example, the

heuristics plus content booklets have fewer pages devoted specifically to the heuristics; the words examination of cases and analogy are underlined in the heuristics only booklets but not in the other heuristics treatment; exercises are included in the heuristics plus content booklets designed to emphasize content only, and these exercises omitted from the heuristics only materials.

Thus two constraints are placed on the design of the booklets, namely attempts to obtain equivalence between vehicles and equivalence between treatments. This results in the following equivalence between all nine booklets: (1) all booklets contain three topics; (2) corresponding topics are models of the same mathematical structure; (3) where possible identical examples and exercises, or translations of these are used in all booklets; and (4) all booklets contain approximately the same number of exercises.

In developing the booklets it was reasonable to assume that all subjects would not progress at the same rate. In particular, the design of the booklets had to accommodate subjects who would not complete a day's work within the allotted period. To do this the material was organized so that most of the new concepts are developed in the first half of a day's work. If a subject completed this segment he was in a position to make a reasonable attempt at the quiz. For subjects who finished early no additional material was provided. It was decided that

these subjects would be allowed to continue with work not associated with the booklets.

Development of the Tests

The tests, one algebraic and the other geometric consisted of novel problems. Furthermore, the nature of the instruments was such as to allow the opportunity for making conjectures by analogy as well as permitting subjects to examine cases.

Algebraic Instrument. Discussions with personnel familiar with the high school mathematics programme, together with an examination of high school textbooks currently employed in the schools indicated that: (a) Grade ten students were familiar with a structural approach to mathematics. In particular, grade ten students had investigated selected field properties such as commutativity, and the existence of inverses; and (b) Grade ten students had been exposed to modulo systems (clock arithmetic).

The algebraic test consisted of the presentation of two mathematical operations developed specifically for the study, followed by the question:

What can you find out about these operations?

The operations are:

'Up-one multiplication' $a @ b = a \times (b + 1)$ (Modulo 7)

'Double addition' $a \# b = 2 \times (a + b)$ (Modulo 7)

(See appendix K for details)

The fact that subjects had been exposed to a structural approach in their mathematical programme assured the investigator that subjects had the necessary core material with which to make analogies, or try cases and draw inferences.

Geometric Instrument. The procedure utilized for the development of this instrument is similar to that followed in the design of the algebraic test.

The subjects were presented with a description of movements on a grid and calculations of the distance between points, followed by the question:

What can you find out about shortest routes and combining shortest routes?

One movement, involving moving up and across the grid is illustrated in figure 17.

Figure 17

Movements on a Grid

Distance between
A and B is 3
segments.

(See appendix L for details)

Statistical Procedures

Identical statistical analyses were performed on both the algebraic and geometric test data. A two-way fixed effects analysis of variance (ANOVA) was employed. The model is:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

where

μ = grand mean

α_i = effect of treatment i

β_j = effect of vehicle j

γ_{ij} = interaction effect of treatment i with vehicle j

ϵ_{ijk} = experimental error of person k

α_i (i = 1, 2, 3) represents the treatment effects, heuristics only (α_1), heuristics plus content (α_2) and content only (α_3). β_j (j = 1, 2, 3) represents the vehicle effects, mathematically neutral (β_1), algebraic (β_2) and geometric (β_3).

Overall F-ratios were computed and tested for significance (at .05 level). Depending on the significance of the overall F-ratios the contrasts corresponding to the hypotheses were tested by the Scheffé Method (Kirk, 1968). Furthermore, the investigator chose to supplement certain of the ANOVA test procedures with a non-parametric method (χ^2). The details and justification for these decisions are found in chapter IV - Analysis of the Data.

CHAPTER IV

Analysis of the Data

The research hypotheses are restated below.

Hypotheses 1 through 3 refer to the treatment effect, while hypotheses 4 and 5 correspond to certain interaction effects. In each of the five hypotheses, the dependent variable is a measure of ability to apply at least one of the heuristics (examination of cases or analogy) to a novel mathematics problem:

Hypothesis 1. A group taught heuristics only (H) will score higher than a group taught content only (C).

Hypothesis 2. A group taught heuristics plus content (HC) will score higher than a group taught content only (C).

Hypothesis 3. A group taught heuristics only (H) will score higher than a group taught heuristics plus content (HC).

Hypothesis 4. The mean score of a heuristics only group (H - N, H - A, H - G) as compared to the mean score of the corresponding content only group (C - N, C - A, C - G) will be independent of the vehicle (instructional material).

Hypothesis 5. The mean score of a heuristics plus content group (HC - N, HC - A, HC - G) as compared to the mean score of the corresponding content only group (C - N, C - A, C - G) will be independent of the vehicle (instructional material).

Grading. Throughout the ten instructional periods different subjects were absent from school. For a subject's test to be graded he had to be present for at least nine of the ten instructional periods. Table I contains the total number of algebraic and geometric tests graded, the percentage of agreement prior to any discussion among judges, and the final percentage agreement.

Table I

Percentage Agreement on Tests

	Total number of tests graded*	Agreement prior to discussion	%	Agreement after discussion	%
Algebraic Test	249	177	71	244	98
Geometric Test	249	175	70	241	96

* 228 subjects wrote both tests

Data Used in Analysis. A subset of the tests on which the judges agreed was used in the analysis. Subjects for whom data from only one test was available were discarded,

leaving data from 218 subjects. The resulting group sizes varied from 21 to 26. In order to equalize the numbers in all nine groups additional data were randomly discarded so that there were 21 subjects per group.

There were two reasons for reducing the total number of subjects included in the analysis. First, in order to draw conclusions concerning both the algebraic and geometric problems it was advisable to restrict the analysis to data based on only those subjects who wrote both tests. Second, the use of a non-orthogonal design (unequal cell sizes), with the associated lack of statistical independence between the overall F-tests, is undesirable since it makes it difficult to interpret results.

Data from 189 subjects were analysed (21 per group). Tables II and III show the summary of the scores on the algebraic and geometric tests, respectively.

Recategorization of Data for Analysis. For the purpose of testing the hypotheses, the data in tables II and III were recategorized according to the following dichotomy:

- 0 - No evidence of either the heuristic of examination of cases or analogy.
- 1 - Evidence of at least one of these heuristics.

The resulting frequencies are shown in table IV. This recategorization of the data was necessary since the

Table II

Summary of Responses on the
Algebraic Test

<u>Group</u>	<u>Number of Subjects in a Category*</u>						<u>Number of Subjects per Group</u>
	0 - No heuristics	1 - Cases only (No pattern)	2 - Cases & Pattern	3 - Analogy only	4 - Both heuristics (Distinct use)	5 - Both heuristics (Combined use)	
H - N	10	1	1	4	-	5	21
HC- N	16	1	-	1	-	3	21
C - N	17	-	1	1	-	2	21
H - A	6	2	3	2	2	6	21
HC- A	12	2	-	1	2	4	21
C - A	16	1	2	-	1	1	21
H - G	10	-	1	2	-	8	21
HC- G	13	-	1	2	-	5	21
C - G	17	2	-	1	-	1	21

Total number of subjects - 189

* Details of the categories can be found on pages 63-4.

Table III
Summary of Responses on the
Geometric Test

<u>Group</u>	<u>Number of Subjects in a Category*</u>						<u>Number of Subjects per Group</u>
	<u>No heuristics</u>	<u>Cases only (No pattern)</u>	<u>Cases & Pattern</u>	<u>Analogy only</u>	<u>Both heuristics (Distinct use)</u>	<u>Both heuristics (Combined use)</u>	
	0	1	2	3	4	5	
H - N	6	6	8	-	-	1	21
HC- N	10	5	4	1	-	1	21
C - N	15	3	3	-	-	-	21
H - A	7	6	5	-	-	3	21
HC- A	8	6	5	-	-	2	21
C - A	14	5	2	-	-	-	21
H - G	9	4	7	-	-	1	21
HC- G	9	6	5	-	-	1	21
C - G	12	6	3	-	-	-	21

Total number of subjects - 189

*Details of the categories can be found on pages 63-4.

Table IV

Mark Assigned for Analysis

<u>Group</u>	<u>Algebraic Test</u>		<u>Geometric Test</u>	
	<u>0*</u>	<u>1**</u>	<u>0*</u>	<u>1**</u>
H - N	10	11	6	15
HC- N	16	5	10	11
C - N	17	4	15	6
H - A	6	15	7	14
HC- A	12	9	8	13
C - A	16	5	14	7
H - G	10	11	9	12
HC- G	13	8	9	12
C - G	17	4	12	9
H	26	37	22	41
HC	41	22	27	36
C	50	13	41	22

* - No evidence of either heuristic.

** - Evidence of at least one of the heuristics.

original 0 through 5 scheme does not represent either an underlying continuum or a totally ordinal scale, and in that form the data were inappropriate for analysis by ANOVA.

The interpretation difficulties were further confounded by the use of dichotomous data. Hsu and Feldt (1969) examined the effect of limited scales (2, 3, 4 and 5 point) on nominal α -levels. The results of their study, which was restricted to equal cell sizes, indicate reasonable control of Type I error for all scales.

With a two point scale an n of 50 plus (per cell) seems indicated by Hsu and Feldt's results for a nominal α of .01 or .10, but not for an α of .05. Although the picture concerning cell size is not entirely clear, on the basis of Hsu and Feldt's results, and the general desirability of equal n's, it would appear advisable to use a cell size larger than 21.

Since it was not feasible to use a larger n, the investigator chose to supplement certain of the ANOVA procedures with a non-parametric method (χ^2). This was done for hypotheses 1 - 3, concerning a main effect, but was not applicable to the interaction hypotheses 4 and 5.

Testing the Hypotheses

The statistical model employed in the analysis of variance is:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

where

μ = grand mean

α_i = effect of treatment i

β_j = effect of vehicle j

γ_{ij} = interaction effect of treatment i with vehicle j

ϵ_{ijk} = experimental error of person k

α_i ($i = 1, 2, 3$) represents the treatment effects: heuristics only (α_1), heuristics plus content (α_2), and content only (α_3).

β_j ($j = 1, 2, 3$) represents the vehicle effects: mathematically neutral (β_1), algebraic (β_2), and geometric (β_3).

The statistical hypotheses corresponding to the research hypotheses are stated below:

<u>Hypothesis</u>	<u>Null Hypothesis</u>	<u>Alternative Hypothesis</u>
H1	$H_{01} : \alpha_1 = \alpha_2$	$H_{11} : \alpha_1 > \alpha_2$
H2	$H_{02} : \alpha_2 = \alpha_3$	$H_{12} : \alpha_2 > \alpha_3$
H3	$H_{03} : \alpha_1 = \alpha_3$	$H_{13} : \alpha_1 > \alpha_3$
H4	$H_{041} : \gamma_{11} - \gamma_{31} = \gamma_{12} - \gamma_{32}$	$H_{141} : \gamma_{11} - \gamma_{31} \neq \gamma_{12} - \gamma_{32}$
	$H_{042} : \gamma_{11} - \gamma_{31} = \gamma_{13} - \gamma_{33}$	$H_{142} : \gamma_{11} - \gamma_{31} \neq \gamma_{13} - \gamma_{33}$
	$H_{043} : \gamma_{12} - \gamma_{32} = \gamma_{13} - \gamma_{33}$	$H_{143} : \gamma_{12} - \gamma_{32} \neq \gamma_{13} - \gamma_{33}$
H5	$H_{051} : \gamma_{21} - \gamma_{31} = \gamma_{22} - \gamma_{32}$	$H_{151} : \gamma_{21} - \gamma_{31} \neq \gamma_{22} - \gamma_{32}$
	$H_{052} : \gamma_{21} - \gamma_{31} = \gamma_{23} - \gamma_{33}$	$H_{152} : \gamma_{21} - \gamma_{31} \neq \gamma_{23} - \gamma_{33}$
	$H_{053} : \gamma_{22} - \gamma_{32} = \gamma_{23} - \gamma_{33}$	$H_{153} : \gamma_{22} - \gamma_{32} \neq \gamma_{23} - \gamma_{33}$

Plan for Analysis. Chapter I contains a model of mathematics. The form of the model, with the distinct components of Discovery and Core led the investigator to state research hypotheses 1 - 3 directionally. Although directional hypotheses are implied by the model, the subsequent review of the literature (Chapter II) shows that at the present time there is little research that either supports or refutes directionality. Furthermore, the initial statement of the hypotheses would not consider alternative hypotheses such as the HC treatment being significantly better than the H treatment, which have important educational implications. Thus for both these reasons, lack of empirical support for directionality and the educational importance of other alternative hypotheses, the investigator tested null hypotheses H_{01} , H_{02} and H_{03} against non-directional alternatives, H_{11} , H_{12} , and H_{13} ,

$$H_{11}: \alpha_1 \neq \alpha_2 \quad H_{12}: \alpha_2 \neq \alpha_3 \quad H_{13}: \alpha_1 \neq \alpha_2$$

Since hypotheses 1 - 3 were specified a priori, it is permissible to test appropriate non-orthogonal contrasts by Dunn's multiple comparison procedure (Kirk, 1968) without first testing for a significant overall main treatment effect. However, due to the limitations imposed by 0 - 1 scoring, the investigator chose to test the hypotheses by the more conservative a posteriori Scheffé procedure (Kirk, 1968). The following is a comparison of the a posteriori

Scheffé and Tukey procedures, and Dunn's Multiple Comparison procedure. Since hypotheses 1 - 3 refer only to pairwise comparisons of row means the discussion is limited to contrasts involving just two row means.

Let ψ represent any population contrast of two means and $\hat{\psi}$ its estimate. Then confidence intervals for the contrasts can be written in the form:

$$\hat{\psi} - \hat{\xi}_1 \leq \psi \leq \hat{\psi} + \hat{\xi}_1$$

where

$$\hat{\psi} = \bar{X}_{i..} - \bar{X}_{i'..}$$

and

For the Scheffé Procedure

$$\hat{\xi}_S = \sqrt{MS_W \left(\frac{1}{n_i} + \frac{1}{n_{i'}} \right) (I-1)_{1-\alpha} F(I-1, N-IJ)}$$

with

MS_W -- within cells mean square

I -- number of treatment levels (row)

J -- number of treatment levels (columns)

N -- total number of subjects

$n_i, n_{i'}$ -- total number of subjects used for computing $\bar{X}_{i..}$ and $\bar{X}_{i'..}$ respectively

For the Tukey Procedure

$$\hat{\xi}_T = 1 - \alpha^{q_{I, N - IJ}} \sqrt{\frac{MS_w}{N/I}}$$

with

$1 - \alpha^{q_{I, N - IJ}}$ -- the $100(1 - \alpha)$ percentile point in the Studentized range distribution with I and $N - IJ$ degrees of freedom

MS_w , N , and I are the same as for Scheffé

For the Dunn Multiple Comparison Procedure

$$\hat{\xi}_D = t'^D_{\alpha/2; C, N - IJ} \sqrt{MS_w \left(\frac{1}{n_i} + \frac{1}{n_{i'}} \right)}$$

with

C -- Number of comparisons to be made

$t'^D_{\alpha/2; C, N - IJ}$ is a coefficient obtained from Dunn's table (Kirk, 1968, p. 551)

MS_w , n_i , and $n_{i'}$ are the same as for Scheffé

In the present study $I = J = 3$, $n_i = n_{i'} = 63$, $N = 189$, $C = 3$, and $\alpha = .05$. Using this information the relative efficiencies of the three procedures were calculated (Kirk, 1968, p. 96) and the ratios of the lengths of the confidence intervals (Kirk, 1968, p.97).

Table V

The Relative Efficiencies and Ratios of Confidence Intervals
for Scheffé, Tukey and Dunn*

<u>Relative Efficiency</u>		<u>Ratio of Confidence Intervals</u>	
$\frac{\hat{\xi}_T^2}{\hat{\xi}_S^2} \times 100 =$	91.99%	$\frac{\hat{\xi}_T}{\hat{\xi}_S} =$	.96
$\frac{\hat{\xi}_D^2}{\hat{\xi}_S^2} \times 100 =$	96.01%	$\frac{\hat{\xi}_D}{\hat{\xi}_S} =$	.98
$\frac{\hat{\xi}_T^2}{\hat{\xi}_D^2} \times 100 =$	95.81%	$\frac{\hat{\xi}_T}{\hat{\xi}_D} =$	.98

* where

$$\hat{\xi}_S^2 = 12.20 (MS_W/63)$$

$$\hat{\xi}_T^2 = 11.22 (MS_W/63)$$

$$\hat{\xi}_D^2 = 11.71 (MS_W/63)$$

The following conclusions can be drawn from the comparison of the Scheffé, Tukey and Dunn's procedures. First, as indicated by the lengths of $\hat{\xi}_S$, $\hat{\xi}_T$, and $\hat{\xi}_D$ the Scheffé procedure produces the longest confidence intervals, and is, therefore, the most conservative. Second, the fact

the relative efficiencies are all in excess of 90 percent indicates that all three procedures have approximately the same control of Type I (α -error) and Type II (β -error) for pairwise comparisons. Thus, the selection of the most conservative Scheffé procedure for the analysis of contrasts does not appear to be too costly in terms of power.

Analysis of Data. Tables VI and VII contain the means and standard deviations for the algebraic and geometric data, respectively.

The overall F-ratios were computed and these results are summarized in Tables VIII and IX. The overall F-ratios indicate that there was no significant interaction on either the algebraic or geometric test. Thus, the null hypotheses corresponding to hypotheses 4 and 5 are accepted, and the investigator concludes that there are no differences that could be attributed to an interaction between treatments and vehicles.

The investigator proceeded to test the three treatment hypotheses H_{01} , H_{02} , and H_{03} using the Scheffé procedure. Tables X and XI contain the .05 simultaneous confidence intervals.

As indicated previously the investigator also tested hypotheses 1 - 3 by a non-parametric (χ^2) method. The results of this analysis are contained in tables XII and XIII.

Table VI
Means and Standard Deviations*
for the Algebraic Test

	Neutral	Algebraic	Geometric	Total
Heuristics only	.5238 (.5118)	.7143 (.4629)	.5283 (.5118)	.5873 (.4963)
Heuristics plus content	.2381 (.4364)	.4286 (.5071)	.3810 (.4976)	.3492 (.4805)
Content only	.1905 (.4024)	.2381 (.4364)	.1905 (.4024)	.2064 (.4071)

* Standard deviations in parentheses

Table VII
Means and Standard Deviations*
for the Geometric Test

	Neutral	Algebraic	Geometric	Total
Heuristics only	.7143 (.4629)	.6667 (.4830)	.5714 (.5071)	.6508 (.4805)
Heuristics plus content	.5238 (.5118)	.6190 (.4976)	.5714 (.5071)	.5714 (.4988)
Content only	.2857 (.4629)	.3333 (.4830)	.3810 (.4976)	.3333 (.4752)

* Standard deviations in parentheses

Table VIII
Overall F-Ratios for the
Algebraic Test

<u>Source of Variation</u>	<u>Degrees of Freedom</u>	<u>Mean Squares</u>	<u>F₀</u>	<u>Probability (F ≥ F₀)</u>
Treatments	2	2.3333	10.7825*	0.00004
Vehicles	2	0.3333	1.5403	0.2171
Interaction	4	0.0714	0.3299	0.8577
Error	180	0.2164		

*Significant at .05 level

Table IX
Overall F-Ratios for the
Geometric Test

<u>Source of Variation</u>	<u>Degrees of Freedom</u>	<u>Mean Squares</u>	<u>F₀</u>	<u>Probability (F ≥ F₀)</u>
Treatments	2	1.7196	7.1432*	0.0010
Vehicles	2	0.0212	0.0897	0.9159
Interaction	4	0.0926	0.3846	0.8195
Error	180	0.2407		

*Significant at .05 level

Table X

Algebraic Test - Simultaneous Confidence Intervals
for Hypotheses 1, 2 and 3

	Difference between means	$\sqrt{MS_{w...}}^{**}$	.05 Confidence Intervals
H1	$\bar{Y}_{1..} - \bar{Y}_{3..} = 0.3819$	0.2047	(0.1772 , 0.5866)*
H2	$\bar{Y}_{2..} - \bar{Y}_{3..} = 0.1429$	0.2047	(-0.0618, 0.3576)
H3	$\bar{Y}_{1..} - \bar{Y}_{2..} = 0.2381$	0.2047	(0.0334 , 0.4428)*

$$** \sqrt{MS_{w...} \left(\frac{2}{n} \right) (3-1) .95 F(2,180)} = \sqrt{\frac{.2164 \times 4 \times 3.05}{63}} = 0.2047$$

*Significant at .05 level

Table XI

Geometric Test - Simultaneous Confidence Intervals
for Hypotheses 1, 2 and 3

	Difference between means	$\sqrt{MS_{w...}}^{**}$	.05 Confidence Intervals
H1	$\bar{Y}_{1..} - \bar{Y}_{3..} = 0.3174$	0.2159	(0.1015 , 0.5333)*
H2	$\bar{Y}_{2..} - \bar{Y}_{3..} = 0.2381$	0.2159	(0.0222 , 0.4540)*
H3	$\bar{Y}_{1..} - \bar{Y}_{2..} = 0.0794$	0.2159	(-0.1365, 0.2953)

$$** \sqrt{MS_{w...} \left(\frac{2}{n} \right) (3-1) .95 F(2,180)} = \sqrt{\frac{.2407 \times 4 \times 3.05}{63}} = 0.2159$$

*Significant at .05 level

Table XII

 χ^2 Analysis for the Algebraic Test

<u>Hypothesis</u>	<u>Groups</u>	<u>χ^2</u>	<u>Probability</u>
H 1	H vs C	19.0989	.00001*
H 2	HC vs C	3.2044	.0734
H 3	H vs HC	7.1718	.0074*

* Significant at .05 level

Table XIII

 χ^2 Analysis of the Geometric Data

<u>Hypothesis</u>	<u>Groups</u>	<u>χ^2</u>	<u>Probability</u>
H 1	H vs C	11.4603	.0007*
H 2	HC vs C	6.2617	.0123*
H 3	H vs HC	2.8349	.3609

* Significant at .05 level

Statistical Conclusions

Before discussing the conclusions in detail it is important to note that the analyses of hypotheses 01 - 03 by the Scheffé Procedure and the χ^2 produced identical statistical results (with $\alpha = .05$).

Figure 18 gives a graphical presentation of the statistical conclusions.

Figure 18

Visual Presentation of Results of Significance

Tests for Hypotheses H_{01} , H_{02} and H_{03}

ALGEBRAIC TEST

GEOMETRIC TEST

Sig - significant at .05 level

NS - not significant at .05 level

There was no significant overall interaction effect on either the algebraic or geometric test. Thus, the contrasts corresponding to hypotheses 4 and 5 were not tested, and null hypotheses H_{04} and H_{05} were accepted.

Interpretation of Results.

The H group scored significantly higher than the C group on both the algebraic and geometric tests (see figure 18). Since the C group was designed to control for possible heuristic ability present in the population prior to experimentation, for incidental learning of heuristic techniques and for factors such as novelty that might affect performance independently of the treatment, the investigator concludes that by focussing instruction on heuristics it is possible to teach students to apply at least one of the heuristics of examination of cases or analogy to novel mathematical problems.

The statistical analyses of H_{02} and H_{03} indicates different results for the algebraic and geometric tests. On the algebraic test the H group performed significantly better than both the HC and C groups, while there was no significant difference between the H and HC groups. The investigator concludes that of the treatments tested, the H treatment is the only effective treatment for teaching students to apply at least one of the heuristics to a novel algebraic problem. On the geometric test both the H and HC

groups performed significantly better than the C group, while no significant difference was found between the H and HC groups. The investigator concludes that both the H and HC treatments are effective in teaching students to apply at least one of the heuristics to a novel geometric problem. Furthermore, there is no statistical evidence to support the conclusion that they are differentially effective. The different conclusions for the two tests are discussed in the next section.

There was no significant overall interaction effect on either the algebraic or the geometric test. This implies that for all three vehicles (N, A, G) the pattern of means of the three treatments (H, HC, C) is essentially the same for the algebraic test, and essentially the same for the geometric test, although the patterns of means are different for the two tests (see figures 19 and 20). The investigator concludes that the vehicle does not seem to be an important consideration in teaching students to apply at least one of the heuristics. Whatever the vehicle, the critical determinant is the approach selected to introduce the heuristics, namely heuristics only, or heuristics combined with content.

Discussion of Results

Hypotheses 1, 2 and 3 refer to differences that can be attributed to the treatments. The statistical analysis enables the investigator to conclude that in terms of average

Figure 19

Mean Scores of Groups on
the Algebraic Test

Overall

Neutral groups only

Algebraic groups only

Geometric groups only

Figure 20

Mean Scores of Groups on
the Geometric Test

Overall

Neutral groups only

Algebraic groups only

Geometric groups only

test scores it is possible to teach students to apply at least one of the heuristics of examination of cases or analogy to novel mathematical problems. In particular, the heuristics only experimental treatment is successful in teaching the students to apply the heuristics to both novel algebraic and geometric problems. The results concerning the effectiveness of the heuristics plus content treatment are different for the two tests. The analysis indicates the heuristics plus content treatment is successful in teaching subjects to apply at least one of the heuristics to a novel geometric problem, but this is not true for the algebraic problem. It is impossible to determine the reasons for these different conclusions. However, the following observations concerning the two tests may, in part, explain the different statistical results.

An examination of tables II and III indicates a different pattern in the application of heuristics on the two tests. On the algebraic test of the 72 subjects who applied at least one heuristic, 54 (75%) applied analogy, either alone (14 subjects) or in conjunction with examination of cases (40 subjects). On the geometric test 98 subjects applied at least one heuristic but only 10 (10%) of them used analogy, and nine of the ten used the heuristic in conjunction with examination of cases. Subjects who had exhibited the ability to employ analogy in the algebraic test could not, or did not, apply the heuristic in the geometric

test. Thus, it appears that the geometric test was more amenable to the application of examination of cases than to the application of analogy. This could be due to the fact that the grade ten subjects were more familiar with algebraic systems, facilitating analogies in an algebraic as opposed to a geometric setting.

The investigator concludes the different results concerning the effectiveness of the heuristics plus content treatment on the tests could be attributed to this treatment better equipping subjects to apply examination of cases than to apply analogy, and that this manifests itself in the geometric test where the problem is more amenable to the application of examination of cases. An examination of the data supports this conclusion. First, on the algebraic test only 18 subjects from the HC group employed analogy, while 29 subjects from the H group applied this heuristic. Second, of the 63 subjects in the HC group there were 20 who had shown no evidence of ability to apply either heuristic in the algebraic test but had applied examination of cases in the geometric test.

One further note can be made concerning the two tests. Figure 21 shows that all three treatment groups (H, HC, C) scored higher on the geometric test than on the algebraic test. There are many possible reasons for the superior results on the geometric test. The algebraic test was administered first and subjects may have discussed that test prior to writing the geometric test. Furthermore, since both tests follow the same general format, involving

Figure 21

Overall Means for Algebraic
and Geometric Tests

Algebraic test

Geometric test

the presentation of a problem followed by the question "What can you find out about...?" there may have been a testing effect (Campbell and Stanley, 1963, p. 5).

Another possibility is that there is something inherent in the geometric problem used on the test that made it simpler than the algebraic problem. Any of these alternatives, or combination of alternatives could account for the better scores on the geometric test.

The original statements of hypotheses 1, 2 and 3 indicated an ordering between the means of the three treatment groups. This ordering is:

$$\text{Mean (H)} > \text{Mean (HC)} > \text{Mean (C)}$$

Although the analysis does not result in statistically significant differences fully justifying this ordering, such an ordering exists on both tests (figure 21).

Furthermore, this ordering also holds within the three vehicles. Figures 19 and 20 show that for the sample used

$$\text{Mean (H - } V_i) > \text{Mean (HC - } V_i) > \text{Mean (C - } V_i)$$

where $V_i = N, A, G$, with the single exception that $\text{Mean (H - G)} = \text{Mean (HC - G)}$.

One further analysis associated with the data has been included in this chapter. Table I contains the information on the number of tests graded. There were 42 subjects who wrote only one test (21 algebraic and 21 geometric). Of these the judges agreed on grades for 20 algebraic and 19 geometric subjects. Tables XIV contains the comparative data for these subjects and the subjects used in the analysis.

Table XIV

Distributions of Data for Subjects Who Wrote Only One Test
and Actual Data Used in the Analysis

	<u>Algebra</u>		<u>Geometry</u>	
	<u>0</u>	<u>1</u>	<u>0</u>	<u>1</u>
H (Data)	26	27	22	41
H (One Test)	4	5	3	4
HC (Data)	41	22	27	36
HC (One Test)	3	1	3	1
C (Data)	50	13	41	22
C (One Test)	5	2	5	3
Total (Data)	117	72	90	99
Total (One Test)	12	8	11	8

The number of subjects who wrote only one test was too small to undertake χ^2 tests for each of the three treatments. However, for each test a χ^2 value was computed between the data used in the analysis (Total(Data)) and data based on subjects who wrote only one test (Total(One Test)).

Table XV

χ^2 Comparisons of Actual Data and Only One Test Data*

<u>Test</u>	<u>Groups</u>	<u>χ^2</u>	<u>Probability</u>
Algebraic	Total (Data)		
	Vs	0.0056	.940
	Total (One Test)		
Geometric	Total (Data)		
	Vs	0.3764	.539
	Total (One Test)		

* with Yates's correction for continuity

This analysis indicates that for both tests the distribution of scores of subjects who wrote only one test (and the judges agreed on the grade) was essentially the same as the distribution of the data used in the analysis.

The following three points are noted. Since all three have educational implications or suggest possible directions for further research they are discussed in detail in the appropriate sections of chapter V.

(1) Of the 72 subjects who used at least one heuristic on the algebraic test, 40 (55%) employed both heuristics in their solution.

(2) Of the 63 subjects who received the heuristics only treatment 26 (41%) did not apply either heuristic to the algebraic problem, while 22 (35%) did not apply either heuristic to the geometric problem.

(3) On the algebraic test the performance of the group that received the heuristics only instruction by the algebraic vehicle (H - A) was better than all the other groups. The H - A group was also second highest on the geometric test.

Finally, one comment is included concerning the interpretation of the results. Conclusions have been drawn concerning the effectiveness of teaching heuristics. It has been assumed that the differences are due to the emphasis on heuristics, either alone or in conjunction with content. However, based on the evidence available in the study, alternative hypotheses that the differences in treatments are due to differential motivational effects of the booklets, effects that are independent of the heuristics, rather than the emphasis on heuristics cannot be rejected.

In one of the pilot studies data were obtained from

subjects concerning their attitude towards the self-instructional booklets. Their comments indicate that most subjects were pleased to complete the booklets, and there were no discernable differences between groups. Furthermore, as was discussed in chapter III every effort was made in designing the materials to make them as equivalent as possible.

Thus, while further research needs to be undertaken into motivation effects associated with the teaching of heuristics, the investigator feels confident that based on the design of the materials, and subjects' comments on the booklets, the success of the H and HC treatments is due to the emphasis on heuristics and not different motivational effects that are independent of the heuristics.

CHAPTER V

Summary and Conclusions

The study investigates whether it is possible to teach students to apply at least one of the heuristics of examination of cases or analogy to novel mathematical problems, and whether learning to apply heuristics can be better accomplished by having the students focus solely on the heuristics, rather than also involve them with the simultaneous learning of mathematical content. Two factors were considered in the design of the study. The first factor concerns employing treatments designed to meet different objectives: one instructional treatment was designed to teach heuristics only; another combined the instruction in heuristics with the teaching of content; and a third treatment utilized content only, without reference to heuristics. The latter treatment was used as an experimental control. The second factor concerns employing distinct instructional vehicles for the treatments, namely algebraic, geometric and mathematically neutral vehicles. Nine groups were utilized in the study corresponding to each of the treatment x vehicle combinations. Self-instructional booklets were developed for each group. The instructional period lasted ten days (one class period per day) at the end of which two tests were administered, one algebraic and the other geometric. Both tests had the

same format, consisting of presenting students with a novel mathematical problem followed by the question:

What can you find out about...?

Conclusions

The results of the data analysis indicate that it is possible to teach students to apply at least one heuristic to a novel algebraic or geometric problem. In particular, the group that received the heuristics only treatment (H) scored significantly higher than the control group (C) on both tests.

The effectiveness of the combined instructional treatment, heuristics plus content (HC), was different for the two tests. On the algebraic test the H group performed significantly better than both the HC and C groups, while no statistically significant difference was found between the HC and C groups. The conclusion is that only the H treatment can be claimed to be effective in teaching students to apply at least one heuristic to a novel algebraic problem. On the geometric test the analysis indicates that the H and HC groups both performed significantly better than the C group, but they were not themselves clearly different in their effectiveness. The conclusion is that both the H and HC treatments are effective in teaching students to apply at least one heuristic to a novel geometric problem, and the evidence

does not permit any conclusion concerning their relative effectiveness.

However, although the analysis does not warrant the general conclusion that the H treatment is more efficacious than the HC treatment in teaching students to apply at least one heuristic to novel mathematical problems, an examination of the overall means (figure 21) shows a trend in that direction. The same trend appears in the parallel examination of the means within the three vehicles (figures 19 and 20). Certainly, if one is solely interested in teaching heuristics, the results on the algebraic test, together with the trends referred to above, indicate that the H treatment is the most efficacious.

An analysis of the pattern of heuristic application on the two tests reveals that while both heuristics were employed on the algebraic test there was little evidence of analogy being used on the geometric test. The investigator suggests the effectiveness of the HC treatment on the geometric test may have been due to the fact that the HC treatment is more effective in teaching examination of cases than teaching analogy, and the geometric test proved more amenable to the application of examination of cases. It is impossible to ascertain the reason for this result concerning the HC treatment. However, the investigator suggests the following possible explanation. Examination of cases can be used to answer questions, while analogy

poses rather than answers questions. With emphasis on both heuristics and content students tend to learn to apply the heuristic that appears most valuable in answering questions, namely the heuristic of examination of cases.

There were no significant differences involving the vehicles. Thus it seems reasonable to conclude that the instructional vehicle employed is not an important factor in whether students learn to apply heuristics.

The following three points are noted. Since all three have educational implications, or suggest possible directions for further research they are discussed in detail in the appropriate sections.

(1) Of the 72 subjects who used at least one heuristic on the algebraic test, 40 (55%) employed both heuristics in their solution.

(2) Of the 63 subjects who received the heuristics only treatment 26 (41%) did not apply either heuristic to the algebraic problem, while 22 (35%) did not apply either heuristic to the geometric problem.

(3) On the algebraic test the performance of the group that received the heuristics only treatment by the algebraic vehicle (H - A) was better than all the other groups. The H - A group was also second highest on the geometric test.

Implications

As a consequence of this study several suggestions can be made concerning the teaching of heuristics. When considering these implications it should be noted that the study considered only the heuristics of examination of cases and analogy. Thus, in the following discussion the term "heuristics" refers to these two particular techniques.

The analysis indicates that it was possible to teach students to apply at least one of the heuristics to novel mathematical problems. Since an important objective of the secondary school mathematics programme is to develop skills in dealing with unfamiliar problem situations, the investigator suggests that units designed to teach heuristics be incorporated into the secondary school mathematics curriculum. If this suggestion is accepted the question arises as to the possible settings that should be employed to develop the heuristics. The study investigated two factors that help answer this question.

First, the analysis suggests that a heuristics only treatment is likely to be superior to a combined treatment, although in this study the scores were significantly better only on the algebraic test. However, based on the trends (see figures 19, 20 and 21) it seems reasonable to conclude that units devoted solely to the teaching of heuristics will be more effective than units that combine instruction in heuristics with instruction in content. To

be most effective the teacher should devote part of the mathematics course solely to developing heuristics. Such an arrangement does not preclude using heuristics in conjunction with other segments of the mathematics curriculum, but the evidence suggests that for the instruction in heuristics to be most effective some time has to be devoted solely to heuristics.

Second, the analysis indicates that the instructional vehicle employed to introduce the heuristics does not affect the students' ability to apply the heuristics. Thus, the selection of an algebraic, geometric or mathematically neutral vehicle does not appear to be critical for developing the heuristics.

It has been recommended that units on heuristics be included in the mathematics curriculum. However, one cautionary note should be added. Although the heuristics only group performed significantly better than the control group, the data indicate that there were a substantial number of students in the heuristics only group who did not apply either heuristic to the novel problems (see conclusions p. 113). These data suggest investigation of the question as to whether all students can benefit from instruction in heuristics, and this question is explored in recommendations for further research.

To summarize, it is recommended that units designed solely to teach students heuristics be incorporated into

the secondary school mathematics curriculum. Furthermore, the evidence available from this study suggests that the vehicle selected does not appear to be of critical importance.

Limitations

This dissertation focuses on certain components of the study of heuristics. The limited selection of heuristics employed and instructional vehicles utilized restricts the generalizability of the conclusions. A further limitation was imposed by the selection of an all male sample. Five limitations are discussed in this section, resulting from: (1) choice of sample; (2) use of self-instructional booklets; (3) selection of tests; (4) choice of instructional vehicles; and (5) selection of heuristics. The investigator has included in this discussion the suggestions for further research that follow directly from the limitations.

With an all male sample it is possible to make supportable statements about the teaching of heuristics to male students in the age range 15 - 17 years (Grade ten), but leaves open the question of how female students will respond to similar treatments. Fennema (1974) reviewed the literature on sex and mathematical achievement. She cited studies involving high school students that favoured males (NLSMA, Y - population), favoured females (Easterday and Easterday), and that found no significant differences

between the sexes (Blushan et. al.). Fennema (1974, p. 137) stated that no conclusion can be reached concerning the relationship between sex and mathematical achievement for high school students. Since the research evidence is ambivalent concerning the role of sex as a factor in mathematical achievement of high school students, the investigator feels that the study should be replicated with females. or with mixed groups in a design which permits the assessment of interaction effects involving sex.

Using self-instructional booklets resulted in certain constraints on the learning situation. For example, teacher-student and peer group interactions, which are an integral part of the classroom environment, were eliminated from the study. It is impossible to assess whether the lack of these interactions had a positive or negative effect on students' performance. It is possible that by allowing students to progress at their own rate, the self-instructional booklets may have helped some students learn to apply the heuristics. Alternatively, the lack of any interaction may have hindered other students. Although the research cited in chapter II indicates that self-instructional materials are an effective means of instruction, further research needs to be undertaken to determine the role of mode of instruction on students' ability to apply heuristics. An additional limitation imposed by the use of booklets is that all the instructional materials were

presented in written form. The role of reading ability in the results of this study is discussed in the recommendations for further research.

The nature of the tests is of critical importance to the study. Each test consists of an open ended mathematical problem, and the students' perception of the problem might affect their ability to apply the heuristics. For example, to apply the heuristic of analogy students must perceive a similiarity between the test problem and some previously experienced system. Although all students had been exposed to appropriate systems (for example, it was possible to make analogies between the test problems and the instructional materials in the booklets) there was no guarantee that an analogy would be made. It is possible that a student who had learned the heuristic of analogy was unable to perceive a similarity between the test problems and his previous experience, and hence unable to apply the heuristic. Because the test problems play a significant role in eliciting student response, the investigator feels that a wide variety of novel test problems must be used to obtain a clear picture of the value of heuristics such as analogy and examination of cases in solving novel problems.

The instructional vehicles selected for the study are algebraic, geometric and mathematically neutral, and the topics were chosen so that the vehicles are "equivalent" (See chapter III). This constraint limits the selection of

topics that could be used to introduce the heuristics. The investigator feels that the study should be replicated with different topics and vehicles.

Finally, only two heuristics, examination of cases and analogy, were selected from the works of Polya (1957), MacPherson (1971b), Rousseau (1971), and Henderson and Smith (1959). Whether students can be taught to apply other heuristics is an open question requiring additional research.

Recommendations for Further Research

The conclusions of this study, together with the limitations, suggest possible directions for further research. Four major questions are discussed in this section. They are:

- (1) What factors affect students' ability to apply heuristics?
- (2) What role does knowledge of core material play in students' ability to apply heuristics?
- (3) Can students be taught to combine heuristics to solve novel problems?
- (4) What instructional modes are most effective in teaching heuristics?

Factors Effecting Heuristic Ability. The fact that a substantial number of students in the heuristics only group (H) did not apply either heuristic to the test problems raises the question as to what factors might be related to the ability to apply heuristics. Science Research Associates High School Placement Test data were available on 59 of the 63 subjects in the H group. These data included measures on mathematical and reading achievement. For each of these measures the H group was divided into three subgroups (approximately one-third in each subgroup), namely high, middle and low achievers. Pairwise χ^2 analyses were undertaken to determine if either of these factors is related to students' ability to apply at least one heuristic. The results of this analysis are shown in tables XVI and XVII. Figures 22 - 27 show the data for the Mathematics and Reading subgroups.

Table XVI

χ^2 Analysis of the Achievement of the Heuristics Only
Group on the Algebraic Test*

		<u>Score</u>		<u>Groups</u>	<u>χ^2 **</u>	<u>Probability</u>
		<u>0</u>	<u>1</u>			
Mathematical Achievement	H	4	15	H , M	0.8731	.3501
	M	8	12	H , L	7.5269	.0061
	L	14	6	M , L	2.5252	.1120

		<u>Score</u>		<u>Groups</u>	<u>χ^2 **</u>	<u>Probability</u>
		<u>0</u>	<u>1</u>			
Reading Achievement	H	5	15	H , M	2.1794	.1399
	M	11	10	H , L	2.5336	.1114
	L	10	8	M , L	0.0154	.9014

		<u>Score</u>		<u>Groups</u>	<u>χ^2</u>	<u>Probability</u>
		<u>0</u>	<u>1</u>			
High or Low on BOTH measures	HH	39	11	HH,LL	2.9936	.0836
	LL	9	6			

* with Yates's correction for continuity

** not independent pairwise groupings.

Table XVII

χ^2 Analysis of the Achievement of the Heuristics Only
Group on the Geometric Test*

		<u>Score</u>		<u>Groups</u>	<u>χ^2**</u>	<u>Probability</u>
		<u>0</u>	<u>1</u>			
Mathematical Achievement	H	5	14	H , M	0.0074	.9314
	M	5	15	H , L	1.4170	.2339
	L	10	10	M , L	1.7067	.1914

		<u>Score</u>		<u>Groups</u>	<u>χ^2**</u>	<u>Probability</u>
		<u>0</u>	<u>1</u>			
Reading Achievement	H	4	16	H , M	0.3728	.5415
	M	7	14	H , L	2.5726	.1087
	L	9	9	M , L	0.5305	.4664

		<u>Score</u>		<u>Groups</u>	<u>χ^2</u>	<u>Probability</u>
		<u>0</u>	<u>1</u>			
High or Low on BOTH measures	HH	2	12	HH,LL	3.3116	.0688
	LL	8	7			

* with Yates's correction for continuity

** not independent pairwise groupings

Figure 22

Achievement of High, Middle and Low Mathematical Achievement
Subgroups of Heuristics Only Group

Algebraic Test

———— High Achievement Subgroup
 - - - - - Middle Achievement Subgroup
 - . - . - Low Achievement Subgroup

Figure 23

Achievement of High, Middle and Low Reading Achievement
Subgroups of Heuristics Only Group

Algebraic Test

———— High Achievement Subgroup
 - - - - - Middle Achievement Subgroup
 - . - . - Low Achievement Subgroup

Figure 24

Achievement of High, Middle and Low Mathematical Achievement
Subgroups of Heuristics Only Group

Figure 25

Achievement of High, Middle and Low Reading Achievement
Subgroups of Heuristics Only Group

Figure 26

Achievement of Subgroups that were High or Low
on Both Mathematics and Reading

Algebraic Test

Figure 27

Achievement of Subgroups that were High or Low
on Both Mathematics and Reading

Geometric Test

The data in tables XVI and XVII indicate that for both mathematical achievement and reading achievement the high group performed best, the low group worst, and the middle group in between. The single exception is on the geometric test where the middle mathematical achievement group performed marginally better than the high achievement group. The superior performance of the high groups as compared to the low groups is reinforced by data from subgroups consisting of students who are high or low on both mathematical and reading achievement.

In both analyses involving students who are high or low on both variables, the probabilities associated with the χ^2 value are less than .1. In the four analyses involving comparisons between high and low subgroups, one associated probability is less than .05 and two others are in the region of .1. Since the analysis is only exploratory no significance level was assigned for testing the χ^2 value. However, the evidence indicates that the high group (on either mathematical or reading achievement) performed considerably better than their corresponding low group. Thus, there is evidence to support the hypothesis that factors such as mathematical and reading achievement effect students' ability to apply at least one heuristic. In particular, the relationship between reading and the ability to apply a heuristic may be due in part to the use of self-instructional booklets in the experiment. It is desirable that further research be undertaken to investigate

the relationship between these factors and the ability to apply heuristics.

Knowledge of Core Material. It is particularly important to undertake research that examines the relationship between students' core knowledge and their ability to apply heuristics. The analysis in the previous paragraphs indicates that mathematical achievement may be a factor effecting the ability to apply heuristics. For example, on the algebraic test the χ^2 value between the high and low mathematical achievement groups had an associated probability of .0061. The special role of previous mathematical experience in applying the heuristic of analogy has already been discussed (p. 123). The performance of the control group shows that just teaching core material (with which analogies could be made on the tests), without specific reference to the heuristic, does not result in much evidence of students' applying analogy (six students on the algebraic test, no students on the geometric test). The evidence of students in the control group using analogy may, in part, reflect ability to apply the heuristic present in the sample prior to experimentation. Research should be carried out to determine the role of knowledge of core material on the ability to apply heuristics.

Combining Heuristics to Solve Problems. In both the algebraic and geometric tests the students were given

total freedom in approaching the problem. There were no instructions to the students to use either, or both of the heuristics in their solutions; although both heuristics could be used in the test situations. An examination of the data reveals that on the algebraic test many students combined the heuristics in solving the problem (See table XVIII and figure 28).

Table XVIII

Analysis of the Application of Zero, One or Two
Heuristics on the Algebraic Test

Group	Number of Heuristics			χ^2*	Probability
	<u>0</u>	<u>1</u>	<u>2</u>		
H	26	16	21	(H , HC) 7.4287	.0244
HC	41	8	14	(H , C) 20.0918	.00004
C	50	8	5	(HC , C) 5.1533	.0760

*Not independent

Since this analysis is only exploratory, no significance level was assigned for testing the χ^2 value. However, the results are similar to those obtained with the 0 - 1 dichotomy employed to test hypotheses 1, 2 and 3. That is, the H group performed best, the HC group worst,

Figure 28

Application of Zero, One or Two Heuristics
on the Algebraic Test

- Heuristics only group (H)
----- Heuristic plus content group (HC)
- . - . - Content only group (C)

and the HC group in between. The data are represented in figure 28.

In view of these findings the investigator feels that further research should be undertaken to consider how best to teach students to combine heuristics to solve novel mathematical problems.

Instructional Modes. The use of booklets resulted in certain constraints on the learning situation. In particular it is noted that booklets eliminated teacher-student and peer group interactions from the experimental process, and this may have hindered certain students in their attempts to learn how to apply heuristics. Furthermore, the analysis of high, middle and low reading achievement students tends to suggest that reading is related to a student's ability to apply at least one heuristic. Further research should employ other instructional modes for the teaching of heuristics. Research should be designed to allow for student-teacher and peer group interactions as part of the experimental procedure: that is, experiments should be conducted in "normal" classroom settings.

An examination of the data indicate that on the algebraic test the H - A group scored higher than any other group. Although there was no significant overall interaction, for the purpose of suggesting possible directions for

further research, the interaction term associated with this group was investigated. Using the approach suggested by Marascuilo and Levin (1970, pp. 397 - 421) γ_{12} was calculated and a 95 percent Scheffé confidence interval formed. The interval is

$$\gamma_{12} = .048 \pm .211$$

and does not contribute significantly to the interaction. This simply confirms the earlier conclusion that in this study there are no significant differences involving the vehicles. However, experiments involving different vehicles and instructional modes may result in the vehicles" being a significant factor. This remains to be determined by further research.

Finally, in view of the possible importance of students' core knowledge in teaching students to apply heuristics, further research should be carried out at other grade levels. If students' core knowledge is an important factor it may be that until a student has a certain breadth of mathematical experience he cannot be taught to use certain heuristics. This could be particularly true of analogy, where he is required to find similarities between novel problems and previous experience. Thus, it may be that effective instruction in heuristics such as analogy cannot be undertaken until a student has reached a certain level of mathematical maturity. It may be that a programme designed to teach heuristics should begin with the

development of examination of cases in the elementary grades, with other heuristics being introduced at later stages.

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APPENDIX A

Heuristics only - Neutral Vehicle

H - N

INTRODUCTION

The only purpose of this booklet is to teach you how to make discoveries about new mathematical topics. The booklet deals with 3 topics, which are:

- 1) How to organize seating plans,
- 2) Tiling problems,
- 3) Electrical Switching Networks.

At first glance these topics do not appear to be mathematical in nature. However, the topics have been chosen to introduce you to problem solving techniques for making mathematical discoveries, and serve no other purpose. When working through this booklet remember that you are trying to learn the problem solving techniques, not the content itself.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

SEATING PLANS

First, let us explain the notation that will be used.

Notation

indicates that a chair is empty

indicates that a chair is occupied. When a chair is occupied, but we do not know the name of the person sitting there; we will use one of the letters, x, y, or z inside the triangle.

indicates that the chair is occupied by the person and we do know their name. For example, suppose there are two lawyers attending a meeting, Mr. Abel (labelled a) and Mr. Brock (labelled b).

Then...

indicates that Mr. Abel is occupying the chair.

indicates that Mr. Brock is occupying the chair.

indicates that the chair is occupied, but we do not know which of the men is sitting there.

indicates that the left chair is occupied by Mr. Abel, and the right seat is empty.

Exercise

A meeting has been arranged between 4 directors of a fish plant. They are Mr. Brace (labelled b), Mr. Drover (d), Mr. Goodyear (g), and Mrs. Murphy (m). The organizers have placed 4 chairs in a row.

Draw diagrams to represent each of the following seating arrangements.

e.g. Chair 1 - Mr. Drover; Chairs 2 and 3 - occupied;
Chair 4 - empty.

The diagram would be

Now draw the diagrams for the following:

- (i) Chair 1 - Mr. Brace; Chair 2 - Mrs. Murphy;
Chairs 3 and 4 - occupied.
- (ii) Chair 1 - empty; Chair 2 - Mrs. Murphy;
Chair 3 - Mr. Brace; Chair 4 - Mr. Goodyear.
- (iii) Chair 1 - Mr. Drover; All other chairs occupied.
- (iv) All 4 chairs empty.
- (v) Chair 1 - Mrs. Murphy; Chair 2 - Mr. Brace;
Chair 3 - Mr. Goodyear; Chair 4 - Mr. Drover.
- (vi) Chair 1 - Mr. Goodyear; Chairs 2 and 3 - empty;
Chair 4 - occupied.

Above is the plan of the small meeting room in a large company's head office. Three branch managers, Mr. Blade, Mr. Johnson and Mr. Hamilton have been invited to talk to a group of head office personnel. Since the audience does not know the three men, you have to place name plates on the desks.

Exercise

The three managers will be seated at the speaker's desks.

How many different ways could you seat the three men? Draw diagrams to represent each of the possible seating plans.

e.g. One plan would be

where b stands for Mr. Blade, etc.

As in the above example, in writing out your solution you need only draw the chairs. There is no need to draw the room and speakers' desks.

Now draw the other diagrams below.

Solution

There are 5 other seating plans, giving a total of 6.

Plan 1. (given in exercise)

Plan 2.

Plan 3.

Plan 4.

Plan 5.

Plan 6.

In your solution to this problem you probably had all the plans listed, although they may have been in a different order. For most people, listing the various plans is a matter of trial and error. That is, you try one plan, and then another, hoping that in the end you will have listed all possible arrangements. It is now that the first problem solving technique can be of help. This problem solving technique is called examination of cases. The first stage of the technique involves the developing a systematic method for writing down all the different cases.

This particular exercise is fairly simple, since it involves only 6 different seating plans. However, it will provide a good example with which to introduce the technique. The next two pages discuss how a systematic approach can be used to list the 6 plans given above.

The problem is to list all possible ways of seating 3 people in 3 chairs. Label the chairs as follows.

Chair 1 Chair 2 Chair 3

Since all 3 chairs are to be occupied, the first step was to place one of the 3 managers in chair 1. You could start with any of the 3, but in this example Mr. Blade was chosen. This gave the following situation.

This left 2 people and 2 chairs to fill. This could be done in 2 ways, leading to plans 1 and 2.

Plan 1.

Plan 2.

We have now considered all possible plans with Mr. Blade sitting in chair 1.

The next move was to place one of the other managers in chair 1. In this example Mr. Johnson was chosen. This gave

Again this left 2 people and 2 chairs to fill, leading to plans 3 and 4.

Plan 3.

Plan 4.

Finally Mr. Hamilton was placed in the first chair, leading to the last 2 plans.

Plan 5.

Plan 6.

We have eliminated all the cases with any of the 3 men sitting in chair 1. But there are only 3 men in the problem, and since the first chair must be occupied we have written down all the cases.

Using a systematic approach to listing cases (in this example the cases were different seating plans) has 2 major advantages:-

(1) You can be sure that all possible cases have been listed. You do not have to worry that some might have been left out.

For example, in the previous exercise all the cases with Mr. Blade in chair 1 were written down before considering anyone else sitting there.

(2) You can be sure that no cases have been repeated. This may not be very important when you have only 6 cases to consider, but if you have 60 different possibilities you do not want to check that they are all different, and that you have not repeated some by mistake.

In the previous exercise the systematic approach assured us that the 6 cases listed were all different and that there were no other possible seating plans. A systematic method will help when you want to examine all cases.

As you go through this booklet, you will have more practice in listing cases by a systematic method.

Since only 3 days are being spent on seating plans this is the last day that will be spent on this topic. The next section will introduce the second of the problem solving techniques, namely analogy, and further develop examination of cases. It may seem that a great deal of time is being spent on this particular step of examination of cases, namely listing cases in a systematic manner. As you work through the other sections you will see the vital importance of this step in solving problems you have not seen before.

In order to make the remainder of today's lesson more realistic the following problem will be considered. However, as you attempt the problems do not forget that the objective of this section is to develop practice with a systematic approach to listing cases and not with the topic of seating plans.

This is part of a map of Africa.

The letters stand for the following countries.

Sierra Leone (s)	;	Liberia (l)	;	Ghana (g)
Ivory Coast (i)	;	Togo (t)	;	Mali (m)
Upper Volta (u)	;	Guinea (gu)		

As you are aware, many developing countries have border arguments with their neighbors. This is true of African countries, so the nations decided to set up The Organization of African States to help the various nations settle disputes. The Organization also helps in many other ways.

Consider the following hypothetical problems that could arise between the countries on the previous page.

Note:- The solutions are not included. You will receive them when you have completed the booklet.

Exercise

In each of the following list all the seating arrangements that would satisfy the conditions. Before listing the arrangements give a brief description of the systematic approach you intend to use. Spend some time thinking about the approach before starting to list the cases.

Try as many of the problems as you can. DO NOT WORRY IF YOU DO NOT COMPLETE ALL THE EXAMPLES. Do as many as you can in the time.

1. A special meeting has been called to discuss the border disputes between the following 4 countries.

Sierra Leone (s)

Liberia (l)

Ivory Coast (i)

Ghana (g)

Representatives of the 4 countries will be seated as follows.

Since these countries are having problems it is advisable not to seat representatives from bordering states together.

Tiling Problems

Today you will start the second topic to be discussed in this booklet, namely tiling problems. The idea of tiling can be considered as a kind of extended jigsaw puzzle, where the question asked is how many ways a particular object can be fitted into a given space.

In the first section you were introduced to the problem solving technique of examination of cases. In this section the second technique, namely analogy, will be used. Furthermore, you will see how examination of cases can be used to test conjectures (educated guesses) developed by analogy. As with seating plans, the topic of tiling is not important. It is just used to introduce analogy.

On the previous page you saw a picture of a decorator with a tile to fit into a wall. His problem is how many different ways can he fit the tile into the wall. Let's consider the following problem.

Problem

A decorator is hired to tile a bathroom. The owner of the house has bought a special triangular tile to fit into the wall. This tile is in fact an equilateral triangle. That is, a triangle with all 3 sides being of equal length. The front of the tile is divided into 3 sections, each being a different colour. The back of the tile is covered with cement so that it will stick in place.

y - yellow

b - blue

r - red

The decorator has completed the wall except for the space where the tile is to fit. He is now faced with a problem - how many different ways can the tile be fitted into the space?

Solution

The tile is in position 2.

Exercise

To save you referring back the 3 positions for the tile are given below.

Position 1

Position 2

Position 3

Now complete the following table.

	Initial Position	Rotation	Final Position
(a)	1	240° clockwise (see below)	
	1	120° clockwise	
	2	120° counterclockwise	
	3	120° counterclockwise	
	2	360° clockwise	

To find the final position in (a) imagine the tile is in position 1. Then think how it moves when rotated through 240° in a clockwise direction. You obtain position 3.

Initial Position

240° clockwise

Final Position

You would place a 3 in the table.

Solution

Initial Position	Rotation	Final Position
1	240° clockwise	3
1	120° clockwise	2
2	120° counterclockwise	1
3	120° counterclockwise	2
2	360° clockwise	2

From the results above it would seem reasonable to conclude that there were no other positions for the tile. However, can the decorator be sure?

Clearly with this particular situation involving a triangular tile decorated on 1 side, he could be reasonably sure. By rotating the tile backward and forward he would soon conclude that there were only 3 positions. However, if he just fits the tile into the wall using a trial and error approach this would not help him in a more complex situation.

We will now discuss a general approach that can be used to attack this problem. In discussing how to approach this problem from a systematic point of view we will lay the base for introducing the second problem solving technique, analogy.

Imagine the decorator picks up the tile and fits it into the wall. Let the base position, that is the first position he tries be the one below.

Base Position

The movement the decorator can use to move from one position to another is a rotation.

Read carefully

EXAMINATION OF CASES

- Step 1 If possible, fix an object as a starting point.
- Step 2 Apply any conditions that may be stated concerning the objects.
- Step 3 Apply movements, operations, or arrange objects, according to the problem. Use a systematic procedure to do this.
- Step 4 Look for a pattern.

Let us see how these steps apply to the two types of problem we have been dealing with, namely seating plans and tiling problems.

Examination of Cases as applied to tiling problems

- Step 1 Fix the tile in a starting position.
- Step 2 There were no conditions other than the tile must fit into the wall.
- Step 3 Tile is rotated in a systematic way. First looking at clockwise, then counterclockwise rotations.
- Step 4 The first 2 rotations give you 3 different positions. The base position and those labelled positions 2 and 3. Looking at the remainder of the table you can see a pattern; the same positions occur again in a definite order. From this you can conclude that no new positions will arise.

Examination of Cases as applied to seating plans

- Step 1 Fix a given person in a seat.
- Step 2 Apply conditions. In some of the problems different people had to sit in particular seats.
- Step 3 Other people were seated in a systematic way.
- Step 4 Had not been discussed and was not appropriate to seating plans.

If necessary the procedure was repeated for different people in different seats.

Read the following carefully

At the beginning of the booklet it was stated that the aim was to introduce you to some problem solving techniques. This page will deal with analogy. We have used analogy on two different occasions.

I. When we were first faced with a new problem. The problem involving the 2-sided tile was introduced. Analogy was used by looking at what approach had been taken with the one sided tile. This led to the idea that rotations might be of value with the new problem.

II. When we tried to extend our knowledge of movements involving the 2-sided tile. We had found the 6 different positions for the tile. We knew that it was possible to move from the base position to positions 2 and 4 with a single movement. We tried to extend our knowledge by asking if it were possible to move from the base position to any other position with a single movement. Again we used analogy which enabled us to find the other movements, which were different rotations and flips.

Notice how in II what we had already found out about the movement of a 2-sided tile led us to use analogy to find out more.

At the moment the idea of analogy is probably vague. Don't worry about this. As you work through the remainder of the booklet, and have more practice with analogy, the idea will become clearer.

Today we will be starting the final section of the booklet. The topic that will be discussed is that of electrical switching networks. You are familiar with switches from your everyday experience but you have probably never considered them as a topic in mathematics. As we develop the topic you will also have practice in applying the problem solving techniques of analogy and examination of cases.

Most switches, whatever their size, shape, colour, etc., have one property in common, namely they are either open or closed. There is no half way stage. You might like to think of a switch as an electrical drawbridge. If the bridge is up (the switch is open) the traffic cannot move (the current cannot flow). If the bridge is down (the switch is closed) the traffic moves (the current flows). The position of the switch is referred to as the state of the switch.

Let A be a switch. The following diagrams will be used in this section.

The state of the switch is unknown. We do not know if the switch is open or closed, and hence we do not know if the current will flow.

The state of the switch is open. The current will not flow.

The state of the switch is closed. The current will flow.

Exercise

Under what conditions will the current flow through this network? You should look at the different combinations of states and draw the networks.

Try to list the cases in a systematic manner.

When using examination of cases to consider different networks, the following results were obtained.

These networks are equivalent.

The current flows if A and B are closed.

These networks are equivalent.

The current flows if A, or B, or both are closed.

Exercise

Can you think of any mathematical operations that remind you of connecting switches?

Hint:- Look at the underlined words.

Solution

If you have found operations that remind you of connecting switches, turn to the next page. If not, the following comment may help.

In the networks on the previous page the switches have been combined(joined). Also the order in which they have been joined does not matter.

That is

Solution continued

You could have chosen many operations that remind you of connecting switches. For example, you could have thought of any of the following.

Addition;	Multiplication;	Union of Sets;	Intersection
$+$	$\times$	$\cup$	of Sets:
			$\cap$

We must decide which analogy to make. The analogy that will be used is with set theory.

Let us consider the definition of Union and Intersection of Sets. If X and Y are sets then

$X \cup Y$ is the set of elements in X or Y , or both.

$X \cap Y$ is the set of elements in X and Y .

If we consider switches labelled A and B we have

Current flows if A or B , or both are closed.

Current flows if A and B are closed.

Exercise

Which network would you draw to correspond to $A \cup B$?
Which one to correspond to $A \cap B$?

Solution

For two switches, A and B, $A \cup B = B \cup A$

Read the following carefully

Before continuing to develop the analogy with set theory, let us review the power of analogy. We were initially faced with a set of problems involving switching networks, a new topic in mathematics. By trying to compare networks with another mathematical system, a system with which we are more familiar, we are now looking at set theory. It is hoped that we will be able to use many of the properties associated with set theory to help solve problems with networks.

Hence, analogy has taken us from a totally new topic, switching networks, into a more familiar topic, set theory. As you work through the remainder of this section you will see how set theory will help us with network problems.

As mentioned above we hope to use the properties of set theory to help with network problems. In particular, we would like to use the properties associated with union and intersection to find equivalent networks. In order to do this we will have to examine the various properties associated with Union and Intersection.

APPENDIX B

Heuristics plus content - Neutral Vehicle

HC - N

INTRODUCTION

There are two purposes to this booklet. To teach you how to make discoveries about new mathematical topics, and to show you how mathematical ideas can help you solve problems taken from areas other than mathematics.

The booklet deals with 3 topics, which are:

- 1) How to organize seating plans,
- 2) Tiling problems,
- 3) Electrical switching networks.

At first glance these topics may not appear to be very mathematical in nature. However, as you work through the booklet you will see how the mathematical ideas developed through the use of discovery techniques will help you deal with these topics.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

SEATING PLANS

You are probably wondering why this topic is important. The following are two examples of where seating plans can play, or played, a role.

At the highest level of diplomacy, approximately one year was spent discussing the seating arrangements for the Paris Peace Talks on Vietnam. The finding of a seating plan that was acceptable to all the parties was essential. This plan had to be found before the talks could start.

The other example is of a more personal nature. Some friends (or maybe your parents) have to arrange the seating plan for a wedding. You would prefer to sit next to Jane, not Tom. There are a large number of guests and many of them have similar requests concerning their own seats. Your friends are faced with the difficult task of trying to organize a seating plan that will keep all the guests happy.

These are only two of many situations where seating plans play an important role. For example, international conferences, civic luncheons, board meeting, etc. quite often involve setting up seating plans. In many cases you are inviting important people who could be offended if they were not sitting in the 'correct' place.

Before looking at seating problems the next page will introduce the notation that will be used in this section.

First, let us explain the notation that will be used.

Notation

indicates that a chair is empty

indicates that a chair is occupied. When a chair is occupied, but we do not know the name of the person sitting there, we will use one of the letters, x, y, or z inside the triangle.

indicates that the chair is occupied by the person and we do know their name. For example, suppose there are two lawyers attending a meeting, Mr. Abel (labelled a) and Mr. Brock (labelled b).

Then...

indicates that Mr. Abel is occupying the chair.

indicates that Mr. Brock is occupying the chair.

indicates that the chair is occupied, but we do not know which of the men is sitting there.

indicates that the left chair is occupied by Mr. Abel, and the right seat is empty.

Exercise

A meeting has been arranged between 4 directors of a fish plant. They are Mr. Brace (labelled b), Mr. Drover (d), Mr. Goodyear (g), and Mrs. Murphy (m). The organizers have placed 4 chairs in a row.

Draw diagrams to represent each of the following seating arrangements.

e.g. Chair 1 - Mr. Drover; Chairs 2 and 3 - occupied;
Chair 4 - empty.

The diagram would be

Now draw the diagrams for the following:

- (i) Chair 1 - Mr. Brace; Chair 2 - Mrs. Murphy;
Chairs 3 and 4 - occupied.
- (ii) Chair 1 - empty; Chair 2 - Mrs. Murphy;
Chair 3 - Mr. Brace; Chair 4 - Mr. Goodyear.
- (iii) Chair 1 - Mr. Drover; All other chairs occupied.
- (iv) All 4 chairs empty.
- (v) Chair 1 - Mrs. Murphy; Chair 2 - Mr. Brace;
Chair 3 - Mr. Goodyear; Chair 4 - Mr. Drover.
- (vi) Chair 1 - Mr. Goodyear; Chairs 2 and 3 - empty;
Chair 4 - occupied.

Above is the plan of the small meeting room in a large company's head office. Three branch managers, Mr. Blade, Mr. Johnson and Mr. Hamilton have been invited to talk to a group of head office personnel. Since the audience does not know the three men, you have to place name plates on the desks.

Exercise

The three managers will be seated at the speaker's desks.

How many different ways could you seat the three men? Draw diagrams to represent each of the possible seating plans.

e.g. One plan would be

where b stands for Mr. Blade, etc.

As in the above example, in writing out your solution you need only draw the chairs. There is no need to draw the room and speakers' desks.

Now draw the other diagrams below.

Solution

There are 5 other seating plans, giving a total of 6.

Plan 1. (given in exercise)

Plan 2.

Plan 3.

Plan 4.

Plan 5.

Plan 6.

In your solution to this problem you probably had all the plans listed, although they may have been in a different order. For most people, listing the various plans is a matter of trial and error. That is, you try one plan, and then another, hoping that in the end you will have listed all possible arrangements. It is now that the first problem solving technique can be of help. This problem solving technique is called examination of cases. The first stage of the technique involves the developing of a systematic method for writing down all the different cases.

This particular exercise is fairly simple, since it involves only 6 different seating plans. However, it will provide a good example with which to introduce the technique. The next two pages discuss how a systematic approach can be used to list the 6 plans given above.

The problem is to list all possible ways of seating 3 people in 3 chairs. Label the chairs as follows.

Chair 1 Chair 2 Chair 3

Since all 3 chairs are to be occupied, the first step was to place one of the 3 managers in chair 1. You could start with any of the 3, but in this example Mr. Blade was chosen. This gave the following situation.

This left 2 people and 2 chairs to fill. This could be done in 2 ways, leading to plans 1 and 2.

Plan 1.

Plan 2.

We have now considered all possible plans with Mr. Blade sitting in chair 1.

The next move was to place one of the other managers in chair 1. In this example Mr. Johnson was chosen. This gave

Again this left 2 people and 2 chairs to fill, leading to plans 3 and 4.

Plan 3.

Plan 4.

Finally Mr. Hamilton was placed in the first chair, leading to the last 2 plans.

Plan 5.

Plan 6.

We have eliminated all the cases with any of the 3 men sitting in chair 1. But there are only 3 men in the problem, and since the first chair must be occupied we have written down all the cases.

Using a systematic approach to listing cases (in this example the cases were different seating plans) has 2 major advantages:-

(1) You can be sure that all possible cases have been listed. You do not have to worry that some might have been left out.

For example, in the previous exercise all the cases with Mr. Blade in chair 1 were written down before considering anyone else sitting there.

(2) You can be sure that no cases have been repeated. This may not be very important when you only have 6 cases to consider, but if you have 60 different possibilities you do not want to check that they are all different, and that you have not repeated some by mistake.

In the previous exercise the systematic approach assured us that the 6 cases listed were all different and that there were no other possible seating plans. A systematic method will help when you want to examine all cases.

As you go through this booklet, you will have more practice in listing cases by a systematic method.

Since only 3 days are being spent on seating plans this is the last day that will be spent on this topic. As mentioned at the beginning of this section seating plans often play an important role in diplomatic circles. We will consider problems that could arise from diplomatic situations.

This is part of a map of Africa.

The letters stand for the following countries.

Sierra Leone (s);	Liberia (l);	Ghana (g)
Ivory Coast (i) ;	Togo (t) ;	Mali (m)
Upper Volta (u) ;	Guinea (gu)	

As you are aware, many developing countries have border disputes with their neighbors. This is true of African Countries, and when arguments occur The Organization of African States tries to help the countries involved settle their dispute. The Organization also helps in many other ways.

Consider the following hypothetical problems that could arise between the countries on the previous page.

Note:- The solutions are not included. You will receive them when you have completed the booklet.

Exercise

In each of the following list all the seating arrangements that satisfy the conditions. Use a systematic approach to list the cases (seating arrangements).

Try as many problems as you can. DO NOT WORRY IF YOU DO NOT COMPLETE ALL THE EXAMPLES. Do as many as you can in the time.

1. A special meeting has been called to discuss the border disputes between the following four countries.

Sierra Leone (s)

Liberia (l)

Ivory Coast (i)

Ghana (g)

Representatives of the four countries will be seated as follows:

Since these countries are having problems it is advisable not to seat representatives from bordering states together.

Tiling Problems

Today you will start the second topic to be discussed in this booklet, namely tiling problems. The idea of tiling can be considered as a kind of extended jigsaw puzzle, where the question asked is how many ways a particular object can be fitted into a given space.

As you work through this section you will be introduced to the second problem solving technique, analogy. Furthermore you will see how these techniques can be combined to help you solve tiling problems.

On the previous page you saw a picture of a decorator with a tile to fit into a wall. His problem is how many different ways can he fit the tile into the wall. Let's consider the following problem.

Problem

A decorator is hired to tile a bathroom. The owner of the house has bought a special triangular tile to fit into the wall. This tile is in fact an equilateral triangle. That is, a triangle with all 3 sides being of equal length. The front of the tile is divided into 3 sections, each being a different colour. The back of the tile is covered with cement so that it will stick in place.

y - yellow

b - blue

r - red

The decorator has completed the wall except for the space where the tile is to fit. He is now faced with a problem - how many different ways can the tile be fitted into the space?

Solution

The tile is in position 2.

Exercise

To save you referring back the 3 positions for the tile are given below.

Position 1

Position 2

Position 3

Now complete the following table.

Initial Position	Rotation	Final Position
(a) 1	240° clockwise (see below)	
1	120° clockwise	
2	120° counterclockwise	
3	120° counterclockwise	
2	360° clockwise	

To find the final position in (a) imagine the tile is in position 1. Then think of how it moves when rotated through 240° in a clockwise direction. You obtain position 3.

You would place a 3 in the table.

Solution

Initial Position	Rotation	Final Position
1	240° clockwise	3
1	120° clockwise	2
2	120° counterclockwise	1
3	120° counterclockwise	2
2	360° clockwise	2

From the results above it would seem reasonable to conclude that there were no other positions for the tile. However, can the decorator be sure?

Clearly with this particular situation involving a triangular tile decorated on 1 side, he could be reasonably sure. By rotating the tile backward and forward he would soon conclude that there were only 3 positions. However, if he just fits the tile into the wall using a trial and error approach this would not help him in a more complex situation.

We will now discuss a general approach that can be used to attack this problem. In discussing how to approach this problem from a systematic point of view we will lay the base for introducing the second problem solving technique, analogy.

Imagine the decorator picks up the tile and fits it into the wall. Let the base position, that is the first position he tries, be the one below.

Base Position

The movement the decorator can use to move from one position to another is a rotation.

From the results it seems reasonable to conclude that there are no other positions for the tile. If you look at the final positions for the tile there is a pattern in the way they appear in the table.

The technique of examination of cases has been of value in helping the decorator be certain that he had all possible positions for the tile. By approaching the problem in a systematic manner (listing the various rotations in a systematic way) and observing that a pattern appeared in the final positions, he could be sure that he had listed all possible positions for the tile.

The components that form the technique of examination of cases are:

- Step 1: If possible fix an object as the starting point.
- Step 2: Apply any conditions that may be stated concerning the objects.
- Step 3: Apply the movements, operations, or arrange objects, according to the problem. Use a systematic procedure to do this.
- Step 4: Look for a pattern.

You will probably have noted that this technique has been used in both the seating plans problems and those involving the tile. All the steps may not be appropriate to all problems. In dealing with seating plans we applied step 2, applying particular seating conditions, but did not use the last step, looking for a pattern. In the tiling problem there are no conditions to apply, but we did look at a pattern.

Today we will be starting the final section of the booklet. The topic that will be discussed is that of electrical switching networks. You are familiar with switches from your everyday experience, but you have probably not considered them as a topic in mathematics. As we develop the topic you will have an opportunity of using the problem solving techniques of analogy and examination of cases.

Most switches, whatever their size, shape, colour, etc., have one property in common, namely they are either open or closed. There is no half way stage. You might like to think of a switch as an electrical drawbridge. If the bridge is up (the switch is open) the traffic cannot move (the current cannot flow). If the bridge is down (the switch is closed) the traffic moves (the current flows). The position of the switch is referred to as the state of the switch.

Let A be a switch. The following diagrams will be used in this section.

The state of the switch is unknown. We do not know if the switch is open or closed, and hence we do not know if the current will flow.

The state of the switch is open. The current will not flow.

The state of the switch is closed. The current will flow.

The following results have been obtained.

These networks are equivalent.

The current flows if A and B are closed.

These networks are equivalent.

The current flows if A or B, or both are closed.

Exercise

Can you think of any mathematical operations that remind you of connecting switches.

Hint: Look at the underlined words.

Solution continued

You could have chosen many operations that remind you of connecting switches. For example, you could have thought of any of the following.

Addition;	Multiplication;	Union of Sets;	Intersection
$+$	$\times$	$\cup$	of Sets:
			$\cap$

We must decide which analogy to make. The analogy that will be used is with set theory.

Let us consider the definition of Union and Intersection of Sets. If X and Y are sets then

$X \cup Y$ is the set of elements in X, or Y, or both.

$X \cap Y$ is the set of elements in X and Y.

If we consider switches labelled A and B we have

Current flows if A, or B, or both are closed.

Current flows if A and B are closed.

Exercise

Which network would you draw to correspond to $A \cup B$?
Which one to correspond to $A \cap B$?

APPENDIX C

Content only - Neutral Vehicle

C - N

INTRODUCTION

The booklet deals with the following three topics:

- 1) How to organize seating plans,
- 2) Tiling problems,
- 3) Electrical switching networks.

At first glance these topics do not appear to be mathematical in nature. However, as you work through the booklet, in particular as you try sections (2) and (3), you will see how mathematical ideas help you solve the problems. At the end of the booklet you should be able to solve problems which deal with any of the three topics listed above.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

SEATING PLANS

You are probably wondering why this topic is important. The following are two examples of where seating plans can play, or played, a role.

At the highest level of diplomacy, approximately one year was spent discussing the seating arrangements for the Paris Peace Talks on Vietnam. The finding of a seating plan that was acceptable to all the parties was essential. This plan had to be found before the talks could start.

The other example is of a more personal nature. Some friends (or maybe your parents) have to arrange the seating plan for a wedding. You would prefer to sit next to Jane, not Tom. There are a large number of guests and many of them have similar requests concerning their own seats. Your friends are faced with the difficult task of trying to organize a seating plan that will keep all the guests happy.

These are only two of many situations where seating plans play an important role. For example, international conferences, civic luncheons, board meetings, etc. quite often involve setting up seating plans. In many cases you are inviting important people who could be offended if they were not sitting in the 'correct' place.

Before looking at seating problems the next page will introduce the notation that will be used in this section.

First, let us explain the notation that will be used.

Notation

indicates that a chair is empty.

indicates that a chair is occupied. When a chair is occupied, but we do not know the name of the person sitting there, we will use one of the letters, x, y, or z inside the triangle.

indicates that the chair is occupied by the person and we do know their name. For example, suppose there are two lawyers attending a meeting, Mr. Abel (labelled a) and Mr. Brock (labelled b).

Then...

indicates that Mr. Abel is occupying the chair.

indicates that Mr. Brock is occupying the chair.

indicates that the chair is occupied, but we do not know which of the men is sitting there.

indicates that the left chair is occupied by Mr. Abel, and the right seat is empty.

Exercise

A meeting has been arranged between 4 directors of a fish plant. They are Mr. Brace (labelled b), Mr. Drover (d), Mr. Goodyear (g), and Mrs. Murphy (m). The organizers have placed 4 chairs in a row.

Draw diagrams to represent each of the following seating arrangements.

e.g. Chair 1 - Mr. Drover; Chairs 2 and 3 - occupied;
Chair 4 - empty.

The diagram would be

Now draw the diagrams for the following:

- (i) Chair 1 - Mr. Brace; Chair 2 - Mrs. Murphy;
Chairs 3 and 4 - occupied.
- (ii) Chair 1 - empty; Chair 2 - Mrs. Murphy;
Chair 3 - Mr. Brace; Chair 4 - Mr. Goodyear.
- (iii) Chair 1 - Mr. Drover; All other chairs occupied.
- (iv) All 4 chairs empty.
- (v) Chair 1 - Mrs. Murphy; Chair 2 - Mr. Brace;
Chair 3 - Mr. Goodyear; Chair 4 - Mr. Drover.
- (vi) Chair 1 - Mr. Goodyear; Chairs 2 and 3 - empty;
Chair 4 - occupied.

Above is the plan of the small meeting room in a large company's head office. Three branch managers, Mr. Blade, Mr. Johnson and Mr. Hamilton have been invited to talk to a group of head office personnel. Since the audience does not know the three men, you have to place name plates on the desks.

Exercise

The three managers will be seated at the speaker's desks.

How many different ways could you seat the three men? Draw diagrams to represent each of the possible seating plans.

e.g. One plan would be

where b stands for Mr. Blade, etc.

As in the above example, in writing out your solution you need only draw the chairs. There is no need to draw the room and speakers' desks.

Now draw the other diagrams below.

Solution

There are 5 other seating plans, giving a total of 6.

Plan 1.				(given in exercise)
Plan 2.
Plan 3.
Plan 4.
Plan 5.
Plan 6.

The order in which you listed the plans is not important. Just check that you have listed all the different plans.

If it did not matter where the men sat there would be no problem in choosing a seating plan. Any of the 6 given above could be used. However, in many problems there are restrictions as to where people sit. For example, you find that Mr. Blade and Mr. Hamilton do not like each other. In this case you try to keep them happy by choosing a plan in which they do not sit next to each other.

Exercise

How many of the 6 arrangements can you choose and still keep Mr. Blade and Mr. Hamilton happy?

List them below.

Since only 3 days are being spent on seating plans this is the last day that will be spent on this topic. As mentioned at the beginning of this section seating plans often play an important role in diplomatic circles. We will consider problems that could arise from diplomatic situations.

This is part of a map of Africa.

The letters stand for the following countries.

Sierra Leone (s);	Liberia (l);	Ghana (g)
Ivory Coast (i);	Togo (t);	Mali (m)
Upper Volta (u);	Guinea (gu)	

As you are aware, many developing countries have border disputes with their neighbors. This is true of African Countries, and when arguments occur The Organization of African States tries to help the countries involved settle their dispute. The Organization also helps in many other ways.

Consider the following hypothetical problems that could arise between the countries on the previous page.

Note:- The solutions are not included. You will receive them after you have completed the booklet.

Exercise

In each of the following give all possible seating plans that will satisfy the conditions. Try as many problems as you can. DO NOT WORRY IF YOU DO NOT COMPLETE ALL THE EXAMPLES. There is no quiz at the end of today's work.

1. A special meeting has been called to discuss the border disputes between the following countries.

Sierra Leone (s)

Liberia (l)

Ivory Coast (i)

Ghana (g)

Representatives of the four countries will be seated as follows:

Since these countries are having problems it is advisable not to seat representatives from bordering states together.

For example, since Ghana and the Ivory Coast border each other, their representatives should not be seated next to each other.

How many different seating arrangements can you find such that no countries with a common border sit together.

e.g.

is an acceptable arrangement.

2. Can the 4 states in 1. be seated round a circular table so that no countries with a common border sit together?

3. Repeat problems 1 and 2 with Sierra Leone replaced by Guinea.

Draw diagrams for the various seating plans. In the case of the circular table draw non-equivalent diagrams.

4. How many different arrangements can you find if the following five countries are involved?

Togo (t)

Ivory Coast (i)

Sierra Leone (s)

Mali (m)

Upper Volta (u)

Again countries with a common border cannot sit together. The table is arranged as follows.

5. Try problem 4 using a circular table.

Tiling Problems

Today you will start the second topic to be discussed in this booklet, namely tiling problems. The idea of tiling can be considered as a kind of extended jigsaw puzzle, where the question asked is how many ways a particular object can be fitted into a given space.

On the previous page you saw a picture of a decorator with a tile to fit into a wall. His problem is how many different ways can he fit the tile into the wall. Let's consider the following problem.

Problem

A decorator is hired to tile a bathroom. The owner of the house has bought a special triangular tile to fit into the wall. This tile is in fact an equilateral triangle. That is, a triangle with all 3 sides being of equal length. The front of the tile is divided into 3 sections, each being a different colour. The back of the tile is covered with cement so that it will stick in place.

y - yellow

b - blue

r - red

The decorator has completed the wall except for the space where the tile is to fit. He is now faced with a problem; how many different ways can the tile be fitted into the space?

Solution

The tile is in position 2.

Exercise

To save you referring back the 3 positions for the tile are given below.

Position 1

Position 2

Position 3

Now complete the following table.

	Initial Position	Rotation	Final Position
(a)	1	240° clockwise (see below)	
	1	120° clockwise	
	2	120° counterclockwise	
	3	120° counterclockwise	
	2	360° clockwise	

To find the final position in (a) imagine the tile is in position 1. Then think how it moves when rotated through 240° in a clockwise direction. You obtain position 3.

Initial Position

240° clockwise

Final Position

You would place a 3 in the table.

Solution

Initial Position	Rotation	Final Position
1	240° clockwise	3
1	120° clockwise	2
2	120° counterclockwise	1
3	120° counterclockwise	2
2	360° clockwise	2

From these results it seems that no other positions for the tile exist. However, on page 39 we asked how the decorator could be reasonably sure that no other positions exist. That is, how he could be reasonably certain that he has all the positions for the tile. The following approach will work.

Imagine the decorator picks up the tile and fits it into the wall. Let the first position he tries be position 1.

Position 1

The movement the decorator can use to move from one position to another is a rotation.

Today you will be starting the final section of the booklet. The topic that will be discussed is that of electrical switching networks. You are familiar with switches from your everyday experience, but you have probably never considered them as a topic in mathematics.

Most switches, whatever their size, shape colour, etc., have one property in common, namely they are either open or closed. There is no half way stage. You might like to think of a switch as an electrical drawbridge. If the bridge is up (the switch is open) the traffic cannot move (the current cannot flow). If the bridge is down (the switch is closed) the traffic moves (the current flows). The position of the switch is referred to as the state of the switch.

Let A be a switch. The following diagrams will be used in this section.

The state of the switch is unknown. We do not know if the switch is open or closed, and hence we do not know if the current will flow.

The state of the switch is open. The current will not flow.

The state of the switch is closed. The current will flow.

You are probably wondering why the topic is important. Before discussing different networks let us consider the following situation.

A firm is involved in manufacturing electrical equipment. They are involved in tendering for jobs with the government, large companies, etc. Clearly the lower their bid the more likely they are to be given the job. They would like to keep the costs as low as possible, and in particular use as little equipment to do the job. Thus, if they can design a system containing only 1,000 switches instead of 2,000 they can save money.

This booklet will not consider networks involving thousands of switches. We will discuss how to reduce the number of switches in a network and still do the same job. Such a reduction would save the firm money and help them reduce their bid. Although we do not discuss large networks the technique used could be used in situations involving a large number of switches.

Solution

(i) A closed; B closed; op(A) open

There is an unbroken path, so the current flows.

(ii) A open; B closed; op(A) closed

There is an unbroken path, so the current flows.

Exercise

Under what conditions will the current flow through this network? You should look at different combinations of states and draw the networks.

Given 2 different switches there are 2 non-equivalent ways they can be connected. Let A and B be the switches.

(1) They can be connected in such a way that the current only flows if both A and B are closed.

Network 1

We examined this network in a previous exercise and found that the current only flows if A and B are closed. If 2 switches are connected in this manner we say that they are connected in series. We will denote network 1 by

$$A \cap B$$

where $\cap$ stands for intersection. The expression $A \cap B$ is referred to as A intersection B.

It should be noted that the following network

Network 2

has been shown to be equivalent to network 1. The notation that would be used for network 2 is $B \cap A$. The letters have been altered to indicate that the order of the switches has changed.

(ii) The switches can be connected so that the current will flow if either A, or B, or both are closed.

If 2 switches are connected in this manner we will say they are connected in parallel. We will denote this network by

$$A \cup B$$

where $\cup$ stands for union, and the expression $A \cup B$ will be referred to as A union B.

APPENDIX D

Heuristics only - Algebraic Vehicle

H - A

INTRODUCTION

The only purpose of this booklet is to teach you how to make discoveries about new mathematical topics. The booklet deals with 3 topics, which are:

- 1) Permutations,
- 2) Mappings,
- 3) Binary Networks.

The topics have been chosen to introduce you to problem solving techniques for making mathematical discoveries, and serve no other purpose. When working through this booklet remember that you are trying to learn the problem solving techniques, not the content itself.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked at returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

PERMUTATIONS

We will consider different permutations of letters, numbers, etc. That is, different ways of arranging objects.

For example, the following would be considered 2 different permutations of the numbers 1,2, and 3.

1 2 3 and 3 1 2

Before discussing the topic in detail, let us introduce the notation that will be used.

Notation

In the following the word object could stand for a number, a letter, etc. It would depend on the particular exercise.

will be used as a placeholder. When drawn like this it will show that the position indicated is empty.

indicates that there is an object in that place, but we do not know what the object is. We will use one of the letters x,y, or z inside the placeholder.

indicates that there is a 1 in the place.

indicates that the letter a is in that place.

An example is given on the next page.

Exercise

Consider the set consisting of b, j, and h.

Using all three letters, how many different possible permutations can you find.

e.g. One permutation would be

b j h

List all the permutations,

Solution

There are 5 other permutations, giving a total of 6.

Permutation 1.	b	j	h	(given in exercise)
Permutation 2.	b	h	j	
Permutation 3.	j	h	b	
Permutation 4.	j	b	h	
Permutation 5.	h	j	b	
Permutation 6.	h	b	j	

In your solution to this problem you probably had all the permutations listed, although they may have been in a different order. For most people, listing the various permutations is a matter of trial and error. That is, you try one permutation, and then another, hoping that in the end you will have listed all possible combinations. It is now that the first problem solving technique can be of help. This problem solving technique is called examination of cases. The first stage of the technique involves the developing of a systematic method for writing down the different cases.

This particular exercise is fairly simple since it involves only 6 different permutations. However, it will provide a good example with which to introduce the technique. The next two pages discuss how a systematic approach can be used to list the 6 permutations given above.

We have eliminated all the cases with any of the 3 letters in position 1. But there were only 3 letters in the set being considered, so we must have written down all possible cases.

Using a systematic approach to listing cases (in this example the cases are permutations) has two major advantages:-

(1) You can be sure that all possible cases have been listed. You do not have to worry that some cases might have been left out.

For example, in the previous exercise all the cases with b in position 1 were written down before considering any other letter in that position.

(2) You can be sure that no cases have been repeated. This may not be very important when you only have 6 cases to consider, but if you had 60 different possibilities you do not want to check that they are all different, and that you have not repeated some by mistake.

In the previous exercise the systematic approach assured us that the 6 cases (permutations) listed were all different and that there were no other possible permutations. A systematic approach will help when you want to examine all cases.

As you go through this booklet you will have more practice listing cases by a systematic method.

Today we are going to discuss permutations involving letters. Furthermore, these letters will be organized in a circular manner. You will recall that the aim of this booklet was to introduce some problem solving techniques. So far, only an introduction to examination of cases has been given, and in particular only the first stage has been discussed, namely systematically listing cases. In dealing with today's material the value of systematically listing cases can be seen more clearly. Particularly when the topic of equivalent permutations is discussed, you will see that without a systematic approach the danger of repeating permutations without realizing it is very high. So the aim of today's material is to give you more practice with systematically listing cases.

Exercise

Consider the set containing the letters a, b, and c.

The 3 places that are to be filled are arranged in a circular manner.

Circular permutation is recorded as

($\triangle P1$, $\triangle P2$, $\triangle P3$)

All three places are occupied. b must be placed next to a. How many different permutations satisfy this condition? (No letter is to be used more than once)

Hint: Fix a in one place and then consider b. Then move a to another place, etc.

Eg. a in P1, b in P2, c in P3 will be recorded as

(a, b, c) - the placeholders have been left out.

List the circular permutations.

Note: Each circular permutation uses each letter once and only once.

Mappings

Today you will start the second topic to be discussed in this booklet, namely mappings. In the first section you were introduced to the problem solving technique of examination of cases. In this section the second technique, namely analogy, will be used. Furthermore, you will see how examination of cases can be used to test conjectures (educated guesses) developed by analogy. As with permutations, the topic of mappings is not important. It is just used to introduce analogy.

Let us start by considering the following problem.

Problem

The set of objects that will be examined consists of

A , B , C

We start with the objects in the order

A B C Position 1

Mapping 1 (M_1) is defined as follows:

A	B	C	
↓	↓	↓	
B	C	A	Position 2

That is, A is mapped to B, B is mapped to C, and C is mapped to A.

A B C is referred to as position 1. When you apply M_1 to this position the letters are moved to position 2.

When we apply M_1 B maps to C. Then applying M_1 again C maps to A. Similarly for the other columns.

When M_1 is applied twice we denote it by $M_1 \& M_1$.

Exercise

To save you referring back the 3 positions and mapping are repeated below:

A B C Position 1

B C A Position 2

C A B Position 3

Now complete the following table.

Initial Position	Mapping	Final Position
2	$M_1 \& M_1$ (see below)	
3	M_1	
3	$M_1 \& M_1 \& M_1$	
1	$M_1 \& M_1 \& M_1$	

(a)

To find the position in (a) consider the following

Position 2

Position 1

B is mapped to C and then to A.
C is mapped to A and then to B, etc.
The final position is A B C,
which is position 1. Therefore
you would place a 1 in the table.

Read carefully

EXAMINATION OF CASES

- Step 1 If possible, fix an object as a starting point.
- Step 2 Apply any conditions that may be stated concerning the objects.
- Step 3 Apply movements, operations, or arrange objects, according to the problem. Use a systematic procedure to do this.
- Step 4 Look for a pattern.

Let us see how these steps apply to the two types of problem we have been dealing with, namely permutations and mappings.

Examination of Cases as applied to mappings.

- Step 1 Fix a position as a starting point.
- Step 2 There were no conditions in this case.
- Step 3 The mapping is applied in a systematic manner.
- Step 4 The first 2 applications (M_1 , and $M_1 \& M_1$) give you 3 different positions. Looking at the remainder of the table you can see a pattern; the same positions occur again in a definite order. From this you conclude that no new positions will arise.

Examination of Cases as applied to permutations.

- Step 1 Fix an object (letter or number) in the starting position.
- Step 2 Apply conditions. In some of the problems particular letters or numbers had to be in particular positions.
- Step 3 Other letters, or numbers, listed in a systematic way.
- Step 4. Had not been discussed and was not appropriate to permutations.

If necessary the procedure was repeated for different letters, or numbers, in different positions.

Today you will start the final section of this booklet. The topic that will be discussed is that of binary networks. As you work through this section remember that the only purpose of this booklet is to introduce you to problem solving techniques for making mathematical discoveries.

Definition

A binary switch is one which can have two values, 0 or 1.

Let A be a binary switch. The following will be used in this section.

— ? — indicates that we do not know if the value of the binary switch is 0 or 1.

— 1 — indicates that the binary switch has the value 1.

— 0 — indicates that the binary switch has the value 0.

A binary network is any collection of binary switches with a single input and output.

e.g.

where A, B, C and D are binary switches. In the above network we do not know the value of any of the binary switches. Therefore, each one is drawn with a ?.

When using examination of cases the following results have been obtained.

(i)

These networks are equivalent.

The networks are complete if A and B both have the value 1.

(ii)

These networks are equivalent.

The networks are complete if A or B, or both have the value 1.

Exercise

Can you think of any mathematical operations that remind you of connecting binary switches?

Hint:- Look at the underlined words.

Solution

If you have found operations that remind you of connecting binary switches turn to the next page. If not, the following comment may help.

In the binary networks on the previous page the switches have been combined (joined). Also the order in which the switches have been joined does not matter.

That is

Solution continued

You could have chosen many operations that remind you of connecting binary switches. For example, you could have thought of any of the following.

Addition;	Multiplication;	Union of Sets;	Intersection
$+$	$\times$	$\cup$	of Sets:
			$\cap$

We must decide which analogy to make. The analogy that will be used is with set theory.

Let us consider the definition of Union and Intersection of sets. If X and Y are sets then

$X \cup Y$ is the set of elements in X, or Y or both.

$X \cap Y$ is the set of elements in X and Y.

If we consider binary switches labelled A and B we have

Exercise

Which binary network would correspond to $A \cup B$?

Which corresponds to $A \cap B$?

Solution

For two switches, A and B, $A \cup B = B \cup A$

Read the following carefully

Before continuing to develop the analogy with set theory let us review the power of analogy. We were initially faced with a set of problems involving binary networks, a new topic in mathematics. By trying to compare networks with another mathematical system, a system with which we are more familiar, we are now looking at set theory. It is hoped that we will be able to use many of the properties associated with set theory to help solve problems with networks.

Hence, analogy has taken us from a totally new topic, binary networks, into a more familiar topic, set theory. As you work through the remainder of this section you will see how set theory will help us with network problems.

As mentioned above we hope to use the properties of set theory to help with network problems. In particular, we would like to use the properties associated with union and intersection to find equivalent networks. In order to do this we will have to examine the various properties associated with Union and Intersection.

APPENDIX E

Heuristics plus content - Algebraic Vehicle

HC - A

INTRODUCTION

There are two purposes to this booklet. To teach you how to make discoveries about new mathematical topics, and to introduce you to some new mathematical topics. The booklet deals with the following 3 topics:

- 1) Permutations,
- 2) Mappings,
- 3) Binary Networks.

The booklet will take 10 days to complete, 3 days are spent on each topic, and 1 day is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make comments that you think will be helpful to you.

PERMUTATIONS

We will consider different permutations of letters, numbers, etc. That is, different ways of arranging objects.

For example, the following would be considered 2 different permutations of the numbers 1, 2, and 3.

1 2 3 and 3 1 2

Before discussing the topic in detail, let us introduce the notation that will be used.

Notation

In the following the word object could stand for a number, a letter, etc. It would depend on the particular exercise.

will be used as a placeholder. When drawn like this it will show that the position indicated is empty.

indicates that there is an object in that place, but we do not know what the object is. We will use one of the letters x, y, or z inside the placeholder.

indicates that there is a 1 in the place.

indicates that the letter a is in that place.

An example is given on the next page.

Exercise

Consider the set consisting of b, j, and h.

Using all three letters, how many different possible permutations can you find.

e.g. One permutation would be

b j h

List all the permutations,

Solution

There are 5 other permutations, giving a total of 6.

Permutation 1.	b	j	h	(given in exercise)
Permutation 2.	b	h	j	
Permutation 3.	j	h	b	
Permutation 4.	j	b	h	
Permutation 5.	h	j	b	
Permutation 6.	h	b	j	

In your solution to this problem you probably had all the permutations listed, although they may have been in a different order. For most people, listing the various permutations is a matter of trial and error. That is, you try one permutation, and then another, hoping that in the end you will have listed all possible combinations. It is now that the first problem solving technique can be of help. This problem solving technique is called examination of cases. The first stage of the technique involves the developing of a systematic method for writing down the different cases.

This particular exercise is fairly simple since it involves only 6 different permutations. However, it will provide a good example with which to introduce the technique. The next two pages discuss how a systematic approach can be used to list the 6 permutations given above.

Today we are going to discuss the idea of circular permutations. When dealing with the idea of equivalent circular permutations you will see how a systematic approach eliminates the danger of repeating a particular circular permutation without realizing it.

Exercise

Consider the set consisting of the letters a,b, and c.

The 3 places that are to be filled are arranged in a circular manner.

Circular permutation is recorded as

($\triangle P1$, $\triangle P2$, $\triangle P3$)

All three places are occupied. How many different circular permutations can you find?

e.g. a in P1 , b in P2, and c in P3 will be recorded as

(a , b , c) - the placeholders have been left out.

List the circular permutations below:

Note: Each circular permutation uses each letter once and only once.

Mappings

Today you will start the second topic to be discussed in this booklet, namely mappings. In the first section you were introduced to the problem solving technique of examination of cases. In this section the second technique, namely analogy, will be used. Furthermore, you will see how these techniques can be combined to solve mapping problems.

Let us start by considering the following problem.

Problem

The set of objects that will be examined consists of

A , B , C

We start with the objects in the order

A B C Position 1

Mapping 1 (M_1) is defined as follows:

A	B	C	
↓	↓	↓	
B	C	A	Position 2

That is, A is mapped to B, B is mapped to C, and C is mapped to A.

A B C is referred to as position 1. When you apply M_1 to position 1 the letters are moved to position 2.

Solution

When we apply M_1 B maps to C. Then applying M_1 again C maps to A. Similarly for the other columns.

When M_1 is applied twice we denote it by $M_1 \& M_1$.

Exercise

To save you referring back the 3 positions and mapping are repeated below:

A B C Position 1

B C A Position 2

C A B Position 3

Now complete the following table.

Initial Position	Mapping	Final Position
2	$M_1 \& M_1$ (see below)	
3	M_1	
3	$M_1 \& M_1 \& M_1$	
1	$M_1 \& M_1 \& M_1$	

(a)

To find the position in (a) consider the following

Position 2

Position 1

B is mapped to C and then to A.
C is mapped to A and then to B, etc.
The final position is A B C,
which is position 1. Therefore
you would place a 1 in the table.

Solution

Initial Position	Mappings	Final Position
Base		2
"		3
"		Base
"		2
"		3
"		Base
"		2
"		3

From the results it seems reasonable to conclude that there are no other positions. If you look at the final positions there is a pattern in the way they appear in the table.

The technique of examination of cases has been of value in making sure that you have all possible positions for the letters. By approaching the problem in a systematic manner (listing the various combined mappings in a systematic way) and observing the pattern in the final positions, you could be sure that all possible positions have been listed.

The components that form the technique of examination of cases are:

Step 1: If possible, fix an object as the starting point.

Step 2: Apply any conditions that may be stated concerning the objects.

Step 3: Apply the movements, operations, or arrange objects, according to the problem. Use a systematic procedure to do this.

Step 4: Look for a pattern.

Today you will start the final section of the booklet. The topic that will be discussed is that of binary networks. As we develop the topic you will have an opportunity of using the problem solving techniques of analogy and examination of cases in solving problems.

Definition

A binary switch is one which can have two values, 0 or 1.

Let A be a binary switch. The following will be used in this section.

A binary network is any collection of binary switches with a single input and output.

e.g.

figure 1

where A,B,C and D are binary switches. In the above network we do not know the value of any of the binary switches. Therefore, each is drawn with a .

When using examination of cases the following results have been obtained.

(1)

These networks are equivalent.

The networks are complete if A and B both have the value 1.

(11)

These networks are equivalent.

The networks are complete if A or B, or both have the value 1.

Exercise

Can you think of any mathematical operations that remind you of connecting binary switches?

Hint:- Look at the underlined words.

Exercise

Find an equivalent network for

Hint:

The expression for this network is

$$(A \cup \text{op}(B)) \cap C \cap (B \cup A)$$

We can rewrite this expression using the commutative properties.
Then use one of the distributive properties and continue.

APPENDIX F

Content only - Algebraic Vehicle

C - A

INTRODUCTION

The booklet deals with the following three topics:

- 1) Permutations,
- 2) Mappings,
- 3) Binary Networks.

At the end of the booklet you should be able to solve problems which deal with any of the three topics listed above.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

Exercise

Consider the set consisting of b, j, and h.

Using all three letters, how many different possible permutations can you find.

e.g. One permutation would be

b j h

List all the permutations.

Solution

There are 5 other permutations, giving a total of 6.

Permutation 1.	b	j	h	(given in exercise)
Permutation 2.	h	j	b	
Permutation 3.	j	h	b	
Permutation 4.	h	b	j	
Permutation 5.	j	b	h	
Permutation 6.	b	h	j	

The order in which you listed the permutations is not important. Just check that you have listed all the different permutations.

Exercise

How many of the 6 permutations can you choose such that b and h are not next to each other?

List them below..

Today we will be dealing with the idea of circular permutations. In discussing these problems you will be introduced to the idea of equivalent circular permutations.

Exercise

Consider the set consisting of the letters a,b, and c.

The 3 places that are to be filled are arranged in a circular manner.

Circular permutation is recorded as

($\triangle_{P1}$, $\triangle_{P2}$, $\triangle_{P3}$)

All three places are occupied. How many different circular permutations can you find?

e.g. a in P1, b in P2 and c in P3 will be recorded as

(a , b , c) - the placeholders have been left out.

List the circular permutations below:

Note: Each circular permutation uses each letter once and only once.

Mappings

Today you will start the second topic to be discussed in this booklet, namely mappings.

Let us start by considering the following problem.

Problem

The set of objects to be examined consists of

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B	C	A	Position 2

That is, A is mapped to B, B is mapped to C, and C is mapped to A.

A B C is referred to as position 1. When you apply M_1 to this position the letters are moved to position 2.

Solution

When we apply M_1 B maps to C. Then applying M_1 again C maps to A. Similarly for the other columns.

When M_1 is applied twice we denote it by $M_1 \& M_1$.

Exercise

To save you referring back the 3 positions and mapping are repeated below:

A B C Position 1

B C A Position 2

C A B Position 3

Now complete the following table.

Initial Position	Mapping	Final Position
2	$M_1 \& M_1$ (see below)	
3	M_1	
3	$M_1 \& M_1 \& M_1$	
1	$M_1 \& M_1 \& M_1$	

(a)

To find the position in (a) consider the following

Position 2

Position 1

B is mapped to C and then to A.
C is mapped to A and then to B, etc.
The final position is A B C,
which is position 1. Therefore
you would place a 1 in the table.

Today you will start the final section of the booklet. The topic that will be discussed is that of binary networks. As we develop the topic you will look at the idea of a complete network, and the idea of equivalent networks.

Definition

A binary switch is one which can have two values, 0 or 1.

Let A be a binary switch. The following will be used in this section.

indicates that we do not know if the value of the binary switch is 0 or 1.

indicates that the binary switch has the value 1.

indicates that the binary switch has the value 0.

A binary network is any collection of binary switches with a single input and output.

figure 1

where A,B,C and D are binary switches. In the above network we do not know the value of any of the binary switches. Therefore, each one is drawn with a .

Consider the pair of binary switches, A and B.

(i) They can be connected as follows:

Network 1

We examined this network in a previous exercise and found that it was complete if A and B had the value 1. If 2 switches are connected in this manner we will denote the network by

$$A \cap B$$

where $\cap$ stands for intersection. The expression $A \cap B$ is referred to as A intersection B.

It should be noted that the following network

Network 2

has been shown to be equivalent to network 1. The notation that would be used for network 2 is $B \cap A$. The letters have been altered to indicate that the order of the switches has been changed.

(ii) They can be connected as follows:

We have also examined this network in a previous exercise and found that it is complete if either A, or B, or both, have the value 1. If 2 switches are connected in this manner we will denote the network by

$$A \cup B$$

where $\cup$ stands for union and the expression $A \cup B$ will be referred to as A union B.

Exercise

Here are some networks. Write the corresponding expressions for them.

and

and combining these two we have

Now try the following.

APPENDIX G

Heuristics only - Geometric Vehicle

H - G

INTRODUCTION

The only purpose of this booklet is to teach you how to make discoveries about new mathematical topics. The booklet deals with 3 topics, which are:

- 1) Arrangements,
- 2) Flips and turns,
- 3) Networks.

The topics have been chosen to introduce you to problem solving techniques for making mathematical discoveries, and serve no other purpose. When working through this booklet remember that you are trying to learn the problem solving techniques, not the content itself.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

Exercise

Consider the set consisting of a triangle, square and circle.

Using all three figures, how many different possible arrangements can you find?

e.g. One arrangement would be

List all possible arrangements.

Solution

There are 5 other arrangements, giving a total of 6.

Arrangement 1. (given in exerci

Arrangement 2.

Arrangement 3.

Arrangement 4.

Arrangement 5.

Arrangement 6.

In your solution to this problem you probably had all the arrangements listed, although they may have been in a different order. For most people, listing the various arrangements is a matter of trial and error. That is, you try one arrangement, and then another, hoping that in the end you will have listed all possible combinations. It is now that the first problem solving technique can be of help. This problem solving technique is called examination of cases. The first stage of the technique involves the developing of a systematic method for writing down the different cases.

This particular exercise is fairly simple since it involves only 6 different arrangements. However, it will provide a good example with which to introduce the technique. The next two pages discuss how a systematic approach can be used to list the 6 arrangements given above.

Finally the circle was placed in position 1, giving the last 2 arrangements.

We have eliminated all the cases with any of the 3 figures in position 1. But there were only 3 figures in the set being considered, so we must have written down all possible cases.

Using a systematic approach to listing cases (in this example the cases were arrangements) has two major advantages:-

(1) You can be sure that all possible cases have been listed. You do not have to worry that some cases might have been left out.

For example, in the previous exercise all the cases with the triangle in position 1 were written down before considering any other figure in that position.

(2) You can be sure that no cases have been repeated. This may not be very important when you only have 6 cases to consider, but if you had 60 different possibilities you do not want to check that they are all different, and that you have not repeated some by mistake.

In the previous exercise the systematic approach assured us that the 6 cases (arrangements) listed were all different and that there were no other possible arrangements. A systematic approach will help when you want to examine all cases.

As you go through this booklet you will have more practice listing cases by a systematic method.

Flips and turns

Today you will start the second topic to be discussed in this booklet, namely flips and turns. In the first section you were introduced to the problem solving technique of examination of cases. In this section the second technique, namely analogy, will be used. Furthermore, you will see how examination of cases can be used to test conjectures (educated guesses) developed by analogy. As with arrangements, the topic of flips and turns is not important. It is just used to introduce analogy.

Problem

This problem will be concerned with an equilateral triangle. That is, a triangle with all 3 sides the same length.

How many different ways can this triangle be fitted into a space of the same size and shape? The triangle can only be rotated, and cannot be turned over.

These are two different ways of fitting the triangle into a space of the same shape and size.

Exercise

In what other position can the triangle be fitted into the space?

Solution

The triangle is in position 2.

Exercise

To save you referring back the 3 positions for the triangle are given below.

Position 1

Position 2

Position 3

Now complete the following table.

	Initial Position	Rotation	Final Position
(a)	1	240° clockwise (see below)	
	1	120° clockwise	
	2	120° counterclockwise	
	3	120° counterclockwise	
	2	360° clockwise	

To find the final position in (a) imagine that the triangle is in position 1. Then think how it moves when rotated through 240° in a clockwise direction. You obtain position 3.

Initial Position

240° clockwise

Final Position

You would place a 3 in the table.

Solution

Initial Position	Rotation	Final Position
1	240° clockwise	3
1	120° clockwise	2
2	120° counterclockwise	1
3	120° counterclockwise	2
2	360° clockwise	2

From the results above it would seem reasonable to conclude that there are no other positions for the triangle. However, can you be sure?

Clearly with this particular situation involving a triangle (that can only be rotated) you can be reasonably sure. By rotating the triangle backward and forward you would soon conclude that there were only 3 positions. However, if you just use a trial and error approach it would not help in a more complex situation.

We will now discuss a general approach that can be used to attack this problem. In discussing how to approach this problem from a systematic point of view we will lay the base for introducing the second problem solving technique, analogy.

Imagine you start by putting the triangle in the appropriate space using one particular position. This will be referred to as the base position.

The movement that can be used to move from one position to another is a rotation.

Read carefully

EXAMINATION OF CASES

- Step 1 If possible, fix an object as a starting point.
- Step 2 Apply any conditions that may be stated concerning the objects.
- Step 3 Apply movements, operations, or arrange objects, according to the problem. Use a systematic procedure to do this.
- Step 4 Look for a pattern.

Let us see how these steps apply to the two types of problem we have been dealing with, namely arrangements, and flips and turns.

Examination of Cases as applied to flips and turns.

- Step 1 Fix the triangle in a starting position.
- Step 2 There were no conditions other than the triangle must fit into a space the same size and shape.
- Step 3 Triangle is rotated in a systematic way. First looking at clockwise, then counterclockwise rotations.
- Step 4 The first two rotations give you 3 different positions. The base position and those labelled positions 2 and 3. Looking at the remainder of the table you can see a pattern; the same positions occur again in a definite order. From this you can conclude that no new positions will arise.

Examination of Cases as applied to arrangements

- Step 1 Fix a figure in a particular position.
- Step 2 Apply conditions. In some problems different figures had to be placed in particular positions.
- Step 3 Other figures were placed in a systematic way.
- Step 4 Had not been discussed and was not appropriate to arrangements.

If necessary the procedure was repeated for different figures in different positions.

Exercise

Under what conditions will this network be complete?
 You should look at the different combinations of conditions
 and draw the networks.

Try to list the cases in a systematic manner.

When using examination of cases to consider different²⁴⁸ networks, the following results were obtained.

(i)

These networks are equivalent.

They are complete if A and B are connected.

(ii)

These networks are equivalent.

They are complete if A, or B, or both are connected.

Exercise

Can you think of any mathematical operations that remind you of connecting switches?

Hint:- Look at the underlined words.

APPENDIX H

Heuristics plus content - Geometric Vehicle

HC - G

INTRODUCTION

There are two purposes to this booklet. To teach you how to make discoveries about new mathematical topics, and to introduce you to some new mathematical topics. The booklet deals with the following 3 topics:

- 1) Arrangements,
- 2) Flips and turns,
- 3) Networks.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make comments that you think will be helpful to you.

ARRANGEMENTS

We will consider different arrangements of geometric figures.

For example, the following would be considered 2 different arrangements of a triangle and square.

Before discussing the topic in detail, let us introduce some notation that could be used.

Notation

_____ will indicate that a particular position is empty.

_____ x _____ indicates that there is a geometric figure in that position, but we do not know what it is. We will use one of the letters x, y, or z for this situation.

_____ ○ _____ indicates that there is a circle in the position

An example is given on the next page.

Exercise

Consider the set consisting of a triangle, square and circle.

Using all three figures, how many different possible arrangements can you find?

e.g. One arrangement would be

List all possible arrangements.

Finally the circle was placed in position 1, giving the²⁵³ last 2 arrangements.

We have eliminated all the cases with any of the 3 figures in position 1. But there were only 3 figures in the set being considered, so we must have written down all possible cases.

Using a systematic approach to listing cases (in this example the cases were arrangements) has two major advantages:-

(1) You can be sure that all possible cases have been listed. You do not have to worry that some cases might have been left out.

For example, in the previous exercise all the cases with the triangle in position 1 were written down before considering any other figure in that position.

(2) You can be sure that no cases have been repeated. This may not be very important when you only have 6 cases to consider, but if you had 60 different possibilities you do not want to check that they are all different, and that you have not repeated some by mistake.

In the previous exercise the systematic approach assured us that the 6 cases (arrangements) listed were all different and that there were no other possible arrangements. A systematic approach will help when you want to examine all cases.

As you go through this booklet you will have more practice listing cases by a systematic method.

Flips and turns

Today you will start the second topic to be discussed in this booklet, namely flips and turns. In the first section you were introduced to the problem solving technique of examination of cases. In this section the second technique, namely analogy, will be used. Furthermore, you will see how these techniques can be combined to solve problems.

Problem

This problem will be concerned with an equilateral triangle. That is, a triangle with all 3 sides the same length.

How many different ways can this triangle be fitted into a space the same size and shape? The triangle can only be rotated, and cannot be turned over.

These are two different ways of fitting the triangle into a space of the same size and shape.

Exercise

In what other position can the triangle be fitted into the space?

Solution

The other position for the triangle is

You will notice that since the triangle can only be rotated a position such as

is impossible.

By rotating the triangle you find the 3 positions that have been given. However, can you be sure that you have all possible positions that can be obtained by rotating.

Soon we will consider the way you could be sure that you have all possible positions. Before doing this the different positions will be labelled.

Position 1

Position 2

Position 3

Exercise

In each of the following fill in the missing vertices or positions.

(i)

(ii)

(iii)

Position _____

(iv)

Position _____

From the results it seems reasonable to conclude that there are no other positions for the triangle. If you look at the final positions for the triangle there is a pattern in the way they appear in the table.

The technique of examination of cases has been of value in helping you be certain that all possible cases have been listed. By approaching the problem in a systematic manner (listing the various rotations in a systematic way) and observing that a pattern appeared in the final positions, you could be sure that all possible positions had been listed.

The components that form the technique of examination of cases are:

Step 1: If possible, fix an object as the starting point.

Step 2: Apply any conditions that may be stated concerning the objects.

Step 3: Apply the movements, operations, or arrange objects, according to the problem. Use a systematic procedure to do this.

Step 4: Look for a pattern.

You will probably have noted that this technique has been used in both the permutation problems and those involving the triangle. All the steps may not be appropriate to all problems. In dealing with permutations (and circular permutations) we applied step 2, applying particular conditions on figures, but did not use the last step, looking for a pattern. In the triangle problem there were no conditions (other than not being allowed to turn it over) to apply, but we did look at a pattern.

When using examination of cases to consider different networks, the following results were obtained.

These networks are equivalent.

They are complete if A and B are connected.

These networks are equivalent.

They are complete if A, or B, or both are connected.

Exercise

Can you think of any mathematical operations that remind you of connecting switches?

Hint:- Look at the underlined words.

You could have chosen many operations that remind you of connecting switches. For example, you could have thought of any of the following.

Addition;	Multiplication;	Union of Sets;	Intersection of Sets
$+$	$\times$	$\cup$	$\cap$

We must decide which analogy to make. The analogy that will be used is with set theory.

Let us consider the definition of Union and Intersection of Sets. If X and Y are sets then

$X \cup Y$ is the set of elements in X or Y or both.

$X \cap Y$ is the set of elements in X and Y.

If we consider switches labelled A and B we have

Complete if A or B, or both are connected.

Complete if A and B are connected.

Exercise

Which network would you draw to correspond to $A \cup B$?

Which one to correspond to $A \cap B$?

The previous example has shown how the notation and properties of set theory can be used to find equivalent networks. On the following pages you will find further examples for you to try. The exercises are NOT easy. They may take a long time and involve many steps.

On one of the two pages following the exercise there is a hint that should help you solve the problem. Only look at the hint AFTER you have tried the problem. You should keep the pages with the different properties handy so that you can refer to them (pages 114-6).

There is no quiz at the end of today's work. Do not worry if you do not complete all the exercises. Just do as many as you can in the time available. Remember that some of them involve many steps to simplify the set theoretic expression.

Exercise

Find an equivalent network for the following.

Note:- The correct set theoretic expression for the network is given on the next page. Check the expression you obtain before trying to simplify it.

APPENDIX I

Content only - Geometric Vehicle

C - G

INTRODUCTION

The booklet deals with the following three topics:

- 1) Arrangements,
- 2) Flips and turns,
- 3) Networks.

At the end of the booklet you should be able to solve problems which deal with any of the three topics listed above.

The booklet will take 10 days to complete. 3 days are spent on each topic, and 1 day at the end is used for review. At the end of most days there is a short quiz. This will be marked and returned the next day.

You may keep the booklet when it is complete. Feel free to write in it. Make any comments that you think will be helpful to you.

ARRANGEMENTS

We will consider different arrangements of geometric figures.

For example, the following would be considered 2 different arrangements of a triangle and square.

Before discussing the topic in detail, let us introduce some notation that could be used.

Notation

will indicate that a particular position is empty.

indicates that there is a geometric figure in that position, but we do not know what it is. We will use one of the letters x, y, or z for this situation.

indicates that there is a circle in the position

An example is given on the next page.

Exercise

263

Consider the set consisting of a triangle, square and circle.

Using all three figures, how many different possible arrangements can you find?

e.g. One arrangement would be

List all possible arrangements.

Solution

There are 5 other arrangements, giving a total of 6.

Arrangement 1. (given in exercise

Arrangement 2.

Arrangement 3.

Arrangement 4.

Arrangement 5.

Arrangement 6.

The order in which you listed the arrangements is not important. Just check that you have listed all the different arrangements.

Exercise

How many of the 6 arrangements can you choose such that the triangle and circle are not next to each other.

List them below.

Flips and turns

Today you will start the second topic to be discussed in this booklet, namely tiling problems.

Let us start by considering the following problem.

Problem

This problem will be concerned with an equilateral triangle. That is, a triangle with all 3 sides the same length.

How many different ways can this triangle be fitted into a space the same size and shape? The triangle can only be rotated, and cannot be turned over.

These are two different ways of fitting the triangle into a space of the same size and shape.

Exercise

In what other position can the triangle be fitted into the space?

The triangle is in position 2.

Exercise

To save you referring back the 3 positions for the triangle are given below.

Position 1

Position 2

Position 3

Now complete the following table.

	Initial Position	Rotation	Final Position
(a)	1	240° clockwise (see below)	
	1	120° clockwise	
	2	120° counterclockwise	
	3	120° counterclockwise	
	2	360° clockwise	

To find the final position in (a) imagine that the triangle is in position 1. Then think how it moves when rotated through 240° in a clockwise direction. You obtain position 3.

Initial Position

240° clockwise

Final Position

You would place a 3 in the table.

Solution

Initial Position	Rotation	Final Position
1	240° clockwise	3
1	120° clockwise	2
2	120° counterclockwise	1
3	120° counterclockwise	2
2	360° clockwise	2

From these results it seems that no other positions for the triangle exist. However, on page 36 we asked how you can be reasonably certain that you have all possible positions for the triangle. The following approach will work.

Let the first position you try use be position 1.

Position 1

The movement that can be used to move the triangle from one position to another is a rotation.

Today you will start the final section of the booklet. The topic that will be discussed is that of networks. As we develop the topic you will look at the idea of a complete network, and the idea of equivalent networks.

Definition

A switch can be either connected or disconnected.

Let A be a switch. The following diagrams will be used in this section.

indicates that we do not know if the switch is connected or disconnected.

indicates that the switch is disconnected. That is, there is a break in the path.

indicates that the switch is connected. That is, there is no break in the path.

A network is any collection of switches with a single input and output path.

e.g.

where A, B, C and D are switches. In the above network we do not know if the switches are connected or disconnected. Therefore, each is drawn with a ?.

Exercise

In these examples there are pairs of networks. In each case you are to find out if the networks are equivalent.

(i)

Network 1

Network 2

(ii)

Network 1

Network 2

Consider the pair of switches, A and B.

(i) They can be connected as follows:

We examined this network in a previous exercise and found that it was complete if A and B are both connected. If 2 switches are connected in this manner we will denote the network by

$$A \cap B$$

where $\cap$ stands for intersection. The expression $A \cap B$ is referred to as A intersection B.

It should be noted that the following network

has been shown to be equivalent to network 1. The notation that would be used for network 2 is $B \cap A$. The letters have been altered to indicate that the order of the switches has been changed.

(ii) They can be connected as follows:

We have also examined this network in a previous exercise and found that it is complete if either A, or B, or both are connected. If 2 switches are connected in this manner we will denote the network by

$$A \cup B$$

where $\cup$ stands for union and the expression $A \cup B$ will be referred to as A union B.

APPENDIX J

Instructions to Subjects

Instructions to students

Over the next few days you will be involved in an experiment. The following is information on the experiment.

During the next 10 days you will spend one mathematics period each day working through booklets. At the beginning of each day you will be given the booklet to work with. You will have a certain number of pages to be covered each day. The booklet contains exercises for you to try, and in most cases the pages following each exercise contain it's solution. Please try an exercise before looking at the solution. At the end of most days there is a short quiz. The complete booklet will be collected at the end of the period, the quiz marked overnight, and returned the next day.

Once the booklets have been collected there will be no further work to be done until the next day. There are many different booklets being used. You will probably have a different one than your friends. It is very important that you DO NOT DISCUSS THE BOOKLET WITH ANYONE. PLEASE DO NOT DISCUSS ANY OF THE WORK.

At the end of the ten days there will be some tests for you to try.

APPENDIX K

Algebraic Test

Read to Students

You are to write down ANYTHING you can find out
NO MATTER HOW LITTLE OR UNIMPORTANT YOU MAY THINK IT IS

If you have any questions about mod. 7 ask.

Note to the Teachers

I will try to get round to all classes.

If a student says to you do you want this (that
is should I do...) say yes.

Name _____

Class 10 - _____

(PLEASE PRINT)

Before attempting the problem on the next page this is to remind you of modulo arithmetic, and what it means.

Modulo 7 (Mod. 7) - (Clock 7)

Given any whole number, to convert it to mod. 7 you will divide it by 7 and consider the remainder.

For example:

$$15 = (2 \times 7) + 1$$

The remainder after dividing 15 by 7 is 1, and we say that

$$15 \longrightarrow 1 \text{ (Mod. 7)}$$

$$13 = (1 \times 7) + 6$$

The remainder after dividing 13 by 7 is 6, and we say that

$$13 \longrightarrow 6 \text{ (Mod. 7)}$$

Numbers less than 6 remain unchanged.

For example:

$$6 = (0 \times 7) + 6$$

The remainder after dividing 6 by 7 is 6, and we say that

$$6 \longrightarrow 6 \text{ (Mod. 7)}$$

You will notice that certain numbers, for example 13 and 6, both go to 6 (mod. 7)

We say that

$$13 \equiv 6 \text{ (Mod. 7)}$$

If you have any problems with mod. 7 please ask before you go onto the problem given on the next page.

Problem

I am going to define 2 new mathematical operations.

Up-One Multiplication

The symbol used for this is @

$$a @ b = a \times (b + 1) \quad (\text{Mod. } 7)$$

For example:

$$2 @ 3 = 2 \times (3 + 1) \quad (\text{Mod. } 7)$$

$$= 8 \quad (\text{Mod. } 7)$$

$$\equiv 1 \quad (\text{Mod. } 7)$$

$$1 @ 1 = 1 \times (1 + 1) \quad (\text{Mod. } 7)$$

$$= 2 \quad (\text{Mod. } 7)$$

Double Addition

The symbol used for this is *.

$$a * b = 2 \times (a + b) \quad (\text{Mod. } 7)$$

For example:

$$2 * 3 = 2 \times (2 + 3) \quad (\text{Mod. } 7)$$

$$= 10 \quad (\text{Mod. } 7)$$

$$\equiv 3 \quad (\text{Mod. } 7)$$

$$0 * 1 = 2 \times (0 + 1) \quad (\text{Mod. } 7)$$

$$= 2 \quad (\text{Mod. } 7)$$

Name (PLEASE PRINT) _____ Class 10 - _____

QUESTION

What can you find out about these operations?

Include ALL your working.

APPENDIX L

Geometric Test

Read to Students

The question should read:

What can you find out about shortest routes and combining shortest routes?

You are to write down ANYTHING you can find out NO MATTER HOW LITTLE OR UNIMPORTANT YOU MAY THINK IT IS

Note to the Teacher

I will try to get round to all classes.

If a student says to you do you want this (that is should I do...) say yes.

Name _____

Class 10 - _____

(PLEASE PRINT)

Attached to this sheet you will find a grid. The following refers to a game that can be played on the grid.

Rules

- 1) To move from a point on the grid to any point that is higher, or at the same level, you may move across the grid in either direction, and/or up the grid; but you may NOT move down the grid.

The following are examples of routes that can be used to go from A to B.

- 2) To move from a point on the grid to any point that is lower, or at the same level, you may move across the grid in either direction, and/or down the grid; but you may NOT move up the grid.

The following are examples of routes that can be used to go from C to B.

- 3) You are looking for the shortest routes between points. The length of a route is the number of segments it takes to go from one point to the other. For example

A total of 3 segments

A total of 7 segments

This path is the shorter of the two.

Shortest routes can be combined. For example, the shortest route from A to B can be combined with the shortest route from B to C. This produces a route from A to C.

please turn over.....

Name (PLEASE PRINT) _____ Class 10 - _____

QUESTION

What can you find out about combining shortest routes?

Include ALL your working.

APPENDIX M
Grading Scheme

GRADING SCHEME

The objective of the grading scheme is to categorize a test according to the evidence that a student used the heuristics (Analogy and Examination of Cases). It is important to note that you are looking for evidence that the student has tried to apply the heuristics in a meaningful way, and not that he has necessarily used them well.

The following are the categories that will be used.

- 0 - No evidence of the use of heuristics.
- 1 - The student has used Examination of Cases ONLY. He has NOT attempted to look for a pattern.
- 2 - The student has examined cases AND looked for a pattern.
- 3 - The student has used Analogy ONLY. There is NO evidence that he has tried to test out his guesses.
- 4 - The student has used Analogy and Examination of Cases, but as DISTINCT techniques. There is NO evidence that the student has combined the two.
- 5 - The student has used both techniques (Analogy and Examination of Cases). Furthermore, he has combined the two in solving problems.

NOTE: When assigning a student's response to a category there must be WRITTEN EVIDENCE of the appropriate response. For example, unless a student has written down some cases, you cannot assume that he has examined cases.

The following are examples of the various categories for the ALGEBRAIC TEST.

0 - No evidence of heuristics

a) Blank Papers.

b) Explanations of how the operations work.

e.g. A verbal description of @.

Add one to the second number and then multiply by the first. Divide your result by 7 and this gives you the answer.

c) A description of Mod. 7.

e.g. A student may say that mod. 7. is like something he did at the beginning of high school.

1 - Examination of Cases ONLY

The student has tried different numbers to see what happens.

e.g. A student would have written down

$$\begin{aligned} 1 @ 3 &= 1 \times (3 + 1) \\ &= 4 \end{aligned}$$

$$\begin{aligned} 2 @ 4 &= 2 \times (4 + 1) \\ &= 10 \\ &= 3 \end{aligned}$$

$$\begin{aligned} 3 @ 1 &= 3 \times (1 + 1) \\ &= 6 \end{aligned}$$

Similarly, a paper would be in this category if he examined cases for the other operation, or both.

NOTE:- There are two possibilities in this examination of cases. The student could have examined them systematically or 'randomly'. If he has examined them systematically mark his response 1A, if randomly 1B.

2 - Examination of Cases and looking for a pattern

Here a student would have tried different numbers and attempted to look for a pattern in the results.

e.g. a)	$1 @ 1 = 1 \times (1 + 1) = 2$	Therefore the answer is
	$2 @ 1 = 2 \times (1 + 1) = 4$	
	$3 @ 1 = 3 \times (1 + 1) = 6$	always even.

NOTE: This conclusion is, in fact, incorrect. However, it does show evidence of a meaningful attempt to apply examination of cases and look for a pattern.

b) Any attempt to examine cases and then draw conclusions a pattern in the results. This may be one of the field properties such as associativity.

c)	$2 @ 3 = 1$	Therefore the order makes a difference.
	$3 @ 2 = 2$	

IMPORTANT: For a paper to fall in this category a student must have examined the cases and then concluded that a pattern exists. The starting point **MUST BE THE EXAMINATION OF CASES**. For example, in c) one would have to assume from the context that the starting point was the examination of cases. If you feel that the starting point is the question it would be categorized under 5.

3 - Analogy

Here a student would have tried to make an analogy with some other mathematical operation or properties.

For a paper to fall into this category the student must make the analogy, but NOT have used examination of cases to test or make the analogy.

e.g. In (a) and (b) the student must not use cases to test his statements.

a) The operation is commutative, like addition.

b) The operation involves the distributive property.

c) Use of analogies with other properties.

d) Use of analogy with the booklets. There has to be a meaningful attempt at analogy. Clearly a statement such as 'this has something to do with permutations' is not an attempt to apply analogy. However, such a statement, together with an explanation as to how it has something to do with permutations could be acceptable. A student would try to raise a question by analogy.

4 - Analogy and Examination of Cases (DISTINCT)

Here a student would have used both heuristics, but have made no attempt to combine them.

e.g.

He has examined cases (see 1)

He has made statements concerning properties (see 3)

NOTE: The two heuristics have been used, but they have been applied independently.

He could have tried different cases and then made a statement (in a different part of the answer) concerning the identity. This would have nothing to do with the cases he examined.

i.e.	$2 @ 1 = 4$	This combination shows that the student has used examination of cases and analogy. However, the cases examined have nothing to do with the statement concerning commutativity.
	$3 @ 1 = 6$	
	$5 @ 5 = 4$	

@ is commutative

5 - Analogy and Examination of Cases (COMBINED)

Here the student would have asked a question and used examination of cases to test it out.

e.g.

a) Is @ commutative? No

$$2 @ 3 \neq 3 @ 2$$

b) Is there an identity for @?

$1 @ 0 = 1$, $2 @ 0 = 2$, etc. Therefore, there is an identity.

NOTE: Although the conclusion concerning the identity is incorrect, it still shows the asking of a question, followed by an examination of cases. The student's problem lies in his conception of the concept of an identity, not in his use of the heuristic.

c) Any other paper than follows the pattern.

Asks question $\longrightarrow$ Tries cases to answer it

This is usually in the form of a question concerning the field properties.

In any paper containing evidence of heuristics one would expect many statements that are not appropriate. You may find evidence of 3 - Analogy ONLY, together with irrelevant statements. It is a matter of shifting through the paper to find out what is appropriate.

The following are examples of the various categories for the GEOMETRIC TEST.

0 - No evidence of heuristics

- a) Blank papers.
- b) Explanations of how to combine routes.
e.g. To combine routes you take the length of the path from A to B say, and then add the length from B to C say.
- c) A description of the rules. Here you are looking at examples where the student has described the rules in his own words.

1 - Examination of Cases ONLY

The student would have tried different paths to see what happens.

e.g.

- a) He has calculated the lengths of different routes.
- b) He has combined different routes.
- c) He has examined different paths between the same two points. (There are many paths between A and B)

NOTE: As with the algebraic test, responses should be categorized as either 1A or 1B.

2 - Examination of Cases and looking for a pattern

Here the student would have examined different paths and looked for a pattern.

- a) He has examined cases and concluded that there is more than one shortest route between two points.
- b) He has examined cases and concluded that combining two shortest routes does not always give a shortest route.
- c) He has examined cases and concluded that the shortest route from X to Y, say, is the same as that from Y to X.

3 - Analogy ONLY

Here a student would have tried to make an analogy with some other mathematical operation or properties.

For a paper to fall into this category the student must make the analogy, but NOT have used examination of cases to test out or make the analogy.

a) He makes statements linking combining routes with networks. Statements concerning the fact that combining routes to form new routes is like combining networks (or switches) to form new networks.

b) Analogies between combining paths and operations such as addition, union, etc.

4 - Analogy and Examination of Cases (DISTINCT)

He could have examined different routes (see 1)

He could have made analogies (see 3)

These would appear in different parts of the students paper with no attempt to use the heuristics together.

5 - Analogy and Examination of Cases (COMBINED)

a) Does combining shortest routes give a shortest route?

NO - followed by a counterexample to prove this statement.

YES - followed by supporting evidence. Note that yes is incorrect. However, the problem is not with the students use of the heuristics, but with his choice of examples.

b) Is the shortest route unique?

NO - followed by counterexample.

c) Is finding shortest route symmetric?

YES - followed by examples.

e.g. Length AB = Length BA

d) Other papers following the pattern

Ask question → Tries cases to test it.

NOTE: When the paper was handed out the students were told that the question should read:

What can you find out about shortest routes, and combining shortest routes?

Include ALL your working.