Re-rooting The Learning Space
Minding Where Children's Mathematics Grow

by

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Abstract

This disquisition presents a qualitative study that investigated the complicit nature of theory and practice in mathematics teaching. Situated within an ecological perspective, this research interrogates the role that theory plays as a cognizing domain in which one's pedagogy of teaching mathematics co-exists and co-evolves. A systemic exploration of mathematics and the teaching and learning of it is conducted and assessed against tenets of complexity, sustainability, languaging, co-emergence, integration, and recursion. This study reveals the impact that theoretical discourses have on the kind of place and the forms of mathematics that are enabled and disabled through the metaphors, perceptions of mathematical understanding, and conceptions of time that are embodied and enacted by the teacher and her students.

This research involved the explication of the teacher's assumed theoretical and practical patterns of teaching mathematics. The expressive forms in which this disquisition is written provide interpretive snapshots that document the teacher's conceptual journey from that of a heavily mechanistic, linear, and hierarchical mindset towards the development of an ecologically coherent theoretical domain for teaching. The classroom vignettes of the teacher, another teacher with whom she collaborated, and the second and third grade students span a course of two and half school years. These vignettes focus on the teacher's work in occasioning ecological forms of teaching, learning, and mathematics in the classroom. The analysis of these episodes revealed stark differences from that of her previous teaching practice not only in the nature of the students' understandings, their ways of acting and being mathematical but also, in the kinds of mathematics that arose during the lessons.

Table of Contents

List of Figures

Acknowledgments

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A Fore-word For The Reader

To understand a living system such as a tree, in an ecologically systemic way means that it is not possible to examine the tree by simply reducing it down to its individual parts or analyzing it from part to whole. Rather, it means that one does indeed need to study the tree's leaves, branches, trunk, root system, and its interaction with the environment but from many different vantage points to make sense of how each part exists in dynamic relationship with the others as an integrated system.

The same can be said about the purpose of this disquisition. It is not meant to be a recipe for how to teach mathematics well or to serve as simply a descriptive account of a teaching practice. It is in essence, a systemic exploration into both the embeddedness and the emergence of theory and practice in mathematics teaching.

Given the nature of this research and the theoretical realm in which it is situated, it was important for the work to be expressed in a form that also possessed an ecological sensibility. Upon first glance, it appears to be a collection of separate compositions. And although each piece is an entity unto itself, the intent was not to render the research as a piecing together of theory, data, and analysis but instead, to bring a multiversal perspective to it and expose the co-existence and co-evolution of theory-and-practice. Thus, the embodiment of this disquisition's thesis is also evident in the organization of the text as a whole and the diversity of writing structures within it.

The organic way in which this disquisition is organized can be likened to a tree: that any one leaf is neither directly connected to the other leaves nor does one need to view them in any particular order yet at the same time, all are interconnected as essential parts of the tree by way of its branches and therefore, are necessary for making the tree a coherent whole. Here, the compositions that were sourced by video and audio taped classroom sessions, journal entries, students' work, and running field notes are not necessarily directly linked to one another or sequential in order. Each piece is considered to be a smaller yet integral system of thinking that in turn forms larger conceptual clusters within an ecological mind-space. And together, these interrelated knowledge systems seek to inform the mathematical learning space.

The different analytic viewpoints are communicated through the following expressive structures:

Metaphors and **visuals** have been used to reflect how it was that I was conceptualizing the theory and my teaching.

Featured quotes or questions positioned on a blank page are the theoretical artifacts that served as provocations for my further research. In this disquisition, they are intended to interrupt the reader's flow and signal a shift or opening of another conceptual space.

Juxtaposition of text with other text visually expresses the recursive and emergent layers of thinking that arose during the research. Sometimes, the exercise of juxtaposition was used to set ideas with or against each other in order to explore what kinds of theoretical tension or generative spaces would arise for my further consideration. While other times, the juxtaposing of ideas enabled the development of relationships between one author's thinking with that of another's.

The use of **black and white** or **colour** for certain texts and images emphasize theoretical underpinnings that I considered to be clearly defined as opposed to those I perceived to be ever-changing and indeterminate. For example, I made use of black and white in the visual-text collage on constructivist notions to convey what I conceived as theory that was "clear-cut" whereas colour was used in the enactive visual-text collage to express theory that I understood as imbued with ecological qualities that were unpredictable, ever-changing, and not so clear cut.

The actual **figuring** of text such as a newspaper article, conversation, free form poem, as well as whether it was organized in a left to right, top to bottom, back and forth, sporadic, or circular manner sought to capture the conceptual and metaphorical essence of the ideas being discussed.

Fading and **bolding** of text functions as the bringing forth of ideas in the development of theory while still preserving the contextual background from which these ideas emerged. This form of writing enables one to see the "double imaging" that was present in my thinking.

It was critical that each piece of writing in some way highlighted the inevitably personal particularities of this research. To do this, different "**characters**" were developed. The characters in this disquisition are my students, a teaching colleague, and myself. In order to analyze and interpret my teaching from multiple perspectives, my character takes on several different "**personalities**". In some of the compositions, I am the main character and describe events as I perceive(d) and experience(d) them either 'in the moment' as they unfolded or by taking on a reflective stance. In other instances, I am another character altogether or am not

present in the piece at all. This allowed me to interpret the research from a connected yet more distanced or 'outside' perspective. In still others, the reader will find me in conversation with another character. These vignettes reveal the ongoing questioning and assessing of the theoretical coherence concerning my research.

In addition to the styles of writing, characters, and personalities, the actual **fonts** of the text help to visually distinguish between the different 'tones' or perspectives taken on in the analysis and interpretation of the work. Finally, this disquisition need not be a front to back, left to right, top to bottom read. In recognizing the various parts of this work here and in the table of contents, it is hoped that the reader will engage with the same spirit as one would exploring a living tree-- perhaps examining its integrated and integral being from its base and climbing up, hanging from a branch and gazing around, leaping from one limb to another, or even peering down at the always emerging whole from a distant hill.
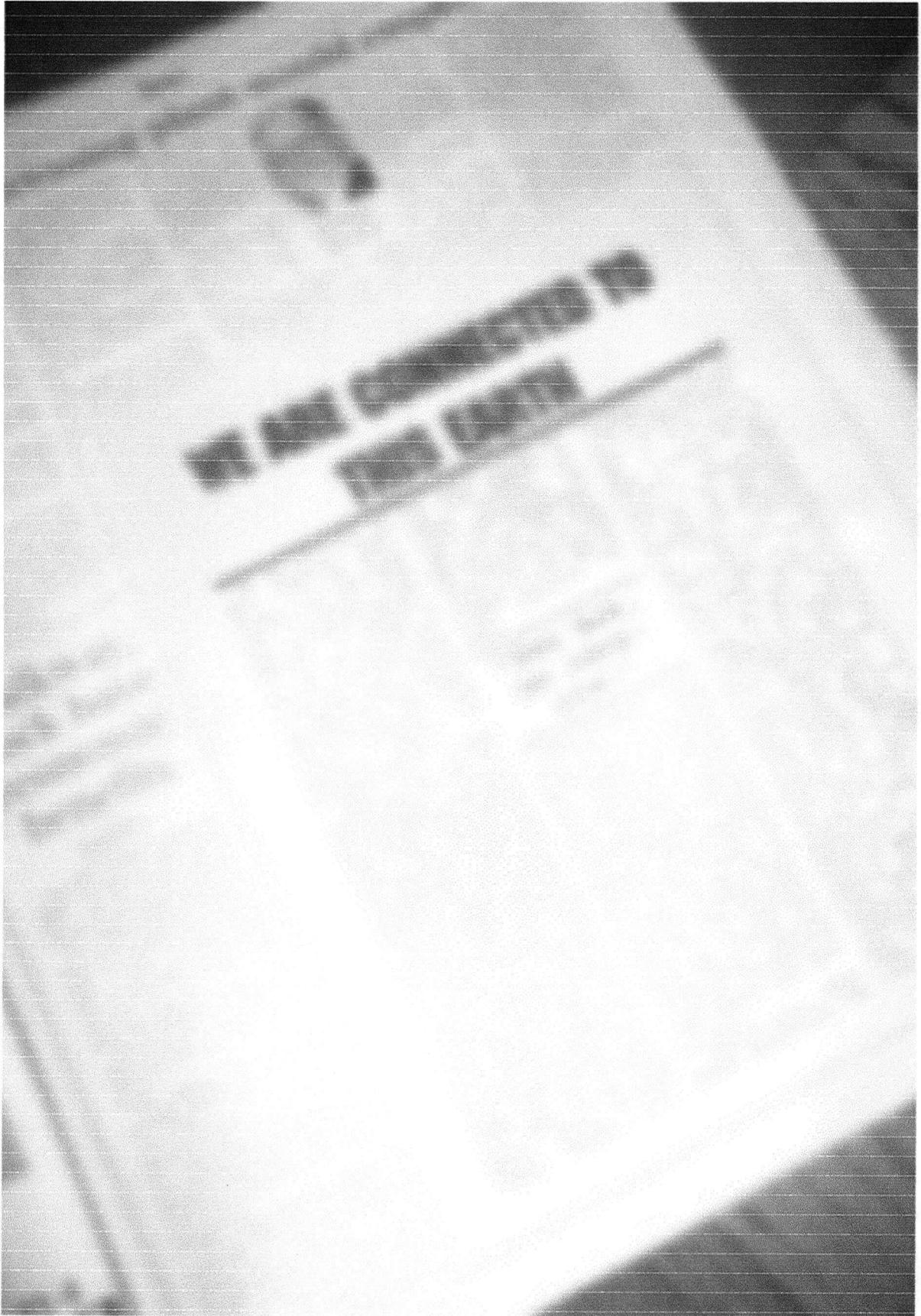
WE ARE CONNECTED TO THIS EARTH

As we continue to pour chemical cocktails into the environment and move fast and furiously from one technological adventure to another, it is no longer a matter of choice but a matter of fact that in order for living systems on the earth to survive, we must live within its limits of sustainability.[1]

Upon our clumsy awakening to the environmental crisis, we are presented with the rude realization that the impact of our actions cannot be 'contained' and the effects of them reach farther and deeper than we ever anticipated. The ongoing devastation of the world's natural and cultural systems makes this point clear: The results of how we live are not only felt by the local human community and our neighboring communities, but what we do affects all that is on this earth with us--the land, water, air, and every living being that depends on these sources for their existence[2]. We are not independent beings. We are part of, connected to, and "just one particular strand in the web of life" explains Capra.[3]

"Yes but, we recycle!"

Deep, integral changes will not take place if our actions to reduce pollution and decrease stress on the earth's natural systems remain rooted in our desires to improve human health and maximize profit at the cost of all other forms of nature.[4] If we are to prevent further damage to the environment it is critical that we change our mechanistic perceptions of the world to ones that are ecological.[5]

Simply put, this means abandoning the re-production of our 'Cartesian-self-assertive-Newtonian' ways of being in order to cultivate a more integrative existence on this earth. It entails re-rooting our thinking so that we may comprehend the world not just in linear, analytical, rational, and reductionist terms, but in ways that are nonlinear, connected, intuitive, and holistic. As well, value needs to be placed on cooperation, quality, and conservation instead of anthropocentric, exploitive, or competitive acts[6] of domination and mindsets that focus on 'the bottom dollar'. This is not a simple matter of 'exchanging' our current ways of living for ecological ones. The mechanistic, anthropocentric traditions that we embody in our culture today have been evolving steadily since the Industrial Revolution and so too will it take take time for

2

ecocentric practices to become taken-for-granted patterns within our thinking, actions, and identities.

QUESTION: But what does this have to do with the teaching and learning of mathematics in the class-room?

ANSWER: EVERYTHING!

For ecocentric thinking to bring about a paradigmatic turn that possesses longevity, depth, and integrity, it cannot be restricted to the domain of 'environmental cleanliness'. It has to become our natural, everyday way of being. Learning to be ecologically mindful cannot be treated as an 'additional component' in children's education but an integral part of each and every classroom. In short, this means that teachers and students of mathematics may not be excused! Perhaps a starting

"Okay, but what does this mean and where would one start?"

place might be to look for spaces in which to propagate nonlinear, connected, fluid, and holistic patterns of thinking mathematically. At the same time, mathematics educators could begin the process of assessing the embedded and taken-for-granted linear and mechanistic rituals that are practised within the field of mathematics education and inside classrooms.

By not excusing ourselves from this task, we can begin working towards reconnecting and developing mathematically ecological ways of being.

Notes

1. M. C. Bateson,1994, 1996; Bowers, 1993, 1995,2000; Suzuki, 1999.
2. M. C. Bateson, 1994, 1996; Bowers, 1993, 1995, 2000, 2001; Suzuki, 1999.
3. Capra, 1993, 1996, p. 7.
4. Fox, 1990; Naess, 1985; 1986; 1996; Orr, 1992, Suzuki, 1997.
5. Bowers 1997, 2003; Capra, 1993, 1996; Snyder, 1990;
6. Capra, 1996; Naess 1985, 1988; Orr, 1992, 1994; Suzuki & Dressel, 1999.
7. As cited in Suzuki, 1997, p. 199.

What is a Tree?

A tree, we might say, is not so much a thing as a rhythm of exchange, or perhaps a centre of organizational forces. Transpiration induces the upward flow of water and dissolved materials, facilitating an inflow from the soil. If we were aware of this rather than the appearance of a tree-form, we might regard the tree as a centre of a force-field to which water is drawn....The object to which we attach significance is the configuration of the forces necessary to being a tree....rigid attention to boundaries can obscure the act of being itself.

-Neil Evernden, The Natural Alien

This redefinition of something as familiar as a tree as at first rings strange. But we can recognize the more-than-tree-form it describes, just as we know that a forest is more than just the trees that grow there, and that our intercourse with the world extends beyond the edges of our skin. Our language falls short of our apprehension because of the way we have been taught to identify the world. We belong to, are made of, that world that surrounds us, and we respond to it in ways beyond knowing.[7]

We are constantly engaged in the flow of interacting but often, it is not until much later that we appreciate the significance of it.

Reflection

A mirror reflects an image
seen as the image is seen.
It does not change the looking.

To reflect on what we do,
or are, is something else.
It reveals what we could not see.[1]

---

[1] Pille Bunnell, 2000.

It happened years ago. Jennifer could not remember when exactly, but at some point as a very young child she was drawn into the enchantment of the "enveloping and sensuous earth".[1] Eyes wide and bright, Jennifer giggles as she stories-out a cluster of her treasures to me. She speaks of wild landscapes just beyond her grandparents' orchard; tunneling on her stomach and disappearing into the tall, sweet grass; lying on her back and watching the night sky for cascading meteorites; and feeling the cool dampness underfoot as she creeps silently and listens hard to find that mysterious chirping cricket. Among these treasures are many more: ones of forests, of the ocean, others storied inside her *ba-chan* [Japanese for grandmother] tanka poems, as well as Chinese proverbs told to her by her *ba-ba* [Cantonese for father].

"I suppose" Jennifer reflects, "because my family life was rooted in a kind of living that looked to nature for metaphors and life lessons, that I also seek to understand the world as a living system that is interrelated to everything else. And I guess that is why I wonder how ecological forms of thinking might help us to better understand this place we call the mathematics classroom."

"Classroom mathematics and ecology? Interrelated? Please, tell me you're kidding!" I gasped.

"I know, I KNOW," she replied and then paused. "But listen" she urged.

Anticipating a long, winding, twisting, turning kind of response, I prepared myself. Straightening my posture, I took a deep breath as Jennifer began taking me down her explanatory path.

"You see," she began, "ecology and classroom mathematics have everything to do with each other."

Jennifer then told me that the word, *ecology* had come from the ancient Greek word, *oikos*. It meant "the family household" and "the maintaining of its daily operations." Eventually, *oikos* was integrated into the term, *oecologie*, coined by Ernst Haeckel in 1866. Described as "the study of the environmental conditions of existence," oecologie was eventually shortened to what we know today as *ecology.*[2]

"I remember exactly what was going on in my mind when I first learned about the history of the word" said Jennifer. "To be quite honest, I hadn't given it much thought at the time because I was preoccupied thinking about something else. Even though I listened to what was being said, it was similar to having to attend to a different matter when you already have your hands full with something else! You see, I was taking a summer graduate course and the professor was explaining to the class how the word ecology came to be. I remember smiling and nodding as I listened and then quickly switching back to my previous thought. It wasn't until much later that I realized the significance of the encounter."

Jennifer explained that her conundrum had been trying to communicate to other people the importance of being ecologically mindful in the mathematics classroom.

"You see, it was easier for me to say what it *didn't* mean. Being ecologically mindful wasn't necessarily about taking environmental issues and making them into mathematical problems. It wasn't about conducting scientific inquiries– you know, taking ecological ways of thinking and using them as a magnifying lens to examine the field of mathematics education and then perform experiments in the classroom. It wasn't about forming a hypothesis, replicating procedures, generating conclusions or formulating a unifying theory that could be transplanted into every classroom. What I was finding it extremely difficult however, was how to explain in simple words what being ecological *did* mean to the mathematics classroom. My descriptions were cumbersome– that knowing and acting were embodied with-and-in one's way of being– or, a mindful comprehension of the integrative, holistic, and nonlinear nature of teacher's practices and children's learning of mathematics. So, while I spent my time trying to sort out the ideas that I viewed as being problematic in developing an ecological sensibility for teaching and learning mathematics" said Jennifer, "I was completely ignorant of the fact that I *did* have a way to express my understanding of ecology and mathematics education! *Oikos*."

Jennifer told me that, upon reflection, it was the term, *oikos* that captured exactly how she understood her mathematics class to be. Here, she explained that she imagined it to be very much like a family household. As the children's teacher, she saw her role as caring for and sustaining the mathematical relationships and interactions of her students.

"So just as environmental thinking focuses on human relationships with nature," Jennifer smiled, "it is a similar focus that I have for my teaching and children's learning of mathematics. It's about examining and assessing the kinds of mathematical relationships, as well as the forms of mathematics that emerge in the classroom, and responding to them in my manners of teaching mathematics".

"And your reason for wanting to be ecologically mindful?"

"My wanting to be an ecologically responsive mathematics teacher comes from caring for how mathematics exists in the classroom, my teaching, and the children's learning of it.[3] It is about being committed to sustaining relations that are not only ecologically coherent in the classroom but also ones that promote a sense of cohesiveness within the larger educational communities."

Jennifer then picked up the book she'd been reading before I had arrived. Opening it to page 78, she read aloud:

> We are living in a time of both creativity and concern about education, and the decisions that are made for the classroom will feed directly into the way graduates

"and children," she added,

> participate in society and the way they impact on the natural

"and social-cultural," she said,

> systems around them.[4]

Bringing our conversation to a close, Jennifer said, "and so you see, the choices we make as mathematics teachers not only affect the kinds of mathematics children learn in school, but equally, the ways in which children are *taught* to learn and the ways they will interact with mathematics outside of school will affect the world they live in. We, mathematics education, and ecology do not exist in separate households but, rather, we share a common space."

Notes

1. Abram, 1996, p. 15.
2. Also, an email correspondence with C. A. Bowers in which we discussed, Donald Worster's (1990) book, *Nature's economy: A history of ecological ideas.* In particular, the definition of *ecology* which is described by the author, p. 191-1922.
3. Naess, 1985, 1986, 1988, 1996; Varela, 1999.
4. M. C. Bateson, 1996, p.78.

THE 3 FACES OF ECOLOGY

According to M. C. Bateson[2], there are three "faces" or realms of ecology: empirical, environmental, and systemic. The author defines empirical ecology as biological, meteorological, and geographical studies that focus on understanding how the planet is changing and how these changes affect the interrelationships of the world's natural systems. The environmental face of ecology is concerned with identifying the level of impact that our ways of living have on the earth's systems. It also involves the development of solutions for environmental problems that will minimize harmful stress on the earth. It is within the systemic realm of ecology where mathematics teaching and learning can be most radically explored. This is because systemic thinking focuses on seeking "the pattern which connects"[3] a system or systems together as interdependent and interacting wholes.

In the field of mathematics education, a "system" could be an individual teacher or a student. It could also be a collective group such as a mathematics class, the school, and so on.

The connecting pattern or patterns that interrelate these systems together as a dynamic whole encompass the forms of knowledge, actions, and identities that are brought into being as a result of the ongoing interactions in the system(s) and the ways in which they are sustained by the system(s).

By focusing on relational qualities, ecological ways of thinking give rise to viewing the world as an integrated whole; a dynamic and fluid network in which all living and social-cultural systems are interconnected. The

world is *not* conceptualized as being composed of a collection of separate entities, but instead, as a highly complex unity in which all systems are interrelated and therefore, interdependent.

It makes sense then, that when looking at mathematics teaching from an ecological perspective, it would be conceived as similar to that of children's mathematical learning.[4] Mathematics teaching as a fluid, complex process implies that it exists always, in relation to the ongoing interactions of the students, the mathematics, and the material and nonmaterial environment of the classroom.[5]

And so it is by taking a systemic perspective from within the conceptual space of ecology that the following query emerges:

> *In what ways can systemic manners of thinking about mathematics education enable forms of teaching and spaces for children's learning of mathematics to possess an ecological sensibility?*

NOTES

1. Stigler & Hiebert, 1997, p. 14.

2. M. C. Bateson, 1996.

3. G. Bateson, 1980, 1991.

4. A. B. Davis, 1996; A. B. Davis, Sumara, & Luce-Kapler, 2000; Kieren, Pirie, & Calvert, 1999; Martin, 1999; Towers, Martin, & Pirie, 2000.

5. Thom, 2002.

6. Kotagama, 1993, p., 120-121.

10

Today we are calling on the nations and the peoples of the world to change personal attitudes and practices. "To adopt the ethic for living sustainably, people must reexamine their values and alter their behavior. Society must promote values that support this ethic and discourage those that are incompatible with the sustainable way of life" (from *Caring for the Earth: A Struggle for Sustainable Living.* Gland, Switzerland: IUCN/UNEP/WWF, 1991).

To whom is this call addressed? The ethic for sustainable living has always been part of our cultures, and the people lived in line with such a way of life. It was the Western materialistic-consumerist strategy, considered the essence of "development," that shattered the foundation of sustainable living that is[6]

Yes, but what gives rise to a systemic, ecological view of the world? or the mathematics classroom for that matter?

Encircling our

classroom

perceptions

mathematics

of

the

the

and world

Our Individual Ways of Being

I think that if we start by looking at how we are as individuals and then connect this to how we exist as collective groups, you'll be able to appreciate why a systemic understanding of the world and the mathematics classroom is really about 'layers of living'. It creates a conceptual space in which we can make visible what often remains invisible-- the co-emergent, complex nature of our

B   ***biological,***

E   ***structure determined,***

I   social,

N   and

G   CULTURAL

Okay, Let's begin!

As humans, we exist in the world as what Maturana and Varela would call, "autopoietic," or self-making systems.[1] We possess both "organization" and "structure". It's our organization that distinguishes you and me as people and not, say, fish or goats! And it's our structure that can be described as the internal dynamics and relations that enable you and me to develop ways of knowing, acting, and being that are uniquely our own. Simply put, your structure is not the same as my structure and it is because of our structural diversity that we can distinguish you and me as being different people.

But how is it that we are structurally different?

Maturana and Varela describe "structural coupling" as the process by which our structures evolve. The changes that occur in our knowing, actions, and identities arise from the recursive interactions between two or more living organisms.

A nice sounding definition, but what does this mean?

Well, if we take this idea of structure and think of a person's understanding of mathematics to be his or her mathematical structure, in a way similar to how

a person's forms of knowing, acting, and being are impacted by life experiences [2]-- a person's mathematical structure too undergoes changes as a result of his or her mathematical interactions.

> And so, because one's structure is dependent on the kinds of mathematical interactions one has and how they then feed into what the person already understands, these differences in experience and impact create differences between one person's mathematical understanding and that of another person's... hence, diversity in mathematical structures.

Yes. And recursively, how we go on then, to teach or learn mathematics will now be shaped by these structural differences. This also means that as a class engages in mathematics, structural coupling is arising in the structures of the individual students and their teacher. The growth that arises from this process is dynamic and continuous. It happens in us moment to moment as we experience human and nonhuman perturbations in the environment.

> So it's the perturbations that make for structural changes?

No, not exactly. It isn't the perturbation that determines how our structures change. And, perturbations only exist if they are perceived by the person as "perturbatory".[3] Rather, it's the individual based on his or her structure, who specifies what will or won't be a perturbation, whether or not coupling will occur, and if so, the kind of internal changes that will arise in his or her structure. Knowing this, we can say that we exist in the world as autopoietic and "structurally determined" systems.[4]

> Would this mean then, that in the mathematics classroom, it is the child who determines based on his or her internal structure, what will and will not serve as occasions for learning mathematics?

Yes, and it's the child's mathematical understandings-- his or her structure, that shapes and is shaped by future understandings.[5]

But what about the teacher? Isn't it the teacher who teaches the class what mathematics to learn?

Of course it should be expected that a teacher attend to children's mathematical learning[6] in ways that are invocative[7] and provocative.[8] However, given a systemic view, we can't assume that the teacher exists as the only source for occasioning children's mathematical perturbations. Engaging in mental reflection about mathematics or taking part in mathematical interactions with the human and nonhuman environment can also serve as possible sources for structural changes to occur.[9]

So, even if a teacher intends to have the class learn, say... a new strategy for adding 3-digit numbers together, it is the child, NOT the teacher who determines if and what kind of learning will arise?

Exactly. And when structural changes do take place, new pathways or relationships emerge and impact on the child's existing mathematical understandings. So it's impossible for us to predetermine how our individual structures will evolve since they are ever-changing because of the coupling process.[10] This is what A. B. Davis, Sumara & Luce-Kapler mean when they say that "learning is DEPENDENT ON, but cannot be DETERMINED BY teaching".[11] Mathematical learning takes place with the environment: as unpredictable yet recursive growth of one's mathematical structure of understandings.

Okay, I can see how we as individuals are shaped by the interactions we have with the environment but it seems to me to be a very inward, insular way to view the mathematics classroom, don't you think? This kind of thinking moves in only one direction-- from the environment to the individual child.

Up to this point it has. However, a systemic view does bring forth a 'co-emergent worldview', if you will, in that it recognizes the interdependence and complex circularity that exists between the environment and living systems. Just as our internal structures are ever-evolving through our interactions with the environment, the environment is also undergoing structural changes. These changes within us and within the larger environment recursively shape

16

what will be possible in terms of future interactions and how each will respond to the other.[12] Lewontin elaborates on this complex circularity when he explains that:

> The organism [living being] and the environment are not actually separately determined. The environment is not a structure imposed on living beings from the outside but is in fact a creation of those beings. The environment is not an autonomous process but a reflection of the biology of the species. Just as there is no organism without an environment, so there is no environment without an organism.[13]

This co-evolution that takes place as we and the environment interact raises an important issue when considering the mathematics classroom. It's not only the environment that shapes the teacher's or a child's mathematical ways of knowing, acting, and being, but it's also the teacher's or child's interactions that affect what future events and responses will take place within the larger classroom.

Each needs the other.

That's right. Now can you begin to see how the world can be viewed as an integrated whole by recognizing the interdependence of living systems and their environments? Life unfolds by way of "natural drift"[14] -- as a result of the recursive interactions between living systems and the environment. This co-evolutionary view of the world differs from other perspectives that project images of evolution as being a linear process of competitive domination where species and their environments are not interdependent but separate from each other.

Yes. A subtle yet important difference, I suspect.

What's more, a systemic, ecological view doesn't portray mathematics teaching or children's mathematical learning as being individualistic and linear in nature. They arise fluidly in relation to each other and with that of the larger environment be it a mathematics class, a school, or even the educational system. An ecological perspective brings attention to understanding interrelationships within the mathematics classroom.

In the beginning of our conversation, you mentioned that we are also social and cultural beings, yes?

Yes, that's right. Keeping in mind what we've discussed in terms of how we are as individuals and how we and the environment co-emerge, let's move outwards to the broader, social realm. By doing so, we can continue to discuss "the pattern which connects" [15] *our living as individuals to our collective actions, identities, and ways of knowing as social systems.*

Our

Social

Days

of

Being

Maturana characterizes social interaction as being:

> *when two or more structure determined systems interact recurrently with each other in a particular medium, they enter in a history of congruent structural changes that follows a course that arises moment after moment contingent on their recurrent interactions, to their own internal structural dynamics, and to their interactions with the medium, and which lasts until... they separate. In daily life, such a course of structural change in a system contingent on the sequence of its interactions in the medium in which it conserves organization and adaptation is called 'drift'.*[16]

In terms of the classroom, this would mean that social mathematical interactions arise when two or more children work mathematically together. Importantly, the learning occasioned from these mathematical interactions not only shapes the further development of each child's understandings but also, the collective understandings of the partner or group and the larger mathematical environment in which the interaction took place. These collective forms of knowledge, actions, and identities and how they're created through social interactions are what Maturana refers to as "drift".

If we understand human social systems to be what Gregory Bateson and Maturana refer to as systems that evolve through the cohesive, collective interactions of the members, then what we know, how we act, and who we are can't be taken as happening only within the realm of the individual. Such growth also needs to be recognized as emerging from our collective manners of living-- the relations that are created through ongoing interactions and that which connects us as interdependent, social beings.

> *And are social phenomena, like our individual structures, unpredictable too?*

Yes. Just as we can't predetermine the evolution of an organism or its environment because they are dynamically interactive, we can't predetermine the collective mathematical activity that will take place in a mathematics classroom. In terms of a class' mathematical learning, it's naturally unpredictable because children's internal and collective structures are constantly changing from moment to moment.[17]

> *I see. So a child isn't only a "structurally determined" learner but he or she is also a member of larger social systems... such as a mathematics class?*[18]

Exactly! And in the classroom it's not only the child but also the children and their teacher interacting at the same time... as individual and collective wholes, responding to environmental perturbations-- shaping and being shaped by the mathematical learning that emerges.[19] Like our individual understandings, collective forms of learning aren't thought of as being transmitted from an external entity. They are constantly emerging and co-emerging through social interactions. Because of this, children's mathematical growth can be described as being "much like paths that exist only as they are laid down in walking."[20]

Can you explain in more detail, the nature of social interactions and what forms they can take on?

In terms of their nature, I think of them as Maturana[21] does... like "conversations in progress". Maturana explains that social interactions can be brief, withdrawn from and then re-entered again or, they can be continuous. It is these "conversations" within social systems that he considers necessary in how it is we come to know and be in the world. Social dynamics are what bind us as a pair or group of living beings together as a collective, social system. Co-emergence takes place as we interact with and in relation to one another... we are able to coordinate and re-coordinate how we think, our actions, 'how we are' basically, in order to maintain cohesive ways of being with one another. In this way, social relationships that keep a collective unity intact can be seen as similar and just as critical to the co-evolution that takes place with individual organisms and their environments.

Maturana's idea of "languaging" is useful because it describes the process by which social systems function and evolve as collective unities.[22] Now, it's important that you don't think of languaging as simply individuals engaged in verbal conversation with themselves or others. Languaging involves the physical, verbal, and mental ways we humans think and interact among one another, but it is also the understandings that arise from such linguistic interactions. It's how we are able to coordinate and recoordinate our ways of being so that we can continue to interact within groups and develop collective forms of knowing. In other words, "languaging" in the mathematics classroom entails the mathematical understandings that emerge from the different ways in which members of the class think and engage mathematically with one another. Because we exist in language and have the potential to be languaging agents, it is possible for new understandings to arise. Knowledge systems evolve then, as a result of our social activity.

So does this mean that it's through the interactions of the class that collective mathematical understandings which are different from personal ones come into being?

Yes. Now, let's talk about our cultural ways of being.

Our

Cultural

Being

to

Tours

23

The practices of teaching and learning school mathematics are two examples of "cultural behaviours" in our Society. Generally speaking, cultural behaviours are social patterns that span generations. It's as a result of our living in the languaging process of "cultural drift" that we establish collective ways of being that are passed on and evolve from generation to generation.

> So cultural ways of being are social phenomena that continue over generations?

Yes. And the historical transformation that happens is a result of the recurrent interactions and languaging between the older and younger members of a cultural group.[23] In the same manner that drift is explained by Maturana and Varela as necessary for us to evolve with the environment and socially with others, so too is cultural drift necessary for the continuity and evolution of cultural systems.

> Okay. I see how cultural drift provides a systemic way for us to understand say, how human-centred and mechanistic social patterns established in the Industrial Revolution have continued into today's culture. But what I don't yet understand is what ecological thinkers such as Bowers, Capra, Naess, and Orr[24] mean when they say that our cultural ways of being shape how we perceive and therefore, exist in the world.

We do much more than simply live in the world-- remember our conversation about coupling?

> Yes.

Well, I believe that connected with this understanding is the fact that **THE WORLD** we bring forth is not only created through our individual and social existence. It is also **SPECIFIED** through our cultural knowledge, actions, and identities. Cultural phenomena, when **CONSERVED**, seamlessly co-emerge from one generation to the next and create a world from our living **WITHIN CULTURAL "COGNITIVE CIRCLES."**[25] These cognitive circles arise from our cultural ways of living and we justify the patterns that they occasion as a matter **"OF TRADITION"** OR, less reflectively and more acceptingly expressed, we say it is simply **"JUST THE WAY THINGS ARE."** In this way, cultural patterns are embodied in our thinking, **THEY BECOME US**, **DISAPPEARING FROM THE SURFACE OF OUR CONSCIOUSNESS.**[26]

> Surely, our cultural actions, beliefs, and identities aren't that invisibly specific!

Hmm... consider the images we as members of Western culture attach to the idea of what it means for a person to be an 'individual'. When we think about what makes a person an individual, often embedded within this is the notion of 'independence'. As a teacher, I find that parents often express to me that it's important for their children as "individuals", to be self-sufficient, able to think for themselves, make independent decisions, and be their own people. In valuing these qualities, we teachers and members of older generations encourage younger generations to develop their independence by providing learning opportunities that focus on the "individual" or "autonomous" child. Within our culture, independence and individuality serve as distinguishing qualities of being successful. They engender a sense of freedom, self-reliance and "standing out from the crowd".

> Well, isn't that what we should be doing? encouraging students to be independent individuals?

Hold on for a moment. Let's contrast this image with what it means be an individual in Japanese and Chinese cultures. Traditionally, within these two cultural circles, the image of an "independent" individual is not the image that comes to one's mind. This is because in Japanese and Chinese cultures, the younger members are taught by the older members that an individual is not defined in terms of self-reliance or self-sufficiency but more in how the individual contributes to the well-being of his or her family. You see, a person's identity exists in the collective sense of the family. This can be seen in how people address one another. Unlike in Western society where we are distinguished on a first name basis such as "Jennifer" or in a first-name-last-name order as "Jennifer Thom", people in Japanese and Chinese

cultures are addressed by their last name or in a last-name-first-name order such as "Thom-san" or "Thom-Jennifer". Identity isn't derived from the validation of one's self but from the respect for the family as a collective whole. Conservation of these relations is carried forth through one's values and actions that foster the well being of the family as a collective whole. Given instances such as these, we can better understand what Maturana[27] speaks of when he says that it's "in the implicit or explicit accepted premises under which their different kinds of discourse, actions, and justification for actions take place" ...that cultures create taken-for-granted and, hence, invisible yet distinctive cognitive circles.

Okay... yes... how cultural beliefs create blinders... that shape how we experience the world... but, if we are truly blind to our cultural ways of being, is it even possible for us to become aware of them?

One way for us to examine just how culturally embedded our lives are is to consider the cultural experiences, beliefs, and values that emerge from the recurrent interactions of a group and metaphorically become what Gregory Bateson[28] and Bowers[29] refer to as cultural "maps".

A map?!

A map. Simply put, a culture's map identifies what its members will and won't perceive as having significance by rooting these features within the culture's temporal, spatial, spoken, written, and symbolic language. Because these cultural distinctions permeate the group's languaging, a culture's map as Neil Postman would say, "does much more than construct concepts about the events and things in the world; it tells us what sorts of concepts we ought to construct".[30]

Can you give me an example of a feature or concept that we create or recreate in our living out of these metaphorical maps?

Just look back at how the idea of the individual is distinguished and played out as a feature of Western and Asian cultural maps. The former very much influences a person to value ways of thinking and actions that enable an identity of independence while the latter, emphasizes a person's connection to her or his family and imbues a sense of interdependence. Same concept-- "the individual", but completely different cultural conceptualizations.

26

Yes, which lead to totally different ways of interacting in the world.

Speaking from a systemic, ecological thinking space, it's here in the envelopment of the cultural realm that we live... nested within our individual and collective layers of knowing and being. And it's here that we dwell in our practices of teaching and learning mathematics. For me, the mathematics classroom is imagined as an integrated space where living, social, and cultural systems co-exist and co-evolve. In an ontological manner, we are living systems within social systems within cultural systems. Encircled once more to include all other living and natural systems on the earth, it is how we humans come to exist as "just one particular strand in the web of life".[31] It's in this way that the world isn't perceived as a collection of separate "parts", but as a dynamic whole; a complex unity of all living and social-cultural systems that are fluidly interconnected and, therefore, necessarily interdependent. And it's here in this conceptual space of knowing that a systemically ecological view of the world and the mathematics classroom resides.

All

natural

systems

Individual

Social

Cultural

Ways of Being

human

and

Notes

1. Maturana, 1987; Maturana and Varela, 1980, 1987.
2. Capra, 1996; Fleischaker, 1990.
3. Kieren and Pirie, 1992.
4. For example, see Maturana, 1988b, p. 36.
5. Kieren and Pirie, 1992; Pirie and Kieren, 1992.
6. A. B. Davis and Sumara, 1997; Martin, 1999; Pirie and Kieren, 1994b; Simmt and A. B. Davis, 1998.
7. Invocative interventions are those which engender folding back-- a recursive form of revisiting to earlier layers of understanding. See Kieren and Pirie 1992; Pirie and Kieren 1994b; Towers 1998.
8. Provocative interventions are those which enable learners to move outwards to new and deeper understandings. See Kieren and Pirie 1992; Pirie and Kieren 1994b; Towers 1998.
9. Simmt and A. B. Davis, 1998; Martin, 1999.
10. Maturana, 1988a, 1988b, 2001; Maturana and Varela, 1987.
11. Emphasis added, A. B. Davis, Sumara and Luce-Kapler 2000, page 64; Pirie and Kieren, 1992.
12. Lewontin, 1983.
13. Lewontin, 1983, as cited in Varela, Thompson, and Rosch, 1996, p. 198.
14. Maturana and Varela, 1987.
15. G. Bateson, 1980, p. 9; 1991.
16. Maturana, 1988b, p. 46.
17. Pirie and Kieren, 1989, 1994a.
18. Bowers and Flinders, 1991; A. B. Davis, 1995.
19. Kieren and Pirie, 1992; Steffe and Tzur, 1994.
20. Varela et al., 1996, p. 205.
21. Maturana, 1988b; 1991; 1995; 1997a, 1997b, 1998; 2001.
22. Maturana 1997a, 1997b, 1998.
23. Maturana, 1988b, 1995, 1997a, 1997b, 1998; Maturana and Varela, 1987.
24. Bowers' 1998 graduate seminar; Capra, 1996; Naess, 1985, 1988; Orr, 1992, 1994.
25. Maturana and Varela, 1987.
26. Maturana, 1988b, 1997a, 1997b, 1998; Ruiz, 1996.
27. Maturana, 1988b, p. 41.
28. G. Bateson, 1972, 1980, 1991.
29. Bowers, 1993, 1995, 1997a, 1997b, 2000.
30. Postman 1995, p. 181.
31. Capra, 1996, p. 7.

THE 3 FACES OF ECOLOGY

According to M. C. Bateson[2], there are three "faces" or realms of ecology: empirical, environmental, and systemic. The author defines empirical ecology as biological, meteorological, and geographical studies that focus on understanding how the planet is changing and how these changes affect the interrelationships of the world's natural systems. The environmental face of ecology is concerned with identifying the level of impact that our ways of living have on the earth's systems. It also involves the development of solutions for environmental problems that will minimize harmful stress on the earth. It is within the systemic realm of ecology where mathematics teaching and learning can be most radically explored. This is because systemic thinking focuses on seeking "the pattern which connects"[3] a system or systems together as interdependent and interacting wholes.

In the field of mathematics education, a "system" could be an individual teacher or a student. It could also be a collective group such as a mathematics class, the school, and so on.

The connecting pattern or patterns that interrelate these systems together as a dynamic whole encompass the forms of knowledge, actions, and identities that are brought into being as a result of the ongoing interactions in the system(s) and the ways in which they are sustained by the system(s).

By focusing on relational qualities, ecological ways of thinking give rise to viewing the world as an integrated whole; a dynamic and fluid network in which all living and social-cultural systems are interconnected. The

Settling In

Thinking about how I (it's me, Jennifer!) might respond to this help wanted ad, I thought it best to 'bring it home' as it were and invite Stigler and Hiebert's question into this ecological thinking space of mine. However, once inside, I soon realized that although this question certainly belonged in the realm of classroom mathematics, it was going to be difficult if not impossible for me to have an open conversation with it! Explained another way, it is like when you spot THE sofa in a furniture store but as soon as you get it home and put it into your living room, the sofa does not look so fabulous anymore. Instead, it is clearly out-of-place because it does not go with any of your existing furniture. For me, (and in spite of the authors' good intentions) this seemed to be the case with bringing this question home; neither the question nor the ecological space suited each other. You see, the manner in which the question is posed:

> How can we **break out of** our **conventional approaches** and imagine **more productive** alternatives? [1]

puts forth for me as a teacher, an 'end-result' mindset of **improving productivity** in the mathematics classroom; the need to **dispose of** or **discard** teaching practices that are perceived to be **old** or **commonplace** and to **acquire new** teaching **tools** so that we may **increase** or **maximize** children's learning of mathematics.

Before one is able to think of possible ways to respond to this question, its linear, disconnecting posture has already "mapped" for us that manners of teaching are commoditizable "relationships", ones that we marry into and, if necessary, divorce ourselves from. What is more, is that within the question's reductionistic confines, Stigler and Hiebert's query closes itself off from the opportunity for deep changes

to

take

root.

You see, even if we changed from one teaching style to another, radical shifts in the mathematics classroom would not likely occur if our thinking continued to be fashioned from the mechanistic pattern of productivity. The persistence of such a mindset is disabling in that it denies the very possibility of Stigler and Hiebert's query being one that provokes teachers to become more integrative and creative.

And because the authors' question does not allow for an examination of the *ecological coherence* of the mathematics classroom, it creates the impossibility for mathematics teaching and learning to be conceived as *holistic, organic, recursive, and co-emergent.*

Determined to reconceptualize Stigler and Hiebert's query within an ecological realm, I set out to find a question that *would* make sense in such a space. The question needed to be one that encouraged a systemic, ecological way of knowing: one that would allow for developing an understanding of how the 'layers' of our living are always and necessarily shaping how we teach mathematics.

Instead of provoking a knee-jerk reaction from the reader or myself, I wanted this question to be taken as an open invitation to look deep and make visible what often remains *in*visible— the cognitive circles and cultural maps we lay down in our paths of teaching. Once brought to the surface, these experiences, beliefs, and values can be examined and assessed in terms of the forms of mathematics teaching that they *enable* or *disable*, the ways in which they become embedded within the mathematical language of the classroom, and the impact they have on how children come to know mathematics. Just then, a similar yet radically differently expressed question invited itself into my thinking:

> *How might we as teachers reconsider our conventional patterns of mathematics teaching? And by doing so, how can we re-seed learning spaces that nurture and sustain children's mathematical growth?*

Not only does this question share Stigler and Hiebert's concern for how mathematics teaching and learning takes place in the classroom, but it also makes sense within a systemically ecological thinking space. The question provides the necessary focal structure for the reconceptualization of mathematics teaching to occur while the ecological mind-space allows a place for such an exploration to unfold. Together they enable examination into how it is that *what we know* and *who we are* emerge and become our manners of teaching mathematics.

Feeling as though I was beginning to settle into this new space of mine, I wondered what to do next. Where might one begin to create openings for ecologically minded ways of teaching and learning mathematics in the classroom? I found myself moving back and forth between reading and pondering Stigler and Hiebert's question and considering how to explore my ecocentric one. It was in the midst of this back and forthing that I realized both of these questions spoke of concern for this place we call "the mathematics classroom". It made sense for me then, that the place to begin was to begin with "place".

Notes

1. Emphasis added, Stigler and Hiebert, 1997, p. 14.

Sense of place complex? We tend not to think so, mainly because our attachments to places, like the ease with which we usually sustain them, are unthinkingly taken for granted. As normally experienced, sense of place quite simply is, as natural and straightforward as our fondness for certain colours and culinary tastes, and the thought that it might be complicated, or even very interesting, seldom crosses our minds...[1]

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[1] Basso, 1996, pp. xiii.

Beginning with Place

Keith Basso's description of the ways in which we create and sustain our relationships with places brings forth just how strong our connections to place are. In doing so, he reminds the reader how taken-for-granted, forgotten, unnoticed, or ignored the actual textures and patterns that make a place a place become. Primary, basic, and essential, sense of place is undeniably and always a critical part of every mathematics classroom.

If we perceive the world as a place in which we live as systems within systems, then the mathematics classroom, as place, is not constituted simply by the presence of four walls, some furniture, a teacher, students, and mathematics. Necessarily, this place includes the relations that evolve from the intermingling of teacher, children, their surroundings, and mathematics. If we think about how we come to know places, then our sense of a mathematical place in the classroom emerges from the spaces in which we perceive mathematics to arise and the forms it takes on. Put another way, the classroom as a mathematical place and how we connect with it not only comes from what we know and feel, but the kind of place it becomes grows out of the interactions we have with it.[1] Thus, "[w]e do not define places; they do not define us. Rather, in dynamic interplay, we come to form together".[2]

As an elementary teacher, I have always been committed to a holistic way of thinking about mathematics in the classroom, one that facilitated my development in teaching mathematics and fostered children's mathematical learning. But it was not until recently after reading Basso's book, *Wisdom sits in places,* that I began to think ecologically about the mathematics classroom as "place". Moreover, as ecological ways of being are not an everyday practice in our society or its educational systems, it is understandable that one would not think of the mathematics classroom in such a manner, much less be able to imagine what an ecologically coherent mathematical place might mean. Sense of place and ideas associated with place do not come about naturally or consciously for us. As a result, they remain hidden or invisibly embedded within our taken-for-granted manners of teaching mathematics.

Through my experiences in gaining a deeper understanding for how my teaching shapes the mathematics classroom and trying to create a sense of place that embodies ecological notions such as recursion, co-emergence, and fluid integration, I have learned that this kind of work cannot be achieved by what we think may be "breaking out of conventional approaches". To do so in the attempt to get rid all that undermines an ecological sense of place in the mathematics classroom would be

35

naive and superficial. Systemic, ecological changes need to begin by first considering what place means to the mathematics classroom. It involves looking deep and engaging in the complex, recursive process of identifying and questioning one's taken-for-granted conventions of thinking about and teaching mathematics–asking ourselves how they contribute to the sense of place that exists in the mathematics classroom.

Why is this so important? Does it **REALLY** matter? I think so. Let me explain by describing the different senses of place for mathematics that I have come to know as both a learner and a teacher. These vignettes chronicle my growing understanding of place. They provide revealing glimpses into how deeply mechanistic, commoditized, linear, and disconnected my common sense of place for the mathematics classroom was and the challenges I faced in making it into a cohesive whole.

Notes

1. Basso, 1996; Camus, 1955; A. B. Davis, 1996.
2. A. B. Davis, 1996, p. 132

Reflections Revealing the Fragility of Classroom Mathematics

It [mathematics] requires silence and neat rows and ramrod postures that imitate its exactitudes. It requires neither joy nor sadness, but a mood of detached inevitability: *anyone* could be here in my place and things would proceed identically.[1]

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[1] Jardine, 1994, p. 109.

A Student's Place

Through my years of elementary schooling, I grew to believe that the spaces where mathematics existed in the classroom, the forms it assumed, and my relationship with it were clearly marked out by my teachers and had little if anything to do with myself or my peers. I came to know that at school, it is the teacher who makes mathematics happen. Just like TV programmers, I would think to myself, teachers always ensured that mathematics began, ended, or reran at exactly the same time each day. They were conscientious not to let mathematics spill into any other programs of study such as science, socials, art, or language. And the only time when mathematics did extend past its designated slot was after school-- if we had not completed our exercises during class.

Our lessons were similar to that of learning to catch a ball. First we would watch the teacher demonstrate how to do the mathematics and then ready or not, the teacher would throw problems up onto the chalkboard or to us in the form of a textbook. Scrambling to catch the mathematics, we would madly record the mess of numbers and symbols in their correct linear fashion, practise, practise, practise, and then hopefully, toss the mathematics correctly back to our teacher. On other days, we would await the moments when timed drills, pop quizzes, and tests became the stage where we performed our proficiency and ability to juggle addition, subtraction, multiplication, and division facts.

I also learned that the teacher liked it best when mathematics happened not with other classmates but rather, silently in our heads, and figuring out solutions should not involve fingers, drawings, or counting! Never questioning but always reproducing, this is how my teachers and we students busily created and maintained a place of anonymity for mathematics in the classroom.

Encountering familiar issues in a strange setting is like returning on a second circuit of a Möbius strip and coming to the experience from the opposite side. Seen from a contrasting point of view or seen suddenly through the eyes of an outsider, one's own familiar patterns can become accessible to choice and criticism. With yet another return, what seemed radically different is revealed as part of a common space.[1]
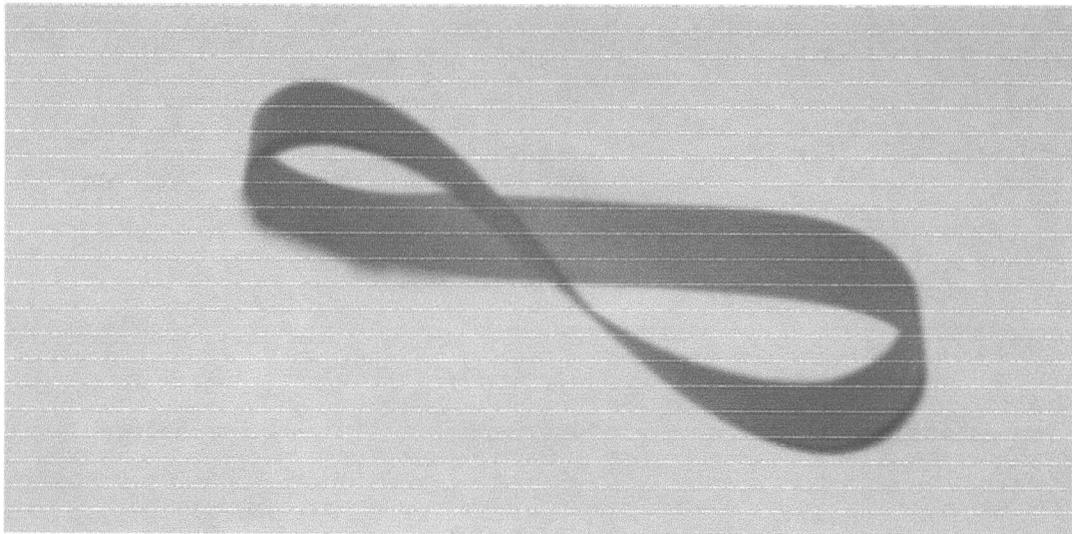
[1] M. C. Bateson, 1994, p. 31.

Back on the Strip: My University Years

Towards the end of completing my undergraduate degree in education at the University of Victoria, I was required to identify my teaching area of concentration. Based on my interests, it was a toss up between the visual arts and mathematics education. I found it difficult to choose one over the other and so, I chose the area that I felt was in the most need of rescuing. The two years of mathematics education and mathematics courses that followed made for what I considered to be two more journeys around the Möbius strip. The first trip, which I have already described, was my childhood experiences learning mathematics in school. The second, from the perspective of a teacher-to-be, and the third, from a learner of mathematics again is what I describe for you now.

During each of these returns, I found myself questioning and shifting my conceptions of what it meant to teach and learn mathematics. However, it was only afterwards that I realized it was on these journeys that I was visiting and revisiting the notion of mathematics' place in the classroom.

From The Side of Pedagogy

Now enrolled in mathematics education and approaching it not from that of a mathematics student but coming at it from the opposite side-- one of pedagogy, -- provided me with the contrasting point of view necessary to reveal taken-for-granted conceptions I held about classroom mathematics.

It began in my first mathematics education class. In this course where the focus was on the teaching of mathematics, the professor engaged us teachers-to-be through modeling possible ways to develop children's conceptual understanding, actually experiencing hands-on minds-on[1] activities for ourselves, and assessing student understanding through video analysis. In doing so, this professor dispelled many of my taken-for-granted assumptions about school mathematics; ones that included it as being an activity of simply "doing tasks and solving problems quickly in one's head"[1] and that "mathematics can be best learned in isolation."[2]

What became apparent was that as teachers we needed to be creative in thinking about our manners of teaching but also in thinking creatively about the mathematics itself. Forms of teaching that communicated to students there are many ways to solve a problem, avoiding what he called "heavy-handed" teaching that implied teaching-by-telling, and developing open-ended activities such as simple games or riddles that draw children into the complexity of the mathematics instead of repelling them from it[3] became important foci in my growth as a teacher. Moreover, this professor made me realize that, just as mathematics should be brought into being through the teacher, the children, and a variety of settings in the classroom, it was also important for teachers to enable learners to develop meaningful connections between the mathematics they study at school and the mathematics that occurs in their daily activities at home[4] and in their community.

Learning from the opposite side of the Möbius strip-- from the perspective of a teacher-to-be-- I began to understand the impact that teachers have on the kinds of mathematics that arise in the classroom. And it was here that I began considering how I might enact a pedagogy that embodied a sense of connectedness for mathematics with the classroom

Notes

1. Liedtke, 1995, p. 52.
2. Liedtke, 1995, p. 56.

3. *Liedtke, 1996-7; Liedtke et al., 1998.*

4. *Liedtke, 2000.*

5. *Rinaldi, 1989.*

Third Time Around

The following September, I found myself taking yet another trip around the Möbius strip. This time again, from the perspective of a learner. Here in two of my mathematics courses I experienced first hand, what it actually felt like when mathematics took place in the dynamic ways that the first professor had described. These professors made it clear to us that we would neither be given nor expected to memorize formulas or procedures. Feeling panicked, my first reaction was that I had registered for the wrong mathematics classes-- how on earth was I to play the game if these professors were not going to show us which mathematics we were to toss back and forth?

Fortunately I persevered, continued to attend the classes, and for the first time in all of my years of learning I became convinced that "real" mathematics was not a game of catch but rather a something is brought into being. Suddenly for me, mathematics no longer took place in an anonymous world but with the world of human and natural contexts. We spent our time examining, questioning, and watching mathematical patterns emerge in different areas such as biology, economics, and everyday life. Learning in this manner provoked and enabled me to explore, devise, and create self-generated methods and mathematical formulations in place.

Common Space

By encountering familiar issues as M. C. Bateson describes and coming at them from the opposite side(s), we experience them as different or new. Then recursively, upon examination and questioning what we think to be distinct events, like the Möbius strip, we come to realize that the apparently disparate issues do not exist on separate planes but rather, exist within a common space.

For me, I realized as a result of moving along what I perceived to be -- completely different planes-- My experiences of elementary mathematics learning, mathematics education classes, and university mathematics courses, was that they existed within a common conceptual space. Whether my experiences were that of a learner or teacher-to-be, they were all situated within the realm of sense of place for mathematics.

As a beginning teacher, I did not enter the classroom with a fixed image in my mind of mathematics as an activity that consisted of teacher demonstrations and student reproductions but instead, an image of mathematics as an ongoing engagement in which children and their teacher "adventure" in a classroom world of knowing together.[5]

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By focusing on relational qualities, ecological ways of thinking give rise to viewing the world as an integrated whole; a dynamic and fluid network in which all living and social-cultural systems are interconnected. The

Many, and perhaps *most* teachers begin their careers with the conviction that they will avoid those teaching practices that they found unhelpful or inappropriate when they were students. .... However, ... most beginning teachers quickly find themselves settling into patterns of teaching that are strikingly similar to the ones they intended to avoid.[1]

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[1] A. B. Davis, Sumara & Luce-Kapler, 2000, p. 41.

Turning The Soil:
The Beginning Years of Teaching

I began teaching second and third grade children in Richmond, British Columbia. When I entered the teaching field, I distinctly remember being eager on one hand, to inspire a more connected sense of place for mathematics in the classroom but on the other hand, careful not to become another "Mrs. Fibonacci".[1] Mrs. Fibonacci (a storybook character) is an elementary school teacher who loves and lives math to such an extreme that she makes learning an unbearable nightmare for the children in her class because everything turns into a mathematical problem for them to solve. For the main character, learning mathematics becomes a "curse" he cannot escape:

> "What if this keeps up for a whole year? How many minutes of math madness would that be?"
> "What's your problem" says my sister.
> "365 days x 24 hours x 60 minutes," I snarl.[2]

Like all beginning teachers, I devoted enormous amounts of time to preparing my lessons. I read journals for mathematics teachers, went to workshops searching for new ideas, collected 'real life' materials to connect the children's mathematics with familiar contexts, and designed interactive mathematical tasks that would engage every child in my class (all the while, being careful not to cast any curses!). But even though the children, their parents, and my colleagues seemed to be pleased with my efforts, I did not feel as if I was accomplishing what I had set out to do. My teaching and the children's mathematical learning still seemed disconnected.

Notes

1. Scieszka and Smith, 1995.
2. Scieszka, and Smith, 1995, p. 27.

What we conserve, what we wish to conserve in our living, is *Insight*, I believe, refers to that depth of understanding that comes by setting experiences, yours and mine, familiar and exotic, new and old, side by side, learning by letting them speak to one another.[1] what determines what can and what cannot change in our lives.[2]

[1] M. C. Bateson, 1994, p. 14.

[2] Maturana, 1997b, p. 5.

In the months that followed, I searched and scrutinized my mathematics program to find the source of my unease. I poured over the curriculum guides to be certain that I was teaching the correct concepts and the skills for the different grade levels. I revised the order and adjusted instructional sequences so that they moved more efficiently. I continued to tweak or elaborate the content of my lessons, depending on the needs of my students. Looking at the program as a whole, I felt that I was engaging the children in mathematical work that enabled their learning to be both "hands-on" and "minds-on" [1], and that I was opening spaces where mathematics could be integrated with other subject areas. Unable to find any obvious problems, I continued to proceed along the current course.

Then, several months later, I started to question the kinds of relationships that existed between my teaching and the children's learning of mathematics. I took a reflective step back and examined my mathematics program for a pattern or patterns that connected the children's mathematical learning spaces together [2] as a whole. In doing so, taken-for-granted ways of teaching began to emerge. I discovered that these were not only rituals unique to myself, but surprisingly, they were matter-of-fact ways of being for my colleagues too... even those of my schoolteachers! For me, these teaching practices had simply become **THE** way to facilitate children's learning in the mathematics classroom. Intrigued with this discovery, I decided to write my conventional manners of being down on paper. As I did this, it became apparent to me just how incredibly matter-of-fact they were and how deeply embedded in my teaching these "shared facilities" [3] had become.

- **Before beginning any lesson, sort the children according to their grade level. Once done, then proceed to teach each group a different mathematics lesson.**

- **When planning the curriculum for the school year, simply divide the mathematical concepts and skills for each grade into ten equal parts. By doing so, you can now allocate one of the ten school months to teaching "addition", one month to "subtraction", another month to "multiplication", and so on, until the end of June.**

- **Always make sure that consistent amounts of time are given to mathematics lessons. Schedule it in regularly each day (e.g., everyday between recess break and lunch hour).**

- **Mathematical concepts should be taught sequentially; from an informal, concrete stage to more formal, abstract ones. Teaching should facilitate the student's (the autonomous child) construction of knowledge in a conceptual to procedural to relational order.**

After reading this "must do list", it was apparent that the sensibility of wholeness and flow that I desired for the mathematics classroom did not exist. Instead, was one that embodied rigid, mechanistic, and disconnected qualities. These could be seen, enacted in my scheduling of lessons at same time everyday, my "taking inventory" of the mathematics curricula and then "packaging" them up into discrete "units" of instruction, teaching separate grade-specific lessons, and my always doing so in a manner that proceeded from the concrete to the abstract. The kind of place that I had intended to root and the one that had actually become embedded were in contradiction to each other. What served as tried and true rituals for teaching mathematics had unthinkingly become that which was furthering the "cultivation of discrete parts without respect or responsibility for the whole".[4] My teaching actions not only dismembered mathematics for the children but, on another level, I had also dismembered mathematics from itself. I say this because one might argue that my efforts to teach for the students' conceptual then procedural then relational knowledge could be viewed as in keeping with facilitating connected understandings. However, despite the fact that I did this in my teaching *within* each of the concepts and procedures, I was still teaching the concepts as separate "parts" and attention was not paid to enabling the students' connections among concepts, procedures or mathematical topics. One might alsoargue that put together, these individual "units" of instruction came to form a complet e mathematics program. This might be true; however, the "units" still

51

werc not fluid or dynamic but rather, discrctc and static.

In taking this reflective step back, I could see that it was not enough for me to design a well put-together mathematics program and I began wondering what I might do in order to engender a clear sense of flow in the mathematics classroom. Even though I could see how some of my invisible or assumed ways of teaching were undermining this, I did not know what kinds of "re-rooting" (conceptual or otherwise) were necessary.

What I had learned however, was just as Basso describes, place is not something that can be taken-for-granted– not even in the mathematics classroom. Place is primary and basic yet at the same time, it *is* far more complex than had originally crossed my mind! If places are indeed created and sustained through interaction, then the mathematics classroom as place, only exists in being. Further, it can be said that what distinguishes one mathematics classroom from another is its sense of place. Together, it is the kinds of mathematics that emerge from one's teaching and from children's learning that become the defining textures and tones of a mathematics classroom.

Notes

1. Liedtke, 1995, p. 51.
2. G. Bateson, 1980, 1991.
3. See "Vacancy" advertisement, p. 43.
4. Berry, 1983, p. 34.

"JUST HAND THEM DOWN THE MATHEMATICS" ... OR NOT?!

WHEN MATHEMATics is imagined and enacted as objectified, static knowledge that is to be traditionally passed down from one generation to the next, the teaching and learning of mathematics is disabled from ever becoming anything else. Under the air of "hand-me-downs", it is easy to understand why mathematics is taught and learned out of a sense of obligation or contempt rather than a sense of open desire or wonder and why, mathematics is all too often considered as that which is to be mastered rather than that which is to be understood. In com-moditizing mathematics, we make absurd the possibility for us as teachers and to those who we teach mathematics to perceive it as anything else but a fixed and inanimate entity. In this way of conceiving mathematics, we make it inconceivable for school mathematics to become something else other than just a collection of hand-me-downs.

The embeddedness of these images within one's taken for granted ways of thinking about mathematics not only make it natural for us to assume mathematics to be an inanimate "thing", but in doing so, displaces mathematics as that which exists "out there". Given this mindset, it is not surprising why a teacher would feel impelled to set the class onto a straight and narrow, one-way course so that the students too, become collectors of mathematics. Given this mindset, it makes sense to ingrain the ritualistic practice of "acquiring" mathematics into school unit and lesson plans, methods of assessment, and enact it in the classroom; product oriented practices which focus on "desired", "expected", or even "measurable" outcomes of instruction--that after instruction, the student will have "mastered" the mathematics taught in the lesson before "moving on" to the next part of the curricular course. Of course, the ways in which children are instructed to take possession of their mathematical hand-me-downs of concepts, skills, and even attitudes may vary. Still, "teaching by telling", engaging students in "hunting for", having them "seek out" "hidden" mathematics within "real" world contexts, and even "explorations" "designed" for children's discovery

53

of mathematics are all examples of teaching and learning forms that keep alive, this tradition of "handing down" of mathematics.

Moreover, when product-oriented ways of thinking about school mathematics are coupled with a "back to basics" mentality, the teaching and learning of mathematics become subjected to the weigh scale of "how much" in regard to the amount of mathematical facts and skills that children are to learn, and little or no emphasis is placed on such things as their mathematical thinking or understanding. Given this mindset, mathematical processes such as those identified by the National Council of Teachers of Mathematics[1] as problem solving, reasoning, communicating, connecting, and representing would likely be deemed "not essential" by most teachers. If viewed as "additional"[2] knowledge, teaching that attends to children's development of mathematical processes would then depend on whether or not the children have acquired first, the *pre*-specified mathematical facts and skills with which to "process" the mathematics.

The point here is that when children are taught to learn mathematics in the tradition of hand-me-downs and as a product oriented matter of collecting, hunting down, or retrieving pieces of knowledge, it creates the impossibility for mathematics to be taught and learned in ways that enable it to arise as living and animate.

Now, identifying the limitations of how mathematics exists in the classroom and the possibility of it becoming something else is all fine and good. But doing so means that the conversation does not end here. Rather, it opens up a whole host of questions that require further interrogation such as:

- How can an ecological way of thinking help us to *reconsider* such taken-for-granted perceptions of classroom mathematics and *re*-imagine a more responsive view for the teaching and learning of it to exist in the classroom?

- What shifts in thinking become necessary in order to reimagine classroom mathematics as being something other than a line of hand-me-downs from teacher to child?

- What could it mean if we assumed mathematics to be "embodied"?

- How could mathematical problem solving, reasoning, communicating, connecting, and expressing be understood as something other than additional knowledge?

Notes

1. NCTM, 2000.
2. Baroody, 1993.

54

Part 2: Settling In(to AnOTHER Space)

Jennifer picked up the newspaper and quickly leafed through it. Slowing down as she came towards the "Letters to the Editor" section, she saw that someone had responded to the "**JUST HAND THEM DOWN THE MATHEMATICS ... OR NOT?!**" article she had been reading.

MOVING THINKING SPACES AND REASSESSING OLD FURNITURE

IN RESPONSE TO LAST WEEK'S ARTICLE: I TOTALLY agree with the author's arguments and the questions are important ones in making positive changes to the math classroom. My concern though, is that real changes can't happen if this job of rethinking and "re-imagining" mathematics in the classroom is approached with the attitude of 'getting rid of' or simply 'adding onto' what's already there!

What the author didn't say was that it's not about taking ecological ways of thinking and coordinating them like new pieces of furniture into a tired and run down living room so that we can update our mind-spaces and have them look more current. It's about moving from invisible and mechanistic places of knowing to ecological ones. It's about rediscovering and assessing the all too familiar furnishings that have been set about (classroom) mathematics, and asking ourselves, "how well do these furnishings go with this space?"

All for opening new spaces,
Joel

"*...getting rid of...*" mumbled Jennifer as she read Joel's letter "*...adding on to what already exists...* no. Definitely not." And so she continued on, reading bits of the letter silently in her head and every so often, sputtering out particular words or phrases.

"Precisely!" Jennifer said with matter of fact certainty. Enabling deep changes in her teaching was not about changing out of certain "approaches" and slipping into

new ones. She agreed with Joel that what she needed to do was to examine her mathematics teaching from where she was now (conceptually) standing. Jennifer wondered what she might see and see differently from an ecological perspective. What kinds of furnishings had become so comfortable and such an integral part of her mathematics teaching that they were now permanent and perhaps, invisible? Jennifer questioned whether they would even suit an ecological mind-space. And more to the point, she was anxious to know what kind of place, what kind of *oikos* she was "mapping" out for her students' mathematics. But where to start?

Jennifer pondered for several days about the specific direction or vantage point she should position her thinking in order to examine these issues. It was only in doing so that she realized there was one theme that kept emerging. It was recursive in the sense that it was the "place" in her thinking, if you will, where Jennifer found herself returning again and again. Moreover, it was not until this moment that she recognized the ever-presence of this location. Here was the place where she existed both *in* and *away* from the classroom. When she described it to me, I immediately named it Jennifer's "in-between space". Not because it was a space of indecision for her but, rather, the in-betweeness had to do with how her teaching and her research[1] co-emerged and co-evolved. For Jennifer, teaching and research neither existed as separate entities nor did they move sequentially from one to the other as she had previously thought.

"As a pre-service and a beginning teacher I understood research to be something that was done at the University that produced theory and in turn, became a *tool* that I could use in my practice of teaching mathematics." she explained. "But now, what comes to mind is an image of teaching and research as continually interacting with one another... they flow into and give rise to one another. This to me is REAL teaching. It's *praxis* and not simply establishing and maintaining of one's teaching *practice.*"

As I listened to Jennifer describe how her view of mathematics teaching and research had changed, I realized that to characterize her in-between space as the location where the two met or intertwined would be to miss the meaning altogether. They did not *meet.* They *were* each other. It was clear that for Jennifer teaching mathematics and her study of it were *inseparably* interconnected. In a complex yet circular manner, Jennifer considered them to be interacting, co-evolving systems– necessary parts of each other. Furthermore, the distinction she made regarding her shift from teaching as a *practice* to teaching as *praxis* reveals that mathematics teaching as praxis is not simply a routine that one performs but instead, requires active *engagement with*; it implies a way of being that is critically reflective and reflexively responsive. And evoked from within an ecological realm is the importance of being ever-mindful of how one's knowing, actions, and identity in

teaching mathematics are firmly rooted in what we have already lived and have become embodied in how we are living, and what we will live. In other words, teaching as praxis acknowledges the simultaneity and the complex circularity of that which *unfolds* from one's teaching is also necessarily *enfolded* with all that interacts with it. This space certainly was not an "in-between" space. It was not a conceptual space located somewhere in the middle of teaching and research. Really, it was anOTHER space.

Jennifer added, "I see this kind of reflexivity as being key in attempting to understand how it is that my teaching and research give rise to each other"

I then asked her, "If you had to describe in your own words, the 'guts' of it all in a nutshell, how might you do so?"

"Simplifying the complex?! Hmm... let's see. I suppose I would have to say that in a nutshell... for me that is, teaching learners mathematics and learning what it means to teach mathematics flow together."

And it was in this spirit and in this other place that Jennifer began the process of bringing her teaching into the foreground and encircling it within ecology.

Notes

1. The term "research" is meant to encompass both theoretical work and work done "in the field."

Jennifer naively assumed that by putting her teaching out in the open and inside the circle of ecology, immediate answers to her questions would be revealed. However, as the days passed, it was only her impatience that became apparent. Frustrated, she picked up a book that she had been reading and turned the page. There in black and white print was the reminder she needed.

> **Letting myself get written by a place.** Bodily scars as the agelines in the droopy skin on the backs of my hands betray. Legibilities of having, once again, lived-through.
> **Sitting squat. Spending time. Waiting. Reserve. Quiet. Composure. Patience. Letting the boredom arrive. Wasting time. Doing nothing with great deliberateness. Collecting dry bones. Boredom: this is one great little demon we have banished from the discourse of authorship and expression and self-annunciation. Deliberately spending time in the old place, feeling through moist weaknesses:**
>> Perception of opportunities requires a sensitivity given through one's own wounds. Here, weakness provides the kind of hermetic, secret perception critical for adaptation to situations. **The weak place serves to open us to what is in the air.** We feel through our pores which way the wind blows. We turn with the wind; trimmers. An opportunity requires... a sense... which reveals the daimon of a situation. The daimon of a place in antiquity supposedly revealed what the place was good for, its special qualities and dangers. The daimon was thought to be a *familiaris* of the place. **To know a situation, one needs to sense what lurks in it.** (Hillman, 1987, p. 161)[1]

Although Jardine's description details how he readies himself to write, his practice of dwelling and "keeping watch" was exactly what Jennifer needed to do. It was obvious to her now that she did not know what aspects of her mathematics teaching needed interrogation and so to go searching for something that you do not know became a ridiculous endeavour. Jennifer decided it best if she let her mind wander back to that "other" place. Dwelling there-- in that place she described where teaching and learning and what it means to teach mathematics flowed together-- she waited patiently, all the while, keeping watch for what "lurked" in it.

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[1] Emphasis added. Jardine, 1999, p. 35.

Surfacing & Noticing

So Jennifer, what came out?

I wouldn't say that anything really CAME OUT... it was more like bubbles making their way to the surface of the water-- and then bursting in the moment of recognition. You know, like, there's one! There's another one! Once you've caught sight of one of them all of a sudden, there they are! And you see them for the first time not because they magically appeared. They'd been there the whole time and were only invisible because you hadn't ever been able to notice them before.

What do you mean? You hadn't ever been able to notice them before?!

Well, even though my focus was on ecology and the mathematics classroom, up until this point, it wasn't specifically about MY teaching.

Does it need to be?

I think at some point it has to be. You see, it was only when I began questioning the kind of place I was creating for my students' learning of mathematics that I saw the need to move deeper-- when I say deeper, I mean delving into the inner layers[1] of my teaching-- not just being attentive to what's developing in my present teaching, but what's already been developed, what's become its inner core or, the roots of my mathematics teaching. Had I not realized this, any growth I made would most likely be superficial because I wouldn't have been considering the whole of my teaching-- I wouldn't have dwelt long enough to notice what was there. So like Joel had mentioned, my ecological ways of being would have been at best, "add ons".

Okay. Just a minute. You figured out you needed to look at the layers of your teaching but you still hadn't figured out what you needed to be noticing, right?

That's right. MORE dwelling! What I did know was that by exploring the layers of my teaching, I might be able to articulate why I was so uneasy about the sense of place I was creating for my students in the mathematics classroom... why it didn't feel right. But still I had no idea what I should be examining in my teaching.

So what did you do?

Well, this time, I took the whole of my mathematics teaching back to where I'd been-- you know, into the space of language and languaging.

Then what?

As I surrounded my thinking with theoretical literature on languaging and the pervasive nature of language, I began by asking myself, So how do these concepts inform my understanding of teaching mathematics?

And...?

Certain key ideas began to pool together. In fact, they were direct quotations from the books and articles I'd been reading.

Such as?

Well, like:

"Language THINKS US as we think within language." [2]

"METAPHOR IS NOT A MERE EMBELLISHMENT; IT IS THE BASIC MEANS BY WHICH ABSTRACT THOUGHT IS MADE POSSIBLE." [3]

The map is NOT the territory. [4]

"...[language] does much more than construct concepts about the events and things in the world: it TELLS us what sorts of concepts we ought to construct." [5]

And these "pooled" together because they were all...

They all had to do with the metaphorical nature of how we think.

60

So it was just these specific authors' works?

Yes and no. It was these four quotations that kept making their way into my mind-space of language and languaging but they weren't the only ideas I was thinking with. They'd emerged from a background of other authors' work such as Sfard[6], Abram[7], Capra[8], M. C. Bateson[9], van Manen[10], Jardine[11], Orr[12], Maturana and Varela[13], Lakoff and Johnson[14].... but yes, it was these specific metavoices that really helped to pinpoint my position of noticing-- so that I could begin to examine my mathematics teaching.

And so what was your position or perspective of noticing? Can you explain it to me?

Let me see. Well for starters, it was directed towards the way in which metaphors become embodied in our forms of knowing, our actions, and our identities. Remember when we were talking about language?

Yes.

By "metaphorical language" I mean, the spoken, written, spatial (how phenomena are portrayed to exist), temporal (how time is conceptualized), and symbolic forms of communication that distinctively structure one's teaching. So briefly, there it is. My point of noticing was to examine the metaphorical "furnishings" if you will, of teaching mathematics and directly related to this, the kind of place I was bringing forth in the classroom.

Do you think that metaphorical forms of communication can really impact one's mathematics teaching in such a profound manner?

Yes I do. Just take a moment to think about it: Theoretical "FOUNDATIONS", instructional "UNITS", conceptual "FRAMEWORKS", "NETWORKS", learning "JIGSAWS", cognitive "STRUCTURES", "SCAFFOLDING", "BUILDING" knowledge... and so on. Metaphors. We are constantly reading them, hearing them, using them, and thinking with them. So usual they become like what Joel wrote, permanent fixtures in one's mind and over time, we no longer notice their presence. Unquestioned, these metaphors become embedded in our taken-for-granted language-- language used to conceptualize mathematics teaching and learning. Language that was directly impacting my teaching, my students' learning, and the kind of mathematics that was emerging.

61

Wait a minute. Going back to what you just said... you claimed that metaphors were affecting the KIND of mathematics that was emerging. Surely, math is math! Metaphors have nothing to do with the nature of mathematics.

Oh but they do! My previous mind-structure predetermined the mathematics content that was to be taught and learned. Because of this, I wasn't aware of or didn't pay any attention to emergent kinds of mathematics[15], integrative kinds of mathematics[16], or individual-within-collective mathematics.[17]

So what?

Well, it does matter. If you're not aware of how metaphorical language is thinking you as you think with it, it's difficult to understand how it functions and becomes an enabling and disabling feature in our manners of thinking and ways of being.

Okay. So make sense of it for me at the classroom level. I still want to know what surfaced for you when you interrogated your teaching. Let's go back to my original question, what came out? What surfaced as a result of your dwelling in this theoretical realm and in this other space of yours?!

What came to the surface of my consciousness were metaphors that carried with them very vivid meanings of how I thought, taught, and identified my role in the mathematics classroom. For the first time, I began to understand how UNconscious these metaphors were.

What do you mean?

Well, because I hadn't realized how pervasive and taken-for-granted they were, the metaphors existed below the surface of my consciousness. They were definitely there but up until this point, I didn't have a theoretical way to examine my teaching and so, it was impossible for me to notice these metaphors. They were simply matter-of-fact ways of conceptualizing and enacting my teaching.

For example, how you MECHANISTICALLY separated and organized the mathematics content for each grade level into teachable UNITS of instruction?

Yes.

And so it was these embedded metaphors that you began to notice rising to the surface?

Exactly! And how it happened was just as I described for you at the start of our conversation-- like bubbles making their way to the surface of the water and punctuated by them bursting as soon as I spotted them. What I also learned as I searched to identify my taken-for-granted metaphors was that they arose from the theoretical languages of my undergraduate mathematics education and Ministry documents[18], teacher texts, and mathematics literature I was working with.[19] You see, it was here that the relationship between my activities in reading and writing and how I envisioned my work in the classroom... that is, planning, teaching, and assessing children's understanding of mathematics became clear. Almost instantly, the metaphors that'd been totally invisible were now so obvious. Because they were visible, I could see the metaphorical images embedded in everything from the way I imagined the mathematics class to my conceptualization of mathematics curricula.

What kinds of metaphors? Give me some examples.

"HANDS-ON MINDS-ON"

"the AUTONOMOUS child"

"CHILD-CENTRED approach"

"The role of the teacher is to FACILITATE learning"

"A GUIDE"   "The teacher is the INITIATOR"
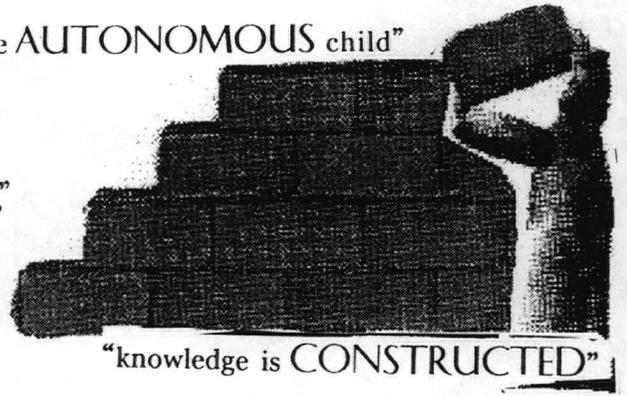
"knowledge is CONSTRUCTED"

"Children are unique INDIVIDUALS"

"REALITY IS CONSTRUCTED by the individual"

"EXTERNAL environment"
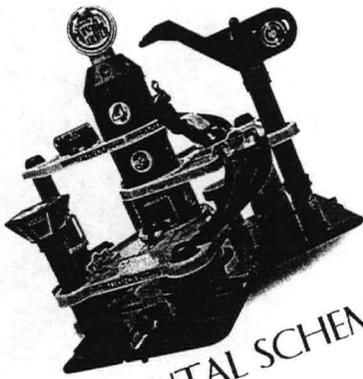
"MENTAL SCHEMAS"

"COMMUNICATING"
"PROBLEM SOLVING"
"REASONING"

"MATHEMATICAL POWER"

"CONNECTIONS"
"REPRESENTATIONS"

"PROCEDURAL understanding"
"CONCEPTUAL understanding"

"SCAFFOLDING"

"INSTRUMENTAL understanding"

"RELATIONAL understanding"

"FORMAL understanding"

"from CONCRETE TO ABSTRACT"

Stage 1: Child's Language
The natural language a child uses to describe the concept in a familiar situation, often a real-world story

| Modeling | → | Creating | → | Sharing |

Stage 2: Material Language
The new language that might be used with concrete or pictorial materials as a child acts out or represents the real-world story

| Modeling | → | Creating | → | Sharing |

Stage 3: Mathematical Language
The use of a few words to record the language that describes the action of the materials. This stage leads to using more specific mathematical language.

| Modeling | → | Creating | → | Sharing |

Stage 4: Symbolic Language
The use of mathematical symbols as an even shorter way of recording action

| Modeling | → | Creating | → | Sharing |

(R. Reuille Irons & C. J. Irons, 1989)

64

mathematics class

"EXTERNAL environment"

"AUTONOMOUS INDIVIDUALS"

"CONSTRUCTING and REconstructing"

their MATHEMATICAL THINKING"

Okay. For instance, take the **MATHEMATICS CLASS** As I imagined it, the class was composed of myself and the students as **AUTONOMOUS INDIVIDUALS**. Anything outside the individual was considered to be part of the **EXTERNAL** environment.

MATHEMATICS itself existed as a **CONNECTED YET FIXED** body of knowledge. It was made up of separate **STRANDS** of algebra, geometry, numbers and operations, measurement and so on.

mathematics

connected-yet-fixed

SEPARATE "STRANDS"

65

mathematical learning

s•e•q•u•e•n•t•i•a•l s•t•a•g•e•s

uni-directional →

abstract ↑

concrete

(triangle with text) hi er arch ical
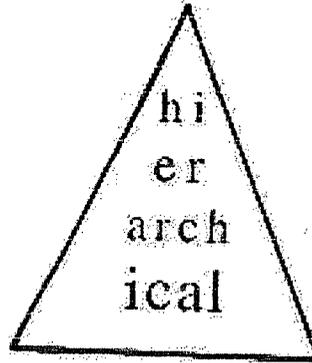
I envisioned meaningful **MATHEMATICAL LEARNING** *to occur in a* **UNI-DIRECTIONAL** *and* **HIERARCHICAL MANNER**; *beginning at an informal, concrete stage and then moving towards* **HIGHER** *levels of more formal, abstract stages of understanding.*

MATHEMATICAL UNDERSTANDINGS

were knowledge **STRUCTURES** *or* **FRAMEWORKS**. *For me, the bigger and more elaborately constructed the structure was, the better the individual's understanding.*

mental KNOWLEDGE STRUCTURES the BIGGER the BETTER

mathematical understanding

Based on the premise that if knowledge was about building structures,**TEACHING** mathematics for me became an activity of directing my students' thinking towards **PRE-DETERMINED** learning outcomes in regards to what they *SHOULD* know and facilitating the ways in which they should **CONSTRUCT** such knowledge.

teaching mathematics

MOVING STUDENTS ON A COURSE towards **PRE-DETERMINED** learning **OUTCOMES** ⟶

curricula
jigsaw puzzles

And as well, **CURRICULA** were **JIGSAW PUZZLES** for teachers to assemble by piecing together concepts and skills set out by the Ministry and other **STANDARD** mathematics documents.

mathematical concepts

mathematical skills
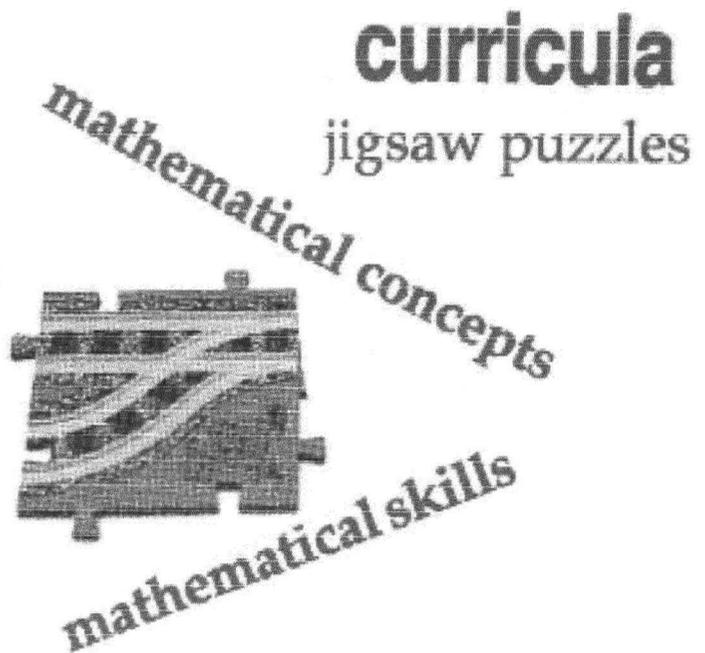
I agree that these are very strong metaphors... mechanistic, linear, and hierarchical ones to be sure. But did they really affect the way you taught mathematics? and if so, how?

Having made them visible, I also asked myself this same question: Were these metaphors simply figures of speech or were they more than that? Were these metaphors truly powerful forms of language? Language that not only shapes how one perceives mathematics teaching and learning to be but also, profoundly impacts how such events COME to be. So I turned my attention-- a little apprehensively, I must admit, towards examining if and how, these metaphors I had identified existed in my forms of teaching.

And....?!

Rather abruptly I came face to face with the notion that all knowing really is doing and all doing really is being! And what's more, how unthinkingly natural it all is.

You see, because I viewed mathematics to be a connected yet fixed body of knowledge and curricula were puzzles to be assembled, my goal in creating an integrated maths program was to connect the different pieces of mathematics together to produce a "logical" and "coherent" picture for the students. In thinking so, it made sense for me to insert their lessons in-- in a piecemeal fashion for an hour each day between recess and lunch. And in keeping with the view that mathematical learning was sequential and hierarchical, because I taught a multi-age class, it made it necessary to sort the children according to their grade level and teach two different lessons. My role as their teacher was to guide each student's learning in a manner that enabled them to construct sturdy frameworks of understanding; ones that began with concrete foundations of experiential knowings upon which more formal, symbolic representations were built. I even remember being asked on several occasions as to how I defined myself as teacher!

And, how did you describe yourself?

I was the children's FACILITATOR... THE initiator of learning opportunities or to stay with the metaphors, the provider of building materials.

After all of this, what was your reaction in realizing that your teaching was indeed enactions of the mechanistic linear, and

hierarchical metaphors that you were thinking with?

If you'd asked me this question before finding this all out, I would've said that I'd probably be shocked, disappointed... even horrified if I were to discover that my teaching contradicted the ecological perspective that I thought I was embracing.

What do you mean? You weren't shocked, disappointed, or horrified? Really. I'd think that burying one's head in the sand so to speak, would be a common reaction.

Strangely enough, it was more of an affirmation... finally being able to recognize the metaphors that had existed for so long beneath the surface of my consciousness and then to see the embodiment of them in my teaching-- relief through affirmation... yes, that's what it was. For some time, I'd had a hunch... a gut feeling that the connected sense of place I was trying to create in the math classroom wasn't quite there... but I couldn't put my finger on it as to why.

But you had, hadn't you? You had identified rituals in your teaching that were linear, mechanistic, and hierarchical?

Sure, I was able to point to teaching actions that I'd unthinkingly inherited and see them as problematic... such as planning a program by dividing and ordering the mathematical concepts and skills for each grade into a September through June sequence! But doing so only indicated forms of teaching I deemed as undesirable. It still didn't provide me with any kind of understanding as to what was giving rise to them or how I might go about creating a more ecological sense of place for my teaching and my students' learning.

Yes, that's right. I agree.

It was only when I moved deeper into my teaching and examined my metaphors... Hmm.... how can I describe the process to you....

For me, this process was very much like "[f]ingering the contaminated wound" [20]-- explicating my metaphors and then watching them fester --how the metaphors were being enacted in my mathematics teaching. Yes, that's an accurate image.

How horrible that sounds!

The image certainly impresses my experience as incredibly uncomfortable-- even painful. And in some ways, it was. Finding out that you're doing exactly the opposite of what you are trying to do is definitely a distressing, uncomfortable mind-space to be in... but at the same time and in a different way, I by no means considered the study of teaching as pathologic.

Although that's how I think most people would interpret your description.

I know... but no. To come to this place in my thinking was critical. The uncomfortableness of it all was not a prompt for identifying and remedying a problem so much as it was an opening for me to arrive at a new place of understanding. You see, I considered the dis-ease of these events to be integral and vital to my growth in teaching mathematics. It was because of dwelling in this mind-space that I was able to grab hold of what I could only before express as being a hunch or a gut feeling and now, I was able to actually put words to it and finally say THERE *it is!*

Making the invisible visible! When you describe your metaphors and explain how they gave rise to your forms of teaching, it really elucidates the point you've been trying to make; that the metaphors and metaphorical patterns with which we think have everything to do with one's teaching of mathematics and the sense of place that is created in the classroom. What also becomes clear is that even though you wanted to create a connected sense of place through creating an integrated mathematics program, the metaphors you unconsciously rooted in your mind critically disabled the possibility for a more organic or ecological kind of integrative mathematics to emerge. The metaphors only allowed for mechanical piecemeal forms of teaching and learning-- definitely not those that are dynamic, flowing, or unpredictably open. It's exactly as you expressed earlier in our last conversation-- that

70

teaching learners mathematics and learning what it means to teach mathematics really do flow together. That all said, going back to Joel's question, now that you'd figured out what "furniture" didn't suit your ecological "living room", how did you go about finding furnishings that would?

Notes

1. Aoki, 1983; 1993; MacDonald and Purpel, 1987; Uhrmacher, 1997.
2. Graduate seminar by C. A. Bowers, July, 1998. Paraphrased from Heidegger, 1962, p. 188-203.
3. Emphasis added, Lakoff and Núñez, 2000, p. 39.
4. G. Bateson, 1972.
5. Emphasis added, Postman, 1995, p. 181.
6. Sfard, 1997, 1998
7. Abram , 1996,
8. Capra, 1993; 1996.
9. M. C. Bateson, 1994, 1996, 2000.
10. van Manen, 1986, 1991.
11. Jardine, 1994, 1999, 2000.
12. Orr, 1992, 1994.
13. Maturana 1988a, 1988b, 1991, 1995, 1997a, 1997b, 1998, 2001; Maturana and Varela 1980, 1987; Varela, 1999.
14. Lakoff and Johnson, 1980, 1999.
15. For example, see page 220, 233, and 322.
16. For example, see page 236 and 250.
17. For examples, see page 155 and 280.
18. For example, Ministry of Education, 1990. As well, up until 1995, each subject area was published by the Ministry as a separate curriculum document.
19. That is, texts and literature which I considered as being situated within "constructivism".
20. Jardine, 1999, p. 37.

Even Deeper

Enactions exposed, Jennifer realized that the metaphors she had rooted in her thinking had *become* her mathematics teaching. It was impossible for her to consider them as merely figures of theoretical speech. Differently, she now understood them to be the "consensual domains" in which her patterns of thinking and forms of teaching mathematics were specified. In a very real way, these metaphors and metaphorical manners were her rituals for place-making in the classroom. Jennifer knew that her taken-for-granted ways of teaching mathematics were not engendering the ecological and fluid forms she wished to enact. Even so, she still felt a sense of awkwardness.

Jennifer had arrived at a new place of knowing in the other space.

It was in these moments of making sense of the limitations of her metaphors and knowing that she wanted to enact ecologically coherent ones that she was also confronted with the fact that one cannot simply change by "exchanging" what one is thinking or doing in the classroom for something else.

AGAIN *I'm reminded that learning what it means to teach mathematics is not an automatic process. It's not smooth, it's not straightforward, and it certainly doesn't appear on demand.*

KEEPING WATCH WHILE DWELLING REQUIRES PATIENCE.

Before Jennifer could begin to re-imagine metaphors that were ecologically sound and work towards rooting them within her classroom praxis, not only did she need to exercise a mindful kind of patience but she also needed to move even deeper into that other space. She had to critically question, assess, and then provoke shifts in her thinking. This included an inquiry into mathematics and mathematical understanding.

How is mathematics conceived? And in doing so, what kinds of being does it become?

PROCLAIMED THE *QUEEN OF SCIENCES* PRISTINE IN ITS SACRED CUSTOMS OF PRECISENESS' AND LINEARITY... MATHEMATICS IS ABSTRACTLY ELOQUENT AND REFINED... IT OFFERS US TRUTHS SO ABSOLUTE SO PURE SO UNAMBIGUOUS... IN ITS PREDICTABILITY AND CERTAINTY AND EXPLICIT INFALLIBILITY, IT IS NOT TO BE QUESTIONED NOR HELD ACCOUNTABLE FOR ANYTHING EXCEPT ITSELF. TREASURED HEIRLOOMS OF LOGIC AND RATIONAL KNOWLEDGE... UNIVERSAL AND TRANSCENDENT... MATHEMATICS EXISTS IN THE REALM OF OBJECTIVITY... IT IS NEUTRAL AND LIVES WITHOUT REGARD FOR US... OUR BELIEFS... OUR VALUES... OUR ACTIONS... OUR CULTURAL WAYS . . . IT LIVES "OUT THERE" AS THE FOUNDATIONS OF THE UNIVERSE

ORDERING AND STRUCTURING THE UNIVERSE

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the
flowers
the snowflakes
the ferns and the trees
creative and beautiful
the stars and the planets
the shells of snails
the orbits
us

·

·

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rooted in our minds

mathematics arises from our biological, social, cultural humanbeing in the world

either... but not only "IN HERE" but not "OUT THERE"

mathematics is not a THING but a way of perceiving... "making visible, the otherwise invisible..."[3]

structure of nonlinear dimensions and never an END PRODUCT

but a SHAPE-SHIFTER of number, of form, of action... whether static or dynamic it is fluid and everchanging... quantitative or qualitative... real or imaginary... visual or mental... purposeful or fanciful... complex and diverse... interacting, shaping, and being shaped...

a science of patterns[2]...

a living, growing system openly multiversal and constitutive... mathematics exists in the realm of (objectivity)[4] ...not there and not neutral... it lives in relation to us...our beliefs... our values... our actions ...our cultural ways[5]...

it lives here... in dynamic relation with our humanbeing.

[1] Lakoff & Núñez, 2000; Wheeler, 1967.
[2] Devlin, 1994, 2000.
[3] Devlin, 2000, p. 92.
[4] Bunnell, 2001; Maturana, 1988b.
[5] A. B. Davis, 1995, 1996, 2001; Jardine, 1994; Lakoff & Núñez, 2000.

Does the way one portrays mathematical understanding matter? And if so, how does it shape one's perception of what it means for learners to understand mathematics?

Theoretical Portraits of Mathematical Understanding

The topic of mathematical understanding continues to be one of critical focus for mathematics educators. As a result, there exists an array of models and interpretations that address aspects of mathematical understanding from the very general to the very specific. Two themes inherent in this particular collection of works are that of *cohesion* and *tension*. Interestingly, there is a general agreement among mathematics educators of what "good" mathematical understanding entails, while at the same time, the ways in which educators portray the nature of mathematical understanding, how it comes to be or should be developed, and the forms that arise create a contrast against one another.

First, several works from perspectives situated within what can be considered to be part of a constructivist realm are showcased. Here, one will get a sense for what it means to frame mathematical thinking and learning within this theoretical discourse and how it portrays understanding as the building and rebuilding of mental schemas. Second, research that seeks to move away from linear or constructivist minded frameworks in order to interpret children's mathematical understanding as more holistic and dynamic are explored. Finally, works that are located within an enactive realm and that strive to illuminate mathematical understanding as being a co-emergently complex phenomenon are examined and discussed.

EXAMINING MATHEMATICAL UNDERSTANDING

FROM A CONSTRUCTIVIST PERSPECTIVE

Model of Intelligence and Forms of Mathematical Understanding

Skemp's (1979) model of intelligence offers a qualitative means for describing individuals' mathematical understanding. The two-level, cybernetic model (see Figure 1) consists of two internal systems: delta-one and delta-two. Delta-one is defined by Skemp as a sensori-motor system that directs an individual's physical mathematical actions based on information received from the external environment. Delta-two serves as the site where construction and reconstruction of an individual's mental mathematical schemas take place. It is this process of schema construction and reconstruction that allows for the mathematical functioning of delta-one to occur. Thus, it is "the construction and testing by delta-two *within* delta-one of the schemas and plans that delta-one must have to do its job" (Skemp, 1979, p. 44). It is here in delta-two where Skemp identifies mathematical understanding as developing. The specific ways in which these internal systems function together is described by Skemp (1978, 1979) as evidenced through one's "instrumental", "relational", and "formal" or, "logical" forms of mathematical understanding.
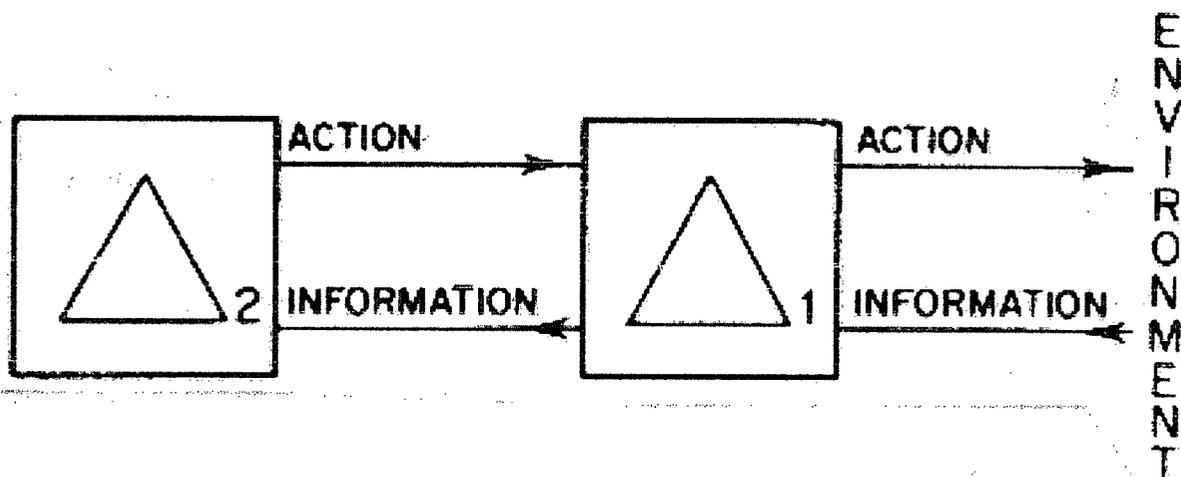

Figure 1. Skemp's two-level cybernetic model of intelligence.

Instrumental Understanding

Skemp explains instrumental understanding as being a function of delta-one. This form of mathematical understanding enables a person to correctly apply previously learned procedures to the solving of mathematical problems. Instrumental understanding in essence, is the learning of "what to do" with the mathematics. It does not however, enable the learner to develop conceptual grasps for interpreting why a method works or what the symbols might mean. (Skemp, 1978, 1979). For example, by remembering the words and the order of the letters in the acronym, "BODMAS" (**B**rackets **O**f **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction), a person can carry out the correct sequence of numerical operations for solving complicated calculations.

The way in which learners are able to develop instrumental mathematical understanding is through rote methods of demonstration and further practising of a particular procedure or set of skills until they become routine. Although the cognitive structures in delta-two that result from instrumental learning enable a person to manipulate mathematical symbols and rules, the person's actions remain restricted because the connections that are formed in the delta-two schemas exist only as relationships between symbols and rules, not among mathematical concepts. The extent to which one is able to apply one's instrumental understanding to different mathematical contexts then remains limited to combining and performing procedures in the prescribed sequence that they were learned.

Relational Understanding

Relational understanding is evidenced by a person who is able to generate appropriate strategies for solving mathematical problems. This form of understanding involves the individual making sense of why particular methods of mathematics may work and why others may not be effective when solving certain problems (Skemp, 1978, 1979). In other words, relational understanding implies the learner's knowing of "what to do" and "why" certain mathematical actions prove to be effective. The manipulation of mathematical concepts and schemas is described by Skemp as a function of delta-one while the individual's conscious or unconscious

reflection of these concepts and schemas takes place in delta-two.

Unlike instrumental understanding, relational understanding gives rise to schemas that connect mathematical concepts with procedures. This form of learning is thought to develop as an individual alternates between activities of interacting mathematically in the external environment and mentally reflecting on these experiences. As relational understanding can only be achieved through the individual's conceptual integration of mathematics, this process requires more time than does instrumental learning through rote methods. However once acquired, relational mathematical understanding is seen to be more flexible because such knowledge is connected to mathematical concepts and not to specific contexts, it can continue to develop. And unlike instrumental learning where an individual recognizes a mathematical problem and then applies and performs a prescribed procedure to solve for it, an individual with relational understanding can derive mathematical procedures through conceptualizing or comprehending the task at hand. Mathematical symbols do not exist simply as abstract objects on which an individual performs actions but rather, they carry meaning for the individual in that the symbols are objects to which conceptual understanding can be attached and enable the construction of connected schemas of concepts and skills.

Formal or Logical Understanding

Skemp (1979) characterizes this form of understanding to be present when an individual consciously connects symbolic mathematical language together with meaningful ideas and logical reasoning. This can occur as either a delta-one activity or in both delta-one and delta-two. If a person possesses delta-one logical understanding, the learner is able to reflect on his or her mathematical actions through an "if... then" type of rationalization; that is, "if I perform the correct methods to solve a given problem, then the result should be correct" type of thinking. On the other hand, Skemp describes logical understanding that takes place in both deltas as being when an individual is able to show through formal mathematical demonstration or proof that the mathematics that has been applied makes sense through inferences that connect the given

premises of the problems to established mathematical axioms or theorems. This type of formal functioning that occurs within delta-two enables the individual to become aware of the connections between delta-one and delta-two activities and establishes consistency between the individual's mathematical schemas and solutions.

Last, within each of these three forms of understanding– instrumental, relational, and logical-formal– there can be "intuitive" and "reflective" dimensions to the learner's mental functioning (Skemp, 1979). Intuitive mathematical functions are characterized as spontaneous processes that occur in delta-one and do not necessarily include the delta-two system. When intuition occurs in both deltas, this gives rise to the unconscious reflection of the individual. However, it is only when the individual is consciously aware of his or her activities in both the first and second deltas that this process can be considered reflective.

Hiebert's Views On Mathematical Understanding

Hiebert and Wearne (1992, 1996) apply a constructivist definition found within cognitive psychology (Brownell, 1935; R. B. Davis, 1984; Hiebert & Carpenter, 1992; Lesh, Post, & Behr, 1987) to define their view of what they consider to be mathematical understanding. They refer to it as the learner's development of mental connections and formation of networks that serve as representations of mathematical ideas. For example, Hiebert and Wearne would consider a well-connected understanding of multi-digit addition to be a network that consists of the child's connected knowledge of concepts regarding place value, basic facts, and the ability to generate effective procedures to deal with the task at hand. They believe the process by which understanding of mathematical ideas occurs is an unpredictable, recurrent, and nonlinear progression. Furthermore, the flexibility of an individual's mathematical understanding is seen as an indication that the learner has constructed mental networks that have many points for external information to enter and to trigger the individual's successful adaptation, acquisition, and retrieval regarding appropriate strategies to solve mathematical problems.

85

Building Bridges to Connect Informal and Symbolic Mathematics with Student Understanding

Hiebert's independent and collaborative research seeks to understand the relationships that exist between children's conceptual understanding of mathematical concepts and their external abilities to recall and modify existing procedures, construct suitable methods, adopt prescribed rules, and use symbolic mathematical language with understanding (Hiebert, 1989; Hiebert & Carpenter, 1992; Hiebert et al., 1996; Hiebert & Wearne, 1993, 1996). A common thread that runs through Hiebert's research concerns itself with previous studies (Carpenter, Hiebert, & Moser, 1983; Lindquist, Carpenter, Silver, & Matthews, 1983; Hiebert & Wearne, 1986) and similar arguments that are raised by other mathematics educators such as Usiskin (1996), Pimm (1987), and Carraher, Carraher, & Schliemann (1987) regarding students' lack of connection between symbolic mathematics found in the classroom and that which occurs in their everyday life. His work emphasizes the need for less formal mathematical representations to serve as a means by which meaningful connections for children's understanding and application of symbolic mathematics can be developed.

As well, Hiebert asserts that in order for students to be able to use the symbolic language of mathematics to their advantage, essential connections regarding their informal, experiential knowledge must be recognized by teachers and made explicit to their students (Hiebert, 1989). Hiebert's framework (1989) (see Figure 2), which identifies three critical sites for linking written symbols with understanding, highlights the necessity for children to develop meanings for the ways in which symbolic mathematics can be used as a powerful language in solving problems. This model makes clear the need for students to "make the symbols work for them" instead of "working with the symbols". Stressing the importance for the learner's utilization and integration of out-of-school mathematical behaviours[1] with school mathematics, Hiebert's model clearly identifies that the final stage of mathematical understanding should not be the learner's ability to perform symbolic mathematics but rather, the child's meaningful understanding for and their ability to reintegrate their use of symbolic mathematics into a variety of settings.

[1] For example, interpreting, judging, devising, estimating, and evaluating.

Figure 2. Summary of Hiebert's (1989) three sites for linking symbolic mathematics with understanding.

Problemetizing Children's Learning of Mathematics

Teaching methods considered to foster rich, connected schemas of mathematical understanding are ones that enable children to "problemetize" their mathematics (Hiebert et al., 1996, 1997). Hiebert et al. distinguish problemetizing mathematics as being different from problem solving approaches to learning that imply teacher demonstration and children's imitation of identifying key words in a problem, selecting an appropriate method, and performing prescribed calculations to solve a task. Rather, problemetizing of mathematics is viewed as facilitating deep mathematical understanding because it focuses on students making sense of and developing meaningful relationships between their mathematical ideas and

mathematical actions. Hiebert et al. (1997) explain this approach to learning mathematics as elucidating:

> ...reflective inquiry as the key to integrating ideas and actions. Problematic situations, and methods of inquiry used to resolve them, elicit ideas *and* actions. This is what distinguishes problemetizing from traditional problem solving in which an acquired procedure is applied. (p. 24)

Within these types of mathematics lessons, the teacher strives to structure learning opportunities that are not only interesting to the students but also introduce to them important mathematics. Students are expected to make sense of the mathematics and methods they employ through discussions led by the teacher that interrogate the effectiveness of particular methods, as well as exploring different ways of representing their understandings through written, verbal, objects, pictorial, symbolic, and informal means of mathematical language (Hiebert et al., 1996, Hiebert & Wearne, 1992, 1993, 1996). As students seek to resolve problematic situations such as determining the difference between 72 and 39, the teacher's facilitating them into into actively generating, adopting, or reflecting on mathematical strategies and ideas allows the students' learning to be "tasks, and discussions... [which] connect with where students are and that are likely to leave an important mathematical residue" (Hiebert et al., 1996, p. 17).

The Construction of Mathematical Understanding

R. B. Davis (1984) explains mathematical understanding in a manner similar to Minsky and Papert's (1972) view; that mathematical understanding is present when an individual is able to integrate a new idea into a larger structure of previously constructed ideas. R. B. Davis (1992) uses the metaphor of assembling a jigsaw puzzle to illustrate his view:

> ...that one assembles ideas in one's mind much as one assembles a jig-saw puzzle.

Each new candidate piece, like each new idea, can be used only if it fits into the aggregate of pieces that have previously been assembled. (p. 228)

R. B. Davis (1992) states that if we consider mathematical proofs or even Skemp's (1978, 1979) "reflective, logical" understanding to be the results of mathematical activity, then mathematical understanding must be taken to be the result of children's working with mathematics. In this way, R. B. Davis (1992) considers mathematics to be a result of children's understandings:

Instead of starting with *mathematical* ideas, and then *applying* them, [teachers] should start with *problems* or tasks, and as a result of working on these problems the children would be left with a residue of *mathematics*... that mathematics is what you have left over after you have worked on problems. (p. 237)

The Teaching of Mathematics

R. B. Davis poses a similar argument to one found in Hiebert et al.'s works (1996; 1997) that stresses that rather than teaching children mathematics through methods of showing and telling, connected understanding can only develop when children have established for themselves a reason for doing mathematics. Solving tasks in this manner provides opportunities for students to decide whether they will employ already established methods or construct mathematical procedures on their own. This is explained below:

Instead of telling students what to do, and leaving them wondering about why one does it this way, the new approach helps students understand the task or the goal, and gives students the responsibility for inventing ways to solve the problem. (R. B. Davis, 1992, p. 238)

In examining the role of the mathematics teacher, R. B. Davis and Vinner (1986) claim that if we believe students "build up" their mathematical schemas through constructing and reconstructing ideas based on their previous experiences, then mathematics teachers play an integral role in the students' learning. On the other hand, the teacher's instructional actions

89

cannot be viewed as those that ultimately determine the ways in which students form their mathematical schemas. Mathematical schemas according, to their view, are assumed to be constructed from and always influenced by the child's previous mathematical experiences.

R. B. Davis and Vinner (1986) raise another issue with respect to student errors in mathematics. They assert that student errors should not necessarily be considered an indicator of lack of understanding but could be, in fact, the student's retrieval or selection of an inappropriate mathematical idea. Since teachers cannot determine what mathematics a child will or will not choose to retrieve, this becomes a responsibility of students to be aware of their mathematical understanding. R. B. Davis and Vinner encourage learning settings that engage students in nonroutine mathematical problems (Schoenfeld, 1985; Silver, 1994) as a way for learners to further construct their mathematical understanding and learn skills in monitoring their mathematical actions (R. B. Davis, 1984; R. B. Davis & Vinner, 1986).

Mathematical Ambiguities

R. B. Davis and Vinner (1986) identify five sources within students' school and out-of-school experiences that can obscure students' conceptual understanding in mathematics. These are as follows: the language of mathematics, assembling mathematical representations from pre-mathematical fragments, building mathematical concepts, the impact of specific examples, and children's misinterpretation of mathematical experiences. They argue that teachers should not try to exclude ideas from contexts outside of mathematics because these, as all other mental representations, serve as a necessary parts in children's assembly of pre-mathematical ideas (R. B. Davis, 1984; Lewin, 1986). So in a manner similar to Sierpinska's argument for understanding the importance of epistemological obstacles and Dubinsky's method of genetic decomposition for mathematics instruction, R. B. Davis and Vinner advise that we should not attempt to prevent children from developing mathematical misconceptions but rather, enable them to

become aware of misconceptions in their thinking and how overcoming them is necessary in being able to make sense of mathematics.

Mathematics as a Language

In North America, the English language is viewed as the linguistic base with which students enter school with and from which they begin to build mathematical ideas. The English language is also identified as a source of many difficulties in terms of students' mathematical understanding (R. B. Davis, 1984; R. B. Davis & Vinner, 1986). Pimm (1987) explores the possible reasons for this confusion.

Pimm (1987) puts forth the notion that just as English has specific ways in which it functions as a language, mathematics too possesses its own linguistic register and has "a set of meanings that is appropriate to a particular function of language, together with the words and structures that express these meanings" (p. 75). He explains that within the mathematics register there are "specialist terms" or, words that hold specific meanings in the context or discipline in which they are functioning. Durkin and Shire (1991) refer to these specialist terms as "lexical ambiguities" of mathematical language and make further distinctions between these words by classifying them into four subcategories– "homonymy" [2], "polysemy" [3], "homophony" [4], and "shifts in applications" [5]

An example illustrating this difference between English and mathematics linguistic registers is observed in the use of the word "any". In ordinary everyday contexts, this word is

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[2] Homonymy describes words that have the same form as in English but imply different meanings in mathematics. For example, the word "leaves" does not signify "leaves" on a tree or the verb "to leave", but rather, describes the subtractive action in mathematics.

[3] Polysemy characterizes mathematical words that may have two or more different but related meanings to their English definitions. For example, the word "product" in English, can be defined as "something that has been made", and in mathematics, takes on a similar meaning, "a quantity obtained by multiplication".

[4] Homophony is defined as two or more distinct words that have identical pronunciation but entirely different meanings– as observed in the words "two", "too", and "to", or "sum" and "some", or "pi" and "pie".

[5] Shifts in applications are similar to what Pimm (1987) describes as "notational metaphors" and these are mathematical symbols that in combination with other symbols, convey particular meanings. Here we can see that the number "5" can be applied in mathematics to communicate the nominal meaning of "the number five", the ordinal meaning of "the fifth number", the cardinal meaning of "1, 2, 3, 4, 5", or the visual representation of "5". For an in depth discussion regarding these lexical ambiguities, please see Durkin & Shire (1991).

most often taken to mean 'some' yet in mathematics, this word implies 'every' such as, "is any odd number prime?" Interpeting this question in a nonmathematical manner, one's answer would most certainly be yes, as seen in the case of the number five or seven. However, comprehending the word as meaning the latter, one would have to answer no, as not all odd numbers such as nine, have only factors of one and itself. A second example is located in our use of number words, that function in mathematics not only as adjectives as in "one" house, but can also exist as nouns such as when we speak of prime "numbers", implying that numbers have distinct qualities. Moreover, given the mathematical fact that "four fours are sixteen", number words also operate as adjectives and nouns.

Building Mathematical Concepts, Specific Examples, and Students' Misinterpretations

Metaphorical usage of English words in mathematics is evident in elementary school when children learn to "carry" when regrouping numbers in addition, to "borrow" when renaming numbers in subtraction, or making reference to the "face" when identifying surfaces of 3-D objects. These metaphors serve as tools for students to think and build images about mathematical ideas and concepts (Pimm, 1985, 1987)[6]. In other words, they are "functioning images... which [can] connect the ideas of mathematics with objects and processes that [students] feel they know and understand" (Pimm, 1987, p.97). Pimm cautions teachers that while metaphors are valuable tools in helping children conceptualize mathematical ideas, it is necessary for teachers to help students to define the usefulness of a metaphor by exploring it in many different contexts. By doing so students can develop an understanding of where and when their use of metaphors is appropriate and when it may be a mathematical act of over generalization.

In the same sense this does not mean that given linguistic ambiguities in mathematics, teachers should try to teach for all possible meanings or misconceptions that may arise when

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[6] Pimm (1985, 1987) identifies three types of metaphors existing in mathematics: "structural", "idiosyncratic", and "standard" or "conventional" metaphors. Structural metaphors refer to the ways in which symbols are arranged together and take on different meanings. Idiosyncratic metaphors are metaphors invented by the user to make sense of a mathematical concept or idea. Standard or conventional metaphors are used and understood by many people. Examples include "having" for positive numbers, "owing" for negative numbers, a mathematical function is a "machine", or an equation is a "balance".

students learn a particular mathematical concept (R. B. Davis & Vinner, 1986). Rather, it is to be expected that at different stages of a child's learning, some aspects of mathematical concepts will be fully developed while other aspects may be partially explored or not at all.

Besides metaphorical overgeneralization, there will be other times when children take specific instances and construct generalizations from them. This phenomenon, that R. B. Davis and Vinner (1986) see as impacting on students' mathematical understanding, can be observed when students learn about the multiplication of whole numbers and form the conclusion that multiplication of any numbers always produces a greater number. While they argue that it is fine for teachers to let learners maintain their partial understanding of a concept as long as the contexts in which the students are applying it is appropriate, R. B. Davis and Vinner also stress the need for teachers to provide settings that provoke children to engage in reconstructing their understanding of a concept. In the example of multiplication, reconstruction of the concept would become necessary when the children begin to work with decimals, fractions, and negative numbers. Finally, R. B. Davis and Vinner (1986) explain that because of the ambiguities that exist in mathematics students may misinterpret what mathematics is being taught and thus, teachers need to be cognizant that children's focus on unnecessary or extraneous aspects of a given concept can also lead to mathematical misconceptions.

Herscovics and Bergeron's Analytic Framework

In their effort towards enabling teachers to teach for children's mathematical understanding, Herscovics and Bergeron (1981, 1982, 1988a, 1988b; Herscovics, 1989) developed their analytical framework (see Figure 3), that has been used to describe key characteristics of particular mathematical concepts such as 'number' (Herscovics & Bergeron, 1988b; Herscovics, Bergeron, & Bergeron, 1986a, 1986b), 'length' and 'surface area' (Héraud, 1988), and algebraic concepts such as 'slope' (Dionne & Boukhssimi, 1988). Herscovics and Bergeron (1988a) assert that the development of individuals' conceptual understanding

should always begin in their physical, concrete world. They advocate for teacher practices to be those that value not only children's written answers but place an equal emphasis on children's thinking processes. Herscovics and Bergeron consider their framework to be a tool that can aid in epistemological analysis of mathematical concepts. Moreover, by accounting for the different components of Herscovics and Bergeron's model when planning instruction for a particular mathematical concept, teachers can design and provide richer mathematical learning settings.
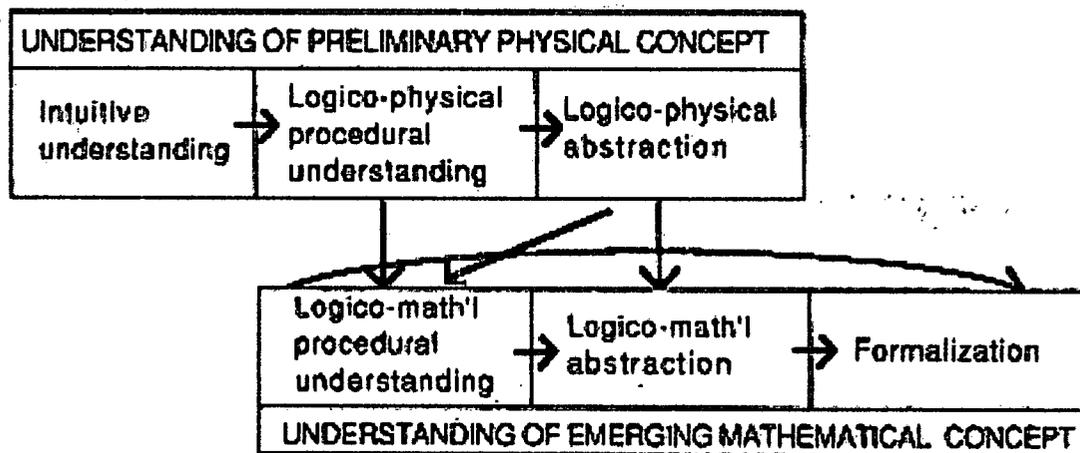
Figure 3. Analytic framework of mathematical processes (Herscovics, 1989).

The model is divided into two partially sequential but non-hierarchical tiers. Herscovics and Bergeron's two-tiered model conceptualizes mathematical understanding as being a partially sequential process that begins first with the individual's intuitive understanding of physical concepts and then develops through a series of levels into an abstract, mathematical concept. The arrows within the model indicate that forms of logico-physical and logico-mathematical abstraction are generated from the individual's preliminary physical concepts. The other arrows show that an individual's understanding of a mathematical concept does not require all three parts within the first tier. This assertion is supported by Herscovics and Bergeron's (1988b) observations of young kindergarten children who are seen to have mastered counting procedures and the formalization of the concept of number but have not yet comprehended all the invariancesregarding quantity and rank.

94

The first tier is identified as "understanding of preliminary physical concepts" and consists of three distinct components of understanding; "intuitive", "logico-physical procedures", and "logico-physical abstraction". Here, intuitive understanding is based on the individual's visual perception, that provides non-numerical approximations. Logico-physical procedural understanding is evidenced by an individual's ability to relate his or her intuitive knowledge through 'physically acting out' mathematical concepts. They describe logico-physical abstraction to be when an individual synthesizes and constructs meaningful relationships such as reversibility or generalizations between physical mathematical concepts (Herscovics, 1989; Herscovics & Bergeron, 1981, 1982, 1983, 1984, 1988a).

The second tier encompasses another three components– "logico-mathematical procedural understanding", "logico-mathematical abstraction", and "formalization", that Herscovics and Bergeron consider to be integral parts of comprehending mathematical concepts (Herscovics, 1989; Herscovics & Bergeron, 1981, 1982, 1983, 1984, 1988a). Herscovics and Bergeron (1988a) define procedural understanding to be when a learner relates preliminary physical concepts that underpin logico-mathematical procedures, such as counting methods for determining quantity or rank by using them appropriately in a given context (Bergeron, Herscovics, Bergeron, 1986; Herscovics et. al, 1986a). Logico-mathematical abstraction refers to the individual's construction of connecting logico-mathematical invariants together with related logico-physical invariants to form generalizations, such as coming to know that the commutativity of addition as a property applies to all pairs of natural numbers (e.g., 4+3 and 3+4 both equal 7) (Herscovics, Bergeron, & Bergeron, 1986b). Finally, formalization is characterized by Herscovics and Bergeron as an individual's activity of axiomatizing and producing mathematical proofs. At an elementary level, children's discovery of axioms and finding logical mathematical justifications would be taken as indicative of formalization. They also consider formalization to include the enclosing of a mathematical notion into a formal definition as

well as the use of mathematical symbolization for such notations. This type of formalization of procedural understanding such as counting can be observed when a child writes out a sequence of digits.

Mathematical Understanding as a Taxonomy

Mathematics educators Pegg and Currie (1998) agree with Piagetian views that assume older children learn in a qualitatively better way than do younger children because they have more developed mental structures. At the same time however, Pegg and Currie take a different stance with respect to observing and analyzing students' mathematical understanding. They support the view put forth by Biggs and Collis (1982) as well as other researchers (Blake, 1978; Hallam, 1967) that different methods rather than ones that generalize students' academic performances based on Piagetian cognitive developmental stages are necessary in order to provide detailed descriptions regarding students' learning within discipline-specific contexts.

The prestructural, unistructural, multistructural, relational, and extended abstract levels in Biggs and Collis' SOLO (Structure of the Observed Learning Outcome) taxonomy are described as being "isomorphic to, but logically distinct from, the stages of preoperational, early concrete, middle concrete, concrete generalization, and formal operational, respectively" (Biggs & Collis, 1982, p. 31). There are also four dimensions within each of the five levels that are used to further categorize student responses. They are as follows: working memory capacity, operations relating task content with cue or question and response, and general, overall structure (see Figure 4). In keeping with Piagetian models, that focus on hypothetical cognitive structures (HCS), SOLO also forms a concrete to abstract framework. In contrast, unlike HCS, that characterizes the individual in terms of age and stage of development, the SOLO taxonomy does not attempt to describe the learner, but rather, the quality of the learner's response(s) within a specific context and, in terms of the theory's levels and dimensions. By adapting elements from Biggs and Collis' (1982) theoretical framework of the SOLO taxonomy,

Pegg and Currie assert that in this way, they are able to analyze students' mathematical understandings in a more detailed manner than allowed by Piaget's developmental stages.

Application of the SOLO Taxonomy to the Analysis of Students' Mathematical Understanding

Just as Biggs and Collis created the SOLO taxonomy because they deemed Piaget's developmental stages as not appropriate for looking at children's understandings, mathematics educators Pegg and Currie (1998) integrate elements of SOLO to further elaborate on the van Hiele theory (van Hiele, 1986; van Hiele-Geldof, 1984) (see Figure 5) in order to analyze children's geometric understanding. Pegg and Currie's main criticism concerning the van Hiele model of geometric thought is that the model "cannot address questions posed outside of the direct notions of properties of figures, class inclusion, and deduction about which the theory is explicit" (1998, p. 334-335). What is common to the SOLO taxonomy and the van Hiele model is that they both derive from Piagetian roots. The difference between the two models is

Developmental base stage with minimal age	SOLO description	1 Capacity	2 Relating operation	3 Consistency and closure	4 Response Structure	
					Cue	Response
Formal Operations (16+ years)	Extended Abstract	*Maximal:* cue + relevant data + interrelations + hypotheses	Deduction and induction. Can generalize to situations not experienced	Inconsistencies resolved. No felt need to give closed decisions—conclusions held open, or qualified to allow logically possible alternatives. (R₁, R₂, or R₃)		
Concrete Generalization (13-15 years)	Relational	*High:* cue + relevant data + interrelations	Induction. Can generalize within given or experienced context using related aspects	No inconsistency within the given system, but since closure is unique so inconsistencies may occur when he goes outside the system		
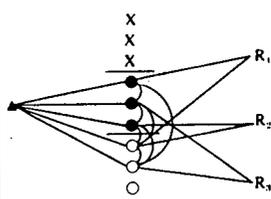
Middle Concrete (10-12 years)	Multistructural	*Medium:* cue + isolated relevant data	Can "generalize" only in terms of a few limited and independent aspects	Although has a feeling for consistency can be inconsistent because closes too soon on basis of isolated fixations on data, and so can come to different conclusions with same data	(diagram)
Early Concrete (7-9 years)	Unistructural	*Low:* cue + one relevant datum	Can "generalize" only in terms of one aspect	No felt need for consistency, thus closes too quickly; jumps to conclusions on one aspect, and so can be very inconsistent	(diagram)
Pre-operational (4-6 years)	Prestructural	*Minimal:* cue and response confused	Denial, tautology, transduction. Bound to specifics	No felt need for consistency. Closes without even seeing the problem	(diagram)

ª Kinds of data used: X = irrelevant or inappropriate; ● = related and given in display; O = related and hypothetical, not given.

Figure 4. Biggs and Collis' (1982) five levels and brief descriptions of each level of the SOLO Taxonomy and the taxonomy's loose correspondence with Piaget's stages of cognitive development.

that by integrating SOLO into the van Hiele model, geometric understanding changes from being conceptualized as whether or not student has "mastered" levels of understanding (Pegg & Davey, 1998), to categorizing students' responses in a polychotomous manner whereby answers can be grouped with similar characteristics and reflect various stages of cognitive growth (Pegg & Currie, 1998).

Given the descriptions above, Pegg incorporates notions of response levels drawn from the SOLO taxonomy (Biggs & Collis, 1982) to the van Hiele theory and, by doing so, elaborates on the level descriptors to enable more inclusive criteria against which to compare and

> **Level 1:** Figures are identified according to their overall appearance. Properties play
>
> no explicit role in this identification process.
>
> **Level 2:** Figures are identified in terms of properties and are considered to be
>
> independent of one another.
>
> ***Level 2A:** Figures are identified in terms of one single property, such as the length of
>
> sides of a figure.
>
> ***Level 2B:** Several properties are identified but exist in isolation of one another.
>
> **Level 3:** Relationship between previously identified properties of a geometric figure
>
> are now established.
>
> ***The** student is able to order the properties so that one or more properties give rise or
>
> imply other properties.
>
> **Level 4:** Deduction is understood and students can develop mathematical proofs.

Figure 5 Pegg's (1997) description of four levels of the van Hiele theory. (*) indicate SOLO-derived adaptations made to the van Hiele theory.[7]

analyze students' geometric understanding. Specifically, Pegg (1997, 1998) and Pegg and Davey (1998) focus on the elaboration of the second and third levels regarding the van Hiele model.

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[7] There is a confusing linguistic mismatching of the levels between the van Hiele model and Pegg's (1997) revision of it. The van Hiele model begins with the basic level of geometric understanding and continues onto levels one through four. Pegg's level one are assumed to be an elaboration to that of van Hiele's basic level. Pegg's level 2A and 2B are taken to be complementary to level one in the van Hiele model. As well, Pegg's level 3 and level 4 are understood as being elaborations of the second and third levels of the van Hiele model.

Here, unistructural responses[8] and multistructural[9] responses are considered to be within the concrete symbolic mode[10] are associated with level 2 thinking (Pegg & Currie, 1998). Level 3 responses are evidenced when students can generate an overview of, or identify the important elements within a task in order to form an appropriate generalization. Here, relational[11] responses of concrete symbolic mode are deemed by Pegg and Currie (1998) as characteristic of level 3 thinking. This particular model of geometric thought has been used with a variety of learners that ranges from primary students looking at basic two-dimensional shapes (Whitland & Pegg, 1999), to secondary (Currie & Pegg, 1997) as well as pre-service primary teachers (Lawrie, 1996).

Mathematical Understanding as Overcoming Epistemological Obstacles

Understanding as Both a Process and an Act

Sierpinska (1990) examines what it means for mathematical understanding to be conceptualized as both a process and an act. She agrees that "[u]nderstanding is achieved *slowly*, along with the accumulation of properties of objects, examples and development of concepts concerning relations between classes of concepts" cited (Lindsay & Norman, 1984, p. 438). In this manner, Sierpinska views mathematical understanding to be the process by which an individual's constant construction and reconstruction of ideas and meanings results in the

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[8] Next to prestructural, unistructural responses are described by Biggs and Collis (1991) as being the second most concrete form of understandings. A unistructural response involves the learner only having to comprehend the given task or question by relating the question with a response that incorporates one of the concepts found within the problem. For example, given a picture of two equilateral triangles and asked to respond in terms of what is the same about the two figures, a unistructural response could be that both figures have three corners, three corners being one of many possible similarities.

[9] Multistructural responses are defined by Biggs and Collis (1991) as when an individual is able to focus on two or more relevant concepts at one time. While considered to be a more sophisticated level of understanding than that of unistructural responses, a second characteristic of multistructural responses is evidenced by the individual's comprehension of the concepts as being separate and not as related ideas. An example found within this level of understanding could be a student who identifies, given a picture of the two equilateral triangles, that both triangles have three corners each and that each of the triangles' three sides are of equal length.

[10] Described by Biggs and Collis (1982) as being when individuals are capable of using and learning symbol systems. Typically, this level of functioning takes place in late primary through secondary school years and requires the individual to be able to internalize and generate representations of objects and events as words or images.

[11] "The relational response requires, in addition (to accessing a number of concepts), an overview of relevant concepts while being able to monitor the process or task from beginning to end, thus allowing for a logically complete conclusion" (Pegg & Currie, 1998, p. 337-338)

100

establishment of connections between mathematical concepts. Sierpinska (1990) also draws on Ricoeur's (1989) notion regarding the dialectic nature of the process of understanding and as acts within a process. Sierpinska (1990) adds that although she agrees with Ricoeur's general idea of the dialectic between an individual's understanding and explaining as "starting with a guess and developing through consecutive validations and modification of the guess" (p. 26), it is difficult to directly apply this model to the comprehension of mathematical concepts. To do so would necessitate the individual's experience in working through a variety of situations, "because the understanding of a concept is not normally reached through reading a single text. It demands being involved in certain activities, problem situations, dialogues and discussions, and the interpretation of many different texts" (Sierpinska, 1990, p. 26). Sierpinska integrates the ideas of both Lindsay and Norman, and Ricoeur, to develop the notion for mathematical understanding to exist as a process and act of constant construction, generalization, and resynthesis of ideas and relationships between concepts through a spiraling "process" of dialectic interpretation.

> *Processes* of understanding are seen as lattices of *acts* of understanding linked by various reasonings (explanations, validations) and a (relatively) 'good' understanding of a given mathematical situation (concept, theory, problem) is said to be achieved if the process of understanding contained a certain number of especially significant acts, namely acts of overcoming obstacles specific to that mathematical situation. (Emphasis added, Sierpinska, 1994, xiv)

A Historico-empirical Approach

Sierpinska (1987, 1990, 1994) explains that in the act of understanding mathematics, new ways of knowing are established. Distinguishing her theoretical work as being different from other models of mathematical understanding that focus on "levels of understanding" (Herscovics & Bergeron, 1988b; Pirie & Kieren, 1989; van Hiele, 1986), "cognitive structures" (Dubinsky & Lewin, 1986; Lesh, Landau, & Hamilton, 1983), or the "dialectic coupling of procedural

and relational forms of understanding" (Sfard, 2000; Skemp, 1978, 1979), Sierpinska classifies her research as being that of a historico-empirical approach. This approach examines students' understanding of mathematics from a perspective that focuses on the "obstacles to understanding encountered both in the history of the development of mathematics and in today's students." (Sierpinska, 1994, p. 120):

> [F]rom the point of view of mathematics education, what is interesting are [sic)]exactly these 'accelerations and regressions' and 'epistemological gaps', as well as 'epistemological obstacles' and difficulties because it is assumed that to learn is to overcome a difficulty. That an equilibrium has to be finally attained– this [sic] is taken as a banality; the problem is that without first destabilizing the student's cognitive structures no process of equilibration will ever occur, i.e., no learning of something radically new will ever occur. (Sierpinska, 1994, p. 121)

Sierpinska argues that in order to improve students' mathematical understanding, teaching should focus on intervenrtions that help students overcome epistemological obstacles.

> So we must introduce the students into new problem situations and expect all kinds of difficulties, misunderstandings and obstacles to emerge and it is our main task as teachers to help the students in overcoming these, in becoming aware of the differences; then the students will perhaps be able to make the necessary reorganizations. (Sierpinska, 1994, p. 122)

Below are specific forms of knowing that Sierpinska sees as impacting on children's mathematical understanding.

Epistemological Obstacles

Sierpinska asserts that it is through the examination of students' acts of understanding that we can interpret thinking processes and epistemological obstacles[12] that are involved in students' construction of meaning regarding mathematical concepts. She makes the point that

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[12] Sierpinska applies Bachelard's (1975, 1983) notion of "epistemological obstacles" to describe an individual's unconscious ways of knowing or understanding that constrain their ability to think about mathematical concepts in general, elaborated, or more abstract ways.

102

although specific methods of measuring students' acts of understanding need to be developed, strategies for teachers to engage students in confronting and overcoming epistemological obstacles also need to be generated.

> [I]nstead of trying to replace the students' 'wrong' knowledge by the 'correct' one, the teacher's effort should be invested into negotiations of meanings with the students, invention of special challenging problems in which a student would experience a mental conflict that would bring to his or her awareness that his or her way of understanding is probably not the only possible one, that it is not universal.
>
> (Sierpinska, 1994, xii)

Furthermore, the partial ordering of a learner's acts of understanding would enable the student's depth of mathematical understanding to be compared against criteria and it could be measured in terms of the number and quality of the acts of understanding demonstrated. As well, the number of epistemological obstacles an individual may need to overcome could then be identified.

Sierpinska (1990) argues that unlike intuitive knowledge, that she describes as "irresistible and certain", rational knowledge in mathematics is acquired through the individual's exercise of rigour and attention. Interrogating and synthesizing perspectives of understanding from Locke (1985), Dewey (1988), and Hoyles (1986), Sierpinska (1990) generates four categories or acts of conceptual mathematical understanding that she deems as necessary for students to experience and use in their studies of mathematics. They are as follows: "identification", "discrimination", "generalization", and "synthesis". Her subsequent work (Sierpinska, 1994) deals with the elaboration of these categories whereby she integrates Vygotski's (1987) theory of intellectual operations. By doing so, Sierpinska forms a more detailed framework that provides descriptions for the acts and processes involved in students' development of mathematical concepts and the types of epistemological obstacles that may occur.

The genesis of concepts in a child, according to Vygotski, is the genesis of his or her intellectual operations such as generalization, identification of features of objects, their comparison and differentiation, and synthesis of thoughts in the form of systems. The very same operations lie at the foundations of understanding. . . . The various genetic forms of these operations, discovered and described by Vygotski, seemed to provide, almost immediately, the possible genetic forms of understanding. Moreover, the theory can be used to explain some of the curious ways in which students understand mathematical notions, and why, at certain stages of their construction of these notions, they simply cannot understand in a different or more elaborate or more abstract way. (Sierpinska, 1994, p. 142-143)

Sierpinska (1994) distinguishes two key tenets within this idea of epistemological obstacles and how they affect student understanding of mathematics. First, cognition is not seen as an accumulative process but, instead, requires the individual's reflection on past mathematical actions in order for their reconstruction of understanding to occur. It is assumed then, that some form of integration and reorganization is required by the individual in order for his or her way of knowing or understanding to move from one level to another. The second assumption is that an individual must rebuild fundamental understandings that give rise to different philosophical considerations in order to overcome an epistemological obstacle. With this process of rebuilding, Sierpinska adds that new knowings can give rise to future epistemological obstacles through our awareness that an obstacle or obstacles exist in our mathematical understanding or, as a result of the resolution of differences. Therefore, obstacles can be viewed as being positive in the sense that we are able to overcome them or, negative in the sense that we acquire them.

[W]e must note that something (a belief, a scheme of thinking) functions as an obstacle often only because either one is unaware of it, or because one does not

question it, treating it as dogma. Overcoming an obstacle does not mean switching to another system of beliefs or another persistent and believed universal scheme of thinking but rather in changing the status of these things to 'one possible way of seeing things', 'one possible attitude', or 'a locally valid method of approaching problems' etc. (Sierpinska, 1994, p. 125)

Sierpinska (1994) makes it clear to the reader that, unlike Vygotski's genetic forms of intellectual operations that are chronologically developmental stages, she distinguishes the four categories as coexisting with one another and to be thought of as stages that one progresses through in childhood and adulthood. So even if an adolescent or an adult was confronted with a new mathematical concept, it would be likely that the individual would be working with a low level of conceptual understanding of generalization and synthesis or perhaps with a vague discrimination between the relevant and the irrelevant features of that particular concept. Moreover, Sierpinska (1994) makes the argument that:

> It seems that one cannot sensibly speak of epistemological obstacles in children before they reach the age of conceptual thinking. Things went easier with the younger children because they did not have to *overcome* epistemological obstacles. The epistemological obstacles still remained to be *constructed.* (p. 158)

Mental Operations: Identification and Discrimination

When an individual begins to identify features of objects and can distinguish them as being either more or less significant in view of some generalization, this can be considered to be a more elaborate form of mental operation (Sierpinska, 1990, 1994). This is illustrated in the following example:

> ... at some point in the process of understanding the topic of equations at the high-school level, the student must identify the simultaneous occurrence of variables

and the equal sign as features characteristic of equations before he or she starts to conceptually think of equations as equality conditions on variables. (Sierpinska, 1994, p. 151)

Chain-Complexes

Sierpinska uses Vygotski's term, "chain-complexes" to describe naive generalizations that precede one's development of conceptual mathematical understanding. This type of understanding can be observed in settings that involve actions of sorting or categorizing. Sierpinska (1994) explains that a chain-complex occurs when "a child... adding objects or pictures of objects to a given model, focuses on the last object added and is satisfied with any link between the new object and this last one, disrespectful of any contradiction that may occur with regard to the previously added objects" (p. 147). She uses an example from Vygotski's research to elucidate this for the reader:

> ... the child may select several objects having corners or angles when a yellow triangle is presented as a model. Then, at some point, a blue object is selected and we find that the child subsequently begins to select other blue objects that may be circles or semicircles. The child then moves on to a new feature and begins to select more circular objects. In the formation of the chained complex, we find these kinds of transitions from one feature to another. (Vygotski, 1987, p. 139)

A second characteristic of chain-complexes is that they usually take place when an individual is developing an understanding for a mathematical concept that involves the notion of equity. In this case, it is not that the individual considers all the attributes of an object as being of equal significance, but rather, that the individual is not able to stay focused on one particular feature for any considerable length of time. "At one moment it can be, for example, the colour, at another, the shape" (Sierpinska, 1994, p. 149). Here it is not possible for a student to abstract common features identified from different contexts or to synthesize them into a

106

mathematical concept because the individuals' processes of understanding and the actual object of their understanding is constantly undergoing change.

Pseudo-concept of generalization

When an individual becomes "aware of the non-essentiality of some assumption, or of the possibility of extending the range of applications" (Sierpinska, 1990, p. 150), this act of mathematical understanding is described by Sierpinska as "generalization". An obstacle that can occur within this category is a "pseudo-concept of generalization" (Sierpinska, 1993, 1994; Sierpinska & Viwegier, 1989), such as when a child identifies geometric shapes according to arbitrary colours so that any object resembling green pattern blocks would be considered squares. This is taken to be a pseudo-concept of generalization because the child is making a generalization but not the mathematical one– that all squares are four-sided polygons.

This epistemological obstacle is different from that of a chain-complex because the learner's way of understanding serves as a more holistic or general manner of thinking about mathematical concepts and does not change from situation to situation. Furthermore, due to the pervasive nature of epistemological obstacles, they cannot be easily abandoned nor replaced without considerable reorganization of one's mathematical understanding.

On Abstraction

When one is able to maintain one's thinking about the same single feature in order to move beyond "complexization" (Vygotski, 1987) and towards the stage of generalization, one also moves closer towards what Sierpinska refers to as true conceptualization. This is preceded by an intermediary phase that she identifies as "potential concepts" (Sierpinska, 1994). Potential concepts are considered as such because it is possible for the individual to develop an abstract understanding of a mathematical concept once the individual is able to abstract the underlying idea or ideas that is at the core of the concrete, factual, or contextual situations.

The Operation of Synthesis: Conceptual Thinking

The formation of a mathematical concept requires the individual to be able to synthesize

features of that concept into a coherent whole. Being able to do so implies that the student can construct mathematical relations between two or more properties, facts, or objects (Sierpinska, 1990, 1994). Sierpinska's research into epistemological obstacles and the role that they play in students' struggle to generate and understand mathematical concepts makes it clear to the reader that students cannot achieve or demonstrate conceptual, mathematical thinking through methods that assume learning by telling. Moreover:

> Concepts cannot be given to the child, ready made, in the verbalized form or symbolic representation. The child has to construct them as generalizations of his or her previous generalizations and it is quite natural that the adolescent's first concepts may bear little resemblance to the fully fledge ones developed by generalizations made by mathematicians in their adult, mature, and often genius lives. And thus they become obstacles to understanding the theories. (Sierpinska, 1994, p. 159)

Mathematical Understanding as APOS via Reflective Abstraction and Genetic Decomposition

The three main areas of research that Dubinsky has explored concern the manners in which individuals construct mental schemas (Cottrill et al., 1996; Dubinsky, 1992a, 1992b), the role of reflective abstraction (Cottrill et al., 1996; Dubinsky, 1992b), and the interrelationship between visual and analytic strategies (Zazkis, Dubinsky, & Dautermann, 1996) in students' development of mathematical concepts. In his collaboration with Cottrill et al. (1996), Dubinsky maintains a view that deep mathematical understanding is characterized by "an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, reconstructing, and organizing in her or his mind, mathematical processes and objects that deal with the situation.[13] Cotrill et al. (1996) argue that mathematics cannot be regarded as a set of static concepts that can be passively acquired by students but rather, sound mathematical understanding necessitates students' active struggle in constructing and reconstructing their own mathematical thinking– their schemas. They make the contention that by not addressing

[13] In keeping with Piagetian views, Cotrill et al. characterize effective mathematical knowledge as being the successful adaptation and accommodation of an individual's schemas." (p. 171).

108

students' incorrect conceptions, teachers reinforce by further embedding the students' misconceptions into their mathematical schemas. Cotrill et al. (1996) describe mathematical knowledge as being a spiraling cycle in which an individual's reflection on mathematical actions, processes, and objects are integrated together to produce mental schemas or networks. Dubinsky (1992b) and Cottrill et al. (1996) refer to this cyclical process as the APOS[14] theory (see Figure 6).

APOS Theory

Actions

Dubinsky (1992b) and Cottrill et al. (1996) explain mathematical actions as "any physical[15] or mental transformation of [mathematical] objects to obtain other [mathematical] objects" (Cottrill et al., 1996, p. 171). These actions can consist of one response or a sequence of connected responses that occur when an individual reacts to a perceived external event. Further still, it is when the individual reflects on his or her mathematical action(s) that their action(s) become a process.

Processes

Cotrill et al. (1996) define a mathematical process to be:

... a transformation of an object (or objects) that has the important characteristic that the individual is in control of. . . in the sense that he or she is able to describe, or reflect on, all of the steps in the transformation without necessarily performing them. (p. 171)

Once constructed, a process can be manipulated and combined with other mathematical processes. For example, once an individual understands that "three add four makes seven", this understanding can be reversed and connected to the process of subtraction, "seven take away four makes three". It is these manipulations together with the individual's reflecting on the

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[14] APOS is an acronym for "actions, processes, objects, schemas".

[15] An example of a physical action could be a student recording or manipulating a mathematical calculation onto paper. A mental action on the other hand, could be a student recalling some mathematical fact such as 6 + 6 = 12 from memory.

REFLECTIVE ABSTRACTION

Interiorization

Action

OBJECTS   PROCESSES

Coordination
Reversal

Encapsulation

Generalization
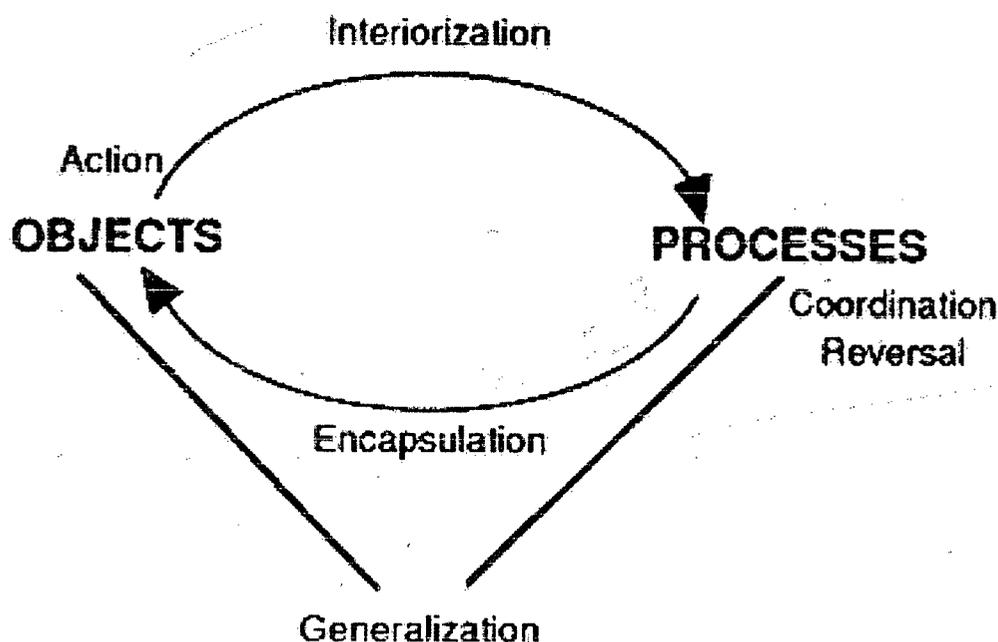
Figure 6. Dubinsky's (1992) model for the cyclical nature of mathematical actions, processes, objects, and schemas.

mathematics at hand that give rise to new processes and can foster the development of relationships between other process constructs to form a schema or, a mathematical object.

Objects

Dubinsky (1992b) and Cottrill et al. (1996) describe mathematical objects as being "constructed through the encapsulation of a process. This encapsulation is achieved when the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformation" (Cottrill et al., 1996, p. 171). The student is able to flexibly move their thinking back and forth between objects and processes of a mathematical idea. Mathematical objects exist as dense and symbolic mathematical schemas with which an individual is able to respond to many different contexts by de-encapsulating a mathematical concept in order to retrieve the appropriate processes or actions.

Schemas

As mentioned above, Dubinsky (1992b) and Cottrill et al. (1996) explain schemas to be coherent mental networks made up of mathematical representations of actions, processes, and objects. Once formed, these networks can also be interrelated with other schemas. Moreover, it is through continual, reflective constructing and reconstructing of mathematical schemas that an individual is able to make sense of, and deal with problematic situations by modifying or developing new mathematical processes, objects, or schemas.

The Role of Reflective Abstraction in Mathematical Understanding

Dubinsky (1992b) regards Piaget's notion of reflective abstraction as playing a critical role in the development of students' mathematical thinking. Moreover, Dubinsky (1992b) supports Piaget's (Beth & Piaget, 1966; Piaget, 1985) view that "first... reflective abstraction has no absolute beginning structure and second, that it continues up on through higher mathematics (Beth & Piaget, 1966, p. 203-208). By keeping in one's mind the APOS model (see Figure 6) while reading the following descriptions regarding the different forms of reflective abstraction, one is able to understand how these types of mathematical abstraction play integral parts in an individual's movement from one stage to the next in developing their mathematical thinking from actions to processes, operations, and schemas (Cottrill, 1996; Dubinsky, 1992a, 1992b).

In contrast to empirical and pseudo-empirical forms of abstraction[16], Dubinsky (1992a, 1992b) considers reflective abstraction to be the most sophisticated. He views reflective abstraction to be necessary for advanced mathematical understanding because it is the process by which students are able to internally coordinate their mathematical actions and form mental mathematical generalizations about external objects or events. Within reflective abstraction, Dubinsky further distinguishes five specific stages of thinking that enable deep mathematical

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[16] Based on Beth and Piaget (1966), Dubinsky (1992a, 1992b) defines empirical abstraction as being the least advanced form of abstract thinking; as one's ability to make generalization(s) regarding the common properties of a collection of objects such as, "all the blocks are blue". Pseudo-empirical abstraction is considered to be a more advanced form of thinking than that of empirical abstraction but not as sophisticated as reflective abstraction. This type of internal construction enables an individual to sort out mathematical properties that are being acted out on a set of objects. For example, a person's action of aligning two sets of objects demonstrates that that person has an understanding of a one-to-one correspondence among the sets of objects.

understanding: "interiorization", "coordination", "encapsulation", "generalization", and "reversal" (see Figure 6).

Interiorization occurs when the learner is able to consciously reflect on mathematical actions and combine them with other actions. Through reflection and *coordinating* two or more mathematical processes together, the individual is able to construct new mathematical objects through *encapsulating* or converting the process(es) into a concept. Once an abstract mathematical object or concept is formed, the learner is able to *generalize* this knowledge and apply it to many different contexts. Further still, this understanding of relationships that exist among mathematical processes, objects, or schemas allows the learner to think flexibly and thus, *reverse* their thinking.[7]

Genetic Decomposition

Dubinsky (1992a, 1992b) and Cottrill et al., (1996) propose "genetic decomposition" as a possible means by which effective teaching methods can foster students' mathematical understanding as framed by the APOS model. Genetic decomposition attempts to identify the elements of thinking that are necessary for students' construction of schemas regarding a particular math concept of say, 'limit'. Dubinsky and his collaborators (Cottrill et al.) characterize genetic decomposition as a cyclical tool that begins with an analysis of the mathematics for a particular concept and then compares it with students' understanding in order to develop a specific sequence for the learning of the mathematical concept. According to Dubinsky, before any instruction takes place, analytic decomposition is necessary. This approach involves breaking down into smaller chunks and then sequencing mathematical problems into specific steps so that students will be led to construct the particular concept (see Figure 7). Methods for instruction are then designed and implemented, and observations are collected regarding the mathematical activities of the students. These observations are then compared against the first genetic decomposition and necessary revisions regarding the sequencing of the mathematical problems

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[7] For instance, by understanding multiplication as the complimentary operation of division we can think about $4 \times 3 = 12$ as being reverse expression of $12 \div 3 = 4$.

or instructional strategies are made. The entire cycle of decomposition, implementation, and comparison is then repeated until no more revisions are needed (see Figure 8).

Preliminary Genetic Decomposition

Our description of what might occur is organized in six steps that occur only very roughly in the given order and with a great deal of "backing and filling" as the student constructs the concept of limit.

1. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.

2. Interiorization of the action of Step 1 to a single process in which $f(x)$ approaches L as x approaches a.

3. Encapsulate the process of Step 2 so that, for example, in talking about combination properties of limits, the limit process becomes an object to which actions (e.g., determine if a certain property holds) can be applied.

4. Reconstruct the process of Step 2 in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \epsilon$.

5. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit. As we indicated in our comments on the literature, applying this definition is a process in which one imagines iterating through all positive numbers and, for each one called ϵ, visiting every positive number, calling each δ this time, considering each value, called x in the appropriate interval, and checking the inequalities. The implication and the quantification lead to a decision as to whether the definition is satisfied.

6. A completed ϵ-δ conception applied to specific situations.

Figure 7. Cottrill et al.'s (1996) example of the genetic decomposition of the concept of limit.
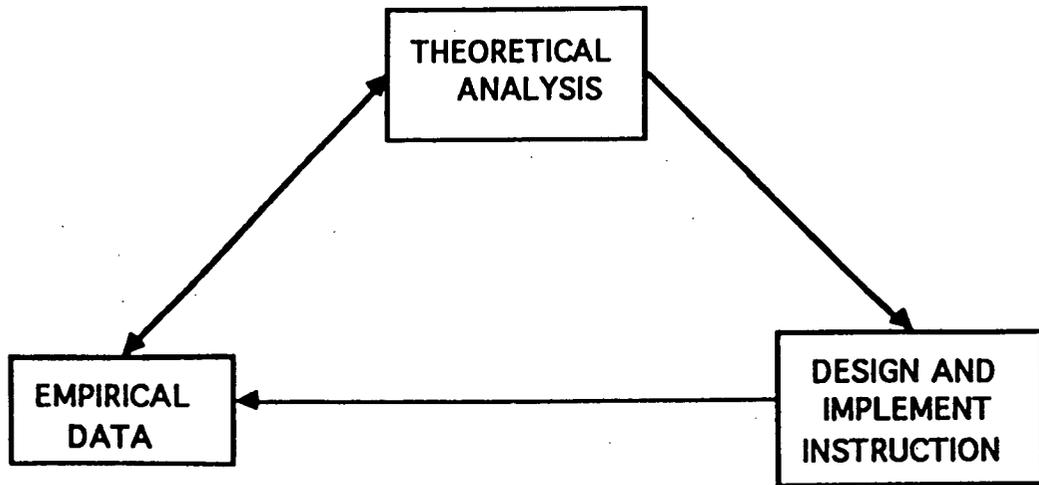
Figure 8. Cottrill et al.'s (1996) diagram of the cycle regarding the process of genetic decomposition, its implementation, and its revision.

CONSTRUCT   RECONSTRUCT   FIT   CONNECT
BUILD UP   PROMBLEMETIZE   INVENT   ADAPT
EQUILIBRIUM   OVERCOME   ACQUIRE   RETRIEVE
ELICITED - BY - THE - FACILITATIVE - TEACHER
DESTABILIZING   CONFRONTING   INDIVIDUAL
INTERNAL   MENTAL   PROCEDURAL   INTUITIVE
REFLECTIVE   INFORMAL   FORMAL
UNPREDICTABLE   LINEAR   NONLINEAR
CONCRETE-TO-ABSTRACT   RECURRENT
HIERARCHICAL   ACTIONS   PROCESSES
OBJECTS   CONCEPTUAL TOOLS   NETWORKS
SCHEMAS   REPRESENTATIONS   JIGSAW
PUZZLES   OBSTACLES   RESIDUE
SEQUENTIAL   MEASURABLE   CATEGORIZABLE
TAXONOMICAL   GENETIC   DECONSTRUCTIBLE

Emphasizing the Dynamic Nature

# of Mathematical Understanding

Visual and Analytic Strategies as Interrelated Qualities of Mathematical Understanding

Taking a different approach from that of investigating mathematical understanding through the development of stages and sequences, Zazkis, Dubinsky, and Dautermann's study (1996) examines the possible relationship(s) between the visual and analytic strategies that undergraduate students employed in solving algebraic problems. The students in this study were given mathematical problems that could be solved using either a "visual" (see Figure 9) or "analytic" approach (see Figure 10). In terms of defining these two different strategies, Zazkis et al. (1996) describe visualization as occurring when one forms a relationship between what one sees in one's mind as a mental construct and that which is experienced through one's senses in the physical environment. Analytic thinking is characterized as being:

> any mental manipulation of objects or processes with or without the aid of symbols.
> [For example,] a biologist who analyzes the nature of a plant through decomposing
> it into its parts, as well as thinking about the relationships among those parts and
> synthesizing them into various other wholes such as leaves, flowers, and seeds.
> Thus we include the naming of parts in our view of analysis, but we also include
> intellectualizing them into various new wholes. (Zazkis et al., 1996, p. 442)

Their definition of mathematical analysis also includes the five previously mentioned forms of reflective abstractions[1] that enable the individual to construct mental representations.

Zazkis et al.'s (1996) research explores previous claims that attribute visual strategies as being less sophisticated (Eisenberg & Dreyfus, 1991; Gollwitzer, 1991; Presmeg, 1986a; Vinner, 1989) and even restrictive to students' mathematical abilities in connecting visual representations to symbolic forms of mathematics (Kruteskii, 1976; Presmeg, 1986a, 1986b). Interestingly, Zazkis et al. (1996) found that the students did not use one or the other of visual or analytic

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[1] Interiorization, coordination, encapsulation, generalization, and reversal.

Our specific situation deals with the dihedral group of order four, denoted D_4, and we will consider students' thinking about two problems: List the elements of this group, and calculate the products, according to the group operation, of pairs of elements. We chose to observe students working with these D_4 problems because (a) each of the interpretations described below represents roughly the same level of mathematical sophistication, (b) both processes are simple enough to be carried out quickly and are therefore manageable during a clinical interview, and (c) the situation itself is complex enough to bring out distinctions in the students' understanding.

The group D_4 can be modeled in two ways. The approach that we take to be highly related to visual thinking is expressed in terms of the symmetries of a square. In this view, the elements of the group are the four rotations of the square around its center—in 0, 90, 180, and 270 degrees, together with four reflections or "flips" (across lines connecting the midpoints of the opposite sides and the two diagonals). The group operation between two symmetries consists of performing one symmetry on a square and then performing the other on the result. Using this approach, a mathematics student might employ a physical model of the square to achieve an understanding of its various rotations and flips. Figure 1 illustrates this method of calculating the product of two symmetries. The student performs a vertical flip followed by a 90-degree clockwise rotation to arrive at the reflection or flip across the diagonal with positive slope.
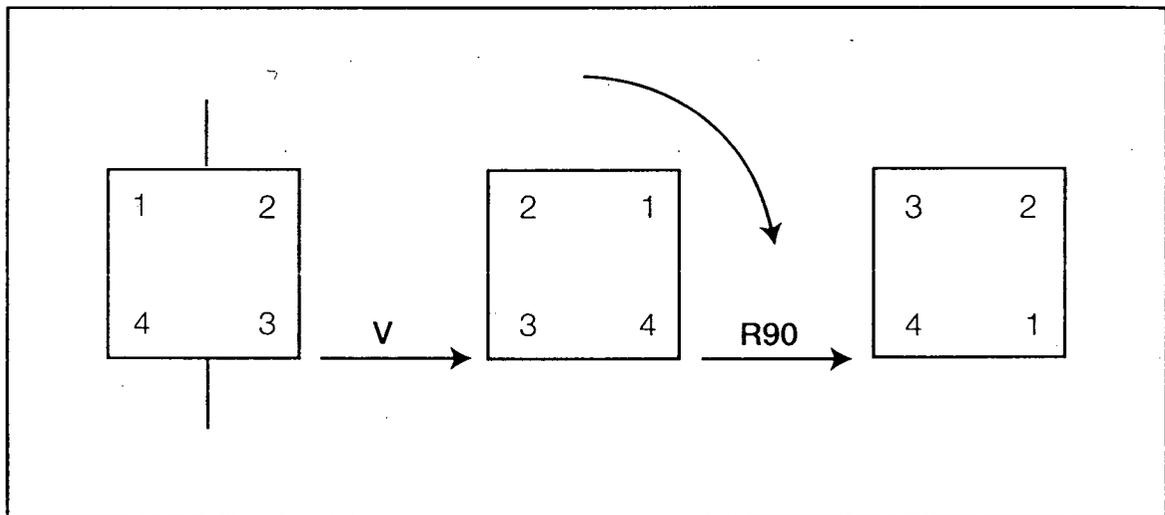
Vertical flip followed by a 90-degree right rotation.

Figure 9. Sample algebraic problem and visual solution (Zazkis et al., 1996).

A second approach to D_4, which we take to be more representative of analytic thinking, expresses the group in terms of permutations of four objects. The group operation in this case consists of applying a specific algorithm to multiply these objects and produce a permutation product. Thus a vertical flip of the square might be represented by the permutation

$$\begin{pmatrix} 1234 \\ 2143 \end{pmatrix}$$

and a 90-degree rotation by the permutation

$$\begin{pmatrix} 1234 \\ 2341 \end{pmatrix}.$$

After multiplying these permutations we get

$$\begin{pmatrix} 1234 \\ 3214 \end{pmatrix},$$

Figure 10. Zazkis et al.'s (1996) example of analytic solution for the same algebraic problem.

strategies but employed a combination of the two approaches. Consequently, they developed an alternative model that does not dichotomize visualization and analysis but puts forth the idea of the two as interrelated. Thus, visual and analytic thinking are "interacting and mutually supporting modes of thinking, rather than as two sides of a coin or as a dichotomy or continuum". (Zazkis et al., 1996, p. 454). Their visualizer-analyzer (VA) model conceptualizes the two problem solving approaches to be interdependent and challenges Piaget's (1977) claim that "some people are particularly visual, others mainly motor, auditory, etc." (p. 684). Moreover, Zazkis et al. align their view to be more in keeping with Clements' (1982a, 1982b) work that explores learners as not only being "visualizers" and "verbalizers" but also, "mixers"– those individuals who "do not have a tendency one way or the other" (Clements, 1982b, p. 34).

Instead of describing student visualization and analysis as being located on opposite ends of a continuum, the VA model (see Figure 11) makes the assumption that:

[i]t could be that a preference for, and difficulties with, visualization (Bishop, 1986; Goldenberg, 1991; Tall, 1991; Vinner, 1989) is no more than an individual's tendency to dwell on one side or another of the triangle, for example, when

119

communicating her or his thinking an individual might be more comfortable drawing pictures or writing formulas, but that does not change the fact that he or she needs analytic thinking in determining what to draw, and he or she eventually constructs a rich mental picture that determines what symbols to write. (Zazkis et al. , 1996, p. 453)

As Zazkis et al. assume visualization and analysis to be intertwined and therefore inseparable, they state that it is not possible to make claims that strive to categorize students' mathematical thinking or prioritize one method over the other. This model serves to encourage mathematics educators to conceptualize visual and analytic thinking as being equally important in students' development of fluid and rich mathematical understanding.
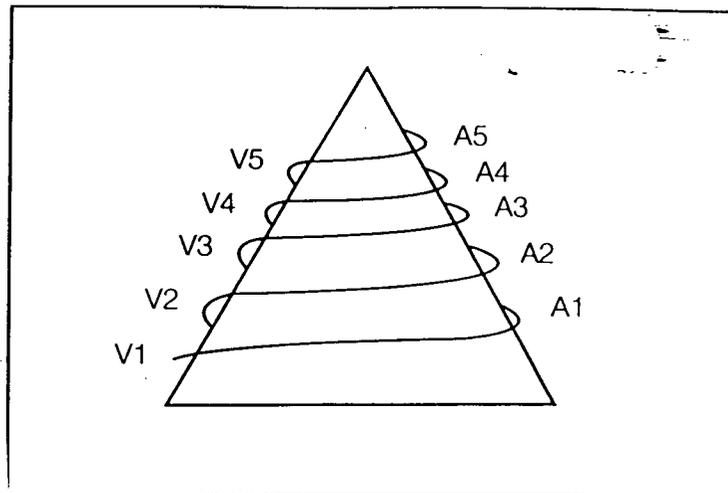

Figure 11. Visualization/Analysis Model (Zazkis et al., 1996).

Schoenfeld's Views on Mathematical Understanding

Schoenfeld (1989a) argues against linear, hierarchical frameworks that characterize children's mathematical knowledge as existing as a series of stages that begin with naive understandings and move progressively towards formal mathematical knowledge. This is because the structure of such frameworks does not highlight the fragmented and unstable nature of children's knowledge structures. Second, Schoenfeld does not conceptualize mathematical understanding as being unidirectional but views it as occurring through a back and forth or

bidirectional manner. Thirdly and in similar vein to Hiebert (1989) and Pimm (1987), Schoenfeld (1989a, 1991) advocates for models that focus on children's mathematical understanding as continuous and not those that "trap" it in structures of linearity. Thus, by describing mathematical understanding as being circular in nature emphasizes mathematical ways of knowing as being constantly reintegrated and giving rise to more complex forms of knowledge.

Moreover, in order for educators to gain an understanding regarding children's development of mathematical concepts, teachers must focus on examining the dynamics of children's mathematical actions as well as how they evolve through language and social interactions (Schoenfeld, 1989a, 1991, 1992, 1996). Mathematics as Schoenfeld describes it, is all about deep, connected understandings; that is, understandings that occur not only within an individual's mind, but also collaboratively and socially through the interacting with others:

> One often thinks of the stereotype, the isolate mathematician alone in his office, struggling to prove a new theorem. This is certainly a part of mathematics, but there is a social aspect of it as well, an aspect that Diaconis captured perfectly. Coming to grips with mathematics involves "talking and explaining, false starts, and the interaction of personalities. "All of it, not the least of which is the challenge of the false starts, is indeed a great joy. (Schoenfeld, 1991, p. 328)

Schoenfeld (1992) applies Ryle's (1949) description to distinguish instrumental knowledge as being "knowledge that" and relational knowledge as "knowing how". He also supports Hiebert's (1985) view that there is not a distinct line or boundary that separates these two forms of mathematical knowledge but rather, each informs and gives rise to the other.

According to Schoenfeld, mathematical understanding that is to be conceptualized as fluid and dynamic cannot be explored through models that assume mathematical development to be a monitonic process that entails the adding on of more knowledge to their knowledge base. Instead, research that aims to describe mathematical understanding as fluid and dynamic demands multiple perspectives that not only address specific issues of mathematical

understanding but also connect individual theories to larger realms within mathematics education:

> ... we need focus and pluralism, and an occasional step back to look at the big picture.... there should be broad diversity in what we look at, and the methods we use to do the looking– 1 don't believe unified theories or methodologies are around the corner– .... We need to work on our descriptions both of the forest and of the trees within. (Schoenfeld, 1989a, p. 116)

A Model for Analyzing Students' Mathematical Understanding

Schoenfeld's (1989a) video analysis examines students' conceptual understanding of mathematics through different levels of detail as it unfolds during problem solving situations. His research explores the possibilities of computer-based learning for enhancing mathematical understanding as it pertains to students' graphing of straight lines (see Figure 12).

Schoenfeld's (1989a) analytic model enables comparisons to be made between students' mathematical structures (the right-hand column) and preestablished mathematical forms (the middle column) as well as defining a student's level of complexity with respect to their conceptual schema(s). As seen in the Figure 12, the left-hand column represents the particular "lens" through which the researcher is examining student interactions; that is, the macro level focuses on the student's mathematical schemas, the middle level examines the entailments of larger schemas, the micro level moves closer in to the connections associated with the entailments of schemas, and the fourth level zooms in to the contexts that give . rise to the student's micro level of understanding. Schoenfeld's leveled structure of analysis enables the researcher to not only deconstruct mathematical concepts and students' conceptual understanding but also to tease out the relationships that exist amongst them.

Consequently, Schoenfeld (1989a, 1991) conceives deep mathematical understanding as being that which allows for flexibility and proficiency because of its rich, well developed micro

level of a knowledge base. Unstable mathematical understandings and misconceptions on the other hand are taken to be a result of an individual's ill-grounded connections at the micro level. Furthermore, Schoenfeld, like R. B. Davis (1992) views formal mathematics as a product or residue of well-connected mathematical understanding. He emphasizes these points below:

If you understand how things fit together in mathematics, there is very little to memorize. That is, the important thing in mathematics is seeing the connections, seeing what makes things tick and how they fit together. Doing the mathematics is putting together the connections and making sense of the structure. Writing down the results-- the formal statements that codify your understanding-- is the end product, rather than the starting place. (Schoenfeld, 1991, p. 328)

Graphs of Straight Lines:
Levels of Analysis and Structure*

Levels:	Traditional View of Subject matter	Our understanding of IN's cognitive structure
1. Macro-organization of knowledge, at the schema level, e.g.:	2-slot schema: L: $y = mx+b$: slope y-intercept	3-slot schema: L: $y = mx+b$: slope x-intercept y-intercept
2. Compiled knowledge: macro-entities and entailments, e.g.:	m is the slope of L m>0: L rises large |m| : L steep (and more...) The point (o,b) is the y- intercept of L... (etc.)	m is the slope of L m>0: L falls large |m| : L flat... the x-intercept has a place in the equation and on the graph... the y-intercept has a place in the equation and on the graph...
3. Fine-grained superstructure supporting domain knowledge: conceptual atoms (nodes) and connections, e.g.:	$m = \frac{y_2 - y_1}{x_2 - x_1}$: $y_2 - y_1$ and $x_2 - x_1$ are directed line segments, so their ratio indicates direction (e.g. + indicating "up, right") and steepness (so much y for so much x). when x=0 in y = mx+b, y = 0x+b = b, so the point (0,b) is (a) on the line, and (b) on the y-axis. Hence it is the y-intercept. [We call this level of structure the "Cartesian connection.]	$m = \frac{y_2 - y_1}{x_2 - x_1}$: but this knowledge is nominal and, while it is used to compute slope, the computation has no graphical entailments. not clear or stable IN knows that "b is the y-intercept," but her understanding is nominal and is not tied to the underlying structure as in the Cartesian connection.
4. The limited applications contexts out of and across which individuals construct the conceptual atoms that are seen at level 3	Void. (In traditional analyses -- and in people who have well developed domain knowledge -- these traces have vanished and the relevant conceptual atoms are the nodes at level 3.)	m is fuzzy, neither stable across contexts nor consistently evoked. Its meanings evolve. The role of the x-intercept evolves over the sessions. In 4 slightly different contexts, IN has 4 different meanings and interpretations for the y-intercept -- ie, the meaning of the term is context- dependent.

*The use of the term "levels" in this context does not presume that the structures discussed are hierarchical, or that they have the customary entailments of hierarchical structures.

Figure 12. An example demonstrating Schoenfeld's (1989a) leveled analysis of regarding the graphing of straight lines and the comparison between that of established mathematics and that of student mathematical understanding.

Problem Solving for the Development of Mathematical Understanding

Schoenfeld (1992) further elaborates on his analytic model (1989a) by taking a step closer and looking at individuals' mathematical knowledge bases. Here he examines what relevant information students draw on during mathematical problem solving, and the ways in which they retrieve and employ this information. For example, given the problem:

> You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using a straightedge and compass, a circle that is tangent to both lines and that as the point P as its point of tangency to one of the lines.
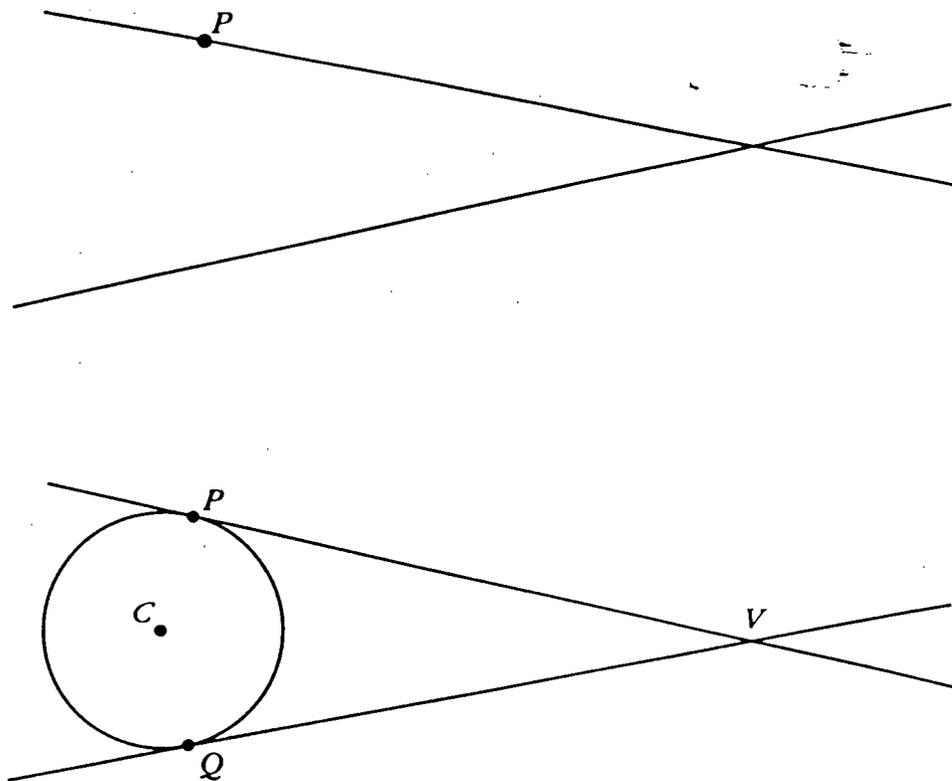

Figure 13. Example of a mathematical problem and its solution used to analyze student problem solving actions (Schoenfeld, 1992).

a student's mathematical understanding can be assessed against the table below:

Degree of Knowledge	of	facts	and	procedures
Does the student:				

		the tangent to a circle is perpendicular to the radius drawn to the point of tangency (true)	a (correct) procedure for bisecting an angle
a. know nothing about b. know about the existence of, but nothing about the details of c. partially recall or suspect the details, but with little certainty d. confidently believe		any two constructible loci suffice to determine the location of a point (true with qualifications) the center of an inscribed circle in a triangle lies at the intersection of the medians (false)	a (correct) procedure for dropping a perpendicular to a line from a point an (incorrect) procedure for erecting a perpendicular to a line through a given point on that line

Figure 14. Partial inventory of an individual's resources for working out the construction problem as described in Figure 13 (Schoenfeld (1992).

Here, informal knowledge is defined as that knowledge that a student brings to bear on a particular problem such as a student's mathematical intuitions and their more formal knowledge consists of mathematical facts, definitions, or algorithmic procedures. The ways in which these forms of knowledge are expressed may vary depending on the individual's confidence or certainty of them.

The Structure of Memory: Access to Resources

Schoenfeld (1992) proposes another model that outlines his conceptualization of our memory system and how the contents of memory are organized, accessed, and processed in a sequential yet somewhat circular manner (see Figure 15). Schoenfeld explains that visual, auditory, and tactile information is received through what he calls as "sensory buffers" or, short term memory. Short term memory is described as the location where the thinking gets done. If

126

sensory information is attended to within one's short term memory, then it is converted into forms that are further developed through the working and long-term memory systems. Working memory is different from that of short and long term memory because not only does one's working memory take in information from these other two sites but it is also where metalevel processes occur and enable one to construct mental representations. In addition to this, within one's working memory, one is able to structure planning, monitor, and evaluate one's
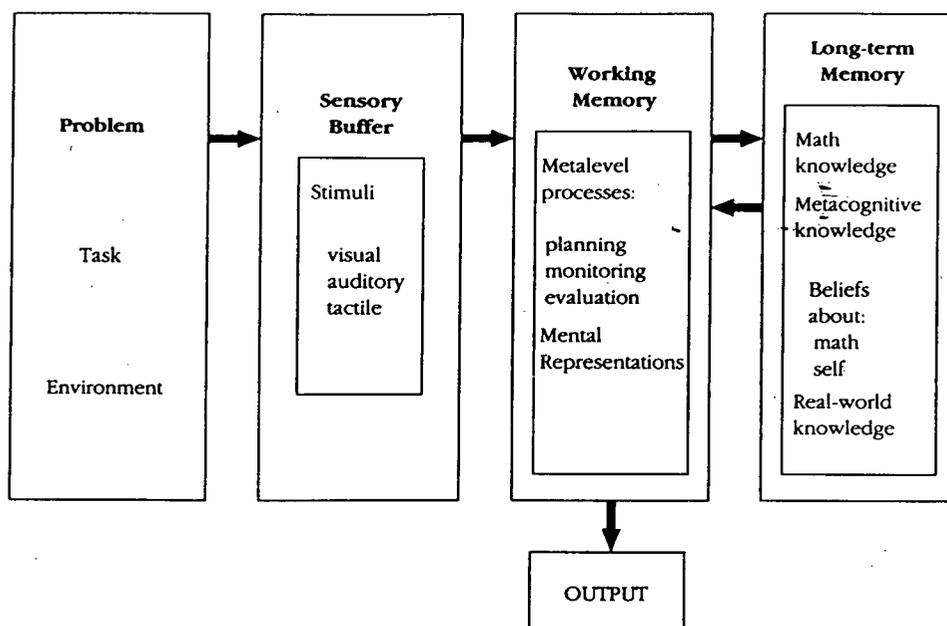

Figure 15. Schoenfeld's (1992) conceptualization of the structure of memory.

mathematical actions. It is this activity that takes place in our working memory that enables us to externalize our mathematical thinking through various physical, written, or verbal forms of expression. Long-term memory system is considered as a "permanent knowledge repository" (Schoenfeld, 1992, p. 351). It is a neural network in which mathematical knowledge functions as nodes and relational knowledge as the strands that connect these nodes together to form the network.

127

Schoenfeld's (1985, 1987, 1989b, 1992) studies that examined and compared students' executive or control skills (see Figure 16) to those of expert mathematicians (see Figure 17) revealed that unlike the students who spent much of their time exploring the mathematical problems, mathematicians spent the majority of their problem solving time making sense of the situation, analyzing, and structuring their exploration; that is, in thinking through the situation at hand, mathematicians produced solutions through generating and implementing devised methods that demonstrated a high level of control and perseverance. In the context of mathematical problem solving, "control" refers to the way in which an individual selects goals and subgoals, monitors, revises, and assesses their progress of a problem solving activity. Control also includes how one makes use of and sense of given or found information in attempts to solve a problem. The second managerial strategy, "perseverance" refers to an individual's intuitive, experiential sense in knowing when to continue with and not give up too soon on a chosen strategy or action but also, knowing when to abandon a particular strategy or action and search for a more effective or useful one.
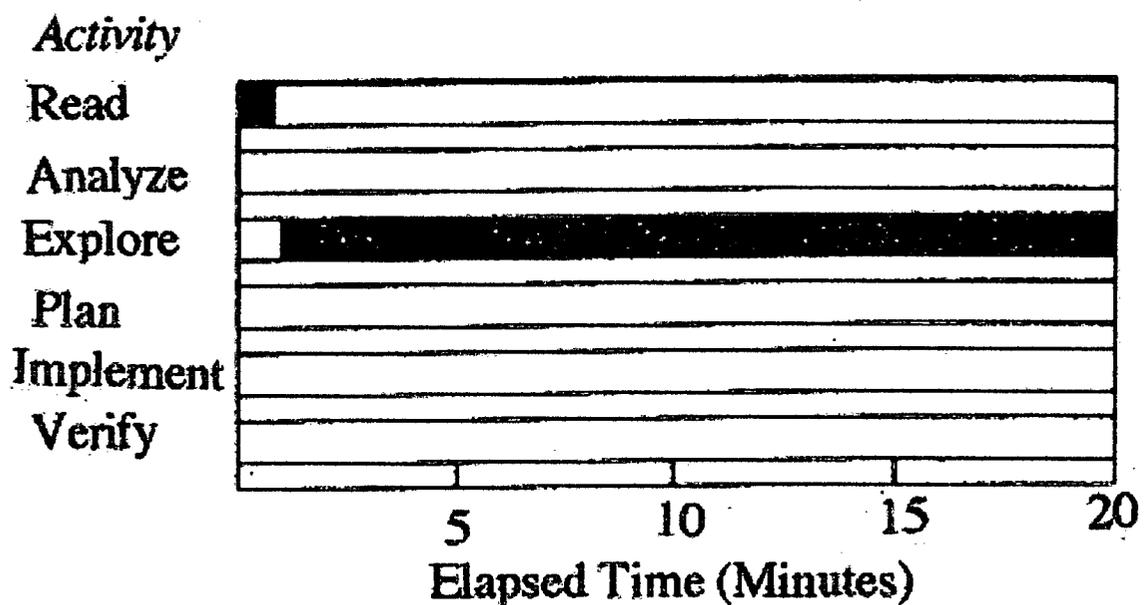

Figure 16. Time-line graph of a typical student attempting to solve a non-standard problem (Schoenfeld, 1992,).
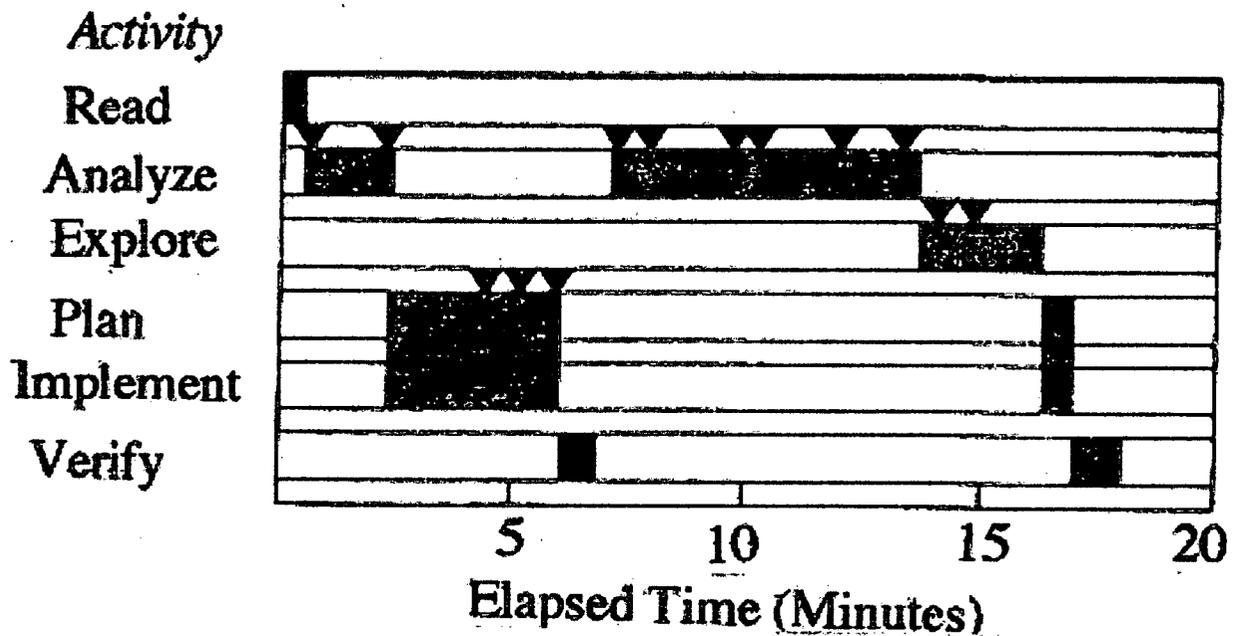
Figure 17. Time-line graph of a mathematician working on a difficult problem (Schoenfeld, 1992).

Based on these observations and implementing instruction that focuses on students' development of control and perseverance in their mathematical thinking (see Figure 18), Schoenfeld's (1989b; 1992) work supports other researchers' (Carraher et. al, 1987; Hart, 1989; Hiebert, 1989; Taplin, 1995) contentions for perseverance and control as two critical qualities necessary for well developed mathematical understanding. Schoenfeld advocates teaching methods that engage students in reflecting and routinely explaining their mathematical actions as well as in providing reasons for why their actions make sense within the given context. In doing so, he states that these managerial skills will then become a natural way of thinking about mathematics and enable more complex mathematical understandings to occur.

129
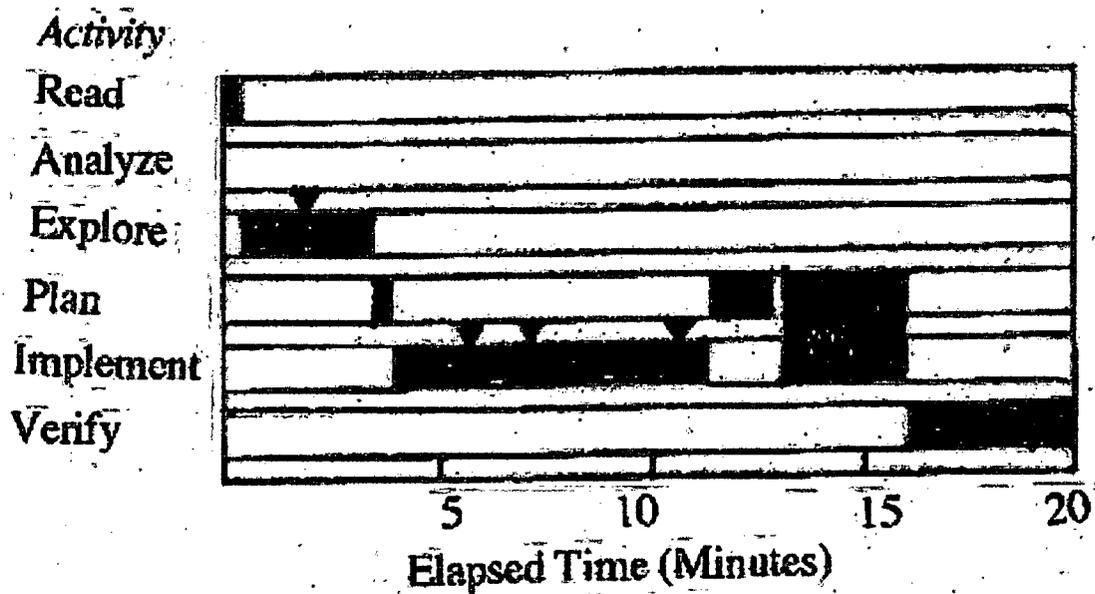
Figure 18. Time-line graph of student's problem solving actions after implementation of instruction that focused on development of metacognitive problem solving skills (Schoenfeld, 1992).

The Metaphorical Nature of Mathematical Understanding: The Work of Anna Sfard

Sfard (1991, 1994, 1998, 2000) examines mathematical understanding as being rooted in and growing from one's use of conceptual metaphors. Metaphors, she explains, not only provide us with a means by which to explain our thinking, but they also shape our ways of understanding and knowing mathematics. This is expressed by Sfard below when she speaks of Reddy's (1978) notion of conduit metaphor and connects it to that of mathematician's conceptions of what it means to understand mathematics:

> Rather than being just tools for a better understanding and memorizing, conceptual metaphors are often the primary source of mathematical concepts. The constitutive role of metaphor has been mentioned explicitly by the mathematicians whom I have interviewed in one of my studies. (Sfard, 1994)

130

In this way or another, all of them made it clear that without a metaphor, a new concept is not a concept at all. They also repeatedly emphasized the indispensability of the metaphor in the subsequent problem-solving process. (Sfard, 1994, as cited in Sfard, 1997, p. 340)

Sfard (1998, 2000) also makes the observation that the two main types of educational metaphors being used today in mathematics education characterize children's mathematical understanding in two different manners; that is, metaphors that describe mathematical understanding as a process of "acquisition" and metaphors that describe mathematical understanding as developed through "participation". Acquistionist metaphors are defined by Sfard as views that describe children's conceptual understanding of mathematics to be a process by which "basic units of knowledge. . . can be accumulated, gradually refined, and combined to form ever richer cognitive structures." (1998, p. 5). Furthermore, Sfard distinguishes that:

The picture is not much different when we talk about the learner as a person who constructs meaning. This approach, which today seems natural and self-evident, brings to mind the activity of accumulating material goods. The language of "knowledge acquisition" and " concept development" makes us think about the human mind as a container to be filled with certain materials and abut the learner as becoming an owner of these materials. (1998, p. 5)

She contrasts the acquisition metaphor with a participation metaphor and states that "the PM [participation metaphor] shifts the focus to the evolving bonds between the individual and others.... Indeed, PM makes salient the dialectic nature of the learning interaction: The whole and the parts affect and inform each other" (1998, p. 5). As well, she also notes that unlike the acquisition metaphor that emphasizes mathematical knowledge as a product of learning and teaching, the participation metaphor amplifies mathematical knowing occurring through ongoing interaction within a mathematical community.

Sfard (1998) recognizes the need for metaphors to be flexible and diverse because "...too

great a devotion to one particular metaphor and rejection of all others can lead to theoretical distortions and to undesirable practical consequences" (p. 5):

> We have to accept the fact that the metaphors we use while theorizing may be good enough to fit small areas, but none of them suffice to cover the entire field. In other words, we must learn to satisfy ourselves with only local sense making. A realistic thinker knows he or she has to give up the hope that the little patches of coherence will eventually combine into a consistent global theory. It seems that the sooner we accept the thought that our work is bound to produce patchwork of metaphors rather than a unified, homogeneous theory of learning, the better for us and for those whose lives are likely to be affected by our work. (Sfard, 1998, p. 12)

Sfard (1998, 2000) does not enter into current (mathematical) educational debates that aim to delineate learning as being conceptualized through either acquistionist or participative metaphors but rather, thinks that we should take the best qualities of both metaphorical ways of thinking and use them not in a divisive manner but in an integrated manner. That is, that discourse should focus on distinguishing contexts in which applications of each approach proves effective. In addition to this, Sfard stresses that even if we wanted to subscribe to framing mathematical understanding as say, solely participatory in nature, due to our cultural embeddedness in acquisitionist language we cannot help but to think acquisitionally, with objects and abstract properties– it is a part of our taken for granted ways of being. Sfard (1997, 1998) states that both metaphorical ways of thinking offer qualities that the other cannot and in doing so, argues that "the most powerful research is the one that stands on more than one metaphorical leg" (Sfard, 1998, p. 11) because these metaphors provide tension from which theories can be interrogated.

Sfard identifies one limitation that can occur when only using a participative approach to teaching mathematics, and that is that this way of thinking about mathematical learning can

lead to the "complete delegitimatization of instruction that is not problem based or not situated in a real-life context" (1998, p. 11) and "[t]his is difficult, when mathematics at some point exists within the symbolic, abstract realm" (Sfard, 1998, p. 36). Conversely, Sfard explains that when applying a solely acquisitional approach to teaching mathematics together with the assumption that learners build their own conceptual understanding of mathematics, the problem of bridging individual and collective knowledge becomes difficult.

Connecting Mathematical Processes of Knowing with Objects of Knowledge: Operational and Structural Conceptions of Mathematics

Instead of describing mathematical understanding as that which exists as either object or action, Sfard's research (1991, 1992, 2000) attempts to bridge this dichotomous gap and establishes the need for the co-existence of both mathematical knowing and knowledge; that is, that "an adequate combination of the AM and the PM would bring to the fore the advantages of them" (1998, p. 11). In describing the conceptual development of mathematics, Sfard (1997) characterizes it as being "a zig-zag movement with our conceptual schemes as constituting an "autopoietic system"; that is, a "system which is continually self-producing" (Maturana & Varela, 1987, p. 355). These qualities regarding mathematical understanding are pervasive elements throughout Sfard's diverse activities of research and reflect the value she holds for both structural or abstract knowledge and operational or context-bound knowings. Sfard seeks to describe the interrelationships that connect mathematical knowledge and knowing by examining the processes that facilitate children's formation of abstract, symbolic, concepts in mathematics.

Sfard's (1991) identifies three hierarchical stages of mathematical conceptions. The stages are referred to as: "interiorization", "condensation", and "reification" (see Figure 19). She defines the interiorization as the stage in which a child is developing an operational concept of the mathematics they are using to perform an action on a given problem. For example, "When I fill each of these three boxes up to the top and count the total number of cubes, I can find out how

much each container holds". All operational concepts that are formed within the interiorization stage are considered to be context-specific knowings.
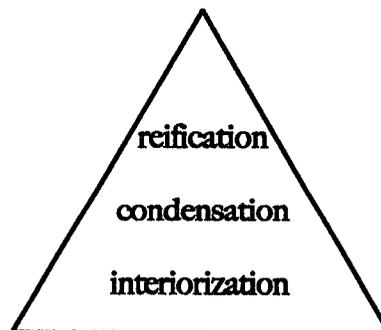

Figure 19. J. S. Thom's (2004) diagram that characterizes Sfard's (1991) configuration of the three stages of mathematical conception.

The second stage– the condensation stage is described by Sfard as being when learners are able to metaphorically, "stand back" and begin to reintegrate or make generalizations about their mathematical understandings. It is this middle stage that elicits an interplay between the synthesis of the child's previous operational mathematical conceptions and move towards the formation of an abstract, structural concept. Using, again, the example of the container, a child may now think, "Each time I filled and counted the number of cubes each of the three containers held. I wonder if there is a way that I can determine how many cubes the containers hold without having to fill each box and count the cubes by ones?"

The final stage of reification is explained by Sfard as the point at which the learner is able to comprehend the mathematical concept– in this case, the volume of rectangular prisms as an "object" or a "thing" that is symbolic, dense, and versatile. So, being able to understand that, "If I want to know how many cubes *any* box can hold, all I need to do is to measure (with cubes) and multiply the length of the box by the width of the box by the height of the box." Hence, by condensing one's knowing of operational, context-specific actions through ongoing analysis, one's mathematical knowledge can become reified into a flexible, structural form.

The Limitations of Metaphors

Sfard (1997, 1998) points out that although metaphors enable us to think about mathematics in powerfully abstract and symbolic ways at the same time, they are also shaped and limited within the confines of our experiential knowledge. Below are ontological obstacles identified by Sfard as they pertain to the integration of metaphors, metaphorical overprojection, and metaphorical confinement. In keeping with Sierpinska and R. B. Davis & Vinner's views, Sfard too considers it necessary for students to overcome these obstacles in order to integrate many different metaphors and develop a sound, stable conceptual knowledge of mathematics.

Integrating Metaphors

Sfard (1991, 1992, 1997) explains that one possible reason for students' difficulties with integrating metaphors is due to their inability to allow certain characteristics of the metaphor to fade into the background in order to integrate new qualities that will extend their knowledge to new or different mathematical situations. For example, when learning about the concept of division, one must, in a sense "forget" one's previous understanding that the operation of division when applied to whole numbers "makes the quotient smaller" in order to develop an understanding for why division "makes the quotient bigger" when working with fractions and decimals.

Metaphorical Overprojection

Instances when metaphorical overprojection take place involve situations where an inconsistency lies in the student's mathematical actions:

> Without abandonment of certain characteristics there may be a danger of a logical incompatibility with the new context or with metaphors contributing to the construction of the new concept. Appropriate modifications, however, are sometimes difficult to perform. Certain characteristics, being a vital component of the source notion, would refuse to go. (Sfard, 1997, p. 368)

She provides the following example of a student who divides the factor 'x - 2' into both sides of

135

the equation, $(x-2)^2(x + 3) = 2(x-2)$. This is considered to be an overprojection of "an equation is a balance" metaphor because when the student performs this operation on both sides of the equation, the student is thinking that equality has been maintained when in actual fact, the root $x = 2$ has been lost. Metaphorical overprojection can also occur when a student tries to integrate two incompatible metaphors such as "number as quantity" with the concept of complex numbers. If students cannot exclude or in some way 'forget' the quantitative quality of numbers, there results an incompatibility that limits and proves problematic to their conceptual understanding of complex numbers.

Metaphorical Confinement

Metaphorical confinement as a third ontological obstacle occurs when a student's metaphor is not broad enough to allow for the development of different, related metaphors or mathematical concepts. This form of confinement in mathematical understanding is present when students can only visualize fractions as being "part(s) of a whole"; with this image, their conceptual understanding is confined and because it cannot be opened to fractions existing as "object(s) within a larger group of objects", or as another way of expressing the divisive action.

*constructs   frameworks   macro middle
micro levels   schemas   short term
working   long term   networks · nodes
product   knowledge   object   action
knowing   interiorize   condense
reintegrate   reify   visual   analytic
individual   collective   social   mental
instrumental   relational   operational
structural   metaphorical   hierarchical
~~hierarchical~~   combining   intertwining
inseparable   fragmented   unstable
continuous   deep   connected   circular
somewhat circular   ~~distinct~~
~~dichotomizable~~   ~~categorizable~~   ~~linear~~
~~sequential~~   sequential   ~~prioritizable~~
~~naive~~ to ~~formal~~   informal   formal
back and forth   zigzag   bidirectional
fluid   dynamic   ~~adding on~~
correspondences   flexible   proficient
organize   manage · access   process
persevere   control   reflect   acquire
construct   participate   interact*

Rethinking Mathematical Understanding

from an Enactive Perspective

Mathematical Understanding as Objects of Personal and Public Forms of Communication

In her recent work Sfard (2000) has shifted her perspective to a more enactive one. Mathematical thinking is now regarded by Sfard (2000) as a form of communication; an integral cognitive process which allows individuals to not only express with others how and what they are understanding about the mathematics at hand but also, mathematical thinking as communication shapes how we individually and collectively make sense of mathematics. She contends that our reasons for communicating are not to establish mathematical objects but rather, mathematical objects are brought into being because we need them to develop conceptual understanding in terms of our own internal thinking and in conversations with others. Sfard (2000) explains that mathematical objects (physical, verbal, mental) arise as a product of our need to communicate; not the other way around: We do not start with mathematical objects and then communicate, we communicate and through this dialogic process, mathematical objects come into being. In keeping with this Sfard (2000) makes the assertion:

> I will argue that the claim of the primacy of communication imposes a literal reversal of this relationship: Instead of being merely helpful in constructing and sharing the knowledge of preexisting mathematical objects, communication and its demands must now be regarded as the primary cause for their existence. (p. 4)

For these reasons Sfard's research (2000) and the collaborative work that she has done with Kieran (Sfard & Kieran, 2000) reveal mathematical communication as having positive, neutral, and even detrimental impact on students' conceptual understanding. Together, Sfard, and Sfard and Kieran's research lessens the gap between our conceptions of students' internal, cognitive thinking as being separate from their interactive and communicative ways of acting. As well, their work brings forth the notion of internal and collective mathematical understanding as being conceived together.

Sfard (1997; 2000) supports Maturana and Varela's (1987) view that cognition in its most encompassing sense, co-evolves from our ways of knowing, our actions, and in our individual and

collective identities. This is evident when Sfard speaks of mathematical understanding as being shaped and evolving within ourselves and with others. Similarly to that of Gadamer (1989) who describes understanding as being like that of a "conversation"– dynamic, unpredictable, and dependent on the conversants, Sfard (2000) too uses this metaphor when she characterizes mathematical thinking as it occurs when individual students work collectively in a larger group:

> Thinking, like conversation between two people, involves turn taking, asking questions and giving answers, and building each new utterance–whether audible or silent, whether in words or in other symbols–on previous ones in such a manner that all are interconnected in an essential way. (p. 5)

Sfard (2000) argues that rather than simply viewing mathematical understanding as being that which exists either in the objective, public realm or in the individual, private realm, we need to also focus on the relationships that emerge between formal mathematics and that which is considered to be informal and embodied; meaning, the connections which take place within individual and collective conversations and the ways in which these interactions affect mathematical understanding.

Mathematical Conceptualization as Complex Circularity

By integrating the latest works of Sfard, which illuminate students' mathematical thinking as being circular and complex with Sfard's model of mathematical conception (1991), the latter shifts from that which was linear in structure, to a view of mathematical understanding as being co-emergent and cyclical (see Figure 20). This co-emerging of theory is possible when we examine the definition for "attended" focus. Sfard (2000) describes this as being the mathematics or mathematical object which arises as the individual or group's subject of conversation. Here as in Sfard's previous model of concept formation, this can be seen as corresponding to the condensation stage. In both cases of attended focus and condensation stage, there is an interactivity which involves the weaving together of several foci. Secondly, the form of mathematics or mathematical thinking which resembles the reification stage can be seen

140

in what Sfard (2000) describes as a "pronounced" focus of conversation; when "the learner[s] can flexibly move back and forth if needed to other realms whereby effective communication mediates these transitions" (p. 33). Thirdly, that which is considered to be the "intended" focus of mathematical communication, specified on an individual level as being each person's interpretation of the pronounced and attended foci, fits with Sfard's operational metaphors located in the interiorization stage.
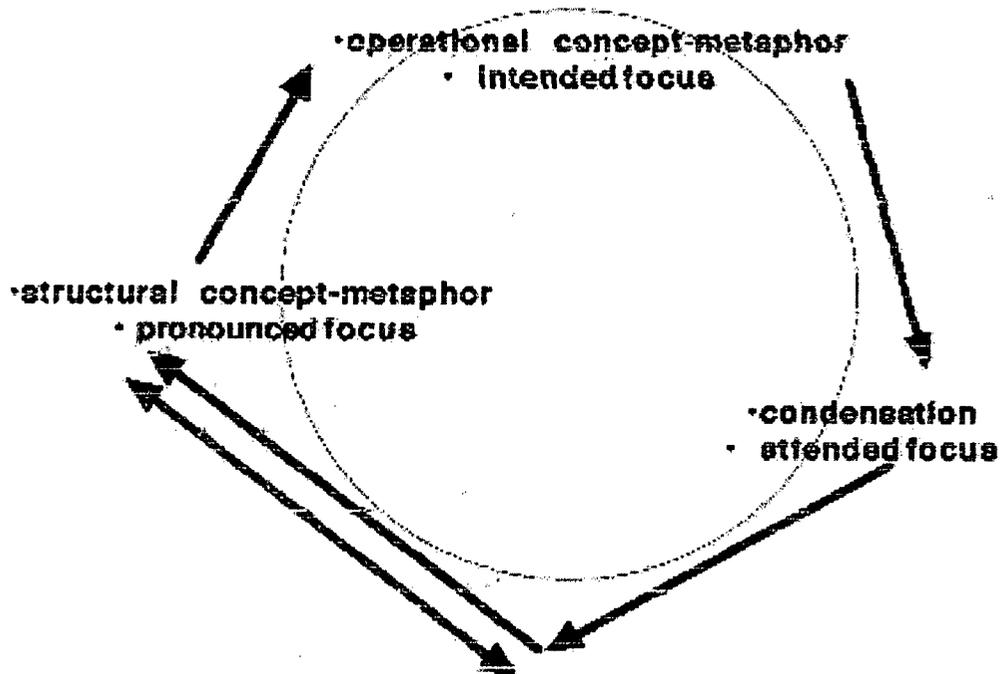

Figure 20. J. S. Thom's (2004) diagram which attempts to integrate Sfard's theories regarding mathematical conception (1991) and mathematical communication (2000).

Mathematical Understanding as Growth: The Pirie-Kieren Model

Pirie and Kieren's cognitive mappings of individuals and groups of students identify mathematical understanding as being simultaneously individual and collective, dynamic, occurring on many levels at once, and revealing qualities of transcendence and recursiveness (see Figure 21 and 22). They consider learners to be autopoietic beings who determine what

phenomena will be experienced as perturbations and who specify the ways in which they structure their mathematical thinking (Kieren & Pirie, 1992). As well, Pirie and Kieren argue that mathematical understanding does not occur as a result of student interactions with others or the
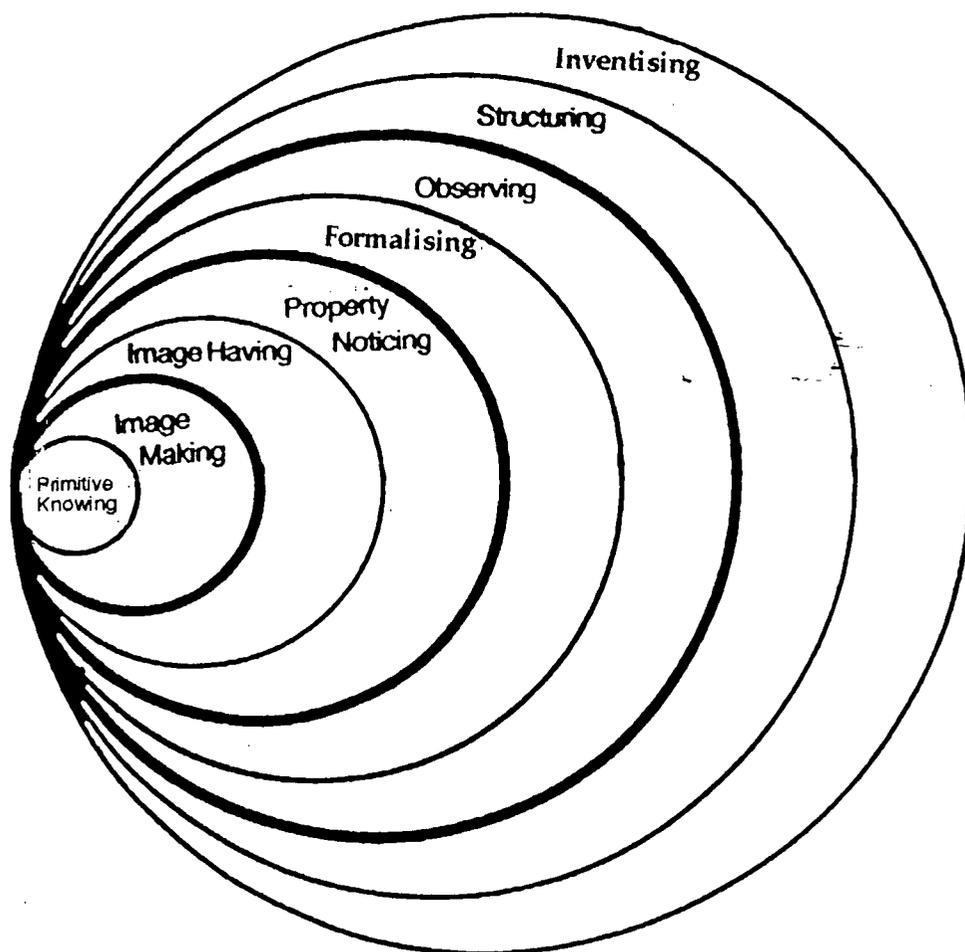

Figure 21. Model of a dynamical theory of the growth of mathematical understanding (Pirie & Kieren, 1994a).

142

Definitions of Levels of Mathematical Growth

Primitive doing or knowing: All the knowledge that a learner or group of learners bring to the particular mathematics and from which all new understandings develop.

Image making: making distinctions in previous knowing and using it in new ways. "Image" not only include physical and verbal forms, but mental representations as well.

Image having: using a mental construct about a topic without performing the particular activities that brought it about.

Property noticing: making note of distinctions, combinations or connections between images, predicting how they might be achieved and recording such relationships.

Formalizing: abstracting a method or common quality from the noted properties which are not dependent on meaningful images.

Observing: reflecting on and coordinating formal activity, expressing coordinations such as theorems.

Structuring: explaining or theorizing one's formal observations in terms of a logical structure.

Inventising: breaking away from preconceptions that brought about previous understanding and creating new questions which might grown into a completely different concept.

Other Features of the Model

• **Folding back:** moving to an inner level in order to extend one's current, inadequate understanding when faced with problems at any level.

• **"Don't need" boundaries:** indicated by the model's bold rings; conveys the idea that beyond the boundary one does not need the specific inner understanding that gave rise to the outer knowing.

• Each level beyond primitive knowing is composed of a complementarity of acting and expressing necessary before one is able move to the next level; **acting** encompasses all previous understanding, and **expressing** gives distinct substance to that particular level.

Figure 22. Definitions of terms and features regarding the Pirie-Kieren model for the growth of mathematical understanding. (Adapted from Stoute, 2000).

environment but rather, comes to be through the structural changes within, between, and among learners and the environment (Kieren, Gordon Calvert, Reid, & Simmt, 1995; Gordon Calvert, 1999; Simmt, 2000). It is this part of the Pirie and Kieren's view on mathematical understanding which emphasizes the notion that mathematical inter-activity as critically important for mathematical learning to grow. Moreover, Pirie and Kieren define mathematical

143

understanding as the embodiment of all verbal, physical, and written acts:

> Mathematical understanding can be characterized as leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further is constrained by those without. (Pirie & Kieren, 1989, p. 8)

A Descriptive Not Prescriptive Model

Pirie and Kieren make it clear that their model is not intended to be used to define or prescribe a particular sequence of static levels which constitute students' mathematical learning but rather, a way of conceptualizing the learning of mathematics as unpredictable and complex phenomena. As well, Pirie and Kieren (1994b) do not distinguish mathematical growth as being monological pathways, or privilege one's fluency to use formal language and mathematical symbols as representing formal mathematical understandings. Mathematical understanding is not only assumed to grow in complexity through the learner or collective unity's outward movement, but also from inward movement or, what they call, folding back to previous levels of knowing. Folding back is not a redoing of what has already been done, but moves the learner or group of learners back to inner levels of mathematical knowings where they will reintegrate understandings as a result of the perturbations experienced in previous outer levels before moving on (Kieren & Pirie, 1991; Martin, 1999; Towers, Martin, & Pirie, 2000). This model also reflects the notion of mathematical knowings existing simultaneously as a product, producer, and process (A. B. Davis, 1995, 1996; A. B. Davis & Sumara, 1997, 2000; Kieren, Simmt, Gordon Calvert, & Reid, 1996; Maturana & Varela, 1987). In this way, Pirie and Kieran advocate for learning settings that encourage students' engagement in folding back in order for the co-emergence of their self-referencing, remembering, and reintegration of mathematical knowings to occur (Pirie & Kieren, 1992).

Thus, the Pirie-Kieren model of mathematical growth (1989) reflects an enactive perspective because it provides a theoretical lens which focuses specifically on the complex, co-emergent, and unpredictable nature of mathematical understanding. Mathematical understanding is viewed as occurring through interrelated, fluid processes and evolving in a fractal-like manner (Kieren, 1990; Pirie & Kieren, 1989; Pirie & Kieren, 1994b). They describe the model's structure as neither hierarchical nor linear. The realms of mathematical knowings found within this model exist as embedded, unbounded circles which are self-similar and compatible with one another. Moreover, the Pirie-Kieren model reflects Maturana and Varela's (1987) axiom of "all doing is knowing, and all knowing is doing" (p. 26) because it locates primitive doing or knowing as being the roots from which all other mathematical knowings emerge (see Figure 23).
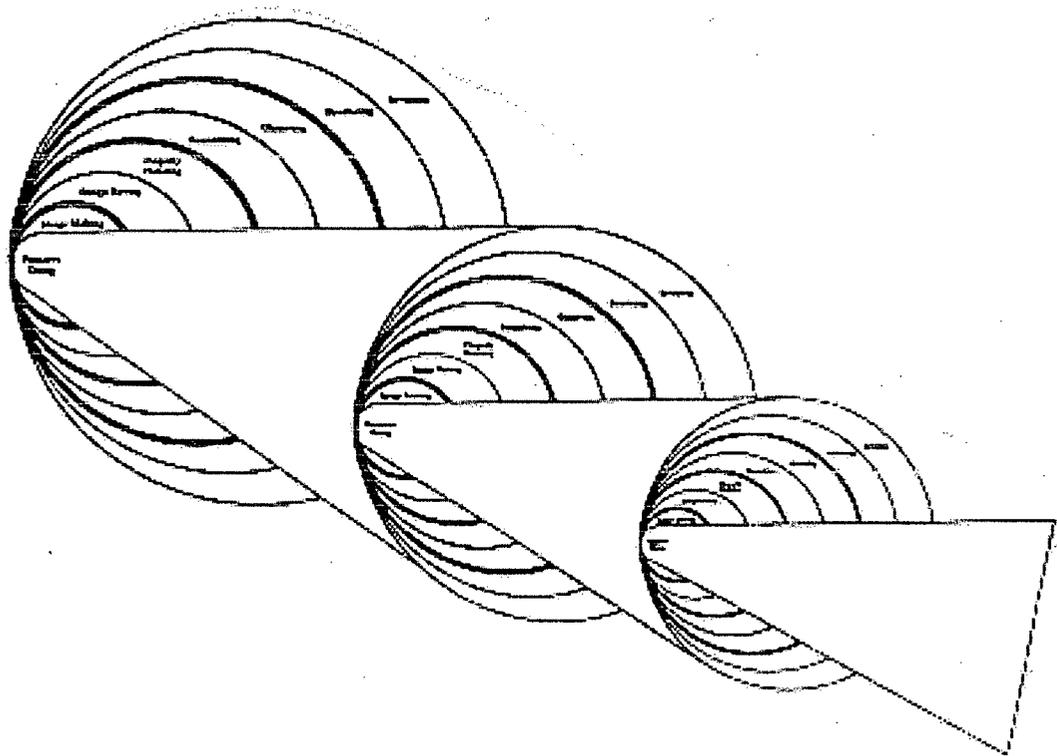

Figure 23. Model illustrating primitive knowing as the source of all other mathematical knowledge (Kieren, 1990).

all doing is knowing, and all knowing is doing[1]
simultaneously individual and collective
personal public private internal external
interrelated embodied co-emergent complex
dynamic ~~static~~ multi-leveled linear complex
circularity transcendent recursive ~~predictable~~
~~monological~~ inward-and-outward audible
silent visible visible hierarchical formal
informal knowing actions identities
communicative objects growth autopoietic
structural changes within between among self-
referencing folding back collecting invocations
provocations fluid reintegration thickened
embedded unbounded self-similar fractal-like
exists only in being inter-actively intra-actively
co-evolving in-relation in-conversation with the
interdependent ~~external~~ environment

---

[1] Maturana & Varela. 1987. p. 26

D espite the diversity of positions taken by mathematics educators regarding the ways in which mathematical understanding is portrayed and the manner in which it develops, it is evident in this literature collection that there is a general consensus among the varying perspectives that "good" mathematical understanding involves the integration of informal and formal mathematical knowledge, that it is flexibly fluid, and that it can be applied to respond to many different situations.

The theoretical portraits located in constructivism emphasize the building of one's mathematical knowledge as schemas, one's progression through specific stages, as well as the maneuvering of one's mathematics over a variety of conceptual obstacles. The positions taken by the authors in the second set of literature are slightly different from those seen in the first as these researchers seek to interpret the dynamic nature of mathematical understanding and explore forms of knowledge as being interrelated phenomena. And finally, in the third grouping of literature, the work of mathematics educators who share an enactive perspective was examined. These more ecological viewpoints serve to highlight the co-emergent and complex nature of mathematical understanding, how it it is individually and collectively brought into being, and the embodied forms in which it exists.

Having taken a good look at the metaphorical furnishings I had arranged comfortably around my teaching and revealed the diversity in how mathematics and mathematical understanding might be portrayed, I now faced the task of deciding whether (and why) they really suited (or did not suit) the ecological mind-space in which I was now dwelling. In order to do this, I had to consider what kinds of thinking my metaphorical furnishings enabled. And if they were not engendering ecologically coherent manners of conceptualizing my mathematics teaching, what metaphorical furnishings would?

New Furniture

So what of my mathematics curricula as jigsaw puzzles metaphor? The metaphor creates the image of a mathematics curriculum as being a set of pieces... like concepts, skills, domains of mathematics.... Complicated maybe, but as such, are 'do- and undo-' able pieces and as a whole-- clearly defined and separately visible. So, when given to the students and the pieces are assembled correctly, they reveal a coherent picture from its interlocking parts. Okay yes, this metaphor is a very 'tidy' and 'systematic' way for a teacher to think about mathematics curricula. But what this metaphor does not do is reveal the ecological qualities that distinguish teaching and learning of mathematics as dynamic and complex.

A. B. Davis, Sumara, and Luce-Kapler[1] make the distinction between complicated and complex thinking when they describe complicated thinking as that which "aims to reduce phenomena to elemental components, root causes, and fundamental laws."[2] They use the example of a clock as a complicated mechanism and state that a complicated understanding of it would involve "detailed knowledge of each of its parts."[3] and how the clock can be disassembled and reassembled. In contrast, a complex comprehension of a clock entails not only an understanding of its parts and the "interdependencies of its parts" but also, the role that is played by the clock is necessarily "embedded in [and thus affects as it is affected by] social and natural environments.[4]

The authors make the important point that something conceptualized in a complicated manner as in the case of the clock, C = A + B which implies that C (i.e., the clock) can be taken apart and put back together again. Thinking in a complex way however, assumes that C depends on "A" and "B" but that at the same time, it exists as something other than just A + B. Take for example a cake, which can be considered to be a complex entity. Once the ingredients have been mixed together and baked, you cannot take it apart again to get back what went into making it. The cake exists as a complex form because of the reaction between the ingredients and its environment.

So even if a teacher managed to design a mathematics program that fit mathematical topics together in a way that produced a complete picture, the curriculum would remain static. It would still be a set of distinct pieces and hence, necessarily a "complicated" NOT "complex" curricular form. The program would only be a product of its parts and not something that possesses the potential for possibilities greater than that[5] or a curriculum that embodies an awareness for the role it plays in the whole of teaching and learning mathematics. What I needed was a complex view of the curriculum. Something more than a complicated one. A new metaphor.

An ecologically coherent metaphor of a mathematics curriculum needs to be one that creates an image of a flexible, dynamic network that co-evolves as a result of the interactions of the teacher, children, and the environment. A mathematics curriculum might then, not be thought of as a commoditized "thing"... that which "prescribes" what teachers are to teach and what children are to learn.[6]


curricula

co-emerges with mathematical learning

mathematical landmarks

geographical map
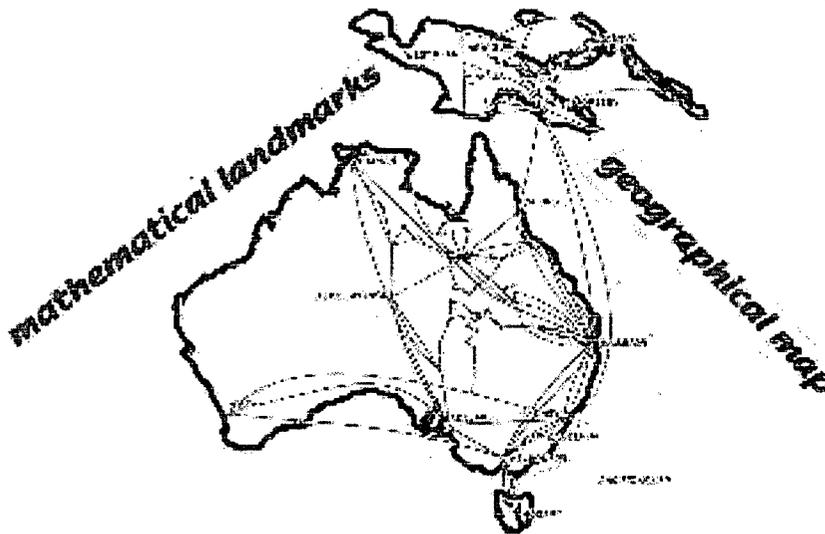
cannot "know" the landscape

G. Bateson's map metaphor[7] works well in that mathematics curricula can be understood in light of an ecological perspective. Envisioned as a map, a teacher can locate mathematical topics, concepts, and skills as important landmarks for the class' learning. Once these locations are marked out, the teacher can then think about how they are connected to one another. Inherent in this is the idea that what cannot be sketched out in advance are the actual paths on which the children will travel to get to the mathematical locations or the understandings they will establish when and after they come to these sites. So although a teacher might be able to mark out the mathematical landmarks, it is impossible to predict the conditions of the ever-changing landscape (i.e., the actual terrain that the class will encounter while engaged in the mathematical studies).

151

In light of this, children's mathematical understanding and learning can be likened to what Varela, Thompson, and Rosch describe as "paths that are laid down in walking"[8]. And like paths, they are rarely if ever straightforward but instead, spread out in several directions and entail twists, turns and switch-backs. Mathematics curricula and learning imagined in this manner distinguishes them as co-emergent phenomena that are brought into being through the students, their teacher, and mathematical settings.

mathematical understanding

and learning

embodied *recursive* *twists* *switch-backs* *turns* *fluid* *likely NOT to be linear* *unpredictable* *dynamic* *paths*
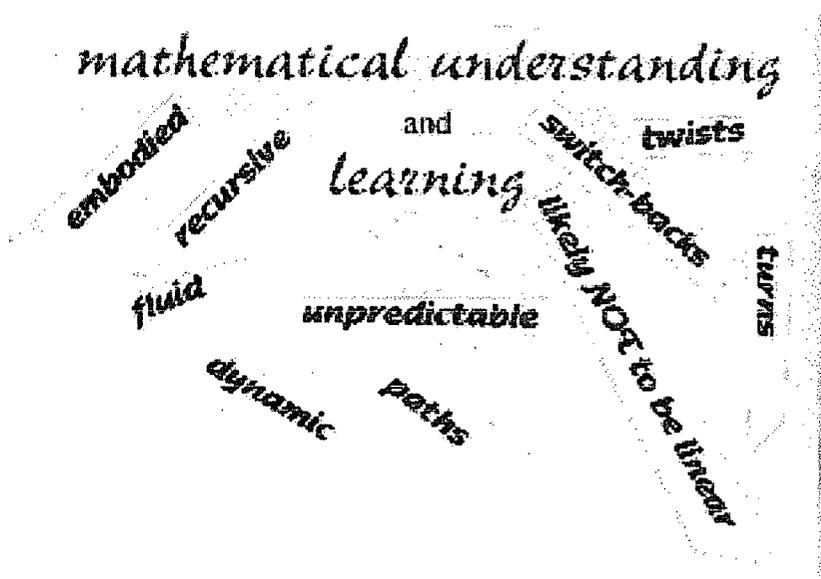
interdependent environment

individual-and-collective

mathematics class

Differently from before, I have come to envision the mathematics class as an image of teacher and children interacting within individual and collective mathematical realms... nested systems but not at all discrete. The larger human and nonhuman environment is not an external entity but an interdependent one.[9] Once a moot point of my earlier thinking, it is now a critical one. Just as children's mathematical knowings impact and shape the classroom environment, the environment now changed, impacts and shapes the understandings that will take place for the children and so on. These realms exist as co-evolving systems-- dynamic and recursively related to one another.
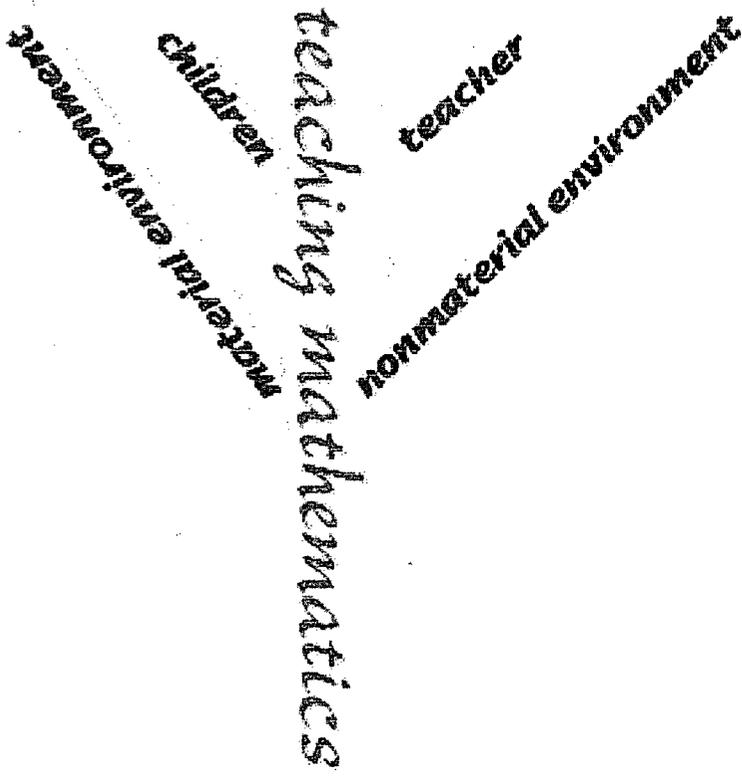
'Teaching' encompasses more than ME as THE teacher; it includes the children as well as the material and nonmaterial environment. Of course, I am a source of provocation and invocation of children's mathematical learning but importantly, I'm not the only one. As "oikos", teaching and learning exist within systemic relationships of difference. When the class is perceived as a system of individual and collective relationships connected with the material and immaterial environment[10], occasions for learning can happen from anything and anyone within this web. When I think about a metaphor for mathematics, instead of notions of "static integration" or "separate strands" of algebra, geometry and so on, I imagine them to be fluid and co-emergent entities that make mathematics a dynamic living whole. In this way, mathematics can be conceived as a 'shape-shifter'-- arising and metamorphosing as it interacts with the contexts of which it is a part. Take for example, a linear equation as one form of mathematics represented algebraically and the equation as a graph as being another... or a pattern as a numerical sequence or as a three dimensional structure.

teaching mathematics

teacher

children

material environment

nonmaterial environment

mathematics

mathematical processes as "language"
physical, verbal, and mental

fluid parts of a dynamic, living whole, changing and taking shape

"source"
unpredictability
playfulness
a beginning open to possibilities

"residue"
what comes to be,
a result of children's learning

Now imaginable are other equally critical differences. Given this mind-space, mathematics can emerge as a "residue" or a "source"[11] of children's learning. It does not have to be 'kept' as a "product" that is produced within the sequential synchronicity of predetermined "outputs" fed by particular "inputs". It is not framed in a deterministic or a predeterministic view of classroom mathematics. Here, Bob Davis'[12] notion of mathematical "residue" serves as a useful beginning point for REimagining mathematics in the classroom. As residue, it is assumed that neither the origins nor the processes by which a particular mathematics arises can be precisely located. It arises from the mathematical languaging of the individual and larger collective(s). Mathematical "processes" such as problem solving, connecting, reasoning, and expressing can be understood as "mathematical language"[13]; that is, they are the physical, verbal, and mental manners in which we can think, (inter)act, and exist mathematically. And, mathematics as a "source" for children's learning embeds a sense of unpredictability and playfulness-- a beginning that is open to all kinds of possibilities.

In a radically different metaphorical manner, focus is not only on the mathematics at hand, not only on mathematics as individual and collective knowing, but also on mathematics as it resides seamlessly and all at once within past, present, and future contexts of children's learning.

Notes

1. A. B. Davis, Sumara, and Luce-Kapler, 2000.
2. A. B. Davis, Sumara, and Luce-Kapler, 2000, pg. 62
3. A. B. Davis, Sumara, and Luce-Kapler, 2000, pg. 62
4. A. B. Davis, Sumara, and Luce-Kapler, 2000, p. 62-63.
5. A. B. Davis and Sumara, 2000; A. B. Davis, Sumara, and Luce-Kapler, 2000; Fels and Meyer, 1997; Meyer and Fels, 1997.
6. Illich, 1971.
7. G. Bateson, 1972; Thom, 2000; in conversation with S. E. B. Pirie, March, 2001.
8. Varela et al., 1996, p. 205.
9. Lewontin, 1983; Varela et al., 1996;
10. A. B. Davis et al., 2000; Martin, 1999; Pirie and Kieren, 1992; Steffe and Tzur, 1994.
11. A further consideration of R. B. Davis' (1992) notion of mathematics as "residue".
12. R. B. Davis, 1992.
13. Maturana, 1997a, 1997b, 1998.

"JUST HAND THEM DOWN THE MATHE- MATICS" ... OR NOT?!

WHEN MATHEMAT- ics is imagined and enacted as objectified, static knowledge that is to be traditionally passed down from one generation to the next, the teaching and learning of mathematics is dis- abled from ever be- coming anything else. Under the air of "hand-me-downs", it is easy to understand why mathematics is taught and learned out of a sense of obli- gation or contempt rather than a sense of open desire or wonder, and why, mathematics is all too often considered as that which is to be mastered rather than that which is to be understood. In com-

moditizing mathe- matics, we make ab- surd, the possibility for us as teachers and to those who we teach mathematics to perceive it as any- thing else but a fixed and inanimate entity. In this way of con- ceiving mathematics, we make it inconceiv- able for school ma- thematics to become something else than just a collection of hand-me-downs.

The embeddedness of these images with- in one's taken for granted ways of thinking about math- ematics not only make it natural for us to assume mathe- matics to be an inani- mate "thing", but in doing so, displaces mathematics as that which exists "out there". Given this mindset, it is not sur- prising why a teacher would feel impelled to set the class onto a straight and narrow, one-way course so that the students too, become collectors of mathematics. Given

this mindset, it makes sense to in grain the ritualistic practice of "acquir- ing" mathematics in- to school mathemat- ics unit and lesson plans, methods of as- sessment, and en- acted in the class- room; product orien- ted practices that fo- cus on "desired", "expected", or even "measurable" out- comes of instruction-- that after instruc- tion, the student will have "mastered" the mathematics taught in the lesson before "moving on" to the next part of the cur- ricular course. Of course, the ways in which children are in- structed to take pos- session of their mathematical hand- me-downs of con- cepts, skills, and even attitudes may vary. Still, "teaching by telling", engaging stu- dents in "hunting for", having them "seek out" "hidden" mathematics within "real" world contexts, and even "explor- ations" "designed" for children's discovery

155

of mathematics are all examples of teaching and learning forms that keep a-live, this tradition of "handing down" of mathematics.

Moreover, when product-oriented ways of thinking about school mathematics are coupled with a "back to basics" mentality, the teaching and learning of mathematics become subjected to the weigh scale of "how much" in regard to the amount of mathematical facts and skills that children are to learn and little or no emphasis is placed on such things as their mathematical thinking or understanding. Given this mindset, mathematical processes such as those identified by the National Council of Teachers of Mathematics[1] as problem solving, reasoning, communicating, connecting, and representing would likely be deemed "not essential" by most teachers. Viewed as "additional"[2] knowledge, teaching that attends to children's development of mathematical processes would depend on whether or not the children have acquired first, the *pre*-specified mathematical facts and skills with which to "process" the mathematics.

The point here, is that when children are taught to learn mathematics in the tradition of hand-me-downs and as a product oriented matter of collecting, hunting down, or retrieving pieces of knowledge, it creates the impossibility for mathematics to be taught and learned in ways that enable it to arise as living and animate.

Now, identifying the limitations of how mathematics exists in the classroom and the possibility of it becoming something else is all fine and good. But in doing so, means that the conversation does not end here. Rather, it opens up a whole host of questions that require further interrogation such as:

- How can an ecological way of thinking help us to *re*consider such taken for granted perceptions of classroom mathematics and *re*-imagine a more responsive view for the teaching and learning of it to exist in the classroom?

- What shifts in thinking become necessary in order to reimagine classroom mathematics as being something other than a line of hand-me-downs from teacher to child?

- What could it mean if we assumed mathematics to be "embodied"?

- How could mathematical problem solving, reasoning, communicating, connecting, and expressing be understood as something other than additional knowledge?

Notes

1. NCTM, 2000.
2. Baroody, 1993.

156

If we take seriously the view that mathematics is an ever-changing entity perceived, created, and embodied as we interact with the world[2], then it does not make sense for mathematics to be conceived as a fixed, inanimate, disconnected "thing" that exists "out there". When we assume that the only mathematics we know or can ever know emerges from our patterns of living as social-cultural beings then, mathematics is not an objective, universal, transcendental reality but a living system that is necessarily constitutive in nature. This means that mathematics takes place in the praxis of living in language.[3] Its coherence is dependent on those who interact with it.

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[2] Lakoff & Núñez, 2000.

[3] Bunnell, 2001; Maturana, 1988b.

Scripting an Unscripted Lesson

The children gathered on the carpet to hear me read the story, *Even Steven and Odd Todd.*[1] The beginning of this lesson was simple– there was no formal introduction, no preamble, not even a motivational hook. I just opened the book and started to read. This was not because I had not come prepared, I had. When I planned how I might share this story with the class, I decided that I did not want to ask the children to *look for, listen for,* or *think about* anything specific. I did not want to preface the book by telling them that it was a humorous tale about two cousins who are clearly different from each other in one particular way. I did not wish to tell them that in my earlier reading of the book I had found out that for Even Steven, life unfolds as patterns of even numbers and for Odd Todd, life is all about odd numbers. It was not that I wanted to "keep" the mathematics from the children but I set out to create a teaching-and-learning space where the class could experience the story *as* it unfolded, in *that* particular moment. I wanted to let the mathematics emerge in a different way than a predetermined, predictable one. In *not* creating a pre-scripted lesson that was about me, "the teacher", identifying "the mathematics" that the children were to find in the story, this lesson remained open to the mathematical possibilities that we might bring forth as we listened and responded to the book as a class.

A moment of digressive thought:

Problem Solving
Reasoning and Proof
Communication
Connections
~~Representation~~ Expressing

If one glances at the NCTM's list of the five "process" standards[2] or reads through the descriptions that accompany them, it is possible for teachers to interpret them as five discrete mathematical "skills". Given this, it is understandable why a teacher might then present them to the class as five separate topics.

It is only when carefully reading through the NCTM's[3] document, *The Principles and Standards for School Mathematics* and making note of the Council's statement, "[p]rocesses can be learned within the Content Standards, and content can be learned within the Process Standards'"[4], that a nonlinear image of mathematics and mathematical processes comes into view. In contrast to linear, mechanistic forms of thinking that would have us imagine mathematical processes to be mechanisms or devices that act as a conduit through which we "transmit" mathematics "into" children, the NCTM proposes that mathematics content and mathematical processes be understood as being reciprocal in nature. The Council takes the position that the process standards are not to be conceived as 'additional knowledge'[5] and certainly not as a "means to a linear end" in mathematics classrooms. However, because this point is communicated very briefly in the document, it does not make prominent, this non-linear and co-emergent image of mathematics and mathematical processes. Consequently, it remains faint and is easily overlooked by the reader who reads from a background of traditionally mechanistic ways of thinking.

Differently, if we set this image within an ecological realm, it immediately conjures up notions such as complexity, circularity, and recursion, all of which help to bring forth an implicit understanding that:

<div align="center">

children's mathematics

arise

can   from and be

shaped by

mathematical processes,

mathematical processes

can

also arise

from

and be

shaped

by

children's mathematics.

</div>

       If we gaze more deeply into the languaging space of ecology, what is brought forth is mathematics content and mathematical processes as enveloping and co-emergent entities. Ecological thinking not only enables an awareness for the complex circularity that exists with-and-in content and processes but also, a mindfulness for the interdependence that defines them as being inseparably part of each other. And it is not that they simply exist in a cyclical sense; that one prompts the formation of the other, but in an integral manner, mathematics content and mathematical processes emerge and evolve together in relation with each other.

       Now if we take the **NCTM***'s process standards and consider them from an ecological space that includes Maturana's definition of "languaging"[6] (and in keeping with this, Sfard's notion of mathematical thinking as "communication"), the processes identified as: problem solving, communicating, reasoning, expressing, and connecting can be understood as being forms of mathematical "language". As explained earlier, Maturana's description of languaging entails how collective unities evolve through their physical, verbal, and mental linguistic manners of thinking, acting, and existing[7]. Mathematical processes as forms of mathematical language are the mathematical patterns of thinking, interacting, and being that enable a class for example, to exist and develop as a collective system. If we think of mathematical language and mathematics metaphorically co-existing and*

<div align="right">160</div>

seamlessly co-emerging with each other, then each is fundamental to the other and in how we teach mathematics and how children learn mathematics.

When classroom mathematics is envisioned as a living system that emerges with the class' mathematical and social-cultural forms of languaging, it also becomes something more than just an end in itself. Mathematics can be thought of as that which arises as the result of children's learning as well as that which serves as a beginning for their learning to occur. Mathematics exists as both a "residue" and a "source" [8]. When mathematics arises from children's mental intra-actions or social inter-actions, it can be understood as being a "residue" or, "outcome of their learning".[9] This contrasts with pedagogical views that assume learning can be prespecified as "learning outcomes". An ecological examination does not focus on what the child SHOULD know but rather, on what mathematics the child actually comes to know. For instance, if a child explores how different sets of objects can be arranged in smaller, equal groups and arrives at an understanding that "division makes smaller", this would be considered to be a mathematical residue of the child's learning.

Notes

1. Cristaldi, 1996.
2. NCTM, 2000.
3. NCTM, 2000. Also, see p. 29.
4. NCTM, 2000, p. 30-31.
5. Baroody, 1993.
6. Maturana, 1998.
7. Maturana, 1997a, 1997b, 1998.
8. As previously described from an ecological perspective on page 150 and then on page 167. Also, see Pirie and Thom, 2001.
9. As previously described from an ecological perspective on page 150 and then on page 167. Also, see R. B. Davis, 1992; Thom and Pirie, 2002.

Mathematical Language, Languaging, & Residues of Learning
(a continuation of Scripting an Unscripted Lesson)

As I continued to read to the story to the class, the children began calling out differences that they were noticing between the two characters:

"Even Steven gets up every morning at eight o'clock sharp."

" Odd Todd likes to get up at nine o'clock sharp."

"Odd Todd rides a tricycle."

"Even Steven has four bicycles– they have two wheels each."

"Even Steven has six cats, eight gerbils, and ten goldfish."

"... and twelve sprinklers in his garden!"

"Odd Todd has five buttons on his jacket and Even Steven has six on his shirt."

While I recorded the children's observations onto a large piece of chart paper Danica looked at what was being written down, glanced away for a moment, and then announced that "Even Steven only likes things that are two, four, six, eight, ten, twelve, and so on... and Odd Todd only likes things that are one, three, five, seven, nine, and eleven".

The whole class nodded and smiled in agreement.

Mark then added, "Even Steven likes *EVEN* numbers".

This was immediately followed by Robby's comment, "and Odd Todd only likes ODD numbers".

However this time, only some of the class nodded or responded with "yeah!" while other children said nothing, looked puzzled, or exclaimed, "what?!"

"Numbers that end in zero, two, four, six, and eight are even numbers and numbers that end with one, three, five, seven, and nine are called odd numbers" stated Mark.

Still, the class reacted with a mix of nods, furrowed brows, and a bunch of "what?!s".

Jumping into the conversation, I agreed with the two boys, "Yes, that's one way of thinking about numbers as being of two different kinds" and then began to push this space that had now been opened by Mark and Robby a little further. Recording the numbers on the chart paper as I spoke, I posed a question to the whole class. " If two, four, six, eight" I began, "and numbers that end with zero, two, four, six, or eight, such as ten that is written as one-*zero*, twelve that is written as one-*two*, thirty-*four* [recording 34]... sixty-*eight* [recording 68] can be described as being *even* and one, three, five, seven, and nine, [recording 1, 3, 5, 7, 9] as well as numbers that end with one, three, five, seven, or nine such as eighty-*one*, forty-*three*, eighty-*five*, sixty-*seven*, twenty-*nine* [recording 81, 43, 85, 67, 29] can be described as being *odd*... *what* is it that makes certain numbers *even* and other numbers *odd*?!... besides just looking at the digit that they end with?!"

Silence. The children said nothing. One second... two seconds... three seconds... waiting. They simply stared back at me, shrugging their shoulders. Michelle leapt into the conversation and conjectured that "*even* numbers– you add two to them... two, four, six, eight, ten".

Moving into her space of thinking, I poked around a bit and said "yes, this is true... but what about one, three, five, seven [pointing to the series that I had already written down on the paper]... aren't you also adding two to one to get three and two to five to get seven...?"

"Oh yeah" Michelle said, smiling.

"As well" I added, "think about Even Steven... did he do everything in a sequence of two then four, then six, then eight...? or Odd Todd, did he do everything in a sequence of one, three, five, seven, and so on?"

"Hmmm... no" Michelle replied, shaking her head from side to side.

I then had the class form a large circle on the carpet. As I moved behind them on the outside of the circle, I randomly asked some of the children to use counters and to build one of the odd or even numbers listed on the chart paper with counters and to place them in the middle of the circle. By doing so, we would be able to continue the exploration but this time, take a closer look at the numbers in their physical form. Although the children did not arrange the numbers they had built in any particular order, except for separating the even numbers from the odd numbers, we soon had physical expressions of 2, 4, 6, 8, 10, 12 and of 1, 3, 5, 7, 9, 11. I asked the class to look at the numbers that had been built, to talk to one another, and to see if there was anything that was "even" about the even numbers or anything that could be considered as "odd" about the odd numbers.

After working with the students on either side of her, Shelby raised her hand to speak and offered this: "The four, it's even because it has two and two. "

I, as well as some of the other students nodded and smiled.

"Yes. That makes sense... we can think of four as being evenly two and two." I said, replying to Shelby's comment.

Taking the idea that the group seemed to be embracing as making some kind of sense to them, I encouraged the class to 'pick it up' and play with it a little more. "Okay, what about the other numbers that we've built here? Can we take Shelby's idea and... Is there anything that we could say about the other numbers, using her idea of 'even twos?'"

Without any talk, the children clustered themselves into smaller working groups, reached into the middle of the circle and set to arranging the counters for each of the even numbers into pairs, by twos. When all the children had finished working, I asked them what sense, if any, they were able to make from what they had done.

They explained to me that two was an "even two", four was "two even twos", six was "three even twos" and so on. And when I posed questions about larger numbers like "what about thirty-six?", I quickly got responses such as "that'd be eighteen even twos".

Noticing that the children had not said or done anything with the set of odd numbers they had built, I pointed to them and asked, "what about these?"

This provoked the children who were sitting nearest the odd numbers to reach into the middle of the circle and begin to move the counters about. Soon comments such as "this one doesn't have a partner", "neither does this one", and "none of them do" began to surface in the conversation as the students arranged each of the numbers by twos. Then, just as I had done with the even numbers, I asked the class what they could say about larger numbers. "What about twenty-nine?" or any odd number. Here, I got replies of "it would have fourteen even twos and one leftover" and "if you put it into partners, one would always be left lonely".

As we sat back and looked at the two sets of numbers now arranged evenly or unevenly by twos, I asked the children if there was any sense in looking at these numbers as being two different kinds, and what sense if any, was there in Mark's claim that you could simply look at the last digit to figure out if it was an even or an odd number. The class agreed that any number could be described as being either odd or even and when asked, some students indicated by nodding their heads that yes, you could just look at the last digit of a number and determine whether it was even or odd. What intrigued me were the different ways of thinking that the children had created for making sense of odd and even numbers.

For instance, Danica explained, "A number bigger than ten, like forty-eight, the forty is groups of ten... even... and the eight is four even twos... it's an even number."

"And forty-nine?" I asked.

"The forty is four groups of ten... that's even, and nine is an odd number because it has four even twos and one leftover, so it's an odd number", Danica explained.

"Are you saying that you don't need to look at the other digits, only the last one to determine if the number is odd or even?" I asked.

"Yes... because they are always even... tens... hundreds... they are always something-zero" she said.

For other children, determining whether a number was even or odd was based on different ways of thinking than Danica's place-value-last-digit notion. Some children concluded that if a number could be "split into equal halves" then it would be an even number and "if it couldn't, then it would be odd". Other children replied that if they could arrive at the number by counting up by twos from zero, the number was even, and if they counted past the number using this rule, then the number would be considered to be odd. Still other children said that through building or visualizing the number as as a set of counters and then "partnering up" the counters by twos, they would be able to determine if a number was odd or even, depending if there was one as a remainder or not.

Reflecting on this lesson, I was reminded of Theodore Roszak's book, *The Cult of Information*.[1] In it, he explains that as humans, it is ideas and not information that we think with; that it is through the integration of patterns that emerge as a result of our lived experience that ideas come to be and from which information arises. And so too, the class integrated their ideas of lived mathematical experiences to form meaningful information about odd and even numbers. They did not blindly accept Mark's mathematical fact that even numbers are numbers that end in 0, 2, 4, 6, 8, and odd numbers are ones that end in 1, 3, 5, 7, or 9, or Shelby's conjecture of "even twos" as "truths". Rather, the class jumped into the mathematical spaces that had been opened and explored these spaces in order to develop individual and collective mathematics that made sense.

Another afterthought: A curious space that was not opened because neither I nor the children thought of it at the time (and the story frames the reader's thinking to assume that all numbers are even or odd) was this:

Is there ever a number that is NOT odd or even?

On another occasion this might have arisen naturally and taken the class to a different mathematical place. Nevertheless, this lesson served as an example of how the mathematics of odd and even numbers was brought forth by the class' mathematical languaging of the story through their physical building of numbers, mental images of numbers as a collection of objects and symbols, as well as powerful metaphors of "even twos", "partners" and "lonely ones".

The mathematical residue that came to be as events of the children's learning were unpredictable and distinctive yet integrated. Not only was there a collective residue of the children's learning; that numbers could be thought in terms of being even or odd, but there were also mathematical knowings within this residue that were also collective and individual in nature. These included the children's notions of place-value-and-looking-at-the-last-digit as well as the patterns they established through their actions of counting and arranging. Like different shades of a colour, each mathematical residue was distinctive while at the same time, each blended with the others, adding depth and dimension to the children's understanding of odd and even numbers.

Notes

1. Roszak, 1994.

Differently from mathematical residue, when mathematics becomes that which occasions children's curiosities, it exists as the source or a place-- a beginning for their further mathematical growth to occur. For example, by posing the question, "Is it true that division always makes things smaller?" enables a source from which many different mathematical directions and spaces for children to take their learning can occur. Mathematics as a source creates openings for children to move mathematically and to deepen their understandings. Given this, children could explore and establish ways of knowing for when and why such a meaning of division would be appropriate and also experience situations in which an understanding of when and why it may not make sense and mark out another place of knowing.

Arisings

(A continuation of Mathematical Language, Languaging & Residues of Learning)

Settling down in front of me again, the children looked up, telling me that they were ready to finish the story. I read on from where we had left off. It was when we came to the part in the story where Even Steven sets aside six pancakes for his lunch and Odd Todd comes along and eats three pancakes that the children got very excited. They stopped gazing at me and turned towards one another, gasping in astonishment.

"He [Odd Todd] just made Even Steven's pancakes odd!" one student exclaimed.

Another child giggled and then whispered to her friend, "Odd Todd is very clever!"

Even after we continued and finished reading the story, the children's chatter about how the even number of pancakes had been "turned into" an odd number of pancakes had not diminished. When I asked them what it was that had them so intrigued about the pancake incident, a flurry of responses came at me:

"I want to see what odd numbers I can make even!"

"Can ALL even numbers be made into odd ones?"

"I wonder how Odd Todd would eat his cousin's 8 pancakes?... so they'd be odd."

"Or Even Steven's ten pancakes?!"   "Or his thirty-six pancakes?!!!"

"I want to find out some other ways Even Steven could eat his twelve pancakes."

"Does an odd number and another odd number always make an even number?"

Here within the same lesson, the mathematical ideas of odd and even numbers emerged again; not as mathematical residue but this time, as mathematical "sources". The children's questions created new places for them to explore and develop their understanding of odd and even numbers. And for most of the two days that followed, the class created smaller working groups and explored the questions that they had posed.

Do the metaphors with which we describe beginnings and ends really matter?

I suppose it is possible one could argue, that even in mechanistic patterns of thinking, mathematics can exist as both a beginning and an end in children's learning. For example, rather than being defined in terms of "source" and "residue", the metaphors of "input" and "output" could be used. By doing so, one might wonder whether in fact this does not make both ways of thinking pretty much the same. And if not, what exactly is the difference?

Yes of course, mathematics CAN be thought in terms of "inputs" and "outputs" of children's learning. However, there are critical differences between these metaphors and the ecological ones of "residue" and "source". You see, the mechanistic metaphor of "output" by its very nature, evokes the idea of (school) mathematics as being a product that is produced or re-produced as a result of a chain of learning events taking place. "Input" conjures up the notion of identifiable, measurable, and even prescribed "ingredients" being used or being "added to" in order to produce a certain mathematical output, result, or product. Together, these metaphors embed a sensibility that assumes specific events that give rise to particular mathematical outcomes can be identified as such and that mathematical outcomes "fueled" by mathematical inputs are predictable and perhaps, can even be predetermined. These ways of thinking about how mathematics exists in children's learning serve only to maintain a confining and reductionistic view of classroom mathematics.

In a very different way, the integrity of the ecological metaphors for mathematics as being a "residue" and a "source" for children's learning is that they do not embed a deterministic or predeterministic view of classroom mathematics. Rather, mathematics as "residue" evokes a notion of something that has come to be, what is left, or what remains as a result of children's learning. It is not a way of thinking that engenders a deterministic stance that presupposes that the origins of mathematical residue can be precisely located or that the process(es) or the languaging acts by which it came to be can be identified, specified, or replicated. The metaphors focus attention on children's mathematics at hand-- the mathematics and mathematical understanding that is emerging. Just as important, is the ecological metaphor of mathematics as a "source" for children's learning. It contrasts with the mechanistic image of mathematics as "input" in that the metaphor of "source" brings with it, a sense of beginning; a beginning that is not concerned with predetermining or predicting what "should" follow but instead, highlights the need to be mindful and open toward the possibilities that "could" unfold-- anticipated or not. Imagining mathematics to be both residue and source infuses a sensibility that is open (as when the ground opens and water springs forth) and ever-changing (as how one can never step into the same river twice). It offers different images

170

and meanings to consider when thinking about how mathematics might exist in the classroom... images and meanings that mechanistic ones cannot.

After having spent considerable time moving deeper into that other mind-space, Jennifer emerged with a new understanding of what it meant to teach mathematics. Her work in re-imagining metaphors that possessed an ecological sensibility was occasioned only because she now knew the two systems of thinking to be what Maturana calls consensual domains[1] . She realized how her previous way of making sense of her teaching and her new ecological one highlighted and diminished particular issues or concepts through their different[2] metaphorical languages. For Jennifer, it was impossible to conceive language as simply a tool for communication.

> *We exist in language and it is through our being in language and languaging that we bring forth metaphors that invisibly and powerfully become our ways of thinking, how we teach, our ways of researching, and ultimately, the kinds of places that are created in the mathematics classroom.*

Notes

1. Maturana, 1988b.
2. "Different" here, does not necessarily mean "incompatible" or "disparate" but rather, "diverse".

"Atmosphere," as the word suggests, is a vaporlike sphere which envelopes and affects everything.... The sense of mood or atmosphere is a profound part of our existence. By it we know the character of the world around us. Mood is a way of knowing and being in the world.... the way in which space is lived and experienced.[1]

[1] van Manen, 1986, p. 32

[E]very classroom, every school contains a certain atmosphere [sense of place]. The question is *not* whether there should be a pervasive atmosphere in the [classroom or the] school, but rather what kind is proper for it, worthy of it.[2]

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[2] Emphasis added, van Manen, 1986, p. 31.

Sense of place.
Place-making.

van Manen's thinking about atmosphere echoes the very notions being explored here of ecological thinking and of place.

Worthiness

The author moves the conversation into the realm of education and highlights the taken for grantedness of atmosphere. He identifies it as being that which profoundly shapes teaching and learning. van Manen makes important, the need to be aware of it, to consider the qualities that make up a particular atmosphere, and to exercise a mindfulness and care for the sense of place we bring forth for and together with our learners. This certainly implicates the examination of one's metaphors and the mathematics teaching and learning occasioned from the embodiment of them. In doing so, van Manen hopes to provoke us to question the worthiness of our actions in relation to the kind of place such rituals in teaching mathematics create.

Because this tacit knowledge influences both *what* and *how* learning occurs-- for all participants.... In some instances, the teacher must keep up with the social reconstituting of taken-for-granted knowledge, and in other instances... the teacher should take a leadership role that can only be fulfilled by modeling and not simply by substituting a new set of taken-for-granted beliefs for the older ones.[3]

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[3] Emphasis added, Bowers & Flinders, 1991, p. 11.

The "leadership role" that Bowers and Flinders speak of is important in developing one's teaching of mathematics. Relating with earlier sentiments, growth as a teacher is not about breaking away from one's entire teaching and replacing it with something else. Not only is the possibility of this questionable, such thinking only exacerbates 'this-or-that', 'either-or' attitudes and reactions. Differently, what Bowers and Flinders argue for is in keeping with what van Manen too seeks. Situated in the mathematics classroom, this entails a teacher's MINDFUL consideration for the KIND of oikos being created for learners through what one chooses to conserve in one's ways of knowing and actions as a mathematics teacher.

For me, this has meant assessing the worthiness of the metaphors embedded in my teaching of mathematics, identifying which ones did not engender an ecological sensibility and re-imagining ones that would. One might assume that doing so should make the enactment of these conceptual shifts natural and effortless given that I was now thinking within a different theoretical system. However, each day that I stepped into the classroom, I was confronted by the fact that...

AGAIN I'm reminded that learning what it
even though I have consciously created "differences"[1] in how I conceive
means to teach mathematics is not an
mathematics teaching, in the classroom-- as part of this place, I am still and
automatic process. It's not smooth, it's not
always will be unconsciously embedded in a web of taken for granted
straightforward, and it certainly doesn't
relationships-- taken for granted language in which and by which I teach
appear on demand.
and children learn mathematics.[2]

The challenge then becomes, which relationships need to be a critical part of one's teaching consciousness?

•

•

•

How might one go about carving out different spaces for teaching and learning?

•

Establish new relationships?

•

Expose a different kind of place for mathematics to grow?

•

•

•

Assuming teaching to be praxis and not a practice imparts the understanding that one can never be separate from the students themselves (and the metaphors they bring to their understanding of place), the events that unfold, or the relationships that exist within the classroom. All co-emerge and co-evolve as living systems WITHIN systems dynamically responding to one another. One's teaching then is not identified in so much WHAT it is but instead, HOW it is-- how it is an interdependent part of the whole. So for me, my work is concerned with how my mathematics teaching relates to the larger whole of embedding an ecological sense of place for mathematics in the classroom.

As well, there is not the anxious temptation to systemATically take my new found metaphors and fit them into the classroom. By situating my study of mathematics teaching within the systemIc realm, the focus becomes grounded in understanding how my metaphors think me (in ecologically coherent ways) as I think within them.[3] Once again moving off the mental line enables me to head towards that "other" space.

oikos...
relationships...
relationships as patterns...
    ... as patterns of difference.

So in the same way that Maturana and Varela speak of knowledge as constitutive in nature; that is, as dynamic structural relationships within a living system and that which distinguishes one living system from another, G. Bateson's[4] concept of mind as connecting patterns of differentiation is also what is at the heart of this work. Focusing on difference in an ecological manner means recognizing difference as a relationship and not as a thing.

"co-emergence"
"non-linear"
"possibilities"
"knowing is doing is being"
"occasioning"
"fluid"
"co-evolution"
"embodied knowings"
"complex circularity"
"individual-and-collective"
"unpredictability"
"sustainability"
"interdependence"
"diversity"
"mindfulness-awareness"
"non-hierarchical"
"recursion"
"holistic"
"knowledge-as-interaction"
"like paths that exist only as they are laid down in walking."

The difference that makes a difference in my teaching of mathematics as learning place centres on the impact that my metaphorical patterns of thinking have on the oikos of it.

So what are the patterns of difference in my new metaphors?

In contrast to the metaphors embedded in my previous conceptualizations of the mathematics class, mathematics, and curricula as a collection of individual "parts", mathematics teaching as a linear chain of events, and mathematical learning and understanding, as building structures, these new metaphors of maps, paths, living systems, residue, and source evoke notions of nonlinearity and unpredictability. They speak of mathematics teaching and learning as complex and recursive growth. They create the image of everything existing in fluid relationship to everything else. They enable the integration of mathematics to be understood not as a product of teaching and learning but as that which happens in the flow of the two. And, that neither mathematics teaching nor the learning of it nor mathematics itself for that matter, exist as "things" but all arise in the midst of dynamic interplay... varying in form and being occasioned in different ways.

These, I would say are the patterns of difference in my metaphors.

Notes

1. G. Bateson, 1972, 1975.
2. Of course, this also includes the taken-for-granted language and languaging of society and particularly how this has influenced the students.
3. Engaging in the examination of how "language thinks us as we think within language."
4. G. Bateson, 1972, p. 318.

Embedding and Rooting an Ecological Sense of Place for Mathematics in the Classroom

What we conserve, what we wish to conserve in our living, is what

Knowing the ground on which I walk. Minding the paths that unfold.

determines what can and what cannot change in our lives.[1]

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[1] (Maturana, 1997b, p. 5)

[c]hildren must feel that... the space, materials, and projects, values and sustains (sic) their interaction and communication.[2]

[2] Rinaldi, 1990 as cited in Edwards, 1993, p. 137.

186

Attending to the Physical Space in the Classroom

Stepping into a classroom is not unlike studying a painting. Both tell of relationships-- the assumptions, ways of knowing, and the kind of experiences that have come to form a classroom's sense of place or the subject matter of a painting. In this manner, albeit through different methods, teachers and artists engage in the act of portraiture. In creating a painting, these relationships are expressed through the media and design of the work. The feeling of coldness and tension for example, might be told through the painter's use of colour and texture whereas coldness and tension in the mathematics classroom is often conveyed through the austere and rigid manner in which the subject is presented by the teacher. Connected to this is how a teacher structures the actual physical space for learning and the types of materials made available for students' use that define the kinds of teaching-and-learning patterns that exist.

Stepping Into Jennifer's Classroom

What is most apparent upon entering Jennifer's classroom is that there are many different areas that make up the space. Like a house with many rooms but without the walls and partitions, this class too has different "rooms". Each of these rooms or what she considers to be physical spaces for learning, are distinct in their purpose and the kinds of mathematical interactions they enable.

At their desks: Working on their own

At their desks: Working with a partner
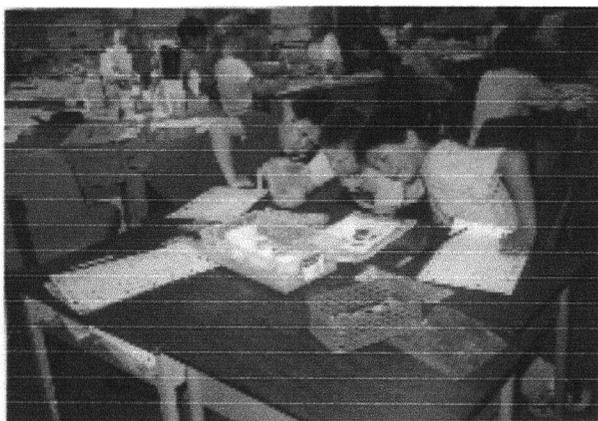
Small group work at their desks
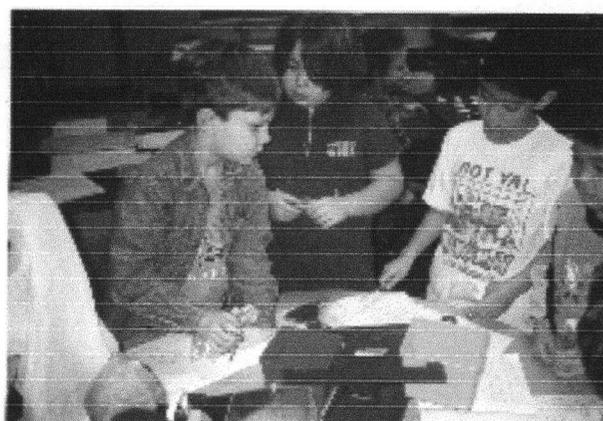
Large group work at the round table

The children's desks are arranged in 2 x 2 groupings in the middle of the room from which the chalkboard, chart board, and overhead projector screen can be seen. Although the desks store student belongings and each bares a child's name, the children frequently move about and use one another's desks. This area allows the students room to work on their own, with a partner, or in small groups. When in larger groups, the children often choose to work at the large round table located across from these desks.

Meeting place
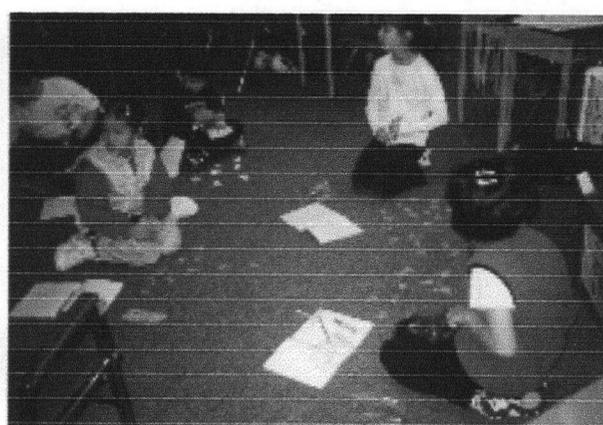
On the floor

Two carpeted spaces can be found on either side of the desk area. The larger of the two is a meeting place of sorts, where Jennifer and her students gather in the morning and at the close of each school day. This is also the place where the class meets to share and examine the mathematics they have been working on. Like the table groupings, the carpet allows the students to work on mathematics on their own, in pairs, as a small group, but also, as one large class. This space, the round

table, and the cluster of desks provide useful places for the students to work with mathematical manipulatives.
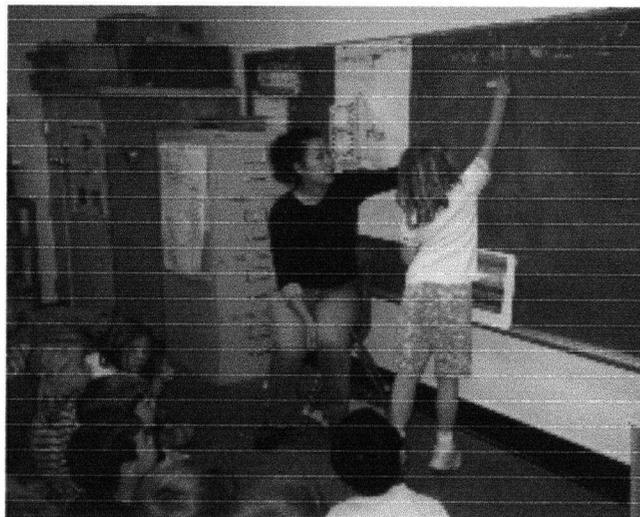
Physical mathematical work at the chalkboard

On the other side of the desk area is the smaller of the two carpeted spaces. This one was created by Jennifer so that she and her students could use the chalkboard, overhead projector, and pocket chart during mathematics lessons. All of the equipment is positioned so that they are at appropriate height for the children to access. This place of gathering is not so much for active, physical investigations of
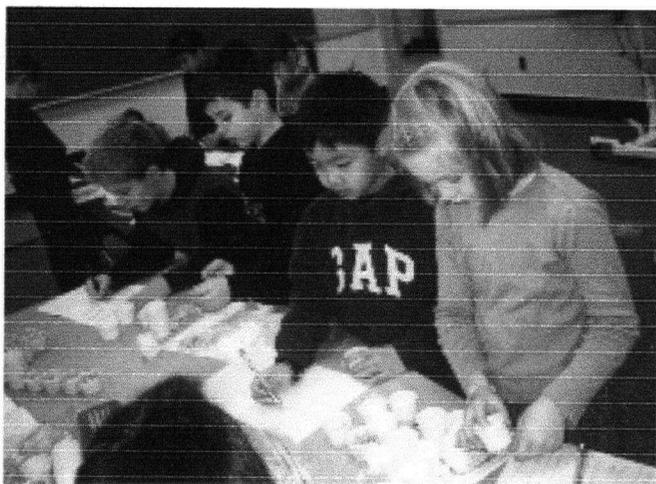
In the "big" room

On the playground

mathematics, but for the class to come and record, display, or bring key mathematical ideas into focus for further studies.

Finally, Jennifer uses still other physical spaces to extend the children's mathematical learning beyond the walls of the classroom. They include an adjoining classroom that provides a large, completely open area in which to work, the children's homes, the natural environment, and a neighboring elementary school. The latter three offer contexts where the students can take the mathematics that they have developed in the classroom and explore the relationships that lie within these other settings.

Materials Matter

Along one of the walls in Jennifer's classroom are three large shelving units: one with blue drawers, one with yellow, and one with red. Each shelving unit contains sixteen drawers, that hold a variety of materials used for mathematics. The blue drawers contain mathematical games and puzzles.

The yellow drawers contain calculators and different kinds of manipulatives for mathematical investigations. They are used by the class as physical objects with which to think about particular mathematics or with which to express their mathematical thinking. They include such items as: as number cubes, tangrams, calculators, zaks[1], geoboards, multi-link cubes, pattern blocks, attribute blocks, base ten blocks, and double-sided counters. There are also many found items that the children have collected from their homes and neighbourhood such as different kinds of rice, pasta, beans, seeds, stones, chestnuts, and buttons. Here, the students make use of these materials for purposes of sorting and classifying, estimating and measuring (i.e., linear, area, volume, mass), computations, as well as number concepts and counting strategies.

The third set of red drawers contain papers of all types: plain, lined, graph, dot, and construction. These are used by the class to build or record their mathematics.

Together, the materials and organization of spaces for learning in this classroom create a portrait that tells of an open, accessible place that nurtures the growth of children's mathematical thinking. The classroom as a physical and expressive form of the relationships that are embedded here, communicates the importance for mathematical interactions to be flexible and diverse.

Notes

1. These are interlocking polygons used for building 3-D structures.

An environment is a living, changing system. More than the physical space, it includes the way *time* is structured and the roles we are expected to play. It conditions how we feel, think, and behave.[1]

[1] Emphasis added, Greenman, 1988, p. 5.

"ABSOLUTE, true and mathematical TIME,
of itself and from its own nature,
flows equably
without regard to anything external,
remains similar and immovable...."[1]

sure in its certainty
TIME lives its own independent existence...

SEPARATE   and   d   i   s   c   o   n   n   e   c   t   e   d ...

created and affected
by
no person
no space
•
•
•
•
•
•
•
•
no-thing...

TIME does not exist with us or because of us...

TIME is just "out there".

... and even though...

TIME is undetectable through human perception[2]...

---

[1] Emphasis added, Newton, 1687 as cited in Koyré, 1958, p. 161-162.
[2] Abram, 1996.

it is **REAL**.

and even though...

TIME is empty... spaceless... invisible... and...

s

h

a

pe

less....

it is **REAL**.

```
TIME is exactly infinite...
it can't be taken a-p-a-r-t
or pressed together..........
it can't be disposed of ......
it just IS................
```

trust it is there.

"Time is just this: the number of a motion with respect to

the prior and the posterior"[1] Time as relative --- as linear --- as

[1] Aristotle, 1969.

straight as an arrow --- flatly fuses with events --- figuring the

spaces --- delineating relationships --- defining   moments ---

stringing beads --- one after another after another after

another --- each as a middle that unfolds from a beginning

and towards an end --- forward moving --- this kind of time

--- these kinds of spaces --- can be measured --- numbered ---

separated --- sorted --- sequenced --- fixed --- located ---

walking the temporal tightrope --- PAST behind us --- that

was then --- PRESENT --- this is now --- FUTURE AHEAD ---

--- how much further .... ?

keep going ....

[The following text is hand-lettered in a spiral, read from the outer edge inward:]

"living takes place in the now and we think, act, and become, moment to moment"[5]. ...transformation and everchanging ... in wholeness and ... and ... indistinguishably ... we live in the changing present in which ... come to form together[7] ...an-and-each-time-and-each-an ... separation ... blending and blurring any separation blurring and blending "Time never becomes long because it originally has no length"[6] ... there are no beginnings, middles, or ends ... "living takes place... Life ... is a circle from childhood to childhood and so it is in everything....[1] ...inseparably, fluidly, The moon does the same, and both are round ... past and future take place now."[4] ... Even the seasons form a great circle in their changing ... "but not along the tightrope of singular pasts, presents, and futures .. inseparably, fluidly, The sun comes forth and goes down again in a circle. Birds make their nests in circles ... our lives unfold in moments"[3] The Wind...whirls and whirls... recursive and cyclical[2] ...Time... whirling in its wholeness ... becomes with space... and as one ... they take place as now ... "our lives unfold in moments" Time

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[1] T. C. McLuhan, 1987, p. 32-33.

[2] Abram, 1996; Bowers, 1997; Maturana, 1995, 2001.

[3] Kabat-Zinn, 1994, p. 4.

[4] Maturana, 2001.

[5] Maturana, 2001.

[6] Heidegger, 1924, p. 14E.

[7] Abram, 1996; Heidegger, 1924, p. 22E.

Enacting Reconceptualizations of Time

Creating a classroom that engenders a systemic sense of place for mathematics requires consideration of how time is understood, how it is enacted through one's teaching, and the impact it has on shaping the spaces where mathematics occurs.

The languaging effect that my unconscious linear image of time had on the structuring of my mathematics program and the kinds of learning spaces that were then actualized is a clear example of this. Distinguished by grade levels, set within clearly marked boundaries of a Monday through Friday, 11:20 am to 12:00 pm and September through June frameworks, I conserved and delineated a place for mathematics in the classroom as a means to an end. The mathematical content of each lesson picked up from where the last one left off and moved the students forward in a concrete to abstract fashion to the next preplanned lesson. Together, these lessons formed instructional "units" that in turn became the year's mathematics program. In the larger scheme of things, each of these programs served as component parts within the K-12 mathematics curriculum.

My linear differencing of time "mapped" in my responsibility as teacher to provide mathematics programs that functioned as links in a curricular chain and progressed the children in their mathematics from one year to the next. In doing so, a straightforward linear time-space of mathematics teaching and learning was created and maintained. Consequently, what my conceptualization of time did not map in was the possibility for time-spaces to embody fluidity, nonlinearity, and recursion-- critical qualities inherent in my new metaphors.

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What if time was enacted as nonlinear, flowing, and recursive?

·

How might it be occasioned in the mathematics classroom?

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What role could it play in re-placing mathematics teaching and learning?

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What kinds of mathematics might then emerge?

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Raising Questions and Questioning the Answers
Linear Time-Spaces for Mathematics in the Classroom

My engagement in wondering WHAT IF? *has revealed all kinds of unnoticed linear and static time-spaces with which I have furnished my mathematics program. Becoming aware of them and bringing them into question has not been unlike my experience in confronting and re-imagining new metaphors.*

ASSESSING THIS FURNITURE: *Why do I organize my mathematics teaching into two separate programs?*

Honestly, I could not come up with any meaningful reasons for why I separated and taught two one year mathematics programs. Sure, I could say that organizing time in this way clearly defined what students were to learn in second and third grade mathematics. I could also say that it served as the base from which I could plan the program overview, units of instruction, and individual lesson plans. However, given the metaphors with which I was now working, none of these explanations of cut and dried efficiency made sense to me anymore. Categorically complicated' and fixed, they were devoid of anything that assumes complexity or strives to be life-giving. Up until now, the issue of time merging with space had never been an important consideration of my teaching. It was not a part of my map of what it means to teach mathematics. Linear ways of structuring time and spaces in which to teach mathematics was what was taken for granted and ritualized by everybody-- administrators, teachers, parents, and the students themselves. But now the issue of time was emerging as a difference of critical importance that would not go away and demanded my attention. Now visible through confronting my temporal ignorance, contemplating its incoherence in my teaching and compounding this with my desire to embed an ecological sense of place in the classroom for my students' mathematics, rethinking my enactment of time proved to occasion new patterns of difference for the coming school year...

213

$\mathbf{I}$ continued to teach second and third grade students but began preparing for the upcoming year of teaching by considering the kinds of temporal patterns necessary for opening teaching and learning spaces that were organic and generative. No longer wanting to teach mathematics in a grade to grade manner but in a way that focused on nurturing an ecological sense of place and the dynamic growth of the children's mathematical understanding, I made the decision to teach a two year program as opposed to two separate one year programs or a single grade class. By doing so, I was able to expand the time-space from ten to twenty months. In contrast to my image of teaching and learning as a linear time-space, I envisioned the first year of the program as being an enveloping and co-emergent layer that grew out of the children's previous inner layers of individual and collective mathematical activity.[2]

The second year was too conceived as a living curricular system and one that would create further layering of the children's mathematics. The Pirie-Kieren model of growth of mathematical understanding was a critical part of the structuring of the program as it locates children's embodied mathematical knowings as being where all new knowledge develops and thus, the place where the two year program would begin.

If time is to be enacted as nonlinear and recursive, then the space in which it is a part must also emerge as such. Thinking systemically about the growth of mathematical understanding necessitates teaching to be a dynamic and responsive activity. Teaching and learning as living systems unfold moment to moment, co-existing and co-evolving in relation to each other.[3] Although the two are viewed as activities that cannot be prescripted, this does not assume that responsive teaching does not require anticipative preparation on the teachers' part or that it is a random activity. As praxis, teaching responsively means being attentive and mindful towards how one's teaching impacts and is impacted by the class' mathematical work.

A mathematics curriculum envisioned as a map, enables teachers to locate
I identified important learning aims that included the goals of the Ministry[4]
mathematical topics, concepts, and skills that are considered to be important
and other documents (e.g., NCTM, BCAMT)[5] for the second and third grades but
landmarks for the class' learning... what cannot be sketched out in advance, are the
instead of sequencing them or categorizing them according to grade levels, I
actual paths that the children will travel... to get to the mathematical locations... or
marked these aims on my two year map as important mathematical sites to be

the understandings they will establish when and after they come to these sites... the explored. As I did this, I also considered contexts that would be open to many map... cannot possibly show the diversity of the landscape... [children's mathematical different kinds of mathematical investigations[6] and ones that the class might revisit paths are rarely if ever, straightforward but instead, spread out in several directions later on in the year or even again in the second year. While I mapped out a and entail twists, turns and switch-backs... Mathematics curricula and learning a curriculum, I was ever mindful that it was just that– a map; distinguishing imagined in this manner distinguishes them as co-emergent phenomena that are pedagogical and mathematical features for teaching, and that the actual forms brought into being through children, their teacher, and mathematical. settings. that my teaching and the students' learning would take on were yet to unfold.

Because it was my intention for the program *not* to have a prescribed teaching sequence, it was important for the learning spaces to be ones that not only had an open flow in terms of mathematical content but also, ones that would encourage the reintegration and renewal of the children's mathematics. To do this, provocative themes were created by myself, with the children themselves, and with other teachers with whom I collaborated. These themes arose from the children's ongoing work and usually came in the form of curious questions or specific topics such as *Who are these things we call numbers?*, *Snowflakes, Number Gymnastics,* and *Mathematics About Me.* So instead of instructional units organized by particular mathematics such as addition, subtraction, or geometry, these themes focused my teaching and the students' work on exploring mathematics as a diverse and interconnected whole.[7]

It was here that I realized how the integrity of this program would be compromised if I continued the ritual of scheduling mathematics into 40 minute daily intervals. Affirming my disbelief that children only have short attention spans, I was inspired by the stories of the Reggio Emilia schools[8] and looked for ways in which I might expand and enable flexible time-spaces to exist in the classroom. I seeded the program with the idea of establishing a place that focused on students' mathematical growth. It made sense then, for these themes to occur in time-spaces that not only allowed for the children to work on ongoing projects, but also ones which enabled the mathematics to shape-shift into different forms of mathematical languaging, to branch off, intersect, and flow from real life contexts into purely mathematical ones and vise versa. Instead of cutting up and inserting mathematics lessons into 40 minute times slots, I opened up larger spaces of time within the day and even entire school days to allow for this.

For the past two months, the children and I have been working on a variety of projects and investigations that focus on their meaning-making of numbers within different contexts. This week, the class watched the film 'Notes on a Triangle.' Reviewing the film on several occasions and from different mathematical perspectives piqued the children's curiosities. The vantage points they marked out-- all of which happened to revolve around the idea of 'three', established the specific mathematical spaces for the children's explorations. Inside these learning spaces, the students worked to develop understandings for the threeness of a triangle, the threeness of particular numbers, why three is considered odd and when it becomes even, the rhythmic pattern of three-four time, and the aesthetic value of three in creating visual art. Here, concepts of addition, multiplication, number theory, patterns, and geometry were brought into being through the class' engagement with the film.

These time-spaces contrast with previous lessons in which I clearly marked beginnings, middles, and ends (i.e., introduction, development, and conclusion). They serve as examples of how mathematics curricula can arise in forms that are generative and embody a sense of flow... where time and space are taken to be inseparable and give rise to one another within to the contextual boundaries of the class' mathematical experiences. Also highlighted is how mathematical concepts emerge from the class in a way that assumes mathematics to be a living system. In these lessons, the mathematics taken up were not explorations in the practice of dissecting and dismembering but quite the opposite. Studying the film as an entity-- as a mathematical form in itself allowed it to be viewed as a source from which further studies could be investigated and connected to one another... keeping mathematics intact and a dynamic whole.

Notes

1. A. B. Davis et al., 2000.

2. This idea of an inside core parallels the Pirie-Kieren (1994b) dynamical model of growth of mathematical understanding which defines primitive knowing as being all the knowledge that a child or group of children bring to the particular mathematics and from which all new understandings develop. It is this embodied mathematical knowledge that I assumed my students possessed upon entering the two year program.

3. M. C. Bateson, 1994.

4. Ministry of Education, 1995.

5. For example, resource material and educational literature published by the National Council for Teachers of Mathematics and British Columbia Association for Mathematics Teachers.

6. In the same manner as the themes in this program, the mathematical investigations were "open ended" in structure. Emphasis was placed on the students' active engagement in problem solving and problem posing within "nonroutine" settings. For example, see Brown and Walter, 1983; Gonzales, 1994; Lesh, 1981; Schoenfeld, 1985; Silver, 1994; Walter and Brown, 1993.

7. Jardine, 1999.

8. New, 1993.

9. National Film Board, 1969.

The 4 Rs of Recursion
Reflecting • Revisiting • Reintegrating • Renewal
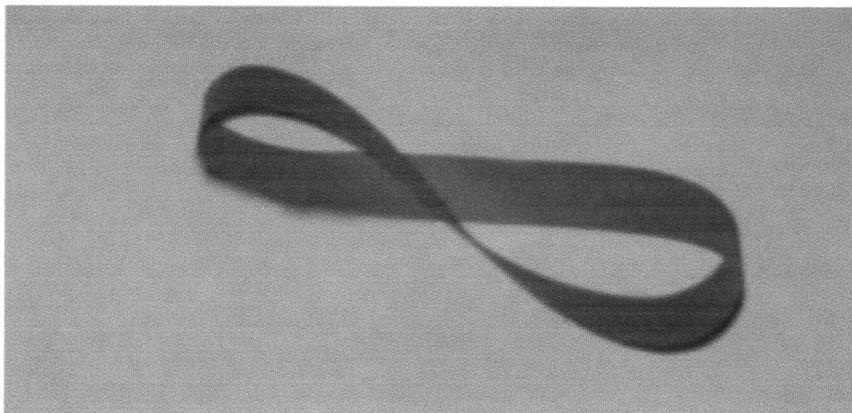
M. C. Bateson's description of learning as traveling along a Möbius strip continues to be a powerful image that prompts me to examine and assess the relationships embedded in my teaching. It also challenges me to consider how I might consciously enable recursive forms of learning in the classroom. If mathematical growth is viewed from an enactive perspective, like other forms of knowing, it is an embodied phenomenon that develops through relating and re-experiencing mathematics from an "opposite side... a contrasting point of view... or seen suddenly through the eyes of an outsider". [1] Because of this, children's learning cannot be achieved through repetitious acts of reproduction or sequential assembly lines of tasks. Doing so implies learning is a matter of practising by redoing what one already knows or taking what one knows and adding to it in a piecemeal manner.

Mathematical growth as a recursive event[2] connotes the actual changing of one's mathematical understanding in ways that are complex and co-emergent. While its evolution possesses qualities of self-similarity based on primitive knowings or inner layers, what it becomes and what it occasions upon each recursion is something qualitatively rather than quantitatively different.

218

Recursion in this sense is not just a better understanding, it is a new understanding that is more than the sum of its parts. Mathematics teaching that provokes recursive forms of learning is more than C=A+B.

Instead of understanding as growing through an additive action, it is seeing the mathematics in a different way. Similar to the change in potential possibilities that occur as a result of structural coupling or drift, this is represented in the Pirie and Kieren model by the "don't need boundaries"; where once beyond a don't need boundary, one does not need the specific inner understanding that gave rise to the outer knowing. For example, a child who takes nine counters, arranges them into pairs, finds out that one is leftover, learns that this distinguishes the number as being "odd" and then repeats these actions to find out that eleven is also an odd number is a qualitatively different kind of understanding than that of a child who works with various numbers and then explains that "any number is odd if you divide it in two equal groups and there is one [whole] leftover". Through making meaningful images for specific numbers, the first child has added to or thickened her understanding by performing the strategy of pairing up the counters and looking for one that is leftover to find out that eleven too is an odd number. The second child however, has explored different numbers and identified a pattern or relationship that can be applied to ANY number to determine whether (or not) it is odd. This child is not making sense of the mathematics in a repetitive manner or on a situation specific basis but conceptualizing the idea of odd numbers in a different and general way-- for ANY instance. This child's thinking comes from a place of formalizing that is located on the Pirie-Kieren model beyond the don't need boundary which separates it from primitive knowing, image making, image having, and property noticing.

So, what does this conception of learning mean for teaching in the classroom?

It places importance on making space for children to reflect on their mathematical patterns of thinking and to revisit their mathematics inside different contexts so that they may critique what they understand from their current place of knowing. Opening learning spaces like this not only allow for students to relate to the mathematics in multiple ways, but it also creates the possibility for occasions in which children can reintegrate and thus renew their mathematical understandings by conceptualizing them differently.

In wanting my students' learning to be recursive[3], one of my aims for the two year program is to establish opportunities for the 4Rs to take place; to engage the children to reflect on, revisit, reintegrate, and renew their mathematical understandings. This means that learning is not just an event which happens when one encounters something for the first time. It occurs when one comes at that something from an opposite side. Taking this image of traveling along a möbius strip and situating it in the context of classroom mathematics where that "something" is mathematics, I am curious as to what an opposite side (or, sides?!) might look like in the mathematics classroom, what forms they might take, and what kinds of understandings will arise.

Notes

1. M. C. Bateson, 1994, p. 14.
2. Kieren and Pirie, 1991.
3. Doll, 1993.

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Making Three Spaces for Recursion

Can You Guess Our Mystery Number?!! ' were mathematical "gifts" created and exchanged with the class' math buddies at a different elementary school. This project brought together several weeks of class investigations in which Jennifer focused on the students' development of different images for thinking with and expressing numbers through the language of: manipulative models, pictorial, informal or formal symbols (verbal or written), and real world situations through dramatization and descriptions. The children worked with a partner to craft a set of clues that would become the riddle for their chosen "mystery" number. Accompanying this list of clues, the students had to also provide eight other numbers in addition to the mystery one as possible choices from which their math buddies could identify the mystery number.

Playful as this project was, Jennifer had taken care in designing it so that the structure of the activity would encourage the students' to reflect, integrate, and reintegrate their conceptual understanding of number. By engaging the children in both the making *and* receiving of the mathematical gifts, Jennifer effectively opened three mathematical spaces for her students' recursion-- first, to reflect and integrate their understanding of number, second, to create a riddle by reintegrating their thinking and taking on a problem posing perspective, and third, to identify their math buddies' mystery number by using their understanding in a problem solving manner.

Before the children set off on the riddle making project, Jennifer gathered the class onto the carpet and asked them to think about the number investigations they had done and what they now understood about numbers. She explained to her students that by collecting[2] these images and recording them as a web on the chart paper, they could then use these to craft their clues. As the children volunteered their ideas about specific numbers, Jennifer wrote them down on the chart paper. For those who found this challenging, Jennifer encouraged them to think about a specific number and then fold back and pick out a particular image that would be true or characteristic of that number.

After the class shared a few examples and nonexamples for certain numbers, Jennifer then prompted the children to see if they could take their specific number image and property notice or express it in a more formal manner; as a descriptor for *any* number. Here, the children produced a variety of images and expressed that any number could be characterized in terms of: the number by which you could *skip count* up to it from zero or skip count down from it to reach zero, whether or not it

fell *between* two given values on the number line, whether the number was *divisible* by a certain value, if it could be arranged into *equal groups*, if the number was *more than (or, how much more than)* or *less than (or, how much less than)* a certain value, whether it was considered to be *even* or *odd*, what the sum of the number's *digits* were, and characteristics of the number in terms of the *place value* of its digits. Once these had been recorded on the chart, Jennifer helped the students to find their partners and a place in the classroom so that they could get started on making the mathematical gift for their buddies.

Entering the second learning space, Mark and Danica left the carpet and went and found a table nearby to begin working on the project. After a brief discussion about the possible numbers that it could be, the two decided that 80 would be their mystery number. Taking a suggestion from one of their classmates, Danica and Mark thought it a good idea to first brainstorm and record as many different kinds of clues as they could think of and then choose the "best" ones for their riddle.

That afternoon, Jennifer asked the class to share the clues they had been working on and to discuss some of considerations they became aware of while working with their partner. She reminded the students to use the web of ideas that they had generated as a reference to help them critique their peers' clues regarding the variety and kinds of images being used. As well as giving clues that began with "The number IS...", some partners had also incorporated clues that started with "The number is NOT..." or "The number does NOT...". When Jennifer commented that this was a clever thing to do, the class agreed and they began discussing why using "not" in their clues encouraged a different kind of thinking for the person solving the riddle. Here, the idea of having to first understand the meaning of the clue and then having to be able to figure out what the opposite of it would be, delighted the children. On returning to their tables, partners giggled to each other as they thought the process but in reverse so that they could craft these clues into their riddles.

Now satisfied with their store of clues, Danica and Mark set to making decisions as to which ones would be used for their riddle. It was here they noticed that some of their clues made it obvious to the receiver what the mystery number was while others were descriptive but not as telling. Mark and Danica sorted their clues and organized in such a way that their riddle opened with general characteristics about the number and then move towards ones that were more specific. The two students also thought it clever to intersperse these with a few that were redundant or unnecessary such as "... is less than 139" after already giving the clue "... is between 2 and 121". During another whole group discussion, the entire class

agreed that the 'sum of the digits in the number' clue was the most telling of them all and that it should appear as the final one. Finally, Danica and Mark moved onto generating the eight other possible numbers for their riddle. They decided that each of the eight numbers had to fit at least two or more but not all of the given clues and they were careful to select only those that satisfied this criteria.

"This way", Mark explained, "Tommy and Ewan will have to think [as to] *why* these aren't the mystery number."

"They might think the mystery number is fifty [because it's between 2 and 121, is an even number, is less than 139, does not have a 4 in the ones place, can be put into groups of 5, and it's more than 10] but it's not because its digits don't make eight", added Danica.

By the close of the afternoon, the Mark and Danica's gift was wrapped in a riddle and ready to give to their math buddies.

Can You Guess Our Mystery Number?!

See if you can find our mystery number. Here are the clues.

The number __is between 2 and 121__

The number __is a even number__

The number __is less than 139__

The number __does not have a 4 in the ones place.__

The number __can be put into groups of 5__

The number __is more than 10.__

The total of all the digits in the number is ____8____

Here are your choices!
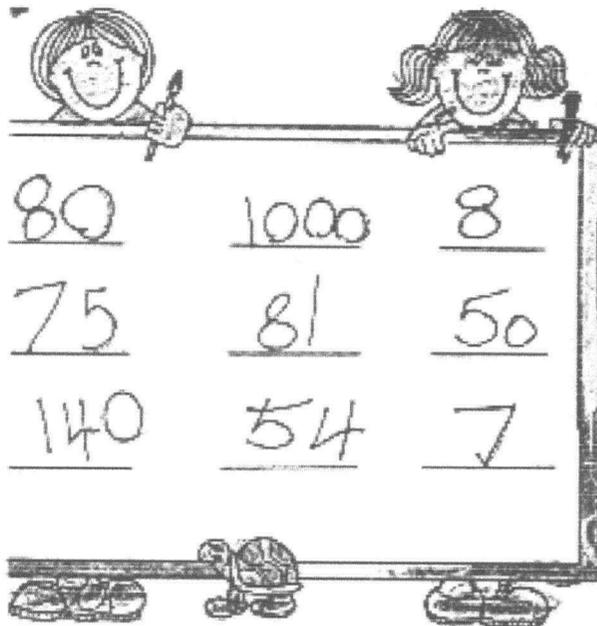
80	1000	8
75	81	50
140	54	7

The secret number is ‑‑‑‑‑
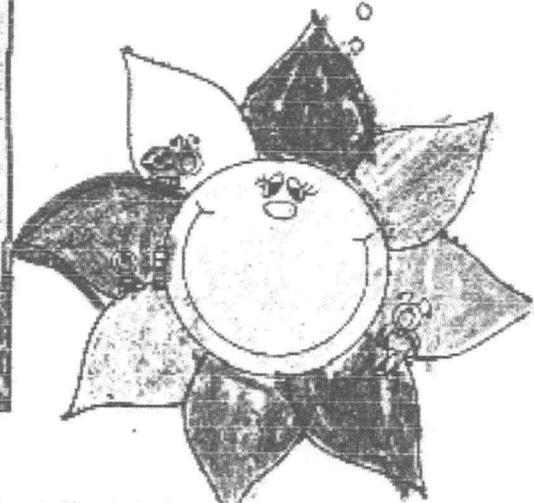
The activity has been adapted from "Find the Secret Number" (Liedtke, 1995).

Danica and Mark's mystery number riddle for Ewan and Tommy

225

One week later, Danica and Mark and Ewan and Tommy received their mystery number riddles from each other. This third learning space that Jennifer created in the structuring of this project required the children to take on a problem solving role as opposed to their previous problem posing one. Not only were each set of partners to identify what they thought the mystery number to be, but they also needed to provide justifications for why the other eight numbers could *not* be the mystery number.

We think your mystery number is __80__ because:

It couldn't be __1000__ because it _is more than 121_

It couldn't be __75__ because it _is an odd number_

It couldn't be __140__ because it _is more than 130_

It couldn't be __54__ because it _has a four in the ones place_

It couldn't be __81__ because it _can't be put into groups of five_

It couldn't be __7__ because it _is less than ten_

It couldn't be __8__ because it _is less than 10_

Finally, it couldn't be __50__ because when you add __5__ and __0__ together, you get a total of __5__ not __8__!

Ewan and Tommy's response to Danica and Mark's riddle

Notes

1. This project was adapted from "Find the Secret Number". See Liedtke, 1983, p. 89.
2. Martin, 1999.

In a different manner, sometimes recursion came as an occasion of renewal. These were times-spaces in which the students revisited a familiar context, thought about the mathematics they had brought forth, and worked on connecting related but different ways of thinking about the mathematics than they had before.

A Snowflake of a Different Kind

As part of the snowflake studies that the class was investigating outdoors, in photographs, on video, and through stories, the children constructed their own Koch snowflake. Each student was to identify an interesting pattern that they observed while creating the snowflake.
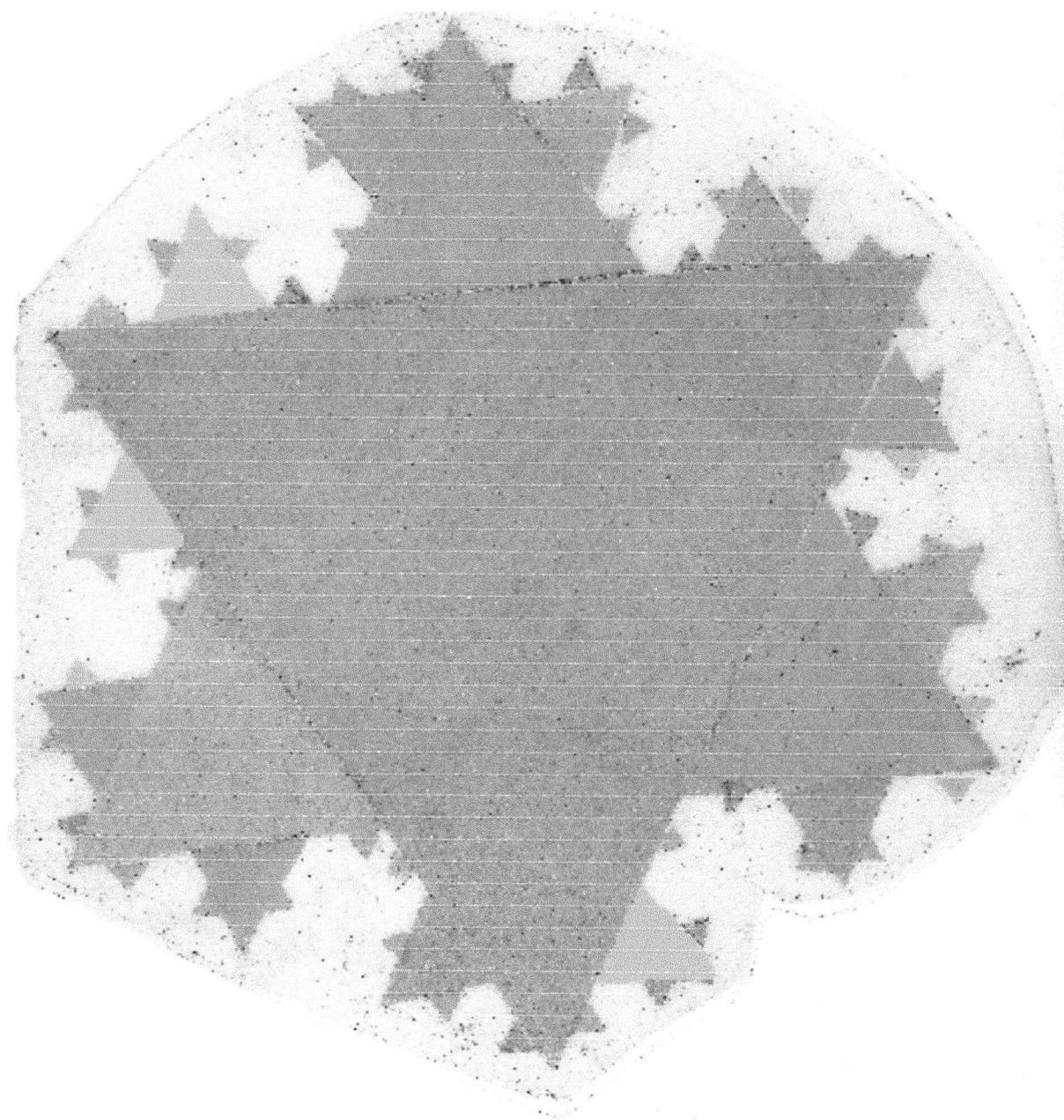
Christina's snowflake

As the students worked on their own to find a pattern in their snowflakes, there was definitely an animate quality in the way the children perceived the mathematical features of the snowflake. Throughout the lesson, students were heard talking to themselves about how the patterns in the snowflake were "growing". An observation like this elucidates the conception that the students held toward their mathematics. It points to the fact that they did not see the purpose of the task to be one of locating "the" answer or as the retrieval of preexistent static mathematics. Communicated in their verbal descriptions of growing patterns is the understanding that this activity was about "keeping watch" for the mathematics to emerge.

Excited about the patterns they had found and fascinated that each one continued on "forever", the students decided to compose a short story that captured the mathematics they saw arising from the snowflake.
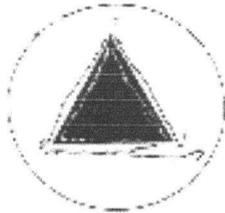
Three sides
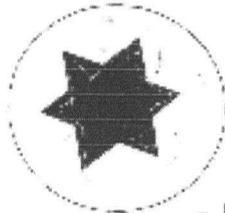
9 more sides
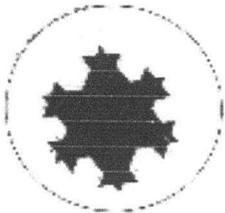
36 more sides

January: Julie's story is about a triangle turned fractal by its ever increasing number of sides.

The pattern of "sides" in Julie's snowflake

"One."
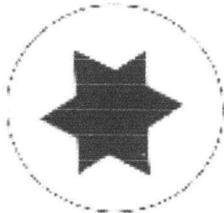
one triangle

"One triangle, one triangle, one triangle."

it grew 3 more triangles

"Two here, here, here, here, here, and here."

it grew 12 more triangles
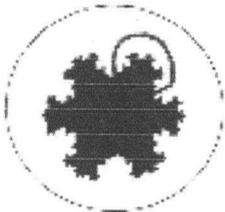
"Eight, eight, eight, eight, eight, eight."

it grew 48 more triangles

The pattern of "triangles" in Shouji's snowflake

January: Clare was fascinated by the "corners" that were emerging in the fractal. When asked to explain what she considered a "corner" to be, she defined it as an "elbow" in the snowflake; where the edge of the fractal "turned direction." Thus, the corners that Clare was counting were located on the perimeter of the fractal.
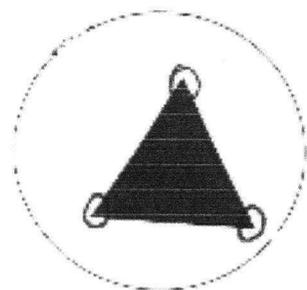
"One, one, and one."

My snowflake began with
threecorners
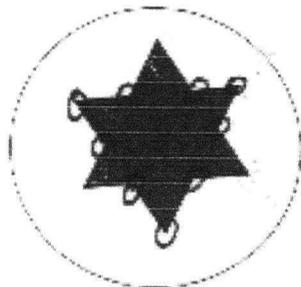
"Three, three, three." [three in between each of the three original "corners"]

Then I saw nine more corners
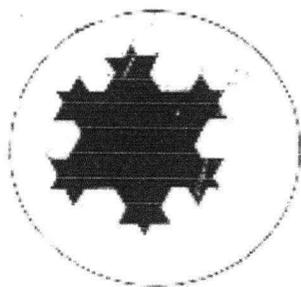
"Then it changes to nine corners and seven corners, nine and seven, and nine and seven." [She arrives at a total thirty-six additional corners by subtracting 3 and 9 from 48]

Then 36 more corners appeared

The pattern of "corners" in Clare's snowflake

232

The class revisited their snowflakes on three other occasions during the school year. For each of these sessions, Shouji, Julie, and Clare considered the total number of triangles, sides, or corners that they had identified previously in each of their fractals. Re-viewing the first and then the subsequent layers of the Koch snowflake, the children worked to develop different ways of thinking and expressing the snowflake's growth incorporating their use of symbols and their knowledge of number operations.
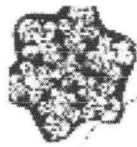
A Snowflake of a Different Kind: A Fascinating Fractal
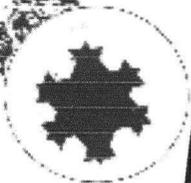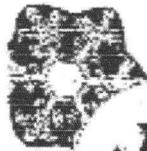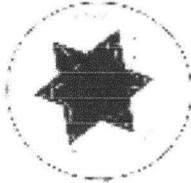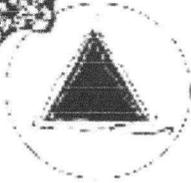
3

1 GrouP of 3 makes 3 si...
$3 - 1 - 1 - 1 = 0$

$2 + 2 + 2 + 2 + 2 + 2 = 12$
6 GrouPs of 2 makes 12
$12 - 2 - 2 - 2 - 2 - 2 - 2 = 0$

$8 + 8 + 8 + 8 + 8 + 8 = 48$
6 Groups of 8 makes 48
$48 - 8 - 8 - 8 - 8 - 8 - 8 = 0$

April: Julie has symbolically expressed the total number of sides for each stage of growth. As well, embodied in her use of repeated addition, is her visualization of the number of sides as being organized into pairs of sides for the second stage and then as six clusters of eight sides each for the third stage.

May: Here, Julie communicates the "groups of" concept through her use of 'everyday' language.

June: The symbols Julie has used here reveal that she has interpreted the snowflake's growth as being a repetitive subtractive action of one side, two sides, and then eight sides each until the total number of sides have been accounted for.

Three more interpretations by Julie

234
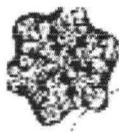
3 Snowflake of a Perfect kind: A Fascinating Fractal

$$1 \times 1 = 1$$
1 group of 1 triangle
$$1 \div 1 = 1$$
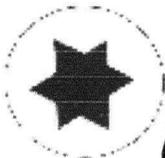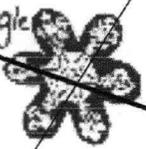
$$1+1+1+1+1+1 = 6$$
6 groups of 1 triangle = 6
$$6 \div 1 = 6$$

$$3+3+3+3+3+3 = 18$$
6 groups of 3 triangles = 18
$$18 \div 3 = 6$$
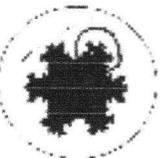
$$11+11+11+11+11+11 = 66$$
6 groups of 11 triangles = 66
$$66 \div 11 = 6$$

April: Shouji's use of symbols express the total number of the triangles as a multiplicative and repetitively additive process.

May: He thinks with the "groups of" image through 'everyday' language.

June: Shouji's symbolic notation describes the fractal's growth as being a divisive action where the total number of triangles split into an increasing number of equal groups. This was clear when he described the process as the snowflake becoming or turning "into" groups of triangles of which they had an equal number "each"-- "One into one triangle, six into one triangle each, eighteen into three triangles each, and sixty-six into eleven triangles each."

Three more interpretations by Shouji

235
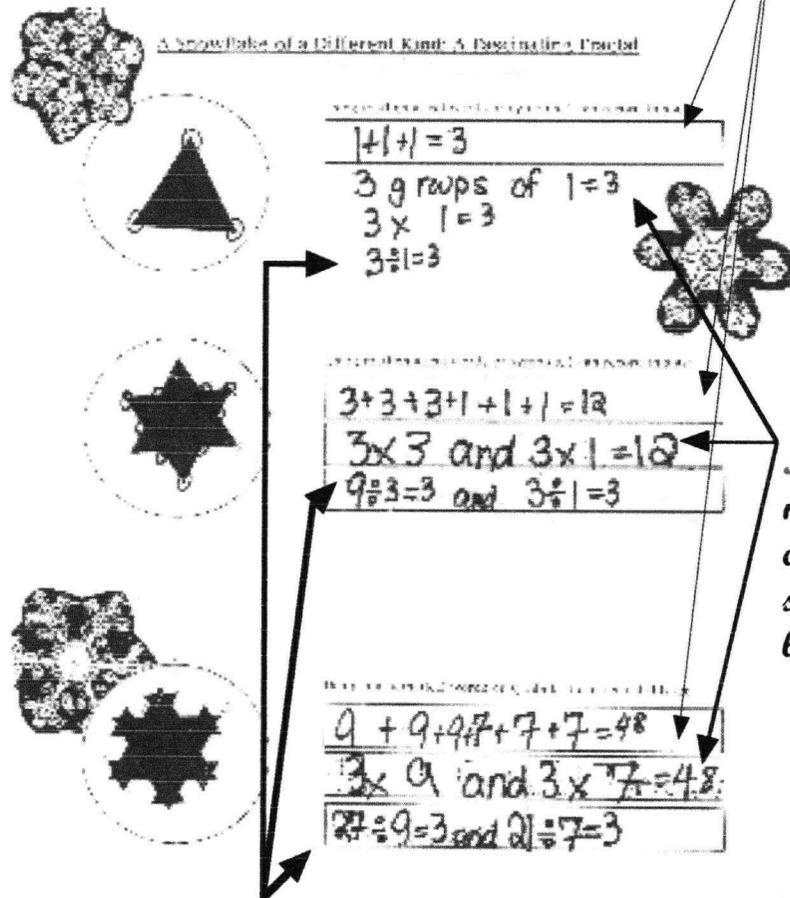
April: Clare's written work communicates her understanding of the fractal's increase of corners as being as process that involves repeated addition.

A Snowflake of a Different Kind: A Fascinating Fractal

$1+1+1=3$

3 groups of $1=3$
$3 \times 1 = 3$
$3 \div 1 = 3$

$3+3+3+1+1+1=12$
3×3 and $3 \times 1 = 12$
$9 \div 3 = 3$ and $3 \div 1 = 3$

May: Thinking through the multiplicative idea of "groups of", Clare made use of both symbols and 'everyday' language.

$9+9+9+7+7+7=48$
3×9 and $3 \times 7 = 48$
$27 \div 9 = 3$ and $21 \div 7 = 3$

June: Clare's use of symbols illustrates her conceptualization of the fractal's corners appearing through the operation of division. Her understanding is viewed as similar to Shouji's explanation but in this context, there are three corners of one each, then twelve corners of which three clusters have three corners each and three clusters have one each, and then forty-eight corners of which three cluster have nine corners each and three clusters have seven each.

Three more interpretations by Clare

236

On still other occasions, the class returned to familiar contexts but investigated them in completely different mathematical ways than they had before.

Re-viewing and Seeing Differently

This Fall, the class watched the film, *Notes on a Triangle.*[1] For the returning students now in the third grade, this was an opportunity for them to re-view it and for the new students in the second grade, it was their first time seeing the film. Like the previous year, the children studied the film by watching it several times over but this time (and, in a very non-linear manner to that of last year) they saw new mathematics coming to life. As a result, they laid down very different learning paths.

For example, some of the children noticed that the triangles in the film "weren't all the same". They watched the film once more and this prompted the class to study the triangles more closely. The children used their fingers to make triangles, they drew them or cut them out of paper, and some went looking around the room gathering them. Having a good collection of triangles, the class worked with a partner or in a small groups to explore what was the same and what was different about them. Afterwards, the class shared their methods for doing this on the carpet. Some students had studied the shapes by moving them about on their desks (i.e., sliding, flipping, rotating, and transposing one on top of the other) to make direct comparisons about the lengths of the sides and the effect that this had on the "shape" of the triangle. Some children used rulers to measure the sides of each triangle while others took the shapes in their hands, turning and feeling each side, surface, and corner. Having gotten to know these triangles so intimately, the children naturally wanted to give them names!
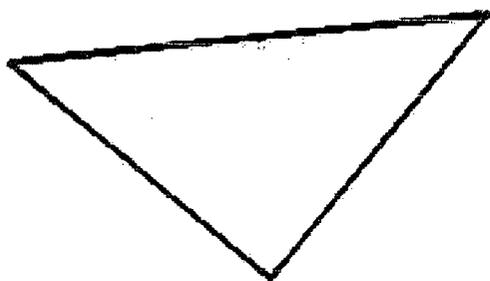
"I call this triangle a[n] **almost all equel triangle** because only two of it's sides are the same." Ethan
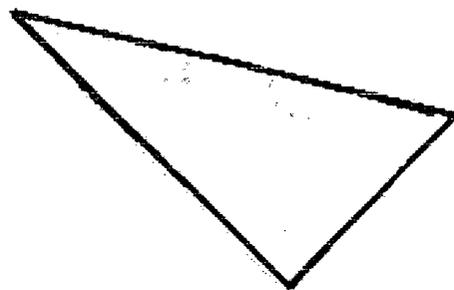
"I call this triangle the **deferent sided triangle** because it has 3 different lengths of sides." Shane
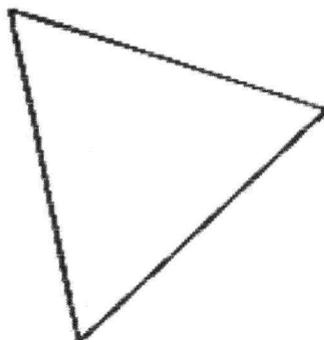
"I call this a **triple side triangle** because It has the same sides." Steven

"they all have 3 corners [and] have 3 sidse." Madelaine

"but they [differ in their] length of the sides." Christina

What is the same about the triangles? What makes each of them different?

The class went on to explore how larger triangles could be composed from using smaller triangles...


equilateral

isosceles
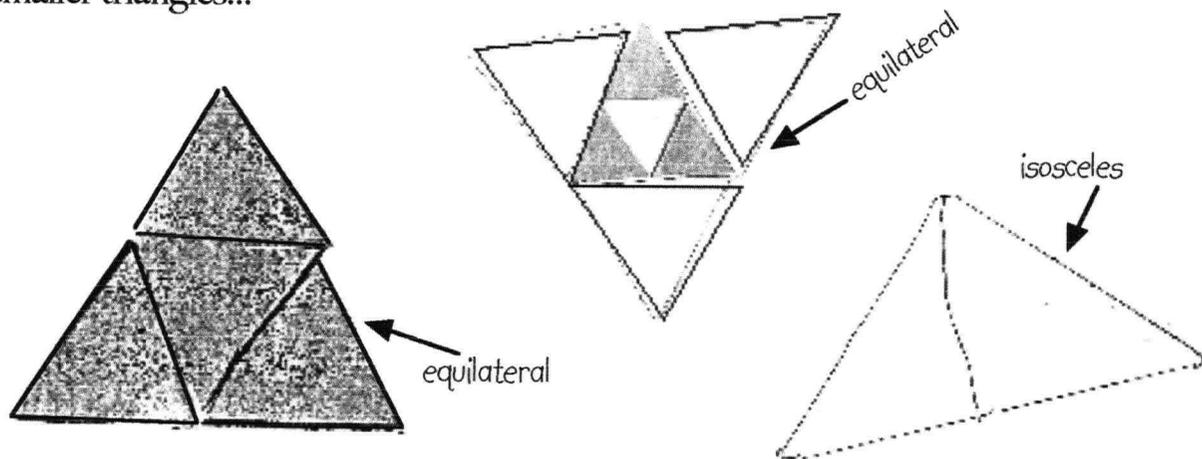
equilateral

Annie's triangles

isoceles
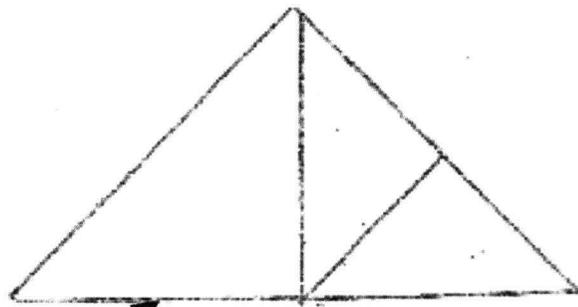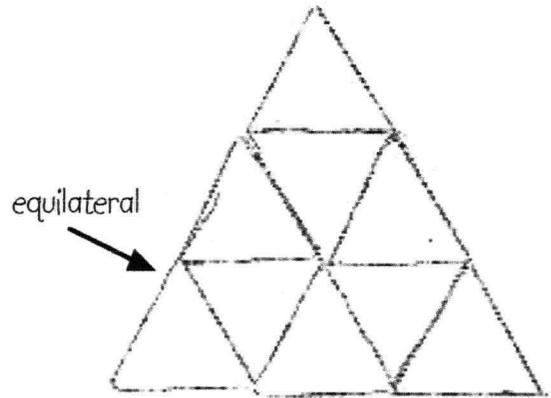
equilateral

I triangle   I trapezoid

Christina's triangles

...and what other shapes could be made from triangles.
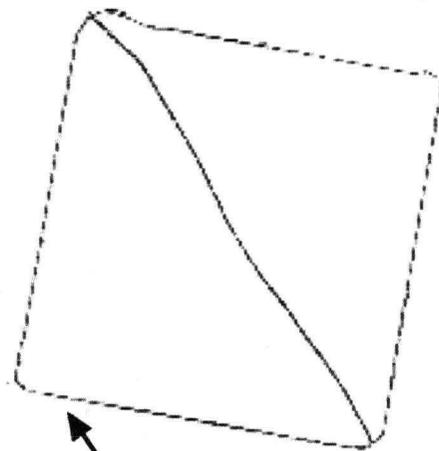
trapezoid

square
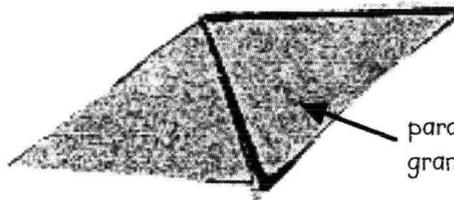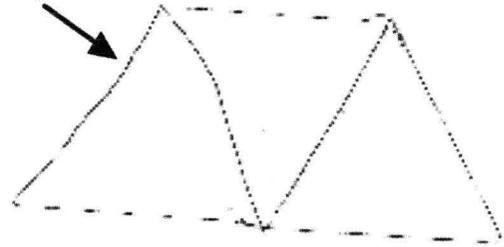
parallelo-
gram

Robby's "other shapes" made from triangles

240
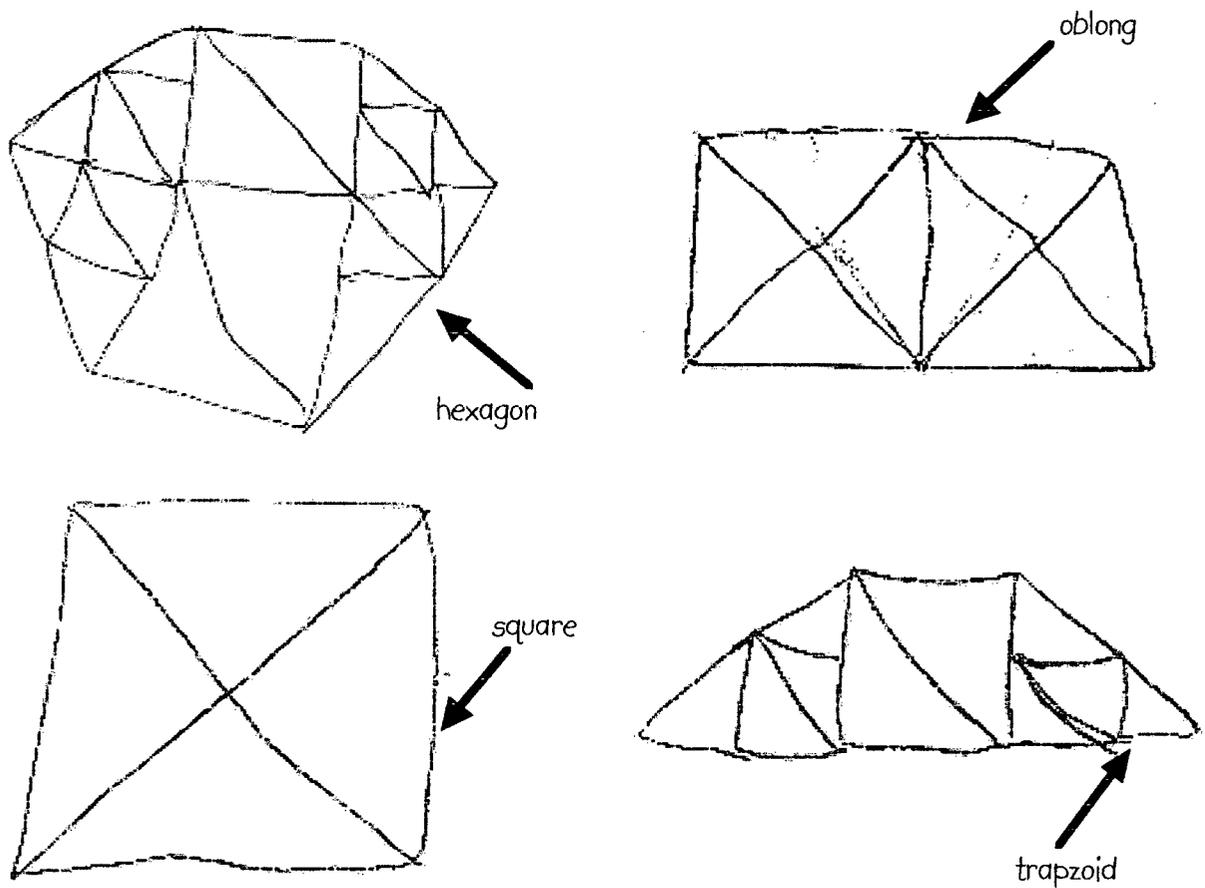
Mac's "other shapes" made from triangles

During the same time that the class was watching the film, five students had been working on another project that involved their construction of many different kinds of "pyramids" made from interlocking cubes. The children were sketching diagrams of the top, side, and bottom views of their structures when they told me that the bottoms of the pyramids looked like triangles:
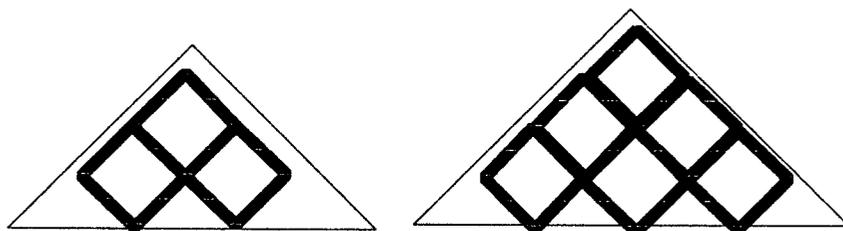
The group shared this with the class and this lead into a study of how numbers too could be considered to be "triangular".
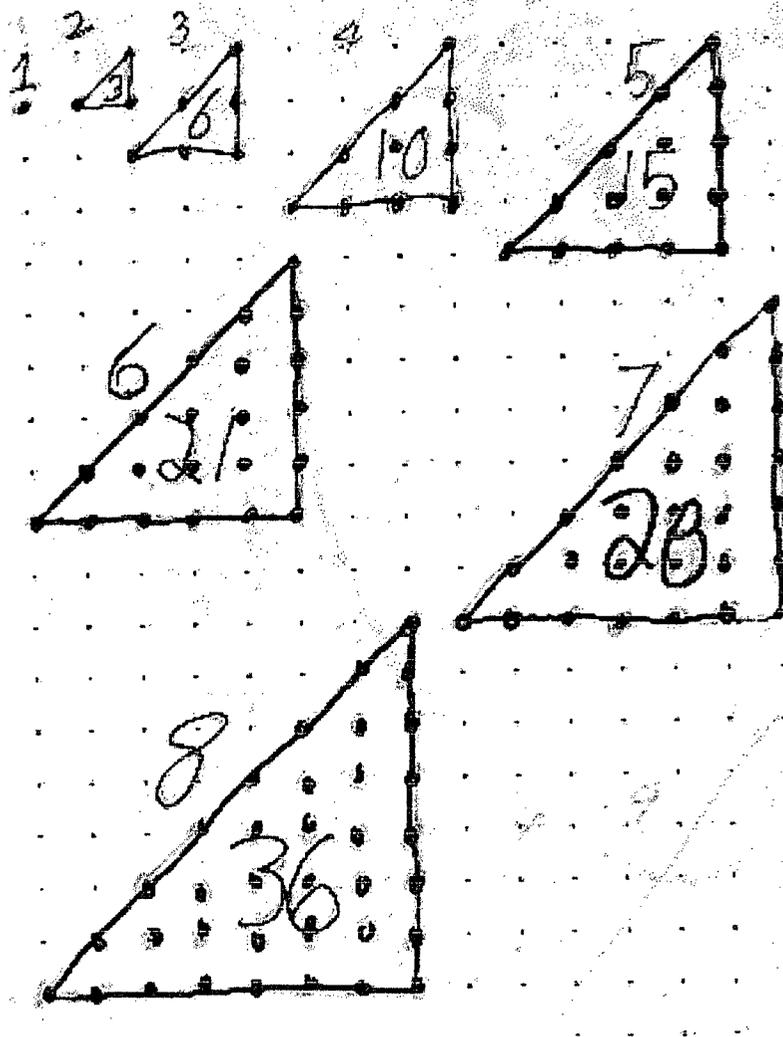
Timothy's diagram of triangular numbers and "how much" they increase each time

"I think the next number will be 45 because... 36 + 9 = 45 so 45 is the next one." Timothy

"I think the next number will be 45 because 28 is 7 more than 21. 36 is eight more than 28. So 45 is 9 more than 36." Steven

"I think the next number will be 45 because the pattern is odd, odd, even, even." Clare

242

"The first has one dot. The second number has two [more] dots. The third you add three, the fourth you add four, the fifth you add five, the sixth you add six, the seventh you add seven, the eighth you add eight. The next one you add nine to it. So thirty-six... thirty-seven, thirty-eight, thirty-nine, forty, forty-one, forty-two, forty-three, forty-four, forty-five."
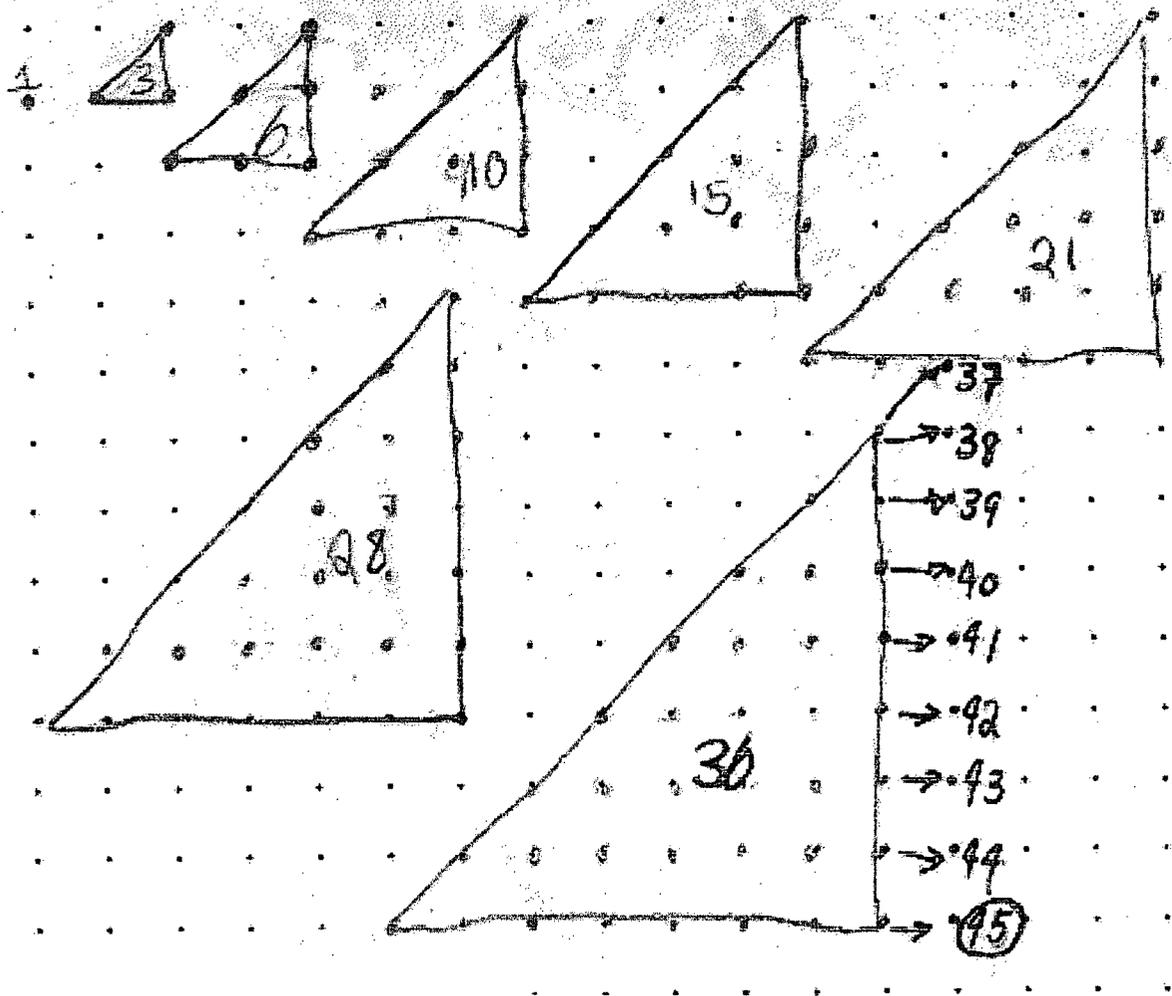
Clare's diagram that shows that the triangular numbers increases by a corresponding column of dots each time.

Notes

1. National Film Board, 1969.

Recursion as Relations:
When Triangles Become Square

(Three months later...)

Last week, I introduced the class to the film, 'Dance Squared.' Since then, we have been re-viewing it and using it as a source from which to occasion their further mathematics. The students have been working together, posing prompts and exploring the geometries of the square. The children also identified numbers that are "square" (spurred on by their interest in triangular numbers). Reflecting on the class' curiosities and making plans for tomorrow's mathematics, I have decided to focus their work on both kinds of numbers. This will be a chance for the children to not only fold back and reflect on what they know about triangular numbers AND square numbers but also, for them (on their own and with the class) to consider each in light of the other. Importantly, had the students' previous work with triangles NOT occasioned their investigation into triangular numbers, this lesson would neither have the same recursive potential nor be appropriate.

(The next day)

Sitting on the carpet, in the middle of the circle, the children helped one another to build the first five triangular numbers with counters. Underneath this, other students worked together to build a row containing the first five square numbers.
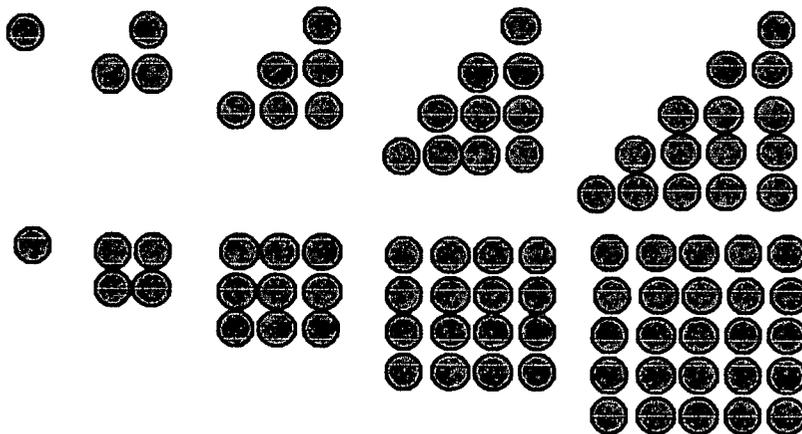

Triangular and square numbers built with counters

As we were doing this, Robby whispered to Mark and struck up a very lively conversation– lots of head-tilting back and forth, smiling, and "yeah!" going on.

I could not hear what they were talking about and so I asked Robby if he would share the conversation with the rest of the class. Robby flashed a bashful smile and then raised his voice to explain what he and Mark had been discussing.

He pointed with his finger to the second and the third triangular number and then to the third square number and told me that, "that number and that number makes this square if you take it and flip it upside down and put it on top of it".
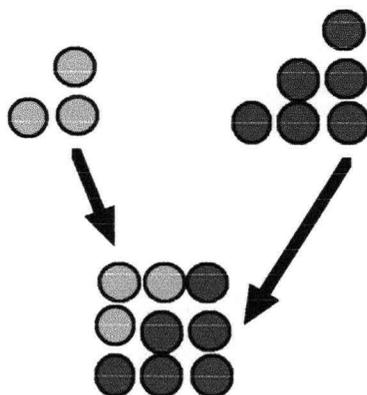
Robby's explanation of how the third square number is created from the second and third triangular numbers

My "WOW!" and the boys' excitement for how they related the two number series together in a spatial way drew several other children into the discussion.

Soon, other students began trying to make sense of this for themselves by talking with one another and displaying similar hand and body gestures to those of Mark and Robby.

Wanting to maintain the focus and momentum of this investigation, I repeated what Robby had said but this time, I also built the numbers with counters as I spoke so that everyone could see the transformation taking place.

"If you take the second and third triangular numbers and put them together like this" I said, demonstrating with the counters just as Robby had indicated earlier, "it is the same as the third square number."

246

This prompted the rest of the class to continue the pattern by combining one triangular number with the one that preceded it in order to produce a square number. Not only did the class do this visually and verbally as Robby and Mark had done but also, arithmetically by adding the two values together, and physically by manipulating the counters of triangular numbers.

After the class produced the second through fifth square numbers from triangular ones, Danny reflected aloud and this time, he related the class' mathematical actions together into a connecting pattern.

"It will be this number" Danny said as he pointed to the third triangular number, "but in the square number". Pointing to the third square number, which was nine, he explained that that is the resulting spatial structure and number when the third triangular number is combined with the second triangular number. Hence, six plus three makes nine.

I encouraged Danny to continue. Pointing with his finger in a left to right direction, beginning with the first triangular number that was one, then moving to the second triangular number that was three, and then to the second square number that was four, he communicated the relationship between the two series of numbers.

"This first number and the second triangular number will make the second square number. And the second and the third triangular number makes the third square number. The third one and the fourth one makes the fourth square number and so on and so on..."

As I listened and watched how Danny was thinking about these numbers, I could see his understanding as also being recursive-- emerging from the mathematics that had already unfolded and at the same time, bringing forth another connection between these numbers. Mark and Robby's observation that arose from their property noticing that the second and third triangular numbers could become the third square number served as a place from which to begin our investigation. The class' further collective work to apply this notion provided several more examples in which this relationship exists. And Danny's understanding revealed yet another quality about the triangular and square numbers. This time, a more formal generative and predictive relationship between them.

I realize now, the impact that one's conceptualization and enactions of time have on the place-making of a mathematics classroom. How time is imagined and enacted very much structures how it is to be experienced by both teacher and students. It is a powerful undercurrent that directly shapes the kinds of mathematical events and relationships that are possible or impossible in the classroom.

I am also learning to let my ecological metaphors think me as I think within them. Doing so provides me with direction and focus to map in recursive notions of time into my teaching of mathematics. The forms of teaching and learning that have emerged so far are a stark contrast with my previous mechanistic ones that created a sense of place where students were expected to assemble specific forms of mathematics before moving on to more complicated lines of mathematical re-production.[2] Instead, these new spaces for learning embody patterns of a temporal difference where past, present, and future exist all-at-once.[3] Time is inseparable from space because it is defined by those spaces in which mathematical experience occurs. The two are not distinct but exist as one co-determining entity of "time-space".[4]

Notes

1. National Film Board, 1961.
2. Abram, 1996; Heidegger, 1962, 1972; Merleau-Ponty, 1968.
3. Heidegger, 1972; Maturana 1995
4. Heidegger, 1962, 1972.

... children... are taught at a tender age that the way to define something is by what it supposedly *is* in itself, not by its relation to other things.[1]

[1] G. Bateson, 1980, p. 18.

To know something is to know what that something is in the way it speaks to us, in the way it relates to us and we to it.[2]

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[2] van Manen, 1986, p. 44.

... the pattern which connects... How are you related to this creature? What pattern connects you to it?[3]

---
[3] G. Bateson, 1980, p. 9.

Might that include this creature we call mathematics?

Creating Patterns That Connect

Interactional Spaces for Mathematics in the Classroom

Insight, I believe, refers to that depth of understanding that comes by setting

In thinking ecologically about the mathematics class, it's impossible for me to imagine my students as "autonomous individuals" anymore. It doesn't make sense within a systemic mind-space for each student to exist as a separate entity, acting on everything else as if everything else was part of the EXTERNAL environment. So even though I continue to recognize each child as an individual, I conceive the children to be individuals within larger collective and environmental systems.

My efforts to nurture children's mathematical growth involves continuing to make spaces for them to explore their mathematical thinking as individuals. But rather than furthering their understanding through just a process of adding on of 'new' mathematical experiences, attention is also cast upon looking deeply and examining the understandings embedded in their mathematics. It's making opportunities for students to not only engage in individual mathematical work but also for them to consider how they are understanding the mathematics by drawing on their mathematical knowings and developing relationships amongst them— in this way, reflecting on what they know and letting these understandings "speak to one another".

experiences... side by side, learning by letting them speak to one another.

Meeting with Mac: A Study of Opposites and Relatedness

Mac and I are sitting at a large table, ready to play a game. Starting with any number of cubes, we are to find at least two possible ways to take away equal groups of cubes until no cubes remain. Mac is to record each of the stories using whatever kind(s) of symbolic notation he wishes to use.

As we play this game, I'm interested in the understandings Mac brings to the task. As well, since he hasn't had any formal lessons involving the operation of division or the "÷" symbol, I'm curious as to how he'll express the mathematical action of repeatedly removing a particular number of cubes. I'm looking for occasions in which I can alert Mac to examine his understandings and engage him in thinking about how he might use what he knows to develop other ways of thinking about the mathematics at hand.

Mac begins the game with a pile of 10 cubes. "I started with ten." He writes the numeral, 10 onto a page in his notebook.

I take away two cubes.

Mac records -2 beside the 10 so that it reads 10-2. "Because we took two away so it's ten minus two."

I take another two cubes away and repeat this three more times.

Each time, Mac records and talks as he works, "minus two, minus two, minus two, minus two! And the answer is... zero!"

Based on what Mac's saying and how he's recording the taking away of groups of two, he views this game as a game of repeated subtraction. He's comfortable and able to independently express the mathematical actions he observes using both verbal mathematical terms and recorded symbols. As well, these forms of mathematical language seem to be an integral part of Mac's thinking because he doesn't watch the events from start to finish and then formulate his equation, he does so in tandem-- as they are happening. In this way, Mac "narrates" the mathematical story as it's unfolding.

Pushing the ten cubes into the middle of the table, Mac starts again but this time, decides to take away groups of five cubes. "Ten minus five...", writes 10-5, removes the remaining five cubes, "minus five...", "equals zero". He finishes his number story by writing down -5=0.

Having completed two ways to remove ten by grouping, I ask Mac to choose a larger number for the next game. He selects twenty to be the number and proceeds to add ten cubes to the ten that are already in the middle of the table. I then ask Mac if he knows of an amount that could be taken away in equal groups from twenty so that zero cubes would remain.

"Hmm." Mac looks at the twenty cubes on the table. He places his elbow on the table and rests his head in his hand. Pursing his lips, he thinks out loud. "Maybe you could do... hmm..." Mac takes his elbow off the table and rests his hand

down beside his notebook. He looks at his fingers and then stares across the room. Curling and uncurling the fingers of his right hand, Mac looks at me, "Hmmm... fives?"

I assume Mac's skip-counting by fives to twenty on his fingers. To be sure, I'll ask him to explain his response.

"And how do you know that fives would be a good one to choose?"

"Because you can count by fives to any number." He demonstrates this to me by taking his pencil and touching it down on the table in two spots horizontal to each other with a good size space between them.

It appears that Mac's thinking with an imaginary horizontal number line along which he skip-counts by fives to twenty. I'm not sure what he means by "any" number and so, I'm going to encourage him to continue.

"Like you can count by fives to twenty. Five, ten, fifteen, twenty." Now touching the two points but this time with his index finger, Mac skip counts by fives, pointing in a left, right, left, and right fashion. He is also moving his head from side to side in a left to right motion-- like a metronome, marking the numbers as he counts and points.

$$5 \qquad 10$$
$$15 \qquad 20$$

Demonstration of Mac's counting action

Now, it's clear to me that Mac is NOT counting along a horizontal line but BETWEEN two points that separate the numbers into those that end with 5 those that end with 0. His counting action might have arisen from the hundred chart that we use often and hangs in the classroom.

The numbers 1-100 are organized on this chart in a 10x10 grid. Because of this, they fall into columns where each number increases by ten as you move to the next one directly below it and so, all of the numbers within a particular column share the same last digit.

It's also possible that Mac's counting method could be a result of the rhythmic pattern that is generated from skip counting aloud.

"I started with twenty, minus five, minus five, minus five, minus five, equals zero." Mac records 20-5-5-5-5=0 into his book...

... while I take away the four groups of five.

"So, that works."

I gather the twenty cubes into the middle of the table again.

Mac begins the next number story. Moving the cubes with his left hand and recording with his right hand, he takes away groups of two cubes each. "Twenty minus two, minus two, minus two, minus two, minus

260

two, minus two, minus two, minus two, minus two, minus two, equals zero. So that works." In similar fashion to his other number stories, Mac records 20-2-2-2-2-2-2-2-2-2-2=0.

Up until now, Mac's used groups of two and five to divide the cubes of ten and twenty. He's doing this by applying the "opposite" process of repeated subtraction. In other words, through repeated addition Mac's skip counting to arrive at the target number and then simply reverses the process by transforming it into a repeated subtraction equation. Since skip counting by multiples of two and five appears to be Mac's strategy of choice, I wonder what other thinking he might bring forth if I ask him to continue working with the number twenty.

"Is there another way?" I place the cubes in front of Mac.

"Hmm... you could do it by tens". Mac quickly writes 20 down into his notebook. "We could do it by tens."

I take away a group of ten cubes.

Mac writes -10 to the right of the numeral 20.

I take away the remaining ten cubes.

"Minus ten". He records -10 and then =0. "Equals zero".

I put the cubes back into the middle of the table.

An attempt to prompt Mac to think of yet another way to evenly divide the twenty cubes.

"Oh..."

"Try by fours." (An impulsive suggestion!)

"Okay." Mac begins talking slowly and taking away groups of four cubes. Each time, he looks to see how many cubes remain.

Mac appears unsure. This is probably because four is NOT a number by which he usually skip counts and therefore it isn't a number he associates with twenty.

Recording as he works, Mac completes the equation, 20-4-4-4-4-4=0.

Pointing to what Mac has recorded, I bring attention to two of his equations. "Now, looking at what you got here for twenties... look at the twenty give away groups of five [20-5-5-5-5=0] and look at the twenty give away groups of ten [20-10-10=0]. Do you notice anything about the number of groups Mac?"

With his pencil, Max points to 20-10-10=0 and 20-5-5-5-5=0. "That this is just double of this".

"Why is that?"

"Because there's two fives in ten so that's four." Mac then demonstrates this by pointing to each "5" in his equation. "There's FOUR fives there and TWO tens and one twenty. This is FOUR fives... TWO tens and one twenty."

By inviting Mac to stand back and reflect on the two number stories, he identifies the two equations as being related.

$$20-10-10=0$$

$$20-5-5-5-5=0$$

Mac's explanation of how 20-10-10=0 is the result of 'doubling up' four groups of five.

Turning Mac's attention to the other two number stories (i.e., 20-4-4-4-4-4=0 and 20-2-2-2-2-2-2-2-2-2-2=0), I ask him what he can tell me that is the same or different about them.

"There's just double twos as there are fours... Pretend I didn't write that [points to 20-2-2-2-2-2-2-2-2-2-2=0] down and I just counted this [points to 20-4-4-4-4-4=0]-- how many fours. There are, one, two, three, four, five. There's five fours. So then, I would just have to double that to see how many twos there'd be if I did twos. So there's ten twos."

Intriguing! As Mac explains how he identifies ten groups of two as being in five groups of four, I get a real sense of the flexibility of his understanding. Taking his previous idea of 'doubling' as a process of multiplying by two, Mac's thinking in this context effectively shape-shifts the process by inversing it. In essence, the doubling now becomes 'halving'.

$$20-4-4-4-4-4=0$$

$$20-2-2-2-2-2-2-2-2-2-2=0$$
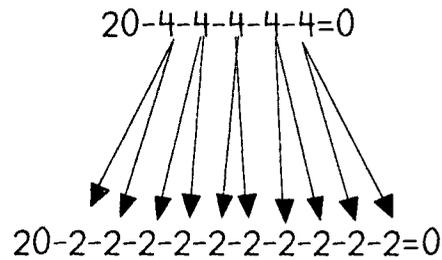
Mac's explanation as to why there are twice as many groups of 2s than groups of 4s

Smiling at Mac, we resume the game. "Okay, let's see if we can use some of that as we go along in this game."

Mac nods his head. "Yes!"

"Pick another number."

"Twenty-four."

I add another four cubes to the pile that is already in the middle of the table.

Mac jumps in and quickly splits the pile of cubes in half or, into two groups of twelve. "Twenty-four minus twelve minus twelve."

He records 24-12-12=0. Mac explains that he doesn't need to count the remaining pile because he "knows" that it will also be twelve.

Again, I want to prompt Mac to reflect on his thinking and consider how it might impact his understanding now.

"... taking that", I point to 24-12-12=0, "can you think of another number story

264

that you could tell?"

Mac looks at the equation. "Groups of two."

"And how many twos would you have?"

Mac examines the equation again and then down to his hands that he is holding palms up. However, they cannot be seen because they are hidden underneath the tabletop.

I tell him to bring them out!

Mac skip counts by twos and keeps track by unfolding one finger at a time until he has six fingers extended. "Two, four, six, eight, ten, twelve. So double six [i.e., groups of two] is... twelve. So I'd have twelve twos. Okay." Mac sets to recording the number story, "Twenty-four minus two, minus two, minus two, minus two, minus two, minus two, minus two, minus two, minus two, minus two, minus two, minus two, equals zero... twelve twos." 24-2-2-2-2-2-2-2-2-2-2-2-2=0.

Mac's carrying forth his understanding from the last task and applying it here to help him determine how many groups of twos are in twenty-four. By making use of the equation that divides twenty-four into half, Mac takes one of the twelves and skip counts by twos to find out that there are six groups of twos in twelve. He then simply doubles the number of twos to get twelve groups of two. Like his other calculations, none of these are done with paper and pencil but through verbal, mental, and physical forms of computation.

"So, what's another way that we could make..." I put the twenty-four cubes back into the centre of the table.

"We could do... try fours!" Mac reaches for the cubes to begin taking away groups of four.

I'm not sure why Mac is reaching for the cubes. Is it because he thinks that since twenty-four is divisible by two that it'll also be divisible by four since four is double of two and he wants to double check this with the cubes?

I ask him to guess how many fours he thinks there will be.

"Hmm..."

Given that Mac doesn't guess or estimate, I take his idea of groups of fours to be a conjecture and not something that Mac "knows" as a fact. I want to help Mac to develop his thinking by connecting something he does know in order to solve for what he's trying to figure out. So instead of letting Mac continue to use the cubes in what I suspect will be a strategy of trial-and-error, I interrupt what he's doing and suggest that he reexamine what he's already done.

"How could you take something you know here..."

Mac moves from the cubes to his notes and uses his equation of 24-2-2-2-2-2-2-2-2-2-2-2-2=0 as his jumping off point. "I could say... I would just take half of these off" pointing to the groups of twos,

"because I know there's twelve of these." Mac covers up six of the groups of twos with his hand so that they are no longer visible and explains to me, "So take half off twelve twos". Through Mac's actions, he demonstrates the need to take away six twos in order to form the groups of fours with the other six twos. "... so just count by twos." Mac proceeds to count the remaining twos. He begins to skip count them by twos and then realizes that he isn't wanting to find out 'how much' remains but rather, 'how many groups' of four there are in twenty-four. He switches his strategy. This time, he takes his pencil and partitions off two twos at a time while he counts the number of fours. "Okay, that'd be one, two, three, four, five, six . So there'd be six... six fours. So one, two, three, four, five, six." Mac keeps track of the number of '-4s' he's writing to produce 24-4-4-4-4-4-4=0 on paper. Finally, Mac double checks his work by taking the pile of 24 cubes and physically dividing it into six groups of four cubes each.

"Okay, taking what you know from the fours, can you do something new?"

"I could do... hmm..." Mac looks at the cubes

in front of him and then looks at me. "Groups of eight?" Mac starts in and implements his doubling up strategy. "So just...", reaching for the piles of four cubes, Mac pushes two clusters together until they are all paired up and are now in groups of eight. "And there'd be three groups."
He picks up his pencil and records 24-8-8-8=0. "... minus eight, minus eight, minus eight, equals zero. There we go!"

"Okay. Is there any other way we could share twenty-four?"

Mac shakes his head. "Hmm. I don't think so." He looks over the three groups of eight cubes in the middle of the table moving his eyes and head in a left to right fashion. "N...nope!"

I push the cubes together so that they are back into a pile of twenty-four.

An attempt to enable Mac to not think with the image of groups of eight but to start anew.

Mac looks at the cubes and sputters out, "one" softly. He takes single cubes away, "You could do one... minus one... so twenty-four minus...." Using his left hand to move the cubes away and his right hand to

record as he acts and tells the number story out, Mac monitors the number of ones being recorded in his equation. He produces 24-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1=0 "And the shortest way of sharing is one group of twenty four." Mac records 24-24=0 down into his book.

Once again, I ask Mac to review what he's done. Because he's been working exclusively with the images of splitting in half and doubling of the numbers, they don't allow for six or three to be possible values by which twenty-four can be divided. I try to open another space for him to work.

"So if you were to partner up the number stories that you were able to figure out from each other... So this one here" [pointing to the 24-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1=0] and then to [pointing to 24-24=0-- the number story Mac had written as the reverse or the opposite story to the previous one]... giving away one whole group... connect them with arrows, what other ones did you..."

"Well, I first did a group of twelve and then from that twelve, I decided to do groups of twos, and from that two, I decided to put them together and I did by fours, and then I just added two fours together and I did eight! So those are all connected." Mac summarizes and connects the number stories with arrows in that order. He also connects 24-2-2-2-2-2-2-2-2-2-2-2-2=0

and 24-4-4-4-4-4=0 as being related "because... four and two... because two twos make up four." Rather than keeping to the number stories about 24, Mac moves on to all of the other equations he has written. He relates 20-5-5-5-5=0 to 20-10-10=0 and 20-2-2-2-2-2-2-2-2-2-2=0 to 20-4-4-4-4-4=0. Framing the equations in terms of doubling and halving, he explains that he doesn't see 10-2-2-2-2-2=0 and 10-5-5=0 as being related and so he leaves them as they are.

Instead of trying to bring him back to twenty-four, I shift the conversation to exploring one last number.

Mac tells me to pick a number that can be divided into two equal groups.

Given what Mac has said and my intent to engage him to think still differently, I want to choose a number that'll divide evenly by two but result in an ODD composite number.

I choose eighteen and ask him if he thinks it is an appropriate choice.

He pauses for a moment. "Yes". With the eighteen cubes in front of him, Mac sets to work. "Okay!" he exclaims. "Let's do...". Mac stops again. He places his hand to his lips as he looks at the cubes.

"What do you know about eighteen?"

Another pause. "... How about nines."
Mac playfully slapping his hand
down on the table.

"What's nine about eighteen?"

"There's two nines in eighteen."

Again, he's using the image of doubling or halving of a number.

He writes 18-9-9=0 onto the page. To
this, he adds what he's coined, the
"shortest" number story. "We could also do
eighteen minus eighteen equals zero" and writes
18-18=0. He follows this with the "longest"
number story. "And we could also do eighteen
minus one...". Mac talks out loud as he
records this on paper and eventually
produces 18-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-
1=0.

I ask Mac to review the three number
stories he's written for eighteen. "Now
looking at any one of these, can you
make up something new?"

"I could... this". Max chooses to rework
18-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1=0.

"Because there's two nines in one eighteen and we could... and this" pointing to 18-9-9=0, "is opposite."

Here, Mac describes how 9+9=18 and 18-9-9=0 are inversely or "opposite[ly]" related.

He points to 18-18=0 and draws an arrow so that now 18-9-9=0 is connected to both 18-18=0 and 18-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1=0. "So you put those together too". Mac looks at me and concludes, "so... that makes it."

Mac appears to be "finished" this game because he's related all of the number stories to one another.

I challenge Mac to see if he can find another number story. Focusing his attention on the string of "ones" in the equation 18-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1=0, I invite him to, "see if you can find some more groups in there".

"We could do groups of two."

Again, putting his doubling strategy to good use!

I acknowledge two as being a possibility for another number story and encourage Mac to make sense of it.

"How do you know you can do groups of two? What tells you that's possible?"

"Because... you can... eighteen is an even number."

"And that means....?"

"As an even number. You can use it and put groups of two in."

"okay."

Mac takes his pencil, "Okay, eighteen minus two, minus two, minus two..." and begins to record his number story. Mac continues writing, while keeping track by skip counting the "twos" he has written aloud "two, four, six... sixteen, eighteen". "Equals zero" he says as he finishes the equation, 18-2-2-2-2-2-2-2-2-2=0. Mac tells me that he can show that nine twos makes eighteen by "just counting the ones and see how many twos there is."

Mac's using the same strategy he developed to determine how many fours there are in twenty-four.

"We can count by two ones at once. One, two, three, four, five, six, seven, eight, nine". Mac counts the nine twos that are made by pairing up two ones at a time. "And one, two, three, four, five, six, seven, eight, nine". He counts and points with his pencil to "prove" that there are indeed, nine twos to make

eighteen in his equation. "Now, we can put these two together too." He draws an arrow to connect 18-2-2-2-2-2-2-2-2-2=0 with 18-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1-1=0 to indicate the relationship of doubling the single units or ones of this even number in order to produce a number story that has groups of twos in it.

Pointing to his first number story, 18-9-9=0, I ask him to look at the "nine and nine. Can nine be told another way? In terms of groups?"

Mac looks down at the equation and thinks out loud. "You can tell it as... three groups of three! Or we can tell it as... groups of nine." Since he's already recorded eighteen shared by two groups of nine, he sets to work on determining how many groups of three there would be.

"What would that look like for eighteen?"

"For eighteen... that would be... there would be six threes in eighteen because there's two nines there." Mac points to 18-9-9=0 in his notebook. "So two nines in eighteen. So one of those two [nines]. There would just be... let's just pretend each of these are three [i.e., three threes] so that would make six." Mac points to the first nine and then the second nine in his

equation. "So six groups of three... so six groups of three!" Mac goes to write this down in his book but stops and looks at me.

Here, the mathematical image Mac is thinking with-- "six groups of three" isn't the same as how he imagined his other equations. I ask Mac how he could reinterpret the story by telling it from the perspective he had expressed the others.

"Eighteen, minus three minus three, minus three... equals zero." Mac records 18-3-3-3-3-3-3=0 into his notebook.

Here's an opportunity for Mac to bring together the image of repeated subtraction that he's been working with and the image he already had for multiplication.

To do this, I repeat something he's said previously. "Now Mac, did you hear what you said? You said, I HAVE SIX GROUPS OF THREE."

"Yes...." Mac continues to look at me, waiting for me to make my point.

"What operation do you think of when you hear GROUPS OF?"

"I think of... I think of... like say, I wanted groups of four." He takes the eighteen cubes and arranges them into four groups of four cubes and a group of two. "A group of four, a group of four, a group of four, and a group of four, and a group of two. Those are groups."

"Okay." I push the counters back into a pile and pick up Mac's pencil and hand it to him. "How would you write six groups of three?"

He takes the pencil, and says, "six" and then stops and asks me, "should I use symbols or write it [i.e., in words]?"

I'm curious what symbols (e.g., + or x) or words (e.g., "and" or "groups of") Mac might use to express how he's thinking about six groups of three.

I tell Mac to record whatever way he wishes to express how he is thinking about it.

"Six times three equals eighteen". Mac writes 6x3=18. He takes a look at this on the paper and then explains to me that it does not make sense with the game we are playing or the other equations he has recorded; that the object of the game is to TAKE AWAY the number in equal groups until there is nothing left. "So that wouldn't work because we're trying to make it equal zero."

Now here's a chance for Mac to consider the complimentary relationship between the mathematical forms of 18-3-3-3-3-3-3=0 and 6x3=18. After all, it's this mathematical back-and-forthing that Mac's been doing while he's been playing the game! Through his actions of doubling and halving, he has been ultimately, determining how many equal groups will make up

the given number. Once he's done this, he then REVERSES his thinking so that it moves it in the opposite or complimentary direction. He does this by expressing the number as being taken away in equal groups until zero is reached.

Mac studies the two equations again. This time, he does not consider them as being unlike each other, but instead, as both communicating eighteen as six groups of three. The difference, he explains is that with 18-3-3-3-3-3-3=0 as opposed to 6x3=18, "we are not plussing" [the groups of three] "we are minussing them.. When you are taking away it would be subtraction. It's just like this [pointing to the multiplication symbol in 6x3=18], but it's subtracting not adding."

In his explanations, Mac not only identifies the inverse relationship between repeated subtraction and multiplication but also, how multiplication can be interpreted as (repeated) addition.

"They're opposite but they are also related". Mac draws an arrow to connect the two equations together.

"Now, what do you mean by they are opposite BUT they are also related?"

"Because there are six groups of three. Well, they are not really opposite because there's six times three and there's six groups of three here except for, this is plussing and this is minussing."

"So that [i.e., plussing and minussing] is opposite then?"

"Yeah but they are also related because there is six groups of three here except we're just taking them away. We're not plussing them." Mac reaches for the cubes on the table and shows me a group of three being ADDED six times to make eighteen and then, in an "opposite-related" manner, how six groups of three can be TAKEN AWAY.
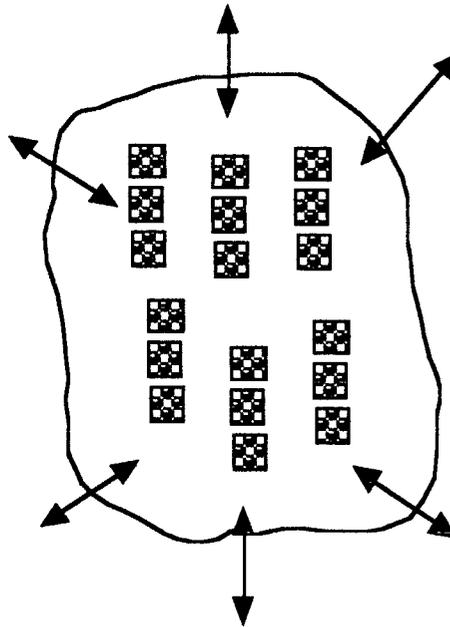
Mac's demonstration of 6x3=18 and 18-3-3-3-3-3-3=0

"Knowing what you know about this [i.e., the opposite relationship of 18-3-3-3-3-3-3=0 and 6x3=18], can you think of any other possibilities?"

Mac employs his doubling strategy one final time to solve for the last number

story. "I could do three groups of six because there's two threes in a six. And I just counted by twos." He shows me that by pairing up the threes in 18-3-3-3-3-3-3=0 is the way he can check to make sure that the groups of three will double up without any remainders. Mac counts the number of groups of six that can be made from the six threes. "So one, two, three. So eighteen minus six, minus six, minus six, equals zero". He records 18-6-6-6=0 into his notebook.

"So you did this one [pointing to 18-3-3-3-3-3-3=0] and then you found out the oppositely-related one [pointing to 6x3=18]— six times three..."

"So, so this one would be related to this one too." Mac draws an arrow to connect them together.

"Do you think you could think of a multiplication sentence that would be the opposite of this one?" [pointing to 18-6-6-6=0]

"Hmm..." Mac pauses for a moment. He rewrites the equation but replaces the subtraction symbols with ones of addition. "Eighteen plus six plus six plus six equals zero."

I attempt to retell and act out the equation that Mac has written using words and my hands to show the additive action. "So if you had eighteen things and then you add six and add six and add six, you would end up with zero."

"No. That doesn't make sense."

I suggest that we use the blocks to try and make sense of it.

Mac agrees.

I mirror the same actions that Mac demonstrated when he showed me how adding and taking away six groups of three were oppositely-related. "So we did eighteen minus six, so I'm doing the action, right, you talk it out."

Mac chimes in as I take away the groups of six "... minus six, minus six .... zero."

"Now, let's do the opposite." Still keeping the eighteen blocks in three groups of six, I perform the opposite action of BRINGING BACK the three groups of six so that they form the original pile of eighteen. "So if we go..."

"Eighteen plus six..."

280

I stop what I am doing and return the group of six cubes back again. "How much are we starting with?"

"Zero." Mac narrates the story I'm acting out. "Zero plus six, plus six, plus six,"

"makes...?"

"Um... eighteen! Okay, zero plus six, plus six, plus six, equals eighteen." Mac writes 0+6+6+6=18.

I ask Mac to look at 18-6-6-6=0 and 0+6+6+6=18 to see if he agrees that they tell opposite stories about eighteen.

"So they're kind of... So...." Max draws an arrow to connect the two number stories together.

"Can you think of a multiplication story using numbers and symbols that's the opposite of eighteen minus six minus six minus six equals zero?"

Mac looks at the addition equation he's just recorded. "Three groups of six equals eighteen!" He then writes 3x6=18. "So I'll write it down... Three minus... umm... times six... three times six equals eighteen." Finally, Mac REALLY finishes (!) the session by drawing an arrow from "this one" [0+6+6+6=18] to [3x6=18] and then

to "this one also" [18-6-6-6=0]". He explains
that the latter is "related because it's opposite".

$$18-3-3-3-3-3-3=0$$
$$6 \times 3 = 18$$
$$18-6-6-6=0$$
$$0+6+6+6=18$$
$$3 \times 6 = 18$$
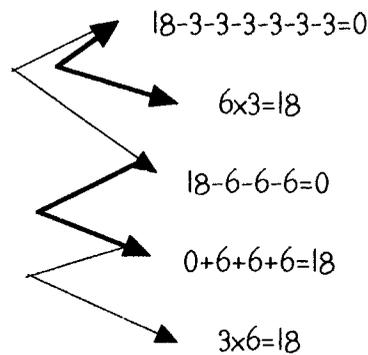
Mac's diagram showing the relationship s between the equations

One of my intentions in working with Mac has been to enable him to develop different ways of seeing and expressing the mathematics he brings forth.... not through prespecified sequences of IMMERSION but through making space for, being attentive to, and responding in that moment to the EMERGENCE of Mac's mathematics.

A second focus has been to open learning spaces for Mac to set his mathematics "side by side".... put his ideas and expressive forms WITH each other and consider the patterns that exist among them. Mac's growth of understanding that arises from these spaces comes about through our interactions and his interplay of making numerical and operational sense of the game. Mac takes numbers and reconfigures them into equal groups through skip counting and physically re-arranging cubes... he folds back on what he now knows and extends his thinking to create yet other solutions by decomposing the number again or recombining the groupings... he examines his actions and develops not only compatible but complementary mathematical forms through making use of different operations.

In the end of our session, Mac creates another conceptual layer that integrates his understanding even more when he effectively brings all five equations together. Importantly, he's not linking one form to another, but

282

comprehending them as interacting systems of knowing 18. By setting his mathematics "side by side", Mac opens a space where he relates and integrates the equations in ways that most definitely, "speak [mathematically] to one another".

Insight, I believe, refers to that depth of understanding that comes by setting ... to hear [and become aware of] multiple points of views, as well as to express and clarify their own-- is not seen as canceling the individual differences, but as a means of identifying them... within the context of the group [or, the larger environment][4] experiences, yours and mine, familiar and exotic, new and old, side by side, learning by letting them speak to one another.[5]

[4] New, 1993, p. 219.
[5] M. C. Bateson, 1994, p. 14.

Thinking systemically about children's mathematical learning raises the issue that as well as nurturing students' individual growth, emphasis should be placed on how their mathematics relates to larger collective realms.[1] Doing so allows students yet another opportunity to integrate different points of viewing and to gain an understanding for how mathematics exists simultaneously within individual and collective spheres of knowing.

Above is a piece of my 'thinking out' on paper. I have lots of these free writings. I call them "free" because I write them very much in the spur of moment. I don't pay attention to the particular words that I'm using or the structure of the sentences. I'm not even writing to anybody. I'm simply letting my thinking out and paper is the medium it happens to happen on. They're also free because when I write each one out I do so on a separate piece of paper and then put it with the rest of them. I don't use them to write from, mainly, I just collect them. And as I think out more, I sift through my free writings and consider how the one I've just written interacts with the others. And from time to time, two or more of these ideas will settle together or when placed with some other, create a tension that enables new thinking to arise. Then I find myself asking, "what does it mean?" And so it goes. Funnily, I'm reminded of my childhood— sitting at the table with my sisters and brother watching the green tea leaves swirl and sink to the bottom of our cups, creating images and each of us trying to decipher the inherent meanings. So, back to my particular piece of free writing. As I read it on paper, it makes sense and it's been integral in my current thinking and teaching. But I've never articulated it any further... out on paper. So, this is the task I've set for myself. Two questions that emerge right away are: What does this fragment of "thinking-out-on-paper" mean for classroom mathematics and what might it look like in the classroom?

An example that immediately comes to mind and highlights the individual and the collective as necessary parts of the mathematics classroom is the lesson in which I was reading the book, *Even Steven and Odd Todd*. Had the lesson only focused on the children's individual ways of thinking about odd and even numbers, it is doubtful that they would have

developed the web of understandings that resulted. As individual knowings, all of the students' responses made some kind of mathematical sense-- that is, rote or conceptual sense. But it wasn't until the students' different ways of thinking and their emerging ideas were brought together that the mathematical tension necessary to occasion their collective sense-making was possible. It was here in the collective realm that the children made collaborative, conceptual sense of why certain numbers are considered to be "odd" and others to be "even", why even and odd numbers end with particular digits, and what exactly is or is not 'two' about them. Had the lesson concluded with each student sharing his or her number images instead of bringing them together and exploring how these ideas were or were not related to one another the children's understanding of odd and even numbers would have most likely existed as a collection of "free" ideas. Had the lesson only focused on the children's thinking as a class, there wouldn't have been the tension created by their individual understandings. Individual and collective understandings are not only necessary parts of the mathematics classroom, they are necessary for the growth of each other. One enables or disables the other.

Notes

1. Cobb, 1999; A. B. Davis and Simmt, 2003; In conversation with S. E. B. Pirie, October 2003.

Connecting as Shape-shifting

Hold on for a minute. Could you go over that again?

I suppose you could say that the mathematical thinking and emergent ideas that lead to a growth of understanding arise in the dynamic of students simultaneously acting as individuals and as collectives.

So to use your favourite word, it's in the flow.

Exactly! And through which mathematical ideas and understandings shape-shift.

Okay... but can you just show me again, an example of this shape-shifting of ideas and understandings.

Well, last week, the children had been given each a small bag of m&ms. During this activity, the students worked on their own: estimating and then finding out how many m&ms there were in the bag, sorting them by colour, graphing, and then determining the divisibility of the coloured sets by whole numbers.

okay.

As I made my way to each of the children while they worked at their desks, I got a bird's eye view of the class' results.

You did, but your students didn't.

No. And they weren't looking and comparing their data with one another. Instead, they extended their initial analysis to general statements from what they had found in their own bag of m&ms.

Like...

Well, like James, for example, observed that for "all colours except green m&ms, there were two or three m&ms." And Holly explained that there were either "lots" of a certain colour or "hardly any" of it. Because these were small

bags of candy, Holly considered "lots" to be four or more and one or two to be "hardly any."

> *So they were generalizing from their own particular instance.*

Yes, and seeing the variations in the children's data, I wondered how I might bring their thinking to a collective realm so that the students could consider these conjectures within larger sets of data and other people's ideas.

> *So, you weren't just moving their thinking into this realm, but their ideas as well.*

That's right.

> *But how would the larger context do more than create a collection of individual ideas?*

Well, I was interested in what ideas might be born from the interaction.

> *... and how they'd shape-shift.*

Exactly. So the next week, the children revisited their work...

Divided into groups of four or five, each student studied his or her bar graph' wrote one statement that reflected a quantitative aspect of their bag of m&ms and shared it with their small group.

## M&M Math
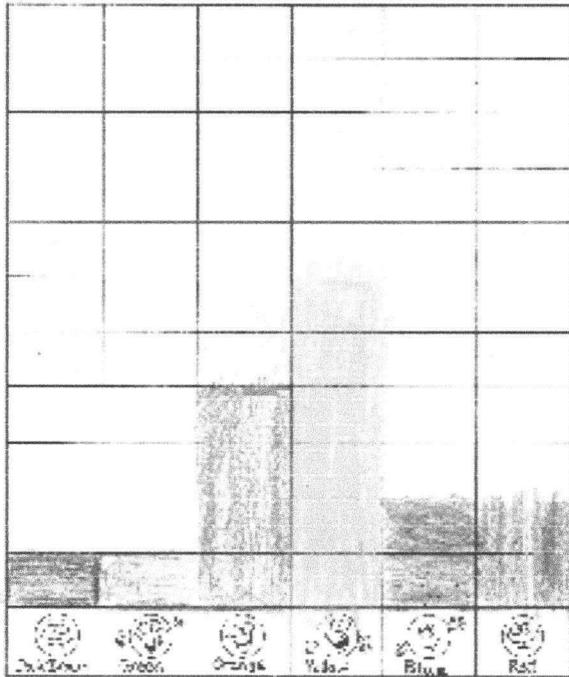
Green and brown don't have very many. (Mark)

M&M Math
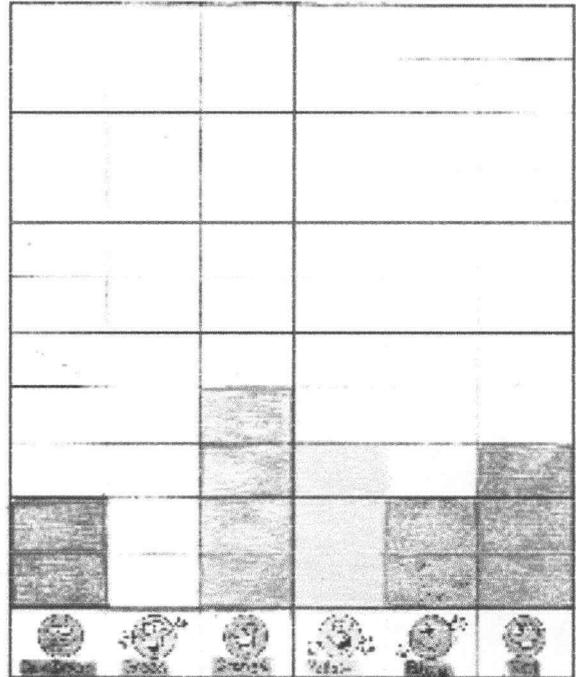
There is 3 reds and 3 yellows. (Danica)

M&M Math
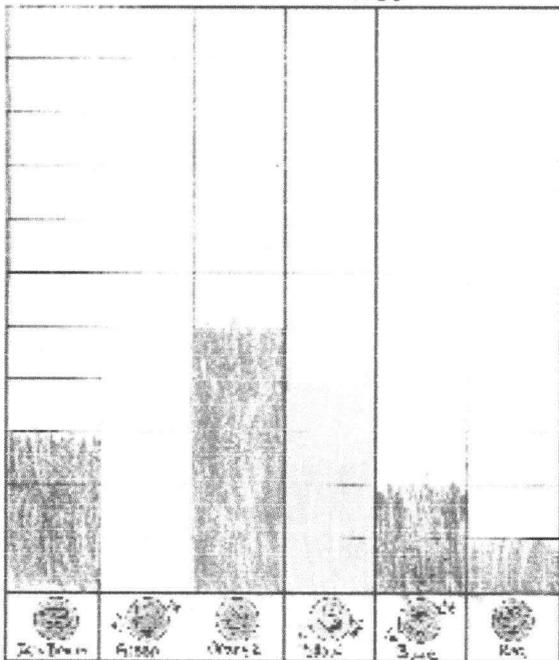
There is no same number of colors.
(Robby)

M&M Math
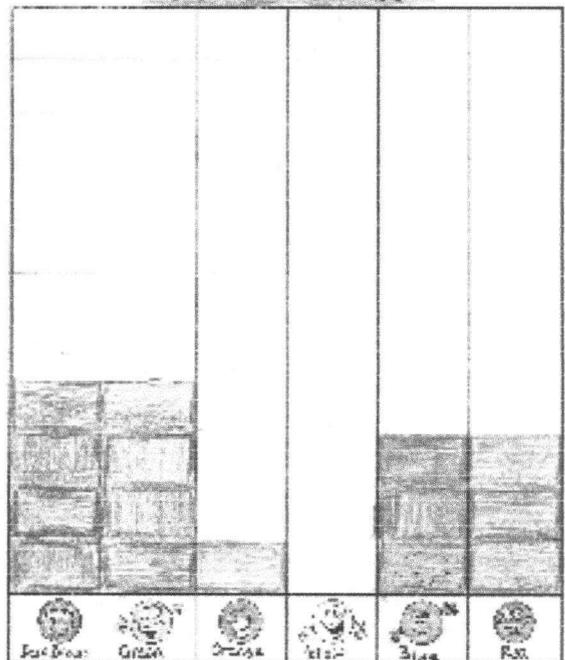
"There is Two pairs of the same numbers
and another That's alone." (Michelle)

... "another that's alone"?

I know! When I asked Michelle to tell me more about her statement, she explained that "because there's four here beside each other" and pointed to the four dark brown m&ms and the four green m&ms. "And three that are the same right there", pointing to the three blue m&ms and the three red m&ms, "... there's one that's all alone... it doesn't have a partner of a different colour" she concluded, pointing to the one orange m&m and then to the yellow m&m column of which there wasn't any.

She's using the "partner" metaphor[2]... not to describe odd and even numbers but the similarities that existed within her sample. What happened next?

The students worked in pairs. Michelle with Danica and Mark with Robby. This time, each student took their "true" statement and examined it in relation to their partner's data.

To see whether or not it made sense for both sets of data.

That's right.

But what if a statement didn't reflect the larger sample?

They were to work out how they might revise it so that it made sense and still maintained its original idea.

Why didn't they just throw away the idea if it was incorrect?

Because I wouldn't let them! That wasn't the point of the lesson. It wasn't about right or wrong ideas but to focus on the notion of re-crafting ideas in the face of other evidence so that the integrity of their original thinking could be maintained.

Oh okay... focusing on how one's thinking and ideas can shift-shape.

Exactly. So when I went to join the two boys...

Mark was comparing Robby's graph against the statement (i.e., **green and brown don't have very many.**) he had recorded on his sheet.

"How are you doing, Mark? Does it make sense?" I asked.

Mark replied "Sort of... because he has three brown and that's not very many and no green."

Having qualified three as "not very many", Mark seemed satisfied to keep his response the same.

I then went to see how Danica and Michelle were coming along with their work. While I was doing this, Mark picked up his pencil and underneath his first statement recorded, **green dosnt have any in Them**. Robby pointed to the sentence and commented that yes, he didn't have any green m&ms in his bag of candy but Mark *did*. When I returned, Mark told me that he had changed his statement to that of only green m&ms. As this did not reflect his *and* Robby's sample, I reiterated that they were now looking at *both* graphs to see if the statement still made sense. Mark immediately shook his head and said no. I then asked them how they might shift it so that it would reflect the larger sample of data.

"Keep the same answer." Mark said.

"So... keep *green and brown don't have very many?*" I asked.

Mark nodded his head and drew an arrow from his first statement down past the second one to show that he had changed his mind.

green and brown don't have very many.

green dosnt have any in Them.

In that episode, even though Robby explained very clearly, why Mark's second conclusion didn't fit for both sets of the data. Robby's attempt to shift his partner's thinking didn't have as much impact as I thought it might've... until you came back and began working with them. Why do you think this happened? What was it about your interjections that enabled Mark to reflect on his work and to see that his statement still made sense?

I suspect first of all, it's because they weren't used to working in this way... having to stay with an idea and push how they're thinking of it even when it doesn't immediately appear to resonate with the new context. Mark didn't think that his statement was true of Robby's data so he shifted his idea to only be about green m&ms. But when he did this, it no longer represented his results! Then, when Robby's pointed out the contraction, it might have been not as effective in prompting Mark to revise his statement because Robby moved from one set of data to the other. My interaction was slightly different in that I focused their attention on looking at the graphs together not separately, and asked them how the idea might be altered to accommodate both bags of m&ms. Not only recognizing the contradiction but also, the opportunity to make new sense within this context.

The boys then examined Robby's statement, **There is no same number of colors.** in light of their data. This time Robby moved his head back and forth only to discover that unlike his graph that had different amounts for every colour, Mark's graph had "... lots of the same... same number".

Interaction of the data occasioning change?

Yes.

Mark had two each of blue and red m&ms and one of each of brown & green. Mark suggested that they change the statement to, "... we have the same number of blue?"

Robby looked at the two graphs but did not say anything.

"Yes" I said, acknowledging Mark's suggestion but added "... then he's switching what his focus is because he's looking at *same amounts of colour*, right?"

Robby looked at his statement and then began thinking aloud, "...the amount of colour that...."

Mark and I turned to see what Robby was trying to work out.

"What do you mean?" I asked Robby.

"Do we have to combine them to get..." Robby asked, taking his pencil and pointing to both of the graphs.

"Yes" I explained, "so you're talking about *same number of colours*, right?"

"Yes" said Robby.

Pointing to the statement on his piece of paper, I paraphrased what he and Mark had found. "You said that there's *NO* same number of colours "I stated, pointing to Robby's graph. "But if you were to look at *two* bags of m&ms" I continued and pointed to the two graphs, "you'd find out that that wasn't so."

Robby shook his head in agreement.

"Because here," I pointed to Mark's graph, "Mark got some [i.e., m&ms] that had the same amount of green and brown and the same amount of blue and red. So, how could you change this idea so that it would make sense for *both* of the graphs?"

You helped them to define the conceptual space they're working in...

The mathematical relationships of the context.

Yes and at the same time, pointed out the need for them to make new sense of that space.

Robby and Mark looked at their graphs. Mark decided that the statement could still maintain the notion of 'same amount' but rather than 'colour', he suggested that perhaps they could look at "same number in both [i.e., bags of m&ms]".

Mark began counting the total number of m&ms in Robby's sample, "three plus five is eight, plus four... is nine" he said as he added the dark brown, orange, and yellow m&ms. "... Is twelve, plus two is fourteen" he continued, adding the number of blue m&ms. "Fifteen" he concluded after adding on the one red m&m.

Robby was also totaling up the m&ms in his sample but in his head, nodding while Mark subtotaled aloud and giving one final nod when he reached fifteen.

Mark then totaled his number of m&ms "... and then you'd go two, four, six, eight, ten, twelve" he said while counting the dark brown, green, and orange, and yellow m&ms by twos and adding the four remaining blue and red m&ms to end with "... sixteen."

"Okay, he's got fifteen and you've got sixteen", I said and looked back and forth at Robby and Mark. "So what could you do with that idea [i.e., 'same amount']? Could you say something about the number of m&ms that would make sense for both of those [pointing to the graphs]?"

"Maybe, not all bags of m&ms have the same number in them." replied Mark.

"What do you think, Robby?" I asked.

Robby nodded his head and said "yes". He then wrote **Not all m and m bags have the same amount of m and ms.** for his second statement.

You can see how the students' thinking progressed in the way they developed the statement. Staying with the notion of 'same amount' created the need for Mark and Robby to reanalyze their data and form a related yet different interpretation.

And the result wasn't the effort of just one individual. What came about was a consequence of mathematical interaction amongst the three of us, all thinking around the idea of 'same amount'.

What were Danica and Michelle doing?

The girls had taken Michelle's idea and "tested" it against both of their graphs. They were smiling and calling me over to see what they'd found out.

"How'd you do?" I asked.

"Good!" Danica beamed. She picked up her graph and showed that "because I have the same as hers... like two two", Michelle's statement remained "true". Danica had the same number of red and yellow m&ms and the same number of blue and dark brown m&ms in her bag.

Michelle supported what Danica had explained, reiterating that "we have two pairs that are the same" and also added, "and then we both have one that's different." This latter statement was in reference to what Michelle had identified as "one that's all alone," which she described as "one that's all alone... it doesn't have a partner of a different colour."

Like Mark, Michelle drew an arrow underneath the first statement to indicate that the statement remained the same when compared against both sets of data. And just as in Robby and Mark's case, Danica and Michelle were also challenged when they came upon examining their graphs in relation to Danica's statement.

Danica commented that "Michelle has three reds, no yellows, and three blues." She then wrote:

Michelle has 3 reds but insted of 3 yellows she has 3 blues.

Both girls agreed that this only made sense for Michelle's graph and that it did not apply to both of the graphs.

So then, what?

Well, just like Robby and Mark, they could say what it wasn't but they weren't sure how to "shift-shape" the idea.

Bringing forth Danica's notion of the 'number of red and yellow m&ms', I repeated what she had already said but focused their attention on the red and yellow m&ms. "And you found out that Michelle has no yellows but the same amount of reds... could you do something... with the idea-- looking at the idea of reds and yellows? What could you say? Both graphs..."

"... have reds in them.... Both graphs have 3 reds." Danica replied.
Michelle and Danica double checked the statement to make sure that it reflected both sets of data. The two girls nodded and concluded "yes!" Danica then recorded:

both gragphs have 3 reds.

Having examined and revised their ideas against their partner's data, I had the children form a working group of four.

I read out the statements that they were to now focus on:

- **green and brown don't have very many.**
- **Not all m and m bags have the same amount of m and ms.**
- **both gragphs have 3 reds.**
- **There is Two pairs of the same numbers and another That's alone.**

As a small group, we reflected on the statements and talked about how two of them had undergone changes while the other two remained the same. The group then began to "test" the statements to see whether they would satisfy all four sets of data. The children laid their graphs out in a row and examined them, beginning with Michelle's claim, **There is Two pairs of the same numbers and another That's alone.**

Mark quickly pointed out that it no longer made sense because it did not reflect his or Robby's graphs. The four children proceeded to study the graphs, saying nothing. To help them, I made the suggestion that they might look to make sense of three of the graphs rather than all four at once.

Robby looked at the girls' graphs, then Mark's, and then his. Here, he corrected Mark by pointing out that Mark's graph was coherent with Michelle's pattern because it too had *two pairs* of colour sets that have the same number of m&ms. He had two of each blue and red m&ms and one of each of dark brown and green m&ms. Robby swept his hand across his graph and explained that it is when they come to his data that the claim does not hold true.

"Mine is the hardest because it doesn't have any pairs. So, it has nothing to do with that" Robby said, pointing to Michelle's statement.

Looking from a different perspective, Danica made the observation that Mark's, Robby's, and her bag of m&ms all have two blue candies. However, she dismisses it because she realizes that Michelle's bag of m&m has three blue m&ms.

Michelle pipes in. "Each of us has four of... well, not of the same colour... but four of *whatever* m&ms."

"Four of *any* colour" clarifies Robby.

"Yes." Michelle agrees.

"Everybody has four of every colour..." Danica concludes.

Michelle mulls this over, "four of...."

"Four of *any* colour" Robby says again, correcting the language of the conclusion so that it would be true for all of the samples.

The group agrees with this and makes the conjecture that a bag of m&ms will have four m&ms of some colour. Michelle is not sure how certain they really are that *every* bag of m&ms will have four candies of a particular colour and hesitates in committing this to paper. She and the other three children decide that they are not completely certain and that the term, "most" needs to be included in their statement. Michelle picks up her pencil and writes:

any
Most bags have four of the ~~same~~ colours.

In that episode, you weren't directly interacting with the students. You were there, but the children seemed to be working on their own... on their own, as a collective entity.

Their thinking and work here is a nice example of what I'd consider to be a collaboratively collective form of shape-shifting.

I think I know what you mean. Each time a new aspect was proposed, it was examined and assessed by the children in terms of whether or not it helped to further define the idea before it was integrated with what they had. Mathematical thinking and mathematics in this collective, shape-shifting manner, is really, a recursive and co-evolving process... emerging thoughts, melding or not with that which already exists and through revision, there's renewal of thinking, of ideas...

The students moved on to Danica's idea of three reds. Scanning the four graphs, Danica shook her head and answered, "no". Neither Mark or Robby had three red candies in their bags of m&ms. Danica adjusted her statement in response to the new data and announced that "everyone has reds".

Robby, Mark, and Michelle all nod and unanimously agree with an enthusiastic "yes!"

Still keeping the idea of red m&ms...

But no longer on a specific quantity...

The new statement now satisfies all four samples.

Danica crafts it a bit more and records,

every body has atleast 1 red.

Robby read out his statement, "Not all m and m bags have the same amount of m and ms." Mark, Robby, and Danica skip count by a combination of twos, ones, and threes to find out that Danica has fourteen m&ms and Michelle has fifteen candies in total. The boys no longer think that the statement makes sense.

Robby justifies this "because Michelle and mine have the same amount."

I reread Robby's statement, emphasizing the language in it, "It says that NOT ALL bags of m&ms have the same amount. Does that mean that *some* bags COULD have the same amount?"

"Yes" replied Robby and Mark. Robby then drew a downward arrow from his second statement to show that the third statement remained the same.

Not all m and m bags have the same amount of m and ms.

$$\downarrow$$

As he did this, I prompted the other children to think about the meaning of the statement and to articulate why the use of the word, "not" made it work. "It wouldn't make sense if Robby said... what?"

"*All* the bags of m&ms have the same amount in them." said Mark.

With this, the group moved on to testing the fourth and final conjecture that was Mark's claim that **green and brown don't have very many.** Immediately, the group saw that it held true for Mark, Robby, and Danica's graphs but not for Michelle's. She had four of each of green and brown candies and they agreed that given the small package of m&ms, four was too many to be considered "not very many".

Wanting to integrate the language used in Robby's statement, Mark thought aloud, "Not all bags of m&ms..."

Danica interjected, "all bags don't have the equal amount of browns in each column. Like... nobody has the same amount in browns. 'cause Mark has one, I have two, Michelle has four, and Robby has three."

Mark agreed with what Danica had said and was about to write "not all the bags have the same number of brown." when Robby spoke up.

"I have another one that he might want to use" he said. "It is... not all bags have very many... not all bags have very many greens."

"But when you look at Michelle's, she has four." argued Mark.

"I said *not all*" responded Robby.

"So.......?" I said.

Mark paused for a moment. "I would say that not all bags have the same number of green and brown in them!" He recorded the revised conclusion:

not all m and m bags have the same number of green and brown.

Another recursion in his thinking that effectively integrated the notions of "not all" and "sameness" together while still keeping the focus on green and brown m&ms.

Adding on one final layer to their analysis, I gave the children two more sets of data to consider:

M&M Math   M&M Math
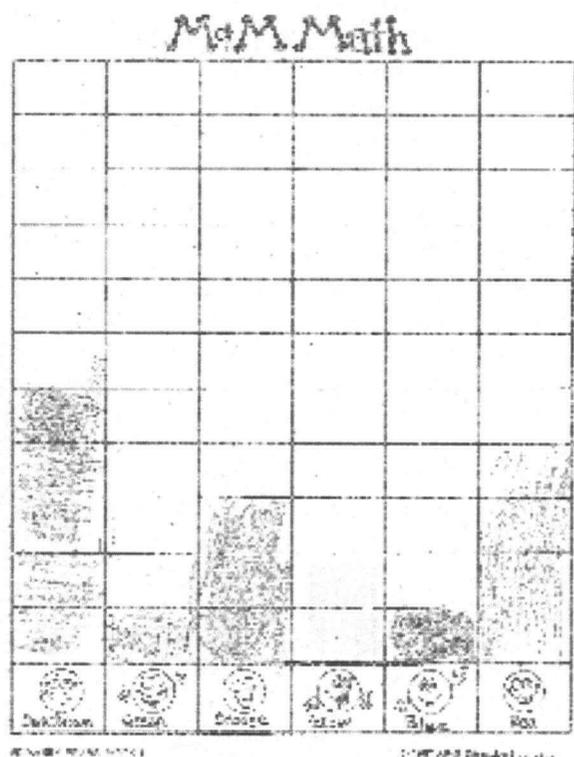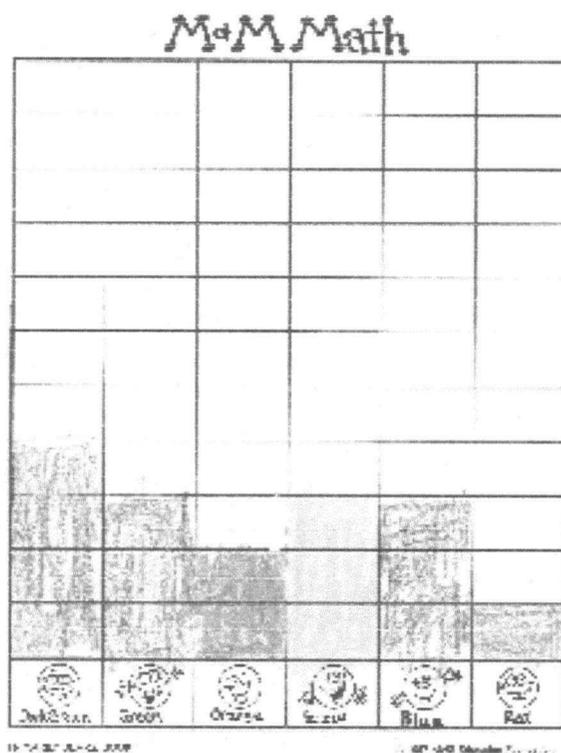
Lara's graph   Amanda's graph

Now with six graphs to consider, the group found that Michelle's statement remained the same. As Lara had four dark brown candies and Amanda had five dark brown and four red m&ms, both samples satisfied the conjecture that most bags would contain four candies of a particular colour.

Danica smiled as she nodded to indicate that her statement was also satisfactory. According to the two new graphs, Amanda and Lara had at least one red m&m.

Robby read his statement aloud, "Not all m and m bags have the same amount of m and ms." Following this, he totaled the number of m&ms for each of the two samples in his head. Robby concluded Lara and Amanda's bags each contained sixteen m&ms. "Does that hold true still?" I asked.

"Yeah, a bit" he replied.

"Why are you saying a bit now. What have you found?!" I asked.

"Fifteen goes with mine and Michelle's and... those ones" he explained as he pointed to Lara and Amanda's graphs, "goes with Mark's".

"So would you want to change your idea or [shift] shape it a little bit?" I asked.

299

"Yes. Most of the bags have fifteen or sixteen" he said and then recorded this as his final statement.

Do you see that? Do you see what's emerging from Robby's thinking?

> *Yes! He's looking at a different, more specific pattern in the data. He didn't identify it as "average" number of m&ms per bag, but that's the distinction he's making what he's getting at-- that of the bags of m&ms, more of them had a total of either fifteen or sixteen candies than not.*

Mark read out his statement and quickly nodded "yes"; that it still made sense for the six sets of data. According to the graphs, both Amanda and Lara's bags of m&ms had different quantities of green and brown when compared with the other four graphs.

Having tested and revised their conjectures across the six samples, the children told me that although they were "pretty sure", they were not completely certain that the four conclusions would hold true for even larger samples of m&ms.

> *I wonder what would make them very certain of their conjectures. Did you ask them?*

Yes, and...

Robby figured that "you'd have to look at as many bags as possible-- twenty to forty" while Michelle was still not convinced that this would be enough samples to be really sure of her statement.

She said that she would need to look at *many* more-- "about twenty-five bags every day for twenty-five days!"

Danica agreed with Robby and explained that a twenty bag data sample would suffice in testing the certainty of whether or not every bag of m&ms had at least one red candy. And Mark said that he would need to test between fifty and seventy-five bags of m&ms to be certain of his conjecture.

Bringing this session to a close, I asked the children to look at their statements and to consider if and how they changed as the number of samples increased.

"Some of them changed and some stayed the same." said Mark.

"Did anyone have an idea that stayed the same for the whole lot?" I asked. All four students shook their heads, "no".

Having established that all of their conjectures went through some kind of revision, I was interested in their thinking on why this happened.

"So everyone's changed at some point. Okay. Why did they change?" I asked.

The group explained that each time they looked at the results of another bag of m&ms, they realized that their conjectures were not always consistent with the new data "because not all m&m bags are the same."

Mark, Michelle, and Danica all noticed subtle shifts in the shape of their conjectures over the course of examining the six samples whereas Robby nodded his head and exclaimed that his work underwent "*HUGE* changes. I started with colours of brown and green and then with all colours (i.e., total number of m&ms in a bag)."

By taking the mathematical patterns that the students identified from each of their graphs and situating them within larger, collective realms Mark, Danica, Michelle, and Robby got firsthand experience in how the mathematical ideas shape-shifted... and how they evolved as a result of the interaction of their individual and group thinking.

And, the integrity of the final conclusions that were born from these interactions... they were really, much more than the sum of the parts that fed into them. I mean, the specific aspects generated by the children as individual and collective agents came together not in a piecemeal, jigsaw puzzle way but in a co-emergent manner. What they gave rise to were very sophisticated ways of thinking about the data.

More specifically and pervasive in all of the conclusions is the understanding of what you can generalize from the data and what you cannot. As well, that growth of understanding isn't always a matter of discarding and replacing ideas, but rather, seeking out patterns... relationships.... and through shape-shifting deeper, more comprehensive notions within a concept can be developed... developing thinking into a greater whole.

Notes

1. This graph was adapted from *AJMS*, 1987, p. 70.
2. This metaphor first emerged while the students were investigating even and odd numbers. See page 161.

Striking a balance between those spaces in which the children and I are involved in direct mathematical interactions and spaces where I am not part of their immediate work is another important aspect of place-making in the classroom. When I am working with a child or group, I am in the midst of, and therefore, part of the co-emergence of mathematics taking place. On the other hand, when I am not directly interacting with the students, I become part of the extended environment. Thus, providing opportunities of a different kind for myself and the children.

For the children, they become responsible for opening and exploring their own mathematical spaces. For me, I am able to attend to and learn about their understandings from a more distant vantage point. The spaces that are created as a result of the children's mathematical intra or inter-actions not only provide me with glimpses into their understandings but also, sources from which to further investigate their thinking and perhaps engage them in folding back or provoke them to extend the mathematics through direct interaction.

Opening Mathematical Spaces Of Their Own

With a blue pencil crayon in hand and a piece of paper to record their mathematics, Sammy makes his way to the round table and pulls up a chair for his best friend, Sam, so that they can work together. Sam sits down and looks around the classroom.

"Sam," Sammy taps Sam on the shoulder.
"Let's do it!" Sammy asks Sam what he
knows about the number, 72.

                              "How about..."

"What equals that number?!" Sammy raises his
right hand in the air. "Oh! I know!"

                              "How 'bout... divided!" Sam leans towards
                              Sammy and clasping his hands together
                              smiles, "... will make it equal..."

"How about times? We could say something that
equals that number."

                              "Yeah!"

"What is it? Don't say anything! Just let me think."
With his chin resting in his hand and his
elbow on the table, Sammy looks in
the opposite direction from Sam. Sammy
then records 8x9=72, which catches Sam
by surprise.

                              "How do you know that?"

"I know the times-table!"

Sam and Sammy are searching for something they already know to be "true" about the number, 72. They don't intend to develop a mathematical description for the number but to re-member or fold back and collect an image-- to search through their mathematical understandings and locate something they know without question, to be a fact about the number. Although Sammy and Sam specify that they are wanting a multiplication fact, they aren't concerned about locating a specific one. Sammy's image of 8×9=72 satisfies this because he simply retrieves it. He hasn't had to engage in any mathematical activity to come up with this fact, he just knows this to be true because of "the times-table!"

"My turn! Seventy-two plus seventy-two..."

"That   number plus that number divided by two equals that number.... Two, two..." Sammy then proceeds to add the two sets of twos in 72 + 72 together.

"It's one hundred forty-four! So it's one-hundred forty-four divided by two!" Sammy attempts to move Sam's thinking along.

Sam pleads with Sammy. "Let me count... Seventy..."

"Multiplied by two."

"... eighty, ninety, one hundred...." Sam counts on, first aloud and then whispering to himself, keeping track of the number of tens he is adding by unfolding a finger each time. Once done, he then writes 144÷2=72.

What's going here? Sam's announces that it's his turn to describe 72 and the boys return to Sam's first idea to include "divided!". They set to work on developing a mathematical expression that uses this operation and now, different aspects of their understanding are being revealed. Sam on the one hand, approaches the task by using repeated addition to find the sum of 72 to 72 and records $144 \div 2 = 72$. From his working with specific numbers and his insistence that Sammy let him skip count on his own to determine what the sum for $72 + 72$ is, I'm assuming that Sam is using what images he has to come up with the mathematical expression. He doesn't appear to "know" this already nor do his actions indicate that he has a more formal understanding of the pattern he is working with... like, 'if you add a number to itself or double it and then divide it by two, you'll end up with the original number.'

Now Sammy on the other hand, is trying to complete Sam's thinking by explaining that, "THAT NUMBER plus THAT NUMBER divided by two equals THAT NUMBER". His thinking is a more formal conceptualization. Where Sam works with specific numbers, Sammy's considering the mathematics from a more distant perspective. He's thinking about how the actions of repeated addition and dividing by two, work for ANY number and then he's applying this understanding to the specific context of 72. As well, Sammy's suggestion that Sam "multiply by two" implies that he understands repeated addition to be related to multiplication.

Pause.
Giggles.
Pause.

Sam suggests taking 72 away from a ridiculously large number.

More giggles.
Pause.

"Aa-she-waa!" Sammy rests his head on his hand. "Oh! Let's say something about it. Let's not do any of the sums."

"It is between eight and something" Sam extends his arms outwards and moves his body in a left to right motion.

Sam's gestures tell me that he's thinking with the image of numbers existing along a horizontal line. He expresses 72 as being located between eight and another supposedly, larger number.

"It's between seventy and eighty." Sammy records It's between seventy and eighty.

"That's a good one." Sam smiles at Sammy.

"Is it odd or even?!" whispers Sammy.

Sam takes the pencil and writes It is a even number. and then states, "It's a two digit number."

"It's a two digit number." Sammy, smiles back at Sam.

Sam writes It is a two digit number.

Sammy's prompt for them NOT to just continue to perform operations on 72 but offer other ways of thinking about 72 reveals that these two students associate numbers as possessing other properties rather than only being a product of arithmetic actions.
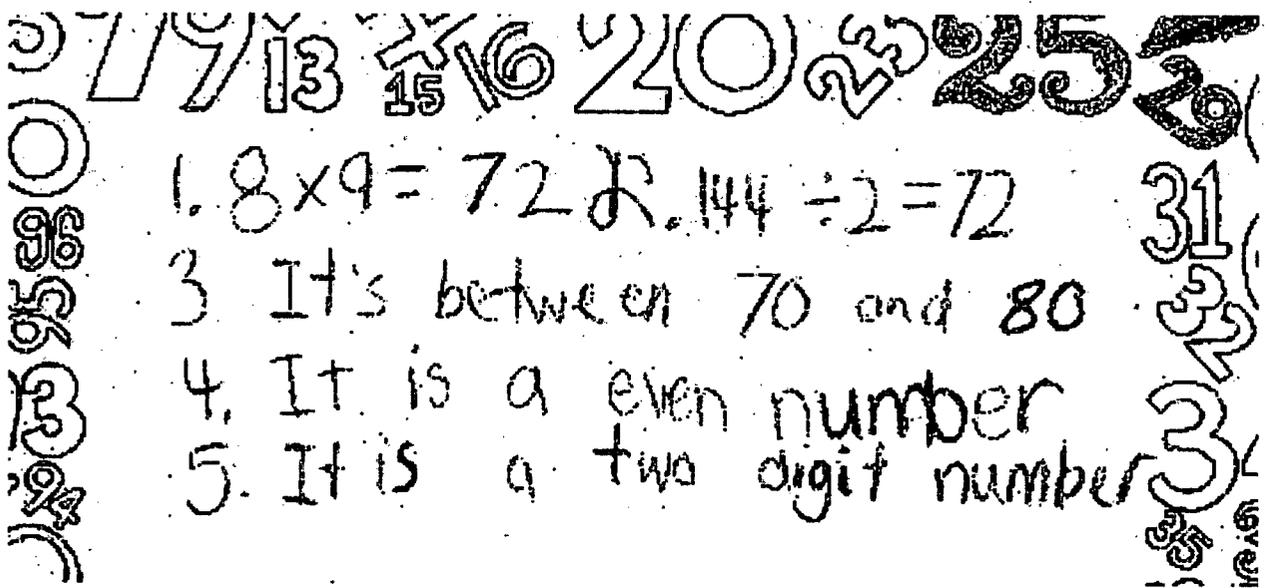
1. $8 \times 9 = 72$ 2. $144 \div 2 = 72$

3. It's between 70 and 80

4. It is a even number

5. It is a two digit number

I approach the two boys. "Sam and Sammy, may I join you?"

Sammy moves the piece of paper between the three of us and begins reading the equations that they have written about 72. He tells me what he has already told Sam-- that he knows 8×9 to be 72 because he knows his times table.

This confirms my previous assessment that he's simply folded back and collected an image. Now, I want to know more about the thinking that gave rise to what Sam has written for #2.

"How did you think of doing one hundred forty-four divided by two equals seventy-two?"

"Because seventy-two plus seventy-two equals one hundred and forty-four. If we divided by two, we'd be cutting one hundred and forty-four in half."

In this one sentence, Sam reveals many aspects about the understandings that are embodied in the mathematical expression. Not only is the equation numerically and operationally "correct", but he's also able to step back and talk reflectively about the relationships and distinctions he's made about the mathematics at hand-- property noticing. He considers repeated addition and possibly, multiplication by "doubling" 72 as being the opposite or inverse action of division. He relates "cutting one hundred and forty-four in half" as being the same as dividing the number by two. Because Sam is focusing on specific values-- 72, 144, and 2, it's not possible to say whether he's got a generalized understanding for what happens when the combination of operations are applied to ANY or ALL numbers.

I ask Sam and Sammy if they've recorded any descriptions other than symbols.

"Yeah" Sam points to #3 on their sheet. "It's between 70 and 80".

"Yeah. We used words."

"Can we move onto that and look at that?"

[reads aloud] "It is between seventy and eighty."

I ask Sam and Sammy why this makes sense to them.

"Because it is."

"Because it's higher than seventy and lower than eighty."

Sammy's image of 72 as "higher" than 70 and "lower" than 80 is similar to Sam's previous bodily gestures. He too is visualizing a number line. Sammy's number line however, is a vertical one on which 72 is located "between" 70 and 80 and "ranked" higher than 70 and lower than 80. But where did this image come from? Is it simply a linguistic one that he learned by rote? Or is it an expression of conceptual number sense? I suspect that it's connected to the game we play in class where the children try to make the highest (or lowest) possible number by placing randomly drawn digits-- 0 through 9 in either the hundreds', tens', or ones' column. In this game, the class creates a VERTICAL number line by recording all of the possible numbers that could have been made with the drawn digits on the chalkboard. This part of the game requires them to identify and describe to their peers whether or not and sometimes, justify why the number is considered to be "higher" or "lower" in relation to the other numbers already recorded.

Moving on, I read out their next description-- "It is an even number" and ask the boys to talk about it.

"Because if it [72] was a seventy-three, it would be odd because the seven and the three are odd....".
Sammy in a more confident tone of voice, "It's [72] even because seventy-two can be divided by nine... it can be... divided into nine, into nine groups." Sammy then points to 8 x 9 = 72 on their paper.

"Okay, so you're saying if seventy-two is even, it can be divided..."

"Because eight times nine means eight groups of nine, equals seventy-two."

"Okay..." I ask the boys what they think of 73. They immediately in unison, tell me that 73 is an odd number.

"It would[n't] be divided into equal groups.... Because it can't be divided by nine, it can't be divided into nine groups."

> I'm interested to see whether Sammy will elaborate on this notion of equal groups by applying his thinking to other numbers, I pose the question, "what if it [the number] was seventy-one or seventy-four?". No response!... "What makes an even number... even?!"

"You can divide it into equal groups."

> "You can count by twos. Two's an even number [unfolds two of his fingers], four [unfolds four of his fingers], six [lays his right hand over his left hand so as to make more' fingers], ... all the even numbers."

Based on the fact that Sam recorded 72 as an even number and Sammy didn't disagree, I might have assumed that they shared the same understanding for why 72 is an even number. However, now that they've revealed what they each mean by "even", it's clear that Sammy and Sam have two COMPLETELY different understandings.

Sammy understands "even" numbers to be those numbers that can be "divided into equal groups". In this case, 72 is considered to be an even number because it can be shared EVENly into eight groups of nine without any remainders. Of course, this is true of even numbers, but it's not this quality that defines them as such. Sammy also explains 73 as being an odd number that "can't be divided [evenly] into groups of nine". Because 73 is both an odd number and a prime number, and as well, its digits--7 and 3 are odd and prime, it's difficult to distinguish what he means by "odd".[1]

311

In contrast, Sam's thinking of even numbers comes from a mathematically different place of knowing than Sammy's. Sam's understanding of what makes a number even or not is in fact, the appropriate one. As he demonstrates for Sammy and me, a number can be distinguished as being even (or not) by whether you can 'arrive' at the number by skip counting by twos.

[reading #5] "It's a two digit number. What do you mean by that?"

"Because there's two numbers."

"It's got tens and ones."

"Okay, what's a ten and what's a one?"

"There's seven tens and..."

"... and there's two ones."

"What does seven tens mean?"

Sam explains that if you take the base ten blocks and use five ten rods and one unit cube, the resulting number is 51. He draws this with his finger.

Sam's drawing

Sammy and Sam explain to me that seven tens can also be thought of as "seven groups of ten" or "seven times ten".

The boys are demonstrating that they know numbers can be identified as having "digits" and in the case of 72, the digits "7" and the "2" make it a number consisting of a total of two digits. At the same time, they are also communicating their understanding of place value. According to Sammy,

72 can be considered as having "tens and ones". Sam points out that 72 has "seven tens" and Sammy finishes by explaining that it also has "two ones". When Sam acts out that 5 "tens" blocks and 1 "units" block would be 51, he reveals the understanding that the first digit of a two digit number refers to the groups of ten in that number and the second digit refers to the number of ones or units it has. The boys' further discussion of how seven tens can also mean "seven groups of ten" or "seven times ten" tells me that they know the left-hand digit in the two digit number as having a value that is ten times greater than the digit on the right.

I ask Sam and Sammy if there is anything else that they know about 72. I leave the table so that they can work on their own again.

"Let's just draw one more. Groups. Groups of what?... I know! Let's draw a groups story and then..."

Sammy grabs the pencil from Sam's hand and moves the paper in front of him. "Let's draw... Let's draw... There's seven tens, right? We can draw seven tens."

Sam pauses. "Oh... we draw ten groups of seven and one group of two."

"SEVEN groups of ten!"

"TEN groups of seven!"

"It's backwards."

"Put ten circles, with each...." Sam draws the diagram with his finger to try and explain his thinking to Sammy.

"It's seven strokes for seven tens, remember?"

"Oh yeah!!"

"One, two, three, four, five, six, seven." Sammy draws and counts out seven large circles.

"One, two, three, four, five, six, seven, eight, nine, ten." Sam draws ten ✗s in the first circle.

Sammy's persistence that Sam's image of ten groups of seven and two ones is backwards and that they should be recording a diagram of seven tens and two ones suggests that Sammy's thinking is firmly grounded in its place-value meaning and thus, doesn't consider the former as being an equally valid way to describe 72. Sam's flexibility to move from one image to the other however, indicates that he does relate the two images as meaning 72.

The two children continue taking turns at drawing the diagram until all seven circles are filled with ten ✗s each and the last circle has two ✗s in it.
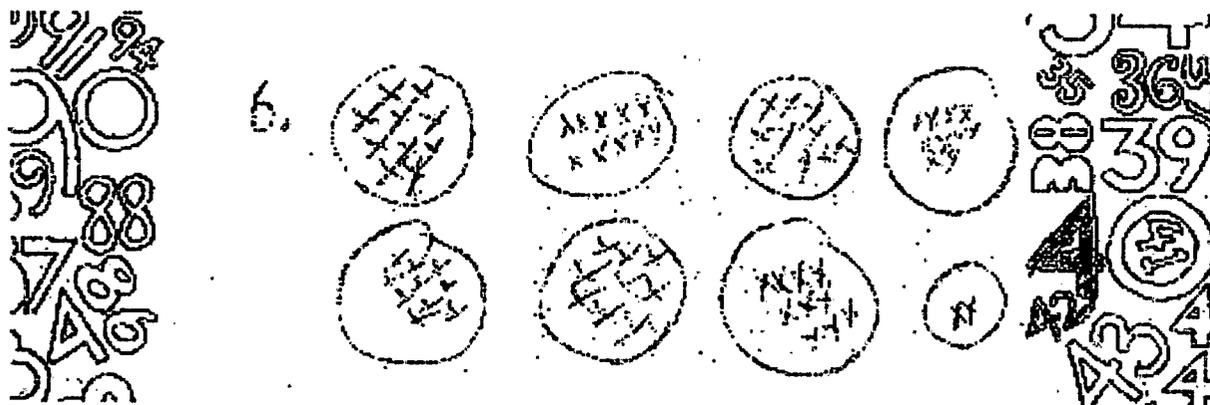
Sam and Sammy's drawing of seven groups of ten and two ones.

I return to Sammy and Sam's table. Sammy's telling Sam that he's thought of another way to describe 72.

Just when Sammy explains that seven tens and two ones make 72,

Sam exclaims, "groups!" and writes 7 groups of 10 and 1 group of 2 makes this number.

I'm curious how they might express their thinking with symbols and so I ask Sam and Sammy if they can think of another way to describe it.

Sammy and Sam talk aloud as Sammy records 10+10+10+10+10+10+10+2=72 onto the paper. "Ten plus ten plus ten plus ten plus ten plus... one, two, three, four, five... plus ten plus ten equals 72."

The students' work demonstrates their fluency in moving from one mathematical form of language to another-- spoken, written, diagrammatic, and symbolic.

Sammy offers another suggestion. "How about minus?"

"How about repeated subtraction?!"

"Oh. I think I know one." Sammy points to 144÷2=72 and says, "How about one hundred forty-five minus seventy-three equals seventy-two? Just make this one higher" as he points to 144.

"Maybe we'll make a repeated subtraction."

"How will that work?"

"One hundred forty-four minus ten..." Sam begins to record 144-10.

"Oh! I know!"

"... minus ten, minus ten..."

"You need four." Sammy keeps track of the number of tens Sam has subtracted from 144.

"minus ten"

"five"

"minus ten"

"six"

"minus ten"

"seven"

"minus two"

"equals..."

Sam finishes the equation.
144-10-10-10-10-10-10-10-2=72.

I point to #9. "How'd you come up with this?"

Sam points to 10+10+10+10+10+10+10+2=72 and tells me "I can read the plus."

Sammy adds: "Forty-two (sic) [presumably, 72] is multiplied twice-- two seventy-twos. [points to 72 in 144÷2=72] And if you minus one of the, take away one of the seventy-twos, but actually don't say something [i.e., 72], do it another way, like this one [sweeps his finger over 10+10+10+10+10+10+10+2=72]. You get seventy-two. Equals another seventy-two [points to =72 in 144 -10-10-10-10-10-10-10-2=72.]

By asking Sammy to explain his thinking reveals yet other sophisticated aspects of his mathematical understanding-- once again, qualities that aren't visible from only looking at what's been written. Sammy explains that instead of writing: 144 - 72 = 72, he'd written 144 - (10+10+10+10+10+10+2) =72. The reason why he didn't record it in this form is simply because he hasn't learned how to use brackets yet. Regardless of this it still doesn't make it impossible for him to think conceptually about the mathematics in this manner! Together with his physical finger pointing to 0+10+10+10+10+10+10+2=72 and verbal reference to "take away ONE of the seventy-twos but don't actually say [it], do it another way, like this one" illustrates that Sammy's very much 'standing back' and reflecting on or observing his formal understanding of the mathematics. He's NOT speaking about the repeated subtraction of ten but IS using a specific case of the distributive law over addition:
- (10+10+10+10+10+10+10+2) = -10-10-10-10-10-10-10-2
Even though Sammy hasn't generated a 'theorem' as such, he is in fact, using a formalized understanding of addition in this context.

I want to know what Sam's thinking and so, I don't react to Sammy's explanation but continue to listen.

"Seventy-two plus seventy-two equals one hundred and forty-four." At the same time, Sam stretches his thumb and fingers around either end of the -10-10-10-10-10-10-10-2

317

and says, "Seventy-two."

"Sam just took another seventy-two by not saying it."

> I repeat what Sammy has just said but in a louder voice to try and bring Sam into the conversation. "**Sam just took another seventy-two by not saying it?**"

"By not... writing it [72] down." Sammy sweeps his fingers in a circle around 144-10-10-10-10-10-10-10-2=72.

> "By not... like... one hundred and forty-four minus... you minus one hundred and forty-four by seventy-two." Sam also sweeps his finger across 144-10-10-10-10-10-10-10-2=72, "and you get seventy-two. Because... because..." and points to 144÷2=72.

"But you also minussed! ... Sam ... Sam also minussed seventy-two but he also... one hundred forty-four minus seventy-two equals seventy-two. He also minussed the seventy-two."

By encouraging them continue, I see that Sam's also thinking of 72 in the same manner as Sammy. Although the boys don't talk in general terms but repeatedly use #8 and #9 as specific examples in their explanation of how they "minussed seventy-two", neither Sam or Sammy speak of "144 minus 10, minus 10, minus ten." and so on. It's clear to me that BOTH students think of subtracting 72 in the form 10+10+10+10+10+10+10+2 as they have expressed in #8.

Moving to their ninth description, I wonder if what Sammy and Sam have recorded here could be true for any number. In other words, can they extend their thinking to a more general understanding of this concept?

"I'd say yes, most of them..."

"Most probably." Sam turns to Sammy. "Pick a number. Pick a number and I'll try."

"Okay. Try the first number we did", referring to the equation II+II+II+II+II+II+II+7+I=85 that they had written earlier. "Eighty-five and..."

"Just pick any number. Just pick a number."

Sam and Sammy are engaging in taking what they know and applying it to a new context. I'm interested in how they might use the mathematics as a "pattern" and not, in a situation-specific manner. Interrupting them, I shift the conversation by bringing Sam and Sammy back to their previous conjecture.

"Sammy, you said that you thought it would work for any number because it was almost opposite. What did you mean by that?"

"It's subtracting instead of adding." Sammy refers me back to what he'd recorded as II+II+II+II+II+II+II+7+I=85.

"... to do this... maybe the number you pick, you add it , you add it with the same number that you... picked... you get the big number..."

"You times it by two! Minus it" (i.e., the number).

"Like eighty-five plus eighty-five equals one hundred

and seventy... and you minus it." Sam points to
$11+11+11+11+11+11+11+7+1$.

"To get rid of the extra half."

After listening to Sam and Sammy explain their thinking again but this time in more general terms, they are definitely demonstrating that what they know isn't specific to particular numbers but a formalized pattern that they've reflected on.

Watching from the edge, I was able to observe Sam and Sammy as they opened up and explored their own mathematical space. Their mathematical activity occasioned opportunities for me to get a close up view of the complex understandings that Sammy and Sam have developed for whole positive numbers.

Their work was not concerned with the generation of as many arithmetic facts as they could think of, but instead, thinking creatively about the number, 72. By folding back, they developed and extended their mathematical ways in which to express it. And as Sam and Sammy did this, one can see how their understandings for whole positive numbers existed as an integrated system of knowing.

Notes

1. In hindsight, had I asked him to explain whether 33, 55, 77, or 75 was even or odd, this might have given me the insight into interpreting his understanding.

Mathematics Beyond the Classroom

I have been working on defining another direction from which to come at G. Bateson's concept of "patterns which connect".[1] I would like to teach collaboratively with another colleague who shares a similar interest and vision for teaching mathematics. As well, because my students will already be working with other students who are younger and older than them and, there is not another second and third grade class in my school, it would be great if the teacher also taught children of the same ages. I would like this collaboration to be an active, ongoing engagement for the teachers and students. I see this as an opportunity not only for sustained cooperative learning-- where long term mathematical relationship can develop for the children's mathematics to take root, connect, and grow but also, an occasion for co-emergent and collaborative teaching.

Two weeks later...

How can I make this happen? More specifically, with whom?! It has to be with a colleague who not only has a passion for teaching mathematics but also someone who envisions children's mathematical learning in a similar manner as I do AND someone who wants to take on the challenge! I have made a list of teachers I have teamed with in the district but none seem appropriate for this particular project.

That afternoon...

I scanned my IN box of emails and noticed that I had received something from my friend, Donna. I double clicked to open the message and found that she had written me a letter asking how my year of teaching had been.

Donna and I first met at the University while completing our Masters degrees in mathematics education. It had been two years since we graduated and we always talked about working together but the opportunity had not presented itself... until now! Not only were Donna and I seeking to teach in ways that focused on the growth of children's mathematical understanding, but upon reading the rest of her email, I discovered that she would also be teaching second and third grade children in the coming school year.

I double clicked on the REPLY icon. Typing a response back to her as fast as I could, I told Donna what I had been up to in terms of my teaching and briefly explained my thinking behind the collaborative project. I then asked her if she was interested in working together and clicked on the SEND icon.

That summer and into the beginning of the school year...

Donna and I began planning the "math buddies" program for our two classes. Using the theme of connecting our students with *their* mathematics, we focused on establishing interactive learning spaces that not only emphasized the children's conceptual connections with mathematics but also, connections that would grow between the two classes and amongst the students as they worked together as one collective group.

Since the program centred on the children's mathematical interactions, we saw it to be a natural context for student-generated problem posing and problem solving projects.

Over the course of two years and despite the fact that Donna and I worked in different school districts, we co-created and taught mathematics classes simultaneously in our own classrooms and together as a group. We communicated to each other through e-mail, phone calls, and meeting each other in person. During these meetings, we reflected on lessons taught and took a closer, critical look at the kinds of teaching and learning that were arising. This gave us a chance to study the students' work and assess their mathematical understandings jointly, thereby, gaining insight from each other's perspective. It also allowed us to flesh out possible lessons and consider the impact that the teaching we were proposing might have on our classes' learning. From here, we sketched out teaching-and-learning settings. Lessons were not prescriptive nor planned far in advance but evolved from the mathematics that surfaced in the classroom(s).

Notes

1. See G. Bateson, 1980, p. 9.

Keeping Our Classes Connected

Even when it was not possible for Donna and I to physically teach together, we took it as an opportunity to find creative ways to keep the children connected. One way we did this was by teaching similar lessons within similar time frames. This provided occasions for sharing and examining the mathematics that was being developed in our classrooms. For example, as both of our classes were working on developing strategies for solving two and three digit computations, Donna and I presented work that children in the other class had produced. Our students were invited to consider how the methods generated by their buddies were the same or different from theirs and whether (or, not) and why the proposed strategies made mathematical sense. Not only did this enable the students to share what they were learning with the other class but it created spaces for them to critique each other's mathematics, to justify their thinking, and to offer new thoughts for their peers to consider.

Robby's "take away one" subtraction strategy

◆ ◆

Mathematical Gifts

The math buddies program was where the students' making and exchanging

exchanging of mathematical gifts began. These gifts contained the conceptual ideas that the children happened to be exploring in class and by wrapping them up in imaginative contexts, the students' offerings turned into riddles, creative problems, and puzzles for their buddies to "play" with. Most often, the children worked with a partner during the posing and the solving of these projects.

Donna and I felt this to be important in order to maximize mathematical communication and collaboration.

The children's first gift to their buddies was a letter introducing themselves. In class, the students had been investigating number concepts and decided that it would be fitting if they introduced themselves in mathematically imaginative terms using the concepts they were learning about, such as "I am in the grade that comes before four and two after Kindergarten" *and* "My favourite number is 12 less than 20 and 5 more than 3". *The children spent a good part of the next day measuring, collecting, and analyzing the numerical data about themselves such as their height, weight, age, shoe size, how many members in their family, favourite number, etc., and then developed mathematical descriptions for them. Donna's students were delighted to receive the letters. This in turn, opened up similar investigations of data and analysis for her students who responded with similar letters for my students to read.*

◆ ◆

"Reflect on the reflection"

Danica drawing reflective images in the "water"

Robby: "Using a mirror helps me to check what I've drawn."

"Reflect on the reflection" was a gift that came out of the classes' ongoing studies of symmetry. Here, Donna's students painted pictures on one half of a piece of paper. They then gave them to my students who had to complete the other half of the image by drawing its reflection in the water.

To solve this puzzle, the students needed to locate the horizontal line of symmetry and then with the aid of a mirror, 'flip' each of the images and paint them in on the other side of the line.

◆ ◆

"Can you build my design?"
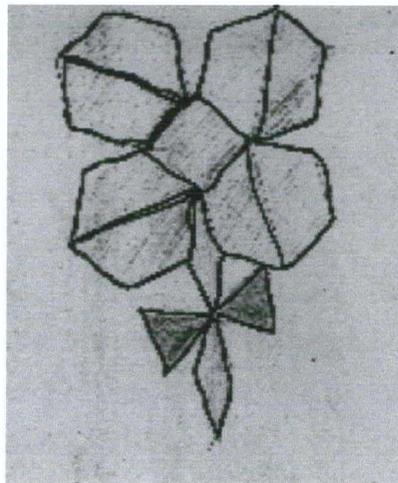
"Flower" pattern block design from Lara, Gregory, Mark, and Brian

1. take a four sided stright square. put it in the middle near the top of the page.

2. take two trapazoids and make it into a hexagon do this four times then put them against the square's sides.

3. take two long skinny dimonds and put them end to end below the flower so it makes a stem

4. Put two triangles poin^{ng}t to the indent of the stem.

5. go and check our drawing.

Lara, Gregory, Mark, and Brian's clues for their block design

Your mystery pattern was fun doing. It was easy because your clues were clear to see. It was a mystery because it was a flower.

A letter from the group's math buddies.

"Can you build my design" was a gift created by my students using pattern blocks to make a geometric design (e.g., a flower). Working in pairs or small groups, the children composed a written set of instructions telling their math buddies how to build the block design and included a picture of what it looked like so that they could compare it with what they had built.

My students applied their understanding of geometry and spatial sense to describe, name, and interpret relative positions in space. For Donna's class who received the challenge, this puzzle required them to visualize, identify, and locate the described positions in space in order to build the design with pattern blocks. Once they had done so, the children compared what they had built with the picture of the problem posers' intended design and wrote back to their buddies explaining what part(s)

of the instructions were clear to understand, what steps were difficult to figure out, and why.

◆ ◆
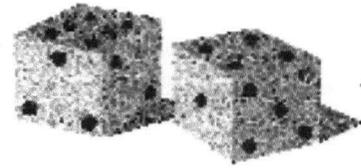
Extraordinary Equations For You!

$5 + 5 + 3 = 13$ ✓

$2 + 2 + 6 + 3 = 13$ ✓

$6 + 6 + 1 = 13$ ✓

$4 + 4 + 5 = 13$ ✓

$4 + 5 + 1 + 2 + 1 = 13$ ✓

$2 \times 5 + 1 + 2 = 13$ ✓

$4 + 4 + 5 + 4 - 4 = 13$ ✓

$4 + 3 + 6 = 13$ ✓

$6 + 2 + 5 = 13$ ✓

$4 + 6 + 3 = 13$ ✓

Carter's number stories about 13.

"Let it roll" was a game my students made for their math buddies. It consisted of a page full of dice showing values of 1-6 and a "target number". Donna's students worked with a partner using as many of the dice to create different number stories about the target number.

The exchanging of gifts and sharing of work gave rise to mathematics that flowed back and forth, connecting our students and bringing our classrooms together. In

this way, the teaching-and-learning that emerged in one classroom naturally became a part of what was taught-and-learned in the other classroom and so on. Very much like a conversation' in progress, the mathematics that emerged existed in the interaction between our two classes and provided an ecological space for growth.

Notes

1. Gadamer, 1989.

◆ ◆

Celebrating Together

As far as Donna and I were concerned, teaching mathematics was neither an exclusively independent activity nor did it need be restricted within the confines of one's classroom. In the math buddies program, one of our aims was to blur and at times, ignore or remove such boundaries we found to be limiting. In this way, we sought to relocate classroom mathematics by planning and teaching lessons in tandem through sharing of students' work, and in the children's exchanging of mathematical gifts. Another way in which we accomplished this was through "celebration days".

Celebration days were when we brought our classes together and entire school days were devoted to engaging in mathematics as a collective group.
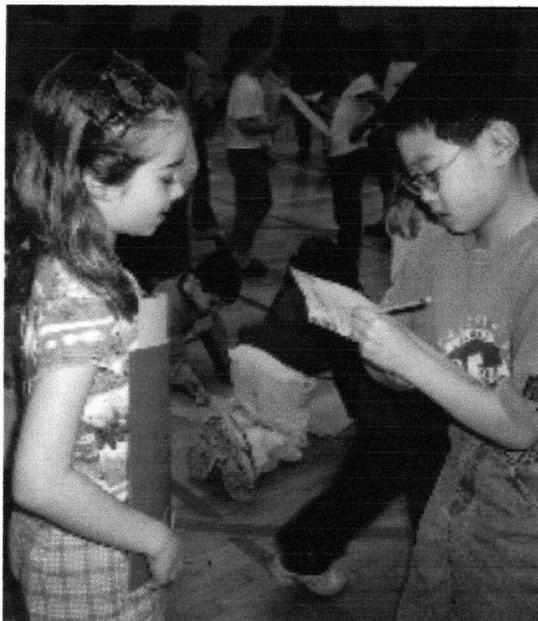
Getting acquainted with one another

This was a scavenger hunt designed for the children to learn about their math buddies' attitudes and thoughts regarding mathematics. The children asked students in the other class questions like, "What do you like about mathematics?", "What do you find challenging about mathematics?", and "Can you think of another way to describe the number, 10?". The collection of questions were generated by Donna and myself as well as both classes of students.

♦ ♦

Teaching our classes together gave Donna and I the opportunity to examine the children's mathematics *as* it emerged and discuss how to plan for subsequent activities. Because we shared similar views on mathematical understanding and considered it to be nonlinear, constantly growing, and ever fluid[1] in nature, the mathematical explorations that we planned needed to be open and responsive to our students' thinking. This meant paying attention to the conversations and questions initiated by the children as they watched a film, read a story, or participated in an activity. Donna and I would then integrate the ideas brought forth by the children into their mathematics for the rest of the day.

In the morning the children had watched a mathematics film called Dance squared[2] and Donna had read a non-mathematical book titled, Selina and the bear paw quilt.[3] Donna and I were curious as to what would happen if we presented the film and the story one after the other and used them as springboards for the students' explorations that afternoon.

Sharing the story of *Selina and the bear paw quilt*

331

Would the children work and produce something with each of their math buddies? Would our classes engage in some kind of collective, whole group mathematics? What would the mathematics entail? Geometric compositions of a square? Tessellations? Quilts?!

The day before, Donna and I collected materials that we thought the children might find useful such as: wallpaper, rulers, scissors, pattern blocks, plastic 2-D shapes of the ones that appeared in the film and the story, chart paper, large pieces of felt material, glue, scrap paper, pencils, a class set of chalkboards, and coloured paper.

The questions and discussion that arose after watching the film and listening to the story created the context that shaped the afternoon's investigation. Over lunch, Donna and I assessed what we thought to be the main themes coming from the children's reactions to the film and the story. Their comments centred around the possible geometric shapes that could be used in composing a square and the shapes needed to make the designs and pictures inside the quilt squares. For instance, "the shapes you would need to make the bear's claw". *Other students spoke about the transformations of two-dimensional figures that would be necessary--* "how you'd need to turn the shape" *in order to make a particular geometric design or picture for a quilt square. And given that the entire discussion was directed toward that of quilts and quilt squares, this naturally became the setting for our students' project.*

That afternoon, the group decided that each of them would work with their math buddy from the other class to plan and produce one square that would then form a quilt made by the entire group. Donna and I explained the two themes that we had drawn from the group's discussion and asked the students how these might be worked into their quilt squares.

"The partners can decide what they want to do... they can do a design or a picture on the quilt square" said one student.

"But they have to do that... they can't just put shapes all over the square... that don't make sense" added another student.

"And, we can use the pattern blocks to figure out how to turn them... as tracers too" explained a third student.

The children dispersed from the middle of the gymnasium floor to find their math buddy and to gather the materials that they needed.

Donna and I made our way around to all of the students either individually or together. Watching, listening, and asking the children to explain what their work entailed, revealed three geometric and spatial strategies that the children were utilizing to plan and make their quilt squares.

Some students used paper and a pencil to sketch their picture and then used the pattern blocks by moving them about. By rotating, flipping, or using them in combination with other blocks, they were able to "fill" the regions of their image with appropriate shapes. Other children chose not to draw their design but used only the pattern blocks as manipulatives with which to plan out and then trace them to make their quilt square.
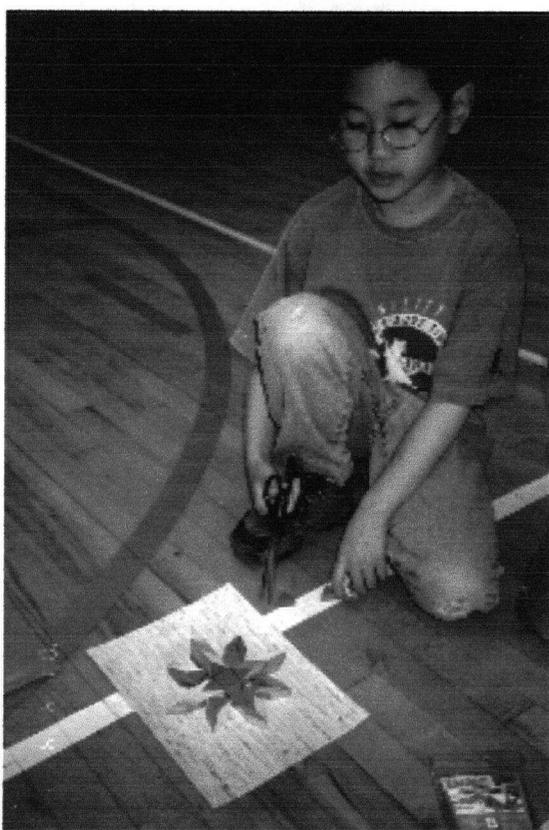
Danny and Jerry's pattern block design

Still, other students relied on their mental image or physical use of a single pattern block to perform transformations.

Clare explains, "If we use the triangle and keep turning it this way again and again until it gets back to the top, it'll make the petals of the flower.

Clare, Julia, and Sean's quilt square design

The students' finished quilt

By the end of the day, the children had created a quilt that integrated the designs and patterns they had experienced through the film, the storybook, and their 'play' with manipulative materials.

"Number gymnastics"

Showing Melissa why 5 is an "odd" not "even" number

During one of our other celebration days, the children helped to create and participate in "number gymnastics". The students selected one of their "favourite" numbers and used it to work through a series of tasks that challenged the students' conceptual thinking and provided Donna and I a chance to observe their flexibility of number sense regarding concepts we had taught during the year.

The mathematical prompts came from a journal that I had kept during the course of the school year. In it contained a collection of curious questions and observations made by Donna, the children, and myself.

15

9

10+3+3+3=19

"Can you create and trace a pattern block design that has the same number of total sides as your number?"

3 out of 3 are green
3 out of 3 are triangles

"Look at your design. What fractions can you find?"

2+2+2+2+2+2+2+2+2+1=19

"Is your number odd or even? Can you show how you know this?"

13-9=4

13

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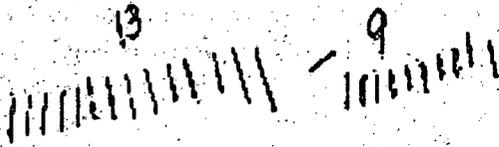
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4

Comparing thirteen and nine

12+7=19

Comparing twelve and nineteen

"Compare your number with __??__ (you choose!) How much larger or smaller is it? Make a diagram or write with symbols to show how you know this."
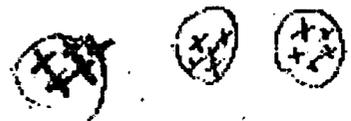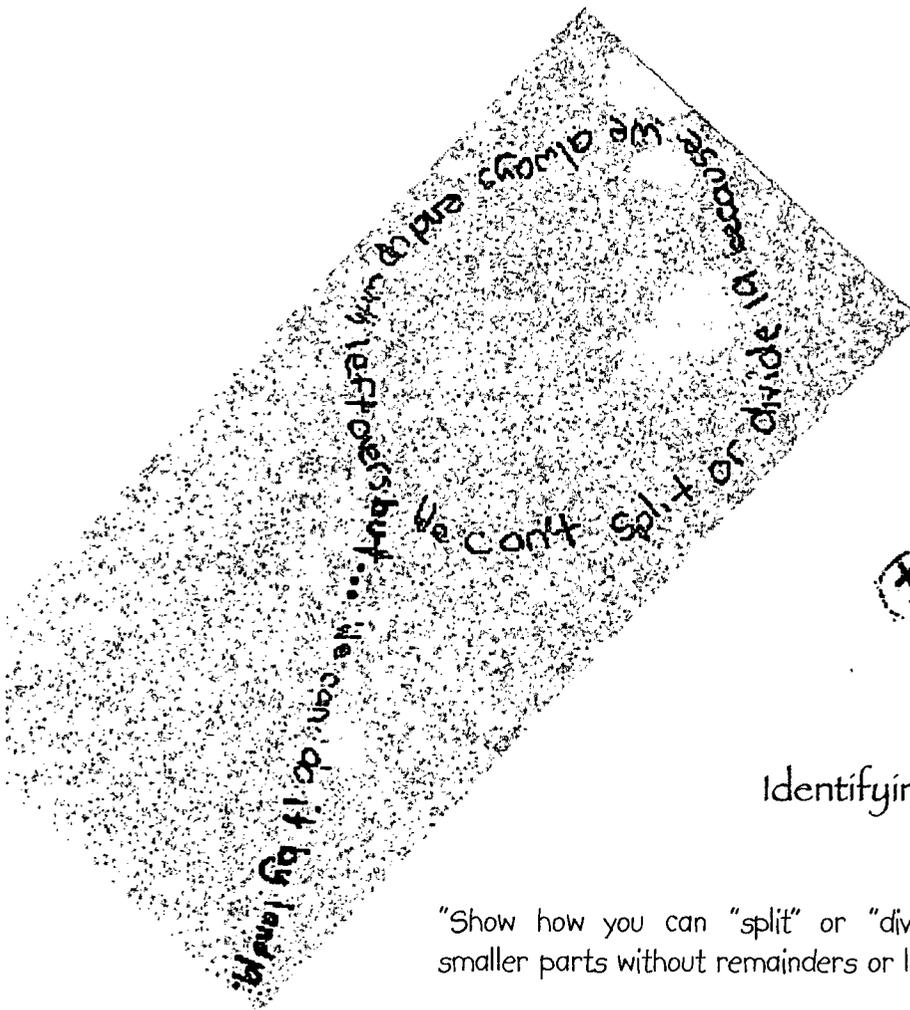
15

Identifying equal "parts" of 15

"Show how you can "split" or "divide" your number into smaller parts without remainders or leftovers."

Identifying equal "parts" of 19

Notes

1. Pirie and Kieren, 1989, 1994b.
2. National Film Board, 1961.
3. Smucker and Wilson, 1995.

Beyond Spaces of Inter-action

Mathematics is not something we have to look up to. It is right in front of us, at our fingertips, caught in the whorl of patterns of skin, in the symmetries of the hands, and the rhythms of blood and breath.[1]

---
[1] Jardine, 1994, p. 112.

Connecting Me, Connecting Us

Classroom mathematics beyond spaces of interaction? What do you mean?

I mean attending and responding to children's mathematical learning by coming at it from a different direction. Instead of focusing on the emergence and development of ideas in relation to the children's interactions, it's about making spaces for them to see how their mathematical ideas and understandings can inform who they are and the contexts of which they are a part.

So it's more on engaging children in thinking about how the mathematics they already know connects to them, to the world, and vice versa rather than their bringing forth of mathematics?

Exactly.

Okay. So why Jardine's thought?

It serves as another reminder that mathematics doesn't have to be something that's "out there". It's "right here" ... quite literally, in you and me... our bodies as examples of mathematics... can't get more personal than that can you?!

Yes. Literally what it means to "be" mathematical.

So it was Jardine's idea that prompted me to see the need to open a learning space that came at the children's learning from a different direction and one that literally connected them with their mathematics.

"Are you a square or an oblong?" I asked the children.

"What?!" was the response I received from my students as they looked at me intrigued but also perplexed by the question I had posed-- What did geometry or more specifically, a square or an oblong have to do with them?! Showing them a copy of Leonardo da Vinci's diagram:
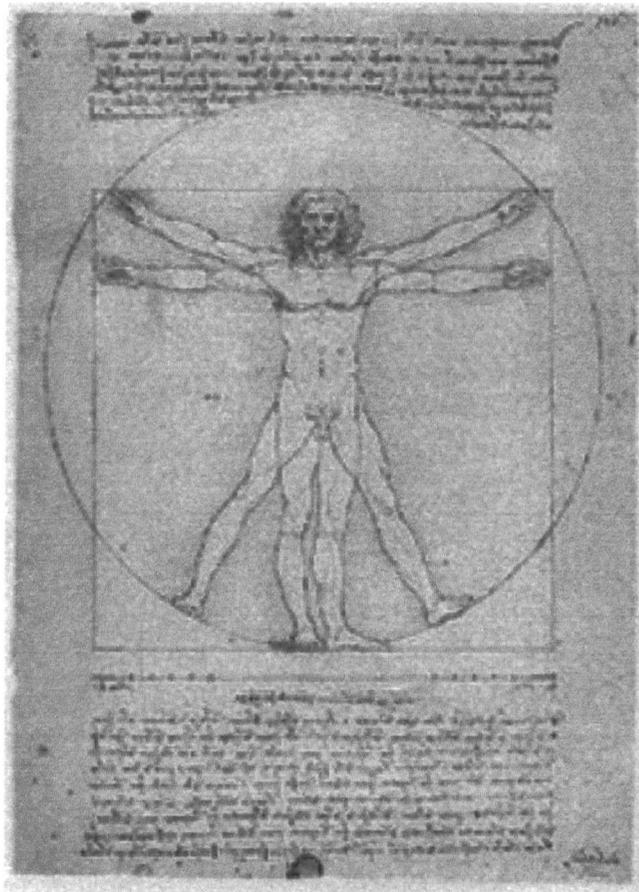
Leonardo da Vinci, *Vitruvian Man*, 1513, dim. 25 x 19.2 cm

I explained to them that he had described a perfectly proportioned "Vitruvian [hu]man" to be shaped like a square; that is, the distance of one's arm span is the same or close to one's head to toe measurement. After we discussed the 'squareness' of the body in the diagram, I turned the conversation to what it might mean if a person was not square but oblong. Giving the students squares and oblongs to physically manipulate, the class then considered the criteria that defined a person as square and compared it to what they knew about oblongs.

The class made the argument that an oblong is different from a square in that when you turn a square so that it "sits" on a flat surface, it "always looks the same".

342
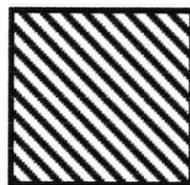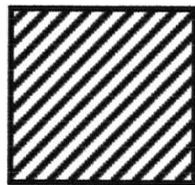
But when you do the same to an oblong, "it doesn't always look the same because oblongs have two longer sides and two shorter sides".
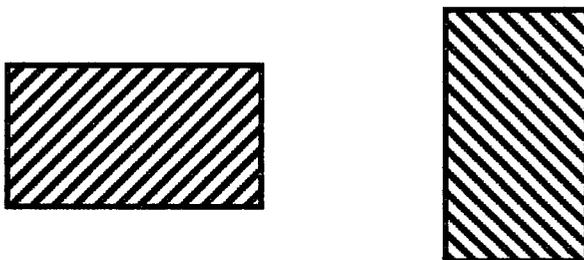
From this, the children made the conjecture that if a person was oblong, he or she might be a "wide", "side to side", or "left to right" oblong. This meant that the person would have to be "wider than they are tall". In addition to this, they explained it was possible that another person could be a "tall" or "up and down" oblong. In this case, the person would have to be "taller than they are wide".

So, here.... squares and oblongs. These are the concepts the children were to think with.

Yes.

And from their verbal and physical descriptions, the children have conceptual meaning for some of what makes a square a square and an oblong an oblong.

And they've taken their spatial understandings-- dimensional properties about squares and oblongs and connected these to dimensions of the body in order to make distinctions between different body shapes.

The class then worked with a partner and carried out the study. They used a tape measure to find out their head-to-toe and arm-to-arm measurements and then rounded the values to the nearest ten centimetres, and then identified what shape they were.
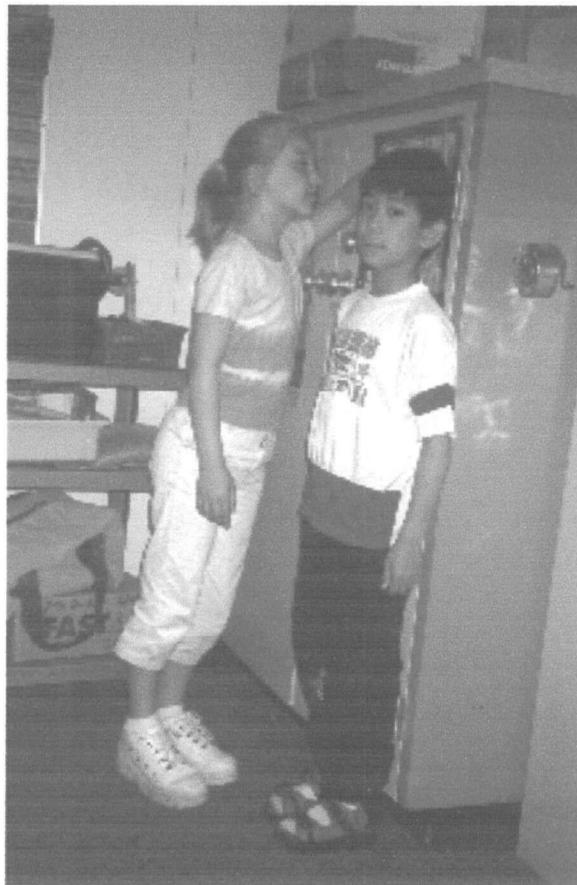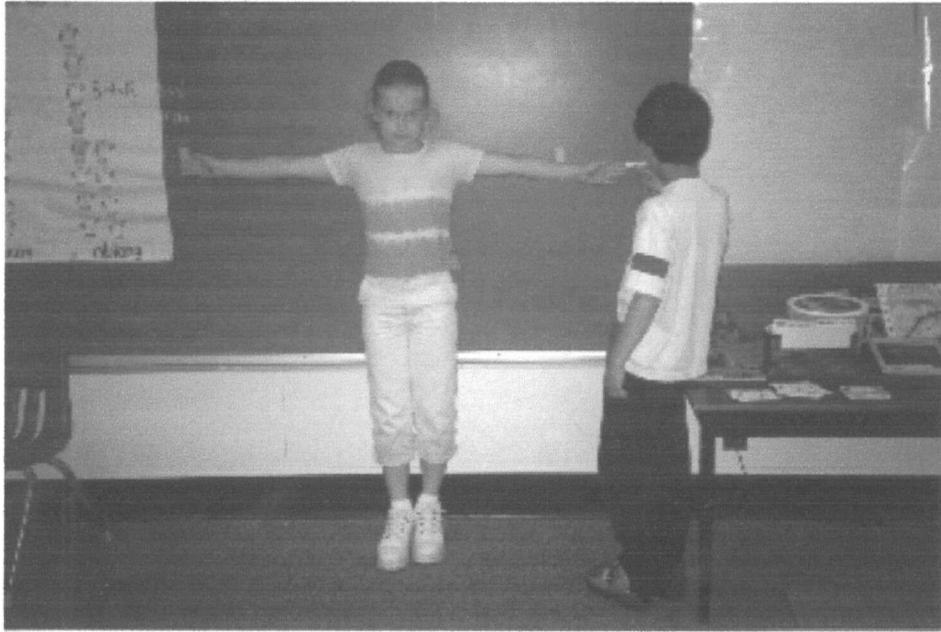
Robby and Madelaine. Are you a square or an oblong?!

Some found out that they were squares while other children discovered that they were tall oblongs. Of a total of 21 students, 9 were square and 12 were tall oblongs. To the children's surprise, there was not one wide oblong in the class.
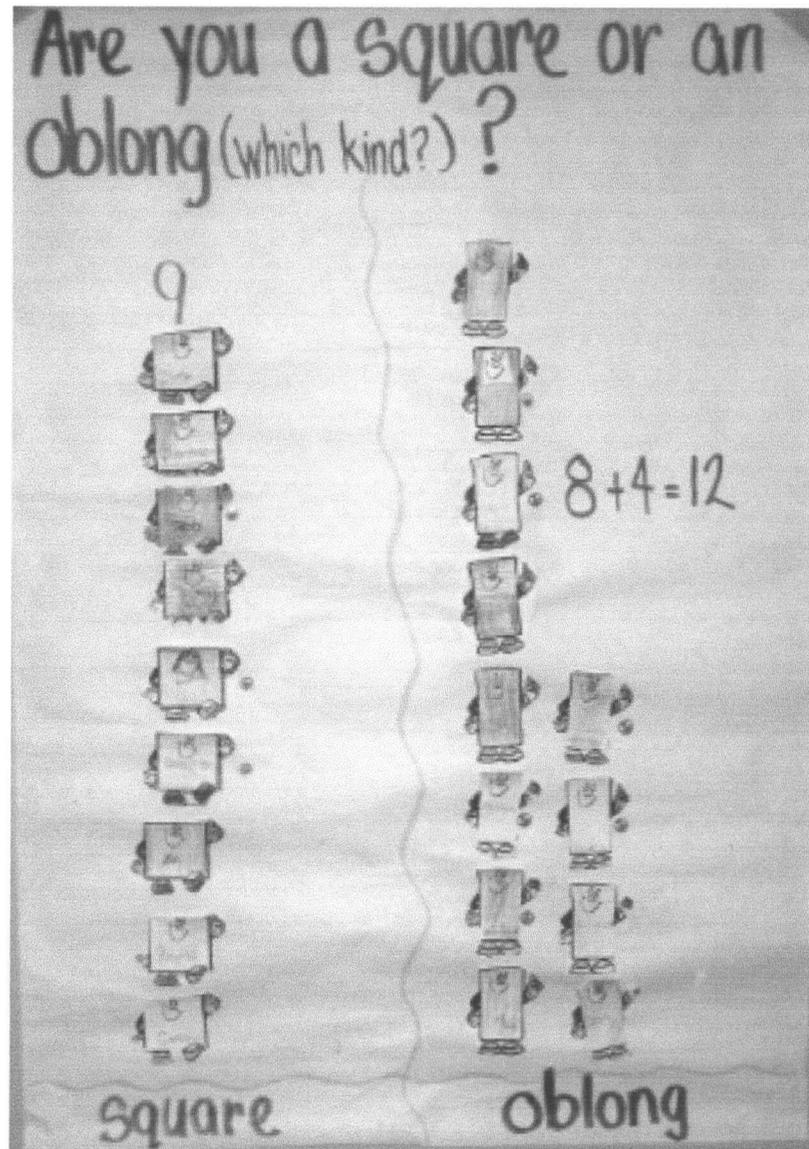
The class graph

Most people are tall oblongs.
    Amie

There are 3 moer oblongs.
  then squares.
    Madelaine

I was exactly 102 cm (armspan).... about 100 cm.
I was exactly 124 cm (head to toe).... about 120 cm.
I am [a tall] oblong.

<div align="center">Annie</div>

I was exactly 118 cm (armspan).... about 120 cm.
I was exactly 129 cm (armspan).... about 130 cm.
I am a[n] up and down ob[l]ong.

<div align="right">Madelaine</div>

I was exactly 120 cm (armspan).
I was exactly 120 cm (head to toe).
[I am a] square.

Sharon

My partner was Holly and she was a [tall]
oblong because she is different both way[s].

<div align="right">Charlotte</div>

Okay. So... yes, they were collecting the data. Yes, they were measuring and rounding numbers to the nearest ten. Yes, the children were analyzing the graph. But that's not what it was about was it? I mean, that's some of what they were learning, but the mathematical space that you opened was for the children to think with the idea of a square and an oblong in a context that hadn't occurred to them. In order for them to distinguish what 2-D shape they were, the children had to first spatially coordinate the two dimensions of length and width, and then compare these against the properties of the 2-D shapes.

Yes, that's right.

The following year we continued the investigation. The eight second grade students who were now in third grade were curious to find out if their body shape had changed or remained the same from the previous year. The entire class wanted to know if there would be more or less oblongs than squares and how this year's results would compare to that of last year's data. Most of all, the children still wondered if it was even possible for someone to be a wide oblong!

So this space that you previously opened has now been re-opened and extended by the children themselves.

... and now it included different contexts in which they could further examine the idea of square and oblong bodies. Their extension of the learning space gave rise to the class' posing of new questions.
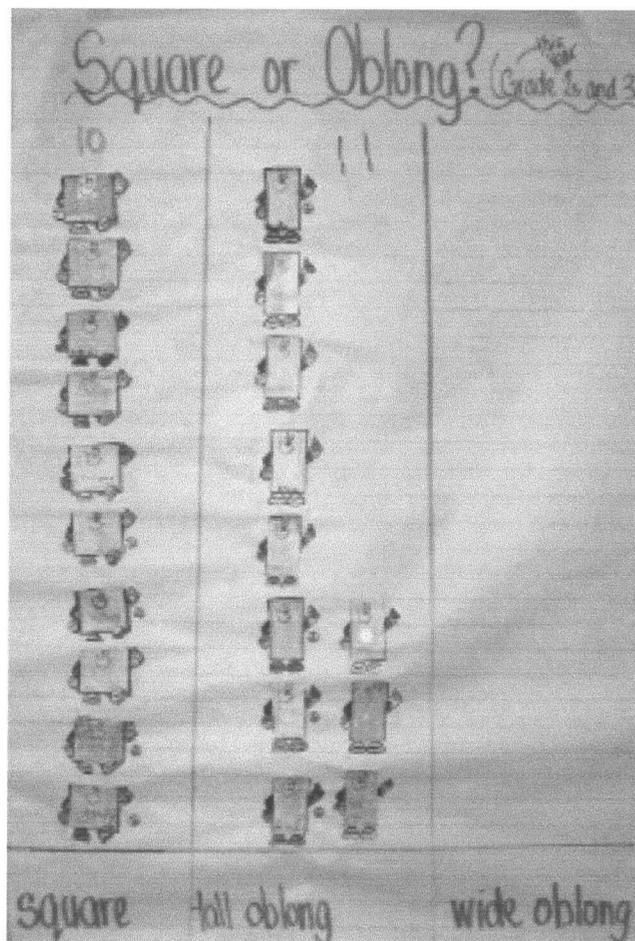
The class graph

Like last year, there was a total of 21 students. After the class collected their new data and analyzed it, they found that 10 of the 21 children were squares, 11 were oblongs, and still, there was not one person who could be considered to be a wide oblong.

Sitting in front of the two graphs, the children began comparing them.

"Last year there were nine [students who were] squares and this year there are ten [students who are] squares. So there's one more square" said Ellie and Michelle.

Mary-Jane identified that "with both of the graphs, there are always more tall oblongs."

Danica and Shelby made the claim that "most people didn't grow from last year. They just stayed the same."

This comment caused a great deal of talk amongst the class as they questioned whether it made sense. According to the students, each and every one of them knew *for sure* that she or he had grown in the past year.

When the class looked at the graphs again, I asked, "Can we tell how much people grew this year?"

Brian shook his head "no" but could not explain why. Shane argued that one would need to "compare the measurements from last year [with this year] and see how much it's changed."

The class came to the conclusion that even though the graphs provided information about the general shapes of students, the graphs did not contain the necessary details for making statements about people's specific measurements.

Picking up on Danica and Shelby's idea-- that most of the returning students had stayed the same shape, I turned the class' attention to another interpretation of the graph. Of the returning students and according to the first graph, two of students had been squares and six had been tall oblongs. This year there were four students who were squares and four who were tall oblongs. I asked the class if the statement that only two of the returning students had changed shape from last year would be an accurate one. Many students were quick to agree because of the difference of two between four and two and, six and four. Others felt that they should be more skeptical and indicated "maybe not" but could not explain why. When I asked the class what they would need to know in order to determine if students remained or changed shape from one year to the next, they said that you would have to find and compare the results of each student. So, each of the returning students located themselves on last year's graph and compared the results with the current graph.

So why did you do this?

Because, what Danica and Shelby had said had caused such a reaction from the students, it needed to be brought around again in order that the class could consider the claim more critically... so that they could take the two sets of data and see what meaning they could make from them.

So not just a superficial reading of the graphs. I mean, not just looking for the obvious facts... like how many of this or that but given the fact that there are so many of this or that, what are the relationships that exist in the data... right?

Yes. Shelby and Danica's statement that most people hadn't grown from last year occasioned the need for the class to go back into the data, see what relevant information they could find, and then stand back and assess the claim by sorting what they knew and what connections existed between the two graphs. In other words, what is specific and lost in the graph such as their names and measurements and what was retained and could be used for comparison (i.e., their shapes).

Michelle saw that she was, "... the same thing as last year (i.e., tall oblong)." and speculated that she had probably grown in both her length and her width but was still taller than she was wide.

Madelaine giggled, "I was a tall oblong and I'm a tall oblong this year".

"I was an oblong last year and now I'm a square this year!" exclaimed Annie.

"I turned from a tall oblong to a square" said Robby.

Shane found that, "Last year I was an oblong and this year I'm a square".

This year Danielle had measured as a tall oblong and attributed the change in her shape to the fact that, "I've grown more than my arms have (stretches out her arms). My legs have grown more because last year I was a square."

Holly said, "last year I was an [tall] oblong and I guess I was a little bit shorter and my arms weren't as long, and this year I'm a square".

As for Mark, last year he had been a square but "I'm a tall oblong this year. I've probably grown more in my head to toe measurement this year."

So by having them go back and compare the data from both years on the actual students themselves as opposed to just looking at the graphs' total number of squares and oblongs...

> *... they found out that in fact, only two of the returning students had remained the same shape; that is, tall oblongs while the other six students had changed. The exact opposite of Danica and Shelby's initial conjecture.*

Exactly.

Still in search of wide oblongs, the children decided that such a body shape might not exist in the second and third grade and that they should go home and measure one of their parents because, perhaps older people were!

> *They've extended this mathematical space again by raising conjectures based on the trends they've located in the data they've collected on themselves.*

Madelaine predicted that "more parents would be square-shaped than anything else".

Mary-Jane had a hunch that "maybe we'll find a couple of... wide oblongs." And the rest of the class anticipated that there would be more tall oblongs than anything else. That weekend, the children collected the measurements of one of their parents and brought the data to class the following week. With it, the class produced a third graph.
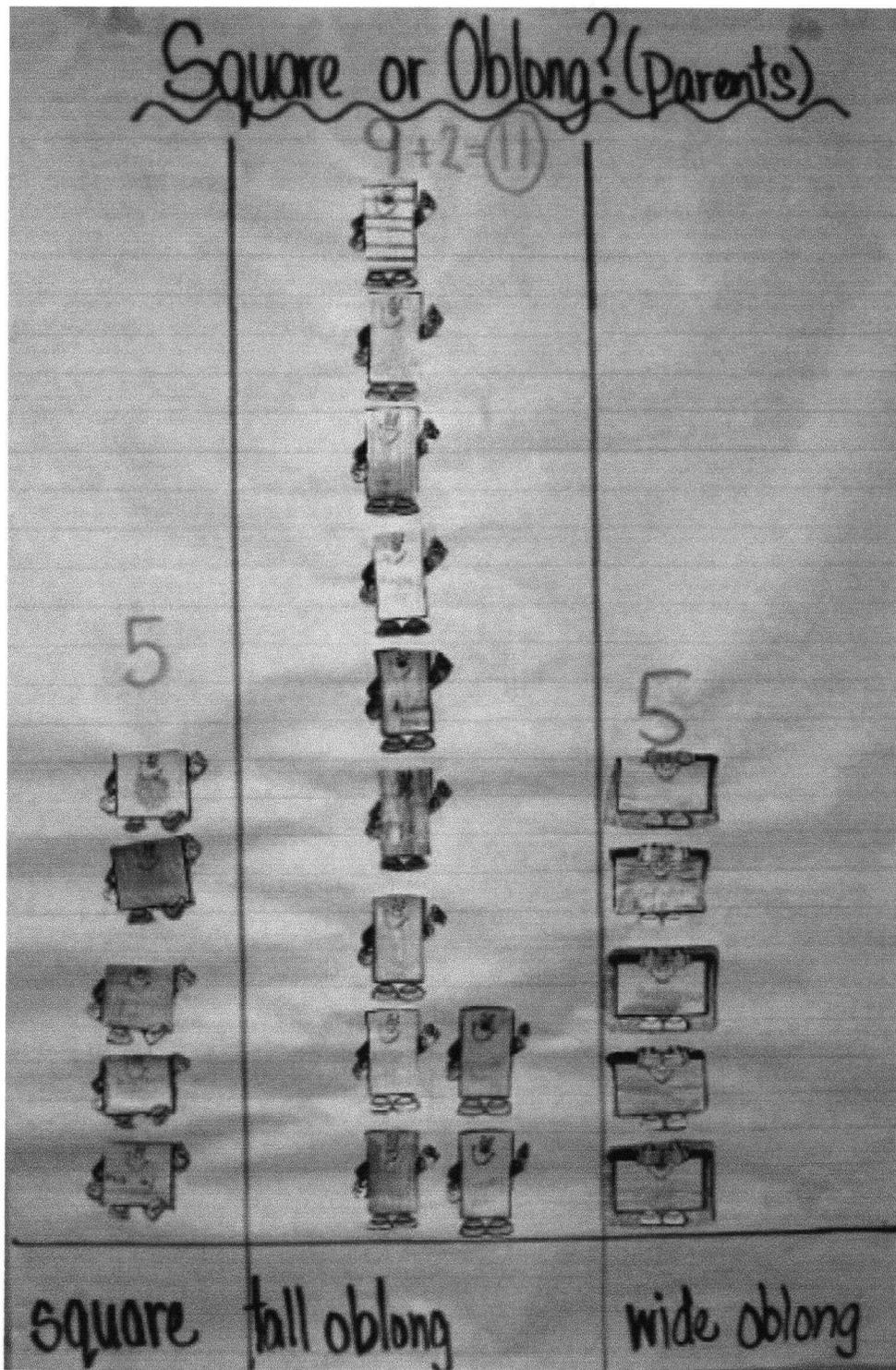
What's the shape of your parent? graph

[Parents that are] "square [or] wide oblong are the same." William.

My parent (mom) is a tell oblong!
Ellie
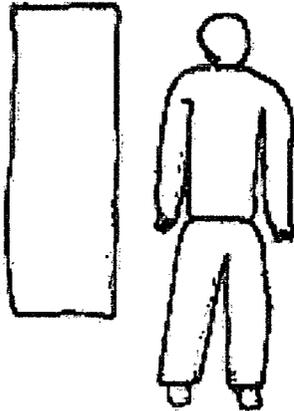
My parent (dad) is a TALL OBLONG!
Shane

My parent (dad) is a wide oblong!
Gregory
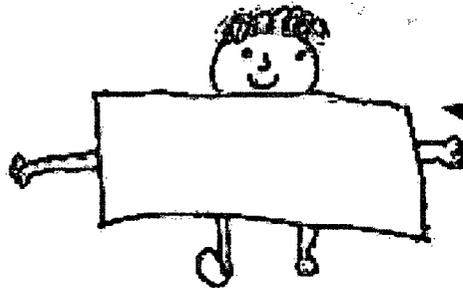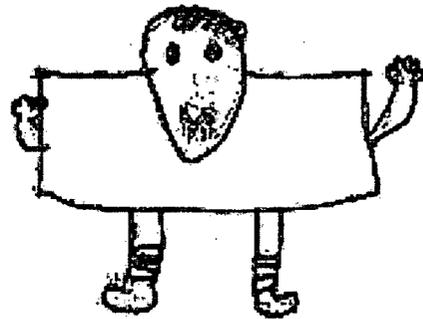
My parent (mom) is a wide oblong!
Lara

My parent (dad) is a square!
Mark
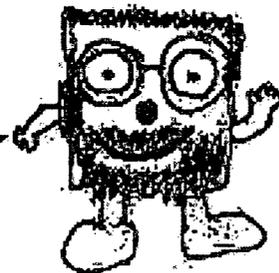
My parent (mom) is a square!
(My dad is a square too!!)
Shelby
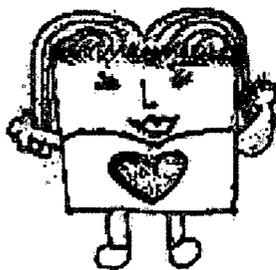
My parent (mom) is a square!
Jerry

352
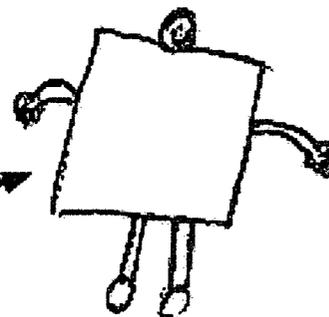
We put this graph with the other two and examined it first by itself and then in relation to the other graphs. Madelaine's prediction that most parents would be squares did not prove to be true. However, the class' "guess" that there would be more tall oblongs than anything else was correct. The children were most excited to discover that Mary-Jane's hunch was a good one and that some parents were indeed, wide oblongs!

Mark then suggested that "you grow taller than your armspan does [as you get]... older."

"If that is true, why did we end up with wide oblongs?" I asked.

Mark shrugged his shoulders and laughing, exclaimed, "maybe they shrunk!" I asked the class what they thought might happen if we continued to measure and graph people's shapes. "Do you think we might... keep finding the pattern that Mark pointed out? That there are always tall oblongs than anything else?"

The students began talking to each other about what group of people they should sample next. Robby thought that the Kindergarten class would be a good group to survey to see if the pattern of tall oblongs would persist "because we've done older people-- our parents, and we've done medium people-- us, so we should do a smaller [i.e., younger] group."

So from the three sets of data that the class has collected, they've identified a pattern of tall oblongs that's consistent with all three graphs.

And now they're moving to a different age group to confirm it.

In addition to engaging the children to think critically and locate relationships in the data, another layer has now emerged in terms of their mathematics. They are now, collecting new data and interpreting in light of the other sets in order to validate the patterns they've found, right?

Yes.

The class agreed with Robby's idea and Keefer predicted that the pattern would continue. Jerry figured that there would be more squares amongst the Kindergartens than any of the other groups surveyed.

Mark pointed out that one should not compare numbers without considering the total number of people in each group-- "it depends how many people are in there [i.e., in the group]."

I acknowledged Mark's comment as being an important one for the class to consider. With all of the groups roughly the same in size (i.e., 21, 21, 19, and 20) the children and I decided that in this case it would not be a major concern.

All of the students predicted that we would definitely find wide oblongs in the Kindergarten group. When I asked them why they thought this would be so, Gregory raised his hand and explained his "theory".

"They're height is really small, the Kindergartens, they're pretty small-- up and down" he said nodding his head, "and they're armspan is probably more." The entire class agreed with Gregory's reasoning and on that note, we left the room to go and measure them.

What shape ARE Kindergarten children?

Wondering...

and finding out!
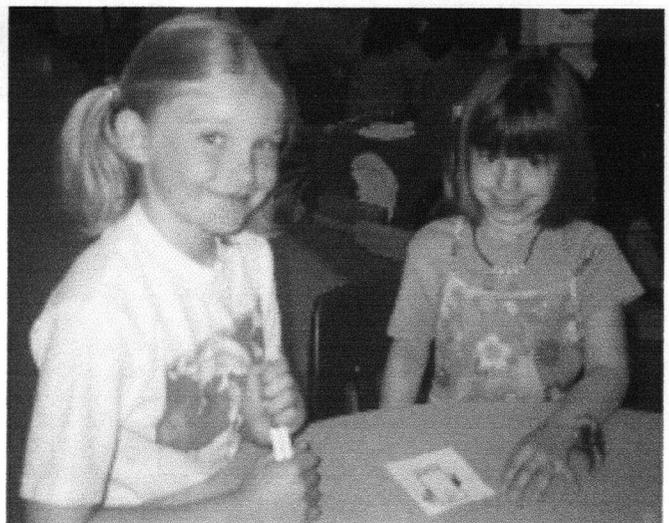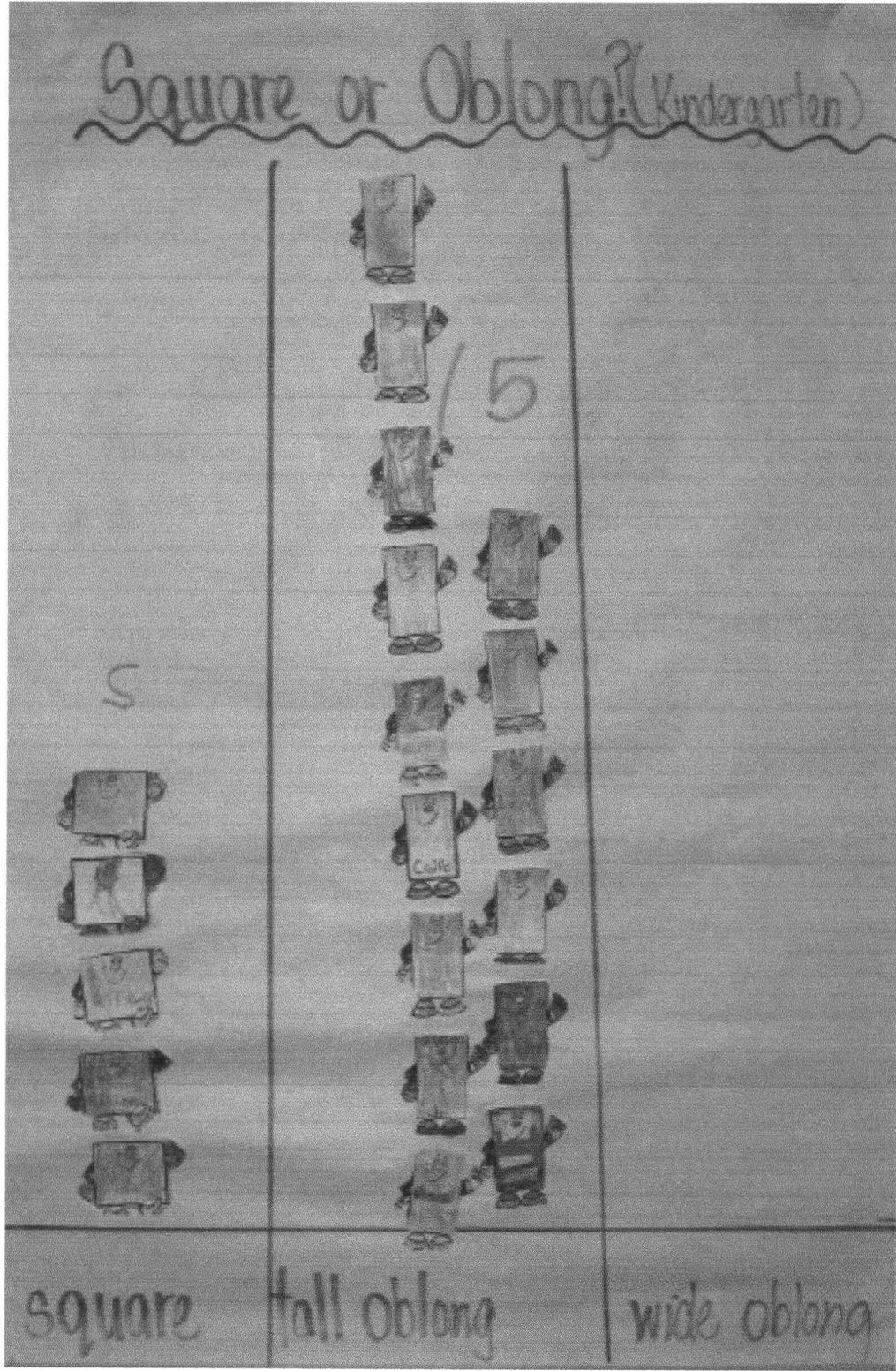
The graph of the Kindergarten class

When we returned to our classroom and hung the new graph with the others, Jerry looked at all four graphs on the chalkboard and said,

"One was right and one was wrong". Here, Jerry's comment was directed at the class' predictions.

"Which one was right?" I asked.

"The tall oblongs" he said. "We thought that they might have more squares but they actually have the same as the parents."

I asked the class, "What was something that was surprising about the Kindergarten's graph?"

Keefer raised his hand and said "We thought that there were less tall oblongs and we thought that there's more wide oblongs."

"And what did we find out?" I asked, prompting Keefer to continue.

"That there's actually more tall oblongs than wide oblongs".

Bringing focus to all four of their graphs and the *Are you a square or oblong?!* study as a whole, I asked the class that given the results of their work, if we cannot assume that Kindergartens will be shorter than their armspans, what kinds of assumptions could we make based on the information they had collected and found in this exploration. In other words, did the mathematics connect us? Or, not?!

Here, making my reason for this mathematical investigation clear to the children...

that the study was for them to develop layers in their analytic thinking-- about how they can interpret data and locate meaningful patterns by working across sets of data.

"People really do come in all shapes and sizes!" noted Madelaine.

A VERY general pattern!

"No kids are wide oblongs" said Gregory.

"Every graph, there are more tall oblongs than anything else." explained Jerry.

And Brian pointed out that according to the graphs, the group in which the most square people were children of the second and third grade. One final mathematical connection that emerged for the class was a pattern about how people's bodies grew.

Robby explained that in Kindergarten, "you're more tall than you are wide".

356

Sarah added, "your armspan grows bigger and it evens out with your height" as you get older.

The class then concluded that the most variety of body shapes happens when you become an adult. You either remain square,

"Or you grow taller [i.e., a tall oblong]" said Mark.

"Or your armspan goes longer than how tall you are." said Robby.

And here, examples of specific relationships or trends that the class has located across the collected data.

I can see now how this learning space that you opened by having the children take already known geometric concepts and applying them to a new context enabled your students to not only to think in mathematical ways but to do so in a generative manner... through the layers of thinking that the children developed; they experienced how their mathematical ideas and understandings informed who they are and connected them with the contexts of which they are a part. From the questions and conjectures they raised about certain patterns that might exist within and across groups of people naturally created the need for your students to plan and generate different sets of data that lead to their activities in critically analyzing the data... that occasioned new insights... and in a recursive way, gave rise to new investigations. Your students connected a real-life phenomenon-- how people grow, to explain their mathematical findings. In essence, bringing the self, the world, and mathematics together. Mathematics wasn't just a set of facts but directly related to them.

Connecting Us to It

In my work to embed an ecological sense of place for mathematics by opening spaces for children to interact and connect with mathematics, I cannot neglect the spaces in which mathematics can connect us to it.[1] By "it", I mean the more-than-human world.[2]

Another of what you'd call, a "free idea"?[3]

Yes. Thinking with G. Bateson's notion of patterns that connect[4] and relating it to what Abram speaks of as the more than human world.

You're speaking of the natural world?

Yes, but not in an objectified or a disconnected manner... an environment that we are connected to and with which we interact.

Right. And so, how do G. Bateson's and Abram's ideas come into play?

In making the need for spaces in which children can interact mathematically with the natural world explicit... and not for the sake of discovering universal truths or facts.

You mean, thinking mathematically and forming relationships through these interactions.

Yes, exactly.

This connects with what you've quoted earlier-- from van Manen[5], it's about enabling children to know the mathematics in a way that it *speaks* to them... here, through contexts that are other than human.

And how they might develop mathematical ways to speak to it... to respond in such settings.

Like what?

T*aking a Closer Look at Snowflakes* was a theme I developed to connect the children with the natural world in mathematical ways. Because my teaching is grounded in the idea that learning can arise from reconsidering that which we take for granted as known terrain, the idea of my students reexamining the familiar snowflake seemed fitting.

With a few snowfalls, the children's dark clothes, magnifying glasses and incorporating videos, photographs, and books about snowflakes, the "small white stuff" became fascinating and complex mathematical pattern makers.

What kinds of patterns?

The class was surprised that there could be so many different geometric shapes in snowflakes-- "diamonds", circles, pentagons, trapezoids, hexagons, squares, oblongs, triangles, "skinny" ovals, flower and star shapes... Here's a diagram made by Isa:

Isa's close-up study of a snowflake
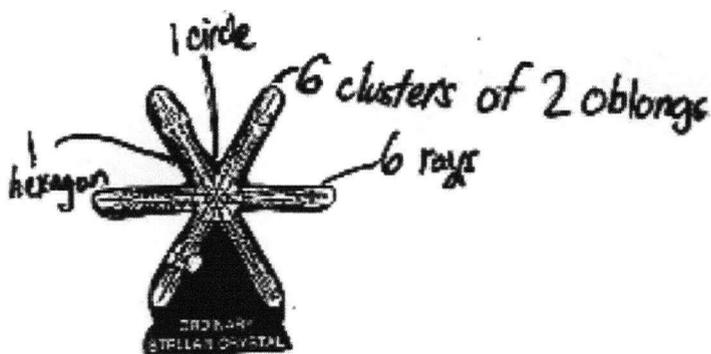
Not only has she identified the geometric shapes within the snowflake but also, the particular organization of them.

This was something that intrigued the class-- the actual number patterns in the arrangement of the snowflakes. As they located new ones, they called them out and I recorded them on a large piece of chart paper.

```
6 sticks           1 hexagon        12 oblongs
6 diamonds         1 star           12 petals
6 circles          1 flower         18 squares
6 pentagons        2 hexagons       18 points
6 trapezoids       120 icicles      12 triangles


6 groups of 2 triangles (6x2) (2+2+2+2+2+2)
6 groups of 4 triangles (6x4) (4+4+4+4+4+4)
6 groups of 3 rays (6x3) (3+3+3+3+3+3)
6 groups of 3 points (6x3) (3+3+3+3+3+3)
6 groups of 2 squares (6x2) (2+2+2+2+2+2)
6 groups of 2 oblongs (6x2) (2+2+2+2+2+2)
6 groups of 3 circles (6x3) (3+3+3+3+3+3)
```

Snowflakes as geometric and numerical pattern makers

Once done, the class looked at the chart to see if they noticed any other patterns. Shouji made the observation that the number of shapes was always either one or an even number. Charlotte added to his statement by explaining that each of the snowflakes had structures that were "different but all groups of six". This she conjectured, was most likely because snowflakes begin with a six sided "germ crystal".

Hence, the multiples of six that the students were thinking of in terms of repeated addition, "groups of, and multiplication. Having now seen these shapes and patterns, in what direction did this study then move?

I challenged the children to create a mathematical response that would capture the dynamic qualities of their particular snowflake. Initially, the students began with verbal and symbolic descriptions that then turned into mathematical poems.

Poetry? In mathematics?

Yes! From the videos we'd watched, the children learned that a snowflake grows outwards from its centre—from the germ crystal that Charlotte

spoke of. This is why the students were always describing the snowflakes from a centre-out perspective. In this manner, the children visualized the formation as an animate and artistic unfolding... poetic in nature. It only made sense then, that their responses be poetic in form. And reflected in each of them is the student's spatial and numerical thinking.

1 small circle in my center
1 of my circles is around another circle
6 small triangles [on] the edge of my largest hexagon
1 hexagon around my 6 triangles
1 medium size hexagon between a big hexagon and a
     small hexagon
6 clusters of 2 squares in my big hexagon
1 big hexagon around 2 small hexagons
6 triangles around a circle
6 rays in my middle
6 clusters of 2 forming a ring
I am telling you this while I am dancing in the air.
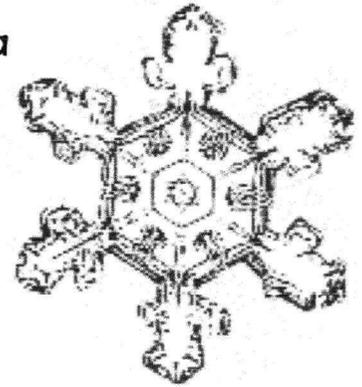
by Isa

1 Flower with 6 petals,
1 Star with 6 points,
1 Hexagon,
12 Little triangles,
6 Big triangles,
Twirling in the garden.
by Sharon

I circle in its centre

two rings of six diamonds around its centre

6 branches from the middle to the edge

6 triangles around its outside

12 bumps on the edge of its largest hexagon...

    6 clusters of 2

    bumps... 6x2=12
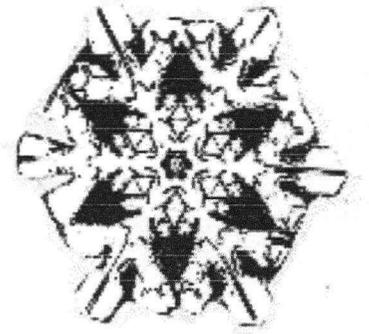
6 pentagons on the tip of its branches

6 clusters of 5 points along its edge... 6x5=30... 5+5+5+5+5+5=30

    filling the air with other dancing snowflakes.

by Shouji

Through their poems, the children communicate the specific shapes, locations, and number relationships they see.

Like in Isa's poem when she writes, "6 clusters of 2 squares in my big hexagon".

Or when Shouji writes, "6 clusters of 5 points along its edge.." and then echoes it two more times but in different ways through his expressions of 6x5=30 and 5+5+5+5+5+5=30. Each of their poems are responses in terms of how the snowflake as a mathematical form is speaking to them but speaking in the poetic language of clusters, flowers, rays, and branches.

Notes

1. G. Bateson, 1980.
2. Abram, 1996.
3. See page 278.
4. See G. Bateson, 1980, p. 9.
5. See van Manen, 1986, p. 44.

I yelled out: "You have to decide *now* which you are– a GIANT, a
WIZARD, or a DWARF!"....
"Where do the Mermaids stand?"
Where do the Mermaids stand?
A long pause. A *very* long pause. "Where do the Mermaids stand?" says I.
"Yes. You see, I am a Mermaid."
"There are no such things as Mermaids."
"Oh, yes, I am one!"
She did not relate to being a Giant, a Wizard, or a Dwarf.... She took it for
granted that there was a place for Mermaids and that I would know just
where.
Well, where DO the Mermaids stand? All the "Mermaids"– all those who are
different, who do not fit the norm and who do not accept the available boxes
and pigeonholes?[1]

[1] Fulghum, 1989, p. 81-82.

Spaces for Unpredictable Mathematics

Having experienced moments of being this little mermaid myself, I am trying in my praxis of teaching to be mindful not to pigeonhole anyone or anything. Given an ecological mind-space, all that I have explored and everything I have come to make sense of, it makes no sense that definitive categories or endpoints in children's mathematical explorations would be desirable or even realistic. If mathematics as well as the teaching and learning of it is to be fluid and responsive, then it is incoherent to assume that children's ways of being mathematical-- and this includes their solutions, would be rigid and absolute.

In making space for children to adventure in their mathematical learning means that there is always to be an element of unpredictability and any "endpoint" remains an open-closure.

> The teacher will often be in the position, unusual for mathematics teachers and uncomfortable for many, of *not knowing*; to work well without knowing all the answers requires experience, confidence, and self-awareness".[1]

It also entails that "teachers must perceive the implications of the students' different approaches, whether they may be fruitful and, if not, what might make them so."[2]

Is A Half Of A Half *Really* A Half?

The class had been busy working on making halves. Jennifer had given them a variety of everyday items such as collections of objects, dollar and coin values, two dimensional shapes, and containers of dry and liquid materials. The children worked in groups of three and four to find as many different ways of 'halving' the items and justifying why despite the appearance of the halves of each of the items looking different that they were in fact, equal.

It was just as the last group was about to finish sharing what they had found in their investigation of a half that Jennifer looked around at the rest of the class and noticed that Sammy sitting very still in a hunkered down and slouched position, staring off into space– lips pursed and concentrating very hard... on something(!) Before she could ask him what he was thinking about, Sammy sat up, leaned his body forward, and pointed his right index finger towards the ceiling.

Looking first directly at Jennifer and then to his classmates with wide eyes and a sense of urgency in his voice, Sammy sputtered out "I wonder... is a half of a half a half?!"

For a brief moment the class was completely silent. No one said or did anything. And then, just as quickly as the children had become quiet, all of a sudden, they were abuzz– turning and looking at one another with furrowed brows and asking "is a half of a half a half?" Shrugging their shoulders, they looked to Jennifer for her response. Recognizing the mathematical playfulness of Sammy's question, she simply raised her hands, shrugged her shoulders, and tossed it back to them.

"IS a half of a half a HALF?!" she said to the class. Jennifer was curious to see the ways in which the children would move inside this unpredictable space that Sammy had opened.

The class dispersed from the carpet and formed small working clusters around the room. Jennifer moved about, helping each group to gather needed materials and so they could begin to explore the question. As she worked with the students, she observed that some groups were busy sketching out diagrams on pieces of paper while others were having a conversation and talking about what a half of a half was. Still others, took hold of actual objects– boxes, containers, sheets of paper, and geometric shapes and then proceeded to draw imaginary lines to make a half of a half. Emanating from all of this activity was a dull roar of "yes it is!" and "no it isn't!"

After some time, the class came together and each group presented what they felt was a convincing "answer" to Sammy's question. Justifications for why a half of a

half *was* a half had to do with the fact that when something is divided into two equal pieces and each one of the pieces is a half of the original half piece. There were other groups however, whose explanations pointed to the fact that when you divide and take a half of a half, what you have taken or what is leftover is one quarter-- *not* a half.

Consequently, the class remained divided-- in half! Grabbing a hold of Sammy's question and through several acts of mathematical tugging, pushing, and pulling apart, the children arrived at two points of viewing and seemed adamant that a half of a half had to be one or the other. Either it *was* a half or it *was not.*

Sammy sat quietly on the carpet for the entire discussion, looking and listening to what his classmates presented. When the conversation came to a grinding halt, Sammy sat up as he had before but instead of opening a mathematical space, he jumped into the thick of it and brought the two confounding interpretations together.

"It is and it isn't." Sammy said. "If you're only looking at the half" he explained, "then it is a half. But if you are looking at the whole then it isn't, it would be a quarter. It depends on how you are looking."

By pointing out that it depends on how one is viewing the half of a half, Sammy effectively transformed it from being a *thing* into a mathematical *relationship* that is contingent on the context in which it is situated. The understanding that his shift in thinking allows for, is that even though one might take an either-or approach and statically define what a half of a half is (or is not), there is also the opportunity for one to consider in a systemic manner, what the half of a half is *part of* (i.e., the half or the whole)-- how it is RELATED. By doing so, one can conceptualize how a half of a half can be both a half and a quarter.

Notes

1. Burkhardt, 1988, p. 18.
2. Burkhardt, 1988, p. 18.

One day, two years later, I found myself sitting on the edge of pessimistic skepticism. It was in this space of mind I began thinking that perhaps, there are only wizards, giants, and dwarfs in the mathematics classroom. Maybe there are no such things as mermaids. So, two years later, I returned to Sammy's question. It was as a little "test" if you will– to see if such spaces really are as curious and unpredictable as I think them to be. If indeed they are then even when reopened, one cannot be certain what will unfold *this* time.

In Search Of Mermaids

I gathered up my students and took them to Sammy's space.

"That's weird!" exclaimed Robby as he and the class were met with the question, *Is a half of a half a half?*

> *Like the students two years before them, the idea of a 'half'-- something so everyday and familiar suddenly became strange and not so familiar.*

The children did not need any coaxing from me to enter this mathematical space. They jumped right in. Once inside, the class proceeded to take a good look around.

"Is a quarter of a quarter a quarter?" asked Danny.

"Is a third of a third a third? Is a tenth of a tenth a tenth?" said Clare.

> *The children were enchanted by the idea of a fraction of a fraction and the linguistic rhythm of it appealed to them too. From here, the class went on to generate several queries.*

Julie wanted to know if a whole of a whole could still be a whole while Ethan wondered about a sixth of a sixth, and Shouji rounded out the list of questions by asking whether or not a fifth of a fifth really was a fifth.

I recorded the children's questions onto a large piece of chart paper and after some discussion as to how we should continue, the class decided to break into smaller groups of three or four and that each group could choose a different question to investigate. The children also agreed that it would be best if they all used a circle shaped pizza to be a common "whole" and for each group to make a poster so that the results of their work could be displayed.

When all of the groups had finished their posters, the class sorted them in order from the greatest fraction– one whole pizza, to the least that was the tenth of a tenth of a pizza. We then taped the posters up across the length of the chalkboard so that everyone could see the entire class' work.

Julie's group arrived at the conclusion that a whole of a whole could not be anything but a whole. Shane's group took the position that a half of a half was *not* a

half but rather, a quarter, because the resulting slice of pizza was one of a total of four pieces.

Wanting to engage the students' thinking further, I asked the children if they agreed with Shane's group's presentation that a half of a half is always a quarter. Robby nodded his head in agreement that a half of a half could be considered to be one quarter as Shane's group had demonstrated but then pointed to the group's poster and also explained that "from a half, it is a half."

> *What initially appeared as "weird" to Robby was now something curious and playful. Like Sammy, Robby too had arrived at a place of knowing where a half of a half could possess two identities. But unlike the class two years ago, this class was dealing with several fractions of fractions and with one that happened to be one whole. And so, Robby's thinking did not become a residual understanding as Sammy's had, but a source that set the class off on a comparison of all seven posters.*

The students agreed with Robby that the half of a half could be viewed in relation to the whole or the half, but they also argued that a whole of a whole could not be anything but a whole because the one pizza was its only reference.

The class moved on to Danny's group's poster. Danny explained that a third of a third could be a third because, "if this is one third (pointing to one of the three pieces of pizza)... and you split it [i.e., into three pieces again] it's like splitting a package into a third."

Isa stared at the resulting third of a third, shook her head, and then, said nothing. Asked to come up to the poster and show the class what she was thinking, Isa offered a second interpretation. With her hand she covered up the thirds of the pizza one at a time, and pointed out, "there would be three, six, nine–pieces. It's one-ninth."

Ethan said that "a quarter of a quarter is a quarter because it's one fourth of a quarter."

And Danny added, "four plus four equals eight and eight plus four... twelve, and twelve plus four is sixteen... a sixteenth."

Mac shared what his group had found out. "Yeah, I think it could be a fifth. You could call it a fifth and it's a fifth of a fifth. So it's just like pretending a fifth is

371

like a whole of a pizza." He walked up to the poster and said, "if you look" pointing to the piece that was a fifth of a fifth, "and you... if you take all the pieces" meaning the entire pizza, "there'd be twenty-five little pieces, so it'd be one twenty-fifth."

Mark nodded his head and followed by explaining his group's poster in a similar manner; that a sixth of a sixth was a sixth "because there's six of them in the sixth."

The conversation was gaining momentum when Mac and Steven made their way to the front of the class. Max pointed to the pizza and exclaimed "there's eighteen [pieces] in one half!"

"And then eighteen on the other side", added Steven.

I joined in, "and eighteen plus eighteen is..."

"Thirty-six!" chimed Steven and Mac.

Smiling, Mac looked at the class and announced, "One thirty-sixth".

> *The class' working through the six of the seven posters in this way and recording the complementary fractions firmly established the two different ways of conceptualizing the resulting fraction: First, as a part of the original part, and secondly, as a part of the greatest whole.*

It was when the class reached the last poster that something different and for me, unexpected, happened. Robby and Danny raised their hands at the same time and both called out that if the entire 'tenth of a tenth' pizza was cut into slices that were all tenths of a tenth, there would be one hundred pieces in total.

Mark said, "There are ten groups of ten and so you have to do ten times ten." he said.

Danny told the class that "if you put this altogether"-- referring to the ten tenths of of a tenth of the pizza, "that would make one tenth" of the whole pizza.
"So that would be another tenth..." he pointed out as he moved around the pizza in a clockwise manner, one tenth of the pizza at a time. "Until you reach to here". By "here", he meant back to the place where you started. "And you would find" Danny said, "that there would be ten of these", pointing to one group of tenths with his finger. "And then there'd also... they're tenths. So, you would find out that there'd be ten of them so you can think of them as ten times ten... they are tenths... so if you split them up like that and you have to have ten pieces to make a whole... one hundredth."

It can be assumed that Robby, Danny, and Mark (who, most clearly demonstrates this) were applying their learned knowledge of whole number multiplication to this context of fractions. They were thinking with a "groups of" notion to account for the total number of pieces (i.e., ten groups of ten pizza slices is one hundred slices altogether).

Just as none of the children had received any formal teaching on identifying fractions within fractions, they also had not worked on creating equivalent fractions, adding, or multiplying them. Yet, these are the understandings that were emerging for Danny.

In his second response, Danny conceptually communicates that putting together (adding) ten one hundredths of a pizza results in a same sized piece (equivalency) as one that is one tenth of a whole pizza. He goes on to demonstrate that in order to determine the size of the individual pieces that arise from sectioning off each tenth into tenths again, one multiplies the tenths (10) by the number of groups of tenths (10) that now exist (10 x 10). This produces the total number of individual pieces in the entire pizza (100) of which each one is one hundredth of the whole (effectively, the reciprocal).

I was about to let the children go when Clare smiled and excitedly waived her hand back and forth.

"Yes, Clare" I said.

Beaming, it was obvious there was something important that she wanted to share with us. Clare came up to the front of the class, pointed to the fraction

denominators on the posters, and beginning with one half and moving through to one tenth, announced:

"Two times two is four, three times three is nine, four times four is sixteen, five times fives is twenty-five, six times six is thirty-six, and ten times ten is one hundred" she chanted. "The numbers work" she concluded, matter of factly.

Upon initial analysis, one might suspect Clare had simply memorized her multiplication tables and recognized it here as a numerical pattern.

Encouraging her to continue, Clare randomly started with the third of a third and then moved to the poster of a tenth of a tenth, She told the class that the reason why the numbers "worked" was because the piece; for example, the *tenth* of a tenth was becoming increasingly smaller.

Clare also pointed out that "if this one" she said referring to the pizza already divided into two halves, "is split in half, then there'd be one group". Clare showed this with her hands cupped around half of the pizza. Here, she demonstrated that "one group" meant one group of two pieces of half of the pizza, "and one group" she said, being the other half of the pizza that also had two slices.

Moving on to further prove her point that the pieces of pizza were becoming multiplicatively smaller, Clare went through the remaining posters in the same way she had the first and verbally highlighted the repetitive additive pattern that was connecting them all.

"There's three, three, and three, [i.e., ninths]
four, four, and four, and four, [i.e., sixteenths]
five, five, five, five, and five, [i.e., twenty-fifths]
six, six, six, six, six, six, and six [i.e., thirty-sixths]
ten, ten, ten, ten, ten, ten, ten, ten, ten, and ten [i.e., one hundredths]."

It is evident that Clare was not merely applying remembered facts but rather, thinking with spatial and numerical ideas all at once. The multiplicative pattern that she identified as "two times two is four, three times three is nine…" is related to the operational and spatial action of making the pieces of pizza repetitively smaller as evidenced in "three, three, three" and "four,

four, four, four..." and so on. Clare also connects repeated addition to multiplication; for example, "three, three, three" with "three times three is nine."

Looking back at the mathematics that defined this (and, any of the other) learning space(s), it could be argued that nothing extra-ordinary happened. In re-rooting the learning space, systemically ecological ways of thinking and doing mathematics had become the class' way of being mathematical.

This was indeed very different from what was previously taken for granted in my teaching. Clearly now was the assumption that the work to be done, the way(s) in which to accomplish it, and the ideas and understandings that were to be realized could not be predicted but only existed in the engagement of bringing mathematics into being.

In the classroom now, making sense of mathematics such as whether a half of a half is really a half had less to do with "what is it?" and more to do with "how" the mathematics, this creature, was speaking to the children-- how it was relating to them and how they were relating to it. Posing questions from a question, moving into, out of, and amongst spatial and numerical realms of thinking, and arriving at different yet compelling places of knowing did not come so much as a surprise but rather as that which is to be expected.

So...

"It's not true… that mermaids do not exist." [2]

[2] Fulghum, 1989, p. 83.

THE 3 FACES OF ECOLOGY

According to M. C. Bateson[2], there are three "faces" or realms of ecology: empirical, environmental, and system-ic. The author defines empirical ecology as biological, meteorological, and geographical studies that focus on understanding how the planet is changing and how these changes affect the interrelationships of the world's natural systems. The environmental face of ecology is concerned with identifying the level of impact that our ways of living have on the earth's

systems. It also involves the development of solutions for environmental problems that will minimize harmful stress on the earth. It is within the systemic relam of ecology where mathematics teaching and learning can be most radically explored. This is because systemic thinking focuses on seeking "the pattern which connects"[3] a system or systems together as interdependent and interacting wholes.

In the field of mathematics education, a "system" could be an individual teacher or a student. It could also be a

collective group such as a mathematics class, the school, and so on.

The connecting pattern or patterns that interrelate these systems together as a dynamic whole encompass the forms of knowledge, actions, and identities that are brought into being as a result of the ongoing interactions in the sys-tem(s) and the ways in which they are sustained by the sys-tem(s).

By focusing on relational qualities, ecological ways of thinking give rise to viewing the world as an integrated whole; a dynamic and fluid network in which all living and social-cultural systems are interconnected. The

Off The Beaten Track

Jennifer sat quietly. "Keeping watch". But this time as she turned and looked on, she saw Stigler and Hiebert's question from the opposite side.[1] Now from the INside of the space, she could see that breaking out[2] of one's conventions is not simply a matter of choice. Walking off the beaten track for Jennifer had proven to be an ongoing and challenging task of re-rooting taken for granted conventions.

In teaching mathematics, she had often heard and observed that "good" teachers were the ones who "moved with the flow". In taking an ecological view however, Jennifer now understood responsive teaching as being much more than just moving with the flow of the classroom and theoretical system or systems. Teaching responsively as praxis, involves continually questioning and responding to the ways in which one's teaching contributes to such a flow; that is, the ecology or the *oikos* of the classroom. Said another way, it means paying attention to the ways in which our forms of teaching are enabled and disabled as a result of our assumed manners of knowing, acting, and being.

"So that's it?" I asked.

Jennifer smiled, and then offered me Gary Snyder's comment– "as an open-closure":

> There is nothing like stepping away from the road and heading into a new part of the watershed. Not for the sake of newness, but for the sense of coming home to our whole terrain. 'Off the trail' is another name for the Way, and sauntering off the trail is the practice of the wild. That is also where– paradoxically– we do our best work. But we need paths and trails and will always be maintaining them. You must first be on the path, before you can turn and walk into the wild.[3]

"By doing so" she said, "we can set Stigler and Hiebert's query and the issues I've raised regarding the need for an ecological sense of place for mathematics in the classroom, side by side."

Jennifer explained to me, that even though we may be tempted to move off of our conventional trails of teaching mathematics and think that our desire to do so will change our direction and move us into the open terrain, we must first become mindfully aware of the taken-for-grantedness that brings ease to our walking of such paths. Only then, will we be able to make thoughtful decisions concerning which paths we should maintain

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Notes

1. M. C. Bateson, 1994, p. 31.
2. Stigler and Hiebert, 1997, p. 14.
3. Snyder, 1990, p. 154.

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