USING CALCULATORS TO TEACH ALGEBRA:
FROM NUMBERS TO OPERATIONS TO STRUCTURE

by

MAUREEN THERESE MURPHY

B.Sc., The University of British Columbia, 1970

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Abstract

The problems of teaching mathematics to unmathematical high school students are described and analyzed in terms of differences in language, especially syntax, between arithmetic and algebra. There follows a detailed description of a demonstrably successful alternative to so-called regular mathematics courses which depends essentially on the use of electronic calculators. Over three years, 274 students in 11 of the author's own mathematics classes, where algebra was taught in this alternative way, were observed for signs of mathematical insight. Relevant anecdotes were recorded, analyzed, and categorized and are discussed so as to justify the claims that, by using electronic calculators, teachers can shift the attention of students from numbers to mathematical operations and from operations to the overall structure of algebraic expressions, and that students who are taught algebra in this way reveal spontaneously, both orally and in writing, that they have an understanding of algebra. Finally, it is suggested that the difficulties encountered by many students in high school algebra could be reduced or eliminated if unmathematical students were taught in this way alone and mathematical students were taught in this way first, before proceeding to so-called regular mathematics courses.
# TABLE OF CONTENTS

Abstract ii

Table of Contents iii

List of Tables v

List of Figures vi

## Chapter One Teaching Mathematics to Unmathematical Students
1. Observable Characteristics 1
2. Underlying Problems 2
3. Attempts at Solutions 4
4. Understanding from Doing 6
5. Successful Practice 7
6. Research Questions 9
7. Consequences 10

## Chapter Two Mathematical Language in Arithmetic and Algebra
1. Same Words, Different Meanings 12
2. Variables 13
3. Operational Structure 14
4. Equals Signs 18
5. Solving Equations 19
6. Conclusion 23

## Chapter Three Syntax of Mathematical Language
1. Syntax of Arithmetic 26
2. Syntax of the Reversal Method 30
3. Syntax of Conventional Algebra 31
4. Comparison of Conventional and Reversal Syntax 33
5. Difficulty of Reification 35
6. Algebra Without Reification 37
7. Conclusion 40

## Chapter Four Role of the Calculator
1. Responsibilities of Calculators and Students 42
2. Understanding from Doing 47
3. Conclusion 54

## Chapter Five Research methods
1. Characteristics of Students Studied 57
2. Basic, "Star," and "Double-Star" Work 58
3. Classroom Activities and Course Requirements 61
4. Quantitative Data About Students Studied 63
5. Data Collection: Attempts at Interviews 65
6. Reasons for Abandoning Interviews 72
7. Collecting Anecdotes 75
8. Recording, Transcribing, and Analyzing Anecdotes 79
Using Calculators to Teach Algebra

| Chapter Six | Expressing Mathematical Insight | 85 |
|             | Communication in Calculator Language | 85 |
|             | Brackets | 88 |
|             | Non-Algebraic Calculator Language | 95 |
|             | Order of Operations | 99 |
|             | Use of Equals Sign | 105 |
|             | Differences Between Input and Execution | 110 |
|             | Conclusion | 120 |

| Chapter Seven | Bridging the Gap Between Arithmetic and Algebra | 122 |
|               | Operations Versus Numbers | 122 |
|               | Transition from Numbers to Letters | 127 |
|               | Nature of Variables | 135 |
|               | Structure of Expressions | 138 |
|               | Groups: Static Algebraic Expressions | 149 |
|               | Authority Behind "The Right Answer" | 162 |
|               | Conclusion | 164 |

| Chapter Eight | Conclusion and Consequences | 165 |
|               | Conclusion | 169 |
|               | Consequences | 177 |

Bibliography | 173 |

Appendix 1 | Mathematics 11A Final Examination | 175 |

Appendix 2 | Students' Records in Previous Mathematics Courses | 179 |

Appendix 3 | Use of the Electronic Calculator | 183 |
|           | Order of Operations | 183 |
|           | Two Different Types of Calculator | 191 |
|           | Non-Algebraic Calculators | 191 |
|           | Algebraic Calculators | 196 |

Appendix 4 | Reversal Method of Solving Equations | 200 |
|           | Formulae | 200 |
|           | Reversing a Formula | 201 |
|           | Brackets | 204 |
|           | Division Lines as Brackets | 204 |
|           | Reversal of Powers | 205 |
|           | Groups | 207 |
|           | Solving for an Exponent | 209 |
|           | Multi-Step Solutions | 210 |

Appendix 5 | Two Methods of Poetic Description | 215 |

Appendix 6 | Outline of Mathematics 8A-11A | 216 |
LIST OF TABLES

Table 1  Conventional vs. Reversal Syntax  33
Table 2  Transformation of Letter Grades  60
Table 3  Actual Letter Grade Distribution  64
Table 4  Codification of Themes  80
Table 2-A  Symbols Used in Students' Records  179
Table 2-B  Records of Students Taught By Terry Baker Between September 1991 and January 1992  180
Table 2-C  Records of Students Taught By Terry Baker Between September 1992 and January 1993  181
Table 2-D  Records of Students Taught by Maureen Murphy Between January 1993 and June 1993  182
Table 6-A  Contents of Mathematics 8A-11A  217
LIST OF FIGURES

Figure 1  Distribution of Letter Grades  65
Figure 2  Distribution of Anecdotes  82
Chapter 1

Teaching Mathematics to Unmathematical Students

The problem of how to teach mathematics to unmathematical students is apparently never-ending. Variation in abilities is taken for granted in elective school subjects such as art, music, and drama, to the point where no one is considered unintelligent because he or she cannot draw, sing, or act. However, it is less taken for granted in mathematics, to the point where mathematics is made a compulsory subject in British Columbia at least up to Grade 11; most educators, mathematical or not, think that all students could be successful in the "regular" mathematics courses if they were taught appropriately; and a great many unmathematical adults feel it necessary to devise excuses for, or to conceal, their lack of success in mathematics at school.

Observable Characteristics

I first encountered the phenomenon of unmathematical students in 1974 as a beginning teacher of General Mathematics 10, whose intended "clients" were students who had had little or no success as determined by examinations in the "regular" mathematics courses. I soon found that they had little of what I, with a background of success in mathematics up to fourth year university honours, could recognize as "understanding" of mathematics.

First, they were, in general, poor at recalling number facts. Second, they had difficulty determining on their own what
mathematical operations to perform on the numbers they were given in a problem so as to obtain the correct answer. Although they could sometimes determine when to add and when to subtract, they could almost never determine when to multiply and when to divide and, given the choice, they nearly always preferred to add rather than multiply, e.g. to determine the number of months in five years.

Third, they could not apply the rules of algebra for simplifying or evaluating expressions or for solving equations. They seemed to have no understanding of what I saw as the structure of arithmetic or algebraic expressions or any of the formulae with which they were provided to help them solve problems.

Underlying Problems

It seemed to me that the problem was primarily one of language and communication. Attempting to teach mathematics to such unmathematical students was like speaking to people who did not understand my language, or rather people who understood the meaning of some of the words I used but not the syntax.

To be sure, the students were all English-speaking: they could isolate in English what they were being told and what they were being asked when they were given a word problem. However, they could not translate the problem into mathematical language, which uses mathematical symbols and syntax, and they could not transform mathematical sentences according to the rules of algebra.
so as to isolate one variable.

To help students translate a word problem into mathematical language, I attempted to devise rules based on what I called "word clues," e.g. "and" is a prompt to add or "less" is a prompt to subtract, but I found myself unable to do so because problems could be worded in English in so many different ways: a word that might signal "add" in one problem could signal "subtract" in another.

The fact that mathematics is a language is suggested by the title of Pimm's book (1987) Speaking Mathematically. Mathematicians are seen by others primarily as people who communicate with each other about certain kinds of things (such as numbers, geometrical shapes, functions, etc.) and therefore make only certain kinds of statements (such as "y is directly proportional to x" or "this triangle is congruent to that one").

Unlike French or German, mathematics is not an alternative language in which to express common ideas, but a language whose purpose is to communicate ideas which could not otherwise be expressed, expressed so exactly, or expressed so economically.

Knowing mathematics involves aural comprehension, speech, reading, and writing. It includes knowledge of spelling, pronunciation, syntax, and vocabulary (Pimm, 1987, p. 3).

It also involves "communicative competence" - knowing how to use language appropriate to context. "A general principle in teaching any kind of communicative competence, spoken or written, is that the speaking, listening, writing or reading should have

Attempts at Solutions

Throughout my teaching career I have continued to teach students who exhibit similar lack of insight. In 1979, I began to teach Consumer Mathematics 11, then one of two terminal courses in mathematics for "unmathematical" students (replaced some years later by Mathematics 11A), where I encountered the same problems as in General Mathematics 10. The contents of the course were markdown and sale price, tax, simple interest, compound interest, mortgages and annuities. Computations in the last three topics were performed with the aid of standard interest and mortgage tables.

Unlike Grade 10 students, these Grade 11 students were not expected to take any further mathematics courses, so I was not accountable to any future teacher. To satisfy myself more than my students, I began to modify the curriculum according to three guiding criteria:

1. I would teach no subject in which the students could not, as a matter of observable fact, remember how to solve the problems.

2. I would keep to a minimum what I gave the students (formulae, data, tables, etc.) in order for them to solve the problems.

3. The course would satisfy me intellectually, and be
recognizable by the students, as a mathematics course.

These criteria sprang from my own perceptions of, and delight in, mathematics as a subject

1. which did not require memorization because its propositions were so obvious and so logically necessary.

2. in which, therefore, one could always be sure of success on exams.

3. which could always provide the stimulation of challenge.

In accordance with these criteria, I gradually introduced three innovations.

The first, in an attempt to solve the problem that so many of my students could not recall number facts, was to encourage them to use electronic calculators, which were just becoming common among high school students. At the same time, calculators would make mortgage and annuity tables obsolete. I argued (implicitly) that it was more economical (in the logical sense) for students to carry around calculators than books of addition, multiplication, mortgage or annuity tables.

However, with the aid of calculators, students could all, immediately, evaluate arithmetic expressions which contained only one operation, e.g. 150 + 75, so I began asking them to evaluate arithmetic expressions containing more than one operation, e.g. 150 + 75 × 3.

In the early 1980s, most of the students owned what I call non-algebraic calculators - calculators which execute the operations of adding, subtracting, etc. in the order in which the
operation buttons are pressed. Therefore, for students to use such non-algebraic calculators successfully, I first had to make them familiar with the conventional order of operations so that they would be able to make the transition from reading an arithmetic expression such as $150 + 75 \times 3$ to pressing the buttons $75 \times 3 + 150 =$ on a calculator.

Second, since my students could not reliably determine what operations to perform on the numbers in a problem so as to answer the question, I began to give them the formulae or equations they needed to solve problems.

However, on the principle of economy, I was unwilling to give them every mathematical variation on each formula. For example, while I was willing to give them the formula $I = Prt$, I was not willing to give them in addition $P = \frac{I}{rt}$, $r = \frac{I}{Pt}$, and $t = \frac{I}{Pr}$.

Accordingly, I made my third innovation: I modified, expanded, and began to teach a new method of solving equations, the germ of which had been suggested to me by a colleague, which used the same language I had used to teach the use of calculators.

**Understanding from Doing**

It will be seen that these three innovations were essentially practical: if I found that my students could not, as a matter of observation, do what I was requiring them to do, I substituted something else. My philosophy was to give the students whatever they needed (albeit a minimum) to be successful: i.e. to pass a
final exam on the problems I was requiring them to solve. My emphasis was on doing rather than on understanding. I argued that anything mathematical I could get my students to do with success, even with the aid of a calculator, was better than nothing.

However, despite this motivation for the changes I made, I gradually began to see and encourage in my students signs of understanding and even creativity.

Apparently, doing, understanding, and creating were linked in ways I had not suspected, as described by Pimm (1995).

**Successful Practice**

Once students had learned how to use calculators efficiently and how to solve an equation for any variable it contained, they could learn to solve word problems for which they were given the relevant formulae.

For the rest of the course, therefore, I set problems in simple interest, compound interest, saving with regular deposits, and financing with regular payments. Appendix 1, comprising the students' final exam, shows the formulae and definitions with which they were provided.

In each problem, students were required

1. to recognize the kind of problem (simple interest, compound interest, saving, or financing).

2. to extract the numerical data and to associate each datum with the appropriate variable.

3. to select the appropriate formula.
4. to solve the formula for the unknown variable.
5. to evaluate the unknown variable.

I have now taught approximately 1000 students this Mathematics 11A course. The students write a one-question quiz every day, a half-hour quiz every three days, four hour-long exams, and a final exam.

Moreover, in 1993, with a few days' notice, I was asked by my vice principal and my department head to teach a similar course to two classes of Grade 8 students who had been identified by their elementary school teachers as likely to have difficulty with the "regular" Mathematics 8 course. I began writing and teaching a similar course to these students, although with some modifications, necessary because of the students' immaturity.

In 1994-95 I taught Mathematics 9A to some of these students as well as others who had not taken my Mathematics 8A course, and in 1995-96 I taught Mathematics 10A to the same students as well as others who had taken only one or even neither of my previous courses. In each case I wrote the course as I taught it.

I have now taught a total of approximately 1100 students in my Mathematics 8A, 9A, 10A and 11A courses. Among those students I have found six who were, in my professional opinion, truly unable to pass. A number of others have failed courses once, taken them again, and passed the second time, or even failed them twice. However, in my opinion and in the opinions of the students themselves - opinions which they often assert publicly in class the second year to help bolster my assertion that my course is
passable - they failed because they did not attend all the classes and/or did not work when they did attend.

Many of these students have had no, or almost no, success in previous mathematics courses. Appendix 2 gives the high school records of three classes of these students, one taught Mathematics 11A by me, the others by another teacher. For example, one student who finished Mathematics 11A with 75% had passed no mathematics courses, modified or otherwise, since Grade 6.

Two other teachers have now used my materials. Appendix 2 gives the results obtained by one of them in two classes: they show that she had the same kind of success with the same kind of students as I have had.

Research Questions

I have found that the observable, testable success of students in my courses requires some emphasis, because again and again when I have offered my material to other teachers, they have declined it on the grounds that it would be too difficult for their students.

However, even when they accept the fact that unmathematical students can be and are successful in these courses, teachers who have not taught from my material often express two reservations about it.

The first is based on a fear that my students are merely repeating by rote, without anything that teachers would recognize
as understanding, rules which are so structured as to make them fool-proof.

The second is based on an impression that my material is so different from the contents of the "regular" courses that it cannot really be dignified with the name of mathematics.

Accordingly, this thesis addresses the following questions:

1. Is it possible to shift the attention of students from numbers to operations by the use of electronic calculators?

2. Is it possible to shift the attention of students from operations to the overall structure of algebraic expressions and equations by the use of electronic calculators?

3. Do the responses of students who have been taught algebra in calculator language show that they understand algebra?

If the answers to these questions are "yes," then not only are my mathematics courses viable alternatives to the "regular" mathematics courses for unmathematical students, but also, perhaps, they can serve as bridges between arithmetic and algebra for students who will be capable, before they leave high school, of passing the "regular" courses.

**Consequences**

Moreover, my findings will have consequences for the study of how mathematics is learned, taught, conceptualized, and performed - a study which interests some educators for its own sake, apart from its application to the design of mathematics courses.

Accordingly, in the rest of this thesis, I
1. analyze how the language of algebra differs from the language of arithmetic.

2. recount how I teach the use of calculators, so as to make the reader familiar with my language.

3. demonstrate how I use the same language in teaching the "reversal" method of solving equations.

4. analyze the syntax of mathematical language of calculators, focusing on the differences between the language I use and the language used in "regular" high school algebra courses.

5. analyze the function and the authority of calculators as I use them.

6. describe more fully the students I teach.

7. report classroom observations illustrating the nature of the students' understanding of the mathematics I teach.

8. argue that my mathematics courses are viable and altogether satisfactory alternatives to the "regular" mathematics courses for unmathematical students.

9. argue that the success of my mathematics courses sheds light on the learning and therefore on the teaching of mathematics.
Pimm (1987) claimed that knowing mathematics involves aural comprehension, speech, reading, and writing. It includes knowledge of spelling, pronunciation, syntax, and vocabulary.

I now turn to a review of the mathematical language encountered by students in high school, focusing on the differences between the language of arithmetic and the language of algebra and how these differences can become stumbling blocks for beginning algebra students.

Same Words, Different Meanings

In an analysis of the role of language in communication in general, von Glasersfeld (1988) said:

...language users must individually construct the meaning of words, phrases, sentences, and texts.... the basic elements out of which an individual's conceptual structures are composed and the relations by means of which they are held together cannot be transferred from one language user to another... (p. 10)

Once we come to see this essential and inescapable subjectivity of linguistic meaning, we can no longer maintain the preconceived notion that words convey ideas or knowledge; nor can we believe that a listener who apparently "understands" what we say must necessarily have conceptual structures that are identical with ours. (p. 11)

For example, Rotman (1991), in a study of students at Lansing Community College in Michigan USA, found that "typical arithmetic courses do little to prepare students to master algebra" (p. 1) and that students who were initially unsuccessful in algebra did
not profit from the college's pre-algebra course, which comprised a review of arithmetic.

It appears that although algebra is perceived by mathematicians to be a generalization of arithmetic, and although many of the symbols used in algebra are the same as those used in arithmetic, differences in the meanings of the language make the two subjects appear to many students to be almost unrelated. Similar, although lesser, misunderstandings occur when people from different English-speaking countries talk to each other or when educated people talk to uneducated; they are "divided by a common language."

Variables

Variables represent for students perhaps the greatest difference between arithmetic and algebra. As they graduate from arithmetic to algebra, letters replace numbers. Pimm (1995) points out that this move "can provoke a sense of real loss" (p. 90) on the part of students.

Kieran (1989) discusses at some length the problems beginning algebra students have with variables, partly because there are so many different kinds of variable. Following Küchemann and Collis (1975), she lists them as follows:

(a) Letter evaluated: The letter is assigned a numerical value from the outset.
(b) Letter not used: The letter is ignored or its existence is acknowledged without giving it a meaning.
(c) Letter used as an object: The letter is regarded as a shorthand for an object or as an object in its
own right.
(d) Letter used as a specific unknown: The letter is regarded as a specific but unknown number and can be operated on directly.
(e) Letter used as a generalized number: The letter is seen as representing, or at least being able to take on, several values rather than just one.
(f) Letter used as a variable: The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values. (p. 42)

Operational Structure

Although letters appear in addition to numbers in the move from arithmetic into algebra, students continue to see the familiar operations signs +, -, x, /, and ÷ (although the x sign, alone among the others, may now be omitted, perhaps for the first time). They also continue to see the same meaning attached to a small number or letter written somewhat above the line - an exponent - indicating the operation of "raising to" a power, which has no conventional symbol of its own.

The difference is that in algebra, unhappily for students, the operations can no longer be carried out. I say "unhappily" because, according to Booth (1984), quoted by Kieran (1989), "Children consider mathematics to be an empirical subject which requires the production of numerical answers" (p. 39). Pimm (1987) notes that in arithmetic, "merely writing 6 + 9" is "perceived as a powerful prompt to carry out the addition" (p. 167).

As Pimm (1995) puts it, "Operation symbols in algebra are virtual and not actual as they seem to be in arithmetic" (p. 88).
In the move from arithmetic to algebra, attention shifts away from the performance of the operations in an expression - known as "getting the answer."

What meaning, therefore, can beginning algebra students attach to the once familiar mathematical operations: +, -, $\times$, $\div$, and exponents? For students who appear in my mathematics classes for the first time, the answer seems to be "none."

Yet an understanding of the operations is key to an understanding of algebra.

In class, Pimm pointed out that when geometrical objects formed on the computer screen by Geometer's Sketchpad are changed, they "retain the memory of how they were constructed." For example, one can construct a line segment, then its perpendicular bisector. If one now "drags" one end of the original line segment, the perpendicular bisector moves as well so as to maintain its identity as the perpendicular bisector of each of the successive new line segments. In this respect, Pimm pointed out, geometrical objects created with this computer program are different from those created with pencil and paper.

In an analogous way, algebraic expressions can be said to retain the memory of the operations that were in some sense performed at their creation and of the order in which these operations were performed. Indeed, the language mathematicians use, and expect their students to use, to describe algebraic expressions bears witness to this memory, as surnames bear witness to the occupations of our ancestors (Cooper, Taylor) and the names
of towns to events in their history, reasons for their existence, etc. (cf. Carrying-Place in Ontario or 100 Mile House in British Columbia).

For example, mathematicians describe $a^2 - b^2$ as the "difference" of two squares, implying that the squaring was done before the subtraction. In explaining that $(a - 2)^2 + (b - 3)^2$ cannot be factored, we are likely to call it the "sum" of two squares, implying that addition was the last operation performed, that squaring immediately preceded it, and that the subtractions can be ignored. In explaining how to solve $(x + 2)(x - 3) = 0$, we note that the left side of the equation is a "product," implying that multiplication was the last operation performed. Moreover, until students know how to distinguish between an algebraic expression which is (in this sense) a product and one which is a sum, they have not learned to "factor" or "simplify."

It appears, then, that although the arithmetic operations $+$, $-$, $\times$, $\div$, and raising-to-a-power cannot be performed with letters as they can with numbers, it is the operations that were performed "virtually" in the construction of an algebraic expression, together with the order in which they were performed, that determines the character of the expression.

In fact, it could be said that the operations and their order are even more important in algebra than in arithmetic. Given the arithmetic expression $10 + 3 \times 2$, we need to know what $+$ and $\times$ mean and the fact that (by convention) the $\times$ is performed before the $+$. When we perform these operations we "get the answer": 16.
But 16 has a fluid rather than a determinate structure. We can describe it equally well as the product of 8 and 2, the sum of 9 and 7, the difference of 20 and 4, the square root of 256, the square of 4, the fourth power of 2, etc. However, the algebraic expression \( m + bc \) is nothing more nor less nor other than the sum of \( m \) together with the product of \( b \) and \( c \). It has no other identity. It cannot be factored. It is not a perfect square. It is not a product or a quotient. Only by invoking what would be looked on as "legal fiction" can we change its identity; for example

\[
m + bc = (m + bc)^2 \cdot \frac{1}{m + bc} = [\sqrt{(m + bc)}]^2
\]

Its identity is completely determined by its structure, and its structure is completely determined by the (virtual) operations which built it and the order in which they were performed.

"The recognition and use of structure" (p. 33) is included by Kieran (1989) among the difficulties encountered by beginning algebra students.

She defines "arithmetic/algebraic structure" in general as "the structure of a system that is comprised of a set of numbers/numerical variables, some operation(s), and the properties of the operation(s)" (p. 33).

She says that since "much of school elementary arithmetic is oriented towards 'finding the answer'" (p. 33), children need not attend to the structure of arithmetic expressions. "However, in algebra, they are required to recognize and use the structure that
they have been able to avoid in arithmetic" (p. 39).

Equals Signs

Besides the substitution of letters for numbers and the relegation of operations to the realm of the virtual, the transition from arithmetic to algebra also includes subtly new meanings for the = sign.

Mathematicians interpret an = sign to mean that whatever is written on the left side of it is in some sense equivalent to whatever is written on the right side.

However, according to Herscovics and Kieran (1980), "students do not view 2 + 3 = 5 as an identity but rather interpret it operationally, as indicated by their reading '2 and 3 make 5'" (p. 573).

Behr et al. (1976, p. 9) suggest that "children do not view sentences like 2 + 3 = 3 + 2 as being sentences about number relationships. They do not see these as indicating the sameness of sets of objects. Indeed, it appears that children consider these as 'do something' sentences."

They also found that some students accepted 2 + 3 = 5 and 3 + 2 = 5 but not 2 + 3 = 3 + 2. The students accepted 2 + 4 as meaningful, but as suggesting that something had to be done. They did not, for example, think of 2 + 4 as being a name for 6 (p. 2).

Significantly, these researchers also found that there is "no evidence to suggest that children change in their thinking about equality as they get older and progress to upper grades; in fact,
the evidence seems to be to the contrary" (p. 10). Similarly, Mevarech and Yitschak (1983), quoted in Kieran (1992), showed that college students often have a "poor" understanding of the meaning of the equals sign (p. 399).

Kieran (1989) also points out the use of the = sign by students to signify that what follows is the result of the calculation that precedes (p. 37). She quotes Vergnaud, Benhadj, and Dussouet (1979), who presented sixth-grade children with the following problem:

In an existing forest, 425 new trees were planted. A few years later, the 217 oldest trees were cut. The forest then contained 1063 trees. How many trees were there before the new trees were planted? (p. 37)

Children typically wrote 1063 + 217 = 1280 - 425 = 855.

Kieran describes this as "writing down the operations in the order in which they are carried out and keeping a running total" (p. 37).

Used in this way, the = sign could be described as "what you write before the answer" or "what you write to show that what follows is the answer."

Solving Equations

With letters instead of numbers, with the prompts suggested by the familiar operation signs impossible to carry out but nevertheless responsible for the structure of expressions, and with a new meaning for the = sign, it is not surprising that students progressing from arithmetic to algebra have difficulty
solving equations, no matter which of the standard methods they are taught.

(I use the words "formula" and "equation" interchangeably, since nearly all equations my students see are formulae; that is, one side of the = sign is occupied by only one variable.)

In an article significantly entitled "Early Learning: A Structural Perspective," Kieran (1989) notes that the structure of an equation "incorporates the characteristics of the structure of expressions..." (p. 34). In particular, she says, the surface structure of an equation is no more than the surface structure of the two expressions and their equality.

She goes on (p. 49) to describe Whitman's (1976) distinction between "formal" and "informal" equation-solving techniques as follows. The equation

\[ 69 - \frac{96}{7 - a} = 37 \]

could be solved "formally" as follows:

\[
\begin{align*}
69 - \frac{96}{7 - a} &= 37 \\
69(7 - a) - 96 &= 37(7 - a) \\
483 - 69a - 96 &= 259 - 37a \\
387 - 69a &= 259 - 37a \\
128 - 69a &= -37a \\
128 &= 32a \\
4 &= a
\end{align*}
\]

or "informally" as follows:

69 minus what number gives 37? — 32
96 divided by what number is 32? — 3
7 minus what number is 3? — 4
Solution: \( a = 4 \)

or by a combination of formal and informal techniques.
Facility with the formal technique certainly requires recognition and understanding of the structure of the equation. However, facility with the informal technique outlined above implicitly requires a knowledge of the order of the operations in the original equation, together (again implicitly) with the knowledge that, after beginning with 37, operation 3 should be "undone" first, operation 2 second, and operation 1 last. The informal technique, therefore, also requires recognition and understanding of the structure of the equation.

In the rest of her 1989 paper (pp. 49-52), Kieran discusses the relative merits of "doing the same thing to both sides" and "transposing," pointing out while to teachers these may be variations of the same method, to students they appear quite different.

She also observes (1992) that "learning to operate on the structure of an equation by performing the same operation on both sides may be easier for students who already view equations as entities with symmetrical balance" (p. 401).

Again, both of these methods require students to understand the structure of the equation; that is, the structure of each side of the equation which is comprised in the operations and their order, together with the equals sign.

Kieran (1992) refers to a limited version of the reversal method as the "undoing" or "working backwards" method, in which, she says, "to solve $2x + 4 = 18$, the student takes the numerical result on the right side and, proceeding in a right-to-left order,
undoes each operation as he/she comes to it by replacing the given operation with its inverse; thus the student is able to operate exclusively with numbers and avoid dealing with the equivalence structure of this mathematical object" (p. 400). (I say "limited" because "proceeding in a right-to-left order" would not always produce a correct solution and because the reversal method is not limited to equations containing only one variable.)

The reversal method per se is referred to in Verse 28 in the Ganita section of Shukla's (1976) edition of the Aryabhatiya, completed by the Hindu mathematician and astronomer Aryabhata in AD 499:

In the method of inversion multipliers become divisors and divisors become multipliers, additive becomes subtractive and subtractive becomes additive.

Example: A number is multiplied by 2; then increased by 1; then divided by 5; then multiplied by 3; then diminished by 2; and then divided by 7; the result (thus obtained) is 1. Say what is the initial number.

Starting from the last number 1, in the reverse order, inverting the operations, the result is $1 \times 7, + 2, \div 3, \times 5, - 1, \div 2$, i.e. 7. (p. 71)

Clearly, this method involves recognition of the order of the original operations, together with the knowledge that, after beginning with the original result, one must "invert" the operations in the "reverse" order. But this is what constitutes what we would call the structure of the equation that corresponds to the original problem:

$$\frac{3 \times \frac{2x + 1}{5} - 2}{7} = 1$$

Similar rules for the inversion method occur in other Hindu
mathematical works AD 628-1356.

An ancient Babylonian problem, quoted in Fauvel (1987), also appears to be set up to be solved by the same method:

I found a stone, (but) did not weigh it; (after) I weighed (out) 6 times (its weight), [added] 2 gin, (and) added one-third of one-seventh multiplied by 24, I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was 4\frac{1}{3} gin (p. 14).

In this problem, at first glance, the \( \frac{8}{7} \) ("one-third of one-seventh multiplied by 24") could be \( \frac{8}{7} \) of the original weight of the stone, \( \frac{8}{7} \) of a gin, or \( \frac{8}{7} \) of a ma-na. However, from the answer supplied, and from comparison with similar problems, it appears that it is \( \frac{8}{7} \) of the stone's weight increased by 2 gin. That is, the stone's weight is first multiplied by 6, then increased by 2 gin, and then multiplied by \( \frac{8}{7} \) to give 1 ma-na. The original weight of the stone, therefore, is calculated by starting with 1 ma-na, dividing by \( \frac{8}{7} \), and subtracting 2 gin.

It appears that the implementation of any method of solving an equation, whether "formal," "informal," "doing the same thing to both sides," "transposing," or reversal, demands knowledge of the equation's structure, i.e., the structure of the expressions on the two sides of the equals sign and their equality.

Conclusion

Although the introduction of letters is the most visible change as students move from arithmetic to algebra, changes in the meanings of operations and the equals sign, together with the
necessity of understanding the structure of algebraic expressions and equations, are also sources of the difficulties so many students have when they move from arithmetic to algebra. These changes and differences are analyzed further in the Chapter 3.

However, to appreciate fully the differences and similarities in syntax among the languages of arithmetic, the calculator, my own algebra courses and the "regular" mathematics courses respectively, the reader must familiarize himself or herself with the language I use in my own courses in teaching students how to use calculators (see Appendix 3) and how to solve equations by the reversal method (see Appendix 4). Both of these appendices are written as if addressed to students, with explanatory notes to the more mathematical reader interpolated in italics.
Although algebra is perceived by mathematicians to be a generalization of arithmetic, and although many of the symbols used in algebra are the same as those used in arithmetic, the two subjects appear to many students to be completely unrelated because of the differences of meaning analyzed in Chapter 2. However, this meaning cannot be divorced from the syntax of mathematical language, which I have not yet considered per se.

The language of the calculator, as illustrated in Appendix 3, is essentially the language of arithmetic, since the purpose of a calculator is to carry out arithmetic operations. The language of the reversal method of solving equations, as illustrated in Appendix 4, is similar.

In this chapter, I will first analyze the syntax of calculator language and the syntax of the language used in the reversal method, in order to show how similar they are. Then I will analyze the syntax of the language conventionally used in "regular" high school algebra courses in order to show how different it is from the language of calculators and the reversal method. Finally, I will argue that the syntax of arithmetic, calculators and the reversal method should have equal status with conventional syntax and be used in teaching students who have difficulty with conventional syntax.

defines "syntax" as "the arrangement of words (in their appropriate forms) by which their connexion and relation in a sentence are shown.... the department of grammar which deals with the established usages of grammatical construction and the rules deduced therefrom" (p. 3212). The fact that "syntax" is an attribute of mathematical language is attested to in Pimm's Speaking Mathematically (1987), in which the title of Chapter 7 is "The syntax of mathematical forms."

Syntax of Arithmetic

Confronted with the following written expression,

\[ 3(4 + 2) + 5 \]

students who have passed my calculator unit behave as follows: reach for a calculator, press the buttons (on an algebraic calculator) \[ 3 \times (4 + 2) + 5 = \] or (on a non-algebraic calculator) \[ 4 + 2 \times 3 + 5 = , \] and write down the result. That is, they obey the "powerful prompt" to do something represented by the operation signs (Pimm 1987, p. 167). They behave as if I had said to them, "Start with 4, add 2, multiply by 3, add 5, press =, and tell me what you get." Conversely, if I told them that I wanted them to "write a question" which, when handed to someone else, would "tell that person" to start with 4, add 2, multiply by 3, add 5, and report the answer, they would write

\[ (4 + 2)3 + 5 \]

(In practice, I do not have them perform the second of these two procedures until they have begun the unit on solving equations.)
We may say, therefore, that one meaning of $3(4 + 2) + 5$ is "start with 4, add 2, multiply by three, add 5, and (implicit in the other instructions) report what you get." Because they are using calculators, this is the meaning I, as a teacher, seek to validate among my students by the language I myself use, the language illustrated in Appendix 3.

Notice that in this language, 2, 3, 4, and 5 are nouns. However, the words used for the operations - "add" and "multiply" - are verbs. Moreover, they are transitive verbs, not intransitive; that is, they are capable of having direct objects. We add, subtract, multiply, and divide numbers and letters. (In contrast, "go" is an intransitive verb; one cannot "go something.")

Transitive verbs, but not intransitive verbs, can be used in the active or the passive voice. For example, if I say "I added four and two" or "Add four and two" (the latter in the imperative, where the understood subject "you" is omitted), I am using the active voice, while if I say "Four and two were added" I am using the passive voice.

There can be a curious difference between the verbalization of "add" and "subtract" on the one hand and that of "multiply" and "divide" on the other. Consider how to verbalize the following.

1) $4 + 2$
2) $4 - 2$
3) $4 \times 2$
4) $4 \div 2$

One way, if the operation signs are read as verbs in the
imperative, is to say, on the one hand, "Add two to four" and "Subtract two from four," but, on the other hand, "Multiply four by two" and "Divide four by two." It is not uncommon for students to say, "Add four by two" or "Subtract four by two," but this is not considered standard language. It appears that the direct object of the verb is the second number when we add or subtract, but the first number when we multiply or divide. One could also say, "Add four and two" and "Multiply four and two," because addition and multiplication are commutative, but "Subtract four and two" and "Divide four and two" are ambiguous.

In practice, I (implicitly) take the symbols +, -, ×, and ÷ to mean "add," "subtract," "multiply by," and "divide by" respectively. The students often substitute the words "plus," "minus," and "times" for the first three respectively, but their use of these words clearly shows that they consider them to be verbs (e.g. "Should I plus this next or times that?" or "So I have to minus this by that.")

As I note in Appendix 4, I use the symbol \( y^x \) to denote the operation of raising to a power. With this convention, I could add the following to the above list

\[
5) \quad 4 \ y^x \ 2
\]

Then "2" would be the object of the operation sign in each of these numbered numerical phrases.

So far I have translated into English only an arithmetic expression. Consider now the following arithmetic equation:

\[
3(4 + 2) + 5 = 23
\]
The operations signs are not now in the imperative mood, because anything they enjoined on the reader has been carried out.

What is written in the equation above could mean that whoever "did the question" (i.e., whoever carried out the operations) started with 4, added 2, multiplied by 3, added 5, and got 23. In this reading of the equation, the operations are verbs in the indicative mood, past tense.

However, it could also mean that when you start with 4, add 2, multiply by 3, and add 5, you get 23. In this reading, the operations are verbs in the indicative mood, present tense.

Moreover, it could also mean that if you started with 4, added 2, multiplied by 3, and added 5, you would get 23. In this reading, the operations are verbs in the conditional mood.

Nevertheless, in all three readings the operations remain verbs.

Notice that the = sign is read "got," "get," or "would get." I also commonly read it "end up with." ("Get" is probably more appropriate for a formula, which you select by its subject with the conscious purpose of calculating a value you want to know, as opposed to an equation, where you simply follow the instructions and are told where you "end up.")

I pass over brackets as a syntactical element in mathematical sentences and expressions because they have the same meaning in algebra as they do in arithmetic (although brackets can have other meanings in algebra as well). If I were to complete the syntactical analysis of arithmetic, I would compare brackets to
the commas around a non-restrictive clause. (Indeed, brackets may often be substituted for these commas in writing). On the calculator, the pressing of brackets around an arithmetic expression may be said to tell the calculator to wait a minute while the operator inserts and the calculator evaluates a parenthetical expression, just as the slight pause which is the oral equivalent of a comma tells the listener to wait a minute while the speaker inserts a parenthetical expression.

Syntax of the Reversal Method

Syntactically, the language of the reversal method of solving equations I used in Appendix 4 is almost identical to the language of arithmetic.

In fact, except for the first paragraph, the section above, "Syntax of arithmetic," could be repeated here word for word simply by changing the numbers to letters.

For example, in Appendix 4 I defined a formula as "a set of mathematical instructions for calculating the value of one variable from the values of other variables." As an example, I might explain to my students that \( n = m + bc \) is a formula for calculating the value of \( n \): it says that if you multiply the value of \( b \) by the value of \( c \) and add the value of \( m \), you will get the value of \( n \), the variable which is the subject of the formula. Alternatively, it says that if you could multiply \( b \) by \( c \) and add \( m \), you would get \( n \).

The letters have replaced numbers as the nouns. However, the
operations, even though "virtual," are still verbs. The only way in which they differ from those of arithmetic is that they can no longer be in the imperative mood or the past tense of the indicative mood, for they cannot be carried out. They can be in the future tense: "If you do this, you will get that," but it is arguable that the most appropriate mood is the conditional: "If you could do this (or were to do this), you would get that."

Syntax of Conventional Algebra

Consider this sentence by Kieran (1989): "... the structure of the expression 3(x + 2) + 5 includes the surface structure, that is, the given ensemble of terms and operations - in this case, the multiplication of 3 by x + 2, followed by the addition of 5 ..." (p. 34).

Notice that, in this sentence, as in "reversal" syntax, 3 and 5 are nouns. However, the similarity ends there.

Notice that x + 2 is also a noun. Moreover, the words used to describe the operations - "multiplication" and "addition" - are nouns as well. Even further, "the multiplication of 3 by x + 2, followed by the addition of 5" is a noun phrase - a noun with modifiers.

Using the terminology of "sums" and "products," one could also say "the sum of 5 and 3(x + 2)" or "the sum of 5 with the product of 3 and (x + 2)" or (rather confusingly) "the sum of 5 together with the product of 3 and the sum of x and 2."

Contrast what I would say to my students: "We are being told
to start with \(x\), then add 2, then multiply by 3, and then add 5."

Even when I discuss what I call *groups*, where I come closest
to treating an algebraic expression as a noun (see Appendix 4), I talk (as long as any individual student finds it necessary) about "the answer" to the expression. For example, I would describe the "3" in the expression \(3(x + 2) + 5\) as being multiplied by "the answer" you get when you add 2 to \(x\) (or would get if you could add 2 to \(x\)).

There is an equally great difference in the syntax of the \(=\) sign between "regular" algebraic methods and the reversal method of solving equations.

Most commonly, mathematicians read the \(=\) sign as "is." "Is" is a copula verb, defined by The Compact Edition of the Oxford English Dictionary (1980) as "that part of a proposition which connects the subject and the predicate" (p. 555). In mathematical sentences such as "\(2 \times 3 = 6\)," both the subject ("\(2 \times 3\)") and the predicate ("6") are nouns.

The predicate of a copula verb is not like the object of a transitive verb: a copula verb does not "act on" its predicate as a transitive verb "acts on" its object. For example, "I will be a teacher" is quite different grammatically from "I will hit a teacher." The copula verb indicates in some sense an equality between the subject and the predicate.

Nevertheless, the subject and the predicate of a copula verb are not interchangeable. For example, the sentence "A cow is an animal" does not mean the same as "An animal is a cow."
Similarly, Schoenfeld and Arcavi (1988) point out that, in a formal sense, the two statements "2 \times 3 = 6" and "6 = 2 \times 3" are equivalent. But in reality, the first statement is about multiplication, the second about factoring. "That is, in context, the equal sign is not read as 'is formally equivalent to'; it is read as 'yields'" (p. 424).

Comparison of Conventional and Reversal Syntax

The differences in syntax between conventional algebra and my courses, described above, are encapsulated in the Table 1.

Table 1

Conventional vs. Reversal Syntax

<table>
<thead>
<tr>
<th>Concept</th>
<th>Conventional</th>
<th>My courses</th>
</tr>
</thead>
<tbody>
<tr>
<td>adding, subtracting, multiplying, dividing, exponentiation</td>
<td>conjunctions or &quot;joining words&quot; denoting relationships between numbers or variables</td>
<td>verbs giving instructions about actions to be performed - &quot;things to be done&quot;</td>
</tr>
<tr>
<td>algebraic or numerical expressions</td>
<td>mathematical &quot;phrases&quot; in which the &quot;nouns&quot; are numbers and variables and the &quot;conjunctions&quot; are operations</td>
<td>sets of instructions telling you what to do with the values of variables</td>
</tr>
<tr>
<td>equals sign</td>
<td>a sign placed between two expressions to denote equality of value</td>
<td>a sign which separates what has to be done from what you end up with</td>
</tr>
<tr>
<td>formulae or equations</td>
<td>statements of the equality of two mathematical phrases</td>
<td>sets of instructions telling you how to calculate the value of a variable from the values of other variables</td>
</tr>
</tbody>
</table>
Sfard (1991) uses the word "operational" to describe the conception of an algebraic expression as a set of instructions and the word "structural" to describe its conception as what I have called a group: an entity, a noun, a "thing" in its own right. She contrasts the two conceptions as follows:

Seeing a mathematical entity as an object means being capable of referring to it as if it were a real thing - a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details.... In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static ... instantaneous, and integrative, the operational is dynamic, sequential, and detailed.

In other words, there is a deep ontological gap between operational and structural conceptions. (p. 4)

She also says that no information is added in a shift from the operational approach to the structural approach; instead the computational processes are "caught into a static construct just as water is frozen in a piece of ice" (p. 25).

Sfard uses the word reification (p. 14) for the transition from seeing expressions "operationally" to seeing them "structurally." Reification is not included in The Compact Edition of the Oxford English Dictionary (1980), but Sfard has constructed it from the Latin words res, meaning "a thing" or "an object," and facere, meaning "to make." To reify an expression means to make it into a thing; in grammatical terms, it means to use a collection of nouns, pronouns, adjectives, verbs, and adverbs as a noun clause in a larger sentence. Fowler (1984)
describes a noun clause as "subordinate words including a subject and predicate, but syntactically equivalent to a noun" (p. 79).

Kieran (1992) says that reification "seems to be a leap: a process solidifies into an object, into a static structure. The new entity is detached from the process that produced it" (p. 392).

**Difficulty of Reification**

Sfard (1991) points out that "unlike material objects ... advanced mathematical constructs are totally inaccessible to our senses - they can be seen only with our mind's eyes.... Being capable of somehow "seeing" these invisible objects appears to be an essential component of mathematical ability; lack of this capacity may be one of the major reasons [why] mathematics appears practically impermeable to so many 'well-formed minds'" (p. 3).

Kieran (1992) agrees with Sfard that there is a need for a "lengthy transition period in moving from operational (procedural) to structural conceptions" (p. 395) and that the leap "is inherently so difficult that there may be students for whom the structural conception will remain practically out of reach whatever the teaching method" (p. 412).

This opinion is supported by the work of researchers who have found that students who have trouble with conventional algebra can nevertheless be successful in courses where they treat operations as "things to be done" rather than relationships among variables.

For example, Sutherland (1991) says that "results from our
own studies of pupils programming in Logo have shown that pupils as young as 10 years old can accept and use 'unclosed' algebraic expressions in a Logo programming context" and that "pupils also seem to accept these 'unclosed' algebraic expressions in a spreadsheet environment" (p. 40), both of which applications treat operations as instructions to the computer.

Another example is the New York City Board of Education's Computer Mathematics: An Introduction (1985), developed for high school students "who are not achieving success in the traditional mathematics program."

The course consists of 60 lessons which use computers and computer programming to help students understand perimeters of polygons, area, the order of arithmetic operations, consumer mathematics, algebraic equations, etc.

"The major focus of the program is on problem-solving" with the teaching of computer programming "intended to provide a model of a problem-solving process in which a problem is broken down into its component parts" (New York City Board of Education 1985).

In the first lesson, students are told that they have "just simulated a computer" when they follow the instructions "Stand up, walk to the front of the room, write your first name on the chalkboard, say 'I love computers,' and return to your seat" (p. 1).

This process is generalized as "we had a plan; you were given instructions in English (a language you understand); you memorized the instructions; when you heard the word RUN, you 'executed'
(followed) the instructions" (p. 2).

After the students type into the computer the formula \( P = 3 \times S \) for the perimeter of an equilateral triangle, they are taught to call the value of \( S \) the "input" and the value of \( P \) the "output" (p. 6).

Algebra Without Reification

Sfard (1991) says, "The careful analysis of textbook definitions will show that treating mathematical notions as if they referred to some abstract objects is often not the only possibility. Although this kind of conception [the "structural"] seems to prevail in modern mathematics, there are accepted mathematical definitions which reveal quite a different approach. Function [for example] can be defined not only as a set of ordered pairs, but also as a certain computational process ... Symmetry can be conceived as a static property of geometric forms, but also as a kind of transformation" (p. 4). The latter type of description is what she calls the "operational."

Sfard emphasizes that the operational and the structural conceptions of the same mathematical notion "are not mutually exclusive. Although ostensibly incompatible (how can anything be a process and an object at the same time?), they are in fact complementary. The term 'complementarity' is used here in much the same sense as in physics, where entities at subatomic level must be regarded both as particles and as waves to enable full description and explanation of the observed phenomena."
Sfard stresses that "unlike 'conceptual' and 'procedural,' or 'algorithmic' and 'abstract,' the terms 'operational and 'structural' refer to inseparable, though dramatically different, facets of the same thing" (p. 9). However, she admits that "there seems to be a consensus that the 'abstract' mathematics deserves the highest esteem" and that "algorithmic" mathematics "is somehow second-rate" (p. 9).

After outlining the historical development of the concepts of various kinds of numbers (integral, rational, negative, imaginary), through processes performed on other, already recognized, kinds of numbers (pp. 11-13), Sfard generalizes the development of each concept thus:

1. the preconceptual stage, at which mathematicians were getting used to certain operations on the already known numbers ...; at this point, the routine manipulations were treated as they were: as processes, and nothing else (there was no need for new objects, since all the computations were still restricted to those procedures which produced the previously accepted numbers).

2. a long period of predominantly operational approach, during which a new kind of number began to emerge out of the familiar processes (what triggered this shift were certain uncommon operations, previously regarded as totally forbidden, but now accepted as useful, if strange); at this stage, the just introduced name of the new number served as a cryptogram for certain operations rather than as a signifier of any "real" object ...

3. the structural phase, when the number in question has eventually been recognized as a fully fledged mathematical object. From now on, different processes would be performed on this new number, thus giving birth to even more advanced kinds of numbers.

She summarizes the historical development of mathematical concepts as follows:
Various processes had to be converted into compact static wholes to become the basic units of a new, higher level-theory. When we broaden our view and look at mathematics ... as a whole, we come to realize that it is a kind of hierarchy, in which what is conceived purely operationally at one level should be conceived structurally at a higher level. Such hierarchy emerges in a long sequence of "reifications," each one of them starting where the former ends, each one of them adding a new layer to the complex system of abstract notions. (p. 16)

In what is far from being a simplistic identification of mathematical history with individual development, Sfard nevertheless suggests that "when a person gets acquainted with a new mathematical notion, the operational conception is usually the first to develop" (p. 16) and introduces the words interiorization, condensation, and reification to describe the stages in a person's concept-development which are analogous to the three historical stages above. This three-phase schema, she says, "is to be understood as a hierarchy, which implies that one stage cannot be reached before all the former steps have been taken" (p. 21).

In a section entitled "Operational Approach: Certainly Necessary, Sometimes also Sufficient," Sfard has this to say:

Apparently, a purely operational way of looking at mathematics would be quite appropriate. Indeed, to an unprejudiced and insightful person, the very notion of "mathematical object" may appear superfluous: since processes seem to be the only real concern of mathematics, why bother about these elusive, philosophically problematic "things," such as infinite sets or "aggregates of ordered pairs"? Theoretically, it would be possible to do almost all the mathematics purely operationally; we could proceed from elementary processes to higher-level processes and then to even more complex processes without ever referring to any kind of abstract objects (p. 23)....
Twentieth-century mathematics, however, seems to be so deeply permeated with the structural outlook that a modern mathematician has to be exceptionally open-minded - indeed, not himself [sic] at all - to realize that from a philosophical (not psychological!) point of view he [sic] could do without "mathematical objects" (p. 24.)....

There is no reason to turn process into object unless we have some high-level processes performed on this simpler process. (p. 31)

Davis and Hersh (1981) point out that our modern preoccupation with the structural is historically "insular":

The mathematics of Egypt, of Babylon, and of the ancient Orient was all of the algorithmic type. Dialectical mathematics - strictly logical, deductive mathematics - originated with the Greeks. But it did not displace the algorithmic. In Euclid, the role of dialectic is to justify a constructions - i.e., an algorithm.

It is only in modern times that we find mathematics with little or no algorithmic content, which we could call purely dialectical or existential. (p. 182)

Sfard (1991) admits that "at certain stages of knowledge formation (or acquisition) the absence of a structural conception may hinder further development" (p. 29) but stresses that, both historically and individually, reification is very difficult.

Conclusion

The language of the calculator is the language of arithmetic, in which operations are carried out and "answers" obtained. This language can be transposed into algebra with only minor changes. When it is, it can be understood by students who have had little or no success in conventional courses. Moreover, unless students are going to progress to higher levels of mathematics (in Sfard's
hierarchical sense), there is no valid mathematical reason for insisting that they do mathematics the conventional way.
Chapter 4
Role of the Calculator

The use of calculators, even today, is frequently condemned by parents who say they want their children to learn to think and by students who say they do not want to take the easy route but rather to understand what they are doing.

I started using calculators for the practical reason that my students could not rely on their memory for number facts, either adding or multiplying. I started asking students to do calculations involving more than one operation because calculations involving just one operation were too easy.

However, I found that students could learn a great deal about the structure of an arithmetic expression by learning how to evaluate it on a calculator.

In this chapter I argue

1. that the responsibilities or tasks handed over by students to calculators are irrelevant to the study of algebra.

2. that understanding need not precede practice.

Responsibilities of Calculators and Students

"Teachers can and do provide electronic calculators," says Pimm (1995, p. 61). "As well as holding number symbols, the electronic calculator also holds procedures over which I have no control ..." (p. 62). By authorizing the use of calculators, he says, the teacher suggests "'you need not attend to this: it is
Certainly, in my courses, students, with my permission, turn over to their calculators responsibility for

1. the meaning of the numbers (how big or small they are, what a decimal point means, and - with some calculators - rounding off a number to a given number of decimal places).

2. knowing what you get when you add, subtract, multiply, or divide two given numbers or raise a number to a power.

3. remembering a number stored in its memory.

4. the execution of the operations.

(If their calculators are algebraic, students would also hand over to the calculator responsibility for the order in which the operations are executed if I did not demand that they continue to number the operations to show their order.)

This handing over of responsibilities gives the calculator authority. It means that the calculator can be used to "prove" certain facts if they come into question or dispute. For example, the fact that the calculator displays 14904 when the buttons 36 $\times$ 414 = are pressed constitutes proof that $36 \times 424 = 14904$.

Of course, the calculator does not acquire this authority untested: students check that it gives 6 for $3 \times 2$, for example. (They often perform such a check spontaneously when they obtain an answer which they think is right but which they are told is wrong in order to make sure that the calculator is working properly, since calculators will give wrong answers when they break down and just before their batteries die.)
In addition, as I pointed out in Appendix 4, I use calculators to justify to students my statements about logarithms and the reverse of the $y^x$ operation.

However, students retain responsibility for
1. pressing the correct digit buttons.
2. pressing the correct operation buttons so that they operate on the correct numbers, i.e. pressing the operations buttons in the conventional order on a non-algebraic calculator or after an algebraic calculator registers an error, and in the written order on an algebraic calculator.
3. recalling a number from the calculator's memory.

As Pimm says, "As a user, my focus then becomes: have I given the machine the correct things in the correct order to have it do what I want?" (Pimm, 1995, p. 62).

After discussing the "relative absence of imagery associated with a calculator," Pimm (1995) warns, "It is an interesting and open question whether this ... is a potential weakness - the mechanisms are opaque and therefore offer very little support - or a potential strength - leaving pupils free to form their own imagery - with regard to using such devices to help gain either numerical fluency or understanding" (pp. 80-81).

Pimm's statement that "the theoretical justification for 'why' the device [the calculator] works is complex and opaque" (p. 83); his question, "What mathematics has been embedded in its construction and how easily available is it to a user?" (p. 83); and his warning that "This degree of opacity with regard to the
method of functioning actually constitutes a shift away from even a calculational aid toward a complete substitution for calculation that is resistant to understanding at any but a very deep level" (p. 84) would be important for my thesis if I were using calculators simply to calculate expressions containing one operation (such as \(12.6 \times 13.2\)).

However, they are irrelevant to my use of the calculator, which has students focus on the order of the operations, not the numbers. (The fact that in their past courses my students have focused on the numbers and not the operations is revealed by the initial tendency of some to put the "ordering numbers" under the numbers in the expression instead of under the operations.) Indeed, for my purposes, the expressions \(2 + 3 \times 6\) and \(1.2 + 3.6 \times 543\) are equivalent. As Pimm acknowledged in class, even at this point in my course the numbers have already "dropped out."

Pimm (1995) also warned that "...if the calculator is there, it determines the syllabus to an extent that may be far from desirable" (p. 84). There is no denying that the calculator has determined the syllabus of my courses.

As far as the algebra which succeeds the calculator unit is concerned, the use of the (non-algebraic) calculator furnishes me with an excuse to have students order the operations in an expression in a manner which reveals the structure of the expression. As far as the rest of the course is concerned, where students calculate interest and mortgage and annuity payments,
etc., the calculator is essential as a calculating device: students must evaluate expressions containing as many as twenty operations and obtain answers in dollars and cents. (When my students reach the longest formulae, in fact, they actually use programmable calculators, handing over to the calculators the further responsibility of repeating given operations in a given order.)

Pimm (1995) distinguishes between computational devices, "primarily concerned with automation and fluency of operation," and pedagogical devices, "about illumination and understanding" (p. 84). "No calculating aid is primarily a pedagogic device - though they all provide untaught lessons nevertheless," he warns (p. 63).

However, there is no question in my mind or, as I shall demonstrate in Chapters 6 and 7, in my students' minds, that when students use calculators in the way I have described they are doing and thinking mathematics. I observe, for example, how quickly their initial jubilation at being allowed to use calculators disappears when they discover how much responsibility they retain.

Used as I use it, a calculator may fairly be called a pedagogical device as well as a computational device, a calculating aid. Since the numbers have already, in effect, "dropped out," students are intellectually ready to substitute letters but retain the structure implied by the operations - that is, to begin algebra as a familiar discussion about a new kind of
Unless the calculator "took care of" computation for them, students would find it impossible to focus on the order of the operations and the structure of the expression, any more than I can focus at the same time on the sound of each chord, the "shape" of each chorus, and the overall structure of Handel's Messiah.

Understanding from Doing

In the context of his Chapter 4, entitled "What Counts as a Number?" Pimm (1995) discusses the simultaneity of and, at the same time, the tension between, creating mathematics, doing mathematics, and understanding mathematics.

I designed my courses so that students could succeed in doing something which both they and I would recognize as mathematics, but I concentrated on having the students do mathematics whether or not they understood it, with very little thought of developing what I would have called insight or understanding in my students and with no thought of their creating it.

Accordingly, I introduced the calculator as a computational aid. I was quite willing to make the "tradeoff between transparency ... and fluency or efficiency ..." (p. 63) described by Pimm (1995). I would have agreed that "any over-reliance on machines can lead to the atrophy of the human faculties involved" (p. 82), but I would have argued that anything mathematical I could get my students to do, even with the aid of a calculator, was better than nothing. In my physics and science courses,
however, I shared with other teachers a "fear of apparent sophistication of performance unrooted in understanding, and the perennial desire ... to be able to read comprehension from successful practice" (p. 77).

Similarly, I began to teach the reversal method as an algorithm for solving equations - or, rather, an algorithm for solving equations for which I could make up fool-proof rules. Again, I would have agreed with Pimm (1995) that "there are two separate aspects of algorithms, namely the 'what to do' and the 'why this is what we do ...'" (p. 69), and I would have said that I was concentrating entirely on having students learn "what to do."

Having thus introduced calculators and the reversal method with the lowest (mathematically speaking) of motives, I have been blessed beyond what I deserved. Besides their success at doing mathematics, I began to see and encourage signs of understanding and even creativity among my students, which I recount in Chapters 6 and 7. What Pimm (1995) calls the "confusion of the two functions" of calculators - the computational and the pedagogical - did not lead, as he warns could happen, "to crossed intentions and erroneous expectations, as well as missed opportunities" (p. 84).

Today, I agree with Pimm that "working at such algorithms can result in meaning" (p. 61); that in fact, "it may well be that some level of practice is required before it even makes sense to ask 'why?'" (p. 76); that "reflecting on the practice may thus
provide some access to the 'embedded' mathematics" (p. 77); and
that "all of these resources (physical, gesture-graphical, and
linguistic) contribute to the meaning" (p. 61). With Valerie
Walkerdine, whom he quotes on p. 75, I "draw attention to the
creation of meaning within practices."

Arising as it does from a background of practice in using
calculators, any mathematical understanding or creativity my
students develop is expressed in terms of calculator language,
making it (perhaps) difficult for anyone unaccustomed to that
language to recognize it. As Pimm (1995) says, "To the extent that
understanding arises from reflection on the practice, particular
hard-won understandings may differ" (p. 86).

It appears that once students find that they can do what they
are being asked to do - once the threat of frequent and prolonged
failure is removed - some of the insight that I missed in them
when I first began to teach does, in fact, emerge.

The evidence is that students frequently and spontaneously
verbalize their own observations about algebraic expressions and
equations. Since the language they use is, like mine, derived
from the language of calculators, I suspect that by using
calculators as I do I have, unwittingly, bypassed a number of
roadblocks for these students.

The insights verbalized by my students, who continue to think
of algebraic expressions as building instructions or, at most, as
entities someone has actually built, may be compared to what a man
sees when he is building a house or looking at a house he has
actually helped to build or has seen others build, as opposed to what someone sees who is familiar with building in general, but who looks at the house for the first time after it is finished.

It may be that we have a bias in favor of understanding before doing because understanding (on the part of somebody, if not ourselves) precedes doing in the logical sense of "precedes." For example, no one could repair a car engine unless somebody had already understood car engines.

However, reflecting on the contributions of the physical, the gesture-graphical and the linguistic to the meaning of numbers, Pimm (1995) says, "It is ... far from obvious whether one should come before another" (p. 62).


This injunction brings to mind the following:

1. Augustine of Hippo (translated in Rettig, 1993) wrote, "Understanding is the recompense of faith. Therefore, seek not to understand so that you may believe, but believe that you may understand" (p. 18).

2. Lewis (1952) wrote, "When you are not feeling particularly friendly but know you ought to be, the best thing you can do, very often, is to put on a friendly manner and behave as if you were a nicer person than you actually are. And in a few minutes, as we have all noticed, you will really be feeling friendlier than you
were. Very often the only way to get a quality in reality is to start behaving as if you had it already" (p. 158).

3. Sfard (1991) quotes Jourdain (1956): "'When logically-minded men objected' to the 'absurd' notions of negative and imaginary (complex) numbers, 'mathematicians simply ignored them and said, 'Go on; faith will come to you.'" Those who could see the inner beauty of the idea thought that the new numbers, 'though apparently uninterpretable and even self-contradictory, must have logic. So they [the numbers] were used with a faith that was almost firm and was justified much later" (pp. 29-30).

4. In the disagreement between observations and the classical wave theory of blackbody radiation, Planck introduced the very unclassical $E \propto f$ in order to bring theory into agreement with observation. However, as Giancoli (1980) put it: "Planck himself was not comfortable with it and reported that he proposed it only to bring theory into accord with experiment; and he said he hoped that a better explanation would soon come forth. Alas, it did not, and Planck's quantum hypothesis has come to be accepted as a fact of nature" (p. 650). In other words, the practice became the new understanding.

5. When my niece Kaetlin was two, I showed her a photograph of our staff when I took her to my school one day. "Actually," I said, "this picture was taken last year." She replied, "So some of them must have retired by now." I was used to such language from her; her mother had always spoken to her in adult terms. A few minutes later, I introduced her to one of the teachers,
saying, "Mr. Harris is going to retire this year." When Harris mouthed to me over Kaetlin's head, "She won't understand that," I asked Kaetlin, "Do you know what 'retire' means?" She answered (in the tone of voice she would have used if I had asked, "Do you know what I want for Christmas?")", "No?" "Then why did you say that some of those teachers must have retired?" I asked. She replied (not deprecatingly, but in the tone of one who is explaining something that should be obvious), "Well, I've heard the word."

I suspect that a great many researchers formulate their questions after they see what kinds of answers are forthcoming. I know that a great many outlines in high school and university writing courses are put together, or at least considerably modified, after the essay has been written. There is a profound truth underlying the remark made by the little girl when she was told to think before she spoke: "How do I know what I think till I see what I say?"

It is far from obvious to me as a physics teacher, for example, whether students should first use the concept of "momentum" or learn its definition. At least once a year, one of my students will ask me what momentum is. The last time, I asked in return, "Do you know what time is?" The student said she did. "What?" I asked. "Tell me." When she had seen that she could not, I pointed out to her that although she knew how to talk about time, tell what time it was, and even solve kinematics problems involving time, she could not define it. I also pointed out that
she had learned what she did know about time through using the concept and observing others use it and suggested that she could acquire the same "feeling" for momentum by using the word, listening to me use the word, and solving the assigned problems on the subject.

Newton defined the momentum of an object as a quantity measured by multiplying the object's mass by its velocity. It is arguable that one reason for his identification of this quantity, measured in this way, is that momentum is a conserved quantity in an isolated system. Did momentum and its conservation spring fully armed from Newton's head? Or was there a fruitful interplay of definition and proof similar to the one described by Lakatos (1976)? If the former, can we induce the same phenomenon in our students? If the latter, what consequences, if any, does the interplay have for teachers of the subject? Should teachers start with the definition of momentum, which is easy to understand but which seems to have no reason or purpose? Should they start with its conservation, before students have a sense of what it is that is conserved? Or should they attempt to paint a picture like an artist whose every brush stroke is "informed" by his sense of what the picture will become?

(According to The Compact Edition of the Oxford English Dictionary (1980), the transitive verb "inform" - apart from the "informing" of a person in the sense of passing on information - means "to give form to; put into form or shape; to give formative principle or determinative character to; hence to stamp, impress,
imbue, or impregnate with some specific quality or attribute; especially to impart some pervading, active, or vital quality to; to imbue with a spirit; to fill or affect (the mind or heart) with a feeling, thought, etc.; to inspire, animate" (p. 1431.).

Conclusion

Although the introduction of letters is the most visible change as students move from arithmetic to algebra, changes in the meanings of operations and the equals sign, together with the necessity of understanding the structure of algebraic expressions and equations, are also sources of the difficulties so many students have when they move from arithmetic to algebra.

The language of the calculator, which is the language of arithmetic, can be transposed into algebra with only minor changes. When it is, it can be understood by students who have had little or no success in "regular" courses. Moreover, unless students are going to progress to higher levels of mathematics (in Sfard's hierarchical sense), there is no valid mathematical reason for insisting that they do mathematics the conventional way.

Nevertheless, two reservations are commonly expressed about this alternative way of teaching algebra: first, that students proceed by rote, without anything that teachers would recognize as understanding; and second, that it does not deserve to be dignified with the name of mathematics.

Accordingly, the observations and anecdotes recounted and analyzed in Chapters 6 and 7 are directed towards demonstrating:
Using Calculators to Teach Algebra

1. that it is possible to shift the attention of students from numbers to operations by the use of electronic calculators.

2. that it is possible to shift the attention of students from operations to the overall structure of algebraic expressions and equations by the use of electronic calculators.

3. that the responses of students who have been taught algebra in calculator language demonstrate that they understand algebra.

It follows not only that my mathematics courses are viable alternatives to the "regular" mathematics courses for unmathematical students, but also that they can serve as bridges between arithmetic and algebra for students who will be capable, before they leave high school, of passing the "regular" courses.

In teaching algebra, which so many students find so difficult, we should be willing to use any method that has a possibility of success, even when its philosophical justification is doubtful. Otherwise we are like the nineteenth century captains who predicted that steamships would never replace sailing ships because they would be the ruin of good seamanship.

We should be more like Shakespeare than Milton, as described in Lewis' essay Variation in Shakespeare and Others (1969). Here Lewis tells of the famous problem presented to Samuel Johnson by Boswell: whether Shakespeare or Milton had drawn the more admirable picture of a man, as revealed in Hamlet's description of his father in Shakespeare's Hamlet and Milton's description of Adam in Paradise Lost (see Appendix 5.)
Lewis says:

The two passages illustrate two radically different methods of poetic description. Milton keeps his eye on the object, and builds up his picture in what seems a natural order. It is distinguished from a prose catalogue largely by the verse, and by the exquisite choice not of the rarest words but of the words which will seem the most nobly obvious when once they have been chosen.... Shakespeare's method is wholly different. Where Milton marches steadily forward, Shakespeare behaves rather like a swallow. He darts at the subject and glances away; and then he is back again before your eyes can follow him. It is as if he kept on having tries at it, and being dissatisfied. He darts image after image at you and still seems to think that he has not done enough. He brings up a whole light artillery of mythology, and gets tired of each piece almost before he has fired it. He wants to see the object from a dozen different angles; if the undignified word is pardonable, he nibbles, like a man [sic] trying a tough biscuit now from this side and now from that." (pp. 74-75)
Using Calculators to Teach Algebra

Chapter 5
Research Methods

Beginning in September 1993 and ending in June 1996, I conducted a study of my own students in Mathematics 8A, 9A, 10A, and 11A, with the overriding purpose of documenting my students' insight into mathematics.

My intent in this chapter is twofold:

1. to describe the characteristics of the students I studied, my classroom activities and course requirements, and the degree of success my students had in my courses, so as to help the reader judge the relevance of my study to other students.

2. to describe and justify my research methods.

Characteristics of Students Studied

Students who take my courses have typically had little or no success in previous mathematics courses, as documented by their school records. They and (usually) their parents have despised of their ever being successful in "regular" courses and of thus meeting the prerequisites for going to university.

Appendix 2, which contains the high school records of the students who took Mathematics 11A from me between January and June 1993 as well as those of two Mathematics 11A classes taught earlier by another teacher using my materials, documents this lack of success.

Students in Mathematics 8A, who have not yet completed any
high school mathematics courses, take the course because they have been recommended to do so by their elementary school teachers.

All of these students typically absent themselves from somewhere between four and five times as many classes as students in my physics classes, who are more academic. Many of their absences constitute outright truancy, as the students would admit, but many others are accounted for, often in advance, with excuses such as, "I had to help my parents move," "I had to go to ICBC," "I have to take my driver's test," "I have to go to the doctor, the dentist, etc." - activities which academic students and their parents would schedule outside school hours, not considering them sufficiently important to make the students miss a class.

They are also typically somewhat unpunctual, prone to leave the classroom for long "bathroom breaks," unwilling to do homework, and disorganized, often appearing in class without books, calculators or writing utensils.

The differences between grades can be summarized by saying that the higher the grade level, the more quickly the students can learn the material and the more complicated the problems they can solve, as detailed below.

Basic, "Star," and "Double-Star" Work

Before I can describe a typical class and the requirements of my courses, I must explain what I mean by "star" and "double-star" work.

It is common practice for teachers to present the same
material to all the students in a class, to examine them all on the same material, and then to assign final letter grades according to the following scheme, which is standard in British Columbia schools:

\[
\begin{align*}
86\% & \leq A \leq 100\% \\
73\% & \leq B < 86\% \\
67\% & \leq C+ < 73\% \\
60\% & \leq C < 67\% \\
50\% & \leq C- < 60\% \\
40\% & \leq D < 50\% \\
0\% & \leq E < 40\%
\end{align*}
\]

However, I found when I first started teaching - in all my courses, not just "alternative" mathematics courses - that I could not mark the extra volume of very difficult work of which my "A" students were capable out of the 14\% allowed between 86\% and 100\%, nor the extra volume of difficult work of which my "B" students were capable out of the 13\% allowed between 73\% and 86\%. Nor, I found, was it reasonable to require my other students even to attempt the difficult work of which the "B" and "A" students were capable. Instead, I started designating all problems - on worksheets, quizzes, and exams - basic, "star," or "double star" and allowing students to omit "star" and "double star" problems if they wished. However, I required students to work on "star" problems if they wanted (and were capable of achieving) a B and to work on both "star" and "double star" problems if they wanted (and were capable of achieving) an A.

With this scheme, I assign letter grades according to Table 2, which shows, as an example, how I assign letter grades on the final exam in Mathematics 11A, which has a possible 108 marks
awarded for basic problems, a possible 60 for "star" problems, and a possible 92 for "double star" problems. (This exam is included in Appendix 1, which also shows how I inform students of the lower cutoff for each letter grade.) The right side of Table 2 shows the conventional letter-grade scheme. In order to report marks to parents, I transform my marks (on the left) to standard marks (on the right) by a continuous series of linear transformations which transform 100% to 100%, 0 to 0, and the lower cutoff for each letter grade to the corresponding standard cutoff.

Table 2

<table>
<thead>
<tr>
<th>Transformation of Letter Grades</th>
</tr>
</thead>
<tbody>
<tr>
<td>108 + 60* + 92** = 260</td>
</tr>
<tr>
<td>.85(260) = 221.0</td>
</tr>
<tr>
<td>108 + 60* = 168</td>
</tr>
<tr>
<td>.85(168) = 142.8</td>
</tr>
<tr>
<td>108</td>
</tr>
<tr>
<td>.85(108) = 91.8</td>
</tr>
<tr>
<td>.70(108) = 75.6</td>
</tr>
<tr>
<td>.50(108) = 54.0</td>
</tr>
<tr>
<td>.25(108) = 27.0</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>100%</td>
</tr>
<tr>
<td>86%</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>73%</td>
</tr>
<tr>
<td>C+</td>
</tr>
<tr>
<td>67%</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>60%</td>
</tr>
<tr>
<td>C-</td>
</tr>
<tr>
<td>50%</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>40%</td>
</tr>
<tr>
<td>E</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>
Classroom Activities and Course Requirements

An hour-long class in any of my mathematics courses typically begins with my "going over" the problems on the "practice quiz" given at the end of the previous day. A "practice quiz" takes about five or ten minutes and typically comprises three problems taken from the day's worksheet, with only the letters or the numbers changed: one basic, one "star" and one "double star." The students do whichever of these problems they choose, but only one. The number of marks awarded for each problem reflects its difficulty.

Usually I ask a student who I know made a mistake on the "practice quiz" to tell me, out loud, how to solve the problem, thus giving me the opportunity to explain the nature of the error to the whole class. Alternatively, in Grade 8, for two classes during one semester, I had students who had solved the problems correctly "go over" them for the class, but I abandoned this practice for various reasons, discussed later in this chapter. This introduction serves to remind the students of what they were doing the previous day and what they will be asked to do today.

Then I tell them which worksheet they will be working on today and what kind of assessment will follow: another "practice quiz" (which serves as a "rewrite" of the first one, with the mark recorded being the higher of the two), or, every three or four days, a quiz, which typically takes between twenty and thirty minutes to complete and typically contains five basic problems, two "star" problems and two "double star" problems. Students may
"rewrite" a quiz (another version) on a pre-announced day after school. Again, the mark recorded is the higher of the two.

For the half an hour in the middle of the class, students typically work on the day's worksheet by themselves or with friends. Each worksheet has three versions, which take three days; each has an answer key supplied and students are encouraged to ask questions of me and/or the teaching aide (if there is one) whenever their answer differs from the one in the key.

I assign no homework, mostly because the students need help so frequently. I have found that if I require them to do homework, the conscientious ones attempt to do the work but get "stuck" or, worse, do it wrong, while the unconscientious ones simply claim that they tried and got "stuck."

At the end of the unit the students write an hour-long exam on the unit, which they may "rewrite," if they wish, on a pre-announced day after school. As always, the mark recorded is the higher of the two.

At the end of the course, there is a two-hour-long final exam.

The students' final marks are calculated by weighting the various components of my assessment as follows: "practice quizzes" 20%, quizzes 20%, unit exams 40%, and the final exam 20%.

I give no marks for so-called "compassionate reasons," arguing that the students' current marks are posted every day and that if they want to improve their marks, they can always get extra help and do "rewrites." Anyone who finishes the course with
less than 49.5% is required to repeat it.

The contents of the five courses - Mathematics 8A, 9A, 10A, and 11A - are similar. The differences reflect the fact that the higher the grade-level, the more quickly the students can learn the material and the more complicated the problems they can solve. For example, I can teach Grade 11 students how to solve equations, including groups and multi-step solutions, in five three-hour-long worksheets, even if they have not taken any of my previous courses. However, to teach Grade 8 students to solve equations with far fewer operations, no groups and only one step takes eight three-hour-long worksheets. Appendix 6 contains an outline of the five courses.

Quantitative Data About Students Studied

Table 3 shows the number of classes I taught in each course between September 1993 and June 1996, the number of students in each class, and the final distribution of letter grades in each class.
Table 3

**Actual Letter Grade Distribution**

<table>
<thead>
<tr>
<th>Course</th>
<th>Dates</th>
<th>A</th>
<th>B</th>
<th>C+</th>
<th>C</th>
<th>C-</th>
<th>D</th>
<th>E</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math. 8A</td>
<td>Sep 93 - Jan 94</td>
<td>3</td>
<td>11</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>Sep 93 - Jan 94</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>Sep 94 - Jan 95</td>
<td>0</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Sep 94 - Jan 95</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>Math. 9A</td>
<td>Sep 94 - Jan 95</td>
<td>0</td>
<td>8</td>
<td>10</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>Sep 95 - Jan 96</td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>Math. 10A</td>
<td>Sep 95 - Jan 96</td>
<td>1</td>
<td>10</td>
<td>10</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>33</td>
</tr>
<tr>
<td>Math. 11A</td>
<td>Jan 93 - Jun 93</td>
<td>0</td>
<td>6</td>
<td>8</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>Sep 93 - Jan 94</td>
<td>1</td>
<td>3</td>
<td>12</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>Jan 95 - Jun 95</td>
<td>2</td>
<td>8</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>Jan 96 - Jun 96</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>27</td>
</tr>
<tr>
<td>Total</td>
<td>Sep 93 - Jun 96</td>
<td>9</td>
<td>65</td>
<td>88</td>
<td>36</td>
<td>33</td>
<td>31</td>
<td>12</td>
<td>274</td>
</tr>
</tbody>
</table>

The graph in Figure 1 shows the distribution of these letter grades.
Considered in conjunction with the above description of my course requirements, Table 3 and Figure 1 are evidence that I am neither unduly lenient nor unduly harsh in evaluating students' achievement and thus support the claim I made in Chapter 1 that my students can and do succeed in these courses, observably and testably.

Data Collection: Attempts at Interviews

I now turn to my attempts to elicit from students responses that would reveal their insight into the mathematics I was teaching them.

By 1993, I had been collecting very informally for about fourteen years comments made by Mathematics 11A students which
had surprised me by the insight they revealed. In September 1993, after I had decided to begin a formal case study, I began making notes on them as they occurred. However, I also envisaged interviewing selected students and videotaping or audiotaping the interviews, perhaps asking the students to solve a problem and discussing it with them as they did so - an idea I eventually abandoned, for reasons given below.

Accordingly, on March 28 1994, I interviewed two students who had successfully completed Mathematics 8A in the first Mathematics 8A course I had ever taught. They had had their last class January 27 1994 and their final exam January 28 1994. Both had worked at the A-level (that is, solved the "double-star" problems) throughout the course but for various reasons had finished with letter grades of B (one with 83% and the other with 81%). I interviewed them for about half an hour and videotaped the interview.

To begin, I asked them to solve the following equation together at the chalkboard:

\[
\frac{3.2 + (x - 1.2)^{4/3}}{13.25} = 2.5
\]

(This was a "double-star" question.) They solved it correctly, the first time, by writing the following on the board:
Using Calculators to Teach Algebra

\[(2.5 \times 13.25 - 3.2)^{3/4} + 1.2 = x = 13.99456778\]

**Check:**
\[
\frac{3.2 + (13.99456778 - 1.2)^{4/3}}{13.25} = 2.5
\]

There was some discussion about where the brackets should go in the reversal column and the fact that they had to calculate the exponents \(3/4\) and, later, \(4/3\), and store the results in the calculator's memory before performing the other calculations. (Both had used non-algebraic calculators.)

The interview satisfied me that the students had retained what they had learned. Although they had seen the equation I gave them on an exam at the end of January, neither of them appeared to recognize it. They were not, therefore, recalling how to do that individual question, but rather remembering and applying general rules.

Indeed, they said as much to each other as they solved the problem. They apologized for stumbling slightly and reminded me that they had not seen "this stuff" since the end of January, but they both agreed that "it comes back to you."

Having been told that I wanted to interview them to convince my colleagues that my students really did understand what they had learned in Mathematics 8A, one of them turned to the camera and said gratuitously:

"It looks difficult, but it's not that difficult really. There's a lot of steps, but there's rules that tell you what to do ... like, there's rules about brackets."

This student also said that he liked this course "because
it's different every day. Last year Math was the same thing day after day after day. It got boring." When I asked whether that was not also true for Mathematics 8A, he replied, "Oh, no. Every question had something new in it, like a new way to catch you ... like brackets in different places." (That, in fact, was true: with five operations in each equation, one of them a power, a great many variations had been possible and I had explored them all.)

Questioned during their solving of the equation about the reason for arranging the operations in the first column in the order they chose, they responded with the rules giving the conventional order of operations: "You have to do what's in brackets before anything else"; "You have to do the power before anything else - unless it's in brackets."

They justified adding 3.2 before dividing by 13.25 as follows: "It has to be the top divided by the bottom." They did not mention explicitly the "bracketing" effect of the long division line.

Nor did they refer to the fact that the order of operations was merely conventional and not a matter of logic.

When asked why they had started with \( x \), they said first that it was because that was what they were solving for. Pressed harder, one of them explained: "You see, we're going to reverse all this [pointing to the operations in the first column] and then \( x \) will be down here [pointing to the bottom of the reversal column] and then all this stuff [pointing to the reversed
operations above x] will tell us how to get x. Then, if we do it correctly, we can put x back in here [pointing to the x in the original equation]."

Asked how he would know whether he had done it correctly, he explained, "It will work out to this [pointing to the 2.5 in the original equation]."

I was also interested to know how well they remembered something they had learned in the first unit of Mathematics 8A, on calculators, but not seen again in the unit on equation-solving because they had not been asked to solve equations that would take more than one step, viz. the fact that "you can add backwards but not subtract." I asked, "I see that you did the plus 3.2 backwards. What would you have done if it had been a minus instead?"

I expected them to explain that "you can't do a minus backwards," but instead one of them said, "The minus would have been over here [pointing to the general area inside the brackets in the original equation]."

I am not sure I understand this answer; the only way I can find meaning for it is to assume that he had realized that any minus signs in equations would always come after the x and not before - a precaution I had had to take in all the equations I had given them so that they would never be faced with a two-step solution.

After the videotaped interview, I questioned the more vocal of the two students again, hoping to have him elucidate why, in
constructing the reversal column, he had reversed both the operations and their order. This interview was audiotaped.

"Because you have to reverse it," he said. "That's how you reverse. If you don't do it that way, it won't work ... it won't check ... you won't get the right answer when you check."

He also said, "Because x has to come at the end ... because x is what you don't know." Pressed still further, and apparently thinking that he was letting me down, he said, "I guess I don't know. I guess I missed that bit. Sorry."

The following week, on April 15 1994, I interviewed three other students who had also completed Mathematics 8A at the end of January and recorded the interview on audiotape. Two of the students I had requested interviews with, one of whom had achieved 94% and the other 86%. The other, who had achieved 73%, came along because she was a friend of one of the others.

I asked them to solve the same equation I had given the first two students. The less able of the two "A" students solved it almost by herself because the attention of the other "A" student kept wandering (the interview took place in a cafeteria full of Physics 11 students rewriting an exam).

At first she solved it wrong, diving by 13.25 before adding 3.2, but finishing the solution, calculating the value of x, substituting it back into the original equation, and finding that it did not "check" before saying, "Something's wrong. Oh, f——, what did I do [looking back over her work]. Oh, f——, I'm supposed to do the add first."
She then scribbled out what she had done and, looking at my tape recorder, said, "You'll erase that, won't you?" before solving the equation again correctly.

When I asked her why her first solution had not been correct, she said, "Because I made a mistake." Asked to describe her mistake, she said, "I did things in the wrong order." Asked why that made a difference, she replied, "Because then you don't get the right answer."

I continued to press her, asking why it was so important to do things in the right order, and she said, "Because if you don't, it won't check."

I abandoned that line of questioning and asked instead, "Can you explain to me why this whole method works?" She answered, "Because it gives you x." I asked why you should start with x and she said, "Because then you'll end up with x." In answer to my questions why you reverse the operations and why you reverse their order, she said twice, "Because that gives you the right answer."

At this point, she said pettishly, "I don't know." I said, "Does this method of finding x make sense to you?" She said, "Oh, yes," in a tone of great assurance. "But you just can't explain it?" "Yeah," she said.

Then the third student - the one who had not been invited - broke in, "Why do you want to know this, anyway?" I explained, as I had already explained to the other two, that some teachers thought that my students did not really understand what they were doing but were merely proceeding by rote and she said indignantly,
"Of course we do!" "Yes, but how can I prove it?" I asked. "Maybe you just memorized what to do." "Memorized all that?" she said in bewilderment. "Anyway," I said, "they would like to know whether you can explain how this method of solving equations works." She looked at me wonderingly and asked, "Why would anybody want to know that?"

Reasons for Abandoning Interviews

I have included a description of the above interview in order to explain why I abandoned this line of research.

Initially, I was disappointed with all the students' responses. They seemed to be familiar with what to do and they could justify what they were doing in an unfamiliar example by general rules, but they could not explain why the rules "worked."

Their principal justification for the rules seemed to be that they led to the "correct" value for x, namely the value that would "check" or satisfy the original equation - what seemed to me a utilitarian justification.

Then I began to wonder what I myself would have answered had I been asked the questions I had asked my students. Why does solving for x involve reversing the operations? Because if you start with x and add 2, you have to subtract 2 to get back to x. Why? Because you won't get back to x if you don't - try it and see! Why does solving for x involve reversing the order of the operations? That, I found, is even more difficult to answer. I explain it to students, in practice, by giving them an example to
show them that it "works." The expressed reaction of most students is (eventually, if not immediately), "Well, of course!"

I had always had the subconscious idea that my justification for this procedure was utilitarian only. But now I began to wonder what other justification there could be. Is it, in fact, possible to explain why performing the opposite operations in the opposite order takes you back from the number you ended with to the number you started with? How would mathematicians "prove" it? Is it true, as I began to suspect, that they would merely invoke mathematical laws that are in reality no more than names for what my students "see"?

I began to think better of my teaching techniques. I explained in Appendix 4 that I justify saying that \( y^{1/x} \) is the reverse of \( y^x \) as follows: If (on a calculator) you press \( 2 \ y^{x} \ 5 \), you get 32. If you now press \( y^x (1 \div 5) = \), you get back to 2.

What other kind of explanation is there? Mathematicians would say that \((y^x)^{1/x} = y\) because the exponents should be multiplied. But what is this rule but an attempt to maintain the laws of exponents? It is possible to explain the fact that \((2^3)^2 = 1024 = 2^{10}\), where the exponents are integers, in terms of multiplication, but what is the comparable meaning of a fractional exponent? Trying to give meaning to a fractional exponent in terms of multiplication is like trying to give meaning to division in terms of subtraction.

When I first started teaching, I looked for "insight" and deplored the fact that so many students operated by rule. Was I
now demanding that students who could "see" should also "prove" what they could see according to mathematical rules?

This is not to denigrate conventional mathematical proof. But as Balacheff (1988) showed, proof, especially proof which satisfies students, can take many forms.

Once students "see" that something is true, perhaps it should be acknowledged that for them there is no more basic reason for its truth than that it can be seen to be true. Certainly this convention exists in Euclidian proof, which in high school is seldom pushed back even as far as the axioms. Certainly teachers are far more satisfied with a genuine "I see!" from a student than from the most glib enunciation of the applicable principle.

I realized that in looking for evidence of mathematical insight and understanding among my students, I might be more successful if I waited for spontaneous utterances by the students - anecdotes such as those I had collected informally over the years.

That meant, of course, that I would be restricted to studying my own students, since students in the few classes of Mathematics 11A taught by other teachers using my materials would not be likely to talk to me spontaneously about mathematics. On the other hand, any spontaneous utterances by my own students would be likely to take place in my own classes (or at least in my own classroom), in the context of the teacher-student relationship, with the result that I could, if I wished, exploit the occasions by asking questions, whether the remark was directed
to me, to a teaching aide, or to another student, without giving the occasion the appearance of a formal interview and thus giving rise to the sorts of problems I have described with interviews.

Collecting Anecdotes

Accordingly, I abandoned all ideas of formal interviews and continued to be alert for spontaneous utterances by my students which revealed mathematical insight or understanding, either because they were original to the students or because, if they were statements they had heard me make, they entailed a significant rewording.

Beginning in September 1994, I made a speech along the following lines to all my mathematics classes very shortly after we had begun the courses and they had begun to find that they could, in fact, do what they were being asked to do: "I am doing my master's degree right now. I'm trying to prove that my students really understand what they're doing and don't just follow the rules blindly." (At this point there would usually be some signs of indignation among the students.) "I know you understand what you're doing, but I have to prove it to other people. One of the ways I know you understand is that quite often you say things about mathematics that are much better than the way I said them. You couldn't do that if you didn't understand what you were talking about. When that happens, if you don't mind, I will make a few notes on what you said so that I will remember it."
(In practice, with so many students waiting for my help at any given moment in a class, I always carry around with me a distinctively coloured piece of paper on which I or they write their names in a list; first, so that I will help them in the order in which they asked, and second, so that they cannot use the fact that they are "waiting for me" as an excuse to stop working - they can continue with the next problem in the assurance that I will get to them in turn. I also use this paper if I need to write so as to demonstrate something to students I am helping, rather than using their notebooks. Accordingly, students very soon became used to seeing me writing on this paper during a class and within two days - at most - stopped trying to see "whether I had written down" anything they had said.)

The students appeared to forget what I had said almost immediately - none of them ever referred to it again unless I brought up the subject - so I am confident that I can discount any possibility of self-consciousness or desire to impress the teacher in the utterances I recorded. Moreover, I argue, the students could not know what I would consider worth recording.

I expected that one particularly fruitful source of anecdotes would be lessons given by individual students to the rest of the class. Accordingly, in two Grade 8 classes between September 1994 and January 1995, I designated as "teachers" each day three students who had scored 100% on the previous day's practice quiz, one for each question, and asked them to explain to the rest of the class from the front of the room, using the overhead
projector, how to do that question correctly.

The students loved the idea and vied with each other all semester long for the privilege of being the "teacher." However, I eventually gave up the practice for a number of reasons:

1. It took more time than I wanted to spend. On average, the students took half an hour to do what I could do in twenty minutes, partly because they wrote so slowly but also because of (what I considered to be) irrelevant comments and questions from the other students.

2. The "teachers" carried their own correctly done quizzes up to the front of the room and simply copied out what they had done, instead of letting the other students see them in the process of solving the problem.

3. I began to see that the goal of the "teachers" was to imitate me as closely as possible - asking the questions I would ask in my words and my tone of voice and using the same colours of ink as I did to write the question, number the operations, draw in the bracket boxes, etc. - instead of expounding the mathematics in their own words.

4. The rest of the students, instead of challenging the "teachers" to explain the mathematics, made comments like, "You're writing it too small," "You're standing in front of the projector," "You're using the wrong colour," etc. Most of the challenges to "teachers" to explain what they were doing came from me.

I also expected that it would be useful to audiotape what
students said and I carried a small audiotape recorder in my briefcase from September 1993 until June 1994. However, I found it impractical to carry the tape recorder in my hand at all times and psychologically impossible (for me) to introduce a tape recorder into an informal, data-rich exchange between a student and me or between two students.

In addition, I expected that it might be revealing to ask students in a questionnaire to write what they thought about the course they had just completed. In fact, I did give such a questionnaire to the Mathematics 11A students I taught between January and June 1993, but their specific comments were all about how much they had enjoyed working "at their own level" (referring to the freedom they enjoyed to omit "star" and "double-star" questions). Moreover, I found, their comments did not distinguish between my ability as a teacher and the nature of the course I was teaching.

Although my students did not keep referring to my study throughout the course, I nevertheless received the impression that they were "on my side" in the project, as the following story demonstrates. On August 16 1996, as I was driving home at about 10:00 p.m., the car in front of me stopped to let off a passenger, who turned out to be a student to whom I had taught Mathematics 10A from September 1995 to January 1996 and Mathematics 11A immediately afterward, from January 1996 to June 1996. He recognized my car and came over to my driver's window. He asked me how my summer was going, and I replied that I was
engaged in writing my thesis. He answered, "Oh, yes ... don't forget to give me credit for anything you put in that I said." I laughed. By that time, the driver of the car, who had stopped on seeing his passenger approach the car behind, had backed up so that his car was parallel to mine on my left. Through his open passenger window, he called, in a pleased tone of voice, "Oh, hi, Miss Murphy!" (I had taught him Mathematics 10A at the same time as the passenger.) The passenger then said to the driver, "I'm just telling her to put my name in her thesis," upon which the driver said, "Yeah ... is there anything about me in it? Are you sure you know how to spell my name?"

Recording, Transcribing, and Analyzing Anecdotes

Between September 1993 and June 1996, I stayed alert for spontaneous utterances by my students which revealed mathematical insight or understanding, either because they were original to the students or because they entailed a significant rewording of what I had said.

Whenever such utterances occurred, I immediately jotted down notes about what had been said and/or done, including the students' names, their grade levels and the letter-grade levels at which they habitually worked. (This was not necessarily the same as their final letter grade, as lack of attendance, lack of attention to the lesson, lack of regular practice in class, missing exams, etc. would often make their final mark lower than it would otherwise have been.) Before the end of each class,
usually while the students were writing the "practice quiz," I amplified these notes in writing from memory.

I thus collected very full notes of 77 anecdotes. On typing them out during June and July 1996, I observed, first, certain repetitions, and, second, twelve distinguishable themes. I coded the themes as shown in Table 4 and ordered them according to increasing depth of mathematical insight, as shown in Table 4.

Table 4

<table>
<thead>
<tr>
<th>No.</th>
<th>Theme</th>
<th>Code</th>
<th>Number of Anecdotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Calculator language</td>
<td>LANG</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>Brackets</td>
<td>BRAC</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>Peculiarities of non-algebraic calculators</td>
<td>NONAL</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>Order of operations</td>
<td>ORDER</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td><em>Equal</em> sign</td>
<td>EQUAL</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>Discrepancies between input and execution</td>
<td>EXE</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>Importance of operations as opposed to numbers</td>
<td>OPER</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>Transition from numbers to letters</td>
<td>LET</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>Nature of variables</td>
<td>VAR</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>Structure of expressions</td>
<td>STRUC</td>
<td>7</td>
</tr>
<tr>
<td>11</td>
<td><em>Groups</em></td>
<td>GROUP</td>
<td>7</td>
</tr>
<tr>
<td>12</td>
<td>Authority</td>
<td>AUTH</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td></td>
<td>53</td>
</tr>
</tbody>
</table>
By eliminating the repetitions among the anecdotes, I reduced the number to 53. Using the above codes, I labelled each anecdote with the single most appropriate code, although many anecdotes could clearly be placed in more than one category. Finally, I ordered the anecdotes within each category in increasing level of insight.

These anecdotes are recounted and analyzed in this order in Chapters 6 and 7. With each anecdote is included, in brackets after the name, the course the student was taking and the letter grade level at which that student habitually worked so that the reader can judge qualitatively how common this kind of insight might be. An insight expressed by a "C" student, for example, might be expected to be common among "B" and "A" students as well, while an insight expressed by an "A" student might be expected to be comparatively rare.

The anecdotes I eventually recorded came from 36 different combinations of students and grades (because 17 of the 53 came from students who had already supplied another anecdote while they were in the same grade). I include the graph in Figure 2, which details the distribution of grade levels and "working" letter grades among the 36 student-grade combinations, in order to give an overall picture of the students from whom the anecdotes came and thus make possible a more reliable assessment of the relevance of my thesis to other mathematics students. Figure 2 shows that while eight anecdotes came from "A" students in Grade 8, the rest came fairly indiscriminately from students working at all levels
in all grades.

Figure 2

Distribution of Anecdotes

In recounting these anecdotes, I have divided them into two chapters. Chapter 6 includes the first six categories in Table 4: calculator language, brackets, peculiarities of non-algebraic calculators, order of operations, equals signs, and discrepancies between input and execution. Chapter 7 includes the other six: importance of operations as opposed to numbers, transition from numbers to letters, nature of variables, structure of expressions, groups, and authority. Chapter 6 can thus be said to deal mainly with students' insights into the mathematical language of arithmetic and calculators, while Chapter 7 deals with their insights into algebra and the transition from arithmetic.
Generalizability and Limits

Following Sfard's (1991) analysis of reification and the comparison in Chapter 3 of my courses with "regular" courses, it appears that my case study of unmathematical students learning at the boundary between arithmetic and algebra has consequences for:

1. research into how mathematics is learned, taught, conceptualized, and performed - a topic which interests some educators for its own sake, apart from its application to teaching.

2. the design of high school mathematics courses for the unmathematical, who can learn algebra without reification.

3. the design of high school mathematics course for the mathematical, who might be able to use algebra without reification to bridge the gap between arithmetic and "regular," or reified, algebra.

The students I studied were those assigned to my Mathematics 8A-11A classes by school counsellors and administrators according to the advice of their previous mathematics teachers, which in turn was based on the students' poor performance in previous mathematics courses.

Since Mathematics 8A-11A courses in general are not accepted by colleges and universities as prerequisites for entry, and since it is well known that those who take these courses are very seldom able to switch successfully to the "regular" courses before they leave high school, students who take these courses, as well as their parents, have abandoned hope of academic post-secondary
I observed all my students without conscious discrimination, although more of my attention was claimed by the more vocal and extroverted. My decision to record or not to record any mathematical exchange was based only on my perception of it as insightful or original.

The so-called Hawthorne effect appeared not to be a factor in my research, as students appeared to forget very soon that I was observing them and could not expect to know what I would find insightful or original.

The effects of the setting can also be largely discounted, as the study's venue was my own classroom, with no one present other than those who would have been present in any case, and I made no conscious difference in my teaching because I was conducting a study; the exchanges I recorded were those that occur naturally in a classroom and were similar to those I had noticed for many years before I began recording them.

Although the courses I taught had been written so that students could be successful, I evaluated their success by means of objective tests according to pre-set standards and they did not pass the course unless they achieved at least 50%.

Two other teachers have each taught my Mathematics 11A course (but not my Mathematics 8A-10A courses) to a number of other classes for a number of years, using my worksheets, quizzes, and exams. Each has found the students' final marks to be similar to mine (see Appendix 2).
Chapter 6
Expressing Mathematical Insight

The anecdotes included in this chapter fall into the first half of my twelve categories: calculator language, brackets, peculiarities of non-algebraic calculators, order of operations, equals signs, and discrepancies between input and execution.

In general, they demonstrate that unmathematical students can become fluent in calculator language, which is a dialect of arithmetic language, and use it to express insight into arithmetic and even algebra.

The brackets after each student's name at the beginning of each anecdote includes the course the student was taking at the time the incident occurred and the level at which the student habitually worked, *e.g.* (Ma 9A - B).

I have tried to indicate wherever I or a student emphasized a spoken word by the use of bold italics thus: *emphasized*.

**Communication in Calculator Language**

By far the most common reaction among students to the news that they are going to be allowed to use calculators is jubilation. Occasionally, a student will announce that he or she is not going to use a calculator because "I want to learn to think for myself." However, both jubilation and renunciation soon become tempered as students realize that they have to tell calculators exactly what to do. Within a few days of the
beginning of a course, they see that calculators are as necessary in their mathematics class as gym strip is in a physical education class. Before long, they are speaking the language of calculators and actually verbalizing what is written on their calculators' keyboards.

In fact, some students, like Alison (Ma 9A - B) actually communicate in writing with me as the teacher by means of calculator language. Asked to solve

$$7x - 6 = 3.95$$

for $x$, she calculated that $x = 1.421428571$ and then wrote

$$(ch) \ 7 \times 1.421428571 - 6 = 3.949999997 \ (mode \ 7,2 = 3.95)$$

"(ch)" means "check." In the "check" she substituted her value for $x$ into the original equation to see whether it "checked"; that is, made the left side of the equation equal to 3.95. She found that it did not; that it came instead to 3.94999997. "Mode 7,2" are the buttons she pushed on her calculator (a Casio) to round off the display to two decimal places. By "mode 7,2 = 3.95," she meant that when she rounded off 3.94999997 to two decimal places, she got 3.95, as she had expected, and so her value for $x$ did, in fact, "check." The comma and the $=$ sign were not buttons she had to press; the comma simply separated the 7 from the 2 and the $=$ sign merely indicated that 3.95 was the result of pressing the buttons "mode 7 2" - a use for the $=$ sign which is extremely common, as I will show later in this chapter.

The fact that calculator language thus enabled Alison to
express very economically what she wanted to say is good evidence that calculator language deserves to be dignified as "mathematical language."

However, students do not use the language of calculators merely to communicate with other people; they also use it to communicate with calculators themselves.

Christine (Ma 11A - C+) showed that she considered herself to be communicating with her calculator when she began using my formula for simple interest, viz.

\[ I = \frac{Prt}{u} \]

where \( P \) represents the principal, \( r \) the interest rate, \( t \) the time, \( u \) the number of time units per year (1 for a time given in years, 12 for months, 52 for weeks, and 365 for days).

Christine: Last year we learned an easier formula for this.

I: Did you? What was it?

Christine: I think it was [writing]

\[ I = Prt \]

I: I put 100 into my formula because the \( r \) is always a percentage. That way you don't have to remember to move the decimal point.

Christine: OK. But why do we need \( u \)?

I: Suppose you leave your money in the bank for 5 years. What are you going to put into the calculator for \( t \)?

Christine: That's not what I asked you.

I: I know, but \( u \) is connected to \( t \) ... [I repeated the
question.]

Christine: Five.

I: OK. Now suppose Kelly leaves her money in the bank for 5 months. What is she going to put into the calculator for t?

Christine: Five ... Oh, but that would give her the same answer ... I see ... you can't put the words on to the calculator ... but you have to tell the calculator what you mean ... Oh, OK.

In other words, you have to tell the calculator what you mean through numbers, because that is the only language it understands.

Brackets

The last two examples illustrate students' translating from English to calculator language, but the reverse can happen as well.

Bob (Ma 10A - A) was encouraging Jack (Ma 10A - C-) to try double-star questions. Accordingly, given the values of the variables (which included p = 10 and q = 1.5), Jack was in the process of evaluating the following expression on his (algebraic) calculator, more or less by following Bob's orders.

\[
Q + ma \left[ \frac{1}{1 + \frac{c}{d}} \right]
\]

He had just punched the large right-hand square bracket.

Jack: [Reciting as he punched the buttons] y to the x, one, divided by ...

Bob: No; you need brackets around here [he drew brackets around the entire exponent].
Jack repeated the part he had done correctly the previous time.

*Jack:* ... [emphatically - he was going to get it right this time] *bracket* y to the x, one, divided by ...

*Bob:* No! No! No! You have to punch the y to the x *before the bracket*!

*Jack:* Why?

*Bob:* 'Cause the y to the x is in here.

He laid his pencil point exactly where the dot is below.

\[ Q + ma \left[ 1 + \frac{c}{d} \right] \]

*Bob:* You have to tell the calculator that you're going to do a y to the x, right? But then you have to tell it to wait while you work out the exponent, 'cause the exponent has to be worked out.

Jack repeated the part he had done correctly all along.

*Jack:* ... y to the x, bracket, one, divided by, ten, times,...

*Bob:* No, you need brackets here as well [drawing in brackets around pq].

*Jack:* [Exasperatedly] What do we need those for?

*Bob:* 'Cause you've got to tell the calculator to wait again. It has to do this [laying his pencil point between the p and the q] before it divides.

*Jack:* [Pushing his paper away] Forget it, man.
Notice how Bob translated from calculator language ("brackets") to English ("wait") and back again: Brackets mean the calculator has to wait and if you want the calculator to wait you have to press brackets. Notice also how necessary the calculator appellation "y to the x" is to Bob's explanation.

In this exchange, Bob was emphasizing communication with the calculator. However, students can also used calculator language to describe mathematical communication with other people, as Tom realized in what follows.

Obviously mimicking my habits as a teacher, Bahman (Ma 9A - B) was explaining to the class how to do yesterday's practice quiz problem, viz. to solve the following equation for y.

\[
\frac{y + b}{a} - p = z
\]

He had elicited from another student that he should write

\[
\begin{align*}
\text{y} & \quad \text{z} \\
+ \text{b} & \quad + \text{p} \\
\div \text{a} & \quad \times \text{a} \\
- \text{p} & \quad - \text{b} \\
\text{z} & \quad \text{y}
\end{align*}
\]

and was now asking Abdul (Ma 9A - C+) whether any brackets should be inserted in the second column.

Bahman: Do we need any brackets?

Abdul: Yes, between the plus and the times.

Bahman: Can you tell us why?

Abdul: [In a sing-song tone] 'Cause you always need brackets if an add or a subtract comes right before a multiply or a divide.

Bahman: [Putting in the brackets, and no longer mimicking me]
But why?

Abdul: I don't know.

Bahman: Can anybody tell me?

Scott (Ma 9A - C): 'Cause an algebraic calculator needs brackets to make it do a times before a minus.


I: But what if someone has a non-algebraic calculator?

Bahman: Huh?

I: Non-algebraic calculators don't need brackets. Does that mean that, say, Kathy doesn't need to put in those brackets?

Bahman: No ... Kathy needs to put in the brackets.

I: Why?

Tom (Ma 9A - C): [After a pause] So anyone else doing the problem knows that she wants them to do the times before the minus.

I: Right! In other words, brackets aren't just to tell calculators what to do. They're to tell people what to do.

Tom: They're to tell people who know the rules what to do.

I: What rules?

Tom: The order of operations.

Here, even before my final summing-up, Tom was talking about communication with people by means of calculator language and not merely communication with a calculator, since he realized that even with her calculator, Kathy would have to write in the brackets.

Of a slightly higher order is the calculator language used by
Bob (Ma 9A - A) to demonstrate to me that, contrary to my instructions, it was not necessary to punch into the calculator the (implied) brackets around the expression above a long division line unless that expression contained a + or a -. Asked to solve

\[ \frac{3x}{6} = 18 \]

for x, he had written

<table>
<thead>
<tr>
<th>x</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>× 3</td>
<td>× 6</td>
</tr>
<tr>
<td>÷ 6</td>
<td>÷ 3</td>
</tr>
<tr>
<td>18</td>
<td>x</td>
</tr>
</tbody>
</table>

Bob: You know how you can prove you don't need brackets around 18 times 6?

I: How?

Bob: You don't need a bracket here, right?

He pointed to the space between "× 6" and "÷ 3" in the second column.

I: Right.

Bob: So when I write this out normally ...

He wrote

\[ \frac{18 \times 6}{3} \]

... that means I don't need brackets here, right? [sketching in brackets around "18 × 6."

I: Right.

Bob: [triumphantly] So that proves I don't have to punch them in on my calculator!

About ten minutes later he addressed the same question again.
Bob: You know that question where I proved you don't need brackets?

I: Yes?

Bob: There's another way to prove it.

I: Is there? How?

Bob: If I don't need brackets here ...

He pointed to the space between "× 6" and "÷ 3" in the second column of his solution for x.

... then this [pointing to the ÷ immediately below the space] doesn't have to be a long line, right?

I: Right.

Bob: So I could write it like this ...

He wrote

18 × 6/3

... right?

I: Right.

Bob: And that kind of division line [pointing to the slash] doesn't need brackets, right?

I: Right.

Bob: So?

I: You're right: you don't need those brackets. I tell students to put them in every time because otherwise they forget when they're really necessary.

Bob: Like when it's a plus or a minus.

Bob's technique was not exactly "proof by calculator," as is shown by his ignoring of the calculator in his second "proof," but
it could certainly be called "proof in calculator language."

From then on, Bob seemed to pride himself on leaving out any brackets he could. On another occasion, he was asked to tell what buttons he would punch on his (algebraic) calculator to calculate the following:

\[
1.5 - \left[ 3.7 + 1.9^3 \left( \frac{1.35^2 + 2}{\frac{1}{2} \left[ 1 + 3.2 \right]^3} \right) \right] \frac{1.795}{1.795}
\]

He had written

\[
1.5 - (3.7 + 1.9 y^3 \times (1.35 x^2 + 2) ÷ (1 ÷ 2) x^2 (1 + 3.2) y^3 ÷ 1.795) x^2 =
\]

and was comparing his answer with mine:

\[
1.5 - (3.7 + 1.9 y^3 \times (1.35 x^2 + 2) ÷ (1 ÷ 2) x^2 \times (1 + 3.2) y^3 \times 1.795) x^2 =
\]

As he saw, he had left out the \(\times\) signs and the two brackets marked with arrows.

**Bob:** Well, I can leave out times signs on my calculator. And I don't need those brackets [pointing to the ones marked above with arrows] 'cause that's just a multiply [pointing to the multiply marked with an arrow]. And I don't really need those brackets, either [pointing to the ones marked with arrows below]

\[
1.5 - (3.7 + 1.9 y^3 \times (1.35 x^2 + 2) ÷ (1 ÷ 2) x^2 \times (1 + 3.2) y^3 ÷ 1.795) x^2 =
\]

**I:** Why not?
Using Calculators to Teach Algebra

Bob: 'Cause that's just a divide [pointing to the + marked with an arrow above] ... just wait a minute ... He ran his finger slowly backward along the lines from the marked + until he reached the + marked above with an arrow.

Bob: ... yes, I do, 'cause of that plus.

I: What about this plus [pointing to the one marked below]? 

\[
1.5 - (3.7 + 1.9)^x \times ( ( 1.35 \times^2 + 2 ) \div ( ( 1 \div^2 ) \times^2 ( 1 + 3.2 )^y \times^3 ) \div 1.795 ) ) \times^2 ) =
\]

Bob: That's already in brackets, see? [pointing to the brackets around (1 + 3.2)].

I: OK. So did you do the question right?

Bob: [Triumphantly] Yep!

I was surprised that he would choose to determine the necessity or otherwise of brackets by looking at the long line of "buttons he would punch" instead of the original numerical expression, which to me is far easier to read. I can explain it only by supposing that the picture in his mind was of what a calculator would do, or would have done, or needed to be told, at each stage along the line.

All of the above anecdotes demonstrate that students can become fluent in calculator language and that calculator language can be used to express mathematical ideas economically - a fundamental characteristic of any mathematical language.

Non-Algebraic Calculator Language

In fact, some students even argued the right of calculators
to determine or modify the language they used, as if they had to communicate with calculators only and not people, as the following two anecdotes demonstrate.

Kyle (Ma 9A - A), who had a non-algebraic calculator, had been asked to solve the following equation for \( x \).

\[
\left( \frac{5 + \frac{x}{3}}{6} \right)^2 = 13
\]

According to what I had taught him, he should have written the following:

\[
\begin{array}{|c|c|}
\hline
x & 13 \\
\hline
\div 3 & y^x 1/2 \\
+ 5 & \times 6 \\
\div 6 & - 5 \\
y^x 2 & \times 3 \\
13 & x \\
\hline
\end{array}
\]

\[
(13^{1/2} \times 6 - 5)3 = x = 49.89992296
\]

Check:

\[
\left( \frac{5 + \frac{49.89992296}{3}}{6} \right)^2 = 13
\]

Instead, he wrote

\[
\begin{array}{|c|c|}
\hline
x & 13 \\
\hline
\div 3 & y^x 1/2 \\
+ 5 & \times 6 \\
\div 6 & - 5 \\
y^x 2 & \times 3 \\
13 & x = 49.89992296 \\
\hline
\end{array}
\]

and then argued with me afterwards thus:

Kyle: How come I didn't get full marks for this question?

I: Because you left out brackets in the second column, you didn't write out the second column normally and you didn't write
Kyle: I don't need brackets on my calculator.

I: I would let you get away with leaving brackets out of the second column if you'd put them in when you wrote it out normally, but you didn't even write it out normally.

Kyle: [Belligerently] Why should I write it out normally when I've got a non-algebraic calculator? Writing it out normally is for algebraic calculators.

I: But this [pointing to the second column] isn't the standard way to write math. No one outside this class would know what it meant.

Kyle: So? No one outside this class has to read it. You know what it means, don't you?

I: Yes.

Kyle: So?

I: But Kyle, I'm trying to teach you math. Part of math is how you write it down. You may have another teacher next year who doesn't understand this way.

Kyle: Aren't you going to teach Math 10A next year?

I: I don't know.

Kyle: If you don't I just won't take Math. I'll leave it until you teach it.

I: You can't do that.

Kyle: And this check [pointing to his long arrow]. Are you telling me nobody would know what that meant?

I: No, I must admit that's pretty clear.
Kyle: So why should I write out the whole of the first equation again with the number for x?

I: [resignedly] OK, I'll give you the marks for the check.

Kyle: [challengingly] I do know how to write out the second column, you know.

I: I know you do. I was disappointed when you lost so many marks for not doing it.

Kyle: So will you let me do it now?

I: OK ... and put the brackets in the second column as well.

In more moderate fashion, Matt (Ma 11A - A) presented me with a similar argument when he was solving compound interest problems. He had been given the values of A, P, n, t and u and was solving this equation

\[ A = P \left[ \frac{r}{n} + 1 \right]^\frac{nt}{u} \]

for r. He wrote

\[
\begin{align*}
\frac{r}{n} & \quad A \\
\div n & \quad \div P \\
\div 100 & \quad y^x 1/(nt/u) \\
+ 1 & \quad - 1 \\
y^x (nt/u) & \quad \times 100 \\
\times P & \quad \times n \\
A & \quad r
\end{align*}
\]

and then reached for his calculator.

I: Aren't you going to put brackets in and write out the second column normally?

Matt: No - I've got a non-algebraic calculator, so it's easier to do it this way [running his finger down the second
I: Could you write it out if you had to?

Matt: Oh yeah - I did it in the formulas unit. But there's no point in this unit - you just want the answer.

I: Of course, you can't really just go down the column, can you? I mean, you have to work out 1/(nt/u) first and put it in the memory.

Matt: Oh yeah, but that's easy. I do that first, and then just go down the column and press memory recall when I get to the y^x.

Both of these students realized that the written notation of the "reversal" column was sufficient and even perhaps more appropriate than conventional notation for a non-algebraic calculator. It can be argued that for students with non-algebraic calculators to "write out the reversal column normally" is indeed superfluous; a student who realizes this can be said to have been mathematically creative and to be justified in defying the conventions, as Kyle did.

Order of Operations

Kyle and Matt were not the only ones to question conventional written notation. Yvonne (Ma 8A - C+) also showed that she would prefer what might be called "sequential" notation when she was asked to write numbers under the operations in this expression to show the order in which they must be done.

\[ 12 + 3 \times 4 \div 6 - 12 \]
Yvonne: [Petulantly] Why do mathematicians write it like this when they want you to do this [pointing to the ×] before this [pointing to the +]?

I: How would you like them to write it?

Yvonne: Write the first one first.

I: I see what you mean. I don't know. I guess I'm so used to thinking of the usual order of operations that I can't imagine it any other way.

Yvonne: Well, it seems dumb to me.

I: [Thinking of "columns"] Later on, in Unit Two, I'm going to let you write it your way, because it is easier sometimes.

Yvonne: Oh, good.

I: Unfortunately, you have to change it back to the "proper" way in your final answer, because nobody else does it the way I do.

Yvonne: [Resignedly] Whatever.

There can be no doubt that conventional written notation, taken in conjunction with the conventional order of operations, does not communicate to many students the structure that mathematical people see at a glance, as Rick (Ma 8A - B) showed when he was asked to (a) number the operations in the following expression, (b) tell what buttons he would punch on his calculator to get the answer, and (c) give the answer to two decimal places.

\[(71.6) \frac{3}{7 + 9}\]

Rick: [handing in his quiz] Can I ask you about a question -
I'm handing it in: I'm not going to change it.

I: Yes.

Rick: First I did it this way ...

He copied the question on to another piece of paper and numbered the operations thus

\[
\frac{3}{7} + \frac{9^2}{3} \]

...But then I remembered you said multiply before divide, so I did it this way.

He erased the 2 and the 3 and wrote them in again, reversing their positions.

I: That's right.

Rick: Well, I thought I'd try it on my calculator - look ...

He ran and got, not his own calculator, which was non-algebraic, but Borzoo's, which was algebraic, and held it so that I could see the display.

... watch [he punched 71.6 x, then stopped]. You don't need brackets around the 3, right?

I: Right.

Rick: 'Cause it's just a single number.

I: Right.

Rick: Now, watch [he punched 3 ÷, then held the calculator in front of me] - see? [The calculator showed 214.8.] That means it's done the times, right?

I: Right.

Rick: So what you said was right.
He laughed deprecatingly, as if he was afraid of seeming rude if he questioned what I had said.

Rick: Of course, I knew that anyway. But when I look at this question [he pointed to the original problem] it looks as though you should do the divide first.

I: Why?

Rick: 'Cause it looks a bit like a fraction.

I: I see what you mean. Actually, in this case you would get the same answer if you did do the the divide first. If I had written the question this way ...

I wrote

\[
\frac{3}{7 + 9} (71.6)
\]

... you would have got exactly the same answer. Try it, if you like.

Rick: [After trying it] Yeah.

I: The reason I tell you to do multiplies and divides left to right is that sometimes - as in a question like this ...

I wrote

\[
3 \div 6 \times 9
\]

it would give you a different answer if you did the multiply before the divide. In this question you have to do the divide before the multiply. So it's easier just to remember to do multiplies and divides left to right.

Rick: OK.

I: Why did you use Borzoo's calculator instead of yours?
Rick: 'Cause mine's non-algebraic.

I: So?

Rick: So it doesn't tell you the order of operations.

After this, Rick would probably have agreed that sequential notation was superior to conventional notation.

This is another example of "proof by calculator." I found it interesting that Rick had accepted the calculator as something he could consult to find out what was right when he was not sure. He was communicating with the calculator in a way I had not seen before: I had taught them that an algebraic calculator "knew" what order the operations should be done in, even when you punched them on to the calculator in the order in which they were normally written; I had never suggested that an algebraic calculator could tell us what order they should be done in. But Rick had apparently assumed that if the calculator "knew" what to do, it could be made to "tell us" what to do.

In telling students that an algebraic calculator "knew" what order the operations should be done in, I had always stressed that the order was merely conventional, an emphasis which perhaps bolstered Kyle's and Matt's expression of preference for the sequential notation of a column.

Many students, I have found, do not understand, even when they know how to apply it, that BEDMAS (a well known acronym for brackets, exponents, division, multiplication, addition, subtraction) has the arbitrary nature of a convention. "You mean I don't have to understand BEDMAS?" a student asked once. "All
these years I've been trying to understand it and it's just an
agreement among a lot of mathematicians? I'm not as dumb as I
thought." He added, humourously, "I feel I can say that here, in
this class ... I mean, it's like Mathematics Anonymous, right?"

Far from accepting the order of operations as "merely"
conventional and, logically, replaceable with any other
convention, Bob (Ma 8A - A) was convinced that he knew why the
conventional order had been chosen.

Bob: I know why they do multiplies and divides before adds
and subtracts.

I: Why?

Bob: Because multiply and divide are higher.

I: What do you mean?

Bob: They're higher.

I: Do you mean they give you higher answers?

Bob: Yes. You learn add and subtract first, and then you
learn multiply and divide. Then you learn powers.

I: I don't see what that's got to do with it.

Bob: You always do higher operations first.

I: Do multiply and divide always give you higher answers?

Bob: [After a pause] They don't have to, I guess. But
they're still higher operations.

Besides the obvious ideas that addition and subtraction
are "easier" than multiplication and division, and that
multiplication can give higher answers than addition, was there in
Bob's mind, perhaps, a sense of a hierarchy among the operations,
perhaps the hierarchy due to the fact that multiplication can be described in terms of addition and powers in terms of multiplication?

Whatever was in Bob's mind about the necessity of BEDMAS, Matt, Kyle and even Yvonne showed signs of creativity in the field of mathematical notation, and I would say that in each case calculators had been the catalyst. (If notation sounds like a minor matter, consider the breakthrough in elementary particle physics made possible by Feynman diagrams.)

**Use of Equals Sign**

As I showed in Appendices 3 and 4, my students use the symbol = as it is used in arithmetic and calculator language and continue to use in this way when they learn how to solve equations by the reversal method.

The following two anecdotes illustrate how students use the = sign to designate "the answer."

When I first started showing students how to solve the following equation for x

\[ 2x + 3 = 8 \]

I wrote on the board

```
start x
* 2
+ 3
end 8
```

*Kyle:* Are you always going to tell us the answer?

*I:* No, we have to work out the answer.

*Kyle:* But if we don't know the answer, how do we know what to
end up with?

I: Oh, I see what you mean. You mean this [pointing to the 8 in the original equation], right?

Kyle: Yeah, the answer.

I: Yes, I'll always tell you what to end up with.

On another occasion, Kyle was solving the following equation:

$$3.16(4.95 + x)^2 = 35.2$$

I: What's the answer, Kyle?

Kyle: 35.2.

I: No, the answer for $x$.

Kyle: Oh ... -1.61244992. I thought you meant the answer.

I: Isn't -1.6124492 the answer?

Kyle: No, that's $x$. 35.2's the answer to the equation. You give us the answer. We have to find out what you started with.

I: I'm asking you a question: what's $x$? Isn't -1.6124492 the answer to that question?

Kyle: Yeah, I guess you could say that. But it would be confusing.

For many students, "confusing" is an exact description of teachers' attempts to change the familiar arithmetic meanings of equals and "the answer."

Another anecdote illustrates an even stranger use of the = sign. I expected my students, given a word problem like the following:

A bank pays simple interest on term deposits at a rate of 10.50%. If you earn $1000.00 in one month, (a) how much have you invested and (b) how much do you end up
with?

together with the following equations and definitions

\[ I = \frac{Prt}{u} \quad \text{A} = P + I \]

- \( P \) principal
- \( r \) interest rate
- \( t \) time
- \( u \) how many time units per year
- \( I \) interest
- \( A \) total amount

to set up a table thus

<table>
<thead>
<tr>
<th>P</th>
<th>r</th>
<th>t</th>
<th>u</th>
<th>I</th>
<th>A</th>
</tr>
</thead>
</table>

and to enter the data from the problem under the appropriate headings before attempting to calculate the values of the unknown variables, which they would then record in the correct place in the table.

In order to make sure that the students knew what they had calculated, I tried to insist that they write out the final answer in the form of a sentence on the grounds that otherwise they were simply asking me to choose the right answers from their table. However, the students tried hard to think of ways around this requirement.

Matt's way (applied to the problem above) was to insert "(a) =" in the table under \( P \) and "(b) =" under \( A \).

However, students do not merely see the symbol = as designating the answer: they also think it should be written before and not after the answer.

I found that students were accustomed to finding the single
number in an equation on the right, as (of course!) the answer should be, so in writing equations I deliberately started alternating between this way

$$3x + 6.5 = 10$$

and this way

$$12 = 14x - 5.4$$

Kathy (Ma 8A - C): [Pointing to the second kind] Why do some of them have the answer first?

I: Because I want you to know that mathematicians often write them that way.

Kathy: But we're not mathematicians.

I: No, but you're studying math.

Kathy: Do we have to do that kind backwards?

I: Well, you have to read it backwards. You always start with $x$, and you always end with the answer, so you read it like this: $x$ times 14 minus 5.4 equals 12.

Kathy: Can you read an equals backwards?

I: Yes.

Kathy: Well, it seems dumb to me.

Given her view of the $=$ sign as what you write before you write the answer, Kathy's questions and comments were reasonable. I always, when I started teaching how to solve equations, used the words "start" and "end" in column 1, but within a few days I would encourage students to leave them out. However, a number of students wrote (and continued to write) $=$ at the end of each column. Others, like Rosa (Ma 11A - A), wrote a long horizontal
line above the last line ("the answer") in each column. Something, students seem to feel, is necessary to distinguish "the answer."

In an extreme example, Sally (Ma 11A - C+) identified "the answer" with an = sign even before she started. I always gave an equation and the variable to solve for in the following form

\[ p = ap_i + b \]

By herself, Sally starting always put a double line under the variable at the right before writing anything else.

* I: Why do you do that?

* Sally: It tells me what I have to start with.

* I: OK.

* Sally: Besides, it looks like an equals sign, and that's what I have to end up with ... you know, equals.

However, as the following anecdote illustrates, I observed some evidence that students could broaden their concept of the = sign and "the answer" that follows it.

For example, Pat (Ma 10A - A) was solving the following ("double-star") equation for x.

\[
\begin{bmatrix}
\frac{m}{k} - \frac{Y}{D}
\end{bmatrix} \uparrow = 1 - \frac{1}{x}
\]

* Pat: What's the answer in this equation?

* I: It depends on what you start with.

* Pat: I'm starting with 1 [pointing to the 1 immediately to the right of the = sign].

* I: So how would you read that equation?

* Pat: What do you mean?
I: [Indicating the whole equation] Read out what it says, starting with 1.

Pat: [Hesitantly] One minus one over x equals ... Is this whole group the answer [sketching a circle around the left side of the equation]?

I: Yes.

Pat: [Slowly] OK.

I: [After a pause during which he appeared to be thinking] What if I had asked you to solve for m? Where would you start then?

Pat: m.

I: What would you end up with?

Pat: This [sketching a circle around the right side of the equation].

I: So it depends on where you start.

Apparently, students do indeed view the = sign as "what you write before the answer." Their view can be modified somewhat, as Pat and Kathy showed, but it is entirely appropriate to arithmetic and calculator language and its extension into algebra via the reversal method of solving equations.

Differences Between Input and Execution

In Chapter 2 I showed that one significant difference between arithmetic and algebra is the fact that the operations in algebra are "virtual" - they cannot be carried out - and yet they still determine the structure of an expression. To be successful in
algebra, therefore, students have to be willing to accept and work with so-called open expressions such as $a + b$, accepting a delay between the writing of the operation and its execution. The execution can be carried out only when numbers are substituted for the letters and the delay may be extended indefinitely. Calculators can play a role in persuading students to accept this delay, as the following anecdotes show.

Jack (Ma 10A - C-) had learned the conventional order of operations and I was now trying to explain to him that the conventional order was not the order in which he should type the following problem on to his algebraic calculator.

$$2(1.01 + 1.001)^3$$

He had been asked to (a) place numbers under the operations to show the order in which they must be done, (b) write down what buttons he would push on his calculator to get the answer, and (c) do the problem on the calculator and write down the answer. He had written

(a) \[ \frac{2(1.01 + 1.001)^3}{1 \quad 2} \]

(b) \[ (1.01 + 1.001)^3 \times 2 = \]

(c) 16.27

and I had marked (b) wrong because he had an algebraic calculator.

Jack: So what's wrong with this? I got the right answer, didn't I?

I: Yes, but you're not using your calculator the way it's meant to be used.

Jack: [challengingly] So?
I: I'm trying to teach you how to use your calculator as efficiently as possible, not just get the right answers.

Jack: [resignedly] OK ... show me how I'm supposed to do this one.

I: On your calculator, you type in the problem exactly as it's written.

I wrote

\[ 2 \times (1.01 + 1.001) \times 3 = \]

Jack: I'm supposed to do this first [pointing to the \( \times \)]?

I: Yes.

Jack: But that's number three.

I: I should have said you type it on to your calculator first. Your calculator won't do it first, but you type it in first.

Jack: But doesn't the calculator do it when I type it in?

I: No; your calculator is programmed not to do a times [pointing to the \( \times \)] when it sees that the next operation is in brackets [pointing to the \( + \)].

Jack: You mean, it waits to see what's coming up next?

I: Yes.

Jack: So brackets mean "wait."

I: Yes; that's exactly what this kind of bracket [pointing to the ( bracket) means. But as soon as you punch this [pointing to the ) bracket] the calculator knows that it can do the +, so it does it.

Jack: [facetiously] The long wait's over.
I: Yes.

Jack: So now it can do the times.

I: Well ... can it? The next operation \( y^x \) is a power. Remember, your calculator is programmed to do operations in the right order even when you type them in in a different order.

Jack: Oh ... I see ... [thoughtfully] a power ... No, I guess it still has to wait ...

I: ... until you press =; when you press = the calculator does all the operations you've typed in. That's why you never press = on an algebraic equation until you get to the end.

Jack: Now I see ... doing an operation doesn't mean it's done.

I: You tell the calculator what to do, but it doesn't always do it right away.

Jack: OK. So why do you tell people with algebraic calculators to put in the order of operations?

I: Because I know you're going to need it in Unit II.

Jack: So why do you teach it now?

I: Because the non-algebraic people do need it now, and it's easier to teach it to everybody at the same time.

Jack: That makes sense.

(Notice how Jack translated between English and calculator language on his own: "Brackets mean wait" and "The long wait's over.")

This delay between the input of an operation by a person on
Using Calculators to Teach Algebra

to an algebraic calculator and the execution of the operation by
the calculator may be the first hint students get that what Pimm
(1987) called "the powerful prompt" (p. 167) of an operation sign
cannot always be obeyed immediately.

A similar delay - although very short - exists even on a non-
algebraic calculator, enabling students to correct mis-punched
operations provided they recognize their error before the
operation is carried out by the calculator.

I had been explaining to a Mathematics 8A class that if
students with non-algebraic calculators punch the wrong operation,
you can correct it by punching in the correct operation
immediately afterward. However, I had warned, students with
algebraic calculators should not attempt the same thing.

(Actually, you can, on an algebraic calculator, safely
replace a power operation with \( \times \) or \( \div \), and \( \times \) or \( \div \) with \( + \) or \( - \).
However, it would not be safe, for example, for students to
attempt to replace \( + \) or \( - \) with \( \times \) or \( \div \) because pressing \( + \) or \( - \)
might cause the calculator to execute previously punched \( + \)'s or \( - \)'s
which would have been further delayed if the correct operation had
been punched.)

However, Bob (Ma 8A - A) wanted to see for himself.

Bob: You said you can't make corrections on an algebraic
calculator, but you can. I just tried it. Look ... 

He punched 2 + \( \times \) 3 = and showed me the 6 on his display.

I: Yes, but you have to be careful.

Bob: Why?
I: [Writing down $11 + 6 \times 5 - 4$] Well, suppose you punched a - there [pointing to the $\times$] by mistake. You couldn't just change it to a $\times$.

Bob: Why not?

I: Well, think about what the calculator would do when you pushed the - by mistake.

Bob: [After a pause] It would do this plus [pointing to the + in the question]. Oh, I see, you couldn't get the 11 and the 6 apart again.

I: Right.

Bob: But if I did a $\times$ there [pointing to the -] I could change it to a $-$. 

I: Why?

Bob: Because the $\times$ would make the calculator do the $\times$ [pointing to the $\times$ between the 6 and the 5], but that's OK, because it's supposed to be done next.

I: [Summarizing] So you have to be careful.

I found his statement that "you couldn't get the 11 and the 6 apart again" particularly interesting; it was almost as though he were seeing that the immediate carrying out of an operation could at times have disadvantages.

The concept of a delay between input and execution of operations evidently stayed in Bob's mind, because he reverted to the point when he and two other students bought programmable calculators at my recommendation. While they were not yet programming them, they had discovered that when they had to
evaluate a numerical expression, the calculator "printed out" the entire expression on the display without any evaluation until they pressed the button marked EXE (there was no button marked =).

Bob (Ma 8A - A): This is great: the calculator doesn't do anything until you tell it to, so you can go back and change something if you've done it wrong.

Borzoo (Ma 8A - A): Even if the calculator's done it, you can still go back and do it again.

Bob: Yes, I know, just punch EXE again.

Borzoo: No, punch this button [pointing to a left-pointing arrow].

Bob: What does that do?

Borzoo: It shows you the whole problem again.

I saw that the ambiguous use of the word "do" was going to cause problems.

I: Doing the problem and executing the problem are not the same thing.

Vince (Ma 8A - A): What d'you mean?

I: First you punch the problem on to the calculator - all the numbers and all the operations. Then, when you're sure you've punched it in right, you tell the calculator to execute the instructions.

Vince: I thought execute meant, you know, electrocute or something.

I: Actually, "execute" means "carry out" or "make happen."

First the judge gives the sentence: he says what should happen to
the criminal. Then, later on, the sentence is **executed**, or carried out. So first you say what should happen to the numbers and then you tell the calculate to execute your instructions.

Vince: Ohhhhh... I see.

Bob: So really, this is like a third type of calculator.

I: What do you mean?

Bob: Non-algebraic, algebraic, and this kind.

I: This kind is algebraic, you know.

Bob: I know, but non-algebraic calculators execute - is that what you say? - every operation as soon as you punch the next operation. Algebraic calculators execute some operations right away... like they don't execute a plus if they see that a times is coming next, so they wait, but if, say, a minus comes next they execute the plus. But this kind of calculator doesn't execute anything until you tell it to.

Again, the distinction between "doing a problem" in the sense of writing it out or typing it on to a calculator like Bob's and "executing the problem" in the sense of carrying out the operations is a foreshadowing of the difference between what Pimm (1995) calls the "virtual" operations (p. 109) in algebraic expressions and the actual or executable operations in arithmetic expressions.

An appreciation of this delay on a calculator can even help students to accept mathematicians' habit of talking about a variable with an undetermined or even indeterminate value ("before they know what it is") as Kyle (Ma 8A - A) implied by his
Using Calculators to Teach Algebra

insistence on the conditional mood for the verbs in the following exchange.

Given that \( m = 2 \), \( a = 6 \), and \( c = 6 \) in the following equation

\[
m(b + a) = c
\]

I had established that it was \( b \) that we were trying to find and was now demonstrating how to find it. I had written

\[
\text{start } b \\
\quad + a \\
\quad \times m \\
\text{end } c
\]

saying as I wrote, "If we start with \( b \), then add \( a \), then multiply by \( m \), we end up with \( c \)."

Kyle: How can you start with \( b \) when you don't know what it is?

I: Why shouldn't we?

Kyle: 'Cause you don't know what \( b \) is, so how can you start with it? Logically, you can't even talk about it until you know what you're talking about. [Cheekily] Always think before you speak, Miss Murphy.

I: But, Kyle, I'll be able to figure out what \( b \) is in a minute. Can't you let me talk about it for a second when you know I'm going to work it out?

Kyle: If you did know what \( b \) was, you could start with it. Then if you did add \( a \) and did multiply by \( m \), we would end up with \( c \).

I: OK.

Kyle: [Challengingly] Well, say that, then.
However, perhaps the ultimate in willingness to delay execution of operations was exhibited by Bob (Ma 8A - A). Given

\[
\left[ \frac{1.8^2 + 3.9}{\left( \frac{5}{6} + \left( \frac{3}{9} - \frac{1}{5} \right)^2 \right)^2} \right]^3
\]

on a quiz, he was required to (a) number the operations to show their conventional order, (b) tell what buttons he would punch on his calculator to get the answer, and (c) give the answer to two decimal places.

Bob: [Having completed parts (a) and (b) of the question] Do I have to punch this out?

I: Yes, if you want the other mark. The question's out of eight. You get one for the order of operations, six for telling me the buttons, and one for the answer.

Bob: 'Cause I don't feel like punching this out.

I: Then you'll lose one mark.

Bob: Why do I have to punch this out?

I: To get the answer.

Bob: But if I've told you how I'm going to punch it out, isn't that the same?

I: No.

Bob: But if I've written down how to punch it out, anyone in the world could do it.

I: So?

Bob: So it's just the same as if I've done it.
Using Calculators to Teach Algebra

I: If the bank sent you a statement, and they didn't tell you how much you actually had at the end of the month, would you be satisfied?

Bob: Yes - if they told me how to calculate it, like I've done.

I: Would you calculate it if they told you how to?

Bob: Yes.

I: Why?

Bob: To find out how much I had.

I: Wouldn't you want to check that the bank had got the same answer?

Bob: I guess so. Anyway, I'm not going to do it - I'm late for a rugby meeting.

Significantly, he called back as he ran off, "Just push EXE."

It was also Bob who, in the middle of the exam at the end of the calculator unit, announced loudly and in a tone of great finality, "I'm - not - going - to punch - any - more - buttons. I just won't. I won't punch another button" - thereby sacrificing a significant number of marks on the exam.

It appears that calculators, by delaying the execution of an operation, can introduce to students the idea that the operation has an existence apart from its arithmetic execution, an idea that is essential to algebra.

Conclusion

Otherwise unmathematical students are capable of becoming
fluent in arithmetic and calculator language. They can be observed to use it in two-way communication with calculators and with other people, to express and explore their insights into arithmetic and algebra, and even to create mathematics for themselves.
Chapter 7

Bridging the Gap Between Arithmetic and Algebra

The anecdotes included in this chapter fall into the second half of my twelve categories: importance of operations as opposed to numbers, transition from numbers to letters, nature of variables, structure of expressions, groups, and authority.

In general, they demonstrate that unmathematical students can be helped to bridge the gap from arithmetic to algebra.

**Operations Versus Numbers**

The fact that one of childish designations for arithmetic is "numbers" is significant, for it is the numbers, not the operations, that first engage the attention of students when they begin arithmetic, as, for example, while they are memorizing the number facts of addition and multiplication. Many of my students, when they first begin numbering operations to show their conventional order, write the ordering numbers under the numbers instead of under the operations until they are corrected.

Learning to use a calculator as I teach it encourages students to stop focusing on the numbers and start focusing on the operations instead. When he first heard a summary of what I teach in my calculator unit and the language in which I teach it, Pimm said tellingly, "The numbers have already dropped out."

Indeed, a number of students do "drop them out" in a very real sense, as the following anecdotes demonstrate.
Kathy (Ma 8A - C) had been asked on a quiz to solve the following equation for $x$:

$$\frac{5x}{3} = 5$$

*Kathy: How come there are two fives in this equation?*

*I: Why shouldn't there be?*

*Kathy: Then how do you know ... Oh, I guess the operations tell you.*

Apparently, she answered her own question by her realization that the two fives were in fact distinguishable from each other. Interestingly, she did not suggest that it was because of their positions in the equation (e.g., left and right of the equals sign, one higher and the other lower on the page, or part of the "question" as opposed to part of the "answer"), but rather because of "the operations" (among which, I suspect, she was including the $\neq$).

For Steve (Ma 11A - B), the numbers had "dropped out" so completely that he absolutely refused to write them down. Given the expression

$$\frac{13.6 + 4.5(1.42 - 0.4199)^4}{\left[\frac{11.2}{1.45} - \frac{13.24}{2.5}\right]^2}$$

on a quiz, he was supposed to tell me what buttons he would punch on his (algebraic) calculator to get the answer. I expected him to write

$$(13.6 + 4.5 \times (1.42 - 0.4199))^4 \div \left(\frac{11.2}{1.45} - \frac{13.24}{2.5}\right)^2 =$$
but instead he wrote

$$\left( + x \left( - y^x \right) \div \left( \div - \div \right) x^2 \right) =$$

I: These aren't the buttons you would punch on your calculator.

Steve: Yes, they are.

I: What about the numbers?

Steve: Well, of course I would punch the numbers - but you know that. You just want to know if I know what order to punch the operations in.

I: What if the same operation comes twice in the same question?

Steve: [Sarcastically] I'll colour them different.

He did.

In rather milder style, Gail (Ma 8A - A), showed that she was ignoring the numbers thus:

Gail: What would I do if ... say you had a plus here, and a minus here, and then maybe a ÷ there ...

She wrote, as she spoke

$$+$$
$$-$$
$$\div$$

... would I need brackets here [pointing to the space between the - and the ÷]?

For Conrad (Ma 11A - C-), who had spent the previous day programming calculators, it was evidently the calculator's way of carrying out programmed operations that impressed him: when Marc, who had missed the previous day's class, said, "What's so great
about programmable calculators?" Conrad replied, "They remember operations." I found it interesting that he did not also include the letters, which he could not type on to his own (non-algebraic) calculator but which he had typed on to the programmable calculator together with the operations the day before. Perhaps he was used to his own calculator remembering numbers and did not think remembering letters to be sufficiently more impressive to make it worth mentioning.

The ability of Bob (Ma 10A - A) to see that not only numbers, but also letters, could be "dropped out" in Pimm's sense led to my own undoing one day. The basic equations students were required to solve for $x$ in Math 10A contained three of the four operations $+, -, \times$, and $\div$, e.g.

$$a = b(x + c) - d$$

or

$$\frac{45x - 54.6}{12.6} = 10$$

In order to give students sufficient practice in solving equations containing letters only, I had made up three worksheets each including 48 basic questions. Since there are only 64 combinations of the three operations, the three worksheets contained repetitions, which I had disguised by changing the letters randomly and, more often than not, by rearranging the questions. (This tactic meant that I did not have to make up a new key for each worksheet.) On one occasion, however, I had not had time to rearrange the questions, as it would have meant
reformatting the worksheet.

Bob: Worksheet 15C is just the same as 15A!

I: No, it isn't.

Bob: Yes it is - look [aligning the two worksheets side by side and pointing from one to the other].

I: Those aren't the same - they're different letters.

Bob: [In the tone of one rallying a child who is trying to pull a fast one] Miss Murphy, they're the same!

I: How can they be the same when they've got different letters?

Bob: Yeah, I know, but they've got the same operations.

I: So are you telling me that ...

I wrote

\[ x + a - b = c \]

... is the same as

I wrote

\[ x - a + b = c \]

... just because they've got the same operations?

Bob: [After a long pause, during which he looked from one to the other of the two equations I had written] No ... because they've changed places. But [returning to the two worksheets] here you didn't change them. You've got the same operations in the same order on both worksheets.

I: [Trying to sound incredulous] And you're telling me that if I have the same operations in the same order it's the same question, even if it's different letters?
Using Calculators to Teach Algebra

Bob: Of course it is! The letters don't matter.

I: [Breaking down and smiling] You're right, but don't tell the others.

By using a calculator, students hand over to the calculator responsibility for anything that depends on the identity of the numbers. It remains their responsibility to give the calculator the correct numbers, but that is easy: every high school student can read numbers and copy them on to a calculator. However, it also remains their responsibility to give the calculator the correct operations in the correct order, and that is new to them; hence their shift of focus from numbers to operations.

Transition from Numbers to Letters

The transition from arithmetic to algebra involves most noticeably a shift from numbers to letters, a shift the students are ready to make, even when no numbers are given for their values, once the numbers have truly "dropped out," as shown in the following anecdotes.

For Mathematics 8A students' first venture into algebra, I had given them a worksheet on which each problem consisted of an expression made up of letters only, e.g.

\[ d(a + bc) \]

and asked them to (a) place numbers under the operations to show the order in which they must be done and (b) tell what buttons they would punch on their calculator to get the answer. These were exactly the same instructions I had given them on the
previous worksheet, except that now there were letters instead of numbers and I was not asking them to calculate the "answers."

Choon (Ma 8A - A): How do we do this?

I: The same as the last worksheet, except that now you can't calculate the answer.

Choon: But how do we do it with letters?

I: Each letter stands for a number. Just do it the same as you would if they were numbers.

Choon: But we can't put letters on a calculator.

Borzoo (Ma 8A - A): Just pretend your calculator has letters on it instead of numbers.

Choon: Oh - OK.

This is not to say that some students, like Mark, did not have reservations about this new kind of mathematics.

Mark (Ma 8A - C-): [Quietly and privately] I don't want to be rude, but isn't this rather dumb?

I: Why?

Mark: I mean, this isn't math.

I: Why not?

Mark: Math is numbers. You can't get the answer with letters.

I: You can if I tell you what numbers the letters stand for.

Mark: Yeah, but you haven't told us.

I: But I could, so this is still math. The more you do math, the more you'll use letters.

Mark: You're kidding.
I: No. Look, here's my Physics 12 book [getting it out of my bag and opening it at random]. See, this page has no numbers on it at all, just letters.

Mark: But how can the kids get the answer?

I: This [I pointed to F = ma] tells the kids that if they want to calculate F, they have to find out what number m is, find out what number a is, and then multiply them together and the answer is what F is.

Mark: [In the tone of "If you say so"] OK.

Students also, I found, had different ideas from me about how to represent variables which had real meanings. For example, when I introduced problems containing regular price and sale price, I explained that although we would like to use "P" for price to remind us of what it meant, we had to have different variables for regular price and sale price.

I: So what we do is use different subscripts. We'll use $P_r$ for the regular price and $P_s$ for the sale price.

Kathy (Ma 8A - C): Why don't you use $R_p$ and $S_p$?

I: We could ... I guess ... but I think that makes it look as though they're totally different variables; don't you?

Kathy: No. I think it looks more like what it means.

I: That's not what mathematicians usually do. For example, in the physics I was just teaching this morning, we had two masses, so we called them [I wrote] $m_1$ and $m_2$, not [I wrote] $1_m$ and $2_m$. Besides - I'm sorry - I've already made up the worksheets.
Kathy: (Resignedly) Whatever.

On the other hand, students soon adopted mathematicians' informal conventions for variables. I had written three worksheets on each new topic in solving equations, and although I had used almost all the letters of the English alphabet, I had developed the habit of asking students to solve for $x$ on Worksheet A, $y$ on Worksheet B, and $z$ on Worksheet C. However, in order to make a particular point, I wrote a worksheet where the first few questions looked like this:

1. $a + b + c + d = e$  
2. $a + b + c + d = e$  
3. $a + b + c + d = e$  
4. $a + b + c + d = e$

with the students asked to solve for the variable at the right.

Yvonne (Ma 8A - C+): [Pointing to the column of variables at the right] What's this?

I: Those are the letters you have to solve for.

Yvonne: [With exaggerated surprise] Solve for $a$ and $b$?

I: Yes; why not?

Yvonne: I guess so ... But we always solve for $x$ or $y$ or $z$.

I: Well, it's good to change.

Yvonne: I guess so ... but it's hard to remember what you're solving for when you keep changing it.

These initial difficulties with letters over, however, students soon began asking whenever they were presented with a new worksheet, "Is it letters or numbers?" In no time at all, they were groaning when it was numbers and cheering when it was
Portia (Ma 8A - C+): I like letters better than numbers.

I: Why?

Portia: 'Cause you can see what you're doing better. Plus you don't have to work out the answer.

The students' desire to cut down on the amount of work, particularly writing, they had to do led to some surprising connections between variables and calculators and some insights into the nature of variables that I found very valuable for my own thinking, as revealed in the next three anecdotes.

I had always insisted that when the students had solved for the value of x in a numerical problem, they check that the value they had obtained actually satisfied the original question and that they show me that they had done so.

One student, asked to solve

\[2 = (3 + z)^7\]

for z, wrote

\[
\begin{align*}
z & \quad \begin{array}{c} 2 \\ + 3 \\ \times 7 \\ \times 2 \\
\end{array} \\
\frac{2}{7} & - 3 = z = -2.714285714 \\
\end{align*}
\]

Check: \(2 = (3 + z)^7\)

The same student, who had a Texas Instrument calculator, when asked to solve

\[(5 + z)(3)(7) = 5\]

for z, wrote
Using Calculators to Teach Algebra

\[
\begin{align*}
&z + 5 \div 7 = \frac{5}{7} - 5 = z = -4.761904762 \text{ (STO)} \\
&x 3 \div 3 = \frac{7}{3} - 5 = z = -4.761904762 \text{ (STO)} \\
&x 7 \div 7 = z \\
&5 z
\end{align*}
\]

('STO" is the button that stores a number in the memory and "RCL" the button that recalls it.) Notice how "RCL" has replaced "z" in the final equation.

Larry (Ma 9A - A), asked to solve the following equation for \(x\),

\[7(x + 2) = 15.5\]

had written the following:

\[
\begin{align*}
x + 2 \div 7 & = \frac{15.5}{7} = x = 0.214285714 \\
x 7 & = 15.5 \\
7 & = x
\end{align*}
\]

He knew that he was now expected to write

\[(\text{Check}) \quad 7(0.214285714 + 2) = 15.5\]

and to check on his calculator that this was in fact true.

Larry: I hate having to write these big numbers [pointing to the 0.21485714] over and over again.

I: Well, if that's what \(x\) works out to, I don't see what else you can do.

Larry: You can't? And you're a math teacher?

I: Larry!

Larry: No, but can't you think of something so we don't have to write it all down? It takes ages, and besides, sometimes I don't copy it right or punch it in right and then I get the wrong answer when I've really got the right answer.
I: What about putting it in your memory?

Larry: Can I?

I: Yes - just push $x \rightarrow M$ when you've got $x$ and then $RM$ when you want $x$ back again. [His calculator was a Sharp.]

Larry: Yeah, but I still have to write it down.

I: Well, you'll have to write it down once or I won't know whether you've got the right answer for $x$, but maybe you could tell me somehow that you've put it in the memory and pulled it out again and then you won't have to write it out in the check.

Larry: OK.

By himself, Larry gradually started writing

\[
\begin{align*}
  x &= 15.5 \\
  + 2 & \quad \div 7 \\
  \times 7 & \quad - 2 \\
  15.5 & \quad x \\
\end{align*}
\]

(Check) \[7(RM + 2) = 15.5\]

Larry: Doing this [he pointed to the $x \rightarrow M$ and the $RM$] is just like letters again.

I: What do you mean?

Larry: It just like changing from $x$ to $M$.

I: I see what you mean.

Larry: So what's the point?

I: Well, it does tell me that you've put the number into your memory and recalled it again.

Larry: [In a "have it your own way" tone of voice] OK - I guess you like $M$ better than $x$ 'cause of your name.

I began to realize for the first time what a very good image of a variable a calculator's memory is: something that can "hold"
a variety of numbers. This image is even better if the calculator is programmable, for then the memories are designated by the letters A-Z, as I realized after the following exchange.

Borzoo (Ma 8A - A), had just suggested to another student, "Pretend your calculator has letters on it instead of numbers." Then he turned to me.

Borzoo: Are there any calculators that do letters?
I: Yes.

Borzoo: Can we buy some?
I: We could, but they're expensive and they're not going to help you that much.

Borzoo: But then we could just punch in the letters.
I: Yes, but you'd still have to tell the calculator what numbers went with the letters before it could calculate the answer.

Borzoo: So? That would be fun!
I: So a programmable calculator won't save you any time - unless you're doing the same question again and again with different numbers. Your worksheet isn't like that - each question is different.

Borzoo: Why don't you make a worksheet for us like that?
I thought of the interest problems I was intending to give them in Unit 3.

I: Later on, I will.

Borzoo: So can we buy calculators with letters now?
I went shopping that night to investigate programmable
calculators and found that a discontinued line - the Casio fx-4500P - was being sold cheaply. Unlike any other programmable calculator I have seen (including, unfortunately, the one that has replaced it), it did not demand that the program contain steps to identify each variable and ask for its value. The program consisted simply of a formula typed in thus

$$I = \frac{PRT}{U/100}$$

When the program was run, the calculator would display \(P?\) as well as the value currently in memory \(P\). The students could press ENTER to accept this value or change it to some other value first. Three students - Borzoo, Bob, and Vince - bought these calculators from me and began to make even deeper connections between calculators and variables, which are described in the next subsection.

**Nature of Variables**

After their initial surprise, the students accepted letters as part of mathematics. They even began to prefer them because they found them easier to write and to copy correctly and because they "didn't have to work out the answer" when the question contained only letters. First "the numbers had dropped out"; now I began to find the identity of the letters "dropping out" as well, as students began to conceptualize variables as having an existence defined by their relation to the rest of an expression, independent of the identity of the letter.

For example, in Mathematics 8A, which I was teaching for the
first time, I was writing three worksheets on each new topic in solving equations, and there was necessarily some repetition. To hide the repetition (for I knew that students would object to "doing the same questions again"), I wrote a program for my computer which would change the letters to other letters according to a pattern which looked random. This entailed first changing the letters to symbols such as ©, ©, ©, ©, ©, ©, etc. and then changing the symbols back to different letters. On one occasion, however, my program failed, and I copied a quiz for the students without realizing that it had hearts (❤) on it. When the students pointed it out to me, I told them to change all hearts to x's.

Mark (Ma 8A - C-): [Cheekily] Why should we?

I: I meant to put x's.

Mark: Why can't we leave them as hearts?

I: You can, I guess.

Mark: I mean, it could be anything.

I: That's true.

Mark: [Getting to work] I like hearts!

At the same time, as mentioned in the last subsection, the three students who had bought programmable calculators began to see connections between letters and calculators which I had never thought of.

One day Vince (Ma 8A - A) and Borzoo (Ma 8A - A) were talking to each other about their programmable calculators, which, when running a program, would display the value currently in each memory (lettered A-Z) and ask for a possible new value at the same
time.

Vince: What's A right now for you?

Borzoo: 10.5.

Vince: Mine's 16.

I: That's not really what the variable A is - it's just what
the calculator has got stored in memory A.

Vince: Same thing. It says A = 16 [showing me].

In my mind, A was the name of a memory which could hold a
value which could be changed at will; in the students' minds, A
was currently 16; I could see why they thought my interjection a
mere quibble.

Bob apparently kept this image of a variable with him right
up to Grade 11. When my Mathematics 11A students were programming
into the school's TI-82's the long formulae for saving and
financing (see Appendix 1), the last two lines of the instructions
I had given them read as follows:

<table>
<thead>
<tr>
<th>Buttons to punch</th>
<th>Calculator shows</th>
</tr>
</thead>
<tbody>
<tr>
<td>PRGM ▶ 3 2nd ALPHA &quot; A &quot; ENTER</td>
<td>:Disp &quot;A&quot;</td>
</tr>
<tr>
<td>PRGM ▶ 3 ALPHA A ENTER</td>
<td>:Disp A</td>
</tr>
</tbody>
</table>

The first line prompts the calculator to display the letter A,
while the second line prompts it to display the number currently
held in memory A.

In programming his calculator, Jack (Ma 11A - C-) had omitted
the quotation marks. Bob, who was used to his own programmable
calculator, was explaining to him why his program wasn't working and why the quotation marks were essential.

Bob: When you put the quotation marks in, it means the letter. When you leave them out, it means the number that letter is.

I: Actually, Bob, it means the number that is in the memory that has that letter.

Bob: Yeah, but the memory's just the same as the letter.

Jack: But you just said, not if it doesn't have quotation marks.

Bob: [turning back to Jack] Don't listen to Miss Murphy ... when you put the quotation marks in, it means the letter ... like this [writing the letter A] ... this letter, right here, right? When you leave them out, it means the number for that letter, OK?

Jack: OK. With quotation marks, it's a letter; without quotation marks, it's a number.

The anecdotes in the last two subsections show that students can not only use calculator language to help each other over the initial difficulties of substituting letters for numbers, despite the fact that most calculators can be given numbers but not letters, but also to build for themselves an image of a variable as high as (e) on the list of Kücheman and Collis (1975).

Structure of Expressions

By requiring my students always to "put in the order of operations," I encourage them to think of algebraic expressions as
instructions which they should follow. As a result, many of them remain mathematically at the level of construction workers who plod through their jobs step by step, with no thought of the finished building, asking themselves at each stage, "What should I do next?" or, as my students would put it, "May I do this before this?" However, a significant number of students begin to behave more like supervisors who, even though they are following instructions, keep the finished building in mind.

One of the latter was Matt (Ma 11A - A), who had been asked to put in numbers to show the order of operations in the following two expressions:

\[
\begin{array}{c}
5 + 6 \\
7 \\
9 \\
3 + 2
\end{array}
\quad \begin{array}{c}
5 + 6 \\
7 \\
9 \\
3 + 2
\end{array}
\]

Matt: These two questions are the same.

I: No, the lengths of the divisions lines are different.

Matt: So how do you do them?

I: Well, first of all, put in the brackets that come with the long division lines, starting with the longest.

Matt: [Doing it] Oh, OK; I never do that because I can just see where the numbers go.

I: In this kind of problem it helps.

After Matt had put in brackets correctly thus:
I put my pencil point on the + in the bottom line and started asking, "May I do this before this?"

Matt: [Interrupting] I don't need to do that. I can see it now. I just needed the brackets.

I: OK.

To continue my construction analogy - Matt behaved like workers who, a former electrician assures me, never read the instructions at the beginning of a job, but only when they reach a point where they cannot "see" what is to be done next.

Another student who clearly began to see the larger picture was Sheila (Ma 11A - B), who was solving the following equation for $a$:

$$\frac{ab + cd}{ef + gh} = j$$

She ended up correctly with

$$\frac{j(ef + gh) - cd}{b} = a$$

Sheila: How come [pointing to the original equation] this has ... you know ... a nice ... kind of ... you know, shape [cupping her hands around the left side of the equation] and when I make $a$ the subject it doesn't?

Apparently, Sheila had noticed the symmetry on the left side of the original equation and the lack of symmetry in the final
equation - not necessarily the symmetry induced by the
commutativity of addition and multiplication, but probably the
physical symmetry of the written expression.

However, Iago (Ma 11A - B) appeared to appreciate a deeper
symmetry, as he showed when he had to solve the following equation
for x:

\[ \frac{mn + np}{x - a} = z \]

Iago: I can't solve for x right away, right?
I: Right.
Iago: So I have to solve for something else first.
I: Right.
Iago: But what?
I: The nearest thing to x that isn't under the divide line.
Iago: Does it matter which?
I: Yes, you have to pick the nearest. Which is the nearest?
Iago: That one [pointing to p] ... I guess.
I: Wouldn't you say that one [pointing to m] is really nearer
to x?
Iago: Not really.
I: Isn't it?

I measured the physical distances from m to x and from p to x
between my finger and thumb.

Iago: [Rallyingly] Come on, Miss Murphy. That's not what you
meant, is it?

I: [Smiling] No. I just wanted to see what you thought.
Iago: But really, which is the nearest?

I: Why did you say p was nearest?

Iago: 'Cause when you read the question, like [pointing as he read] m n plus n p divided by x minus a, p comes right before x.

I: So why did you ask me?

Iago: I wondered if it really made any difference.

I: Actually, in this case, you could start with m, n, or p and it wouldn't make any difference. Do you know how you can tell?

Iago: How?

I: Suppose I wrote the question like this.

I wrote

$$\frac{mn + pn}{x - a} = z$$

I: Would that mean the same thing, or is that different?

Iago: It's different.

I: How?

Iago: You turned these two around [pointing to the p and the n].

I: If I gave you numbers for p and n, would it make any difference to the answer?

Iago: Oh, I see ... no.

I: Why not?

Iago: 'Cause 2 times 3 is the same as 3 times 2.

I: Right. So p and n are interchangeable in this question.

Iago: So it doesn't matter whether I start with p or n?
I: Right. Could you start with m?

Iago: [After a pause] Yes. I guess so.

I: How do you know?

Iago: 'Cause I could write it ...

He wrote

\[
\frac{pn + mn}{x - a} = z
\]

... and that would work out the same when you put numbers in.

I: So if you can change around two letters without making any difference to the answer, then it doesn't matter which of those two letters you start with.

Iago: OK.

Dave (Ma 11A - B) made observations similar to Iago's, but in a slightly different context. He had just solved the equation

\[
\frac{a + b}{c + d} - \frac{m - n}{j - k} = p
\]

for a and b, obtaining

\[
\left[ p + \left[ \frac{m - n}{j - k} \right] \right] (c + d) - b = a
\]

and

\[
\left[ p + \left[ \frac{m - n}{j - k} \right] \right] (c + d) - a = b
\]

Dave: Look, these are both the same except the a and the b changed places [pointing alternately to the b and the a in his second answer].
I: Yes; it's neat, isn't it?

Dave: Does that always happen?

I: No.

Dave: How do I know when it's going to happen?

I: Why do you want to know?

Dave: 'Cause then I can just write down the answer.

I: OK. Look at the original equation. Would it make any difference if I changed the a and the b around?

Dave: What do you mean? ... Oh, no, I guess not.

I: If the a and the b are interchangeable in the original equation, then they're interchangeable in the answers, too.

Dave: Oh ... OK!

Kyle (Ma 8A - A) noticed not only this kind of symmetry, but also the asymmetry introduced by division. The students had been using the following equation to solve simple interest problems for I, P, r, or t when they were given the values of the other variables.

$$I = \frac{Prt}{u}$$

(The variable "u" takes the value 1, 12, 52, or 365, depending on whether the time is given in years, months, weeks, or days.)

I noticed that Kyle was sitting apparently doing nothing.

I: Kyle, have you finished your work?

Kyle: I don't have to do any more. They're all the same.

I: No, they're not. What do you mean?

Kyle: Well, look.
He showed me his paper, on which he had written, in the course of solving for \( P, r, \) and \( t \):

\[
\frac{1100u}{t} = P
\]

\[
\frac{1100u}{r} = P
\]

\[
\frac{1100u}{r} = t
\]

Kyle: See? They're all the same, so I'm not going to do any more. But how come you never ask us to find \( u \)?

I: Well, it wouldn't be very realistic, would it? I mean, if I tell you \( t \), of course you know what \( u \) is. And if I ask you to find \( t \), of course I have to tell you what units I want the time in. So I never really have to ask you for \( u \).

Kyle: I made up a problem like that ...

I: Did you?

Kyle: ... but I couldn't solve for \( u \).

I: Why not?

Kyle: 'Cause when you start with \( u \) you have to divide backwards.

I: Good for you! I wasn't going to teach you how to do that kind until Grade 10.

Kyle: [obviously pleased with his own acumen] Hey, wow, man! So how do you do it?

I: You use one of the other equations you've already worked
out. [Looking over his work] See, in these equations [I pointed to the ones he had derived] u doesn't come after a divide; it comes after a multiply, and you can multiply backwards. So you could solve these for u. But I'm not going to give you any questions like that.

Kyle: OK ... [to his neighbour] Hey, did you hear? I figured out a Grade 10 problem!

Bob (Ma 10A - A) developed an ability to see the structure of an expression that was unusual in my students. It began when he complained that he was bored and I challenged him to solve the equations on that day's worksheets as I myself did, without writing down either of the two columns.

Accordingly, asked to solve the following equation for b

\[ P = 1 + a \left[ 1 + a \left[ 1 + a(b + 1)^m - c \right] \right] \]

he wrote in the order of operations thus

\[ P = 1 + a \left[ 1 + a \left[ 1 + a(b + 1)^m - c \right] \right] \]

and then wrote

\[
\left[ \begin{array}{c}
\frac{P - 1}{a} \\
-1
\end{array} \right]^{1/p} + d - 1 + c - 1
\]

\[
\left[ \begin{array}{c}
a \\
-1
\end{array} \right]^{1/n} + c - 1
\]

\[
\left[ \begin{array}{c}
a \\
-1
\end{array} \right]^{1/m} + c - 1
\]

saying to himself as he wrote, "End up with P, so start with P,
Using Calculators to Teach Algebra

minus 1, divided by a, brackets, y to the x 1 over p, plus d, minus 1, divided by a . . ." Here he started to write the symbol "/", but scratched it out, saying "No, there's a minus there . . . and a plus, so I need a long one . . . brackets, y to the x one over n, plus c, minus 1, long divided by a, brackets . . . you know, you can practically guarantee that if there's a y to the x coming up, you'll need brackets . . . y to the x 1 over m, minus 1, equals b."

I: Practically guarantee?
Bob: Seems like it.
I: Why?
Bob: 'Cause the calculator won't do anything before y to the x unless you put brackets.
I: "Anything" or "practically anything"?
Bob: Anything, I guess.
I: So are you always going to need brackets?
Bob: Yes . . . No, not if it's a single letter before the y to the x.
I: Good! That's worth remembering.

(Notice again how Bob expressed his insight into the need for brackets in calculator language.)

The equations I challenged Bob to solve in this way had only a single letter as the object of each operation. Unfortunately for his marks, he then attempted to solve every equation, even those including groups and multi-step solutions, in the same way, and was not successful. As a result, his final mark in both Grade 10 and Grade 11 was B instead of A. I attempted on a number of
occasions to persuade him to write down the columns at least for quizzes and exams, pointing out (truthfully) that even I, a teacher, could not easily solve a multi-step problem in my head and would at least write down the intermediate equations, but he always answered that he "should be able to do it" and continued to try.

Audrey (Ma 8A - A) saw perhaps more deeply into the structure of equations, as opposed to expressions, than any other Grade 8 student. For fun, I made up the following equation for her class one day when some of them said they had finished and were bored

\[
\frac{[\frac{x}{y}]}{\frac{1}{h}} - p = M
\]

and told them to solve it for \( x \). They soon found

\[(Mur + p)^h y = x\]

and appreciated the joke. Without any more help from me, some of them then started to try doing the same thing with their own names, and several succeeded, apparently by trial and error. The next day Audrey arrived in class with a similar problem already made up and asked if she could put it on the overhead projector for everyone to solve. I agreed, and she then tried to explain to the class how she had constructed the equation.

Audrey: You have to make up the answer first.

I: Then what do you do?

Audrey: Then you reverse it.

I: And what does that give you?
Audrey: That gives you the question.

I: When you've made up the answer, what do you solve it for?

Audrey: [After a pause] The first letter in your name.

Kakal (Ma 8A - A): I've got two k's in my name ... Oh, I guess I should make the first one a capital.

I: Why?

Kakal: 'Cause if I don't you won't know which k I mean.

I: And when you've got the question, you give it to somebody else. What do you tell them to solve for?

Audrey: x.

Apparently - to continue my construction metaphor - if you build and take down enough houses step by step, seeing only one step at a time, you begin to see more than one step at a time and finally the whole structure. The same thing happens - as I can attest - in following an unfamiliar knitting or bead-weaving pattern. Eventually the unit of the pattern is reified, to use Sfard's word, and it is ready to become a unit in a larger pattern.

Groups: Static Algebraic Expressions

The closest students come in my courses to reifying expressions is in their treatment of what I call groups (see Appendix 4), in which, at the very least, students have to consider \( a + b \) as "the answer to \( a + b \)."

Even before I speak of groups, however, I notice a few students reifying numerical expressions for the purposes of
calculators. For example, Kyle (Ma 8A - A) was calculating the answer to the following:

\[
3 - \left[ \frac{3}{6.9 - 4} \right]^5
\]

On his (non-algebraic) calculator, he had already punched out

\[
6.9 - 4 \text{ M+ 3 } \div \text{ MR MC } \times = = = \text{ M+ 3 } - \text{ MR MC M+ 3 } \div
\]

when he stopped.

I: Go on.

Kyle: [After a pause] Oh ... of course, that's in the memory.

I: What is?

Kyle: That [pointing with his pencil approximately towards the lower part of the question].

I: What exactly?

Kyle: [In an exasperated tone] That!

He very deliberately drew a heavy black line around the expression thus

\[
3 - \left[ \frac{3}{6.9 - 4} \right]^5
\]

Kyle: [sarcastically] OK? Got it now?

If I had been describing the same "thing" to him, I would have said, "The answer to that," as I drew the line. By leaving out the italicized words, Kyle had clearly made a "thing" of the entire expression. Perhaps he could ignore its internal structure because he had already performed those operations, turning the
boxed expression into a single number - clearly a "thing" - and storing it in his calculator's memory.

Identifying "what we have to keep together as a group" proved to be difficult for many students, who at first did not realize that what constituted a group in any given expression depended on what variable they were solving for. Hoping to help them see this point, I made up a worksheet on which they had to solve the same equation for a number of different variables. For example, they had to solve the following equation first for a, then for b, then for c and then for d:

\[ a(b + c) + d = e \]

Kelly (Ma 10A - C+): [in an exasperated tone] How do you know when it's a group and when you can break it up?

I: Which question are you on?

Kelly: This one [pointing to the one which asked her to solve the equation for a]. I said this is a group [sketching a circle around \( b + c \)] but Christine said it's this [sketching a circle around \( a(b + c) \)]. Which is it?

I: It depends on what letter you're solving for.

Kelly: [With surprise] It does?

I: Yes. What's Christine solving for?

Christine (Ma 10A - C+): d.

I: Well, Christine's right for her question and you're right for your question.

Kelly: How come?

I: Put in the order of operations.
Kelly wrote

\[ a(b + c) + d = e \]

\[ 2 \quad 1 \quad 3 \]

I: What are you starting with?

Kelly: \( a \)

I: OK; start with \( a \).

She wrote

\[ a \]

I: Now what?

Kelly: Times, right?

I: Right.

She wrote

\[ a \times \]

I: Times what?

Kelly: I don't know.

I: That times was operation 2, right?

Kelly: Right.

I: Then any operation which has a lower number than 2 is already supposed to be done by the time you get to it, and that means it has to be kept in a group.

Kelly: So \( b + c \) is a group.

I: Yes.

Kelly: Not \( b + c + d \).

I: No, because the plus \( d \) is operation number three and three is supposed to be done after two, so you write plus \( d \) on the next line.
Kelly finished the problem correctly.

Kelly: Don't go away. Let me do the one Christine was doing.

She wrote

\[ d + \]

Kelly: This [pointing to the + she had just written and looking back at the original equation] ... is number 3. So

[sketching a circle around a(b + c) in the original equation] all this has to be kept as a group.

I: Right.

Kelly: Wow!

She finished that problem correctly as well.

Kelly: So if I start with a letter that's inside a group, the group gets broken up.

I: Yes.

Kelly: But if I start outside a group, it doesn't get broken up.

I: Yes. What you call a group depends on which letter you're starting with.

Kelly: Now I got it!

In this exchange, Kelly learned to identify as a group any expression inside which the operations had lower numbers than the operations outside. She showed no signs of reifying groups, but rather identified them "operationally."

In contrast, Larry (Ma 10A - C+), who always wrote very slowly and neatly, showed signs of reifying algebraic expressions
in order to cut down on the amount of writing he had to do. He was solving the following equation for $v_1$:

$$\frac{a_1 v_1 + a_2 v_2}{a_2 v_2 - a_3 v_3} = M$$

Larry: Can't I just put a square instead of writing a group out all the time?

I: Show me what you mean.

He had written

$$\begin{align*}
& v_1 \\
& \times a_1 \\
& + a_2 v_2 \\
& \div M
\end{align*}$$

and now he pointed to the square.

Larry: 'Cause the square is just like brackets anyway.

I: Yes, that's OK, provided you don't forget what's inside the square. And of course you have to write it out in your final answer so I can tell.

Larry: Can I put a box instead of this group, too (pointing to $a_2 v_2$)?

I: You can, but won't you get mixed up about which box is which?

Larry: I'll use a triangle for the other one.

He eventually developed a system of drawing circles, rectangles and triangles around groups in the original equation that he could see were going to remain unchanged. In the two columns he would draw only the shapes; in the final equation he would write out what the shapes contained or represented.
Unfortunately, in multi-step problems, he often attempted to avoid identifying the contents of the shape in the intermediate equations, and thus became confused in the later stages of the solution.

Larry's *reification* of the expressions inside his shapes fits the description given by Sfard (1991) exactly: "Seeing a mathematical entity as an object means ... being able to recognize the idea 'at a glance' and to manipulate it as a whole, without going into details...."

In contrast, Kelly viewed a *group* as a "potential rather than actual entity, which comes into existence upon request in a sequence of actions" (Sfard, 1991, p. 4).

The following two samples of students' work, which give clear evidence of students' *reification* of *groups*, are typical of a very large number of others. On the quiz from which they were derived, for example, seven other students out of a class of 33 also left out brackets in intermediate steps.

Asked to solve the following equation for $x$

$$b\left[\frac{b}{x - p + n} - c\right] = a$$

Alison (Ma 10A - B) should have written for the first step

$$\begin{array}{c}
b \\
\div (x - p + n) \\
\div b \\
\frac{a}{b} \\
+ c \\
\frac{a + c}{b} \\
\times (x - p + n) \\
\times b \\
\frac{a + c}{b} \\
\times (x - p + n) = b \\
\end{array}$$

but she left out one set of brackets thus:
Using Calculators to Teach Algebra

\[ \frac{b}{a} \div (x - p + n) \div b \]
\[ - c + c \]
\[ \times b \times (x - p + n) \]
\[ \frac{a}{b} + c(x - p + n) = b \]

However, at the beginning of the next step she wrote

\[ \frac{x}{b} \]
\[ - p \]
\[ + n \]
\[ \times \left[ \frac{a}{b} + c \right] \]
\[ \frac{1}{b} \]

just as if she had not omitted the brackets. She ended up with the correct expression for x, so I gave her full marks. However, as I handed back her quiz, I laid it down on the desk in front of her, pointed immediately to the result of her first step, and asked her to "put in the order of operations." She wrote

\[ \frac{4a}{b} + c(x - p + n) = b \]

as I had expected. When I pointed out the discrepancy between this order and what she had written in the next step, she at first appeared confused and then said, dismissing the problem, "Well, I guess I knew what I was doing yesterday!"

Similarly, Rosa (Ma 11A - A), asked to solve

\[ \left[ \frac{e^2}{x} \right] - \frac{n}{y} \]
\[ \frac{t/S}{z} = b \]

for y, wrote the following (notice especially how she alternates between leaving out brackets and putting them in, even around the same expression, but notice also her use of long horizontal lines
Using Calculators to Teach Algebra

157

to separate "what she does" from "what she gets"):

\[
\begin{align*}
\frac{e/x^2}{n/y} & - \frac{t/S}{b} = b(t/S) + \frac{n/y}{e/x^2} \\
\frac{\frac{e/x^2}{n/y} + \frac{t/S}{b}}{e/x^2} & = \frac{[e/x^2 - b(t/S)]y = n}{e/x^2 - b(t/S)} \\
\frac{y}{n} & = \frac{n}{\frac{e/x^2 - b(t/S)}{y}} \\
\end{align*}
\]

It seems that students mentally reify groups as they are writing them, even if they do not write in the brackets that are necessary to identify the expression as a group in the next step. Provided they read what they have written immediately afterward, the group is still reified in their minds and they read what they meant, not the conventional interpretation of what they wrote.

Similarly, one of my friends, who is a newspaper editor, told me that it is common, when he complains that a sentence is not clear, for the writer of the sentence to read it aloud with what he calls "oral punctuation" which makes the meaning clear, like the student who is said to have written "John where Jane had had had had had had had had had had had had had had had had the approval of the examiner" and to have read it thus: "John, where Jane had had 'had,' had had 'had had'; 'had had' had had the approval of the examiner." Like Alison and Rosa, the writer knows what he or she meant when the
sentence was written and continues to read it that way for some time.

In practice, students will explicitly use "oral punctuation" to reify groups, as Nikki (Ma 11A - C+) did when I asked her, from the front of the classroom, to tell me, from her desk, how to solve the following equation for $p$.

$$\frac{p}{b + c} = N$$

*Nikki: Start with $p$ ...*

*I wrote*

$$p$$

*Nikki: Multiply by $b$ plus $c$ ...*

*I did not know whether she wanted to me to write*

$$p \times b + c$$

or

$$p \times (b + c)$$

*I: Tell me exactly what to write.*

*Nikki: I told you.*

*I: I'm not sure whether you mean times $b$ plus $c$, on two lines, or times $b$ plus $c$ in brackets on one line.*

*Nikki: I told you. I said times [short pause] $b$ plus $c$ [saying "b-plus-c" quickly as though it were one word].*

One of the most significant reifications by student occurred in a new context when Matt (Ma 11A - A) had to solve the following equation for $x$: 
Using Calculators to Teach Algebra

\[ \frac{a + b}{x - m} + \frac{c - d}{y - b} = z \]

Matt: Can I start with this whole group [sketching a circle around "a + b"]?

I: Yes.

Matt: OK.

I had not suggested this technique to the class, having identified groups operationally, as having lower operation numbers than the operation just performed. Matt finished the problem on his own.

Matt: But I got a different answer. [Looking at the answer sheet] No I didn't ... yes I did.

I: Let me see.

My answer was

\[ \frac{b + a}{z - \frac{c - d}{y - b}} + m = x \]

while Matt's answer was

\[ \frac{a + b}{z - \frac{c - d}{y - b}} + m = x \]

Matt: [Before I could start to explain] Oh, I guess it doesn't make any difference.

I: Why not?

Matt: Because a plus b is the same as b plus a.

I: Right.

Matt: [Curiously] But why did you get b plus a?

I: I was following my own rules very carefully. I suppose
most students would call $b$ the closest thing to $x$ that you can start with, and so I started with $b$. But you started with a plus $b$, all together, so your answer looks a bit different.

Matt wrote

$$b + a$$

Matt: So your way, $b$ and $a$ got separated.

I: Right.

Matt: But how did they get back together again?

I: In the next step. See, my answer to the first step would be ...

I wrote

$$\left[ z - \frac{c-d}{y-b} \right] (x - m) - a = b$$

I: Then I would go ...

I wrote

$$\frac{x}{ } - \frac{m}{ } + a$$

See?

Matt: [Slowly] I see.

I left him examining the question.

Matt: [After a while] I think my way's better.

I: Why?

Matt: 'Cause it doesn't take as many steps.
I: Show me.

Matt: Here ...

He pointed to the end of the first column of his second step, which looked like this:

\[
x - m \\
\times [ ] \\
a + b
\]

... 'cause I didn't have to break up this group [sketching a circle around \(a + b\)]. You had minus \(a\) and then ended up with \(b\) on another line.

I: Yes. The bigger the group you start with, the fewer operations you have to reverse.

Matt: 'Cause you just end up putting them back together again in the next step anyway.

I: Right. Just make sure that you don't start with a group in the last step.

Matt: Why not? Oh ... 'cause then you wouldn't end up with the letter you're supposed to.

It would be difficult to argue that such students have not taken the "leap" whereby "a process solidifies into an object, into a static structure" and becomes "detached from the process that produced it," as Kieran said (1992, p. 392).

Nevertheless, mathematics educators must not forget the need for a "lengthy transition period in moving from operational (procedural) to structural conceptions" (Kieran, 1992, p. 395). Nor can we forget people like Kelly, for whom the leap "is
inherently so difficult" that "the structural conception will remain practically out of reach whatever the teaching method" (Kieran, 1992, p. 412).

Authority Behind "The Right Answer"

Sometimes students know, simply and beyond argument, even beyond the question of marks, that what they have written or said is true; they know that they are in touch with a reality or a truth which is independent of the teacher: it does not depend on the teacher for validation. Occasionally such students reveal their inner certainty by arguing with the teacher, as the following anecdote illustrates.

I had been teaching my first Grade 8 classes that they could do additions and multiplications backwards, but not subtractions and divisions. Their worksheet looked like this:

Without using your calculator, tell whether you will get the right answer (R) or the wrong answer (W) to the question if you press the following buttons on your calculator.

<table>
<thead>
<tr>
<th>Question</th>
<th>Buttons</th>
<th>Answer R or W</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 142.45 + 35.2</td>
<td>35.2 + 142.45 =</td>
<td></td>
</tr>
<tr>
<td></td>
<td>142.45 + 35.2 =</td>
<td></td>
</tr>
</tbody>
</table>

Kyle (Ma 8A - A) got zero on the quiz.

I: [After handing it back to him] I thought you understood this!

Kyle: I do! This [stabbing his pencil on to the paper for emphasis] ... is ... one ... hundred ... percent ... correct.

I: It's not; it's one hundred percent wrong.
Kyle: [Stabbing at the paper] *If this is the question, and I press these buttons, I'll get the right answer.*

I: *So why did you put W?*

Kyle: [Slowly, as if trying to teach me] *W ... R ... I... T ...E.*

I: *[Wondering whether he was trying to hoax me] So how do you spell wrong?*

Kyle: [In the same manner] *R ... O ... N ... G ... But [as the bell for the end of class rang] I don't care. I know I got it all right.*

I: *I see what the problem is. But didn't you notice when you checked the answers on the back of the worksheet that you were getting them wrong?*

Kyle: I didn't look. *I knew I was getting them right.*

I: *Well, I'll give you 100% for the math - but you'd get zero for spelling.*

Kyle: I don't care. *[Striking a pose] I know, and God knows.*

(The following year, I changed "R" and "W" on the worksheet to check marks and x's.)

Annoying as such episodes may be, teachers should value them as signs that the student has truly learned whatever is in question and no longer needs the teacher. Science teachers have the same experience when students tell them what they actually observed in an experiment instead of asking, "Is that what's supposed to happen?"
Conclusion

Calculators can help students to shift their focus from numbers to operations and to make the transition from numbers to letters which is demanded as they move from arithmetic to algebra. Moreover, calculators can help students develop an image of a variable as an entity whose numerical value can vary.

Moreover, the language of calculators as I employ it in solving equations by the reversal method can enable students to understand and describe verbally the structure of algebraic expressions and can even help some students to make the leap called reification.
Chapter 8
Conclusion and Consequences

I share with other teachers what Pimm (1995) describes as a "fear of apparent sophistication of performance unrooted in understanding, and the perennial desire ... to be able to read comprehension from successful practice" (p. 94).

Accordingly, in "regular" mathematics classes I have tried to teach for understanding, not just practice. On the whole I have failed, not only to teach the majority of my students to understand, but also even to complete the curriculum.

I remember well the plea uttered by a Mathematics 11 student to the effect that she was going to be a nurse, that she needed credit for Mathematics 11 to enter nursing school, and that please, would I stop trying to make her understand mathematics and just tell her what she had to do to pass.

In practice, most high-school mathematics teachers make, to various degrees, the compromise this student was pleading for. The result is described by Kieran (1992):

In reporting the results of the fourth mathematics assessment of seventh- and eleventh- grade U.S. students by the National Assessment of Educational Progress (NAEP), Brown et al. (1988) concluded that "Secondary school students generally seem to have some knowledge of basic algebra and geometric concepts and skills. However, the results of this assessment indicate, as the results of past assessments have, that students often are not able to apply this knowledge in problem-solving situations, nor do they appear to understand many of the structures underlying these mathematical concepts and skills (pp. 346-347)."

However, to cover their lack of understanding, it
appears that students resort to memorizing rules and procedures and they eventually come to believe that this activity represents the essence of algebra. Brown et al. reported that a large majority of the students in the NAEP study felt that mathematics is rule-based, and about half considered that learning mathematics is mostly memorizing. These findings are not restricted to the NAEP evaluation. They have been reported in countless studies conducted in other countries. (p. 390)

Who needs to be able to do high-school algebra as it is taught in the "regular" courses? Not nurses, typesetters, or bankers. Not average citizens, who do not even balance their cheque books, count their change, or fill out their own income tax returns. Certainly not students for whom the essence of algebra is "memorizing rules and procedures," who feel that "mathematics is rule-based" and consider learning mathematics to be "mostly memorizing" (Kieran, 1995, p. 390).

Certainly not a newspaper layout person I know, who calculates the percentage of its original size to which a photograph should be reshot in order to fill the space allotted on something that looks like a circular slide rule (I have never actually examined it). On one occasion, her instructions to the photographer who was going to do the reshotting read "50%, then 92%" instead of the usual single percentage. When I inquired why, he explained that the original photograph had been so large and the space allotted so small that she had said the two numbers would not fit on her slide rule. She had not known what to do, so he had suggested that she halve the size of the original first. "Doesn't she realize that this is 46%?" I asked him. "Probably
not, he replied. "She probably thinks I'll shoot it first at 50% and then reshoot the result at 92%.

Certainly bankers do not need high school algebra. As treasurer of the North Vancouver High School Education Foundation, I have to calculate the simple interest I can expect to be paid into our account from some thirty-two investments in about twelve financial institutions annually. Eleven of them always pay exactly the amount I expect; the twelfth is always slightly too high or too low, apparently at random, by up to two percent of what I expect, and no one in the bank or its head offices knows why.

Handy (1994) probes the meaning of intelligence and lists nine forms - factual, analytical, linguistic, spatial, musical, practical, physical, intuitive, and interpersonal (pp. 204-205) - none of which, he says, is "necessarily connected with any other" (p. 203). When I read out this list and gave examples of each kind of intelligence to students in two Mathematics 8A classes and a Mathematics 9A class in October 1994, the students were interested, but they had only two distinct responses - both of them, I think, tragic: "Can I have a copy of that list to show my parents?" and "What if you don't have any of those?"

Handy says, of analytical intelligence, "When this intelligence is combined with factual intelligence, examinations come easy. When we describe someone as an intellectual, it is often this combination we have in mind" (p. 204) - that is, a combination of just two of the nine.
I myself do not shine at athletics or art. Yet I never felt put down in school because of these inabilities. It must be very different for students who do not shine in mathematics, if one can judge from the excuses made by adults for lack of success in mathematics: "I had a bad teacher in Grade Five," "I missed a section of Grade Nine and never understood anything afterward," "I never saw how I could use math." How often do people admit openly, without shame or excuse, that they "cannot do math?"

One of the people who does is my school principal, who told me that he had simply memorized his way through mathematics courses. With his permission, I tell my students at the beginning of each course that some otherwise intelligent people are not good at mathematics and cite him as an example. It evidently makes an impression on them; he tells me that a significant number later mention to him what I have told them.

I explain to my physics students that they cannot get A in my courses unless they can apply the principles they have learned to solve problems they have not seen before, because physics is not primarily a "memory" subject, but an "understanding" or "thinking" subject. Although it is a prestigious subject, students know that they are not all expected to be good at physics. Credit in Physics 11 or 12 is not required for high-school graduation or for further training except in related fields. If students cannot succeed in physics, they need not take the courses.

But students are required to pass Mathematics 8-11 courses in order to graduate from high school in British Columbia. The
implication is that any "normal" person can succeed in the "regular" math courses (despite the fact that a large percentage of them simply memorize rules).

If we want our mathematics courses to be primarily "thinking" courses and at the same time want them to be passable by students who clearly (on all other grounds) deserve to graduate from high school, then we must allow students to work at the level at which they can think mathematically and acclaim them for doing so.

For example, when one of my Grade 9 students worked out for herself that if she knew the price of one item, she could "add two more on" to find the price of three, she was thinking mathematically. It would be pointless to argue that a Grade Nine student should be able to do better than that. To tell her to "multiply by three" would be to substitute my rule for her thinking.

Conclusion

1. It is possible to shift the attention of students from numbers to operations by the use of electronic calculators.

2. It is possible to shift the attention of students from operations to the overall structure of algebraic expressions and equations by the use of electronic calculators.

3. The responses of students who have been taught algebra in calculator language show that they understand algebra.

The mathematics courses I have designed use calculator language, which is a specialized form of the language of
arithmetic, and continue to use this language in solving equations. Students who are unsuccessful in "regular" mathematics courses can perform this mathematics successfully.

Moreover, they understand it and can verbalize their own mathematical insights. They can even create it. Despite the fact that the emphasis of my courses is on successful practice, students from all grades and all levels of achievements show by what they write and say spontaneously that they have an understanding of what they are doing. Moreover, they often know when they have "done it right" without waiting for my approval - evidence that they do not think they are simply obeying arbitrary, meaningless rules.

The experience of students who take my mathematics courses is summed up in the following letter, written to me by a girl a full semester after she had finished Mathematics 8A (the spelling and punctuation are hers):

Dear Miss Murray: How are you? I'm doing fine. My second semester has been alright. Even though I havn't been all that good I would like to thank you, for what you did for me. I'm talking about making math look so easy. I had a real good time last semester in math class. It was on of the classes, I looked forward to going too, last semester. I also did hate math before, all those numbers seemed to be boring, for me. When I first came into your class, I thought it would be a long and boring class and semester. After a couple of classes, I started understanding it. Then it started becoming simple. I never thought that I would ever get a B in math. I always got C or lower. I would really like to thank you for all your help.
Consequences

Mathematics as I teach it, which may be called operational algebra, can be offered to students as an end in itself; that is, as mathematical activity that is worth engaging in for its own sake, like skiing, because it is enjoyable and because, like any other ability, especially trained ability, it enhances life. It appears possible that, with further research, my success could be generalized to other students who have had little or no success in previous "regular" mathematics courses.

There is much less evidence that my courses could serve as a bridge from arithmetic to "regular" algebra. First, I have never attempted to use them in this way. Second, I know of only two students who have switched from my courses to "regular" courses. Both of them were very successful in Mathematics 8A, one finishing with 94% and the other with 83%. The first one has since passed Mathematics 9, 10, and 11 with marks slightly above 50%, while the second has barely passed Mathematics 9 and 10, having failed each course once before he passed it, and is currently failing Mathematics 11.

Nevertheless, taken in conjunction with Sfard's (1991) analysis of reification, described in Chapter 3, the signs of reification among some of my students suggest that further research along these lines might be profitable.

Accordingly, I claim that the results of my case study have consequences for:

1. research into how mathematics is learned, taught,
conceptualized, and performed - a topic which interests some educators for its own sake, apart from its application to teaching.

2. the design of high school mathematics courses for the unmathematical, who can learn algebra without *reification*.

3. the design of high school mathematics course for the mathematical, who might be able to use algebra without *reification* to bridge the gap between arithmetic and "regular," or *reified,* algebra.
Bibliography


## Simple Interest

\[ I = \frac{Prt}{u} \]

\[ A = P + I \]

## Compound Interest

\[ A = \left( \frac{r}{n} \right)^{(nt/u)} + 1 \]

\[ I = A - P \]

\[ e = \left( \left( \frac{r}{n} \right)^{(nt/u)} - 1 \right) \cdot 100 \]

## Saving

\[ A = \left( \frac{r}{n} \right)^{(nt/u)} + 1 - 1 \] \[ \cdot \left[ \frac{100m}{r} + \frac{m/n + 1}{2} \right] \cdot d \]

## Financing

\[ p = \frac{(C - D)r/100}{(n-m)r/200 - m} \left[ \frac{1}{\left( \frac{r}{n} \right)^{(nt/u)} - 1} \right] \]

\[ I = \text{pmt}/u - C + D \]

\[ A = I + C \]

\[ a = \left( \frac{(n-m)r}{200 - m} \right) \cdot \frac{100p}{r} \cdot C - D \] \[ \cdot \left[ \left( \frac{r}{n} \right)^{(nt/u)} \right] - 1 \] + C - D

- P principal
- d deposit
- C cost
- D downpayment
- r interest rate
- t total time
- T time already spent paying
- u how many time units per year
- n how many times interest is compounded per year
- m how many deposits or payments per year
- p payment
- I interest (finance charge)
- A total amount
- a amount still owing
- e equivalent simple interest rate
Instructions

1. Round off interest rates to two decimal places and times to zero decimal places.
2. Show your working as neatly as you can in the space provided.
3. In any question where you use a program in your calculator, you must show how to change the subject, if that would be necessary, if you want full marks.
4. Don't forget $ signs and % signs.

Part 1

Do the problem on your calculator and write down the answer to the number of decimal places indicated in the brackets.

1. \[
\begin{array}{c}
\frac{2.3}{1.5} + \frac{1.8}{2.7}
\end{array} \] (3)

2. \[
\frac{(13.75 - 12.702)^9}{13.6 - 1.35} \] (2)

3. \[
1.1 + 2.1\left[1.5 + 0.7\left[0.85 + 0.1(1 + 1/14)^9\right]\right] \] (0)

4. \[
\frac{(103 + 1.4)^2}{1.3 - 1.6 + 1.7} - \frac{1.2 - 1.5 + 1.6}{1.407} \] (0)

Part 2

Evaluate the variable whose value is not given and round off your answer to two decimal places.

5. \[
A = \frac{b}{q} + c(a - b) \quad A = 15, a = 3.4, b = 1.2, q = 15
\]

6. \[
c = \frac{g - h}{b} \quad c = 132.56, g = 1.4, h = 4.5
\]

7. \[
r = \frac{a + 2b}{b - c} + \frac{a - 2b}{b - d} \quad r = 2.953, a = 2, b = 3.4, d = 9
\]

8. \[
H = \frac{1}{1 + \frac{1}{1 + \frac{1}{c}}} \quad H = 7.95
\]
Part 3

9. If you invest $200.00 for 8 mo and end up with $216.45, (a) how much interest do you make? and (b) at what interest rate is your money deposited?

10. If you leave $750.00 in the bank at 15.00% from May 1, 1990 to Jan 25, 1991, (a) how much do you make? and (b) how much do you end up with?

11. Jenny puts $1500.00 in a savings account paying 7.25% compounded monthly. (a) How much does she have in the account after 6 mo? (b) How much interest has she made?

12. Find the present value of a T-bill that will be worth $3000.00 in 256 da if it pays 8.75% compounded daily.

13. Find the equivalent simple interest rate of 15.25% compounded weekly.

14. A certain savings account pays 9.50% compounded monthly. How much must you pay into the account every half month if you want to end up with $25 000.00 in 5 a?

15. You pay $50.00 every month into your savings account, which pays you 8.75% compounded quarterly. How much do you have in total by the end of 10 a?

16. You borrow $5 000.00 and pay it back in monthly installments of $250.00 each. If you are being charged 15.00% interest compounded monthly, calculate how much you still owe when you have been paying for 15 mo.

17. You make a downpayment of $25 000.00 on a house which costs $150 000.00. You pay the rest in monthly installments over 24 a at an interest rate of 12.50% compounded semiannually. Find (a) the size of each payment, (b) the finance charge, and (c) the total bill.

*18. You deposit $350.00 in the bank on Feb 1, 1992. You get 12.00% interest and you want to make $50.00. On what date should you take the money out?

*19. You can buy a lot for $80 000.00 or $10 000.00 now and another $100 000.00 in 5 a. If the banks are paying 11.50% compounded quarterly, which is the better deal? How much do you save now by taking the better deal?

*20. Show which interest rate on your savings account is better over 1 a - 10.25% compounded semiannually or 10.50% compounded daily.
*21. You buy a house worth $185,000.00 for $35,000.00 down. You amortize your mortgage over 25 years, making monthly payments and being charged 14.00% interest compounded semiannually. How much do you still owe after 10 years?

**22. After 7 years in an account which compounds interest quarterly, your principal has tripled. Find the rate at which it was invested.

**23. How many years and days does $3000.00 have to be invested at 10.25% compounded daily to make $1200.00 interest?

**24. What interest rate compounded semiannually is equivalent to 12.00% compounded monthly?

**25. You pay $750.00 twice a month to pay back a loan of $120,000.00 on which the interest rate is 15.00% compounded monthly. Find how many years and months it takes you to (a) reduce your debt to $60,000.00 and (b) pay back the loan.

**26. If you deposit $150.00 every half month in an account that pays 8.75% compounded monthly, how many years and months does it take to accumulate $25,000.00?
Appendix 2

Students' Records in Previous Mathematics Courses

Tables 2-B, 2-C, and 2-D contain the high school mathematics records of three classes of Mathematics 11A students, one taught by me, the others by another teacher in my school using my materials. Students' records are divided from each other by horizontal lines and each designated by a different letter.

These records show the lack of success experienced by these students in previous courses. Failing grades are printed in bold type and the symbols •, ♥, ♦, and ♣ are used to designate courses other than the "regular" mathematics courses.

Letter grades are to be interpreted according to the following scheme, which is standard in British Columbia schools:

- $86\% \leq A \leq 100\%$
- $73\% \leq B < 86\%$
- $67\% \leq C+ < 73\%$
- $60\% \leq C < 67\%$
- $50\% \leq P < 60\%$
- $0\% \leq F < 50\%$

Table 2-A explains the meanings of the other symbols.

Table 2-A

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>course work incomplete</td>
</tr>
<tr>
<td>N</td>
<td>no mark assigned</td>
</tr>
<tr>
<td>W</td>
<td>withdrawn from course</td>
</tr>
<tr>
<td>SG</td>
<td>standing granted</td>
</tr>
<tr>
<td>F+P, SG+P</td>
<td>second mark obtained at summer school</td>
</tr>
<tr>
<td>♣</td>
<td>introductory course interpolated between &quot;regular&quot; courses</td>
</tr>
<tr>
<td>♥</td>
<td>modified course</td>
</tr>
<tr>
<td>♦</td>
<td>taken at an alternative school</td>
</tr>
<tr>
<td>♣</td>
<td>for &quot;special&quot; students</td>
</tr>
</tbody>
</table>
Table 2-B

Records of Students Taught By Terry Baker Between September 1991 and January 1992

<table>
<thead>
<tr>
<th>Grade 8</th>
<th>Grade 9</th>
<th>Grade 10</th>
<th>Grade 11</th>
<th>Ma 11A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Ma 8 B</td>
<td>Ma 9 B</td>
<td>Ma 10 C</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Ma 8 F</td>
<td>Ma 9 P</td>
<td>Ma 10 P</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Ma 8 C</td>
<td>♥Ma 9C F</td>
<td>Ma 10 C</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>Ma 8 53%</td>
<td>♥Ma 9AB N</td>
<td>♥Ma 10A P</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>♥Ma 8C C+</td>
<td>♥Ma 9A F</td>
<td>♥Ma 10A P</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>♥Ma 8C P</td>
<td>Ma 9 F</td>
<td>♥Ma 10A C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ma 8 F</td>
<td>Ma 9 F</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ma 8 P</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>Ma 8 P</td>
<td>Ma 9 P</td>
<td>♥Ma 10B F</td>
<td>Ma 10 C</td>
</tr>
<tr>
<td>H</td>
<td>♥Ma 8C C</td>
<td>♥Ma 9C C</td>
<td>♥Ma 10A P</td>
<td>Ma 11A F</td>
</tr>
<tr>
<td>I</td>
<td>Ma 8 F</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>J</td>
<td>Ma 8 F</td>
<td>♥Ma 9C C</td>
<td>♥Ma 10A C</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>Ma 8 P</td>
<td>Ma 9 P</td>
<td>Ma 10 45%</td>
<td>♥Ma 11 F</td>
</tr>
<tr>
<td>M</td>
<td>♥Ima8 P</td>
<td>♥Ma 9A C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td></td>
<td>Ma 9 F</td>
<td>♥Ma 10A P</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ma 9 F+P</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>Ma 8 SG</td>
<td>Ma 9 SG</td>
<td>♥Ma 10A P</td>
<td>Ma 10 F</td>
</tr>
<tr>
<td>P</td>
<td>♥Ma 8C C</td>
<td>♥Ma 9A C</td>
<td>♥Ma 10A P</td>
<td></td>
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<tr>
<td>Q</td>
<td>Ma 8 P</td>
<td>Ma 9 P</td>
<td>Ma 10 F</td>
<td>Ma 10 F</td>
</tr>
<tr>
<td>R</td>
<td>♥Ima8 C</td>
<td>♥Ma 9A C</td>
<td>♥Ma 10A P</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>Ma 8 61%</td>
<td>Ma 9 40%</td>
<td>Ma 10 P</td>
<td>Ma 11 F</td>
</tr>
<tr>
<td>T</td>
<td>♥Ma 8R A</td>
<td>♥Ma 9AB P</td>
<td>♥Ma 10A P</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ma 8 P</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>Ma 8 F</td>
<td>♥Ma 9AB P</td>
<td>♥Ma 10A P</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W</td>
<td>♥Ma 8C C</td>
<td>♥Ma 9C P</td>
<td>♥Ma 10A SG</td>
<td>Ma 11A F</td>
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<tr>
<td>X</td>
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<td>♥Ma 10A F</td>
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<tr>
<td>Y</td>
<td>Ma 8 F</td>
<td>Ma 9 P</td>
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<tr>
<td>Z</td>
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<td>Ma 9 F</td>
<td>Ma 10 F</td>
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<td></td>
<td>Ma 8 P</td>
<td>Ma 9 P</td>
<td>♥Ma 10A C</td>
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</tr>
<tr>
<td>a</td>
<td>♥Ma 8C P</td>
<td>♥Ma 9A F</td>
<td>♥Ma 10A C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ma 8 SG</td>
<td>♥Ma 9A F</td>
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### Table 2-C

Records of Students Taught By Terry Baker Between September 1992 and January 1993

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<thead>
<tr>
<th>Grade 8</th>
<th>Grade 9</th>
<th>Grade 10</th>
<th>Grade 11</th>
<th>Ma 11A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Ma 8 B</td>
<td>Ma 9 C+</td>
<td>Ma 10 F</td>
<td>C+</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ma 10 P</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>♦Ma 7/8 C</td>
<td>♦Ma 9A SG</td>
<td>♦Ma 10A C+</td>
<td>C</td>
</tr>
<tr>
<td>Ma 8 F</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ma 8 P</td>
<td></td>
<td></td>
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<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>♦Ma 8C P</td>
<td>♦Ma 9A F</td>
<td>♦Ma 10A P</td>
<td>C</td>
</tr>
<tr>
<td></td>
<td>♦Ma 9A F</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>Ma 8 F</td>
<td>♦Ma 10A SG</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>Ma 8 C</td>
<td>Ma 9 F</td>
<td>♦Ma 10A B</td>
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</tr>
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<td></td>
<td>Ma 9 I</td>
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<td></td>
</tr>
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<td>Ma 9 W</td>
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<tr>
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<td>Ma 8 P</td>
<td>Ma 9 SG</td>
<td>♦Ma 10A C</td>
<td>C+</td>
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<td>I</td>
<td>Ma 8 50%</td>
<td>♦Ma 9A 66%</td>
<td>♦Ma 10A 46%</td>
<td>♦I Ma11 F</td>
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<tr>
<td>J</td>
<td>Ma 8 C</td>
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Table 2-D

Records of Students Taught by Maureen Murphy Between January 1993 and June 1993

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Appendix 3
Use of the Electronic Calculator

In order to make the reader familiar with my calculator language, I here present the lessons as I would present them to the students in written form. In practice, I present them orally, and consequently they involve a great deal of questioning and repetition.

Whenever it has seemed appropriate or necessary to explain to the mathematical reader the reasoning behind my instructions, I have included it in italics.

Order of Operations

What is the answer to the following problem?

\[ 2 + 5 \times 3 \]

If you do the addition first, you get 21. If you do the multiplication first, you get 17. However, mathematicians do not like to have more than one right answer, so they have agreed among themselves always to do a multiplication before an addition.

If mathematicians want you to do the addition before the multiplication, they put brackets around the + sign and the two numbers that are being added, thus:

\[ (2 + 5) \times 3 \]

Mathematicians place brackets around an operation to make us do one operation before another when we would not otherwise do so. Brackets are not operations; we cannot "do" brackets. Rather,
brackets tell us that we must do the operations inside them before the operations outside them. Nor do brackets mean "multiply."

Mathematicians have had to make a number of other rules like the one about addition and multiplication in order to ensure that every problem has only one right answer.

Mathematicians call anything you can do to a number, like add, subtract, multiply, divide, or raise to a power, an operation. You have to obey the rules about the order of two operations when the two operations involve the same number, as the + and the \( \times \) do in the example above: they both involve the 5. In other words, you have to obey the rules when two operations are adjacent, or next to each other. However, you do not have to obey the rules when two operations have another operation between them. For example, in

\[
2 \times 3 + 10^2
\]

it does not matter whether you do the multiplication first or the power first; as long as you do them both before the + you will get the same answer, 106.

I have found that students interpret BEDMAS, an acronym for brackets, exponents, division, multiplication, addition, subtraction, which they have usually been taught by the time they reach Grade 11, to mean that in any given expression there is a unique permitted order, determined by going through the problem from left to right and top to bottom "doing the brackets," then going through the problem again the same way doing the multiplications, etc.
For example, when confronted with the expression
\[
\frac{(1.009 + 0.001)^{44}}{(1.1 - 0.95)^2} \cdot \frac{1.1^2}{1.395 + 1.4759}
\]
students tend to the operations in the order indicated by the italicized numbers thus:

\[
\begin{align*}
(1.009 + 0.001)^{44} & \quad \text{1} \\
(1.1 - 0.95)^2 & \quad \text{2} \\
1.395 + 1.4759 & \quad \text{9}
\end{align*}
\]

If a non-algebraic calculator, which executes operations in the order in which the operation buttons are pressed, is employed to evaluate this expression by performing the operations in this order, 1.009 will have to be added to 0.001 and the result written down or stored in the calculator's memory; then 0.95 must be subtracted from 1.1 and the result written down (most non-algebraic calculators still have only one memory); then the first result must be raised to the 44th power and the result written down or stored, etc.

Instead, I wanted students, in order to utilize their calculators most efficiently (that is, with a minimum of writing) to perform the operations in the following order:

\[
\begin{align*}
(1.009 + 0.001)^{44} & \quad \text{7} \\
(1.1 - 0.95)^2 & \quad \text{4} \\
1.395 + 1.4759 & \quad \text{9}
\end{align*}
\]

so that nothing would have to be written down except the final
answer.

When you are deciding which of two adjacent operations to do first, always do the last one first if you are allowed to, because that works out best for a calculator, but you must obey the following rules:

1. If one operation is inside brackets and the other is outside, you must do the one inside brackets first.
2. If one operation is a power and the other is not, you must do the power first.
3. If one operation is a multiply or divide and the other is an add or subtract, you must do the multiply or divide first.
4. If both operations are add or subtract, you must do them left to right.
5. If both operations are multiply or divide, you must do them left to right.

It will be seen that rules 4 and 5 are more stringent than is necessary to ensure "one right answer." For example, 12 + 6 - 2 = 16 whether the addition or the subtraction is performed first. However, 12 - 6 + 2 = 8 if the subtraction is performed first and 12 - 6 + 2 = 4 if the addition is performed first. The situation is similar with multiplication and division. Rather than teach students when the order makes a difference and when it does not, I have found it easier to write simple rules that always apply and make special provisions for more knowledgeable or insightful students when necessary.

Before I can do an example, I must mention the following
Using Calculators to Teach Algebra

points:

1. Mathematicians often leave out multiplication signs. Multiplication signs are the only ones they leave out, and therefore you can assume that there is an invisible multiplication sign between any two numbers which have no operation sign between them.

2. Leaving out a multiplication sign between two numbers could make the two numbers run together and be misunderstood. Putting in a raised dot like this \( \cdot \) is dangerous because it can be mistaken for a decimal point. To avoid confusion, mathematicians often put in brackets. For example, if they do not want to write

\[ 2 \times 3 \]

they cannot write simply

\[ 2 3 \]

but must write

\[ 2(3) \]

or

\[ (2)(3) \]

or, less commonly

\[ (2)3 \]

instead. However, these brackets are not "real" brackets; they have no operation inside them, but only a single number. They are "fake" brackets, which you can leave out if you put the multiplication sign back in.

3. The symbol / means the same as the symbol \( \div \). You read it from left to right.
4. The horizontal line in, for example, $\frac{1}{2}$ means the same as
the symbol $\div$. You read it from top to bottom.

5. A long division line like this one

\[
\frac{2 + 4}{3 - 6}
\]

does the job of brackets as well as meaning divide. The line has
invisible brackets attached to its ends above and below it like
this

\[
\frac{(2 + 4)}{(3 - 6)}
\]

so the expression above means the same as

\[(2 + 4) \div (3 - 6)\]

It will be seen that the upper brackets would not be
necessary if the addition in this example were replaced with a
power, a multiplication or a division. On the other hand, the
lower brackets could be omitted only if the subtraction were
replaced with a power. Again, I have found it easier to teach
students one simple rule.

6. Evaluating an expression means doing the operations in
it. We cannot do the numbers. If I gave you this problem on a
test

\[283\]

you would not know what to do. It is only when I write, for
example

\[283 + 45\]

that I am telling you to do something. Therefore, if we are
numbering the operations to show their order, we place the numbers
indicating the order of operations under the *operations* and not under the *numbers*. However, there is no symbol for a power like the symbols for the other operations (+, -, *, ÷, /, —), so we simply place the number indicating the order of a power near the exponent.

The example that follows shows how these rules are applied. It may appear tedious, but you will probably find that you will soon be able to do it without asking yourself all the questions.

**Example**

\[
\left(\frac{2 + 3^2}{1 - 7}\right)^7
\]

To place numbers under the operations to show the order in which they must be done, first put in the brackets implied by the long division line.

\[
\left[\frac{(2 + 3^2)}{(1 - 7)}\right]^7
\]

Look at the last pair of operations only.

\[
\left[\frac{(2 + 3^2)}{(1 - 7)}\right]^7
\]

*Ask yourself:* May I do the power before the -?

*Answer:* No, because the power is outside brackets and the - is in.

Look at the next pair of operations.

\[
\left[\frac{(2 + 3^2)}{(1 - 7)}\right]^7
\]
Ask yourself: May I do the - before the ÷?
Answer: Yes, because the - is inside brackets and the ÷ is not.

Do: Put 1 under the -.

\[
\left[ \frac{2+3^2}{1-7} \right]^7
\]

Look at the last pair of operations.

\[
\left[ \frac{(2+3^2)}{1-7} \right]^7
\]

Ask yourself: May I do the power before the ÷?
Answer: No, because the power is outside brackets and the ÷ is in.

Look at the next pair of operations.

\[
\left[ \frac{(2+3^2)}{1-7} \right]^7
\]

Ask yourself: May I do the ÷ before the power?
Answer: No, because the ÷ is not in brackets and the power is.

Look at the next pair of operations.

\[
\left[ \frac{(2+3^2)}{1-7} \right]^7
\]

Ask yourself: May I do the power before the ÷?
Answer: Yes, because powers must be done before ÷.

Do: Put 2 over the power, 3 over the ÷, 4 at the side of the ÷, and 5 under the power.

\[
\left[ \frac{2}{(2+3^2)} \right]^5
\]
Two Different Types of Calculator

Press the following buttons, in this order, on your calculator: 2, +, 3, \times, 5. What answer do you get?

If you got 25, your calculator is non-algebraic. If you got 17, your calculator is algebraic.

You have to learn to use your calculator. Non-algebraic calculators work differently from algebraic calculators and you have to use them differently to get the right answer. If you have an algebraic calculator, skip the instructions for non-algebraic calculators.

Non-Algebraic Calculators

Before I can teach you how to use a non-algebraic calculator, I have to teach you how to do a power.

Most non-algebraic calculators do not have a power button (usually designated $y^x$, $x^y$, or $a^x$). The only non-algebraic calculators with power buttons are business calculators, which tend to be more expensive.

Example

To calculate $2^5$, type $2 \times = \ldots \times =$. That is, tell the calculator that you are starting with 2; then tell it that you want to multiply. However, since you are going to multiply by 2 four more times, you do not have to press 2 again. Just press $=$ four times.
Using Calculators to Teach Algebra

Example

To calculate $1.009^{**}$, first divide 192 by 2 until you reach an odd number. Just press $192 \div 2 = = ...$ and write down the numbers you get thus:

$\begin{align*}
192 \\
96 \\
48 \\
24 \\
12 \\
6 \\
3
\end{align*}$

Now press $1.009 \times = =$. That will calculate $1.009^3$ (the 3 is the 3 on the bottom line). For each of the other lines, press $\times =$. Altogether, you will press $1.009 \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = \times = 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Using Calculators to Teach Algebra

1.009\#. Then press $x = $ to go up to 10, then $x \text{ MR}$ to go over to 11, then $x = $ to go up to 22, then $x = $ to go up to 44, etc. Altogether, you will press $1.009 \text{ M+} x = = = x = x \text{ MR} x = x = x \text{ MR} x = x = = =$. (I am indebted for this last technique to Val Greig, who served as an aide in my classes for a number of years.)

Here are the rules for using a non-algebraic calculator:

1. Number the operations as you have been taught and then perform the operations in that order.

2. You may do additions and multiplications forward (reading left to right) or backward (right to left) because that does not change the answer. For example, if you do

\[
2 \times 3
\]

on your calculator, you get the same answer as if you do

\[
3 \times 2
\]

If you do

\[
2 + 3
\]

on your calculator, you get the same answer as if you do

\[
3 + 2
\]

Therefore, if you find yourself wanting to do an addition or a multiplication backward, do so.

3. However, you must do subtractions forward (reading left to right) and divisions forward (reading ÷ or / left to right and — top to bottom) or you will get the wrong answer. For example, if you do

\[
8 \div 2
\]
on your calculator, you get 4, but if you do

\[ 2 \div 8 \]

you get 0.25. Similarly, if you do

\[ 8 - 2 \]

you get 6, but if you do

\[ 2 - 8 \]

you get -6, which looks similar to 6 but means something very different. (Which would you rather have as your bank balance?) Therefore, if you find yourself wanting to do a subtraction or a division backwards, stop, store what you have done so far in the calculator's memory, go to the number in front of the subtraction or division sign, and do the subtraction or division forward, recalling what you put in the memory when you need it.

4. Also, if you find that you have to skip over one operation to do another operation first, stop, store what you have done so far in the calculator's memory, do the next operation in order, and recall what you put in the memory when you need it.

5. To store what you have done in your calculator's memory, press the \( M+ \) button. You do not have to press = first; the \( M+ \) button does the equals for you. However, the memory must be empty. If there is a number in the calculator's memory already, pressing \( M+ \) will add the number you are trying to store to the number that is already there.

6. To recall a number from your calculator's memory, press \( MR \).

7. To clear your calculator's memory, press \( MC \). On many
calculators, MC is on the same button as MR. With this kind of calculator, you should get into the habit of always pressing MC immediately after you have pressed MR; simply press the same button twice. On such a calculator, clearing the memory entails recalling the stored number first, and recalling the stored number entails losing the currently displayed number. Only if the memory is cleared immediately after a stored number is recalled can the memory be used twice in the same problem.

Example

Calculate

\[
\frac{2 + 3^2}{1 - 7}
\]

The order of operations is as follows:

\[
\left(\frac{\frac{3}{2}}{\left(\frac{2 + 3^2}{1 - 7}\right)}\right)^5
\]

Therefore, on a non-algebraic calculator, you would press the following buttons to get the answer:

\[
1 - 7 \text{ M+} 3 \times = + 2 \div \text{ MR MC x } = = = = =
\]

The reason why I ask students to do later operations first if possible can now be made clear; if you numbered the operations thus

\[
\left(\frac{2 + 3^2}{1 - 7}\right)^5
\]

you would start by pressing the following buttons:
3 \times = + 2 M+ 1 - 7 \ldots

at which point you would be faced with doing the division backwards.

Algebraic Calculators

Notice that an algebraic calculator has two power buttons, one marked $y^x$, $x^y$, or $a^x$ and the other marked $x^2$. Notice that it also has two bracket buttons, marked ( and ).

Use your power buttons as follows.

Example

To calculate $2^2$, press $2 \times^2$. Use this button only if the exponent is 2.

Example

To calculate $1.009^{192}$, type $1.009 \ y^x \ 192 =$. Use this button for every exponent except 2.

Here are the rules for using an algebraic calculator.

1. In general, type the expression on to your calculator just as it is written, from top to bottom and from right to left.

2. However,
   (a) do not type in any fake brackets (that is, brackets around a single number).
   (b) do type in any unwritten multiplication signs. (This may not be necessary in most circumstances on some calculators, but it is safer to type it in.)
(c) do type in the brackets that come with long division lines.

Example

Calculate

\[
\frac{2 + 3^2}{1 - 7}
\]

First put in the brackets that come with the long division line.

\[
\frac{(2 + 3^2)}{(1 - 7)}
\]

Then, on an algebraic calculator, you would press the following buttons to get the answer:

\[
( ( 2 + 3 \times x^2 ) \div ( 1 - 7 ) ) \times 7 =
\]

It will be seen that students do not need to know the conventional order of operations to use an algebraic calculator. However, I insist that students with algebraic calculators insert numbers indicating the order of operations as outlined above for two reasons. One is that they need to be able to order the operations correctly when I teach them to solve equations. The second is that the better students need to know the order of operations in order to evaluate long expressions in the midst of which an algebraic calculator can indicate an error, as shown in the following example.
Example

\[3.6 + 5.2 \left[ 12 - 6 \left[ 3.1 - \frac{6.1}{371} \right] \right]\]

The order of operations is as follows:

\[3.6 + 5.2 \times \frac{12 - 6 \times \left[ 3.1 - \frac{6.1}{2 \times 371} \right]}{5} \]

1. Type the problem on to your calculator as usual, but watch the display for an error message and note the exact point in the calculation at which the error message appears. Assume that the error message appears immediately after you type in the second subtraction sign. Note this point.

\[3.6 + 5.2 \times ( 12 - 6 \times ( 3.1 - 6.1 \div 371 ) ) = \]

You have typed the operations numbered 6, 5, 4, and 3 on to your calculator, but your calculator has not done them yet. It knows it cannot do operation 6 until it has done operation 5, and it cannot do operation 5 until it has done operation 4, etc. Instead, it has to remember these operations. When it gives you the error message, it is telling you that it cannot remember any more operations.

2. Clear your calculator. Now back up from the error point, say to the previous bracket, and, beginning there, underline a natural unit from the middle of the problem, as shown.

\[3.6 + 5.2 \times \frac{12 - 6 \times \left[ 3.1 - \frac{6.1}{2 \times 371} \right]}{5} \]
A natural unit includes only consecutive operations and may not omit one of a pair of brackets.

3. Type in the underlined part of the problem and store the answer in your calculator's memory.

\[
( 3.1 - 6.1 \div 371 ) M
\]

4. Type in the whole problem again, recalling the stored part from the memory when you get to it.

\[
3.6 + 5.2 \times ( 12 - 6 \times MR ) =
\]
In order to make the reader familiar with the language of the
reversal method of solving equations, I here present the lessons
as I would present them to the students in written form. In
practice, as with calculator lessons, I present them orally, and
consequently they involve a great deal of questioning and
repetition.

Again, whenever it has seemed appropriate or necessary to
explain to the mathematical reader the reasoning behind the
instructions, I have included it in italics.

Formulae

If you put some money in the bank, the bank will pay you
interest. The amount of interest the bank pays you depends on how
much money you put in the bank (called the "principal"), what the
interest rate is, and how much time you leave the money in the
bank.

To calculate how much interest they owe you, the bank starts
with the principal, multiplies by the interest rate, and then
multiplies by the time.

If we write the principal as $P$, the interest rate as $r$, the
time as $t$ and the interest as $I$, we can write down the formula the
bank uses thus:

$$P \times r \times t = I$$
Using Calculators to Teach Algebra

201

Usually we leave out the x's because they look like x's, and so the formula looks like this:

\[ Prt = I \]

Any formula is a set of mathematical instructions for calculating the value of one variable from the values of other variables.

Since this is a formula for calculating \( I \), \( I \) is called the subject of the formula.

We can use this formula as follows. Suppose I put $200 in the bank at 10% interest for 5 years. That means \( P = 200 \), \( r = 0.10 \) (because it is a percentage) and \( t = 5 \). The formula tells us what to do with these numbers: on our calculators, we press \( 200 \times 0.10 \times 5 = \) and we get 100. That is the value for \( I \).

\( I = 100 \). The amount of interest the bank will pay us is $100.

The advantage of using letters in a formula is that the formula still works even when we change the numbers. We call the letters \( P \), \( r \), \( t \), and \( I \) variables because their values - the numbers we substitute for them - can vary.

Reversing a Formula

However, we can also use the formula \( Prt = I \) in another way.

Suppose I tell you that I left my money in the bank at an interest rate of 10% for 5 years and got $100 interest and ask you what my principal must have been. In this case, I am telling you that \( r = .10 \), \( t = 5 \) and \( I = 100 \) and asking you what \( P \) must be.

You know that \( P \) must be 200, because that is the value of \( P \) that gave us 100 for \( I \) in the first place. But how could we work
out the value of P if we did not know it already? What if I told you that \( r = .10 \), \( t = 5 \), and \( I = 200 \); what would \( P \) have to be then?

Here is a method by which you can \textit{change the subject} of a formula.

To see how it works, calculate the following without using your calculators: Start with 2 ... add 3 ... multiply by 4 ... subtract 5 ... divide by 5 ... add 6 ... you end up with 9. We can write this as follows:

\[
\begin{align*}
\text{start} & \quad 2 \\
+ & \quad 3 \\
\times & \quad 4 \\
- & \quad 5 \\
\div & \quad 5 \\
+ & \quad 6 \\
\text{end} & \quad 9 
\end{align*}
\]

\textit{I write the operations in a vertical column instead of a horizontal row because students, having been taught the conventional order of operations, are unwilling to perform the operations in the given order if I write them all on the same horizontal line. I write "end" instead of "=" to distinguish 9 from the other numbers. In shape, "=" looks very like an operation sign.}

Now undo what you have just done by starting at the end, doing the opposite operations in the opposite order, and ending up where you started. Start with 9 ... subtract 6 (because subtract is the opposite of add) ... multiply by 5 (because multiply is the opposite of divide) ... add 5 ... divide by 4 ... subtract 3 ... and you end up with 2. We can write this as follows:
We can use the same reasoning to change the subject of the formula $I = Prt$ from $I$ to $P$. The order of operations in the original formula is as follows:

$$Prt = I$$

The original formula says that if we start with $P$, then multiply by $r$, and then multiply by $t$, we will end up with $I$. We can write this as follows:

$$\begin{align*}
\text{start } P \\
\times r \\
\times t \\
\text{end } I
\end{align*}$$

Then we can write the reverse as follows:

$$\begin{align*}
\text{start } P \\
\times r \\
\times t \\
\frac{\text{end } I}{\div r}
\end{align*}$$

Notice that the second, or reverse, column tells us what to do with $I$, $t$, and $r$ to get $P$. However, this is not the usual way of writing formulae; the usual way is as follows:

$$\frac{I}{\frac{t}{r}} = P$$

This is a formula for calculating $P$. We have changed the subject of the original formula from $I$ to $P$. If we do on our calculators what the formula tells us to do with the values above:
100 ÷ 5 ÷ .1 =, we get 200, as we expected.

**Brackets**

Make b the subject of this formula.

\[ A = \frac{b}{a} + c \]

First put in the order of operations.

\[ A = \frac{b}{a} + c \]

<table>
<thead>
<tr>
<th>start b</th>
<th>reverse</th>
<th>brackets</th>
<th>write normally</th>
</tr>
</thead>
<tbody>
<tr>
<td>÷ a</td>
<td>A</td>
<td>[ A ]</td>
<td>(A - c)a = b</td>
</tr>
<tr>
<td>+ c</td>
<td>x a</td>
<td>x a</td>
<td>b</td>
</tr>
<tr>
<td>end A</td>
<td>b</td>
<td>b</td>
<td></td>
</tr>
</tbody>
</table>

Decide where to put brackets according to the following rule:

Look down the operations in the reverse column. Wherever a + or a - is immediately above a x or a ÷, put a line between the + or - and the x or ÷ and make it into a box around the top of the column. You must put brackets around whatever is in the box, so that the + or the - and all the operations which come before it are inside brackets and the x or the ÷ is left outside. These brackets are necessary to make people do the + or - before the x or ÷.

**Division Lines As Brackets**

Make b the subject of this formula.

\[ p = a ; b - c \]

First put in the order of operations.
Using Calculators to Teach Algebra

\[ p = \frac{a \cdot b - c}{7^2} \]

\[ \begin{array}{cccc}
\text{start } b & \text{reverse } & \text{brackets } & \text{write normally }\\
\times a, & p & + c & a_1 \\
- c & \div a, & + c & \frac{p + c}{a_1} = b \\
\text{end } p & b & b & \\
\end{array} \]

If the operation immediately after the + or - is a ÷, then a long division line will do the job of brackets. Be sure you make the division line long enough to go under everything that is supposed to be in brackets.

You may leave out the words "start" and "end."

Reversal of Powers

Make \( a \) the subject of this formula.

\[ R = \left[ \frac{Q\left[ (a - b)^2 + 3 \right]^{1/m} + n}{c} \right]^a \]

First put in the order of operations.

\[ R = \left[ \frac{5}{c} \right]^a \left[ \frac{Q\left[ (a - b)^2 + 3 \right]^{1/m} + n}{6} \right]^8 \]
Write $y^x$ for power operations in the columns because there is no other symbol for a power and that is what the calculator calls it. However, do not write $y^x$ when you write it out normally: simply write the exponent in a raised position.

$y^x$ and $y^{1/x}$ are the reverses of each other. To see this, punch the following buttons: $2 \, y^x \, 5$. The answer you get is 32.

Now punch $y^x \, (1 \div 5) =$. You get back to 2.

When there are powers in the formula, you also need brackets under the following circumstances: Look down the operations in the reverse column. Wherever any operation is immediately above a $y^x$, put a line between the other operation and the $y^x$ and make it into a box around the top of the column. You must put brackets around whatever is in the box, so that the other operation and all the operations which come before it are inside brackets and the $y^x$ is left outside. These brackets are necessary to make people do the other operation before the $y^x$. 
Make $a$ the subject of this formula.

$$A = a(b + c) + d$$

First put in the order of the operations.

$$A = a(b + c) + d$$

<table>
<thead>
<tr>
<th>reverse</th>
<th>brackets</th>
<th>write normally</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$A$</td>
<td>$A - d$</td>
</tr>
<tr>
<td>$x (b + c)$</td>
<td>$-d$</td>
<td>$\frac{A - d}{(b + c)} = a$</td>
</tr>
<tr>
<td>$+ d$</td>
<td>$\div (b + c)$</td>
<td>$\frac{A}{b + c} = a$</td>
</tr>
<tr>
<td>$A$</td>
<td>$a$</td>
<td></td>
</tr>
</tbody>
</table>

If you want to make $a$ the subject, you must start with $a$. But if you start with $a$, you have to write the $x$ next. What does the original formula tell you to multiply by? $b$? No, by the time you do the multiply (operation 2), $b$ is supposed to have been already added to $c$ (operation 1). $c$? No; there is the same objection. By the time you do the multiply, $b$ and $c$ no longer exist; they have been added together. All there is now is the answer to $b + c$.

Does that mean you cannot start with $a$ and then multiply? The answer is that you can, provided somebody has already worked out the answer to $b + c$ for you and stored it in your memory, ready for you to multiply $a$ by.

You do not know the answer to $b + c$, because you do not know what $b$ and $c$ are. So write $b + c$ to tell people you want them to add $b$ and $c$ (that is, get the answer to $b + c$) and put brackets around it to make sure people do the addition and get the answer to $(b + c)$ before they multiply by it.
When you reverse, do not change the $+$ in $(b + c)$, but only
the $\times$ in front of it. In the first column you multiplied by the
answer to $(b + c)$; in the reverse column you must divide by the
same number, namely the answer to $(b + c)$.

When you write out the reverse column normally, you will find
that the long division line can do the job of the brackets around
the $(b + c)$, so you can leave out the brackets.

We call $(b + c)$ in this example a group. Groups can have any
number of operations inside them, as the next example shows. In
future, when you are writing out the first column, whenever you
come to an operation whose number is lower than the operation you
have just written down, keep the group containing the lower-number
operation on the same line, in brackets.

When you reverse, do not change any operations inside the
group. When you write out the new formula normally, take out any
brackets that you can take out without changing what the formula
means.

Groups can be of any size. For example, make $m$ the subject
of this formula.

\[
A = \frac{a + c}{e + g} + \frac{j + m}{p + r}
\]

First put in the order of the operations.

\[
A = \frac{5}{6} a + \frac{2}{3} c + \frac{1}{7} j + \frac{1}{4} m
\]
Using Calculators to Teach Algebra

209

\[ m + j \div (p + r) - j \]

\[ \frac{a + c}{e + g} \times (p + r) \]

\[ A \]

\[ \frac{a + c}{e + g} \]

\[ \text{reverse brackets write normally} \]

\[ A - \frac{a + c}{e + g} \]

\[ (p + r) - j = m \]

---

Solving for an Exponent

Make \( n \) the subject of this formula.

\[ P = 100 \left[ (1 + rt)^n - 1 \right] \]

Put in the order of the operations.

\[ P = 100 \left[ \frac{(1 + rt)^n - 1}{2} \right] \]

First identify the group that the exponent "applies to" - the group that is being multiplied by itself \( n \) times - and make that group together with the exponent the subject first. This group contains all the operations to the left of the exponent which are supposed to be done before the power.

\[ (1 + rt)^n \quad \text{reverse brackets write normally} \]

\[ \frac{P}{100} + 1 = (1 + rt)^n \]

Now take the logarithm of the group on the left side of the equals sign and divide it by the logarithm of the group on the right side without the exponent. (We usually write "log" instead of "logarithm.") That gives you the exponent you want.
Using Calculators to Teach Algebra

\[ \frac{\log \left( \frac{P}{100} + 1 \right)}{\log(1 + rt)} = n \]

It is very difficult to understand this example. However, the following may help. To see what the log button on your calculator does, punch the following buttons: 10 y^x 3 =. The answer is 1000. Now punch your log button. The answer is 3. You can see that the log button tells you what exponent you used to get 1000. Whenever you want to make an exponent the subject, you will have to use your log button.

**Multi-Step Solutions**

Make a the subject of this formula.

\[ A = bc - ad \]

Put in the order of the operations.

\[ A = \frac{bc}{2} \cdot \frac{ad}{3} \]

If you start with a, you will multiply it by d ... but then you would have to do the subtraction backwards, and that is not what the original formula says. Subtractions have to be done forward, from left to right. But that means you cannot start with a right away. Instead, you have to do an extra step.

Step 1: start with the nearest letter to a that is not after the subtraction.
Using Calculators to Teach Algebra

Step 2: Now use this formula to make \( a \) the subject. Notice that the \(-\) in the original formula which made it impossible to start with \( a \) has now been reversed to a \(+\), and \(+\)'s can be done backwards. First put in the order of the operations for the new formula.

\[
\frac{2 + 1}{b} = \frac{A + ad}{b} = c
\]

<table>
<thead>
<tr>
<th>reverse</th>
<th>brackets</th>
<th>write normally</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c \times b )</td>
<td>( A )</td>
<td>( A + (ad) )</td>
</tr>
<tr>
<td>(- (ad) \div b )</td>
<td>( A )</td>
<td>( A + (ad) )</td>
</tr>
<tr>
<td>( A \div c )</td>
<td>( b )</td>
<td>( b )</td>
</tr>
</tbody>
</table>

The same thing will happen with letters after divisions, because divisions cannot be done backwards either. In future, if the letter you want to make the subject comes after a \(-\) sign, a \(/\) sign or a \(\div\) sign, or if it comes under a long division line, you cannot make it the subject in one step. Instead, choose the nearest variable which does not have this problem and make that letter the subject instead. Then use the new formula to make the original letter the subject.

If the letter you want to make the subject comes after one or more subtractions and one or more divisions, it may take three or more steps, as you will see if you make \( b \) the subject of this
Using Calculators to Teach Algebra

212

Put in the order of operations.

\[ V = \frac{d + c}{a - \frac{1}{b}} \]

Put in the order of operations.

b is after a / sign, after a - sign, and under a long division line. In other words, there are three "problems" with making it the subject. If one "problem" means two steps, then three "problems" mean four steps. The nearest letter to b that is not after a subtraction or a division is c, so Step 1 is to make c the subject first.

\[ V(a - \frac{1}{b}) - d = c \]

Put in the order of operations for this new formula.

The nearest letter to b that is not after a subtraction or a division in this new formula is a, so Step 2 is to make a the subject.

\[ \frac{c + d}{V} + \frac{1}{b} = a \]

Put in the order of operations for this new formula.
The nearest letter or, in this case, number, to \( b \) that is not after a subtraction or a division in this new formula is \( 1 \), so Step 3 is to make \( 1 \) the subject.

\[
\begin{align*}
\frac{c + d}{v} + \frac{1}{b} &= a \\
\text{reverse} & \quad \text{brackets} & \quad \text{write normally} \\
\frac{1}{b} &= a - \frac{c + d}{v} \\
+ \frac{c + d}{v} & \times b \\
a & \times b \\
\text{Put in the order of operations for this new formula.} \\
\left[ a - \frac{c + d}{v} \right] b = 1
\end{align*}
\]

Finally, the letter you were trying to make the subject, \( b \), no longer comes after a subtraction or a division. It may look as though it comes after the subtraction, but the letters in brackets will be a group if we start with \( b \), so you will not have to "do" the subtraction at all; if it is in a group, you assume that it has been done.

\[
\begin{align*}
\text{reverse} & \quad \text{brackets} & \quad \text{write normally} \\
b & \times \left[ a - \frac{c + d}{v} \right] \\
1 & \div \left[ a - \frac{c + d}{v} \right] \\
\text{none} & \quad \text{none} & \quad \frac{1}{a - \frac{c + d}{v}} = b
\end{align*}
\]

It will be noticed that this method of solving equations would not work if the variable being solved for occurred more than
once in the formula. That difficulty, however, does not arise in
the formulae I use in the rest of the course.
In March 1781, Thrale and Boswell presented Samuel Johnson with a problem: had Shakespeare or Milton drawn the more admirable picture of a man? The passages produced on either side were Hamlet's description of his father (Hamlet, III, iv, 55) and Milton's description of Adam (Paradise Lost, IV, 300).

The two passages are reproduced here.

Hamlet's Description of His Father

See, what a grace was seated on this brow; Hyperion's curls, the front of Jove himself, An eye like Mars, to threaten and command, A station like the herald Mercury New-lighted on a heaven-kissing hill, A combination and a form indeed, Where every god did seem to set his seal, To give the world assurance of a man.

Milton's Description of Adam

His fair large front and eye sublme declar'd Absolute rule; and hyacinthine locks Round from his parted forelock many hung Clustering, but not beneath his shoulders broad.
In the following outline of the courses I have designed, * designates the work attempted by more able students (those capable of achieving B) and ** designates the work attempted by the most able students (those capable of achieving A).

The terminology of this outline is explained in Appendices 3 and 4.

In order to save space, "operation" has been abbreviated "op." For example, what is written under "Solving Equations" for "Ma 10A" means that the "star" work in Mathematics 10A involves solving equations comprising four operations and groups containing one operation, with up to three steps required for the solution, but there are no powers which have to be reversed and no logarithms; e.g., solving the following equation for x:

\[
\frac{a + b}{m - x} + c = n
\]
Using Calculators to Teach Algebra

Table 6-A

Contents of Mathematics 8A-11A

<table>
<thead>
<tr>
<th></th>
<th>Calculators</th>
<th>Solving Equations</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ma 8A</strong></td>
<td>evaluating numerical expressions with 2 operations</td>
<td>2 ops.; no power reversal, groups, multi-step, logs</td>
<td>markdown tax</td>
</tr>
<tr>
<td></td>
<td>*evaluating numerical expressions with 3 operations</td>
<td>*3 ops.; no power reversal, groups, multi-step, logs</td>
<td></td>
</tr>
<tr>
<td><strong>Ma 9A</strong></td>
<td><strong>evaluating numerical expressions with ≤ 6 operations including complex fractions</strong></td>
<td>**4 ops., power reversal; no groups multi-step, logs</td>
<td>markdown tax simple interest</td>
</tr>
<tr>
<td></td>
<td>evaluating numerical expressions with ≤ 3 operations</td>
<td>3 ops., groups with 1 op.; no power reversal, multi-step, logs</td>
<td></td>
</tr>
<tr>
<td></td>
<td>*evaluating numerical expressions with ≤ 4 operations</td>
<td>*4 ops., groups with 1 op.; no power reversal, multi-step, logs</td>
<td></td>
</tr>
<tr>
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<td>**7 ops., power reversal, groups with ≥ 1 op., ≤ 4 steps; no logs</td>
<td>markdown tax simple interest compound interest</td>
</tr>
<tr>
<td></td>
<td>evaluating numerical expressions with ≤ 3 operations</td>
<td>3 ops., groups with 1 op., two steps; no power reversal, logs</td>
<td></td>
</tr>
<tr>
<td></td>
<td>*evaluating numerical expressions with ≤ 4 operations</td>
<td>*4 ops., groups with 1 op., ≤ 3 steps; no power reversal, logs</td>
<td></td>
</tr>
<tr>
<td><strong>Ma 11A</strong></td>
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<td>**15 ops., power reversal, multi-op. groups, ≤ 5 steps, logs</td>
<td>markdown tax simple interest compound interest saving with regular deposits financing with regular payments</td>
</tr>
</tbody>
</table>