

STUDENTS' CONCEPTIONS OF MATHEMATICS,
AND THEIR CONCEPTIONS OF LEARNING,
KNOWING, AND SELF IN MATHEMATICS

by

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ABSTRACT

This study examined the conceptions of mathematics, and conceptions of learning, knowing, and self in mathematics held by two typical students in an intact Algebra 12 class. Their learning of logarithms served as the context for this study.

The students' various conceptions were largely integrated, and while there were some differences in the views which made up their conceptions, there were many similarities. These included their views that: mathematics was a set of truths, handed down to them by their teachers, in the form of rules and procedures to get answers for the questions encountered in class; knowing and learning in mathematics were undifferentiated from that in other subjects at school, the memorization of unrelated facts; it was the teacher's job to make learning easy for students by presenting new material slowly and in a step-by-step manner; students played a passive role in the learning process; and they "understood" mathematics when they got the right answers for the questions encountered in class. Furthermore, the students lacked confidence in themselves in mathematics and believed that their learning and success depended upon factors which were largely beyond their control.

The following conclusions were drawn from this analysis: students' views on the nature of mathematics and on learning and knowing mathematics influence the kind of mathematics knowledge that they construct by shaping the goals that they set for themselves in the process of learning mathematics, profound but subtle failures in communication which hamper students' learning can occur between teachers and students in the mathematics classroom, students' views can play a major role in limiting or enhancing students' participation within the mathematics classroom and therefore their opportunities for learning and success, students may have unresolved conflicts amongst their views which may be a detriment to their learning, and students may hold views about knowing and learning which they constructed in contexts outside of the mathematics classroom and which are inappropriate for learning mathematics.

Fundamental changes are needed in mathematics programming and teaching to address students' conceptions relating to mathematics, and mathematics learning and knowing so that their command of mathematics can be improved.

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DEDICATION

This thesis is dedicated to my students.

CHAPTER ONE: INTRODUCTION

In the course of learning mathematics a child develops his own ideas, views, and beliefs about mathematics which can be represented as his conception of mathematics. This conception of mathematics may be regarded as a developing conceptual system of interrelated ideas, beliefs, emotions, and views concerning mathematics and learning mathematics that directs and controls his mathematical behaviour, how he learns, and what he understands. From this point of view a child's observable mathematical behaviour may only be interpreted and explained to the extent that his underlying conception is understood. (Erlwanger, 1975, pp. 166-167)

Thompson (1992) describes conceptions as general and somewhat integrated mental structures encompassing both specific beliefs and any aspect of knowledge that bears upon one's experience, such as meanings, ideas, concepts, theories, propositions, expectations, rules, mental images, etc. Students construct their own conceptions of mathematics and a wide range of conceptions relating to mathematics from their mathematics experiences at school, and these conceptions in turn play an important role in their ongoing understanding, learning, and performance within the mathematics classroom. However, these learning outcomes remain, for the most part, outside of the spotlight in mathematics education and are seldom a major concern for teachers as they endeavour to "cover the content" of the mathematics curriculum.

In an attempt to provide new and useful insights on these matters, this study examines two Algebra 12 students' conceptions of mathematics; conceptions of learning, knowing, and self in mathematics; and interrelations between these conceptions from a constructivist perspective. The students' learning of a new

mathematics topic, namely logarithms, will serve as the context for this.

Background for this study

In this section the nature and significance of students' conceptions will be discussed. This will be followed by an overview of the related mathematics education literature. This section will end with a brief summary of the research findings on students' conceptions relating to mathematics.

The nature and significance of students' conceptions

Each student constructs and reconstructs a wide range of complex, integrated, idiosyncratic, and epistemologically legitimate conceptions relating to mathematics on an ongoing basis as he or she negotiates his or her mathematics classroom experience (Confrey, 1991). These conceptions have, in turn, been conceptualized in various ways by mathematics education researchers, for example: as a "belief system" (Frank, 1985), as a "network of beliefs" (Schoenfeld, 1983), as a "mathematical world view" (Silver, 1982), and as "conceptions of mathematics and mathematics learning" (Confrey, 1984). In addition to mathematics content knowledge, one's conceptions relating to mathematics and mathematics learning can include consciously or unconsciously held ideas, beliefs, expectations, feelings, and theories about: the nature of mathematics itself, the value of learning mathematics, the roles of the learner and teacher in the learning process, the relation of school mathematics to the real

world, the social milieu of the classroom, and the possibility of one's success in learning mathematics (Cobb, 1985, 1986; Erlwanger, 1975; Confrey, 1980, 1984, 1990a; Frank, 1985). Furthermore,

alternative conceptions research [in a number of disciplines] has documented students' beliefs indicating that they enter instruction with conceptual configurations that are culturally embedded; are tied into the use of language; and connected to other concepts; have historical precursors; and are embedded in a cycle of expectation, prediction, and confirmation or rejection. For students... it appears that the course of learning is not a simple process of accretion, but involves progressive consideration of alternative perspectives and the resolution of anomalies. (Confrey, 1990a, p. 32)

The conceptions that students bring to the classroom are a most significant part of their cognition in that they set the context for their interpretation of experience and actions. In the mathematics classroom for example, students abstract from their mathematics experiences over time to form relatively stable sets of mathematics related conceptions. These, in turn, influence how students approach, comprehend, and follow through on mathematical tasks including problem solving; how they study, learn, and remember mathematics; and even how and when they attend to mathematics instruction (Garofalo, 1989; Novak, 1983; Glaser, 1984; Thompson, 1992; Putnam, 1987). Schoenfeld (1985) asserts that "the effects of belief systems or mathematical world views... pervade virtually all mathematical behavior" (p. 184).

While students' conceptions will be coherent and make sense from their own perspectives, they may not always appear to be logical or coherent to other individuals. Even students with good school grades may have views which do not correspond with

the more informed views of their teachers or disciplinary experts (Erlwanger, 1975; Frank, 1985; Confrey, 1990a; Putnam, 1987; Underhill, 1988). It is appropriate to speak of students' "alternative conceptions" rather than "misconceptions," because as Cobb (1986) explains,

the beliefs students construct, the overall goals they establish, and the contexts in which they do mathematics are their attempts to find a viable way of operating in the classroom. They are expressions of students' underlying rationalities, of the way they try to make sense of classroom life. In short, students' beliefs about mathematics are their attempted solutions to problems that arise as they interact with the teacher and their peers. (p. 8)

Therefore, in order to be effective, a teacher needs to figure out what a student's rationality might be, regardless of how it may appear at first. (Cobb, Yackel & Wood, 1988)

The conceptions of mathematics and mathematics learning that one constructs, in turn, act to guide, filter, and shape the individual's subsequent mathematics understanding and learning (Confrey, 1984). These conceptions serve as a deep-seated, often unconscious, and powerful force operating behind the scenes within each mathematics classroom (Borasi, 1990). Therefore, teachers need to make themselves aware of these conceptions and address them in their teaching. From his study, Erlwanger (1975) concluded that in the mathematics classroom

evaluation and diagnosis based on tests, an occasional interview, or brief conferences appears to be inadequate in revealing [students'] conceptions. The teachers [in Erlwanger's study] often misunderstood and misjudged the nature of the children's understanding and progress, and the adequacy of their learning experiences. (p. 158)

Schoenfeld (1987) asserts further that traditional mathematics

classroom practice, with its emphasis on a narrow range of written work done by individual students, does not provide much opportunity for students to reveal the full extent of their mathematics and mathematics-related knowledge, nor for mathematics teachers to assess thoroughly the full range and depth of each student's understanding.

An overview of the literature on conceptions relating to mathematics

Erlwanger's "Case Studies of Children's Conceptions of Mathematics," published in 1975, marked the beginning of a now established mathematics education research focus on students' and teachers' beliefs and conceptions relating to mathematics. In this pioneering work, which is still being cited in the literature, Erlwanger demonstrated clearly the need for mathematics educators to pursue this line of enquiry (Confrey, 1986). His grade six research subject, Benny, is well known within mathematics education for the "unmathematical," counterproductive, and unsuspected notions (from a mathematics educator's perspective) that he had constructed while he appeared to be proceeding "successfully" through an individualized mathematics instruction program.

Subsequent to Erlwanger's study, much of the literature on students' beliefs or conceptions relating to mathematics has been written from a cognitive science or information-processing perspective, and most often as a byproduct of research whose primary focus is something else, such as the study of mathematical problem solving (for example, Davis, 1984;

Schoenfeld, 1985; Silver, 1985; Lester & Garofalo, 1987). Research which examines students' beliefs relating to mathematics has resulted, in part, from the growing consensus that mathematics content knowledge alone cannot satisfactorily account for their mathematics related performance. Specific factors, such as metacognitive knowledge and belief systems about mathematics (Schoenfeld, 1985) have become recognized widely within mathematics education research as an important part of the "cognitive science" equation to explain and predict mathematics related performance.

In the last decade, "constructivism" has become established in mathematics education as an alternative to the cognitive paradigm (Lerman, 1989). The rise in prominence of constructivism within mathematics education is evidenced by the following: Constructivist views on the teaching and learning of mathematics (Davis, Maher & Noddings, 1990), Epistemological foundations of mathematical experience (Steffe, 1991), and Radical constructivism in mathematics education (von Glasersfeld, 1992). A constructivist perspective presumes that one's conceptions of mathematics and conceptions of mathematics learning are parts of an all encompassing, integrated and inseparable whole which plays a most significant role in the dialectic and generative process of establishing, delimiting, and transforming the very domain and contexts within which one's ongoing mathematics cognition can occur (Cobb, 1985, 1986, 1990a). In contrast, the cognitive science perspective treats specifics of an individual's beliefs of mathematics and

mathematics learning as another category of variables to explain or model his or her mathematics performance (e.g., Schoenfeld, 1985, 1987; Lester, 1985; Mayer, 1985). Despite this and other significant differences between the perspectives used within mathematics education research there has developed a wide ranging consensus that students' beliefs and conceptions relating to mathematics learning and mathematics itself have a profound effect on their understanding, learning, and mathematical performance.

A summary of the research on students' conceptions relating to mathematics

In recent years, research on students' beliefs or conceptions relating to mathematics has become widely recognized as an important topic for study with great potential to contribute to mathematics education (Thompson, 1992). This recognition has come, in part, from research results on students' "misconceptions" which have found that

in learning certain key concepts in the curriculum, students were transforming in an active way what was told to them, and those transformations often led to serious misconceptions. Misconceptions were documented to be surprisingly pervasive and resilient. (Confrey, 1990a, p. 19)

An interesting example of some of the conceptions held by mathematics students is provided by Schoenfeld (1988). In his study of a "well-taught" grade ten geometry class, Schoenfeld found (from his perspective as a mathematician and mathematics educator) that while the class performed well on the state-wide final examination

the students gained at best a fragmented sense of the subject matter and understood few if any of the connections that tie together the procedures that they had studied. More importantly, the students developed perspectives regarding the nature of mathematics that were not only inaccurate, but likely to impede their acquisition and use of other mathematical knowledge.
(p. 145)

Schoenfeld (1985, p. 184) comments that "the implications [of the research on students' beliefs or conceptions] for pedagogy are serious and deep, and much more work needs to be done in this area."

Background summary

In summary, a wide range of conceptions play a significant role in setting the context for each student's understanding, learning, and performance in mathematics. These conceptions are a significant and often hidden source of difficulty in learning mathematics. Each student's set of conceptions forms a stable and idiosyncratic whole which is coherent and rational from the student's perspective, and these conceptions can be surprisingly different from those of the student's teachers. Furthermore, the full range of conceptions that can affect each student's learning and performance is somewhat inaccessible to teachers during conventional classroom activities. Despite the recognized significance of conceptions relating to mathematics and mathematics learning in the learning of mathematics, little research is available which has examined individual students' conceptions as an integrated whole and from an emic or student's perspective.

The problem

Students' conceptions of mathematics, and their conceptions of learning, knowing, and self in mathematics substantially define their subjective "world-views" as players within the mathematics classroom. These world-views, in turn, can have a profound effect on students' success in mathematics. Mathematics educators and their students would benefit greatly from insights on these matters.

Given the constructivist perspective of this study, the researcher, in attempting to construct an understanding of students' conceptions, is obliged to consider that the very meaning of each individual's (both the student's and the researcher's) terms of reference and the relationships between these terms are themselves constructed from within each individual's own conceptual frame of reference. Meanings must therefore be negotiated and, at best, the meanings or understandings that one constructs can only be considered to have a viable fit with the perspectives of others¹. Furthermore, it is accepted that there are many aspects of one's own conceptions that are outside of one's conscious awareness. Therein lies the challenge for the researcher. It can be only after much patience, astute observation, negotiation, reflection, and active accommodation, that a researcher can construct a viable understanding of a student's conceptions. As Denzin (1978) reminds us:

¹ From a constructivist perspective, an individual's beliefs can, at best, only be "taken-as-shared" with the beliefs of others (Cobb, Yackel & Wood, 1992).

The researcher who has not yet penetrated the world of the individuals being studied is in no firm position to begin developing predictions, explanations, and theories about that world. (p. 39)

In launching an inquiry into students' conceptions, it therefore must be the researcher's task to interact sensitively and responsively with each student participant. It is also the researcher's obligation to actively extend him- or herself beyond his or her own existing understandings, recognizing all the while that this will be an imperfect exercise. Furthermore, in making claims from such an inquiry, the researcher is obliged to remain consistent with the perspective upon which the inquiry has been based.

Research questions

This study provides answers to the following questions for each of two members of a single Algebra 12 class within the context of the logarithms and exponentials unit of study prescribed by the British Columbia mathematics curriculum.

1. What were the students' conceptions of mathematics?
 - 1.1 How were these conceptions evidenced in the students' learning of logarithms?
2. What were the students' conceptions of learning in mathematics?
3. What were the students' conceptions of knowing in mathematics?
4. What were the students' conceptions of self in mathematics?
5. What interrelations were there between the students' conceptions of mathematics, and their conceptions of learning, knowing, and self in mathematics?

These research questions have been designed to work as an integrated whole. Research questions one, two, and three are highly interrelated and serve to facilitate our understanding of the students' epistemologies relating to mathematics. Research question four considers the students' views of themselves as mathematics students. This includes their expectations regarding their abilities and success in learning and doing mathematics. Research question five was posed to focus on interrelationships amongst the results from the first four research questions. This will help the reader to construct a more holistic perspective of the students.

Research setting

When the field work for this study was carried out the Algebra 12 unit on logarithms and exponentials provided an opportunity to study students' learning of an entirely new mathematics topic (logarithms) without the influence of the students' prior conceptions of this specific topic. Logarithms were covered for the first time in Algebra 12 in the British Columbia Mathematics Curriculum (1978), and students who had studied this topic previously (i.e. in other courses such as science or in an earlier attempt at Algebra 12) were identified easily by the researcher and excluded from the study. Furthermore, the abstract nature of logarithms was such that students were unlikely to have learned about them from their experiences outside of the mathematics classroom prior to Algebra 12. This provided an excellent setting in which to study the

ways in which students went about learning this topic, their emerging understandings of this topic, their expectations about this topic and about learning this topic, their prior conceptions relating to mathematics in general, and their expectations of themselves in the learning process.

There were other advantages for focusing on students' learning of logarithms in Algebra 12 as well. This unit of study was concise, and many students in grade twelve had a level of maturity and an ability to reflect which enabled them to discuss readily their thinking and their work within an interview setting. Also, after twelve years of mathematics instruction, these students had become experts at "playing the game" of school mathematics, and therefore an understanding of their conceptions is helpful for understanding some of the more general learning outcomes of school mathematics.

Information sources

A number of information sources were used in this study. These included copies of the students' written work on a general algebra survey prepared by the researcher and completed by the students prior to the unit of study, the teacher's regular unit test, and a logarithms survey designed by the researcher and completed as a follow-up to the unit of study, copies of the two students' written work and notes completed during the unit of the study, a video recording of the classroom proceedings for the duration of unit, audio or video recordings of interviews with the students individually, copies of the students' written work

done during the interviews, and field notes taken by the researcher throughout the information gathering process. Over the course of several interviews with each of the two students, interim analyses provided the basis for specific lines of inquiry in subsequent interviews. Transcriptions were constructed from the audio and video recordings, and these were analyzed in concert with the students' written work to address the specific research questions for each of the two students. Many of the information sources from the entire class and from each of the two students served to contextualize subsequent events and information sources for the researcher both during the field work phase and final analysis phase of the study. Data analysis methodology consistent with the constructivist perspective of this study was used to construct trustworthy descriptions of each of the two students' conceptions.

Limitations of this study

This study was limited by its focus, the amount of time spent collecting data, and the number of participants. While the analysis is in direct reference to the two students who participated, this in itself is not a limitation or "trade-off" from a naturalist or constructivist perspective (Lincoln & Guba, 1985; Guba & Lincoln, 1989). Mathematics educators will be able to use the authentic and detailed description of these two students to understand other situations when they decide it is fitting.

The significance of this study

The outcomes from this study have the potential to be useful for improving mathematics educators' understandings of their students, the learning process, and the range of learning outcomes that may result from their teaching. These, in turn, can contribute to the improvement of the design, implementation, and assessment of mathematics programs. Thompson (1992) asserts the importance of this in stating that:

We [mathematics educators] must develop a sensitivity for the many subtle ways in which unintended messages and meanings might be communicated to our students.... We must find ways of helping teachers become aware of the tacit rules and beliefs that operate in their classrooms and help them examine their consequences.
(p. 142)

This study also has the potential to demonstrate the usefulness of, and need for more research in mathematics education and for mathematics teaching practice which recognizes students' unique perspectives on mathematics, on their learning of mathematics, and of themselves as players within the mathematics classroom.

CHAPTER TWO: THEORETICAL PERSPECTIVE AND REVIEW OF THE RESEARCH LITERATURE

This chapter provides a selective review of the literature pertaining to this study. It is divided into two main sections. The first of these deals with constructivism, the theoretical perspective of this study, with particular emphasis on the significance of students' conceptions from this perspective; the second of these main sections reports research findings on students' beliefs and conceptions relating to mathematics and mathematics learning. Research findings reported in the latter section are critiqued from a constructivist perspective.

A constructivist perspective

Research perspectives which recognize the predominant role and scope of each individual's world-view in his or her understanding, learning, and performance have developed concurrently in the past decade both within and outside of mathematics education. For example, the educational psychologist Nicholls, in "Students as educational theorists" (1992) explains that:

rather than speaking of mere accuracy or logic as the mark of good theory or interpretation, we might speak of the usefulness or adaptiveness of scientific interpretations. As for any given lay interpretation, usefulness cannot be judged from any abstract, absolute position because there is no such position [emphasis added]. Rather, the value or adaptiveness of any scientific or lay interpretation depends on one's purposes or... one's point of view. Students, like scientists, approach their work with different purposes. The concepts they employ, the data they construct and the way they interpret it can be understood in terms of these purposes. That is to say, students' interpretations of the different aspects of

school are closely bound up, in a rational fashion, with their goals or concerns.... Thought and action are seen as rational expressions of an individual's goals. What can appear irrational to an observer will, if considered in the light of the individual's purposes or intentions, appear rational. (p. 269)

The constructivist paradigm, in particular, is based upon the following principles or assumptions:

1. Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognizing subject.
2. a. The function of cognition is adaptive, in the biological sense of the term, tending toward fit or viability [with one's experiential world];
 - b. Cognition serves the subject's organization of the experiential world, not the discovery of ontological reality. (Von Glasersfeld, 1990, p. 22-23).²

These assumptions have ontological, epistemological and, in turn, methodological implications which are fundamentally irreconcilable with the conventional positivist paradigm as it is commonly formulated at present (Guba & Lincoln, 1989).

Von Glasersfeld (1990) argues that these basic principles emerge quite clearly from the work of Jean Piaget, and refers to Piaget as "the great pioneer of the constructivist theory of knowing" (p. 22). Confrey (1990c) describes Piaget's contribution to constructivism and further suggests implications of this perspective for understanding the conceptions of others, and for teaching:

². In this citation, von Glasersfeld (1990) refers to the principles put forward as indicative of a "radical constructivist perspective," a label that he uses interchangeably with the label "constructivist perspective" (e.g. von Glasersfeld, 1987). The assumptions cited here are also entirely consistent with "constructivism" (without the "radial" connotation) as described elsewhere (e.g. Kilpatrick, 1987; Guba & Lincoln, 1989).

Piaget provided an essential key to a constructivist perspective on teaching in his work, wherein he demonstrated that a child may see a mathematical or scientific idea in quite a different way than it is viewed by an adult who is expert or experienced in working with the idea. These differences are not simply reducible to missing pieces or absent techniques or methods; children's ideas also possess a different form of argument, are built from different materials, and are based on different experiences. Their ideas can be qualitatively different, which can sometimes mean that they make sense only within the limited framework experienced by the child and can sometimes mean they are genuinely alternative. To the child, they may be wonderfully viable and pleasing. They will not be displaced by any simple provision of the 'correct method,' for, by their existence for the child, they must have served some purpose. Before children can change such beliefs, they must be persuaded that the ideas are no longer effective or that another alternative is preferable. (p. 108-109)

The implications of constructivism for education and research are most significant. A constructivist perspective holds that individuals cannot have direct or unmediated knowledge of an external or objective "reality." Instead, each person interprets his or her subjective reality through the "cognitive lenses" (Confrey, 1990c) of his or her prior experiences, current understandings or conceptual structures, and current goals; all within the contexts which he or she constructs and reconstructs on an ongoing basis. These, in turn, determine the character of the individual's experiences. In building meaning, one constructs a fit with one's experience in order to resolve contradictions and inconsistencies--to make sense of that experience, and to restore stability to one's own reality, rather than an exact match of "the real world," which is considered impossible. The interpretations and actions of another individual should, therefore, be considered to be

sensible, rational, and purposeful from that person's perspective. However, this might not be immediately obvious to the observer who is similarly constrained by his or her own "cognitive lenses" when attempting to make sense of this. When communicating with one another, individuals actively construct their own contextually based meanings of the words and actions of others, each in an effort to seek and maintain a meaningful sense of the other. The teacher or researcher, when interpreting a student's performance, must therefore seek a detailed understanding of the various ways that students interpret particular situations, keeping in mind that students' constructions have played some useful purpose for them and will therefore not be displaced readily by new learning experiences. This requires that the teacher or researcher must negotiate meanings with students and seek to accommodate their understandings in a manner far more radical than that required in most everyday situations. To conclude, constructivism is essentially a philosophical perspective about the limits of human knowledge which has profound implications for the notions of truth, objectivity, understanding, communicating, learning, and teaching (Cobb & Steffe, 1983; Cobb, 1986, 1988; Cobb, Yackel & Wood, 1992; Confrey, 1990c, 1991; von Glasersfeld, 1990; Steffe, 1988; Guba & Lincoln, 1989).

Cobb (1986) uses the analogy of Kuhn's (1970) notion of scientific paradigm to explain the most significant role and effect of one's prior knowledge, or "cognitive lenses," in the interpretation or construction of one's experience:

A paradigm or world view restricts the phenomenological field accessible to scientific investigation. The general context within which the scientist operates therefore constrains what can count as a problem and as a solution. Analogously, the general contexts within which children do mathematics [including their conceptions of mathematics and their conceptions of learning mathematics] delimits [sic] both what can be problematic and how problems can be resolved. (p. 2)

Labinowicz (1985) affirms this succinctly in stating "we see what we understand rather than understand what we see" (p. 7).

Context from a constructivist perspective

Context, from a constructivist perspective, "is the cognizing subject's own construction and is distinguished from the situations or settings that the observer might isolate in the subject's environment" (Cobb, 1990a, p. 201). While, for a student at school, context includes his or her construction of the situation and setting, it also includes his or her construction of the less immediate social context including classroom and social norms and his or her own purposes or own agenda. Context delimits the student's phenomenological field, what can be problematic, what can be considered to be a goal, and how problems can be resolved or goals achieved (Cobb, 1986, 1990a; Confrey, 1991).

Constructions held by people are born out of their experience with and interaction with their contexts. Indeed, the tie is so close that one can easily argue... that constructions create the context that the constructors experience and are in turn given life by that erected context. (Guba & Lincoln, 1989, p. 60)

Lave (1988; citing work published by Lave, Murtaugh & de la Rocha, 1984) provides an example of the powerful influence that context can have on one's perception of a mathematics task and

subsequent performance in her study and comparison of adults' "best-buy" arithmetic calculations done in situ when shopping in a supermarket with traditional arithmetic calculations on a set of pencil and paper tasks. In the in situ best-buy calculations, Lave's adult participants actively transformed these problems using invented strategies with great flexibility. However, these same adults approached the school-like arithmetic pencil and paper tasks with no perception of control or choice in the problem solving process.

[Performing the set of pencil and paper arithmetic tasks] was supposed to be a relaxed, certainly not a test-like occasion, at home, with a staff person from the AMP [Adult Math Project] who had gotten to know the participants during initial interview sessions. However, we were not successful in removing the evaluative sting from the occasion. Participants did not believe our claim that this was 'not a test in the usual sense.' They reacted to the request that we be allowed to observe their math procedures with comments of 'ok, teacher,' by clearing the work space, and by talking about not having studied math for a long time. Common requests were phrased as, 'May I rewrite problems?' and 'Should I...?' (p. 54)

Lave's participants scored an average of fifty-nine percent correct on the set of pencil and paper general arithmetic exercises, and fifty-seven percent correct on a set of pencil and paper ratio comparison problems ("isomorphic," to use Lave's term, with the best-buy problems in the supermarket). In dramatic contrast to these scores, Lave's participants scored an average of ninety-eight percent correct, virtually error free, on their in situ supermarket calculations!

Central to a constructivist perspective is the presumption that the constructions of others (e.g. students), including contexts, have integrity, sensibility and rationality from their

perspectives (Confrey, 1980, 1990c; Cobb, 1986). Confrey (1980) asserts that "rational in this context means that for that particular student, the reason justified the response" (p.404). And, while this might not be apparent immediately to an observer, it therefore must be the teacher's or researcher's obligation to construct a detailed understanding of students' rationality and the various ways that they interpret and construct their own sense of particular situations when working with them. It also follows, from a constructivist perspective, to speak of learners' "alternative conceptions," rather than "misconceptions" or "developmentally primitive conceptions," which imply that deviations from the norm are inadequate or in error. Within the contexts that students construct, their conceptions are legitimate expressions of their attempts to make sense of, or reconcile their experiences, both inside and outside of the classroom (Confrey, 1991). As students gain experience and reflect upon this experience, including experience within the formal domain of mathematics in school, previously separate contexts become integrated and otherwise contradictory and inconsistent sets of beliefs become reconciled. As a result, students' constructions become increasingly powerful and useful.

Constructivism and other perspectives

Some confusion surrounds the label "constructivism" in that it is used by many both inside and outside of in mathematics education for a multitude of eclectic theoretical positions which inconsistently combine assumptions of both "constructivism," as

described earlier, and "representational views of mind" or "transmissional views of learning" (Cobb, Yackel & Wood, 1992). Philosophically, constructivism stands in distinct opposition to positivist, objectivist, realist, deterministic, or representational views of knowledge, learning, and understanding (Guba & Lincoln, 1989). Unreconcilable theoretical and practical difficulties for these two sets of perspectives or paradigms, from a constructivist point of view

stem from the dualistic nature of the basic underlying metaphor [of the representational view]. At the outset, mathematics in students' heads [internal representations] is separated from mathematics in their environment [external representations that are transparent for the expert]. (Cobb, Yackel and Wood, 1992, p. 14)

The "learning paradox" (Bereiter, 1985) exemplifies the irreconcilable problem with eclectic positions which simultaneously hold that all knowledge is constructed by each individual and that mathematics is objectively "out there" in the world:

If one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced as the one to be learned.... The learning paradox does apply where--as in being introduced to rational numbers, for example--learners must grasp concepts or procedures more complex than those they already have available for application. (p. 202)

Cobb, Yackel and Wood (1992) put this another way:

The assumption that students will inevitably construct the correct internal representation from the materials presented implies that their learning is triggered by the mathematical relationships they are to construct before they have constructed them. How then, if students can only make sense of their worlds in terms of their internal representations, is it possible for

them to recognize mathematical relationships that are developmentally more advanced than their current internal representations? (p. 5)

Von Glasersfeld (1987) coined the term "trivial constructivism" to refer to these eclectic positions, which assert that knowledge is constructed while maintaining that this knowledge is of the world "out there."

Constructivism also stands in distinct contrast to information-processing approaches to cognition. The "strong research program" in information-processing research emphasizes the creation of operable computer programs to model human cognition. Alternatively, the "weak information-processing program" does not limit itself to the computer processing metaphor. It focuses instead on the creation of production systems to explain cognition and, in particular, students' "systematic errors" in mathematics performance (Cobb, 1987, 1990a, 1990b; Confrey, 1991). Problematic student performance from an information-processing perspective is considered to result from incomplete knowledge or "misconceptions"--inadequacies in the student's cognitive process. The possibility for legitimate alternative perspectives is seldom considered and it is the researcher (or teacher) who judges the appropriateness of a student's performance from his or her own perspective. "No learning theory is seen as necessary to account for these errors [from this perspective], beyond the recognition that the errors represent overgeneralizations on the part of the students" (Confrey, 1990a, p 42). Furthermore, information-processing theories are limited to specific and delineated domains of

mathematical performance, fail to address the important role of context in mathematical cognition, and do not treat the social aspects of mathematical learning in school (Cobb, 1990b). As was the case with representational views of mind discussed earlier, the label "constructivism" has been used in the mathematics education literature to describe perspectives which are explicitly or implicitly based upon information-processing approaches to cognition (for example, Schoenfeld, 1987; Davis & Maher, 1990. For further discussion see Cobb, 1987). Confrey (1990a) makes the observation that an interesting quality of the research on systematic errors is that it continues to evolve toward a constructivist perspective "but is inhibited in doing so by its epistemological naivete" (p. 37).

Implications of a constructivist perspective for the study of students' conceptions of mathematics and their conceptions of mathematics learning

Recognition of alternative perspectives and the predominance of context obliges the constructivist researcher to search actively and creatively beyond his or her own conceptual and contextual frameworks, students' more obvious mathematical performance indicators, and the immediate pedagogical setting of the mathematics classroom when attempting to understand students' mathematics performance and mathematics learning (Confrey, 1991). This demands active and deliberate decentering and accommodation (in a Piagetian sense) on the researcher's part, and the use of an open-ended hermeneutic approach, in order to develop authentic understandings of the conceptions of others. All the while it is

recognized that understandings constructed by the researcher, including those constructed from the negotiation of meanings with students, are themselves tentative working hypotheses which fit, but cannot match, the researcher's experience. At best, meanings can only be taken-as-shared (Cobb, Yackel & Wood, 1992). This in turn delimits and characterizes the claims that can be made about students' conceptions. In contrast, other theoretical perspectives, such as information-processing, allow the researcher to interpret and assimilate meaning readily from the performances of others in terms of his or her own existing conceptual frame of reference. This severely limits the possibility that alternative and more authentic understandings of students' conceptions will be constructed by the researcher.

It follows also, from the argument above, that in order to interpret the meaning of students' actions and engage students in the negotiation of meanings, the constructivist teacher or researcher must, in effect, construct case studies or models of individual students' understandings relating to mathematics (Confrey, 1990c; Cobb & Steffe, 1983). In doing so, the investigator must endeavour to formulate viable models of, and accept as epistemologically legitimate, the alternative mathematical and related perspectives and contexts which students construct (Cobb & Steffe, 1983; Confrey, 1990b, 1991). Furthermore, it follows that students' conceptions of mathematics and conceptions of mathematics learning are fundamental parts of their contexts for learning mathematics, and it is essential that these conceptions be investigated, in order to inform one's

understanding of students' mathematical understanding, learning, thinking, and performance. As Rogoff (1984) asserts, "context is an integral aspect of cognitive events, not a nuisance variable" (p. 3).

Finally, a constructivist perspective in mathematics education research provides a philosophical basis for knowledge, truth, objectivity, communication, and learning which is consistent with, and informed by, current perspectives on these matters in other social sciences, and in both the philosophy of science and the philosophy of mathematics (see Cobb, 1986, 1990a; Cobb, Yackel & Wood, 1992; Nicholls, Cobb, Yackel, Wood & Wheatley, 1990). This foundation serves to inform the researcher in locating and interpreting evidence of students' conceptions, including evidence which has traditionally been considered to be outside the domain of mathematical cognition such as the research of classroom norms (Yackel, Cobb & Wood, 1991), students' emotional acts and motivation (Cobb, Yackel & Wood, 1989), and students' personal theories about mathematics (Nicholls, Cobb, Yackel, Wood & Wheatley, 1990).

Limits of a constructivist perspective in research

From a constructivist perspective "human knowledge consists of a series of constructions, which precisely because they are humanly generated, are problematic, that is, indeterminate, unsettled, and ambiguous" (Guba & Lincoln, 1989, p. 68), and as Confrey (1990c) affirms, a constructivist perspective is essentially about the limits of one's knowledge. Therefore,

constructivism has significant implications for research methodology and kinds of claims that can be made by a researcher investigating students' conceptions.

From a constructivist perspective, certainty, truth, and objectivity are not possible (von Glasersfeld, 1990). For this reason, research from a constructivist perspective cannot aim to provide reasons or explanations for specific aspects of students' mathematical performance in an antecedent-consequence manner, nor provide general prescriptions to address problems in students' learning. Furthermore, notions from the conventional positivist scientific enquiry paradigm such as variable, design, control, cause, generalize, discover, identify, correlate, and the like are all in one sense or another inappropriate (Guba & Lincoln, 1989, p. 76). Instead, constructivist researchers attempt to build tentative models of individual students' conceptual frameworks, their goals, and expectations on the basis of sustained inquiry and negotiation so that they can, in turn, construct viable and authentic understandings of particular students within specific contexts (Cobb & Steffe 1983; Cobb 1986). Constructivist research does not have as its goal "the traditional one of accurate prediction, but instead, the careful description and validation of a patterned set of interrelationships" (Magoon, 1977, p. 668).

The generalizability of findings concerning students' conceptions from specific cases to other students is not possible from a constructivist perspective because of the inherent "context-stripping," to use Guba and Lincoln's (1989) term, that

occurs when this is attempted. An individual's conceptions are ideographic and inextricably tied to the specific contexts which he or she has constructed. These conceptions cannot be taken to be shared by others without an in-depth examination of the perspectives of these other individuals, or the "receiving context" to use Guba and Lincoln's (1989) description. However, a constructivist perspective does recognize that, at times, an individual's conceptions can have a viable fit with those of others, within specified contexts, and can be taken-as-shared for those individuals.

These claims about the limitations of a constructivist perspective should not be taken to suggest that findings about students' conceptions are not useful. Studying students' conceptions from a constructivist perspective provides rich, authentic, and trustworthy interpretations of particular cases. With this knowledge, the researcher and others will have local understanding of these particular cases, and in turn, informed perspectives for the interpretation of other cases. This empowers the researcher and others in devising solutions to problems with local meaning and utility (Guba & Lincoln, 1989).

Review of the mathematics education research literature

This review of the research literature on students' conceptions relating to mathematics and mathematics learning is divided into three parts. The first part reports on research which exemplifies the importance of students' conceptions relating to mathematics and mathematics learning. The second

part reports large scale questionnaire-type studies which have investigated specific beliefs held by students relating to mathematics and learning mathematics. The third part reports case studies which have investigated in detail students' conceptions relating to mathematics and mathematics learning.

Research which exemplifies the significance of students' conceptions relating to mathematics and mathematics learning

As mentioned in chapter one, Earlwanger's "Case Study of Children's Conceptions of Mathematics" (1975) marked the beginning of a now established research focus on students' beliefs and conceptions relating to mathematics. It had been Earlwanger's original purpose to assist pupils working in an individual instruction program to resolve their misunderstandings with the mathematics content that they were learning and to determine the nature of their difficulties. After working with a particular Grade six student named Benny, however, Earlwanger discovered that even students who were making good progress within this program, as evidenced by the amount of work that they had completed and their test results, could have inappropriate beliefs relating to the specific content that they were learning and about mathematics itself. Earlwanger's entire study of children's conceptions of mathematics and their conceptions of learning mathematics, and of the relationship of these beliefs to the particular individual instruction program which the students had experienced from the second grade grew out of his surprise at Benny's unexpected ideas.

After a semester of in-class observations, discussions, and

interviews with two children in each of Grades 4, 5, and 6 in the individual mathematics instruction program Earlwanger concluded that:

each child had developed a conception which appeared to function as a relatively stable cohesive system of interrelated ideas, beliefs and views about mathematics and the learning of mathematics. Aside from individual variation, the children revealed conceptions that were unanticipated and different from an adult view of mathematics. This included a view of mathematics as a set of rules for making arcane marks on paper; various views about the purpose of learning mathematics, individualization and the role of the teacher and workbooks [within the individualized mathematics program], tests, review, and progress in mathematics; and a variety of beliefs about rules in mathematics programs, the relation between rules and answers, and so on. (p. 157)

Earlwanger discussed further the inappropriateness of these views for students' learning of mathematics and was critical of the particular individual instruction program and the conventional evaluation methods used (pencil and paper testing), citing these as the cause of the students' inappropriate conceptions and the reason that these had gone unnoticed by the classroom teachers. For the purpose of this study, suffice it to say that Earlwanger established the study of students' conceptions relating to mathematics as a critical area for research to develop educators' understandings of students' performance and learning in mathematics.

Cobb (1985), from his work with a small number of primary level students over a two year period (Grades one and two), concluded that students' beliefs about mathematics and their motivations for doing mathematics were intimately related. Descriptions of the beliefs and motivations of two students were

presented to support this claim. The first student, Scenetra, viewed mathematics as a set of unrelated rules for solving unrelated problems. For her, understanding why a method did not work was not important, getting the correct answer was most important, and this was her sole criterion for judging the appropriateness of her efforts. When faced with a genuine problem, she frequently focused on superficial features of the problem statement or would give a sequence of different answers in an effort to get the answer that the teacher wanted. Also prominent was her concern not to appear incompetent to others as she worked through the tasks assigned. In short, Scenetra was preoccupied with success and was missing the substance of the mathematics that was being covered.

In contrast to Scenetra the second child reported by Cobb was Tyrone who enjoyed playing around with numbers for its own sake. He frequently used information from previously completed problems as he solved new problems or determined the appropriateness of his answers for these, and he would seek out opportunities to do this on his own. In his view, mathematics was full of connections and he sought these out as an end in itself rather than engaging in mathematics to look smart.

From his analysis, Cobb suggested that individuals have an "increasingly global hierarchy of anticipations corresponding to specific conceptual structures or problem representations, heuristics, and beliefs" (1985, p. 111). These beliefs and anticipations about mathematics and doing mathematics seem to be intimately interrelated with the students' motivations when

engaging in mathematics.

In subsequent studies, Nicholls, Cobb, Wood, Yackel, and Patashnick (1990); and Nicholls, Cobb, Yackel, Wood, and Wheatley (1990) report their validation of an assessment instrument with Grade two students which distinguishes between "task orientation" and "ego orientation" towards mathematics. Task orientation, like Tyrone's approach to mathematics,

involved the goals of working hard, figuring things out, and collaboration; the beliefs that success requires effort, interest, collaboration, and attempts to understand; and the beliefs that learning mathematics improves people's ability to make sense of things and to explain themselves. (Nicholls, Cobb, Yackel, et al., 1990, p. 146)

In contrast, ego orientation, like Scenetra's approach to mathematics,

was not associated with these beliefs but was ... associated with beliefs that success depends on superior ability and attempts to beat one's peers. (Nicholls, Cobb, Wood, et al., 1990, p. 118-119)

These researchers assert that as early as the second grade, students have constructed coherent and encompassing conceptions of mathematics and conceptions of learning mathematics which have most significant implications for the way in which they engage in mathematics learning and doing mathematics in the classroom.

Kloosterman (1991) administered a set of questionnaires and standardized mathematics achievement tests to 400 Grade seven students to determine how four specific beliefs about how mathematics is learned were correlated with mathematics achievement. This was based on the assumption that students' beliefs are key to understanding their performance. The four

beliefs that Kloosterman examined were:

- 1) self-confidence in learning mathematics,
- 2) attributional style in mathematics (that is the attribution of academic success in mathematics or lack thereof),
- 3) effort as a mediator of mathematical ability, and
- 4) failure as an acceptable phase in the learning of mathematics. (p. 5)

Kloosterman claims that a quantitative analysis of the data revealed that a "latent construct, namely beliefs about how mathematics is learned is significantly positively correlated with achievement in mathematics" (p. 10) and, therefore, that students' beliefs about mathematics learning are related to their achievement. No further details were reported.

While the studies reported above have all indicated the importance of students' beliefs or conceptions relating to mathematics for their learning, none of these studies have investigated the range and nature of beliefs or conceptions that students in mainstream mathematics programs hold relating to mathematics and mathematics learning. This will be considered in the following sections.

Large scale assessments of students' beliefs relating to mathematics and mathematics learning

The largest study to be reported here is the 1988 National Assessment of Educational Progress carried out in the United States (reported by: Kouba, Brown, Carpenter, Lindquist, Silver, & Swafford, 1988a, 1988b; Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988a, 1988b). As part of this project a "representative" national sample of 2000 students at each of the Grade 3, 7 and 11 levels was assessed on a wide range of

mathematics content areas. A number of additional items were included on the test to determine students' attitudes or beliefs relating to: mathematics as a subject in school; mathematics and oneself, including their perceptions of themselves as learners of mathematics; mathematics and society; and mathematics as a discipline. Only a small part of these attitudinal items were included in the assessment of the Grade three students.

The results of the NAEP study which are significant for this study are those for students at both the Grade seven and eleven levels as follows:

- 1) Approximately two thirds of the students thought that they were as good at mathematics as their classmates.
- 2) Roughly three quarters of the students indicated that they usually "understood" [emphasis added] what was going on in their mathematics lessons.
- 3) Over four fifths thought that mathematics was rule based, while close to this same proportion indicated that learning in mathematics requires lots of practice following rules.
- 4) About half of the students reported that learning in mathematics was mostly memorization while at the same time, approximately ninety percent of the students indicated that knowing how to solve a problem was as important as getting the solution, and close to this proportion indicated that knowing why an answer was correct is as important as getting the correct answer.
- 5) Approximately two thirds of the students indicated that mathematics helps a person to think logically.
- 6) Most students lacked an understanding of what mathematicians do or that mathematics is a dynamic cohesive discipline. (Brown et al., 1988b)

From these results, Brown et al. (1988b) suggest that "students appear to have an emerging view of mathematics as a process but also hold the view that rules are still important" (p. 346).

Kouba and McDonald (1991) examined the operational

definitions or conceptions of mathematics held by students from the kindergarten to Grade six levels, recognizing that these conceptions are a hidden but important factor affecting students' confidence in mathematics and their understanding of the perceived usefulness of mathematics, which in turn are important factors affecting students' success. These researchers developed a set of 54 descriptions depicting a variety of mathematical and non-mathematical activities. The students in the study, over 1200 students from 20 different schools, responded "yes" or "no" to each of these to indicate whether they thought the depiction was of a person doing mathematics or not. The mathematics activities included: number and numeration, arithmetic operations, measurement and geometry, and probability and statistics. An example of one of the "non-mathematics" activities was the following: "Ellen read a book about trees" (p. 106). Subsets of these tasks were given to all of the students in each of a number of intact classes, and the regular classroom teachers led their own classes through the questionnaire and whole class interviews about the content of the questionnaire afterwards. During the follow-up interviews individual students explained their answers to many of the items on the questionnaire. These discussions were audio recorded and used to supplement the students' questionnaire responses as the data in the analysis.

The results indicated that the kindergarten and Grade one students responded in a random-like manner to the items and, from their comments recorded on tape, it was determined that they did

not understand many items on the questionnaire. From the results of the Grade 2 to 6 students, however, Kouba and McDonald constructed three views of mathematics which they believed could explain many of the students' decisions regarding their classification of the activities described. These views were:

1) Narrow domain--that mathematics consists exclusively of counting and arithmetic operations, 2) Exclusive domain--that school subjects are mutually exclusive, that one cannot do two different (subject) activities at once, and that mathematics is only done at school, and 3) Upwardly shifting domain--that an activity is mathematics only until it becomes known or automatic to the students. (The implication here is that some of the students expected mathematics to be difficult always.) Kouba and McDonald state that these beliefs can potentially limit the basis upon which students value or decide the usefulness of mathematics, and furthermore, that these beliefs imply that some students may disregard their prior knowledge and success in mathematics believing that mathematics must be difficult.

Of the three large scale studies reported thus far (including Kloosterman, 1991), only the last (Kouba & McDonald, 1991) provided an opportunity for students to explain their understandings of the particular questionnaire items to which they responded. The results from the other two, in contrast, were based on the assumption that the student participants shared the same meanings for each of the questions as the researchers. Without some indication of the students' perspectives, interpretation of meanings such as the following should be

considered problematic when interpreting this data: "to be as good at mathematics as other students," "to understand mathematics," "to know why an answer is correct," "to know how to solve a problem in mathematics," and "to think logically." The meanings that students construct for these and other phrases on questionnaires may not always correspond with the meanings constructed for them by researcher(s).

Furthermore, the questionnaire studies which deal with different sets of beliefs (Kloosterman, 1991; Brown et al., 1988a, 1988b) did not indicate that consideration had been given to determining whether the questionnaire items correspond to the ways in which students conceptualized mathematics and learning mathematics, nor considering how students' beliefs are held in concert, nor considering interrelations amongst students' beliefs. Without consideration of these, mathematics educators have a limited perspective with which to understand the significance of the beliefs reported above for students' understanding of, and performance in, mathematics.

Case studies of students' beliefs or conceptions relating to mathematics and mathematics learning

A number of case studies which examine students' beliefs in greater detail than the large scale studies cited earlier will now be reported. While the focus of each of these studies is different, students' beliefs or conceptions relating to mathematics and/or mathematics learning are prominent in each.

Frank (1985, 1988) studied the beliefs relating to mathematics of four junior high school students who were enrolled

in a summer enrichment program in mathematical problem solving using computers at Purdue University. It was her purpose to investigate and explain the role of students' beliefs relating to mathematics in their mathematical problem solving. Frank interviewed each of the participants four to six times during the two weeks that they were at Purdue. Each interview consisted of a conversation with the student about mathematics in an effort to determine his or her beliefs relating to it, and explanations from each student as he or she worked through a number of non-routine mathematics problems. A problem solving test and mathematics beliefs questionnaire given to all 27 students in the class on the first day of the course was also used in the analysis. From a subsequent analysis of the written information from the students and transcripts generated from the interviews, Frank constructed five categories of students' beliefs which all appeared to influence the students' performance as they attempted to solve mathematics problems. These categories were beliefs about one's ability to do mathematics, beliefs about mathematics as a discipline, beliefs about where mathematics knowledge comes from, beliefs about solving mathematics problems, and beliefs about how mathematics should be taught and learned.

In particular, Frank found that all of the students believed that they would be able to solve routine-looking problems, but lacked confidence in their ability to solve problems that appear to be nonroutine. These nonroutine problems were perceived by the students as "extra-credit" and not really mainstream mathematics problems. In their view mathematics was a set of

facts, rules, and procedures learned to get the right answers, and answers in mathematics were dichotomized as either completely right or completely wrong without a middle ground and, not surprising, these students expected that mathematics problems could be solved quickly in a few steps. These students also viewed the mathematics teacher as the source and authority for mathematics, and they learned mathematics mostly by memorization. Finally, for these students, "understanding" mathematics meant being able to get correct answers to the problems that they encountered.

Confrey and Lanier (1980) also focused on students' problem solving, specifically a number of processes that students used to solve mathematics problems as well as their conceptions of mathematics. The participants in this study were in a general level Grade 9 program. These researchers reported that while there was an idiosyncratic component to each student's conception of mathematics, there were a number of characteristic assumptions about mathematics, or themes, which extended across the students who were interviewed. These themes included:

- 1) That the primary aim in mathematics was to get answers, and that the process was important only in that it was instrumental to this aim.
- 2) That mathematics was a set of symbols and rules for manipulating these symbols, and if the students encountered difficulties they were left with little recourse.
- 3) That the teacher or the textbook was the authority and arbiter of truth in mathematics. The students seldom expressed a view that their solutions were correct on the basis of their own sense making.

Confrey and Lanier assert that these conceptions are contributing factors to the students' lack of confidence and their sense of

powerlessness in mathematics and, as such, an impediment to students' intellectual growth.

In another study, Confrey (1984) examined the conceptions of mathematics held by female secondary school students who were enrolled in a summer mathematics enrichment program. It was the aim of this study to assess the success of the summer program in broadening these students' conceptions of mathematics. From the analysis of a questionnaire given to the students on the first day of the course, interviews with students, and diaries kept by the students, Confrey determined that the students held complex and often contradictory sets of beliefs and expectations about mathematics. In particular, Confrey reported that the students were "relying upon the teacher, focusing on answers primarily, imitating examples from the text, memorizing, and learning rules and formulae in isolation" (p. 19). Furthermore, these students did not make a clear distinction between mathematics as a discipline and mathematics as portrayed by the implemented school curriculum, saw mathematics as dichotomized into being completely right or completely wrong, were unwilling to ask or answer questions within the classroom setting, lacked confidence in their own answers and ideas, and felt a great deal of pressure with the pace at which they had been required to learn mathematics. Confrey concluded that wide ranging interventions would be needed to change the full range of conceptions held by students and that a better understanding of students' behavior in mathematics learning is needed so that those aspects of students' classroom experience which contributed to their performance in

mathematics can be addressed.

Schoenfeld also investigated students' beliefs in an effort to better understand their problem solving performance. In his well known book Mathematical Problem Solving, Schoenfeld (1985) presented a cognitive science view of the role of students' beliefs as a variable when explaining their mathematical problem solving performances. He also presented his preliminary findings from two related studies of students' beliefs. Subsequently (1988 and 1989), he reported in detail the results of these two studies, the first an in-depth single class (Grade 10 geometry) case study over the course of a school semester, and the second a larger scale (n=230) assessment of the beliefs of high school students using a questionnaire. It was Schoenfeld's intent to examine the ways that students' conceptions of mathematics impact upon their mathematical activity.

In the single class case study, Schoenfeld (1988) found that despite being well taught and performing in the top 15% of the New York State Regents mathematics exam, "the students gained at best a fragmented sense of the subject matter and understood few if any of the connections that tie together the procedures that they had studied" (p. 145). More specifically, Schoenfeld concluded from both studies that:

- 1) The students had learned to separate mathematical proof and discovery in mathematics. From the students' perspectives, proofs (geometric proofs in particular) were confirmation of what was already known and not a process by which they could come to make discoveries on their own. Furthermore, many students had an empirical conception of geometric proof, that is, constructions were correct if and only if they looked right.

2) The students had learned that the form of a mathematical argument (in the case of geometric proofs, the use of a standardized two-column format) was at least as important as its content.

3) The students believed that problems in mathematics could be solved in just a few minutes or not at all. More specifically, the students indicated on the questionnaire that on average, solving a typical homework problem should take 2 minutes 12 seconds, and no one indicated a time greater than five minutes. In response to a question asking what was a reasonable amount of time to work on a problem before deciding that it was impossible, the average time given was 12 minutes and no student indicated that more than 20 minutes should be spent.

4) The students saw themselves as the passive consumers of mathematics handed down from others. In particular, many students believed that mathematics in school was mostly facts and procedures to be memorized, that solving mathematics problems depended on knowing the rules, and that good teaching in mathematics included making sure that students knew how to apply rules.

Schoenfeld, in an effort to argue the "generalizability" of these findings, stated that his findings were consistent with the findings from the Third National Assessment of Educational Progress (NAEP, 1983) study which found, among other things, that approximately half of the teenagers (13- and 17-year-olds) agreed that mathematics was mostly memorization, and that nine in ten students agreed that there is always a rule to follow in mathematics. These 1983 NAEP findings are similar to the 1988 NAEP findings (Brown et al., 1988a and 1988b) reported above.

Oaks (1987) studied the conceptions of mathematics, conceptions of mathematics learning, and the affective constructs relating to the learning of mathematics held by students who were attending a number of colleges in New York State. In the first of three preliminary studies reported in her doctoral dissertation, seven pairs of students who had just completed a

Year one algebra course were videotaped while attempting to solve a number of algebra problems. Oaks concluded that in each case the students were recalling memorized algorithms in attempting to solve these problems and in many instances they seemed to be unaware of the questions they were trying to answer or the goals of the problems they were trying to solve. Oaks concluded further that a significant number of students

view mathematics in a meaningless, algorithmic manner. ...they do not appear to realize there are other discourses for mathematics; this is not their [emphasis original] way of doing mathematics--it is the [emphasis original] way to do mathematics for everyone. As a result they look at behavior not consistent with this mathematical belief as not doing "real" mathematics. (p. 47-48)

In a second preliminary study, Oaks collected questionnaire data on students' attitudes and beliefs relating to mathematics from 144 college students who were enrolled in Year one mathematics at a number of different colleges. From a preliminary analysis, Oaks reported that many of these students equated "understanding mathematics" with knowing how to solve the problems that they encountered in their mathematics classes, which in the student's view was separate from "memorization." "Memorization" for many of these students was restricted to the memorization of rules and formulas--

many do not seem to consider learning an algorithm (even if it is by rote) to be memorization. For example, many appear to define "working for understanding" as committing to memory all of the steps for solving a quadratic equation by practicing until one knows "how to do it." They then define the ability to perform such an algorithm as "understanding."
(p. 50)

In short, getting the right answers meant, for many students,

that they "understood" the mathematics content and that they were learning in an appropriate manner. Oaks concluded that these students lived in a different "mathematical world" than their teachers and that this has significant implications for understanding the difficulties faced by them in college level mathematics.

In a third preliminary study, Oaks analyzed the journals written by 44 college students who were enrolled in Year one mathematics. This analysis indicated that many of these students' conceptions of mathematics were limited to the material presented in class and their performance in class. Many journals also indicated that students relied on their teachers for their learning, that many of them blamed their lack of success on their instructors, that students considered test scores and getting right answers to be most important, that students focused on remembering correct algorithms when trying to solve problems, and that many students were puzzled about the causes of their poor grades in mathematics.

Oaks used the findings from these preliminary studies to establish the focus for more in-depth case studies with each of four students who were enrolled in a remedial first year college mathematics program. Four students (two male, and two female) were selected from those who had successfully completed at least one year of college, and who had completed at least one term of first year mathematics in different classes. All of these students had also completed a mathematics journal in their one term of mathematics instruction and were willing to participate

in the study. These students as a group also had experienced different levels of success in their Year one mathematics course. Oaks conducted three interview sessions, 1 to 1.5 hours in length, with each of the four participants. During the first round of interviews the students were asked to respond to a fixed set of questions about their background in mathematics, mathematics in general, their mathematics teachers, learning mathematics, tests in mathematics, and doing problems. During the subsequent interviews, specific sets of questions were generated for each of the students based on the preliminary analysis of each previous interview and the student's journal in order to clarify statements that he or she had made earlier. A common set of questions was also presented to each participant during the second and third round of interviews so that their responses could be compared, and during their final interviews the students were asked to react to a series of statements about mathematics such as "Mathematics is essentially: learning to solve problems,... a game,... drill,... like following a recipe,... a way of thinking" (p. 75). Each of the interviews was recorded on audio tape and transcripts constructed from these recordings served as data in the final analysis. Both the students' statements in response to the interviewer's questions and their behavior during the interviews were interpreted by Oaks in this analysis.

The analysis of the students' responses indicated that all four of the students had little knowledge of mathematics as a discipline and of mathematicians apart from their classroom

experiences. They believed that mathematics was doing problems, that it was cumulative in nature, that each problem in mathematics had at least one exact algorithm for its solution, and that their primary goal in doing mathematics was to find such algorithms and to apply them to get the correct answers. Furthermore, each student relied upon the teacher and/or the textbook as his or her source for these algorithms.

The students also had significant differences in their conceptions of mathematics and mathematics learning as well. Oaks presents two distinct sets of beliefs or conceptions relating to mathematics held by the students. Three of the four students, Patricia, Roland, and Katie viewed algebraic problems as meaningless strings of symbols. For example, they were unable to discuss the meaning of an equation, its context or derivation, only how to solve it. In Oaks own words: "They appear to have little knowledge of the system of mathematics--they do not seem to recognize the structure of axiom, theorem, and proof and do not appear to view reasoning as an appropriate mathematical activity" (p. 356). On the other hand, the fourth student, Andrew, speculated about proof and conjecture in mathematics and viewed mathematics as a system of axioms, definitions, theorems and proofs, all of which were a crucial part of learning mathematics for him. Andrew used his own deductive reasoning to solve problems, he was able to verify his own solutions, viewing this process as a crucial part of doing a problem, and he recognized that knowledge of the goals and meaning of problems were important.

There were significant differences between the students with respect to their conceptions of mathematics learning as well. The group of three students, identified earlier, were passive participants in the learning process who depended on being lead by their teachers. Their focus during mathematics lessons was on the "right way" of doing things, and locating the correct algorithm was the most important activity for them when doing a problem. These students had rote and algorithmic conceptions of theorems, formulas, methods of solution, and concepts--of mathematics itself. "Understanding" meant little more than knowing how to get the right answers, and they were motivated to avoid mathematics. They lacked confidence in their abilities, they had low expectations for success in their mathematics learning, their efforts in learning mathematics were focused on committing solutions to memory, they attributed their failure to uncontrollable factors such as their ability and the nature of mathematics, and they lacked the means to improve their performance in mathematics. In short, these students' conceptions of mathematics did not allow for their success. Andrew, in stark contrast, sought to develop his conceptual understanding when learning mathematics. Learning mathematics included meaning and derivation in addition to learning how to solve problems. As a result, Andrew was able to derive methods of solution on his own, evaluate his own performance, and feel confident in his own abilities and in control of his own learning.

Oaks asserts further that Patricia, Katie, and Rolland's conceptions of mathematics and of mathematics learning had dire consequences for them by preventing their progress and their success in learning mathematics.

Their focus on algorithms dictates that they resist any efforts on the part of others to show them that conceptual understanding is necessary in doing mathematics.... They do not appear to view proof and derivation as important and therefore ignore it when it is presented to them. (p. 377-378)

Andrew on the other hand was, in Oaks' words, "one of the rare cases of success" (p. 380) among student in a college remedial mathematics program. While he indicated that in high school his conceptions were similar to those of the other three students,

he appears to have experienced a change in conception which allows him to finally be successful at it [mathematics]. This change caused him to search for meaning in mathematics by paying close attention to derivation, theorem/proof, etc.. As a result he knows the goal of a question, is able to generate a correct method of solution, and can verify his work. (p. 381)

Summary of the research literature review

The research presented in the previous section exemplifies the significance and scope of the conceptions relating to mathematics that students develop. These, in turn, constitute the context for students' learning of, and success or failure in, mathematics. Students establish their world views of mathematics and mathematics learning as early as the second grade and their views can be significantly different from and not obvious to their teachers.

There were a number of themes within the research results presented above. Many students view mathematics as a set of

rules and procedures to be handed down from their teachers and learned in a rote manner. Many students lack confidence in their abilities as independent learners and users of mathematics, for many students mathematics is little else than the activities that they have experienced in their mathematics classes, and for many students, understanding mathematics means being able to get the right answers for routine problems. Most distressing is the conclusion that many students hold views of mathematics and mathematics learning which not only hinder their success in mathematics, but views which result in their ignoring alternative approaches to mathematics and mathematics learning, approaches which would be helpful for them.

The findings from the case studies suggest that a number of claims made in the large scale NAEP questionnaire study need to be examined further. Many of the assertions made by Brown et al. (1988b) about students' beliefs appear to be less promising for mathematics education in light of these insights, than if taken at face value. In particular, the claim that students think they are as good at mathematics as their classmates, should be reconsidered in light of the conclusion that many students have conceptions of mathematics which are limited to following rules and procedures. Many students may not recognize that others are working at different and more mathematically powerful conceptual levels than they are. Other conclusions, for example, that students usually "understand" mathematics, that students view knowing how to solve a problem as important, and that students think that mathematics helps one to think logically, should also

be reconsidered in light of students' limited perspectives of mathematics and mathematics learning as reported in the case studies. Oaks (1987) addresses this very point in the conclusion to her study. She asserts that the students in her study seem to have taken on the "myth of the classroom" (citing a paper presented by Schoenfeld in 1985 and published subsequently in 1989)--they have a tendency to repeat statements that they have heard in their mathematics classes regarding the usefulness of mathematics and what is necessary to be successful in learning mathematics, without these ideas being an integral part of their own conceptions of mathematics and mathematics learning.

The research outcomes that have been reported in this chapter leave a number of questions unaddressed. A specific example is from the report by Brown et al. (1988b). While many students indicated that knowing why an answer is correct is important, no details have been provided to indicate what the students mean by this nor if their meanings correspond with those understood by mathematics educators. In more general terms, the research in mathematics education presented here does not address the conceptions of mathematics, conceptions of mathematics learning, or conceptions of self as doers and learners of mathematics held by successful students in mainstream high schools. These matters will be addressed in the following chapters.

CHAPTER THREE: RESEARCH METHOD AND THE CONTEXT OF THE CASES

This chapter outlines the research method of this study and background information relating to the case analyses which follow in chapter four. The research method used in this study is based upon principles of naturalistic inquiry³ (Lincoln & Guba, 1985), and principles of constant comparative analysis (Strauss, 1987).

Choice of the target class

It was my goal in this study to focus on two students in a single intact and reasonably typical Algebra 12 class. All of the participants, the teacher of the class and the students, were volunteers. The participating teacher was selected to ensure that he had a strong mathematics background, had experience teaching the Algebra 12 course, and was regarded within his school as being a competent teacher⁴. It was also necessary that the class to be studied was accessible for this investigation (i.e. location, scheduling of the logarithms unit, accommodation of the researcher by the teacher and school administration).

³ The authors Guba and Lincoln use the terms "naturalistic" or "naturalist" (Lincoln & Guba, 1985), and later "constructivism" or "constructivist" (Guba & Lincoln, 1989) to refer to the same paradigm. In the references to the work of these authors, these two sets of terms are taken here to be synonymous. (For more information on this point refer to Guba & Lincoln, 1989, p. 19.)

⁴ While the gender of the teacher who was selected to participate in the study was male, this was not a criterion for selection.

Prior to the teaching and my observation of the exponentials⁵ and logarithms unit each student in the class was given the opportunity to choose the degree to which he or she would participate in the study. Parent or guardian consent was also obtained for those students under 18 years of age (the legal age of consent). All of the students in the class agreed to participate in video taping of the classroom proceedings during the unit, and all but two indicated their willingness to participate in the subsequent interview process if selected to do so.

Selection of the two students, about whom the case analyses were constructed, was based upon an interim analysis of the information available during the field work. This was to ensure that these participants had learned something of the mathematics content covered during the unit, and that together they were likely to provide a good deal of information during the interview process.

Field work

Information was collected from a variety of sources during the study. Results from the ongoing analysis of this information served to continually refocus my interactions and information collection with the participants, and to contextualize subsequent events for me. An overview of the events, information sources

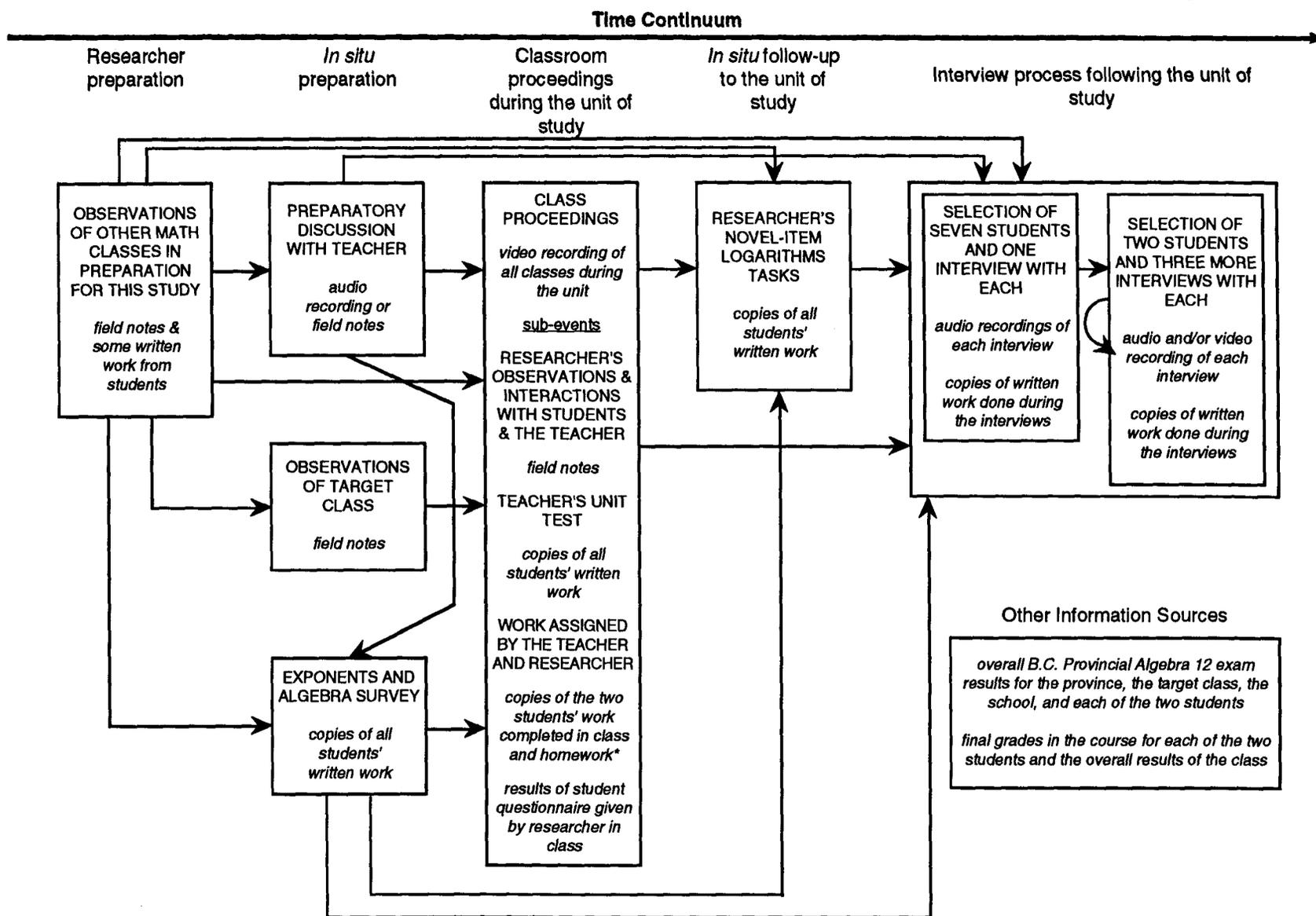
⁵ The Algebra 12 course of study (British Columbia Ministry of Education, 1978) and the classroom teacher who participated in this study used the term "exponential" to refer to powers and exponential functions.

and the process of event contextualization during field work is provided in Figure 3.1. Each box in this figure signifies an event or set of events pertinent to this study. Information sources used in the analysis are indicated in lower case letters, and each arrow on the diagram indicates where one event or set of events has served to contextualize a subsequent event or set of events in the analysis process. A more detailed description of these events and this process follows in this section. Upon completion of the field work for this study, the information from all sources was analyzed in concert to construct the case studies.

Researcher preparation

To prepare for the field work with the target class in this study, I made preliminary observations in a number of British Columbia Lower Mainland secondary school mathematics classrooms. During this time I refined the focus of the research topic, developed a set of novel-item logarithm tasks, piloted these in two Algebra 12 classes, and rehearsed observation and interview techniques. Field notes were also kept during this process and students' work on three different sets of the novel-item logarithms tasks was collected. This set of events served to familiarize me with some of the views relating to mathematics, mathematics learning, and the topic of logarithms that secondary level mathematics students construct. I was also able to draw upon my own experience as a secondary school mathematics teacher in preparation for this study.

Figure 3.1 Events, information sources, and the process of event contextualization for the researcher during field work



***NOTE:** Copies of the two students' work and notes completed during the course of the unit were obtained after they had been selected for the final round of interviews.

In situ preparation

Upon selection of the participating teacher and the target class, a number of activities served to prepare the participants, including myself, for this study. Informal discussions were held with the classroom teacher to develop a rapport and to negotiate a productive working relationship for the duration of the enquiry. The teacher's assumptions regarding the students' prerequisite knowledge for the logarithms unit, and his views relating to the teaching of the unit, the intended learning outcomes for the unit, and the anticipated areas of difficulty for the students were also discussed. Details of these discussions will follow later in this chapter. Collectively, the teacher and I also decided upon our modus operandi for use during the unit to facilitate the information collection process and to ensure a minimum of disruption for the class. For example, it was decided that I would videotape the class proceedings from the back corner of the room and that discussions between the teacher and individual students during the seat-work part of the lessons would not be recorded, the teacher would address students during the course of the lessons using their first names so that their identities could be determined readily from the audio and video recordings, and the teacher would try to get the students to repeat themselves whenever their questions or answers during the lessons were not clear, so that the dialogue could be understood from the audio and video recordings. Plans were made regarding the scheduling of the exponents and algebra survey that was to be completed by the students during class time prior to the

logarithms unit. An ongoing dialogue with the teacher continued throughout the study. This enabled me to draw upon the teacher's experience and perspective as I came to know the class. For example, it was during these discussions that the teacher indicated that there were two "Honours" Algebra 12 classes specifically for students who scored highly in the Grade 11 mathematics course, and that this had the effect of skimming off the better students from the remaining few Algebra 12 classes at the school (including the target class), that the students were "a little shy" on the first day of the video taping, but by the second day they were responding as usual in class, and that it was a major goal of the teacher to have the students "verbalize" the rules for logarithms during the lessons on this topic.

I was introduced to the class as a "student" from UBC doing a project, and my observations of the class began approximately one week prior to the start of the unit on exponentials and logarithms. This served to acquaint me with the class, and I developed a friendly rapport with many of the students in the class within a matter of days.

Finally, a paper-and-pencil exponent and algebra survey with 35 tasks including three tasks that could not be solved was completed by all of the students in the class prior to the start of the unit. This set of tasks was intended to serve as a general survey of individual students' background mathematics knowledge prior to their study of the logarithms unit, knowledge which the teacher considered as prerequisite for this new topic. In addition to finding the answers for this set of items, the

students were required to indicate their sense of familiarity with each type of task. The students were also told that some of the questions could not be solved and for these they should write "cannot be solved." Copies of the two students' completed exponents and algebra surveys can be found in Appendix 1.

Classroom proceedings during the unit of study

The teacher proceeded to teach the exponentials and logarithms unit in his usual manner while I videotaped the classroom proceedings and recorded the class dialogue on audio tape. I did not intervene in the teaching process in any manner other than to videotape the teacher and participating students during the classroom activities relating to this unit of study. My field notes from this process included a record of the teaching materials and blackboard notes from class and the material assigned from the textbook. An episode from the classroom video recording was selected for use in the follow-up interviews with individual students to facilitate discussion with them about their perspective of the classroom proceedings during the unit.

The students appeared to be comfortable with my presence in the classroom and they appeared to interact with me on a more casual basis than they did with their regular mathematics teacher. In comparison with my first few days in the classroom prior to the videotaping, I noticed no appreciable difference in the students' interaction with the teacher, nor a change in the students' behaviour during the taping in the classroom.

Furthermore, the students' behaviour during the study was not unlike that in the many other secondary school mathematics classrooms that I had observed in my career as a mathematics teacher.

On day ten, the final day of the unit, the teacher administered the regular unit test. Photocopies of each student's test paper were retained for future reference. Copies of the two target students' unit test papers can be found in Appendix 2.

In situ follow-up to the unit of study

In the class which immediately followed the completion of the unit test, each student completed the logarithms survey which I had prepared. Calculators were not permitted for the survey. Copies of the two target students' work on this survey are included in Appendix 3. The particular items on this survey were selected on the basis of the results from previous field trials which I had conducted with Algebra 12 students at another school, as mentioned earlier. These items required the use of the concepts that the teacher had covered during the unit, but in written tasks that were different from those which the students had encountered in class or in the homework that had been assigned. In the preliminary trials with these tasks, students were found to respond readily to most of these tasks, and to provide a variety of responses in doing so. Like the exponents and algebra survey, the logarithms survey also contained a small number of tasks that could not be solved and the students were

instructed to indicate this when they thought it was fitting. The results from the logarithms survey served to complement the results from the teacher's regular unit test in providing evidence of individual students' views of logarithms for subsequent analysis and follow-up discussion in the interviews.

Interviewee selection and the interview process following the unit of study

Ten students in the class were not considered for the interviews because they had been absent for one or more of the nine teaching days of the exponentials and logarithms unit. Two of the remaining students had indicated from the outset of the study that they did not wish to be considered for the interviews, and one other of the remaining students was not considered because he was repeating the Algebra 12 course. This left a group of 13 students from which the researcher selected seven students for the first round of interviews at the end of the unit. Students were selected who had some knowledge of logarithms as demonstrated by their participation in class during the unit and a passing grade on the unit test given by the teacher, and who, as a group, demonstrated a wide range of achievement on the unit test. These criteria were intended to provide me with the opportunity to interview students with a variety of understandings relating to logarithms, while ensuring at least a minimum level of student content understanding, so that we could discuss their understandings of logarithms during the interviews.

The first round of interviews was conducted at the school on

the Monday and Tuesday following the unit test (the previous Friday) and the researcher's logarithms survey (written in class on the Monday morning). Each of these interviews lasted approximately 30 to 45 minutes.

The objective of the first round of interviews with each of the seven students was to determine some of the views that these students had developed about logarithms, and to select two students for the subsequent round of interviews and case studies. A common set of items from the unit test and the logarithms survey, upon which the students had revealed a wide range of performance, was used to initiate episodes of dialogue with each of them during the interview. This, in part, was to facilitate my comparison of these students. The students were instructed to explain, as best they could, their original solutions to these items and reasoning for their answers. They were not shown their original work on these items during this interview. I interjected mostly to seek clarification from the students or to prompt them to keep talking during each episode. In this way, the students were able to explain themselves thoroughly and direct the course of the conversation. I endeavored to probe each student's conceptions further by presenting them with additional mathematics tasks and related questions based upon the students' initial responses, in a systematic manner to clarify my own understanding of each student's understanding.

The final selection of the two students for further in-depth investigation was based upon the results of the first interview. Two students were selected who:

- 1) had been successful in developing some knowledge of the logarithms content presented in class,
- 2) between them exhibited differences in their knowledge of logarithms and their approach to learning mathematics,
- 3) were accessible to the researcher for the subsequent series of interviews, and
- 4) demonstrated an ability and a willingness to reflect upon and explain their understandings freely and in detail during the interviews.

This last point was vital because, given the limited amount of time that was available to spend with these students, it was most important that the interviews provide me with a rich source of useful information about the students' views.

The second to fourth rounds of interviews were held during the six weeks which followed the end of the exponentials and logarithms unit in class. Each interview lasted from 35-75 minutes and all but one interview was conducted at the school. The exception was the second part of James' second interview⁶, which was conducted at his home.

During the second round of interviews, work from the unit test and the logarithms survey which had not been dealt with in the first interview and parts of the pre-unit exponents and algebra survey were discussed. In these interviews the students were given photocopies of their original unit test (graded) and their original exponents and algebra survey, and logarithms

⁶ The student James, one of the final interviewees, had to excuse himself from the second interview before I was able to complete the planned agenda for the meeting. A follow-up interview was held a few days later to complete the discussion. For this reason, while James participated in a total of five interviews, the numbering of these interviews and the materials which correspond to these interviews are identified by the numbers I, IIa, IIb, III, and IV in the description of his case. Keri, the other student in this study, participated in the planned four interviews (I, II, III, and IV).

survey papers (ungraded), to which they could refer. In addition, questions about the students' conceptions which arose from a preliminary analysis of the first interview were pursued, using related mathematics tasks which I had selected. In both of these cases the students were observed as they talked through their work, and I prompted them for further clarification when I deemed it necessary. The selection of the follow-up mathematics tasks used in the second round of interviews was based upon my analysis of the mathematics content knowledge needed to complete the individual items from the unit test and the logarithms survey. New tasks were introduced, including some generated in situ, which isolated specific aspects of the original test and survey tasks so that each student's performance could be observed in a variety of situations and discussed with them. No attempt was made to change the way that the students worked as they completed the mathematics tasks at any time during the interviews, although it was inevitable that the students would, at times, recognize and even resolve some of their own impasses as a result of their reflection during these discussions.

An excerpt from the video recordings of the classroom teaching during the unit was selected for use during the third round of the interviews. The selected excerpt, from day seven of the unit, was of the teacher taking up a set of logarithm review questions at the blackboard after the students had been given some time to work on these tasks at their seats. These tasks dealt with most of the logarithm ideas that had been covered by the teacher to that point in the unit. In the interview this

excerpt served as the basis for discussion with each student about his or her individual perspective of the particular classroom activity shown in the excerpt, and how he or she directed his or her own activity during this teaching/learning episode. The students were asked to explain their own perspective of the video episode, and not to worry about trying to remember exactly what they were thinking during the original lesson. The students were encouraged to comment whenever they wished, at which time the video playback was paused. Occasionally, the videotape was rewound and a part of the video recording re-viewed. The logarithm questions that the teacher was taking up in video recording, and the solutions to these, had been written out on paper ahead of time to facilitate the discussion of these during the interviews. The subsequent discussions in the third round of interviews stemmed from the students' initial comments during their viewing of the video recording.

The fourth round of interviews served as a follow-up to the first three interviews and dealt with a pot pourri of different topics in an attempt to clarify and explore further the students' intended meanings from the earlier discussions. Direct and indirect questioning was used to initiate episodes of dialogue relating to the students' conceptions of mathematics and mathematics learning, and additional points which stemmed from this discussion were also pursued. To investigate their conceptions of learning mathematics, for example, the students were asked to describe what they would do if they had to teach a

mathematics class, and under what circumstances they learned mathematics best themselves. To investigate their conceptions of mathematics and their conceptions of knowing mathematics, for example, the students were asked to explain what they thought the differences would be between the understandings of mathematics held by 1) a mathematician, 2) a secondary school mathematics teacher, and 3) an Algebra 12 student.

Throughout the interview process with each of the students, written work done by the students and my own written work used in the discussions with them was retained, and each interview was either audio- or videotaped. After the two students for the final round of interviews had been selected, their completed written work and notes from class for the unit were obtained and photocopies retained for further reference.

Other information sources

At the end of the school year, the provincial exam grades and term grades for each of the two students who participated in the complete round of interviews, and the corresponding means for the Algebra 12 population of the target school as a whole, were obtained from the teacher of the class. The province wide Algebra 12 exam mean grade was also obtained.

Follow-up data analysis

Upon completion of the field work, detailed verbatim transcriptions were made from the audio and video recordings for all of the interviews with each of the two students who

participated in the set of four interviews. General outlines of the events in each class during the unit were also made from the classroom recordings. These outlines included a summary of what was taught by the teacher, cross-referenced with the notes which I had copied from the blackboard during each of the lessons, a record of many of the questions from students pertaining to the mathematics content, and the teacher's responses to these questions. The interview transcripts served as the primary source of data in the final analyses. These were supplemented by field notes taken from the discussions with the teacher; the field notes taken from the classroom observations; copies of the two students' unit tests; copies of the two students' exponents and algebra survey, and logarithms surveys; copies of the two students' written work from their notebooks; and the written work done by the two students during the interviews.

As mentioned at the beginning of this chapter, the analysis of the data in this study was based on principles of naturalistic inquiry (Lincoln & Guba, 1985; Guba & Lincoln, 1989). A key feature of the process was a form of constant comparative analysis (Strauss, 1987; Lincoln & Guba, 1985) which was used in the analysis of the interview texts. Constant comparative analysis is a continually developing and reiterative process of analyzing data, constructing categories from the data, reanalyzing the data, reconstructing the categories, and finally the construction of the description of the case(s). In this study it involved:

- 1) identifying episodes within the text of potential use in the analysis,

- 2) categorizing and coding the episodes identified in step 1,
- 3) writing a preliminary analysis/description of the categories generated from step 2 in the form of specific assertions about specific views held by each of the student participants,
- 4) a recursive process of comparing and contrasting the assertions and the corresponding data from step 3, including the analysis of the interrelations between assertions in an effort to further delineate, integrate, and organize them,
- 5) examining the secondary data sources for supporting or conflicting evidence,
- 6) rewriting a description of each student's views, including a description of the interrelations between them,
- 7) checking the analysis in step 6 by re-examining the original interview transcripts, for the purpose of checking for contradictions to the assertions made,
- 8) the construction of the final descriptions of the students and, finally,
- 9) writing the case descriptions with supporting evidence from the data.

To illustrate this process, an example of a category that was constructed and later integrated with other categories was the category "testwiseness" for the student James. James spoke in detail about his approach to writing the teacher's logarithms test, including his analysis of the test questions to determine which items would be straightforward to complete and which items were worth the most marks so that he would not "waste his time" on items worth only a small number of marks and that he didn't know how to do right off. Subsequently, I concluded that James viewed mathematics and understanding mathematics, in general, as the straightforward application of rules with a minimum of

thought and effort. His expectation while writing his test, to get answers quickly, was only a part of this larger picture and thus not a separate feature of his conceptions of mathematics and knowing in mathematics.

A new category, the students' conceptions of self in mathematics (which corresponds to research question number four) emerged from the analysis after the analysis process was well underway. While this had not been a focus during the information collection process in the field, there was a significant amount of information relating to this topic to warrant its development as a theme in the final analysis.

Trustworthiness in this study

A number of features of this study and the report of this study have been designed to help establish its trustworthiness⁷. To establish "credibility," or the viability of the match between the constructed realities of the individuals being investigated and those realities as represented by the researcher and attributed to those individuals (Guba and Lincoln, 1989), this study has employed a variety of information sources, a contextualization process for the researcher during the field work, and the use of the constant comparative method in the construction of the case descriptions. To facilitate the "transferability" of the results from these cases to other cases, or more specifically, "to provide a sufficient base to permit a

⁷ A detailed description and rationale for establishing trustworthiness in constructivist inquiry is provided by Guba and Lincoln, 1989.

person contemplating application in another receiving setting to make the needed comparisons of similarity" (Guba and Lincoln, 1989, p. 359-360), a detailed description of the cases is provided. Furthermore, the video- and audiotapes, and written records used in this study provide a permanent record of the information sources used, so that the claims from this analysis may be verified to ensure that they are rooted in contexts and persons apart from this researcher. This satisfies the trustworthiness criteria of "dependability" and "confirmability" as described by Guba and Lincoln (1989).

The context of the cases

This section will provide background information relating to the cases to be presented in chapter four. This includes a description of the researcher's beliefs concerning the learning of logarithms; information about the Algebra 12 course, the particular unit of study, the target school, the target class, the physical setting of the class, and the participating teacher; consideration of the researcher-phenomena interaction during the data collection; and a description of the implementation of the exponentials and logarithms unit by the teacher. The purpose of this information is to help situate the cases and the analysis for the reader.

The researcher as an instrument in naturalistic research

A most significant feature of naturalistic research is recognition of the researcher as an instrument in the information gathering process and the construction of a case. Therefore, it

is useful for the researcher's beliefs to be explicitly stated as part of the analysis, so that the reader can consider their significance.

In addition to the description of the constructivist paradigm outlined both in chapters two and three, this researcher holds a number of beliefs about knowing mathematics which have guided this inquiry. First, it is my belief that the most important objective of mathematics at school is to develop students' autonomy as thinkers and problem solvers. Second, and pursuant to the first belief, it is my view that students (ideally) would have learned to do the following from an introductory unit on logarithms, such as the one in Algebra 12:

- 1) recognize and apply algebraic patterns for logarithms as generalizations of numeric cases,
- 2) understand patterns or interrelationships between different aspects of logarithms;
- 3) apply a "number sense" about logarithms and logarithmic functions, including being able to estimate the values of various forms of logarithm expressions without a calculator;
- 4) evaluate expressions in which logarithms are integrated with other mathematics concepts both with and without a calculator;
- 5) understand interrelationships between logarithms and other mathematics concepts, most notably exponential equations;
- 6) communicate effectively about logarithms using conventional terminology; and
- 7) apply/use logarithm concepts flexibly in problem solving applications in both familiar and unfamiliar contexts.

It was on this basis that I assessed the students' learning of logarithms in this study.

The Algebra 12 course

The Algebra 12 course (British Columbia Ministry of Education, 1978) was designed for those students who intended to pursue mathematics at the post-secondary level. On average, approximately 40 percent of the grade 12 students in British Columbia enrolled in Algebra 12. The content of the course included: 1) conics and quadratic systems, 2) exponentials and logarithms, 3) polynomial and rational functions, 4) sequences and series, 5) complex numbers, 6) systems of linear systems in three variables, 7) quadratic relations, and 8) trigonometry.

The exponentials and logarithms unit of study

This study focused on the exponentials and logarithms unit of the Algebra 12 course with a particular focus on logarithms. The target class covered this unit during the fourth and fifth weeks of the course immediately following three weeks on conics and quadratic systems. "Behavioral objectives" for the exponentials and logarithms unit were outlined on a hand-out given to the students by the teacher and discussed with them in class. These objectives are listed below.

Students should be able to:

- 1) convert terms in exponential form to radical form and evaluate,
- 2) convert terms in radical form to exponential form and simplify using the laws governing real exponents,
- 3) solve equations of the form $x^{a/b} = c$,
- 4) simplify expressions involving rational and irrational exponents,
- 5) graph exponential functions of the form $y = b^x$ (b does not equal zero or one), and describe their

properties,

- 6) solve exponential equations with special bases,
- 7) convert from logarithmic to exponential form in order to determine the base, logarithm, or argument of a logarithmic equation,
- 8) understand intuitively the meaning of a logarithmic expression,
- 9) graph a logarithmic function and understand its relation to its accompanying inverse exponential function,
- 10) determine the common logarithms of positive real numbers using a calculator.
- 11) determine the antilogarithm of a real number using a calculator,
- 12) express the fundamental operations of multiplication, division, and exponentiation in terms of logarithmic operations,
- 13) solve logarithmic equations using the logarithmic properties of multiplication, division and exponentiation,
- 14) express logarithmic equations in terms of ordinary fundamental operations which do not employ logarithmic expressions,
- 15) solve logarithmic equations,
- 16) solve exponential equations.

(Students were expected to use scientific calculators during this unit.)

The target school and class

This study was conducted in the spring semester of 1990 at an urban secondary school in the Lower Mainland of British Columbia. The teacher of the target class described the population served by the school as lower middle to middle class. The school had a student population of approximately 1,200.

The target class consisted of 26 students, 12 girls and 14 boys. Three of the students were repeating the course for a second time and none of the students in the class appeared to have any difficulty with English, the language of instruction.

Previously, all of the students in the Grade 11 mathematics classes at this school had written the same unit tests and final exam. Therefore, some degree of equivalent mathematics content coverage prior to Algebra 12 could be assumed for all of the students in the target class. There was also some degree of equivalence with respect to the students' prior achievement in that there were two concurrent Algebra 12 honours classes at the school for those students who scored a high B or an A in the Algebra 11 course. The teacher suggested that the results from the first unit test (conics) also supported this conclusion--only one student achieved an A standing on this test.

The physical setting for the class

The classroom was arranged in rows of single desks facing a wall of blackboards, with the teacher's desk at the front of the room. There was a bulletin board by the teacher's desk for "school business." The British Columbia provincial mathematics test results from the previous year were posted on this bulletin board as well as a list of tutoring in mathematics available to students both in and out of the school. Also posted was a notice from a local community college advertising a workshop to help Grade 12 students prepare for their British Columbia provincial exams. The heading "Good provincial exam results are important"

was displayed most prominently on this notice.

The teacher's background in teaching

At the time of the study, the teacher from whose class the student participants were chosen had over 20 years of experience as a mathematics teacher, had taught the Algebra 12 course for 17 years, had served the British Columbia Ministry of Education for many years on committees relating to the annual provincial Algebra 12 examinations, and had been the Mathematics Department head at his present school for the previous eight years. Prior to joining this school, the teacher had completed an undergraduate degree in mathematics, had taught mathematics at both the elementary and secondary levels in a number of school boards in the province, and had completed graduate course work in mathematics education. This teacher was regarded highly by the school principal and was most willing to participate in the study.

The teacher's perspective of teaching the exponentials and logarithms unit

The teacher expected Algebra 12 students be able to do the following from their previous mathematics courses: evaluate exponential expressions including those with integer exponents and fractional bases, solve algebraic equations up to and including quadratics, graph equations, apply transformations to the equations and the graphs of functions including translations and reflections over each of the axes and the line $y = x$, and apply the exponent laws. The teacher reported that the students

would need some review of the graphing of inverse functions by reflecting a given function over the line $y = x$ because only a small amount of time had been devoted to this in Algebra 11.

The teacher also indicated that he thought the hardest part of logarithms for the students would be the combination of "concepts such as $\log_a a^a = \underline{\quad}$." In contrast, he expected the easiest part of logarithms to be "anything with rote manipulation."

It was the teacher's view that learning mathematics provided students with the opportunity to develop deductive and inductive reasoning skills, and that the purpose of learning logarithms, in particular, was to enable students to apply these concepts in calculations in science courses. It was also his view that this topic was of medium difficulty for Algebra 12 students relative to the other topics in the course.

The teaching of the exponentials and logarithms unit

The lessons during the exponentials and logarithms unit were very similar from day to day. Therefore, a detailed description of a typical lesson is provided to give the reader some sense of the students' classroom learning experience. The lesson selected was from Day 6 of the unit. It was atypical only in that the teacher spent the first 30 minutes of the 75 minute period explaining the answers and marking scheme of the previous unit test while the students followed along with their graded test papers. The 45 minute lesson on logarithms which followed will now be described.

The teacher began the lesson by writing two questions on the

blackboard as follows:

- 1) Locate the two integers between which the logarithm in base 3 of 537 must lie.
i.e. $\underline{\quad} < \log_3 537 < \underline{\quad}$
- 2) If $\text{Log } m = 5$ and $\text{Log } n = 4$, find
 - a) $(\log m)(\log n) =$
 - b) $\log (mn) =$

The students worked quietly and individually at their seats on these tasks for a few minutes before the solutions were explained in detail by the teacher at the blackboard. This was followed by the teacher giving the answers for the homework aloud and then explaining the solutions to the following three homework questions in detail on the blackboard:

State the logarithmic equation you would use to compute... the following:

$$N = (-0.08125)(0.2317)$$

Determine the solution set of each equation.

$$\log 9x + \log x = 4$$

$$\log_6 n - \log_6 (n - 1) = \log_6 3.$$

The teacher then introduced the new topic for the day, "The logarithm of an exponential," using the example

$$N = 100^3.$$

By rewriting 100 as 10^2 , and then writing each side as a base ten exponential, the teacher demonstrated the pattern for the logarithm of an exponential to the students by writing the following sequence on the blackboard;

$$\begin{aligned} N &= 100^3. \\ &= (10^2)^3 \\ 10^{\text{Log } N} &= (10^{\text{Log } 100})^3 \\ 10^{\text{Log } N} &= 10^3 \text{Log } 100 \\ \text{Log } N &= 3 \log 100 \end{aligned}$$

$$\begin{aligned}
 \text{Log } N &= 3 \quad (2) \\
 \text{Log } N &= 6 \\
 N &= 10^6 \\
 N &= 1\,000\,000
 \end{aligned}$$

During this process the teacher asked the question, "What does the rule of exponents say about raising one power to another power?", and then stated that, "Any rule that works for exponents just happens to work for logarithms." No further discussion of the relationship between the logarithm rules and the exponent rules ensued, nor was this discussed at any other time during the unit. The teacher then wrote the logarithm rule on the blackboard for students to copy into their notes, "The logarithm of an exponential is equal to the product of the exponential and the logarithm of the base." The solutions for two related examples were also presented:

$$N = (38.6)^4 \quad \text{and} \quad N = \sqrt[3]{(38.6)^5},$$

and finally 20 questions from the textbook were assigned for homework in the final minutes of the period. Each of the assigned homework questions required students to determine and simplify the logarithm equation that would be used to evaluate a number raised to an exponent, such as:

$$\sqrt[5]{79600}, \quad (0.04881)^{1/6}, \quad \text{and} \quad (-.0.009043)^7.$$

The teacher insisted that students include the following three steps in their answers: 1) Set the expression equal to the variable N , 2) convert the equality in step 1 into a logarithmic equation by making each side the argument of a common logarithm, and finally 3) simplify the side of the equation containing the logarithm of an exponential by applying the rule given for the logarithm of an exponential, as he had done in the examples on

the blackboard.

The teacher worked through the teaching examples at a brisk pace, talking through each step of these examples on the blackboard by asking questions such as: "What am I asking?", "What kind of equation is this?", "This $[\log mn]$ is the log of a ... [waiting for a student to give the missing word.]?", "How do you say them again?", "Can the logarithm of a number be negative? ... The argument can't be negative, what else can't be negative?", and "What does the rule of exponents say about raising one power to another power?" Most of these questions required single word or short answers which individual students called out. At other times groups of students would respond in unison. The teacher emphasized the conventional mathematics terminology and rules relating to the content being covered. In this particular lesson the terms: characteristic, mantissa, common logarithm, logarithm of product, and product of logarithms were stressed; and the rule, "The log of a product is equal to the sum of the log of its factors," was repeated many times. The teacher also stressed the importance of distinguishing between different types of expressions, for example: "This [in reference to $(\log m)(\log n)$ written on the blackboard] is not the log of a product, this is the other way around--product of logs. They're different, ...very different, yes?"

Another feature of the lesson was the teacher's warnings against the potential errors that students could make with the content being covered. The teacher's comments to the class in this regard included, "I don't want you to do what?", and "Do you

know why I'm asking you? Because I don't know if you'll know what to do. Half the people in the room might solve that by doing what?"

It is important to note that the responses given by students during the lessons, typically short answers in response to direct questions from the teacher, revealed little of their sense making of this mathematics content. Similarly, most of the questions from the students dealt with procedural aspects of the work being demonstrated on the blackboard by the teacher.

The rapport between the teacher and students was positive, and the students seemed to accept that the teacher was in firm control of the class. There were no behaviour problems in the class, and the teacher's instructions were followed promptly and accepted willingly.

The coverage of content during the unit

A summary of the new content that was covered during the unit is provided on a day-by-day basis. Logarithms were introduced on the fourth day of the unit after three days on exponentials. The unit test followed on Day 10.

Day 1: Fractional exponents were introduced, including how to switch expressions with fractional exponents from exponential form to radical form, and how to evaluate these. Emphasis was placed on the terms: radicand, root index, and root indices.

Day 2: Solving exponential equations was introduced including equations which resulted in a quadratic equation.

Day 3: Exponential functions of the form $y = 2^x$ were introduced, including the graphing of these; the constraints on the base, the domain, and the range of these functions; and the implications of simple transformations of, in the teacher's words, "the mother equation" ($y = 2^x$) for both the equation and the graph. The teacher stressed the usefulness in graphing exponential functions by starting with the "mother function" and then applying transformations to this, one after another, to obtain the required curve on a graph.

Day 4: Logarithms were introduced by the teacher presenting a number of logarithms and their values and having students guess the pattern. He then explained that logarithms were exponents and that to evaluate a logarithm, one would set the given logarithmic equal to a variable, rewrite the logarithm as an exponential equation, and then solve for the unknown. Logarithmic functions were described as the inverse of exponential functions and the steps for graphing logarithmic functions by first writing their equivalent exponential "mother function" and then reflecting this over the line $y = x$ were demonstrated. The teacher also identified the domain, the range, and the constraints for the base of a logarithmic function with reference to the graph.

Day 5: Common logarithms were introduced, including logarithmic expressions with negative values, and how to estimate the values of logarithms with non-integer arguments by rewriting them as exponentials. The rules for finding

the logarithm of a product and logarithm of a quotient were also introduced as well the evaluation of logarithmic expressions involving a sum or difference of logarithms.

Day 6: The rule for finding the logarithm of an exponential, including the exponential of a radical was introduced. The terms characteristic and mantissa were emphasized.

Day 7: Combined operations in logarithmic expressions including the order of operations for these types of expressions were introduced. The students were also given review sheets with over forty questions to start preparing for their unit test.

Day 8: The use of logarithms to solve exponential equations was introduced and the teacher demonstrated the application of this using the compound interest formula. Evaluating logarithms with bases other than ten using common logarithms was also introduced.

Day 9: Solving logarithmic equations with bases other than ten was introduced. The teacher also read through the "Core Objectives" handout that had been given to the students at the beginning of the unit and correlated each of the objectives with specific items on the review sheets for the students. A few items from the review sheet were then taken up on the blackboard.

Student evaluation in the Algebra 12 course

The most prominent feature of the student evaluation in the Algebra 12 course was the province-wide final exam which determined 50 percent of each student's final grade. The teacher reported that each of the unit tests (which together determined the students' term mark) had been designed to help prepare students for the final exam. In particular, the proportion of multiple choice and open-ended questions was selected to match that of the final exam, and the types of individual questions and the time given for these was similar to previous provincial exams.

CHAPTER FOUR: THE CASES

This chapter contains the case analyses for two Algebra 12 students, James and Keri (pseudonyms). It begins with a short description of the students to help situate them for the reader. Each of the three sections which follow detail the students' conceptions relating to one of the research questions. These sections begin with a general description, in the form of an assertion, about the conceptions of both students, followed by supporting evidence from the field data and a more in-depth discussion of the specific conceptions for each of the individuals. The next section of the chapter deals with the interrelation of the students' views and conceptions by examining the ways in which their views support one another. Finally, the differences between James' and Keri's conceptions will be summarized.

Quotations taken from the interview transcripts are identified within this analysis by a Roman numeral to indicate the particular interview with that student, followed by an Arabic number to indicate the page number from the interview transcript. Some of the verbatim quotations have been edited, for example to improve upon the original grammar, omit repetition, or omit the researcher's interjections which were intended to keep the student talking. The purpose of this editing has been to make the quotations easier for the reader to follow. The use of three ellipsis points (...) indicates that the interview dialogue went on to some other topic briefly, and that this has been omitted.

The interpretations presented in these cases are best thought of as tentative working hypotheses, and each case is necessarily limited by the information that was collected. Excerpts from the interviews are provided to serve as examples only, and were not in themselves the only source of information upon which the conclusions are based. Also, it should be noted that the interrelated nature of the research questions resulted in some overlap of the supporting information presented and the related discussion.

Background of the two students

At the time of the study, James was about to turn 18 years of age, and was planning to enter college the following September with the goal of completing a university degree. James had no plans regarding a specific post-secondary program. He worked part-time at the local McDonald's Restaurant where he was a crew-trainer.

Mathematics and sciences were a prominent part of James' Grade 11 and 12 program. He reported that he had achieved a grade of 82% (a high B standing) in Algebra 11 with little difficulty and he was easily maintaining an A in Grade 11 Business Mathematics which he was taking concurrently with Algebra 12. James was also maintaining a B average in his science courses (over the two years) which included Biology 11, Earth Science 11, Physics 11, Chemistry 11 and 12, and a computer course at the Grade 11 level.

James chose to take Algebra 12 because he was pleased with his achievement in Algebra 11 and thought that Algebra 12 would be similar. While he did not expect to use the mathematics that he learned in Algebra 12 later on, he expected that success in Algebra 12 would look good on his report and indicate to others that he was a hard worker (IV, 35). James was surprised to find that Algebra 12 was as difficult as it was and was disappointed with his mid-semester grade of C-minus. During the study he reported that it was his goal to attain a 70% (letter grade C-plus) in the course.

Keri was 17 years and 5 months old at the time of the study. She had no specific plans beyond Grade 12 other than to take a year off from school after which she planned to return to some form of a post secondary program. Keri also worked part-time at a local community recreation centre where she served as a receptionist and cashier.

Mathematics and science were less prominent in Keri's Grade 11 and 12 program in comparison with James' program. Her mathematics related courses included two science courses--Biology 11, and Physics 11, and a Grade 11 computer course. She reported that she had achieved a B standing in the Algebra 11 honors class with little difficulty.

Keri chose to take Algebra 12 because she enjoyed learning mathematics. In particular, she liked the challenge that learning mathematics offered--she especially liked figuring things out, and she was interested in learning about new topics (I, 35). Keri's interest in doing mathematics was evidenced by

her persistence in continuing to reflect upon points of difficulty outside of class and after each of the interviews during this study in an attempt to resolve these for herself. As a result, her facility with logarithms continued to develop over the course of this study. Furthermore, she stated that the exponents and algebra survey completed in class prior to the logarithms unit was a useful indicator of what she remembered from Algebra 11 and of what she would have to "try harder to remember" (II, 6). Keri reported that Algebra 12 was much more difficult for her than Algebra 11.

In comparison with the other Algebra 12 students at their school, James' achievement at the end of the semester was average. His provincial Algebra 12 exam mark of 67% was 0.1 standard deviations below the mean for the school, and his term mark of 66% (calculated from the results of seven unit tests) was 0.07 standard deviations below the school mean. For further comparison, his exam mark was 0.2 standard deviations above the mean for the entire province. Keri's achievement at the end of the semester was slightly below average. Her provincial Algebra 12 exam mark of 62% was 0.24 standard deviations below the school mean, and her term mark of 60% (calculated from the results of seven unit tests) was 0.30 standard deviations below the school mean. Her exam mark was 0.08 standard deviations below the mean for the entire province. These scores were reported to the researcher by James' and Keri's teacher after the conclusion of the semester. During the study the teacher reported that James and Keri were two of only eight students in the class who had

completed all of the assigned homework during the unit on exponentials and logarithms. James' unit test result of 63% was close to the class mean of 65% while Keri's result of 80% was the highest grade achieved in the class, with the exception of a single higher grade by a student who was repeating the course.

The students' conceptions of mathematics

Assertion 1: The students viewed mathematics as a set of truths, more specifically procedures and rules, which were handed down to them by their teacher for use to answer "questions" in mathematics. Mathematics knowledge was undifferentiated from other kinds of knowledge which students learned at school--unconnected truths to be memorized, with the exception that this knowledge was applied to get answers instead of just reported back on tests, as was the case in their other subjects at school. Finally, the students viewed their taking of mathematics at the Algebra 12 level as serving purposes outside of the field of mathematics.

James' conception of mathematics and the evidence of this in his learning of logarithms

James believed that mathematics was a set of "truths" in the form of procedures and rules which increased in difficulty as one progressed in mathematics, and that these were passed on from mathematics teachers to their students. This became evident in a discussion about the mathematics knowledge of a mathematician and that of a high school mathematics teacher. When asked to speculate about the mathematics knowledge that a mathematician--a person who could invent mathematics--would have, James stated that "the mathematician should know everything..." or in other words, know all of the rules (IV, 26). When asked to speculate about the mathematics that a secondary school mathematics teacher

would have to learn at university prior to becoming a teacher, James answered that a teacher would have to know "really tough and hard" topics so that he or she would be able to teach the easier ones such as logarithms (IV, 27). For this perspective it would be inappropriate for a student to question or challenge the ideas presented in class by the teacher who is mathematically superior because of his or her experience with tough and hard topics. Alternative perspectives in general, and the views of students in particular, had little place in mathematics for James.

James' sense of mathematics, as rules and procedures passed on to students from the teacher, was evident in his discussion of logarithms. Most prominent was the set of formulas or rules that he used for evaluating the logarithm of a product, logarithm of a quotient, and logarithm of a power. James repeatedly made reference to these rules as he worked through logarithm tasks during the interviews, for example: "First I think of those rules that the teacher gave us..." (I, 9), "Cause it all comes back to those rules..." (I, 10), "I remember there's another rule about..." (I, 10), "I think that's what the rules are, if I got them right" (I, 10), and "those rules that he's [the teacher's] got" (III, 9).

An analysis of James' work on the unit test, the researcher's logarithms survey, and his discussion of these and other logarithms tasks during the interviews indicated that he relied almost exclusively on a small set of rules or procedures to evaluate expressions containing logarithms. These included

procedures for evaluating expressions containing a single logarithm, both with and without a calculator, and his approach for evaluating expressions with more than one logarithm.

When required to evaluate an expression containing a single common logarithm, James expected to use, and relied upon using, his calculator. For example, when asked to evaluate the following item which was presented to him for the first time during the interviews:

$$\log \sqrt[5]{10^3} =$$

James started by rewriting the argument of this logarithm as a power. Then he used his calculator to divide three by five (= 0.6), calculated the value of $10^{0.6}$, and then calculated the common logarithm of this (I Ib, 1). Item B3 from the researcher's logarithms survey also illustrates James' expectation to use his calculator when required to evaluate an expression containing a single logarithm. Initially, a calculator had not been available for this item, and James had left it blank:

$$10^{\log 3} =$$

James' first comment when presented with this expression in the interviews was to indicate that he could evaluate it if he had a calculator. When asked if he could do it without a calculator, James indicated that he could not (I, 2). A third example was item A of the researcher's logarithm survey. When asked to explain his estimates for this set of common logarithms with arguments between 1 and 100, James' response was, "I didn't have a calculator so I couldn't punch it in exactly" (I, 1). At no time during the discussion of any of these items did James

indicate that he had considered evaluating them without his calculator. The only exceptions to using his calculator were the tasks of evaluating the logarithms of 10 and 100 which James could evaluate by inspection.

When James was required to evaluate a logarithmic expression without a calculator, or when he encountered logarithm expressions which could not be evaluated using a calculator, his strategy was to set the given logarithm expression equal to a variable, and then rewrite this logarithm equation as an exponential equation. During the interviews, for example, when James was attempting to solve unit test item 11;

$$\text{If } \log_b 2 = c \text{ and } \log_b 3 = d, \text{ then } \log_b 12 = ?,$$

in a step-by-step manner, rather than by making an educated guess as he explained that he had done from the multiple-choice answers during his writing of the unit test (I, 5), he began by rewriting each of the given logarithm relationships in terms of their corresponding exponential equations as follows:

$$b^c = 2, \quad b^d = 3, \quad \text{and} \quad b^? = 12 \quad (\text{I, 4 \& IIa, 17})$$

Elsewhere, when James attempted to solve unit test item 9,

$$\text{If } \log_{2x} 16 = 2, \text{ then } x = ?$$

he rewrote the logarithm in the form of its corresponding exponential equation, $(2x)^2 = 16$, and arrived at his answer of $x = \pm 2$.

James' initial estimates for item A from the researcher's test (attempted without a calculator) and the subsequent discussion relating to this item during the interviews provide further evidence that he relied upon using a calculator to

evaluate or estimate the values of common logarithms with arguments other than powers of ten. This item required James to match base 10 logarithm expressions with arguments between 1 and 100 by drawing lines between them. A line was drawn between $\log 1$ and 0.0 as an example. James reported that he had determined the logarithms for 10 and 100 right away, and guessed that the horizontal line pattern found for these two expressions and the example given would be appropriate for all of the other expressions on the page. When asked if he knew of any patterns amongst the pairs of the logarithm values within the given set he indicated that he did not (I, 1).

A summary example provides further evidence that James expected to use his calculator to evaluate logarithms, and relied on rewriting logarithms as their corresponding exponential equations to evaluate them when his calculator was not available. To set the context for this example, it should be noted that earlier in this interview session James had indicated that 125 equals five times five times five (IIb, 4). In the present example (IIb, 6-7), James was told that the common logarithm of 5 was 0.7, and asked to use this to determine the value of $\log 125$. James' first step was to ask if he could use a calculator (which he could not so that it could be determined if he would apply his knowledge of the rules for the logarithm of a product or logarithm of a power in this context). Then he expressed his disbelief, a number of times, that the value of the log of 5 was 0.7, indicating that he thought the value of this should be five instead. Then he rewrote $\log 5 = .7$ and $\log 125 = x$, as $10^{.7} = 5$

and $10^x = 125$ respectively (interview I Ib work sheet 1a), in an attempt to find a solution. James finally indicated that the value of $\log 125$ was between two and three before he gave up on the task. Following this, James was asked to determine the value of $\log 50$, again without a calculator. He made no progress towards finding the value of this either (I Ib, 8).

James' procedure for evaluating expressions containing more than one logarithm was as follows:

- 1) use a calculator to evaluate directly any part of the expression that could be done so,
- 2) apply the logarithm rules for the logarithm of a product, the logarithm of a quotient, and the logarithm of a power to whatever remained un-evaluated after step one,
- 3) evaluate the logarithms which remained from step two using a calculator, and
- 4) perform any arithmetic which was left to be done.

This four step approach was revealed when James was asked to categorize and explain how he would work through various types of logarithmic expressions (i.e. III, 9-12).

Together, the examples cited above indicate that James' approach to evaluating logarithmic expressions was largely procedural. When the logarithm tasks that he was attempting did not correspond with his repertoire of rules and procedures, James had little success.

For James, the purpose of mathematics was to get the answers to mathematics "questions." This became evident in the discussion relating to a number of items from the unit test. When discussing unit test item 11 for example,

$$\text{If } \log_b 2 = c \text{ and } \log_b 3 = d, \text{ then } \log_b 12 = ?,$$

James was asked how this item was different from other mathematics questions that he had seen. His response; "Well, normally there's just an equation and you solve it" (IIa, 18). In another example, when asked to simplify the expression

$$\log A^3,$$

James wrote

$$3 \log A = \quad (\text{interview work sheet IIb, 2})$$

And, when asked about the equals sign that he had written he replied,

I thought maybe an x or something was coming, then I realized that was it. If I had a number here [in place of the variable A], then I could put an equal sign, and give the answer. But since it's just an A, I guess there isn't really any answer. (IIb, 11)

Elsewhere in the interviews James' process view of mathematics was evident as he described the process of moving from "question" to "answer" as moving forwards, and from "answer" to "question" as moving backwards (IV, 17-18).

James' view of mathematical knowledge was largely undifferentiated from his view of other kinds of knowledge learned in formal learning situations both in and out of school. While more discussion of this will follow in the section on James' view of learning mathematics, suffice it to say that he viewed mathematics much like the content of his other courses at school, as facts or truths to be memorized, although the content of mathematics was applied in the process of answering "mathematics questions" in a way that other school content was not.

James viewed mathematics as a largely unconnected body of knowledge. For James this unconnectedness was apparent on a number of levels. On one level, James' concept of logarithms was largely unconnected or unassociated with other mathematics concepts such as exponentials. When asked about the reason for preceding the study of logarithms with exponentials for example, James commented that "a log is really an exponent, [logarithms] went back to exponents" (IV, 9) as had been explained in class by the teacher. Here James was referring to the point that a logarithm could be rewritten as an exponential expression. This, however, was largely the extent to which James saw these two topics as related. The absence of further conceptual links between the rules of logarithms and exponents was evidenced when James explained the usefulness of starting the topic of logarithms with exponents. He replied;

[By] giving the exponents, he's [the teacher] relating something harder [logarithms] to something easy [exponents], so it doesn't seem as hard. (IV, 9-10)

And again:

Well, when he brings up the exponents he's kind of easing you into it slowly, he uses it as developing your mind to cope with the logs. First he'll go through the exponents, and then slowly bring [you] into the logs, but then after you get that sort of system or whatever you want to call it, then you forget about the exponents. You just--you already know the logs because, it's built in your mind. (IV, 10)

When asked specifically if he thought the topic of logarithms was related to the topic of exponents or something separate, James replied:

I'd say separate. I just used these [in reference to the logarithm rules]. I didn't think of [exponents], at the beginning I did, but later on I didn't think of

exponents. I just thought logs and that was it. (IV, 9)
 Elsewhere, James reiterated his belief that the logarithm rules which he had learned were not related to exponents (IV, 15).

On a second level, the topic of logarithms itself was an unconnected body of rules and procedures for James. This was most evident when James encountered unfamiliar forms of logarithm tasks. James, for example, did not relate his knowledge of the logarithm rules learned as algebraic patterns to tasks which involved numbers. Examples were items B4 and B7, from the researcher's logarithms survey;

$$B4) \quad \log 25 + \log 4 =$$

$$B7) \quad \log 360 - \log 6^2 =$$

on which James made no progress either during his first attempts in class or subsequently when item B4 was discussed in the interviews (I, 3). As mentioned earlier, when attempting to evaluate $\log 125$ and $\log 50$, given that $\log 5$ equals .7, and that 125 equals five times five times five (IIb, 6-8), James did not apply his rule for the logarithm of a product or logarithm of a power while he was able to apply these rules in algebraic contexts. Similarly, when estimating the values for common logarithms on the researcher's logarithms survey (item A) James stated that he did not know of a relationship or pattern between pairs of logarithms in this item, such as logarithm of a product relating $\log 2$ and $\log 20$, $\log 4$ and $\log 40$, and $\log 5$ and $\log 50$ (I, 1). A review of the work presented in class and the tasks assigned for homework indicated that logarithm tasks of this type had not been covered by the teacher. In comparison,

James successfully applied the rules for the logarithm of a product and logarithm of a quotient to simplify algebraic logarithmic expressions in a number of items on the unit test (i.e. items 15b, 16b, 16c, and 18), and he was able to write the logarithm expressions for given algebraic expressions which included combinations of multiplication, division and exponentiation for other unit test items (i.e. 17a, and 17b). Items 17b and 18 are provided to indicate the degree of sophistication of the algebraic logarithm tasks that James answered appropriately:

- 17b) Write a logarithmic equation to represent the following equation [sic]:

$$\frac{\sqrt{5\pi}}{\sqrt[3]{1.08}}$$

for which James wrote,

$$\log N = 1/2(\log 5 + \log \pi) - 1/3 \log 1.08$$

and

- 18) If the logarithmic equation for a calculation is illustrated below, write the original calculation without logarithms:

$$\log V = \log 4\pi + 3 \log r - \log 3$$

for which James wrote,

$$V = \frac{(4\pi)(r^3)}{3}$$

These forms of the logarithm tasks had been covered by the teacher in class during the unit of study.

Further evidence of James' view of logarithms as a largely unconnected collection of rules and procedures was evidenced by his encounter with unfamiliar logarithm tasks in class. The following are examples:

If $\text{Log } r = 4$, $\text{Log } t = 3$, $\text{Log } w = 8$

Find $[\text{Log } r]^{\text{Log } t} =$

$$\sqrt[\text{Log } t]{\text{Log } w} =$$

During the subsequent review of the classroom video recording of this episode, James reported that he had not known how to do these tasks even though he was able to evaluate the following familiar expressions as part of the same exercise:

$$(\text{Log } r)(\text{Log } t) =$$

$$\text{Log } [r^{\text{Log } t}] =$$

$$\text{Log } [w/t] =$$

$$\text{Log } [rt] =$$

$$\frac{\text{Log } w}{\text{Log } r} =$$

(III, 7-13)

James also indicated that while he was often unsure of the rule for the logarithm of a power while he worked through other logarithm tasks during the interviews, he had not attempted to relate this rule to any of the other logarithm rules which he was sure of, such as the logarithm of a product, to verify his conjectures about the rule for the logarithm of a power (I, 10; IIa, 19; IIb, 11). The only connection among the various logarithm rules that James reported was on a lexicographic level. When asked what was similar about the logarithm rules, James responded, "All the name things, they sound all the same."

(IV, 7)

Finally, James believed that the mathematics content in Algebra 12 was not of any direct use for him outside of the classroom. In particular, James saw little application for the content being covered in Algebra 12. He stated that, "Most of

the stuff [the content being covered in mathematics], you'll never use again the rest of your life" (IV, 33). When asked about the purpose for studying mathematics in high school James indicated that it was to exercise the mind--to make thinking hard easier for students in the future:

I think it--it'll make you smarter. Math uses logic too, so your logic will be better. But most of the stuff, you'll never use again the rest of your life. It makes you think, so, you've gone through math like this and it really makes you think. Then the stuff in later life, will be better--a lot simpler, it won't be as hard to think about them as much--as someone who didn't take math. They wouldn't be used to using their mind that much, and it would be [more] difficult.
(IV, 33)

James also believed that his having taken Algebra 12 would look good to others in the future.

Having Algebra 12 as a credit for Grade 12 means you have worked, because Algebra 12 is a tough course and if you passed it then you must be good. You must be a hard worker if you passed... [James then described the usefulness of taking other courses, and that he would not have taken Algebra 12 if he wasn't planning to go on to post secondary education] I'm not going to use this [mathematics] later on, but it'll look good, if I do pass it, it'll look good on my report. (IV, 35)

In other words, a major goal for James in learning mathematics was to impress others.

Keri's conception of mathematics and the evidence of this in her learning of logarithms

Keri believed that mathematics was a set of "truths" in the form of rules and procedures which were created by mathematicians to get answers for mathematics "questions" (III, 28-29). These truths were to be accepted without question by students as Keri indicated in the following:

That's the way that it's been always laid out, and that's the rules of it. I've always kind of just accepted them the way that we've been taught, instead of like, some people will just, well "I have to know why, I want to know why, why, why?" I've just more or less accepted them [the rules] all the time. (III, 9)

When asked to elaborate on what she meant by "some people asking why" in mathematics, Keri responded:

Well, I can give you an example of when I wouldn't [ask why in mathematics]. Like, you know our rules that we have, right? [referring here to the rules for logarithms] He [the teacher] would say, "The quotient of the log, that equals this, and this equals that." I wouldn't go and ask, well why does the quotient of the logs equal the log of the quotient sort of thing, like why does that equal that? I wouldn't ask that, 'cause it's been proven. You know mathematicians have researched and done it and this is what they've come up with. (III, 9)

This view of the origin and nature of mathematics implies that alternative perspectives and interpretations are not part of mathematics for students. Either one knows a "truth," procedure, or rule that mathematicians have come up with, or one doesn't.

During the interviews, Keri spoke of specific rules or procedures which she used for logarithms. These included rules and procedures for evaluating logarithms and solving exponential equations (I, 7); graphing logarithmic functions (II, 19); determining the conditions for the argument of a logarithm expression (I, 32); and evaluating expressions such as the logarithms of a product, quotient, and power (III, 17). Further evidence of Keri's rules- and procedures-based approach to logarithms was her response when asked how a particular set of review exercises done in class could have been made easier for students. She replied:

I think that if I had to do them again, I'd have the rules written out, like on one sheet or something. Or

if I had the examples, like you know how we had the pages of notes, and we had done them at different times, put it all onto like a page, so I could have that beside me and go, ok, well this is this type so you apply these rules to that type. (III, 4)

Elsewhere she reaffirmed her procedural approach to logarithms in describing the confidence she felt about her answer to a logarithm task: "I'm confident because I've done everything step-by-step the way I know how to" (II, 14). In other words, being able to complete a mathematics task in a step-by-step, procedural manner indicated to Keri that she was on the right track.

Keri was, at times, uncertain of what to do when performing logarithm tasks, yet she was very persistent. She would often perform a variety of different procedures, one after another, in order to discover a sequence of procedures which would lead to an answer. In attempting to solve for "x" in the equation

$$10^{\log 3} = x$$

for example, Keri started by writing the following:

$$10^{(3=10^x)} = x$$

which she described as taking the equation "out of a log and [putting it] into exponential form" in an effort to evaluate the log 3 because, as she explained, the bases of both sides of the given equation were not the same. Then Keri rewrote the original expression as

$$\log 3 (\log 10) = \log x$$

and explained, "I'm using the rule that you bring your exponent down and you add a log on [to the base 10 on the left side of the equation], and you add a log to the other side." Then Keri

rewrote the equation as:

$$\log 3 = \frac{\log x}{\log 10}$$

and remarked that in this form it was still necessary to find the logs of each of 3, x and 10. She proceeded to rewrite the expression as

$$(3 = 10^x) = \frac{(x = 10^x)}{(10 = 10^x)}$$

where the newly introduced x's represented three different values. At this point Keri gave up on trying to evaluate these individual logarithms and focused on estimating the value of log 3 explaining that it would be between 1 and 2 ($10^1 < 3 < 10^2$). Then Keri reconsidered the equation

$$\log 3 (\log 10) = \log x$$

and decided that $\log 3 (\log 10) = \log 3^{1 \cdot \log 10}$, then that log 10 equals one, and then that $\log 3^1 = \log 3$. She then rewrote the expression as

$$\log 3 = \log x$$

and concluded that $3 = x$ (II, 32-33; interview II notes p. 2).

When asked, in retrospect, what it was about this task that was difficult for her, Keri provided further evidence of her procedural focus in performing this logarithm task: "I was concentrating so much on the steps that I wasn't concentrating on how to do it.", (II, 34) and "I was so caught up in all that [the procedures that she was doing to find an answer], I didn't realize what I was writing down" (II, 35).

Keri also believed that of mathematicians, mathematics teachers, and students, mathematicians were the only ones who

created new rules and procedures in mathematics, or who were able to do this. To be able to do the job of a mathematician, Keri explained that:

You'd have [to know] about all of them [the rules] and then take what you know and take a question and try and apply a rule to it, [a rule] that you do know, and if that rule doesn't work, maybe apply another one.
(III, 27)

and

You'd have to, if you want then to make rules of logs, you'd have to know what's already been found out about them, and what is already a rule and then come up with some, some sort of question or something that doesn't apply to that rule and then try to make a [new] rule.
(III, 28)

More specifically, Keri believed that new rules were created by mathematicians to answer questions that were encountered in applications, for example in electronics or physics, for which the existing rules did not apply--when mathematicians "wanted to answer the unexplained answers" (III, 29). For Keri, mathematics was made to match or represent "real world" phenomena. Therefore, the possibility of mathematics developing out of "pure mathematics," as it would be understood by a member of the mathematics community, was not part of mathematics for her.

Keri's view of mathematics was further evidenced when she explained what she thought mathematics was about: "So that you can get the answers. That's a lot of what it is" (IV, 26). Getting answers was the primary objective in mathematics for her, and details regarding the methods used to get answers, more general discussion of mathematics topics, and the terminology of mathematics were subordinate to this. Keri explained that, at times, when she had shared the details of her methods for solving

questions in class, her teachers had told her that she should have done her work "a bit differently" even though her answers were correct. From Keri's perspective using different methods which resulted in the right answers should not have mattered and she did not understand, nor seek to understand, what her teachers were getting at when they critiqued her work. As a result, Keri chose to avoid explaining her solutions in class, preferring instead to compare her final results with the teacher's answers on the blackboard and to attend to the teacher's explanations only when her answers did not match (III, 8-9). Keri's comments when asked about the need for, or usefulness of mathematics terminology provide further indication of her focus on getting answers in mathematics. This was evident in her emphatic statement during the interviews: "Well I know how to do it! I'll recognize it when I see it, so I don't really have to know that [for example] this is a 'quadratic,' and this is a 'quadratic formula.'" (IV, 22)

Keri also believed that all questions in mathematics have answers. She refused to accept that items could not be solved, believing that when she got stuck answering a question she probably had not seen the particular type of task before and that this was the reason for her inability to get an answer for it. This was evident in Keri's reply when asked about the answer "cannot be solved" (which she refused to use) on the exponents and algebra survey:

[The response "cannot be solved" meant that] I hadn't seen it before but I'm going to try to do it to try and figure it out, more than saying, well, this can't be done. So to try and see where it can be done and I'm

going to figure out a way so it can be done. I think that's how it went in my mind, I didn't want to have that feeling, well, this can't be done. If it's a math question, it should be able to be done. (II, 5)

Keri's solution to item B8 from the researcher's logarithms survey,

$$\log (-100) = \underline{\quad}$$

provides another example in which she refused to accept that an item could not be answered. Her written work for this item follows:

$$\begin{aligned} -100 &= 10^x \\ -1(10^2) &= 10^x \\ -(2) &= x \\ \log (-100) &= -2 \end{aligned}$$

While Keri indicated that she thought a negative argument was not "allowed" in logarithms, she devised a method to get around this so that she could obtain an answer, which she explained as follows:

We had always been told that you're not suppose to have a negative argument. And so then we had one and it was in brackets so I think, okay, we're not suppose to, but we do, so maybe I'll just try and work it out the same way, just in case. Okay, then if I take out the negative [that is in the line $-1(10^2) = 10^x$], then it won't be a negative argument at the time. (I, 16)

The example just described also illustrates Keri's uncertainty about the applicability of the rules and procedures for logarithms which she had learned. Keri was also uncertain of connections amongst the rules and procedures of logarithms, and her knowledge of logarithms was mostly separate from, and unconnected with, other mathematics concepts. For example, in the discussion relating to her work evaluating the expression

$$\log \sqrt[5]{10^3} \quad (\text{II, 11})$$

with a calculator, Keri indicated that she was perplexed by her

answer of 0.6 after she had indicated confidently that the argument of the logarithm was equal to $10^{0.6}$. She believed that she had made a mistake because it was "so easy to do" (II, 12) which indicates that she had not made a prior connection between the exponent of a base ten power and the logarithm of such a power. Elsewhere, Keri indicated her uncertainty with the equivalence of two different methods for finding the logarithm of a product, given the logarithms for the factors of the argument. The first method, using a calculator, which Keri had devised herself, was to find the antilogarithms of each factor, use these values to calculate their product, and then find the logarithm of this product (I, 25-26; II, 36; III, 13; III, 19). The second method, which Keri used more frequently as she became familiar with logarithms during the course of the study, was to employ the rule given by the teacher--the logarithm of a product is equal to the sum of the logarithms (of the factors) (III, 13). Her uncertainty with the equivalence of these two methods was indicated by her comments: "If it figures out the same?" (II, 37) and

For me it was more difficult, I don't know why, but I had these [the teacher's rules for evaluating logarithmic expressions], and I had to remember that they did mean this [her own methods]! It was more difficult for me to remember that. (IV, 19)

Further evidence of Keri's largely unconnected view of logarithms became evident during the third interview when Keri explained her approach for solving a set of logarithms tasks which she had encountered in class. Two unfamiliar problems were contained within the set of problems:

$$(\log r)^{\log t} = \underline{\quad} \quad \text{and} \quad \sqrt[\log t]{\log w} = \underline{\quad},$$

given that $\log r = 4$, $\log t = 3$, and $\log w = 8$

Both of these problems could be evaluated by the substitution of the given logarithm values, and Keri had earlier performed substitutions in logarithm tasks successfully (tasks which had not included powers or radicals), and she had also evaluated powers and radical expressions successfully in contexts which had not included these types of substitutions. Keri's perspective on these unfamiliar forms of logarithm tasks was indicated in her remarks:

I think the confusion for me was that since some of the questions we hadn't done already, and so going through and doing them, it confused you of what you were supposed to do or not supposed to do. There were some questions on here that we hadn't seen before and then you go down and you do them, and then trying to apply the rules that you could remember to the ones that we hadn't seen before. (III, 3)

In other words, Keri was unsure of the appropriateness of the rules which she had learned earlier for these new and unfamiliar forms of logarithm tasks. She expected to be shown how to do every type of logarithm task that she would encounter and in this instance did not proceed on tasks which she had not been shown how to do. Similarly, Keri made no progress (without using a calculator), given that $\log 5 = 0.7$, in attempting to evaluate $\log 50$ (II, 23-24); and when evaluating $\log_b 12$, given that $\log_b 2 = c$, and $\log_b 3 = d$ (unit test item 11; I, 24-25). As mentioned earlier in the discussion of James' conception of mathematics, the task of evaluating expressions of this exact type, where the argument was a numerical product given the logarithms of its factors, had not been covered by the teacher

during the unit of study. Keri, like James, was unable to evaluate these numerical logarithmic expressions which were far simpler than the elaborate algebraic logarithmic expressions which both students had performed correctly on the unit test (e.g. unit test items 17a, 17b, & 18). Keri could apply her procedures for logarithms in complex logarithm tasks like those shown by the teacher in class, but could not apply these same procedures on her own in less complex problems of slightly different forms, or in different contexts (from the perspective of a member of the mathematics community).

When asked about connections between the logarithm rules and the rules for exponents Keri stated that she hadn't thought of the exponent rules when she had done logarithms (IV, 20). And, when the similarity of the rules for the multiplication, division and exponentiation of powers and the rules for the logarithms of products, quotients and powers was explained to her by the interviewer during the last interview, Keri remarked: "I never realized exponentials related to logs like that!" (IV, 27) The connections that she had constructed between exponentials and logarithms were, 1) that a logarithm was an exponent, and 2) that logarithms could be rewritten as exponential equations when evaluating them. (Both of these had been told to her by the teacher.) One example of Keri's application of the latter procedure is particularly interesting and indicative of her view that different procedures in logarithms were separate or unconnected (II, 25-24). Given that $\log 5 = 0.7$, Keri's task was to estimate $\log 50$ without a calculator. While she began by

considering the idea that fifty is ten times five, she promptly disregarded this relationship and proceeded to estimate the value of $\log 50$ using her knowledge that the square root of ten is approximately 3.2 as a bench mark. Keri then explained that $10\sqrt{10}$ or $10^{1.5}$ is approximately 32, and from this point that $10^{1.7}$ would be closest to 50. Towards the end of this process, Keri explained:

Okay, 'cause 1.6 would probably, it would make it [$10^{1.6}$] a little bit more [than 32] but not quite 50. 1.6 would make it only a little bit more, like say 40 or something, or whatever that would work out to, but not quite enough. Then if you had 8 [$10^{1.8}$] or 9 [$10^{1.9}$] then it would be too much. It would be too close to 100. So it would have to be in-between, like one and a half and 2. So 1.7 is the closest one in-between those two. (II, 24)

When asked to look back at the original expression and given information, and to consider if she could have arrived at the same solution by any other means, Keri indicated that she could not (III, 24). Elsewhere, Keri explicitly stated that her learning of logarithms was unconnected with other topics that she had learned in mathematics.

If you keep going on [in mathematics], there may just be--I'm sure there is some way that they're inter-related. We just haven't learnt it yet. (IV, 27)

For Keri, mathematics in Algebra 12 was a largely unconnected body of rules and procedures. Making connections between ideas or topics was not part of mathematics for students in her mind.

Keri's view of mathematics content was largely undifferentiated from her view of the type of content in her other subjects at school. While this will be discussed in detail in the section on Keri's conception of learning mathematics which

follows, suffice it to say here that she viewed mathematics as a collection of facts or truths to be memorized, with the added feature that in mathematics students had also to apply this knowledge (rules and procedures) to complete mathematics tasks (I, 32-33; III, 16; IV, 7-8 & 31).

Keri expected that completing mathematics tasks in general, and logarithms tasks in particular, would be difficult, even after she had completed the study of logarithms in her Algebra 12 class. Keri explained that when she attempted to evaluate item B11 on the logarithms survey which required her to indicate which numbers have (base ten) logarithm values between 3.0 and 4.0, she left it blank because she thought it was too easy, and that this could not be so (I, 6). In other words, Keri had taken her interpretation, that the task was very easy, to indicate that she was mistaken in her understanding of it. As discussed earlier, when asked to evaluate

$$\log \sqrt[5]{10^3} \text{ (II, 11)}$$

Keri was surprised with her result of 0.6, expressing doubt in this answer. She worked through her solution a number of times before she finally accepted it. In her words, "It just doesn't seem like it should be like that because it is so easy to do it that way, but it looks so complicated when you look at it."

(II, 12) For Keri, the form of an answer and the effort required to complete an answer in mathematics were important indicators of the appropriateness of her work.

While Keri enjoyed studying mathematics in school, she did not understand why mathematics was required for all students to

Grade 11, as was the case in British Columbia.

I never understood it before [laughs]. I didn't mind math, but, you learn all this algebra stuff and then you'll never use it. But, I've noticed more and more as you do go out like, some of the simplest basic things that you can use, and [but] why they teach us so much of it?, I don't really know if they think well, because--it probably will. I mean when you get older and stuff, you probably will be using it more but, to a point I don't know if it's necessary, I mean to learn it all the way up to Grade 11 or something, maybe Grade 10 would be enough. (IV, 11-12)

Keri recognized the usefulness of more elementary mathematics for all students but in her view the mathematics beyond this level was learned, for the most part, so that it could be applied in science and technology (III, 29). Keri thought that the specific mathematics content she was learning in Algebra 12 might be applicable in the workplace when she finished school.

The students' conceptions of learning in mathematics

Assertion 2: The students believed in a rote approach to learning mathematics--that learning mathematics involved the transmission of information from teacher to students, and that students in turn were to memorize and practice applying this information. The teaching of mathematics was best done slowly and in an easy step-by-step manner, and the students' role in this process was a passive one, to attend to and accept what the teacher said. This view of learning mathematics was no different from the students' views of learning in other subjects at school.

James' conception of learning in mathematics

James believed in a rote approach to formal learning; that learning in a number of domains, including mathematics, involved the transmission of information from teacher to student, and subsequently, memorization and practice applying this information

by students. Specifically, James believed that the best method for learning something, including the content of mathematics and his other subjects at school, was by saying it, seeing it, and doing it. This was the approach which he had learned and used at the local McDonald's Restaurant where, as a crew trainer, he taught new employees the kitchen procedures (III, 30-32). During the interview discussions James spontaneously critiqued his mathematics teacher's method of teaching from this perspective, indicating that the students in his mathematics class were not provided with enough opportunity to see the logarithms rules as they were learning this topic. James thought that it would be better if his mathematics teacher posted the rules that were being learned on the wall in the classroom for the students to refer to when needed (III, 30-31).

In mathematics class James focused entirely on learning what his teacher wrote on the blackboard which, for the most part, consisted of examples worked out in detail. He believed that any information not written on the blackboard, only explained verbally by his teachers, was not important. In his words:

The things he [the teacher] takes time to write on the board are obviously important. The stuff he just blabs out, he just tells you, usually aren't that important. So I usually pay attention to the stuff he actually puts on the board, I copy down myself. I do that in every class. If he actually takes the time to write it out then you better know it. (IV, 33)

Furthermore, James chose to disregard mathematics related terminology (i.e. terms such as "sum," "difference," and "product") and his teacher's verbal explanations and elaborations relating to the topics being covered. He believed that the

terminology associated with mathematics in general, and with logarithms in particular, was confusing and of little use for students, even though he made repeated reference to the logarithm rules as he worked through various tasks in the interviews, as mentioned earlier. James was particularly critical of the logarithm rules which the teacher emphasized in class. James complained: "He's got so many of those little sayings!" (III, 5), and, in another instance when reviewing his teacher's description of a logarithm rule during the interviews, he described it mockingly as "his little riddle" (III, 2). When asked about the difficulty that he experienced in learning the logarithm rules, James confirmed that he had made little sense of the terminology and explanations given by his teacher:

I can remember the first part by just looking at the question, [such as] log of a product. [James is referring here to the left side of the equation $\text{Log}(ab) = \text{Log } a + \text{Log } b$ as an example]. But then, he [the teacher] goes equals the blah-de-dah-de-dah. I just can't remember the end part. (III, 19)

Elsewhere James stated his view more explicitly; "I really don't like those words that he uses, I don't, 'cause they confuse you" (III, 14).

The student's role in James' view of the learning process was as an intellectually passive receiver of information. The most telling evidence of this was James' explanation of his experience attending to his teacher during mathematics lessons:

When the teacher's doing it [teaching a lesson] in the classroom you're just kind of sitting and waiting to see what he puts up next, and you're not really thinking. You're just kind of sitting there, watching and copying and watching and copying. (IV, 21)

And, when the teacher asked questions of students as new work or

reviewed material was presented at the blackboard, James often avoided taking an active role in class; "As soon as he asks a question I pretend I'm writing, 'cause sometimes they'll [teachers will] nab you 'cause they think you weren't listening" (IV, 30). James reported unequivocally, that he hated being asked questions by the teacher unless he had his hand raised (IV, 28).

Further evidence of James' intellectually passive role in the learning process came in a discussion relating to the usefulness of the teacher providing examples on the blackboard. When asked what it was about having an example that was most useful for him, James explained that he mimicked these in a rote manner to complete his work, as follows:

What numbers go where. You look at the question and you look at the numbers and you look at the answer, you see where the numbers go. Just match the numbers with the places in the question. (I, 12)

James thought that his Algebra 12 classmates worked in a similar manner: "They'd probably do the same thing, look at another question and see where the numbers go" (I, 12).

From James' perspective, the explanations in his mathematics textbook were unnecessarily difficult and of no use to students. This was evidenced in his reply when asked if he referred to his textbook for help when doing his homework or studying:

No, I don't read the textbooks. They're too confusing. They use language that's not really used and you don't really know. I like when the teacher takes the explanations out of textbooks and interprets it and writes down a new one on the board, and [the] same with the solutions. Sometimes the solutions in the textbook are--there's an easier way. I like when they [teachers] show you the easier and more understandable way. (I, 14)

James believed further that it was the teacher's job to modify the explanations from the textbook to make them easier for students to do. As an example, he cited the following:

Well, back in conics I remember there's one equation type question that will [would] take a whole page to do, of all these different steps, but then the teacher showed us a lot easier one that just took maybe three lines. (IV, 26)

In James' view, presenting new rules and procedures to students in a slow, easy, and step-by-step manner helped to ensure that these explanations were easy for students. It was also important that his teacher present new rules and procedures one at a time, not moving on until students could apply each rule or procedure successfully. From James' perspective this helped students feel confident when learning new content, which was crucial for their success in learning and doing mathematics (III, 19 & 25; IV, 10).

James was also critical of his teacher's presentation and explanation during mathematics lessons of common errors made by students because it gave students too much to have to remember:

You're not supposed to ever show them [students] the wrong way because that might stick in people's heads. They might do that... You're not supposed to do that, 'cause say on a test, if they're [students are] thinking about it, they've got two things now (hits desk to emphasize his point) instead of one. (III, 15)

It can be concluded from this that James considered each example or idea that the teacher presented in class as a separate entity, and not as something which students would attempt to associate or integrate with their prior knowledge. Learning mathematics, for James, was the accrual of a largely unconnected body of information.

Given that he did not use the textbook, as mentioned earlier, it was concluded from an examination of James' written work from the unit on exponentials and logarithms that he relied almost exclusively on the definitions and examples of procedures that he had copied from the blackboard for resolving difficulties with his homework, and as the basis for studying for tests and exams--he had written nothing else in his notes whatsoever during the entire unit. The information which James had copied from the blackboard was very similar to that which was required of him on his mathematics tests and exams, specifically the solutions to problems and the occasional definition.

Finally, practice was an essential part of learning mathematics in James' view. When he compared the learning of mathematics with learning in his other school subjects, James indicated that there was an important element of memorization in all of them, but for mathematics in particular, students also had to be able to deal with equations that had been "switched around," before applying the information that they had memorized (IV, 32). When asked to estimate how much of his understanding came from listening to a lesson in class and how much came from doing his homework, James indicated the relative importance that he placed on practicing the procedures that had been learned in class. His response: "I'd say about 30 percent, 70 percent. 70 at home, 30 in class" (IV, 22). He explained further the important role of practicing what had been shown in class:

Well, usually--a lot of times I won't--I won't get it when the teacher's explaining it, but once I take it home and try them, and have an example [from class to follow], and do them, then it clicks in. (IV, 21)

Keri's conception of learning in mathematics

Keri believed that learning in many of her subjects at school, including mathematics, was by rote, the memorization of facts and rules, although in mathematics there was an additional component of applying this information to get answers in a way that was different from other subjects. The following quote summarizes Keri's perspective on learning in her school subjects other than mathematics:

In a lot of other courses [other than mathematics] it's just memorization, you have one kind of statement, and you have this statement, and you mix them together.... In history [for example], you know the facts and, what can you do? You can't apply them to anything. That's what happened, and that's all that you need [to know]. You have to just memorize, and then, learn how to recall them. It's more to me like, learn it, memorize it. Understand that this came before this, and that's it. (IV, 7-8)

Keri also cited Chemistry and English as examples of other school subjects where learning was mostly memorization (IV, 7-8).

Keri, like James, believed that the students' role in the process of learning mathematics is a passive one, to accept what they are told by the teacher and not to wonder why things are as they are (IV, 10). Learning mathematics for Keri involved following the teacher's instructions: "I follow the rules that are given to me (IV, 14)," "this is how the teacher says how to do it" (III, 9), and "it's mechanically taught to us" (III, 2), or in other words a transmission view of learning. Elsewhere, she made a number of references to the memorization of rules in mathematics (I, 32-33; III, 16; IV, 31), and to the need to practice applying these rules (IV, 23; III, 26). Keri also indicated that, at times, she was unable to make sense of the

rules or explanations provided by her mathematics teacher, and that this was an inevitable part of doing mathematics for her (III, 2; III, 9). When asked to explain her thinking as she worked through an unfamiliar mathematics task, for example, Keri stated, "Well, this is math. This is what we do here, and some things don't make sense, but you do them" (II, 16). And, when asked about the logarithm rules provided by her teacher she indicated that she had accepted what she was told without question: "I never really question why you'd do it" (III, 16), and "Well, this is how it is" (IV, 10). Keri's acceptance of this way of learning and knowing mathematics was confirmed by her comment that she liked how she was being taught in mathematics (IV, 24).

Another aspect of learning and doing mathematics for Keri was her expectation that the teacher would show her every possible form of question or task she needed to know for every topic that was covered (II, 16). In other words, Keri believed that she was entirely dependent upon her teacher for learning and her success in mathematics. She believed that students were dependent on their teachers to show them how to apply known mathematics rules in new and unfamiliar contexts as well as to teach them new content. For example, Keri expressed uncertainty in applying her knowledge of exponents in the following item;

$$a^{-3} \times 1/a^? = a^{-2}$$

(II, worksheet 2; II, 5)⁸

⁸ The variable "x" used in this item on the original interview worksheet was changed here to the variable "a" to avoid confusion with the multiplication sign in this type written text.

even though she had given the correct response, negative one. She stated that her uncertainty with her answer stemmed from having not seen an example like this (with a negative exponent in the denominator) before: "They're [the rules of exponents] clear but they [her mathematics teachers] haven't showed every possibility to me" (II, 16). Elsewhere, when she got confused while completing a logarithm task she attributed the problem to not being able to recall what she had been told to do; "I couldn't remember how he [the teacher] had figured it out exactly" (I, 2) and "I couldn't quite remember how he [the teacher] had said how to find it" (I, 27). Keri also thought that any connections between topics in mathematics would be taught to her by her mathematics teachers: "If you keep going on [in mathematics], there may just be [connections between topics]. I'm sure there is some way that they're inter-related, we just haven't learnt it yet" (IV, 27).

From Keri's perspective, mathematics teachers, like their students, were dependent on others (in the teachers' case mathematicians) to make the rules, to teach them new content, and to teach them how to apply mathematics rules in new and unfamiliar contexts. Keri described her mathematics teachers' abilities to create, learn, and do mathematics in the following episode:

They [mathematics teachers] don't have to necessarily have the ability to teach themselves, to learn it, new stuff--like, to learn--if say a new rule comes in or something.... They have to, if there were rules that were made by mathematicians, then they would have to have the ability to learn those new rules, but they wouldn't have to have the ability to come up with them. That wouldn't necessarily, not really be up to them.

They wouldn't have the ability to make the rules, to decipher between a question, they wouldn't have to have that. They'd have to have the ability to learn them, understand them, to teach us. But not that extra little bit. A teacher couldn't really "wing" a problem. They'd have to be taught. (III, 28-29)

Keri believed that mathematicians made the rules and passed them on to teachers, who, in turn, passed them along to students, and that only mathematicians could learn, create, or discover mathematics on their own. In Keri's view, mathematics teachers learned mathematics by rote, as rules and procedures, and this was what they passed on to their students.

Keri, again like James, preferred and expected that mathematics be taught slowly, using simple terms, and in a step-by-step manner. It was also important that teachers move gradually from simple problems to more difficult ones, and that no step be left unexplained in this process (III, 24-25).

The students' conceptions of knowing in mathematics

Assertion 3: The students believed that they understood mathematics when they were able to apply the rules and procedures provided by the teacher to get the right answers for the questions encountered in class.

James' conception of knowing in mathematics.

James believed that, for students, knowing mathematics meant remembering what the teacher had said or done in class, and understanding mathematics meant being able to apply the rules and procedures provided by the teacher to perform the tasks encountered in homework exercises and on tests. In an interview discussion relating to his understanding of the rules for

evaluating the logarithm of a product, a quotient and a power (the logarithm rules), James explained his view of understanding in mathematics.

Ok, when I'm saying understanding I mean I know what he's [the teacher's] doing. I don't know why these do this [in reference to the processes carried out by the teacher in working through a question]. I understand that's what you're going to have to do. I just don't understand the why--the why part I--why you'd put the three part out front of the log here [for an expression like $\log x^3$]. Like why, I don't know. I just know you have to do that. (III, 24)

Understanding mathematics, for James, also included ease and proficiency with the rules and procedures covered in class:

When I can just do a question with hardly thinking, just--just run right through it. Just kind of sit there, go through it, and get the right answer. Then I know I, I know it. Right there. (IV, 27)

While understanding why the rules were as they were was not part of understanding mathematics for James, he recognized this as another type or level of understanding. He made reference to this second type of understanding in the following:

Well if you have the formulas memorized, you don't necessarily really need to understand it, as long as you can plug it in and get to the right answer. A lot of things [in mathematics] people don't understand why it happens. It just does [emphases added] because you're shown that's what it does. It's better if you understand why but, like the stuff we're doing now, I don't understand it, but I know the formula. I just know it does that, but I don't understand why. (III, 24)

Keri's conception of knowing in mathematics.

Keri, like James, believed that understanding in mathematics meant being able to apply the rules and procedures provided by the teacher to get answers for the questions encountered in class

and on tests (III, 26). In Keri's mind, the terminology of mathematics was not needed for getting answers, and it was therefore not important, as discussed earlier (IV, 22). Knowing why the rules worked as they did was not part of her understanding of mathematics either, as Keri explained in the following:

A lot of times when they say this is how you do it in math you don't understand well how could they have come up with something like that? Like it works but how do they come up with it? (III, 27)

Keri reiterated this view elsewhere: "I never really question why you'd do it" (III, 16), and "You don't really need to know the reason for it" (III, 10). Furthermore, Keri believed that it was inappropriate for students to ask why the rules in mathematics worked as they did. This was evidenced in Keri's description of helping her younger and curious cousin with her mathematics. Keri complained that her cousin would always ask "Why do you do that?," and "Why is that?" From Keri's perspective students should simply accept mathematics as it is presented to them by their teachers (III, 9).

When asked when it might be useful for a student to know why the rules worked the way they did in mathematics, Keri indicated that this was removed from the needs of secondary school mathematics students:

If I was going to be going on to something that was more into the field of logs [for example] then I'd want to know why it's derived that way. But as long as I know that if I follow the rules that are given to me for the questions that I have, then I'll be satisfied. I'll be satisfied with that. I won't have to know why they're derived to that. (IV, 14)

Part of Keri's rationale for this position was indicated in her

assertion that getting answers for her homework or on tests was her most important focus in mathematics;

When I'm on a test I'm not going to be thinking, well, this is how they derived logs and this is the way that they're going to be doing this. I'll just want to know the rules. I want to know how I'm going to apply this to this. How I derive it--that's not going to be what I'm going to be thinking about on a test, or when I'm doing my homework. I'm not necessarily going to be thinking this is how it came about. I want to know how am I going to be able to do it, more than why, or why it came [to be] the way it is. (IV, 15)

Keri's point in this quotation regarding the types of tasks given to students for homework and on tests was confirmed for the logarithms unit by an analysis of the tasks assigned for homework during the unit and the tasks on the unit test. The only types of tasks that were assigned were textbook exercises which required students to apply rules or procedures to find "answers" in numerical or algebraic form.

When asked when she might ask "why?" in mathematics, Keri indicated that asking why for her related to identifying which features of a mathematics question or task were critical in determining the rule or procedure to use to find the answer. She explained this as follows; "This is your example, this is your rule, this is why this rule applies to this one. That's what I mean by the why of it" (IV, 9).

The students' conceptions of self in mathematics

Assertion 4: In general, the students lacked confidence in themselves when learning and doing mathematics. They also believed that their learning and success in mathematics depended upon factors which were largely beyond their control.

Assertion 4 was, for the most part, based upon information which has already been discussed. This information will be reviewed and reframed here to address the focus of this section.

James' conception of self in mathematics

James lacked confidence in his ability to understand why the rules of mathematics worked as they did. As was discussed in the section on his conception of knowing in mathematics, James believed and accepted that he did not understand why the rules worked as they did in mathematics. He did, however, acknowledge that it would be better if one did understand mathematics in this way (III, 24).

James also lacked confidence in his ability to deal with unfamiliar tasks in mathematics, that is, to figure out things on his own when he had not been shown what to do by his teacher. When faced with unfamiliar logarithm tasks in class, for example, James made no effort to try them, stating that he did not know how to do them (III, 7). Elsewhere, when he was asked to evaluate logarithms without a calculator, he promptly stated that he could not (I, 1 & 2). (James was accustomed to using his calculator for this kind of task.) Similarly, when James had been taught how to do a procedure in class but forgot what his teacher had said, he was uncertain of the appropriateness of his

work (e.g. I, 2 & 10).

James believed that his own ideas relating to the content being covered in mathematics were not relevant. As was described in the section on his conception of learning in mathematics, James copied into his notes only that which his teacher had written on the blackboard. Had he thought that his own perspective on, or interpretation of the content being covered was important, it is reasonable to expect that he would have supplemented the teacher's notes with his own comments or ideas. Further evidence to support this view came from the classroom video recording--James did not offer any ideas of his own relating to the content being covered during class, and elsewhere he indicated that he expected that he would be unable to make sense of the teacher's explanations in class (IV, 21). Another example of this was James' statement that his teacher's explanation of common errors made by students was confusing for students (III, 15). In other words, James believed that students were unable to deal with this. In mathematics class he focused almost exclusively on copying from the blackboard, and he avoided participating in any form of dialogue with the teacher during lessons (IV, 21 & 30).

Related to James' lack of confidence in himself was his belief that he and other students were dependent upon their teachers to learn mathematics. James relied almost exclusively upon what his teacher wrote on the board in class to learn mathematics (IV, 33), and he did not refer to his textbook to learn new material or for help when he encountered difficulties

(I, 14). And, as mentioned earlier, James believed that his success and confidence in learning mathematics depended upon being taught slowly and in a step-by-step manner, something over which he had little control (III, 25). Furthermore, he believed that his performance on tests depended upon the teacher designing tests with easy questions first "to build up [students'] confidence" (IV, 17).

Finally, when difficulties or uncertainty were encountered while performing mathematics tasks, James attributed this to not being able to remember what he had been told by the teacher (e.g. I, 10; IIa, 4). Again, this was something over which he believed he had little direct control; either he remembered what he had been told or he did not.

Keri's conception of self in mathematics

Keri was confident in her answers for familiar looking mathematics tasks if they worked out in a step-by-step manner (II, 14). When Keri was unable to answer a question it bothered her, and she was very persistent in trying to get an answer (e.g. II, 23-24; 32-33). She would often continue to think about particular tasks after class, or in the case of this study, after an interview (e.g. I, 5 & 9). Keri's expectation that she would be able to get the correct answers for familiar looking tasks was so strong that at times she would consciously violate rules so that she could arrive at an answer (e.g. I, 16). To put this another way, Keri was confident that she could get the correct answers for the tasks assigned in class and would persist in

attempting to do so, but at times this confidence overrode her making or maintaining sense of the mathematics involved. This is not surprising considering that Keri expected that mathematics would not always make sense to her (II, 16; III, 2 & 9).

Keri's confidence in doing mathematics was limited to those tasks which she was able to work out in a step-by-step manner. If a complicated looking task worked out too easily for her, however, Keri was uncertain about the appropriateness of her work (e.g. I, 6; II, 12 & 36). And, while Keri would attempt unfamiliar looking problems, she was not confident in her answers for these types of tasks (III, 3). She expected that her teacher would show her every possible type of problem that she might encounter (II, 16), and when she encountered an unfamiliar form of problem she was uncertain of the applicability of or interrelationships between the rules and procedures which she knew (I, 14; II, 37; III, 3; IV, 27). She also lacked confidence in her ability to make sense of mathematics rules or procedures on her own (III, 9). In her view only mathematicians could do or create mathematics without being led by someone else (III, 28-29).

Like James, Keri believed that students were dependent upon their teachers for their success in learning mathematics. Mathematics learning for Keri was the transmission of information from teachers to students, as indicated earlier, and students' success in this process depended upon their being taught slowly, in a step-by-step manner, and in simple terms (III, 24-25). Her success also depended upon the thoroughness of the teacher's

explanations for applying each new rule and procedure (II, 16). Keri believed further that success in doing mathematics tasks depended upon being able to remember what she had been told by her teacher (I, 2 & 27). Both the teacher's manner of teaching and Keri's own ability to remember what she had been told by the teacher were crucial for her success in mathematics and these were things over which she had little direct control.

Interrelations within and between the students' conceptions

Assertion 5: The students' conceptions of mathematics, and their conceptions of learning, understanding and self in mathematics were, as a whole, largely integrated.

The analysis presented thus far has been based upon the information sources (e.g., interview transcripts, audio and video recordings, and the students' written work during the unit of study and the interviews) which were made or collected during the field work for this study. In this section, many of the individual views which made up each of the students' conceptions of mathematics, and their conceptions of learning, knowing, and self in mathematics will be used to support one another and to show how the views and conceptions of each student were interrelated.

Many of the individual views that have been described in the previous sections of this chapter support, and are therefore an integral part of, other views within and/or across the concept domains which are the focus of this study. Thus, each of the student's individual conceptions is integrated to some degree, and many of their views are part of two or more of their

conceptions, which serves to integrate their conceptions as a whole.

Interrelations within and between James' conceptions

The ways in which James' individual views supported one another will now be described. All but two of his views which related to his conception of mathematics (that mathematics increased in difficulty as one advances in the study of mathematics, and that the purpose of Algebra 12 level was to exercise the mind) were interrelated with his other views in this analysis. While this description includes interrelations amongst James' views both within and across concept domains, it is organized by connections across concept domains to emphasize the ways in which the students' conceptions were integrated.

Interrelations which served to integrate James' conception of mathematics and his conceptions of learning and knowing in mathematics.

James' view that mathematics was a set of rules and procedures to get answers supported his view that "understanding" in mathematics was synonymous with getting the right answers for the tasks or "questions" which were encountered in homework and on tests. This view of mathematics also supported James' procedural or rote approach to learning--that practice was important for learning and understanding in mathematics, and his view that the teacher's explanations, elaborations (other than how to use the rules and procedures to get answers), and the terminology of mathematics were not useful for students.

Interrelations which served to integrate James' conceptions of learning and knowing in mathematics, and his conception of self in mathematics.

James' view that his teacher's explanations, elaborations (other than how to use the rules and procedures to get answers), and use of terminology were confusing and of little use for students, supported his view that understanding mathematics was synonymous with getting the right answers, and the view that he was unable to deal with, or make sense of these explanations from the teacher.

James' view that the explanations in the course textbook were removed from the needs and abilities of students supported his view that explanations like those in the textbook were unimportant--that students needed only to "understand" how to get answers for tasks like those which had been demonstrated by the teacher in class, and that it was the teacher's job to modify the content to be learned to make it easy for students and to make them feel confident. Both James' view of the explanations in the textbook and his view of the teacher's job to modify the course content supported his view that he was entirely dependent upon his teacher to learn mathematics.

James' view that knowing mathematics was synonymous with remembering what the teacher had said or done in class supported his views that mathematics was learned by mimicking the teacher, and he was entirely dependent upon his teacher to learn mathematics and for his success in mathematics. These views, in turn, supported the view that he had little, if any, direct control over his learning and success in mathematics--it depended

upon the actions of his teacher and his ability to remember what had been done or said, and the view that he lacked confidence in his ability to make sense of mathematics and do unfamiliar looking problems on his own.

James' lack of confidence in his ability to make sense of mathematics and do unfamiliar looking problems on his own supported his view that he was entirely dependent upon his teacher to learn mathematics and for his success in mathematics, his view that mathematics was learned by mimicking the teacher's examples, and his view that students played a passive role in the learning process.

Interrelations which served to integrate all three of James' conception domains.

James' view that mathematics was a largely unconnected set of truths, much like the content of his other subjects at school supported his views that students learned mathematics by transmission from the teacher; knowing mathematics was synonymous with remembering what the teacher had said or done in class; students played a passive role in the process of learning mathematics; and his view that he had little, if anything, to contribute to his learning and knowledge of mathematics. Within this set of views, James' view that mathematics was learned by transmission lent further support to his views about "knowing mathematics" and students' passive role in learning mathematics, and his view about having little to contribute supported his view of playing a passive role in the process of learning mathematics. These views, in turn, lent support for his views that he was

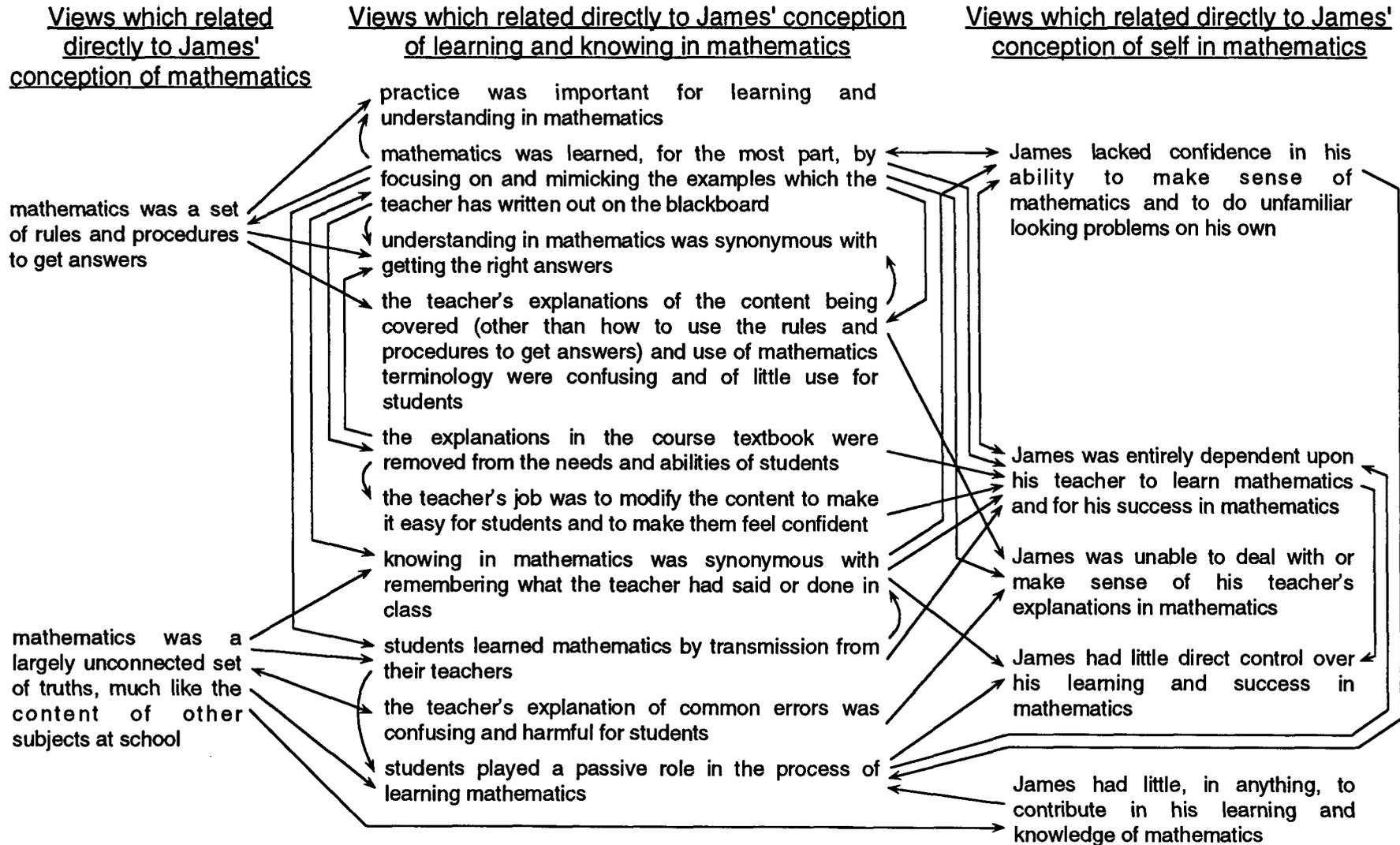
entirely dependent upon his teacher to learn mathematics and for his success in mathematics, and his view that he had little direct control over his learning and success in mathematics.

James' critical view of his teacher's explanation of common errors supported the view that he was unable to deal with, or make sense of the teacher's explanations. It also supported his view that mathematics was an unconnected collection of truths.

James' reliance on mimicking the examples which the teacher had written out on the blackboard to learn mathematics, and his reliance on this information when studying for tests, supported a wide range of his other views. These included his views that mathematics was learned by transmission, knowing mathematics was synonymous with being able to do tasks like those demonstrated by the teacher in class, understanding mathematics was synonymous with being able to get the right answers, practice was important for learning and understanding in mathematics, explanations (both those of the teacher and those in the textbook) were unimportant for learning mathematics, mathematics was a set of rules and procedures, he was unable to deal with or make sense of his teacher's explanations in mathematics, he lacked confidence in his ability to make sense of mathematics on his own, he was dependent upon his teacher to learn mathematics, and in turn he had little direct control over his learning and success in mathematics.

The interrelations of James' conceptions are illustrated in Figure 4.1. Each arrow in the diagram indicates where one of James' views supported another view as described above.

Figure 4.1 The integration of James' views and conceptions



James' other views about mathematics

mathematics increased in difficulty as one advanced in the study of mathematics

the purpose of learning mathematics at the Algebra 12 level was to exercise the mind

Interrelations within and between Keri's conceptions

Keri's views were interrelated with one another to a greater extent than James' views. For example, James' views relating to mathematics were not interrelated at all, whereas all but one of Keri's views relating to mathematics were related to her view that mathematics was created by mathematicians to fit "real world" phenomena. For Keri, this single view played a pervasive and unifying role in shaping most of her conceptions relating to mathematics. All of Keri's individual views which were described earlier in this chapter were interrelated with other of her views in this analysis. As was the case in the description of the interrelations of James' views, this description of the interrelations between Keri's views is organized by connections across concept domains to emphasize these connections.

Interrelations which served to integrate Keri's conception of mathematics and her conception of self in mathematics.

Keri's view that mathematics was created by mathematicians to fit "real world" phenomena supported many other of her views which made up her conception of mathematics. These include her views that mathematics in Algebra 12 was a set of unconnected truths, much like the content of other subjects at school, to be accepted without question; mathematics was learned to be applied elsewhere, e.g. in science and technology; and all questions in mathematics had answers. Both this last view and the initial view in this paragraph supported another of Keri's views of mathematics, her view that mathematics was a set of rules and procedures to get answers. Keri's view about the origin of

mathematics also supported her view that she was entirely dependent upon others to learn and know mathematics.

Interrelations which served to integrate Keri's conception of mathematics, and her conceptions of learning and knowing in mathematics.

Keri's view that mathematics was learned to be applied elsewhere, e.g. in science and technology, supported her view of mathematics as a set of rules and procedures to get answers. It also supported her view that understanding mathematics was synonymous with getting the right answers.

Interrelations which served to integrate Keri's conceptions of learning and knowing in mathematics, and her conception of self in mathematics.

Keri's view that students learned mathematics by rote memorization and practice supported another of her views of learning in mathematics, that students played a passive role in the process of learning mathematics. This view, in turn, supported her views that, at times, it was inevitable that students would be unable to make sense of the rules or explanations provided by their teachers, and that she lacked confidence in her answers for unfamiliar mathematics tasks. In turn, these supported Keri's view that mathematics teaching was best done in a detailed, step-by-step manner using simple terms.

Interrelations which served to integrate all three of Keri's conception domains.

Keri's view that mathematics, for her, was a set of unconnected truths to be accepted without question, much like the content of her other subjects at school, supported her views that students learned mathematics by transmission, rote memorization, and practice. Together these supported her view that students played a passive role in the process of learning mathematics. In turn, these views support her view that students learned best when they were taught in a detailed, step-by-step manner using simple terms, she lacked confidence in her answers for unfamiliar mathematics tasks, she was entirely dependent upon others to learn and know mathematics, and she had little direct control over her learning and success in mathematics. Keri's view that she was entirely dependent upon others lent further support for her views that students played a passive role in the process of learning mathematics and that she had little direct control over her learning and success in mathematics.

Keri's view that mathematics was a set of rules and procedures to get answers supported her views that students learned mathematics by rote memorization and practice and that the terminology of mathematics was not important for students. All of these views in turn supported her view that understanding in mathematics was synonymous with getting the right answers and, in turn, that she was confident in her answers for familiar tasks only if they worked out in a step-by-step manner.

Keri's view that doing mathematics was difficult supported her views that students learned mathematics best when they were

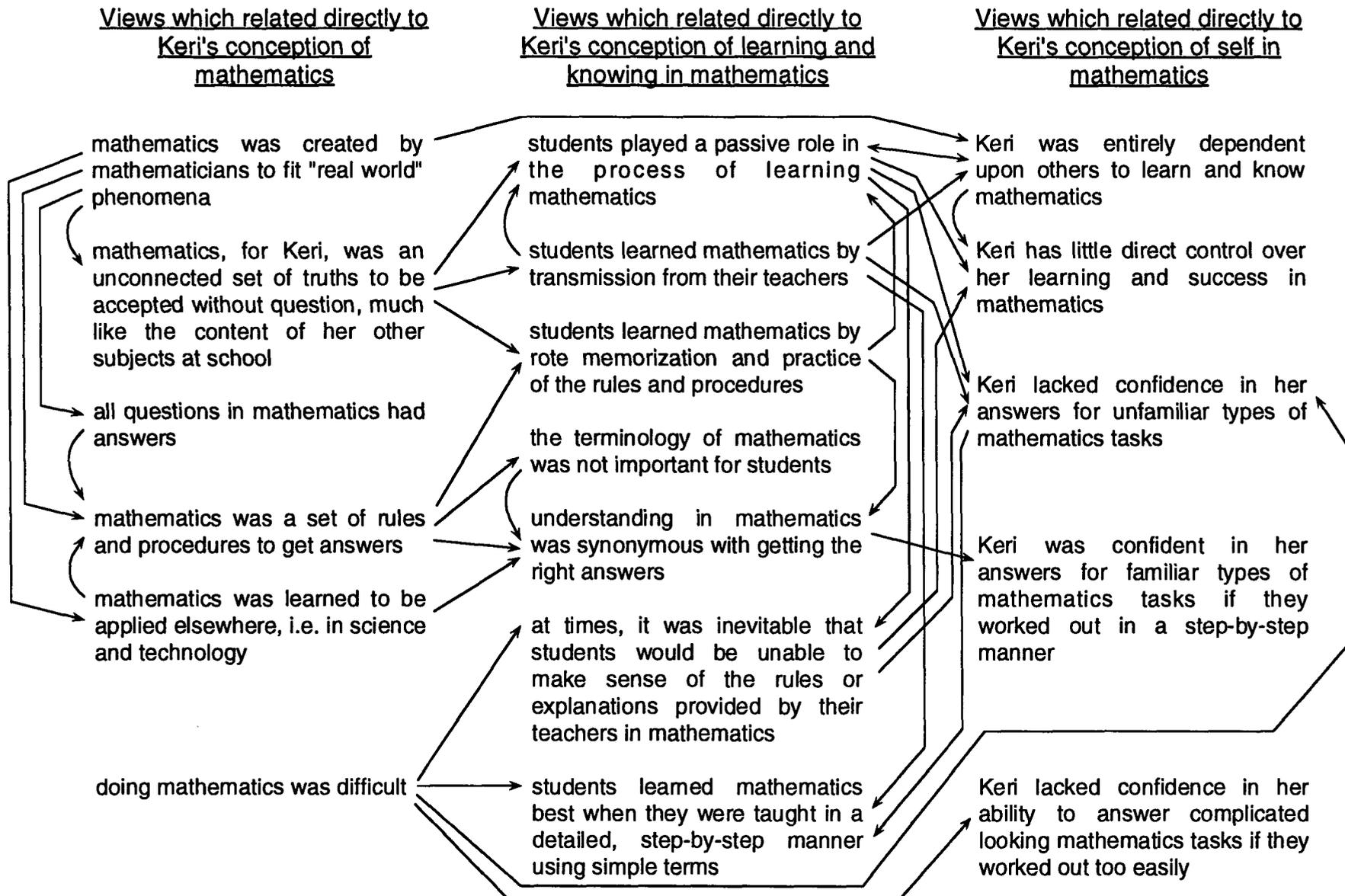
taught in a detailed, step-by-step manner using simple terms; she lacked confidence in her ability to answer complicated looking mathematics tasks if they worked out too easily; she lacked confidence in her answers for unfamiliar looking mathematics tasks; and, at times, it was inevitable that students would be unable to make sense of the rules and explanations provided by their teachers. This last point, in turn, supported Keri's view that she was powerless as a mathematics learner--she had little direct control over her learning and success in mathematics, and it lent further support to her view that she lacked confidence in her ability to make sense of unfamiliar looking mathematics tasks on her own. The interrelations of Keri's conceptions which have been described above are illustrated in Figure 4.2. As was the case in Figure 4.1, each arrow in the diagram indicates where one view supported another.

Differences between James' and Keri's conceptions

The use of assertions as organizers in the previous sections of this chapter focused attention on the many similarities in the two students' conceptions. It would be inappropriate to leave this discussion without consideration of the ways in which the students' views differed so these will now be dealt with to provide some balance to this discussion. Comparisons will be reported only for those aspects of the students' conceptions for which corresponding views have been constructed and reported for both of the students.

Keri's conception of mathematics was shaped and integrated

Figure 4.2 The integration of Keri's views and conceptions



largely by her view of the origin of mathematics knowledge. James, on the other hand, did not indicate that he held any beliefs about mathematics which served to unify or integrate his conception of mathematics. While both students experienced the mathematics that they were learning in Algebra 12 as an unconnected set of rules and procedures, Keri believed that mathematics was a connected body of information and that these connections would be taught to her at some later time. Keri believed that mathematics might play a direct and significant role in her future career. She believed that she was learning mathematics so that she might put it to use later in life. James, on the other hand, saw no use for the mathematics content that he was learning. He was taking mathematics to exercise his mind and to look good to others.

Both James and Keri expected that they would be unable to understand their teacher's explanations in class. Keri expected that she would not understand from time to time, while James made no attempt to attend to the teacher's explanations on a regular basis, expecting that he would not understand these at all. The students' views of their mathematics teacher differed as well. Keri preferred and expected her teacher to explain things slowly and in a step-by-step manner, and she was generally content with the way that she was being taught in Algebra 12. James, while reporting that he also expected his teacher to explain things slowly and in a step-by-step manner, was critical of many aspects of the teaching in his Algebra 12 class. In criticizing his teacher's emphasis on rules for doing logarithms, James

contradicted his view that when doing logarithms it all came down to the rules. No contradictions between any of Keri's views were evident in this analysis.

Both students lacked confidence in their abilities to do unfamiliar mathematics problems. James often decided that he could not do unfamiliar looking problems even before making any attempt to answer them. Keri, in contrast, would often try to get an answer for unfamiliar types of problems, but lacked confidence in her results.

From the analysis and description of the integration of Keri's views and conceptions presented earlier, it is evident that her conception of mathematics also played a large part in shaping and integrating her views about learning and knowing mathematics, and her view of herself in mathematics. Every single one of the individual views which made up her conceptions of learning and knowing in mathematics, and her conception of self in mathematics was supported by at least one of her views which related to her conception of mathematics.

From the analysis and description of the integration of James' views and conceptions, it is evident that his conceptions of learning and knowing mathematics played a large part in shaping and integrating most of his views about mathematics and about himself in mathematics. Many of his views relating to learning and knowing in mathematics were connected with his conception of mathematics and/or his conception of himself in mathematics. Two of James' views, that mathematics increased in difficulty as one advances in the study of mathematics, and that

the purpose of learning mathematics at the Algebra 12 level was to exercise the mind, were not integrated with any other of his views in this analysis; and, as mentioned earlier, two of James' views, that he hated his teacher's emphasis on rules in logarithms, and that logarithms were little more than a set of rules, were contradictory.

CHAPTER FIVE: SUMMARY, CONCLUSIONS, AND IMPLICATIONS

In this study two Algebra 12 students' conceptions relating to mathematics, and learning, knowing, and self in mathematics were examined in detail using qualitative research methodology consistent with a constructivist perspective. The logarithms unit in their Algebra 12 course served as the context for the study. Observations of these students in class during the unit of study, the students' regular written work done during the unit, their written work for tasks prepared by the researcher, and a number of follow-up interviews after the unit served to inform this analysis.

Summary of the results

In this section the views that made up James' and Keri's conceptions of mathematics, and their conceptions of learning, knowing and self in mathematics will be summarized. These conceptions were largely integrated.

The students viewed mathematics as a set of procedures and rules, which were handed down to them by their teacher for use to answer "questions" in mathematics. Mathematics knowledge was largely undifferentiated from other kinds of knowledge which they learned at school--unconnected truths to be memorized. They also viewed their taking of mathematics at the Algebra 12 level as serving purposes outside of the field of mathematics.

The students believed in a rote approach to learning mathematics--that learning mathematics involved the transmission of information from teacher to students, and that students in

turn were to memorize and practice applying this information. The teaching of mathematics was best done slowly and in a step-by-step manner to make it easy for students, and their role in this process was a passive one, to attend to and accept what the teacher said. This view of learning mathematics was no different from their views of learning in other subjects at school. They also believed that they "understood" mathematics when they were able to apply the rules and procedures provided by the teacher to get the right answers for the questions encountered in class.

Finally, the students lacked confidence in themselves when learning and doing mathematics. They also believed that their learning and success in mathematics depended upon factors which were largely beyond their control.

A comparison of the students' views of mathematics with those reported in the research literature

James' and Keri's views of the nature and content of mathematics were much like those reported for other mathematics students in the research literature. Two aspects of students' views as reported in this study were unlike those reported in any of the earlier studies. These were their views on the nature of knowledge in mathematics in comparison with knowledge in their other subjects at school, and in Keri's case, the origin or source of knowledge in mathematics.

Most of the studies reported earlier indicated that the student participants had little sense of mathematics apart from their classroom experiences--they lacked an understanding of the work of mathematicians and of mathematics as a discipline, and

they relied upon their teacher or the textbook as their source of knowledge and authority in their learning of mathematics. Similarly, James and Keri relied upon their teacher as their source of mathematics knowledge, although Keri was unique in that she had a well articulated and integrated set of views relating to mathematics as originating with mathematicians who created it to fit "real world" phenomena and passed this knowledge on to mathematics teachers who, in turn, passed it on to their students.

A number of earlier studies reported that students viewed mathematics as a largely unconnected set of truths. Only two students, Tyrone in Cobb's (1985) study and Andrew in Oak's (1987) study, viewed mathematics as a connected body of knowledge. Both Frank (1985) and Confrey (1984) indicated further that their students viewed answers in mathematics as either completely right or completely wrong. James and Keri both experienced mathematics as an unconnected set of truths, although Keri believed that mathematics was connected and that she would eventually be taught these connections. While James' and Keri's focus on getting the right answers in mathematics and their view of mathematics as truths were consistent with the view of answers in mathematics as completely right or completely wrong, their specific views of the nature of answers in mathematics are open to conjecture.

Most of the earlier studies indicated that students viewed mathematics as comprised of rules, procedures, symbols, and/or algorithms used to get answers. James' and Keri's shared view,

that mathematics was a set of rules and procedures to get answers, was consistent with this.

Only two studies reported students' views on the purpose of learning or doing mathematics. The 1988 NAEP study (Brown et al., 1988b) indicated that approximately two thirds of the students surveyed thought that learning mathematics helped one to think logically, and Cobb's (1985) student Scenetra had the immediate goal of trying to look smart to others when she performed mathematics tasks in the classroom. Keri's view of learning mathematics--as a set of rules and procedures to be applied in science or technology was unlike the views of the students reported in the previous research literature, whereas James' view was similar. He viewed mathematics as a way of exercising his mind--to make him smarter and to improve his logic. James, in a less immediate way than Scenetra, was also trying to impress others in that he expected that Algebra 12 would look good to others on his school transcript.

One can infer Confrey's (1984) students' difficulty with mathematics from her description of their concern about the pressure they felt with the pace of their mathematics courses. In Kuba and McDonald's (1991) study, many of the elementary level students believed that an activity was mathematics only until it became known or automatic to them. In other words, for these students mathematics was difficult by definition. Both James and Keri viewed mathematics as difficult, and James reported being confused by many aspects of the teaching in his mathematics class.

A comparison of the students' views of learning and knowing in mathematics with those reported in the research literature

James' and Keri's views which made up their conceptions of learning and knowing in mathematics were consistent with many of the students' views reported in the research literature. James also held a number of specific views relating to teaching methodology and the usefulness of the information provided for learning mathematics that had not been examined in the other studies.

Most of students in the studies reported earlier described learning and doing mathematics primarily as imitating examples, memorization, following rules, and practice. This view, which implies a transmission view of learning, was similar to the view of learning and doing mathematics shared by James and Keri. In contrast, two of the students reported earlier had very different views of learning and doing mathematics. Cobb's (1985) grade two student Tyrone enjoyed playing around with numbers as an end in itself and regularly sought out connections between the various problems that he worked on. For Tyrone, mathematics was an active process of making sense of things for himself. Oak's (1987) student Andrew also viewed learning and doing mathematics as an active process which included speculation, trying out and verifying his own ideas, reasoning things out for himself, and making connections.

Cobb's (1985) student Scenetra; Frank's (1985) students; and three of Oaks' (1987) students, Patricia, Rolland, and Katie viewed "understanding" in mathematics as synonymous with being able to solve the problems encountered in class, a view also

shared by James and Keri. Related to this view was James' and Keri's shared view that the terminology of mathematics was not important, James' view that his teacher's explanations of the content and his explanation of common errors were confusing and of little use for students, and his view that the explanations in the textbook were removed from the needs of students. Oaks, in her survey of 144 college students reported that these students viewed the process of developing one's "understanding" in mathematics as involving two separate steps: 1) the memorization of an algorithm, and 2) practice using the algorithm until it could be performed to get correct answers on a consistent basis. James' view of "understanding" in mathematics was much like this view; it meant that he could apply rules and procedures to get the right answer. Furthermore, James' had a view of "knowing mathematics" that was distinct from "understanding." "Knowing mathematics," for James, was synonymous with remembering what the teacher had said or done in class.

Another view of "understanding mathematics" was reported in the research literature in contrast to that just mentioned. Oaks' (1987) student Andrew sought to develop his "conceptual understanding" when learning mathematics. This included developing meaning for problems, developing his knowledge about the goals in solving particular types of problems, and using mathematical derivation to construct new knowledge, as well as learning how to solve problems.

Confrey (1984), Schoenfeld (1988), and Oaks (1987) all reported that the students in their studies viewed their role in

the process of learning mathematics as a passive one, as the consumers of the mathematics presented by their teacher. Confrey reported further that the students in her study were unwilling to ask or answer questions within the classroom setting. James and Keri viewed their role in the classroom in much the same way, and like Confrey's students, they avoided taking part in dialogue with the teacher during their mathematics lessons.

Related to the students' view of their role in the classroom was their view of the teacher's role. Many of the studies reported earlier indicated that students viewed their mathematics teacher as their source and authority for mathematics knowledge. Schoenfeld (1988) reported further that the students in his study believed that it was their teacher's job to make sure that students knew how to apply the rules that were taught. Both James and Keri viewed their mathematics teacher as the source of their mathematics knowledge, and felt that it was the teacher's responsibility to present the content in a manner which made learning easy for them. James believed that the examples provided by his teacher on the blackboard in class were very important for his learning and success in mathematics. He had strong negative views, however, regarding the usefulness of the Algebra 12 textbook and certain types of his teacher's explanations in class as discussed above. None of the research literature reviewed dealt with students' views of the information provided by the teacher and teaching materials in regular mathematics programs. While Earlwanger's (1975) report did include students' views of specific learning materials, the focus

of his study was on students in an unconventional, individualized mathematics program in which there was little presentation of information by the teacher.

A comparison of the students' views of self in mathematics with those reported in the research literature

James' and Keri's views relating to their conceptions of self in mathematics were also consistent with those reported for many other students in the mathematics education research literature. The present study is unique in that a description of specific conditions which determined a student's (Keri's) confidence when doing mathematics is provided. While both James and Keri lacked confidence in their abilities to do unfamiliar looking mathematics problems, there was a significant difference in their approach to these types of problems--Keri would often make an attempt to get an answer while James would often not.

Frank (1985) reported that the students in her study were confident in their ability to do familiar looking types of problems; Confrey and Lanier (1980) reported that the students in their study lacked confidence generally in mathematics; Confrey (1984) reported that the students in her study lacked confidence in their own answers and ideas in mathematics; and three of the four students in Oaks' (1987) study, Patricia, Rolland, and Katie had low expectations for their success in learning mathematics. The only exception to this was Andrew in Oaks' study who was confident that he could make sense of mathematics on his own. Both James and Keri lacked confidence in their ability to answer unfamiliar types of problems, and James also lacked confidence in

his ability to make sense of mathematics.

Confrey and Lanier (1980) and Oaks (1987) reported on students' sense of control over their learning and success in mathematics. The students in both cases had a sense of powerlessness in mathematics, and in Oaks' third preliminary study students blamed their mathematics teacher for their lack of success. Further evidence of students' sense of their lack of control in learning mathematics can be inferred from the reports that many students relied upon the teacher as their source of mathematics knowledge, and did not see themselves as playing an active role in making sense of mathematics for themselves. Like many of the students reported in the research literature, James and Keri believed that they were entirely dependent upon their teachers for learning and knowing mathematics, and had little direct control over their own learning and success in mathematics.

Confrey and Lanier (1980), and Confrey (1984) examined students' views of their own ideas in mathematics, and in both cases the students did not value them. In contrast, Oaks' (1987) student Andrew had learned that he was able to figure things out for himself and verify his own conjectures in mathematics. In this study James indicated that he had very little, if anything, to contribute to his own learning and knowledge of mathematics. Keri's view of her own ideas was not part of the analysis in this study.

A comparison of the interrelation of the students' conceptions with other reports in the research literature

All of the studies reported earlier which examined a wide range of students' beliefs indicated, at least implicitly, that students' views relating to different aspects of learning mathematics were part of a coherent and integrated system of beliefs. Earlwanger (1975) described students as having a cohesive system of interrelated ideas; Cobb (1985) reported that students' beliefs about mathematics and doing mathematics were intimately related; and Nicholls, Cobb, Wood, et al. (1990) reported that students had coherent and encompassing conceptions of mathematics and learning mathematics. These views are entirely consistent with the analysis of James' and Keri's conceptions in this study.

Conclusions

The results from this study indicate that students' conceptions of mathematics, and their conceptions of learning, knowing, and self in mathematics play important roles in their learning of mathematics. These conceptions impact upon students' learning of mathematics by influencing the kinds of goals they construct for themselves in learning mathematics, shaping the meanings that students construct in the classroom, influencing how students participate in the learning process, and, at times, even distracting them from the learning process. A more detailed explanation of these conclusions follows.

Students' views on the nature of mathematics and their views on learning and knowing in mathematics influence the kind of

mathematics knowledge that they construct by shaping the goals that they set for themselves in the process of learning mathematics. Both James and Keri, for example, viewed mathematics as an unconnected set of rules and procedures and "understanding" in mathematics as being able to apply these rules and procedures to get the right answers for problems like those encountered in their mathematics class. From this perspective of mathematics, there was little else to be learned. Furthermore, both students expected that the teacher would show them everything that they needed to know, and expected and accepted that having difficulty with the teacher's explanations was part of learning mathematics. Therefore, James and Keri had no reason to consider how else and where else new material might be applied, nor to consider on their own how it might be connected with their prior mathematics knowledge, nor to seek clarification from the teacher when things didn't make sense.

Profound but subtle failures in communication which hamper students' learning can occur between teachers and students in the mathematics classroom when their views differ and the negotiation of meaning does not occur. If, for example, a mathematics teacher were to explain a new concept or topic to James or Keri intending for them to construct connections between it and their knowledge of other mathematics topics, and afterwards the student's response was, "Yes, I understand," the teacher could easily misinterpret this to mean that the student understood it as the teacher had intended. Similarly, Keri's asking "Why?" in mathematics, which for her was done to ascertain which features

of a problem were critical in determining what rule or procedure to use, could be misinterpreted by her mathematics teacher to indicate that she was asking about the underlying reasons for a rule or procedure working as it did. And, given Keri's view that in Algebra 12 she didn't need to know anything about underlying reasons of this sort, one would expect Keri to be dissatisfied and possibly frustrated with her teacher's efforts to help her in such a situation. The same kinds of possibilities for communications failure occur in the assessment of students' understandings when there are limited opportunities for the negotiation of shared meaning between students and their teacher. While James and Keri were successful in evaluating complicated algebraic expressions involving logarithms on their unit test, for example, they did not view algebraic representations in this context as a generalization of numeric cases. Therefore, James' and Keri's success on such tasks could be misinterpreted to indicate that they had facility in evaluating numeric expressions involving logarithms as well, which was not the case.

Students' views on learning, knowing, and self in mathematics can play a major role in limiting or enhancing students' participation within the mathematics classroom, and therefore their opportunities for learning and success. James' views' relating to the learning of mathematics limited his participation in the classroom and limited the sources of information that were available to him when learning mathematics and studying for tests. Specifically, James chose to systematically disregard certain of his teacher's explanations in

class, avoided dialogue with the teacher during lessons, did not use his textbook as a resource in his learning, did not bother to attempt certain types of problems on his own, and blamed his teacher for his confusion in learning logarithms. These views limited James' opportunities to learn and improve his performance in mathematics. Keri's views, in contrast, gave her confidence that she could be successful when attempting to complete familiar looking mathematics tasks, and she would often persist in working on these during and after class.

Students' views relating to mathematics are most often consistent. However, students may have unresolved conflicts amongst their views which may be a source of frustration and distraction for them, and therefore a detriment to their learning. James, for example, disliked his teacher's emphasis on rules for logarithms, while expressing the views that logarithms in particular, and mathematics in general, consisted of little more than rules and procedures. This was an inescapable but needless source of frustration and distraction for James in his learning of logarithms.

Students' conceptions which guide them in their learning of mathematics are shaped by many things, including their views and expectations which arise from their learning experiences outside of the mathematics classroom. Therefore, students may hold views about knowledge and learning which are inappropriate for learning mathematics and which hinder them in learning mathematics. Both James' and Keri's views of knowledge and learning in mathematics were largely undifferentiated from their views of knowledge and

learning in their other subjects at school, and in James' case, his view of learning was undifferentiated from his view of learning to cook hamburgers at McDonalds. In both cases, the students viewed mathematics as a collection of unrelated truths to be accepted without question. These views, which most likely had been fitting for them elsewhere, played a part in impeding their development of a more connected and conceptual view of mathematics.

A comparison of the conclusions from this study with those reported in the research literature

Most of the conclusions from the studies reported earlier were general ones emphasizing the important role of a wide range of students' views relating to mathematics in their learning of mathematics, and the need to expand the scope of curriculum and assessment in mathematics to encompass these learning outcomes. And, while many recommendations were made to improve mathematics teaching, Oaks' (1987) study was the only one which was similar to this one and the only one whose conclusions were constructed in a manner which facilitated their comparison with the conclusions in this study. It should be noted however, that many of the findings, conclusions, and implications in the other research reports are consistent with those presented in this study.

The most prominent conclusions in the research literature are those which deal with the role of students' beliefs in their learning of mathematics. Most of the studies reported that students' views relating to mathematics played an important role

in their learning and success in mathematics; furthermore, the studies reported by Cobb (1985) and Nicholls, Cobb, Wood, et al. (1990) indicated that students develop conceptions relating to mathematics as early as the second grade. Confrey and Lanier (1980) reported further that many students' conceptions were an impediment to their intellectual growth, and similarly, Oaks (1987) reported that for three of the students in her study, their rote conceptions of mathematics did not allow for their success. These conclusions are consistent with the conclusion reported here, that students' views influence the kind of mathematics knowledge that they construct.

Oaks' (1987) position regarding classroom discourse is similar to the conclusion reached here, that communication in the classroom should be considered problematic because students sometimes use mathematics terminology differently than their teachers. She also reported that some of the students in her study had conflicting views relating to their learning of mathematics. These students believed that hard work would result in their success in mathematics. Their experiences indicated otherwise, however. These students' views, that they should be successful in mathematics because they had worked hard, also played a role in limiting their participation in learning mathematics--they responded by avoiding it. While Oaks' students' conflicting views were unlike those reported for James, the need to resolve conflicts amongst students' views was common to both cases. And, similar to the conclusion reported here, Oaks' students' views also played a major role in limiting their

participation in learning mathematics.

None of the research reported in the review of the literature addressed the point that students' conceptions which guided them in their learning of mathematics may have resulted from their learning experiences outside of the mathematics classroom and may therefore be inappropriate for their learning of mathematics.

Implications

There is a widely shared view within the mathematics education literature that students' conceptions relating to mathematics need to be addressed in order to improve their performance in mathematics. And, given that students' conceptions are largely integrated, are self supporting, and play a fundamental role in students' interpretation of their experience, fundamental changes are needed in students' mathematics learning experiences to bring about wholesale changes in their views of mathematics and learning.

The scope of mathematics teaching and evaluation must be expanded to address the wide range of conceptions that students construct and which impact upon their subsequent learning and performance in mathematics. This will require that teachers examine their own views relating to mathematics and develop an understanding of the role of their own and their students' conceptions within the mathematics classroom, the ways that their teaching impacts upon the development of students' conceptions, and new teaching and evaluation methodologies. Students will

need learning experiences which challenge their inappropriate views relating to mathematics and provoke the development of more helpful beliefs and conceptions. Students will also need to examine and better understand their own learning and performance in mathematics; examine and develop their own mathematics epistemologies; and consider the ways that knowledge and learning in mathematics are similar to, and dissimilar from, that in their other subjects at school so that they will approach learning in each in an appropriate manner. The allocation of time and effort within mathematics programs must be adjusted to facilitate reflection and the negotiation of meaning amongst the players in the mathematics classroom on an ongoing basis, and to facilitate making the adjustments that will be necessary to move to this new approach to teaching and learning both for teachers and students.

Mathematics curriculum designers and policy makers must also reconsider and redefine standards for success in school mathematics based upon the view that a wide range of learning is needed, apart from content knowledge, for students to develop their facility and confidence in applying mathematics and in problem solving outside of the classroom. Meeting these new standards would likely require that less content be covered, especially in the early stages of implementation, to facilitate students' development of deeper understandings of the content that is covered as well as mathematics related conceptions that would help to facilitate their future success in mathematics.

Teachers must also become involved in an ongoing dialogue to reflect upon and develop their views about teaching and learning,

and to negotiate consensus about the kinds of mathematics programs that are needed to address students' wide ranging beliefs relating to mathematics. Teachers must become active participants in the rebuilding of a new mathematics education culture if this change is to last. All the while, students, their parents, and the other players in education must be persuaded to buy into this new view of school mathematics as well.

Implications for research

The results from this study are tied in many ways to the context in which the research was done. To develop our understanding of students' conceptions relating to mathematics, further research is needed which examines other students' conceptions within similar contexts, students' conceptions relating to the study of different topics in mathematics, the conceptions held by students of different ages, the longitudinal development of students' conceptions, and the conceptions of students in different types of mathematics programs. Furthermore, the role of students' experiences outside of the mathematics classroom in shaping their conceptions relating to mathematics and learning mathematics needs to be examined, as well as students' conceptions relating to learning, knowing, and self in the wider school context.

Research is also needed which examines alternative teaching and evaluation methods which might contribute to students' development of more appropriate conceptions, as well as research

which examines alternative approaches for implementing new mathematics curriculum.

Epilogue

Undoubtedly, a radical shift or revolution is needed to bring about the substantial changes and improvements in mathematics education that the research on students' conceptions suggests. Piecemeal efforts have not, and will not, work. Success stories, like those of Tyrone (Cobb, 1985) and Andrew (Oaks, 1987) are all too rare in the research literature, but they offer some direction and indication that students' conceptions can be developed for the better.

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APPENDIX 1: THE STUDENTS' EXPONENTS AND ALGEBRA SURVEYS

James' exponents and algebra survey: page 1 of 4

Exponent Review This WILL NOT be graded. name James

Each of the following questions has two parts. First, circle the letter which represents the statement that best represents your experience with each type of problem. Your choices are:

- A. I have seen this type of problem before and know how to solve it
 B. I have seen this type of problem before but I am not absolutely sure what to do
 C. I have not seen this type of problem before.

Then solve each problem if you can and show all of your steps. Some questions cannot be solved. For these items write "Cannot Be Solved" beside them & explain in a few words your reason why.

Cross out any mistakes rather than erase them so that your teacher can see what you did.

DO NOT use a calculator.

Solve the following

1) $2,700 \div 3^3$ (A) B C

= 100

2) $5^3 \times 2^3$ (A) B C

= 1000

3) $(4.76 \times \pi)^0$ (A) B C

= 1

4) If $x=5$ A B (C)

then $x^{\frac{x}{2}} =$ _____

5) -9^2 (A) B C

= 81

6) $\left(\frac{3}{4}\right)^3$ (A) B C

= $\frac{27}{64}$

James' exponents and algebra survey: page 2 of 4 page 2
 remember A - I have seen this type of problem before and know how to solve it.
 B - I have seen this type of problem before but I am not absolutely sure what to do.
 C - I have not seen this type of problem before

$$7) 10^3 + 10^2 \quad \textcircled{A} B C$$

$$= \underline{1100}$$

$$8) \left(\frac{2}{5}\right)^{-2} \frac{5^2}{2^2} \quad \textcircled{A} B C$$

$$= \underline{25/4}$$

$$9) -4^3 \times 4^{1/2} \quad A \textcircled{B} C$$

$$= \frac{-64 \times 2}{-128}$$

$$10) 3 \left(\frac{a}{2b}\right)^2 3 \frac{a^2}{4b^2} \quad \textcircled{A} B C$$

$$= \underline{3 \frac{a^2}{4b^2}}$$

$$11) (\sqrt{x})^3 \quad \textcircled{A} B C$$

$$= \underline{x\sqrt{x}}$$

$$12) \frac{3}{2}(m+2) = 2(2m-5) \quad \textcircled{A} B$$

$$\frac{3}{2}m + 2 = 4m - 10$$

$$2\frac{1}{2}m = \frac{-12}{2\frac{1}{2}}$$

$$m = \underline{-4.8}$$

$$13) \frac{3l-7}{8+l} = \frac{1}{2} \quad A B \textcircled{C}$$

$$l = \underline{\hspace{2cm}}$$

$$14) x^2 - 11x + 30 = 0 \quad A B C$$

$$(x-6)(x-5) = 0$$

$$x = \underline{x=6 \quad x=5}$$

$$15) 2x^2 - 5x - 3 = 0$$

$$A \textcircled{B} C$$

$$x = \underline{(2x+1)(x-3)}$$

James' exponents and algebra survey: page 3 of 4

- remember
- A. I have seen this type of problem before and know how to solve it ^{page 3}
- B. I have seen this type of problem before but I am not absolutely sure what to do
- C. I have not seen this type of problem before.

Fill in the box to make each equation true (if you can)

16) $5^{\boxed{4}} = 625$ (A) B C 17) $3^{\boxed{E}} = \frac{1}{3}$ (A) B C

18) $(-4)^2 = \boxed{16}$ (A) B C 19) $6^{\boxed{0}} = 1$ (A) B C

20) $1^{\boxed{\text{can't}}} = 2$ A B C

21) $\sqrt{10} = 10^{\boxed{\frac{1}{2}}}$ A (B) C

22) $8^{\boxed{7}} = 2^{21}$ A (B) C

23) $3^{\boxed{-2}} = \frac{1}{9}$ (A) B C

24) $\left(\frac{1}{3}\right)^{\boxed{-1}} = 3$ (A) B C

25) $10^{\boxed{\text{can't}}} = 0$ A B C

26) $2^{\boxed{\text{can't}}} = -4$ A B C

27) $\pi^4 \cdot \pi^7 = \pi^{\boxed{11}}$ (A) B C
multiply ↗

28) $\left(\frac{1}{2}\right)^{\boxed{-4}} = 16$ (A) B C

29) $\frac{x^3}{x^3} = x^{\boxed{-2}}$ (A) B C

30) $5^7 \times 5^{\boxed{-5}} = 25$ (A) B C

31) $10^3 x^6 = (\boxed{10x^3})^3$ (A) B C

32) $\sqrt[10]{8^3} = 8^{\boxed{\quad}}$ A (B) C

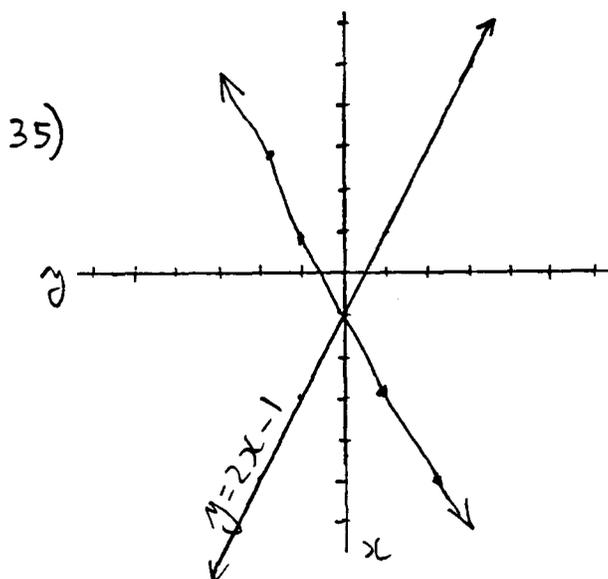
33) $\frac{1}{n^{\boxed{\quad}}} \times n^{-2} = n^3$ A (B) C

page 4

Spend 2-3 minutes on each of the following.

- 34) Does squaring a positive number always result in a larger number? Explain in a few words.

No because if you square 1 you get 1 which is not larger



Explain in a few words what is meant by the term "inverse of a function."

Use the function equation and graph in your explanation and draw the inverse of $y = 2x - 1$ on the same graph if you can.

for inverse you switch x, y .

$$x = 2y - 1$$

Exponent Review This WILL NOT be graded. name Keri

Each of the following questions has two parts. First, circle the letter which represents the statement that best represents your experience with each type of problem. Your choices are:

- A. I have seen this type of problem before and know how to solve it
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Then solve each problem if you can and show all of your steps. Some questions cannot be solved. For these items write "Cannot Be Solved" beside them & explain in a few words your reason why.

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Solve the following

1) $2,700 \div 3^3$ (A) B C

= 100

2) $5^3 \times 2^3$ (A) B C

= 1000

3) $(4.76 \times \pi)^0$ (A) B C

= 1

4) If $x=5$ (A) B C

then $x^{\frac{2}{2}} =$ 125

5) -9^2 A (B) C

= 81

6) $\left(\frac{3}{4}\right)^3$ (A) B C

= $\frac{27}{64}$

Keri's exponents and algebra survey: page 2 of 4

remember page 2
 A- I have seen this type of problem before and know how to solve it.
 B- I have seen this type of problem before but I am not absolutely sure what to do.
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7) $10^3 + 10^2$ (A) B C

$= 1100$

8) $\left(\frac{2}{5}\right)^{-2}$ A (B) C

$= \frac{25}{4}$

9) $-4^3 \times 4^{\frac{1}{2}}$ A B (C)

$=$

10) $3\left(\frac{a}{2b}\right)^2$ A (B) C

$= \frac{3a^2}{4b^2}$

11) $(\sqrt{x})^3$ (A) B C

$= x\sqrt{x}$

12) $\frac{3}{2}(m+2) = 2(2m-5)$ (A) B C
 $3m+6 = 8m-20$
 $26 = 5m$

$m = \frac{26}{5}$

13) $\frac{3l-7}{8+l} = \frac{1}{2}$ A B C

$2(3l-7) = 8+l$
 $6l-14 = 8+l$
 $7l = 22$

$l = \frac{22}{7}$

14) $x^2 - 11x + 30 = 0$ (A) B C
 $(x-6)(x-5)$

$x = 6 \text{ or } 5$

15) $2x^2 - 5x - 3 = 0$

A B C

$x = \frac{-1}{2}, 3$

Keri's exponents and algebra survey: page 3 of 4

- remember
- A. I have seen this type of problem before and know how to solve it ^{page 3}
- B. I have seen this type of problem before but I am not absolutely sure what to do
- C. I have not seen this type of problem before.

Fill in the box to make each equation true (if you can)

16) $5^{\boxed{4}} = 625$ (A)BC 17) $3^{\boxed{-1}} = \frac{1}{3}$ (A)BC

18) $(-4)^2 = \boxed{16}$ (A)BC 19) $6^{\boxed{0}} = 1$ (F)BC

20) $1^{\boxed{2}} = 2$ A B (C) 21) $\sqrt{10} = 10^{\boxed{1/2}}$ A B (C)

22) $8^{\boxed{7}} = 2^{21}$ A B (C) 23) $3^{\boxed{-2}} = \frac{1}{9}$ A (B)C

24) $(\frac{1}{3})^{\boxed{-1}} = 3$ (A)BC 25) $10^{\boxed{0}} = 0$ A B (C)

26) $2^{\boxed{-4}} = -4$ A B (C) 27) $\pi^4 \cdot \pi^7 = \pi^{\boxed{11}}$ (A)BC
multiply

28) $(\frac{1}{2})^{\boxed{-4}} = 16$ A (B)C 29) $\frac{x^3}{x^3} = x^{\boxed{-2}}$ A (B)C

30) $5^7 \times 5^{\boxed{-9}} = 25$ A (B)C 31) $10^3 x^6 = (\boxed{10x^3})^3$ A (B)C

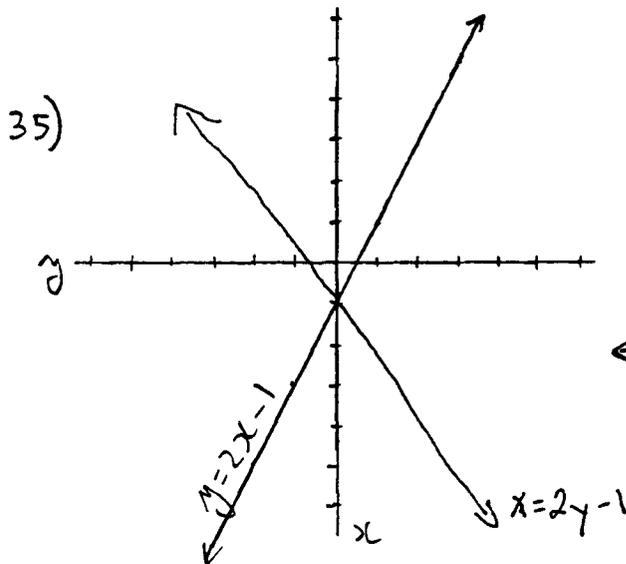
32) $10\sqrt{8^3} = 8^{\boxed{1/7}}$ A B (C) 33) $\frac{1}{n^{\boxed{5}}} \times n^{-2} = n^3$ A (B)C

page 4

Spend 2-3 minutes on each of the following.

- 34) Does squaring a positive number always result in a larger number? Explain in a few words.

If you square a positive number that is not a fraction then it is larger if it is a fraction it is smaller.



Explain in a few words what is meant by the term "inverse of a function."

Use the function equation and graph in your explanation and draw the inverse of $y = 2x - 1$ on the same graph if you can. You switch the "x" and "y" in your question for an inverse variation.

APPENDIX 2: THE STUDENTS' UNIT TESTS

James' unit test: page 1 of 5

James

NAME _____

UNIT 2
EXPONENTIAL FUNCTIONS
 AND
LOGARITHMS

DATE Mar 9/90

$$\frac{96}{153} = \frac{48}{76}$$

1. A. Classify each of the following as true or false:

a) $2^{2/3} \cdot 2^{4/3} = 4$ b) $(2^{2/3})^{4/3} = 4$

c) $2^{2/3} \cdot 3^{2/3} = 6^{2/3}$

2M 1a) True //
 2M 1b) False // $\frac{6}{6}$
 2M 1c) True // $\frac{6}{6}$

1. B. Which one of the following is not true?

d) $2^1 < 2^{\sqrt{3}} < 2^2$ e) $(\sqrt{2})^x \in \text{Reals}$

f) $(\pi)^{\sqrt{2}} \in \text{Reals}$ g) $\left(\frac{1}{2}\right)^x = 2^{-x}$, where $x \in \text{Reals}$ 5M 1B. H // $\frac{2}{2}$

h) $\left(-\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-x}$, where $x \in \text{Reals}$

2. Simplify the following: (express answers in radical form)

a) $\sqrt[3]{4} \cdot \sqrt[3]{4} =$ b) $\sqrt[3]{4} \cdot \sqrt[3]{32} =$ c) $\sqrt{\left(\frac{4}{9}\right)^3} =$ 2M 2a) $\sqrt{1} = 2$ //

d) $\frac{\sqrt{2}}{\sqrt{2}} =$ e) $\sqrt[3]{4} \cdot \sqrt[3]{16} =$ f) $-\left(\frac{8}{27}\right)^{-2/3} =$ 3M 2b) $2 \sqrt{2^3} = 2^3$ //

g) $\sqrt[3]{t^2} \cdot \sqrt[3]{t^3} =$ h) $(\sqrt{5^{12}})^{-1/2} =$ 2M 2c) $\frac{5}{27}$ //

2M 2d) $\frac{3 \cdot 2^2 \cdot 0/2}{21}$ //

3M 2e) 2 //

3M 2f) $-\frac{5\sqrt{2}}{2} = -\frac{3}{2} \cdot \frac{2}{3}$ //

3M 2g) $1 + \frac{12}{t} + 5 \cdot \frac{2}{3}$ //

3M 2h) $\frac{1}{5}$ //

3. Solve:

a) $x^{-2/3} = 9$

b) $t = 3 + 2t^{1/2}$

2M 3a) $\frac{1}{27}$ //

c) $5^{x+1} = 5^{3x+5}$

d) $9^{3x^2} = 3^{-x+2}$

4M 3b) $t = 9, t = -1$ //

2M 3c) $x = -2$ //

4M 3d) $t = \frac{1}{2}, t = \frac{4}{3}$ //

36
41

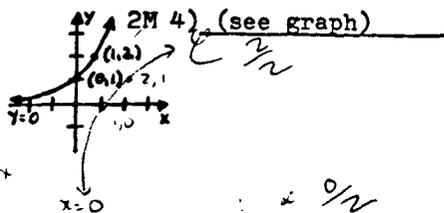
James' unit test: page 2 of 5

James

page 2

UNIT 2

4. Given the graph of $f = (x,y): y=2^x$, sketch the graph of the inverse function on the same set of axes.



5. In Question 4, the formula for the inverse function is:

- a) $y = 2^x$ b) $y = x^2$ c) $y = \log_2 x$
- d) $y = \log_x 2$ e) $x = \log_2 y$

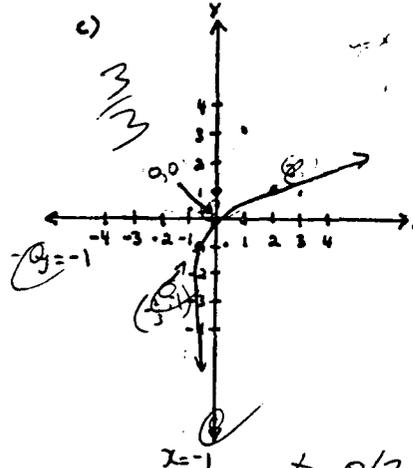
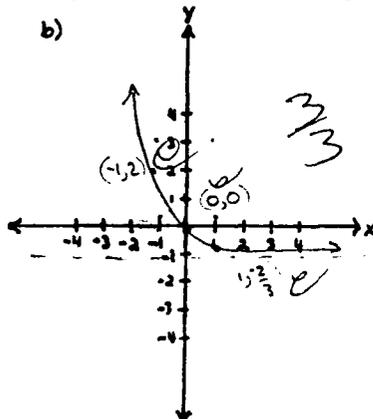
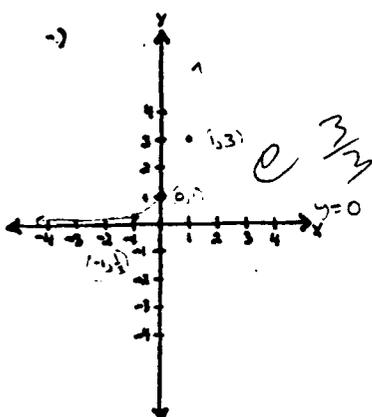
2M 5) $E(x = \log_2 y)$

6. Graph each of the following on the grid provided:

- a) $y = 3^x$ b) $y = 3^{-x} - 1$ c) $y = \log_3(x+1)$

Note: label at least 3 points and all asymptotes.

3M 6a) $y = 3^x$
 3M 6b) $y+1 = 3^{-x}$
 3M 6c) $y = \log_3(x+1)$



7. Simplify:

a) $5\sqrt{3} \cdot 5\sqrt{27} =$ b) $(\sqrt{2}\sqrt{3}) - \sqrt{3} =$

2M 7a) $5^9 = 1953125$
 3M 7b) $\frac{\sqrt{2}}{4} = \frac{1}{4}\sqrt{2}$

8. Which one of the following is not true?

- a) $\log_9 3 = \frac{1}{2}$ b) $3^{\log_3 3} = 3$ c) $\log_3 3^3 = 3$

- d) $\log_{\sqrt{2}}(-8) = -6$ e) If $\log_2 x = \log_2 3$, then $x=3$

2M 8) d
 16/20

James' unit test: page 3 of 5

age 3 James UNIT 2

9. If $\log_{2x} 16 = 2$, then $x = ?$

- a) 2 b) -2 c) ± 2 d) Undefined

3M 9) $C \times 0$

10. Solve for "x":

a) $\log_{25} x = -\frac{3}{2}$ b) $\log_x \left(\frac{1}{9}\right) = -2$

3M 10a) $x = 0.008$

3M 10b) $x = \frac{1}{3} \times 0$

3M 10c) $x = 1.503$

c) $\log_9 27 = x$

d) $\frac{1}{2}x^{-2/3} = 8$

3M 10d) $x = 0.088$

Handwritten notes:
 $\frac{2}{3} \left(\frac{1}{6} \right)^{-3} = 8$
 $\frac{2}{3} \left(\frac{1}{6} \right)^{-3} = 8$
 $x = .125$
 [Completed during the interviews with a calculator]

11. If $\log_b 2 = c$ and $\log_b 3 = d$, then $\log_b 12 = ?$

- a) $c + d$ b) cd c) $2cd$ d) $2c + d$

e) $c^2 + d$

2M 11) $b \times 0$

12. Using Question 11, $\log_b \frac{9}{8} = ?$

- a) $\frac{2d}{3c}$ b) $\frac{3c}{2d}$ c) $2d - 3c$ d) $3d - 2c$

e) $d^2 - c^3$

4M 12) $b \times 0$

13. $\log_3 18 - \log_3 2 + \log_3 3 = ?$

3M 13) 1×0

14. The solution set of $\log(x-1) + \log(x+2) = 1$ is ?

- a) $\{3\}$ b) $\{4, 3\}$ c) $\{-4\}$ d) $\{-4, 3\}$ e) \emptyset

4M 14) $E \times 0$

15. Without referring to tables, determine the value of "n" or "x":

a) $\log_{10}(x+4) + \log_{10}(x+1) = 1$

4M 15a) $\phi \times 0$

b) $\log_3 n = \log_3 4 + \log_3 12 - \log_3 16$

3M 15b) $N = \frac{3}{3}$
 $\frac{9}{35}$

James' unit test: page 4 of 5

page 4 James UNIT 2

15. c) $\log_{10} x - \log_{10} 3 = 1 - \log_{10} 2$

5M 15c) $\frac{KX = 60}{\frac{0}{5}}$

16. If $\log M = 6$ and $\log N = 3$, evaluate the following:

a) $(\log M)(\log N) = (6)(3)$

2M 16a) $\frac{18}{//}$

b) $\log(MN) = 6 + 3$

2M 16b) $\frac{9}{//}$

c) $\log\left(\frac{M}{N}\right) =$

2M 16c) $\frac{3}{//}$

d) $\frac{\log M}{\log N} =$

2M 16d) $\frac{2}{//}$

e) $\log(N^3) =$

2M 16e) $\frac{1.413}{//}$

f) $\log(\sqrt{M}) =$

2M 16f) $\frac{2.45}{//}$

17. Write a logarithmic equation to represent the following calculations:

a) $N = \frac{50,200}{(0.637)(149)}$

$\log N = \log 50,200 - (\log 0.637 + \log 149)$
3M 17a) $\underline{\hspace{2cm}}$

b) $N = \frac{\sqrt{5\pi}}{\sqrt[3]{1.08}}$

$\log N = \frac{1}{2}(\log 5 + \log \pi) - \frac{1}{3} \log 1.08$
4M 17b) $\underline{\hspace{2cm}}$

18. If the logarithmic equation for a calculation is illustrated below, write the original calculation without logarithms:

$\log V = \log 4r + 3 \log r - \log 3$

3M 18) $V = \frac{(4r)(r^3)}{3}$

$V = \frac{4\pi r^2}{3}$

19) Solve for "T":

$\log_2 \sqrt[3]{T} + \log_2 \sqrt[3]{T} = 5$

5M 19) $T = 64 \left(\frac{2}{5}\right)$

?

?

$\frac{23}{32}$

James' unit test: page 5 of 5

8. 5 James UNIT 2

20) If $\log A = 3$ and $\log B = 4$, find $\frac{\log(AB^3)}{\log(A^2B)}$

5M 20) $\frac{5.3}{0}$

21) Solve for "x": $3 \log x = 2 \log 8$

3M 21) $x = 4 \sqrt{\frac{1}{3}}$

22) Solve for "n": $\log_6 N = 2 - \log_6(N+5)$

4M 22) $N = 4 \quad N = -9$

23) Classify each of the following as true or false, given that $\log 1.24 \approx 0.0934$.

a) $\log 124 \approx 2.0934$

1M 23a) True ✓

b) The characteristic of $\log 124$ is 0.0934

1M 23b) False ✓

c) The mantissa of $\log 12.4$ is 2.

1M 24c) False ✓

d) $\log 0.00124 \approx 7.0934 - 10$

1M 24d) True ✓

e) $10^{1.0934} \approx 12.4$

1M 25e) True ✓

24) If $\log 13.4 \approx 1.1271$, then $\log 1340 \approx ?$

2M 24) 3.1271

25) If $x^3 = 4$, then $\log x = ?$

3M 25) $0.201 \sqrt[3]{3}$

26) Which one of the following is not true?

a) $\log_5 7 = \frac{\log_3 7}{\log_3 5}$ b) $\log_5 7 \cdot \log_7 5 = 1$ c) $\log_5 5^5 = 5$

d) $5^{\log_5 5} = 1$ e) $\log 5^7 = 7 \cdot \log 5$

3M 26) $E. \frac{0}{3}$

$\frac{12}{25}$

Keri's unit test: page 1 of 5

NAME Keri

UNIT 2
EXPONENTIAL FUNCTIONS
AND
LOGARITHMS

DATE March 9/00

$\frac{121}{153} = \frac{61}{76}$
①

1. A. Classify each of the following as true or false:

a) $2^{2/3} \cdot 2^{4/3} = 4$ b) $(2^{2/3})^{4/3} = 4$
c) $2^{2/3} \cdot 3^{2/3} = 6^{2/3}$

2M 1a) True //
2M 1b) False // 4
2M 1c) False // 6

1. B. Which one of the following is not true?

d) $2^1 < 2^{\sqrt{3}} < 2^2$ e) $(\sqrt{2})^x \in \text{Reals}$
f) $\left(\frac{\pi}{\pi}\right)^{\sqrt{2}} \in \text{Reals}$ g) $\left(\frac{1}{2}\right)^x = \left(\frac{1}{8}\right)^x$, where $x \in \text{Reals}$ 5M 1B. // // 2
h) $\left(-\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-x}$, where $x \in \text{Reals}$

2. Simplify the following: (express answers in radical form)

a) $\sqrt[4]{4} \cdot \sqrt[4]{4} = 2$ b) $\sqrt[4]{4} \cdot \sqrt[5]{32} = 2^{6/5}$ c) $\sqrt{\left(\frac{4}{9}\right)^3} = \frac{8}{27}$ 2M 2a) 2 // 2
d) $\frac{\sqrt{2}}{\sqrt{2}} = 1$ e) $\sqrt[4]{4} \cdot \sqrt[4]{16} = 2$ f) $-\left(\frac{8}{27}\right)^{-2/3} = -\frac{27}{8}$ 3M 2b) $\sqrt{2^3} = 2\sqrt{2}$ //
g) $\sqrt[3]{\frac{2}{8}} \cdot \sqrt[4]{\frac{3}{12}} = \frac{1}{2}$ h) $(\sqrt{5})^2 - \sqrt{2} = 3 - \sqrt{2}$ 2M 2c) $\frac{8}{27}$ //
2M 2d) $3\sqrt{2}$ //
3M 2e) 2 //
3M 2f) $\frac{27}{8}$ //
3M 2g) $\frac{1}{2}$ //
3M 2h) $3 - \sqrt{2}$ //

3. Solve:

a) $x^{-2/3} = \frac{9}{5}$
 $x = \left(\frac{5}{9}\right)^{3/2}$

c) $5^{x+1} = 5^{3x+5}$
 $x+1 = 3x+5$
 $-4 = 2x$
 $-2 = x$

*b) $t = 3 + 2t^{1/2}$
 $t - 3 = 2t^{1/2}$
 $(t-3)^2 = 4t$
 $t^2 - 6t + 9 = 4t$
 $t^2 - 10t + 9 = 0$
 $(t-1)(t-9) = 0$
 $t = 1$ or $t = 9$

d) $9^{3x^2} = 3^{-x+2}$
 $6x^2 = -x+2$
 $6x^2 + x - 2 = 0$
 $(3x+2)(2x-1) = 0$
 $x = -\frac{2}{3}$ or $x = \frac{1}{2}$

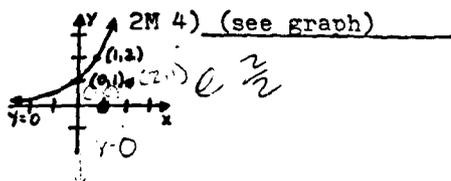
2M 3a) $t = \frac{1}{2}$ //
4M 3b) $t = 9$ // 2
2M 3c) $x = -2$ //
4M 3d) $x = -\frac{2}{3}$ or $\frac{1}{2}$ //

$\frac{31}{41}$

Keri's unit test: page 2 of 5

page 2 *Keri* UNIT 2

4. Given the graph of $f = (x,y): y=2^x$, sketch the graph of the inverse function on the same set of axes.



5. In Question 4, the formula for the inverse function is:

- a) $y = 2^x$ b) $y = x^2$ c) $y = \log_2 x$
- d) $y = \log_x 2$ e) $x = \log_2 y$

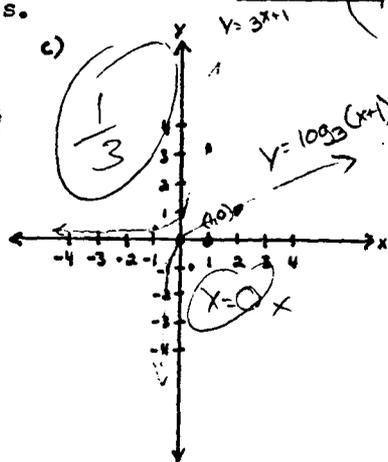
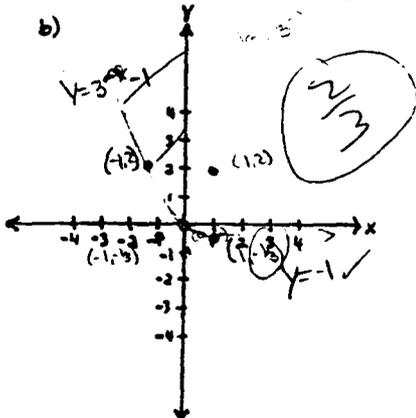
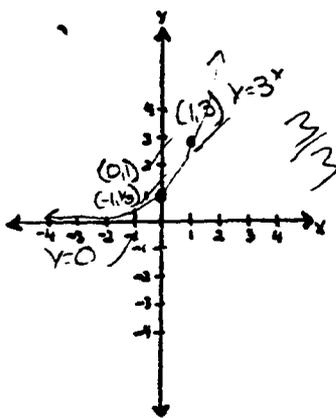
2M 5) C

6. Graph each of the following on the grid provided:

- a) $y = 3^x$ b) $y = 3^{-x} - 1$ c) $y = \log_3(x+1)$

Note: label at least 3 points and all asymptotes.

3M 6a) _____
 3M 6b) _____
 3M 6c) _____



7. Simplify:

a) $5\sqrt{3} \cdot 5\sqrt{27} = 9$
 $\sqrt{3} \cdot 3 \cdot 3 \cdot 3$

b) $(\sqrt{2} \cdot \sqrt{3})^{-3} =$
 $(2 \cdot 3)^{-3/2} = 2\sqrt{2}$

2M 7a) 9
 3M 7b) 2\sqrt{2}

8. Which one of the following is not true?

- a) $\log_9 3 = \frac{1}{2}$ b) $3^{\log_3 3} = 3$ c) $\log_3 3^3 = 3$

- d) $\log_{\sqrt{2}}(-8) = -6$ e) If $\log_2 x = \log_2 3$, then $x=3$

2M 8) D

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9. If $\log_{2x} 16 = 2$, then $x = ?$

- a) 2 b) -2 c) +2 d) Undefined

3M 9) $C \times \frac{0}{3}$

10. Solve for "x":

a) $\log_{25} x = -\frac{3}{2}$ b) $\log_x \left(\frac{1}{5}\right) = -2$

$x = 25^{\frac{3}{2}}$
 $x = 125^{\frac{3}{2}}$
 $x = 125 \sqrt{125}$

$1/5 = x^{-2}$
 $x^2 = 5$
 $x = \sqrt{5}$

3M 10a) $x = 125^{\frac{3}{2}} = 125 \sqrt{125}$

3M 10b) $x = 5^{\frac{2}{3}}$

3M 10c) $x = \frac{3}{2} e^{\frac{3}{3}}$

3M 10d) $x = \sqrt{16} = 4 = \frac{3}{3}$

c) $\log_9 27 = x$ d) $\frac{1}{2} x^{-2/3} = 8$

$27 = 9^x$
 $3^3 = 3^{2x}$
 $3 = 3^{2x}$
 $1 = 2x$
 $x = \frac{1}{2}$

$x^{-2/3} = 16$
 $x = 16^{\frac{3}{2}}$
 $x = 64$

11. If $\log_b 2 = c$ and $\log_b 3 = d$, then $\log_b 12 = ?$

- a) $c + d$ b) cd c) $2cd$ d) $2c + d$

e) $c^2 + d$ $2 = \frac{c}{b}$ $12 = b^d$

2M 11) $C \times \frac{0}{2}$

12. Using Question 11, $\log_b \frac{9}{8} = ?$

- a) $\frac{2d}{3c}$ b) $\frac{3c}{2d}$ c) $2d - 3c$ d) $3d - 2c$

e) $d^2 - c^3$

4M 12) $E \times \frac{0}{4}$

13. $\log_3 18 - \log_3 2 + \log_3 3 = ?$

$\log_3 \left[\frac{18}{2}\right] \cdot 3 = \log_3 [27] = x$

$27 = 3^x$
 $3^3 = 3^x$
 $3 = x$

3M 13) $x = 3 e^{\frac{3}{3}}$

14. The solution set of $\log(x-1) + \log(x+2) = 1$ is ?

- a) {3} b) {4, 3} c) {-4} d) {-4, 3} e) \emptyset

$\log[(x-1)(x+2)] = 1$
 $(x^2 + x - 2) = 10$
 $x^2 + x - 12 = 0$
 $(x+4)(x-3) = 0$

$x = -4, 3$ 4M 14) $A e^{\frac{4}{4}}$

15. Without referring to tables, determine the value of "n" or "x":

a) $\log_{10}(x+4) + \log_{10}(x+1) = 1$

$\log_{10}[(x+4)(x+1)] = 1$
 $\log_{10}(x^2 + 5x + 4) = 1$
 $x^2 + 5x + 4 = 10$
 $x^2 + 5x - 6 = 0$
 $(x+6)(x-1) = 0$

4M 15a) $x = 1 e^{\frac{4}{4}}$

b) $\log_3 n = \log_3 4 + \log_3 12 - \log_3 16$

$\log_3 n = \log_3 \left[\frac{4 \cdot 12}{16}\right]$ $n = 3$

3M 15b) $n = 3 e^{\frac{3}{3}}$

$\frac{26}{35}$

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15. c) $\log_{10} x - \log_{10} 3 = 1 - \log_{10} 2$

$\log_{10} x - \log_{10} 3 + \log_{10} 2 = 1$
 $\log_{10} \frac{x \cdot 2}{3} = 1$
 $\frac{x \cdot 2}{3} = 10$
 $x = \frac{15}{2}$

5M 15c) $x = \frac{15}{2}$

16. If $\log M = 6$ and $\log N = 3$, evaluate the following:

a) $(\log M)(\log N) =$

2M 16a) $18 //$

b) $\log(MN) =$

2M 16b) $9 //$

c) $\log\left(\frac{M}{N}\right) =$

2M 16c) $3 //$

d) $\frac{\log M}{\log N} =$

$\frac{12}{12}$ 2M 16d) $2 //$

e) $\log(N^3) =$

2M 16e) $9 //$

f) $\log(\sqrt{M}) =$

2M 16f) $3 //$

17. Write a logarithmic equation to represent the following calculations:

a) $N = \frac{50,200}{(0.637)(149)}$

$\log N = \log 50,200 - [\log 637 + \log 149]$

3M 17a) _____

b) $N = \frac{\sqrt{5\pi}}{\sqrt[3]{1.08}}$

$\log N = \frac{1}{2}(\log 5 + \log \pi) - \frac{1}{3} \log 1.08$

4M 17b) _____

18. If the logarithmic equation for a calculation is illustrated below, write the original calculation without logarithms:

$\log V = \log 4r + 3 \log r - \log 3$

$\log V = \log \left[\frac{4r(r^3)}{3} \right]$ $V = \frac{4r^4}{3}$

3M 18) $V = \frac{4r^4}{3}$

19) Solve for "T":

$\log_2 \sqrt{T} + \log_2 \sqrt[3]{T} = 5$

5M 19) $T = 64$

$\log_2 \left[T^{1/2} \cdot T^{1/3} \right] = 5$

$\log_2 \left[T^{5/6} \right] = 5$

$T^{5/6} = 2^5$
 $T = 2^6$
 $T = 64$

$\frac{32}{32}$

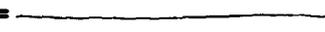
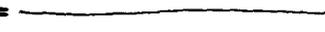
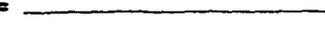
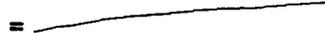
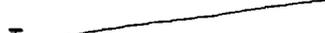
APPENDIX 3: THE STUDENTS' LOGARITHMS SURVEYS

James' logarithms survey: page 1 of 2

Log Problems Algebra 12name James

DO NOT use a calculator for these problems.

A Draw a straight line from each log expression on the left to what you think its approximate value is, on the right.
The first one has been done for you.

log 1 =		0.0
log 2 =		0.1
log 3 =		0.2
log 4 =		0.3
log 5 =		0.4
log 6 =		0.5
log 7 =		0.6
log 8 =		0.7
log 9 =		0.8
log 10 =		0.9
		1.0
		1.1
log 20 =		1.2
		1.3
log 40 =		1.4
		1.5
log 50 =		1.6
		1.7
log 75 =		1.8
		1.9
log 100 =		2.0
		2.1
		2.2
		2.3
		2.4
		2.5

continued...

James' logarithms survey: page 2 of 2

Part B

James

Some of the log expressions below have been chosen because they cannot be solved. If an expression cannot be solved write "cannot be solved" beside it. Also include a few words to indicate why you believe that this is so.

If an expression can be solved then give your answer and show your work.

1. $\log (80 / 2^3) = ($

2. $\log_2 2 =$ can't do
1 to any exponent is 1

3. $10^{\log 3} =$

4. $\log 25 + \log 4 =$

5. $\log 0 =$

6. $\log_{0.5} 8 =$

7. $\log 360 - \log 6^2 =$

8. $\log (-100) =$ can't solve
no neg. argument

9. $\log_{\frac{100}{2^2}} 100 = 4$

10. $\log_4 1/2 = .$

11. What numbers have logarithm values between 3.0 and 4.0 ?

$$1000 \rightarrow 10000$$

12. What numbers have logarithm values between 0.0 and 0.5 ?

$$1 \rightarrow$$

13. What numbers have logarithm values between -1 and 1/3 ?

14. Estimate to the nearest interger the value of

$$\log \left(\frac{86 \times 110}{9.8} \right) = 3$$

Keri's logarithms survey: page 2 of 2

Part B

Keri

Some of the log expressions below have been chosen because they cannot be solved. If an expression cannot be solved write "cannot be solved" beside it. Also include a few words to indicate why you believe that this is so.

If an expression can be solved then give your answer and show your work.

1. $\log(80 / 2^3) = 1$

$$3 \log 10 = \log 3$$

$$3 = \frac{\log 3}{\log 10} \quad 3 = 3/10$$

3. $10^{\log 3} = 3$

5. $\log 0 = \emptyset$
 $0 > 10$

7. $\log 360 - \log 6^3 = 1$

9. $\log 100^{\log 100} = 41$

11. What numbers have logarithm values between 3.0 and 4.0 ?

12. What numbers have logarithm values between 0.0 and 0.5 ?

13. What numbers have logarithm values between -1 and 1/3 ?

14. Estimate to the nearest interger the value of

$$\log \left(\frac{86 \times 110}{9.8} \right) =$$

2. $\log_2 2 = \emptyset$

$$(25 = 10^x) + (4 = 10^x)$$

4. $\log 25 + \log 4 = 2$

$$\log[(25)(4)] = \log 100 = \log 10^2$$

$$\log(100) = 100 = 10 \times 100 = 10^2$$

6. $\log_{0.5} 8 =$

$$8 = 0.5^x$$

8. $\log(-100) = -2$

$$-100 = 10^x$$

$$-1(10^2) = 10^x$$

$$-(2) = x$$

10. $\log_4 1/2 = -1/2$

$$1/2 = 4^x$$

$$(2^{-1}) = 2^{2x} \quad -1 = 2x$$